Magnetohydrodynamics

- Magnetic field is present everywhere in the universe, starting from the Earth, Sun, interstellar space, galaxies etc.
- Hence electromagnetic forces need to be included in the equations of fluid mechanics if the fluid includes charged particles. e.g., plasma.
- The plasma or fluid is on an average electrically neutral.
- Even though plasma is electrically neutral, the electrons and ions move at different speeds giving a net current.

- Consider a small volume dV containing electrical charges moving with velocity $\mathbf{v}$. We can define the charge and current densities

$$\rho_e \, dV = \sum_{dV} e_i$$

$$\mathbf{J} \, dV = \sum_{dV} e_i \mathbf{v}_i$$

Thus, even if $\rho_e = 0$, $\mathbf{J}$ can be nonzero.

- In the presence of electromagnetic field $\mathbf{E}, \mathbf{B}$, where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is magnetic field, the force on individual particles is given by

$$\mathbf{F}_i = \frac{e_i}{c}\mathbf{v}_i \times \mathbf{B} + e_i\mathbf{E}$$

which gives the total force on volume element as

$$\mathbf{F}\, dV = \left(\frac{\mathbf{J} \times \mathbf{B}}{c} + \rho_e \mathbf{E} \right) \, dV$$

which is the Lorentz force which needs to be added to the eq of fluid mechanics.

- The equations may also need to be supplemented by the Maxwell's equation if the fluid is electrically conducting. This is because the fluid will have a back-reaction on $\mathbf{E}$ and $\mathbf{B}$.

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\left(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right)$$

- If the conductivity is high the charge density ρ_e would be negligible. Considering dimensional estimates, we get

$$E \sim \frac{BL}{cT} = B\frac{V}{c}$$

$$\rho_e \sim \frac{E}{4\pi L}$$

$$J \sim \frac{Bc}{4\pi L}$$

$$\frac{\rho_e E}{JB/c} \sim \frac{E^2/(4\pi L)}{B^2/(4\pi L)} = \frac{V^2}{c^2}$$

This is small if $V \ll c$. Similarly, the ratio of displacement current to J is

$$\frac{E/T}{4\pi J} \sim \frac{BV/(cT)}{Bc/L} = \frac{V^2}{c^2}$$

- Thus the Lorentz force is given by

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \frac{(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}}{4\pi}$$

Using the vector identity

$$(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} = \mathbf{B} \cdot \boldsymbol{\nabla}\mathbf{B} - \frac{1}{2}\boldsymbol{\nabla}(B^2)$$

The second term is like the pressure term and we can identify $B^2/(8\pi)$ as the magnetic pressure. The first term can be regarded as a tension term arising from stretching of lines of force. The ratio of the magnetic to gas pressure decides which force dominates.

The Ohm's law

- The Ohm's law gives the relation between the electric field and the current density

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)$$

This can be obtained by considering the motion of charged particle constituting the fluid. Consider a fully ionised plasma consisting of electrons and ions. For simplicity we assume that ions have single positive charge. Since the plasma is electrically neutral, the number densities $n = n_i = n_e$. The average velocity of ions $\mathbf{v}_i$ essentially defines the fluid velocity $\mathbf{v}$. The velocity of electrons is $\mathbf{v}_i + \mathbf{v}_e$. The current density is given by

$$\mathbf{J} = n_i e \mathbf{v}_i - n_e e(\mathbf{v}_i + \mathbf{v}_e) = -ne\mathbf{v}_e$$

- The total momentum lost by electrons due to collisions with ions per unit volume per unit time is

$$\frac{n_e m_e \mathbf{v}_e}{\tau_{ei}}$$

where τ_{ei} is the mean collision time. This drag force per electron can be equated to the electromagnetic force on electrons to get

$$-e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) = \frac{m_e \mathbf{v}_e}{\tau_{ei}} = -\frac{\mathbf{J} m_e}{\tau_{ei} n_e e}$$

which gives conductivity

$$\sigma = \frac{e^2 n_e \tau_{ei}}{m_e}$$

- The basic assumption here is that τ_{ei} is much smaller than the time scale of the problem. Further it is also assumed that the cyclotron frequency $\omega_e = eB/(m_e c)$ is small compared to $1/\tau_{ei}$.

- For partially ionised gas we need to consider the neutral atoms also with the respective collision time and velocity.

The Induction equation

- Using the Ohm's law

$$\mathbf{J} = \frac{c}{4\pi}\boldsymbol{\nabla} \times \mathbf{B} = \sigma\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right)$$

which give

$$c\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta\boldsymbol{\nabla} \times \mathbf{B}$$

where $\eta = c^2/(4\pi\sigma)$ is the resistivity. Substituting this in the Maxwell eq we get the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) = -\boldsymbol{\nabla} \times (\eta\boldsymbol{\nabla} \times \mathbf{B})$$

- By comparing the first and last term we can see the dimensionless diffusivity $\eta T/L^2$ is usually small in astrophysical conditions as L is large. We can define the magnetic Reynolds number as the reciprocal of this. In the ideal MHD we can neglect η. Just like the inviscid case there will be a boundary layer, referred to as the current sheet where η needs to be included.

- The diffusivity gives the Ohmic decay time-scale

$$T_o = \frac{L^2}{\eta}$$

over which the magnetic field will decay if there is no dynamo action to generate the field.

- This equation is similar to vorticity equation for fluid flow

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}) = \nu \nabla^2 \boldsymbol{\omega}$$

  Thus the magnetic flux tubes behaves in the same way as vortex tubes. Expanding the curl we can write the induction equation for the case of constant η as

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{v} + \eta \nabla^2 \mathbf{B}$$

The Alfven Waves

- Consider an ideal incompressible fluid with zero resistivity and viscosity

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{v} - \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{B}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = \frac{\mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B} - \frac{1}{2} \boldsymbol{\nabla} (B^2)}{4\pi\rho} - \frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla} \Psi$$

- Consider an equilibrium situation with uniform $\mathbf{B}_0$. The pressure gradient would balance the gravity. Now consider a small perturbation about the equilibrium state, $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, $P = P_0 + P_1$, where the perturbations are small.

- Linearising the equations in perturbations we get

$$\frac{\partial \mathbf{B}_1}{\partial t} = \mathbf{B}_0 \cdot \boldsymbol{\nabla} \mathbf{v}$$

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathbf{B_0} \cdot \boldsymbol{\nabla} \mathbf{B}_1 - \boldsymbol{\nabla}(\mathbf{B}_0 \cdot \mathbf{B}_1)}{4\pi\rho} - \frac{\boldsymbol{\nabla} P_1}{\rho}$$

Now taking the divergence of the second eq. gives

$$\nabla^2 \left(P_1 + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{4\pi} \right) = 0$$

The relevant solution is

$$P_1 + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{4\pi} = 0$$

- We can take $\mathbf{B}_0 = (0, 0, B_0)$ to get the equations

$$\frac{\partial \mathbf{B}_1}{\partial t} = B_0 \frac{\partial \mathbf{v}}{\partial z}$$
$$\frac{\partial \mathbf{v}}{\partial t} = \frac{B_0}{4\pi\rho} \frac{\partial \mathbf{B}_1}{\partial z}$$

Differentiating the first eq. with t and using the 2nd eq. we get

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \frac{B_0^2}{4\pi\rho} \frac{\partial^2 \mathbf{B}_1}{\partial z^2}$$

- This is a wave equation with speed $V_A^2 = B_0^2/(4\pi\rho)$. The waves are known as Alfven waves and V_A is the Alfven speed. The sound speed in the fluid is given by $c_s^2 = \Gamma_1 P/\rho$. Thus the ratio

$$\frac{V_A^2}{c_s^2} = \frac{B^2}{4\pi\Gamma_1 P}$$

which is comparable to the ratio of magnetic to gas pressure.

- The solution of the wave equation would be of the form

$$\mathbf{B}_1 = \mathbf{b}\exp(i\omega t - ik_z z)$$

where $\mathbf{b}$ is a constant vector and $\omega^2 = k_z^2 V_A^2$ is the dispersion relation, which can be obtained by substituting this form in the equation. Further, since $\boldsymbol{\nabla}\cdot\mathbf{B}_1 = 0$, the z-component of $\mathbf{b}$ must be zero. Thus the wave is transverse. Magnetic tension provides the restoring force, just like tension in a string gives rise of transverse waves. In the presence of viscosity and resistivity these waves would be damped.

- Consider an incompressible fluid with constant viscosity, ν and resistivity, η. The eqs are given by

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{v} - \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{B} + \eta \nabla^2 \mathbf{B}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = \frac{\mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B} - \frac{1}{2} \boldsymbol{\nabla} (B^2)}{4\pi\rho} - \frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla} \Psi + \nu \nabla^2 \mathbf{v}$$

- Again using the same basic state as before and perturbing the variables and linearising the eqs

$$\frac{\partial \mathbf{B}_1}{\partial t} = \mathbf{B}_0 \cdot \boldsymbol{\nabla} \mathbf{v} + \eta \nabla^2 \mathbf{B}_1$$

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathbf{B_0} \cdot \boldsymbol{\nabla} \mathbf{B}_1 - \boldsymbol{\nabla} (\mathbf{B}_0 \cdot \mathbf{B}_1)}{4\pi\rho} - \frac{\boldsymbol{\nabla} P_1}{\rho} + \nu \nabla^2 \mathbf{v}$$

Now taking the divergence of the second eq. gives

$$\nabla^2 \left(P_1 + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{4\pi} \right) = 0$$

The relevant solution is

$$P_1 + \frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{4\pi} = 0$$

- We can take $\mathbf{B}_0 = (0, 0, B_0)$ to get the equations

$$\begin{aligned} \frac{\partial \mathbf{B}_1}{\partial t} &= B_0 \frac{\partial \mathbf{v}}{\partial z} + \eta \nabla^2 \mathbf{B}_1 \\ \frac{\partial \mathbf{v}}{\partial t} &= \frac{B_0}{4\pi\rho} \frac{\partial \mathbf{B}_1}{\partial z} + \nu \nabla^2 \mathbf{v} \end{aligned}$$

- This is an eq with constant coefficients and hence looking for exponential solution of the form

$$\mathbf{B}_1 = \mathbf{b}\exp(i\omega t - ik_z z)$$
$$\mathbf{v} = \mathbf{V}\exp(i\omega t - ik_z z)$$

gives

$$\omega\mathbf{b} = -B_0 k_z \mathbf{V} + i\eta k_z^2 \mathbf{b}$$
$$\omega\mathbf{V} = -k_z \frac{B_0}{4\pi\rho}\mathbf{b} + i\nu k_z^2 \mathbf{V}$$

- Which gives the dispersion relation

$$\begin{vmatrix} \omega - i\eta k_z^2 & B_0 k_z \\ k_z \frac{B_0}{4\pi\rho} & \omega - i\nu k_z^2 \end{vmatrix} = 0$$

or

$$\omega^2 - i\omega k_z^2(\nu+\eta) - \nu\eta k_z^4 - V_A^2 k_z^2 = 0$$

and

$$\omega = \frac{ik_z^2(\nu+\eta) \pm \sqrt{-(\eta+\nu)^2 k_z^4 + 4\nu\eta k_z^4 + 4V_A^2 k_z^2}}{2}$$

Assuming ν and η to be small we get the approximate value

$$\omega \approx V_A k_z + i\frac{(\nu+\eta)k_z^2}{2}$$

which is a damped wave.

- Again since $\nabla \cdot \mathbf{b} = 0$ and $\nabla \cdot \mathbf{V} = 0$ we get $b_z = 0$ and $v_z = 0$ and the waves are transverse.

The Alfven Theorem

- The Alfven Theorem states that for ideal MHD the magnetic flux crossing a surface is conserved, or the magnetic flux tubes are frozen in the fluid. This is similar to the circulation for vorticity.

- Consider an open surface S bounded once by a closed curve, C which is in the surface. The flux through the surface is
$$\Phi = \int_S \mathbf{B} \cdot \mathbf{n} \, dS$$

- The flux can change due to change in $\mathbf{B}$ along the surface, or due to flux coming in from the boundary

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS + \oint_C \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l}$$
$$= \int_S \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{n}\, dS + \oint_C \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l}$$

Using Stokes theorem we get

$$\frac{d\Phi}{dt} = \oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} + \oint_C \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l} = 0$$