Homework 3

Saturday, February 13, 2021 11:02 AM

1)

1. If P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically and give a geometric interpretation.

P orthogonal projector $\Rightarrow P^2 = P = P^*$

w.t.s $\quad (I-2P)^*(I-2P) = I$

$= I^2 - 2P - 2P^* + 4P^*P$

$= I - 2P - 2P + 4P^2$

$= I - 4P + 4P = I \quad \blacksquare$

This implies $(I-2P)^* = (I-2P)^{-1}$

Thus $(I-2P)$ is unitary.

2)

2. Given $A \in \mathbb{R}^{m\times n}$ with $m \geq n$. Show that A^*A is nonsingular if and only if A has full rank.

$(\rightarrow)$ A^*A is nonsingular, $\operatorname{rank}(A^*A) = n$

$(A^*A)x \neq 0, \; x \neq 0$

$A^*(\underline{Ax}) \neq 0$

$Ax \neq 0 \Rightarrow A$ has full rank

$(\leftarrow)$ A has full rank $\quad \operatorname{rank}(A) = n$

$Ax \neq 0, \; x \neq 0 \qquad$ w.t.s: $\; A^*Ax \neq 0, \; x \neq 0$

$Ax \neq 0, x \neq 0$ w.t.s. $A^*Ax \neq 0, x \neq 0$

Left multiply both sides by A^* $A^*Ax \neq A^*0 = 0$

$A^*Ax \neq 0$

thus A^*A is nonsingular.

3)

3. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad (1)$$

Answer the following questions by hand calculation.

(a) What is the orthogonal projector P onto range(A), and what is the image under P of $v = [1, 2, 3]^T$?

(b) Do the same for B.

(1) $P = A(A^TA)^{-1}A^T$

(2) $P = QQ^T$

a)

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1}} \Bigg)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P$$

$$P_v = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

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b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$b_1 \ b_2$

let $v_1 = b_1$

$$v_2 = b_2 - \frac{\langle b_1, b_2 \rangle}{\langle b_1, b_1 \rangle} b_1$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$ next we normalize

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix} = Q$$

$Q Q^T$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix} = P$$

$$P_v = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

4)

4. Let $P \in \mathbb{R}^{m\times m}$ be a nonzero projector.

(a) Show that $||P||_2 \geq 1$, with equality if and only if P is an orthogonal projector.

(b) If P is an orthogonal projector, then P is positive semi-definite with its eigenvalues are either zero or 1.

a) 1^{st} w.t.s $||P||_2 \geq 1$

we know $||A||_2 = \sup_{||x||_2=1} ||Ax||_2$

$$||P||_2 = \sqrt{\lambda_{max}(PP^T)}$$

$||P||_2 = \sup_{||x||_2=1} ||Px||_2$ $\quad \exists\, v$ s.t $Pv = v$

then we know the following

$$||P||_2 = \sup_{||x||_2=1} ||Px||_2 \geq ||Pv||_2 = 1 \implies \boxed{||P||_2 \geq 1}$$

by def. of sup.

($\Leftarrow$) assume P is ortho. Proj. $\implies P = P^2 = P^T$

Consider the SVD of P $\quad P = U\Sigma V^T$

Consider the SVD of P $P = U\Sigma V^T$

Thus $PP^+ = P^2 = P = U\Sigma V^T V \Sigma U^T = U\Sigma^2 U^T$

This implies P Has both singular Values Σ and Σ^2 the only time this is true is if $\sigma = 1$ or 0 and since P is non-zero at least one singular value is non-zero thus $\|P\|_2 = 1$

$(\rightarrow)$ I could not figure out a Proof for this direction.

b) $Px = \lambda x \Rightarrow P^2 x = \lambda Px$ and $P^2 x = \lambda x$

$\lambda Px = \lambda x$ $\lambda^2 x = \lambda x$ $\lambda^2 x - \lambda x = 0$

$$\lambda(\lambda x - x) = 0 \Rightarrow \boxed{\begin{array}{c}\lambda = 0\\ \lambda = 1\end{array}} \quad \text{or } \lambda x = x$$

Since $P = P^T$ P is P.S.D

5)

5. Consider the matrices in Problem 3.

(a) Using any method you prefer, determine a reduced QR factorization $A = \hat{Q}\hat{R}$ and a full QR factorization $A = QR$ by hand calculation.

(b) Do the same for B.

a) $$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$a_1 \quad a_2$

$\langle a_1, a_2 \rangle = 0$ ✓

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Q = \begin{array}{c} \begin{matrix} q_1 & q_2 \end{matrix} \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \end{array}$$

$$R = \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}$$

$$QR = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$ reduced Q-R

Full QR same q_1, q_2 but now we want q_3 s.t.

$\langle q_1, q_3 \rangle = 0$

$\langle q_2, q_3 \rangle = 0$

$\langle q_3, q_3 \rangle = 1$

$$q_3 = \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \end{bmatrix}$$

$\frac{1}{\sqrt{2}} q_{31} + \frac{1}{\sqrt{2}} q_{33} = 0$

$q_{32} = 0$

$q_{31} = -q_{33}$ $\quad q_{33} = 1$

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$q_3 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$q_{31} = -q_{33}$

$q_{32} = 0$

$q_{33} = q_{33}$

$q_{33} = 1$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \hat{R} \\ \hline 0 \quad 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

b) $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reduced QR

$$\hat{Q} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \qquad Q^T B = \hat{R}$$

$$\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

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Full qr

$Q = [q_1 | q_2 | q_3]$ $\quad q_1, q_2$ same from $\hat{Q}$

$\langle q_1, q_3 \rangle = 0 \qquad \frac{1}{\sqrt{2}} q_{31} + \frac{1}{\sqrt{2}} q_{33} = 0$

$\langle q_2, q_3 \rangle = 0 \qquad \frac{1}{\sqrt{3}} q_{31} + \frac{1}{\sqrt{3}} q_{32} - \frac{1}{\sqrt{3}} q_{33} = 0$

$\langle q_3, q_3 \rangle = 1$

$q_{31} + q_{33} = 0$

$q_{31} + q_{32} - q_{33} = 0$

$q_{31} + q_{33} =$

$q_{32} - 2q_{33}$

$q_{31} = q_{33}$

$q_{32} = -2q_{33}$

$q_{33} = q_{33}$

let $q_{33} = 1$

$$q_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{normalize} \quad \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$$QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

6)

6. Let $A \in \mathbb{R}^{m\times m}$ with $m \geq n$, and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.

(a) Show that A has rank n if and only if all the diagonal entries of $\hat{R}$ are nonzero.

(b) Suppose $\hat{R}$ has k nonzero diagonal entries for some k with $0 \leq k < n$. What does this imply about the rank of A? Exactly k? At least k? At most k? Give a precise answer and prove it.

$A \in \mathbb{R}^{m\times n}$ $m \geq n$, $A = \hat{Q}\hat{R}$ reduced QR factorization

$\hat{r}_{ij} = \langle q_i, a_j \rangle$ w.t.s. $r_{ii} \neq 0$ iff $\hat{R}$ has non zero diagonal entries.

a) To show we need to show

$\text{rank}(A) = \text{rank}(\hat{Q}\hat{R}) = \text{rank}(\hat{R})$

let $w_1, \dots, w_k$ be a basis for the nulity of $\hat{R}$

$\Rightarrow \dim(\text{null}(\hat{R})) = k$ So we know $\hat{R} w_i = 0$ for $i = 1, \dots, k$

then $\hat{Q}\hat{R} w_i = 0$ for $i = 1, \dots, k$

This implies $\dim(\text{null}(\hat{Q}\hat{R})) \geq k$

Next suppose that $\dim(\text{null}(\hat{Q}\hat{R})) > k$ we will prove by contradiction this isnt true. Suppose $\exists\, v \neq 0$

s.t $\hat{R}v \neq 0$ and $\hat{Q}\hat{R}v = 0$ but since the cols of Q are L.I. we know that $\hat{Q}y = 0 \Rightarrow y = 0$ So the above implies

$\hat{R}v = 0$ $\Rightarrow\Leftarrow$ thus we have shown

$\dim(\text{null}(\hat{Q}\hat{R})) \leq k$

We have shown $\dim(null(\hat{Q}\hat{R})) \geq k$ and

$\dim(null(\hat{Q}\hat{R})) \leq k$ which implies

$\dim(null(\hat{Q}\hat{R})) = k = \dim(null(\hat{Q}\hat{R}))$

By Rank-null thm

$Rank(A) = n - \dim(null(\hat{Q}\hat{R})) = n - \dim(null(\hat{R})) = Rank(\hat{R})$

Thus it has been shown $Rank(A) = Rank(\hat{R})$

Since $Rank(A) = n$ $Rank(\hat{R}) = n$ $\blacksquare$

b) Using the above proof

$Rank(A) = Rank(\hat{R})$ If $Rank(\hat{R}) = k$

then $Rank(A) = k$ $\blacksquare$

7) **7.** Let $A \in \mathbb{R}^{m \times n}$. Determine the exact numbers of floating point additions, subtractions, multiplications, and divisions involved in computing the factorization $A = \hat{Q}\hat{R}$ in the Modified Gram-Schmidt algorithm in Chapter 3 of the lecture note.

Out loop n times

$\|V_i\| \Rightarrow 2n-1$ ops

$(3n-1)n$ = total ops outer loop

$\boxed{3n^2 - n}$

$q_i \Rightarrow n$ ops

inner loop $(n)(n-i)$ times

$r_{ik} \Rightarrow 2n-1$ ops

$v_k \Rightarrow 2n$ ops

$(4n-1)(n)(n-i)$

$(4n^2-n)\sum_{i=1}^{n}(n-i)$

$= (4n^2-n)(n^2-n)\frac{1}{2}$

$= (4n^4 - 5n^3 + n^2)\frac{1}{2}$

$= 2n^4 - \frac{5}{2}n^3 + \frac{1}{2}n^2$ total ops inner loop

$2n^4 - \frac{5}{2}n^3 + \frac{1}{2}n^2 + 3n^2 - n$

$= 2n^4 - \frac{5}{2}n^3 + \frac{7}{2}n^2 - n$ total ops both loops

8) **8.** Determine the (i) eigenvalues, (ii) determinant, and (iii) singular values of a Householder reflector. For the eigenvalues, give a geometric argument as well as an algebraic proof.

$H \in \mathbb{R}^{n \times n}$

i) $H = H^* = H^{-1} \Rightarrow H^2 = I$

$H = I - 2^* v^T v$

$Hv = v - 2vv^Tv = v - 2v = -v$

at least 1 eigen value is -1

Consider w where $w \perp v$

$Hw = w - 2^* vv^T w = w$ since

the $\dim(\text{span}(v)^\perp)$ is $n-1$

the dim(span(v)) ...

we know there are $n-1$ eigen values of 1 and 1 eigenvalue of -1

ii) Since $\det(H)$ is equal to the product of the eigen values we know

$\det(H) = -1$ since its

$$(1^{n-1})(-1) = -1.$$

iii) Since H is involutory

$$\sigma_i(H) = \sqrt{\lambda_i(H^T H)} = \sqrt{\lambda_i(I)} = 1$$

thus all singular values are equal to 1.

9) **9.** Gram-Schmidt Process: Let

$$v_1 = \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ \varepsilon \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \varepsilon \end{bmatrix},$$

where ε is given to satisfy that its floating point operation $1+\varepsilon^2$ becomes 1 in computer arithmetic.

(a) Apply the classical Gram-Schmidt method and show that the computed vectors are not numerically orthogonal, i.e., computed vectors have dot products much larger than ε.

(b) Apply the Modified Gram-Schmidt method and show that the computed vectors are numerically orthogonal, i.e., computed vectors have dot products $= \mathcal{O}(\varepsilon)$.

a) $U_1 = V_1$

$$q_1 = \frac{u_1}{\|u_1\|} \qquad q_1 = \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix}$$

$$u_2 = V_2 - \frac{V_1 V_2}{\|V_1\|^2} V_1$$

$$u_2 = \begin{bmatrix} 1 \\ 0 \\ \varepsilon \\ 0 \end{bmatrix} - \frac{1}{1+\varepsilon^2} \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\varepsilon \\ \varepsilon \\ 0 \end{bmatrix}$$

$$u_3 = V_3 - \frac{V_1 V_3}{\|V_1\|^2} V_1 - \frac{V_2 V_3}{\|V_2\|^2} V_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \varepsilon \end{bmatrix} - \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \varepsilon \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -\varepsilon \\ -\varepsilon \\ \varepsilon \end{bmatrix}$$

$$u_1 \cdot u_2 = \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -\varepsilon \\ \varepsilon \\ 0 \end{bmatrix} = -\varepsilon^2 = 0$$

$$u_1 \cdot u_3 = \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -\varepsilon \\ -\varepsilon \\ \varepsilon \end{bmatrix} = -1 - \varepsilon^2 = -1 \neq 0 \quad \text{not orthogonal}$$

b) $r_{11} = \|V_1\| = 1 + \varepsilon^2 = 1$

$$q = \frac{1}{1} \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix} \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix}$$

b) $r_{11} = \|v_1\| = 1$

$$q_1 = \frac{1}{1}\begin{bmatrix}1\\ \varepsilon\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1\\ \varepsilon\\ 0\\ 0\end{bmatrix}$$

$$r_{12} = q_1 \cdot v_2 = \begin{bmatrix}1\\ \varepsilon\\ 0\\ 0\end{bmatrix} \cdot \begin{bmatrix}1\\ 0\\ \varepsilon\\ 0\end{bmatrix} = 1$$

$$V_2 = V_2 - r_{12} q_1 = \begin{bmatrix}1\\ 0\\ \varepsilon\\ 0\end{bmatrix} - \begin{bmatrix}1\\ \varepsilon\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}0\\ -\varepsilon\\ \varepsilon\\ 0\end{bmatrix}$$

$$r_{13} = q_1 \cdot V_3 = \begin{bmatrix}1\\ \varepsilon\\ 0\\ 0\end{bmatrix} \cdot \begin{bmatrix}1\\ 0\\ 0\\ \varepsilon\end{bmatrix} = 1$$

$$V_3 = V_3 - r_{13} q_1 = \begin{bmatrix}1\\ 0\\ 0\\ \varepsilon\end{bmatrix} - \begin{bmatrix}1\\ \varepsilon\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}0\\ -\varepsilon\\ 0\\ \varepsilon\end{bmatrix}$$

$$r_{22} = \|V_2\| = \sqrt{2}\,\varepsilon = 0$$

$$q_2 = \frac{V_i}{r_{ii}} = \begin{bmatrix}0\\ -1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{bmatrix}$$

$$r_{23} = q_2 \cdot V_3 = \begin{bmatrix}0\\ -1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{bmatrix} \cdot \begin{bmatrix}0\\ -\varepsilon\\ 0\\ \varepsilon\end{bmatrix} = \frac{\varepsilon}{\sqrt{2}}$$

$$V_3 = V_3 - \frac{2}{\sqrt{2}} q_2 = \begin{bmatrix}0\\ -\varepsilon\\ 0\\ \varepsilon\end{bmatrix} - \frac{\varepsilon}{\sqrt{2}}\begin{bmatrix}0\\ -1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{bmatrix} = \begin{bmatrix}0\\ -\frac{\varepsilon}{2}\\ -\frac{\varepsilon}{2}\\ \varepsilon\end{bmatrix}$$

$$r_{33} = \|v_3\| = \sqrt{\frac{6}{4}\varepsilon^2} = \frac{\sqrt{6}}{2}\varepsilon$$

$$q_3 = \frac{v_3}{r_{33}} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$q_1 \cdot q_2 = \frac{-\varepsilon}{\sqrt{2}}$$

$$q_2 \cdot q_3 = \frac{-1}{\sqrt{12}} + \frac{1}{\sqrt{12}} = 0$$

$$q_1 \cdot q_3 = \frac{-\varepsilon}{\sqrt{6}}$$

q_1, q_2, q_3 are $O(\varepsilon)$ orthogonal.

10)

10. Let P_j be the $m \times m$ orthogonal projector defined by $P_j = I - Q_{j-1}Q_{j-1}^T$, where Q_{j-1} is a matrix whose k-th column is an orthogonal vector q_k, $1 \le k \le j-1$. Each q_k is an n-vector satisfying $q_k \perp q_\ell$ for $k \ne \ell$. Prove that

$$P_j = P_{\perp q_{j-1}} P_{\perp q_{j-2}} \cdots P_{\perp q_1}, \quad j = 2, 3, \ldots, m, \tag{2}$$

where $P_{\perp q_k} = I - q_k q_k^T$.

$$P_{\perp q_k} \; I - q_k q_k^T$$

W.t.s $\quad P_j = P_{\perp q_{j-1}} P_{\perp q_{j-2}} \cdots P_{\perp q_1} \quad j = 2, \ldots, m$

W.T.S.

$$\Rightarrow I - Q_{j-1}Q_{j-1}^T = (I - q_{j-1}q_{j-1}^T)(I - q_{j-2}q_{j-2}^T)\ldots(I - q_1 q_1^T)$$

$$(I - q_{j-1}q_{j-1}^T) \cdots\cdots (I - q_1 q_1^T)$$

$$= I - q_{j-1}q_{j-1}^T - \dots - q_1 q_1^T$$

$$q_{j-1}q_{j-1}^T + \dots + q_1 q_1^T = Q_{j-1}Q_{j-1}^T \quad ?$$

$$= \begin{pmatrix} q_1 & \cdots & q_{j-1} \end{pmatrix} \begin{pmatrix} - q_1 - \\ \vdots \\ - q_{j-1} - \end{pmatrix} = Q_{j-1}Q_{j-1} \quad \checkmark$$

11) **11.** Let $A \in \mathbb{R}^{m\times n}$. Show that $\text{cond}(A^T A) = \left(\text{cond}(A)\right)^2$.

SVD of A

$A = U \Sigma V^T$ U, V orthogonal

$A^T A = V^T \Sigma^T U^T U \Sigma V^T = V \Sigma^T \Sigma V^T$

$\Sigma^T \Sigma$ is a product of diagonal matrices which will result in a nxn diagonal matrix with the entries being squared.

thus the condition number for

thus the condition number for

$A^T A$ is $K(A^T A) = \dfrac{\Sigma_{11}^2}{\Sigma_{nn}^2}$

Then the condition number for

A is $K(A) = \dfrac{\Sigma_{11}}{\Sigma_{nn}}$

Square both sides

$$(K(A))^2 = \frac{\Sigma_{11}^2}{\Sigma_{nn}^2}$$

Thus $K(A^T A) = (K(A))^2$

$A^T A$ $\quad A = QR$

$A^T = R^T Q^T$

$$R^T Q^T Q R = R^T R$$