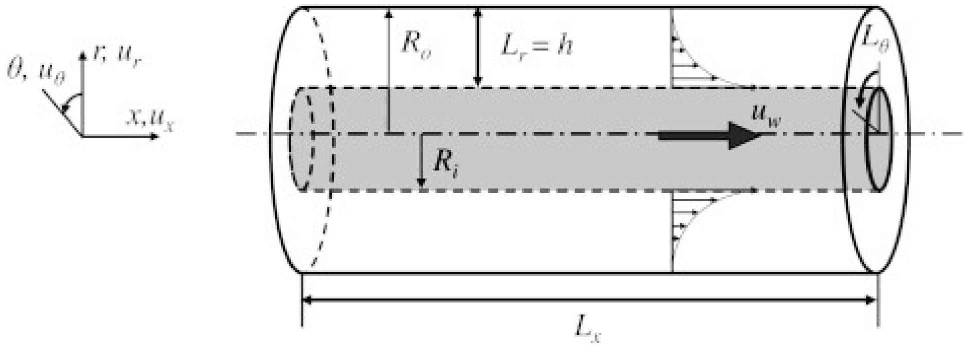
Stability of Couette Flow

- Couette flow is the flow between two coaxial cylinders rotating about their axis with different rotation rates. There is no external force.



- We use cylindrical polar coordinates (r, θ, z). In these coordinate system the unit vectors are not constant and

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \qquad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$

- The eq of motion for incompressible, inviscid fluid are

$$\frac{\partial v_r}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} v_r - \frac{v_\theta^2}{r} = -\frac{\partial}{\partial r}\left(\frac{P}{\rho}\right)$$
$$\frac{\partial v_\theta}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} v_\theta + \frac{v_r v_\theta}{r} = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{P}{\rho}\right)$$
$$\frac{\partial v_z}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} v_z = -\frac{\partial}{\partial z}\left(\frac{P}{\rho}\right)$$

where

$$\mathbf{v} \cdot \boldsymbol{\nabla} = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}$$

- The eq of continuity

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

- We look for steady axisymmetric flow, $\partial / \partial t = 0$, $v_r = 0$, $v_z = 0$, $v_\theta(r)$. This would give

$$\frac{v_\theta^2}{r} = \frac{1}{\rho} \frac{\partial P}{\partial r}$$

which gives the pressure to balance the centrifugal force. There is no restriction on the form of v_θ.

- The θ component of the eq for axisymmetric flow can be written as

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = 0$$

or

$$\frac{\partial r v_\theta}{\partial t} + v_r \frac{\partial r v_\theta}{\partial r} + v_z \frac{\partial r v_\theta}{\partial z} = 0$$

This is equivalent to

$$\frac{d r v_\theta}{dt} = \frac{d r^2 \Omega}{dt} = 0$$

which is essentially conservation of angular momentum $L = r^2 \Omega$ per unit mass.

- If viscosity is included then the θ component of the eq would give

$$(\nabla^2 \mathbf{v})_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2}$$
$$- \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} = 0$$

Considering the steady state as specified earlier we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} = 0$$

or

$$\frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\right) = 0$$

- Integrating once we get

$$\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} = \frac{1}{r}\frac{\partial r v_\theta}{\partial r} = 2A$$

which gives the general solution

$$v_\theta = \Omega r = Ar + \frac{B}{r}$$

or

$$\Omega = A + \frac{B}{r^2}$$

We assume this form even for inviscid case.

- Rayleigh has given the stability criterion for the inviscid case as
$$\frac{d}{dr}(\Omega r^2)^2 > 0$$

- Rayleigh did not do a stability analysis. His arguments are based on minimum energy. Consider two rings of matter at radius r_1 and r_2 (with $r_2 > r_1$) of equal height and radial extent to match the mass, i.e., $2\pi r_1 dr_1 = 2\pi r_2 dr_2 = dS$. The angular momentum L_1 and L_2 would remain the same after interchange because of the conservation. The change in KE due to the interchange is given by

$$\left[\left(\frac{L_2^2}{r_1^2} + \frac{L_1^2}{r_2^2}\right) - \left(\frac{L_1^2}{r_1^2} + \frac{L_2^2}{r_2^2}\right)\right] = (L_2^2 - L_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

- The second factor is positive. If $L_2^2 > L_1^2$, then the KE increases, and the interchange requires source of energy, implying stability. On the other hand if $L_2^2 < L_1^2$ energy is released by interchange, which implies instability. Thus stability requires L^2 to be monotonically increasing.

- For stability we can use the discriminant

$$\Phi(r) = \frac{1}{r^3}\frac{d}{dr}(r^2\Omega)^2 = \frac{2}{r}\Omega\frac{d}{dr}(r^2\Omega) > 0$$

  everywhere. For the general solution

$$\Phi(r) = \frac{2}{r}\left(A + \frac{B}{r^2}\right)2Ar = 4A\left(A + \frac{B}{r^2}\right)$$

- Now we consider the fluid between two cylinders with radius R_1 and R_2 ($R_2 > R_1$) and rotating at frequency Ω_1 and Ω_2. We take the general form of rotation to get

  $$\Omega_1 = A + \frac{B}{R_1^2}, \qquad \Omega_2 = A + \frac{B}{R_2^2}$$

  These can be solved for A, B, we further define

  $$\eta = \frac{R_1}{R_2}, \qquad \mu = \frac{\Omega_2}{\Omega_1}$$

  with $\eta < 1$. This gives the solution

  $$A = -\Omega_1 \left(\frac{\eta^2 - \mu}{1 - \eta^2} \right), \qquad B = \Omega_1 R_1^2 \frac{1 - \mu}{1 - \eta^2}$$

- With this solution the discriminant

$$\Phi(r) = \frac{2}{r}\left(A + \frac{B}{r^2}\right)(2Ar) = 4A\left(A + \frac{B}{r^2}\right)$$

It is clear that $\phi(r) > 0$ if $AB > 0$. Using the solution

$$AB = -\Omega_1^2 R_1^2 \frac{(1-\mu)(\eta^2-\mu)}{(1-\eta^2)^2}$$

Thus $AB > 0$ gives

$$(1-\mu)(\mu-\eta^2) > 0$$

If $\eta^2 < \mu < 1$ then it is clearly satisfied.

- For $AB < 0$ it is possible to have stability if $|A| > |B|/r^2$ for all $R_1 < r < R_2$. If this condition is satisfied at $r = R_1$ it will be true in the entire range. Now

$$\frac{AR_1^2}{B} = -\frac{\eta^2 - \mu}{1 - \mu} = -1 + \frac{1 - \eta^2}{1 - \mu}$$

  This quantity should be < -1. This would be true if $\mu > 1$. Thus the stability is realised for $\mu > \eta^2$. If $\mu < 0$ this condition is not satisfied, thus we need both cylinders to rotate in the same direction.
- The system is unstable for $\mu < \eta^2$

- We consider the stability of inviscid Couette flow defined by $v_r = v_z = 0$ and $v_\theta = r\Omega = V(r)$. Now $V(r)$ is arbitrary function. In the steady state the pressure would balance the centrifugal force. Now we consider the velocity $(v_r, V + v_\theta, v_z)$ and $P/\rho = P_0/\rho + p_1$.

- The basic eq are

$$\frac{\partial v_r}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} v_r - \frac{v_\theta^2}{r} = -\frac{\partial}{\partial r}\left(\frac{P}{\rho}\right)$$

$$\frac{\partial v_\theta}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} v_\theta + \frac{v_r v_\theta}{r} = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{P}{\rho}\right)$$

$$\frac{\partial v_z}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} v_z = -\frac{\partial}{\partial z}\left(\frac{P}{\rho}\right)$$

- The linearised eq are

$$\frac{\partial v_r}{\partial t} + \frac{V}{r}\frac{\partial v_r}{\partial \theta} - 2\frac{V}{r}v_\theta = -\frac{\partial p_1}{\partial r}$$

$$\frac{\partial v_\theta}{\partial t} + \frac{V}{r}\frac{\partial v_\theta}{\partial \theta} + \left(\frac{V}{r} + \frac{dV}{dr}\right)v_r = -\frac{1}{r}\frac{\partial p_1}{\partial \theta}$$

$$\frac{\partial v_z}{\partial t} + \frac{V}{r}\frac{\partial v_z}{\partial \theta} = -\frac{\partial p_1}{\partial z}$$

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}v_r + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

- Boundary conditions $v_r = 0$ at $r = R_1$ and $r = R_2$

- Here the coefficients are functions of r and it is not possible to get an analytic solution. Looking for solutions of the form

  $$f(r)\exp(i(pt + m\theta + kz))$$

  for all quantities we get

  $$\begin{aligned} i\sigma v_r - 2\Omega v_\theta &= -\frac{dp_1}{dr} \\ i\sigma v_\theta + \left(\Omega + \frac{d(r\Omega)}{dr}\right) v_r &= -\frac{im}{r} p_1 \\ i\sigma v_z &= -ikp_1 \\ \frac{dv_r}{dr} + \frac{v_r}{r} + \frac{im}{r} v_\theta + ikv_z &= 0 \end{aligned}$$

where

$$\sigma = p + m\Omega = \frac{1}{i}\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \theta}\right)$$

- Eliminating v_θ and v_z using the respective components of eq of motion we get

$$-\sigma^2 v_r + 2\Omega\left(2\Omega + r\frac{d\Omega}{dr}\right)v_r = -i\sigma\frac{dp_1}{dr} - \frac{2im\Omega}{r}p_1$$

$$i\sigma\frac{1}{r}\frac{d(rv_r)}{dr} - \frac{im}{r}\left(2\Omega + r\frac{d\Omega}{dr}\right)v_r = -\left(\frac{m^2}{r^2} + k^2\right)p_1$$

- First consider the axisymmetric perturbation only with $m = 0$ and $\sigma = p$ and further define $v_r = ip\xi_r$, with ξ_r as the displacement. Noting that

$$2\Omega\left(2\Omega + r\frac{d\Omega}{dr}\right) = 4\Omega^2 + 2r\Omega\frac{d\Omega}{dr} = \Phi(r)$$

we get

$$(p^2 - \Phi(r))\xi_r = \frac{dp_1}{dr}$$
$$\frac{1}{r}\frac{d(r\xi_r)}{dr} = \frac{k^2}{p^2}p_1$$

- Eliminating p_1 gives

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(r\xi_r)}{dr}\right) - k^2\xi_r = -\frac{k^2}{p^2}\Phi(r)\xi_r$$

- This is a Sturm–Liouville problem with eigenvalues k^2/p^2, which are positive if $\Phi(r) > 0$. This can be shown by multiplying the eq by $r\xi_r$ and integrating over r to get

$$\int \left[r\xi_r \frac{d}{dr}\left(\frac{1}{r}\frac{d(r\xi_r)}{dr}\right) - k^2 r\xi_r^2 \right] dr = -\frac{k^2}{p^2}\int \Phi(r) r\xi_r^2 \, dr$$

Integrating by parts the first term we get

$$\frac{p^2}{k^2} = \frac{\int \Phi(r) r\xi_r^2 \, dr}{\int \left[\frac{1}{r}\left(\frac{d(r\xi_r)}{dr}\right)^2 + k^2 r\xi_r^2 \right] \, dr}$$

Thus if $\Phi(r) > 0$, $p^2 > 0$ implying stability.

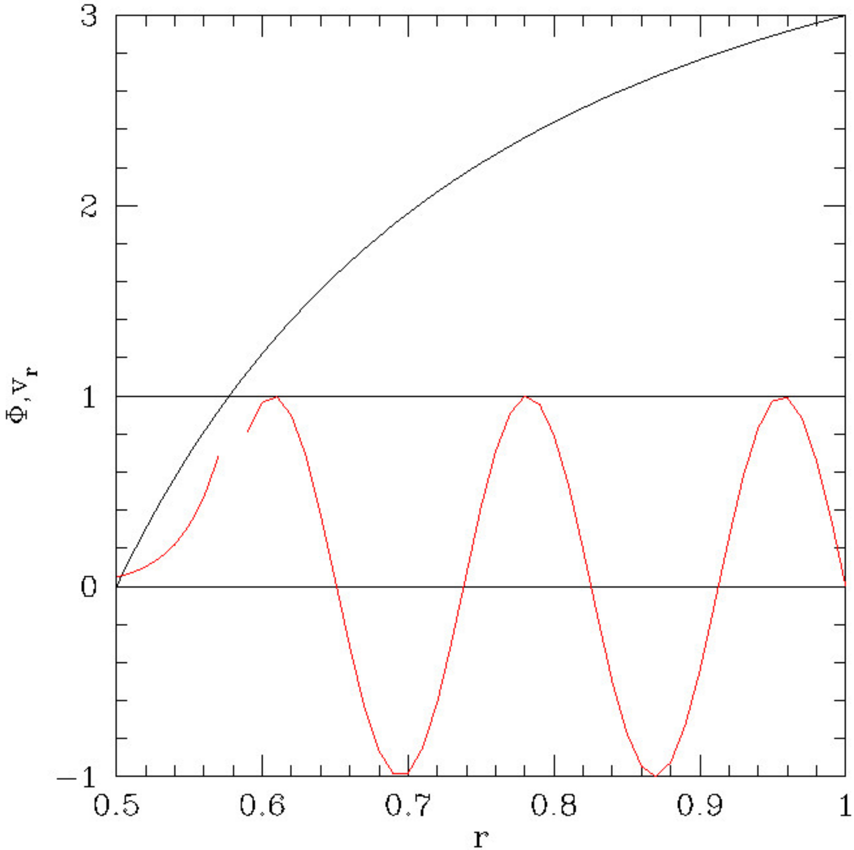
- To complete the proof we need to demonstrate instability if $\Phi(r) < 0$ anywhere in the region. For this we apply WKB type of approximation and assume that locally $\xi_r = \exp(ik_r r)$. Substituting in the eq and ignoring derivatives of r, we get

$$k_r^2 + k^2 = \frac{k^2}{p^2}\Phi(r)$$

If $\Phi(r) > 0$ then $p^2 > 0$ giving stability. Further,

$$k_r^2 = \frac{k^2}{p^2}(\Phi(r) - p^2)$$

and $k_r^2 > 0$ if $\Phi(r) > p^2$. In this region the solution is oscillatory and exponential outside this region. The fig is schematic.



3
2
1
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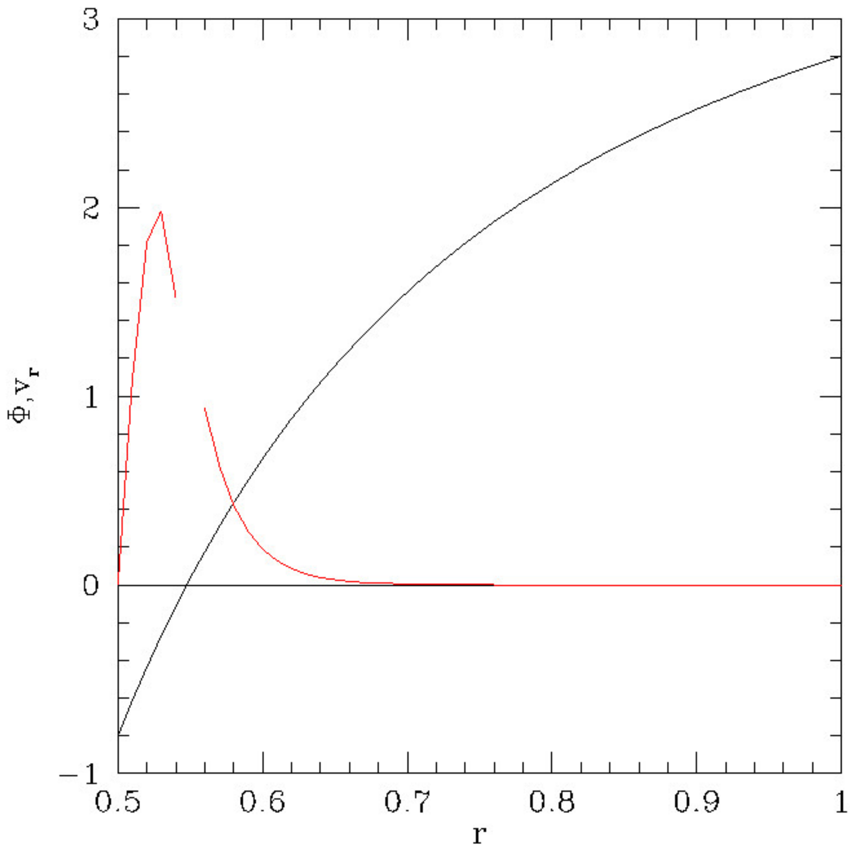
−1
0.5
0.6
0.7
0.8
0.9
1
r
Φ, V_r

- Now let us assume that $\phi(r) < 0$ in some region and in that case both positive and negative values of p^2 are allowed. To demonstrate we need to show that negative values of p^2 are possible. From the integral it means we should look for solutions which are dominating in region where $\Phi(r) < 0$ and is small elsewhere. Since $k_r^2 \propto k^2$ the magnitudes can be made sufficiently large by choosing k to be large. This enables the solution to be localised in a small region of r.

- If $p^2 < 0$, then $k_r^2 > 0$ when $\Phi < 0$ (and $|p|^2 < |\Phi(r)|$) and $k_r^2 < 0$ outside that region. Thus the solution is oscillatory when $\Phi < 0$ (and $|p|^2 < |\Phi(x)|$) and exponential when $\Phi > 0$.



3
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0.7
0.8
0.9
1
r
Φ, v_r

- Now we try to consider the non-axisymmetric case. We rewrite the equations using $v_r = i\sigma\xi_r$, and noting that $\sigma = p + m\Omega$, which is now a function of r, to get the eqs

$$(\sigma^2 - \Phi)\xi_r = \frac{dp_1}{dr} + \frac{2m\Omega}{\sigma r} p_1$$

$$\sigma^2 \frac{1}{r} \frac{d(r\xi_r)}{dr} + \sigma m \frac{d\Omega}{dr} \xi_r - \frac{m\sigma}{r} \left(2\Omega + r\frac{d\Omega}{dr} \right) \xi_r = \left(\frac{m^2}{r^2} + k^2 \right) p_1$$

$$\sigma^2 \frac{1}{r} \frac{d(r\xi_r)}{dr} - \frac{2m\sigma\Omega}{r} \xi_r = \left(\frac{m^2}{r^2} + k^2 \right) p_1$$

- Once again we neglect derivatives of r and apply WKB type of approximation with form

$$v_r \exp(ik_r r), \qquad p_1 \exp(ik_r r)$$

to get the eq

$$\left(\sigma^2 ik_r - \frac{2m\sigma\Omega}{r}\right)\xi_r - \left(\frac{m^2}{r^2} + k^2\right)p_1 = 0$$
$$(\sigma^2 - \Phi)\xi_r - \left(ik_r + \frac{2m\Omega}{\sigma r}\right)p_1 = 0$$

Further, since $\sigma = p + m\Omega$, the second term in the diagonal elements would be small as compared to k_r and hence we neglect it. For nontrivial solution the determinant should be zero giving the dispersion relation

$$\sigma^2 k_r^2 + \left(\frac{m^2}{r^2} + k^2 \right) (\sigma^2 - \Phi) = 0$$

which gives

$$\sigma^2 \left(k_r^2 + \frac{m^2}{r^2} + k^2 \right) = \Phi \left(\frac{m^2}{r^2} + k^2 \right)$$

which would give stability for $\Phi(r) > 0$ as before.

Rayleigh–Taylor Instability

- Rayleigh–Taylor Instability and the Kelvin–Helmholtz instabilities are instabilities at the interface between two fluids. Rayleigh–Taylor instability is when a denser fluid is overlying a lighter fluid. The Kelvin–Helmholtz instability is due to shear at the interface. These are the two most common instabilities in nature.

- Using energy arguments we can show that heavier fluid lying above a lighter fluid is always unstable in the inviscid case. By interchanging the fluid elements in the two layers the potential energy would be reduced and hence kinetic energy will be released. Benard convection is also an example.

- We consider only plane-parallel interface in Cartesian coordinates, with gravity in z direction and the interface is at $z = 0$. We consider fluid to be incompressible.

- To start with we take the case of an inhomogeneous fluid with density and viscosity varying with height. The eq of motion is

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = -\boldsymbol{\nabla} P - \mathbf{g}\rho + \boldsymbol{\nabla} \cdot \sigma$$

where the stress tensor

$$\sigma_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

- For the incompressible fluid we have $\nabla \cdot \mathbf{v} = 0$ This gives the ith component of stress term as

$$\mu \nabla^2 v_i + \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \frac{\partial \mu}{\partial x_j}$$

- We assume that in the steady state ρ and μ depend only on z.
- For an incompressible fluid we have

$$\nabla \cdot \mathbf{v} = 0$$

The equation of continuity gives

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$$

- In the steady state $\partial/\partial t = 0$, $\mathbf{v} = 0$ which gives

$$\frac{dP_0}{dz} = -g\rho$$

which gives the pressure P_0.

- The state is perturbed to $\mathbf{v}$, $P = P_0 + P_1$ and $\rho = \rho_0 + \rho_1$ which gives the eq in perturbed quantities

$$\rho\frac{\partial v_x}{\partial t} = -\frac{\partial P_1}{\partial x} + \mu\nabla^2 v_x + \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\frac{d\mu}{dz}$$

$$\rho\frac{\partial v_y}{\partial t} = -\frac{\partial P_1}{\partial y} + \mu\nabla^2 v_y + \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\frac{d\mu}{dz}$$

$$\rho\frac{\partial v_z}{\partial t} = -\frac{\partial P_1}{\partial z} - g\rho_1 + \mu\nabla^2 v_z + 2\frac{\partial v_z}{\partial z}\frac{d\mu}{dz}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\frac{\partial \rho_1}{\partial t} = -v_z\frac{d\rho_0}{dz}$$

- We seek solutions of the form

$$f(z)\exp(ik_x + ik_y y + nt)$$

with $k^2 = k_x^2 + k_y^2$ and $D = d/dz$ to get

$$ik_x P_1 = -n\rho v_x + \mu(D^2 - k^2)v_x + (D\mu)(ik_x v_z + Dv_x)$$
$$ik_y P_1 = -n\rho v_y + \mu(D^2 - k^2)v_y + (D\mu)(ik_y v_z + Dv_y)$$
$$DP_1 = -n\rho v_z - g\rho_1 + \mu(D^2 - k^2)v_z + 2(D\mu)Dv_z$$
$$ik_x v_x + ik_y v_y = -Dv_z$$
$$n\rho_1 = -(D\rho_0)v_z$$

multiplying the 1st eq by $-ik_x$ and 2nd by $-ik_y$ and adding. Using continuity eq we get

$$k^2 P_1 = [-n\rho + \mu(D^2 - k^2)]Dv_z + (D\mu)(D^2 + k^2)v_z$$

Eliminating ρ_1 from the 3rd eq gives

$$DP_1 = [-n\rho + \mu(D^2 - k^2)]v_z + \frac{g}{n}(D\rho)v_z + 2(D\mu)Dv_z$$

Eliminating P_1 gives the required eq in v_z

$$D\left[\left(\rho - \frac{\mu}{n}(D^2 - k^2)\right) Dv_z - \frac{1}{n}(D\mu)(D^2 + k^2)v_z\right] =$$

$$k^2\left[-\frac{g}{n^2}(D\rho)v_z + \left(\rho - \frac{\mu}{n}(D^2 - k^2)\right) v_z - \frac{2}{n}(D\mu)Dv_z\right]$$

- The boundary conditions are $v_z = 0$ and $Dv_z = 0$ on the bounding surfaces if the extent is finite in z. The 2nd condition comes from $v_x = v_y = 0$ at the boundary. In the simplest case with interface, we do not have any boundary as the fluid extends to infinity on both sides. In that case also v_z should tend to zero at large distances from the interface.

- We first consider the inviscid case $\mu = 0$ and the eq becomes

$$D(\rho D v_z) - \rho k^2 v_z = -\frac{k^2}{n^2} g (D\rho) v_z$$

with $v_z = 0$ at boundaries.

- This is again a Sturm-Liouville problem and

$$\frac{n^2}{k^2} = \frac{g \int (D\rho) v_z^2 \, dz}{\int \rho [(D v_z)^2 + k^2 v_z^2] \, dz}$$

- This is similar to the eq for Couette flow with $D\rho$ playing the role of $\Phi(r)$. In this case we have defined the time variation as $\exp(nt)$ and hence $n^2 < 0$ for stability. Thus for stability $D\rho$ should be negative everywhere, or ρ should be decreasing with height.

- On the other hand if $D\rho > 0$ anywhere in the layer, the system is unstable.

- Now we consider two fluids of constant density separated by an interface at $z = 0$. The z extent is infinite on either side of boundary. $\rho = \rho_1$ for $z < 0$ and $\rho = \rho_2$ for $z > 0$.
- In both regions the eq becomes

$$(D^2 - k^2)v_z = 0$$

which gives a general solution

$$v_z = Ae^{kz} + Be^{-kz}$$

This should vanish at both $\pm\infty$. Further the solution should be continuous at $z = 0$ giving (assume $k > 0$)

$$v_z = \begin{cases} Ae^{kz} & \text{for } z < 0 \\ Ae^{-kz} & \text{for } z > 0 \end{cases}$$

- Integrating the original eq across the interface we get

$$\int_{-\epsilon}^{\epsilon} \left(D(\rho D v_z) - \rho k^2 v_z \right) \, dz = -\frac{k^2}{n^2} g \int_{-\epsilon}^{\epsilon} (D\rho) v_z \, dz$$

which gives

$$\Delta_i(\rho D v_z) = -\frac{k^2}{n^2} g \Delta_i(\rho) v_z$$

where Δ_i denotes the jump across the interface. This gives

$$-k(\rho_2 + \rho_1) = -\frac{k^2}{n^2} g(\rho_2 - \rho_1)$$

- The dispersion relation becomes

$$n^2 = gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)$$

  For $\rho_2 > \rho_1$, $n^2 > 0$ and the system is unstable. Further, the growth rate n is proportional to $k^{1/2}$.

- Here we have neglected the effect of surface tension at the interface. If that is included the modes with large k or small length scale are damped and the growth rate is maximum at some k, but instability remains.

- For $\rho_2 < \rho_1$, $n^2 < 0$ and the solution is oscillatory, which represents a surface gravity wave propagating along the interface.

- If $\rho_2 = 0$, i.e. there is only one fluid layer, then

$$\omega^2 = -n^2 = gk$$

which is the dispersion relation for surface gravity waves.

Rayleigh–Taylor Instability with surface tension

- We will now consider the effect of surface tension. This provides a difference in pressure at the two sides of the interface. The pressure difference is given by

$$\Delta P = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

  where T is the surface tension and R_1, R_2 are the radii of curvature along the principal axes parallel to the surface. The sign convention is that $R_1 > 0$ if the centre is on the positive side of z.

- For a curve $y = f(x)$, the curvature is given by

$$\frac{1}{R} = \frac{d\phi}{ds} = \frac{f''}{(1+f'^2)^{3/2}} \qquad (\tan\phi = f')$$

- If we assume that the displacement ξ_z at the interface $z = 0$ is small then the force per unit area is given by

$$P_T = T\left(\frac{\partial^2 \xi_z}{\partial x^2} + \frac{\partial^2 \xi_z}{\partial y^2}\right) = -k^2 T \xi_z$$

  Noting that $d\xi_z/dt = v_z$ or $\xi_z = v_z/n$ gives

$$P_T = -\frac{k^2}{n} T v_z$$

  This is the additional force (per unit area) that is acting at the interface in addition to the pressure gradient.

- Consider the inviscid case where the eqs become

$$DP_1 = -n\rho v_z + \frac{g}{n}(D\rho_0)v_z - \frac{k^2}{n}Tv_z\delta(z)$$
$$P_1 = -\frac{1}{k^2}n\rho Dv_z$$

where $\delta(z)$ is the Dirac delta function. Eliminating P_1 we get

$$D(\rho Dv_z) = k^2\left(\rho v_z - \frac{g}{n^2}(D\rho)\right)v_z + \frac{k^4}{n^2}Tv_z\delta(z)$$

- Integrating over the interface we get

$$\Delta_i(\rho Dv_z) = -\frac{gk^2}{n^2}(\rho_2 - \rho_1)v_z + \frac{k^4}{n^2}Tv_z$$

Using the solution

$$v_z = \begin{cases} e^{kz} & \text{for } z < 0 \\ e^{-kz} & \text{for } z > 0 \end{cases}$$

we get

$$-k(\rho_2 + \rho_1) = -\frac{k^2}{n^2} g(\rho_2 - \rho_1) + \frac{k^4}{n^2} T$$

which gives the dispersion relation

$$n^2 = gk \left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} - \frac{k^2 T}{g(\rho_2 + \rho_1)} \right)$$

which gives instability when

$$k^2 < \frac{g(\rho_2 - \rho_1)}{T} = k_c^2$$

- Further the growth rate would be maximum where $dn^2/dk = 0$ which gives

$$\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} - \frac{3k^2 T}{g(\rho_2 + \rho_1)} = 0$$

which gives

$$k^2 = \frac{(\rho_2 - \rho_1)g}{3T} = \frac{1}{3}k_c^2$$

and the maximum growth rate is given by

$$n_{\max}^2 = g\frac{k_c}{\sqrt{3}}\frac{2}{3}\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} = \frac{2g^{3/2}(\rho_2 - \rho_1)^{3/2}}{T^{1/2}3^{3/2}(\rho_2 + \rho_1)}$$

- For $\rho_2 < \rho_1$ the system is stable and we get surface gravity modes with frequency given by

$$\omega^2 = gk\left(\frac{\rho_1 - \rho_2}{\rho_2 + \rho_1} + \frac{k^2 T}{g(\rho_2 + \rho_1)}\right)$$

- For a single fluid layer, $\rho_2 = 0$ we get

$$\omega^2 = gk + \frac{k^3 T}{\rho_1}$$

For small values of k this is the usual dispersion relation for the surface gravity waves, while at large k surface tension modifies the frequency.