Homework 2

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February 11, 2021

1 Part 2

1.1 Problem 1

Using the Schur decomposition we can write $A = QUQ^T$ where A is symmetric U is upper triangular and Q is unitary Thus $Q^TAQ = U$ this implies that A and U are similar. Next if we transpose both sides we get the following

$$(Q^TAQ)^T = U^T$$

$$Q^TAQ = U^T$$

Since $U = U^T$ we know that U is symmetric but it was given by the Schur decomposition that U was upper triangular thus that implies U is diagonal. Also Since U is similar to A we know they share the same eigenvalues.

1.2 Problem 2

($\rightarrow$)Assume A$\in \mathbb{S}^n_{++}$ then we know for some eigenector $x \neq 0$

$$Ax = \lambda x > 0$$

$$x^TAx = \lambda \|x\|^2 > 0$$

$$\frac{x^TAx}{\|x\|^2} = \lambda > 0$$

($\leftarrow$)Assume A$\in \mathbb{S}^n$ and all $\lambda_i > 0$ by spectral theorem $A = PDP^T$ where P is an orthonormal basis in $\mathbb{R}^m$

$$P = \begin{bmatrix} P_1| & P_2| & ...| & P_m \end{bmatrix}.$$

Let $x = c_1P_1 + c_2P_2 + ... + c_mP_m$. Want to show that for any $x \neq 0$

$$x^TAx > 0$$

By the following steps

$$x^TPDP^Tx$$

$$= (c_1\|P_1\|^2|...|c_m\|P_m\|^2)D(c_1\|P_1\|^2|...|c_m\|P_m\|^2)^T$$
$$= (\lambda_1 c_1\|P_1\|^2|...|\lambda_m c_m\|P_m\|^2)(c_1\|P_1\|^2|...|c_m\|P_m\|^2)^T$$
$$= (\lambda_1 c_1\|P_1\|^4 + ... + \lambda_m c_m\|P_m\|^4)$$

Since all $\lambda_i > 0$ then for any $x \neq 0$ $x^T Ax > 0$ Therefore A is positive semidefinite.

1.3 Problem 3

They must be positive. Let A be Positive definite then $x^T Ax > 0$ then $\forall x \neq 0$ let $x = e_i$ the ith standard basis vector. Then $e_i^T Ae_i = A_{ii} > 0$. Thus we have shown that for any diagonal entry in a positive definite matrix must be positive.

1.4 Problem 4

a)

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A12 \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A12 \\ -A_{21}A_{11}^{-1}A_{11} + A_{21} & -A_{21}A_{11}^{-1}A_{11} + A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A12 \\ 0 & -A_{21}A_{11}^{-1}A_{11} + A_{22} \end{bmatrix}$$

b) In equation 3 we know that A_{11} has an LU-decomposition after you preform n-steps of gaussian elimination. Thus we can write $A_{11} = LU$. If we left multiply equation 3 by the following

$$\begin{bmatrix} L^{-1} & 0 \\ B & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} U & C \\ 0 & D \end{bmatrix}$$

We need to find what B is equal to in order to eliminate A_{21} thus we want to solve

$$BLU = -A_{21}$$
$$B = -A_{21}U^{-1}L^{-1}$$

so then our final matrix multiplication will be

$$\begin{bmatrix} L^{-1} & 0 \\ -A_{21}U^{-1}L^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} U & C \\ 0 & A_{22} - A_{21}U^{-1}L^{-1}A_{12} \end{bmatrix} = \begin{bmatrix} U & C \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

1.5 Problem 5

For problem 5 we want to show that no row swapping occurs when a matrix is strictly column diagonally dominant if we show that the resulting matrix after the first step of Gaussian elimination is diagonally dominant. The First step of the Gaussian elimination will clearly not swap rows. Let $b_{j,k} = a_{j,k} - a_{j,1}\frac{a_{1,k}}{a_{1,1}}$. Where $b_{j,k}$ is the entries after the first step of GE $j > 1$

$$\sum_{j=2, j\neq k}^{n} |b_{j,k}| = \sum_{j=2, j\neq k}^{n} |a_{j,k} - a_{j,1}\frac{a_{1,k}}{a_{1,1}}|$$

We can break this summation up using the triangle inequality.

$$\leq \sum_{j=2,j\neq k}^{n} |a_{j,k}| + \sum_{j=2,j\neq k}^{n} |a_{j,1}| \frac{|a_{1,k}|}{|a_{1,1}|}$$

Next we know that

$$|a_{k,k}| > \sum_{j=1,j\neq k}^{n} |a_{j,k}| = |a_{1,k}| + \sum_{j=2,j\neq k}^{n} |a_{j,k}|$$

This implies that

$$\sum_{j=2,j\neq k}^{n} |a_{j,k}| < |a_{k,k}| - |a_{1,k}|$$

Thus we can make the follow substitution. After this step I had gotten stuck on what to-do next so I searched on google and found a stack exchange article that had the full solution although it had the full solution the crucial part for me was know that

$$\sum_{j=2,j\neq k}^{n} \frac{|a_{j,1}|}{|a_{1,1}|}$$

could be replaced with

$$(1 - \frac{|a_{k,1}|}{|a_{1,1}|})$$

$$\leq |a_{k,k}| - |a_{1,k}| + |a_{1,k}| \sum_{j=2,j\neq k}^{n} \frac{|a_{j,1}|}{|a_{1,1}|}$$

Another fact we can use is

$$1 > \sum_{j=1}^{n} \frac{|a_{j,k}|}{|a_{k,k}|} = \frac{|a_{k,1}|}{|a_{1,1}|} + \sum_{j=2,j\neq k}^{n} \frac{|a_{j,1}|}{|a_{1,1}|}$$

This implies that

$$1 - \frac{|a_{k,1}|}{|a_{1,1}|} > \sum_{j=2,j\neq k}^{n} \frac{|a_{j,1}|}{|a_{1,1}|}$$

thus we can make the following substitution

$$\leq |a_{k,k}| + |a_{1,k}| + |a_{1,k}|(1 - \frac{|a_{k,1}|}{|a_{1,1}|})$$

From here we can distribute and simplify

$$|a_{k,k}| - |a_{1,k}| \frac{|a_{k,1}|}{|a_{1,1}|}$$

Then from here we can use the triangle inequality

$$\leq |a_{k,k} - a_{1,k}\frac{a_{k,1}}{a_{1,1}}|$$

$$\leq |b_{k,k}|$$

Now that we know the matrix is still diagonally dominant we can continue to do GE to and know there will be no row swaps. Here is the link to the mentioned stack exchange: Problem 5 Help