Homework 6

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Problem 1

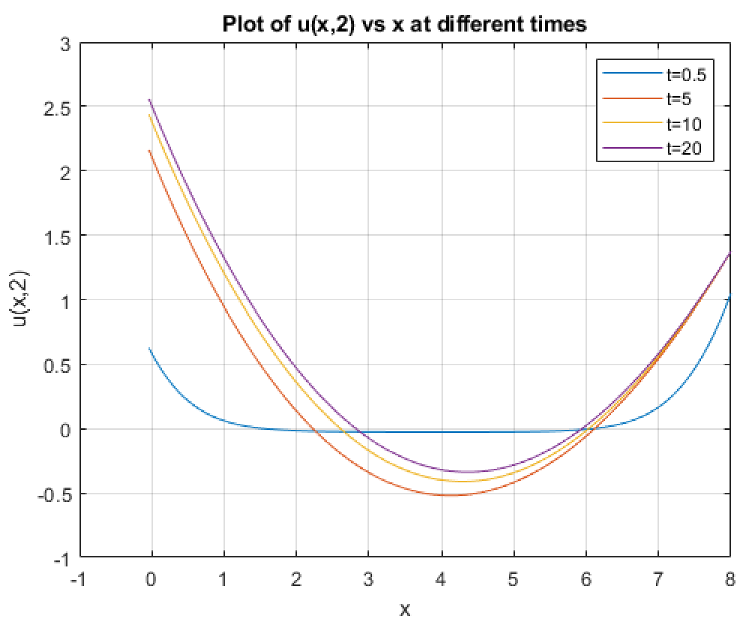
Submitted as a hand written pdf attached at the end of this report.

Problem 2

Problem 2 using the IBVP similar to problem 6 in homework 6 we had to modify the grid to properly plot the problem.

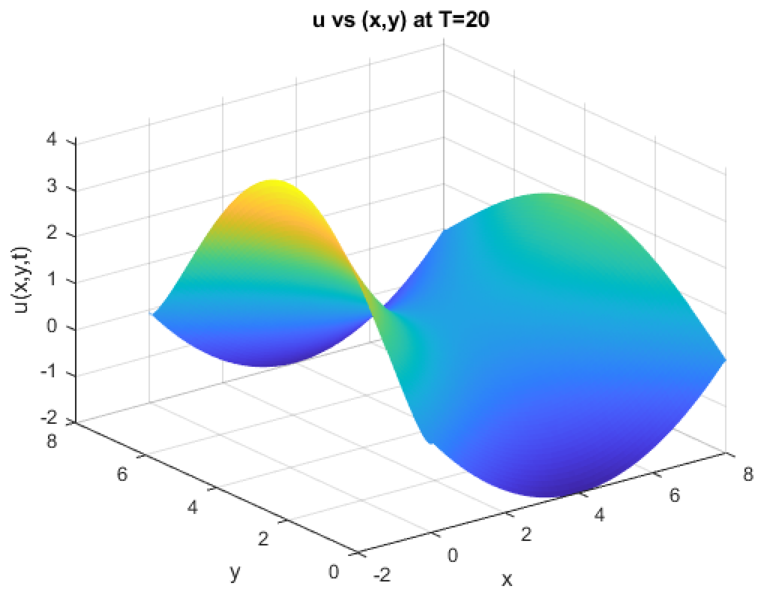
Part 1

Part 1 asked us to plot $u(x, 2)$ when $t = 0.5, 5, 10, 20$ shown below.



Part 2

Part 2 asked us to plot $u(x, y)$ when $t = 20$ as a surface pictured below.

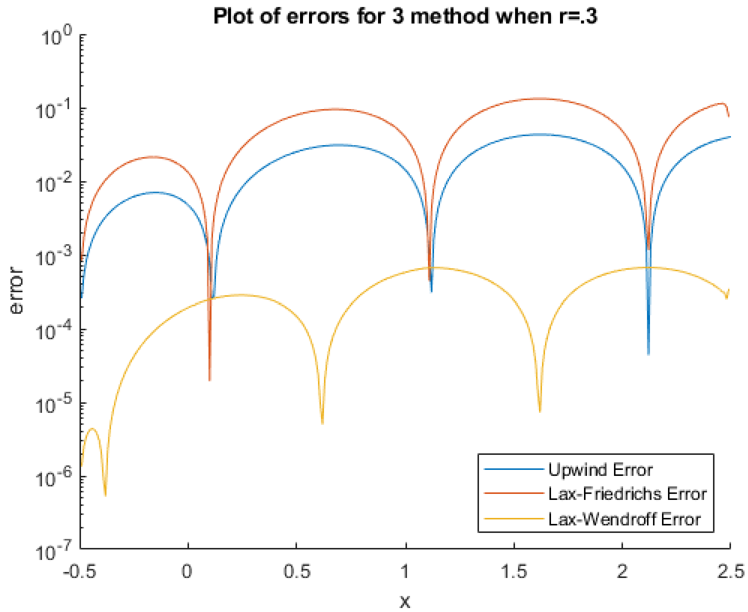


Problem 3

Problem 3 we solve an IBVP using three different methods Upwind, Lax-Friedrichs, and Lax-Wendroff.

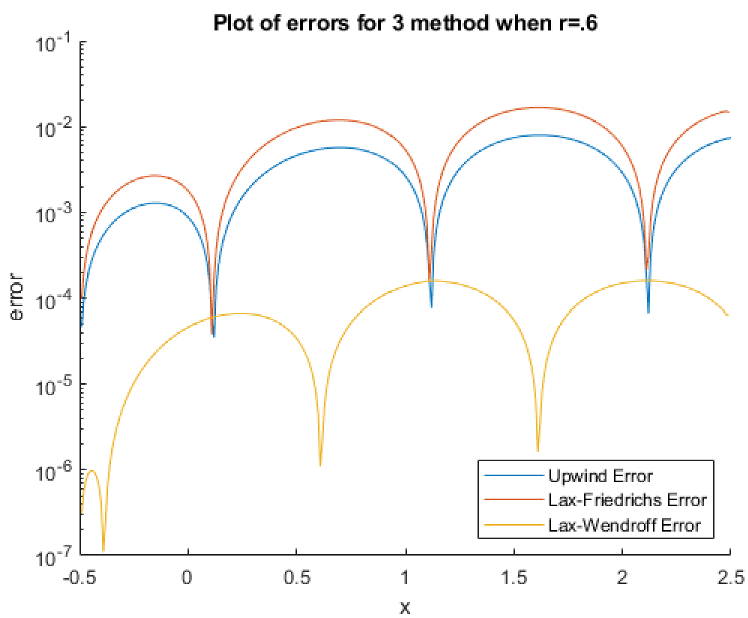
Part 1

Part 1 asks us to plot the error vs x for the 3 methods at $t = 1.08$ in one graph. One when $r = 0.3$ and one when $r = 0.6$ shown below.



Part 2

Part 2 asks us which r value and method has the smallest error. From the graphs we see that when $r = 0.6$ the Lax-Wendroff gives smallest error.

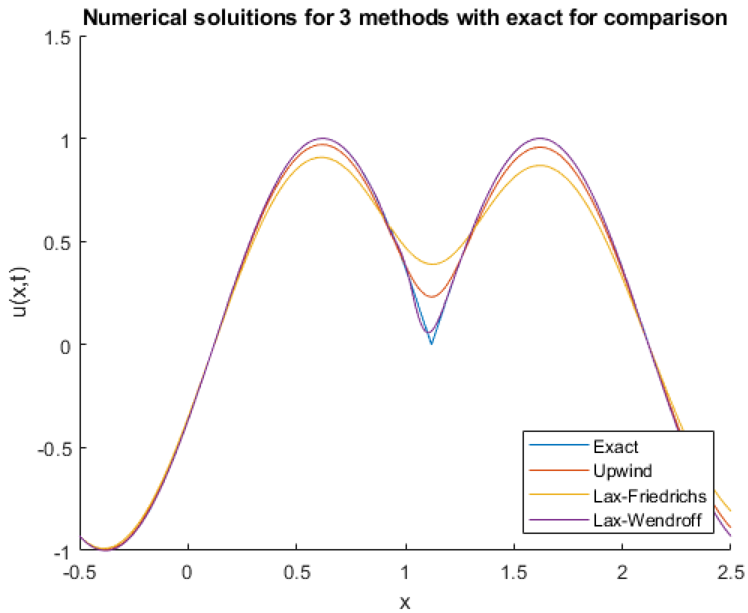


Problem 4

Using the same IBVP from problem we modify the initial value.

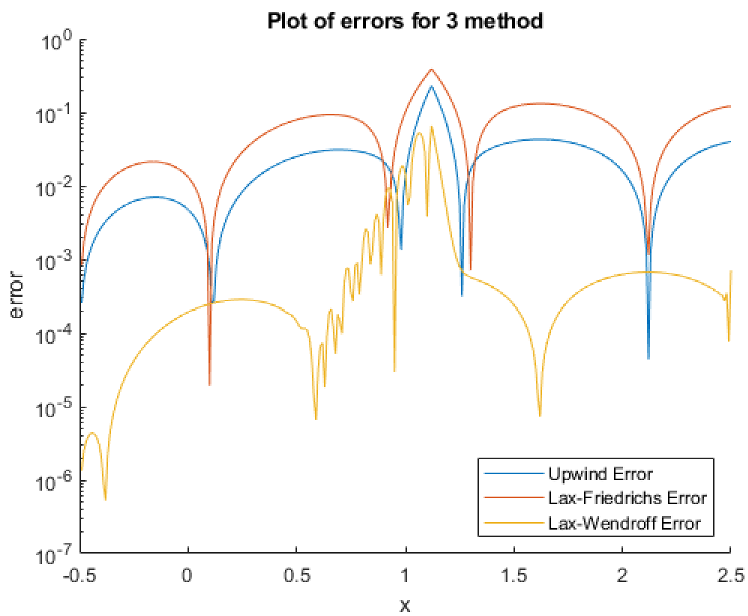
Part 1

Part 1 asks us to plot the numerical solutions of the 3 methods mentioned above vs the exact solution at $t = 1.08$.



Part 2

Next in part 2 we plot error vs x for the three methods at $t = 1.08$.



Part 3

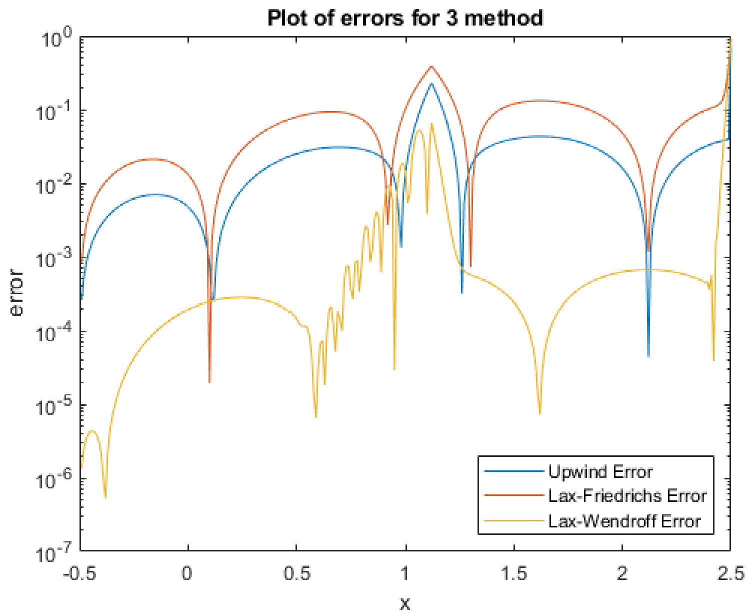
We can see that as the numerical methods approach the cusp in the exact solution that is where maximum error occurs.

Problem 5

Again we continue with the IBVP from problem 4 but this time change the ad hoc boundry condition.

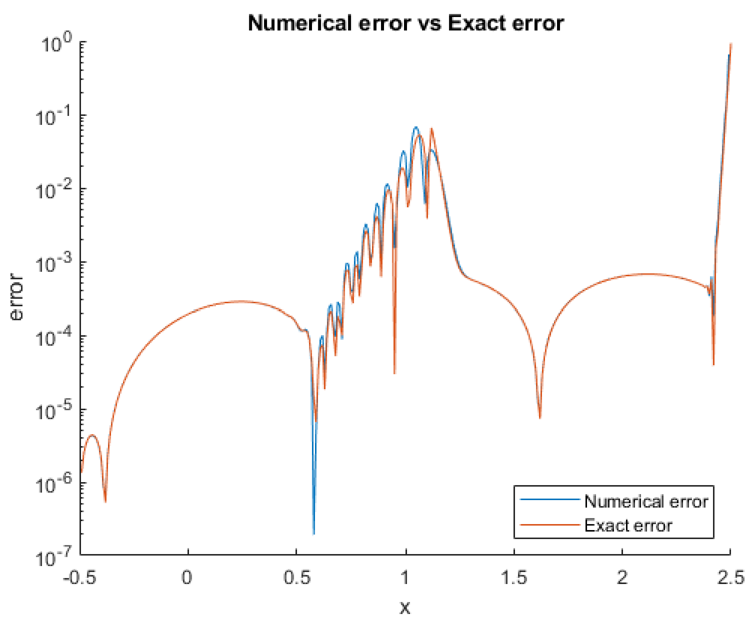
Part 1

Part 1 we plot the error of the 3 methods vs x at time $t = 1.08$ looking at the graph below we can see that the BC does not affect the deep interior of computational region far away from $x_N = 2.5$.



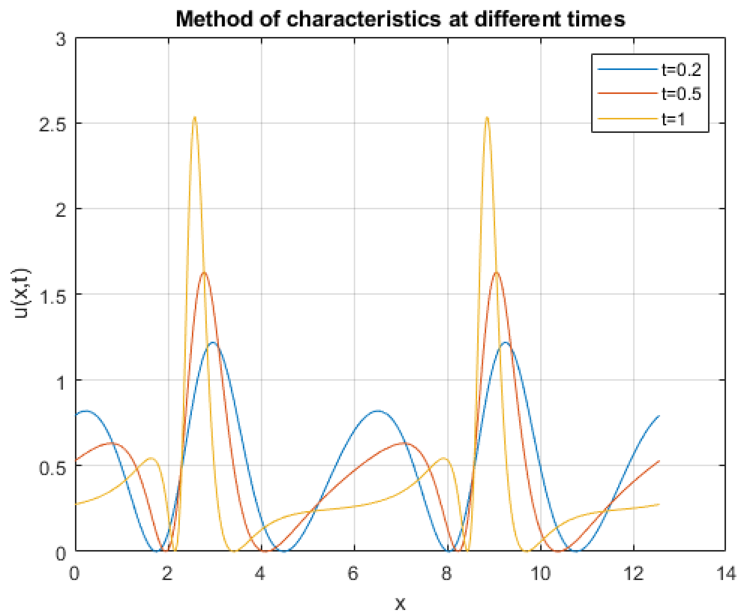
Part 2

Part 2 we are only looking at Lax-Wendroff and we plot the exact error vs the numerical error.



Problem 6

Problem 6 asks us to solve an IVP linear hyperbolic PDE with variable coefficients by using the method of characteristics. We calculted u(x,t) at three different time levels $t = 0.2, 0.5, 1$. We were then asked to plot $u(x,t)$ vs x for the 3 different time levels.



Homework 7

Friday, May 14, 2021 10:06 AM

Problem 1 (Theoretical)

1)

Consider the Lax-Friedrichs method for solving $u_t + a\, u_x = 0$

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

On the RHS, we write $\frac{u_{i+1}^n + u_{i-1}^n}{2}$ as $u_i^n + \underbrace{\frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}_{\text{Added viscosity}}$.

We know that the Lax-Friedrichs method has too much added viscosity. So we consider a modified version of Lax-Friedrichs

$$u_i^{n+1} = u_i^n + \frac{q}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad 0 \le q \le 1 \qquad \text{(LF-2)}$$

Part 1: Find the modified PDE of (LF-2).

Part 2: Find the modified PDE of the implicit upwind method

$$u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Hint: Expanding around (x_i, t_{n+1}) will make it easier.

Part 1

$$w(x_i,t_{n+1}) - w(x_i,t_n) = \frac{q}{2}\big(w(x_{i+1},t_n) - 2w(x_i,t_n) + w(x_{i-1},t_n)\big) - \frac{a\Delta t}{\Delta x}\big(w(x_{i+1},t_n) - w(x_{i-1},t_n)\big)$$

Expand around (x_i, t_n)

$$LHS = w_t\Big|_{(x_i,t_n)} \Delta t + w_{tt}\Big|_{(x_i,t_n)} \frac{(\Delta t)^2}{2} + O\left((\Delta t)^3\right)$$

$$RHS = \frac{q}{2}\left[w_{xx}\Big|_{(x_i,t_n)} (\Delta x)^2 + O\left((\Delta x)^4\right)\right] - a\frac{\Delta t}{\Delta x}\left[w_x\Big|_{(x_i,t_n)} \Delta x + O\left((\Delta x)^3\right)\right]$$

divide by Δt

$$w_t = -a w_x + \frac{q}{2} w_{xx} \frac{(\Delta x)^2}{\Delta t} - w_{tt}\frac{\Delta t}{2} + O\left((\Delta t)^2\right) + O\left((\Delta x)^2\right) + \frac{O\left((\Delta x)^4\right)}{\Delta t}$$

$$w_t = -a w_x + O(\Delta t + \Delta x)$$

$$w_{tt} = (-a w_x)_t + O(\Delta t + \Delta x) = -a(w_t)_x + O(\Delta t + \Delta x)$$

$$= -a(-aw_x)_x + O(\Delta t + \Delta x) = a^2 w_{xx} + O(\Delta t + \Delta x)$$

$$w_t = -aw_x + \left[\frac{q}{2}\frac{(\Delta x)^2}{\Delta t} - \frac{a^2 \Delta t}{2}\right] w_{xx} + O\left((\Delta t)^2 + (\Delta x)^2\right)$$

eq for $w(x,t)$

$$w_t = -a w_x + \sigma w_{xx}$$

where $\sigma = \frac{\Delta x}{2}\left[q r^{-1} - a^2 r\right]$

Part 2 $u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right)$, $r = \frac{\Delta t}{\Delta x}$

$$w(x_i, t_{n+1}) - w(x_i, t_n) = -ar\left(w(x_i, t_{n+1}) - w(x_{i-1}, t_{n+1})\right)$$

Expand around (x_i, t_{n+1})

RHS: $w_t\Big|_{(x_i,t_{n+1})} \Delta t - w_{tt}\Big|_{(x_i,t_{n+1})} \frac{(\Delta t)^2}{2} + O((\Delta t)^3)$

LHS: $-a\frac{\Delta t}{\Delta x}\left[w_x\Big|_{(x_i,t_{n+1})} \Delta x - w_{xx}\Big|_{(x_i,t_{n+1})} \frac{(\Delta x)^2}{2} + O((\Delta x)^3)\right]$

divide by Δt

$$w_t = -a\, w_x + \frac{a\Delta x}{2} w_{xx} + \frac{\Delta t}{2} w_{tt} + O\left((\Delta x)^2 + (\Delta t)^2\right)$$

$$w_t = -a w_x + O(\Delta t + \Delta x)$$

$$w_{tt} = (-a w_x)_t + O(\Delta t + \Delta x) = -a\,(w_t)_x + O(\Delta t + \Delta x)$$

$$= -a(-a w_x)_x + O(\Delta t + \Delta x) = a^2 w_{xx} + O(\Delta t + \Delta x)$$

$$w_t = -a w_x + \left[\frac{a\Delta x}{2} + \frac{a^2 \Delta t}{2}\right] w_{xx} + O\left((\Delta x)^2 + (\Delta t)^2\right)$$

$$w_t = -a w_x + \sigma w_{xx}$$

where $\sigma = \frac{a\Delta x}{2}\left[1 + a r\right]$