Homework 1

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1 Problem 1

Assume A is upper-triangular thus A^* is lower-triangular. Since A is unitary we know $AA^* = I$. We also know that the inverse of an upper-triangular unitary matrix must also be upper triangular. Since $A* = A^{-1}$ and we know that A^* is lower-triangular and A^{-1} is upper-triangular this implies that $A*$ is diagonal thus A is diagonal. The same logic holds for lower-triangular and unitary matrices.

2 Problem 2

a) Since we know that A is invertible and $\lambda \neq 0$ we know the following:

$$Av = \lambda v \implies v = \lambda A^{-1}v \implies \frac{1}{\lambda}v = A^{-1}v$$

Thus $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

b) If λ is an eigenvalue of AB then the following is true. $ABx = \lambda x$ $(x \neq 0)$. Let $z = Bx$ $Az = \lambda x$ -¿ $BAz = \lambda Bx = BAz = \lambda z$ thus it has been shown that BA and AB have the same eigenvalues.

c) We want to show that $\det(A - \lambda I) = \det(A^* - \lambda I)$ Since $A \in \mathbb{R}^{n\times n}$, $A^* = A^T$ So we only need to show that A^T and A have the same eigenvalues or $\det(A - \lambda I) = \det(A^T - \lambda I)$. We can show this by the following

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$$

So we see that A^T and A have the same eigenvalues.

3 Problem 3

a) Since we know that A is hermitian we know that $A = A^*$ By the following:

$$\langle Ax, y\rangle = \langle \lambda x, y\rangle = \lambda\langle x, y\rangle$$

$$\langle x, A^*y\rangle = \langle x, Ay\rangle = \langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle$$

This implies $\lambda = \bar{\lambda}$. This is only true when λ has only a real part.

b) Let $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ where $\lambda_1 \neq \lambda_2$ Since $A = A^*$ the following holds.

$$\langle x, Ay \rangle = y^* A^* x = y^* A x = y \lambda_1 x = \lambda_1 \langle x, y \rangle$$

we also have

$$\langle x, Ay \rangle = \langle x, \lambda_2 y \rangle = \lambda_2 \langle x, y \rangle.$$

Therefore we have $\lambda_1 \langle x, y \rangle = \lambda_2 \langle x, y \rangle \implies (\lambda_1 - \lambda_2)\langle x, y \rangle = 0$ since $\lambda_1 \neq \lambda_2 \implies \langle x, y \rangle = 0$

4 Problem 4

a) $Ax = \lambda x$ and $(Ax = \lambda x)^* = x^* A^* = \bar{\lambda} x^*$ left multiply the second equation with the first and we get the following

$$x^* A^* A x = \bar{\lambda}\lambda x^* x \implies \|x\|^2 = |\lambda|^2 \|x\|^2 \implies |\lambda|^2 = 1 \implies |\lambda| = 1$$

b) Disproof

$$\|A\|_F = \sqrt{\operatorname{tr}(AA^*)} = \sqrt{\operatorname{tr}(I)} \neq 1$$

unless $A \in \mathbb{C}^{1\times 1}$

5 Problem 5

a)

$$\langle Ax, y \rangle = \langle \lambda x, y \rangle = \lambda \langle x, y \rangle$$

$$\langle x, A^* x \rangle = \langle x, -Ax \rangle = \langle x, -\lambda x \rangle = -\bar{\lambda} \langle x, y \rangle$$

$\lambda = -\bar{\lambda}$ Thus λ must be only imaginary.

b) The eigenvalues of $I - A$ are equal to $1 - \lambda$ where λ is the eigenvalues of A where $\lambda = bi$ thus $1 - \lambda \neq 0$

c) Multiply B and B^* to show if B is unitary

$$BB^* = (I-A)^{-1}(I+A)(I-A)(I+A)^{-1} = (I-A)^{-1}(I-A)(I+A)(I+A)^{-1} = I$$

The middle terms can commute since $(I + A)(I - A) = I - AA$ and $(I - A)(I + A) = I - AA$ Thus we know $(B^*)^{-1} = B \implies [(B^*)^{-1}]^* = B^* \implies (B^{**})^{-1} = B^* \implies B^{-1} = B^*$

6 Problem 6

$\rho(A) =] \max\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\} = \lambda_{max}$. Want to show $\rho(A) \leq \|A\| = \max_{\|x\|=1} \|Ax\|$. If $\rho(A) = |\lambda^*|$ and u is an eigenvector corresponding to λ^* where $\|u\| = 1$ then

$$\|A\| \leq \|Au\| = \|\lambda^* u\| = \|\lambda^*\|.$$

7 Problem 7

a) $\|A\|_2 = \sqrt{\lambda_{max}(AA^*)} = \sqrt{\lambda_{max}(uv^*vu^*)} = \sqrt{\|v\|_2^2\lambda_{max}(uu^*)}$ The only non-zero eigenvalue of the matrix uu^* is $\|u\|_2^2$ so the equation reduces to $\sqrt{\|v\|_2^2\|u\|_2^2} = \|u\|_2\|v\|_2$. Thus it has been shown.

b) $\|A\|_F = \sqrt{\operatorname{tr}(AA^*)} = \sqrt{\operatorname{tr}(uv^*vu^*)} = \sqrt{\|v\|_2\operatorname{tr}(uu^*)} = \sqrt{\|v\|_2^2\|u\|_2^2} = \|v\|_2\|u\|_2$.

8 Problem 8

a) Let $U\Sigma V^*$ be the SVD of B Thus $A = QU\Sigma(QV)^*$. This is also a singular value decomposition for A because (QU) and $(QV)^*$ are unitary matrices.

b) Disproof. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Assume B and A are unitarily equivalent Thus we have the following

$$B = QAQ^* \implies \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = Q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}Q^* \implies I = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\rightarrow\leftarrow$ Thus A,B are not unitarily equivalent.

9 Problem 9

a) $J_{(x)} = (1, 1)$ a row vector. Using the infinity-norm to find the condition number

$$\|J\|_\infty = 2$$
$$\|x\|_\infty = \max(|x_1|, |x_2|)$$
$$\|f(x)\|_\infty = |x_1 + x_2|$$
$$\frac{\|J\|_\infty\|x\|_\infty}{\|f(x)\|_\infty} = \frac{2\max(|x_1|, |x_2|)}{|x_1 + x_2|}$$

Becomes ill-conditioned when $x_1 \to -x_2$

b) Let $x_2 = \alpha x_1$ then the function becomes $g(x) = \alpha x^2$. From here we see $J_{(x)} = 2\alpha x$. Using the infinity-norm we have

$$\|J\|_\infty = |2\alpha x|$$
$$\|x\|_\infty = |x|$$
$$\|f(x)\|_\infty = |\alpha x^2|.$$

We find that our condition number is

$$\frac{\|J\|_\infty\|x\|_\infty}{\|f(x)\|_\infty} = \frac{|2\alpha x||x|}{|\alpha x^2|} = 2.$$

There is no point where the condition number becomes ill-conditioned.

c) $J = 9(x-2)^8$ Using the infinity-norm to find the condition number $\frac{\|J\|_\infty\|x\|_\infty}{\|f(x)\|_\infty} = \frac{9(x-2)^8|x|}{|x-2|(x-2)^8} = \frac{9|x|}{|x-2|}$ Becomes ill-conditioned as $x \to 2$

10 Problem 10

c) We see that as we approach 2 where the function becomes ill-conditioned we see the expanded function starts to have round off errors.