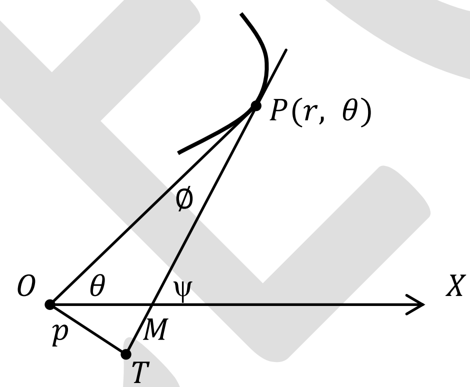
Module-1: Differential Calculus - 1
Polar curves, angle between the radius vector and the tangent, angle between two curves. Pedal equations. Curvature and Radius of curvature - Cartesian, Parametric, Polar and Pedal forms. Problems. **Self-study:** Center and circle of curvature, evolutes and involutes. **(RBT Levels: L1, L2 and L3)**

Note:

- $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
- $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

- $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.
- $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$, $\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$.

Polar curves:



O is the pole, OX is the initial line, OP the radius vector , PT is the tangent to the curve at P .

And $OT = p$.

In ΔOPM , $\psi = \theta + \phi$.

1. Prove that $\tan\boldsymbol{\phi} = \boldsymbol{r}\frac{\boldsymbol{d\theta}}{\boldsymbol{dr}}$.

Proof: Since $x = r\cos\theta$, $y = r\sin\theta$

Slope of the tangent $= \tan\psi = \frac{dy}{dx} = \frac{dy/dr}{dx/dr}$

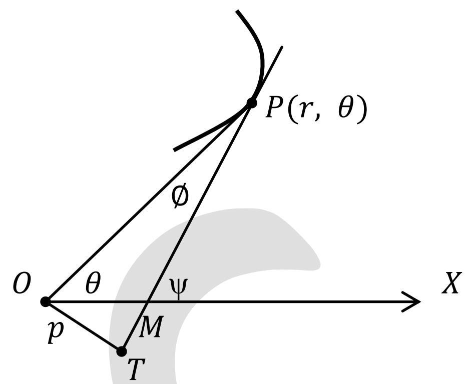
$$= \frac{\sin\theta + r\cos\theta\frac{d\theta}{dr}}{\cos\theta - r\sin\theta\frac{d\theta}{dr}} = \frac{\tan\theta + \boldsymbol{r}\frac{\boldsymbol{d\theta}}{\boldsymbol{dr}}}{1-\tan\theta\ \boldsymbol{r}\frac{\boldsymbol{d\theta}}{\boldsymbol{dr}}} \qquad \cdots\cdots\cdots\cdots (1)$$

But $\psi = \theta + \phi \implies \tan\psi = \tan(\theta+\phi) = \frac{\tan\theta + \tan\emptyset}{1-\tan\theta\ \tan\emptyset} \qquad \cdots\cdots\cdots\cdots (2)$

By (1) and (2) $\boldsymbol{\tan\phi = r\frac{d\theta}{dr}}$.

2. Prove that $\boldsymbol{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$.

Proof:



In ΔOPT $\frac{OT}{OP} = \sin\phi \implies \frac{p}{r} = \sin\phi$ *or* $p = r\sin\phi$.

$$p = r\sin\phi \implies \frac{1}{p^2} = \frac{1}{r^2}\ \operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}\ (1+\cot^2\phi) \qquad (\tan\phi = r\frac{d\theta}{dr} \implies \cot\phi = \frac{1}{r}\frac{dr}{d\theta})$$

$$= \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] = \boldsymbol{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}.$$

Note: 1. Angle between the two polar curves is $|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2|$

Find $\tan\phi_1 = \frac{r}{r_1}$ for the first curve and $\tan\phi_2 = \frac{r}{r_1}$ for the second curve

And if $\tan\phi_1 . \tan\phi_2 = -1$. Then angle of intersection is $\frac{\pi}{2}$.

2. Equation involving only $\boldsymbol{p}$ *and* $\boldsymbol{r}$ is called **pedal equation.**

To find the pedal equation, find $\boldsymbol{\frac{r_1}{r}}$ and use it in $\boldsymbol{\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{r_1}{r}\right)^2\right]}$ and then eliminate θ .

Problems:

1. Find the angle between the following two curves.

a) $r = a(1-\sin\theta)$, $r = b(1+\sin\theta)$

$r = a(1-\sin\theta)$	$r = b(1+\sin\theta)$
Diff. w.r.to θ we get, $r_1 = a(-\cos\theta)$	Diff. w.r.to θ we get, $r_1 = b(\cos\theta)$
$\therefore \tan\phi_1 = \frac{r}{r_1} = -\frac{(1-\sin\theta)}{\cos\theta}$	$\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{(1+\sin\theta)}{\cos\theta}$

$$\Rightarrow \tan\phi_1 . \tan\phi_2 = -\frac{(1-\sin^2\theta)}{\cos^2\theta} = -1$$

Hence angle between them is $\frac{\pi}{2}$.

b) $r^n = a^n \cos n\theta$, $r^n = b^n \sin n\theta$.

$r^n = a^n \cos n\theta$

Diff. w.r.to θ we get,

$nr^{n-1} r_1 = -na^n \sin n\theta$

Or $r^n \frac{r_1}{r} = -a^n \sin n\theta$

$\therefore \tan\phi_1 = \frac{r}{r_1} = -\cot n\theta$

$r^n = b^n \sin n\theta$

Diff. w.r.to θ we get,

$nr^{n-1} r_1 = nb^n \cos n\theta$

Or $r^n \frac{r_1}{r} = b^n \cos n\theta$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \tan n\theta$

$$\Rightarrow \tan\phi_1 . \tan\phi_2 = -\cot n\theta \tan n\theta = -1$$

Hence the angle of intersection is $\frac{\pi}{2}$.

c) $r = \frac{2a}{(1-\cos\theta)}$, $r = \frac{2b}{(1+\cos\theta)}$

$r(1-\cos\theta) = 2a$

Diff. w.r.to θ we get,

$r_1(1-\cos\theta) + r\sin\theta = 0$

Or $r\sin\theta = -r_1(1-\cos\theta)$

$\therefore \tan\phi_1 = \frac{r}{r_1} = -\frac{(1-\cos\theta)}{\sin\theta}$

$r(1+\cos\theta) = 2b$

Diff. w.r.to θ we get,

$r_1(1+\cos\theta) - r\sin\theta = 0$

Or $r\sin\theta = r_1(1+\cos\theta)$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{(1+\cos\theta)}{\sin\theta}$

$\Rightarrow \tan\phi_1 . \tan\phi_2 = -\frac{(1-\cos^2\theta)}{\sin^2\theta} = -1.$ Hence the angle of intersection is $\frac{\pi}{2}$.

d) $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$

$r = \sin\theta + \cos\theta$

Diff. w.r.to θ we get,

$r_1 = \cos\theta - \sin\theta$

$\therefore \tan\phi_1 = \frac{r}{r_1} = \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta} = \frac{1+\tan\theta}{1-\tan\theta}$

$= \tan\left(\theta + \frac{\pi}{4}\right)$

$\Rightarrow \quad \phi_1 = \theta + \frac{\pi}{4}$

$r = 2\sin\theta$

Diff. w.r.to θ we get,

$r_1 = 2\cos\theta$

$\therefore \tan\phi_2 = \frac{r}{r_1} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$\Rightarrow \quad \phi_2 = \theta$

Hence the angle of intersection $= |\phi_1 - \phi_2| = \frac{\pi}{4}$.

2 . Find the pedal equation of the following curves.

a) $r = a(1-\sin\theta)$

Diff. w.r.to θ we get, $r_1 = a(-\cos\theta)$

$\therefore \ \frac{\boldsymbol{r_1}}{\boldsymbol{r}} = -\frac{\cos\theta}{(1-\sin\theta)}$

$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{\cos\theta}{1-\sin\theta}\right)^2\right] = \frac{1}{r^2}\left[\frac{2(1-\sin\theta)}{(1-\sin\theta)^2}\right] = \frac{\boldsymbol{2a}}{r^3}$

Hence Pedal equation is $\boxed{r^3 = 2ap^2}$.

b) $r^n = a^n \cos n\theta$

Diff. w.r.to θ we get, $nr^{n-1}r_1 = -na^n \sin n\theta$

Or $r^n \frac{\boldsymbol{r_1}}{\boldsymbol{r}} = -a^n \sin n\theta$

$\therefore \frac{r_1}{r} = -\tan n\theta \quad \Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \tan^2 n\theta\right] = \frac{1}{r^2\cos^2 n\theta}$

$\Rightarrow p = r\cos n\theta \quad \Rightarrow$ Pedal equation is $\boxed{pa^n = r^{n+1}}$

c) $r(1-\cos\theta) = 2a$

Diff. w.r.to θ we get, $r_1(1-\cos\theta) + r\sin\theta = 0$

Or $r\sin\theta = -r_1(1-\cos\theta) \quad \Rightarrow \frac{r_1}{r} = -\frac{\sin\theta}{(1-\cos\theta)}$

$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{\sin\theta}{1-\cos\theta}\right)^2\right] = \frac{1}{r^2}\left[\frac{2(1-\cos\theta)}{(1-\cos\theta)^2}\right] = \frac{\boldsymbol{1}}{ar}$

Hence Pedal equation is $\boxed{p^2 = ar}$.

d) $r = m\theta$

Diff. w.r.to θ we get, $r_1 = m \quad \Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{m^2}{r^2}\right]$

Hence Pedal equation is $\boxed{r^4 = [r^2 + m^2]p^2}$

Derivative of arc length: $\frac{ds}{dx} = \sqrt{1 + y_1^2} = \sec\psi$, $\frac{ds}{dy} = \sqrt{1 + \frac{1}{y_1^2}} = \operatorname{cosec}\psi.$

Parametric: $\frac{ds}{dt} = \sqrt{\dot{\boldsymbol{x}}^2 + \dot{\boldsymbol{y}}^2}$ where $\dot{\boldsymbol{x}} = \frac{dx}{dt}$, $\dot{\boldsymbol{y}} = \frac{dy}{dt}$

Polar form: $\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2} = r\operatorname{cosec}\emptyset$, $\frac{ds}{dr} = \sqrt{\left(\frac{r}{r_1}\right)^2 + 1} = \sec\emptyset$,

Therefore $\sin\emptyset = r\frac{d\theta}{ds}$ and $\cos\emptyset = \frac{dr}{ds}$.

Curvature $\boldsymbol{K} = \frac{d\psi}{ds}$, **Radius of curvature** $\boldsymbol{\rho} = \frac{ds}{d\psi}$.

Radius of curvature in Cartesian form: $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$

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Proof: We know that $\tan\psi = y_1$ or $\psi = \tan^{-1} y_1$.

Differentiating both sides w.r.t. x, $\frac{d\psi}{dx} = \frac{1}{1+y_1^2} \cdot \frac{dy_1}{dx} = \frac{y_2}{1+y_1^2}$.

$$\rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{d\psi} = \sqrt{1+y_1^2} \cdot \frac{1+y_1^2}{y_2} = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} .$$

Radius of curvature in Parametric form:

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} \qquad \text{Where } \dot{x} = \frac{dx}{dt}, \ \dot{y} = \frac{dy}{dt}, \ddot{x} = \frac{d^2x}{dt^2}, \ddot{y} = \frac{d^2y}{dt^2} .$$

Radius of curvature in Polar form: $\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$.

Proof: We know that $\tan\phi = \frac{r}{r_1}$, diff. w. r. t. θ we get $\sec^2\emptyset \frac{d\phi}{d\theta} = \frac{r_1^2-rr_2}{r_1^2}$

$$\frac{d\phi}{d\theta} = \frac{r_1^2-rr_2}{r_1^2\left[\left(\frac{r}{r_1}\right)^2+1\right]} = \frac{r_1^2-rr_2}{r^2+r_1^2} .$$

But $\psi = \theta + \phi$, therefore $\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2-rr_2}{r^2+r_1^2} = \frac{r^2+2r_1^2-rr_2}{r^2+r_1^2}$.

Finally $\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta} \cdot \frac{d\theta}{d\psi} = \sqrt{r^2+r_1^2} \cdot \frac{r^2+r_1^2}{r^2+2r_1^2-rr_2} = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$.

Radius of curvature in Pedal form: $\rho = r\frac{dr}{dp}$.

Note: i) If x-axis is a tangent to a curve at $(0,\ 0)$, then ρ at $(0,\ 0) = \lim_{x\to 0} \frac{x^2}{2y}$.

ii) If y-axis is a tangent to a curve at the origin, then ρ at $(0,\ 0) = \lim_{x\to 0} \frac{y^2}{2x}$.

Problems: 1. Find the radius of curvature at $(\frac{3a}{2},\ \frac{3a}{2})$ of the Folium $x^3 + y^3 = 3axy$.

Differentiating with respect to x, we get

$$3x^2 + 3y^2y_1 = 3a(xy_1 + y) \implies y_1 = -\frac{x^2-ay}{y^2-ax} \quad \text{........}(i)$$

$$\therefore y_1 \text{ at } (\frac{3a}{2},\ \frac{3a}{2}) = -1 .$$

Differentiating (i) , $y_2 = -\frac{(y^2-ax)(2x-ay_1)-(x^2-ay)(2yy_1-a)}{(y^2-ax)^2}$

$$\therefore y_2 \text{ at } (\frac{3a}{2},\ \frac{3a}{2}) = -\frac{32}{3a} .$$

Hence ρ at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{3a}{8\sqrt{2}}$.

2. Find the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$

$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$.

$$x = a(\theta + \sin\theta)\ , \qquad y = a(1 - \cos\theta)$$

$$\Longrightarrow \quad \dot{x} = a(1 + \cos\theta)\ , \qquad \dot{y} = a\sin\theta$$

$$\text{And} \quad \ddot{x} = -a\sin\theta \ , \qquad \ddot{y} = a\cos\theta$$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(a^2(1+\cos\theta)^2+a^2\sin^2\theta)^{\frac{3}{2}}}{a^2(1+\cos\theta)\cos\theta+a^2\sin^2\theta}$$

$$= a2^{\frac{3}{2}}\sqrt{(1+\cos\theta)} = 4a\cos\frac{\theta}{2}.$$

3. For the cardioid $r = a(1+\cos\theta)$, show that ρ^2 is proportional to r .

$$r = a(1+\cos\theta) \ \Longrightarrow r_1 = -a\sin\theta \quad \text{and} \quad r_2 = -a\cos\theta$$

$$\therefore \ \rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{(a^2(1+\cos\theta)^2+a^2\sin^2\theta)^{\frac{3}{2}}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta}$$

$$= \frac{a[2(1+\cos\theta)]^{\frac{3}{2}}}{3(1+\cos\theta)}$$

$$\Longrightarrow \quad \rho^2 = \frac{8a^2(1+\cos\theta)^3}{9(1+\cos\theta)^2} = \frac{8a}{9}r, \qquad \text{and hence} \qquad \rho^2 \propto r\ .$$

4. Find the radius of curvature for $p^2 = ar$.

$$p^2 = ar \Longrightarrow r = \frac{p^2}{a}$$

$$\therefore \frac{dr}{dp} = \frac{2p}{a} = 2\sqrt{\frac{r}{a}} \qquad \Longrightarrow \rho = r\frac{dr}{dp} = \frac{2r\sqrt{r}}{\sqrt{a}}\ .$$

5. Find the radius of curvature of the curve $r^n = a^n\cos n\theta$.

Solution: Given curve is $r^n = a^n\cos n\theta$

Diff. w.r.to θ we get,

$$nr^{n-1}r_1 = -na^n\sin n\theta$$

$$\text{Or} \quad r^n\,\frac{r_1}{r} = -a^n\sin n\theta$$

$$\therefore \frac{r_1}{r} = -\tan n\theta \qquad \Longrightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\,[1+\tan^2 n\theta\,] = \frac{1}{r^2\cos^2 n\theta}$$

$$\Longrightarrow p = r\cos n\theta \qquad \Longrightarrow \text{Pedal equation is } \boxed{pa^n = r^{n+1}}$$

Diff. w.r.to r we get , $\frac{dp}{dr}a^n = (n+1)r^n$

Therefore $\rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}}$.

Or

Given curve is $r^n = a^n \cos n\theta$

Diff. w.r.to θ we get,

$$nr^{n-1}r_1 = -na^n \sin n\theta$$

$$\therefore \quad r_1 = -r \tan n\theta$$

$$r_2 = -r_1 \tan n\theta - nr \sec^2 n\theta$$

$$= r \tan^2 n\theta - nr \sec^2 n\theta$$

$$\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{r^3(1+\tan^2 n\theta)^{\frac{3}{2}}}{r^2+2r^2\tan^2 n\theta - r(r\tan^2 n\theta - nr\sec^2 n\theta)}$$

$$= \frac{r\sec^3 n\theta}{1+\tan^2 n\theta + n\sec^2 n\theta} = \frac{r\sec n\theta}{n+1}$$

$$= \frac{ra^n}{(n+1)r^n} = \frac{a^n}{(n+1)r^{n-1}} .$$

6. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point $(1,\ 1)$.

Solution: $x^4 + y^4 = 2$

Differentiating with respect to x, we get

$$4x^3 + 4y^3y_1 = 0 \implies y_1 = -\frac{x^3}{y^3} \quad \text{........} (i) \quad \therefore y_1 \text{ at } (1,\ 1) = -1 .$$

Differentiating (i) with respect to x, we get,

$$y_2 = -\frac{y^3 3x^2 - x^3 3y^2 y_1}{y^6} \qquad \therefore y_2 \text{ at } (1,\ 1) = -6 .$$

Hence ρ at $(1,\ 1) = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$.

7. Find the radius of curvature of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at any point $(x,\ y)$.

Solution: $y_1 = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \implies y_1^3 = -\frac{y}{x} \implies xy_1^3 = -y \implies 3xy_1^2y_2 + y_1^3 = -y_1$

$$\implies 3xy_1y_2 + y_1^2 = -1 \implies y_2 = -\left(\frac{1+y_1^2}{3xy_1}\right)$$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{(1+y_1^2)^{\frac{3}{2}}}{-\left(\frac{1+y_1^2}{3xy_1}\right)} = -3xy_1(1+y_1^2)^{\frac{1}{2}}$$

$$= 3x\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\left(\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{2}} = 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}} = 3\sqrt[3]{axy} .$$

Or Parametric equation is $x = a\cos^3 t \quad and \quad y = a\sin^3 t$

$\implies \quad \dot{x} = -3a\cos^2 t\ \sin t$, $\qquad \dot{y} = 3a\sin^2 t\cos t$

$\ddot{x} = -3a(-2\cos t\sin^2 t + \cos^3 t)$, $\qquad \ddot{y} = 3a(2\sin t\cos^2 t - \sin^3 t)$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(3a)^3\cos^3 t\ \sin^3 t}{-9a^2\cos^2 t\ \sin t(2\sin t\cos^2 t-\sin^3 t)+9a^2\sin^2 t\ \cos t(2\sin t\cos^2 t-\sin^3 t)}$$

$$= \frac{3a\cos^3 t\ \sin^3 t}{-\cos^2 t\ \sin^2 t} = 3a\cos t\sin t = 3a\left(\frac{x}{a}\right)^{\frac{1}{3}}\left(\frac{y}{a}\right)^{\frac{1}{3}} = 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}} = 3\sqrt[3]{axy}$$

8. Find the radius of curvature of the curve $r^n = a^n\sin n\theta$.

Solution: $r^n = a^n\sin n\theta$

Diff. w.r.to θ we get, $nr^{n-1}r_1 = na^n\cos n\theta$

Or $r^n\,\frac{\boldsymbol{r_1}}{\boldsymbol{r}} = a^n\cos n\theta$

$\therefore \frac{r_1}{r} = \cot n\theta \quad \Longrightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\,[1+\cot^2 n\theta\,] = \frac{1}{r^2\sin^2 n\theta}$ (3)

$\Longrightarrow\ p = r\sin n\theta \quad \Longrightarrow$ Pedal equation is $\boxed{pa^n = r^{n+1}} \Longrightarrow \frac{dp}{dr} = \frac{(n+1)r^n}{a^n}$

$\therefore \rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}}$.

9. Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ at the point $(a,0)$.

$y^2 = \frac{a^2(a-x)}{x}$

$y^2x = a^3 - a^2x$

Differentiating w.r.t x

$2xyy_1 + y^2 = -a^2 \qquad \Longrightarrow y_1 = -\frac{a^2+y^2}{2xy}$.

Since $y_1 = \infty$ at $(a,0)$, therefore $\rho = \frac{1}{x_2}$.

$x_1 = -\frac{2xy}{y^2+a^2}$ then $x_1 = 0\ at\ (a,0)$

$(y^2+a^2)x_1 = -2xy$.

Again differentiating w.r.t y

$(y^2+a^2)x_2 + 2yx_1 = -2x - 2yx_1$

$\Longrightarrow\ (y^2+a^2)x_2 = -2x - 2yx_1 - 2yx_1$

Then $x_2 = -\frac{2}{a}$ at $(a,0)$

$\therefore \rho = \frac{1}{x_2} = -\frac{a}{2}$

$\therefore$ The radius of curvature of the given curve is $\frac{a}{2}$.

10. Find the radius of curvature of the curve $x = a\log(\sec t+\tan t)$, $y = a\sec t$.

Given that, $x = a\log(\sec t+\tan t)$, $y = a\sec t$.

$\Longrightarrow \qquad \dot{x} = a\frac{\sec t\ \tan t+\sec^2 t}{\sec t+\tan t} = a\sec t$, $\qquad \dot{y} = a\sec t\ \tan t$

$\ddot{x} = a\sec t\ \tan t$, $\ddot{y} = a(\sec^3 t + \sec t\tan^2 t)$

$$\therefore \quad \rho = \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{a^3(\sec^2 t+\sec^2 t\tan^2 t)^{\frac{3}{2}}}{a^2\sec^2 t\,(\sec^2 t+\tan^2 t)-a^2\sec^2 t\tan^2 t}$$

$$= \frac{a\sec^6 t}{\sec^4 t} = a\ \sec^2 t.$$

Exercise:

a)

Find the angle between the following pair of curves.

1). $r^2 = a^2\operatorname{cosc} 2\theta$ and $r^2 = b^2\sec 2\theta$, 2). $r = ae^{\theta}$ and $re^{\theta} = b$.

3) $r^n = a^n\cos n\theta$, $r^n = b^n\sin n\theta$ 4) $r = \frac{2a}{(1-\cos\theta)}$, $r = \frac{2b}{(1+\cos\theta)}$.

5) $r = a(1-\sin\theta)$, $r = b(1+\sin\theta)$, 6). $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$.

b) Find the pedal equation of the following curves.

1. $r^2 = a^2\sec 2\theta$ 2. $r = a(1-\sin\theta)$

3. $r^n = a^n\cos n\theta$ 4. $r = m\theta$

5. $r = ae^{\theta\cot\propto}$.

c)

1. Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y} = 4$ at the point where it cuts the line $y = x$.

2.Find the radius of curvature at $(\frac{3a}{2}, \frac{3a}{2})$ of the Folium $x^3+y^3 = 3axy$

3 Find the radius of curvature of the curve $x^4+y^4 = 2$ at the point $(1, 1)$.

d)

Find the radius of curvature of the following curves

1. $xy^3 = a^4$ at the point (a, a) 2. $y = 4\sin x - \sin 2x$ at $(\frac{\pi}{2}, 4)$

3. $r = a(1+\cos\theta)$ 4. $x = a\log(\sec t + \tan t)$, $y = a\sec t$

f) 1. If ρ_1 and ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $a(1+\cos\theta)$ which passes through the pole, show that $\rho_1^2+\rho_2^2 = \frac{16a^2}{9}$.

Self-study:

Centre of curvature: A point $\boldsymbol{C}$ on the normal to any point $\boldsymbol{P}$ of a curve at a distance $\boldsymbol{\rho}$ from it, is called center of curvature.

Circle of curvature: A circle with center $\boldsymbol{C}$ (center of curvature at $\boldsymbol{P}$) and radius $\boldsymbol{\rho}$ is called circle of curvature Or osculating circle at $\boldsymbol{P}$.

Centre of curvature at any point $P(x,\ y)$ on the curve $y = f(x)$ is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}\ , \qquad \bar{y} = y + \frac{1+y_1^2}{y_2}\ .$$

Evolute: The locus of the center of the curvature for a curve is called its **evolute** and the curve is called an **Involute** of its evolute.

Problems:

1. If the center of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at one end of the minor axis lies at the other end, then show that the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$.

Solution: Given that the center of curvature at $(0,\ b)$ is $(0,-b)$. Therefore $\bar{y} = -b$.

For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $y_1 = -\frac{b^2x}{a^2y}$, and $y_2 = -\frac{b^2}{a^2}\left(\frac{y-xy_1}{y^2}\right)$.

At the point $(0,\ b)$, $y_1 = 0$, and $y_2 = -\frac{b}{a^2}$.

$$\bar{y} = y + \frac{1+y_1^2}{y_2} \Longrightarrow\ -b = b - \frac{a^2}{b}\ \Longrightarrow 2b^2 = a^2 \ \text{ or } \ \frac{b^2}{a^2} = \frac{1}{2}\ .$$

Hence eccentricity $e = \sqrt{1-\frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$.

2. Find the coordinates of the center of curvature at any point of the parabola $y^2 = 4ax$. Hence find its evolute.

Solution: Differentiating with respect to x, we get

$$2yy_1 = 4a \ \Longrightarrow\ y_1 = \frac{2a}{y} \quad \text{........} (i)$$

Differentiating (i), $y_2 = -\frac{2ay_1}{y^2}\ \Longrightarrow y_2 = -\frac{4a^2}{y^3}$.

Centre of curvature at any point is $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x + \frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{\frac{4a^2}{y^3}} = x + \frac{y^2+4a^2}{2a} = 3x+2a$,

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = y - \frac{\left(1+\frac{4a^2}{y^2}\right)}{\frac{4a^2}{y^3}} = y - \frac{y(y^2+4a^2)}{4a^2} = -\frac{y^3}{4a^2}\ .$$

To find the evolute, $(\bar{y})^2 = \frac{y^6}{16a^4} = \frac{(4ax)^3}{16a^4} = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3$ or $(\bar{y})^2 = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3$.

Therefore the locus of $(\bar{x},\ \bar{y})$ i.e., evolute, is $27ay^2 = 4(x-2a)^3$.

Assignment questions:

g) Find the coordinates of the center of curvature at $(2,\ 1)$ on the parabola $x^2 = 4y$.

h) Find the coordinates of the center of curvature at any point of the parabola $x^2 = 4ay$. Hence find its evolute.

i) Show that the equation of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.