Homework 6

Thursday, May 6, 2021 10:28 AM

1)

Problem 1 (Theoretical)

Consider the Lax-Friedrichs method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Part 1: Carry out von Neumann stability analysis.

Part 2: Use Taylor expansion to find the local truncation error e_i^n.

Find coefficients of $(\Delta t)^2$, $(\Delta t)(\Delta x)$ and $(\Delta x)^2$ terms in e_i^n.

Part 3: Answer the two questions below.

For fixed $\frac{\Delta t}{\Delta x} = r$, write $\frac{e_i^n}{\Delta t}$ in terms of Δx only. Do we have $\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0$?

For fixed $\frac{\Delta t}{(\Delta x)^2} = c$, write $\frac{e_i^n}{\Delta t}$ in terms of Δx only. Do we have $\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0$?

Part 1

$$U_i^n = \rho^n \exp(\sqrt{-1}\,\xi i \Delta x)$$

$$\rho^{n+1} \exp(\sqrt{-1}\,\xi i \Delta x) = \frac{\rho^n \exp(\sqrt{-1}\,\xi (i+1)\Delta x) + \rho^n \exp(\sqrt{-1}\,\xi (i-1)\Delta x)}{2}$$

$$- \frac{ar}{2}\left(\rho^n \exp(\sqrt{-1}\,\xi (i+1)\Delta x) - \rho^n \exp(\sqrt{-1}\,\xi (i-1)\Delta x)\right)$$

$$\rho = \frac{\exp(\sqrt{-1}\,\xi \Delta x) + \exp(-\sqrt{-1}\,\xi \Delta x)}{2}$$

$$- \frac{ar}{2}\left(\exp(\sqrt{-1}\,\xi \Delta x) - \exp(-\sqrt{-1}\,\xi \Delta x)\right)$$

$$\rho = \cos(\xi \Delta x) - ari\sin(\xi \Delta x)$$

For stability we need $|\rho| \leq 1 + C\Delta t$

$$|\rho| = \sqrt{\cos^2(\xi \Delta x) + a^2 r^2 \sin^2(\xi \Delta x)}$$

$|\rho| > 0$ for any ar

$$|\rho| = \sqrt{1 - \sin^2(\xi\Delta x) + a^2r^2\sin^2(\xi\Delta x)}$$

$$= \sqrt{1 + (a^2r^2 - 1)(\sin^2(\xi\Delta x))}$$

let $ar > 1$ then $a^2r^2 > 1$

let $w = a^2r^2 - 1 > 0$

thus

$$\sqrt{1 + w\sin^2(\xi\Delta x)} > 1$$

and also let

$ar < -1$ then $a^2r^2 > 1$

$u = a^2r^2 - 1 > 0$

thus

$$\sqrt{1 + u\sin^2(\xi\Delta x)} > 1$$

Finally $|ar| < 1$ then $a^2r^2 < 1$

$V = a^2r^2 - 1 < 0$

$$\sqrt{1 + v\sin^2(\xi\Delta x)} \leq 1$$

We see for stability $|\rho| \leq 1$ it must be that $-1 \leq ar \leq 1$

Part 2

$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{ar}{2}\left(u_{i+1}^n - u_{i-1}^n\right), \quad r = \frac{\Delta t}{\Delta x}$

expand around (x_i, t_n)

$$e_i^n = U(x_i, t_{n+1}) - \frac{1}{2}\left[U(x_{i+1}, t_n) + U(x_{i-1}, t_n)\right] + \frac{ar}{2}\left[U(x_{i+1}, t_n) + U(x_{i-1}, t_n)\right]$$

Expand

$$U(x_i, t_{n+1}) = U + U_t \Delta t + \frac{U_{tt} \Delta t^2}{2} + O(\Delta t^3)$$

Expand

$$U(x_{i+1}, t_n) + U(x_{i-1}, t_n) = 2U + U_{xx} \Delta x^2 + O(\Delta x^4)$$

Expand

$$U(x_{i+1}, t_n) - U(x_{i-1}, t_n) = 2U_x \Delta x + O(\Delta x^3)$$

Sub in

$$\cancel{U} + U_t \Delta t + \frac{U_{tt} \Delta t^2}{2} - \cancel{U} - \frac{U_{xx} \Delta x^2}{2} + a \Delta t U_x + \Delta t\, O(\Delta x^2) + O(\Delta x^4) + O(\Delta t^3)$$

$$\underbrace{U_t \Delta t + a \Delta t U_x}_{=0 \text{ from problem}} + \frac{U_{tt} \Delta t^2}{2} - \frac{U_{xx} \Delta x^2}{2} + \Delta t\, O(\Delta x^2) + O(\Delta x^4) + O(\Delta t^3)$$

$$e_i^n = \frac{1}{2}\left(U_{tt}\Delta t^2 - U_{xx}\Delta x^2\right) + \Delta t\, O(\Delta x^2) + O(\Delta x^4) + O(\Delta t^3) \quad \blacksquare$$

Part 3

$$\frac{e_i^n}{\Delta t} = \frac{1}{2}\left(U_{tt}\Delta t - U_{xx}\frac{\Delta x^2}{\Delta t}\right) + O(\Delta x^2) + \frac{O(\Delta x^4)}{\Delta t} + O(\Delta t^2)$$

when

$$\Delta t = r\Delta x \Rightarrow \frac{1}{2}\left(U_{tt}\, r\Delta x - \frac{U_{xx}}{r}\Delta x\right) + O(\Delta x^2) + \frac{O(\Delta x^3)}{r} + O(r^2\Delta x^2)$$

$$\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = 0$$

When

$$\Delta t = c\Delta x^2$$

$$\Rightarrow \frac{1}{2}\left(U_{tt}\, c\Delta x^2 - U_{xx}\, c^{-1}\right) + O(\Delta x^2) + \frac{O(\Delta x^2)}{c} + O(c^2\Delta x^4)$$

$$\lim_{\Delta x \to 0} \frac{e_i^n}{\Delta t} = \frac{-U_{xx}\, c^{-1}}{2} \neq 0 \quad \blacksquare$$

Problem 2 (Theoretical)

Carry out von Neumann stability analysis on each of the methods below

i) the BTCS method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = u_i^n - \frac{ar}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

ii) the implicit upwind method for the case of $a > 0$

$$u_i^{n+1} = u_i^n - ar\left(u_i^{n+1} - u_{i-1}^{n+1}\right), \qquad r = \frac{\Delta t}{\Delta x}$$

Hint: Examine $|1/\rho|^2$.

2 i)

$$\rho^{n+1}\exp(\sqrt{-1}\,\xi i\Delta x) = \rho^n \exp(\sqrt{-1}\,\xi i\Delta x) - \frac{ar}{2}\rho^{n+1}\left(\exp(\sqrt{-1}(i+1)\xi\Delta x) - \exp(\sqrt{-1}(i-1)\xi\Delta x)\right)$$

$$\rho = 1 - \frac{ar\rho}{2}\left(\exp(\sqrt{-1}\,\xi\Delta x) - \exp(-\sqrt{-1}\,\xi\Delta x)\right)$$

$$\rho = 1 - ar\rho i \sin(\xi\Delta x)$$

$$\rho + ar\rho i \sin(\xi\Delta x) = 1$$

$$\rho = \frac{1}{1 + ari\sin(\xi\Delta x)}$$

$$|\rho| = \frac{1}{\sqrt{1 + a^2 r^2 \sin^2(\xi\Delta x)}}$$

Since the denominator is always greater than or equal to 1

this implies $|\rho| \leq 1$ so we have stability

ii)

$$\rho^{n+1}\exp(\sqrt{-1}\,\xi i\Delta x) = \rho^{n}\exp(\sqrt{-1}\,\xi i\Delta x) - ar\,\rho^{n+1}\exp(\sqrt{-1}\,\xi i\Delta x)\left(1 - \exp(-\sqrt{-1}\,\xi\Delta x)\right)$$

$$\rho = 1 - ar\rho\left(1 - \exp(-\sqrt{-1}\,\xi\Delta x)\right)$$

$$\rho = \frac{1}{1 + ar\left(1 - \cos(\xi\Delta x) + i\sin(\xi\Delta x)\right)}$$

$$|\rho|^2 = \frac{1}{(1+ar-ar\cos(\xi\Delta x))^2 + (ar\sin(\xi\Delta x))^2}$$

$$= \frac{1}{1+2ar-2ar\cos(\xi\Delta x)+(ar)^2-2(ar)^2\cos(\xi\Delta x)+(ar)^2\cancel{\cos^2}(\xi\Delta x)+(ar)^2\cancel{\sin^2}(\xi\Delta x)}$$

$$= \frac{1}{1+2ar-2ar\cos(\xi\Delta x)+2a^2r^2-2a^2r^2\cos(\xi\Delta x)}$$

$$= \frac{1}{1+2ar(1-\cos(\xi\Delta x)+ar-ar\cos(\xi\Delta x))}$$

$$\frac{1}{|\rho|^2} = 1+2ar(1-\cos(\xi\Delta x)+ar-ar\cos(\xi\Delta x))$$

$$= 1+2ar(1-\cos(\xi\Delta x)+ar(1-\cos(\xi\Delta x)))$$

$$= 1+2ar(1-\cos(\xi\Delta x))(1+ar)$$

Since $a > 0$

$$\Rightarrow \frac{1}{|\rho|^2} \geq 1 \Rightarrow |\rho| \leq 1$$

Thus we have shown that the implicit upwind method is unconditionally stable.