Homework 2

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Problem 1-3

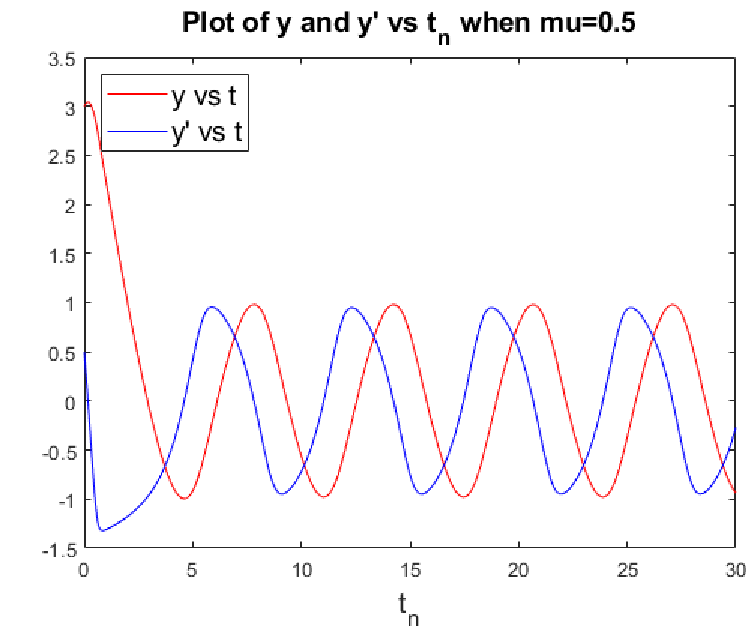
Submitted as a hand written pdf attached at the end of this report.

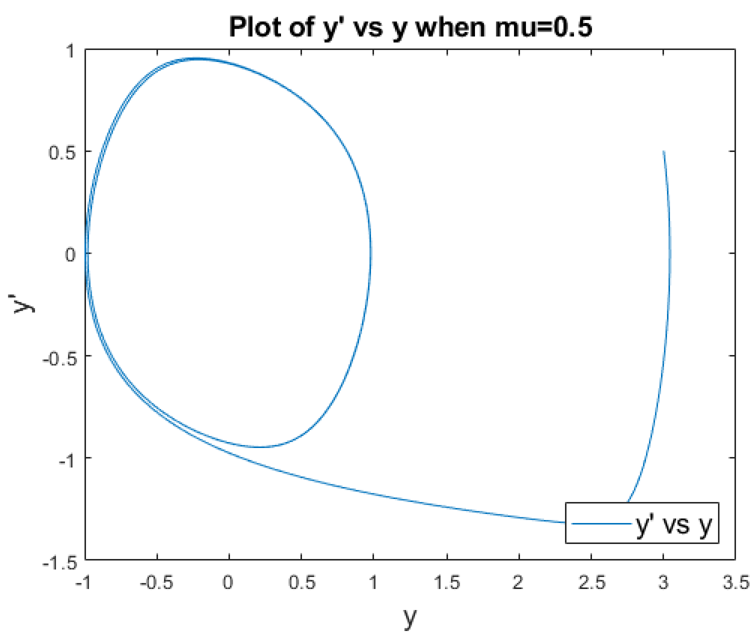
Problem 4

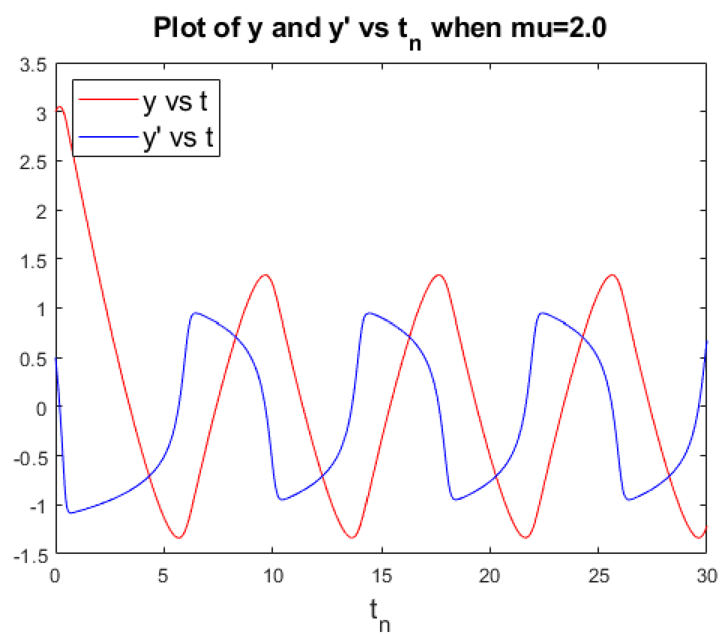
In problem 4 we implemented classic 4th order Runge-Kutta method to solve the following IVP

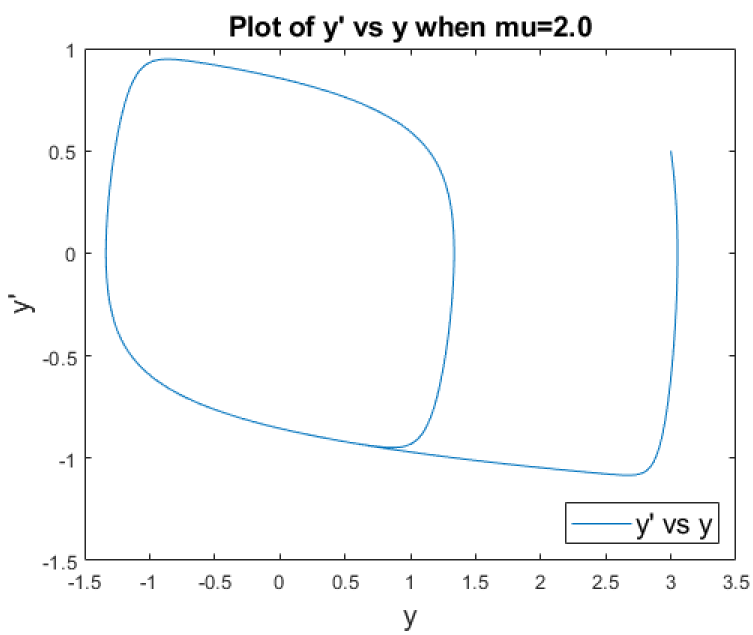
$$y'' - \mu(2 - \exp(y'^2))y' + y = 0$$
$$y(0) = y_0, \quad y'(0) = v_0$$

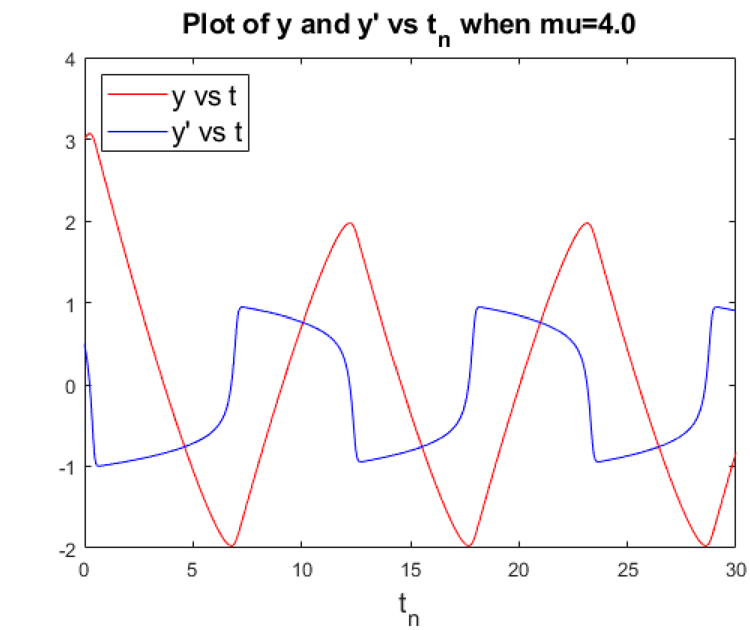
using $y_0 = 3, v_0 = 0.5$ and $h = 0.025$. Before we can use RK4 we convert the IVP to a first order system. We are asked to solve the IVP to T = 30 for 3 different μ values being $\mu = 0.5, 2, 4$. Below are 6 figures grouped by μ one is y(t) vs t and y'(t) vs t and the other is y'(t) vs y(t)

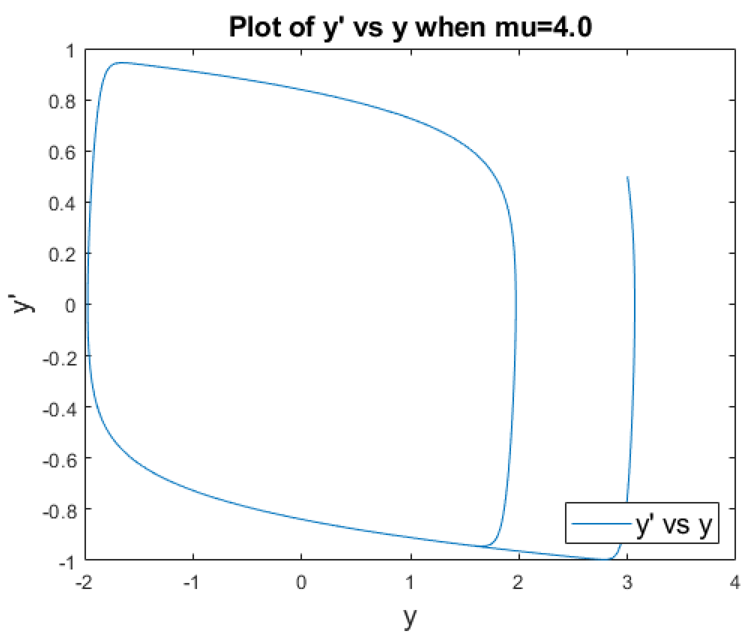

Plot of y and y' vs t_n when mu=0.5
y vs t
y' vs t
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
t_n


Plot of y' vs y when mu=0.5
1
0.5
0
-0.5
-1
-1.5
y'
y' vs y
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
y


Plot of y and y' vs t_n when mu=2.0
y vs t
y' vs t
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
t_n


Plot of y' vs y when mu=2.0
1
0.5
0
-0.5
-1
-1.5
y'
y' vs y
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
y



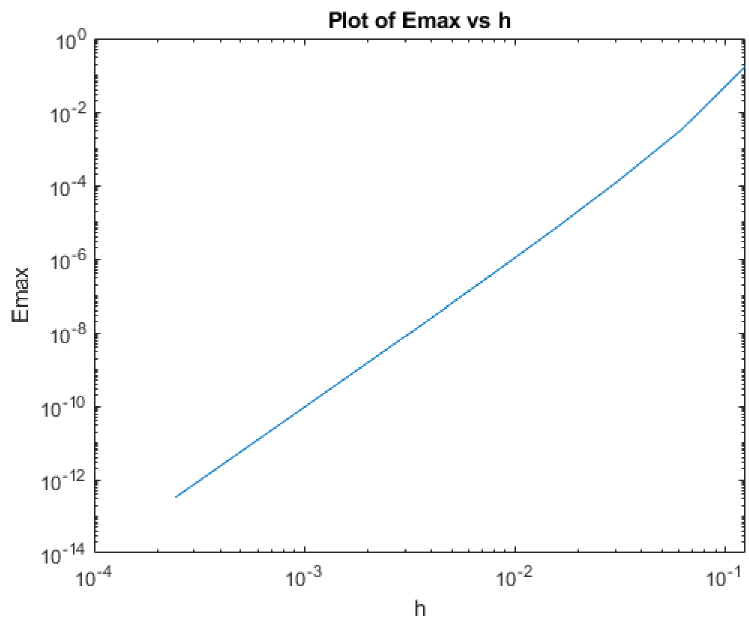


Problem 5

In problem 5 we were asked to solve the IVP in problem 4 using $y_0 = 3$ and $v_0 = 0.5$ but this time $\mu = 4$ and we vary the time stepsize $h = \frac{1}{2^3}, \frac{1}{2^4}, ..., \frac{1}{2^{13}}$ We used the numerical solutions to estimate the error in the numerical solutions.

Part 1

In part 1 we look at the max error for each time step size. Below is the plot of E_{max} vs h

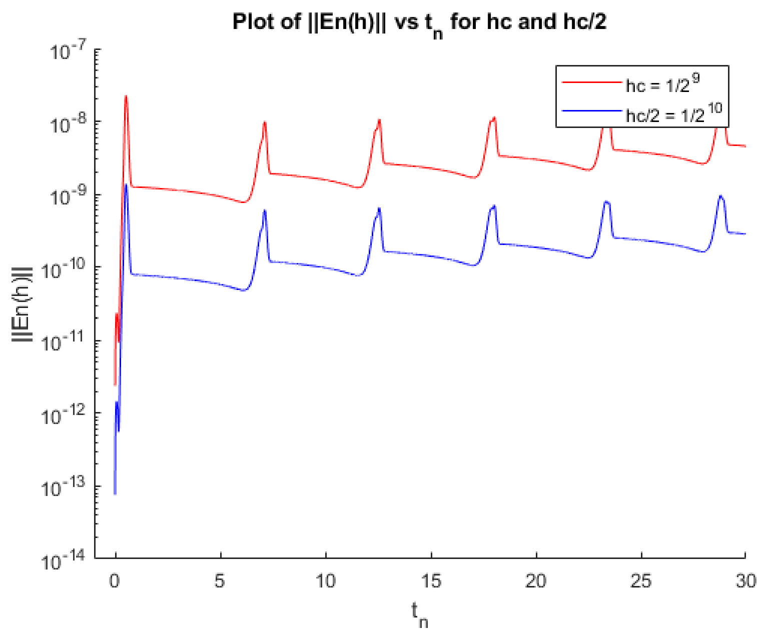


Part 2

In part 2 we needed to find a step-size such that $E_{max}(h) < 5 \times 10^{-8}$. that value was

$$h_c = \frac{1}{2^9}$$

Below is the plot of $\|E_n(h)\|$ vs t_n with time step size h_c and $\frac{h_c}{2}$

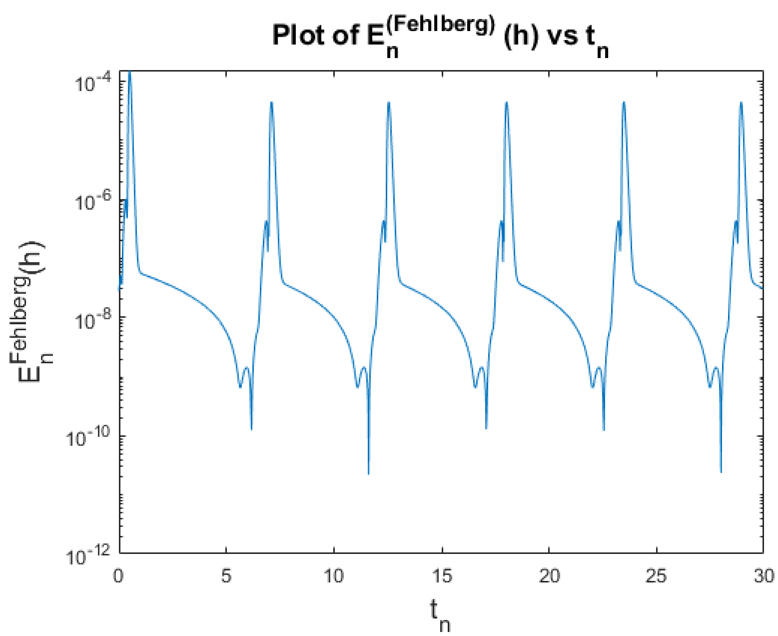


Problem 6

In Problem 6 we program the fehlberg 45 method. Using this method we once again solve the IVP to $T = 30$ from problem 4. We use $y_0 = 3, v_0 = 0.5\mu = 4$ and our step size $h = 0.025$.

Part 1

Part 1 asks us to calculate $E_n^{Fehlberg}(h)$ and make a plot of $E_n^{Fehlberg}(h)$ vs t_n. The plot of that is below.

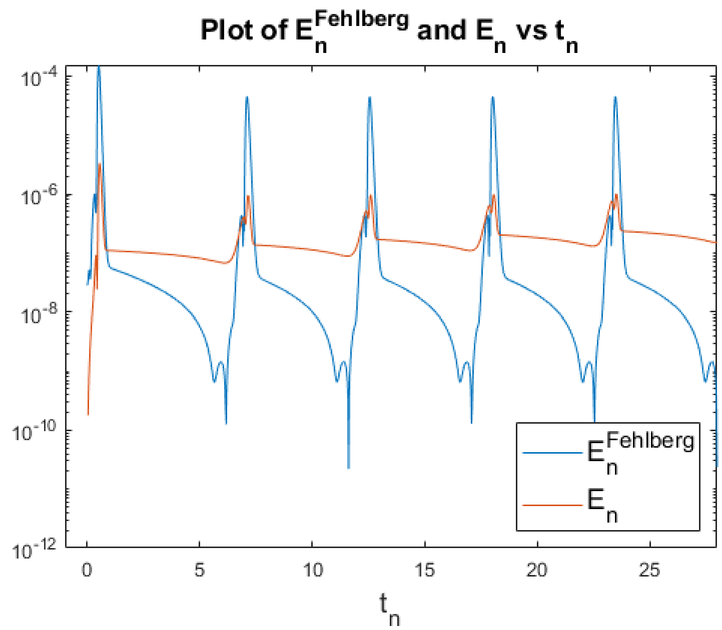


Part 2

Finally in part 2 we used

$$E_n(h) = \frac{1}{1-(0.5)^5}\|w_n(h) -_{2n} (\frac{h}{2})\|$$

to calculate the error then we plot $E_n^{Fehlberg}(h)$ vs t_n and $E_n(h)$ vs t_n in one figure to compare them. from the figure below we can see that E_n is more consistent with error than $E_n^{Fehlberg}(h)$ is.


Plot of $E_n^{Fehlberg}$ and E_n vs t_n
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0
5
10
15
20
25
t_n
$E_n^{Fehlberg}$
E_n

Homework 2

Wednesday, April 7, 2021 10:34 AM

Problem 1 (Theoretical)

Suppose k satisfies the equation

$k = h\exp(1+k)$ where h is a small quantity

Recall the approach of iterative expansion we used in lecture.

Start with $k = O(h)$. Expand k iteratively into

$k = a_1 h + a_2 h^2 + \cdots$

Find the coefficients a_1 and a_2.

1) $K = O(h)$

$K = h\exp(1 + O(h))$

$K = h\exp(1)\,\exp(O(h))$

$K = a_1 h + a_2 h^2 + \cdots$

$he\,\exp(O(h)) = a_1 h + O(h^2)$

$he\,(1 + O(h)) = a_1 h + O(h^2)$

$eh + \cancel{O(h^2)} = a_1 h + \cancel{O(h^2)}$

$\boxed{a_1 = e}$

$K = eh + O(h^2)$

$K = h\exp(1 + eh + O(h^2))$

$$K = he\exp(eh)\exp(O(h^2))$$

$$eh\exp(eh)\exp(O(h^2)) = eh + a_2h^2 + O(h^3)$$

$$eh\exp(eh)(1 + O(h^2)) = eh + a_2h^2 + O(h^3)$$

$$he(1 + eh + O(h^2)) + O(h^3) = eh + a_2h^2 + O(h^3)$$

$$eh + a_2h^2 + \cancel{O(h^3)} = eh + e^2h^2 + \underbrace{O(h^3) + O(h^3) + O(h^3)}_{O(h^3)}$$

$$\cancel{eh} + a_2h^2 = \cancel{eh} + e^2h^2$$

$$\boxed{a_2 = e^2}$$

$$\boxed{K = eh + (e^2+e)h^2 + \dots}$$

Problem 2 (Theoretical)

Consider the Runge-Kutta (RK) method specified by the Butcher tableau below

$$\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Write out the method in the form of $k_1 = \cdots$, $k_2 = \cdots$, ..., $u_{n+1} = u_n + \cdots$

Support $f(u, t)$ satisfies $|f(u,t) - f(v,t)| \le C|u - v|$ for all u, v, and t.

Part 1: Show that the method is stable.

Part 2: Use Taylor expansion to show $e_n(h) = O(h^2)$.

Part 3: What is the order of its global error $E_N(h)$?

2) This is a 2-Stage method

$$K_1 = h \cdot f(U, t)$$

$k_2 = h \cdot f(U + \frac{1}{2}k_1, t + \frac{1}{2}h)$

$U_{n+1} = U_n + \frac{1}{2}k_1 + \frac{1}{2}k_2$

i) want to show method is stable

$f(u,t)$ is LC

$$U_{n+1} = U_n + \frac{h}{2}\left[f(U_n, t_n) + f(U_n + \tfrac{1}{2}hf(U_n,t_n), t_n + \tfrac{1}{2}h)\right]$$

$$V_{n+1} = V_n + \frac{h}{2}\left[f(V_n, t_n) + f(V_n + \tfrac{1}{2}hf(V_n,t_n), t_n + \tfrac{1}{2}h)\right]$$

To show if a method is stable we want to show

$$|U_{n+1} - V_{n+1}| \leq (1 + C \cdot h)|U_n - V_n|$$

$$|U_{n+1} - V_{n+1}| = \Big| U_n + \frac{h}{2}\left[f(U_n,t_n) + f(U_n + \tfrac{1}{2}hf(U_n,t_n), t_n + \tfrac{1}{2}h)\right] - V_n - \frac{h}{2}\left[f(V_n,t_n) + f(V_n + \tfrac{1}{2}hf(V_n,t_n), t_n + \tfrac{1}{2}h)\right]\Big| \quad (1)$$

$$\leq |U_n - V_n| + \frac{h}{2}\Big| f(U_n,t_n) - f(V_n,t_n) + f(U_n + \tfrac{1}{2}hf(U_n,t_n), t_n + \tfrac{1}{2}h) - f(V_n + \tfrac{1}{2}hf(V_n,t_n), t_n + \tfrac{1}{2}h)\Big| \quad (2)$$

Because $f(U_n, t_n)$ is LC

we have

$$|f(U_n,t_n) - f(V_n,t_n)| \leq C|U_n - V_n|$$

and

$$\left|f\left(U_n+\tfrac{1}{2}hf(U_n,t_n),t_n+\tfrac{1}{2}h\right)-f\left(V_n+\tfrac{1}{2}hf(V_n,t_n),t_n+\tfrac{1}{2}h\right)\right| \le c\left|U_n+\tfrac{1}{2}hf(U_n,t_n)-V_n-\tfrac{1}{2}hf(V_n,t_n)\right|$$

$$\le c|U_n-V_n| + \frac{c}{2}h|f(U_n,t_n)-f(V_n,t_n)|$$

$$\le c|U_n-V_n| + \frac{c^2h}{2}|U_n-V_n|$$

Subbing into (2)

$$|U_{n+1}-V_{n+1}| \le |U_n-V_n| + \frac{h}{2}\left[c|U_n-V_n| + \left(c+\frac{c^2h}{2}\right)|U_n-V_n|\right]$$

$$\le |U_n-V_n| + ch+\frac{c^2h^2}{4}|U_n-V_n|$$

$$\le \left(1+ch+\frac{c^2h^2}{4}\right)|U_n-V_n|$$

for small h $h^2\approx 0$

$\Rightarrow$

$$|U_{n+1}-V_{n+1}| \le (1+ch)|U_n-V_n|$$

Thus the method is stable

ii) $U_{n+1} = U_n + \frac{1}{2}h\left(f(U_n,t_n) + f(U_n + \frac{1}{2}h f(U_n,t_n), t_n + \frac{1}{2}h\right)$

$$e(h) = U(t_n+h) - U(t_n) - \frac{h}{2}\left[f(U(t_n),t_n) + f(U(t_n) + \frac{1}{2}h f(U(t_n),t_n), t_n + \frac{1}{2}h)\right]$$

$$U(t_n+h) = U(t_n) + U'(t_n)h + U''(t_n)\frac{h^2}{2} + \cdots$$

$$U'(t_n) = f(U(t_n),t_n)$$

Sub in we get

$$e(h) = \frac{h}{2}f(U(t_n),t_n) - \frac{h}{2}f(U(t_n) + \frac{1}{2}h f(U(t_n),t_n), t_n + \frac{1}{2}h) + O(h^2)$$

Expand $f(U(t_n) + \frac{1}{2}h f(U(t_n),t_n), t_n + \frac{1}{2}h)$

$$= f(U(t_n),t_n) + O(h)$$

$$\Rightarrow e(h) = \cancel{\frac{h}{2}f(U(t_n),t_n)} - \cancel{\frac{h}{2}f(U(t_n),t_n)} + \underbrace{\cancel{\frac{h}{2}O(h)}}_{O(h^2)\ \text{combine}} + O(h^2)$$

$$e(h) = O(h^2) \blacksquare$$

iii) from lecture notes 2 Pg. 12 theorem

iii) from lecture notes 2 Pg. 12 theorem

$e_n(h) = O(h^{p+1})$ then $E_N(h) = O(h^p)$

Since in Part 2 we showed

$e_n(h) = O(h^2)$ thus $E_N(h) = O(h)$.

3)

Problem 3 (Theoretical)

Recall that in lecture, we carried out polynomial interpolation based on 3 points and used the polynomial interpolation to derive

- 3-step Adams-Bashforth method and
- 2-step Adams-Moulton method

Carry out polynomial interpolation based on 2 points and use it to derive

- 2-step Adams-Bashforth method and
- 1-step Adams-Moulton method

$x_1 = -1 \qquad x_2 = 0$

$y_1 = g(x_1) \qquad y_2 = g(x_2)$

$$P_1 = \prod_{K=1,\ k\neq j}^{2} \left(\frac{x - x_k}{x_1 - x_k} \right) = -x$$

$$P_2 = x+1$$

$$P(x) = y_1 \frac{x}{9} + y_2(-x-1)$$

$S = 1$

explicit $r = s+1 = 2$

implicit $r = s = 1$

implicit $r = s = 1$

$$\{t_n, t_{n+1}\} \to \{x_1 = -1, x_2 = 0\}$$

$$P^{(t)}(t) = p^{(x)}\left(\frac{t - t_{n+1}}{h}\right)$$

$$U_{n+2} = U_{n+1} + \int_{t_{n+1}}^{t_{n+2}} p^{(t)}(t)\,dt = U_{n+1} + \int_{t_{n+1}}^{t_{n+2}} p\left(\frac{t - t_{n+2}}{h}\right) dt$$

$$= U_{n+1} + h\int_0^1 p(x)\,dx \qquad x = \frac{t - t_{n+1}}{h}$$

$$\int_0^1 P_1(x)\,dx = \int_0^1 -x\,dx = -\frac{1}{2}$$

$$\int_0^1 P_2(x)\,dx = \int_0^1 x + 1 = \frac{3}{2}$$

$$\int_0^1 P(x)\,dx = \frac{1}{2}(-y_1 + 3y_2)$$

2-Step Adams-Bashforth

$$U_{n+2} = U_{n+1} + \frac{h}{2}\left[-f(U_n, t_n) + 3\,f(U_{n+1}, t_{n+1})\right]$$

$$U_{n+1} = U_n + \int_{t_n}^{t_{n+1}} p^{(t)}(t)\,dt = U_n + \int_{t_n}^{t_{n+1}} p\left(\frac{t - t_{n+1}}{h}\right) dt$$

$$= U_n + h\int_{-1}^{0} P(x)\,dx \quad , \qquad x = \frac{t - t_{n+1}}{h}$$

$$\int_{-1}^{0} P(x)\,dx$$

$$\int_{-1}^{0} P_1(x)\,dx = \int_{-1}^{0} -x\,dx = \frac{1}{2}$$

$$\int_{-1}^{0} P_2(x)\,dx = \int_{-1}^{0} x+1\,dx = \frac{1}{2}$$

$$\int_{-1}^{0} P(x)\,dx = \frac{1}{2}(y_1 + y_2)$$

1 - Step <u>Adams - Moulton</u>

$$U_{n+1} = U_n + \frac{h}{2}\left(f(U_n, t_n) + f(U_{n+1}, t_{n+1})\right)$$