Shocks

- Because of nonlinear terms in the eq of motion, the solution can develop a discontinuity under certain conditions leading to shocks. We first consider nonlinear waves.

- Consider a compressible but inviscid fluid in one dimension

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} = -\frac{\partial P}{\partial x}$$
$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = -\rho \frac{\partial v}{\partial x}$$

- We assume that the fluid is adiabatic $P \sim \rho^{\gamma}$ and sound speed $c^2 = \gamma P/\rho \sim \rho^{\gamma-1}$. Which gives

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{2}{\gamma - 1}\frac{1}{c}\frac{dc}{dt} = -\frac{\partial v}{\partial x}$$

and

$$\frac{1}{P}\frac{\partial P}{\partial x} = \frac{2\gamma}{\gamma - 1}\frac{1}{c}\frac{\partial c}{\partial x}$$

or

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{2\gamma}{\gamma - 1}\frac{P}{\rho c^2}c\frac{\partial c}{\partial x} = \frac{2}{\gamma - 1}c\frac{\partial c}{\partial x}$$

- This gives the eqs in v and c

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{2}{\gamma - 1}c\frac{\partial c}{\partial x}$$

$$\frac{2}{\gamma - 1}\left(\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x}\right) = -c\frac{\partial v}{\partial x}$$

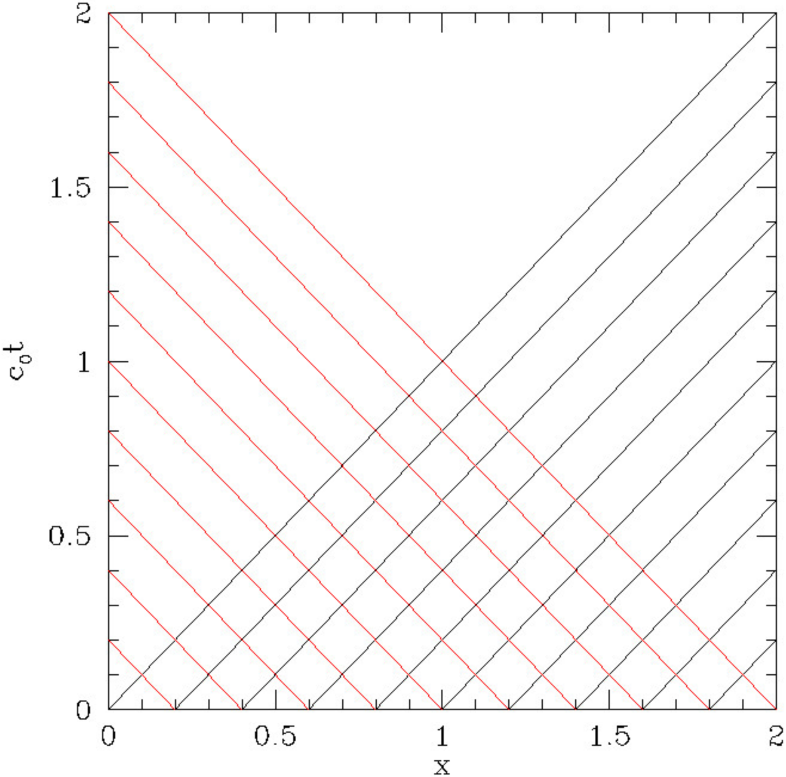
- Adding and subtracting these eqs give

$$\left(\frac{\partial}{\partial t} + (v+c)\frac{\partial}{\partial x}\right)\left(v + \frac{2}{\gamma - 1}c\right) = 0$$
$$\left(\frac{\partial}{\partial t} + (v-c)\frac{\partial}{\partial x}\right)\left(v - \frac{2}{\gamma - 1}c\right) = 0$$

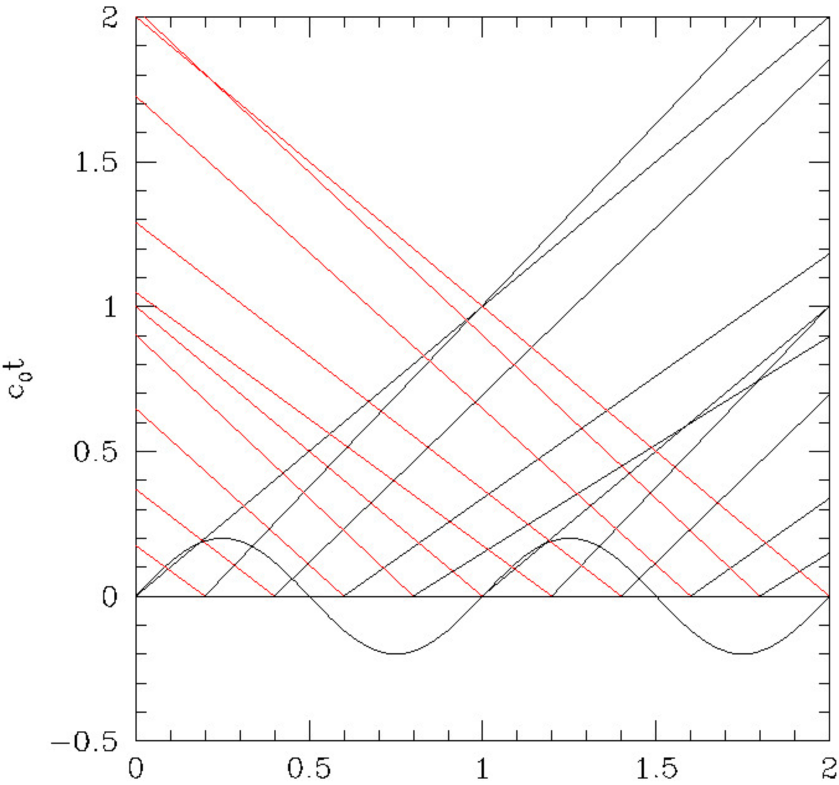
- These are the characteristics of the hyperbolic eq. defined by the equation

$$\left(\frac{dx}{dt}\right)_{\pm} = v \pm c$$

These define the region where the solution can be found as well as find the solution by propagating the variables along the characteristics.



Characteristics for unperturbed fluid



Characteristics for fluid with acoustic wave

- If the characteristics intersect then the solution would be multiple valued and hence unphysical.
- To illustrate this consider time evolution of an acoustic wave which is excited in the fluid. To get the initial condition we solve the linearised eqs, with v and $c = c_0 + c_1$. We assume the undisturbed state to have $c_0 = 1$. The eqs are

$$\frac{\partial v}{\partial t} = -\frac{2}{\gamma - 1} c_0 \frac{\partial c_1}{\partial x}$$
$$\frac{2}{\gamma - 1} \frac{\partial c_1}{\partial t} = -c_0 \frac{\partial v}{\partial x}$$

- Since these are eqs with constant coefficients, we look for solutions of the form

$$v(x,t) = v \exp(i\omega t - ikx), \qquad c_1(x,t) = c \exp(i\omega t - ikx)$$

to get

$$\omega v = \frac{2}{\gamma - 1} k c_0 c, \qquad \frac{2}{\gamma - 1} \omega c = k c_0 v$$

which gives the dispersion relation $\omega^2 = k^2 c_0^2$.

- Substituting, $\omega = k c_0$ in the first eq we get

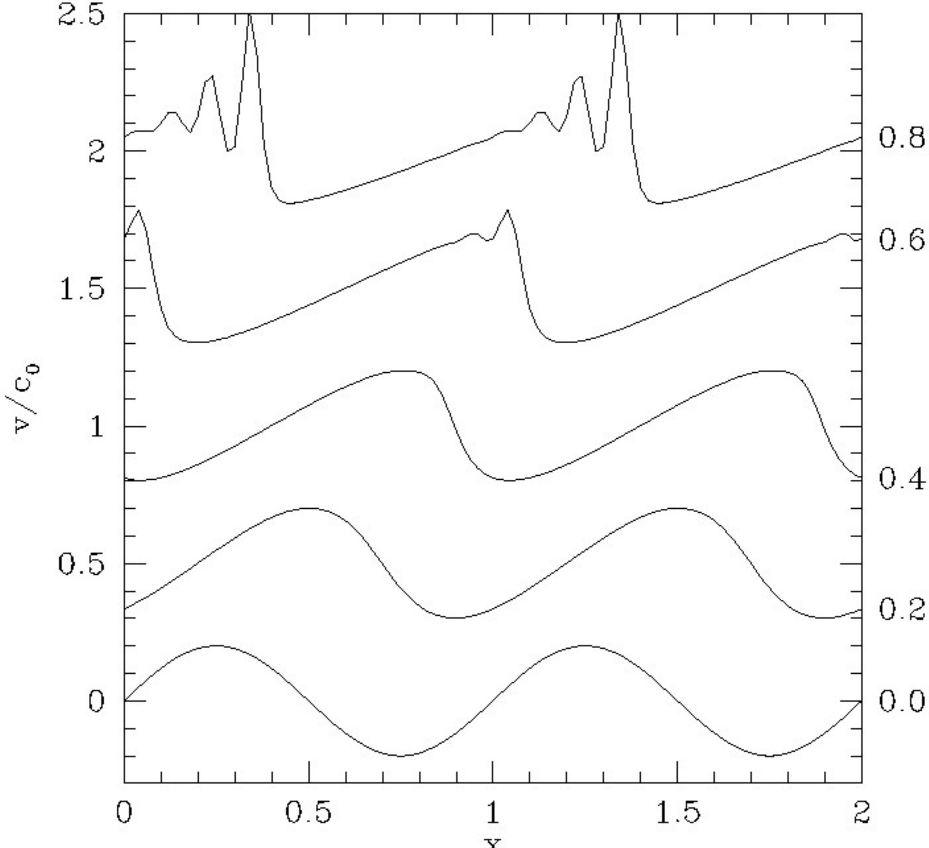
$$v = \frac{2}{\gamma - 1} c$$

we use $\gamma = 5/3$ and $k = 2\pi$ to get the initial perturbation

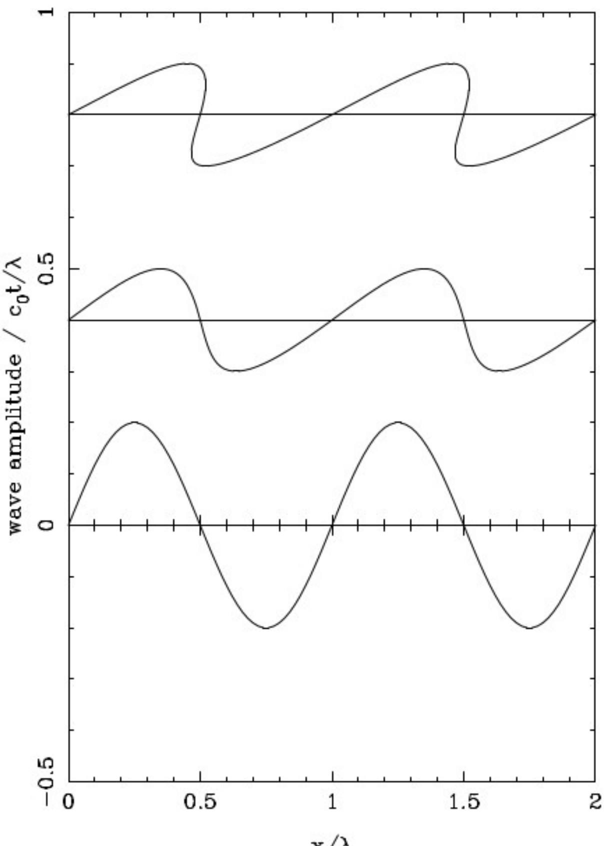
$$v(x, 0) = a \sin(2\pi x), \qquad c(x, 0) = 1 + \frac{a}{3} \sin(2\pi x)$$

- Using this initial conditions we solve the nonlinear eq numerically and the solution is shown in the figure at different times.

- The waves steepen to a shock and after that numerical solution has some oscillations. Instead if the characteristic eqs are used then the solution becomes multiply defined after the time when the characteristics cross.



2.5
2
1.5
1
0.5
0
0.8
0.6
0.4
0.2
0.0
v/c_0
0
0.5
1
1.5
2
x

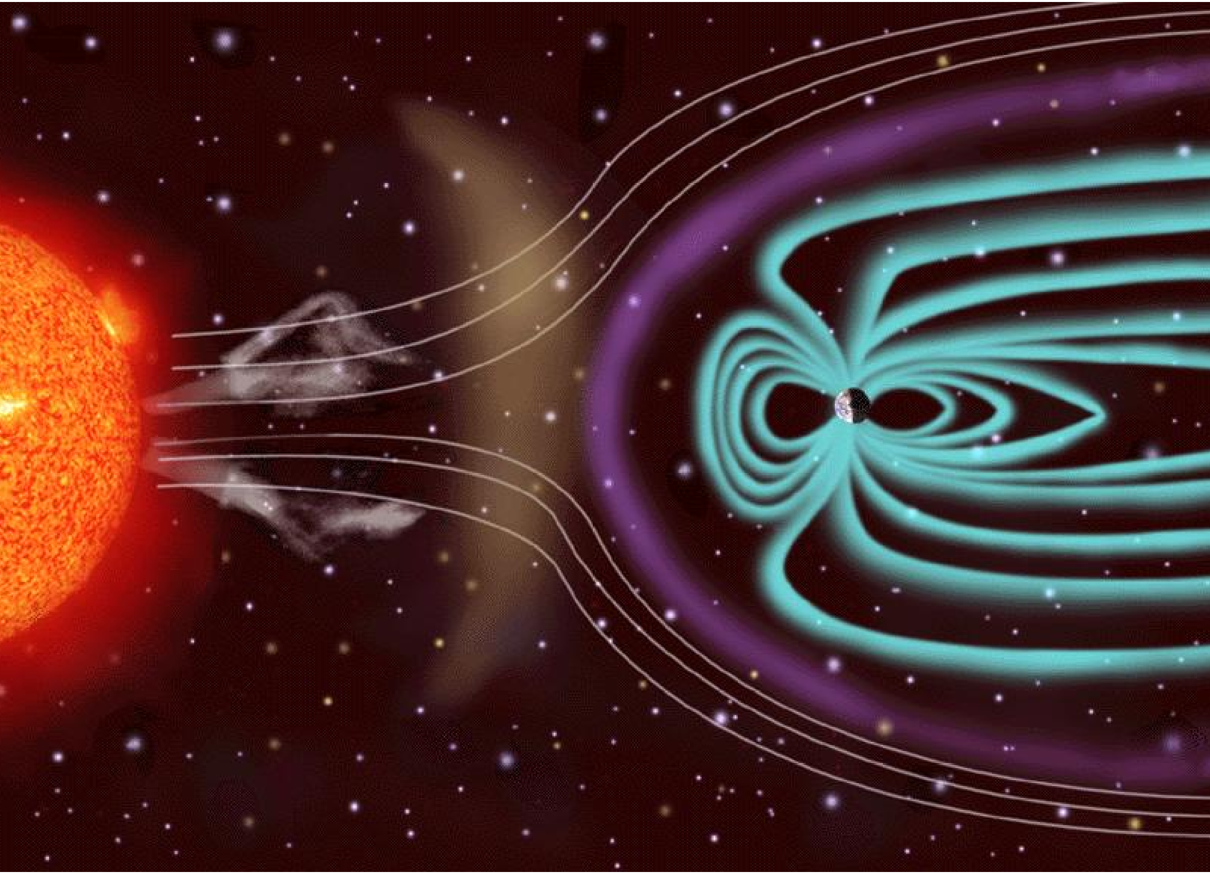


wave amplitude / c_0t/λ
1
0.5
0
−0.5
0
0.5
1
1.5
2
x/λ

- To avoid multiply defined nature of solution, we need to introduce a discontinuity in v and c at some point, which gives rise to a shock.

- Almost all cases where the inviscid fluid has compression would lead to formation of shock. For an inviscid fluid the shock thickness is zero so it is actual discontinuity. In the above calculations, the oscillations occur due to numerical limitations before the discontinuity develops.

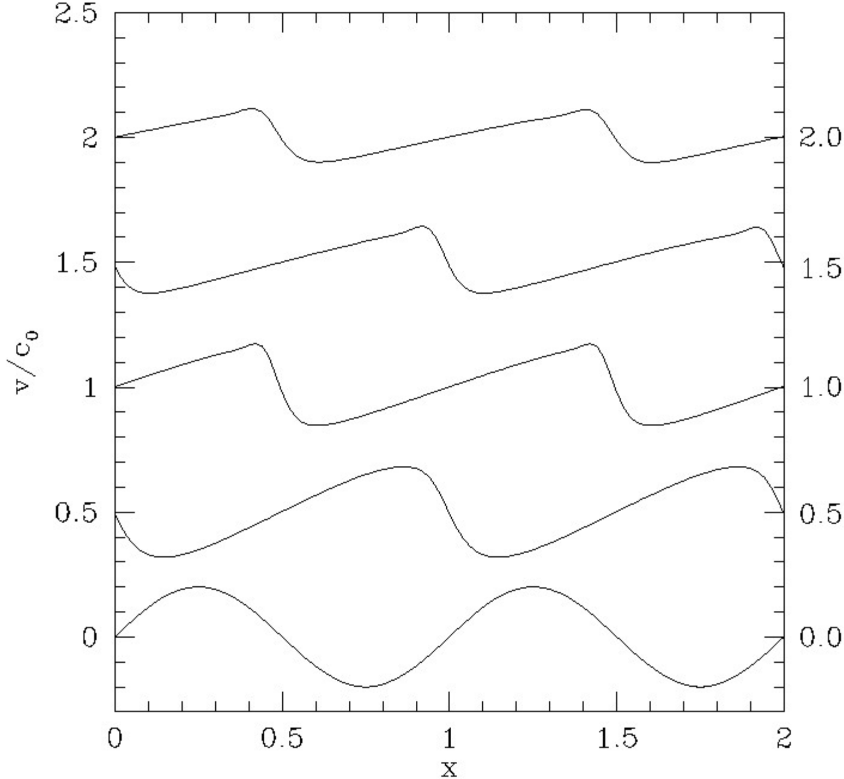
- Other cases of shock formation occur when the velocity v has a transition from subsonic to supersonic, or when an object is moving at supersonic speed through a medium. This is because the surrounding medium doesn't get a warning of incoming object.



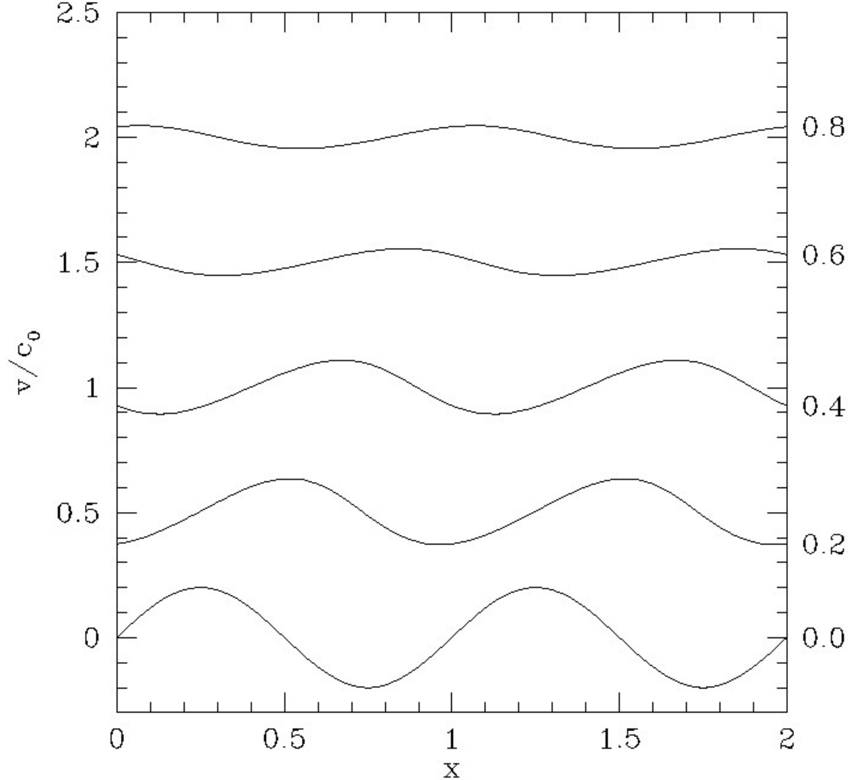
- If we include viscosity, the viscous stresses would not allow discontinuity as the stress will blow up in that case and the shock would have finite thickness, δ, such that $\nu v/\delta^2$ would balance the offending nonlinear term v^2/δ. Comparing the two gives the shock thickness

$$\delta = \frac{\nu}{v}$$

- In most cases the viscosity is very small and in practice it is difficult to resolve the shock region and hence it can still be considered as a discontinuity. Sometimes it is easier to introduce numerical viscosity such that the shock thickness is larger than the mesh spacing.



$\nu = 0.01$



$\nu = 0.1$

- These calculations used a mesh spacing of 0.02 in x and amplitude of $v = 0.2$ in some units. Hence, if the shock thickness is less than or comparable to this the numerical solution will not be able to resolve it and the result would be similar to inviscid case. For $\nu = 0.01$ the shock thickness is of order of $\nu/v \approx 0.05$, thus the effect of viscosity is visible.

- The viscosity also damps the wave. The damping rate can be calculated using linear theory. Thus we add the viscosity term to the eq of motion

$$\frac{\partial v}{\partial t} = -\frac{2}{\gamma - 1} c_0 \frac{\partial c_1}{\partial x} + \frac{4}{3} \nu \frac{\partial^2 v}{\partial x^2}$$

- Substituting the same form we get

$$\omega v = \frac{2}{\gamma - 1} k c_0 c_1 + i\frac{4}{3} k^2 \nu v, \qquad \frac{2}{\gamma - 1}\omega c_1 = k c_0 v$$

which gives the dispersion relation

$$\omega^2 = k^2 c_0^2 + i\frac{4}{3} k^2 \omega \nu$$

Assuming $\nu k \ll 1$ this gives

$$\omega = \pm k c_0 + \frac{2}{3} i k^2 \nu$$

which gives a damping time scale of $3/(2\nu k^2)$.

- For our calculations, $\nu = 0.01$, $k = 2\pi$ we get the damping time of about 4, which is consistent with the results shown. When $\nu = 0.1$ the damping time scale would be 0.4, which is also consistent with the results.

Shocks in ideal fluid

- We consider an ideal compressible fluid with no dissipation except possibly in the shock region. Thus on either side of the shock $P/\rho^\gamma = \mathrm{const}$, though the constant could be different on the two sides of shock as the shock may add some entropy. Further, there is no external force.

- Shocks typically occur in gases when the velocities are comparable to sound speed. Typical molecular viscosity $\nu \sim l v_m$, where l is the mean free path and v_m is the mean speed of molecules, which is typically comparable to sound speed c. Thus, the Reynold's no. $R_e = Lv/\nu = Lv/lc$, where L is the typical length scale in the system and v the gas speed. Thus for $v \sim c$, $R_e = L/l \gg 1$. Hence the fluid is almost inviscid.

- We assume that the shock is perpendicular to x-axis and at $x = 0$, while all quantities are only a function of x. Basically, we are in a frame which is moving with the shock. Further, we assume that the velocity is in $x - z$ plane. Further, $v_x > 0$, i.e. the fluid is moving in the positive x-direction.

- We first obtain the required conservation eq, starting from the basic eqs in Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial P}{\partial x_i}$$

$$\rho \frac{\partial C_v T}{\partial t} + \rho v_j \frac{\partial C v T}{\partial x_j} + P \frac{\partial v_j}{\partial x_j} = 0$$

- Taking appropriate combinations of the continuity eq with other eqs we get the eq for conservation of momentum and energy

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_j v_i}{\partial x_j} + \frac{\partial P}{\partial x_i} = 0$$

$$\frac{\partial \rho(\frac{v^2}{2} + u)}{\partial t} + \frac{\partial \rho v_j(\frac{v^2}{2} + h)}{x_j} = 0$$

where $u = C_v T$ is the internal energy and $h = u + P/\rho$ is the enthalpy.

- Applying these eq to the fluid on either side of the shock and noting that only derivatives w.r.t. x will contribute, we get

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_x}{\partial x} = 0$$

$$\frac{\partial \rho v_x}{\partial t} + \frac{\partial (\rho v_x^2 + P)}{\partial x} = 0$$

$$\frac{\partial \rho v_z}{\partial t} + \frac{\partial (\rho v_x v_z)}{\partial x} = 0$$

$$\frac{\partial \rho (\frac{v^2}{2} + u)}{\partial t} + \frac{\partial \rho v_x (\frac{v^2}{2} + h)}{\partial x} = 0$$

where for perfect gas EOS.

$$u = \frac{P}{(\gamma - 1)\rho}, \qquad h = \frac{\gamma P}{(\gamma - 1)\rho}$$

- All the conservation eq have the form

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Integrating them over the shock $(-\epsilon, \epsilon)$ we get

$$F_2 - F_1 = 0$$

where subscripts 1 and 2 refer to $x < 0$ and $x > 0$.

- For shock with infinitesimal thickness this would be valid as Q and its temporal derivative are finite. For shock of finite thickness it will be valid if the system is in steady state, $\partial/\partial t = 0$.

- Thus we get the connection conditions across the shock which are known as the Rankine–Hugoniot jump conditions

$$\rho_1 v_{x1} = \rho_2 v_{x2} = J$$
$$\rho_1 v_{x1}^2 + P_1 = \rho_2 v_{x2}^2 + P_2$$
$$\rho_1 v_{x1} v_{z1} = \rho_2 v_{x2} v_{z2}$$
$$\rho_1 v_{x1} \left(\frac{v_{x1}^2 + v_{z1}^2}{2} + h_1 \right) = \rho_2 v_{x2} \left(\frac{v_{x2}^2 + v_{z2}^2}{2} + h_2 \right)$$

  where, J is the mass flux across the shock.

- We expect $\rho_2 > \rho_1$, which means $v_{x1} > v_{x2}$. Combining the 1st and 3rd eq

$$v_{z1} = v_{z2} = v_z$$

With these the last eq gives

$$\frac{v_{x1}^2}{2} + h_1 = \frac{v_{x2}^2}{2} + h_2$$

- If we define the specific volume, i.e, the volume that contains unit mass $V = 1/\rho$, then

$$V_1 = \frac{1}{\rho_1}, \qquad V_2 = \frac{1}{\rho_2}$$

Now the conservation of x-component of momentum give

$$P_1 + J^2 V_1 = P_2 + J^2 V_2$$

or

$$J^2 = \frac{P_2 - P_1}{V_1 - V_2}$$

- The conservation of energy gives

$$\frac{1}{2}J^2V_1^2 + h_1 = \frac{1}{2}J^2V_2^2 + h_2$$

Using the expression for enthalpy we get

$$J^2(V_1^2 - V_2^2) = \frac{2\gamma}{\gamma - 1}(P_2V_2 - P_1V_1)$$

- Eliminating J between these two eqs gives

$$\frac{\gamma}{\gamma - 1}(P_2V_2 - P_1V_1) = \frac{1}{2}(V_1 + V_2)(P_2 - P_1)$$

- We can define the shock compression ratio

$$r = \frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{v_{x1}}{v_{x2}}$$

Using $V_1 = rV_2$ in the earlier eq we get

$$r = \frac{(\gamma+1)P_2 + (\gamma-1)P_1}{(\gamma+1)P_1 + (\gamma-1)P_2}$$

- We can also define the shock pressure ratio $R = P_2/P_1$ which gives

$$r = \frac{(\gamma+1)R + (\gamma-1)}{(\gamma+1) + (\gamma-1)R}$$

or

$$R = \frac{(\gamma+1)r - (\gamma-1)}{(\gamma+1) - r(\gamma-1)}$$

- Since $R > 0$ we get constraints on r as

$$\frac{\gamma-1}{\gamma+1} < r < \frac{\gamma+1}{\gamma-1}$$

For $\gamma = 5/3$ this gives the range $(1/4, 4)$.

- The ratio for all other quantities can be expressed in terms of these, e.g., for the sound speed

$$\frac{c_2^2}{c_1^2} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{R}{r}$$

- For weak shock, $r \approx 1$ and $R \approx 1$. We can write

$$R - 1 = \frac{2\gamma(r-1)}{(r+1) - \gamma(r-1)} \approx \gamma(r-1)$$

Using

$$\Delta P = P_2 - P_1 = (R-1)P_1, \qquad \Delta\rho = \rho_2 - \rho_1 = (r-1)\rho_1$$

we get

$$\Delta P = c_1^2 \Delta\rho$$

Similar to adiabatic perturbations in sound waves.

- For strong shocks we expect $R \gg 1$ which gives

$$r \to \frac{\gamma + 1}{\gamma - 1} = r_{\max}$$

which is the upper limit on r as beyond this $R < 0$. For perfect gas EOS, $\gamma = 5/3$ and $r_{\max} = 4$.

- It would be more interesting to get a relation for r and R in terms of basic variables for $x < 0$. For this purpose we can write the Rankine–Hugoniot jump conditions in terms of the sound speed $c^2 = \gamma P/\rho$

$$\rho_1\left(v_{x1}^2 + \frac{c_1^2}{\gamma}\right) = \rho_2\left(v_{x2}^2 + \frac{c_2^2}{\gamma}\right)$$

$$\frac{v_{x1}^2}{2} + \frac{c_1^2}{\gamma - 1} = \frac{v_{x2}^2}{2} + \frac{c_2^2}{\gamma - 1}$$

using the ratios, r, R and defining the Mach number $M_{n1} = v_{x1}/c_1$ for normal component of velocity. $M_{n1} = M_1 \cos\theta$, whree M_1 is the actual Mach No. and θ is the angle of inclination to the normal.

- Dividing the eqs by c_1^2, We get

$$M_{n1}^2 + \frac{1}{\gamma} = r\left(\frac{M_{n1}^2}{r^2} + \frac{1}{\gamma}\frac{R}{r}\right)$$

$$\frac{1}{2}M_{n1}^2 + \frac{1}{\gamma - 1} = \frac{M_{n1}^2}{2r^2} + \frac{1}{\gamma - 1}\frac{R}{r}$$

Eliminating R between these we get r and then R as

$$r = \frac{(\gamma+1)M_{n1}^2}{2 + (\gamma - 1)M_{n1}^2}$$

$$R = 1 + \frac{2\gamma}{\gamma + 1}(M_{n1}^2 - 1)$$

- Shocks tradionally exist only for $M_{n1} > 1$. For $M_{n1} = 1$, $r = R = 1$ and there is no discontinuity. As $M_{n1} \to \infty$, $r \to r_{\max}$ and $R \to \infty$. For $R > 0$ we need

$$M_{n1}^2 > \frac{\gamma - 1}{2\gamma}$$

which gives a lower limit of 0.2.

Strong Shocks

- We had shown that in the limit of strong shocks,

$$\frac{P_2}{P_1} = R \to \infty, \qquad \frac{\rho_2}{\rho_1} = r \to \frac{\gamma+1}{\gamma-1}$$

These can also be derived directly from the Rankine–Hugoniot jump conditions, by taking $P_2 \gg P_1$

$$\rho_1 v_{x1} = \rho_2 v_{x2} = J$$

$$\rho_1 v_{x1}^2 = \rho_2 v_{x2}^2 + P_2$$

$$\frac{1}{2} v_{x1}^2 = \frac{1}{2} v_{x2}^2 + \frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2}$$

- Using the first eq in the other two we get

$$v_{x1} - v_{x2} = \frac{P_2}{J}, \quad v_{x1}^2 - v_{x2}^2 = \frac{2\gamma}{\gamma-1} \frac{P_2 v_{x2}}{J}$$

eliminating P_2 from these gives

$$v_{x1} + v_{x2} = \frac{2\gamma}{\gamma - 1} v_{x2}$$

Noting that $r = \rho_2/\rho_1 = v_{x1}/v_{x2}$ we get

$$r = \frac{\gamma + 1}{\gamma - 1}$$

which is the limiting value for strong shock obtained earlier.

- With this we get

$$P_2 = \rho_1 v_{x1}(v_{x1} - v_{x2}) = \frac{2}{\gamma + 1} \rho_1 v_{x1}^2$$

Contact discontinuities

- The Rankine–Hugoniot jump conditions

$$\rho_1 v_{x1} = \rho_2 v_{x2} = J$$

$$\rho_1 v_{x1}^2 + P_1 = \rho_2 v_{x2}^2 + P_2$$

$$\rho_1 v_{x1} v_{z1} = \rho_2 v_{x2} v_{z2}$$

$$\rho_1 v_{x1} \left(\frac{v_{x1}^2 + v_{z1}^2}{2} + h_1 \right) = \rho_2 v_{x2} \left(\frac{v_{x2}^2 + v_{z2}^2}{2} + h_2 \right)$$

also allow for another solution when $v_{x1} = v_{x2} = 0$, which gives

$$P_2 = P_1, \quad \rho_2 \neq \rho_1, \quad v_{z2} \neq v_{z1}$$

- These are referred to as contact discontinuities, where there is no mass flux across the shock and pressure is continuous. However, there can be arbitrary discontinuity in ρ and v_z. The discontinuity in v_z would give rise to Kelvin–Helmholtz instability. Thus we can assume v_z to be continuous.