Homework 7

Friday, May 14, 2021 10:06 AM

Problem 1 (Theoretical)

1) Consider the Lax-Friedrichs method for solving $u_t + a\, u_x = 0$

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

On the RHS, we write $\frac{u_{i+1}^n + u_{i-1}^n}{2}$ as $u_i^n + \underbrace{\frac{1}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}_{\text{Added viscosity}}$.

We know that the Lax-Friedrichs method has too much added viscosity. So we consider a modified version of Lax-Friedrichs

$$u_i^{n+1} = u_i^n + \frac{q}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}, \quad 0 \le q \le 1 \qquad \text{(LF-2)}$$

Part 1: Find the modified PDE of (LF-2).

Part 2: Find the modified PDE of the implicit upwind method

$$u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Hint: Expanding around (x_i, t_{n+1}) will make it easier.

Part 1

$$w(x_i,t_{n+1}) - w(x_i,t_n) = \frac{q}{2}\big(w(x_{i+1},t_n) - 2w(x_i,t_n) + w(x_{i-1},t_n)\big) - \frac{a\Delta t}{\Delta x}\big(w(x_{i+1},t_n) - w(x_{i-1},t_n)\big)$$

Expand around (x_i,t_n)

$$LHS = w_t\Big|_{(x_i,t_n)}\Delta t + w_{tt}\Big|_{(x_i,t_n)}\frac{(\Delta t)^2}{2} + O\left((\Delta t)^3\right)$$

$$RHS = \frac{q}{2}\left[w_{xx}\Big|_{(x_i,t_n)}(\Delta x)^2 + O\left((\Delta x)^4\right)\right] - \frac{a\Delta t}{\Delta x}\left[w_x\Big|_{(x_i,t_n)}\Delta x + O\left((\Delta x)^3\right)\right]$$

divide by Δt

$$w_t = -a w_x + \frac{q}{2} w_{xx}\frac{(\Delta x)^2}{\Delta t} - w_{tt}\frac{\Delta t}{2} + O\left((\Delta t)^2\right) + O\left((\Delta x)^2\right) + \frac{O\left((\Delta x)^4\right)}{\Delta t}$$

$$w_t = -a w_x + O(\Delta t + \Delta x)$$

$$w_{tt} = (-a w_x)_t + O(\Delta t + \Delta x) = -a(w_t)_x + O(\Delta t + \Delta x)$$

$$= -a(-a w_x)_x + O(\Delta t + \Delta x) = a^2 w_{xx} + O(\Delta t + \Delta x)$$

$$w_t = -a w_x + \left[\frac{q}{2}\frac{(\Delta x)^2}{\Delta t} - \frac{a^2 \Delta t}{2}\right] w_{xx} + O\left((\Delta t)^2 + (\Delta x)^2\right)$$

eq for $w(x,t)$

$$w_t = -a w_x + \sigma w_{xx}$$

where $\sigma = \frac{\Delta x}{2}\left[q r^{-1} - a^2 r\right]$

<u>Part 2</u> $u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right)$, $r = \frac{\Delta t}{\Delta x}$

$$w(x_i, t_{n+1}) - w(x_i, t_n) = -ar\left(w(x_i, t_{n+1}) - w(x_{i-1}, t_{n+1})\right)$$

Expand around (x_i, t_{n+1})

RHS: $w_t\Big|_{(x_i,t_{n+1})} \Delta t - w_{tt}\Big|_{(x_i,t_{n+1})} \frac{(\Delta t)^2}{2} + O\left((\Delta t)^3\right)$

LHS: $-a\frac{\Delta t}{\Delta x}\left[w_x\Big|_{(x_i,t_{n+1})} \Delta x - w_{xx}\Big|_{(x_i,t_{n+1})} \frac{(\Delta x)^2}{2} + O\left((\Delta x)^3\right)\right]$

divide by Δt

$$W_t = -a\, W_x + \frac{a\Delta x}{2} W_{xx} + \frac{\Delta t}{2} W_{tt} + O\left((\Delta x)^2 + (\Delta t)^2\right)$$

$$W_t = -a W_x + O(\Delta t + \Delta x)$$

$$W_{tt} = (-a W_x)_t + O(\Delta t + \Delta x) = -a\,(W_t)_x + O(\Delta t + \Delta x)$$

$$= -a(-a W_x)_x + O(\Delta t + \Delta x) = a^2 W_{xx} + O(\Delta t + \Delta x)$$

$$W_t = -a W_x + \left[\frac{a\Delta x}{2} + \frac{a^2 \Delta t}{2}\right] W_{xx} + O\left((\Delta x)^2 + (\Delta t)^2\right)$$

$$W_t = -a W_x + \sigma W_{xx}$$

where $\sigma = \frac{a\Delta x}{2}\left[1 + a r\right]$