

Fluid Mechanics

- Fluid is loosely defined as a substance that continually deforms under an applied shear stress. Fluid could be a liquid, which has a free surface, or a gas/plasma without surface. A liquid has a defined volume, while gas tends to fill the available space.
- Although, fluid deforms under shear, they do offer some resistance to shear in the form of viscosity. Most ordinary liquids have low viscosity, but substances with high viscosity can resist shear to a varying extent, depending on time-scale. E.g., tar, putty, mud etc., which may be difficult to classify. Another, class of materials which can also flow is powder.

- The usual fluid where viscous stress is proportional to strain are referred to as Newtonian fluids, as opposed to non-Newtonian fluids where the stress is not proportional to strain. The study of non-Newtonian fluid is also referred to as Rheology.
- The fluids do show resistance to normal force, i.e., pressure. Fluids can be classified as compressible or incompressible. Generally, liquids can be approximated as incompressible, where the density is independent of pressure. Gases are typically compressible, but if the thickness of the layer is small as compared to the scale height of pressure, then it may also be approximated as incompressible. Study of incompressible fluids is sometimes referred to as hydrodynamics.

- Applicability of fluid dynamics: Fluids typically consist of many independent atoms/molecules which move collectively. These can be treated as fluid if

mean free path \ll length-scale of system

- For a plasma consisting of charged particles, in the presence of magnetic field, the particles would all spiral around the magnetic field lines. This can be treated as a fluid provided

$$\text{gyroradius} = \frac{mv_{\perp}}{|q|B} \ll \text{length-scale of system}$$

- For a gas of particles with number density, n and cross section σ

$$l = \frac{1}{n\sigma}$$

For the air in our room $l \approx 0.3 \mu\text{m}$ ($n \approx 3 \times 10^{19} \text{ cm}^{-3}$, $\sigma = 10^{-15} \text{ cm}^2$).

For solar atmosphere $n \approx 2 \times 10^{16} \text{ cm}^{-3}$, $l \approx 1 \text{ cm}$.

For interstellar medium $n \approx 1 \text{ cm}^{-3}$, $l \approx 10^{16} \text{ cm}$.

- The fluid is typically described by average local properties, density, ρ , pressure, P , Temperature, T , coefficient of viscosity, μ , and an equation of state (EOS) connecting P, T, ρ . The dynamics is described by the local velocity, \mathbf{v} . The average has to be taken over a volume of size much larger than the mean free path ($V \gg l^3$).

- There are two ways to describe the fluid elements. The first is the Eulerian description where all quantities are functions of \mathbf{r}, t or in Cartesian coordinates, x, y, z, t , e.g., $\rho(x, y, z, t)$.
- Alternately, we can use the Lagrangian description, where we follow each fluid element in time. The fluid elements can be labelled by their initial position $\mathbf{r}_0 = (x_0, y_0, z_0)$. In this case the position \mathbf{r} of the fluid element is also a function of initial position and time, $\mathbf{r}(\mathbf{r}_0, t)$. The velocity is now defined as

$$\mathbf{v} = \lim_{\delta t \rightarrow 0} \frac{\mathbf{r}(\mathbf{r}_0, t + \delta t) - \mathbf{r}(\mathbf{r}_0, t)}{\delta t} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial t}.$$

The acceleration is given by

$$\frac{d\mathbf{v}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\mathbf{v}(\mathbf{r}_0, t + \delta t) - \mathbf{v}(\mathbf{r}_0, t)}{\delta t}.$$

To translate this to Eulerian description we can write the first term in the numerator as

$$\mathbf{v}(\mathbf{r} + \delta\mathbf{r}, t + \delta t) = \mathbf{v}(\mathbf{r}, t) + \frac{\partial \mathbf{v}}{\partial t} \delta t + \delta\mathbf{r} \cdot \nabla \mathbf{v}$$

Now taking the limit we get

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}.$$

Similarly, we can write for other scalar quantities

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P.$$

Equations of Fluid Mechanics

- Conservation of mass: Consider a fixed volume V enclosed by a closed surface S of arbitrary shape. The mass of fluid enclosed is given by

$$m(t) = \int_V \rho \, dV.$$

The mass can change due to change in ρ or due to material flowing into or out of the volume across the surface. The rate of fluid flow across the surface is given by

$$\int_S \rho \mathbf{v} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot (\rho \mathbf{v}) \, dV$$

by using Gauss theorem. Conservation of mass requires that the increase or decrease in mass is equal to the total mass flowing in through the surface

$$\frac{\partial}{\partial t} \left(\int_V \rho dV \right) = - \int_V \nabla \cdot (\rho \mathbf{v}) dV$$

or

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$$

for all V . This gives the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0$$

Using the definition of Lagrangian derivative

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0.$$

- For incompressible fluid density is constant and the continuity eq reduces to $\nabla \cdot \mathbf{v} = 0$.

- Conservation of momentum or Equation of motion: The momentum of a fluid element is given by

$$\int_V \rho \mathbf{v} \, dV$$

Following argument similar to that for continuity equation we can write the rate of change of momentum. The difference is that rate has to be equated to the force and can be nonzero. The rate of change of momentum is

$$\int_V \left(\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) \right) dV$$

where the second term can be written in Cartesian coordinates as

$$\frac{\partial}{\partial x_i} (\rho v_i v_j) = \frac{\partial \rho v_i}{\partial x_i} v_j + \rho v_i \frac{\partial v_j}{\partial x_i} = \nabla \cdot (\rho \mathbf{v}) \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

- This gives the integrand in the previous eq

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \rho}{\partial t} \mathbf{v} + \nabla \cdot (\rho \mathbf{v}) \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

The two middle terms cancel because of continuity eq. giving the eq. of motion

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = \rho \frac{d\mathbf{v}}{dt} = \mathbf{F}$$

where \mathbf{F} is the force.

- The force has two components, one external force, e.g., from gravity or electromagnetic forces or due to rotation if the equations are written in a rotating frame. Second is the force from molecular or atomic forces acting on fluid elements. This can be considered as stress on fluid element. The stress tensor σ_{ij} is the i th component of the force per unit area acting across a surface with normal in j th direction.
- The stress is due to frictional forces between different fluid particles as they move w.r.t. each other. Thus the stress depends on the velocity gradient and for Newtonian fluids we assume that the stress is proportional to the strain, i.e., the velocity gradient.

- The diagonal elements of the stress tensor defines the force along normal to the surface, which is referred to as the pressure. Further, if pressure is isotropic, i.e. all diagonal elements are equal to P , then $\sigma_{ij} = -P\delta_{ij}$ or $P = -\frac{1}{3}\sigma_{ii}$. The force along i th direction is

$$\int_S \sigma \cdot \mathbf{n} \, dS = \int_S \sigma_{ij} n_j \, dS = \int_V \frac{\partial \sigma_{ij}}{\partial x_j} \, dV = - \int_V \frac{\partial P}{\partial x_i} \, dV$$

For isotropic stress tensor the eq. of motion is given by

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla P$$

- For static fluid, $\mathbf{v} = 0$ and we get the eq of hydrostatic equilibrium

$$\nabla P = \rho \mathbf{g}$$

- For fluid in motion the stress would not be isotropic and the stress tensor can be decomposed into two components, $\sigma_{ij} = -P\delta_{ij} + f_{ij}$. The second part can be written in terms of the strain tensor, which can be decomposed

$$\frac{\partial v_i}{\partial x_j} = e_{ij} + \eta_{ij}$$

where

$$e_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$
$$\eta_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$$

The antisymmetric component η_{ij} has the same form as $\nabla \times \mathbf{v}$. We can define vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and write

$\eta_{ij} = \epsilon_{ijk}\omega_k$. Ignoring time for the moment, the change in velocity

$$\mathbf{v}(\mathbf{r} + \delta\mathbf{r}) = \mathbf{v}(\mathbf{r}) + \delta\mathbf{r} \cdot \nabla \mathbf{v} = \mathbf{v}(\mathbf{r}) + \delta\mathbf{r} \cdot \mathbf{e} + \delta\mathbf{r} \cdot \boldsymbol{\eta}$$

Here the first term is uniform translation, second is pure strain, while the third which can be written as $\delta r_j \epsilon_{ijk} \omega_k = \delta\mathbf{r} \times \boldsymbol{\omega}$, is due to rotation, with rotation rate $\boldsymbol{\omega}$. Further the trace of strain tensor $e_{ii} = \nabla \cdot \mathbf{v}$. From the continuity eq. it is clear that this is the relative change in volume

$$e_{ii} = -\frac{1}{\rho} \frac{d\rho}{dt} = \frac{1}{V} \frac{dV}{dt}$$

- Ignoring the translation and rotation, which would not cause any stress in the fluid, we consider the pure strain given by symmetric component of strain tensor. For Newtonian fluids the stress is given by

$$\sigma_{ij} = \mu_{ijkl} e_{kl}$$

where μ is the viscosity tensor of rank 4. For an isotropic fluid the stress should be independent of translation and rotation and the tensor can be written as

$$\mu_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

Alternately, we can say that only possible symmetric component of the strain are e_{ij} and $e_{kk} \delta_{ij}$. This can be added to the pressure $-P \delta_{ij}$. Thus the stress is given by

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$$

- The trace of this tensor should vanish as the average is already included in the pressure. Thus giving

$$3\lambda e_{kk} + 2\mu e_{kk} = 0$$

or $\lambda = -(2/3)\mu$. Thus the stress is given by

$$\sigma_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3}\mu (\nabla \cdot \mathbf{v}) \delta_{ij}$$

- However, the pressure force does not depend on the strain and hence there is a justification of additional term given by λ which is referred to as the bulk viscosity, ζ giving the stress

$$\sigma_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \left(\zeta - \frac{2}{3}\mu \right) (\nabla \cdot \mathbf{v}) \delta_{ij}$$

- From the continuity eq.

$$e_{kk} = \nabla \cdot \mathbf{v} = -\frac{1}{\rho} \frac{d\rho}{dt}$$

Thus the bulk viscosity gives the resistance to compression. In most cases the bulk viscosity is neglected. For incompressible fluids it is irrelevant.

- The stress which operates on surface can be transformed to volume force by taking divergence giving the equation of motion, which is also known as the Navier–Stokes equation for the case where the external force per unit mass is \mathbf{F}

$$\rho \frac{dv_i}{dt} = \rho F_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial v_k}{\partial x_k} \right]$$

For incompressible fluid $\nabla \cdot \mathbf{v} = 0$, and with constant μ we get the equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = \rho \mathbf{F} - \nabla P + \mu \nabla^2 \mathbf{v}$$

- To get the equation for conservation of energy we can multiply the eq. of motion by v_i and sum over i to get

$$\rho \frac{d}{dt} \left(\frac{1}{2} v^2 \right) = \rho v_i F_i - v_i \frac{\partial P}{\partial x_i} + v_i \frac{\partial \sigma_{ij}}{\partial x_j}$$

Integrating over a volume V and following the arguments we used earlier while deriving the momentum eq. we get

$$\begin{aligned} \frac{1}{2} \int_V \frac{\partial}{\partial t} (\rho v_i^2) dV &= \int_V \rho v_i F_i dV - \frac{1}{2} \int_S \rho v_i^2 v_j n_j dS \\ &+ \int_V P \frac{\partial v_i}{\partial x_i} dV - \int_S P v_j n_j dS \\ &- \int_V \sigma_{ij} \frac{\partial v_i}{\partial x_j} dV + \int_S \sigma_{ij} v_i n_j dS \end{aligned}$$

The LHS is rate of change of KE. The first term on RHS is the work done by external force. The 2nd term gives the rate at which the energy flows out of the volume. The other surface integrals give the rate at which the stresses work at the surface.

- The eq of energy conservation can be written as

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t}(\rho v^2) + \nabla \cdot \left(\frac{1}{2} \rho v^2 \mathbf{v} + P \mathbf{v} - \boldsymbol{\sigma} \cdot \mathbf{v} \right) \\ = \rho \mathbf{v} \cdot \mathbf{F} + P \nabla \cdot \mathbf{v} - \boldsymbol{\sigma} \cdot \nabla \mathbf{v} \end{aligned}$$

- The volume integral over the pressure term give

$$-P \frac{1}{\rho} \frac{d\rho}{dt} = P \frac{1}{V} \frac{dV}{dt}$$

which per unit volume give PdV which is the work done by pressure. Similarly, the stress term gives

$$\sigma_{ij} \frac{\partial v_i}{\partial x_j} = \sigma_{ij} e_{ij} = (2\mu e_{ij} - \frac{2}{3}\mu \delta_{ij} e_{kk}) e_{ij}$$

which gives the rate of viscous dissipation

$$\Phi = 2\mu e_{ij}^2 - \frac{2}{3}\mu (e_{jj})^2$$

It can be shown that this a positive definite because

$$\begin{aligned} \Phi = & 4\mu(e_{12}^2 + e_{23}^2 + e_{31}^2) \\ & + \frac{2}{3}\mu \left((e_{11} - e_{22})^2 + (e_{22} - e_{33})^2 + (e_{33} - e_{11})^2 \right) \end{aligned}$$

- The above derivation does not account for the change in the internal energy of the fluid. E.g., when the gas is heated the internal energy increases as the KE of molecules increases. Similarly, the viscous dissipation also would cause heating, thus increasing the internal energy of the fluid.
- To take this into account we need to add term for internal energy as well as the heat flux into the system. The heat flux from the surface is given by

$$\int_S K \nabla T \cdot \mathbf{n} dS = \int_V \nabla \cdot (K \nabla T) dV$$

where, T is temperature and, K the thermal conductivity.

- The internal energy is given by $E = \rho C_v T$, C_v is the specific heat at constant volume. This expression is valid when C_v is constant. The rate of change in internal energy is given by heat flux and work done by stress

$$\frac{\partial}{\partial t}(\rho C_v T) + \nabla \cdot (\rho C_v T \mathbf{v}) = \nabla \cdot (K \nabla T) + \Phi - P \nabla \cdot \mathbf{v}$$

This gives the energy eq in the usual form

$$\rho \frac{dC_v T}{dt} + P \nabla \cdot \mathbf{v} = \Phi + \nabla \cdot (K \nabla T)$$

- The LHS can also be written in terms of entropy S_e as $\rho T (dS_e/dt)$.

- If the external force is conservative, it can be written as a gradient $F = -\nabla\Psi$, which is independent of time. Then

$$v_i F_i = -v_i \frac{\partial \Psi}{\partial x_i} = -\frac{d\Psi}{dt}$$

In this case the potential energy Ψ can be added to KE to get conservation of total energy.

- The Boussinesq approximation: Assume that the fluid is incompressible and density is independent of P , but it can depend on T , through a coefficient of thermal expansion

$$\rho = \rho_0[1 - \alpha(T - T_0)]$$

Further, this is used only in the term for external force. Everywhere else we have $\nabla \cdot \mathbf{v} = 0$. This is usually valid for liquids. For a perfect gas $\rho \sim P/T$ and it is difficult to justify this approximation. But it may still be useful when the thickness of the layer is much smaller than the scale heights.