Jean's Instability with Magnetic Field

- We assume a constant magnetic field, $\mathbf{B_0}$ as the initial state. Since Lorentz force is zero the steady state remains the same. The induction eq is added

$$\frac{\partial B}{\partial t} = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{v})$$

- The perturbation eqs, with $\mathbf{B} = \mathbf{B_0} + \mathbf{B_1}$ are

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\rho_0 \nabla \Psi - \nabla P_1 - \frac{1}{4\pi}[\nabla(\mathbf{B}_0 \cdot \mathbf{B}_1) - \mathbf{B}_0 \cdot \nabla \mathbf{B}_1]$$

$$\frac{\partial B_1}{\partial t} = \mathbf{B}_0 \cdot \nabla \mathbf{v} - \mathbf{B}_0(\nabla \cdot \mathbf{v})$$

- Take divergence of eq. of motion

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \nabla^2 \rho_1 + 4\pi G \rho_0 \rho_1 + \frac{1}{4\pi} \nabla^2 (\mathbf{B}_0 \cdot \mathbf{B}_1)$$

- Take wave travelling in z direction and $\mathbf{B}_0 = (0, B_y, B_z)$ and look for solutions of form $\exp(\omega t + ikz)$ and $\mathbf{B}_1 = (b_x, b_y, b_z)$.

- Then $\nabla \cdot \mathbf{B}_1 = 0$ gives $b_z = 0$ and $\mathbf{B}_0 \cdot \mathbf{B}_1 = B_y b_y$. The y-components of eq. of motion and induction eq. give

$$\omega b_y = ikB_z v_y + \frac{B_y}{\rho_0} \omega \rho_1$$

$$\omega \rho_0 v_y = \frac{B_z}{4\pi} ikb_y$$

- Eliminating v_y we get

$$\omega^2 b_y = -k^2 \frac{B_z^2}{4\pi\rho_0} b_y + \frac{B_y}{\rho_0}\omega^2\rho_1$$

Using this to eliminate b_y we get the dispersion relation

$$\omega^2 = 4\pi G\rho_0 - k^2 c_s^2 - k^2 \frac{B_y^2}{4\pi\rho_0} \frac{\omega^2}{\omega^2 + k^2 \frac{B_z^2}{4\pi\rho_0}}$$

which gives

$$\omega^4 + (k^2 \frac{B_0^2}{4\pi\rho} + k^2 c_s^2 - 4\pi G\rho_0)\omega^2 + k^2 \frac{B_z^2}{4\pi\rho_0}(k^2 c_s^2 - 4\pi G\rho_0) = 0$$

- Using the Alfven speed V_A and angle θ between magnetic field and z-axis we get

$$\omega^4+(k^2V_A^2+k^2c_s^2-4\pi G\rho)\omega^2+k^2V_A^2\cos^2\theta(k^2c_s^2-4\pi G\rho)=0$$

- If $4\pi G\rho > k^2c_s^2$ one root with $\omega^2 > 0$ giving instability. This is the same condition for Jean's instability. If $\cos\theta = 0$ or $B_z = 0$ or the field is normal to direction of wave then the stability condition will have one additional term in $k^2V_A^2$.

- Since stability has to be achieved for all perturbations, $\cos\theta \neq 0$ would give the stability condition.

Convection: Effect of rotation

- We consider the problem of convection in a rotating fluid. If we consider the eq in a rotating frame, the continuity eq and energy eq would remain the same. Only in the eq of motion there would be additional forces due to rotation.

- Consider a rotating frame (x, y, z) which is rotating about the z-axis with a rotation rate Ω w.r.t. an inertial frame (ξ, η, ζ) with ζ-axis coinciding with z-axis. The transformation is given by

$$x = \xi \cos(\Omega t) + \eta \sin(\Omega t)$$
$$y = -\xi \sin(\Omega t) + \eta \cos(\Omega t)$$
$$z = \zeta$$

- Differentiating these eq twice we can show that there would be additional acceleration given by

$$2\mathbf{v} \times \boldsymbol{\Omega} + \boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\Omega} \times \mathbf{r}|^2\right)$$

  which are respectively the Coriolis and Centrifugal acceleration.

- We assume that $\boldsymbol{\Omega}$ is constant. If fluid is at rest in the rotating frame, then the Coriolis force vanishes. The Centrifugal force depends only on the position and hence this term will not be perturbed.

Waves in a rotating fluid

- We look for solution of the form $\exp(i(pt - \mathbf{k}\cdot\mathbf{r}))$ for an incompressible fluid. The continuity eq. give $\mathbf{k}\cdot\mathbf{v} = 0$. Note that we are not linearising the eq. We are considering an infinite system.

- The eq. of motion gives

$$(ip + \nu k^2)\mathbf{v} + i(\mathbf{v}\cdot\mathbf{k})\mathbf{v} = -i\mathbf{k}\Pi + 2\mathbf{v}\times\mathbf{\Omega}$$

where

$$\Pi = \frac{P_1}{\rho}$$

The nonlinear term vanishes.

- Use $n = p - ik^2\nu$ and take dot product with $\mathbf{v}$, $\mathbf{k}$ and $\mathbf{\Omega}$ we get

$$
\begin{aligned}
n\mathbf{v}\cdot\mathbf{v} &= 0 \\
k^2\Pi + 2i\mathbf{k}\cdot(\mathbf{v}\times\Omega) &= 0 \\
n\mathbf{v}\cdot\mathbf{\Omega} + \Pi\mathbf{k}\cdot\mathbf{\Omega} &= 0
\end{aligned}
$$

- We consider wave propagating along z-axis $\mathbf{k} = (0, 0, k)$ and take $\mathbf{\Omega} = (\Omega_x, 0, \Omega_z)$. The continuity eq give $kv_z = 0$ and hence $v_z = 0$. Thus the waves are transverse.

- The eqs are

$$
\begin{aligned}
n(v_x^2 + v_y^2) &= 0 \\
k\Pi &= 2iv_y\Omega_x \\
nv_x\Omega_x &= -k\Pi\Omega_z
\end{aligned}
$$

The last two eq give

$$\Pi = -\frac{n}{k}\frac{\Omega_x}{\Omega_z}v_x, \qquad v_y = i\frac{n}{2\Omega_z}v_x$$

- Which gives the dispersion relation

$$n\left(1 - \frac{n^2}{4\Omega_z^2}\right) = 0$$

or $n = \pm 2\Omega_z = \pm 2\Omega\cos\theta$ and the dispersion relation is

$$p = n + i\nu k^2 = \pm 2\Omega\cos\theta + i\nu k^2$$

Thus the waves are damped. Now $v_y = \pm i v_x$ and $v_z = 0$.

- Thus these are transverse and circularly polarised waves. In the limit of $\nu = 0$, the phase speed

$$V = \frac{p}{k} = \frac{2\Omega}{k}\cos\theta$$

and the group velocity is given by

$$\frac{dp}{dk_i} = \pm 2\frac{d}{dk_i}\left(\frac{k_j\Omega_j}{k}\right) = \pm 2\left(\frac{\Omega_i}{k} - \frac{\Omega_j k_j}{k^3}k_i\right) = \pm\frac{2}{k^3}\mathbf{k}\times(\boldsymbol{\Omega}\times\mathbf{k})$$

- To understand the concept of group velocity consider a wave packet at time $t = 0$ defined by

$$a(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk$$

By superposition principle, the wavepacket at time t is given by

$$a(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx-\omega t)} \, dk$$

- Linearising the dispersion relation give

$$\omega(k) \approx \omega_0 + (k - k_0)\omega_0'$$

where $\omega_0 = \omega(k_0)$ and ω_0' is the derivative at the same point. This gives

$$a(x,t) = e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} A(k) e^{i(k-k_0)(x-\omega_0' t)} \, dk$$

Convection in Rotating Fluid

- We assume that the axis of rotation is also along z-axis. The steady state will now have some x, y dependence as pressure has to balance the centrifugal force. This could introduce another approximation as we may neglect the variation in P. But P term vanishes when curl of eq is taken.

- The eq of continuity and energy remain the same. The eq of motion in perturbed quantities is

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\boldsymbol{\nabla} P_1 + \nu\nabla^2\mathbf{v} - \mathbf{g}\alpha T_1 + 2\mathbf{v}\times\boldsymbol{\Omega}$$

- Taking curl of eq. of motion we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \alpha g \left(\frac{\partial T_1}{\partial y}, -\frac{\partial T_1}{\partial x}, 0 \right) + \nu \nabla^2 \boldsymbol{\omega} + 2\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \mathbf{v}$$

Taking the curl once again

$$\frac{\partial}{\partial t} \nabla^2 \mathbf{v} = g\alpha \left(-\frac{\partial^2 T_1}{\partial x \partial z}, -\frac{\partial^2 T_1}{\partial y \partial z}, \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} \right)$$

$$+ \nu \nabla^4 \mathbf{v} - 2\boldsymbol{\Omega} \cdot \boldsymbol{\nabla} \boldsymbol{\omega}$$

- The z component of the two equations gives

$$
\begin{aligned}
\frac{\partial \omega_z}{\partial t} &= \nu \nabla^2 \omega_z + 2\Omega \frac{\partial v_z}{\partial z} \\
\frac{\partial}{\partial t} \nabla^2 v_z &= g\alpha \left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} \right) + \nu \nabla^4 v_z - 2\Omega \frac{\partial \omega_z}{\partial z} \\
\frac{\partial T_1}{\partial t} &= \beta v_z + \kappa \nabla^2 T_1
\end{aligned}
$$

These are 3 eq in 3 unknowns.

- We can first consider the case where there is no dissipation, $\nu = \kappa = 0$. In that case T_1, ω_z can be easily eliminated by differentiating the first eq. w,r,t. t to get

$$\frac{\partial^2}{\partial t^2}\nabla^2 v_z = g\alpha\beta\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) - 4\Omega^2\frac{\partial^2 v_z}{\partial z^2}$$

We can look for solution of form $v_z = \exp(pt + i(k_x x + k_y y + k_z z))$ to get the dispersion relation

$$p^2(k_x^2 + k_y^2 + k_z^2) = g\alpha\beta(k_x^2 + k_y^2) - 4\Omega^2 k_z^2$$

- If $p^2 > 0$ we will have convection, otherwise we get oscillatory modes. For convection we need

$$4\Omega^2 k_z^2 < g\alpha\beta(k_x^2 + k_y^2)$$

- Thus for large rotation, convection tends to be suppressed. However, the system is infinite in x, y directions and hence there is no limit on k_x, k_y. On the other hand k_z has a lower limit of π/d, but there is no upper limit. Thus the system is unstable on some length scale.

- If $\beta = 0$ we get the wave solution considered earlier. This term did not arise earlier as we assumed ρ to be constant. The use of Boussinesq approximation produces this additional term which can give convection.

- Boundary conditions: Now we have an additional eq for vorticity, but that is not independent of velocity and the boundary conditions for vorticity can be obtained from those on velocity. Thus for free boundary condition

$$\frac{\partial \omega_z}{\partial z} = \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial y \partial z} = 0$$

  Similarly, on rigid boundary $\omega_z = 0$.

- We look for solutions of the form

$$v_z = v(z) \exp[i(k_x x + k_y y) + pt]$$
$$\omega_z = \omega(z) \exp[i(k_x x + k_y y) + pt]$$
$$T_1 = \theta(z) \exp[i(k_x x + k_y y) + pt]$$

with $k^2 = k_x^2 + k_y^2$ to get

$$p\left(\frac{d^2}{dz^2} - k^2\right)v = -g\alpha k^2\theta + \nu\left(\frac{d^2}{dz^2} - k^2\right)^2 v - 2\Omega\frac{d\omega}{dz}$$

$$p\omega = 2\Omega\frac{dv}{dz} + \nu\left(\frac{d^2}{dz^2} - k^2\right)\omega$$

$$p\theta = \beta v + \kappa\left(\frac{d^2}{dz^2} - k^2\right)\theta$$

- Again we define dimensionless variables as before. We use d as the unit of length and d^2/ν as the unit of time to get the dimensionless wave-number $a = kd$ and dimensionless growth rate $\sigma = pd^2/\nu$. $D = d\, d/dz$ is the dimensionless derivative to get

$$(D^2 - a^2)(D^2 - a^2 - \sigma)v = \left(\frac{g\alpha d^2}{\nu}\right) a^2\theta + \left(\frac{2\Omega d^3}{\nu}\right) D\omega$$

$$(D^2 - a^2 - \sigma)\omega = -\left(\frac{2\Omega d}{\nu}\right) Dv$$

$$(D^2 - a^2 - \mathcal{P}\sigma)\theta = -\frac{\beta}{\kappa} d^2 v$$

where $\mathcal{P} = \nu/\kappa$ is the Prandtl no.

- In this case elimination is not easy, and it is not possible to find analytic conditions when the transition of instability occurs through secular modes. Thus we assume that the transition to instability happens through $\sigma = 0$ then the eqs become

$$(D^2 - a^2)^2 v = \left(\frac{g\alpha d^2}{\nu}\right) a^2\theta + \left(\frac{2\Omega d^3}{\nu}\right) D\omega$$

$$(D^2 - a^2)\omega = -\left(\frac{2\Omega d}{\nu}\right) Dv$$

$$(D^2 - a^2)\theta = -\frac{\beta}{\kappa} d^2 v$$

- Now we can eliminate θ and ω to get

$$(D^2 - a^2)^3 v = -Ra^2 v - TD^2 v$$

Where R is the Rayleigh no. and the Taylor number

$$T = \frac{4\Omega^2 d^4}{\nu^2}, \qquad R = \frac{g\alpha\beta d^4}{\kappa\nu}$$

- We again look for solutions of form $v = \sin(n\pi z)$ to get

$$Ra^2 = (n^2\pi^2 + a^2)^3 + Tn^2\pi^2$$

It is clear that minimum value of R would be achieved for $n = 1$. Further, we can take $a^2 = \pi^2 x$. Thus the transition to instability occurs when

$$R = \frac{\pi^4}{x}\left((1+x)^3 + \frac{T}{\pi^4}\right)$$

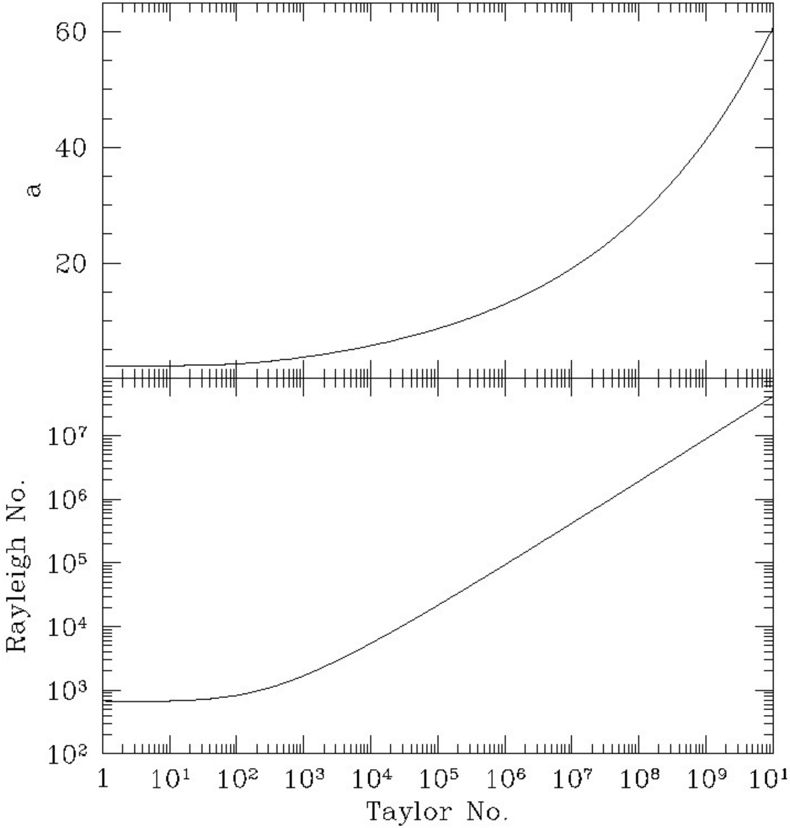
This gives $R(x,T)$. For instability, R has to be greater than this limiting value. The minimum of R over all permissible values of x would give the critical value.

$$\frac{dR}{dx} = -\left(1+\frac{T}{\pi^4}\right)\frac{1}{x^2} + 3 + 2x = 0$$

or

$$2x^3 + 3x^2 - 1 - \frac{T}{\pi^4} = 0$$

This eq has at least one positive root. It turns out that there is only one positive root which gives the required value of x. It can be verified that the second derivative is positive at this point, Hence it is a minimum and the corresponding value of R is shown in the figure.



60
40
20
a
10^7
10^6
10^5
10^4
10^3
10^2
Rayleigh No.
1
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
10^10
Taylor No.

- As $T \to \infty$ we get the solution

$$x_{\min} \approx \left(\frac{T}{2\pi^4}\right)^{1/3}$$

$$R_{\text{crit}} \approx 3\pi^4 \left(\frac{T}{2\pi^4}\right)^{2/3} = 8.7T^{2/3}$$

$$a_{\min} \approx \left(\frac{\pi^2 T}{2}\right)^{1/6} = 1.3T^{1/6}$$

- To complete the solution we can find θ and ω from the respective eqs. To find the horizontal components of velocity we can use the corresponding components of the eq. of motion coupled with that of the vorticity eq. Looking for solutions of the form $\exp(ia_x x + ia_y y)$ which gives

$$(D^2 - a^2)^2 v_x = \left(\frac{g\alpha d^2}{\nu}\right) ia_x D\theta + \frac{2\Omega d^3}{\nu} D\omega_x$$

$$(D^2 - a^2)\omega_x = -\frac{2\Omega d}{\nu} Dv_x - \frac{\alpha g d}{\nu}\frac{\partial\theta}{\partial y}$$

we get

$$(a^2 - D^2)^3 v_x = R\frac{\partial^2}{\partial x \partial z} v_z + \frac{4\Omega^2 d^4}{\nu^2} D^2 v_x + \frac{2\Omega d^4 \alpha g}{\nu^2}\frac{\partial^2\theta}{\partial y \partial z}$$

- Using $v_z = \sin(\pi z)\cos(a_x x)\cos(a_y y)$ we get

$$\theta = \frac{\beta d^2}{\kappa(a^2+\pi^2)}\sin(\pi z)\cos(a_x x)\cos(a_y y)$$

$$\omega_z = \frac{2\Omega d\pi}{\nu(a^2+\pi^2)}\cos(\pi z)\cos(a_x x)\cos(a_y y)$$

$$((a^2+\pi^2)^3+T\pi^2)v_x = -Ra_x\pi\cos(\pi z)\sin(a_x x)\cos(a_y y)$$
$$-\frac{2\Omega d^6\alpha g\beta}{\nu^2\kappa(a^2+\pi^2)}\pi a_y\cos(\pi z)\cos(a_x x)\sin(a_y y)$$

Recognising that the factor multiplying v_x is Ra^2 we get

$$v_x = -\frac{a_x\pi}{a^2}\cos(\pi z)\sin(a_x x)\cos(a_y y)$$
$$-T^{1/2}\frac{\pi a_y}{a^2(\pi^2+a^2)}\cos(\pi z)\cos(a_x x)\sin(a_y y)$$

Onset of Convection as Overstability

- If the onset of convection happens through overstability σ would be pure imaginary at the onset. We can eliminate θ and ω from the eqs to get one eq in v. First eliminating ω

$$(D^2 - a^2)(D^2 - a^2 - \sigma)^2 v = -TD^2 v$$
$$+ \left(\frac{g\alpha d^2}{\nu}\right) a^2 (D^2 - a^2 - \sigma)\theta$$

Now eliminating θ to get

$$(D^2 - a^2 - \mathcal{P}\sigma)[(D^2 - a^2)(D^2 - a^2 - \sigma)^2 + TD^2]v =$$
$$- Ra^2(D^2 - a^2 - \sigma)v$$

Using the solution $v_z = \sin(\pi z)$ we get the dispersion relation

$$(\pi^2 + a^2 + \mathcal{P}\sigma)[(\pi^2 + a^2)(\pi^2 + a^2 + \sigma)^2 + \pi^2 T] = \\ Ra^2(\pi^2 + a^2 + \sigma)$$

Since σ is pure imaginary we define

$$a^2 = \pi^2 x, \quad \sigma = i\sigma_1\pi^2, \quad R = \pi^4 R_1, \quad T = \pi^4 T_1$$

to get

$$(1+x+i\mathcal{P}\sigma_1)[(1+x)(1+x+i\sigma_1)^2+T_1] = R_1 x(1+x+i\sigma_1)$$

- Since all quantities other than σ_1 are real and we wish to find condition when σ_1 is real, we assume that to be the case and separate the real and imaginary parts of the equation and equate them

$$(1+x)[(1+x)((1+x)^2-\sigma_1^2)+T_1]-2\mathcal{P}\sigma_1^2(1+x)^2 = R_1 x(1+x)$$

$$\mathcal{P}[(1+x)((1+x)^2-\sigma_1^2)+T]+2(1+x)^3 = R_1 x$$

Removing the factor $(1+x)$ from first eq and equating the two expressions for $R_1 x$ we get an equation which can be solved for σ_1^2

$$\sigma_1^2 = \frac{T_1}{1+x}\frac{1-\mathcal{P}}{1+\mathcal{P}} - (1+x)^2$$

and using this we get

$$R_1 = \frac{2(1+\mathcal{P})}{x}\left((1+x)^3 + \frac{\mathcal{P}^2}{(1+\mathcal{P})^2}T_1\right)$$

- Since $\sigma_1^2 > 0$ we must have

$$(1+x)^3 < T_1 \frac{1-\mathcal{P}}{1+\mathcal{P}}$$

This gives the critical value x^* where $\sigma_1 = 0$. At this point

$$R_1^* = \frac{2T_1}{x^*(1+\mathcal{P})}$$

- Further if $\mathcal{P} > 1$ then it is not possible to have $\sigma_1^2 > 0$. Thus for $\mathcal{P} > 1$ the transition to instability should occur through secular modes. Only for smaller values it is possible to have transition through oscillatory modes. In fact, for $x^* > 0$ we must have

$$T_1 \frac{1-\mathcal{P}}{1+\mathcal{P}} > 1$$

which give

$$\mathcal{P} < \frac{T_1 - 1}{T_1 + 1}$$

Thus for $T_1 < 1$ it is not possible to satisfy this condition.

- As usual we try to find the minimum value of R_1 w.r.t. x, but this is acceptable only if $x_{\min} < x^*$, otherwise we can consider x^* as the minimum.

$$\frac{dR_1}{dx} = -\frac{1}{x^2}\left(1 + \frac{\mathcal{P}^2}{(1+\mathcal{P})^2}T_1\right) + 3 + 2x = 0$$

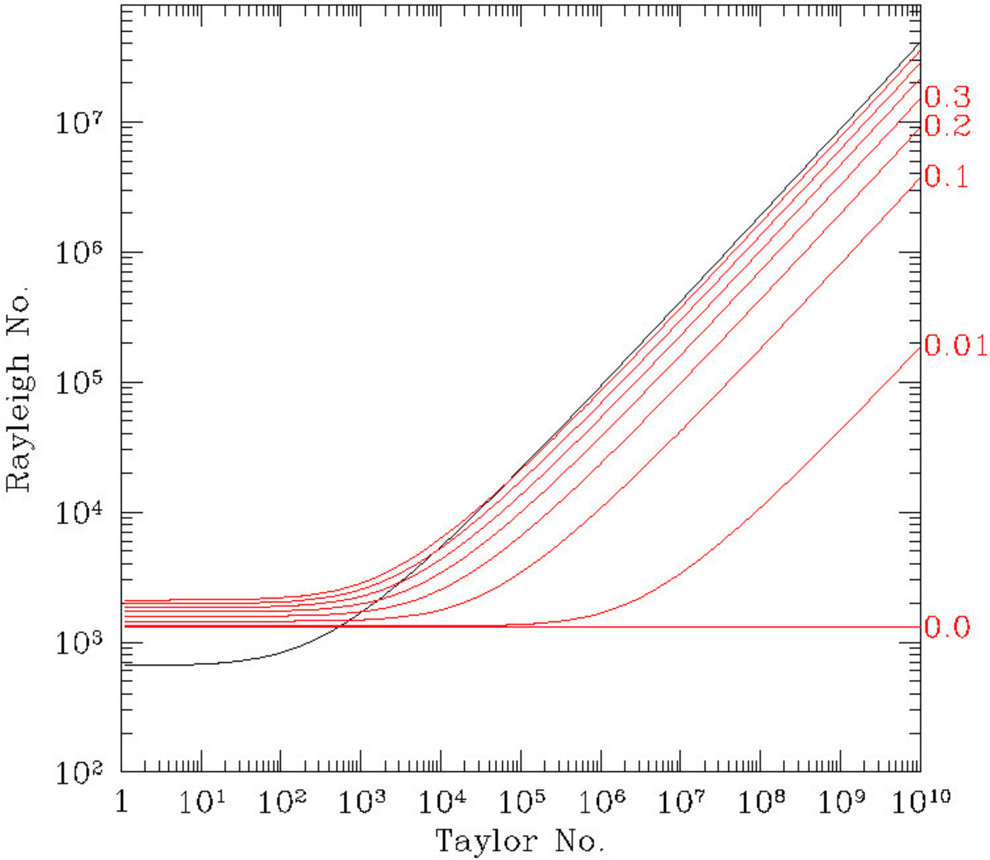
or

$$2x^3 + 3x^2 - 1 - \frac{\mathcal{P}^2}{(1+\mathcal{P})^2}T_1 = 0$$

which is similar to the earlier eq., except that this also depends on $\mathcal{P}$.

- The figure shows $R_{\rm crit}$ for different values of $\mathcal{P}$ along with that for the case where transition happens through $\sigma = 0$. At low values of T the $R_{\rm crit}$ for $\sigma = 0$ is below that for the oscillatory instability and hence the transition would happen through secular modes. While at large T the oscillatory modes give lower $R_{\rm crit}$, at least for low $\mathcal{P}$ and hence the transition would occur through oscillatory modes. $x_{\rm min} < x^*$ for most cases except for $T < 30$ (for some $\mathcal{P}$) where the transition anyway occurs through stationary convection.

- For $\mathcal{P} = 0$ the critical R is independent of T and its value is twice that for the case of no rotation.



Rayleigh No.
Taylor No.
1
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
0.3
0.2
0.1
0.01
0.0

- In the asymptotic limit as $\mathcal{P}^2 T \to \infty$ the critical values are

$$x_{\min} = \left(\frac{\mathcal{P}^2 T}{2\pi^4(1+\mathcal{P})^2}\right)^{1/3}$$

$$R_{\text{crit}} = 6\pi^4(1+\mathcal{P})\left(\frac{\mathcal{P}^2 T}{2\pi^4(1+\mathcal{P})^2}\right)^{2/3}$$

$$a_{\min} = \left(\frac{\pi^2\mathcal{P}^2 T}{2(1+\mathcal{P})^2}\right)^{1/6}$$

- The asymptotic ratio of critical value of R for the oscillatory case to that for secular case is

$$2\mathcal{P}\left(\frac{\mathcal{P}}{1+\mathcal{P}}\right)^{1/3}$$

This ratio exceeds unity around $\mathcal{P} = 0.675$. Thus above this the transition to instability should occur through secular modes.