

Problem 1 (Theoretical)Suppose k satisfies the equation

$$k = h \exp(1+k) \quad \text{where } h \text{ is a small quantity}$$

Recall the approach of iterative expansion we used in lecture.

Start with $k = O(h)$. Expand k iteratively into

$$k = a_1 h + a_2 h^2 + \dots$$

Find the coefficients a_1 and a_2 .

i) $K = O(h)$

$$K = h \exp(1 + O(h))$$

$$K = h \exp(1) \exp(O(h))$$

$$K = a_1 h + a_2 h^2 + \dots$$

$$\underbrace{h \exp(O(h))} = a_1 h + O(h^2)$$

$$h e (1 + O(h)) = a_1 h + O(h^2)$$

$$eh + O(h^2) = a_1 h + O(h^2)$$

$a_1 = e$

$$K = eh + O(h^2)$$

$$K = h \exp(1 + eh + O(h^2))$$

$$k = h \cdot e^{\exp(eh)} \exp(O(h^2))$$

$$eh \exp(eh) \exp(O(h^2)) = eh + \alpha_2 h^2 + O(h^3)$$

$$eh \exp(eh)(1 + O(h^2)) = eh + \alpha_2 h^2 + O(h^3)$$

$$he(1 + eh + O(h^2)) + O(h^3) = eh + \alpha_2 h^2 + O(h^3)$$

$$eh + \alpha_2 h^2 + O(h^3) = eh + e^2 h^2 + O(h^3) + \underbrace{O(h^3) + O(h^3)}_{O(h^3)}$$

$$eh + \alpha_2 h^2 = eh + e^2 h^2$$

$$\alpha_2 = e^2$$

$$k = eh + (e^2 + e)h^2 + \dots$$

Problem 2 (Theoretical)

Consider the Runge-Kutta (RK) method specified by the Butcher tableau below

c^T	A	b
	$\begin{array}{ cc } \hline 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \\ \hline \end{array}$	$\begin{array}{ c } \hline \frac{1}{2} \\ \hline \end{array}$

Write out the method in the form of $k_1 = \dots$, $k_2 = \dots$, ..., $u_{n+1} = u_n + \dots$

Support $f(u, t)$ satisfies $|f(u, t) - f(v, t)| \leq C|u - v|$ for all u, v , and t .

Part 1: Show that the method is stable.

Part 2: Use Taylor expansion to show $e_n(h) = O(h^2)$.

Part 3: What is the order of its global error $E_N(h)$?

2) This is a 2-stage method

$$k_1 = h \cdot f(u, t)$$

$$k_2 = h \cdot f(U + \frac{1}{2}k_1, t + \frac{1}{2}h)$$

$$U_{n+1} = U_n + \frac{1}{2}k_1 + \frac{1}{2}k_2$$

i) want to show method is stable

$f(v, t)$ is LC

$$U_{n+1} = U_n + \frac{h}{2} \left[f(U_n, t_n) + f\left(U_n + \frac{1}{2}h f(U_n, t_n), t_n + \frac{1}{2}h\right) \right]$$

$$V_{n+1} = V_n + \frac{h}{2} \left[f(V_n, t_n) + f\left(V_n + \frac{1}{2}h f(V_n, t_n), t_n + \frac{1}{2}h\right) \right]$$

To show if a method is stable we want to show

$$|U_{n+1} - V_{n+1}| \leq (1 + Ch) |U_n - V_n|$$

$$|U_{n+1} - V_{n+1}| = |U_n + \frac{h}{2} \left[f(U_n, t_n) + f\left(U_n + \frac{1}{2}h f(U_n, t_n), t_n + \frac{1}{2}h\right) \right] - V_n - \frac{h}{2} \left[f(V_n, t_n) + f\left(V_n + \frac{1}{2}h f(V_n, t_n), t_n + \frac{1}{2}h\right) \right]| \quad (1)$$

$$= |U_n - V_n| + \frac{h}{2} |f(U_n, t_n) - f(V_n, t_n) + f\left(U_n + \frac{1}{2}h f(U_n, t_n), t_n + \frac{1}{2}h\right) - f\left(V_n + \frac{1}{2}h f(V_n, t_n), t_n + \frac{1}{2}h\right)|$$

$$\leq |U_n - V_n| + \frac{h}{2} |f(U_n, t_n) - f(V_n, t_n)| + |f\left(U_n + \frac{1}{2}h f(U_n, t_n), t_n + \frac{1}{2}h\right) - f\left(V_n + \frac{1}{2}h f(V_n, t_n), t_n + \frac{1}{2}h\right)| \quad (2)$$

Because $f(v, t)$ is LC

$$|f\left(U_n + \frac{1}{2}h f(U_n, t_n), t_n + \frac{1}{2}h\right) - f\left(V_n + \frac{1}{2}h f(V_n, t_n), t_n + \frac{1}{2}h\right)|$$

We have

$$|f(U_n, t_n) - f(V_n, t_n)| \leq C |U_n - V_n|$$

and

$$\begin{aligned} \left| f\left(U_n + \frac{1}{2}hf(U_n, t_n), t_n + \frac{1}{2}h\right) - f\left(V_n + \frac{1}{2}hf(V_n, t_n), t_n + \frac{1}{2}h\right) \right| &\leq c \left| U_n + \frac{1}{2}hf(U_n, t_n) - V_n - \frac{1}{2}hf(V_n, t_n) \right| \\ &\leq c |U_n - V_n| + \frac{c}{2}h |f(U_n, t_n) - f(V_n, t_n)| \\ &\leq c |U_n - V_n| + \frac{c^2 h}{2} |U_n - V_n| \end{aligned}$$

Subbing into (2)

$$\begin{aligned} |U_{n+1} - V_{n+1}| &\leq |U_n - V_n| + \frac{h}{2} \left[c |U_n - V_n| + \left(c + \frac{c^2 h}{2}\right) |U_n - V_n| \right] \\ &\leq |U_n - V_n| + ch + \frac{c^2 h^2}{4} |U_n - V_n| \\ &\leq \left(1 + ch + \frac{c^2 h^2}{4}\right) |U_n - V_n| \\ \text{for small } n \quad h^2 \approx 0 \end{aligned}$$

\Rightarrow

$$|U_{n+1} - V_{n+1}| \leq (1 + ch) |U_n - V_n|$$

Thus the method is stable

$$\text{ii) } U_{n+1} = U_n + \frac{1}{2}h \left(f(U_n, t_n) + f\left(U_n + \frac{1}{2}h f(U_n, t_n), t_n + \frac{1}{2}h\right)\right)$$

$$e(h) = U(t_n + h) - U(t_n) - \frac{h}{2} \left[f(U(t_n), t_n) + f\left(U(t_n) + \frac{1}{2}h f(U(t_n), t_n), t_n + \frac{1}{2}h\right)\right]$$

$$U(t_n + h) = U(t_n) + U'(t_n)h + U''(t_n) \frac{h^2}{2} + \dots$$

$$U'(t_n) = f(U(t_n), t_n)$$

Sub in we get

$$e(h) = \frac{h}{2} f(U(t_n), t_n) - \frac{h}{2} f\left(U(t_n) + \frac{1}{2}h f(U(t_n), t_n), t_n + \frac{1}{2}h\right) + O(h^2)$$

$$\text{Expand } f\left(U(t_n) + \frac{1}{2}h f(U(t_n), t_n), t_n + \frac{1}{2}h\right)$$

$$= f(U(t_n), t_n) + O(h)$$

combine
 $O(h^2)$

$$\Rightarrow e(h) = \frac{h}{2} f(U(t_n), t_n) - \frac{h}{2} \cancel{f(U(t_n), t_n)} + \cancel{\frac{h}{2} O(h)} \rightarrow O(h^2)$$

$$e(h) = O(h^2)$$

iii) from lecture notes 2 Pg. 12 theorem

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$$e_n(h) = O(h^{p+1}) \text{ then } E_N(h) = O(h^p)$$

since in part 2 we showed

$$e_n(h) = O(h^2) \text{ thus } E_N(h) = O(h).$$

3)

Problem 3 (Theoretical)

Recall that in lecture, we carried out polynomial interpolation based on 3 points and used the polynomial interpolation to derive

- 3-step Adams-Bashforth method and
- 2-step Adams-Moulton method

Carry out polynomial interpolation based on 2 points and use it to derive

- 2-step Adams-Bashforth method and
- 1-step Adams-Moulton method

$$x_1 = -1 \quad x_2 = 0$$

$$y_1 = g(x_1) \quad y_2 = g(x_2)$$

$$P_1 = \prod_{k=1, k \neq j}^2 \left(\frac{x - x_k}{x_j - x_k} \right) = \frac{-x}{x_j - x_k}$$

$$P_2 = \frac{x+1}{x_j - x_k}$$

$$P(x) = y_1 \frac{x}{x_j - x_k} + y_2 \frac{(-x-1)}{x_j - x_k}$$

$$S=1$$

$$\text{explicit } r=S=1$$

$$\text{implicit } r=S=1$$

implicit $r=s=1$

$$\{t_n, t_{n+1}\} \rightarrow \{x_1 = -1, x_2 = 0\}$$

$$P^{(t)}(t) = P^{(x)}\left(\frac{t-t_{n+1}}{h}\right)$$

$$U_{n+2} = U_{n+1} + \int_{t_{n+1}}^{t_{n+2}} P^{(t)}(t) dt = U_{n+1} + \int_{t_{n+1}}^{t_{n+2}} P\left(\frac{t-t_{n+1}}{h}\right) dt$$

$$= U_{n+1} + h \int_0^1 P(x) dx \quad x = \frac{t-t_{n+1}}{h}$$

$$\int_0^1 P_1(x) dx = \int_0^1 -x dx = -\frac{1}{2}$$

$$\int_0^1 P_2(x) dx = \int_0^1 x+1 dx = \frac{3}{2}$$

$$\int_0^1 P(x) dx = \frac{1}{2} (-y_1 + 3y_2)$$

2-Step Adams-Basforth

$$U_{n+2} = U_{n+1} + \frac{h}{2} \left[-f(U_n, t_n) + 3 f(U_{n+1}, t_{n+1}) \right]$$

$$\overbrace{U_{n+1} = U_n + \int_{t_n}^{t_{n+1}} P(t) dt} = U_n + \int_{t_n}^{t_{n+1}} P\left(\frac{t-t_{n+1}}{h}\right) dt \\ = U_n + h \int_{-1}^0 P(x) dx, \quad x = \frac{t-t_{n+1}}{h}$$

$$\int_{-1}^0 P(x) dx$$

$$\int_{-1}^0 P_1(x) dx = \int_{-1}^0 -x dx = -\frac{1}{2}$$

$$\int_{-1}^0 P_2(x) dx = \int_{-1}^0 x+1 dx = \frac{1}{2}$$

$$\int_{-1}^0 P(x) dx = \frac{1}{2}(y_1 + y_2)$$

1 - Step Adams-Moulton

$$U_{n+1} = U_n + \frac{h}{2} (f(U_n, t_n) + f(U_{n+1}, t_{n+1}))$$