Convection: Effect of Magnetic Field

- We consider the problem of convection in the presence of a uniform magnetic field. For an incompressible fluid the eq. of motion is modified and the induction eq needs to be added to the system. The continuity eq and the energy eq remain the same.

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} - \boldsymbol{\nabla}\left(\frac{P}{\rho}\right) + \nu\nabla^2\mathbf{v} - \frac{1}{8\pi\rho}\boldsymbol{\nabla}(B^2) + \frac{1}{4\pi\rho}\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B}$$

$$\frac{\partial\mathbf{B}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{B} = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{v} + \eta\nabla^2\mathbf{B}$$

- Since in the steady state $\mathbf{B}$ is uniform there is no Lorentz force and steady state remains the same as for the Benard problem. For simplicity we assume that the magnetic field is also in z-direction.

- The eq of continuity and energy remain the same. The magnetic field is perturbed as $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ and eq of motion and the induction eq in perturbed quantities are

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\boldsymbol{\nabla} P_1 + \nu \nabla^2 \mathbf{v} - \mathbf{g}\alpha T_1 - \frac{1}{4\pi\rho}\boldsymbol{\nabla}(\mathbf{B}_0 \cdot \mathbf{B}_1)$$
$$+ \frac{1}{4\pi\rho}\mathbf{B}_0 \cdot \boldsymbol{\nabla}\mathbf{B}_1$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \mathbf{B}_0 \cdot \boldsymbol{\nabla}\mathbf{v} + \eta \nabla^2 \mathbf{B}_1$$

- Taking curl of eq. of motion we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \alpha g \left(\frac{\partial T_1}{\partial y}, -\frac{\partial T_1}{\partial x}, 0 \right) + \nu \nabla^2 \boldsymbol{\omega} + \frac{1}{4\pi\rho} \mathbf{B}_0 \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \times \mathbf{B}_1)$$

- Taking the curl once again

$$\frac{\partial}{\partial t} \nabla^2 \mathbf{v} = g\alpha \left(-\frac{\partial^2 T_1}{\partial x \partial z}, -\frac{\partial^2 T_1}{\partial y \partial z}, \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} \right)$$

$$+ \nu \nabla^4 \mathbf{v} + \frac{1}{4\pi\rho} \mathbf{B}_0 \cdot \boldsymbol{\nabla} (\nabla^2 \mathbf{B}_1)$$

- The z component of the two equations with $\mathbf{B}_1 = (b_x, b_y, b_z)$ gives

$$\frac{\partial b_z}{\partial t} = \eta \nabla^2 b_z + B_0 \frac{\partial v_z}{\partial z}$$

$$\frac{\partial}{\partial t} \nabla^2 v_z = g\alpha \left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} \right) + \nu \nabla^4 v_z + \frac{B_0}{4\pi\rho} \frac{\partial}{\partial z} (\nabla^2 b_z)$$

$$\frac{\partial T_1}{\partial t} = \beta v_z + \kappa \nabla^2 T_1$$

These are 3 eq in 3 unknowns.

- We can first consider the case where there is no dissipation, $\nu = \kappa = \eta = 0$. In that case T_1, b_z can be easily eliminated by differentiating the first eq. w,r,t. t to get

$$\frac{\partial^2}{\partial t^2}\nabla^2 v_z = g\alpha\beta\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) + \frac{B_0^2}{4\pi\rho}\frac{\partial^2}{\partial z^2}(\nabla^2 v_z)$$

We can look for solution of form $v_z = \exp(pt + i(k_x x + k_y y + k_z z))$ to get the dispersion relation

$$p^2(k_x^2 + k_y^2 + k_z^2) = g\alpha\beta(k_x^2 + k_y^2) - V_A^2 k_z^2(k_x^2 + k_y^2 + k_z^2)$$

- If $p^2 > 0$ we will have convection, otherwise we get oscillatory modes. For convection we need

$$V_A^2 k_z^2 < g\alpha\beta\frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2} < g\alpha\beta$$

- Thus for large magnetic field, convection tends to be suppressed. The system is infinite in x, y directions and hence there is no limit on k_x, k_y. On the other hand k_z has a lower limit of π/d, but there is no upper limit. Thus the system is unstable if

$$\pi^2 V_A^2 < g\alpha\beta d^2$$

  Since $\eta = 0$ the field lines are frozen with fluid.

- If $\beta = 0$ we get the Alfven waves.

- Boundary conditions on magnetic field: If the surrounding medium is vacuum, then B should be continuous across the boundary and if we assume that the field outside is maintained at the initial value, then $\mathbf{B}_1 = 0$ at both boundaries.
- We look for solutions of the form

$$\begin{aligned} v_z &= v(z) \exp[i(k_x x + k_y y) + pt] \\ b_z &= b(z) \exp[i(k_x x + k_y y) + pt] \\ T_1 &= \theta(z) \exp[i(k_x x + k_y y) + pt] \end{aligned}$$

with $k^2 = k_x^2 + k_y^2$ we get

$$p\left(\frac{d^2}{dz^2}-k^2\right)v = -g\alpha k^2\theta + \nu\left(\frac{d^2}{dz^2}-k^2\right)^2 v$$
$$+\frac{B_0}{4\pi\rho}\frac{d}{dz}\left(\frac{d^2}{dz^2}-k^2\right)b$$
$$pb = \eta\left(\frac{d^2}{dz^2}-k^2\right)b + B_0\frac{dv}{dz}$$
$$p\theta = \beta v + \kappa\left(\frac{d^2}{dz^2}-k^2\right)\theta$$

- Again we define dimensionless variables as before. We use d as the unit of length and d^2/ν as the unit of time to get the dimensionless wave-number $a = kd$ and dimensionless growth rate $\sigma = pd^2/\nu$. $D = d\, d/dz$ is the dimensionless derivative

$$(D^2 - a^2)(D^2 - a^2 - \sigma)v = \left(\frac{g\alpha d^2}{\nu}\right) a^2\theta - \left(\frac{B_0 d}{4\pi\rho\nu}\right) D(D^2 - a^2)b$$

$$(D^2 - a^2 - \sigma\mathcal{P}_1)b = -\left(\frac{B_0 d}{\eta}\right) Dv$$

$$(D^2 - a^2 - \mathcal{P}\sigma)\theta = -\frac{\beta}{\kappa} d^2 v$$

where $\mathcal{P} = \nu/\kappa$ is the Prandtl no. and $\mathcal{P}_1 = \nu/\eta$ is the magnetic Prandtl no.

- In this case elimination is not easy, and it is not possible to find analytic conditions when the transition of instability occurs through secular modes. Thus we assume that the transition to instability happens through $\sigma = 0$ then the eqs become

$$(D^2 - a^2)^2 v = \left(\frac{g\alpha d^2}{\nu}\right) a^2\theta - \left(\frac{B_0 d}{4\pi\rho\nu}\right) D(D^2 - a^2)b$$

$$(D^2 - a^2)b = -\left(\frac{B_0 d}{\eta}\right) Dv$$

$$(D^2 - a^2)\theta = -\frac{\beta}{\kappa} d^2 v$$

- Now we can eliminate b to get

$$(D^2 - a^2)^2 v - \left(\frac{B_0^2 d^2}{4\pi\rho\nu\eta}\right) D^2 v = \left(\frac{g\alpha d^2}{\nu}\right) a^2\theta$$

We define another dimensionless no.

$$Q = \frac{B_0^2 d^2}{4\pi\rho\nu\eta} = \frac{V_A^2 d^2}{\nu\eta}, \quad R = \frac{g\alpha\beta d^4}{\kappa\nu}$$

and eliminate θ to get

$$(D^2 - a^2)[(D^2 - a^2)^2 - QD^2]v = -Ra^2 v$$

- We again look for solutions of form $v = \sin(n\pi z)$ to get

$$Ra^2 = (n^2\pi^2 + a^2)^3 + Qn^2\pi^2(n^2\pi^2 + a^2)$$

It is clear that minimum value of R would be achieved for $n = 1$. Further, we can take $a^2 = \pi^2 x$. Thus the transition to instability occurs when

$$R = \frac{\pi^4}{x}\left((1 + x)^3 + \frac{Q}{\pi^2}(1 + x)\right)$$

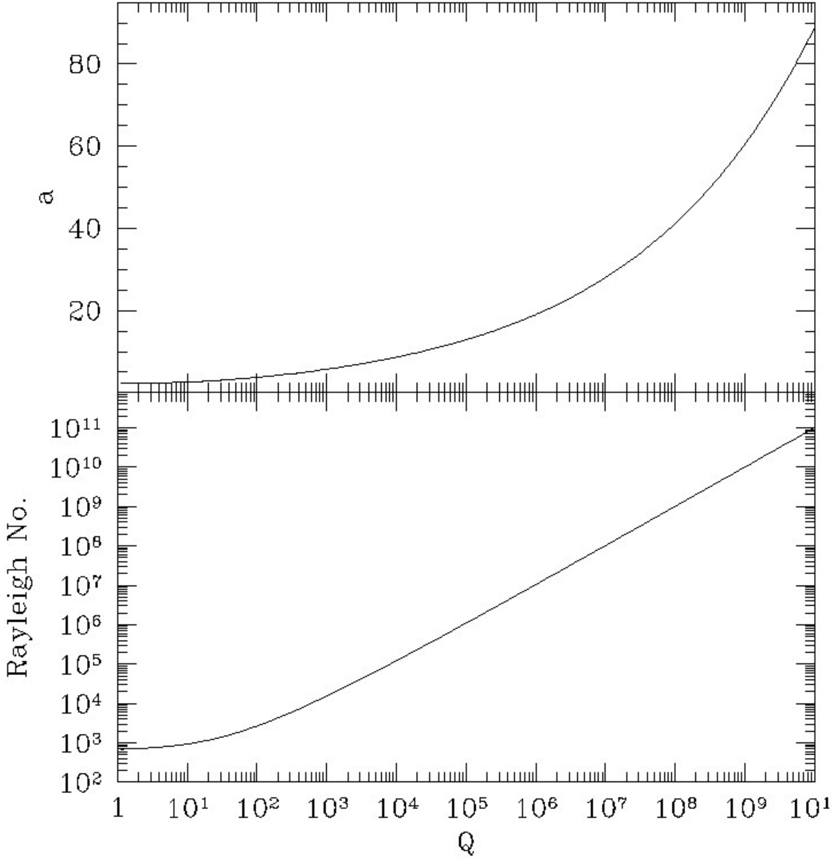
This gives $R(x, Q)$. For instability, R has to be greater than the minimum of R over all permissible values of x

$$\frac{dR}{dx} = -\left(1 + \frac{Q}{\pi^2}\right)\frac{1}{x^2} + 3 + 2x = 0$$

or

$$2x^3 + 3x^2 - 1 - \frac{Q}{\pi^2} = 0$$

This eq has at least one positive root. It turns out that there is only one positive root which gives the required value of x. It can be verified that the second derivative is positive at this point, Hence it is a minimum and the corresponding value of R is shown in the figure.



80
60
a
40
20
10^{11}
10^{10}
10^{9}
10^{8}
Rayleigh No.
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
1 10^{1} 10^{2} 10^{3} 10^{4} 10^{5} 10^{6} 10^{7} 10^{8} 10^{9} 10^{10}
Q

- As $Q \to \infty$ we get the solution

$$x_{\min} \approx \left(\frac{Q}{2\pi^2}\right)^{1/3}$$

$$R_{\text{crit}} \approx \pi^2 Q = 9.9Q$$

$$a_{\min} \approx \left(\frac{\pi^4 Q}{2}\right)^{1/6} = 1.9Q^{1/6}$$

- To complete the solution we can find θ and b from the respective eqs. To find the horizontal components of velocity we can use the corresponding components of the eq. of motion coupled with that of the induction eq. Looking for solutions of the form $\exp(ia_x x + ia_y y)$ which gives

$$(D^2 - a^2)^2 v_x = \left(\frac{g\alpha d^2}{\nu}\right) ia_x D\theta - \frac{B_0 d}{4\pi\rho\nu} D(D^2 - a^2) b_x$$

$$(D^2 - a^2) b_x = -\frac{B_0 d}{\eta} D v_x$$

Eliminating b_x gives

$$(D^2 - a^2)^2 v_x = \left(\frac{g\alpha d^2}{\nu}\right) ia_x D\theta + \frac{B_0^2 d^2}{4\pi\rho\nu\eta} D^2 v_x$$

Eliminating θ we get

$$(D^2 - a^2)[(D^2 - a^2)^2 - QD^2]v_x = -R\frac{\partial^2}{\partial x \partial z} v_z$$

- Using $v_z = \sin(\pi z)\cos(a_x x)\cos(a_y y)$ we get

$$\theta = \frac{\beta d^2}{\kappa(a^2 + \pi^2)} \sin(\pi z)\cos(a_x x)\cos(a_y y)$$

$$b_z = \frac{B_0 d\pi}{\eta(a^2 + \pi^2)} \cos(\pi z)\cos(a_x x)\cos(a_y y)$$

$$(a^2 + \pi^2)((a^2 + \pi^2)^2 + Q\pi^2)v_x = - Ra_x\pi\cos(\pi z)\sin(a_x x)\cos(a_y y)$$

Recognising that the factor multiplying v_x is Ra^2 we get

$$v_x = -\frac{a_x \pi}{a^2} \cos(\pi z) \sin(a_x x) \cos(a_y y)$$

which is the same as that for Benard problem.

- However, the solution for b_z doesn't satisfy the boundary condition $b_z = 0$. Consider the eq.

$$(D^2 - a^2) b_z = -\frac{B_0 d}{\eta} D v_z$$

The solution above is the particular solution. In addition there would be solutions of the homogeneous part. Considering only the part that depends on z the solution is

$$b_z = \frac{B_0 d \pi}{\eta(a^2 + \pi^2)} \cos(\pi z) + A_1 \cosh(az) + A_2 \sinh(az)$$

- Using the boundary condition $b_z = 0$ at $z = 0$ gives

$$A_1 = -\frac{B_0 d\pi}{\eta(a^2 + \pi^2)}$$

while $b_z = 0$ at $z = 1$ gives

$$A_2 \sinh(a) = \frac{B_0 d\pi}{\eta(a^2 + \pi^2)}(\cosh(a) + 1)$$

Thus giving the solution

$$b_z = \frac{B_0 d\pi}{\eta(a^2 + \pi^2)} \Bigg(\cos(\pi z) - \cosh(az)$$

$$+ \frac{\cosh(a) + 1}{\sinh(a)} \sinh(az) \Bigg) \cos(a_x x) \cos(a_y y)$$

Onset of Convection as Overstability

- If the onset of convection happens through overstability σ would be pure imaginary at the onset. We can eliminate θ and b from the eqs to get one eq in v. First eliminating b we get

$$(D^2 - a^2)[(D^2 - a^2 - \sigma)(D^2 - a^2 - \mathcal{P}_1\sigma) - \frac{B_0^2 d^2}{4\pi\rho\nu\eta} D^2]v$$

$$= \left(\frac{g\alpha d^2}{\nu}\right) a^2 (D^2 - a^2 - \mathcal{P}_1\sigma)\theta$$

now eliminating theta we get the final eq in v

$$(D^2 - a^2 - \mathcal{P}\sigma)(D^2 - a^2)[(D^2 - a^2 - \sigma)(D^2 - a^2 - \mathcal{P}_1\sigma) - QD^2]v_z = -Ra^2(D^2 - a^2 - \mathcal{P}_1\sigma)v_z$$

- Using the solution $v_z = \sin(\pi z)$ we get the dispersion relation

$$(\pi^2 + a^2 + \mathcal{P}\sigma)(\pi^2 + a^2)[(\pi^2 + a^2 + \mathcal{P}_1\sigma)(\pi^2 + a^2 + \sigma) \\ + \pi^2 Q] = Ra^2(\pi^2 + a^2 + \mathcal{P}_1\sigma)$$

Since σ is pure imaginary we define

$$a^2 = \pi^2 x, \quad \sigma = i\sigma_1\pi^2, \quad R = \pi^4 R_1, \quad Q = \pi^2 Q_1$$

to get

$$(1 + x + i\mathcal{P}\sigma_1)(1 + x)[(1 + x + i\mathcal{P}_1\sigma_1)(1 + x + i\sigma_1) + Q_1] \\ = R_1 x(1 + x + i\mathcal{P}_1\sigma)$$

- Since all quantities other than σ_1 are real and we wish to find condition when σ_1 is real, we assume that to be the case and separate the real and imaginary parts of the equation and equate them to zero

$$\begin{aligned}
&(1+x)[(1+x)^2 - \sigma_1^2\mathcal{P}_1 + Q_1] \\
&\qquad - \mathcal{P}\sigma_1^2(1+x)(1+\mathcal{P}_1) = R_1 x \\
&\mathcal{P}(1+x)[(1+x)^2 - \sigma_1^2\mathcal{P}_1 + Q_1] + (1+x)^3(1+\mathcal{P}_1) \\
&\qquad = R_1 x \mathcal{P}_1
\end{aligned}$$

Equating the two expressions for $R_1 x$ we get an equation which can be solved for σ_1^2

$$\sigma_1^2 = \frac{1}{\mathcal{P}_1^2}\left(\frac{\mathcal{P}_1 - \mathcal{P}}{1+\mathcal{P}} Q_1 - (1+x)^2\right)$$

and using this we get

$$R_1 = \frac{(1+\mathcal{P}_1)(\mathcal{P}+\mathcal{P}_1)}{\mathcal{P}_1^2}\frac{1+x}{x}$$
$$\left((1+x)^2 + Q_1\frac{\mathcal{P}^2}{(1+\mathcal{P})(\mathcal{P}+\mathcal{P}_1)}\right)$$

- Since $\sigma_1^2 > 0$ we must have

$$(1+x)^2 < Q_1\frac{\mathcal{P}_1-\mathcal{P}}{1+\mathcal{P}}$$

This gives the critical value x^* where $\sigma_1 = 0$.

- Further if $\mathcal{P} > \mathcal{P}_1$ then it is not possible to have $\sigma_1^2 > 0$. Thus in this case the transition to instability should occur through secular modes. Only for smaller values it is possible to have transition through oscillatory modes. In fact, for $x^* > 0$ we must have

$$Q_1 \frac{\mathcal{P}_1 - \mathcal{P}}{1 + \mathcal{P}} > 1$$

which give

$$\mathcal{P} < \frac{Q_1 \mathcal{P}_1 - 1}{Q_1 + 1}$$

Thus for $Q_1 \mathcal{P}_1 < 1$ it is not possible to satisfy this condition.

- As usual we try to find the minimum value of R_1 w.r.t. x, but this is acceptable only if $x_{\min} < x^*$, otherwise we can consider x^* as the minimum.

$$\frac{dR_1}{dx} = -\frac{1}{x^2}\left(1 + \frac{\mathcal{P}^2}{(1+\mathcal{P})(\mathcal{P}+\mathcal{P}_1)} Q_1\right) + 3 + 2x = 0$$

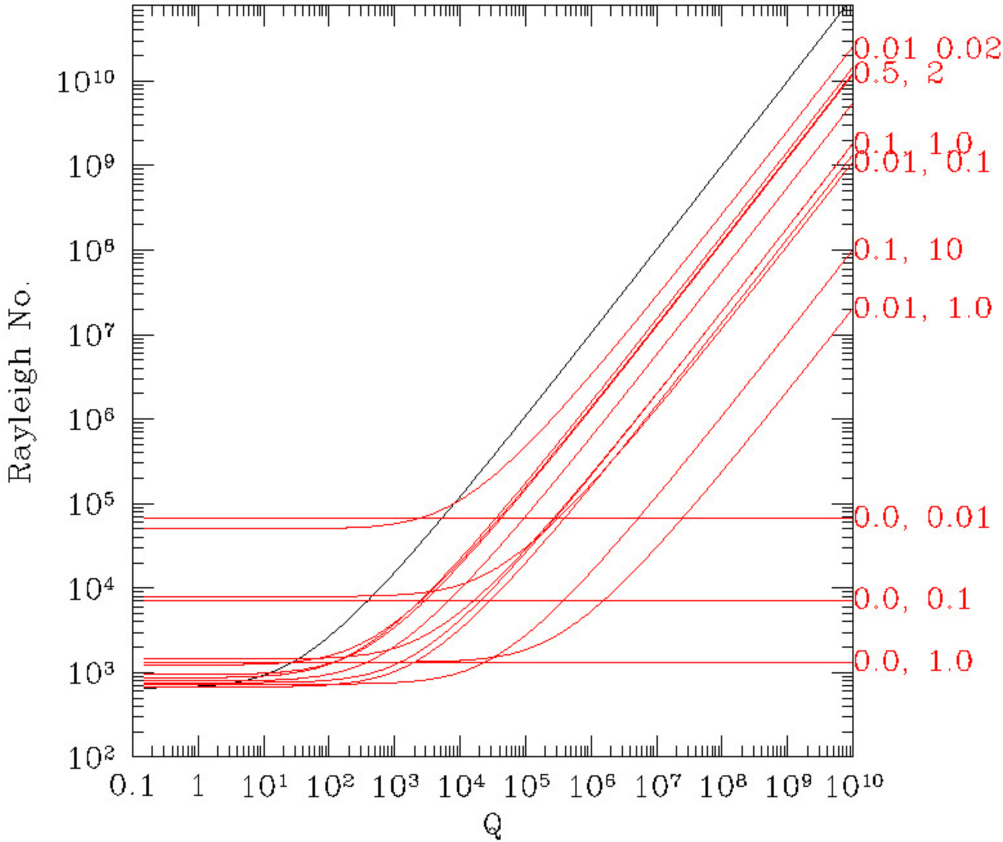
or

$$2x^3 + 3x^2 - 1 - \frac{\mathcal{P}^2}{(1+\mathcal{P})(\mathcal{P}+\mathcal{P}_1)} Q_1 = 0$$

which is similar to the earlier eq except that this also depends on $\mathcal{P}$ and $\mathcal{P}_1$.

- The figure shows $R_{\rm crit}$ for different values of $\mathcal{P}, \mathcal{P}_1$ along with that for the case where transition happens through $\sigma = 0$. At low values of Q the $R_{\rm crit}$ for $\sigma = 0$ is below that for the oscillatory instability and hence the transition would happen through secular modes. While at large Q the oscillatory modes give lower $R_{\rm crit}$, for a range of Prandtl Nos and hence the transition would occur through oscillatory modes. $x_{\rm min} < x^*$ for all cases where the transition is through oscillatory modes.

- For $\mathcal{P} = 0$ the critical R is independent of Q and its value is $1 + 1/\mathcal{P}_1$ times that for the case of no magnetic field.


Rayleigh No.
Q
0.1
1
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
0.01 0.02
0.5, 2
0.1, 1.0
0.01, 0.1
0.1, 10
0.01, 1.0
0.0, 0.01
0.0, 0.1
0.0, 1.0

- In the asymptotic limit as $\mathcal{P}^2 Q \to \infty$ the critical values are

$$x_{\min} = \left(\frac{\mathcal{P}^2 Q}{2\pi^2 (1+\mathcal{P})(\mathcal{P}+\mathcal{P}_1)} \right)^{1/3}$$

$$R_{\mathrm{crit}} = \pi^2 Q \left(\frac{\mathcal{P}^2 (1+\mathcal{P}_1)}{\mathcal{P}_1^2 (1+\mathcal{P})} \right)$$

$$a_{\min} = \left(\frac{\pi^4 \mathcal{P}^2 Q}{2(1+\mathcal{P})(\mathcal{P}+\mathcal{P}_1)} \right)^{1/6}$$

$$\sigma_1 = \frac{1}{\pi \mathcal{P}_1} \left(\frac{Q(\mathcal{P}_1 - \mathcal{P})}{1+\mathcal{P}} \right)^{1/2}$$

- The asymptotic ratio of critical value of R for the oscillatory case to that for secular case is

$$\frac{(1+\mathcal{P}_1)\mathcal{P}^2}{(1+\mathcal{P})\mathcal{P}_1^2}$$

This ratio exceeds unity when $\mathcal{P} > \mathcal{P}_1$, which is ruled out as $\sigma_1^2 < 0$. Thus for all values of $\mathcal{P}_1 > \mathcal{P}$, or $\kappa > \eta$, asymptotically the transition would take place through oscillatory modes.