

Homework 4

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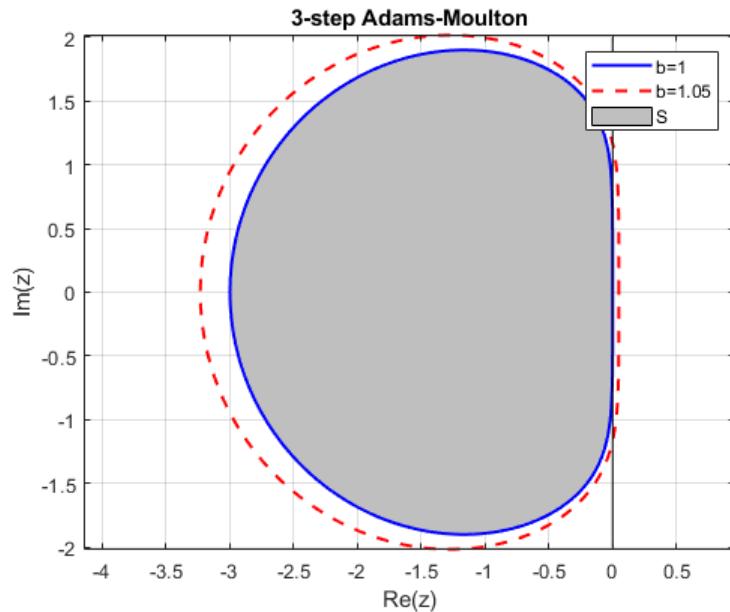
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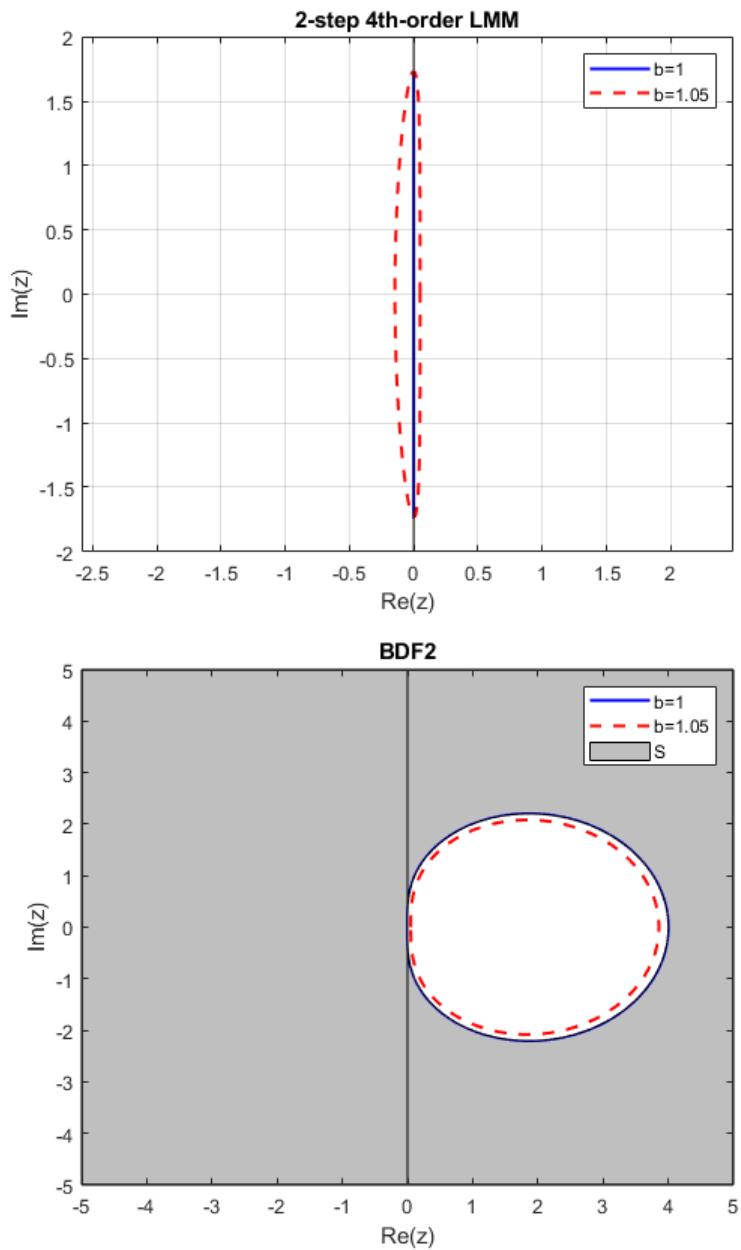
Problem 1

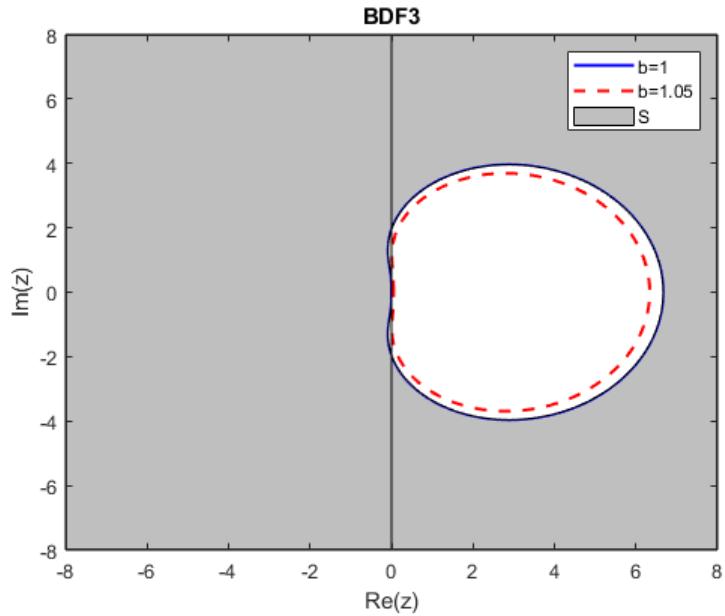
Submitted as a hand written pdf attached at the end of this report.

Problem 2

In problem 2 we are asked to plot the region of absolute stability for the following four methods 3-step Adams-Moulton, 2-step 4th-order LMM, BDF2, and BDF3. Here are the results







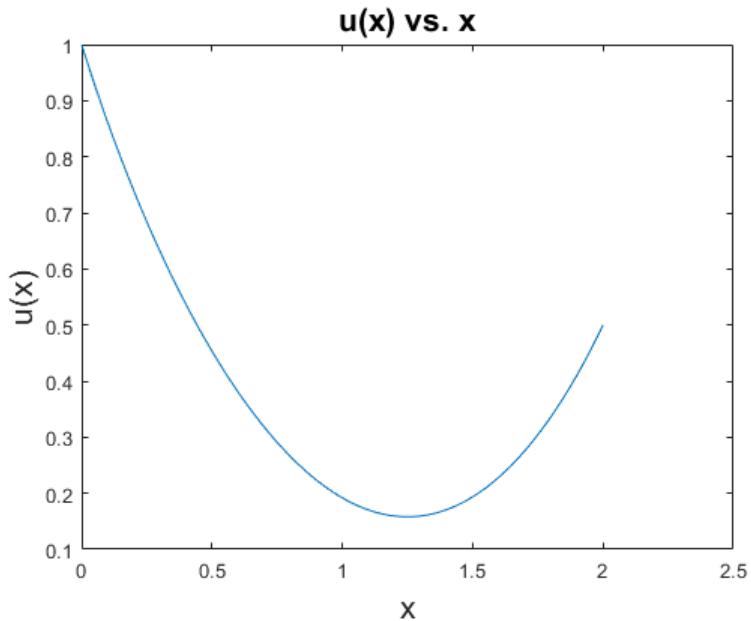
Problem 3

In problem 3 we implemented the shooting method to solve the following BVP

$$u'' - (1 + 0.5u'^2)u = \sin x$$

$$u(0) = 1, \quad u(2) = 0.5$$

We started $v_0 = -1$ and the value of v found was $v = -1.4851$. Below is a plot of $u(t)$ vs. x .



Problem 4

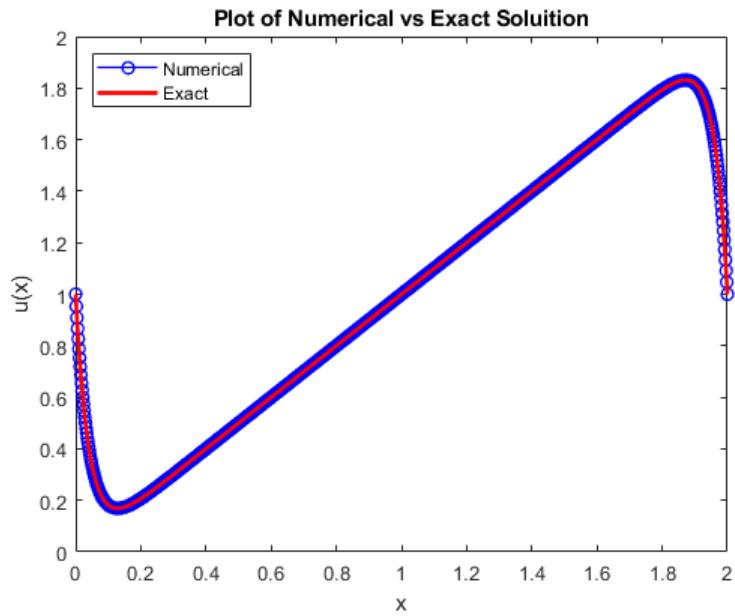
In Problem 4 we used the FDM to solve the following BVP

$$u'' - 625u = -625x$$

$$u(0) = 1, \quad u(2) = 1$$

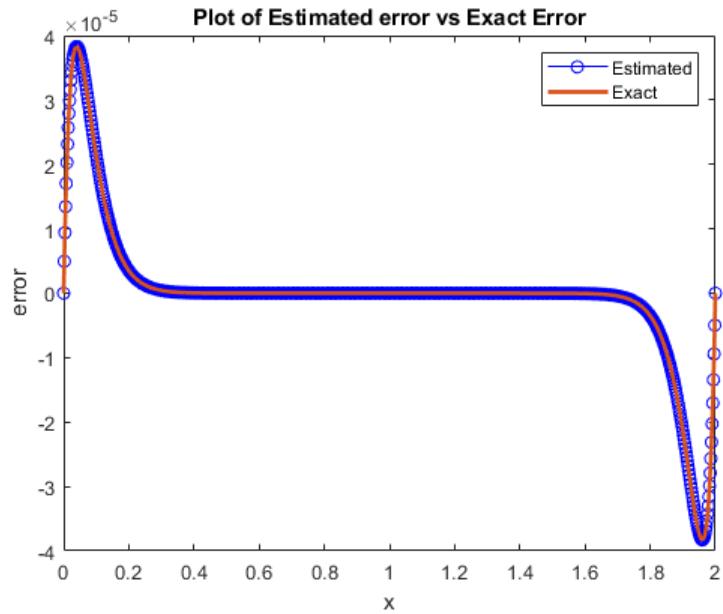
Part 1

For part one we were simply asked to plot the numerical solution $u(x)$ vs x and the exact solution $u(x)$ vs x on one figure shown below.



Part 2

In part 2 we estimated the error of the numerical solution from part 1 and plot it on the same figure as the exact error. Figure shown below.



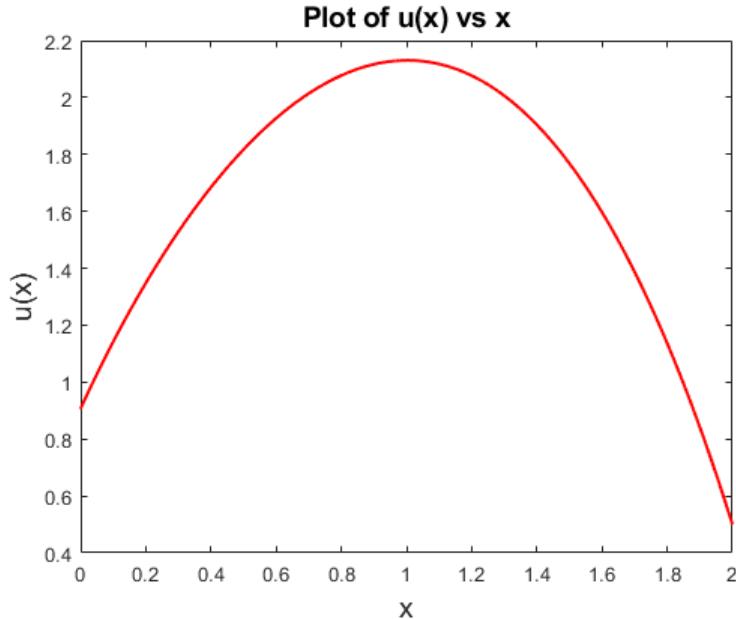
Problem 5

Problem 5 had us edit FDM to solve the following BVP

$$u'' - (1 + \exp(-\sin x))u = -5 - (\sin x)^2$$

$$u'(0) = 2.5, u(2) = 0.5$$

. The change was given to us in the homework problem so we only had to implement it in matlab. Below is a plot of numerical $u(x)$ vs x .



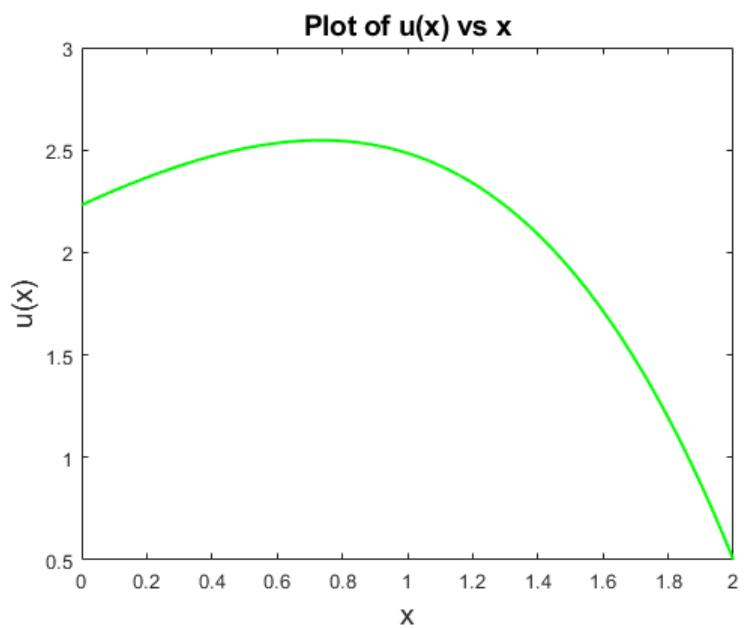
Problem 6

Problem 6 had us edit FDM similar to problem 5 to solve the following BVP

$$u'' - (1 + \exp(-\sin x))u = -5 - (\sin x)^2$$

$$u(0) - u'(0) = 1.5, u(2) = 0.5$$

. The derivation of the discretization for FDM is attached at the end of this report. Below is a plot of numerical $u(x)$ vs x .



Problem 1 (Theoretical)

Consider the one-stage implicit RK method described by Butcher tableau

$$\begin{array}{c|cc} c^T & A \\ \hline b & \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

Part 1: Show that it has second order accuracy.

Hint: Check the conditions on second order accuracy.

Part 2: Derive its stability function $\phi(z)$.

Part 3: Show that it is A-stable, but not L-stable.

Part 1

internal condition

$$c_i = \sum_{j=1}^p a_{ij} \Rightarrow \frac{1}{2} = \frac{1}{2} \checkmark$$

Condition for first order

$$\sum_{i=1}^p b_i = 1 \Rightarrow 1 = 1 \checkmark$$

Condition for the second order

$$\sum_{i=1}^p b_i c_i = \frac{1}{2} \Rightarrow 1 \cdot \frac{1}{2} = \frac{1}{2} \checkmark$$

Part 2

Rk form of BT

$$k_1 = h f(U_n + \frac{1}{2} k_1, t_n + \frac{h}{2})$$

$$U_{n+1} = U_n + h f(U_n + \frac{1}{2} k_1, t_n + \frac{h}{2})$$

$$f(u) = \gamma u \quad k_1 = h \gamma (U_n + \frac{1}{2} k_1)$$

$$\varepsilon = h\gamma \quad k_1 = \varepsilon U_n + \frac{\varepsilon}{2} k_1$$

$$k_1 - \frac{\varepsilon}{2} k_1 = \varepsilon U_n$$

$$k_1 = \frac{\varepsilon U_n}{1 - \frac{\varepsilon}{2}}$$

$$U_{n+1} = U_n + \frac{\varepsilon U_n}{1 - \frac{\varepsilon}{2}}$$

$$U_{n+1} = \phi(\varepsilon) U_n = \left(1 + \frac{\varepsilon}{1 - \frac{\varepsilon}{2}}\right) U_n$$

$$\phi(\varepsilon) = 1 + \frac{\varepsilon}{1 - \frac{\varepsilon}{2}}$$

Part 3

$$|\phi(\varepsilon)| < 1 \quad \forall \operatorname{Re}(\varepsilon) < 0$$

$$|\phi(z)| = \left| 1 + \frac{z}{1 - \frac{z}{2}} \right| < 1$$

$$= \left| \frac{1 - \frac{z}{2} + z}{1 - \frac{z}{2}} \right| < 1$$

$$= \left| \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}} \right| < 1$$

$$= \left| 1 + \frac{z}{2} \right| < \left| 1 - \frac{z}{2} \right| \quad \text{Let } z = a+bi$$

$$= \left(1 + \frac{a+bi}{2} \right) \left(1 + \frac{a+bi}{2} \right) < \left(1 - \frac{a+bi}{2} \right) \left(1 - \frac{a+bi}{2} \right)$$

$$\left| 1 + \frac{a+bi}{2} + \frac{a+bi}{2} + \frac{a^2+b^2}{4} \right| < \left| 1 - \frac{a+bi}{2} - \frac{a+bi}{2} + \frac{a^2+b^2}{4} \right|$$

$$\left| 1 + a + \cancel{\frac{a^2+b^2}{4}} \right| < \left| 1 - a + \cancel{\frac{a^2+b^2}{4}} \right|$$

$$1 + 2a < 1$$

$$2a < 0$$

$$\alpha < 0$$

$$\text{thus } |\phi(z)| < 1 \quad \forall \operatorname{Re}(z) < 0$$

So this method is A-stable \blacksquare

To check if L-stable

$$\phi(z) \rightarrow 0 \quad \text{as } z \rightarrow \infty$$

$$\lim_{z \rightarrow \infty} \left(1 + \frac{z}{1 - \frac{z}{2}} \right) = 1 + \lim_{z \rightarrow \infty} \frac{2z}{2-z} = 1 - 2 = -1 \blacksquare$$

Thus method is not L-stable.

Problem 6 (Theoretical and computational)

6)

Consider a slightly different version of the BVP in Problem 5.

$$\begin{cases} u'' - (1 + \exp(-\sin x))u = -5 - (\sin x)^2 \\ u(0) - u'(0) = 1.5, \quad u(2) = 0.5 \end{cases} \quad (\text{P6})$$

Design the discretization of the finite difference method (FDM). Implement it in Matlab.
Solve the BVP (P6) with $N = 1000$.

Plot the numerical $u(x)$ vs x .

Need to discretize the BC

$$U(0) - U'(0) = 1.5$$

$$\frac{U_1 + U_0}{2} - \frac{U_1 - U_0}{h} = \alpha \Rightarrow$$

$$(2+h)U_0 - (2-h)U_1 = 2\alpha h$$

Solve for V_0

$$V_0 = \frac{2\alpha h + (2-h)V_1}{2+h}$$

$$\frac{V_{i+1} - 2V_i + V_{i-1}}{h^2} + P_i \frac{V_{i+1} - V_{i-1}}{2h} + q_b V_i = r_i$$

At $i := 1$

$$\frac{V_2 - 2V_1 + V_0}{h^2} + P_1 \frac{V_2 - V_0}{2h} + q_b V_1 = r_1$$

$$\frac{4 - 2h + (2h - h^2)P_1 + 2q_b h^2(2+h)}{2h^2(2+h)}$$

$$\frac{V_2 - 2V_1}{h^2} + \frac{2\alpha h + (2-h)V_1}{h^2(2+h)} + P_1 \left(\frac{V_2}{2h} - \frac{2\alpha h + (2-h)V_1}{2h(2+h)} \right) + q_b V_1 = r_1$$

$$V_2 \left(\frac{1}{h^2} + \frac{P_1}{2h} \right) + \left[-\frac{2}{h^2} + \frac{(2-h)}{h^2(2+h)} - \frac{P_1(2-h)}{2h(2+h)} \right] V_1 \rightarrow \left[\frac{2h}{h^2(2+h)} + \frac{2hP_1}{2h(2+h)} \right] \alpha = r_1$$

$$V_1 \left(\frac{1}{h^2} + \frac{P_1}{2h} \right) + \left[\frac{2}{h^2} - \frac{(2-h)}{h^2(2+h)} - \frac{P_1(2-h)}{2h(2+h)} \right] = r_1$$

$$U_2 \left(\frac{1}{h^2} + \frac{P_1}{2h} \right) + \left(\frac{2 - P_1 h}{h(z+h)} \right) \alpha + \left[\frac{-4 - 6h - 2P_1 h + P_1 h^2}{2h^2(z+h)} + q_1 \right] \quad U_1 = r_1$$

$$U_2 \left(\frac{1}{h^2} + \frac{P_1}{2h} \right) + \left[\frac{-4 - 6h - 2P_1 h + P_1 h^2}{2h^2(z+h)} + q_1 \right] \quad U_1 = r_1 \left(\frac{2 - P_1 h}{h(z+h)} \right) \alpha$$

$$g_1 = r_1 - \left(\frac{2 - P_1 h}{h(z+h)} \right) \alpha$$

$$\alpha_{11} = \frac{-4 - 6h - 2P_1 h + P_1 h^2 + q_1 (2h^2(z+h))}{2h^2(z+h)}$$