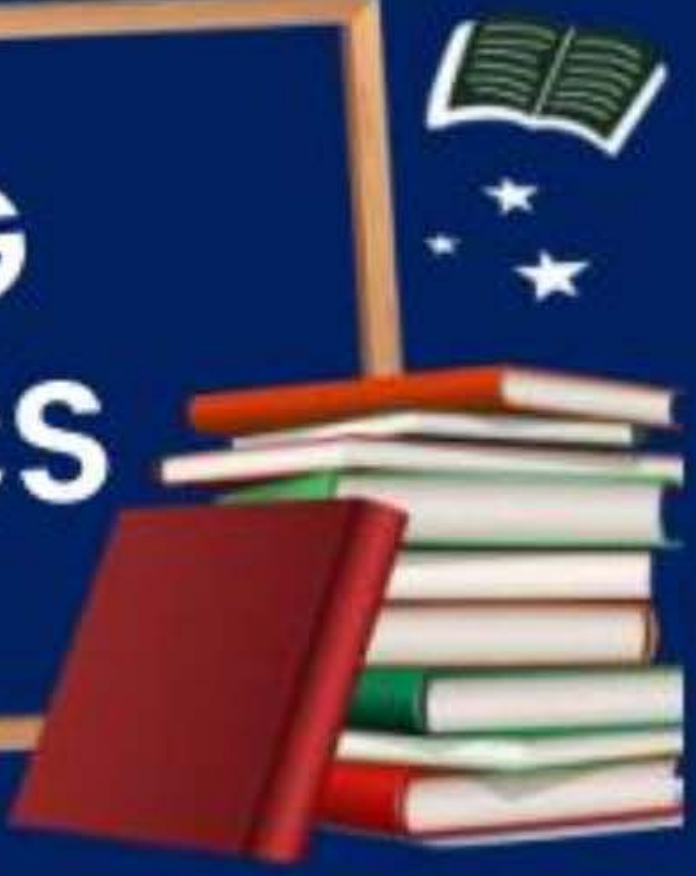
ALL BRANCHES

ENGINEERING MATHEMATICS



Lecture No.-2

Determinant & Its Properties

By- Chetan Sir

Topics to be Covered

- PROPERTIES OF DETERMINANTS
- ADJOINT OF MATRIX
- INVERSE OF A MATRIX
- CONJUGATE OF A MATRIX
- CONJUGATE TRANSPOSE OF A MATRIX

Minimum number of multiplications :-

$$A_{3\times 5} \qquad B_{5\times 2} \qquad C_{2\times 3}$$

$(AB)C \Rightarrow (AB)_{3\times 2}\, C_{2\times 3} \Rightarrow 3\times 5\times 2 + 3\times 2\times 3$

$30 + 18 \Rightarrow \boxed{48}$

$(BC)A \Rightarrow (BC)_{5\times 3}\, A_{3\times 5} \Rightarrow 5\times 2\times 3 + 5\times 3\times 5$

$30 + 75 \Rightarrow 105$

$(CA)B \Rightarrow (CA)_{2\times 5}\, B_{5\times 2} \Rightarrow 2\times 3\times 5 + 2\times 5\times 2$

$30 + 20 \Rightarrow 50$

Minimum no. of multiplications

$(AB)C = 48$

MINORS OF MATRIX

The determinant value of the square matrix obtained from the original matrix of any order by the omission of the rows and columns is called a minor of a matrix. For example

3×3 minor $\Rightarrow \Delta$

2×2 minors $\Rightarrow M_{11}, M_{12} \ldots$

1×1 minors $\Rightarrow a_{ij}$

If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 7 & 1 \\ 8 & 2 & 7 & 3 \end{bmatrix}_{3 \times 4}$ is a matrix of order 3×4.

Then minors of A are $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 7 \\ 8 & 2 & 7 \end{vmatrix}$; $\begin{vmatrix} 0 & 3 \\ 8 & 2 \end{vmatrix}$ & $\begin{vmatrix} 7 & 1 \\ 7 & 3 \end{vmatrix}$ etc.

$3\times3 \rightarrow [C_1 \ C_2 \ C_3] \quad [C_1 \ C_2 \ C_4] \quad [C_2 \ C_3 \ C_4] \quad [C_1 \ C_3 \ C_4]$

Total no. of 3×3 matrices $= {}^3C_3 \cdot {}^4C_3 = 1\times4 = 4$

Total no. of 2×2 matrices $= {}^3C_2 \cdot {}^4C_2 = 3\times6 = 18$

Total no. of 1×1 matrices $= {}^3C_1 \cdot {}^4C_1 = 3\times4 = 12$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}}$$

Ex:-

$$\begin{vmatrix} 1 & 0 & 3 \\ 7 & 1 & 0 \\ 5 & -1 & 2 \end{vmatrix}_{3\times3} \Rightarrow$$

$M_{11} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$ $\qquad M_{21} = \begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix}$

$M_{12} = \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14$ $\qquad M_{22}$ $\qquad M_{32} =$

$M_{13} = \begin{vmatrix} 7 & 1 \\ 5 & -1 \end{vmatrix} = -12$ $\qquad M_{23}$

$A_{32} = ?$

[COFACTORS OF A MATRIX]

The cofactors of the element a_{ij} is defined as

$$A_{ij} = (-1)^{i+j}|M_{ij}|$$

where $|M_{ij}|$ is the determinant obtained by deleting the i^{th} row j^{th} column from given matrix.

$$\begin{vmatrix} 1,1 & 1,2 \\ 2,1 & 2,2 \end{vmatrix} \Rightarrow \begin{vmatrix} + & - \\ - & + \end{vmatrix}$$

$$\begin{vmatrix} 1,1 & 1,2 & 1,3 \\ 2,1 & 2,2 & 2,3 \\ 3,1 & 3,2 & 3,3 \end{vmatrix} \Rightarrow \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\Delta = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

$$= 1 \times 2 + 0 \times -14 + 3 \times -12$$

$$\boxed{\Delta = -34}$$

$$A_{11} = (-1)^{1+1} |M_{11}| = +(2) = 2$$

$$A_{12} = (-1)^{1+2} |M_{12}| = -(14) = -14$$

$$A_{13} = (-1)^{1+3} |M_{13}| = +(-12) = -12$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 7 & 0 \end{vmatrix} = -(-21) = 21$$

[PROPERTIES OF DETERMINANTS]

1. $|A'| = |A|$ $\quad\quad |A^T| = |A|$
2. If we interchange any two rows or columns then sign of determinants change.
3. If any 2 rows or any 2 columns in a determinant are identical (or proportional) then the value of the determinant is zero.
4. Multiplying a determinant by K means multiplying the elements of only one row (or one column) by K.

e.g. $$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

then $$2|A| = 2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = 8 - 12 = -4 = 2(-2)$$

PROPERTIES OF DETERMINANTS

5. If elements of a row in a determinant can be expressed as the sum of two or more elements then the given determinant can be expressed as the sum of 2 or more determinants.

e.g. $$|A| = \begin{vmatrix} a+c & b+d \\ e & f \end{vmatrix} = \begin{vmatrix} a & b \\ e & f \end{vmatrix} + \begin{vmatrix} c & d \\ e & f \end{vmatrix}$$

*

6. If we apply the operations like $R_i \to R_i + KR_j$ or $C_i \to C_i + KC_j$, the value of the determinant remains unchanged.

$\boxed{\begin{array}{ccc} R_1 & \to & R_1 + 4R_2 \\ \Delta & & \Delta \end{array}}$ $\boxed{\begin{array}{ccc} C_2 & \to & C_2 - 2C_1 \\ \Delta & & \Delta \end{array}}$ $\boxed{\begin{array}{ccc} R_2 & \to & R_2 - R_3 + 3R_1 \\ \Delta & & \Delta \end{array}}$ $\boxed{\begin{array}{ccc} R_1 & \to & 5R_1 - R_3 \\ \Delta & & 5\Delta \end{array}}$ $\boxed{\begin{array}{ccc} C_2 & \to & 4C_2 - 6C_1 \\ \Delta & & 4\Delta \end{array}}$

PROPERTIES OF DETERMINANTS

7. Thus, the determinant: is the sum of products of elements of any row and their corresponding cofactors. i.e.,

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}$$

where i represents rows. This expansion is called the expansion by cofactors.

Similarly, we can, expand the determinant as the sum of the product of elements of any column and the corresponding cofactors of the elements of the same column. That is,

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

PROPERTIES OF DETERMINANTS

8. This is expansion has a very important property that sum of products of elements of the i^{th} row and the corresponding cofactors of the elements of the i_1^{th} row is zero, that is, $a_{i1}A_{i_1 1} + a_{i2}A_{i_1 2} + a_{\ 3}A_{i_1 3} = 0$, in the case of the determinant of general 3×3 matrix where $i \neq i_1$. Thus, if we choose $i = 1, i_1 = 3$

$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$$

$= a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 0$

PROPERTIES OF DETERMINANTS

$$a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}(a_{12}a_{23} - a_{13}a_{22}) - a_{12}(a_{11}a_{23} - a_{13}a_{21}) + a_{13}(a_{11}a_{22} - a_{12}a_{21})$$

$$= a_{11}a_{12}a_{23} - a_{11}a_{13}a_{22} - a_{12}a_{11}a_{23} + a_{12}a_{13}a_{21} + a_{13}(a_{11}a_{22}) - a_{13}a_{12}a_{21} = 0$$

The same result holds good for the column expansion. That is $a_{1j}A_{1j_1} + a_{2j}A_{2j_1} + a_{3j}A_{3j_1} = 0$ where $j \neq j_1$

9. $|ABCD| = |A||B||C||D|$

$|A^n| = |A|.|A|.....n\text{ times} = |A|^n$

DETERMINANT OF A MATRIX

→ Determinant is defined only for square matrix.

→ Δ = Sum of products of any row/column elements × Corresponding cofactors

0 = Sum of products of any row/column elements × Non-corresponding cofactors

(2) $\begin{vmatrix} 1 & 0 & 5 \\ -1 & 2 & 3 \\ 3 & 6 & 7 \end{vmatrix} \qquad \begin{vmatrix} 3 & 6 & 7 \\ -1 & 2 & 3 \\ 1 & 0 & 5 \end{vmatrix} \rightarrow \begin{vmatrix} 6 & 3 & 7 \\ 2 & -1 & 3 \\ 0 & 1 & 5 \end{vmatrix}$

$1(14-18)+5(-6-6)$

$-4-60$

$\Delta=-64$

$\Delta = 64$

$\Delta = -64$

③ $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 5 & 4 \end{vmatrix}$ (Identical)

$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 0 & -6 \end{vmatrix}$ (Proportional)

$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$

$$\boxed{\Delta = 0}$$

④ a) $\begin{vmatrix} 5 & 0 & 10 \\ 4 & 8 & -4 \\ 3 & 6 & 9 \end{vmatrix} = 5 \times 4 \times 3 \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix}$

b) $\begin{vmatrix} 5 & 0 & 0 \\ 10 & -5 & -10 \\ 15 & 20 & 10 \end{vmatrix} = 5^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & 4 & 2 \end{vmatrix} = 5^3(-2-(-8)) = 5^3 \times 6$

c) $[A]_{3\times3} \xrightarrow[\text{by } 4]{\text{Multiply}} \begin{bmatrix} 4a_{11} & 4a_{12} & 4a_{13} \\ 4a_{21} & 4a_{22} & 4a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{bmatrix} = 4^3 \begin{vmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \end{vmatrix} = 4^3 \times 5 = 320$

$\downarrow$

$\Delta = 5$

$$*\Rightarrow \quad [A]_{n\times n} \xrightarrow[\text{by } K]{\text{Multiply}} [KA]_{n\times n} \qquad \left\{ |KA| \rightarrow K^n |A| \right\}$$

$$(\Delta) \longrightarrow K^n \Delta \qquad A_{n\times n}$$

(5)

$$\begin{vmatrix} 1 & 1+x & 1+x^2 \\ 1 & 1+y & 1+y^2 \\ 1 & 1+z & 1+z^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1+x^2 \\ 1 & 1 & 1+y^2 \\ 1 & 1 & 1+z^2 \end{vmatrix} + \begin{vmatrix} 1 & x & 1+x^2 \\ 1 & y & 1+y^2 \\ 1 & z & 1+z^2 \end{vmatrix}$$

$$= 0 + \begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix} + \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)$$

⑥ $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$

$$(y-x)(z-x) \begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$$

$$(y-x)(z-x)(z-y)$$

$$\Rightarrow (x-y)(y-z)(z-x)$$

$\begin{cases} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 - 2R_1 \end{cases}$

- This should not be done otherwise Δ will change.
- Never row & column operations are applied simultaneously.

$$\begin{vmatrix} \boxed{1} & 2 & 3 \\ 2 & 5 & 0 \\ 3 & 2 & 1 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{vmatrix} 1 & 2 & 3 \\ 0 & \boxed{1} & -6 \\ 0 & -4 & -8 \end{vmatrix} \xrightarrow{R_3 \to R_3 + 4R_2} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & -32 \end{vmatrix}$$

$\Delta = -32$

NOTE :- For diagonal, scalar, identity, UTM, LTM :-

Δ = Product of diagonal elements

Ex :-

$$A = \begin{vmatrix} 0 & 0 & 0 & -5 \\ 1 & 0 & 2 & -3 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 3 \end{vmatrix} \quad \begin{vmatrix} 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 3 \end{vmatrix} \quad \begin{vmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

$\Delta = -15$ $\qquad$ $\Delta = 15$ $\qquad$ $\Delta = 1 \times 1 \times 3 \times -5 = -15$

(9) $|A^3| = |A|^3 = (-15)^3 = -3375$

ADJOINT OF MATRIX

Let $A = [a_{ij}]_{n\times n}$ be a square matrix and A_{ij} is the cofactor of a_{ij}. Then the transpose of the matrix B of cofactors of the elements of A is known as the adjoint of A and is denoted by Adj A i.e. Adj A = B′ where B = A_{ij} = cofactor matrix of A.

Thus, if $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Cofactor matrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

[ADJOINT OF MATRIX]

Then $$adj\, A = transpose\ of \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\text{cofactor matrix}} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where A_{ij} denotes the cofactors of the corresponding element a_{ij}.

$$\boxed{\text{Adj } A = [\text{Cofactor matrix}]^T}$$

RESULTS ON ADJOINT OF A MATRIX

If $A = [a_{ij}]_{n \times n}$ be a square matrix of order n, then

$A.(AdjA) = |A|.I_n = (adjA).A.$

where I_n is the identity matrix of the same order as A.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{3} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} = \Delta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$$

[INVERSE OF A MATRIX]

Definition: Let a be a square matrix of order n. If there is a matrix B such that

$$A.\mathrm{B} = \mathrm{B.A.} = \mathrm{I}, \text{ then}$$

B is called the inverse of the matrix A and denotes by A^{-1}. Thus, if A is square matrix of order n, then A^{-1} is also a square matrix of order n.

$$\boxed{A^{-1} = \frac{adjA}{|A|}}$$

$$A(\text{adj}\,A) = |A|\,I_n$$

Multiply by A^{-1};

$$A^{-1}A(\text{adj}\,A) = |A|\,A^{-1}I_n$$

$$I_n(\text{adj}\,A) = |A|\,A^{-1}$$

$$(\text{adj}\,A) = |A|\,A^{-1}$$

$$\boxed{A^{-1} = \frac{\text{adj}\,A}{|A|}}$$

Properties of inverse & Adjoint :-

1. $(AB)^{-1} = B^{-1}A^{-1}$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

2. $A(\text{Adj } A) = |A| I_n$

3. $\text{Adj}(\text{Adj } A) = |A|^{n-2} A$

$n \rightarrow$ Order of square matrix.

4. $|A^{-1}| = \dfrac{1}{|A|}$

5. $|\text{Adj } A| = |A|^{n-1}$

6. $|\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2}$

7. $|\text{Adj}(\text{Adj}(\text{Adj } A))| = |A|^{(n-1)^3}$

$$\begin{vmatrix} 1 & 1 & 2 \\ 5 & 0 & 6 \\ 7 & 8 & 3 \end{vmatrix}$$

SARRUS RULE for Δ.

$$\begin{array}{ccccc} 1 & 1 & 2 & 1 & 1 \\ 5 & 0 & 6 & 5 & 0 \\ 7 & 8 & 3 & 7 & 8 \end{array}$$

$(0+42+80)-(0+48+15)$

$\Delta =$

Ex:-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

i) $Adj\ A$

ii) A^{-1}

iii) $|A^{-1}|$

iv) $|Adj\ A|$

v) $|Adj(Adj\ A)|$

vi) $|A^2|$

vii) $|A^3|$

Thank you

GW Soldiers !