CTFT and Sampling

EGC 113

Duality

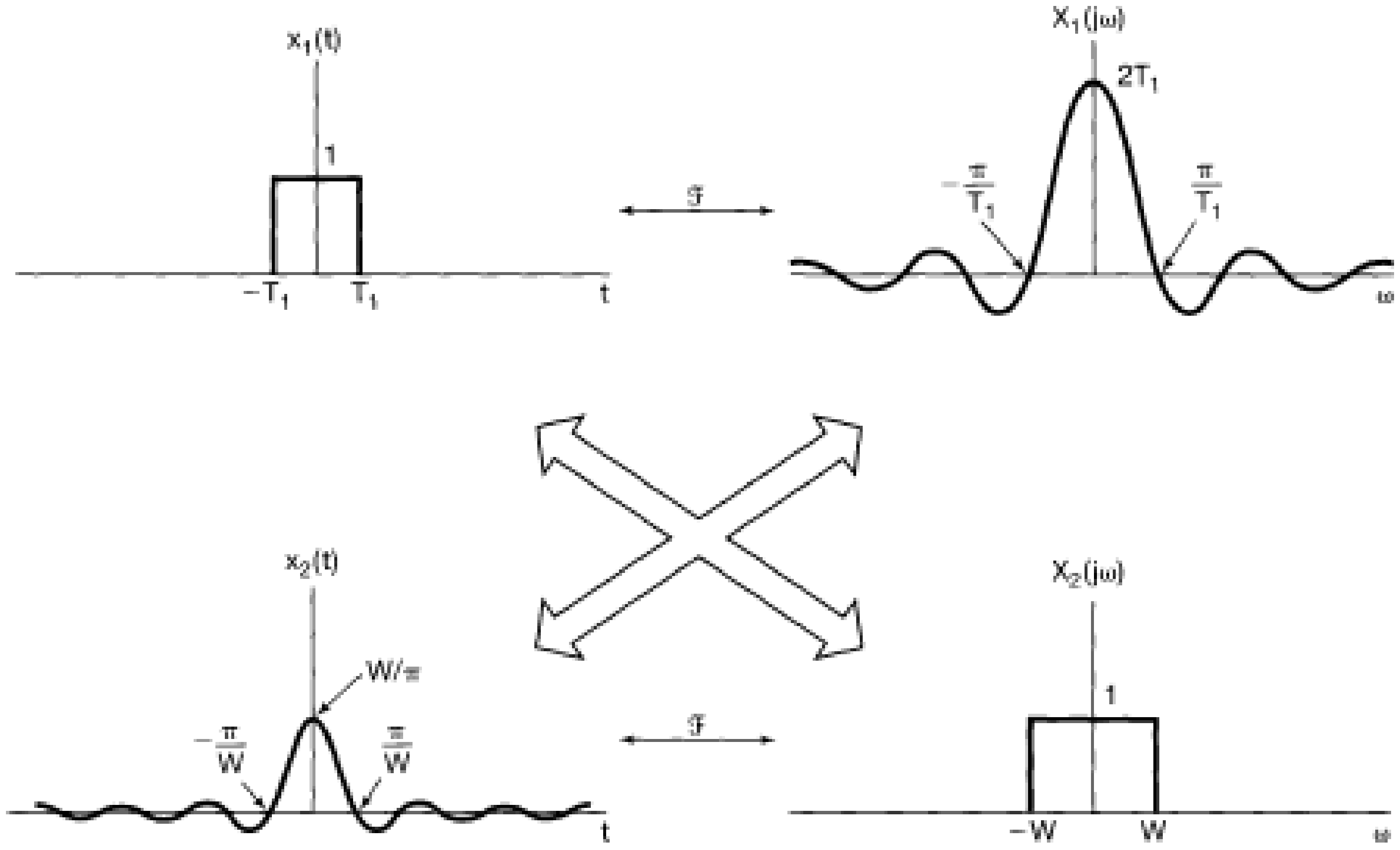


Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

Frequency response of continuous-time LTI systems

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad y(t) = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \qquad y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) X(\omega) e^{j\omega t} d\omega$$

- LTI systems characterized by differential equations:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Examples of such systems: Filters

Example

- Consider a stable LTI system defined by differential equation below. Determine the impulse response h(t).
- $\frac{dy(t)}{dt} + ay(t) = x(t)$
- $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$

Example

4.14. Consider a signal $x(t)$ with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

1. $x(t)$ is real and nonnegative.
2. $\mathcal{F}^{-1}\{(1 + j\omega)X(j\omega)\} = Ae^{-2t}u(t)$, where A is independent of t.
3. $\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = 2\pi$.

Determine a closed-form expression for $x(t)$.

4.15. Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

1. $x(t)$ is real.
2. $x(t) = 0$ for $t \leq 0$.
3. $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{R}e\{X(j\omega)\}e^{j\omega t} \, d\omega = |t|e^{-|t|}$.

Determine a closed-form expression for $x(t)$.

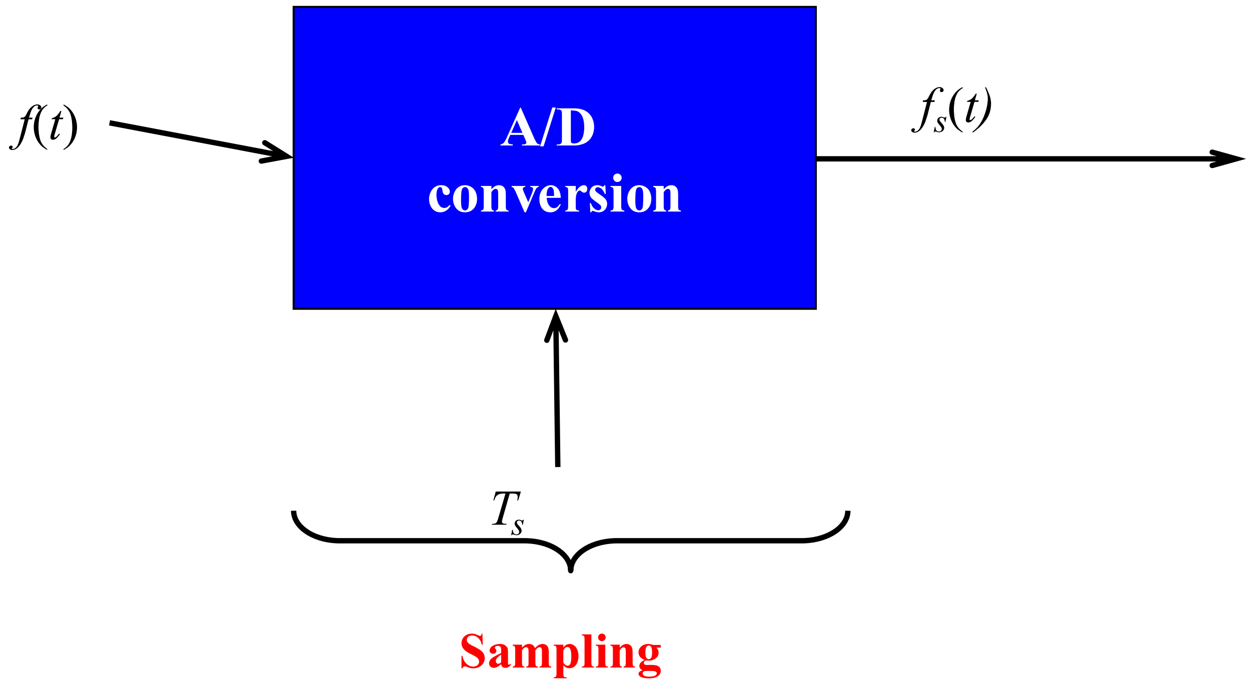
Contents

Sampling Theory

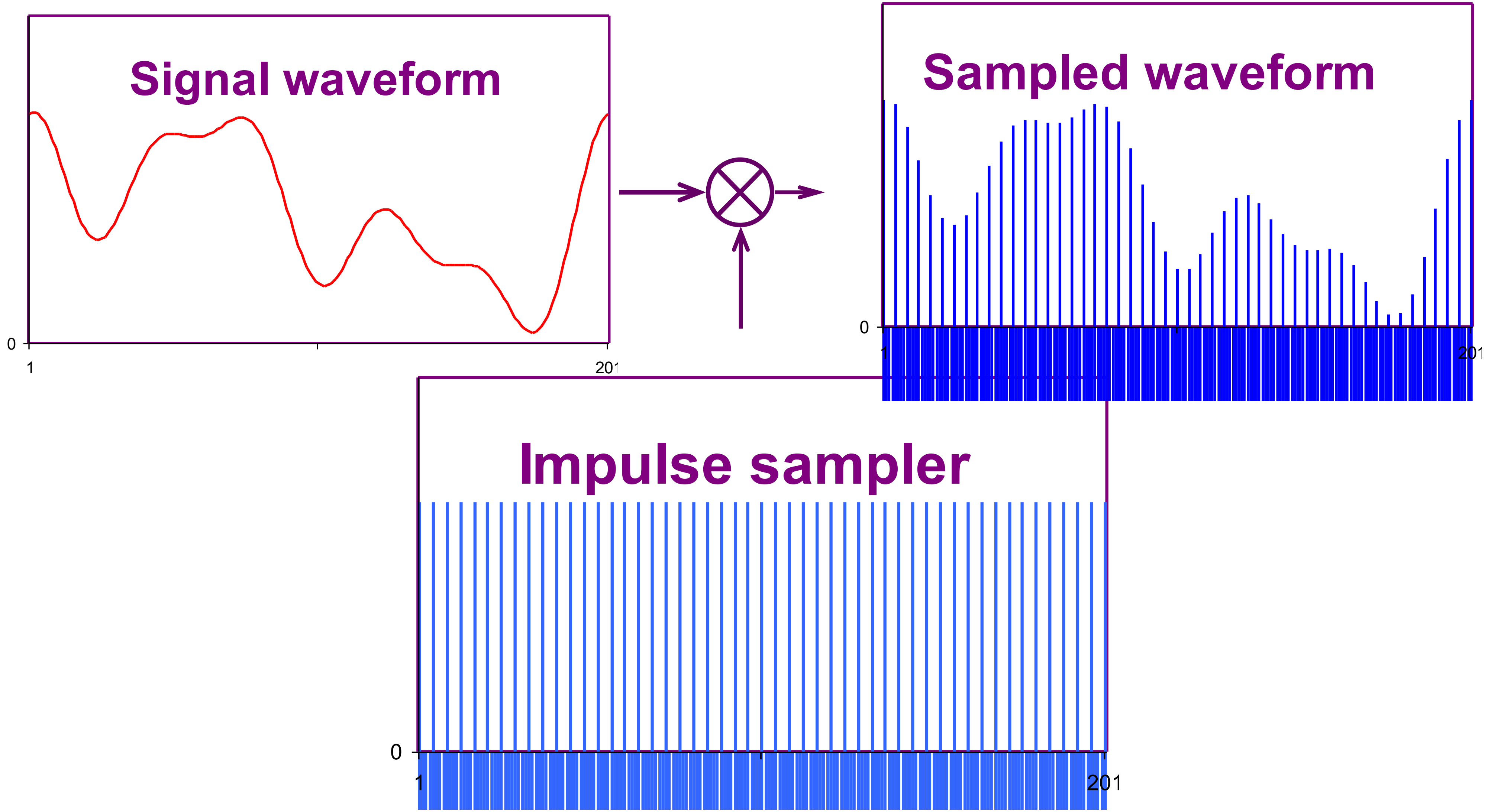
- In many applications it is useful to represent a signal in terms of sample values taken at appropriately spaced intervals.
- The signal can be reconstructed from the sampled waveform by passing it through an ideal low pass filter.
- In order to ensure a faithful reconstruction, the original signal must be sampled at an appropriate rate as described in the ***sampling theorem***.
  - A real-valued band-limited signal having no spectral components above a frequency of f_{max} Hz is determined uniquely by its values at uniform intervals spaced no greater than $T_s = \frac{1}{2f_{max}}$ seconds apart.

Sampling Block Diagram

- Consider a band-limited signal f(t) having no spectral component above f_{max} Hz.

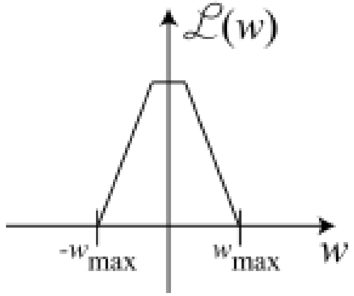


Impulse Sampling



Sampling Theorem

- Let's start by considering a band-limited signal $l(t)$. A band-limited signal is a signal with no spectral content above a frequency ω_{max}.



- Let's denote $l(t)$as the continuous signal and its sampled version $l[n] = l(nT_s)$ is the discrete signal obtained by sampling. It is clear that by sampling some information will be lost. If no information is lost, we should be able to recover the original signal from its sampled version by doing some sort of interpolation.

- We are interested in knowing if that is true and if yes, what are the conditions for that.

Sampling Theorem

- The sampling theorem states that for a signal to be perfectly reconstructed from a set of samples, the sampling frequency $\omega_s = {}^{2\pi}/_{T_s}$ must be $\omega_s > \omega_{\max}$, where $\omega_{\max}$ is the maximum frequency (also referred to as bandwidth) present in the input signal.

- The same theorem can be stated in terms of periods: the sampling period, $\mathrm{T_s}$, has to be
- $\mathrm{T_s} < \mathrm{T_{min}}/2$, i.e., smaller than half of the period of the highest frequency component, $\mathrm{T_{min}} = {}^{2\pi}/_{\omega_{\max}}$.

- We are interested in determining how the Fourier transform of $l(t)$ is related to the Fourier transform of $l[nT_s]$.

Sampling Theorem

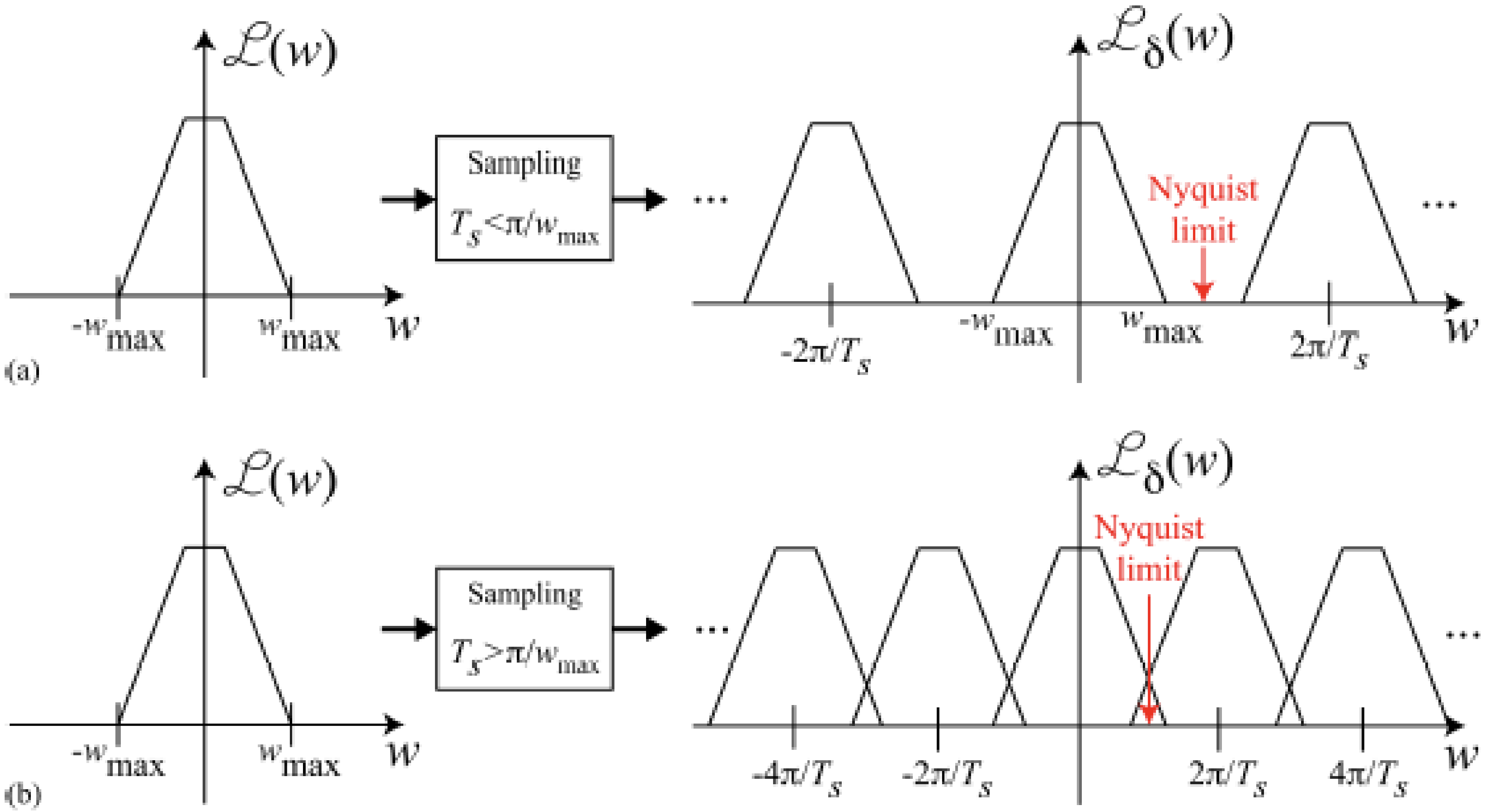
- Let's start writing a model of the sampling process by revisiting the **impulse train** or **Dirac comb**. The delta train, $\delta_{T_s}(t)$, is a periodic signal, with period T_s, composed of delta impulses centered at times nT_s. It is defined as:
- $\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$
- Sampled version of $l(t)$ can be written as,
- $l_\delta(t) = l(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} l_s[n]\delta(t - nT_s)$

- CTFT of $l_\delta(t)$ is written as,
- $L_\delta(\omega) = \int_{-\infty}^{\infty} l_\delta(t) e^{-j\omega t} dt$

Sampling Theorem

- Fourier transform of the impulse trains is:
- $\Delta_{T_s}(\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T_s}\right)$

- Multiplication in time domain is convolution in frequency domain:
- $L_\delta(\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} L\left(\omega - \frac{2\pi k}{T_s}\right)$
- It can be seen that $L_\delta(\omega)$ is built as an infinite sum of translated copies of $\mathrm{L}(\omega)$ and each copy is centered at $\frac{2\pi k}{T_s}$. For small value of T_s, the copies will be far enough from each other. As T_s starts increasing the copies will get closer and high frequency content will start affecting low frequency components, resulting in aliasing.

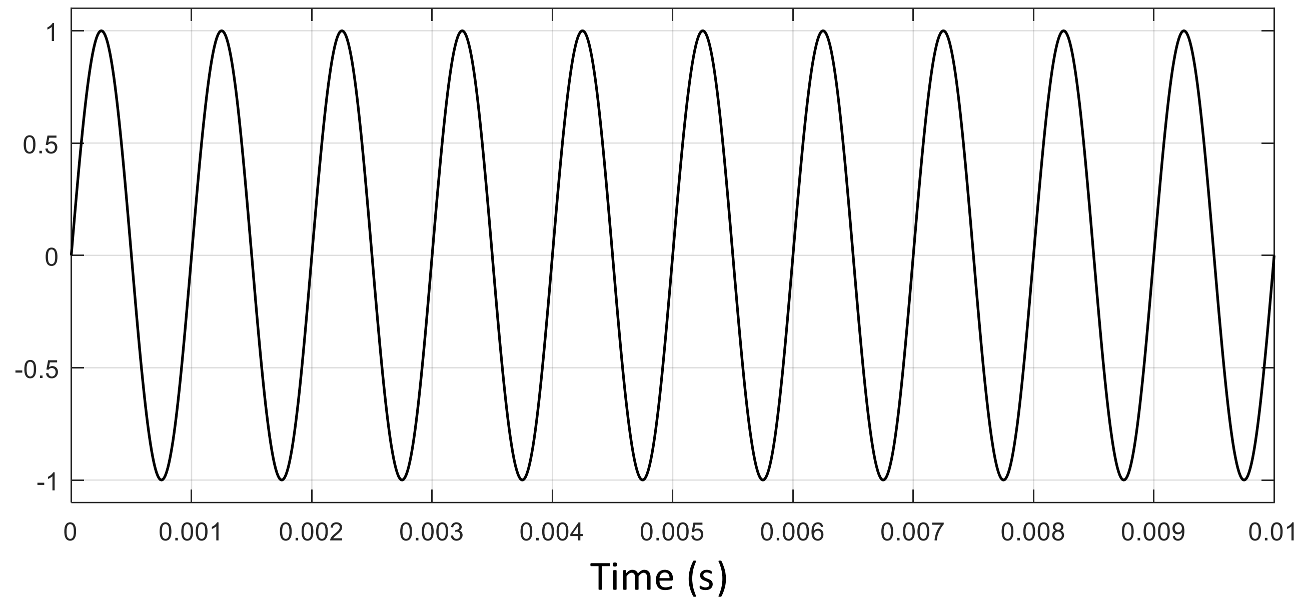


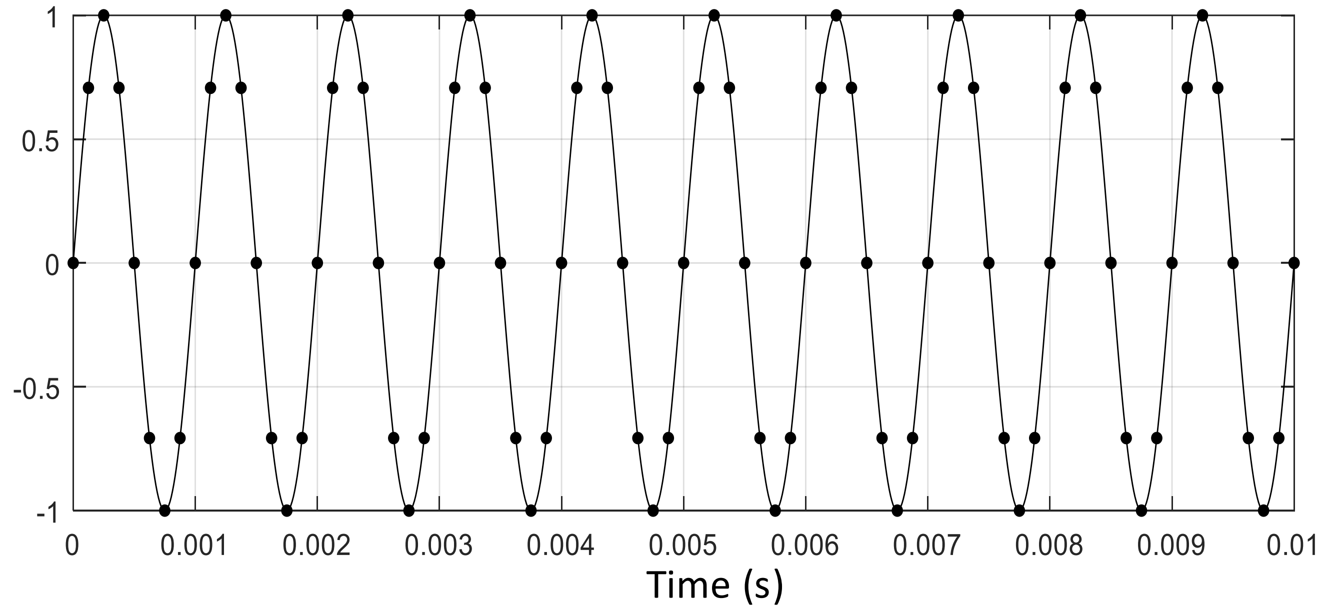
(a)
(b)
Sampling
Nyquist limit

Sampling - Example

- Consider $x(t) = \sin(2\pi f_0 t)$ with $f_0 = 1kHz$

Sampling frequency $f_s = \frac{1}{T_s} = 8kHz$





$$x(t) = \sin(2\pi f_0 t)$$

$$\boldsymbol{x(nT_s) = x(t)\delta_{Ts}(t)\Big|_{t=nT_s}}$$

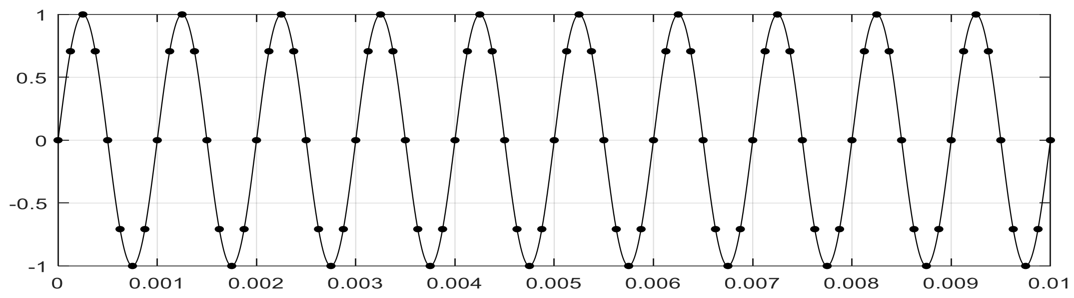
$$x(nT_s) = \sin(2\pi f_0 n T_s)$$
$$x[n] = \sin\left(2\pi \frac{f_0}{f_s} n\right)$$
$$= \sin(\Omega_0 n)$$

Discrete frequency

$$\boldsymbol{\Omega = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}}$$

Nyquist-Shannon Sampling theorem

Theorem: *If a signal $x(t)$ contains no frequency greater than f_{max} Hz, it is completely determined by giving its ordinates at a series of points spaced $1/(2f_{max})$ seconds apart.*



Sampling frequency, $f_s \geq 2f_{max}$

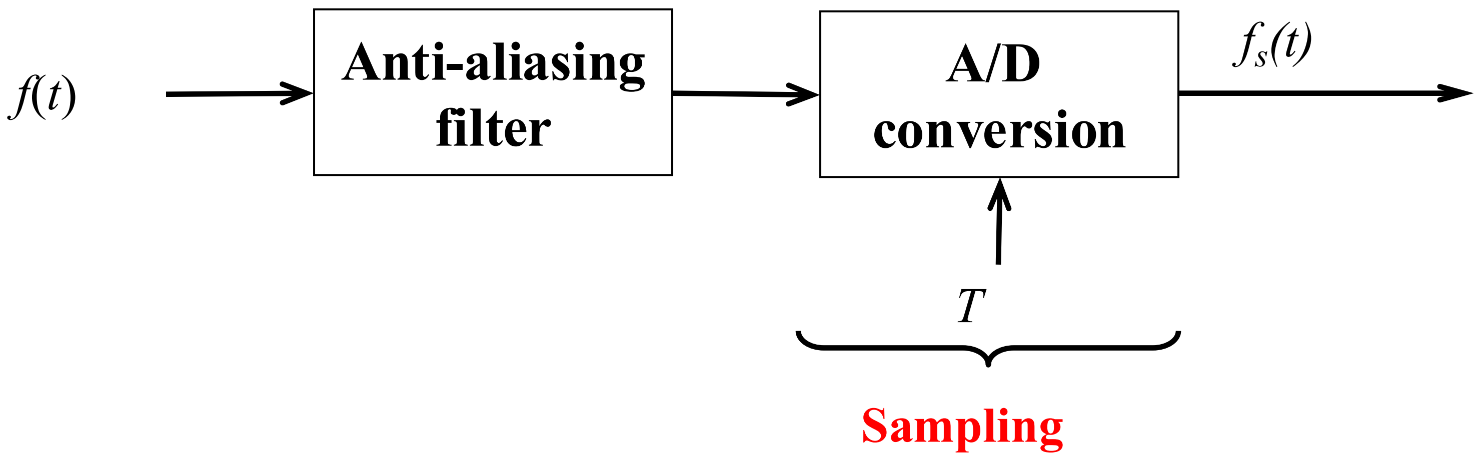
WHY?

<u>Approaches to obtain frequency-limited signal $x(t)$</u>

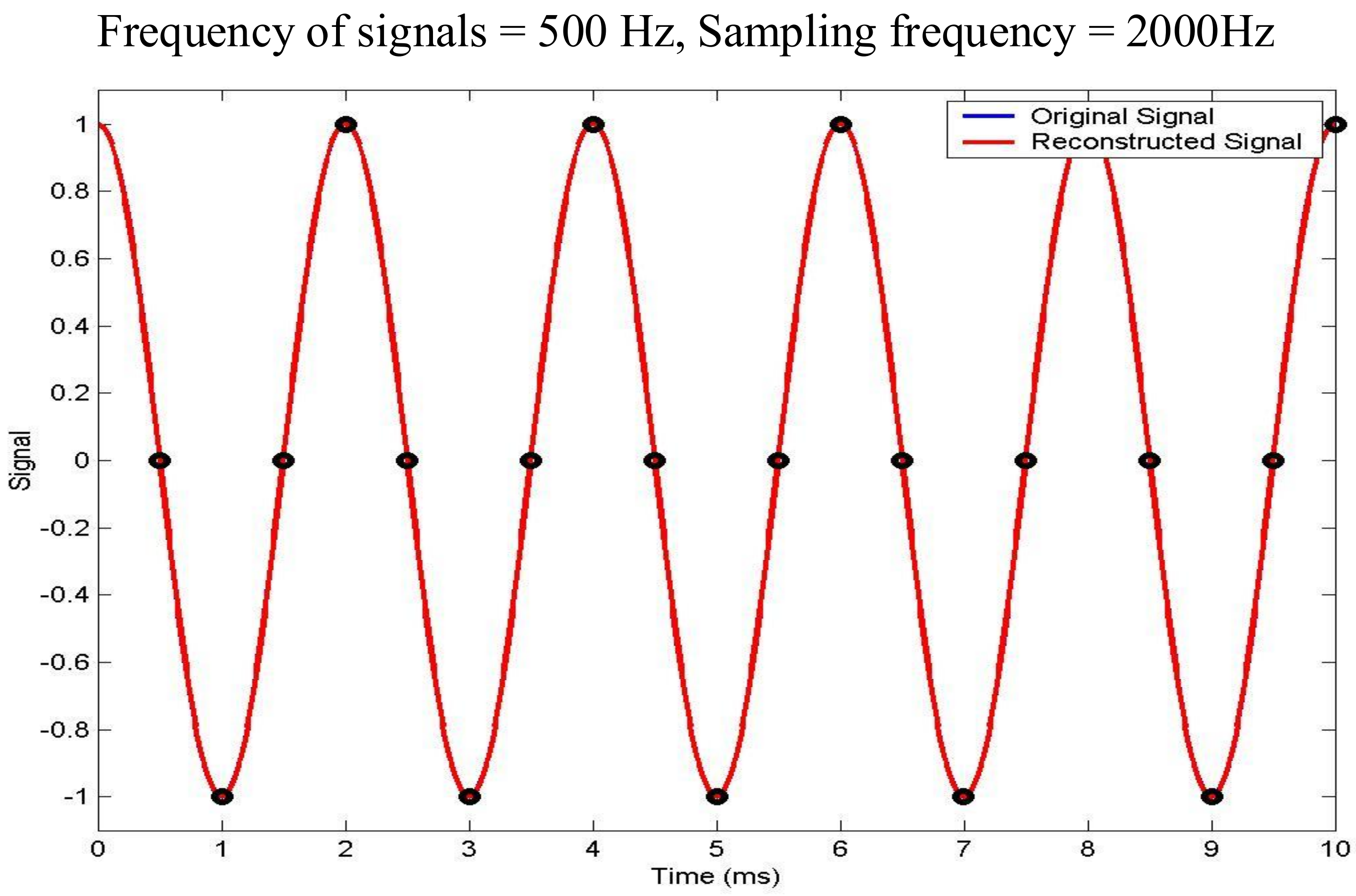
1. If frequency contents of $x(t)$ are known, $f_s \geq 2f_{max}$
2. If sampling frequency f_s is fixed, the signal is filtered such that $f_{max} < \frac{f_s}{2} Hz$

How to Avoid Aliasing?

- Band-limiting signals (by filtering) before sampling.
- Sampling at a rate that is greater than the Nyquist rate.

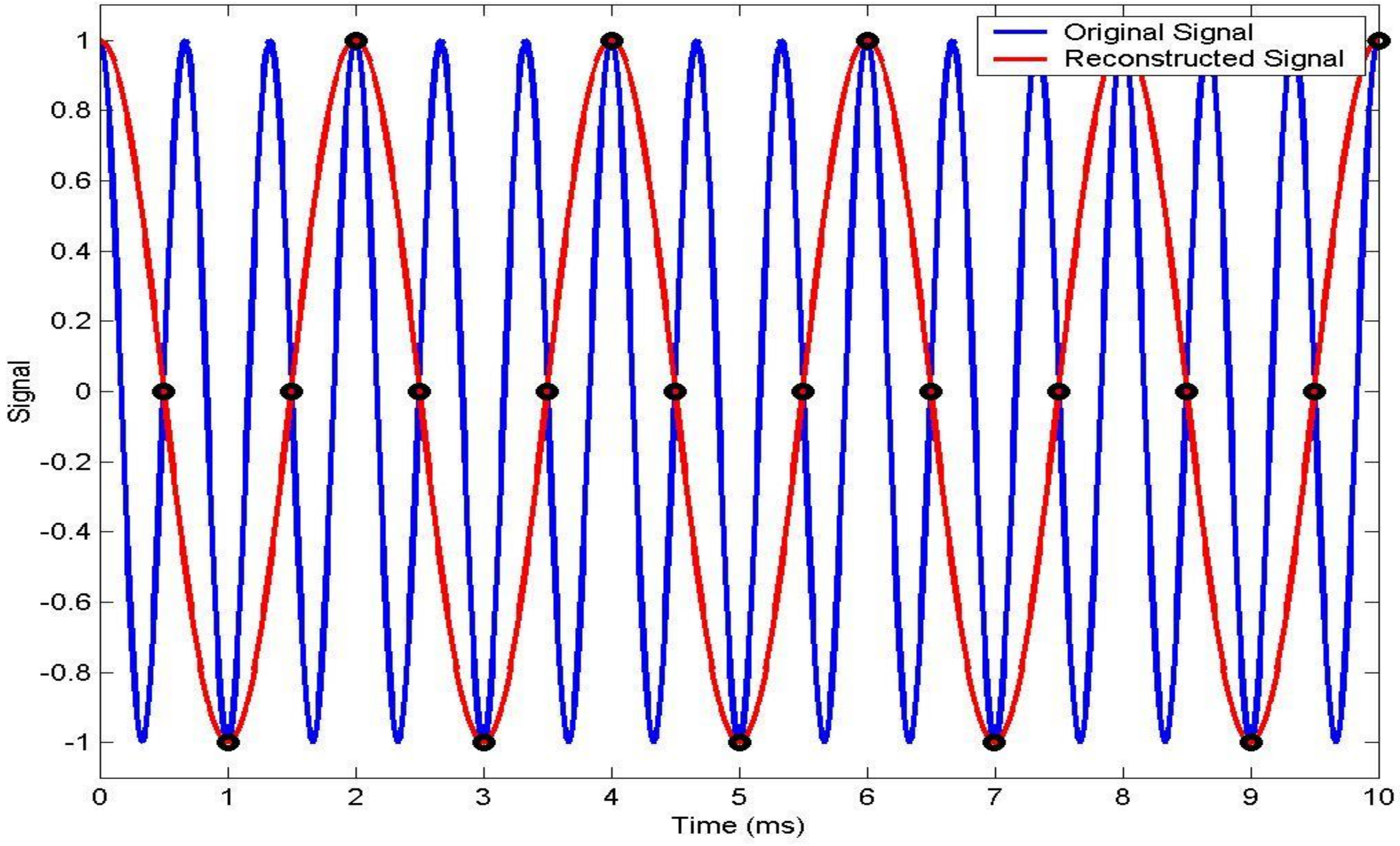


Aliasing of Sinusoidal Signals



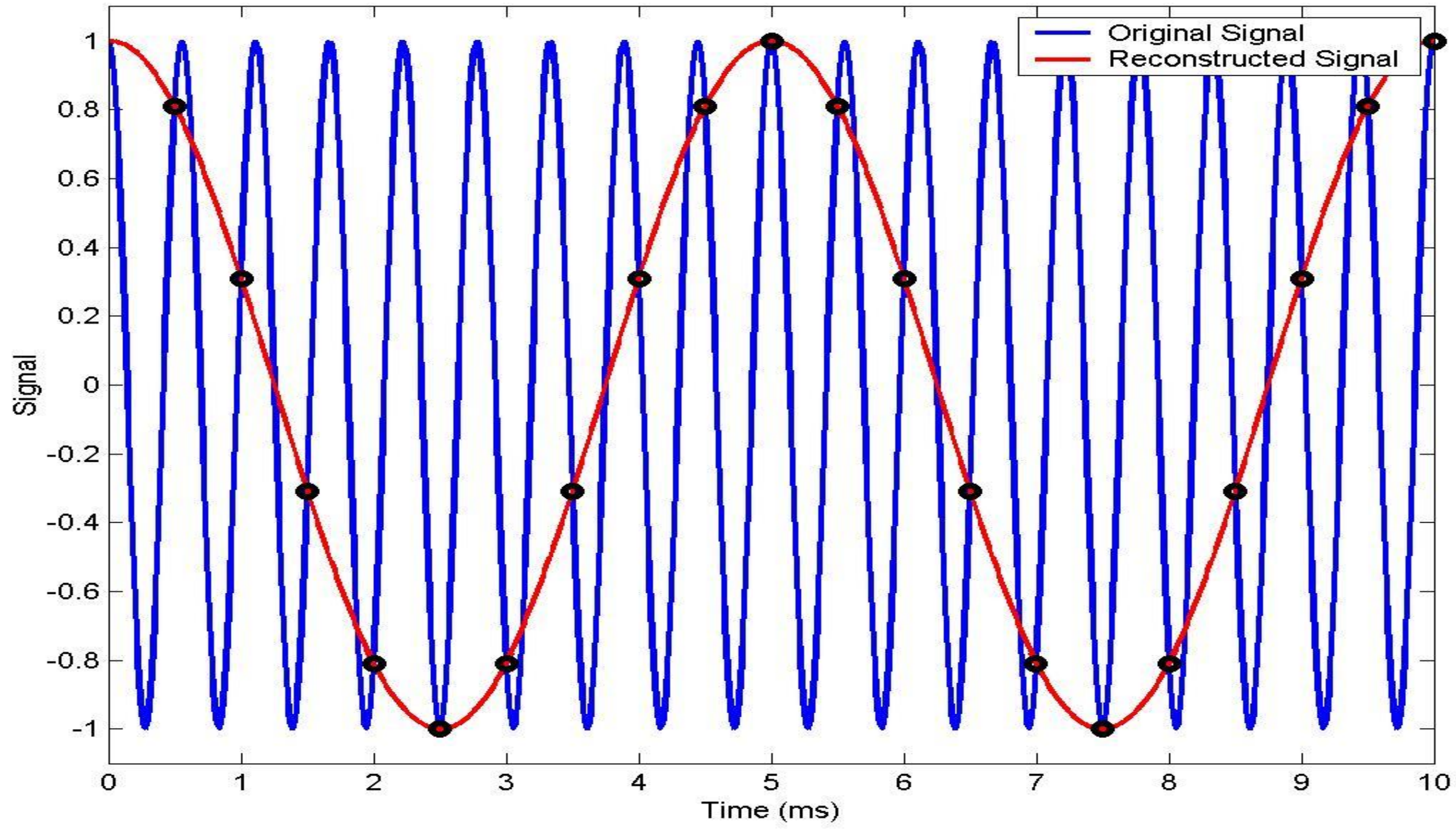
Aliasing of Sinusoidal Signals

Frequency of signals = 1500 Hz, Sampling frequency = 2000Hz



Aliasing of Sinusoidal Signals

Frequency of signals = 1800 Hz, Sampling frequency = 2000Hz



Digital Frequencies

- The range of digital frequencies are limited to:
- $0 \leq f \leq 1; or \ -\frac{1}{2} \leq f \leq \frac{1}{2}$: Linear frequency
- $-\pi \leq \Omega \leq \pi$: Angular frequency
- Actual values will depend upon the sampling frequency. E.g for $f_s = 1000Hz$, the mapping will be:

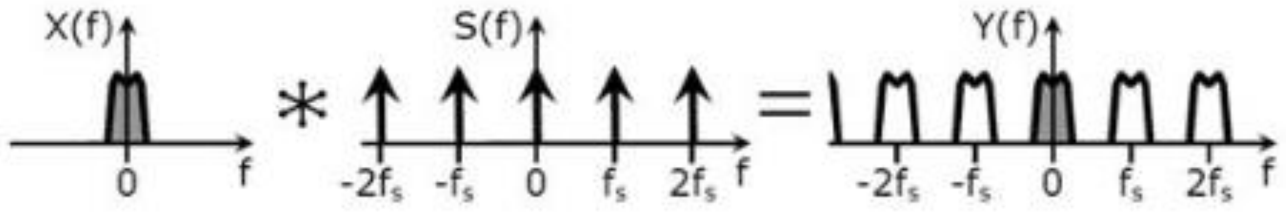
Actual Frequency	Digital Linear frequency	Ω
500 Hz	1/2	π
250 Hz	1/4	$\pi/2$
100 Hz	1/10	?

Sampling

- Sampling in TD is multiplication of two signals

$$\boldsymbol{x(nT_s) = x(t)\delta_{Ts}(t)\Big|_{t=nT_s}}$$

- Convolution in frequency domain



Reconstruction

Ideal Reconstruction

When the Nyquist condition is satisfied, we know that it is possible to reconstruct the original continuous signal from the samples. We just need to apply a box filter that has a constant gain for all the frequencies inside $w \in [-\pi/T_s, \pi/T_s]$, and 0 outside. The phase of the filter should be zero. That is,

$$H(w) = \begin{cases} \frac{T_s}{2\pi} & \text{if } w \in [-\pi/T_s, \pi/T_s] \\ 0 & \text{otherwise} \end{cases} \tag{20.1}$$

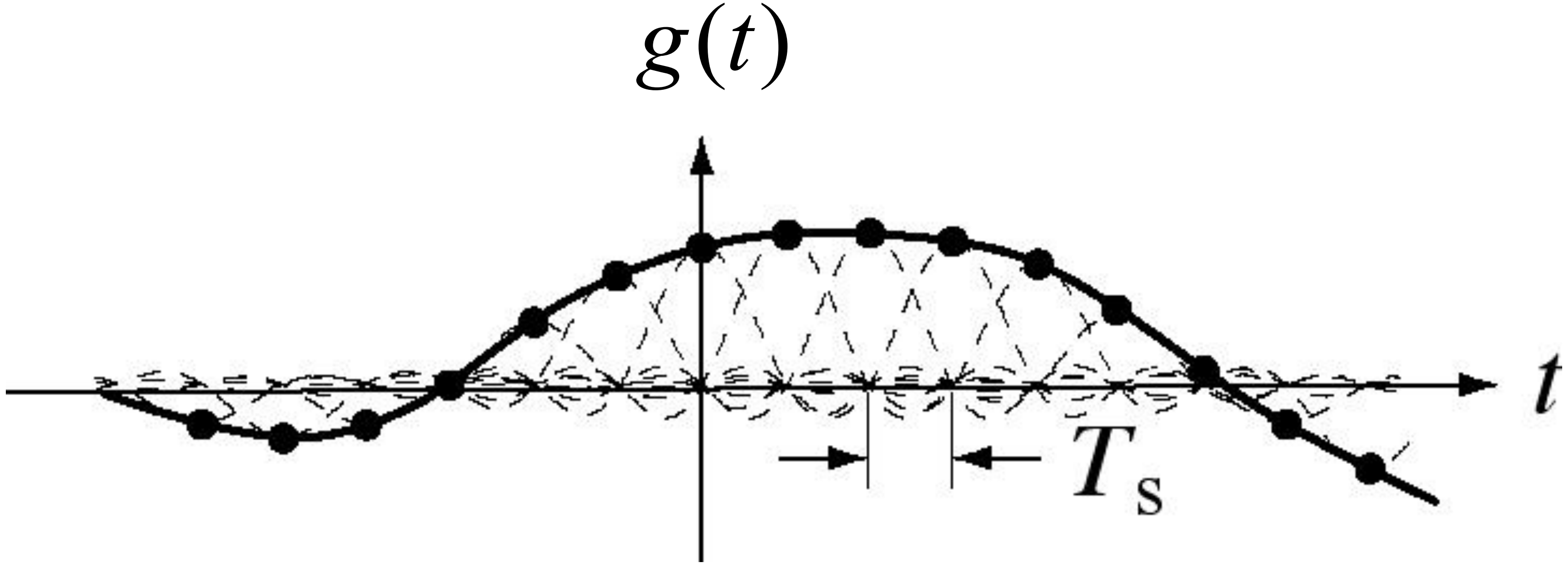
The impulse response of such a filter is $h(t) = \text{sinc}(t/T_s)$ where the sinc function is:

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Interpolation

If the sampling is at exactly the Nyquist rate, then

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$



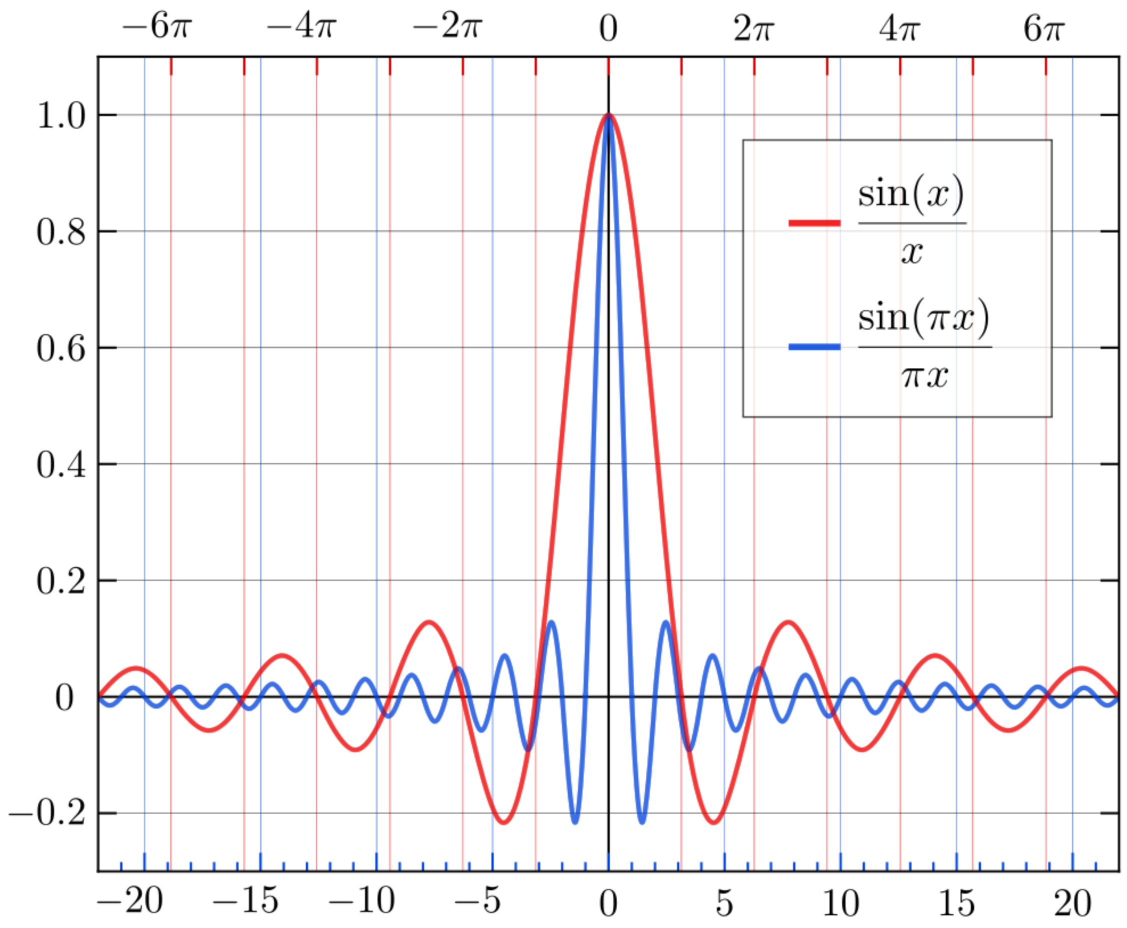
Prerequisites

- Fourier transform pair:

$$rect\left(\frac{t}{T}\right) \longleftrightarrow Tsinc\left(\frac{\omega T}{2}\right)$$

$$\frac{\omega_s}{2\pi} sinc\left(\frac{\omega_s t}{2}\right) \longleftrightarrow rect\left(\frac{\omega}{\omega_s}\right)$$

$$sinc(x) = \frac{\sin x}{x}$$

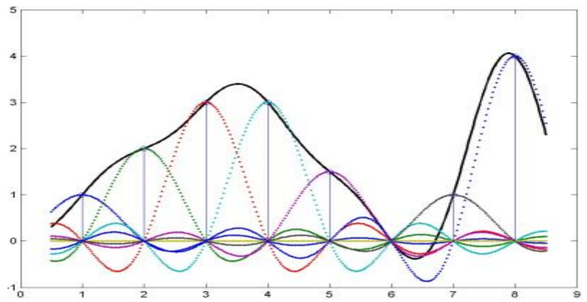


Reconstruction

- Reconstruction
  - 0^{th}- and first-order reconstruction
  - Ideal filter

$$H(\omega) = T_s rect\left(\frac{\omega}{\omega_s}\right)$$

$$h(t) = sinc\left(\frac{\pi t}{T_s}\right)$$



  - Sinc interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) sinc\left(\frac{\pi(t - nT_s)}{T_s}\right)$$

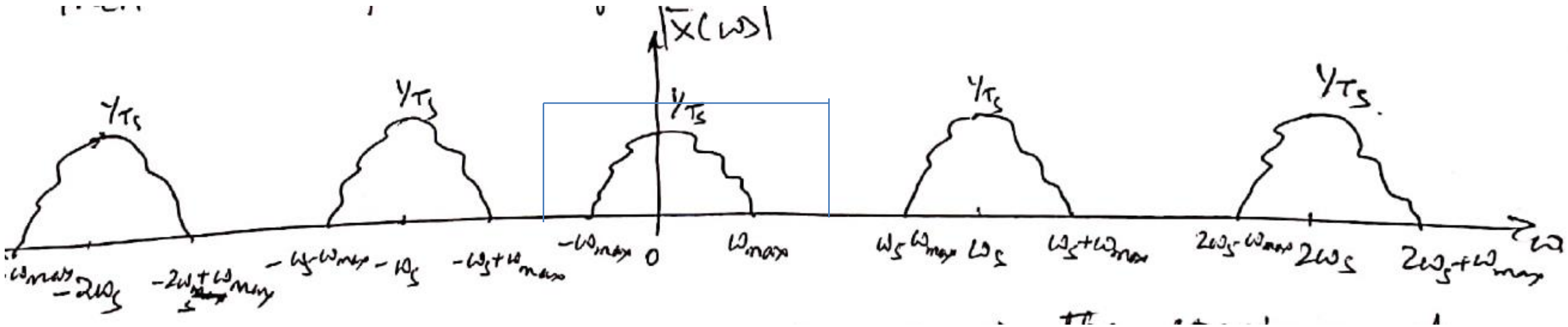
$$x(nT_s) \rightarrow \boxed{h(t)} \rightarrow x(t)$$

Reconstruction

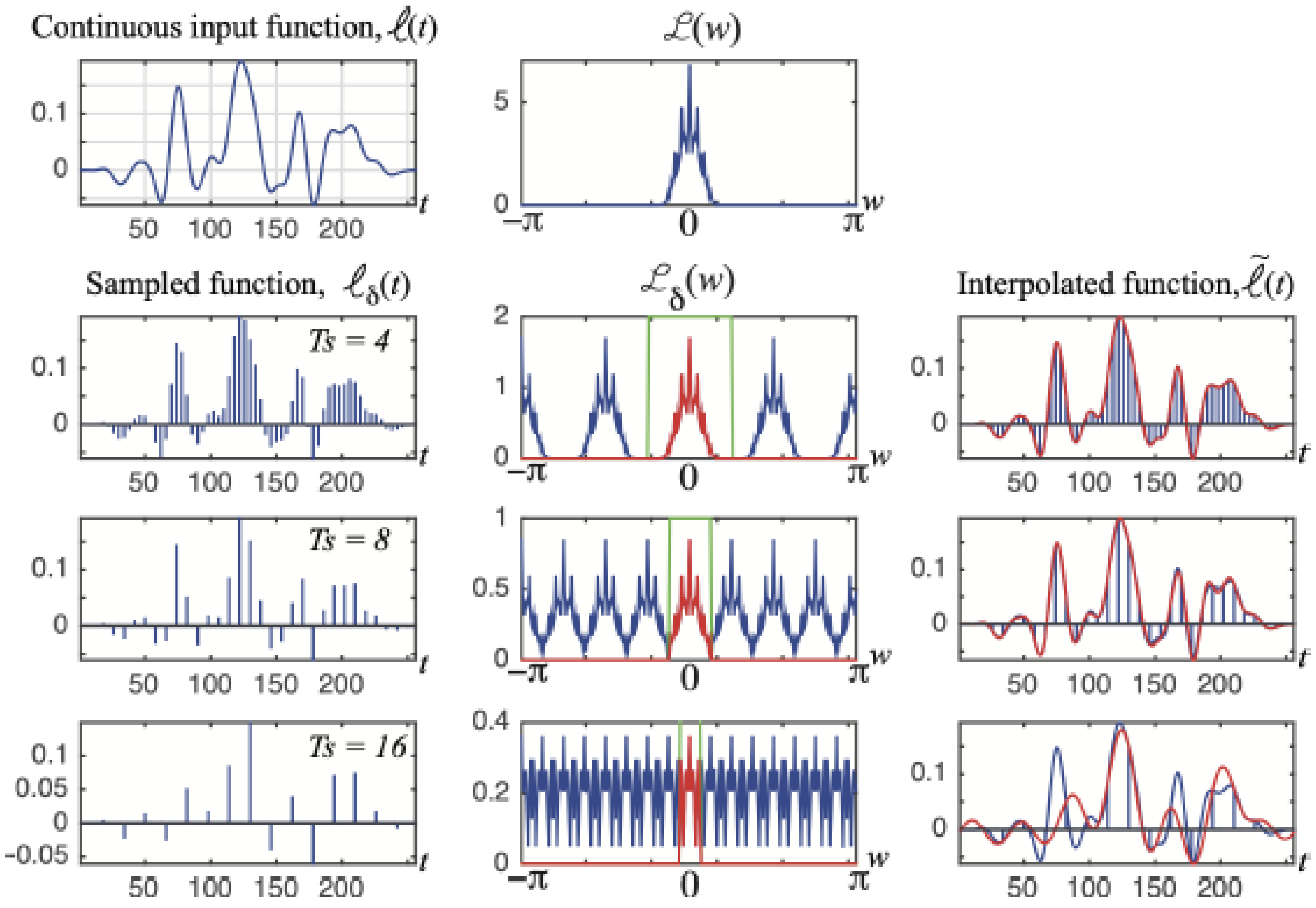
- Perfect reconstruction with sinc interpolation

$$rect\left(\frac{t}{T}\right) \longleftrightarrow Tsinc\left(\frac{\omega T}{2}\right)$$

$$\frac{\omega_s}{2\pi} sinc\left(\frac{\omega_s t}{2}\right) \longleftrightarrow rect\left(\frac{\omega}{\omega_s}\right)$$



Example



Continuous input function, $\ell(t)$
$\mathcal{L}(w)$
Sampled function, $\ell_\delta(t)$
$\mathcal{L}_\delta(w)$
Interpolated function, $\tilde{\ell}(t)$
Ts = 4
Ts = 8
Ts = 16

Spectral sampling

- Dual of temporal sampling
- Temporal sampling vs Spectrum sampling
  - Condition for recovery
  - Reconstruction
- Relation between CTFS and spectrum samples

Question

- If $x(t) = \cos 2\pi f_0 t$ is sampled at a sampling frequency f_s, draw the magnitude spectrum $|X(\omega)|$ of the sampled signal $x(nT_s)$
  1. $f_0 = 1\text{kHz}, f_s = 4\text{kHz}$
  2. $f_0 = 3\text{kHz}, f_s = 4\text{kHz}$
  3. $f_0 = 5\text{kHz}, f_s = 4\text{kHz}$

Questions

- If $f_{\max\_x}$ is the maximum frequency of $x(t)$, what is the maximum frequency of $y(t) = x(t)\cos 2\pi f_0 t$?

- If $f_{\max\_x}$ and $f_{\max\_h}$ are the maximum frequency components of $x(t)$ and $h(t)$, what will be the f_{max} for $y(t) = x(t) * h(t)$?

Example

- Consider an input signal x(t) and impulse response h(t),
  $x(t) = \frac{sin\omega_i t}{\pi t}$ and $\mathrm{h}(t) = \frac{sin\omega_c t}{\pi t}$

- Determine y(t)