Homework 4

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1)

Problem 1 (Theoretical)

Consider the one-stage implicit RK method described by Butcher tableau

$$\begin{array}{c|c} c^T & A \\ \hline & b \end{array} = \begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

Part 1: Show that it has second order accuracy.

Hint: Check the conditions on second order accuracy.

Part 2: Derive its stability function $\phi(z)$.

Part 3: Show that it is A-stable, but not L-stable.

Part 1

internal condition

$$c_i = \sum_{j=1}^{p} a_{ij} \Rightarrow \frac{1}{2} = \frac{1}{2} \checkmark$$

Condition for first order

$$\sum_{i=1}^{p} b_i = 1 \Rightarrow 1 = 1 \checkmark$$

Condition for the second order

$$\sum_{i=1}^{p} b_i c_i = \frac{1}{2} \Rightarrow 1 \cdot \frac{1}{2} = \frac{1}{2} \checkmark$$

Part 2

RK form of BT

$$k_1 = hf\left(U_n + \tfrac{1}{2}k_1,\ t_n + \tfrac{h}{2}\right)$$

$$U_{n+1} = U_n + hf\left(U_n + \tfrac{1}{2}k_1,\ t_n + \tfrac{h}{2}\right)$$

$f(U) = \gamma u$ $$k_1 = h\gamma\left(U_n + \tfrac{1}{2}k_1\right)$$

$z = h\gamma$ $$k_1 = zU_n + \frac{z}{2}k_1$$

$$k_1 - \frac{z}{2}k_1 = zU_n$$

$$k_1 = \frac{zU_n}{1 - \frac{z}{2}}$$

$$U_{n+1} = U_n + \frac{zU_n}{1 - \frac{z}{2}}$$

$$U_{n+1} = \phi(z)\,U_n = \left(1 + \frac{z}{1 - \frac{z}{2}}\right)U_n$$

$$\boxed{\phi(z) = 1 + \frac{z}{1 - \frac{z}{2}}}$$

Part 3

$$|\phi(z)| < 1 \quad \forall\ \mathrm{Re}(z) < 0$$

$$|\phi(z)| = \left|1 + \frac{z}{1-\frac{z}{2}}\right| < 1$$

$$= \left|\frac{1-\frac{z}{2}+z}{1-\frac{z}{2}}\right| < 1$$

$$= \left|\frac{1+\frac{z}{2}}{1-\frac{z}{2}}\right| < 1$$

$$= \left|1+\frac{z}{2}\right| < \left|1-\frac{z}{2}\right|$$ Let $z = a+bi$

$$= \left(1+\frac{a-bi}{2}\right)\left(1+\frac{a+bi}{2}\right) < \left(1-\frac{a-bi}{2}\right)\left(1-\frac{a+bi}{2}\right)$$

$$1+\frac{a+bi}{2}+\frac{a-bi}{2}+\frac{a^2+b^2}{4} < 1-\frac{a+bi}{2}-\frac{a-bi}{2}+\frac{a^2+b^2}{4}$$

$$1+a+\cancel{\frac{a^2+b^2}{4}} < 1-a+\cancel{\frac{a^2+b^2}{4}}$$

$$1+2a<1$$

$$2a<0$$

$a < 0$

thus $|\phi(z)| < 1 \quad \forall \ Re(z) < 0$

So thss method is A-stable ▪

To check if L-stable

$\phi(z) \to 0$ as $z \to \infty$

$$\lim_{z\to\infty}\left(1 + \frac{z}{1-\frac{z}{2}}\right) = 1 + \lim_{z\to\infty} \frac{2z}{2-z} = 1-2 = -1$$ ▪

Thus method is not L-stable.

6)

Problem 6 (Theoretical and computational)

Consider a slightly different version of the BVP in Problem 5.

$$\begin{cases} u'' - \left(1+\exp(-\sin x)\right)u = -5-(\sin x)^2 \\ u(0)-u'(0)=1.5, \quad u(2)=0.5 \end{cases} \qquad \text{(P6)}$$

Design the discretization of the finite difference method (FDM). Implement it in Matlab.
Solve the BVP (P6) with N = 1000.

Plot the numerical $u(x)$ vs x.

Need to discretize the BC

$U(0) - U'(0) = 1.5$

$$\frac{U_1 + U_0}{2} - \frac{U_1 - U_0}{h} = \alpha \quad \Rightarrow$$

$$(2+h)U_0 - (2-h)U_1 = 2\alpha h$$

Solve for U_0

$$U_0 = \frac{2\alpha h + (2-h)U_1}{2+h}$$

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + P_i \frac{U_{i+1} - U_{i-1}}{2h} + q_i U_i = r_i$$

At $i=1$

$$\frac{U_2 - 2U_1 + U_0}{h^2} + P_1 \frac{U_2 - U_0}{2h} + q_1 U_1 = r_1$$

$$\frac{4 - 2h + (2h - h^2) P_1 + 2q_1 h^2 (2+h)}{2h^2(2+h)}$$

$$\frac{U_2 - 2U_1}{h^2} + \frac{2\alpha h + (2-h)U_1}{h^2(2+h)} + P_1\left(\frac{U_2}{2h} - \frac{2\alpha h + (2-h)U_1}{2h(2+h)}\right) + q_1 U_1 = r_1$$

$$U_2\left(\frac{1}{h^2} + \frac{P_1}{2h}\right) + \left[-\frac{2}{h^2} + \frac{(2-h)}{h^2(2+h)} - \frac{P_1(2-h)}{2h(2+h)}\right]U_1 + \left[\frac{2h}{h^2(2+h)} + \frac{2hP_1}{2h(2+h)}\right]\alpha = r_1$$

$U_2\left(\frac{1}{\ }+\frac{P_1}{\ }\right) + \left(2 - P_1 h \right) \ldots \left[-4 - 6h - 2P_1h + P_1h^2 \ldots\right.$

$$U_2\left(\frac{1}{h^2}+\frac{P_1}{2h}\right)+\left(\frac{2-P_1h}{h(2+h)}\right)\alpha+\left[\frac{-4-6h-2P_1h+P_1h^2}{2h^2(2+h)}+q_1\right]U_1=r_1$$

$$U_2\left(\frac{1}{h^2}+\frac{P_1}{2h}\right)+\left[\frac{-4-6h-2P_1h+P_1h^2}{2h^2(2+h)}+q_1\right]U_1=r_1-\left(\frac{2-P_1h}{h(2+h)}\right)\alpha$$

$$g_1=r_1-\left(\frac{2-P_1h}{h(2+h)}\right)\alpha$$

$$a_{11}=\frac{-4-6h-2P_1h+P_1h^2+q_1\left(2h^2(2+h)\right)}{2h^2(2+h)}$$