Homework 1

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Problem 1 (Theoretical)

Suppose E_n satisfies the recursive inequality

$$E_{n+1} \leq (1+Ch)E_n + h^2 \quad \text{for } n \geq 0$$
$$E_0 = 0$$

where $C > 0$ is a constant independent of h and n.

Derive that $E_N \leq \frac{e^{CT}-1}{C} h$ for $Nh \leq T$

1) $(1+Ch)^{-(n+1)} E_{n+1} \leq ((1+Ch)E_n + h^2)(1+Ch)^{-(n+1)}$

$$(1+Ch)^{-(n+1)} E_{n+1} \leq (1+Ch)^{-n} E_n + h^2 (1+Ch)^{-(n+1)}$$

$$(1+Ch)^{-(n+1)} E_{n+1} - (1+Ch)^{-n} E_n \leq h^2 (1+Ch)^{-(n+1)}$$

Next sum from $n=0$ to $n=N-1$ where $E_0 = 0$

we get

$$(1+Ch)^{-N} E_N \leq h^2 \sum_{n=0}^{N-1} (1+Ch)^{-(n+1)}$$

From HW#0 $\sum_{n=0}^{N-1} r^{n+1} = r \frac{1-r^N}{1-r}$

$$\leq h^2 \cdot (1+ch)^{-1} \left(\frac{1-(1+ch)^{-N}}{1-(1+ch)^{-1}} \right)$$

$$\leq \frac{h}{c}\left(1-(1+ch)^{-N}\right)$$

Multiply by $(1+ch)^N$ and use $1+ch \leq e^{ch}$

$$E_N \leq \frac{h}{c}\left((1+ch)^N - 1\right) \leq \frac{h}{c}\left(e^{chN}-1\right) \leq \frac{e^{cT}-1}{c} h$$

where $T \geq Nh$

Hence we have

$$E_N \leq \frac{e^{cT}-1}{c} h \quad \blacksquare$$