# Continuous Distributions

1. Uniform Distn
2. Exponential Distn
3. Gamma ,,
4. Beta ,,
5. Normal ,,

## 1 Uniform Distn

For $x \sim U(a,b)$ or $R(a,b)$

$$f_X(x) = \frac{1}{b-a} I_{a<x<b}$$

$$F_X(x) = \begin{cases} 0 & , x \le a \\ \frac{x-a}{b-a} & , a < x < b \\ 1 & , x \ge b \end{cases}$$

# $\mu_1' = E(x) = \frac{a+b}{2}$.

# $var(x) = E(x^2) - E^2(x) = \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2$

$$= \frac{(b-a)^2}{12}$$

# $E(x^r) = \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}$

# Let $x$ be a continuous r.v. with PDF $f(x)$, then $Y = F(x) = \int_{-\infty}^{x} f(t)\,dt$ has uniform distn over $[0,1]$

# $MGF = M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$

# $X \sim U(0,n)$, $n \in \mathbb{N}$. Let $Y = X - [X]$. Then $Y \sim U(0,$

## (2) Exponential Distn

For $X \sim \text{Exp}$ with mean $\theta$,

$$f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} I_{0<x<\infty}$$

$$E(X) = \theta.$$

$$E(X^r) = \int_0^\infty r x^{r-1} (1-F(x))\,dx = \underline{|r}\, \theta^r$$

$$\text{var}(X) = \theta^2.$$

# Lack of Memory Property :-

$X \sim \text{Exp}$ then $P[X > a+b \mid X > a] = P[X > b]$

for $a>0$, $b>0$.

# Only Exponential Distn possesses Lack of Memory Property.

# If $X$ is a rv $\ni$ $P[X \geq 0] = 1$ and $X$ is absolutely continuous with df $F(x) \ni$

$$\frac{F'(x)}{1-F(x)} = \text{a constant}, \quad \forall\ x \geq 0$$

distn :

then $X$ must follow a exp

[Iff Condition]

# Let $X_i \sim$ Exponential Distn with mean $\theta$, $i = 1(1)n$ independently. Then

$$Y = \min_{i=1(1)n} \{X_i\} \sim \text{Exp with mean } \frac{\theta}{n}.$$

# If $X \sim Exp$ with mean $\frac{1}{\theta}$, then median$(x) = \frac{\ln 2}{\theta}$.

and $E[|X-a|] \geq \frac{\ln 2}{\theta}$ $[\because MD_{median} \leq MD_{x}]$

## Shifted Exponential Distn

$$f_X(x) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} I_{x>\mu}$$

# $F_X(x) = \begin{cases} 0 & \text{if } x \leq \mu \\ 1 - e^{-\frac{x-\mu}{\sigma}} & \text{if } x > \mu \end{cases}$

# The $p$th quantile $\xi_p$ is,
$\xi_p = \mu - \sigma \ln(1-p)$

# $E(X-\mu)^r = \sigma^r \lfloor r = \sigma^r \Gamma(r+1)$

$E(X) = \mu + \sigma$

$V(X) = \sigma^2$.

# Doesn't possess lack of memory prop$^n$

## Double Exponential Distn / Laplace Distn

$X \sim D.E. (\mu, \sigma)$

$\Rightarrow f_X(x) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}} I_{x \in \mathbb{R}}$

where $\mu \in \mathbb{R}$, $\sigma > 0$

$E(X) = \mu$ and $\mu_{2r-1} = 0$.

$$\mu_{2r} = \sigma^{2r} \lfloor 2r$$

# $F_X(x) = \begin{cases} \frac{1}{2} e^{+\frac{x-\mu}{\sigma}} & \text{if } x \le \mu \\ 1 - \frac{1}{2} e^{-\frac{x-\mu}{\sigma}} & \text{if } x > \mu \end{cases}$

# For $z \sim DE(0, 1)$, $f_Z(z) = \frac{1}{2} e^{-|z|}$, $z \in \mathbb{R}$

$$M_Z(t) = \frac{1}{1-t^2}, \quad |t| < 1$$

# For $X \sim DE(\mu, \sigma)$

$$M_X(t) = e^{t\mu} \frac{1}{1 - t^2\sigma^2}, \quad |t| < \frac{1}{\sigma}$$

# ③ Beta Distribution

$$f_X(x) = \frac{x^{m-1}(1-x)^{n-1}}{B(m,n)} \, I_{0<x<1} \quad , \quad \begin{matrix} m>0 \\ n>0 \end{matrix}$$

# Let G be the Geometric Mean of $B_1(m,n)$. Then,

$$\ln_e G = \frac{\partial}{\partial m} \ln \frac{\Gamma(m)}{\Gamma(m+n)}$$

\# $E(X) = \dfrac{a}{a+b}$ $\quad X \sim B_1(a,b)$

$V(X) = \dfrac{ab}{(a+b)^2(a+b+1)}$

$V(X) \leq \dfrac{1}{4}$

$\text{mode} = \dfrac{a-1}{a+b-2}$

\# MGF of ~~$B_1(a,b)$~~ $B_1(a,b)$ exists.

\# If $X \sim B_1(a,b)$ then

$$E\left[\left\{b - \frac{(a-1)(1-X)}{X}\right\} g(X)\right] = E\left[(1-X)g'(X)\right]$$

Suppose $X \sim B_2(m, n)$

$$f_X(x) = \frac{x^{m-1}}{(1+x)^{m+n}} \frac{1}{B(m,n)} \quad I_{x>0}$$

Then $\frac{X}{1+X} \sim B_1(m, n)$

$$E(X) = \frac{m}{n-1}$$

$$V(X) = \frac{m(n+m-1)}{(n-1)^2(n-2)}$$

MGF of $X$ doesn't exist.

# ① Gamma Distribution

$$f_X(x) = \frac{\theta^n}{\Gamma n} e^{-\theta x} x^{n-1} \quad I_{x>0}$$

$$\theta > 0$$
$$n > 0$$

Suppose $x \sim \text{Gamma}(n, \theta)$

$$E(x) = \frac{n}{\theta}$$

$$V(x) = \frac{n}{\theta^2}$$

$$M_x(t) = \left(\frac{\theta}{\theta - t}\right)^n$$

$$HM = \frac{n-1}{\theta}, \text{ if } n > 1$$

$$\text{mode}(x) = \frac{n-1}{\theta}$$

\# If $x \sim P(\lambda)$ then

$$P[X \geq k+1] = \Gamma_\lambda(k+1)$$

where $$\Gamma_\lambda(k+1) = \frac{\int_0^\lambda e^{-t} t^k \, dt}{\Gamma(k+1)}$$

\# If $x_1 \sim \text{Gamma}(n_1, \theta)$, $x_2 \sim \text{Gamma}(n$

then $x_1 + x_2 \sim \text{Gamma}(n_1 + n_2, \theta)$

# ⑤ Normal Distribution

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} I_{x \in \mathbb{R}}$$

mean = median = mode = $\mu$

$\mu_{2r-1} = 0$

$\mu_{2r} = \sigma^{2r}(2r-1)(2r-3)\cdots 5.3.1$

$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$E|X-\mu| = \sqrt{\frac{2}{\pi}}\,\sigma$

$$\frac{MD_X(\mu)}{SD_X} = \sqrt{\frac{2}{\pi}} \quad \text{(Geary's Ratio)}$$

# For $x > 0$

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) \le 1 - \Phi(x) \le \frac{\phi(x)}{x}$$

# $$\lim_{x \to \infty} \frac{x\{1 - \Phi(x)\}}{\phi(x)} = 1$$

# If $X \sim N(\mu, 1)$ then,

$$E\left\{\frac{1-\Phi(X)}{\phi(X)}\right\} = \frac{1}{\mu}.$$

# $X \sim N(0,1) \Rightarrow E[X] = -\frac{1}{2}$

# If $X \sim (\mu, \sigma^2)$, $\mu \neq 0$ then $E\,\Phi(X) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$

# $\int_{-\infty}^{\infty} \int x\,\phi(x)\,dx = -\phi(x)$

# Chi-Square

$X \sim \chi^2_n \Leftrightarrow X \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$

$$f_X(x) = \frac{e^{-x/2}\, x^{n/2 - 1}}{\Gamma\frac{n}{2}\; 2^{n/2}} \quad I_{x>0}$$

$n>0$

$E(\chi^2_n) = n$

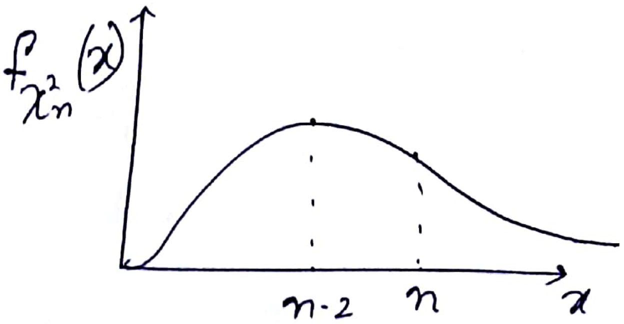
$V(\chi^2_n) = 2n$

$M_{\chi^2_n}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{\frac{n}{2}} = \left(\frac{1}{1-2t}\right)^{n/2}, \; t<\frac{1}{2}$

$\text{mode}(\chi^2_n) = n-2 \quad \text{if } n>2$

# Skewness

$$Sk = \frac{\text{mean} - \text{mode}}{\text{s.d.}} = \sqrt{\frac{2}{n}} > 0$$




# If $X \sim \chi^2_2$ and $p = P[X > x_0]$ then,

$x_0 = -2\ln p$

# $P[\chi^2_n > n] < \frac{1}{2}$ $\quad$ [$\because$ mean > median]

# $x_i \overset{iid}{\sim} N(0,1) \Rightarrow \sum_1^n x_i^2 \sim \chi^2_n$.

# $X \sim \chi^2_{n_1}$ and $Y \sim \chi^2_{n_2}$ independently

$\Rightarrow X+Y \sim \chi^2_{n_1+n_2}$

$\Rightarrow \frac{X}{X+Y} \sim B_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$

$\Rightarrow \frac{X}{Y} \sim B_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$

# T Distribution

$T \sim t_n \Rightarrow f_T(t) = \frac{1}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)} \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \; I_{t \in \mathbb{R}}$

$t_1 \equiv C(0,1)$

\# $T \sim t_n \Rightarrow \mu_r'$, $r \geq n$ doesn't exist.

\# T is a symmetric distn.

$\mu_{2r-1} = 0$

\# $E(T) = 0$, $V(T) = \frac{n}{n-2}$ if $n > 2$

\# $X, Y \overset{iid}{\sim} N(0,1) \Rightarrow \frac{X}{|Y|} \sim C(0,1)$.

o Let X and Y be independently distributed rvs where $X \sim N(0,1)$ and $Y \sim \chi^2_n$

Define $t = \frac{X}{\sqrt{\frac{Y}{n}}}$

Then $t \sim t_n$

o Suppose $X \sim \chi^2_m$, $Y \sim \chi^2_n$ } independently

Then $F = \frac{X/m}{Y/n} \sim F_{m,n}$

# F - distribution

$F \sim F_{n_1, n_2}$

$$\Rightarrow f_F(f) = \frac{\left(\frac{n_1}{n_2} f\right)^{\frac{n_1}{2}-1} \left(\frac{n_1}{n_2}\right)}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2} f\right)^{\frac{n_1+n_2}{2}}} I_{f>0}$$

# $\frac{n_1}{n_2} F = \frac{\chi^2_{n_1}}{\chi^2_{n_2}} \sim B_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$

# $E(F) = \frac{n_2}{n_2 - 2}$ if $n_2 > 2$

$$V(F) = \frac{n_2^2(2n_1 + 2n_2 - 4)}{n_1(n_2-2)^2(n_2-4)} \quad \text{if } n_2 > 4.$$

$$\text{mode}(F) = \frac{n_2(n_1-2)}{n_1(n_2+2)} \quad \text{if } n_1 > 2$$

$$t_n^2 \overset{D}{=} F_{1,n}$$

# $F \sim F_{n_1, n_2} \Rightarrow \frac{1}{F} \sim F_{n_2, n_1}$

# If $\xi_p$ is the $p$th quantile of $F_{n_1, n_2}$
and $\xi_p'$ " " $p$ th " " $F_{n_2, n_1}$
then $\xi_p' = \frac{1}{\xi_{1-p}}$

# Bivariate Normal Distribution

$(X,Y) \sim BN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}}$$

$I(x,y) \in \mathbb{R}^2$.
$|\rho| < 1$

$\Rightarrow X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

$$Y \mid X = x \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \sigma_2^2(1-\rho^2)\right)$$

\# $(X,Y) \sim BN$ then $X$ and $Y$ are independently distributed iff $\rho(X,Y) = 0$.

\# $$M_{X,Y}(t_1, t_2) = E\left(e^{t_1 X + t_2 Y}\right)$$
$$= e^{t_1\mu_1 + t_2\mu_2 + \frac{1}{2}\left\{t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 t_1 t_2\right\}}$$