

**ALL BRANCHES**

# ENGINEERING MATHEMATICS



Lecture No.-3

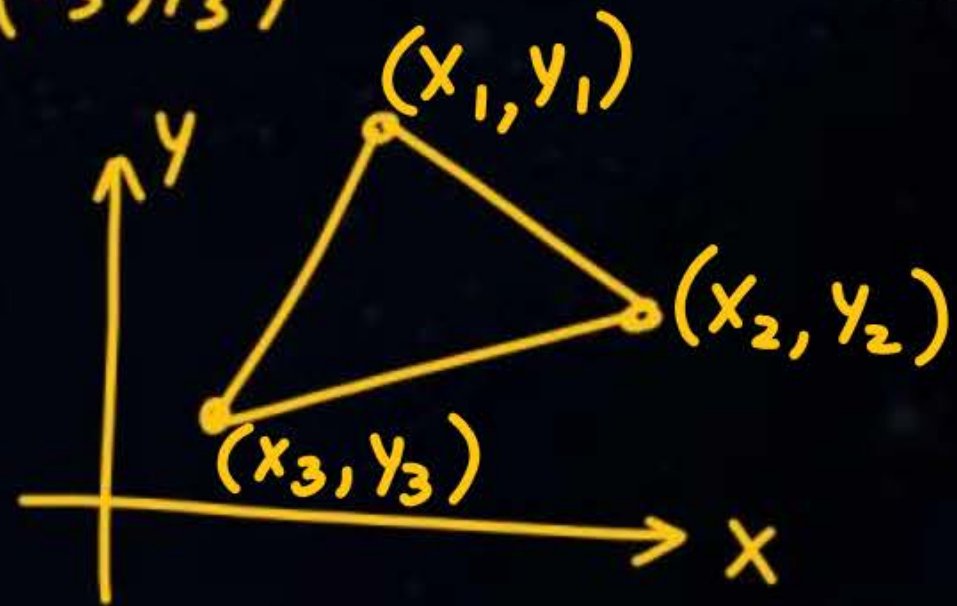
**Linear Algebra**



**By- Chetan Sir**

Let triangle have vertices  $(x_1, y_1)$   $(x_2, y_2)$   $(x_3, y_3)$

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$



Ex:- Triangle having corners  $(3, 0)$   $(4, 1)$  and  $(10, 3)$   
Find its area = ?

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 1 & 3 & 0 \\ 1 & 4 & 1 \\ 1 & 10 & 3 \end{vmatrix}$$



# [ INVERSE OF A MATRIX ]

**Definition:** Let  $A$  be a square matrix of order  $n$ . If there is a matrix  $B$  such that

$$A \cdot B = B \cdot A = I, \text{ then}$$

$B$  is called the inverse of the matrix  $A$  and denoted by  $A^{-1}$ . Thus, if  $A$  is square matrix of order  $n$ , then  $A^{-1}$  is also a square matrix of order  $n$ .

$$A^{-1} = \frac{\text{adj}A}{|A|}$$

$A^{-1}$  exist if  $|A| \neq 0$

$$A \cdot (\text{Adj } A) = |A| I$$

$$A^{-1} A (\text{Adj } A) = |A| A^{-1} I \quad (\text{Pre multiply by } A^{-1})$$

$$I (\text{Adj } A) = |A| A^{-1}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

Properties of inverse & Adjoint of Matrix:-

1.  $(AB)^{-1} = B^{-1} A^{-1}$

$$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$$

$$(ABCD)^{-1} = D^{-1} C^{-1} B^{-1} A^{-1}$$

2.  $A (\text{Adj } A) = (\text{Adj } A) \cdot A = |A| I_n$



$$3. \text{Adj}(\text{Adj} A) = |A|^{n-2} A$$

$$4. |A^{-1}| = \frac{1}{|A|}$$

$$5. |\text{Adj} A| = |A|^{n-1}$$

$$6. |\text{Adj}(\text{Adj} A)| = |A|^{(n-1)^2}$$

$$7. |\text{Adj}(\text{Adj}(\text{Adj} A))| = |A|^{(n-1)^3}$$

$n \rightarrow$  Order of square matrix

NOTE:- Inverse of diagonal matrix is reciprocal of diagonals elements.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

Ex:-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{3 \times 3}$$

$n \rightarrow 3$

$$\text{Cofactor matrix} = \begin{bmatrix} -1 & 0 & -1 \\ +1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$$

$$i) \text{Adj} A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$ii) A^{-1} = \frac{\text{Adj} A}{|A|} = \frac{1}{-2} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$|A| = 1(0-1) - 0 + 1(-1-0) = -1 - 0 - 1 = -2$$



$$\text{iii) } |A^{-1}| = 1/|A| = 1/-2 = -\frac{1}{2}$$

$$\text{iv) } |\text{Adj } A| = |A|^{n-1} = (-2)^{3-1} = (-2)^2 = 4$$

$$\text{v) } |\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2} = (-2)^{(3-1)^2} = (-2)^4 = 16$$

$$\text{vi) } |A^2| = |A|^2 = (-2)^2 = 4$$

$$\text{vii) } |A^3| = |A|^3 = (-2)^3 = -8$$

Shortcut for 2x2 matrix:-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

SARRUS RULE for  $\Delta$ :-

$$\begin{vmatrix} 1 & 1 & 0 \\ 5 & 0 & 4 \\ 2 & 3 & 6 \end{vmatrix}$$

$$\Delta = (0 + 8 + 0) - (0 + 12 + 30)$$

$$8 - 42$$

$$\Delta = -34$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{3 \times 3} \quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\text{Cofactor matrix} = \begin{bmatrix} -1 & 0 & -1 \\ +1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$$

$$\text{i) } \text{Adj } A = \begin{bmatrix} -1 & +1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$|A| = 1(0-1) + 0 + 1(-1-0)$$

$$|A| = -1 + 0 - 1 = -2$$

$$\text{ii) } A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{2} \begin{bmatrix} -1 & +1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\text{iii) } |A^{-1}| = \frac{1}{|A|} = \frac{1}{-2}$$

$$\text{iv) } |\text{Adj } A| = |A|^{n-1} = (-2)^{3-1} = (-2)^2 = 4$$

$$\text{v) } |\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2} = (-2)^{(3-1)^2} = (-2)^4 = 16$$

$$\text{vi) } |A^2| = |A|^2 = (-2)^2 = 4$$

$$\text{vii) } |A^3| = |A|^3 = (-2)^3 = -8$$



$$A = \begin{bmatrix} 1 & 6 \\ -2 & 5 \end{bmatrix} \quad \begin{vmatrix} + & - \\ - & + \end{vmatrix} \quad \text{Cof. matrix} = \begin{bmatrix} 5 & 2 \\ -6 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{17} \begin{bmatrix} 5 & -6 \\ 2 & 1 \end{bmatrix}$$


$$A^{-1} = \frac{1}{17} \begin{bmatrix} 5 & -6 \\ +2 & 1 \end{bmatrix}$$



$N \rightarrow 1, 2, 3, \dots$

$W \rightarrow 0, 1, 2, 3, \dots$

$Z \text{ or } I \rightarrow -3, -2, -1, 0, 1, 2, 3, 4, \dots$

$R \rightarrow$  

Complex  $\rightarrow$  Real + Imaginary



Root of -ve number



$$\sqrt{-25}$$

$$\sqrt{-1} \sqrt{25} = 5i$$

## [CONJUGATE OF A MATRIX]

Let  $A$  be any  $m \times n$  matrix having complex numbers as its elements. The matrix of order  $m \times n$  which is obtained from  $A$  by replacing each element of  $A$  by its conjugate is called the conjugate of  $A$  denoted by  $\bar{A}$ . Thus if

$$A = [a_{ij}]_{m \times n}, \bar{A} = [\bar{a}_{ij}]_{m \times n} \text{ where } \bar{a}_{ij} \text{ is the conjugate of } a_{ij}.$$



# [CONJUGATE OF A MATRIX]

**NOTE:** If  $A$  real matrix, then  $\bar{A} = A$

e.g. If  $A = \begin{bmatrix} 1+2i & i \\ 3 & 2-3i \end{bmatrix}$ , then  $\bar{A} = \begin{bmatrix} 1-2i & -i \\ 3 & 2+3i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1-2i & -i \\ 3 & 2+3i \end{bmatrix}$$

(Real matrix)  $A = \begin{bmatrix} 5 & 0 \\ -1 & -2 \end{bmatrix} \rightarrow \bar{A} = \begin{bmatrix} 5 & 0 \\ -1 & -2 \end{bmatrix}$

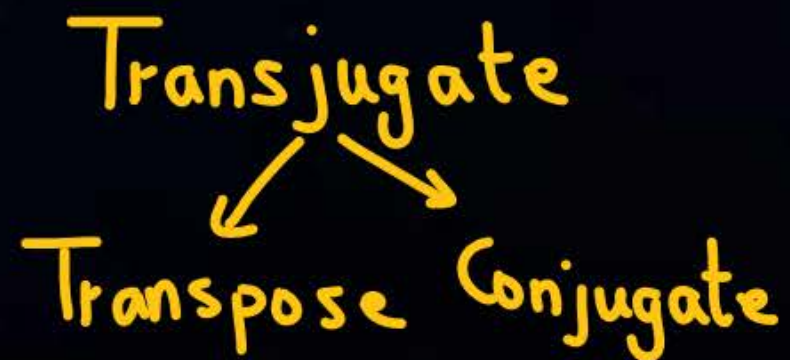
# [CONJUGATE TRANSPOSE OF A MATRIX]

The conjugate of the transpose of a matrix  $A$  is called the conjugate transpose of  $A$  and denoted by  $A^*$ . Thus if  $A = [a_{ij}]$ , then

$$A^* = (\bar{A}') = (\bar{A})' = [\bar{a}_{ji}]$$

$$A^* = (\bar{A})' = \overline{(A')} = A^\theta$$

Clearly the conjugate of the transpose is the same as the transpose of the conjugate.

Transjugate  
  
 Transpose Conjugate

$$(A + B)^* = A^* + B^*$$

$$(AB)^* = B^* A^*$$

$$(A^*)^* = A$$



# [ CONJUGATE TRANSPOSE OF A MATRIX ]

NOTE: (i)  $(A^*)^* = A$

(ii) If  $A$  is real matrix, then  $A^* = A'$

e.g. If  $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$  then  $A' = \begin{bmatrix} 1+2i & 4-5i & 8 \\ 2-3i & 5+6i & 7-8i \\ 3+4i & 6-7i & 7 \end{bmatrix};$

$$A^* = (\bar{A}') = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$$

# Topics to be Covered

A diagram consisting of a vertical line with arrowheads at both ends, and a horizontal arrow pointing to the right, originating from the middle of the vertical line.

**SPECIAL TYPES OF MATRICES**



# [SPECIAL TYPES OF MATRICES]

## IDEMPOTENT MATRIX

A square matrix  $A$  is said to be idempotent if  $A^2 = A$ .

E.g:  $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

- If  $A \rightarrow$  idempotent, then  $(I - A)$  is idempotent.
- If  $A$  and  $B$  are idempotent, then  $(A + B)$  is idempotent when  $AB = O$

$$A.A = A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

# [ INVOLUTORY MATRIX ]

A square matrix  $A$  is said to be involutory matrix if  $A^2 = I$  (unit matrix).

$$A = A^{-1}$$

E.g:  $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$



# [NILPOTENT MATRIX]

A square matrix  $A$  such that  $A^k = 0$  where  $k$  is the least positive integer, is called the nilpotent matrix of index  $k$ .

E.g:  $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

Ex:-  $A \neq 0$

$A^2 \neq 0$

$A^3 \neq 0$

$A^4 = 0$

then  $A$  is nilpotent of index 4.

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$A$  is nilpotent matrix of index 2.

# Square matrices

## Real Matrices

→ Symmetric matrix

$$A = A^T$$

→ Skew-symmetric matrix

$$A = -A^T$$

→ Orthogonal matrix

$$AA^T = I$$

## Complex Matrices

→ Hermitian matrix

$$A = A^*$$

→ Skew-Hermitian matrix

$$A = -A^*$$

→ Unitary matrix

$$AA^* = I$$



# [SYMMETRIC MATRIX]

$$A = A^T$$

$$A \rightarrow a_{ij}$$

$$A^T \rightarrow a_{ji}$$

A square matrix  $A = [a_{ij}]_{n \times n}$  is said to be symmetric if  $A = A'$

i.e.  $a_{ij} = a_{ji}$  for all values of  $i$  and  $j$

e.g. If  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix}$ , then

$$A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix} = A, \quad \begin{matrix} 3 \times 3 \\ n=3 \end{matrix}$$

- Symmetric matrix is symmetrical about diagonal.

Let  $A_{n \times n}$  be any matrix,

$\rightarrow A + A^T$  is always symmetric

$\rightarrow AA^T$  is always symmetric

- Maximum number of unique elements

$$= n + \frac{n^2 - n}{2} = \frac{n(n+1)}{2}$$

$$2 \times 2 \rightarrow 3 \quad 3 \times 3 \rightarrow 6 \quad 4 \times 4 \rightarrow 10$$

# [SKEW-SYMMETRIC MATRIX]

$$A = a_{ij} \quad A^T = a_{ji}$$

$$A = -A^T$$

A square matrix  $A = [a_{ij}]_{n \times n}$  is said to be skew-symmetric if  $A = -A'$

i.e.  $a_{ij} = -a_{ji} \forall i \text{ \& } j$ , e.g. If  $A = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & 6 \\ -4 & -6 & 0 \end{bmatrix}$ , then

$$A' = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & -6 \\ 4 & 6 & 0 \end{bmatrix} = -A.$$

- Sum of all elements = 0
- Diagonal elements are always 0.
- $\Delta$  of odd order skew-symmetric matrix is 0.
- $A_{n \times n}$  is any matrix,  $A - A^T$  is always skew-symmetric



# **[SKEW-SYMMETRIC MATRIX]**

## NOTE:

1. If  $A_{n \times n}$  such that  $A$  is skew symmetric and  $n$  is odd then  $|A| = 0$  (always).
2. Every square matrix can be uniquely expressed as the sum of the symmetrical and skew-symmetrical matrices i.e.  $A = \left(\frac{A+A'}{2}\right) + \left(\frac{A-A'}{2}\right) = P + Q$  where  $P$  is symmetric and  $Q$  is skew symmetric.
3. Symmetric matrix is symmetrical about leading diagonal.

# **(SKEW-SYMMETRIC MATRIX)**

E.g: Show that every diagonal element of a skew-symmetric matrix is necessarily zero.

Solution: Let A be any skew-symmetric matrix i.e.

$$a_{ij} = -a_{ji} \quad \forall \text{ i and j} \quad \dots(i)$$

For diagonal elements of a matrix, we can put  $i = j$  in (i)

i.e., 
$$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$$

$\Rightarrow$  every element in the principal diagonal is necessarily zero.



## **[SKEW-SYMMETRIC MATRIX]**

E.g: Write the following matrix as the sum of a symmetric and a skew-symmetric matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$$

If  $A$  is any square matrix, then symmetric and skew-symmetric matrices are  $\frac{1}{2}(A + A')$  and  $\frac{1}{2}(A - A')$  respectively and  $A$  can be written as

# [SKEW-SYMMETRIC MATRIX]

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$

$$\Rightarrow A + A' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 6 \\ 10 & 6 & 0 \end{bmatrix} \dots(i)$$



## **(SKEW-SYMMETRIC MATRIX)**

$$\text{and } A - A' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix} \quad \dots(\text{ii})$$

Adding (i) and (ii), we get

$$2A = (A + A') + (A - A') = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 6 \\ 10 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}$$

# [SKEW-SYMMETRIC MATRIX]

$$\Rightarrow A = \frac{1}{2} \overbrace{\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 6 \\ 10 & 6 & 0 \end{bmatrix}}^{A+A^T} + \frac{1}{2} \overbrace{\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}}^{A-A^T} = \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 0 \end{bmatrix}}_{\text{Symmetric}} + \underbrace{\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}}_{\text{Skew-symmetric}}$$

$A = \text{symmetric matrix} + \text{skew - symmetric matrix.}$



# [ORTHOGONAL MATRIX]

$$A A^T = I$$

$$A^T = A^{-1}$$

A square matrix  $A$  is called an orthogonal matrix if the product of matrix  $A$  with its transpose matrix  $A'$  is an identity matrix, i.e.  $AA' = I$

E.g: Show that the matrix  $A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$  is orthogonal.

$$A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \Rightarrow A' = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$R_1 = \left[ \frac{1}{3} \quad \frac{2}{3} \quad \frac{2}{3} \right]$$

$$R_2 = \left[ \frac{2}{3} \quad \frac{1}{3} \quad -\frac{2}{3} \right]$$

$$|R_1| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$$

# [ORTHOGONAL MATRIX]

$$\therefore AA' = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence A is an orthogonal matrix.

Properties:-

$$\begin{aligned} \rightarrow R_1 \cdot R_2^T &= 0 & \rightarrow C_1 \cdot C_2^T &= 0 \\ \rightarrow R_2 \cdot R_3^T &= 0 & \rightarrow C_2 \cdot C_3^T &= 0 \\ \rightarrow R_1 \cdot R_3^T &= 0 & \rightarrow C_1 \cdot C_3^T &= 0 \\ \rightarrow |R_1| &= |R_2| = |R_3| = |C_1| = |C_2| = |C_3| = 1 \end{aligned}$$

If  $X_1$  and  $X_2$  are 2 vectors;  
then they are orthogonal if

$$X_1 \cdot X_2^T = 0$$

$$R_1 \cdot R_2^T \\ \left[ \frac{1}{3} \quad \frac{2}{3} \quad \frac{2}{3} \right] \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$



# [ORTHOGONAL MATRIX]

## NOTE:

1. The value of determinant of an orthogonal matrix is either 1 or -1.

Proof : Let A be any orthogonal matrix i.e.

$$A'A = I \Rightarrow |A'A| = |I| \Rightarrow |A'| \cdot |A| = 1 \quad (\because |I| = I)$$

$$\Rightarrow |A| \cdot |A| = 1 \quad (\because |A'| = A)$$

$$\Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$$

2. If A and B are orthogonal matrices then AB and BA are also orthogonal.

# [HERMITIAN MATRIX]

$$A \rightarrow a_{ij} \quad \overline{A^T} = \overline{a_{ji}}$$

A square matrix  $X$  is called Hermitian if  $A = A^*$ . Thus, a square matrix  $A = [a_{ij}]$  is Hermitian if  $a_{ij} = \overline{a_{ji}} \forall i \text{ and } j$

$$A = (\overline{A^T})$$

E.g: The matrix  $A = \begin{bmatrix} 1 & 2+i & 3+2i \\ 2-i & 3 & -3i \\ 3-2i & 3i & -2 \end{bmatrix}$  is Hermitian.

$$a_{12} = \overline{a_{21}} \quad a_{13} = \overline{a_{31}} \quad a_{23} = \overline{a_{32}}$$

→  $A + A^*$  is always Hermitian matrix

→ Diagonal elements are always real.



# [SKEW-HERMITIAN MATRIX]

A square matrix  $A$  is said to be skew-Hermitian if  $A^* = -A$ . Thus, a square matrix  $A = [a_{ij}]$  is skew-Hermitian if  $a_{ij} = -\bar{a}_{ji} \forall i$  and  $j$

E.g: The matrix  $A = \begin{bmatrix} i & 2+i & 3+2i \\ -2+i & 3i & -3i \\ -3+2i & -3i & 0 \end{bmatrix}$  is skew-Hermitian.

$$a_{12} = -\bar{a}_{21} \quad a_{13} = -\bar{a}_{31} \quad a_{23} = -\bar{a}_{32}$$

→  $A - A^*$  is always skew-hermitian matrix.

→ Diagonal elements are always 0 or imaginary.

# **(SKEW-HERMITIAN MATRIX)**

For  $A' = \begin{bmatrix} i & -2+i & -3+2i \\ 2+i & 3i & -3i \\ 3+2i & -3i & 0 \end{bmatrix}$  and

$$A^* = \bar{A}' = \begin{bmatrix} -i & -2-i & -3-2i \\ 2-i & -3i & 3i \\ 3-2i & 3i & 0 \end{bmatrix} =$$

$$-\begin{bmatrix} i & 2+i & 3+2i \\ -2+i & 3i & -3i \\ -3+2i & -3i & 0 \end{bmatrix} = -A$$



# **[SKEW-HERMITIAN MATRIX]**

## NOTE:

1. If A and B are two matrices such that these are conformable for addition then  $\overline{(A + B)} = \bar{A} + \bar{B}$
2. If A and B are two matrices conformable for multiplication, then  $(AB) = \bar{A} \cdot \bar{B}$
3. If A and B are any two matrices conformable for addition, then  $(A + B)^* = A^* + B^*$
4. If A and B are any two matrices conformable for multiplication, then  $(AB)^* = B^* A^*$
5. The diagonal elements of a Hermitian matrix are necessarily real.

## **[SKEW-HERMITIAN MATRIX]**

6. The diagonal elements of a skew-Hermitian matrix are either purely imaginary or zero.
7. Every square matrix can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix.  $A = \left(\frac{A+A^*}{2}\right) + \left(\frac{A-A^*}{2}\right) = P + Q$  where  $P$  &  $Q$  are Hermitian and skew-Hermitian matrices respectively.
8. Every square matrix can be uniquely expressed as  $P + iQ$  where  $P$  and  $Q$  are Hermitian.

$$A = \left(\frac{A + A^*}{2}\right) + i \left(\frac{A - A^*}{2i}\right) = P + iQ$$



## **[SKEW-HERMITIAN MATRIX]**

9. Every Hermitian matrix  $A$  can be uniquely expressed as  $P + iQ$  form, where  $P$  and  $Q$  are real symmetric and real skew-symmetric. Let  $A$  be a Hermitian matrix i.e.  $A^* = A$ . Now we can write

$$A = \frac{1}{2}(A + \bar{A}) + i \left[ \frac{1}{2i}(A - \bar{A}) \right]$$

where  $P = \frac{1}{2}(A + \bar{A})$  and  $Q = \frac{1}{2i}(A - \bar{A})$

Here  $P$  and  $Q$  are real symmetric and real skew-symmetric matrices respectively.

# [UNITARY MATRIX]

A square matrix  $A$  is said to be unitary matrix if  $A^* A = I = AA^*$

$$A A^* = I$$

NOTE:

If the matrix  $A$  is real then  $\bar{A} = A$ ,  $A^* = A'$ , so we can write  $A'A = I = AA'$ . Thus a unitary matrix on a field of real numbers is also an orthogonal matrix.

NOTE :-  $|A A^*| = |I|$

$$|A| |\bar{A}'| = 1$$

$$|A| |\bar{A}| = 1$$

$$|A| |A| = 1$$

$$|A| = 1$$

$$|A| = |A^T|$$

$$|A| = |\bar{A}|$$



# [UNITARY MATRIX]

E.g: Show that the matrix  $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$  is unitary.

Solution : If  $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$  then  $\bar{A} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

So that  $A^* = (\bar{A})' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

# [UNITARY MATRIX]

So that  $A^* = (\bar{A})' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}$$

$$\therefore A^* A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence A is unitary matrix.



# [UNITARY MATRIX]

**Theorem:** The modulus of the determinant of a unitary matrix is unity.

**Proof:** Let  $A$  is unitary matrix i.e.

$$A^* A = I \Rightarrow |A^* A| = |I| \quad \text{[taking determinant on both sides]}$$

$$\Rightarrow |A^*| |A| = 1 \Rightarrow |(\bar{A})'| |A| = 1 \quad [\because |\bar{A}'| = |\bar{A}|]$$

$$\Rightarrow |(\bar{A})| |A| = 1 \Rightarrow |A|^2 = 1 \quad [\because |\bar{A}| = |A|]$$

# [UNITARY MATRIX]

$$\Rightarrow |A| = 1$$

$\Rightarrow$  The modulus of the determinant of unitary matrix is unity.

E.g: Express  $\begin{bmatrix} -2+3i & 1-i & 2+1 \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$  as the sum of a Hermitian and a Skew-Hermitian matrix.



# [UNITARY MATRIX]

If  $A$  is any square matrix, then we can write

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$

Where  $\frac{1}{2}(A + A^*)$  is a Hermitian matrix and  $\frac{1}{2}(A - A^*)$  is a Skew-Hermitian matrix.

# [UNITARY MATRIX]

Let  $A = \begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$

Then  $\bar{A} = \begin{bmatrix} -2-3i & 1+i & 2-i \\ 3 & 4+5i & 5 \\ 1 & 1-i & -2-2i \end{bmatrix}$

$\therefore (\bar{A})' = A^* = \begin{bmatrix} -2-3i & 3 & 1 \\ 1+i & 4+5i & 1-i \\ 2-i & 5 & -2-2i \end{bmatrix}$



# [UNITARY MATRIX]

$$\text{Now } \frac{1}{2}(A + A^*) = \frac{1}{2} \begin{bmatrix} -4 & 4 - i & 3 + i \\ 4 + i & 8 & 6 - i \\ 3 - i & 6 + i & -4 \end{bmatrix} = \begin{bmatrix} -2 & 2 - \frac{1}{2}i & \frac{3}{2} + \frac{1}{2}i \\ 2 + \frac{1}{2}i & 4 & 3 - \frac{1}{2}i \\ \frac{3}{2} - \frac{1}{2}i & 3 + \frac{1}{2}i & -2 \end{bmatrix}$$

which is Hermitian matrix.

$$\text{Again } \frac{1}{2}(A - A^*) = \frac{1}{2} \begin{bmatrix} 6i & -2 - i & 1 + i \\ 2 - i & -10i & 4 + i \\ -1 + i & -4 + i & 4i \end{bmatrix} = \begin{bmatrix} 3i & -1 - \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \\ 1 - \frac{1}{2}i & -5i & 2 + \frac{1}{2}i \\ -\frac{1}{2} + \frac{1}{2}i & -2 + \frac{1}{2}i & 2i \end{bmatrix}$$

# [UNITARY MATRIX]

which is Skew-Hermitian matrix.

$$\text{Thus } A = \begin{bmatrix} -2 & 2 - \frac{1}{2}i & \frac{3}{2} + \frac{1}{2}i \\ 2 + \frac{1}{2}i & 4 & 3 - \frac{1}{2}i \\ \frac{3}{2} - \frac{1}{2}i & 3 + \frac{1}{2}i & -2 \end{bmatrix} + \begin{bmatrix} 3i & -1 - \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \\ 1 - \frac{1}{2}i & -5i & 2 + \frac{1}{2}i \\ -\frac{1}{2} + \frac{1}{2}i & -2 + \frac{1}{2}i & 2i \end{bmatrix}$$

where the first matrix on the R.H.S. is Hermitian and the second matrix is skew-Hermitian.



Thank you

**GW**  
*Soldiers !*

