Vorticity and Circulation

- Fluid flow can be classified as laminar or turbulent. Laminar flow is defined as a flow where the velocity changes smoothly with $\mathbf{r}, t$. Flow with low velocity generally tend to be laminar. Viscosity also plays a role with large viscosity favouring laminar flow. (Stability)

- On the other hand turbulent flows tend to have strong variations, including presence of eddies. In a highly turbulent flow there would be eddies with a large range of length and time-scales, which makes it difficult to model the flow.

- irrotational flow is defined as a flow with zero vorticity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v} = 0$.

- Consider an incompressible fluid with conservative external force, which can be written as a gradient. Using the vector identity

$$\mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = \boldsymbol{\nabla}(\frac{1}{2}v^2) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})$$

we can write the equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla}(\frac{1}{2}v^2) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = -\boldsymbol{\nabla}\Psi - \boldsymbol{\nabla}\left(\frac{P}{\rho}\right) + \nu \boldsymbol{\nabla}^2 \mathbf{v}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

- Taking the curl of this equation, assuming ν to be a constant, we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}) = \nu \nabla^2 \boldsymbol{\omega}$$

Noting that all terms involve $\boldsymbol{\omega}$, if $\boldsymbol{\omega} = 0$ initially, it will remain zero and the flow would be irrotational. Thus vorticity cannot be generated in the interior of fluid, but is produced at the boundary and is advected inside. An advantage of this equation is that it doesn't involve the external force and pressure.

- Using the identity

$$\boldsymbol{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\mathbf{v} - \mathbf{v} \cdot \boldsymbol{\nabla}\boldsymbol{\omega} + \mathbf{v}(\boldsymbol{\nabla} \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}(\boldsymbol{\nabla} \cdot \mathbf{v})$$

with $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0$, we get

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}$$

The last term gives diffusion of vorticity

- For a compressible inviscid fluid, when we take the curl of eq we would get an extra term

$$-\boldsymbol{\nabla} \times \left(\frac{1}{\rho} \boldsymbol{\nabla} P \right) = \frac{1}{\rho^2} \boldsymbol{\nabla} \rho \times \boldsymbol{\nabla} P$$

- If s is the entropy which by EOS $s(P, \rho)$ and hence $\boldsymbol{\nabla} s$ is a linear combination of gradients of ρ, P. Thus taking scalar product with $\boldsymbol{\nabla} s$ gives the vorticity eq

$$\begin{aligned} \boldsymbol{\nabla} s \cdot \frac{\partial \boldsymbol{\omega}}{\partial t} &= \boldsymbol{\nabla} s \cdot \boldsymbol{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}) \\ &= -\boldsymbol{\nabla} \cdot [\boldsymbol{\nabla} s \times (\mathbf{v} \times \boldsymbol{\omega})] \end{aligned}$$

where we have used the identity

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\nabla} \times A - \mathbf{A} \cdot \boldsymbol{\nabla} \times B$$

Using

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

The RHS of eq becomes

$$\begin{aligned}
&- \boldsymbol{\nabla} \cdot [(\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s)\mathbf{v}] + \boldsymbol{\nabla} \cdot [(\mathbf{v} \cdot \boldsymbol{\nabla} s)\boldsymbol{\omega}] \\
&= -(\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s)\boldsymbol{\nabla} \cdot \mathbf{v} - \mathbf{v} \cdot \boldsymbol{\nabla}(\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s) + \boldsymbol{\omega} \cdot \boldsymbol{\nabla}(\mathbf{v} \cdot \boldsymbol{\nabla} s)
\end{aligned}$$

- For inviscid fluid the flow is adiabatic as there is no dissipation. Thus

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} s = 0$$

Thus

$$\mathbf{v} \cdot \boldsymbol{\nabla} s = -\frac{\partial s}{\partial t}, \qquad \boldsymbol{\nabla} \cdot \mathbf{v} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

This gives

$$\boldsymbol{\nabla} s \cdot \frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s)\frac{1}{\rho}\frac{d\rho}{dt} - \mathbf{v} \cdot \boldsymbol{\nabla}(\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s) - \boldsymbol{\omega} \cdot \left(\frac{\partial}{\partial t}(\boldsymbol{\nabla} s)\right)$$

$$\frac{\partial}{\partial t}(\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s) + \mathbf{v} \cdot \boldsymbol{\nabla}(\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s) - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s)\frac{1}{\rho}\frac{d\rho}{dt} = 0$$

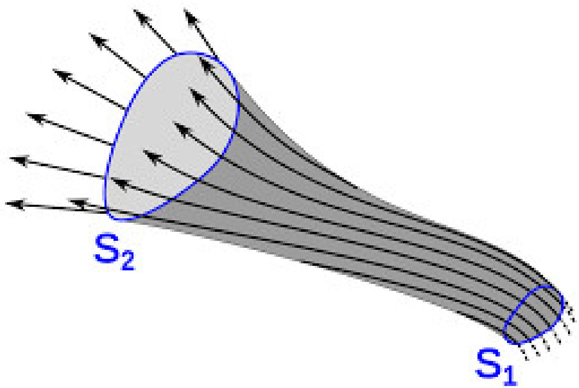
$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s}{\rho}\right) = 0$$

This gives the conserved quantity in this case.

- If $\boldsymbol{\nabla} s = 0$ then there is no entropy gradient and hence $\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P = 0$ and the fluid is said to be barotropic.

- Consider a vortex tube as shown in the figure which is bounded by a surface that is parallel to vorticity $\boldsymbol{\omega}$. If S_1 and S_2 are the surfaces marking the two ends of the vortex tube, then over a closed surface

$$\oint_S \boldsymbol{\omega} \cdot \mathbf{dS} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0$$



- Considering the volume bounded by the two surfaces S_1 and S_2 and the vortex tube. At the surface of the vortex tube $\boldsymbol{\omega} \cdot dS = 0$ and hence

$$\int_{S_1} \boldsymbol{\omega} \cdot \mathbf{dS_1} = \int_{S_2} \boldsymbol{\omega} \cdot \mathbf{dS_2} \ ,$$

  where the normal $\mathbf{dS_1}$ and $\mathbf{dS_2}$ are directed along the direction of ω. Thus the vorticity flux across the cross-section of the tube is conserved.

- In 2D space (x, y)

$$\mathbf{v} = (v_x, v_y, 0) \qquad \text{and} \quad \frac{\partial}{\partial z} = 0$$

So $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ is perpendicular to $\mathbf{v}$ (along z axis) and

$$\boldsymbol{\omega} \cdot \boldsymbol{\nabla} \mathbf{v} = \left(\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z} \right) \mathbf{v} = 0$$

to get

$$\frac{d\boldsymbol{\omega}}{dt} = \nu \nabla^2 \boldsymbol{\omega}$$

For inviscid fluid the vorticity is conserved in 2D.

- For irrotational flow $\boldsymbol{\nabla} \times \mathbf{v} = 0$ and hence we can write $\mathbf{v} = \boldsymbol{\nabla}\phi$. Thus there is only one independent component of velocity. For incompressible fluid

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \nabla^2 \phi = 0$$

  The potential satisfies the Laplace eq.

- Alternately we can use stream function, $\boldsymbol{\psi}$

$$\mathbf{v} = \boldsymbol{\nabla} \times \boldsymbol{\psi}$$

  which satisfied the continuity eq. $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and for irrotational flow we get

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\psi}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\psi}) - \nabla^2 \boldsymbol{\psi} = 0$$

This still requires 3 independent components.

- For 2D flows we need only ψ_z as

$$\mathbf{v} = \boldsymbol{\nabla} \times \boldsymbol{\psi} = \left(\frac{\partial \psi_z}{\partial y}, \; -\frac{\partial \psi_z}{\partial x}, \; \frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y} \right)$$

Since $v_z = 0$ in 2D, only ψ_z is needed and hence stream function is useful in 2D, flows, including axisymmetric flow in spherical polar coordinates (r, θ, ϕ), where $v_\phi = 0$ and $\frac{\partial}{\partial \phi} = 0$. The same applies for 2D polar coordinates (r, θ, z), where flow is independent of z. In this case $\boldsymbol{\nabla} \cdot \boldsymbol{\psi} = 0$.

- This form is valid even if the flow is not irrotational. In that case ψ_z can be determined by solving the vorticity eq. The eq of continuity is automatically satisfied and

$$\boldsymbol{\omega}_z = -\nabla^2 \psi_z$$

other components of vorticity are zero and $\boldsymbol{\omega} \cdot \boldsymbol{\nabla} = 0$.

- Thus the vorticity eq becomes

$$\frac{d\omega_z}{dt} = \nu \nabla^2 \omega_z$$

- For a steady flow in a compressible fluid, the continuity eq gives

$$\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

and we can define the stream function by $\rho \mathbf{v} = \boldsymbol{\nabla} \times \boldsymbol{\psi}$.

- If the flow and steady state (ρ) are in 2D, i.e., $v_z = 0$ and $\partial/\partial z = 0$, then again only ψ_z is needed and

$$\rho v_x = \frac{\partial \psi_z}{\partial y}, \qquad \rho v_y = -\frac{\partial \psi_z}{\partial x}$$

- e.g., steady axisymmetric meridional flow in spherical polar coordinates (r, θ, ϕ) with $v_\phi = 0$ and $\partial/\partial\phi = 0$ we can define

$$\rho v_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \psi), \qquad \rho v_\theta = -\frac{1}{r} \frac{\partial r \psi}{\partial r}$$

It can be verified that this satisfies

$$\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\rho \sin\theta v_\theta) = 0$$

A rotational component $v_\phi(r, \theta)$ can also be added without disturbing the above expression for v_r, v_θ.

- Bernoulli equation: For potential flow in inviscid, incompressible fluid, the Navier–Stokes eq

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla}\left(\frac{v^2}{2}\right) - \mathbf{v} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}\left(\frac{P}{\rho} + \Psi\right)$$

gives

$$\boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2\right) = -\boldsymbol{\nabla}\left(\frac{P}{\rho} + \Psi\right)$$

which gives

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{P}{\rho} + \Psi = f(t)$$

- For hydrostatic case $\mathbf{v} = 0$ and $\frac{\partial}{\partial t} = 0$ we get

$$P + \rho\Psi = c$$

  For gravity $\Psi = gh$

- For steady flow with no external force

$$P + \rho\frac{v^2}{2} = c$$

- For steady flow with constant gravity in z direction

$$\frac{1}{2}\rho v^2 + P + \rho g z = c$$

  The first term is the dynamic pressure

Circulation

- It is customary to define the strength of a vortex tube by

$$\int_S \boldsymbol{\omega} \cdot \mathbf{n} \, dS$$

  over an open surface which is bounded by a closed curve lying on the surface and linking it once.

- Using Stokes theorem we can write this as circulation

$$C = \oint_L \mathbf{v} \cdot dl$$

- For rotation, $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ and circulation about a ring with centre along the axis of rotation and radius R.

$$C = \oint \mathbf{v} \cdot \mathbf{dl} = \int_0^{2\pi} \Omega R^2 \, d\theta = 2\pi \Omega R^2$$

  Thus the circulation is 2π times the angular momentum per unit mass, or 2Ω times the area of the circle.

- Advantage of circulation is that it can be computed without knowing the axis of rotation.

- To calculate the time derivative of C, we need time derivative of $\mathbf{dl}$. For Lagrangian derivative the curve moves with the fluid. Using $\delta\mathbf{l} = \mathbf{r} + \delta\mathbf{r} - \mathbf{r}$, then at time $t + \delta t$

$$\begin{aligned}\delta\mathbf{l}(t+\delta t) &= (\mathbf{r} + \delta\mathbf{r} + (\mathbf{v} + \delta\mathbf{v})\delta t) - (\mathbf{r} + \mathbf{v}\delta t) \\ &= \delta\mathbf{r} + \delta t\delta\mathbf{v}\end{aligned}$$

Now $\mathbf{v} \cdot \delta\mathbf{v} = (1/2)\delta v^2$ and integral of that over a closed contour would vanish.

$$\frac{dC}{dt} = \oint \frac{d\mathbf{v}}{dt} \cdot d\mathbf{l}$$

For incompressible fluid with conservative external force, using the equation of motion we get

$$\frac{dC}{dt} = \oint d\mathbf{l} \cdot \boldsymbol{\nabla}\left(-\Psi - \frac{P}{\rho}\right) + \nu \oint (\nabla^2\mathbf{v}) \cdot d\mathbf{l}$$

Noting that integral over closed curve of gradient would vanish we get

$$\frac{dC}{dt} = \nu \oint (\nabla^2 \mathbf{v}) \cdot d\mathbf{l} = -\nu \oint (\boldsymbol{\nabla} \times \boldsymbol{\omega}) \cdot d\mathbf{l}$$

Thus the circulation can change only through viscous diffusion of vorticity across the closed curve.

- For inviscid fluid the circulation is conserved, Kelvin's theorem.

- This can be written as

$$\frac{d}{dt} \int_S \boldsymbol{\omega} \cdot \mathbf{n} dS = -\nu \oint \boldsymbol{\nabla} \times \boldsymbol{\omega} \cdot d\mathbf{l}$$

Reynolds No.

- Reynolds No. is the ratio of inertial term to viscous force. The inertial term

$$\rho \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} \sim \frac{\rho V^2}{L}$$

Viscous force

$$\mu \nabla^2 \mathbf{v} \sim \frac{\mu V}{L^2}$$

Hence, Reynolds No.

$$R = \frac{\rho V L}{\mu}$$

For small R the flow is laminar, while for large R it is turbulent. The critical value depends on the system.

- When R is large the fluid may be considered as inviscid in the interior, but viscosity term is higher order and hence additional boundary conditions are required. Near the boundaries viscosity needs to be included, giving a boundary layer. If d is the thickness of boundary layer then equating the two terms would give

$$R\left(\frac{d}{L}\right)^2 \approx 1$$

- For $R \ll 1$ we can neglect the inertial term. For incompressible fluid with no external force

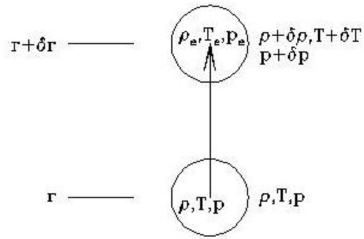
$$\boldsymbol{\nabla} P = \mu \nabla^2 \mathbf{v}$$

Convection

- Consider a fluid element moving from radius r to $r + \delta r$ The external ρ, T, p become

$$\rho + \delta\rho = \rho + \frac{d\rho}{dr}\delta r$$

$$p + \delta p = p + \frac{dp}{dr}\delta r$$



The element reaches pressure equilibrium

$$p_e = p + \frac{dp}{dr}\delta r$$

If there is no exchange of energy then the density is

$$\rho_e = \rho + \left(\frac{d\rho}{dp}\right)_s \delta p = \rho + \frac{1}{\Gamma_1}\frac{\rho}{p}\frac{dp}{dr}\delta r$$

where

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho} \right)_s$$

- For stability $\rho_e > \rho + \delta\rho$ or

$$\frac{1}{\Gamma_1} \frac{\rho}{p} \frac{dp}{dr} \delta r > \frac{d\rho}{dr} \delta r$$

or

$$\frac{1}{\Gamma_1 p} \frac{dp}{dr} - \frac{1}{\rho} \frac{d\rho}{dr} > 0$$

Which is the Ledoux criterion for convective stability.

- The frequency of oscillation is given by N.

$$N^2 = g \left(\frac{1}{\Gamma_1 p} \frac{dp}{dr} - \frac{1}{\rho} \frac{d\rho}{dr} \right)$$

The Brünt Väisälä frequency or buoyancy frequency.