Dissipation in a shock

- We can calculate the change in entropy across the shock. For a perfect gas the specific entropy (entropy per unit mass) is given by

$$s = C_v \ln(P/\rho^\gamma)$$

Thus the change in entropy across the shock

$$\begin{aligned} ds = s_2 - s_1 &= C_v[\ln(P_2/P_1) - \gamma \ln(\rho_2/\rho_1)] \\ &= C_v(\ln R - \gamma \ln r) \end{aligned}$$

- In the weak shock limit we can write this as

$$
\begin{aligned}
ds &= C_v(\ln[1+(R-1)] - \gamma\ln[1+(r-1)]) \\
&\approx C_v\left((R-1) - \frac{1}{2}(R-1)^2 + \frac{1}{3}(R-1)^3\right. \\
&\qquad \left. -\gamma(r-1) + \frac{1}{2}\gamma(r-1)^2 - \frac{1}{3}\gamma(r-1)^3\right)
\end{aligned}
$$

- For weak shocks we can expand $(R-1)$ in terms of $(r-1)$ using

$$
\begin{aligned}
R &= \frac{(\gamma+1)r - (\gamma-1)}{(\gamma+1) - (\gamma-1)r} \\
R-1 &= \frac{2\gamma(r-1)}{(r+1) - \gamma(r-1)} = \frac{2\gamma(r-1)}{2 + (1-\gamma)(r-1)}
\end{aligned}
$$

Expanding this in a Taylor series for $r-1$ gives

$$R-1 \approx \gamma(r-1)\left(1-\frac{1-\gamma}{2}(r-1)+\frac{(1-\gamma)^2}{4}(r-1)^2\right)$$
$$(R-1)^2 \approx \gamma^2(r-1)^2\left[1-(1-\gamma)(r-1)\right]$$

- Substituting this in expression for ds we get

$$ds \approx C_v\gamma\frac{\gamma^2-1}{12}(r-1)^3$$

which is positive if $r>1$. Thus for dissipation in shock we must have $r>1$.

- We need to show that $ds > 0$ for $r > 1$. For this we can take the derivative

$$\frac{d(ds)}{dr} = C_v \left(\frac{1}{R} \frac{dR}{dr} - \frac{\gamma}{r} \right)$$

$$= C_v \left(\frac{\gamma + 1}{(\gamma + 1)r - (\gamma - 1)} + \frac{\gamma - 1}{(\gamma + 1) - r(\gamma - 1)} - \frac{\gamma}{r} \right)$$

$$= C_v \frac{\gamma(\gamma - 1)[(\gamma + 1)r^2 - 2(\gamma - 1)r + \gamma + 1]}{r[(\gamma + 1)r - (\gamma - 1)][(\gamma + 1) - r(\gamma - 1)]}$$

which is positive for $1 < r < r_{\max}$. This follows from the fact that the numerator is always positive, as the roots of quadratic are complex. This shows that $ds > 0$ for all admissible values of $r > 1$.

- For

$$\frac{\gamma - 1}{\gamma + 1} < r < 1$$

  $ds < 0$ (because $ds = 0$ at $r = 1$) and hence these are not physical.

- For strong shocks ($M_{n1} \gg 1$) we have

$$r = \frac{\gamma+1}{\gamma-1}, \qquad R = \frac{2\gamma M_{n1}^2}{\gamma+1}$$

and

$$ds = C_v(2\ln M_{n1} + \ln(2\gamma) - \ln(\gamma+1) - \gamma\ln(\gamma+1) + \gamma\ln(\gamma-1)) = C_v(2\ln M_{n1} - 2.087)$$

Thus for $M_{n1} > 2.85$, $ds > 0$

- For $r > 1$, the Mach No. $M_{n1} > 1$ or $v_{x1} > c_1$. On the other side of shock

$$M_{n2}^2 = \frac{v_{x2}^2}{c_2^2} = \frac{1}{rR} M_{n1}^2$$

Using the relations

$$r = \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2}, \quad R = \frac{2\gamma M_{n1}^2 - (\gamma-1)}{\gamma+1}$$

we get

$$rR = \frac{M_{n1}^2\left(2\gamma M_{n1}^2 - (\gamma-1)\right)}{2+(\gamma-1)M_{n1}^2}$$

which gives

$$M_{n2}^2 - 1 = \frac{M_{n1}^2}{rR} - 1 = -\frac{(\gamma+1)(M_{n1}^2-1)}{2\gamma M_{n1}^2 - (\gamma-1)}$$

Thus $M_{n2} < 1$ or the flow is subsonic on the other side of the shock. Hence shock region is the transition region from supersonic to subsonic flow.

- Coming to the increase in entropy, which should be due to dissipation in the shock, which now should have a finite thickness, δ. Let us assume that viscosity is the only dissipative agent, which gives the dissipation

$$\Phi = 2\mu e_{ij}^2 - \frac{2}{3}\mu(e_{jj})^2$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

- In our case only non-zero derivative is w.r.t. x and ignoring the tangential velocity which is continuous across the shock, the only derivative that contributes is $\frac{\partial v_x}{\partial x}$ which gives

$$\Phi = \frac{4}{3}\mu e_{11}^2 = \frac{4}{3}\mu\left(\frac{\partial v_x}{\partial x}\right)^2$$

- Here $\Phi = dE/dt$ is the rate of energy dissipation. Thus $\Phi dx = (dx/dt)dE = vdE$ and the energy dissipated is given by

$$\begin{aligned}\rho T ds &= \frac{4}{3}\frac{\mu}{v_x}\int_{-\delta/2}^{\delta/2}\left(\frac{\partial v_x}{\partial x}\right)^2 dx \approx \frac{4}{3}\frac{\mu}{v_x}\delta\frac{(v_{x1}-v_{x2})^2}{\delta^2} \\ &\approx \frac{4\nu}{3\delta}\rho v_{x1}\frac{(r-1)^2}{r^2}\end{aligned}$$

- This can be used to estimate the shock thickness for a given viscosity. To get the variation inside shock we need to solve the eq of motion with viscous term. For weak shock $ds \propto (r-1)^3$ and hence $\mu/\delta \propto (r-1)$.

- For strong shocks $\rho T C_v = P/(\gamma-1)$ also varies across the shock and this expression would need to be integrated. An approximate value may be obtained by using P_2 to get

$$
\begin{aligned}
\rho T ds &= \frac{P_2}{\gamma-1}(2\ln M_{n1} - 2.087) \\
&= \frac{2}{\gamma^2-1}\rho_1 v_{x1}^2 (2\ln M_{n1} - 2.087)
\end{aligned}
$$

which can be used to estimate the thickness of shock.

- Neglecting constants of order unity we get

$$\rho v_{x1}^2 \approx \frac{\nu \rho v_{x1}}{\delta}, \quad \delta \approx \frac{\nu}{v_{x1}}$$

which is same as what we obtained earlier using a completely different argument.

- Similar result would be obtained for dissipation due to thermal conductivity, which gives dissipation

$$\delta K \left(\frac{\partial T}{\partial x}\right)^2 \approx \frac{K}{\delta}(T_2 - T_1)^2 = \frac{K}{\delta} T_1^2 \left(\frac{R}{r} - 1\right)^2$$

- In these cases the dissipation is basically due to collisions between atoms as both viscosity and thermal conductivity are due to collisions between atoms in a gas. If the shock thickness is less than the mean free path, then collisions would not happen and it is called a collisionless shock, where the dissipation is due to plasma processes like dissipation of waves.

MHD Shocks

- In the presence of magnetic field the equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \left(P + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} \mathbf{B} \cdot \nabla \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{v})$$

$$\nabla \cdot \mathbf{B} = 0$$

Now the total energy and the flux includes the magnetic energy

$$U = \frac{1}{2}\rho v^2 + \frac{P}{\gamma - 1} + \frac{B^2}{8\pi}$$

$$\mathbf{F_u} = \frac{1}{2}\rho v^2 \mathbf{v} + \frac{\gamma}{\gamma - 1} P \mathbf{v} + \frac{B^2}{4\pi} \mathbf{v} - \frac{1}{4\pi} (\mathbf{v} \cdot \mathbf{B}) \mathbf{B}$$

- Combining these eq with continuity eq etc we can get the conservation eqs.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j} (B_i v_j - B_j v_i) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho \mathbf{v}\mathbf{v} + \left(P + \frac{B^2}{8\pi} \right) I - \frac{1}{4\pi} \mathbf{B}\mathbf{B} \right) = 0$$

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F_u} = 0$$

- Considering the shock perpendicular to x-axis and magnetic field and velocity in only x and z component with variation only along x we get the eq

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_x)}{\partial x} = 0$$

$$\frac{\partial B_x}{\partial x} = 0$$

$$\frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x}(v_x B_z - v_z B_x) = 0$$

$$\frac{\partial \rho v_x}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x v_x + P + \frac{B_z^2 - B_x^2}{8\pi}\right) = 0$$

$$\frac{\partial \rho v_z}{\partial t} + \frac{\partial}{\partial x}\left(\rho v_x v_z - \frac{1}{4\pi} B_x B_z\right) = 0$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho v^2 v_x + \frac{\gamma P v_x}{\gamma - 1} + \frac{B_z^2}{4\pi} v_x - \frac{B_x B z}{4\pi} v_z\right) = 0$$

- Considering the jump across the shock we get the jump conditions

$$\rho_1 v_{x1} = \rho_2 v_{x2}$$

$$B_{x1} = B_{x2}$$

$$v_{x1} B_{z1} - v_{z1} B_{x1} = v_{x2} B_{z2} - v_{z2} B_{x2}$$

$$\rho_1 v_{x1}^2 + P_1 + \frac{B_{z1}^2 - B_{x1}^2}{8\pi} = \rho_2 v_{x2}^2 + P_2 + \frac{B_{z2}^2 - B_{x2}^2}{8\pi}$$

$$\rho_1 v_{x1} v_{z1} - \frac{B_{z1} B_{x1}}{4\pi} = \rho_2 v_{x2} v_{z2} - \frac{B_{z2} B_{x2}}{4\pi}$$

$$\frac{1}{2}\rho_1 (v_{x1}^2 + v_{z1}^2) v_{x1} + \frac{\gamma}{\gamma - 1} P_1 v_{x1} + \frac{B_{z1}^2 v_{x1} - B_{z1} B_{x1} v_{z1}}{4\pi}$$

$$= \frac{1}{2}\rho_2 (v_{x2}^2 + v_{z2}^2) v_{x2} + \frac{\gamma}{\gamma - 1} P_2 v_{x2} + \frac{B_{z2}^2 v_{x2} - B_{z2} B_{x2} v_{z2}}{4\pi}$$

- These eqs are difficult to solve, so we consider a special case where $B_z = 0$ which gives

$$
\begin{aligned}
\rho_1 v_{x1} &= \rho_2 v_{x2} \\
B_{x1} &= B_{x2} \\
\rho_1 v_{x1} v_{z1} &= \rho_2 v_{x2} v_{z2} \\
\rho_1 v_{x1}^2 + P_1 &= \rho_2 v_{x2}^2 + P_2 \\
v_{x1}\left(\frac{1}{2}\rho_1 v_{x1}^2 + \frac{\gamma}{\gamma-1}P_1\right) &= v_{x2}\left(\frac{1}{2}\rho_2 v_{x2}^2 + \frac{\gamma}{\gamma-1}P_2\right)
\end{aligned}
$$

which are the Rankine–Hugoniot jump conditions for non-magnetic case.

- Thus we consider the case where $B_x = 0$ to get

$$\rho_1 v_{x1} = \rho_2 v_{x2}$$
$$v_{x1} B_{z1} = v_{x2} B_{z2}$$
$$\rho_1 v_{x1} v_{z1} = \rho_2 v_{x2} v_{z2}$$
$$\rho_1 v_{x1}^2 + P_1 + \frac{B_{z1}^2}{8\pi} = \rho_2 v_{x2}^2 + P_2 + \frac{B_{z2}^2}{8\pi}$$
$$v_{x1}\left(\frac{1}{2}\rho_1 v_{x1}^2 + \frac{\gamma P_1}{\gamma - 1} + \frac{B_{z1}^2}{4\pi}\right) = v_{x2}\left(\frac{1}{2}\rho_2 v_{x2}^2 + \frac{\gamma P_2}{\gamma - 1} + \frac{B_{z2}^2}{4\pi}\right)$$

From the first three eqs we get

$$v_{z1} = v_{z2}, \qquad r = \frac{\rho_2}{\rho_1} = \frac{B_{z2}}{B_{z1}} = \frac{v_{x1}}{v_{x2}}, \qquad R = \frac{P_2}{P_1}$$

- Substituting these in the last two eqs we get

$$\rho_1 v_{x1}^2 + P_1 + \frac{B_{z1}^2}{8\pi} = \frac{1}{r}\rho_1 v_{x1}^2 + P_1 R + r^2 \frac{B_{z1}^2}{8\pi}$$

$$r\left(\frac{1}{2}\rho_1 v_{x1}^2 + \frac{\gamma P_1}{\gamma - 1} + \frac{B_{z1}^2}{4\pi}\right) = \frac{1}{2r}\rho_1 v_{x1}^2 + \frac{\gamma R P_1}{\gamma - 1} + r^2 \frac{B_{z1}^2}{4\pi}$$

Dividing by P_1 and noting that $\gamma M_1^2 = \rho_1 v_{x1}^2 / P_1$ and $\beta_1 = B_{z1}^2/(8\pi P_1)$ we get

$$R = 1 + \gamma M_1^2 \left(1 - \frac{1}{r}\right) + \beta_1 (1 - r^2)$$

$$R = r + \frac{\gamma - 1}{2} M_1^2 \left(r - \frac{1}{r}\right) + \frac{2(\gamma - 1)}{\gamma} \beta_1 (r - r^2)$$

- Equating the two values of R gives a quadratic in r after removing a factor of $(r-1)$

$$f(r) = 2(2-\gamma)\beta_1 r^2 + \gamma[2\beta_1 + 2 + (\gamma-1)M_1^2]r - \gamma(\gamma+1)M_1^2 = 0$$

This has one positive and one negative root for r. Only the positive root is admissible. Further $f(0) < 0$ and $f((\gamma+1)/(\gamma-1)) > 0$. Thus the root is in $(0, r_{\max})$.

- For weak shocks $r \approx 1$ we can write

$$R-1 = (r-1)\left(\frac{\gamma}{r}M_1^2 - \beta_1(r+1)\right) \approx (r-1)(\gamma M_1^2 - 2\beta_1)$$

- Considering the entropy change across the shock

$$ds \approx C_v[(R-1) - \gamma(r-1)]$$

For $ds > 0$ we need

$$\gamma M_1^2 - 2\beta_1 > \gamma, \qquad M_1^2 > 1 + \frac{2\beta_1}{\gamma} = 1 + \frac{V_{A1}^2}{c_1^2}$$

This gives

$$v_{x1}^2 > c_1^2 + V_{A1}^2$$

- In the strong shock limit $f(r) = 0$ gives

$$r \approx \frac{\gamma + 1}{\gamma - 1}, \qquad R \approx 1 + \frac{2\gamma M_1^2}{\gamma + 1}$$

which is the same as non-magnetic case.

Blast Waves

- If there is a sudden release of energy in a small volume, e.g., nuclear explosion or supernova explosions, the surrounding fluid heats up and expands. This expansion is supersonic and drives a shock wave through surrounding medium which is known as blast wave. The shock compresses the surrounding medium and propagates outwards.

- If an explosion releases kinetic energy E, which may be a small fraction of total energy released. If the mass of ejecta is m_e then

$$E = \frac{1}{2} m_e v^2, \qquad v = \sqrt{\frac{2E}{m_e}}$$

- Because of high velocity the bubble will expand rapidly, sweeping up more surrounding fluid. This is referred to as the free expansion phase as the velocity remains approximately constant.
- If the density of surrounding fluid is ρ_s and the radius of the bubble is R then the total mass involved is

  $$m(t) = m_e + \frac{4\pi}{3}\rho_s R^3$$

  Because of increase in mass the velocity would reduce, but it would still remain supersonic for some time.
- After sufficient time the swept up mass would dominate the total mass and the velocity is now given by

  $$v = \sqrt{\frac{3E}{2\pi\rho_s}}\, R^{-3/2}$$

The velocity decreases with increasing radius and this phase is known as the Sedov–Taylor phase.

- The transition between the free expansion phase and Sedov–Taylor phase happens gradually. An estimate for this can be obtained when ejecta mass equals the swept up mass. The radius, R_d at which this happens is known as the deceleration radius

$$\frac{4\pi}{3}\rho_s R_d^3 = m_e, \qquad R_d = \left(\frac{3m_e}{4\pi\rho_s}\right)^{1/3}$$

- After the Sedov–Taylor phase the bubble begins to cool and energy dissipation is significant and KE can not be considered as constant.

- Although we have said KE is conserved, it is the total energy which includes internal energy that is conserved

$$E = \frac{1}{2} m \left(\frac{dR}{dt} \right)^2 + \frac{4\pi}{3} R^3 \rho C_v T$$

Assuming that the resulting shock is strong we can write the second term as

$$\frac{m}{\rho} \frac{P_2}{\gamma - 1} = m \frac{2}{\gamma^2 - 1} v_1^2$$

Combining the two terms we get

$$E = m \left(\frac{dR}{dt} \right)^2 \left(\frac{1}{2} + \frac{2}{\gamma^2 - 1} \right) = C_\gamma m \left(\frac{dR}{dt} \right)^2$$

where constant

$$C_\gamma = \frac{\gamma^2 + 3}{2(\gamma^2 - 1)} \approx 1.625$$

- Thus we can write

$$E = C_\gamma m(t) \left(\frac{dR}{dt}\right)^2 = C_\gamma \frac{4\pi}{3} R^3 \rho_s \left(\frac{dR}{dt}\right)^2 = \mathrm{const}$$

which gives

$$R^{3/2} \frac{dR}{dt} = \left(\frac{3E}{4\pi C_\gamma \rho_s}\right)^{1/2}$$

which is similar to eq we obtained earlier.

- We can try a power law solution $R \propto t^\alpha$ which gives

$$v = \frac{dR}{dt} = \alpha \frac{R}{t} \propto t^{\alpha-1}$$

Substituting this in the above eq we get

$$\frac{R^{5/2}}{t} = \text{const}, \qquad R \propto t^{2/5}$$

Thus $\alpha = 2/5$ and

$$R(t) = \bar{C} \left(\frac{E}{\rho_s} \right)^{1/5} t^{2/5}$$

where

$$\bar{C} = \left(\frac{75}{16\pi C_\gamma} \right)^{1/5} \approx 0.98$$

- With this solution $v \propto t^{-3/5}$ and pressure $P_2 \propto v^2 \propto t^{-6/5}$. The Sedov–Taylor phase applies for $R \gg R_d$ until the radiative losses become important.