Kelvin–Helmholtz Instability

- Kelvin–Helmholtz instability is for two fluids moving with respect to each other. We assume that all variations in steady state are along z-axis, which is the vertical direction and the velocity is in the horizontal direction.

- To start with we take the case of an inhomogeneous fluid with density and velocity varying with height. We consider an inviscid and incompressible fluid.

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\boldsymbol{\nabla} P + \mathbf{g}\rho$$
$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \mathbf{v}$$

- We assume that in the steady state ρ and $\mathbf{v}$ depend only on z. Further $\partial/\partial t = 0$, $\mathbf{v} = \mathbf{U}$ only in horizontal direction, which gives

$$\frac{dP_0}{dz} = -g\rho_0$$

which gives the pressure P_0.

- The state is perturbed to $\mathbf{U} + \mathbf{v}$, $P = P_0 + P_1$, and $\rho = \rho_0 + \rho_1$ which gives the eq in perturbed quantities

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{U} \cdot \boldsymbol{\nabla} \mathbf{v} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{U} = -\boldsymbol{\nabla} P_1 + \mathbf{g}\rho_1$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

$$\frac{\partial \rho_1}{\partial t} + \mathbf{U} \cdot \boldsymbol{\nabla} \rho_1 + \mathbf{v} \cdot \boldsymbol{\nabla} \rho_0 = 0$$

- We have not included surface tension, which will be added later at the interface when we separate the eq in components. Similarly, we will need an equation connecting v_z and the displacement ξ_z at the interface.

- We seek solutions of the form

$$f(z)\exp(ik_x x + ik_y y + int)$$

with $k^2 = k_x^2 + k_y^2$ and $D = d/dz$. We assume that $\mathbf{U} = (U, 0, 0)$.

- The eq in component form with surface tension added are

$$i\rho_0(n + k_x U)v_x + \rho_0(DU)v_z = -ik_x P_1$$
$$i\rho_0(n + k_x U)v_y = -ik_y P_1$$
$$i\rho_0(n + k_x U)v_z = -DP_1 - g\rho_1 - k^2 T\xi_z\delta(z)$$
$$ik_x v_x + ik_y v_y = -Dv_z$$
$$i(n + k_x U)\rho_1 = -(D\rho_0)v_z$$
$$i(n + k_x U)\xi_z = v_z$$

- Multiplying the 1st eq by $-ik_x$ and 2nd by $-ik_y$ and adding we get using the continuity eq

$$i\rho_0(n + k_x U)Dv_z - ik_x\rho_0(DU)v_z = -k^2 P_1$$

- Eliminating ρ_1 and ξ_z from the z-component of eq of motion and using $\sigma = n + k_x U$, we get

$$i\sigma\rho_0 v_z = -DP_1 - i\frac{g}{\sigma}(D\rho_0)v_z + i\frac{k^2}{\sigma}Tv_z\delta(z)$$

- Eliminating P_1 we get

$$D[\rho_0\sigma Dv_z - \rho_0 k_x(DU)v_z] - k^2\rho_0\sigma v_z$$
$$= g\frac{k^2}{\sigma}\left((D\rho_0) - \frac{k^2}{g}T\delta(z)\right)v_z$$

- Now we consider two fluids with density ρ_1, ρ_2 and velocity U_1, U_2 with interface at $z = 0$ in the steady state. Within each layer the eq becomes

$$(n + k_x U)(D^2 - k^2)v_z = 0$$

$n = -k_x U$ represents a wave travelling along x-axis with velocity U. This cannot be satisfied in both layers.

- Neglecting that we get the general solution

$$v_z = Ae^{kz} + Be^{-kz}$$

This should vanish at both $\pm\infty$.

- Further ξ_z should be continuous at $z = 0$, thus v_z/σ should be continuous across the interface. Noting that $\sigma = n + k_x U$ has discontinuity at $z = 0$ gives

$$v_z = \begin{cases} A\sigma_1 e^{kz} & \text{for } z < 0 \\ A\sigma_2 e^{-kz} & \text{for } z > 0 \end{cases}$$

where

$$\sigma_1 = n + k_x U_1, \qquad \sigma_2 = n + k_x U_2$$

- Integrating the eq across the interface we get

$$\int_{-\epsilon}^{\epsilon} \left(D(\rho_0 \sigma D v_z - \rho_0 k_x (DU) v_z) - \rho_0 k^2 \sigma v_z \right) \, dz$$

$$= k^2 g \int_{-\epsilon}^{\epsilon} \left((D\rho_0) - \frac{k^2}{g} T \delta(z) \right) \frac{v_z}{\sigma} \, dz$$

which gives

$$\Delta_i(\rho_0 \sigma D v_z) = k^2 g \left(\Delta_i(\rho_0) - \frac{k^2}{g} T \right) \frac{v_z}{\sigma}$$

where Δ_i denotes the jump across the interface. This gives

$$-k(\rho_2 \sigma_2^2 + \rho_1 \sigma_1^2) = -k^2 g \left((\rho_1 - \rho_2) + \frac{k^2}{g} T \right)$$

- Defining

$$\alpha_1 = \frac{\rho_1}{\rho_1+\rho_2}, \qquad \alpha_2 = \frac{\rho_2}{\rho_1+\rho_2}$$

the dispersion relation becomes

$$\alpha_2(n+k_xU_2)^2+\alpha_1(n+k_xU_1)^2 = gk\left((\alpha_1-\alpha_2)+\frac{k^2T}{g(\rho_1+\rho_2}\right)$$

Noting that $\alpha_1+\alpha_2=1$, this gives a quadratic

$$n^2+2k_x(\alpha_1U_1+\alpha_2U_2)n+k_x^2(\alpha_1U_1^2+\alpha_2U_2^2)$$
$$-gk\left((\alpha_1-\alpha_2)+\frac{k^2T}{g(\rho_1+\rho_2)}\right)=0$$

The discriminant of the quadratic is

$$k_x^2(\alpha_1 U_1 + \alpha_2 U_2)^2 - k_x^2(\alpha_1 U_1^2 + \alpha_2 U_2^2)$$
$$+ gk\left((\alpha_1 - \alpha_2) + \frac{k^2 T}{g(\rho_1 + \rho_2)}\right)$$

The first two terms give

$$k_x^2\left(U_1^2\alpha_1(\alpha_1 - 1) + U_2^2\alpha_2(\alpha_2 - 1) + 2U_1U_2\alpha_1\alpha_2\right)$$

Noting that $\alpha_1 - 1 = -\alpha_2$ and $\alpha_2 - 1 = -\alpha_1$ we get the solution of the quadratic

$$n = -k_x(\alpha_1 U_1 + \alpha_2 U_2) \pm$$
$$\left[gk\left((\alpha_1 - \alpha_2) + \frac{k^2 T}{g(\rho_1 + \rho_2)}\right) - k_x^2\alpha_1\alpha_2(U_1 - U_2)^2\right]^{1/2}$$

- If n is real the system is stable as we have oscillatory solution. If n is complex, the system would be unstable.

- We first consider the case when surface tension is absent ($T = 0$) and the solution is given by

$$n = -k_x(\alpha_1 U_1 + \alpha_2 U_2) \pm \left[gk(\alpha_1 - \alpha_2) - k_x^2 \alpha_1 \alpha_2 (U_1 - U_2)^2\right]^{1/2}$$

- When $k_x = 0$ i.e, waves perpendicular to direction of velocity, we get

$$n = \pm \left[gk(\alpha_1 - \alpha_2)\right]^{1/2}$$

which is a surface wave, when $\alpha_1 > \alpha_2$ or $\rho_1 > \rho_2$. For $\rho_2 > \rho_1$ is is unstable (Rayleigh–Taylor Instability).

- For other directions, instability occurs when $\rho_2 > \rho_1$ for all k. Thus we only consider the case when $\rho_1 > \rho_2$. In that case for instability

$$k > \frac{g(\alpha_1 - \alpha_2)}{\alpha_1 \alpha_2 (U_1 - U_2)^2 \cos^2 \theta}$$

where θ is the angle between the wave vector and direction of flow $(k_x = k \cos \theta)$.

- The minimum value of k occurs when $\theta = 0$, i.e., in the direction parallel to the flow and is given by

$$k_{\min} = \frac{g(\alpha_1 - \alpha_2)}{\alpha_1 \alpha_2 (U_1 - U_2)^2}$$

. Thus the system is unstable.

Effect of Surface Tension

- For Rayleigh–Taylor instability we had seen that the surface tension stabilises the modes with large k. Thus in this case we would expect that surface tension may stabilise the system.
- Considering the worst case when $k = k_x$, i.e,, $k_y = 0$ we can get the stability condition. It can be easily seen that if it is stable for $k_y = 0$, it would be stable also when $k_y \neq 0$. This gives the condition

$$k^2\alpha_1\alpha_2(U_1 - U_2)^2 < gk\left((\alpha_1 - \alpha_2) + \frac{k^2 T}{g(\rho_1 + \rho_2)}\right)$$

or

$$\alpha_1\alpha_2(U_1 - U_2)^2 < g\left(\frac{\alpha_1 - \alpha_2}{k} + \frac{kT}{g(\rho_1 + \rho_2)}\right)$$

- The RHS is minimum when

$$\frac{\alpha_1 - \alpha_2}{k^2} = \frac{T}{g(\rho_1 + \rho_2)}$$

or

$$k_{\min} = \left(\frac{(\alpha_1 - \alpha_2) g(\rho_1 + \rho_2)}{T} \right)^{1/2}$$

and the minimum value of RHS is

$$2 \left(\frac{gT(\alpha_1 - \alpha_2)}{\rho_1 + \rho_2} \right)^{1/2}$$

- This implies that the instability can be suppressed if

$$(U_1 - U_2)^2 < \frac{2}{\alpha_1 \alpha_2} \left(\frac{gT(\alpha_1 - \alpha_2)}{\rho_1 + \rho_2} \right)^{1/2}$$

- If the system is unstable, then the imaginary part of n for $k = k_{\min}$ is given by

$$
\begin{aligned}
&\left(k_{\min}^2 \alpha_1 \alpha_2 (U_1 - U_2)^2 - 2 g k_{\min} (\alpha_1 - \alpha_2)\right)^{1/2} \\
&= \left(\frac{(\alpha_1 - \alpha_2) g (\rho_1 + \rho_2) \alpha_1 \alpha_2 (U_1 - U_2)^2}{T} \right. \\
&\qquad \left. - \frac{2 g^{3/2} (\alpha_1 - \alpha_2)^{3/2} (\rho_1 + \rho_2)^{1/2}}{T^{1/2}} \right)^{1/2} \\
&= \left[\frac{(\alpha_1 - \alpha_2) g (\rho_1 + \rho_2)}{T} \right]^{1/2} \\
&\qquad \left(\alpha_1 \alpha_2 (U_1 - U_2)^2 - 2 \left[\frac{g T (\alpha_1 - \alpha_2)}{\rho_1 + \rho_2} \right]^{1/2} \right)^{1/2}
\end{aligned}
$$

- For air over sea-water, $\alpha_2 = 0.00126$ and $T = 74$ dyn cm^{-1}, we have stability if

$$|U_1 - U_2| < 6.5 \text{ m s}^{-1} = 23.4 \text{ kmPH}$$

At this critical value, $k_{\min} = 3.7$ cm^{-1} and the wavelength is $2\pi/k_{\min} = 1.7$ cm. Further, $n = \alpha_2 k_{\min} U_2 = 3$ s^{-1}. The speed of propagation $n/k = 0.82$ cm s^{-1}.

- If $\rho_1 = \rho_2$, i.e, there is no difference in density, then the surface tension would also not be present and the system is always unstable.
- In the absence of surface tension, the system is unstable for arbitrary small difference $|U_1 - U_2|$. Surface tension puts a limit on this difference, below which the system is stable.

Effect of Rotation

- We assume a constant rotation Ω about the z-axis and the velocity U is in the rotating frame. Considering the eq in rotating frame the Coriolis force would be added to the perturbation eq. The centrifugal force will introduce a pressure gradient in the horizontal direction in the steady state, but will not appear in the perturbed eq.

- The perturbed eq of motion in horizontal directions are

$$i\rho_0(n + k_x U)v_x + \rho_0(DU)v_z - 2\rho_0\Omega v_y = -ik_x P_1$$
$$i\rho_0(n + k_x U)v_y + 2\rho_0\Omega v_x = -ik_y P_1$$

other eq remain the same

- Multiplying the 1st eq by $-ik_x$ and 2nd by $-ik_y$ and adding we get using the continuity eq

$$i\rho_0(n + k_x U)Dv_z - ik_x\rho_0(DU)v_z + 2\rho_0\Omega\omega_z = -k^2 P_1$$

where the z-component of vorticity

$$\omega_z = ik_x v_y - ik_y v_x$$

- Multiplying the 1st eq by $-ik_y$ and 2nd by ik_x and adding we get

$$i\rho_0(n + k_x U)\omega_z = ik_y\rho_0(DU)v_z + 2\rho_0\Omega Dv_z$$

- Eliminating ω_z between the two eq we get

$$i\rho_0\sigma\left(1 - \frac{4\Omega^2}{\sigma^2}\right)Dv_z - i\rho_0\left(k_x + \frac{i2k_y\Omega}{\sigma}\right)(DU)v_z = -k^2 P_1$$

- Eliminating P_1 from the z-component of eq of motion we get

$$D\left[\rho_0\sigma\left(1-\frac{4\Omega^2}{\sigma^2}\right)Dv_z-\rho_0\left(k_x+\frac{2ik_y\Omega}{\sigma}\right)(DU)v_z\right]$$

$$-k^2\rho_0\sigma v_z=g\frac{k^2}{\sigma}\left((D\rho_0)-\frac{k^2}{g}T\delta(z)\right)v_z$$

- Now we consider two fluids with density ρ_1, ρ_2 and velocity U_1, U_2 with interface at $z = 0$ in the steady state. Within each layer the eq becomes

$$\rho_0 \sigma \left(1 - \frac{4\Omega^2}{\sigma^2}\right) D^2 v_z = k^2 \rho_0 \sigma v_z$$

- Assuming $2\Omega < |\sigma|$ the general solution is given by

$$v_z = Ae^{\kappa z} + Be^{-\kappa z}$$

where

$$\kappa^2 = \frac{k^2}{1 - \frac{4\Omega^2}{\sigma^2}}$$

- This solution should vanish at both $\pm\infty$. Further ξ_z should be continuous at $z = 0$, thus v_z/σ should be continuous across the interface. Noting that $\sigma = n + k_x U$ has discontinuity at $z = 0$ gives

$$v_z = \begin{cases} A\sigma_1 e^{\kappa_1 z} & \text{for } z < 0 \\ A\sigma_2 e^{-\kappa_2 z} & \text{for } z > 0 \end{cases}$$

where

$$\kappa_1^2 = \frac{k^2}{1 - \frac{4\Omega^2}{\sigma_1^2}} \quad \kappa_2^2 = \frac{k^2}{1 - \frac{4\Omega^2}{\sigma_2^2}}$$

$$\sigma_1 = n + k_x U_1, \qquad \sigma_2 = n + k_x U_2$$

- If $2\Omega > |\sigma|$ then it would represent oscillatory solution in z and it is not possible to satisfy the boundary conditions.

- Integrating the eq across the interface we get

$$\Delta_i \left[\rho_0 \sigma \left(\frac{k^2}{\kappa^2} \right) D v_z \right] = k^2 g \left(\Delta_i(\rho_0) - \frac{k^2}{g} T \right) \frac{v_z}{\sigma}$$

where Δ_i denotes the jump across the interface. This gives

$$-\frac{1}{\kappa_2} \rho_2 \sigma_2^2 - \frac{1}{\kappa_1} \rho_1 \sigma_1^2 = -g \left((\rho_1 - \rho_2) + \frac{k^2}{g} T \right)$$

- If $k_x = 0$ the dispersion relation is independent of U and $\sigma_1 = \sigma_2 = n$ and $\kappa_1 = \kappa_2$. This is the same relation for the Rayleigh–Taylor instability

$$n^2 (\rho_1 + \rho_2) \left(1 - \frac{4\Omega^2}{n^2} \right)^{1/2} = -gk \left((\rho_2 - \rho_1) - \frac{k^2 T}{g} \right)$$

- Dividing by $\rho_1 + \rho_2$ and taking the square of the eq we get

$$n^4 - 4\Omega^2 n^2 - g^2 k^2 \left(\delta - \frac{k^2 T}{g(\rho_1 + \rho_2)}\right)^2 = 0$$

or

$$\frac{n^2}{4\Omega^2} = \frac{1 \pm (1 + n_0^2)^{1/2}}{2}$$

where

$$n_0 = \frac{gk}{4\Omega^2}\left(\delta - \frac{k^2 T}{g(\rho_1 + \rho_2)}\right)$$

- The $+$ sign gives $n^2 > 4\Omega^2$ which is acceptable. The $-$ sign gives $n^2 < 0$ which is also feasible. The choice is to be fixed by looking at the original eq before squaring.

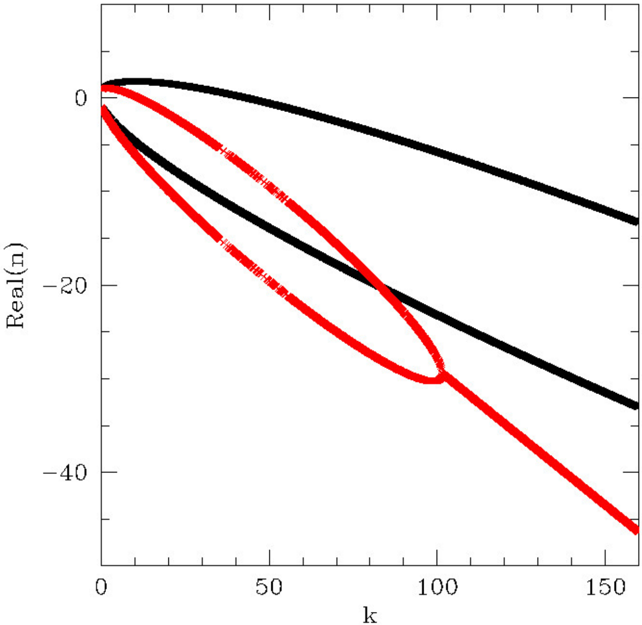
- The quadratic in n^2 is obtained by squaring the original eq. There we had assumed that the square root on LHS should be positive. Thus for $n_0 > 0$, $n^2 < 0$ and we will have instability. In that case the $-$ sign should be used for solution. For $n_0 < 0$, $n^2 > 0$ and hence the $+$ sign should be used.

- For large k the surface tension would stabilize the modes as $n_0 < 0$. The condition is identical to that obtained for non-rotating case in Rayleigh–Taylor instability. The value of n would be different in the two cases.

- Now we consider the case $k_x \neq 0$ but take $T = 0$. Following the non-rotating case we expect the worst case when $k_y = 0$ so we consider only that case $k_x = k$.
- The dispersion relation is given by

$$\frac{1}{\kappa_2}\rho_2\sigma_2^2 + \frac{1}{\kappa_1}\rho_1\sigma_1^2 = g(\rho_1 - \rho_2)$$

  For the given system, $g, \rho_1, \rho_2, U_1, U_2$, it is possible to solve this equation for n as a function of k.
- If we take the asymptotic limit $k \to \infty$, $\kappa \to k$ and the dispersion relation is the same as that for non-rotating case. Thus we expect instability for large k. It turns out that for small k the eq has only real roots, giving stability, but for large k the roots are complex and the system is unstable.



$n(k)$ in units of 2Ω for $(\rho_1 - \rho_2)/(\rho_1 + \rho_2) = 0.1$ and $(U_1, U_2) = (0.1, 0.2)$ (black points) and $(0.2, 0.4)$ (red points). There are two roots which merge at large k as it becomes a pair of complex conjugate roots.

Magneto-acoustic waves

- We consider waves in a compressible fluid in the presence of a magnetic field. For simplicity we assume the magnetic field to be uniform in equilibrium state which is also assumed to be homogeneous, i.e., no gravity. We also neglect all dissipation (inviscid and perfectly conducting) i.e., $\mu = 0$, $\eta = 0$.
- The governing eq are

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P - \frac{1}{8\pi} \nabla (B^2) + \frac{1}{4\pi} \mathbf{B} \cdot \nabla \mathbf{B}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{v})$$

- We assume that in the steady state ρ and $\mathbf{B}$ are constant. Further we take $\mathbf{B}$ in x-direction.
- The state is perturbed to $\mathbf{v}$, $P = P_0 + P_1$, $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ and $\rho = \rho_0 + \rho_1$ which gives the eq in perturbed quantities

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} P_1 - \frac{1}{4\pi} \boldsymbol{\nabla}(B_0 b_x) + \frac{1}{4\pi} B_0 \frac{\partial \mathbf{b}}{\partial x}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{b}}{\partial t} = B_0 \frac{\partial \mathbf{v}}{\partial x} - \mathbf{B}_0 (\boldsymbol{\nabla} \cdot \mathbf{v})$$

- We seek solutions of the form

$$\exp(ik_x x + ik_y y + ik_z z + i\omega t)$$

with $k^2 = k_x^2 + k_y^2 + k_z^2$. Further, since the perturbation are adiabatic $P_1 = c^2 \rho_1$.

- The eq in component form are

$$
\begin{aligned}
\rho_0 \omega v_x &= -k_x P_1 - \frac{k_x}{4\pi} B_0 b_x + \frac{k_x}{4\pi} B_0 b_x = -k_x P_1 \\
\rho_0 \omega v_y &= -k_y P_1 - \frac{k_y}{4\pi} B_0 b_x + \frac{k_x}{4\pi} B_0 b_y \\
\rho_0 \omega v_z &= -k_z P_1 - \frac{k_z}{4\pi} B_0 b_x + \frac{k_x}{4\pi} B_0 b_z \\
\omega \rho_1 &+ k_x \rho_0 v_x + k_y \rho_0 v_y + k_z \rho_0 v_z = 0 \\
\omega b_x &= k_x B_0 v_x - B_0 (k_x v_x + k_y v_y + k_z v_z) \\
&= -B_0 (k_y v_y + k_z v_z) \\
\omega b_y &= k_x B_0 v_y \\
\omega b_z &= k_x B_0 v_z
\end{aligned}
$$

- The first eq can be used to eliminate P_1 and ρ_1

$$P_1 = -\frac{\omega \rho_0 v_x}{k_x}, \qquad \rho_1 = -\frac{\omega \rho_0 v_x}{c^2 k_x}$$

while the last three eqs can be used to eliminate the components of the magnetic field to get

$$\omega v_y = \frac{k_y \omega}{k_x} v_x + \frac{B_0^2}{\omega 4\pi \rho_0}\left(k_y(k_y v_y + k_z v_z) + k_x^2 v_y\right)$$
$$\omega v_z = \frac{k_z \omega}{k_x} v_x + \frac{B_0^2}{\omega 4\pi \rho_0}\left(k_z(k_y v_y + k_z v_z) + k_x^2 v_z\right)$$
$$-\frac{\omega^2}{c^2 k_x} v_x + k_x v_x + k_y v_y + k_z v_z = 0$$

- After some simplifications and defining the Alfven speed

$$V_A^2 = \frac{B_0^2}{4\pi\rho_0}$$

these eq are

$$(k_x^2 c^2 - \omega^2) v_x + c^2 k_x k_y v_y + c^2 k_x k_z v_z = 0$$

$$\omega^2 k_y v_x + k_x \left(V_A^2 (k_x^2 + k_y^2) - \omega^2 \right) v_y + k_x k_y k_z V_A^2 v_z = 0$$

$$\omega^2 k_z v_x + k_x k_y k_z V_A^2 v_y + k_x \left(V_A^2 (k_x^2 + k_z^2) - \omega^2 \right) v_z = 0$$

- These are 3 linear eq in 3 unknowns and for nontrivial solutions the determinant of matrix should vanish, which gives the dispersion relation

$$\begin{vmatrix} c^2 k_x^2 - \omega^2 & c^2 k_x k_y & c^2 k_x k_z \\ \omega^2 k_y & k_x (V_A^2 (k_x^2 + k_y^2) - \omega^2) & k_x k_y k_z V_A^2 \\ \omega^2 k_z & k_x k_y k_z V_A^2 & k_x (V_A^2 (k_x^2 + k_z^2) - \omega^2) \end{vmatrix}$$

- Expanding the determinant we get the dispersion relation

$$\left(\omega^2 - V_A^2 k_x^2\right)\left(\omega^4 - \omega^2 k^2 (c^2 + V_A^2) + c^2 V_A^2 k^2 k_x^2\right) = 0$$

- The first factor gives Alfven waves travelling along the magnetic field. Substituting $\omega^2 = k_x^2 V_A^2$ in the first two eqs gives

$$k_x^2 (c^2 - V_A^2) v_x + c^2 k_x k_y v_y + c^2 k_x k_z v_z = 0$$
$$k_x^2 k_y V_A^2 v_x + k_x k_y^2 V_A^2 v_y + k_x k_y k_z V_A^2 v_z = 0$$

or

$$k_x (c^2 - V_A^2) v_x + c^2 (k_y v_y + k_z v_z) = 0$$
$$k_x V_A^2 v_x + V_A^2 (k_y v_y + k_z v_z) = 0$$

- Eliminating $k_y v_y + k_z v_z$ e get

$$k_x(V_A^2(c^2 - V_A^2) - V_A^2 c^2)v_x = -k_x V_A^4 v_x = 0$$

or $v_x = 0$ and $k_y v_y + k_z v_z = 0$. Thus these are transverse waves ($\mathbf{k} \cdot \mathbf{v} = 0$). Further, for this solution $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and hence $\rho_1 = 0$ and $P_1 = 0$. Hence the perturbations are incompressible. Further $b_x = 0$ and $V_A \mathbf{b} = B_0 \mathbf{v}$. The magnetic field perturbations are also transverse.

- The second factor can be solved to get

$$\omega^2 = \frac{1}{2}\left(k^2(c^2+V_A^2) \pm \sqrt{k^4(c^2+V_A^2)^2 - 4c^2V_A^2k^2k_x^2}\right)$$

  It can be shown that the discriminant is always positive as it is

$$(c^2-V_A^2)^2k^4 + 4c^2V_A^2k^2(k_y^2+k_z^2)$$

- It can easily seen that both roots for ω^2 are positive and hence they represent magneto-acoustic waves. These are referred to as the fast and slow mode.

- In the absence of magnetic field ($V_A = 0$) the dispersion relation reduces to $\omega^2 = k^2 c^2$, which are the acoustic modes. It can be shown that the displacement is along the direction of propagation, i.e, the waves are longitudinal. In this case the eqs give

$$\frac{v_x}{k_x} = \frac{v_y}{k_y} = \frac{v_z}{k_z}$$

- For an incompressible fluid $c^2 \to \infty$ and the dispersion relation gives $\omega^2 = k_x^2 V_A^2$ which is the Alfven wave considered earlier.

- For waves travelling perpendicular to the magnetic field ($k_x = 0$) the dispersion relation gives

$$\omega^2 = (c^2 + V_A^2)(k_y^2 + k_z^2)$$

which is the fast mode. The slow mode has zero frequency in this case.

- For waves travelling along the magnetic field ($k = k_x$) the dispersion relation gives,

$$\omega^2 = c^2 k_x^2 \qquad \text{and} \qquad \omega^2 = V_A^2 k_x^2$$

these are respectively, the acoustic and Alfven waves.

- for $V_A^2 \ll c^2$, the discriminant can be approximated by

$$k^4 (c^2 - V_A^2)^2 \left(1 + 4 \frac{k_y^2 + k_z^2}{k^2} \frac{V_A^2}{c^2} \right)$$

which gives the roots

$$\omega^2 \approx c^2 k^2 + V_A^2 (k_y^2 + k_z^2), \quad V_A^2 k_x^2$$