Rayleigh–Taylor Instability

- Now we consider the viscous case with interface. Now the viscosity μ_1 and μ_2 on the two sides is also constant.

- In both regions the eq becomes

$$D\left[\rho - \frac{\mu}{n}(D^2 - k^2)\right] Dv_z = k^2 \left[\rho - \frac{\mu}{n}(D^2 - k^2)\right] v_z$$

  Since ρ and μ are constants on the two sides we can divide by ρ and define the kinematic viscosity $\nu = \mu/\rho$ to get the eq

$$\left[1 - \frac{\nu}{n}(D^2 - k^2)\right](D^2 - k^2)v_z = 0$$

Thus the solutions would be of the form

$$\exp(\pm kz) \qquad \text{and} \qquad \exp(\pm qz)$$

where

$$q^2 = k^2 + \frac{n}{\nu}$$

- If $q^2 > 0$, we can take the positive sign for q. Since $v_z = 0$ at $z = \pm\infty$ the solution is of the form

$$v_z = \begin{cases} w_1 = A_1 e^{kz} + B_1 e^{q_1 z} & \text{for } z < 0 \\ w_2 = A_2 e^{-kz} + B_2 e^{-q_2 z} & \text{for } z > 0 \end{cases}$$

where A_1, A_2, B_1, B_2 are constants and

$$q_1 = \sqrt{k^2 + \frac{n}{\nu_1}}, \qquad q_2 = \sqrt{k^2 + \frac{n}{\nu_2}}$$

- Across the interface at $z = 0$ we should have continuity of v_z as well as v_x, v_y, by continuity eq this gives Dv_z should be continuous. Further the stresses σ_{xz} and σ_{yz} should also be continuous giving

$$i(k_x \sigma_{xz} + k_y \sigma_{yz}) = \mu[D(ik_x v_x + ik_y v_y) - k^2 v_z]$$

which gives continuity of $\mu(D^2 + k^2)v_z$. This would give 3 conditions on the 4 constants.

$$A_1 + B_1 - A_2 - B_2 = 0$$

$$kA_1 + q_1 B_1 + kA_2 + q_2 B_2 = 0$$

$$2k^2 \mu_1 A_1 + \mu_1 (k^2 + q_1^2) B_1 - 2k^2 \mu_2 A_2 - \mu_2 (k^2 + q_2^2) B_2 = 0$$

- The 4th conditions is provided by integrating the eq across the interface at $z = 0$

$$\left[\rho_2 - \frac{\mu_2}{n}(D^2 - k^2)\right] Dw_2 - \left[\rho_1 - \frac{\mu_1}{n}(D^2 - k^2)\right] Dw_1$$
$$= -g\frac{k^2}{n^2}(\rho_2 - \rho_1)v_z - 2\frac{k^2}{n}(\mu_2 - \mu_1)Dv_z$$

The 1st term inside $[\cdots]$ on LHS is

$$\rho_2(-kA_2 - q_2B_2) - \frac{\mu_2}{n}B_2(q_2^2 - k^2)(-q_2) = \rho_2(-kA_2)$$

we get the eq

$$\rho_2(-kA_2) - \rho_1(kA_1) + R(A_1 + B_1) + C(kA_1 + q_1B_1) = 0$$

where

$$R = g\frac{k^2}{n^2}(\rho_2 - \rho_1), \qquad C = 2\frac{k^2}{n}(\mu_2 - \mu_1)$$

- The 4 linear homogeneous eq need to be solved. For non-trivial solution the determinant should be zero.

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ k & q_1 & k & q_2 \\ 2k^2\mu_1 & \mu_1(k^2+q_1^2) & -2k^2\mu_2 & -\mu_2(k^2+q_2^2) \\ -k\rho_1 + R + Ck & R + Cq_1 & -k\rho_2 & 0 \end{vmatrix}$$

- Eliminating the 1st row using $C_2 - C_1$, $C_4 - C_3$ and $C_3 + C_1$

$$\begin{vmatrix} q_1 - k & 2k & q_2 - k \\ \rho_1 n & -2k^2(\mu_2 - \mu_1) & -\rho_2 n \\ C(q_1 - k) + k\rho_1 & -k(\rho_1 + \rho_2) + R + Ck & k\rho_2 \end{vmatrix}$$

- Chandrasekhar has suggested the simplification $\nu_1 = \nu_2 = \nu$ which gives $q_1 = q_2 = q$ and $\mu_2 - \mu_1 = \nu(\rho_2 - \rho_1)$. The determinant becomes

$$\begin{vmatrix} q - k & 2k & q - k \\ \rho_1 & -C & -\rho_2 \\ C(q - k) + k\rho_1 & -k(\rho_1 + \rho_2) + R + Ck & k\rho_2 \end{vmatrix}$$

- This gives the dispersion relation

$$(q-k)^2C^2 + (q-k)[-k(\rho_1+\rho_2)^2 + R(\rho_1+\rho_2) \\ - 2Ck(\rho_2-\rho_1)] - 4k^2\rho_1\rho_2 = 0$$

Dividing by $k^2(\rho_1+\rho_2)^2$ and define

$$\delta = \frac{\rho_2-\rho_1}{\rho_1+\rho_2}, \qquad y = \frac{q}{k}$$

δ is referred to as Atwood no.

$$\frac{R}{\rho_1+\rho_2} = \frac{gk^2\delta}{n^2}, \quad \frac{C}{\rho_1+\rho_2} = \frac{2k^2\nu\delta}{n}$$

using these we get

$$(y-1)^2\frac{4k^4\nu^2}{n^2}\delta^2+(y-1)\left(\frac{gk}{n^2}\delta-4\frac{k^2\nu}{n}\delta^2-1\right)-(1-\delta^2)=0$$

Using the definition of q we get

$$y^2-1=\frac{n}{k^2\nu}$$

which gives

$$n=\nu k^2(y^2-1)$$

Substituting this gives the dispersion relation in terms of y

$$4\frac{(y-1)^2}{(y^2-1)^2}\delta^2-y+(y-1)\left(\frac{Q}{(y^2-1)^2}\delta-\frac{4}{y^2-1}\delta^2\right)+\delta^2=0$$

where the dimensionless parameter

$$Q = \frac{g}{k^3 \nu^2}$$

Multiplying by $(y^2 - 1)(y + 1)$ gives

$$y^4 + y^3(1 - \delta^2) - y^2(1 - 3\delta^2) - y(1 + 3\delta^2) + \delta^2 - Q\delta = 0$$

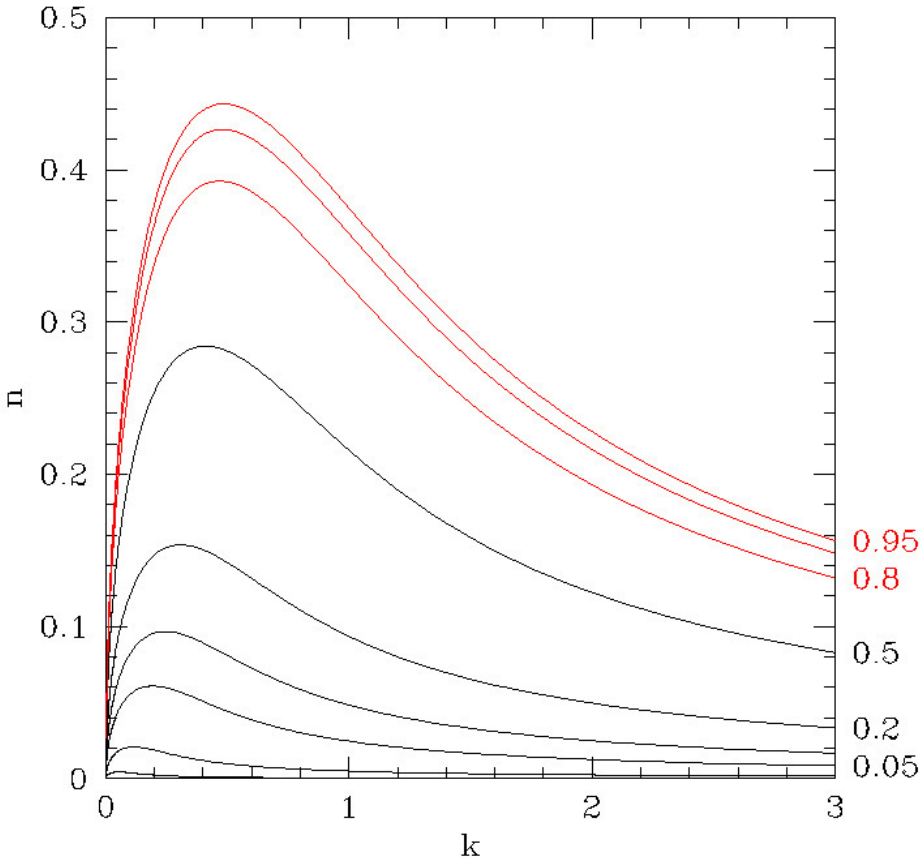
- At $y = 1$ the LHS is $-Q\delta$. Thus for $\delta > 0$, one root of the polynomial is $y > 1$ which is the value we want. For a given value of Q

$$k = \left(\frac{g}{\nu^2}\right)^{1/3} \frac{1}{Q^{1/3}}$$

and

$$n = k^2\nu(y^2 - 1) = \left(\frac{g^2}{\nu}\right)^{1/3} \frac{y^2 - 1}{Q^{2/3}}$$

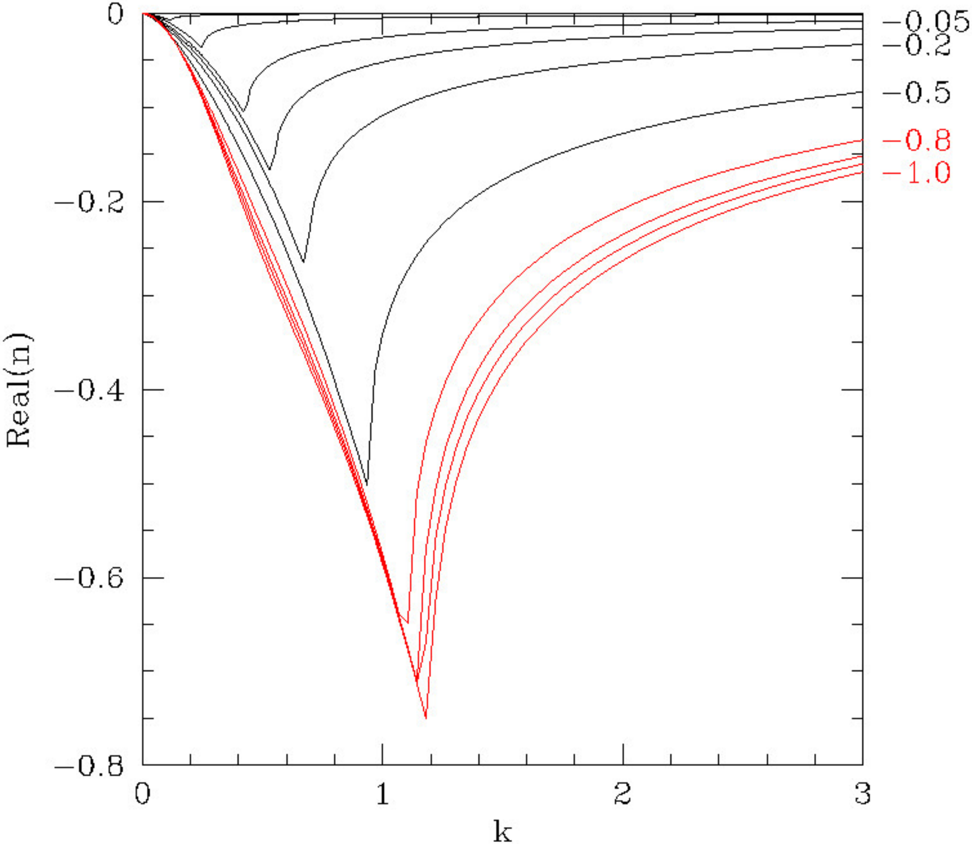
Thus we get k in units of $(g/\nu^2)^{1/3}$ and n in units of $(g^2/\nu)^{1/3}$ and the results are shown in the figure for different values of δ.



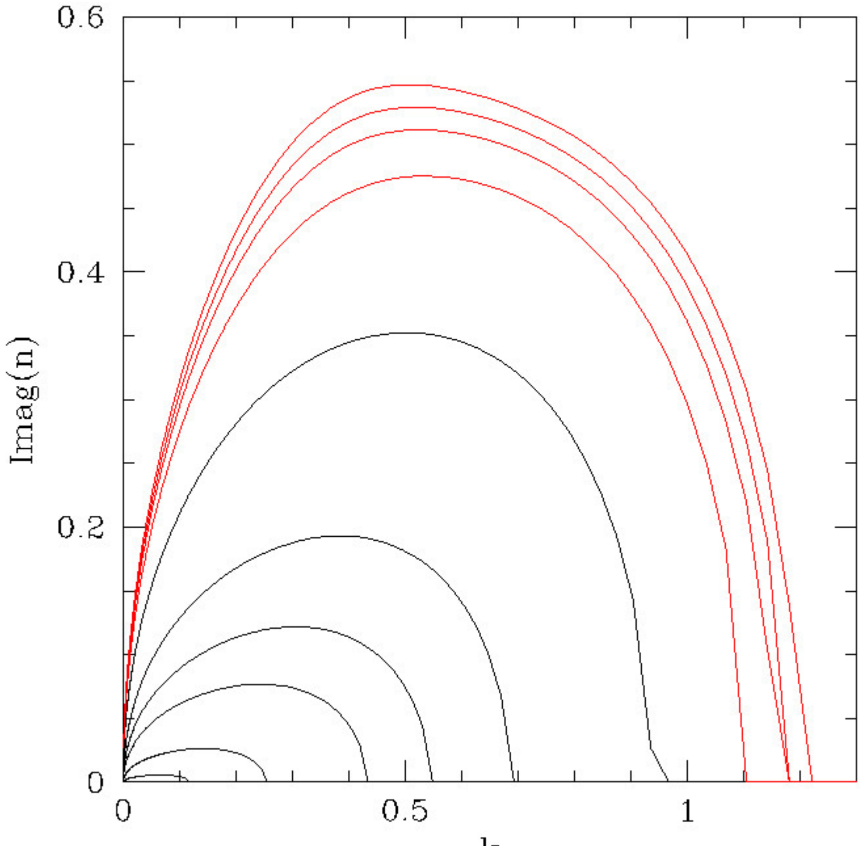
0.5
0.4
0.3
0.2
0.1
0
n
0
1
2
3
k
0.95
0.8
0.5
0.2
0.05

- For $0 < \delta < 1$ there is always a root with $n > 0$ and the system in unstable. The k value for the maximum in $n(k)$ gives the dominant mode and the length scale.

- For $\delta < 0$, the roots can become complex and hence we need to look at only those roots for $\mathrm{Real}(y) > 0$ as the exponent q is assumed to have positive real part for the boundary conditions to be satisfied. With these constraints there is no root with $\mathrm{Real}(n) > 0$ and hence the system is stable. The figure shows the root with largest real part.



0
−0.2
−0.4
−0.6
−0.8
Real(n)
0
1
2
3
k
−0.05
−0.2
−0.5
−0.8
−1.0


0.6
0.4
0.2
0
Imag(n)
0
0.5
1
k

Effect of Magnetic Field

- We consider the effect of uniform horizontal magnetic field on Rayleigh–Taylor instability for inviscid and perfectly conducting fluid.
- To start with we take the case of an inhomogeneous fluid with density varying with height.

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}=-\boldsymbol{\nabla}P-\mathbf{g}\rho+\frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{B}=\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{v}$$

- We assume that in the steady state ρ depends only on z. The equation of continuity gives

$$\frac{\partial\rho}{\partial t}+v_z\frac{d\rho}{dz}=0$$

- In the steady state $\partial/\partial t = 0$, $\mathbf{v} = 0$ which gives

$$\frac{dP_0}{dz} = -g\rho$$

which gives the pressure P_0, which is the same as non-magnetic case as the field is force-free.

- The state is perturbed to $\mathbf{v}$, $P = P_0 + P_1$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ and $\rho = \rho_0 + \rho_1$ which gives the eq in perturbed quantities

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla P_1 - \mathbf{g}\rho_1 + \frac{1}{4\pi}(\nabla \times \mathbf{B_1}) \times \mathbf{B_0}$$
$$\frac{\partial \mathbf{B}_1}{\partial t} = \mathbf{B}_0 \cdot \nabla \mathbf{v}$$

- We seek solutions of the form

$$f(z)\exp(ik_x x + ik_y y + nt)$$

with $k^2 = k_x^2 + k_y^2$ and $D = d/dz$. We assume that $\mathbf{B}_0 = (B_0, 0, 0)$. The induction eq gives

$$n\mathbf{B}_1 = ik_x B_0 \mathbf{v}$$

which can be used to eliminate $\mathbf{B}_1$ from the eq of motion to get

$$n\rho_0\mathbf{v} = -\boldsymbol{\nabla} P_1 - \mathbf{g}\rho_1 + \frac{ik_x B_0}{4\pi n}(\boldsymbol{\omega} \times \mathbf{B}_0)$$

where $\boldsymbol{\omega}$ is the vorticity. The eq in component form are

$$ik_x P_1 = -n\rho_0 v_x$$

$$ik_y P_1 = -n\rho_0 v_y + \frac{ik_x B_0^2}{4\pi n}\omega_z$$

$$DP_1 = -n\rho_0 v_z - g\rho_1 + \frac{ik_x B_0^2}{4\pi n}(ik_x v_z - Dv_x)$$

$$ik_x v_x + ik_y v_y = -Dv_z$$

$$n\rho_1 = -(D\rho_0)v_z$$

- Multiplying the 1st eq by $-ik_y$ and 2nd by ik_x and adding we get

$$\left(\rho_0 n + \frac{k_x^2 B_0^2}{4\pi n}\right)\omega_z = 0$$

which gives $\omega_z = 0$ or

$$n^2 = -\frac{k_x^2 B_0^2}{4\pi\rho_0} = -k_x^2 V_A^2$$

where V_A is the Alfven speed. This is the dispersion relation form Alfven waves, but since ρ_0 is not constant. This condition cannot be satisfied everywhere. Thus $\omega_z = 0$.

- Eliminating v_x and v_y from the continuity equation gives

$$\rho_0 n D v_z = -k^2 P_1$$

which gives

$$v_x = \frac{ik_x}{k^2} D v_z, \qquad v_y = \frac{ik_y}{k^2} D v_z$$

- Thus we can eliminate all other variables from the z-component of eq of motion to get

  $$\rho_0 n v_z = \frac{B_0^2 k_x^2}{4\pi n k^2}(D^2 - k^2)v_z + \frac{g}{n}(D\rho_0)v_z + \frac{n}{k^2}D(\rho_0 D v_z)$$

  The required eq is

  $$D(\rho_0 D v_z) + \frac{k_x^2 B_0^2}{4\pi n^2}(D^2 - k^2)v_z - k^2 \rho_0 v_z = -\frac{g k^2}{n^2}(D\rho_0)v_z$$

  where we have used

  $$k^2 i k_x (i k_x v_z - D v_x) = -k_x^2 (k^2 v_z - D^2 v_z) = k_x^2 (D^2 - k^2) v_z$$

- If $k_x = 0$ or the wave propagating perpendicular to the magnetic field this eq reduces to the non-magnetic case and the result will be the same.

- Thus we only consider the case when $k_x \neq 0$ for two fluids separated by interface at $z = 0$. In both regions the eq becomes

$$\rho_0(D^2 - k^2)v_z + \frac{k_x^2 B_0^2}{4\pi n^2}(D^2 - k^2)v_z = 0$$

or

$$\rho_0 \left(1 + \frac{k_x^2 B_0^2}{4\pi \rho_0 n^2}\right)(D^2 - k^2)v_z = 0$$

- We are considering the case when $n^2 > 0$. In case $n^2 < 0$ the quantity in parenthesis may vanish at some layer, giving a singularity at that layer. This happens when we are considering waves propagating in the medium.

- In both regions the eq becomes

$$(D^2 - k^2)v_z = 0$$

which gives a general solution

$$v_z = Ae^{kz} + Be^{-kz}$$

This should vanish at both $\pm\infty$. Further the solution should be continuous at $z = 0$ giving (assume $k > 0$)

$$v_z = \begin{cases} Ae^{kz} & \text{for } z < 0 \\ Ae^{-kz} & \text{for } z > 0 \end{cases}$$

- Integrating the original eq across the interface we get

$$\int_{-\epsilon}^{\epsilon} \left(D(\rho_0 D v_z) + \frac{k_x^2 B_0^2}{4\pi n^2}(D^2 - k^2) v_z - \rho_0 k^2 v_z \right) \, dz$$

$$= -\frac{k^2}{n^2} g \int_{-\epsilon}^{\epsilon} (D\rho_0) v_z \, dz$$

which gives

$$\Delta_i(\rho_0 D v_z) + \frac{k_x^2 B_0^2}{4\pi n^2} \Delta_i(D v_z) = -\frac{k^2}{n^2} g \Delta_i(\rho_0) v_z$$

where Δ_i denotes the jump across the interface. This gives

$$-k(\rho_2 + \rho_1) + \frac{k_x^2 B_0^2}{4\pi n^2}(-2k) = -\frac{k^2}{n^2} g(\rho_2 - \rho_1)$$

- The dispersion relation becomes

$$n^2 = gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} - \frac{k_x^2 B_0^2}{2\pi(\rho_1 + \rho_2)gk}\right)$$

For $\rho_2 > \rho_1$, $n^2 > 0$ for

$$k_x^2 < \frac{(\rho_2 - \rho_1)2\pi gk}{B_0^2}$$

For larger k_x, $n^2 < 0$ and the perturbations are oscillatory. Nevertheless, the system is unstable to perturbations with small k and hence is unstable.

- The effect of horizontal Magnetic field is similar to that of surface tension as the horizontal magnetic field provides tension force. The equivalent surface tension is given by

$$T = \frac{B_0^2 \cos^2 \theta}{2\pi k}$$

where θ is the angle between magnetic field and the propagation direction defined by k.

Magnetic Buoyancy

- If the horizontal magnetic field varies with height, then the magnetic pressure would also be effective. We will consider the case of no dissipation, i.e., $\nu = \eta = 0$, but consider a compressible fluid (Acheson, 1979, Solar Phys., 62, 23).

- In the equilibrium state, $\partial/\partial t = 0$, $\partial/\partial x = 0$, $\partial/\partial y = 0$ and $\mathbf{v} = 0$, the total pressure should balance gravity

$$\frac{d}{dz}\left(P + \frac{B^2}{8\pi}\right) = -g\rho$$

- In a compressible fluid we can define the Buoyancy frequency or the Brünt Väisälä frequency (see fm2.pdf)

$$N^2 = g\left(\frac{1}{c^2\rho}\frac{dP}{dz} - \frac{1}{\rho}\frac{d\rho}{dz}\right)$$

This can be used to write dP/dz in terms of $d\rho/dz$

$$\frac{dP}{dz} = c^2\frac{d\rho}{dz} + \frac{N^2c^2\rho}{g}$$

- Instead of doing a formal stability analysis we would just consider that a flux tube of cross-section Δ at height z is lifted without twisting it to a height $z + dz$. If we define a notation where any property ϕ inside the flux-tube changes to $\phi + \delta\phi$. While due to stratification the change is $\phi + d\phi$. Since the mass per unit length $\rho\Delta$ and the magnetic flux $B\Delta$ should be conserved. Hence the quantity B/ρ is conserved, giving

$$\frac{\delta B}{B} = \frac{\delta \rho}{\rho}$$

- The dynamical time-scale is much smaller than the thermal time-scale and hence we can consider adiabatic perturbation with no energy exchange. The (total) pressure balance would be achieved

$$\delta P + \frac{1}{4\pi} B \delta B = dP + \frac{1}{4\pi} B dB$$

Using the adiabatic approximation

$$c^2 \delta\rho + \frac{B^2}{4\pi\rho} \delta\rho = (c^2 + V_A^2)\delta\rho = dP + \frac{1}{4\pi} B dB$$

- If $\delta\rho < d\rho$, the system is unstable, which gives

$$dP + \frac{1}{4\pi} B dB < (c^2 + V_A^2) d\rho$$

Dividing by dz and eliminating dP/dz we get

$$c^2 \frac{d\rho}{dz} + \frac{N^2 c^2 \rho}{g} + \frac{B^2}{4\pi\rho} \frac{\rho}{B} \frac{dB}{dz} < (c^2 + V_A^2) \frac{d\rho}{dz}$$

dividing by $V_A^2 \rho$ which gives

$$\frac{1}{\rho} \frac{d\rho}{dz} - \frac{1}{B} \frac{dB}{dz} > \frac{N^2 c^2}{g V_A^2}$$

or

$$-\frac{d}{dz} \left(\ln \left(\frac{B}{\rho} \right) \right) > \frac{N^2 c^2}{g V_A^2}$$

- In the absence of magnetic field we recover the well-known Ledoux stability criterion $N^2 > 0$ for stability. For $N^2 < 0$ we get convection.

- If we consider the stability of a magnetic flux-tube in non-magnetic fluid. Then the magnetic field in external medium is dropped and we get for instability

$$\frac{d\ln\rho}{dz} > \frac{N^2c^2}{gV_A^2}$$

Since ρ is decreasing with height the derivative is negative and this condition can be satisfied only if $N^2 < 0$, i.e., inside the convection zone.