Recurrence Relation

* Problem solving Approach

Components of a Recurrence Relation:

i) Base Case: ~~what~~ when we stop or know the answer to

ii) Recursive Formula: what we continue doing

* Towers of Hanoi

For n disk	m moves
0	0
1	1
2	3
3	7
.	.
.	.
.	.
k	$2^k - 1$

For $n=2$, → when solving T_n, we know sol^n of T_{n-1}

S-1: Move top $n-1$ disks to middle (aux) towers (T_{n-1})

S-2: Move largest disk to destination (final tower) (1)

S-3) " ~~mid~~ top $n-1$ disk from aux to dest (T_{n-1})

$$\therefore T_n = T_{n-1} + 1 + T_{n-1}$$

$$\therefore T_n = 2T_{n-1} + 1 \longrightarrow \text{Open Form Solution}$$

↳ Getting n doesn't help

$$= 2(2T_{n-2}+1)+1$$

$$\cancel{= 2T_{n-2} + 2 + 1}$$

$$= 2^2(2T_{n-3}+1) + 2 + 1$$

$$= 2^3(2T_{n-4}+1) + 4 + 2 + 1$$

$$\vdots$$

$$= 2^{n}(2T_0+1) + 2^{n-2} + \dots + 4 + 2 + 1$$

$$= 2^{n}(0+1) + 2^{n-2} + \dots + 4 + 2 + 1$$

$$= 2^{n-1} + 2^{n-2} + \dots + 4 + 2 + 1$$

$$= \frac{2^n - 1}{2 - 1}$$

$$\therefore T_n = 2^n - 1$$

*Using Mathematical Induction:

Basis: $T_0 = 2^0 - 1 = 0$

Hypothesis: Let, $\cancel{2^n}$ $T_n = 2^n - 1$ / $T_k = 2^k - 1$

we have to prove,

Induction: $T_{k+1} = 2^{k+1} - 1$

$\therefore T_{k+1} = \cancel{2(2T_k + 1) + 1}\ 2T_k + 1$

$= 2(2^k - 1) + 1$

$= 2^{k+1} - 1$

C-2, W-1 24/06/25

TOH Variation

[i] Double Tower of Hanoi:

S-1 : Move top n-1 pairs of disks to aux tower (DT_{n-1})

S-2 : Move largest ~~two~~ pair of disks to dest tower (2)

S-3 : Move top n-1 pairs of disks from aux to dest tower (DT_{n-1})

$DT_n = DT_{n-1} + 2 + DT_{n-1}$

$\therefore DT_n = 2DT_{n-1} + 2 = 2(DT_{n-1} + 1)$

$= 2(2DT_{n-2} + 2) + 2$

$= 2^2 DT_{n-2} + 2^2 + 2$

$= 2^2(2DT_{n-3} + 2) + 2^2 + 2$

$= 2^3 DT_{n-3} + 2^3 + 2^2 + 2$

$\vdots$

$= 2^n DT_{n-n} + 2^n + \dots + 2^3 + 2^2 + 2$

$= 2^n(0) + 2^n + \dots + 2^3 + 2^2 + 2$

$= 2^n + \dots + 2^3 + 2^2 + 2$

$= 2(2^{n-1} + \dots + 2^2 + 2 + 1)$

$\therefore D_0T_n = 2(2^n - 1) = 2 \times$ Single Tower of Hanoi

Self Study : TTOH

★ Using ~~Mathematicl~~ Mathematical Induction:

\# Prove using Mathematical Induction that, for DTOH with n pairs of Disks, minimum number ~~of~~ $2(2^n - 1)$ moves are required

Basis: For, $n = 0$, $DT_0 = 2(2^0 - 1) = 0$

Hypothesis: Let, $DT_n = 2(2^n - 1)$ for $n = n$

Induction: $DT_{n+1} = 2 DT_n + 2$

$= 2 \times [2(2^n - 1)] + 2$

$= 2(2^n - 2) + 2$

$= 2(2^n - 2 + 1) = 2(2^n - 1)$ (Proved)

Lines in the Plane : Max number of regions using n lines

1 Line	2 Region
2 "	4 "
3 "	7 "
4 "	11 "

$\therefore T_n = T_{n-1} + n$

Here,

$T_0 = 1$

$$\therefore T_n = T_{n-1} + n = (T_{n-2} + n - 1) + n$$
$$= \{T_{n-3} + (n-2)\} + 2n - 1$$
$$= \{T_0 + n - (n-1)\} + n(n-1) - 1 - 2 - \ldots$$
$$= 1 + 1 + (n$$

C-3, W-1 26/05/25

Lines in Surface:

$$L_n = L_{n-1} + n \text{ (open)}$$
$$= L_{n-2} + (n-1) + n$$
$$= L_{n-3} + (n-2) + (n-1) + n$$
$$\vdots$$
$$\cancel{= L_{n-n} + \{n}$$
$$= L_0 + 1 + 2 + 3 + \dots + (n-1) + n$$
$$= 1 + \frac{n(n+1)}{2} \text{ (closed)}$$

* Using Mathematical Induction:

Basis: $L_0 = 1 + \frac{0(0+1)}{2} = 1$

Hypothesis: Let, $L_{n-1} = 1 + \frac{n(n-1)(n-1+1)}{2} = 1 + \frac{(n-1)n}{2}$

Induction: $L_n = L_{n-1} + n$
$$= 1 + \frac{(n-1)n}{2} + n$$
$$= \frac{2 + n^2 - n + 2n}{2} = \frac{n^2 + n + 2}{2}$$
$$= \frac{n(n+1)}{2} + \frac{2}{2} = \frac{n(n+1)}{2} + 1 \text{ (Proved)}$$

Variations

If nth line passes through $(n-1)$th intersection point, $L_n = 2n$

Zig:

No. of Zigs (n)	No. of Lines $(2n)$	Zigs Region Z_n	Lines Region (L_{2n})	Los $(L_{2n} - Z_n)$
0	0	1	1	0
1	2	2	4	2
2	4	7	11	4

$\therefore Z_n = L_{2n} - 2n$ = Line Region − Numbers of Lost Region

$$= \frac{2n(2n+1)}{2} + 1 - 2n$$

$$= 2n^2 + n + 1 - 2n$$

$$\therefore Z_n = 2n^2 - n + 1$$

Self: Inductive Proof

* Zigzag :

No. of Zigzags (n)	No. of Lines ($3n$)	Zigzag Region (ZZ_n)	Lines Region (L_{3n})	Regions Lost ($L_{3n} - ZZ_n$)
0	0	1	1	0
1	3	2	7	5
2	6	12	22	10

$$\therefore ZZ_n = L_{3n} - 5n$$

$$= \frac{3n(3n+1)}{2} - 5n + 1$$

$$= \frac{9n^2 + 3n - 10n + 2}{2} = \frac{9n^2 - 7n + 2}{2}$$

$$\therefore ZZ_n = \frac{9n^2 - 7n + 2}{2}$$

* W or M Shape:

n	$4n$	W_n	L_{4n}	$L_{4n} - W_n$
0	0	1	1	0
1	4	2	11	9
2	8	19	37	18

$$W_n = \boxed{L_{4n} - 9n}$$

$$= \frac{4n(n+1)}{2} - 9n + 1$$

$$= 8n^2 + 2n - 9n + 1$$

$$\therefore W_n = 8n^2 - 7n + 1$$

* Using Mathematical Induction:

Basis: $W_0 = 8(0)^2 - 7(0) + 1 = 1$ $\quad \cancel{8(n-1)^2 - 7(n-1) + 1}$

Hypothesis: $W_{n-1} = \cancel{8(n-1)^2 - 7(n-1) + 1}$ $\quad \cancel{8n^2 - 7n + 1}$

~~Induction:~~ $\cancel{W_n = L_{4n} - 9n =}$

C-4, W-2 30/6/25

Open form of lines in planes (variations)

ZZ:

$n-2$	$n-1$	n	ZZ_n
-1	0	1	2
0	1	2	12
1	2	3	31
2	3	4	59

Regions increased	
1	10 − (−1)
10	10 − 0
19	20 − 1
28	30 − 2

$= 10(n-1) - (n-2)$

↳ to find use

$ZZ_n - ZZ_{n-1}$

$\therefore ZZ_n = ZZ_{n-1} + 9n - 8$ (open)

* Mathematical Induction:

Basis: $ZZ_1 = \frac{9(1)^2 - 7(1) + 2}{2} = 2$

Hypothesis: $ZZ_{n-1} = \frac{9(n-1)^2 - 7(n-1)+2}{2}$

Induction: $ZZ_n = ZZ_{n-1} + 9n - 8$

$$= \frac{9(n-1)^2 - 7(n-1) + 2}{2} + 9n - 8$$

$$= \frac{9n^2 - \cancel{18n} + 9 - 7n + 7 + 2 + 18n - 16}{2}$$

$$= \frac{9n^2 - 7n + 2}{2}$$

= (Proved)

* Check for Zig and W

C-5, W-2 01/07/25

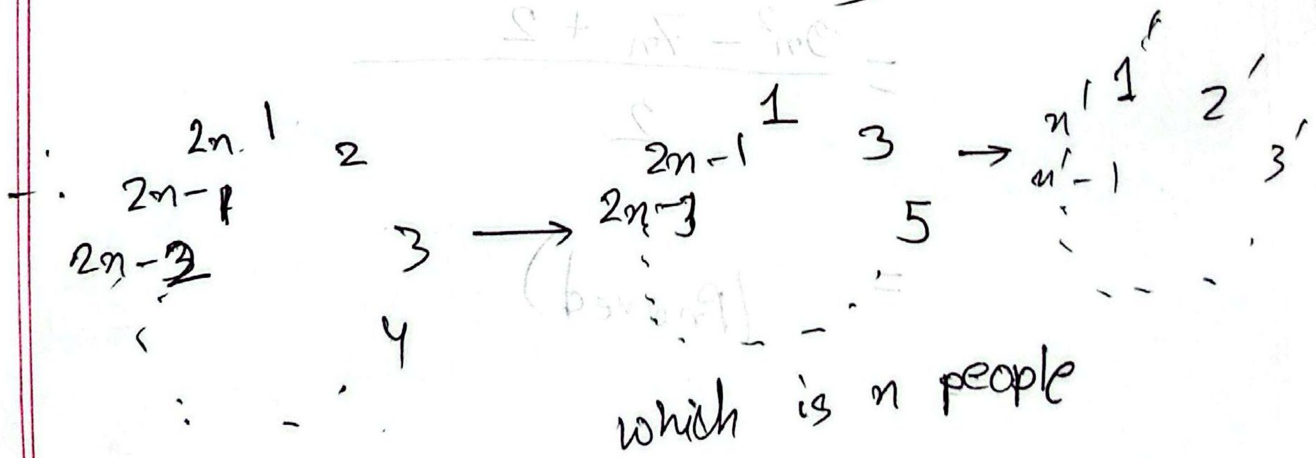
Josephus Problem

* In a cyclic fashion, find the surviving person

$J(n) \longrightarrow$ Survivor ~~for~~ for n numbers of people

For $2n$ numbers of People

After 1st round $\longrightarrow$ all even places are missing



which is n people

$\therefore J(n)$ placed person shifts to $2J(n)-1$

$\therefore J(2n) = 2J(n) - 1$
(even case)

$J(1) = 1 \longrightarrow$ 1 person always survives

(Base case)

∴ For $2n+1$ people:

Same as ~~even~~ $2n$

Except 1 dies

$\therefore 2n+1-n-1 = n$

$\therefore J(n)$ ~~= 2~~ is placed at ~~2(J~~ $2J(n)+1$

$\therefore J(2n+1) = 2J(n)+1$

(odd case)

$n = 25$

$J(25) = 2J(12)+1 = 2(2J(6)-1)+1$

$= 4(2J(3)-1)-2+1$

$= 8J(3)-4-1$

$= 8(2J(1)+1)-5$

$= 8(2+1)-5$

$= 19$

~~Open~~ Closed form:

$$J(2^m + l) = 2l + 1 \; ; \; [2^m < n]$$

Here,

n	1	2	3	4	5	6	7	8	9	...
$J(n)$	1	1	3	1	3	5	7	1	3	...

2^n will always reset to 1

All the outputs are odd

* Using Mathematical Induction: $J(n) = 2l+1$ where $n = 2^m + l$

Basis: Induction is on m. If $m = 0$, then $l = 0$

$\therefore J(2^m + l) = 2l + 1$

or, $J(2^0 + 0) = 2(0) + 1$

$\therefore J(1) = 1$

Hypothesis:

Let,

$J(2^{m-1} + c) = 2c + 1$

~~Hypothesis and~~ Induction:

If $2^m + l = 2n$ (Even) and $m > 0$

$\therefore J(2^m + l) = \cancel{2}(2J(2^{m-1} + \frac{l}{2})$; $[\because J(2n) = 2J(n) - 1]$

$= 2(2\frac{l}{2} + 1) - 1$; [Assuming $c = \frac{l}{2}$]

$= 2l + 2 - 1$

$= 2l + 1$

If $2^m + l = 2n + 1$ (odd) and $m > 0$

Now, l must be odd $\rightarrow$ 2^m always will be even; odd + even = odd

$\therefore 2^m + l$ = even + odd = odd

Let,

$l = 2n + 1$

$$\therefore J(2^m + l) = J(2^m + 2n + 1)$$

$$= J(2\underbrace{(2^{m-1} + n)}_{n} + 1) \rightarrow J(2n+1)$$

$$= 2J(2^{m-1} + n) + 1$$

$$= 2(2n+1) + 1 \text{ ; [Hypothesis]}$$

$$= 2l + 1 \text{ ; } [\because l = 2n+1]$$

C-6, W-2 03/07/25

Binary Properties of Josephus Problem:

~~When~~ Let,

$$n = (b_m\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)_2$$

$$J(n) = (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, b_m)_2$$

$$\therefore l = (0\, b_{m-1}\, b_{m-2} \ldots b_1\, b_0)$$

$$\therefore 2l = (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, 0)_2$$; [left shift = ×2]

$$\therefore 2l+1 = (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, 1)_2$$

$$= (b_{m-1}\, b_{m-2} \ldots b_1\, b_0\, b_m)_2$$

Radix Based Properties of Josephus Problem:

Generalized Version of Josephus:

$f(1) = \alpha$

$f(2n) = 2f(n) + \beta$

$f(2n+1) = 2f(n) + \gamma$

* We solve using Repertoire Method:

~~When n = 1,~~ For $f(2)$,

$f(2n) = f(2 \times 1) = 2f(1) + \beta = 2\alpha + \beta$

For $f(3)$,

$f(2n+1) = f(2 \times 1 + 1) = 2f(1) + \gamma = 2\alpha + \gamma$

$\therefore f(n)$ will always be,

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

Let,

$\alpha = 1,\ \beta = 0,\ \gamma = 0$

$\therefore f(n) = A(n) + 0 + 0 = A(n) \ldots\ldots (i)$

$\therefore A(1) = 1 = \alpha$;

$A(2n) = 2A(n) + \beta = \overset{2A(n)}{\cancel{\alpha}} + \beta = 2A(n)$

$A(2n+1) = 2A(n) + \cancel{0}\gamma = 2A(n)$

* $A(8) = 2A(4) = 2(2(A(2))) = 2(2(2A(1)))$

$= 2^3 = 8$

* $A(9) = \cancel{2(A}\ 2A(2\times 4 + 1) = 2A(4) = 2^3 = 8$

$\therefore$ for this recurrence, $A(n) = 2^m$

Let,

$f(n) = 1$

$\therefore f(1) = 1 = \alpha$

$\therefore f(2n) = 2f(n) + \beta$

or $1 = 2 + \beta$

$\therefore \beta = -1$

$\therefore f(2n+1) = 2f(n) + \gamma$

or, $1 = 2 + \gamma$

$\therefore \gamma = -1$

$\therefore 1 = A(n) - B(n) - C(n) \quad \ldots \quad (ii)$

Let,

$f(n) = n$

$\therefore f(1) = 1 = \alpha$

$\therefore f(2n) = 2f(n) + \beta$

or, $2n = 2n + \beta$

$\therefore \beta = 0$

$\therefore f(2n+1) = 2f(n) + \gamma$

or, $2n + 1 = 2n + \gamma$

$\therefore \gamma = 1$

$\therefore n = A(n) + C(n) \quad \ldots \quad (iii)$

or, $2^m + 2 = A(n) + C(n)$

or, $2^m + 2 = 2^m + C(n)$

$\therefore C(n) = 2$

From eq^n (ii)

$1 = 2^m - B(n) - 2$

$\therefore B(n) = \cancel{1 + 2 + 2^m} \; 2^m - 2 - 1$

$\therefore f(n) = 2^m \alpha + (2^m - 2 - 1)\beta + 2\gamma$

Putting $\alpha = 1, \beta = -1, \gamma = 1$ (Josephus)

$f(n) = 2^m - 2^m + 2 + 1 + 2$

$= 2l + 1$

Radix based solution of recurrence:

We can only use this iff,

$$f(j) = \cancel{\alpha j}\ \alpha_j \ ; [1 \leq j < d]$$

$$f(dn+j) = cf(n) + \cancel{\beta j}\ \beta_j \ ; [0 \leq j < d]$$

$f(1) = \alpha_1$; if $d = 3$, $c = 10$

$f(2) = \alpha_2$

$f(3n+0) = 10f(n) + \beta_0$

$f(3n+1) = 10f(n) + \beta_1$

$f(3n+2) = 10f(n) + \beta_2$

If $c = d = 2$

$f(1) = \alpha_1$

$f(2n+0) = 2f(n) + \beta_0$

$f(2n+1) = 2f(n) + \beta_1$

If we add $\alpha_1 = 1$, $\beta_0 = -1$, $\beta_1 = 1$, we get josephus

CamScanner

Here,

for $f(n) = (\cdot - \quad - \quad - \quad)_d$

$= (\alpha_m \; b_m \; - - - -)_c$

Sums

A Sum of any series

$a_0 + a_1 + \dots + a_{n-1} + a_n$ → three dot

$\sum_{i=0,\, k \le n} a_k$ ⟶ Sigma Notation

$\boxed{\sum_{k=1}^{n} a_k \longrightarrow \text{Delimited Form}}$

$S_n = \sum_{k=0}^{n} a_k$

$S_0 = a_0$

$\therefore S_n = \underbrace{a_0 + a_1 + a_2 + \cdots + a_{n-1}}_{S_{n-1}} + a_n$

$\therefore S_n = S_{n-1} + a_n$

For any general series,

Let,

$R_0 = \alpha$

$R_n = R_{n-1} + \beta + \gamma n$

Case 1:

Let,

$R_n = 1$

$\therefore R_0 = 1$

$\vdots$

$R_{n-1} = 1$

$\therefore R_0 = 1 = \alpha$

Case 2: $R_n = n$ to find $B(n)$

Case 3: $R_n = n^2$ " " $C(n)$

$\vdots$

$$R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma$$

↳ for math no need to derive, just use it

[Follow notes]

$S_n = \sum_{k=0}^{n} (a+bk)$

Here,

$S_0 = a$

$$S_n = a + (a+b) + (a+2b) + \dots + \underbrace{\cancel{(a+nb)}}_{}\{a+(n-1)b\} + (a+nb)$$

$$= S_{n-1} + (a+nb)$$

By comparing the eq^{n}s with general sum eq^n, we get,

$\alpha = a$

$\beta = a$

$\gamma = b$

$$\therefore R_n = a + an + \frac{n^2+n}{2} b$$

$$= a(n+1) + \frac{n^2+n}{2} b$$

+ [More math in lecture notes]

[4] Converting Recurrence to sum:

* We assume somethings:
- we always have to multiply something with recurrence.

We assume we know how to find it.

* TOH:

$T_0 = 0$

$T_n = 2T_{n-1} + 1$

we multiply both sides with 2^{-n}

$\therefore \frac{T_0}{2^0} = \frac{0}{2^0} = 0$

$\therefore \frac{T_n}{2^n} = \frac{2T_{n-1}}{2^n} + \frac{1}{2^n}$

$= \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$

Let, $S_n = \frac{T_n}{2^n}$

$\therefore S_n = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$

$= S_{n-1} + \frac{1}{2^n}$

$= S_{n-2} + 2^{-(n-1)} + 2^{-n}$

$= S_{n-3} + 2^{-(n-2)} + 2^{-(n-1)} + 2^{-n} = S_{n-n} + 2^{-\{n-(n-1)\}} + \ldots$

$\vdots$

$= S_0 + 2^{-1} + 2^{-2} + \ldots + 2^{-n}$

$$= \sum_{k=1}^{n} 2^{-k}$$

Now,

$$S_n = \sum_{k=1}^{n} 2^{-k}$$

$$\text{or, } S_n + 2^0 = \sum_{k=1}^{n} 2^{-k} + 2^0$$

$$\text{or, } S_n = \left(2^0 + 2^{-1} + 2^{-2} + \dots + 2^{-n}\right) - 1$$

$$= \frac{1\left\{1 - \left(\frac{1}{2}\right)^{n+1}\right\}}{1 - \frac{1}{2}} - 1 \quad ; \left[S_n = \frac{a(1-r^n)}{1-r}\right]$$

$$= \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} - 1$$

$$= 2\left(1 - \frac{1}{2^{n+1}}\right) - 1$$

$$\text{or, } \frac{T_n}{2^n} = 1 - \frac{1}{2^n}$$

$$\therefore T_n = 2^n - 1 \longleftarrow \text{upto sum}$$

C-8, W-3 08/07/25

SUMS

For a

$$a_n T_n = b_n T_{n-1} + C_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$$

$$\therefore S_n = \frac{a_{n-1} a_{n-2} a_{n-3} \cdots a_1}{b_n b_{n-1} b_{n-2} \cdots b_2}$$

summation factor

*Index Change : $\sum_{k=1}^{n} 2^{-k+1} \longrightarrow$ let, $j = k-1$

when $k=1$, $j=0$

" $k=n$, $j=n-1$

$$\therefore \sum_{k=1}^{n} 2^{-k+1} = \sum_{j=0}^{n-1} 2^{-j}$$

$$= \sum_{k=0}^{n-1} 2^{-k}$$

Quick Sort: $C_0 = 0$

$$C_n = n+1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k \rightarrow C_n \text{ is number of comparisons}$$

Here,

$$\cancel{C_{n-1} =} \quad nC_n = n(n+1) + 2\sum_{k=0}^{n-1} C_k$$

$$\therefore (n-1)C_{n-1} = \cancel{(n-1)(n-1+1)} + 2\sum_{k=0}^{n-1-1} C_k$$

$= n^2 + n$

$(n-1)^2 + n^2 - 1$

$\cancel{= (n-1)n}$

$$\therefore nC_n - (n-1)C_{n-1} = n^2 + n + 2\sum_{k=0}^{n-1} C_k$$

$$- n^2 - \cancel{2n} + n$$

$$\vdots$$

$$\therefore nC_n = (n+1)C_{n-1} + 2n$$

$\therefore a_n = n$

$b_n = n+1$

$d_n = 2n \rightarrow$ usually c_n

$$\therefore S_n = \frac{a_{n-1} \cdots\cdots a_1}{b_n \cdots\cdots b_2} = \frac{\cancel{(n-1)}(n-2)\cdots 3.2.1}{(n+1)\cancel{(n)}\cancel{(n-1)}\cdots 3}$$

$$= \frac{2\cdot 1}{n(n+1)}$$

* Harmonic Series: $\sum_{k=1}^{n} \frac{1}{k}$

Here,

$$C_n = 2(n+1)\sum_{k=1}^{n} \frac{1}{k+1}$$ → derivation from provided notes

Now,

$$\sum_{k=1}^{n} \frac{1}{k+1} = \cancel{1} \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n+1}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) + \frac{1}{n+1} - 1$$

$$= H_n - \frac{n}{n+1}$$

$$\therefore C_n = 2(n+1)H_n - 2n$$

* Index Replacement: ~~using~~

Must be in $\sum_{k=0}^{n}(a+bk)$ form

- Replace ~~n~~ k by n-k
- becomes $a(n+1)+\frac{n(n+1)}{2}b$

$$\#\ \sum_{k=0}^{10}(2k+1) \longrightarrow a=1$$

$b=2$

$n=10$

$$\therefore (11)+\frac{10(11)}{2}(2) = 11+110$$

$$= 121$$

C-9, W-3 10/07/25

Perturbation Method:

Given,

$$S_n = \sum_{0\le k\le n} a_k \rightarrow$$ convert it to closed form

↳ open because we can't put n to find value

$$S_{n+1} = S_n + a_{n+1} = S_0 + \sum_{1\le k\le n+1} a_k$$; Step 1: Express in this from

$$= a_0 + \sum_{1\le k+1\le n} a_{k+1}$$

$$\boxed{S_n + a_{n+1} = a_0 + \sum_{0\le k\le n} a_{k+1}}$$

$k+1 = 0$
$k = 0$
$k+1 = n+1$
$k = n$

↳ Main formula for Perturbation Method

$$= \cancel{a_0 + S_n}$$

Find $S_n = \sum_{0 \le k \le n} a x^k$ in closed form

$$S_n = \sum_{0 \le k \le n} a x^k$$

$$S_n + a x^{n+1} = a x^0 + \sum_{0 \le k \le n} a x^{k+1}$$

$$= a + \cancel{a \sum x'} \sum_{0 \le k \le n} a x^k \cdot x$$

$$= a + x \sum_{0 \le k \le n} a x^k$$

$$S_n + a x^{n+1} = a + x S_n$$

$$\therefore S_n = \frac{a - a x^{n+1}}{1 - x}$$

$$\therefore S_8 = \frac{3 - 3 \cdot 5^9}{1 - 5} = 1464843$$

$S_n = \sum k \cdot 2^k$

Here,

$$S_n + (n+1) \cdot 2^{n+1} = 0 \cdot 2^0 + \sum_{0 \le k \le n} (k+1) 2^{k+1}$$

$$= \sum (k 2^{k+1}) + \sum 2^{k+1}$$

$$= \sum k \cdot 2^k \cdot 2 + \sum 2^k \cdot 2$$

$$= 2 \sum_{0 \le k \le n} k \cdot 2^k + \frac{2 - 2 \cdot 2^{n+1}}{1-2}$$

$$S_n + (n+1) 2^{n+1} = 2S_n + \frac{2 - 2 \cdot 2^{n+1}}{1-2}$$

⋮

$$\cancel{\therefore S_n = (n+1) \cdot 2^{n+1} + 2 - 2^{n+1}}$$

$$\therefore S_n = 2^{n+1}(n-1) + 2$$

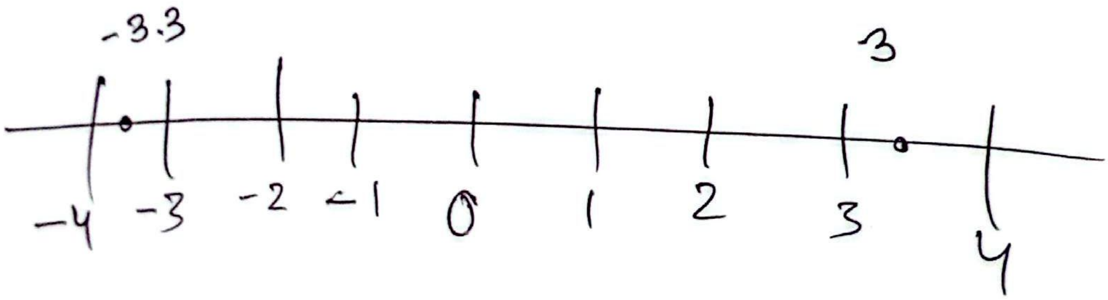
$\star$ $S_n = \sum k x^k =$

Chap-3 : Integer Functions

Floor: $\lfloor 3.3 \rfloor = 3$, $\lfloor -3.3 \rfloor = -4$

Celing: $\lceil 3.3 \rceil = 4$, $\lceil -3.3 \rceil = -3$

Mod:



General form of eq^n: $\left\lfloor \frac{n}{m} \right\rfloor \times m + n \bmod m = n$

C-10, W-4 14/07/25

Necessary Rules: for $x \in \mathbb{R}, n \in \mathbb{Z}$

~~1. $\lfloor x \rfloor = n \Leftrightarrow n \leq$~~

1. $\lfloor x \rfloor = n \Leftrightarrow n \leq x < n+1$

2. $\lfloor x \rfloor = n \Leftrightarrow x-1 < n \leq x$

3. $\lceil x \rceil = n \Leftrightarrow n-1 < x \leq n$

4. $\lceil x \rceil = n \Leftrightarrow x \leq n \leq x+1$

5. $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

6. $n \leq x \Leftrightarrow n \leq \lfloor x \rfloor$

7. $x \leq n \Leftrightarrow \lceil x \rceil \leq n$

#1.

Let,

$x = 3$

$\therefore \lfloor 3 \rfloor = 3$

$\therefore n = 3$

For R.H.S,

$3 \leq 3 < 3+1$

$\therefore 3 \leq 3 < 4$

$\therefore$ when $x \in \mathbb{Z}$, the formula holds

Let,

$x = 3.5$

$\lfloor x \rfloor = \lfloor 3.5 \rfloor = 3 = n$

$\therefore$ For R.H.S,

$3 \leq 3.5 < 3+1$

$\therefore 3 \leq 3.5 < 4$

$\therefore$ when x is floating, the formula holds

$\therefore$ 1 is always true

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Q. If n is an m-bit integer, then prove that $m = \lfloor \lg n \rfloor + 1$ → To find how many bits are needed to express a decimal in binary

Here,

The smallest m bit number is 2^{m-1}

" largest m " " " $2^m - 1$

$\therefore 2^{m-1} \leq n \leq 2^m - 1$

or, $2^{m-1} \leq n < 2^m$

or, $\lg(2^{m-1}) \leq \lg n < \lg(2^m)$

or, $m - 1 \leq \lg n < m$

or, $m \leq \lg n + 1 < m + 1$

or, $m = \lfloor \lg n + 1 \rfloor$; $[\lfloor x \rfloor = n \Leftrightarrow n \leq x < n+1]$

~~or, $\lfloor \lg n \rfloor +$~~

$\therefore m = \lfloor \lg n \rfloor + 1$; $[\lfloor x + n \rfloor = \lfloor x \rfloor + n]$

Q. Prove that, $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor$; $x \in \mathbb{R}$, $x \geq 0$

Let,

x be an integer

$\underline{L.H.S} = \lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor = R.H.S$

~~Let,~~

If x ~~be a~~ is not an integer,

Let,

$m = \lfloor\sqrt{\lfloor x\rfloor}\rfloor$

or, $m \leq \sqrt{\lfloor x\rfloor} < m+1$

or, $m^2 \leq \lfloor x\rfloor < (m+1)^2$

or, $\cancel{m^2 \leq x}$ $m^2 \leq x < (m+1)^2$; [Rule 6]

or, $m \leq \sqrt{x} < (m+1)$

$\therefore \lfloor\sqrt{x}\rfloor = m$

$\therefore \lfloor\sqrt{x}\rfloor = \lfloor\sqrt{\lfloor x\rfloor}\rfloor$

$\therefore L.H.S = R.H.S$

(Proved)

* $n = \left\lfloor \frac{n}{m} \right\rfloor * m + n \bmod m$

$$\boxed{\therefore n \bmod m = n - \left\lfloor \frac{n}{m} \right\rfloor * m}$$

Case 1: $5 \bmod 3 = 5 - \left\lfloor \frac{5}{3} \right\rfloor * 3$

$= 5 - 1 * 3$

$= 2$

Case 2: $-5 \bmod 3 = -5 - \left(\left\lfloor \frac{-5}{3} \right\rfloor * 3 \right)$

$= -5 - \left(- \left\lceil \frac{5}{3} \right\rceil * 3 \right)$

$= -5 - (-2 * 3)$

$= -5 + 6$

$= 1$

* Special case: $n \bmod \overset{m=0}{0} = n$

C-11, W-4 15/07/25

Prove that $c(x \bmod y) = cx \bmod cy$

L.H.S $= c(x \bmod y)$

$= c\left(x - y * \left\lfloor \frac{x}{y} \right\rfloor\right)$; $\left[x \bmod m = x - m * \left\lfloor \frac{x}{m} \right\rfloor \right]$

$= cx - cy * \left\lfloor \frac{x}{y} \right\rfloor$

$= cy - cy * \left\lfloor \frac{cx}{cy} \right\rfloor$

$= cx \bmod cy$;

Prove that, $(x \bmod ny) \bmod y = x \bmod y$

Case 1: Let, $y = 0$

L.H.S $= (x \bmod n \cdot 0) \bmod y$

$= (x \bmod 0) \bmod y$

$= x \bmod y$

B.

Case 2: Let $y \neq 0$,

L.H.S $= (x \bmod ny) \bmod y$

$= \left(x - ny * \left\lfloor \frac{x}{ny} \right\rfloor\right) \bmod y$

$$= \left(x - ny * \left\lfloor \frac{x}{ny} \right\rfloor\right) - y * \left\lfloor \frac{x - ny * \left\lfloor \frac{x}{ny} \right\rfloor}{y} \right\rfloor$$

$$= \left(x - ny * \left\lfloor \frac{x}{ny} \right\rfloor\right) - y \left\lfloor \frac{x}{y} - \overbrace{\frac{n\cancel{y}}{\cancel{y}} \left\lfloor \frac{x}{ny} \right\rfloor}^{\text{integer}} \right\rfloor$$

$$= x - ny \left\lfloor \frac{x}{ny} \right\rfloor - y \left\lfloor \frac{x}{y} \right\rfloor + ny \left\lfloor \frac{x}{ny} \right\rfloor$$

$$= x - y \left\lfloor \frac{x}{y} \right\rfloor$$

$$= x \bmod y$$

$$= R.H.S$$

C-12, W-5 21/07/25

Number Theory

~~K \ M~~

$K \backslash m \longrightarrow m =$ dividend $\longrightarrow \frac{m}{K} =$ some integer

$K =$ divison

$GCD(m,n) = K \mid K \backslash m$ and $K \backslash n$

$\downarrow$

K_{max}

$LCM(m,n) = K \mid m \backslash K$ and $K \backslash n$

$\downarrow$

K_{min}

GCD : Steins Formula:

1. $G(0, v) = v$
2. If u and v are both even then,
   $g(u, v) = 2g(u, v)$
3. If u is even and v is ~~and v are~~ ~~both~~ odd then,
   $g(u, v) = g(\frac{u}{2}, v)$

4. If u is odd but v is even,

$$g(u,v) = g\left(u, \frac{v}{2}\right)$$

5. If u and v are both odd,

i) if $u \geq v$

$$g(u, v) = g\left(\frac{u-v}{2}, v\right)$$

ii) if $u < v$,

$$g(u, v) = g\left(\frac{v-u}{2}, v\right)$$

GCD (12, 18)

$= 2GCD(6, 9)$

$= 2GCD(3, 9)$

$= 2GCD\left(\frac{9-3}{2}, 9\right)$

$= 2GCD(3, 3)$

$= 2GCD\left(\frac{3-3}{2}, 3\right)$

$= 2GCD(0, 3)$

$= 2 \times 3$

$= 6$

Euclid's Formula:

~~GCD:~~

$$GCD(u, v) = \begin{cases} GCD(0, v) = v, & \text{if } u = 0 \\ GCD(u, v) = GCD(v \bmod u, u); & \text{if } u > 0 \end{cases}$$

GCD(12, 18)

$= GCD(18 \bmod 12, 12)$

$= GCD(6, 12)$

$= GCD(12 \bmod 6, 6)$

$= GCD(0, 6)$

$= 6$

LCM:

$$m * n = GCD(m,n) * LCM(m,n)$$

$$\therefore LCM = \frac{m * n}{GCD(m,n)}$$

↳ use this if LCM using Euclid

[1] Prime Numbers: If divisor is 1 and only itself

[2] Composite Numbers: If more than 2 divisors

Fundamental Theorem of Arithmetic:

There is only one way to write ~~an~~ n as a product of primes in non decreasing order (proof by contradiction)

Let,

n can be expressed as, in two ways product of primes . . .

$\therefore n = p_1 * p_2 * p_3 * \ldots * p_m$

and, $n = q_1 * q_2 * q_3 * \ldots * q_m$ and $p_1 < q_1$

or, ~~and p~~ $p_1 * (p_2 * p_3 * \ldots * p_m) = q_1 * q_2 * q_3 * \ldots * q_m$

or, $p_2 * p_3 * \ldots * p_m = \dfrac{q_1 * q_2 * q_3 \cdots q_m}{p_1}$

Here, L.H.S is an integer

$\therefore$ R.H.S must also be an integer

but this can not be the case as q_1 is a prime and cannot be divided by p_1

$\therefore$ ~~$p_1 \not> q_1$~~ $p_1 \not< q_1$

Similarly, we can prove it for $p_2, p_3 \ldots p_m$

$\therefore p_1 = q_1$. There is only one way to . . .

Every positive integers can be expressed as

$n = \prod_{p} p^{n_p}$ (~~Proof~~ Product of a primes something)

↳ self

$12 = (2^2 \times 3^1 \times 5^0 \times 7^0 \times \cdots)$

$18 = (2^1 \times 3^2 \times 5^0 \times 7^0 \times \cdots)$

$19 = (2^0 \times 3^0 \times 5^0 \times 7^0 \times \cdots 19^1 \times \cdots)$

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C-13, W-5 22/07/25

Any positive integer > 1 is a prime or product of prime

OR, Every composite number has a prime divisor

Let,

Smallest number which is not a prime or product of prime is n

$\rightarrow$ 2 to $(n-1)$ is a prime or product of prime

$\therefore$ We can say

$n = a \times b$, where a is a composite number
b " " " "

Now,
if a, b are factors of n, then $2 \leq a, b < n$

Hence, if a, b are both primes,
then n becomes products of primes

Again, if a, b are neither primes,
then a and b can be expressed as ~~divided into~~ product of primes
$\therefore$ the n becomes products of primes

$\therefore$ Our assumption contradicts that n is a product of prime

$\therefore$ There is no such n.

(Proved)

Euclid's Theorem: There are infinitely many prime numbers:

Let,

there are ~~a~~ a finite number of primes.

$\therefore S = \{2, 3, 5, 7, \dots, P_k\}$

Again, let,

$A = (2 \times 3 \times 5 \times 7 \times \dots \times P_k) + 1$

A is not divisible by any of the prime numbers 2, 3, 5, 7, ... P_k

because A-1 is " " the " " "

$\therefore$ A cannot be expressed as product of primes

$\therefore$ A is prime ~~&~~ greater than P_k

(Proved)

*Euclid Number:

$$e_n = e_0 * e_1 * e_2 * \dots * e_{n-1} + 1$$

↳ open form

$e_0 = 1$

$\therefore e_1 = e_0 + 1$

$= 1 + 1 = 2$

Usually they are primes
But not always

$\therefore e_5 = 1807 = 13 * \cancel{18}139$

→ For closed form:

$e_n = e_0 * e_1 * \dots * e_{n-1} + 1$

or, $e_n = e_1 * e_2 * \dots * e_{n-1} + 1$

Let, $e_1 * e_2 * \dots * e_{n-1} = P_{n-1}$

$\therefore e_n = P_{n-1} + 1$

$\therefore P_n = e_1 * e_2 * \dots * e_n$

Or, $P_n = P_{n-1} \cdot e_n$

Or, $P_n = (P_{n-1})(P_{n-1}+1) \dots (i)$

Now,

$e_n = P_{n-1} + 1$

$P_{n-2} = e_{n-1} - 1$

$\therefore e_{n-1} = P_{n-2} + 1$

$= P_{n-2} \cdot e_{n-1} + 1$

$= (e_{n-1} - 1)(e_{n-1}) + 1$

$= e_{n-1}^2 - e_{n-1} + 1$

* **Sieve of Erast o thenes:** Finding primes in any specific range

2 - 25

Let,

$25 = n$

$\therefore \sqrt{n} = 5$

Here,

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~

~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17

~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ 23 ~~24~~ ~~25~~

Step 1: Remove all Mults of 2

Step 2: Remove all Mults of the next uncrossed number

Step 3: Repeat until (after) uncrossed $= \sqrt{n}$

* Marsene Number : $2^p - 1$ (where p is a prime)

Not all Marsene Numbers are prime

But the ones that are, are called Marsene Prime

→

C-14, W-5 24/07/23

[1] Factorial Factors:

$\mathcal{E}_p(n!)$ = max^m power of prime p is n!

$$\mathcal{E}_2(4!) = 4! = 1\times 2\times 3\times 4$$
$$= 1\times 2\times 3\times 2\times 2$$
$$= 2^3\times 3$$
$$= 3$$

→ how many of p we can find

*Generarized version:

$\mathcal{E}_2(10!) \longrightarrow p = 2$, $n = 10$

Until $p^k \le n$

1 to n

	1	2	3	4	5	6	7	8	9	10	Powers of 2
Divisible by 2		x		x		x		x		x	$\lfloor \frac{10}{2} \rfloor = 5$
" " 4				x				x			$\lfloor \frac{10}{4} \rfloor = 2$
" " 8								x			$\lfloor \frac{10}{8} \rfloor = 1$
Powers of 2	0	1	0	2	0	1	0	3	0	1	8

$2^4 = 16 >$ n, so we take $2^3 = 8$

$\therefore \mathcal{E}_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots + \left\lfloor \frac{n}{p^k} \right\rfloor$; $p^k \leq n$ in $n!$

* For $\mathcal{E}_2(n)$,

$\mathcal{E}_2(n) = n - v_2(n)$ where v_2 is the function of finding number of 1 in binary

* For Upper bound:

$$\mathcal{E}_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots \left\lfloor \frac{n}{p^k} \right\rfloor$$

$$\leq \frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \dots = \frac{n}{p}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right)$$

$$= \frac{n}{p}\left(\frac{1}{1-\frac{1}{p}}\right)$$

$$= \frac{n}{p}\left(\frac{p}{p-1}\right)$$

$$\mathcal{E}_p(n!) \leq \boxed{= \frac{n}{p-1}}$$

Actual ans is smaller

[#] Relative Primality: $m \perp n$

~~whe~~ $\rightarrow \gcd(m, n) = 1$

***Stern-Brocot Tree:** Binary tree where,

if left ancestor $= \frac{m}{n}$

if right " $= \frac{m'}{n'}$

then, child (mediant) $= \frac{m+m'}{n+n'}$

$\rightarrow$ we find relative primes using this

* Always start with $\frac{0}{1}$ and $\frac{1}{0}$

* Stern Brocot tree always follows $m'n - mn' = 1$

Basis: Initially, $\frac{m}{n} = \frac{0}{1}$, $\frac{m'}{n'} = \frac{1}{0}$

$\therefore m'n - mn' = 1 \times 1 - 0 \times 0 = 1$

Hypothesis: Let, $m'n - mn' = 1$ is true for two consecutive fractions $\frac{m}{n}$ and $\frac{m'}{n'}$

Induction: The new mediant $= \frac{m+m'}{n+n'}$

First case: $m' = m + m'$, $n' = n + n'$

$\therefore (m+m')n - m(n+n') = mn + m'n - mn - mn' = m'n - mn' = 1$

Again, $m = (m+m')$, $n = n + n'$

~~$(m'(n+n$~~

$\therefore \frac{m}{n} < \frac{m+m'}{n+n'} < \frac{m'}{n'}$; $\frac{m}{n} < \frac{m'}{n'}$ and all values are ~~a~~ nonnegative

Here,

$$\frac{m+m'}{n+n'} - \frac{m}{n} = \frac{mn + m'n - mn - mn'}{n(n+n')} = \frac{1}{n(n+n')} > 0$$

$\therefore \frac{m}{n} < \frac{m+m'}{n+n'}$

Again,

$$\frac{m'}{n'} - \frac{m+m'}{n+n'} = \frac{m'n + m'n' - mn' - m'n'}{n(n+n')} = \frac{1}{n(n+n')} > 0$$

$\therefore \frac{m+m'}{n+n'} < \frac{m'}{n'}$

(Proved)

C-15, W-6 28/07/23

Binomial Coefficients

use this for $\binom{-n}{k}$

$$\binom{n}{k} = {}^nC_k = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} ;\ k>0$$

from n elements
we choose k in each subset

$$\# \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} ;\ k>0$$

$$\text{L.H.S} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n \times (n-1)(n-2)\dots(n-k+1)(n-k)(n-k-1)\dots 2\cdot 1}{k!\cdot(n-k)(n-k-1)\dots 2\cdot 1}$$

$$= \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

$\binom{n}{0} = 1$, $\binom{n}{1} = n$

$\binom{n}{k} = 0$ if $k<0$ or $k>n$

* Pascal's Triangle :

$$\binom{n}{k} = \binom{n}{n-k} \longrightarrow \text{Symmetry identity}$$

$$\underline{R.H.S} = \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

$$= L.H.S$$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \longrightarrow \text{Absorption Identity}$$

$$\underline{L.H.S} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)!}{k(k-1)!\,(n-1-k+1)!}$$

$$= \frac{n}{k}\,\frac{(n-1)!}{(k-1)!\{(n-1)-(k-1)\}!}$$

$$= \frac{n}{k}\cdot \cancel{\frac{(n-1)!}{(k-1)!\,(n}} \binom{n-1}{k-1}$$

~~$k(n-k)\binom{n}{k} = (n-k)\binom{n}{n-k}$~~

$$k(n-k)\binom{n}{k} = n\binom{n-1}{k}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \longleftarrow \text{Addition formula}$$

C-16, W-6 29/07/25

$\binom{5}{3} = \cancel{\binom{5}{2} + \binom{4}{2}}\ \binom{4}{3} + \binom{4}{2}$; [Addition]

to find this

Use this

$$\sum_{0<k\le n} \binom{n+k}{k} = \binom{n+n+1}{n}$$

$$\sum_{0\le k<n} \binom{k}{m} = \binom{n+1}{m+1}$$

$$\# \sum_{k\le n} \binom{m+k}{k} = \binom{m+n+1}{n} ; \cancel{[k\le n]}$$

$$L.H.S = \sum_{\alpha\, k\le n} \binom{m+k}{m+k-k} = \sum_{-m<k\le n} \binom{m+k}{m} = \sum_{0\le k\le m+n} \binom{k}{m}$$

$$= \binom{m+n+1}{m+1} = \binom{m+n+1}{m+n+1-m-1}$$

$$= \binom{m+n+1}{n} = R.H.S$$

(Proved)

C-17, W-7 04/07/25

Probability Models

For $S = \{H, T\}$

$\therefore P(H) = \frac{1}{2}$

$$P(\text{favorable event}) = \frac{|\text{favorable event}|}{|\text{total event}|}$$

* For ~~eve~~ sample spaces E, F

$\cancel{P(E \cap F) =}$

Also written as $P(EF)$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$\cup \to$ OR

$$\therefore P(E \cap F) = P(E) + P(F) - P(E \cup F)$$

$$P(E \cup F)^c = 1 - P(E \cup F)$$

→ For mutually exclusive,

$$P(E \cup F) = P(E) + P(F)$$

General Formula :

$$P(E_1 \cup E_2 \cup \ldots E_n)$$

$$= \sum P(E_i) - \sum P(E_i E_j) + \sum P(E_i E_j E_k) + \ldots (-1)^n \sum P(E_1 E_2 E_n)$$

Conditional Probability:

Probability of E given F

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Independent:

$$P(EF) = P(E)P(F)$$

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C-18, 10-~~8~~7 07/07/25

Bayes Rule/Formula:

$$P(E|F) = \frac{\overbrace{P(F|E)}^{\text{Likelihood}} \cdot \overbrace{P(E)}^{\text{Evidence}}}{\underbrace{P(F)}_{\text{Prior Probability}}}$$

$P(E|F)$ → Posterior Probability

Here,

$$P(F) = P(EF) + P(E^c F)$$
$$= P(F|E) \cdot P(E) + P(F|E^c) \cdot P(E^c)$$

→ only when two cases

General Form:

$$P(F_j|E) = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n}\{P(E|F_j)P(F_j)\}} = \frac{P(EF_j)}{P(E)}$$

$P(F_j|E)$ → For j-th event

$$\boxed{* \ \frac{(n-1)!}{n!} \text{ for finding out wrong}}$$