Magnetohydrodynamics (cont)

- The Lorentz force

$$\frac{1}{4\pi}(\mathbf{\nabla} \times \mathbf{B}) \times \mathbf{B} = \frac{1}{4\pi}\left(\mathbf{B} \cdot \mathbf{\nabla}\mathbf{B} - \frac{1}{2}\mathbf{\nabla}(B^2)\right)$$

- In Cartesian coordinates we can write, the ith component

$$\frac{1}{4\pi}\left(B_j \frac{\partial B_i}{\partial x_j} - \frac{1}{2}\frac{\partial B^2}{\partial x_i}\right)$$

Noting that $\mathbf{\nabla} \cdot B = 0$ we can write it as

$$\frac{1}{4\pi}\frac{\partial}{\partial x_j}\left(B_j B_i - \frac{1}{2}B^2\delta_{ij}\right)$$

which can be used to define the magnetic stress tensor, or the Maxwell stress as

$$M_{ij} = \frac{1}{8\pi}B^2\delta_{ij} - \frac{1}{4\pi}B_iB_j$$

- Using this definition the Lorentz force can be written as $-\mathbf{\nabla} \cdot \mathbf{M}$.
- The viscous stress tensor is referred to as the Reynold stress, which is given by $P\delta_{ij} - \sigma_{ij}$ or

$$P\delta_{ij} - \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{2}{3}\mu\delta_{ij}\frac{\partial v_k}{\partial x_k}$$

Taylor Proudman Theorem

- Steady slow motion of a perfectly conducting incompressible plasma permeated by a uniform magnetic field must be two-dimensional, with no variation along the field direction

- The induction equation gives

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0 = \mathbf{B} \cdot \nabla \mathbf{v}$$

  which shows $\mathbf{v}$ has no variation in direction of $\mathbf{B}$.

- The same eq is obtained in Ferraro's law of isorotation, which considers axisymmetric system with velocity due to rotation alone.

- Consider the cylindrical coordinate system (r, θ, z) with steady axisymmetric magnetic field and velocity due to rotation about z axis

$$\mathbf{B} = (B_r(r,z), 0, B_z(r,z)), \quad \mathbf{v} = (0, r\Omega(r,z), 0)$$

then

$$\mathbf{v} \times \mathbf{B} = (r\Omega B_z, 0, -r\Omega B_r)$$

and

$$\boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) = \left(0, \frac{\partial}{\partial z}(r\Omega B_z) + \frac{\partial}{\partial r}(r\Omega B_r), 0\right)$$

Equating it to zero we get

$$rB_z \frac{\partial \Omega}{\partial z} + r\Omega \frac{\partial B_z}{\partial z} + rB_r \frac{\partial \Omega}{\partial r} + \Omega \frac{\partial r B_r}{\partial r} = 0$$

Using $\nabla \cdot \mathbf{B} = 0$ we get the eq

$$\mathbf{B} \cdot \nabla \Omega = 0$$

- Hence the rotation rate is constant along the field line.
- If the same analysis is applied to vorticity equation with the same velocity one would get $\partial \Omega / \partial z = 0$. Thus the rotation rate is constant along cylinders.

- For compressible fluid the Induction equation can be written as

$$\frac{\partial \mathbf{B}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) = \eta \nabla^2 B$$

or

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{B} - \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{v} + \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{v}) = \eta \nabla^2 B$$

using the continuity eq.

$$\frac{d\mathbf{B}}{dt} - \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{v} - \mathbf{B} \frac{1}{\rho} \frac{d\rho}{dt} = \eta \nabla^2 B$$

or

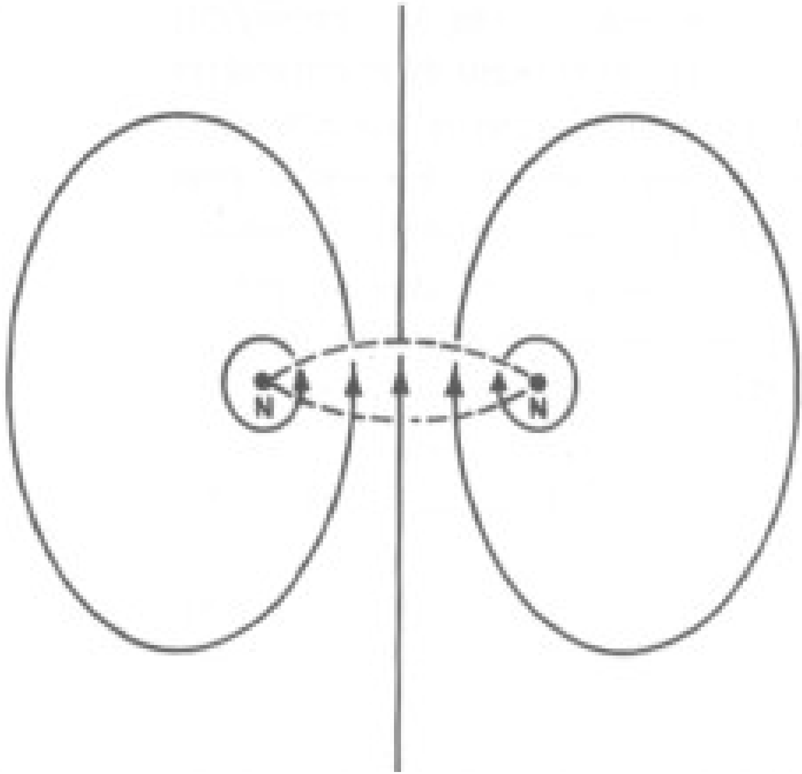
$$\frac{d}{dt}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho} \cdot \boldsymbol{\nabla} \mathbf{v} + \frac{\eta}{\rho} \nabla^2 B$$

Cowling's anti-dynamo theorem

- A steady axisymmetric magnetic field vanishing at infinity can not be maintained by dynamo action.

- Consider a spherical polar system (r, θ, ϕ). Axisymmetry implies $\partial/\partial\phi = 0$. Decompose the field in terms of poloidal and toroidal components $\mathbf{B} = \mathbf{B}_p + B_\phi \hat{\boldsymbol{\phi}}$.

- Because of symmetry the field lines of $\mathbf{B}_p$ are the same in each meridional plane ($\phi = c$) passing through the axis. The field lines are closed since the field is confined to finite volume. Each plane has two neutral points on either side of axis where $\mathbf{B}_p = 0$



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- Start with Ohm's law

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)$$

Taking the integral along a closed circle passing through the neutral points

$$\oint J_\phi \, dl = \sigma \oint \mathbf{E} \cdot \mathbf{dl} + \frac{\sigma}{c} \oint \mathbf{v} \times \mathbf{B} \cdot \mathbf{dl}$$

The 2nd term on RHS vanishes as $\mathbf{B}_p = 0$ and B_ϕ is parallel to $\mathbf{dl}$. Using Stokes theorem the first term give

$$\int_S \nabla \times \mathbf{E} \cdot \mathbf{n} \, dS = -\frac{1}{c} \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, dS = 0$$

However, J_ϕ is non-zero as field lines are circling the neutral point. Thus there is a contradiction.

- Even for non steady field it can be shown that the field will decay with time for incompressible fluid. We use cylindrical polar coordinates (r, θ, z) and Axisymmetry requires $\partial/\partial\theta = 0$. The poloidal components are given by

$$\mathbf{B}_p = \frac{\boldsymbol{\nabla}\psi \times \hat{\boldsymbol{\theta}}}{r} = \frac{1}{r}\left(-\frac{\partial\psi}{\partial z}, 0, \frac{\partial\psi}{\partial r}\right)$$

$$\mathbf{v}_p = \frac{\boldsymbol{\nabla}\phi \times \hat{\boldsymbol{\theta}}}{r} = \frac{1}{r}\left(-\frac{\partial\phi}{\partial z}, 0, \frac{\partial\phi}{\partial r}\right)$$

where ψ, ϕ can be considered as stream functions. This satisfies $\boldsymbol{\nabla} \cdot \mathbf{B}_p = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{v}_p = 0$.

- Consider the toroidal (θ) component of Ohm's law

$$J_t = \sigma\left(E_t + \frac{\mathbf{v}_p \times \mathbf{B}_p \cdot \hat{\boldsymbol{\theta}}}{c}\right)$$

From the $\nabla \times \mathbf{E}$ equation it can be verified that

$$E_t = -\frac{1}{rc}\frac{\partial \psi}{\partial t}$$

while

$$\begin{aligned}
(\mathbf{v}_p \times \mathbf{B}_p)_\theta &= \frac{1}{r^2}(\nabla\phi \times \nabla\psi)_\theta \\
&= \frac{1}{r^2}\left(\frac{\partial\psi}{\partial r}\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial z}\right)
\end{aligned}$$

Similarly

$$\begin{aligned}
J_t &= \frac{c}{4\pi}(\nabla \times \mathbf{B}_p)_\theta \\
&= -\frac{c}{4\pi r}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right)
\end{aligned}$$

This gives the equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial r}\frac{\partial \psi}{\partial z}\right)$$
$$+\eta\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right)$$

Multiply by ψ and integrate over the entire space. The first term on RHS gives

$$2\pi \int\int \psi\left(\frac{\partial \psi}{\partial r}\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial r}\frac{\partial \psi}{\partial z}\right)\, drdz$$

Integrating by parts we get

$$2\pi \int\int \left[-\phi\frac{\partial}{\partial z}\left(\psi\frac{\partial \psi}{\partial r}\right) + \phi\frac{\partial}{\partial r}\left(\psi\frac{\partial \psi}{\partial z}\right)\right] drdz = 0$$

The last term gives

$$2\pi \int\int r\psi \left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} \right) drdz$$

Integrating by part the 1st and last term gives

$$-2\pi \int\int \left(\frac{\partial r\psi}{\partial r}\frac{\partial \psi}{\partial r} + r\left(\frac{\partial \psi}{\partial z}\right)^2 \right) dr\ dz$$

which gives

$$-2\pi \int\int r\left[\left(\frac{\partial \psi}{\partial r}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 \right] dr\ dz$$

This gives the final eq.

$$\frac{\partial}{\partial t}\int \psi^2 \, dV = -2\eta \int |\boldsymbol{\nabla}\psi|^2 \, dV$$

Thus ψ^2 decreases with time and the field would decay.

- Similarly, using θ component of the induction eq we can show that the toroidal component, B_t also decays with time.

Force Free Magnetic Field

- If magnetic pressure is much larger than the gas pressure, the Lorentz force term in the eq. of motion dominates and is larger that all other terms. Hence the field configuration must be such that this term vanishes giving the force free field

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$$

  This gives

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}$$

  where α is a scalar function. If α is a constant, then taking the curl of this eq give

$$\nabla \times (\nabla \times \mathbf{B}) = \alpha \nabla \times \mathbf{B} = \alpha^2 \mathbf{B}$$

or

$$(\nabla^2 + \alpha^2)\mathbf{B} = 0$$

- A simpler force-free field is given by the potential field $\mathbf{B} = \boldsymbol{\nabla}\psi$, which gives the eq.

$$\nabla \cdot \mathbf{B} = \nabla^2\psi = 0$$

- This is useful in estimating the magnetic field in solar corona using the photoshperic magnetic field as the boundary condition.

- We can define helicity of magnetic field or any other vector field by

$$H = \int_V (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{B}\, dV$$

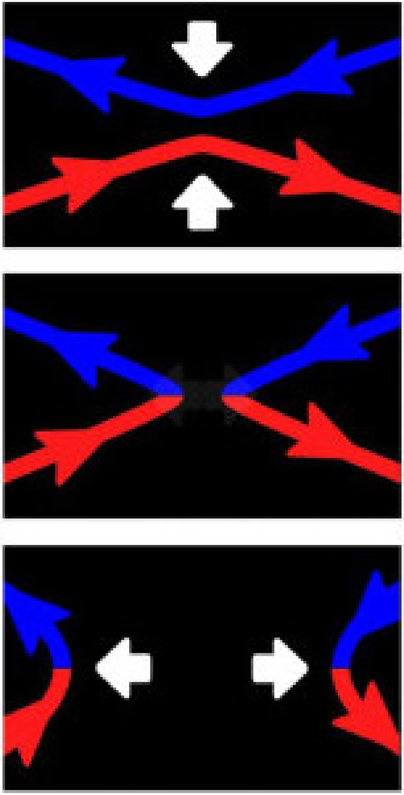
  For velocity this describes a helical motion $\boldsymbol{\omega} \cdot \mathbf{v}$. For pure rotation this will vanish, but if a component parallel to axis of rotation is added, it becomes non-zero.

- For a force free magnetic field, the integrand is αB^2.

Magnetic Reconnection

- Magnetic reconnection is the breaking and reconnecting magnetic field lines in a conducting medium. Typically two oppositely directed field line approach each other and reconnect to change the magnetic field configuration releasing magnetic energy.

- Magnetic reconnection is implied in many astrophysical phenomenon, e.g., solar flares, magnetic storms in Earth's magnetosphere, magnetars and possibly Fast Radio Bursts.



Credits: Center for Visual computing, Univ. of California Riverside

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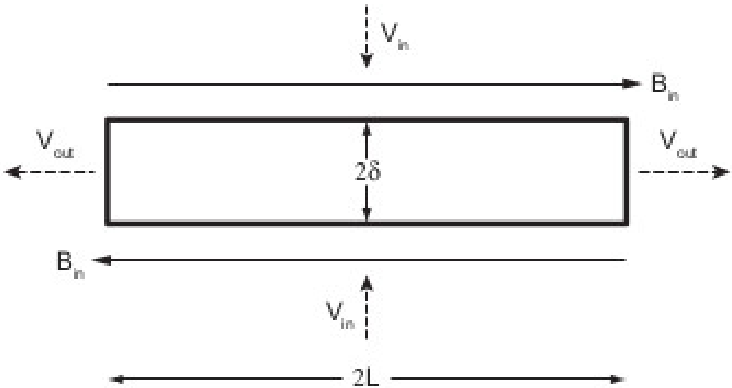
- Magnetic reconnection is strictly not possible in a perfectly conducting plasma as the magnetic field lines are frozen with the fluid.

- As the field lines approach each other, the magnetic field gradient increases and the length scale reduces. At some stage the resistivity becomes important and leads to reconnection. The magnetic field gradient leads to strong currents and formation of current sheet where resistivity needs to be included,

- Magnetic reconnection leads to release of magnetic energy and change in magnetic topology. This can happen on a very short time-scale leading to sudden release of energy.

The Sweet-Parker model provides the simplest description of resistive magnetic reconnection



- Elongated current sheet of half-length L and half-width δ
- Characteristic inflow velocity V_{in} and magnetic field B_{in}
- Characteristic outflow velocity V_{out}
- Uniform density ρ and resistivity η

- Resistivity is ignored outside the current sheet
- Conservation of mass

$$LV_{\rm in} \sim \delta V_{\rm out}$$

- Conservation of energy: Magnetic energy to kinetic energy flux across the length L

$$LV_{\rm in}\left(\frac{B_{\rm in}^2}{8\pi}\right) \sim \delta V_{\rm out}\left(\frac{\rho V_{\rm out}^2}{2}\right)$$

- Combining these eq gives

$$V_{\rm out}^2 \sim V_A^2 = \frac{B_{\rm in}^2}{4\pi\rho}$$

- Current density can be estimated from Ohm's law and the Ampere's law to get

$$J \sim \frac{\sigma}{c} V_{\rm in} B_{\rm in} \sim \frac{c}{4\pi} \frac{B_{\rm in}}{\delta}$$

This gives

$$V_{\rm in} \sim \frac{\eta}{\delta}$$

- Combining with earlier estimate

$$\frac{V_{\rm in}}{V_A} \sim \frac{\delta}{L} \sim \frac{\eta}{\delta V_A} \sim \frac{1}{S^{1/2}}$$

where the Lundquist number

$$S = \frac{L V_A}{\eta} = \frac{V_A}{V_{\rm in}} R_m$$

where R_m is the magnetic Reynolds number. S is typically 10^9 to 10^{20} is astrophysical situation.

- We can write all quantities in terms of S

$$\delta^2 = \frac{\eta L}{V_A} = \frac{L^2}{S}$$

Further,

$$V_{\rm in} \sim V_A \frac{L}{\delta} \frac{1}{S} \sim \frac{V_A}{S^{1/2}}$$

- Magnetic reconnection converts magnetic energy to KE. The outflow will interact with surrounding medium and would ultimately generate heat through viscous dissipation.