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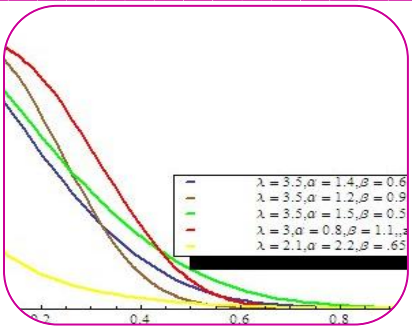

TESTING AND CONFIDENCE INTERVALS FOR PARAMETERS OF EXPONENTIATED EXPONENTIAL DISTRIBUTION

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ABSTRACT

Gupta et al.(1998) have introduced an Exponentiated Exponential Distribution (EE) with shape and scale parameters. In the present article likelihood ratio test for the parameters of Exponentiated Exponential Distribution is provided along with study of their performance. Asymptotic confidence intervals based on normal and lognormal distribution and bootstrap confidence intervals have been obtained. Some simulation experiments are performed to assess performance of confidence intervals. The proposed confidence intervals have been illustrated with the real life data due to Lawless (1982).

KEYWORDS: *Exponentiated Exponential Distribution; Likelihood ratio test ; Confidence interval ; Bootstrap technique.*

1. INTRODUCTION

Gupta et al (1998) introduced an Exponentiated Exponential (EE) distribution as a generalization of the standard exponential distribution. Gupta and Kundu named EE distribution as Generalised Exponential (GE) and studied quite extensively several properties of GE distribution (Gupta and Kundu 1999, 2001a, 2001b, 2002).

The two parameter EE distribution is defined by the cumulative distribution function (cdf)

$$F_E(x,\alpha,\lambda) = \left(1-e^{-\lambda x}\right)^{\alpha} \quad ; \qquad \alpha, \lambda, x > 0 \quad \ldots\ldots\ldots\ldots(1.1)$$

The corresponding density function is

$$f_E(x,\alpha,\lambda) = \alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x} \quad ; \qquad \alpha, \lambda, x > 0 \quad \ldots\ldots\ldots\ldots\ldots(1.2)$$

where λ and α are the scale and shape parameters respectively. When α equals one , it coincides with the exponential distribution with mean $1/\lambda$. If $\alpha \le 1$, the density function is strictly decreasing and for $\alpha > 1$, it has a unimodal shape. The shape of the density function of EE distribution for different α can be found in Gupta and Kundu (2001a).

The maximum likelihood estimators (MLE's) of two parameters α and λ and their asymptotic property have been studied in detail by Gupta and Kundu (2001), in which it is stated that asymptotic confidence intervals of $\underline{\theta}$ = (α, λ)` can be obtained by using the asymptotic normality result as $\sqrt{n}$ $(\hat{\theta}-\theta)$ $\rightarrow N_2$ (0 , $I^{-1}(\underline{\theta})$) ,where $I(\underline{\theta})$ is the Fisher information matrix and

$\hat{\underline{\theta}}$ =$(\hat{\alpha},\hat{\lambda})$` are the mle of $\underline{\theta}$ and $I^{-1}(\underline{\theta})$ is estimated by $I^{-1}(\hat{\underline{\theta}})$. In this paper we simulate MLE's and assess their distribution for finite sample size based on Anderson Darling and Kolmogorov-Smirnov statistics. Testing of scale and shape parameter using likelihood ratio test is given in section 2. In section 3 we propose asymptotic confidence intervals (CI) using normal and lognormal distribution. We also use Bootstrap technique to provide confidence intervals namely Percentile CI and Normal CI for both the parameters. Simulation results are presented in section 4 regarding performance of confidence intervals through their lengths and coverage. In section 5, we provide test for exponentiality and confidence intervals for real life data due to Lawless (1982) and conclusions are given in section 6.

2. TESTING OF SCALE AND SHAPE PARAMETER

Let X_i , i = 1,2,3,.....n be a random sample drawn from EE(α , λ) then the likelihood function is

$$L(\alpha,\lambda/\underline{x}) = (\alpha\lambda)^n \prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{\alpha-1} e^{-\lambda\sum x_i} \qquad ....(2.1)$$

The maximum likelihood estimators of two parameters α and λ be $\hat{\alpha}$ and $\hat{\lambda}$ which is solution of normal equations given in detail by Gupta and Kundu(2001) as

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-\lambda x_i}\right)=0 \qquad \text{and}$$

$$\frac{n}{\lambda}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)}-\sum_{i=1}^{n}x_i=0 \qquad ....(2.2\ )$$

In the following we consider testing of parameters α and λ based on likelihood ratio.

Case-1: $H_0 : \lambda=\lambda_0$ against $H_1: \lambda\neq\lambda_0$ when α is known (say α_0) .
The likelihood function under H_0 is

$$L_0(\alpha_0,\lambda_0/\underline{x}) = (\alpha_0\lambda_0)^n \prod_{i=1}^{n}\left(1-e^{-\lambda_0 x_i}\right)^{\alpha_0-1} e^{-\lambda_0\sum x_i}$$

and $\hat{\lambda}$ is mle of λ which is solution of normal equation

$$\frac{n}{\lambda}+(\alpha_0-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)}-\sum_{i=1}^{n}x_i=0$$

Then the likelihood ratio test is rejects H_0 if $\Lambda_1(\underline{x})=\dfrac{L_0(\alpha_0,\lambda_0/\underline{x})}{L(\alpha_0,\hat{\lambda}/\underline{x})}<c_1$

which is equivalent to reject H_0 , if -2 log ($\Lambda_1(\underline{x})$) > c_1` and
-2 log ($\Lambda_1(\underline{x})$) follows chi-square distribution with 1 d.f. when H_0 is true.

Case –2 : $H_0 : \lambda=\lambda_0$ against $H_1: \lambda \neq \lambda_0$, when α is not known .
The likelihood function under H_0 is

$$L_0(\alpha,\lambda_0/\underline{x}) = (\alpha\lambda_0)^n \prod_{i=1}^{n}\left(1-e^{-\lambda_0 x_i}\right)^{\alpha-1} e^{-\lambda_0 \sum x_i} ,$$

The maximum likelihood estimators of α under H_0 is $\hat{\alpha}_0$ obtained as

$$\hat{\alpha}_0 = \frac{-n}{\sum_{i=1}^{n}\ln\left(1-e^{-\lambda_0 x_i}\right)}$$

and mle's $\hat{\alpha}$ and $\hat{\lambda}$ are obtained from equations (2.2) . The likelihood ratio test rejects null hypothesis if

$$\Lambda_2(\underline{x}) = \frac{L_0(\hat{\alpha}_0,\lambda_0/\underline{x})}{L(\hat{\alpha},\hat{\lambda}/\underline{x})} < c_2$$

which is equivalent to reject H_0 , if -2 log ($\Lambda_2(\underline{x})$) > c_2` and
-2 log ($\Lambda_2(\underline{x})$) follows chi-square distribution with 1 d.f. when H_0 is true .
Table 2.1 represents power of test based on $\Lambda_1(\underline{x})$ & $\Lambda_2(\underline{x})$ for λ_0=1.
Case-3 : $H_0 : \alpha = \alpha_0 =1$ against H_1: $\alpha \neq 1$ and λ is known (say λ_0).
The likelihood function under H_0 is

$$L_0(\alpha_0,\lambda_0/\underline{x}) = (\lambda_0)^n e^{-\lambda_0 \sum x_i}$$

And mle $\hat{\alpha}$ of α is obtained as $\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n}\ln\left(1-e^{-\lambda_0 x_i}\right)}$

The likelihood ratio test indicates that one should reject H_0 if $\Lambda_3(\underline{x}) = \frac{L_0(\alpha_0,\lambda_0/\underline{x})}{L(\hat{\alpha},\lambda_0/\underline{x})} < c_3$ which is equivalent to reject H_0 if
-2 log ($\Lambda_3(\underline{x})$) > c_3` and -2 log ($\Lambda_3(\underline{x})$) follows chi-square distribution with 1 d.f. when H_0 is true.
Gupta and Kundu (1999) showed that above test is uniformly most powerful unbiased test.
Case-4 : *To test $H_0 : \alpha = \alpha_0 =1$ against H_1: $\alpha \neq 1$, λ is unknown.*
The likelihood function under H_0 is

$$L_0(\alpha_0,\lambda/\underline{x}) = (\lambda)^n e^{-\lambda \sum x_i}$$

Under H_0 mle of λ is $\hat{\lambda}_0 = \frac{1}{\bar{x}}$ and mle's $\hat{\alpha}$ and $\hat{\lambda}$ are obtained from equations (2.2) .The likelihood ratio test rejects H_0 if $\Lambda_4(\underline{x}) = \frac{L_0(\alpha_0,\hat{\lambda}_0/\underline{x})}{L(\hat{\alpha},\hat{\lambda}/\underline{x})} < c_4$

which is equivalent to test reject H_0 if -2 log ($\Lambda_4(\underline{x})$) > c_4` and
-2 log ($\Lambda_4(\underline{x})$) follows chi-square distribution with 1 d.f. when H_0 is true.

Table 2.1 : Power of test based on $\Lambda_1(x)$ & $\Lambda_2(x)$ when λ_0=1 and n=25 (α is unknown)

λ	Power of test based on $\Lambda_1(x)$		Power of test based on $\Lambda_2(x)$	
	α=2	α=5	α=2	α=5
1.0	0.0395	0.030638	0.05537	0.06654
1.2	0.1655	0.279152	0.159202	0.190913
1.4	0.482	0.748741	0.34086	0.427736
1.6	0.801	0.943	0.545605	0.670639
2.0	0.98	0.998	0.839445	0.931744
2.6	1	0.9995	0.980985	0.998596
3.0	1	1	0.995688	1

Table 2.2 : Power of test based on $\Lambda_3(x)$ & $\Lambda_4(x)$ when α_0=1 and n=25 (λ is unknown)

α	Power of test based on $\Lambda_3(x)$		Power of test based on $\Lambda_4(x)$	
	λ=1	λ=3	λ=1	λ=3
1.0	0.0535	0.0535	0.05603	0.0695
1.5	0.5005	0.499	0.341761	0.336765
2.0	0.9255	0.9305	0.727632	0.697767
2.5	0.9985	0.997	0.907435	0.912579
3.0	1	1	0.972334	0.975864
4.0	1	1	0.999048	0.997067

3. CONFIDENCE INTERVAL

Asymptotic Confidence Intervals.

Based on random samples $X_1, X_2 \ldots X_n$ our problem is to obtain a two-sided confidence interval for parameters λ and α .

That is for given δ>0, the problem is to find $L_i(X)$ & $U_i(X)$ i=1,2 such that

$$\Pr[\,L_1(X) < \lambda < U_1(X)] \geq 1-\delta \quad \text{and} \quad \Pr[\,L_2(X) < \alpha < U_2(X)] \geq 1-\delta$$

In literature there are various techniques to find confidence interval of desired confidence level (1- δ). However popular techniques depend on either suitable pivotal quantity or existence of optimum test for testing a two sided alternative about the parameter of interest. In the present situation, maximum likelihood estimator of λ and α do not have closed form density for finite sample. Also finite sample distributions of the MLE's are difficult to obtain in either case. Therefore we are left with only asymptotic distribution of the MLE's. Confidence intervals are computed assuming that sampling distribution of the MLE is approximately normal. Estimating standard errors of distribution of $\hat{\lambda}$ and $\hat{\alpha}$, we get confidence intervals for λ and α respectively are

$$\hat{\lambda} \pm t\,S.E.(\hat{\lambda}) \quad \text{and} \quad \hat{\alpha} \pm t\,S.E.(\hat{\alpha}) \qquad \ldots\ldots\ldots\ldots\ldots(3.1)$$

where t is 100(1- δ/2)th upper percentile of t-distribution with the appropriate number of degrees of freedom .

Table-3.1: Anderson Darling and Kolmogorov -Smirnov statistic values for various values of λ and α , when n=25.

(λ , α)		Anderson Darling		Kolmogorov Smirnov Dn	
		Normal Dist.	Lognormal Dist.	Normal Dist.	Lognormal Dist.
λ= .5 α= .5	$\hat{\lambda}$	82.99	8.98	0.101	0.029
	$\hat{\alpha}$	62.77	8.98	0.085	0.033
λ= 0.5 α= 2	$\hat{\lambda}$	10.85	0.23	0.047	0.011
	$\hat{\alpha}$	65.79	8.54	0.118	0.047
λ= 1 α= 2	$\hat{\lambda}$	11.38	0.36	0.053	0.018
	$\hat{\alpha}$	58.24	5.30	0.109	0.037
λ= 3 α= 2	$\hat{\lambda}$	14.14	0.55	0.054	0.017
	$\hat{\alpha}$	61.10	9.01	0.119	0.048
λ= 1 α= 3	$\hat{\lambda}$	27.82	1.03	0.127	0.013
	$\hat{\alpha}$	147.0	14.6	0.054	0.045

In order to assess normality of $\hat{\lambda}$ and $\hat{\alpha}$, we conducted simulation experiment. We generate 4000 samples each of size 25 from EE(α, λ) for various λ and α . For each sample we obtain mle $\hat{\lambda}$ and $\hat{\alpha}$. Empirical distribution of $\hat{\lambda}$ and $\hat{\alpha}$ is compared with normal and lognormal distribution (Refer to Graph 3.1 to 3.8). Details are summarized in the Table 3.1. We observed that distribution of mle is closed to lognormal distribution than normal distribution for all the combinations of λ and α (except λ =0.5 and α=2) considered in simulation study.

In the light of results reported in Table 3.1, we assume the sampling distribution of mle to be lognormal.

Then the lower and upper confidence limits for λ and α are computed as (Refer Burnham et al .1987 :212)

$\hat{\lambda}_L = \hat{\lambda}\ /c_1$ and $\widehat{\theta}_U = \hat{\lambda}\ .c_1$

$\hat{\alpha}_L = \hat{\alpha}\ /c_2$ and $\widehat{\theta}_U = \hat{\alpha}\ .c_2$ ....(3.2)

where $c_1 = \exp(Z_{\partial/2}\sqrt{\log_e(1+(CV(\hat{\lambda}))^2)})$

and $c_2 = \exp(Z_{\partial/2}\sqrt{\log_e(1+(CV(\hat{\alpha}))^2)})$

where Z_δ is the upper δ^{th} quantile of the standard normal distribution.

BOOTSTRAP CONFIDENCE INTERVALS.

In case of asymptotic distribution of MLE's for each of the parameters , asymptotic variance of the $\hat{\lambda}$ is not in the closed form (Please refer Gupta and Kundu (2001a)). Therefore we use resampling technique to estimate it. We obtain a bootstrap sample denoted by $X_1^*, X_2^*, \ldots, X_n^*$ of size n , where this sample is obtained from the original sample $X_1, X_2, \ldots, X_n$, using simple random sampling with replacement. We obtain such 2000 bootstrap samples. Using each of bootstrap sample we compute the MLE $\hat{\lambda}_i^*$ and $\hat{\alpha}_i^*$, i=1,2,….2000. These 2000 values of MLE's can be used to estimate variances of the MLE's of λ and α and to obtain bootstrap normal confidence interval (BNCI) and percentile confidence interval (BPCI)(Please refer to Efron and Tibshirani (1986) for more details). Using asymptotic normal distribution of $\hat{\lambda}$ and $\hat{\alpha}$ the (1-δ)100% BNCI of λ and α respectively are

$$\hat{\lambda} \pm Z_{\delta/2} S_{\lambda}^* \quad \text{and} \quad \hat{\alpha} \pm Z_{\delta/2} S_{\alpha}^* \qquad \ldots\ldots\ldots\ldots(3.3)$$

where quantities S_{λ}^* and S_{α}^*, are bootstrap estimators of the standard deviation of $\hat{\lambda}$ and $\hat{\alpha}$ respectively . Using the ordered values of $\hat{\lambda}_i^*$ and $\hat{\alpha}_i^*$ the δ percentage and the (1-δ) percentage points are used to obtain the BPCI of λ and α. These are given by

$$\left(\hat{\lambda}^*_{(B\delta/2)}, \hat{\lambda}^*_{(B(1-\delta)/2)}\right) \quad \text{and} \quad \left(\hat{\alpha}^*_{(B\delta/2)}, \hat{\alpha}^*_{(B(1-\delta)/2)}\right) \qquad \ldots\ldots\ldots\ldots\ldots(3.4)$$

where $\hat{\lambda}^*_{(k)}$ and $\hat{\alpha}^*_{(k)}$ are the lower k th sample percentile of the ordered values of $\hat{\lambda}_i^*$ and $\hat{\alpha}_i^*$ respectively. For example, for δ = .05, a 95 % confidence interval with B=2000, confidence intervals are

$$\left(\hat{\lambda}^*_{(50)}, \hat{\lambda}^*_{(1950)}\right) \quad \text{and} \quad \left(\hat{\alpha}^*_{(50)}, \hat{\alpha}^*_{(1950)}\right).$$

4. NUMERICAL RESULTS.

In this section we perform some simulation experiments to observe how proposed confidence intervals work for finite sample size. We perform simulation experiment by considering the following different model parameters.

Model 1: λ =.5 and α =.5 Model 2: λ =0.5 and α =2 Model 3: λ =1 and α =2
Model 4: λ =3 and α =2 Model 5: λ =1 and α =3 .

For each model parameters we compute the MLE's of λ and α and their confidence intervals. We repeat this process 4000 times and compute average length of confidence interval and their coverage probability. The results are reported in Table-4.1. It is observed from Table -4.1 that the lognormal confidence interval behaves similar to normal confidence interval as the average length of confidence interval and their coverage probability are almost same. Bootstrap normal and percentile confidence intervals are reported in Table -4.2.

Table-4.1: Normal and Lognormal based average length of C.I. of α and λ and their coverage probability. (δ =.05 , n=25)

Model	Normal based Average Length of C.I. (Coverage) λ	α	Lognormal based Average Length of C.I. (Coverage) λ	α
λ= .5 α= .5	0.80359 (93.96%)	0.558063 (93.89%)	0.804328 (94.37%)	0.540135 (94.90%)
λ= .5 α= 2	0.495429 (95.30%)	3.536543 (95.30%)	0.502477 (94.45%)	3.721537 (95.90%)
λ= 1 α= 2	0.975145 (93.80%)	3.404553 (93.68%)	1.009272 (94.91%)	3.135684 (93.75%)
λ= 3 α= 2	2.9653568 (94.85%)	3.463691 (95.05%)	3.026081 (93.85%)	3.640599 (94.80%)
λ= 1 α= 5	0.851943 (93.27%)	12.55058 (93.95%)	0.847267 (94.78%)	11.05199 (94.24%)

Table : 4 .2 Confidence Interval for α , λ and n=25 , δ=.05. (n_b =2000)

	Bootstrap Confidence Interval	λ Lower	Upper	Length	α Lower	Upper	Length
λ=.5 α=.5	Normal	0	1.608046	1.608046	0.167858	0.811639	0.643781
	Percentile	0.342315	2.307791	1.965476	0.354017	0.968991	0.614974
λ=.5 α=2	Normal	0.325017	0.693446	0.368429	0.620507	2.506628	1.886121
	Percentile	0.391512	0.768453	0.376941	1.078931	2.896897	1.817966
λ=1 α=2	Normal	0.541517	1.656947	1.11543	0.566967	4.589679	4.022712
	Percentile	0.750249	1.85764	1.107391	1.707955	5.626163	3.918208
λ=3 α=2	Normal	2.758753	7.146357	4.387604	0.480952	12.64504	12.16408
	Percentile	3.226457	7.541739	4.315282	3.294066	14.18604	10.89197
λ=1 α=5	Normal	0.597184	1.146235	0.549051	0.665514	8.45606	7.790546
	Percentile	0.653347	1.20822	0.554873	2.3955639	9.91759	7.521951

5. Application to real-life data.

The data given here arose in tests on endurance of deep groove ball bearings (Lawless 1982). The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are 17.88, 28.92, 33.00, 1.12, 45.60, 48.80, 51.84, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.88, 84.12, 93.12, 8.64, 05.12, 105.12, 105.84, 127.92, 128.04 and 173.40.

Gupta and Kundu (2001) have shown that the above data is better fitted by EE distribution than Weibull and Gamma distributions. The MLE's are $\hat{\lambda}$ =0.0323 and $\hat{\alpha}$ =5.284382. We compute three types of confidence intervals as Normal based CI, Bootstrap normal CI and Percentile bootstrap CI. These have been tabulated in Table 5.1. Likelihood ratio test for exponentiality is carried out for this data , we get results for case -4 as -2log ($\Lambda_4(\underline{x})$) =16.92701 therefore we rejects H_0 at 5% l.o.s.

Table 5.1- Confidence interval for data due to Lawless (1982)

Confidence Interval	λ			α		
	Lower	Upper	Length	Lower	Upper	Length
Normal based	0.024417	0.040184	0.015767	3.124717	7.444046	4.319329
Bootstrap Normal	0.018636	0.045965	0.027329	0	12.48005	12.48005
Bootstrap Percentile	0.02395	0.051338	0.027388	3.158019	15.39011	12.23209

6. CONCLUSION :

In this article we provide the asymptotic confidence intervals of two parameter EE distribution based on normal and lognormal distribution and it is clear that lognormal confidence interval is the alternative to normal confidence interval as the average length of confidence interval and their coverage probability is almost same. The MLE's are better fitted to lognormal distribution than normal distribution based on Anderson Darling and Kolmogorov-Smirnov statistics. We also provide confidence intervals based on MLE's using bootstrap technique. It is observed that bootstrap normal and bootstrap percentile confidence intervals are almost same as far as interval length is concerned.

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