FWC22084

ASSIGNMENT- MATRICES
CONICS

Contents

- Problem
- Solution
- Construction

Problem:

If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$,then the value of m is:

Solution:

The equation of given hyperbola is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + \mathbf{f} = \mathbf{0}$$

where

$$\mathbf{V} = \begin{pmatrix} \frac{1}{24} & 0 \\ 0 & -\frac{1}{18} \end{pmatrix} \tag{1}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

$$\mathbf{f} = -1 \tag{3}$$

The equation of the normal is

$$\mathbf{n}^\top \mathbf{x} = \mathbf{C}$$

whose normal vector is

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix}, \mathbf{C} = 7\sqrt{3}$$

Let us consider a tangent perpendicular to given normal with normal vector $\mathbf{n_t}$,

$$\mathbf{n_t} = \begin{pmatrix} 1 \\ m \end{pmatrix}$$

Given the normal vector $\mathbf{n_t}$, the tangent points of contact to conic are given by

$$\mathbf{q}_i = \mathbf{V}^{-1}(\kappa_i \mathbf{n}_t - \mathbf{u}), i = 1, 2 \tag{4}$$

$$\text{where } \kappa_i = \pm\sqrt{\frac{f_0}{\mathbf{n}_t^\top \mathbf{V}^{-1}\mathbf{n}_t}} \tag{5}$$

Substituting points of contact of tangent(eq 4) in the normal equation ,

$$\mathbf{n}^\top(\mathbf{q}_i) = \mathbf{C}$$

$$\mathbf{n}^\top(\mathbf{V}^{-1}(\kappa_i \mathbf{n}_t - \mathbf{u})) = 7\sqrt{3}\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\mathbf{n}^\top(\mathbf{V}^{-1}\kappa_i \mathbf{n}_t) = 7\sqrt{3}$$

$$\kappa_i(\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}_t) = 7\sqrt{3}$$

$$\kappa_i \begin{pmatrix} -m & 1 \end{pmatrix} \left(-24 \times 18\right) \begin{pmatrix} \frac{-1}{18} & 0 \\ 0 & \frac{1}{24} \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = 7\sqrt{3}$$

$$\kappa_i \left(-24 \times 18\right) \begin{pmatrix} \frac{1}{18} & \frac{m}{24} \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = 7\sqrt{3}$$

$$\kappa_i \left(-24 \times 18\right) \left(\frac{m}{18} + \frac{m}{24}\right) = 7\sqrt{3}$$

$$\kappa_i \left(-24 \times 18\right) \left(\frac{42m}{24*18}\right) = 7\sqrt{3}$$

$$-\kappa_i m = \frac{\sqrt{3}}{6} \tag{6}$$

Value of κ_i:

$$\kappa_i = \pm\sqrt{\frac{f_0}{\mathbf{n}_t^\top \mathbf{V}^{-1}\mathbf{n}_t}}$$
$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f$$
$$f_0 = -f$$
$$\kappa_i = \pm\sqrt{\frac{1}{\mathbf{n}_t^\top \mathbf{V}^{-1}\mathbf{n}_t}}$$
$$\kappa_i = \pm\sqrt{\frac{1}{(-24\times 18)\begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}\frac{-1}{18} & 0\\ 0 & \frac{1}{24}\end{pmatrix}\begin{pmatrix}1\\ m\end{pmatrix}}}$$
$$\kappa_i = \pm\frac{1}{6}\sqrt{\frac{1}{-12\begin{pmatrix}\frac{-1}{18} & \frac{m}{24}\end{pmatrix}\begin{pmatrix}1\\ m\end{pmatrix}}}$$
$$\kappa_i = \pm\frac{1}{6}\sqrt{\frac{1}{-12\left(-\frac{1}{18}+\frac{m^2}{24}\right)}}$$
$$\kappa_i = \pm\frac{1}{6}\sqrt{\frac{1}{-2\left(-\frac{1}{3}+\frac{m^2}{4}\right)}}$$

Substituting value of κ_i in eq 6,

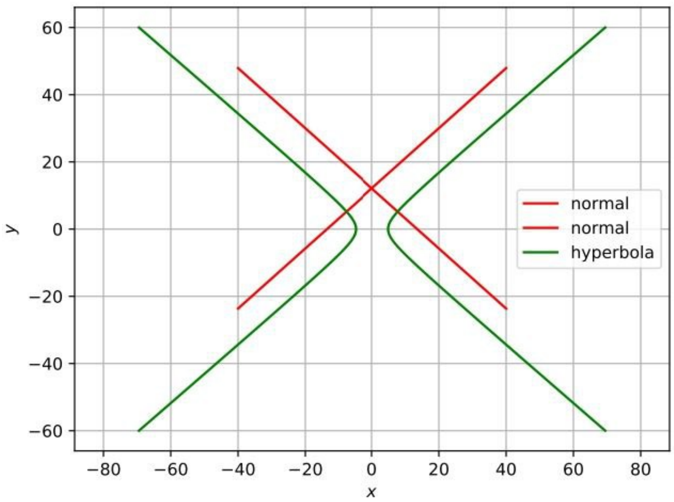
$$-\kappa_i m = \frac{\sqrt{3}}{6} \pm \frac{1}{6}\sqrt{\frac{1}{-2\left(-\frac{1}{3}+\frac{m^2}{4}\right)}}m = \frac{\sqrt{3}}{6}$$
$$\pm\sqrt{\frac{1}{2\left(-\frac{1}{3}+\frac{m^2}{4}\right)}}m = \sqrt{3}$$

Squaring on both sides,

$$\frac{m^2}{\frac{2}{3}-\frac{2m^2}{4}} = 3$$
$$m^2 = 2 - \frac{3m^2}{2}$$
$$\frac{5m^2}{2} = 2$$
$$m = \pm\frac{2}{\sqrt{5}}$$

Construction: The input parameters for this construction are

Symbol	Value
n	$\begin{pmatrix}-\frac{2}{\sqrt{5}}\\ 1\end{pmatrix}$
V	$\begin{pmatrix}\frac{1}{24} & 0\\ 0 & -\frac{1}{18}\end{pmatrix}$
u	$\begin{pmatrix}0\\ 0\end{pmatrix}$
q	$\begin{pmatrix}0\\ 7\sqrt{3}\end{pmatrix}$
C1	$7\sqrt{3}$
f	-1



The below python code realizes the above construction:
https://github.com/reshma0639/FWC-Assignment-1/blob/main/matrices/line.py