Important Formulas in Estimation of $\psi(\theta)$

① If an UMVBUE T exists, then its given by

$$T = \psi(\theta) \pm \frac{\psi'(\theta)}{I(\theta)} \frac{\partial}{\partial\theta} \ln f(x;\theta)$$

[In case the family of distn satisfies all the regularity conditions.]

Here $I(\theta)$ is Fisher's Information.

② Equality holds in RCLB when

$$\frac{\partial}{\partial\theta} \ln f(x,\theta) = k(\theta)\{T(x) - g(\theta)\}$$

③ If $x_1, x_2, \ldots, x_n$ are iid having common pdf $f(x,\theta)$ then

$$V_\theta(T) \geq \frac{(g'(\theta))^2}{n E\left\{\frac{\partial}{\partial\theta} \ln f(x,\theta)\right\}^2} = \frac{(g'(\theta))^2}{-n E\left(\frac{\partial^2 \ln f(x,\theta)}{\partial\theta^2}\right)}$$

④ $I(\theta) = E\left\{\frac{\partial}{\partial\theta} \ln f(x,\theta)\right\}^2$

⑤ An unbiased estimator T having finite variance is said to be UMVUE of $g(\theta)$ iff $\mathrm{cov}_\theta(T, e) = 0$ where $E_\theta(e) = 0$ and $V_\theta(e^2) < \infty$.

Least Square Estimate is same as Maximum Likelihood Estimate under a Gaussian Model.

Suppose a random sample $x_1, x_2, \ldots x_n$ is drawn from –

① $U(0, \theta)$ then $x_{(n)}$ is complete and sufficient

② $U(-\theta, \theta)$ " $\max |x_i|$ " " " "

③ $e^{-(x-\theta)} I_{x>\theta}$ " $x_{(1)}$ " " " "

④ Discrete $U(1, N)$ " $x_{(n)}$ " " " "

and $\dfrac{x_{(n)}^{\,n+1} - (x_{(n)}-1)^{n+1}}{x_{(n)}^{\,n} - (x_{(n)}-1)^{n}}$ is UMVUE for

⑤ $U(\theta_1, \theta_2)$ then $x_{(1)}$ and $x_{(n)}$ is Complete and sufficient

UMVUE $\left(\dfrac{\theta_1+\theta_2}{2}\right) = \dfrac{x_{(1)}+x_{(n)}}{2}$ and UMVUE $\left(\dfrac{\theta_2-\theta_1}{2}\right) = \dfrac{n+1}{n-1}\,\dfrac{x_{(n)}-x_{(1)}}{2}$

① Let $f(x|\theta)$ be the pmf / pdf of a sample X. Suppose there exists a funcⁿ $T(x)$ ∋ for every two sample points x and y, the ratio $f(x|\theta)/f(y|\theta)$ is const. as a funcⁿ of θ iff $T(x) = T(y)$. Then $T(x)$ is a minimal sufficient statistic for θ.

Maximum Likelihood Estimators :-

Suppose $x_1, x_2, \dots, x_n$ be a r.s. of size n from

(i) $Bin(1, p)$; $p \in (0,1)$ then $mle(p) = \bar{x}$

$mle(p)$ doesn't exist when $\sum x_i = 0$ or n

(ii) $P(\lambda)$; $\lambda > 0$ the $mle(\lambda) = \bar{x}$.

(iii) $N(\mu, \sigma^2)$; $\mu \in \mathbb{R}, \sigma > 0$ then $mle(\mu) = \bar{x}$ and

$mle(\sigma^2) = \frac{1}{n} \sum_1^n (x_i - \bar{x})^2$.

(iv) $DE(\mu, \sigma)$; $\mu \in \mathbb{R}, \sigma > 0$ then $mle(\mu) = \tilde{x}$ = sample median

and $mle(\sigma) = \frac{1}{n} \sum_1^n |x_i - \tilde{x}|$

(v) $\frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} I_{x > \mu}$; $\mu \in \mathbb{R}, \sigma > 0$ then

$mle(\mu) = x_{(1)}$, $mle(\sigma) = \bar{x} - x_{(1)}$.

For $\sigma = \mu$, $mle(\mu) = x_{(1)}$.

If MLE exists it will be a function of the non-trivial sufficient statistic (if exists).

Under the regularity conditions in CR inequality, if MVBUE T of θ exists, then T is the MLE of θ.

Suppose $x_1, x_2, \ldots, x_n$ be a r.s. of size n from the pdf

(i) $e^{-(x-\theta)} I_{x>\theta}$ then

$$P\left[x_{(1)} + \frac{1}{n}\ln\alpha \le \theta \le x_{(1)}\right] = 1-\alpha$$

(ii) $\frac{1}{2} e^{-|x-\theta|} I_{x\in\mathbb{R}}$ then

$$P\left[x_{(1)} \le \theta \le x_{(n)}\right] = 1 - \frac{1}{2^{n-1}}$$

(iii) $\frac{1}{\beta} e^{-\frac{x}{\beta}} I_{x>0}$ then

$$P\left[\frac{2\Sigma x_i}{\chi^2_{\frac{\alpha}{2};2n}} \le \beta \le \frac{2\Sigma x_i}{\chi^2_{1-\frac{\alpha}{2};2n}}\right] = 1-\alpha$$

and $$P\left[\frac{n\, x_{(1)}}{-\ln\alpha} \le \beta \le \infty\right] = 1-\alpha$$