III Laplace Transformation:

* Transformation: Change one state to another
* Integral Transformation
* Operator function is called Kernel (Function)
* Transforms space variable to another constant term using Kernel

Kernel

$$\underbrace{F(s)}_{\text{Transformation}} = \int_{\alpha}^{\beta} K(s,t)\, f(t)\, dt, \quad -\infty \le \alpha, \beta \le \infty$$

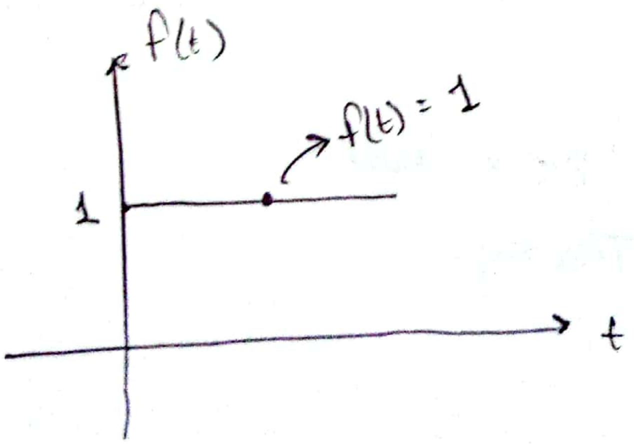
function of t

* NOT Laplace

* Let, f be a function defined for $t \ge 0$ and satisfies certain conditions to be named later.

$\therefore$ The Laplace Transformation of f is defined as:

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$



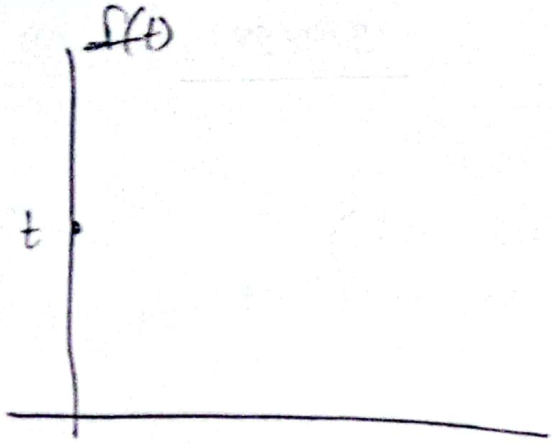
$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$= \int_0^{\infty} e^{-st} \cdot 1\, dt$$

$$= \left[-\frac{1}{s} e^{-st}\right]_0^{\infty}$$

$$= -\frac{1}{s}\left(\frac{1}{e^{\infty}} - 1\right)$$

$$= \frac{1}{s}$$



$d e^{-st}$

$$\int_0^\infty e^{-st}\, t\,dt$$

$$\cancel{= \int_0^\infty e^{-st}\, t \cdot \frac{d(e^{-st})}{-se^{-st}}}$$

$$\cancel{= -\frac{1}{s}}$$

$$= \int_0^\infty t \times \frac{e^{-st}}{-s} - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s}\,dt$$

$$= 0 - 0 + \left[\frac{e^{-st}}{-s^2}\right]_0^\infty$$

$$= \frac{1}{s^2}$$

$dv = e^{-st}$

$\therefore v = -\frac{1}{s}e^{-st}$

$u = t$

$\therefore du = dt$

$\cancel{u\int v\,dv - \int \frac{d(u)}{dx}}$

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u\int v\,dx\right\}dx$$

H.W

$t^2,\ t^3,\ t^n,\ \sin t,\ \cos t$

(C-4)(W-3) 21/04/24

Find $f(t) = e^{at}$ for $t \geq 0$

Solⁿ: Let $L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$

$$= \int_0^\infty e^{t(a-s)}\, dt$$

$$= \left[\frac{e^{t(a-s)}}{a-s}\right]_0^\infty$$

$$= \left(\frac{e^{-\infty}}{a-s} - \frac{e^{0(a-s)}}{a-s}\right)$$

$$= 0 - \frac{1}{a-s}$$

$$= \frac{1}{s-a}$$

$$\therefore L\{e^{-2t}\} = \frac{1}{s+2}$$

$\boxed{4}$ $f(t) = \sin at$

Euler Formula:

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$(+) \Rightarrow e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$(-) \rightarrow e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$L\{\sin at\} = \int_0^\infty e^{-st} \cdot \sin at \, dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2i}\left(e^{iat} - e^{-iat}\right) dt$$

$$= \frac{1}{2i}\int_0^\infty e^{(ia-s)t} - e^{-(ia+s)t} \, dt$$

$$= \frac{1}{2i}\left[\frac{e^{(ia-s)t}}{ia-s} - \frac{e^{-(ia+s)t}}{-(ia+s)}\right]_0^\infty$$

$$= \frac{1}{2i}\left[-\frac{1}{ia-s} - \frac{1}{ia+s}\right]$$

$$= \frac{-1}{2i} \times \frac{ia-s+ia+s}{(ia)^2-(s)^2}$$

$$= \frac{1}{\cancel{2i}} \times \frac{\cancel{2i} \times a}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\oplus\ e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\ominus\ e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

hyperbolic function

↑ called

complex conjugate → $e^{i\theta} + e^{-i\theta} = 2i\sinh\theta$

$$L\{\cosh at\} = \int_0^\infty e^{-st}\cosh at\, dt$$

$$= \int_0^\infty e^{-st} \times \frac{1}{2}\left(e^{iat} - e^{-iat}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{t(ia-s)} - e^{-t(ia+s)}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{t(ia-s)}}{ia-s} - \frac{e^{-t(ia+s)}}{ia+s}\right]_0^\infty$$

$$= \frac{1}{2}\left[-\frac{1}{ia-s} - \frac{1}{ia+s}\right]$$

$$= \frac{1}{2}\left(\frac{-ia - s - ia + s}{-a^2 - s^2}\right)$$

$$= \frac{-2ia}{2(-a^2-s^2)} = \frac{ia}{s^2+a^2}$$

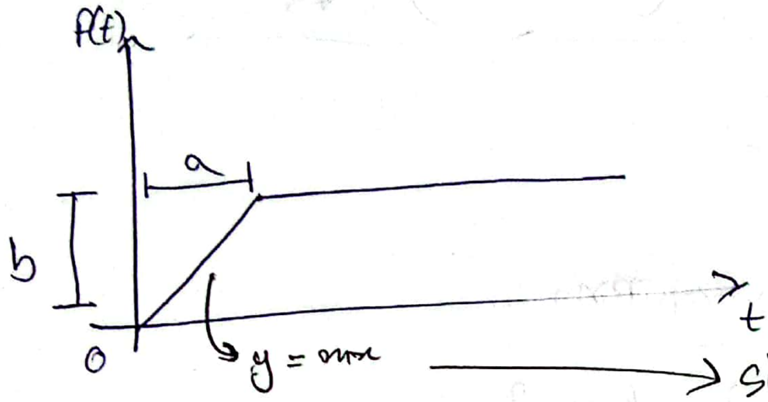
H.W: $L\{\sinh at\}$

$L\{\cosh at\}$

Example 12: Perform the Laplace transformation on the ramp illustrated below:

→ 1. Find mathematical Form

→ 2. Do Laplace



→ slope $= \frac{b-0}{a-0} = \frac{b}{a}$

From the graph, we have,

$$f(t) = \begin{cases} \frac{b}{a}t \; ; 0 \le t \le a \\ b \; ; t \ge a \end{cases}$$

→ ~~pei~~ $\boxed{\text{piecewise function}}$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^{a} e^{-st} f(t)\,dt + \int_a^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^{a} e^{-st} \frac{bt}{a}\,dt + \int_a^{\infty} e^{-st}\, b\,dt$$

$$= \frac{b}{a}\int_0^{a} t e^{-st}\,dt + b\int_a^{\infty} e^{-st}\,dt$$

$$\cancel{= \frac{b}{a}\Big[\quad \frac{t}{s^2}}$$

$$= \frac{b}{a}\left(\left[t \cdot -\frac{1}{s} e^{-st} \right]_0^a - \int_0^a -\frac{1}{s} e^{-st} \, dt \right) ; \quad \begin{array}{l} u = t \\ u = t \\ du = dt \end{array} \Bigg| \begin{array}{l} dv = e^{-st} \\ v = -\frac{1}{s} e^{-st} \end{array}$$

$$+ b \left[\frac{e^{-st}}{-s} \right]_a^{\infty}$$

$$= \frac{b}{a} \left(a \left(-\frac{1}{s} e^{-as} \right) \right) + \frac{1}{s^2} \left(e^{-as} - 1 \right) + \frac{b}{s} e^{-as}$$

$= \;$ Do at home

Linearity of Laplace transform:

$$L\{f(t) + g(t)\} = L\{f(t)\} + L\{g(t)\}$$

$f(t) = \underbrace{5e^{2t}}_{L\{f(t)\}} - \underbrace{3\sin(4t)}_{L\{g(t)\}}$, for $t \geq 0$

, Do @ home

LC-5(W-4) 28/04/24

Laplace Transformation

[I] Properties:

* Linearity
* Change of scale
* Time shift ; [x is time ; Domain is time]
* "Frequency" or s-plane shift; [$t \rightarrow e^{t}$ // e^{at} // whatever t]
* Multiplication by t^n
* Integration
* Differentiation

P.T.O

[1] Linearity

$$L\{c_1 f_1(t) + c_2 f(t)\} = c_1 F_1(s) + c_2 F_2(s)$$

*check prev class

*check slide

[2] Change of Scale

$\hookrightarrow L\{\sin t\} = \frac{1}{s^2+a^2}$

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$\therefore L\{\sin at\} = ?$

Proof:

$$L\{f(at)\} = \int_0^{\infty} f(at) e^{-st} dt$$

Let, $u = at$, $t = \frac{u}{a}$

$$dt = \frac{1}{a} du$$

still ∞

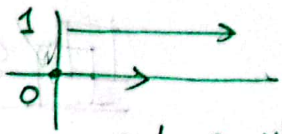
$$\therefore \frac{1}{a} \int_0^{\frac{\infty}{a}} f(u) e^{-\left(\frac{s}{a}\right)u} du$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

Example:

*fix and do slide

II Time Shift



unit step function

has only 2 outputs (0 or 1)

$$u(t_0 - t_0) = \begin{cases} 1; & t \geq 0 \\ 0; & t < 0 \end{cases}$$

$$L\{f(t-t_0)\boxed{u(t-t_0)}\} = e^{-st_0}F(s)$$

∴ this will be 0 or 1

Proof:

$$L\{f(t-t_0)\,u(t-t_0)\}$$

$$= \int_0^{\infty} f(t-t_0)\,u(t-t_0)\,e^{-st}\,dt$$

$$= \int_{t_0}^{\infty} f(t-t_0)\,e^{-st}\,dt$$

Let,

$u = t - t_0$, $t = u + t_0$

$$\therefore \int_0^{\infty - t_0} f(u)\,e^{-s(u+t_0)}\,du$$

$$= e^{-st_0}\int_0^{\infty} f(u)\,e^{-su}\,du = e^{-st_0}F(s)$$

Example

$$L\{e^{-a(t-10)}u(t-10)\}$$

$$= \frac{e^{-10s}}{s+a}; \quad L(e^{-at}) = \frac{1}{s+a}$$

S-Plane (Frequency) Shift

$$L\{e^{-at}f(t)\} = F(s+a)$$

Proof:

$L\{e^{-at}f(t)\}$

$= \int_0^\infty e^{-at} f(t) e^{-st}\,dt$

$= \int_0^\infty f(t) \cdot e^{-(s+a)t}\,dt$

$= F(s+a)$

Example:

Multiplication by t^n

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Proof:

$L\{t^n f(t)\}$

$= \int_0^\infty t^n f(t)$

Example:

$L\{t^2 \sin at\}$ * clean up at home

Let, $L\{\sin at\} = \frac{a}{s^2+a^2} = F(s)$

$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$

$= 1 \cdot \frac{d}{ds}\left\{-a(s^2+a^2)^{-2} \cdot 2s\right\}$

$= -2a\left(s^2+a^2\right)^{-2} \cdot 2(s^2+a^2)^{-3}$

$= \cancel{2}as(s^2+a^2)^{-3} \cdot 2(s^2+a^2)^{-2}$

Inverse Transformation

~~* If F(s)~~

Laplace :

$$f(t) \longrightarrow \boxed{L.T} \longrightarrow F(s)$$

Inverse :

$$f(t) \longleftarrow \boxed{I.L.T} \longleftarrow F(s)$$

$$f(t) = 1$$

$$L\{f(t)\} = L\{1\} = \frac{1}{s} = F(s)$$

$$L^{-1}\{F(s)\} = L^{-1}\left(\frac{1}{s}\right) = 1 = f(t)$$

$\cancel{L\{}$ $Y(s) =$

Linearity of Inverse

*everything that works for L.T, works for Inverse.

Example 1: $Y(s) = \frac{2}{s}$

Ans. $L^{-1}\{Y(s)\} = L^{-1}\{\frac{2}{s}\} = 2L^{-1}(\frac{1}{s})$

$= 2 \cdot 1 = 2$ ~~(Ans)~~

$\therefore y(t) = 2$

Example 2: ~~$Y(s) = \frac{2}{s}$~~ $Y(s) = \frac{3}{s-5}$

Ans. $L^{-1}\{Y(s)\} = L^{-1}\{\frac{3}{s-5}\} = 3L^{-1}\{\frac{1}{s-5}\}$

$= 3e^{5t}$

$\therefore y(t) = 3e^{5t}$ (Ans)

Example 3: $Y(s) = \frac{6}{s^4} = \frac{3!}{s^{3+1}}$

Ans. ~~$L\{t^n\}$~~ $L^{-1}\{Y(s)\} = L^{-1}\{\frac{6}{s^4}\} = L^{-1}\{\frac{3!}{s^{3+1}}\}$

$= t^3$

$\therefore y(t) = t^3$ (Ans)

Example 4: $Y(s) = \frac{8}{s^3}$

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{4\times 2!}{s^{2+1}}\right\} = 4L^{-1}\left(\frac{2!}{s^{2+1}}\right)$$

$$= 4t^2$$

$\therefore y(t) = 4t^2$

Example 5: $Y(s) = \frac{4s+1}{s^2+9}$

$\cancel{L^{-1}\{Y(s)\} = L^{-1}}$

$$Y(s) = \frac{4s+1}{s^2+9} = \frac{4s}{s^2+9} + \frac{1}{s^2+9}$$

$$= 4\left(\frac{s}{s^2+3^2}\right) + \left(\frac{3}{s^2+3^2}\right)\cdot\frac{1}{3}$$

$\therefore L^{-1}\{Y(s)\} =$

$\therefore y(t) = 4\cos 3t + \frac{1}{3}\sin 3t$

Example 6: $Y(s) = \frac{4s+1}{s^2-9}$

Example 7: $Y(s) = -\frac{10}{(s+1)^3}$

$\therefore Y(s) = -5\left(\frac{2!}{(s+1)^{2+1}}\right)$

$\therefore L^{-1}\{Y(s)\} = -5t^2e^{-t}$

C-6 (10-5)

05/05/24

Transforms of Derivative

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \quad ; \quad \left[\int uv\,dt = u\int v\,dt - \int\left\{\frac{d}{dt}(u)\int v\,dt\right\}dt\right]$$

$$= \left[e^{-st}\int f'(t)\,dt\right]_0^\infty - \int_0^\infty \left\{\frac{d}{dt}(e^{-st})\int f'(t)\,dt\right\}dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \left[e^{-st}\int f''(t)\,dt\right]_0^\infty - \int_0^\infty \left\{\frac{d}{dt}(e^{-st})\int f''(t)\,dt\right\}dt$$

$\int f''(t)\,dt$

$$= 0 - f'(0) - \int_0^\infty -se^{-st} f'(t)\,dt$$

$$= s\left(sF(s) - f(0)\right) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

$\therefore L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''$

$\therefore L\{f^n(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots f^{n-1}(0)$

Consider:

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3$$

Taking Laplace Transformation on both the sides,

$$L\{y''\} + 5L\{y'\} + 6L\{y\} = L\{0\}$$; [Linearity]

or, $\{s^2Y(s) - \cancel{s} \; sy(0) - y'(0)\}$

$+ 5\{\cancel{sF(s)} \; \cancel{s(} \; sY(s) - \cancel{s}y(0)\} + 6Y(s) = 0$

or, $\cancel{s^2} \; Y(s)\{s^2+5s+6\} - y(0)\{s-5\} - y'(0) = 0$

or, $Y(s)(\cancel{s^2+}(s+3)(s+2) - y(0)(s-5) - y'(0) = 0$

$= 2(s+5) + 3 \quad = 2s + 10 + 3$

$$\therefore Y(s) = \frac{2s - 13}{s^2 + 5s + \cancel{18}6} = \frac{2s-13}{(s+3)(s+2)}$$

Now, taking inverse Laplace,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2s+3}{(s+3)(s+2)}\right\}$$

(Thumb rule is allowed)

$$\frac{2s+13}{(s+3)(s+2)} = \frac{A}{(s+3)} + \frac{B}{(s+2)}$$

$s = -2$; $B = 9$

$s = -3$, $A = 7$, ~~$B = 9$~~

$\therefore$

$$L^{-1}\{Y(s)\} = L^{-1}\left\{-\frac{7}{s+3}\right\} + L^{-1}\left\{\frac{9}{s+2}\right\}$$

$$\therefore y(t) = -7e^{-3t} + 9e^{-2t}$$

C-5 (W-6)

12/05/24

Convolution Theorem

[#] Example:

$L\{f(t)\} = L\{1\} = \frac{1}{s}$,

$L\{g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$

$\therefore L\{f(t)g(t)\} \neq L\{$

$$L^{-1}\{H(s)\} = h(t)$$

$$= \int_0^t f(t-\tau)\,g(\tau)\,d\tau \longrightarrow \text{Convolution Integration}$$

$$= \int_0^t f(t)\,g(t-\tau)\,d\tau$$

[Proof in Slide]

Example 2:

Let, $f(t) = t$

$g(t) = \sin 2t$

$H(s) = F(s)G(s)$

$\therefore L^{-1}\{H(s)\} = h(t)$

$$= \int_0^t (t-\varphi)\sin 2\varphi \, d\varphi$$

$$= t\int_0^t \sin 2\varphi - \int_0^t \varphi \sin 2\varphi \, d\varphi$$

$$= -t \cdot \frac{\cos 2\varphi}{2}\Big|_0^t + \frac{\varphi \cos 2\varphi}{2}\Big|_0^t + \int_0^t 1\left(-\frac{\cos 2\varphi}{2}\right)d\varphi$$

$$= -t\left(\cancel{\frac{\cos 2t}{2}} - \frac{1}{2}\right) + \cancel{\frac{t\cos 2t}{2}} - \frac{\sin 2\varphi}{4}\Big|_0^t$$

$$= \frac{t}{2} - \frac{\sin 2t}{4} \text{ (Ans.)}$$

Example 3 :

$$H(s) = \frac{2}{s^2(s-2)}$$

Let,

$F(s) = \frac{2}{s^2}$ $\quad f(t) = 2t$

$G(s) = \frac{1}{s-2}$ $\quad g(t) = e^{2t}$

$$\therefore L^{-1}\{H(s)\} = \int_0^t 2(t-\tau)2e^{2\tau}d\tau$$

$$= 2\int_0^t te^{2\tau}d\tau - 2\int_0^t \tau e^{2\tau}d\tau$$

$$\cancel{= 2t\int_0^t e^{2\tau}d}$$

$$= 2t \cdot \frac{1}{2}e^{2\tau}\Big|_0^t - \cancel{2\int_0^t \tau e^{2\tau}d\tau}\ 2\left[\frac{\tau e^{2\tau}}{2}\right]_0^t$$

$$- 2\int_0^t \frac{e^{2\tau}}{2}d\tau$$

$$= te^{2t} - t - te^{2t} + \frac{e^{2\tau}}{4}\Big|_0^t$$

$$= t + \frac{e^{2t}}{2} - \frac{1}{2}$$

C-7 (10-7) 19/05/24

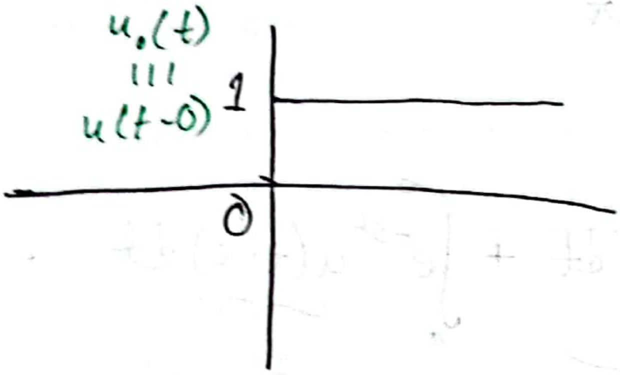
Heavside's Unit Step

* Heaviside Function:

$$u(t) = \begin{cases} 0; & t < 0 \\ 1; & t \geq 1 \end{cases}$$

* piecewise function

$$\equiv u(t) \equiv u_0(t) \equiv u(t-0)$$



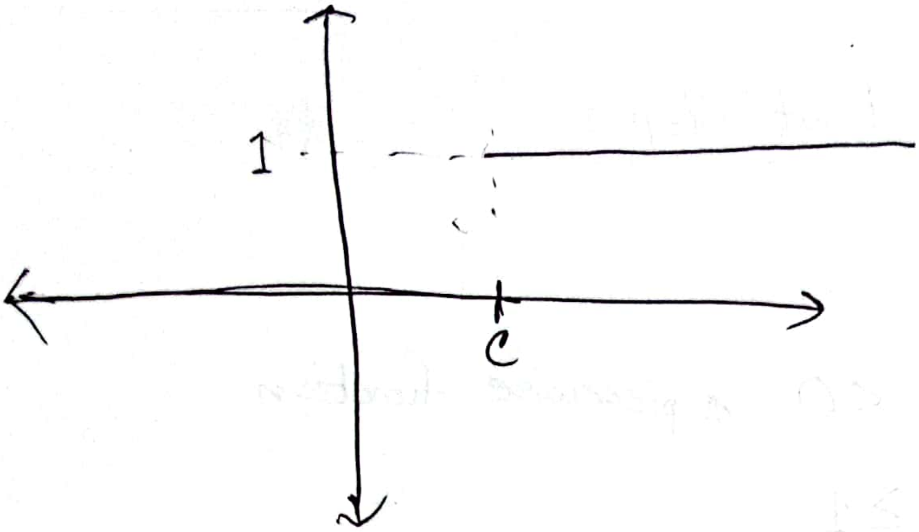
$$u(t) = \begin{cases} 0; & t < 0 \\ 1; & t \geq 0 \end{cases} = 0 \cdot \{u(t-0)\} + 1 \cdot \{u(t-0)\}$$

$$= u(t)$$

$$\therefore L\{u(t)\} = \int_0^\infty e^{-st} u(t)\, dt$$

$$= \int_0^\infty e^{-st} \cdot 1\, dt \; ; \; [u(t) = 1 \text{ when } t \geq 0]$$

$$= \frac{1}{s}$$



Here,

$$u(t) = \begin{cases} 0 ; & t \leq c \\ 1 ; & t \geq c \end{cases}$$

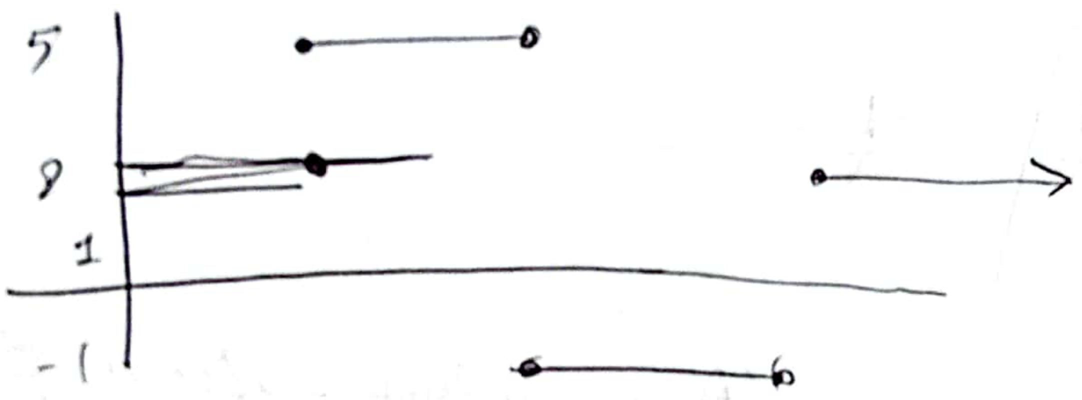
$$L\{u(t-c)\} = \int_0^\infty e^{-st} u(t-c)\,dt$$

$$= \underbrace{\int_0^c e^{-st} \underbrace{u(t-c)}_{0}\,dt}_{0} + \int_c^\infty e^{-st} \underbrace{u(t-c)}_{1}\,dt$$

$$= \int_c^\infty e^{-st}\,dt = \left[\frac{1}{s} e^{-st}\right]_c^\infty \to 0 - \left(-\frac{1}{s}e^{-ct}\right) = \frac{e^{-cs}}{s}$$

Example 4:

$$f(t) = \begin{cases} 2; & 0 \le t < 4 \\ 5; & 4 \le t < 7 \\ -1; & 7 \le t < 9 \\ 1; & t \ge 9 \end{cases} \qquad \begin{aligned} &= 2\{u(t-0) - u(t-4)\} \\ &+ 5\{u(t-4) - u(t-7)\} \\ &+ (-1)\{u(t-7) - u(t-9)\} \\ &+ 1\{u(t-9)\} \end{aligned}$$



$$\therefore f(t) = 2u(t) + 3u(t-4) - 6u(t-7) + 2u(t-9)$$

$$\therefore L\{f(t)\} = \cancel{2\int_0^\infty e^{-st} f(t)\,dt}$$

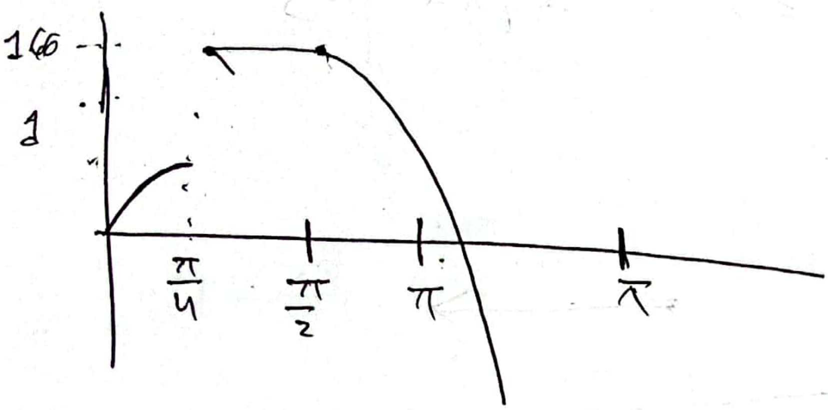
$$= 2L\{u(t)\} + 3L\{u_4(t)\} - 6L\{u_7(t)\} + 2L\{u_9(t)\}$$

$$= 2 \cdot \frac{1}{s} + 3 \cdot \frac{e^{-4t}}{s} - 6 \cdot \frac{u^{-7t}}{s} + 2 \cdot \frac{u^{-9t}}{s}$$

$$= \frac{1}{s}\left(2 + 3e^{-4t} - 6u^{-7t} + 2u^{-9t}\right)$$

Example 2:

$$f(t) = \begin{cases} \sin t \;;\; 0 \le t < \frac{\pi}{4} \;;\; \sin t\{u(t-0) - u(t-\frac{\pi}{4})\} \\ \qquad + \{\sin t + \cos(t-\frac{\pi}{4})\} \\ \sin t + \cos(t-\frac{\pi}{4}) \;;\; t \ge \frac{\pi}{4} \;;\; \{u(t-\frac{\pi}{4})\} \end{cases}$$



$\therefore$ $f(t) = u_0 t \sin t - \cancel{u_{\frac{\pi}{4}}(t)\sin t} + \cancel{u_{\frac{\pi}{4}}(t)\sin t} + u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4})$

$$= u_0(t)\sin t + u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4})$$

$\therefore L\{f(t)\} = L\{u_0(t)\sin t\} + L\{u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4})\}$

$$= L\{\sin t\} + L\{\cos(t-\frac{\pi}{4})\,u(t-\frac{\pi}{4})\}$$

time shift

$$= \frac{1}{s^2+1} + e^{-\frac{s\pi}{4}} \cdot \frac{s}{s^2+1}$$

C-8(W-8) 26/05/24

~~Complex Variable~~

Solution of DE Using LT

$2y'' + y' + 2y = g(t)$; $y(0) = 0$, $y'(0) = 0$

where,
$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t \le 20 \\ 0, & 0 \le t \le 5 \text{ and } t \ge 20 \end{cases}$$

Here,
$$g(t) = 1\{u(t-5) - u(t-20)\}$$
$$= u_5(t) - u_{20}(t)$$

$\therefore 2y'' + y' + 2y = u_5(t) - u_{20}(t)$

or, $2L\{y''\} + L\{y'\} + 2L\{y\} = L\{u_5(t)\} - L\{u_{20}(t)\}$

or, $2\{s^2Y(s) - sy(0) - y'(0)\} + \{sY(s) - y(0)\}$

$- \cancel{2y'(0)}\ 2Y(s) = \dfrac{e^{-5s}}{s} - \dfrac{e^{-20s}}{s}$

or, $Y(s)\{2s^2 + s + 2\} - y(0)(2s+1) - y'(0) = \frac{1}{s}(e^{-5s} - e^{-20s})$

or, $(2s^2 + s + 2)Y(s) = \dfrac{e^{-5s} - e^{-20s}}{s}$

$$\therefore Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}$$

Let,

$$\frac{\cancel{e^{-5s} - e^{-20s}}\ 1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$\text{or, } \cancel{e^{-5s} - e^{-20s}}\ 1 = A(2s^2+s+2) + (Bs+C)(s)$$

$$= \cancel{2As\ A.2s^2+}$$

$$= s^2(2A+B) + s(A+C) + 2A$$

$\therefore 2A = 1$ | $2A+B=0$ | $A+C=0$

$\therefore A = \frac{1}{2}$ | $\therefore B = -1$ | $\therefore C = -\frac{1}{2}$

$\therefore$ We will get,

$$\therefore Y(s) = e^{-5s} - e^{-20s}\left(\frac{1}{2s} + \frac{-s - \frac{1}{2}}{2s^2+s+2}\right)$$

$$= \left(e^{-5s} - e^{-20s}\right)\left(\frac{1}{2s} - \frac{2s+1}{4s^2+2s+4}\right)$$

$$= \frac{e^{-5s}}{2s} - \frac{e^{-20s}}{2s} - e^{-5s}\frac{2s+1}{4s^2+2s+4} + \frac{e^{-20s}(2s+1)}{4s^2+2s+4}$$

$$= \frac{1}{2s}\left(e^{-5s} - e^{-20s}\right) - \frac{e^{-5s}(2s+1)}{\{(2s)^2+2.2s.\frac{1}{2}+\frac{1}{4}\} - \frac{1}{4}+4} + \frac{e^{-20s}(2s+1)}{\{(2s)^2+2s+\frac{1}{4}\} - \frac{1}{4}+4}$$

$$= \frac{1}{2s}\left(e^{-5s} - e^{-20s}\right) - \frac{e^{-5s}(2s+1)}{}$$

* Multiple Equations - System of Equation

* Circuit related question will be in exam

$$0.01\frac{di_2}{dt} + 5i_2 + 5i_3 = 100$$

$$0.0125\frac{di_3}{dt} + 5i_2 + 5i_3 = 100$$

$\therefore \frac{d}{}$

C-11(10-9) 02/06/24

Complex Numbers System

$* \; i = \sqrt{-1}$, $z = a + ib$

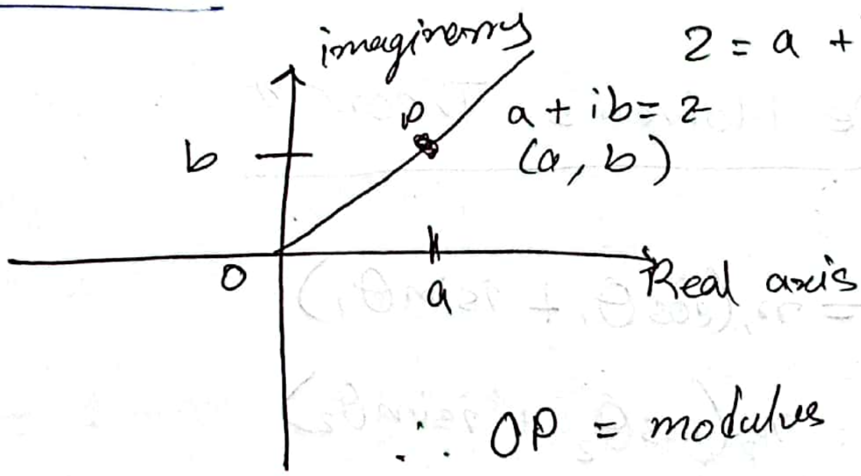
$a \rightarrow$ Real part or $Re(z)$

$ib \rightarrow$ Imaginary part $Im(z)$

* If 2 ~~imaginary~~ complex numbers are equal, their real and imaginary numbers are also equal.

* Absolute value : (distance from the [illegible])

$z = a + ib$



$\therefore$ OP = modulus

$\therefore \; OP = \sqrt{a^2 + b^2}$

$\therefore \; |a + ib| = \sqrt{a^2 + b^2}$

* De Moivre (?) Theorem:

$$(\cos\theta + i\sin\theta)^n = \cancel{\cos\theta +} \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$$

$w^n = z$

or, $w = z^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}}$

$= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

$\cos\theta = \cos(2n\pi + \theta)$

$= r^{\frac{1}{n}}\left(\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right)$; $[k = 0, 1, \dots n]$

De Moiver's Theorem

Let, $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and, $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\therefore \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

$\therefore z^n = r^n(\cos n\theta + i\sin n\theta)$

Euler's Theorem

Using Mc Laurin's Theorem,

$$\cos \cancel{\theta} x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\sin \cancel{\theta} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$z = x + iy$$

$$= r\cos\theta + ir\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

$$\therefore e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + i\left(x - \frac{x^3}{3!} + \cdots\right)$$

$$= \cos\theta - i\sin\theta$$

C-12(W-10) # Complex Function 23/06/24

* **Function:** A relation between two sets of numbers under certain condition

$$f : A \longrightarrow B \; ; \; f(x) = x + 1$$

* Model is function

* **Complex Numbers:**

A complex function is a function f whose domain and range are subsets of $\mathbb{C}$

$$f : A \longrightarrow B \; ; \; A, B \subseteq \mathbb{C}$$

* Elementary Functions :

a. Polynomial : Exponent is positive integers

$$\omega = a_0 z^n + a_1 z^{n-1} + \cdots a_n = P(z)$$

b. Rational : Numerator & Denominator must be complex (Den $\neq 0$)

$$w = \frac{P(z)}{Q(z)}$$

c. Exponential : Exponential is complex

$$a^z = e^{z \ln a}$$

d. Trigonometry : Trig function having complex

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{e^{iz} - e^{-iz}}{i\,(e^{iz} + e^{-iz})}$$

and all its friends.

e. Hyperbolic Function:

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

and all its friends.

f. Logarithmic Function:

$$w = \ln z = \ln r + i(\theta + 2k\pi) ; \; k = 0, \pm 1, \pm 2$$

g. Inverse Trigonometric Function:

$$\sin^{-1} z = \frac{1}{i} \ln \left(iz + \sqrt{1 - z^2} \right)$$

$$\cos^{-1} z = \frac{1}{i} \ln \left(z + \sqrt{z^2 - 1} \right)$$

$$\tan^{-1} z = \frac{1}{2i} \ln \left(\frac{1 + iz}{1 - iz} \right)$$

4. Inverse Hyperbolic:

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right)$$

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$\boxed{4}$ Terminology:

Distance : $|z - z_0|$

Circle : $|z - z_0| = r$

Interior of Circle : $|z - z_0| < r$

Exterior " " : $|z - z_0| > r$

Annulus : $r_1 < |z - z_0| < r_2$

Neighbourhood : All points in $|z - z_0| < r$

Boundary Point : Point not in interior and exterior point

Open Set : Does not contain boundary points.

Closed set : Contains everything inside boundary points

Connected Set : If two sets ~~have~~ can be connected by a polygonal line they are connected set.

Domain : Open and connected set

Bounded region : Region enclosed by a circle.

Compact region : Closed and bounded region.

Limit Function of Complex Variable

$f(x) \longrightarrow L$

when $x \longrightarrow L$

$$\lim_{x \to a} f(x) = L$$

$|f(x) - L| < \varepsilon$; [ε very small, positive numbers close to zero]

if, $0 < |x - a| < \delta$; [δ " " " " " "]

and, $\delta = f(\varepsilon)$

*Slide Math : DIY

+(03/07/24)

Prove that $\lim_{z \to 1+i} (2+i)z = 1+3i$; Goal : $|x - a| < \cancel{f(\varepsilon)}\ \delta$

$\lim_{x \to a} f(x) = L$; $= |z - (1+i)| < f(\varepsilon)$

Let, $|f(z) - L| < \varepsilon$

or, $|(2+i)z - (1+3i)| < \varepsilon$

or, $|2+i| \times \left| z - \frac{1+3i}{2+i} \right| < \varepsilon$; [Force common $|2+i|$]

or, $\sqrt{2^2 + 1^2} \times \left| z - \frac{(1+3i)(2-i)}{(2+i)(2-i)} \right| < \varepsilon$

or, $\sqrt{5} \times \left| z - \frac{2 - i + 6i + 3}{5} \right| < \varepsilon$

$\cancel{2}$. $\cancel{B}$

or, $\sqrt{5}\left|z - \frac{5+5i}{5}\right| < \cancel{\infty}\ \varepsilon$

$\therefore |z-(1+i)| < \frac{\varepsilon}{\sqrt{5}} = \delta$; [~~Anything~~ Any function of ε is δ]

(Proved)

C-14 (W-11) 03/07/24

* Continuity In Terms of Imaginary Numbers

* A real function is continuous when,

i) $f(x)$ at $x=a$ exists

ii) $\lim_{x \to a} f(x)$ existists

iii) $\lim_{x \to a} f(x) = f(a)$

* A complex number is continuous when,

$|f(z) - f(z_0)| < \varepsilon$ and $|z - z_0| < \delta$

$\therefore$ ~~f(z) - f(z0~~ $f(z)$ is continuous at z_0 if,

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Show that the function $f(z)$ defined by,

$$f(z) = \begin{cases} \frac{Re(z)}{z} ; & z \neq 0 \\ 0 ; & z = 0 \end{cases}$$ is not continuous at $z=0$

Here,

~~f(z) =~~ At $z=0$, $f(z)$ exists

For limit,

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{Re(z)}{z}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{x + iy}$$

~~Now~~ Along Y-axis,

$$\lim_{y \to 0} \left| \lim_{x \to 0} \frac{x}{x + iy} \right| = \lim_{y \to 0} \frac{0}{0 + iy}$$

$$= \lim_{y \to 0} 0$$

$$= 0$$

~~Again~~ Along X-axis,

$$\lim_{x \to 0} \left| \lim_{y \to 0} \frac{x}{x + iy} \right| = \lim_{x \to 0} \frac{x}{x + 0}$$

$$= \lim_{x \to 0} \frac{x}{x}$$

$$= 1$$

$\because$ The function has different value at $z=0$,

$\therefore$ The function ~~is not differentiable~~ does not have limit at $z = 0$

$\therefore$ " " " " continuous at $z = 0$.

(~~Proved~~ Showed)

Discuss the continuity of $f(z)$ at origin,

$$f(z) = \begin{cases} \frac{\bar{z}}{z} ; & z \neq 0 \\ 0 ; & z = 0 \end{cases}$$

Here,

$f(z)$ exists at $z=0$

$\therefore$ ~~Along X-axis~~, Now,

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - iy}{x + iy}$$

$\therefore$ Along X-axis,

$$\lim_{x \to 0} \left| \lim_{y \to 0} \frac{x - iy}{x + iy} \right| = \lim_{x \to 0} \frac{x}{x} = 1$$

$\therefore$ Along Y-axis,

$$\lim_{y \to 0} \left| \lim_{x \to 0} \frac{x - iy}{x + iy} \right| = \lim_{y \to 0} \frac{-iy}{iy} = -1$$

$\because$ The function has different values at the origin,

$\therefore$ The function does not have limit at the origin

$\therefore$ The function ~~does not~~ is not continuous at origin.

Example 4: (a) Prove that $f(z)=z^2$ is continuous at z_0

(b) Prove that, $f(z)=\begin{cases} z^2 \cancel{\neq z_0}, & z\neq z_0 \\ 0, & z=z_0 \end{cases}$, where $z_0\neq 0$, is discontinuous at $z=z_0$

~~(a)~~

(a) (ε, δ)

$\vdots$

$$|z^2 - z_0^2| < \varepsilon$$

or, $|z+z_0||z-z_0| < \varepsilon$

or, $\cancel{x+x_0}\ \sqrt{(x+x_0)^2+(y+y_0)^2}\cdot \dfrac{|z-z_0|}{\cancel{\sqrt{(x-x_0)^2+(y-y_0)^2}}} < \varepsilon$

$\therefore |z-z_0| < \dfrac{\varepsilon}{\sqrt{(x+x_0)^2+(y+y_0)^2}} = \delta$

$\therefore f(z)$ is continuous at $z=z_0$.

Ex-5.

$$f(z) = \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$$

$$= \frac{(3z^4 - 2z^3 + 8z^2 - 2z + 5)(z+i)}{(z-i)(z+i)}$$

$$= \frac{(3z^4 - 2z^3 + 8z^2 - 2z + 5)(z+i)}{z^2 + 1}$$

$$\therefore \cancel{f(i) = (3 + 2i + 8 - 2i + 5)(2i)}$$

$$= \frac{3z^5 - 2z^4 + 8z^3 - 2z^2 + 5z + 3z^4 i - 2z^3 i + 8z^2 i - 2zi + 5i}{z^2 + 1}$$

Do at home

$$= \frac{(z^2+1)(3z^3}{}$$

$$= \frac{3z^5 - 2z^4 + 8z^3 - 2z^2 + 5z + 3z^4 \cdot i - 2z^3 \cdot i + 8z^2 \cdot i - 2z \, i + 5i}{z^2 + 1}$$

$$=$$

$$= \frac{\cancel{(z^2+1)}(3z}{\cancel{(z^2+1)}}$$

$$= \frac{3z^5 + 3z^3 + 5z^3 + 5z - 2z^4 - 2z^2 + 3z^4 \cdot i + 3z^2 \cdot i + 5z^2 i + 5i}{(z^2+1)}$$

$$= \frac{(z^2+1)3z^3 + (z^2+1)5z + (z^2+1)(-2z) + (z^2+1)(3z^2 i) + (z^2+i + (z^2+}{z^2+1}$$

Complex Functions ~~at~~ as Mapping

[Check Slide + Book]

$z^2 = 4ay$

$$= \frac{(z^2+1)(3z^3+5z-2z^2+3iz^2-2iz+5i)}{(z^2+1)}$$

$$= 3z^3+5z-2z^2+3iz^2-2z+5i$$

$$= (z+i)(3z^2 - 2z+5)$$

$$\therefore \lim_{z\to i} f(z) = \lim_{z\to i}(z+i)(3z^2-2z+5)$$

$$= 2i(-3-2i+5)$$

$$= 4-4i$$

$\therefore$ The limit exists at $z = i$

$\therefore$ The $f(z)$ is continuous at $z = i$.

(Ans)

$(\ iy)^2$

$= 1 + 2iy - y^2$

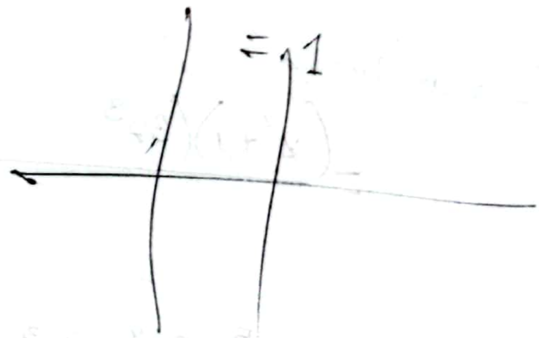
$= (1-y^2) + (2y)\cdot i$

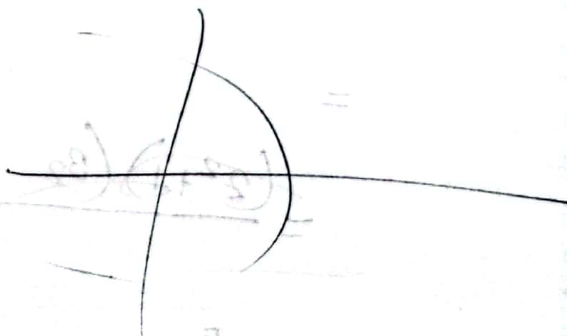
$(z, f(z))$

$v = 2y \quad \therefore y = \frac{v}{2}$

$\therefore u = 1 - \frac{v^2}{4}$

$= 1$



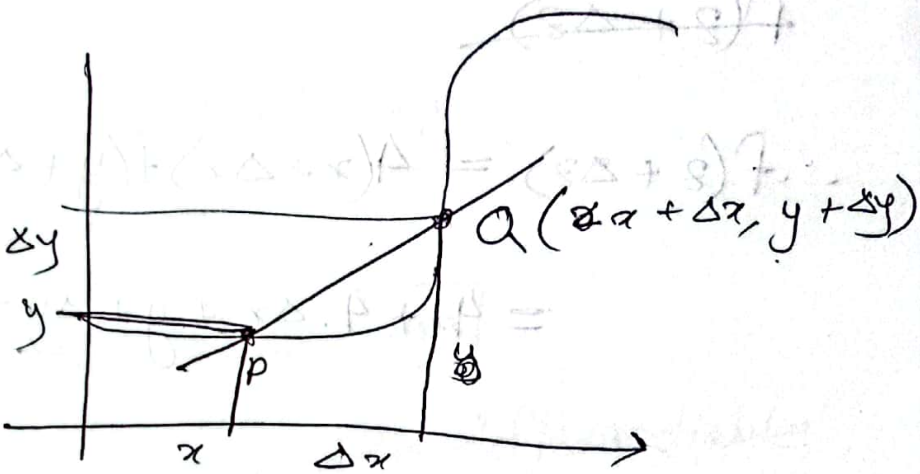


C-15 (W-12) 07/07/24

Analytic Function

* If a function is differentiable at any point, then the function will be analytic at that point.

$y = f(x)$



$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\therefore \omega + \Delta\omega = f(z + \Delta z) = f(z + z_0)$

$\therefore \Delta\omega = f(z + z_0) - f(z_0)$

$\therefore \frac{\Delta\omega}{\Delta z} = \frac{f(z + z_0) - f(z_0)}{\Delta z}$

$\therefore \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + z_0) - f(z_0)}{\Delta z}$

variation

$\therefore$

$f(z) = 4x + y + i(4y - x)$; discuss $\frac{df}{d(z)}$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

~~$f(z+\Delta z)$~~

$$\therefore f(z+\Delta z) = 4(x+\Delta x) + (y+\Delta y) + i\{4(y+\Delta y) - (x+\Delta x)\}$$

$$= 4x + 4.\Delta x + y + \Delta y + i(4y + 4\Delta y - x - \Delta x)$$

$$\therefore f(z+\Delta z) - f(z) = 4.\Delta x + \Delta y + i(4\Delta y - \Delta x)$$

$$\therefore \frac{df}{dz} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{4\Delta x + \Delta y + i(4\Delta y - \Delta x)}{\Delta x + i\Delta y}$$

Along X-axis, (put $\Delta y = 0$)

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{4\Delta x - i\Delta x}{\Delta x} = 4 - i$$

Along Y-axis, (put $\Delta x = 0$)

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{\Delta y + 4i\Delta y}{i\Delta y} = \frac{1+4i}{i}$$

$$= \frac{i-4}{-1}$$

$$= 4 - i$$

~~Taki~~ Along $y = mx$,

$$\lim_{\cancel{y \to mx}} \frac{df}{dz} = \lim_{\Delta x \to 0} \frac{4\Delta x + m\Delta x + i(4m\Delta x - \Delta x)}{\Delta x + i \cancel{\Delta m} m \Delta x}; \quad [\Delta y = m\Delta x]$$

$$= \lim_{\Delta x \to 0} \frac{4 + m + i(4m - 1)}{1 + im}$$

$$= \cancel{\lim} \frac{4(1 + im) - i(1 + im)}{1 + im}$$

$$= 4 - i; \quad [\text{free from } m]$$

~~∴ It is not dependent on m, it is differentiable~~

~~∴ It is~~

∴ Along all different path $\frac{df}{dz} = (4 - i)$ which is unique.

∴ $f(z)$ is differentiable at point z

$f(z) = \begin{cases} \dfrac{x^3 y(y-ix)}{x^6+y^2} ; & z \neq 0 \\ 0 ; & z = 0 \end{cases}$, then discuss $\dfrac{df}{dz}$ at $z=0$

$$\therefore \frac{df}{dz} = \lim_{z \to 0} \frac{f(z+z_0) - f(z_0)}{z} \longrightarrow \frac{f(0+z) - \cancel{f(0)}\ 0}{x+iy}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\frac{x^3 y(y-ix)}{x^6+y^2} - 0}{x+iy}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y(y-ix)}{(x^6+y^2)(x+iy)}$$

Along $y = mx$,

$$\lim_{x \to 0} \frac{x^3(mx)(mx-ix)}{(x^6+m^2x^2)(x+imx)} = \lim_{x \to 0} \frac{mx^2(m-i)}{(1+im)(x^4+m^2)}$$

$$= 0$$

$$= \lim_{x \to 0} \frac{(mx^4)(mx - ix)}{(x^7 + imx^7 + m^2x^3 + im^2x^3)}$$

$$= \lim_{x \to 0} \frac{m^2x^4 - imx^5}{x^7 + imx^7 + m^2x^3 + im^2x^3}$$

Along $y = x^3$,

$$\lim_{x \to 0} \frac{x^3 \cdot x^3(x^3 - ix)}{(x^6 + x^6)(x + ix^3)} = \lim_{x \to 0} \frac{x^6 (x^3 - ix)}{\cancel{(2x^6)}(x + ix^3)}$$

$$= \lim_{x \to 0} \frac{x^2 - i}{2(1 + ix^2)} = \frac{-i}{2}$$

$\therefore$ Along different axis, value is different

$\therefore$ It is not differential

C-16 (W-12)

10/07/24

Analytic at a Point

and neighbourhood → left limit, right limit exists

Def^n: If differentiable at z_0, it is analytic at that point.

If every point ~~at n~~ is differentiable, it is analytic at a domain

Holomorphic: Within a ~~fu~~ domain if a function is analytic, it is holomorphic (synonym of analytic)

Entire Function: Analytic at every point z in complex function

Theorem:

(i) A polynomial function $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is an ~~ϕ~~ entire function where n is a non negative integer,

(ii) A rational function $f(z) = \dfrac{p(z)}{q(z)}$ is analytic in any domain contain no point where $p(z_0) = 0$

Singular Point: If a function is not analytic at ~~a~~ any point, that point is a singular point.

L'Hôpital :

For any function: $\cancel{\lim_{z \to z_0} \frac{f(z_0)}{g(z_0)} = \frac{f'(z_0)}{g'(z_0)}}$

where $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$,

then, $\lim_{z \to z_0} \frac{f(z_0)}{g(z_0)} = \frac{f'(z_0)}{g'(z_0)}$

\# $f(z) = z^2 - 4z + 5$

$g(z) = z^3 - z - 10i$

$\therefore f(2+i) = (2+i)^2 - 4(2+i) + 5$

$= \cancel{4}\ 3 - 2i - 8 - \cancel{8}\ 4i + 5$

$= 0$

$\therefore g(2+i) = (2+i)^3 - (2+i) - 10i$

$= 0$

$\therefore \lim_{z \to 2+i} \frac{f(2+i)}{g(2+i)} = \frac{0}{0}$

$\therefore \lim_{z \to 2+i} \frac{f(2+i)}{g(2+i)} = \frac{f'(2+i)}{g'(2+i)}$

$= \frac{2i}{9 - 6i}$

$\therefore f'(z) = 2z - 4$

$\therefore f'(2+i) = 4 + 2i - 4$

$= 2i$

$\therefore g'(z) = \cancel{3z}\ 3z^2 - 1$

$= 3(2+i)^2 - 1$

$\cancel{= 8 - 6i \neq 0}$

$= 8 -$

27. $f(z) = \frac{iz^2 - 2z}{3z+1-i}$

Given that,

$$f(z) = \frac{z^2 i - 2z}{3z + 1 - i}$$

If, $3z + 1 - i = 0$,

$$\therefore z = \frac{i-1}{3}$$

$\therefore \frac{i-1}{3}$ is a singular point of $f(z)$

$\therefore$ at $z = \frac{i-1}{3}$, $f(z)$ is not analytic.

Cauchy-Riemann Equation

If ~~f~~ $f(z) = u(x,y) + iv(x,y)$ and at differentiable at a point $z = x+iy$, then at z the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

+ Check proof & verification

∴ If equation does not satisfy C-R equations, then the function is not analytic.

Necessary Condition for C-R:

$f(z)$ is analytic if

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial u}$ (ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, provided $\frac{\partial u}{\partial x} \cdots \frac{\partial v}{\partial y}$ exist

Sufficient:

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ continuous

C-17 (W-13) 14/07/24

C-R in Polar Form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \; ; \; \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Show that $w = e^z$ is analytic in complex plane. Hence find $\frac{dw}{dz}$.

$u = e^x \cos y \; , \; v = e^x \sin y$

$\therefore \frac{\partial u}{\partial x} = \cancel{e^x \cos y + e^x \sin y} \; e^x \cos y$

$\therefore \frac{\partial u}{\partial y} = -e^x \sin y$

$\therefore \frac{\partial v}{\partial x} = e^x \sin y$

$\therefore \frac{\partial v}{\partial y} = e^x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ CR equation is satisfied

$\therefore$ w is an analytic function.

$$\therefore \frac{dw}{dz} = \underbrace{\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}}_{\text{with respect to } x \text{ and } y}$$

$w = e^z$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x . e^{iy}$$

$$= e^{x+iy}$$

$$= e^z$$

Harmonic Function:

If $f(z) = u + iv$

where,

$f(z)$ is analytic

and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$,

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$,

then $f(z)$ is called harmonic.

Laplace Equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

∴ If ~~f(z) i~~ $f(z)$ is analytic and harmonic, then u and v are harmonic conjugate of each other.

$u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic but not conjugate harmonic.

$\therefore \frac{\partial u}{\partial x} = 2x$

$\therefore \frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y$

$\therefore \frac{\partial^2 u}{\partial y^2} = -2$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial v}{\partial x} = \frac{y}{x^2+y^2}$

$= \cancel{\frac{(x^2+y^2)\cdot 0 - y\, 2x}{}}$

$= \frac{-(2xy)}{(x^2+y^2)^2}$

$\therefore \frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(-2y) - 2(-2xy)(x^2+y^2)^{2}}{(x^2+y^2)^4}$

$$\frac{\partial v}{\partial y} = \cancel{\frac{2xy}{(x^2+y^2)^2}}$$

$$\therefore \cancel{\frac{\partial^2 v}{\partial y^2} = \frac{}{(x^2+y^2)}}$$

$$\frac{\partial v}{\partial y} = \frac{x^2+y^2 - 2y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2 \cdot (-2y) - 2(x^2-y^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\cancel{(x^2+y^2) \cdot (-2y)} - 4(x^2+y^2)(x^2-y^2)y - \cancel{(x^2+y^2)(2y)} + 4x(x^2+y^2)}{(x^2+y^2)^4}$$

$$\vdots$$

$$= 0$$

$\therefore$ They are harmonic

Here,

$$\frac{\partial u}{\partial x} = 2x \neq \frac{\partial v}{\partial y}$$

$\therefore$ ~~f(z) is~~ They are not conjugate harmonics.

Milne Thompson Method

$z = x + iy$, $\bar{z} = x - iy$

$\therefore x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$

$\therefore f(z) = u(x, y) + iv(x, y)$

$= u\left(\dfrac{z + \bar{z}}{2}, \dfrac{z - \bar{z}}{2i}\right) + iv\left(\dfrac{z + \bar{z}}{2}, \dfrac{z + \bar{z}}{2i}\right)$

If $z = \bar{z}$,

then $f(z) = u(z, 0) + iv(z, 0)$; [this follows CR]

Case-1: If u is given,

step 1 : Find $\dfrac{\partial u}{\partial x}$ and equate it to $\phi_1(x, y)$

" 2 : Find $\dfrac{\partial u}{\partial y}$ " " " " $\phi_2(x, y)$

" 3 : Replace x with z and y with 0 to get $\phi_1(z, 0)$

" 4 : " ~~y~~ " z " y " 0 " " $\phi_2(z, 0)$

" 5 : Find $f(z)$ by the formula $f(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\} dz$

Case 2: When v ~~x~~ is given.

Step 1: Find $\frac{\partial v}{\partial x}$ and equate it to $\psi_2(x,y)$

" 2: " $\frac{\partial v}{\partial y}$ " " " " $\psi_1(x,y)$

" 3: Relpace x with z and y with 0 to get $\psi_1(z,0)$

" 4: " " " " $\psi_2(z,0)$

" 5: Find $f(z)$ using $f(z) = \int \{\psi_1(z,0) + i\,\psi_2(z,0)\}dz + C$

#Slide Math

C-18(10-13) 15/09/24)

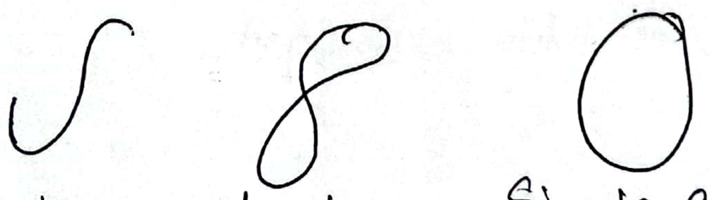
Definite Integral

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k \to R$$

$y = f(x)$, $x = x(t)$ → Parameter
(independent variable of a dependent variable)

→ t is a parameter of y $\boxed{\text{NOT } x}$



Simple Closed Simple Closed

→ Curve

(a) $\int_c xy^2\,dx$ (b) $\int_c xy^2\,dy$ (c) $\int xy^2\,ds$

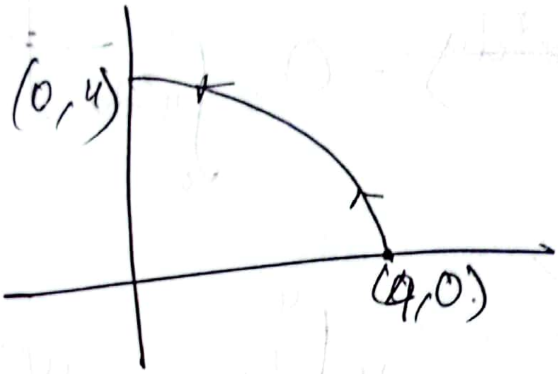
where C is a quarter circle defined by

$x = 4\cos t$, $y = 4\sin t$; $0 \le t \le \frac{\pi}{2}$

$x = 4\cos t$

$y = 4\sin t$

$\therefore x^2 + y^2 = \cancel{a}\ 16$



(a) $\int_C xy^2\,dx$ | $x = 4\cos t$, $\therefore dx = -4\sin t\,dt$

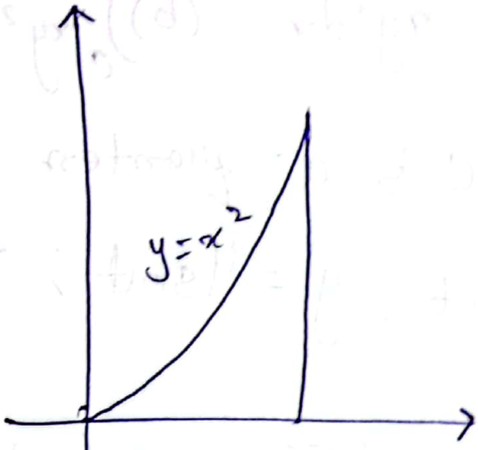
$$= \int_0^{\frac{\pi}{2}} (4\cos t)(4\sin t)^2 (-4\sin t\,dt)$$

$$= \int_0^{\frac{\pi}{2}} 4\cos t\,(16\sin^2 t)(-4\sin t\,dt)$$

$$= -\int_0^{\frac{\pi}{2}} 256\sin^3 t\cos t\,dt = -\int_0^{\frac{\pi}{2}} 256\sin^3 t \times \cancel{\cos t} \times \frac{d(\sin t)}{\cancel{\cos t}}$$

$$= -256\left[\frac{1}{4}\sin^4 t\right]_0^{\frac{\pi}{2}}$$

$\oint_c y^2dx - x^2dy$



Along x-axis,

$$I_1 = \int_0^2 y^2dx - \cancel{x^2dy} - 0 = \int_0^2 y^2dx = 0$$

Along $x=2$,

$$I_2 \int_0^4 -x^2dy = -4\int_0^4 dy = -4[y]_0^4 = -16$$

Along $y=x^2$,

$$\int_4^0 y^2dx - x^2dy$$

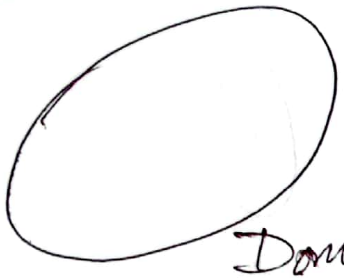
$y = x^2$, $\therefore y^2 = x^4$ | $y = x^2$, $\therefore dy = 2x\,dx$

$$= \int_4^0 x^4dx - x^2(2x\,dx)$$

C-20 (W-14) 22/09/24

Cauchy Integral Theorem and Cauchy Integral Theorem Formula

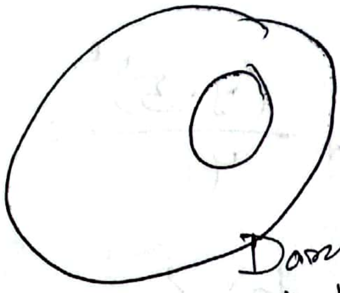
Simply:



Domain

~~Fully~~ has no holes

Multiply:



Domain with holes (undefined)

$c \rightarrow$ Curve

Cauchy theorem: $\oint_c f(z)\,dz = 0$

* Contour: Closed curve

(i) Function f is analytic

(ii) " " " simply

(iii) f' is continuous in D then

↳ update: Cauchy Goursat said (iii) is not necessary

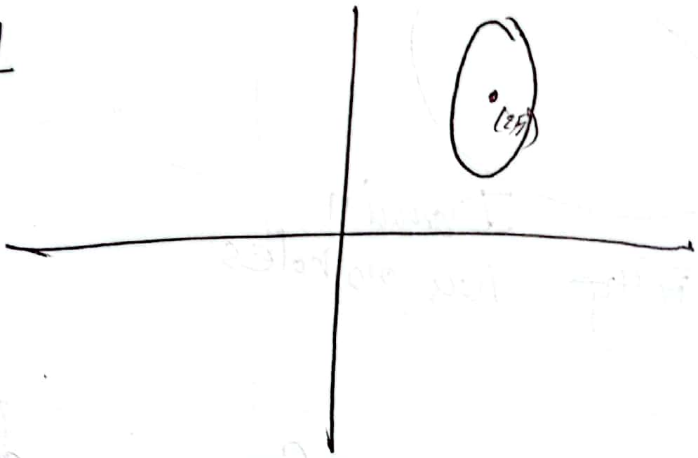
\# $\oint_C \frac{dz}{z^2}$ when ~~centre~~ contours C is ellipse $(x-2)^2 + \frac{(y-5)^2}{4} = 1$

Here, $f(z) = \frac{1}{z^2}$

$\therefore$ At $z=0$, $f(z) = \cancel{\frac{1}{0^2}}$. is not analytic

Now,

$$\frac{(x-2)^2}{1} + \frac{(y-5)^2}{4} = 1$$



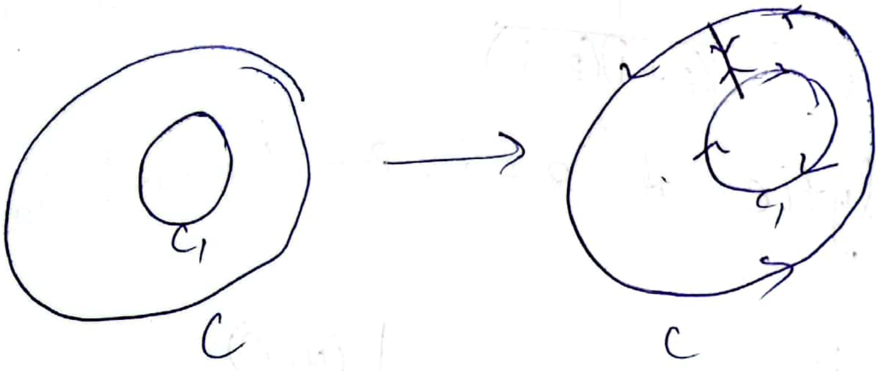
$\therefore a = 1,\ b = 2$

$\therefore b > a$

$\therefore$ ~~z=~~ $f(z)$ is analytic in the curve

$$\therefore \oint_C \frac{dz}{z^2} = 0 \quad \text{(Ans)}$$

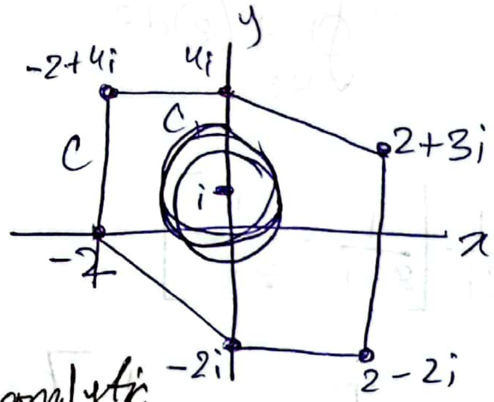
Cauchy - Goursat Theorem for multiply connected



$$\therefore \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz \left[+ \oint_{C_2} f(z)\,dz \;\cdots\cdots \right]$$

if exists

$\oint_C \frac{dz}{z-i}$ where C is



$$f(z) = \frac{1}{z-i}$$

at $z = i$, $f(z)$ ~~of~~ is not analytic

$= (0, 1)$

$|z-i| = 1$

or, $z = i + e^{i\theta}$

or, $dz = i e^{i\theta} d\theta$

$$\therefore \oint_C \frac{dz}{z-i} = \oint_{C_1} f(z)\,dz$$

$$= \oint_{C_1} \frac{i e^{i\theta} d\theta}{z-i}$$

$$= \oint_0^{2\pi} \frac{i e^{i\theta}}{e^{i\theta}} d\theta = i \oint_0^{2\pi} d\theta = 2\pi i$$

(Ans)

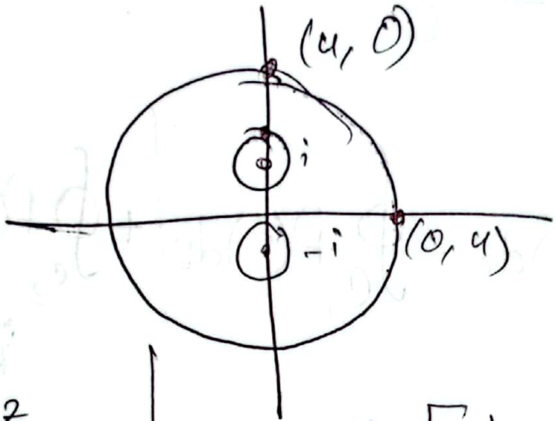
CamScanner

Example 3:

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$\therefore$ Not analytic at $z = i$, $z = -i$

Here, $|z| = 4$ is



$$\therefore \oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$

$$f(z) = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$= \oint_{C_1} \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right] + \oint_{C_2} \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$= \oint_{C_1} \frac{1}{2i} \times \frac{1}{z-i} - \oint_{C_1} \frac{1}{2i} \times \frac{1}{z+i} + \oint_{C_2} \frac{1}{2i} \times \frac{1}{z-i} - \oint_{C_2} \frac{1}{2i} \times \frac{1}{z+i}$$

$$= 2\pi i - 0 + 0 - 2\pi i$$

$$= 0 \text{ (Ans)}$$

Cauchy's Integral Formula:

1st theorem: $f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$

point where not analytic

2nd theorem: $f^n(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz$