FWC22084

ASSIGNMENT- MATRICES
CIRCLE

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Problem:

If a circle passes through the points of intersection of the coordinate axis with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$,Find the value of λ .

Solution:

The parametric equation of a circle is

$$\mathbf{x}^\top \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + \mathbf{f} = \mathbf{0} \tag{1}$$

.

The circle passes through the intersection points of the given lines and coordinate axis.

Equations of given lines and coordinate axis are

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = \textbf{-3} \tag{2}$$

$$\begin{pmatrix} \lambda & -1 \end{pmatrix} \mathbf{x} = \textbf{-1} \tag{3}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{4}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{5}$$

Intersection point 1:(p1)

solving equation 2 and 4 gives us p1

$$\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 2 & 3 \end{array}\right)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 3 \end{array}\right)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{2} R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\implies \mathbf{x} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

$$\mathbf{p_1} = \mathbf{x} = k_1 \mathbf{e_2}$$

Intersection point 2:(p2)

solving equation 2 and 5 gives us p2

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 0 \end{array}\right)$$

$$\implies \mathbf{x} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$\mathbf{p_2} = \mathbf{x} = k_2 \mathbf{e_1}$$

Intersection point 3:(p3)

solving equation 3 and 4 gives us p3

$$\begin{pmatrix}\lambda & -1\\ 1 & 0\end{pmatrix}\mathbf{x} = \begin{pmatrix}-1\\0\end{pmatrix}$$
$$\xleftrightarrow{R_1\leftarrow\frac{1}{\lambda}R_1}\left(\begin{array}{cc|c}1 & -\frac{1}{\lambda} & -\frac{1}{\lambda}\\ 1 & 0 & 0\end{array}\right)$$
$$\xleftrightarrow{R_2\leftarrow R_2-R_1}\left(\begin{array}{cc|c}1 & -\frac{1}{\lambda} & -\frac{1}{\lambda}\\ 0 & \frac{1}{\lambda} & \frac{1}{\lambda}\end{array}\right)$$
$$\xleftrightarrow{R_1\leftarrow R_2+R_1}\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & \frac{1}{\lambda} & \frac{1}{\lambda}\end{array}\right)$$
$$\xleftrightarrow{R_2\leftarrow\lambda R_2}\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 1\end{array}\right)$$
$$\implies \mathbf{x} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\mathbf{p_3} = \mathbf{x} = k_3\mathbf{e_2}$$

Intersection point 4:(p4)
solving equation 3 and 5 gives us p4

$$\begin{pmatrix}\lambda & -1\\ 0 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}-1\\0\end{pmatrix}$$
$$\xleftrightarrow{R_1\leftarrow R_2+R_1}\left(\begin{array}{cc|c}\lambda & 0 & -1\\ 0 & 1 & 0\end{array}\right)$$
$$\xleftrightarrow{R_1\leftarrow\frac{1}{\lambda}R_1}\left(\begin{array}{cc|c}1 & 0 & -\frac{1}{\lambda}\\ 0 & 1 & 0\end{array}\right)$$
$$\implies \mathbf{x} = \begin{pmatrix}-\frac{1}{\lambda}\\0\end{pmatrix}$$

$$\mathbf{p_4} = \mathbf{x} = k_4\mathbf{e_1}$$

Substituting all the four intersection points in the circle equation gives us the following equations,

$$k_1^2 + 2k_1\mathbf{e_2}^\top\mathbf{u} + f = 0 \tag{6}$$
$$k_2^2 + 2k_2\mathbf{e_1}^\top\mathbf{u} + f = 0 \tag{7}$$
$$k_3^2 + 2k_3\mathbf{e_2}^\top\mathbf{u} + f = 0 \tag{8}$$
$$k_4^2 + 2k_4\mathbf{e_1}^\top\mathbf{u} + f = 0 \tag{9}$$

To find 'u'and 'f' :

sloving equations 6, 7 and 8 gives us

$$\begin{pmatrix}2k_1\mathbf{e_2}^\top & 1\\ 2k_2\mathbf{e_1}^\top & 1\\ 2k_3\mathbf{e_2}^\top & 1\end{pmatrix}\begin{pmatrix}u\\f\end{pmatrix} = \begin{pmatrix}-k_1^2\\-k_2^2\\-k_3^2\end{pmatrix}$$
$$\left(\begin{array}{ccc|c}0 & 3 & 1 & -\frac{9}{4}\\ -6 & 0 & 1 & -9\\ 0 & 2 & 1 & -1\end{array}\right)$$
$$\xleftrightarrow{R_1\leftarrow R_1-R_3}\left(\begin{array}{ccc|c}0 & 1 & 0 & -\frac{5}{4}\\ -6 & 0 & 1 & -9\\ 0 & 2 & 1 & -1\end{array}\right)$$
$$\xleftrightarrow{R_2\leftarrow -\frac{R_2}{6}}\left(\begin{array}{ccc|c}0 & 1 & 0 & -\frac{5}{4}\\ 1 & 0 & -\frac{1}{6} & \frac{3}{2}\\ 0 & 2 & 1 & -1\end{array}\right)$$
$$\xleftrightarrow{R_1\leftarrow R_1+R_2}\left(\begin{array}{ccc|c}1 & 1 & -\frac{1}{6} & \frac{1}{4}\\ 1 & 0 & -\frac{1}{6} & \frac{3}{2}\\ 0 & 2 & 1 & -1\end{array}\right)$$
$$\xleftrightarrow{R_2\leftarrow R_2-R_1}\left(\begin{array}{ccc|c}1 & 1 & -\frac{1}{6} & -\frac{1}{4}\\ 0 & -1 & 0 & \frac{5}{4}\\ 0 & 2 & 1 & -1\end{array}\right)$$
$$\xleftrightarrow{R_3\leftarrow R_3+2R_2}\left(\begin{array}{ccc|c}1 & 1 & -\frac{1}{6} & \frac{1}{4}\\ 0 & -1 & 0 & \frac{5}{4}\\ 0 & 0 & 1 & \frac{3}{2}\end{array}\right)$$
$$\xleftrightarrow{R_1\leftarrow R_1+R_2}\left(\begin{array}{ccc|c}1 & 0 & -\frac{1}{6} & \frac{6}{4}\\ 0 & 1 & 0 & -\frac{5}{4}\\ 0 & 0 & 1 & \frac{3}{2}\end{array}\right)$$
$$\xleftrightarrow{R_1\leftarrow R_1+\frac{R_3}{6}}\left(\begin{array}{ccc|c}1 & 0 & 0 & \frac{7}{4}\\ 0 & 1 & 0 & -\frac{5}{4}\\ 0 & 0 & 1 & \frac{3}{2}\end{array}\right)$$
$$\implies \mathbf{u} = \begin{pmatrix}\frac{7}{4}\\-\frac{5}{4}\end{pmatrix}$$
$$f = \frac{3}{2}$$

To find the value of λ:
substituting values of 'u' and 'f' in equation 9 gives us λ,

$$k_4^2 + 2k_4\mathbf{e_1}^\top\mathbf{u} + f = 0$$

Solving the quadratic equation gives us

the k_4 value as

$$k_4 = \frac{-2\mathbf{e}_1^\top \mathbf{u} \pm \sqrt{(2\mathbf{e}_1^\top \mathbf{u})^2 - 4f}}{2}$$

$$k_4 = \frac{-\frac{7}{2} \pm \sqrt{\frac{25}{4}}}{2}$$

$$k_4 = \frac{-\frac{7}{2} \pm \frac{5}{2}}{2}$$

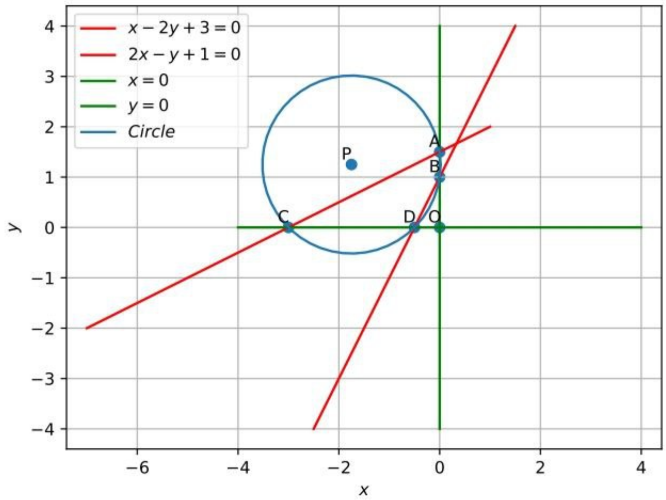
$$k_4 = -\frac{1}{\lambda} = \frac{-1}{2}, -3$$

As $\lambda = \frac{1}{3}$ is already a point,

$$\lambda = 2$$

Construction: The input parameters for this construction are

Symbol	Value
n1	$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
n2	$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
n3	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
n4	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
C1	-3
C2	-1
C	0
$\mathbf{u}$	$\begin{pmatrix} \frac{7}{4} & -\frac{5}{4} \end{pmatrix}$
f	$\frac{3}{2}$



The below python code realizes the above construction:
https://github.com/reshma0639/FWC-Assignment-1/blob/main/matrices/line.py