

# Homework 2

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April 14, 2021

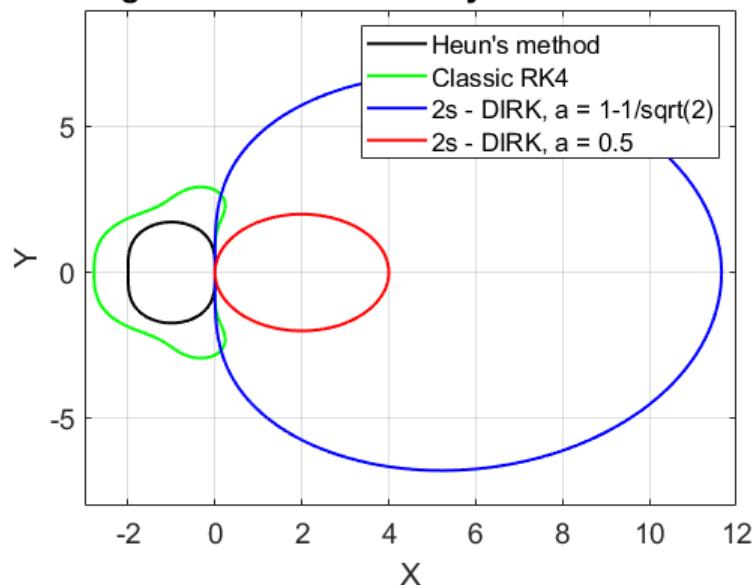
## Problem 1-3

Submitted as a hand written pdf attached at the end of this report.

## Problem 4

In Problem 4 we were asked to plot the region of absolute stability for the following methods Predictor-corrector method (Heun's method), Classic 4th order Runge-Kutta method (RK4) and the 2s-DIRK method with  $\alpha = 1 - 1/\sqrt{2}$  and  $\alpha = -.5$ . We derived the stability function for the first 2 methods in problem 1 and the stability function for 2s-DIRK method was derived in problem 2. Below is a figure of the 4 contours

**Region of absolute stability for four methods**



## Problem 5

In problem 5 we were asked to solve an IVP to  $T = 30$  using the 2s-DIRK method using  $\alpha = 1 - 1/\sqrt{2}$ .

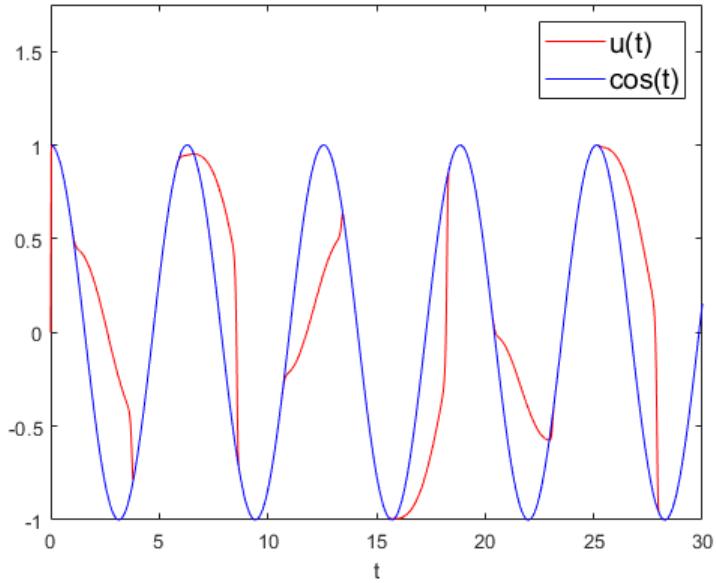
$$u' = -(0.5 + \exp(20 \cos(1.3t))) \sinh(u - \cos(t))$$

$$u(0) = 0$$

### Part 1

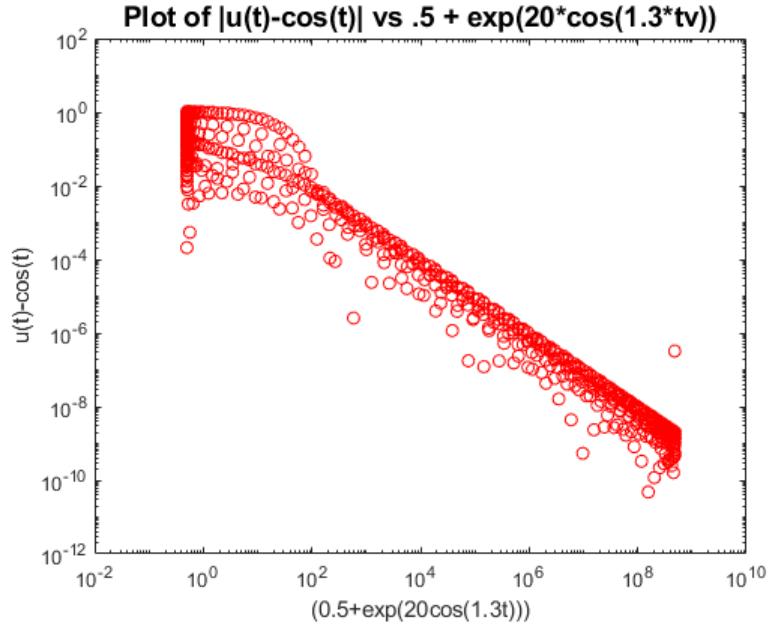
In part 1 using a step size of  $h = 2^{-5}$  we plot the numerical solution  $u(t)$  vs  $t$  and  $\cos(t)$  vs  $t$  in the same figure inserted below. We can see that the numerical solution does not closely follow  $\cos(t)$ .

**Plot of  $u(t)$  vs  $t$  compared with  $\cos(t)$  vs  $t$**



### Part 2

For part 2 we were asked to plot  $|u(t) - \cos(t)|$  vs  $(0.5 + \exp(20 \cos(1.3t)))$  for  $t \in (0, 30]$ . Below is a figure of the plot. From the plot we can see that for a horizontal value of  $10^5$  or greater  $u(t)$  follows  $\cos(t)$  under  $10^{-6}$  accuracy.



## Problem 6

In Problem 6 we once again solve the IVP from problem 5 this time using different step-sizes  $h = \frac{1}{2^3}, \frac{1}{2^4}, \dots, \frac{1}{2^8}$ . We will calculate numerical error estimation for both 2s-DIRK method with  $\alpha = 1 - 1/\sqrt{2}$  and backward euler method.

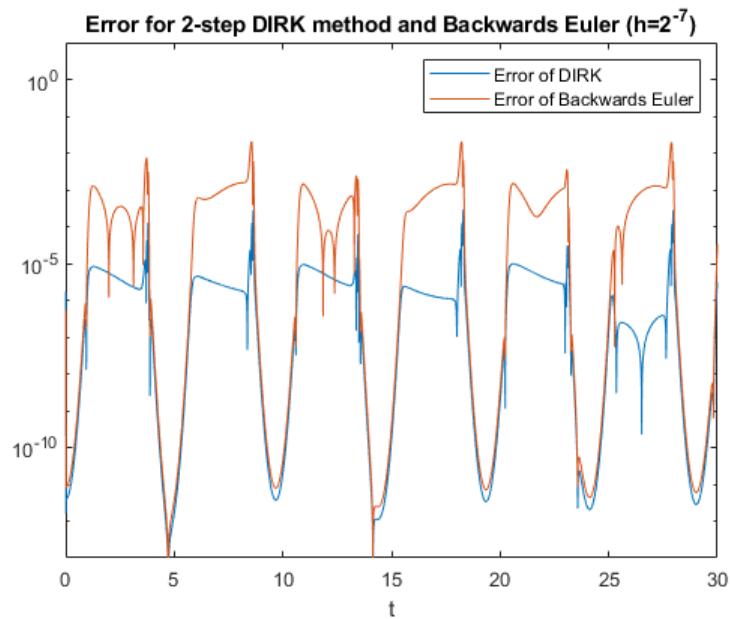
### Part 1

Part 1 asks us to calculate and plot the numerical estimation for both methods when  $h = \frac{1}{2^5}$ . Below is a figure of the plot.



## Part 2

In part 2 we followed part one except we plotted for  $h = \frac{1}{2^7}$  instead of  $h = \frac{1}{2^5}$ .



**Problem 1 (Theoretical)**

**Part 1:** Derive the stability function  $\phi(z)$  for each of the two RK methods below

- o Predictor-corrector method (Heun's method)
- o Classic 4-th order Runge-Kutta method (RK4)

**Hint:** Check your expression of  $\phi(z)$  with the theorem we studied.

**Theorem:** If an RK method is  $p$ -th order accurate, then it must satisfies

$$\phi(z) = e^z + O(|z|^{p+1}) \quad \text{for any complex } z \text{ with small } |z|$$

**Part 2:** Study the zero-stability for each of the two LMMs below

- o  $u_{n+2} - 2u_{n+1} + u_n = hf(u_{n+1}, t_{n+1}) - hf(u_n, t_n)$
- o  $u_{n+2} - u_n = h \left[ \frac{1}{3}f(u_{n+2}, t_{n+2}) + \frac{4}{3}f(u_{n+1}, t_{n+1}) + \frac{1}{3}f(u_n, t_n) \right]$

Part 1 To find a stability function we apply  
 $U' = \gamma U$  to Heun's method and RK4  
 $U_{n+1} = \phi(z)U_n$  where  $z = h\gamma$

B-T for Heun's method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \xrightarrow{\text{RK form}} \begin{cases} k_1 = hf(U_n, t_n) \\ k_2 = hf(U_n + k_1, t_n + h) \\ U_{n+1} = U_n + \frac{1}{2}k_1 + \frac{1}{2}k_2 \end{cases}$$

$$f(U) = \gamma U$$

Plugging into RK form

$$\begin{aligned} k_1 &= h\gamma U_n = zU_n \\ k_2 &= h\gamma f(U_n + zU_n) = h\gamma(U_n + zU_n) = zU_n + z^2U_n \\ U_{n+1} &= U_n + \frac{1}{2}zU_n + \left(\frac{1}{2}z + \frac{1}{2}z^2\right)U_n \end{aligned}$$

$$U_{n+1} = \left(1 + z + \frac{1}{2}z^2\right)U_n$$

$$\boxed{\phi(z) = 1 + z + \frac{1}{2}z^2}$$

Next for RK4

BT  $f(U) = \gamma U$

$$\begin{array}{c|cccc}
 0 & 0 & 0 & 0 & 0 \\
 \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
 \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
 1 & 0 & 0 & 1 & 0 \\
 \hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
 \end{array}
 \quad \xrightarrow{\text{R-K form}}
 \quad \left\{
 \begin{array}{l}
 k_1 = h f(v_n) \\
 k_2 = h f(v_n + \frac{1}{2}k_1) \\
 k_3 = h f(v_n + \frac{1}{2}k_2) \\
 k_4 = h f(v_n + k_3) \\
 v_{n+1} = v_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4
 \end{array}
 \right.$$

$$K_1 = h \gamma v_n = z v_n$$

$$K_2 = h \gamma (v_n + \frac{z}{2} v_n) = z v_n + \frac{1}{2} z^2 v_n = (z + \frac{z^2}{2}) v_n$$

$$K_3 = h \gamma (v_n + \underbrace{(z + \frac{1}{2} z^2) v_n}_{z}) = z v_n + (\frac{z^2}{2} + \frac{z^3}{4}) v_n = (z + \frac{z^2}{2} + \frac{z^3}{4}) v_n$$

$$K_4 = h \gamma (v_n + (z + \frac{z^2}{2} + \frac{z^3}{4}) v_n) = (z + z^2 + \frac{z^3}{2} + \frac{z^4}{4}) v_n$$

$$v_{n+1} = v_n + \left( \frac{z}{6} + \frac{z}{3} + \frac{z^2}{6} + \frac{z}{3} + \frac{z^2}{6} + \frac{z^3}{12} + \frac{z}{6} + \frac{z^2}{6} + \frac{z^3}{12} + \frac{z^4}{24} \right) v_n$$

$$v_{n+1} = \left( 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} \right) v_n$$

$$\boxed{\phi(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}}$$

Part 2

$$v_{n+2} - z v_{n+1} + v_n = h f(v_{n+1}, t_{n+1}) - h f(v_n, t_n)$$

$$\rho(\xi) = \xi^2 - 2\xi + 1 \quad \sigma(\xi) = \xi - 1$$

Find roots of  $\rho(\xi) = 0$

$$(\xi - 1)(\xi - 1) = 0 \quad \xi = 1, \text{ with multiplicity 2}$$

Thus this LMM is not zero stable because  
the root condition is not satisfied  
"all roots with multiplicity above 1 satisfy  $|g| < 1$ "

$$U_{n+2} - U_n = \frac{1}{8}hf(U_{n+2}, t_{n+2}) + \frac{4}{3}hf(U_{n+1}, t_{n+1}) + \frac{1}{3}f(U_n, t_n)$$

$$\rho(\xi) = \xi^2 - 1 \quad \sigma(\xi) = \frac{1}{3}\xi^2 + \frac{4}{3}\xi + \frac{1}{3}$$

Find roots of  $\rho(\xi) = 0$

$$\xi^2 - 1 = 0 \Rightarrow \xi_1 = 1 \quad \xi_2 = -1 \quad \text{two simple roots where } |\xi_i| \leq 1 \text{ thus this LMM}$$

is zero-stable.

#### Problem 2 (Theoretical)

Consider the Runge-Kutta method described by Butcher tableau

Butcher tableau:	$\begin{array}{c cc} \alpha & \alpha & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array}$
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where  $\alpha > 0$ . This is called a 2s-DIRK (2-Stage Diagonally Implicit Runge-Kutta) method.

The matrix A of a DIRK method is lower triangular so  $\{k_1, k_2, k_3, \dots\}$  can be solved sequentially. The first row of A gives an equation on  $k_1$  without involving  $\{k_2, k_3, \dots\}$ . The second row of A gives an equation on  $k_2$  without involving  $\{k_3, \dots\}$  where  $k_1$  is already known. This is in contrast to a fully implicit Runge-Kutta where  $\{k_1, k_2, k_3, \dots\}$  has to be solved simultaneously in a joint system.

Part 1: Show that method is second order for  $\alpha = 1 - 1/\sqrt{2}$ .

Hint: check the internal consistency condition, the condition for first order and the additional condition for second order.

Part 2: Apply the 2s-DIRK to solving  $u' = \gamma u$ .

Derive the expressions for  $k_1, k_2$  and the stability function  $\phi(z)$ .

Part 1

BT

BT	$\begin{array}{c cc} \alpha & \alpha & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array}$
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To show above method is second order for  $\alpha = 1 - 1/\sqrt{2}$

check the internal condition, the first order order condition, and additional condition for second order

## Initial condition

$$C_i = \sum_{j=1}^2 a_{ij} \quad i=1,2$$

$$C_1 = \alpha \quad \sum_{j=1}^2 a_{1j} = \alpha \quad \checkmark$$

$$C_2 = 1 \quad \sum_{j=1}^2 a_{2j} = 1 \quad \checkmark$$

## First order condition

$$\sum_{i=1}^2 b_i = 1 \quad b_1 = 1 - \alpha$$

$$b_2 = \alpha$$

$$b_1 + b_2 = 1 - \alpha + \alpha = 1 \quad \checkmark$$

## Second order condition

$$\sum_{i=1}^2 b_i C_i = \frac{1}{2} \quad b_1 = 1 - \alpha \quad C_1 = \alpha$$

$$b_2 = \alpha \quad C_2 = 1$$

$$\sum_{i=1}^2 b_i C_i = \alpha - \frac{\alpha}{2} + \alpha = 2\alpha - \frac{\alpha}{2} \quad \alpha = 1 - \frac{1}{\sqrt{2}}$$

$$\Rightarrow 2 - \frac{2}{\sqrt{2}} - \left(1 - \frac{2}{\sqrt{2}} + \frac{1}{2}\right)$$

$$2 - \sqrt{2} - 1 + \sqrt{2} - \frac{1}{2} = \frac{1}{2} \quad \checkmark$$

We have shown that this method is

$\sqrt{2}$  order when  $\alpha = 1 - \frac{1}{\sqrt{2}}$   $\blacksquare$

Part 2

$$\alpha = 1 - \lambda^2$$

$$\begin{array}{c|cc}
 \alpha & \alpha & 0 \\
 1 & 1-\alpha & \alpha \\
 \hline
 & 1-\alpha & \alpha
 \end{array} \rightarrow \text{RK form} \left\{
 \begin{array}{l}
 k_1 = h f(U_n + \lambda k_1, t_n + \Delta h) \\
 k_2 = h f(U_n + (1-\alpha)k_1 + \alpha k_2, t_n + h) \\
 U_{n+1} = U_n + (1-\alpha)k_1 + \alpha k_2
 \end{array}
 \right.$$

Next we find stability Function

$$f(u) = \gamma u$$

$$k_1 = h \gamma f(U_n + \lambda k_1) = \gamma U_n + \gamma \lambda k_1 \Rightarrow k_1 - \gamma \lambda k_1 = \gamma U_n$$

$$k_1 = \frac{\gamma}{1-\lambda} U_n$$

$$k_2 = h \gamma f(U_n + (1-\alpha)k_1 + \alpha k_2) = \gamma \left( U_n + (1-\alpha) \frac{\gamma}{1-\lambda} U_n + \alpha k_2 \right)$$

$$k_2 = \left( \gamma + (1-\alpha) \frac{\gamma^2}{1-\lambda} \right) U_n + \gamma \alpha k_2$$

$$\frac{k_2}{1-\lambda} = \left( \gamma + (1-\alpha) \frac{\gamma^2}{1-\lambda} \right) U_n = \boxed{\frac{\gamma(1-\lambda) + (1-\alpha)\gamma^2}{(1-\lambda)^2} U_n}$$

$$U_{n+1} = U_n + (1-\alpha) \frac{\gamma}{1-\lambda} U_n + \frac{\alpha \left( \gamma + (1-\alpha) \frac{\gamma^2}{1-\lambda} \right)}{1-\lambda} U_n$$

$$U_{n+1} = \left( 1 + \frac{\gamma - \gamma \lambda}{1-\lambda} + \frac{\gamma \alpha + (\alpha - \alpha^2) \gamma^2}{(1-\lambda)^2} \right) U_n$$

$$= \left( 1 + \frac{z}{1-z\alpha} + \frac{z^2\alpha - z^2\alpha^2}{(1-z\alpha)^2} \right) u_n$$

$$\phi(z) = 1 + \frac{z}{1-z\alpha} + \frac{z^2\alpha - z^2\alpha^2}{(1-z\alpha)^2}$$

$$\begin{aligned} &= \frac{1 - z^2\alpha + z^2\alpha^2 + z - z^3\alpha + z^2\alpha - z^2\alpha^2}{(1 - z\alpha)^2} \\ &= \boxed{\frac{1 - z^2\alpha + z}{(1 - z\alpha)^2}} \end{aligned}$$

$$k_1 = \frac{z}{1-\alpha z} u_n, \quad k_2 = \frac{(1-\alpha z)z + (1-\alpha)z^2}{(1-\alpha z)^2} u_n$$

$$\phi(z) = \frac{1+(1-2\alpha)z}{(1-\alpha z)^2}$$

Part 3: Suppose the 2s-DIRK is A-stable for  $\alpha = 1 - 1/\sqrt{2}$  (see Problem 4 below).

Show that it satisfies the second condition of L-stability.

### Part 3

Suppose 2s-DIRK is A-stable for  $\alpha = 1 - \frac{1}{\sqrt{2}}$

$$\lim_{z \rightarrow \infty} \frac{1 + (1 - z\alpha)z}{(1 - \alpha z)^2} = 0$$

By definition if an RK method is

A-stable and  $\phi(z) \rightarrow 0$  as  $z \rightarrow \infty$

Then the method is L-stable.

**Problem 3 (Theoretical)**

Consider the implicit 2-step method below

$$u_{n+2} - u_n = h \left[ \frac{1}{3} f(u_{n+2}, t_{n+2}) + \frac{4}{3} f(u_{n+1}, t_{n+1}) + \frac{1}{3} f(u_n, t_n) \right]$$

Part 1: Use Taylor expansion to show  $e_n(h) = O(h^5)$ .

Hint: Expand everything around  $t_{n+1}$ .

Part 2: The stability polynomial is

$$\pi(\xi, z) = (\xi^2 - 1) - z \left( \frac{1}{3}\xi^2 + \frac{4}{3}\xi + \frac{1}{3} \right)$$

Consider  $z = -\varepsilon$  with small  $\varepsilon > 0$ . We examine the two roots of  $\pi(\xi, -\varepsilon)$ .

Show that the two roots  $\xi_1(\varepsilon)$  and  $\xi_2(\varepsilon)$  have the expansions

$$\xi_1(\varepsilon) = 1 - \varepsilon + O(\varepsilon^2), \quad \xi_2(\varepsilon) = -\left(1 + \frac{\varepsilon}{3}\right) + O(\varepsilon^2)$$

Therefore,  $z = -\varepsilon$  is NOT in the region of absolute stability.

Remarks:

- This method demonstrates that the Dahlquist barrier on accuracy of implicit LMMs:  $p \leq r + 2$  is actually attainable.
- Although this method has the 4th order, it is not practically useful. When applied to solving  $u' = -u$ , the numerical solution contains a decaying mode (corresponding to the exact solution), and an oscillating and exponentially growing mode (which will eventually ruin the numerical solution). This is similar to the situation with the 2-step midpoint method.

Part 1

$$U_{n+2} - U_n = h \left[ \frac{1}{3} f(U_{n+2}, t_{n+2}) + \frac{4}{3} f(U_{n+1}, t_{n+1}) + \frac{1}{3} f(U_n, t_n) \right]$$

$$e_n(h) = U(t_{n+1}-h) - U(t_{n+1}+h) + \frac{h}{3} f(U(t_{n+1}+h), t_{n+1}+h)$$

$$+ \frac{4}{3} h f(U(t_{n+1}), t_{n+1}) + \frac{h}{3} f(U(t_{n+1}-h), t_{n+1}-h)$$

Since this is a LMM we know that

$$f(u_n + jh) = U'(t_n + jh) \quad \forall j$$

$$e_n(h) = U(t_{n+1}-h) - U(t_{n+1}+h) + \frac{h}{3} U'(t_{n+1}+h) + \frac{4h}{3} U'(t_{n+1}) + \frac{h}{3} U'(t_{n+1}-h)$$

Next we expand around  $t_{n+1}$

$$U(t_{n+1}-h) = U(t_{n+1}) - h U'(t_{n+1}) + \frac{h^2}{2!} U''(t_{n+1}) - \frac{h^3}{3!} U'''(t_{n+1}) + \frac{h^4}{4!} U^{(4)}(t_{n+1}) + O(h^5)$$

$$U(t_{n+1}+h) = U(t_{n+1}) + h U'(t_{n+1}) + \frac{h^2}{2!} U''(t_{n+1}) + \frac{h^3}{3!} U'''(t_{n+1}) + \frac{h^4}{4!} U^{(4)}(t_{n+1}) + O(h^5)$$

$$U(t_{n+1}+h) = U(t_{n+1}) + hU'(t_{n+1}) + \frac{h^2}{2!}U''(t_{n+1}) + \frac{h^3}{3!}U'''(t_{n+1}) + O(h^4)$$

$$U(t_{n+1}-h) = U(t_{n+1}) - hU'(t_{n+1}) + \frac{h^2}{2!}U''(t_{n+1}) - \frac{h^3}{3!}U'''(t_{n+1}) + O(h^4)$$

Sub in expansions into  $e_n(h)$

$$e_n(h) = -2hU'(t_{n+1}) - 2\frac{h^3}{3!}U^{(3)}(t_{n+1}) + \frac{h}{3} \left[ 2U'(t_{n+1}) + h^2U^{(4)}(t_{n+1}) + O(h^4) \right]$$

$$+ \frac{4h}{3}U'(t_{n+1}) + O(h^5)$$

$$= -2hU'(t_{n+1}) - \cancel{\frac{h^3}{3}U^{(3)}(t_{n+1})} + \frac{2h}{3}U'(t_{n+1}) + \cancel{\frac{h^3}{3}U^{(4)}(t_{n+1})} + \frac{4h}{3}U'(t_{n+1}) + O(h^5)$$

$$= -2hU'(t_{n+1}) + \cancel{\frac{6h}{3}U'(t_{n+1})} + O(h^5)$$

$$= O(h^5)$$

Part 2

$$\Pi(\xi, \varepsilon) = (\xi^2 - 1) - \varepsilon \left( \frac{1}{3}\xi^2 + \frac{4}{3}\xi + \frac{1}{3} \right)$$

Small  $\varepsilon > 0$

$$\Pi(\xi, -\varepsilon) = \xi^2 - 1 + \frac{\varepsilon}{3}\xi^2 + \frac{4\varepsilon}{3}\xi + \frac{\varepsilon}{3}$$

To find the roots of the  
above equation apply the quadratic equation

$$\xi = \frac{-4\epsilon \pm \sqrt{12\epsilon^2 + 36}}{6 + 2\epsilon}$$

$$\xi_1 = \frac{-4\epsilon + 2\sqrt{3(\epsilon^2 + 3)}}{6(1 + \frac{\epsilon}{3})} = \frac{-4\epsilon + 6\sqrt{\frac{\epsilon^2}{3} + 1}}{6(1 + \frac{\epsilon}{3})}$$

$$\xi_2 = \frac{-4\epsilon - 2\sqrt{3(\epsilon^2 + 3)}}{6(1 + \frac{\epsilon}{3})} = \frac{-4\epsilon - 6\sqrt{\frac{\epsilon^2}{3} + 1}}{6(1 + \frac{\epsilon}{3})}$$

Want to expand around  $\epsilon = 0$   
denominator first

$$\alpha = -1$$

$$\frac{1}{6} \left(1 + \frac{\epsilon}{3}\right)^{-1} = \frac{1}{6} \left[1 - \frac{\epsilon}{3} + \frac{\epsilon^2}{9} + O(\epsilon^3)\right]$$

now numerator

$$6 \left(1 + \frac{\epsilon^2}{3}\right)^{1/2} = 6 \left[1 + \frac{\epsilon^2}{6} + O(\epsilon^4)\right]$$

Thus

$$\xi_1 = \left(-4\epsilon + 6 + \frac{\epsilon^2}{6}\right) \left(\frac{1}{6} - \frac{\epsilon}{18} + \frac{\epsilon^2}{54}\right)$$

$$= -\frac{4}{6}\epsilon + O(\epsilon^2) + O(\epsilon^2) + 1 - \frac{6\epsilon}{18} + O(\epsilon^2)$$

$$= -\frac{2}{3}\epsilon - \frac{1}{3}\epsilon + 1 + O(\epsilon^2)$$

$$= 1 - \epsilon + O(\epsilon^2) \quad \checkmark$$

$$\zeta_2 = (-4\epsilon - 6 - \frac{\epsilon^2}{6}) \left( \frac{1}{6} - \frac{\epsilon}{18} + \frac{\epsilon^2}{54} \right)$$

$$= -\frac{4\epsilon}{6} - 1 + \frac{6\epsilon}{18} + O(\epsilon^2)$$

$$= -\frac{2}{3}\epsilon + \frac{1}{3}\epsilon - 1 + O(\epsilon^2)$$

$$= -1 - \frac{\epsilon}{3} + O(\epsilon^2) \quad \checkmark$$