Homework 5

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Problem 1-2

Submitted as a hand written pdf attached at the end of this report.

Problem 3

In problem 3 we are asked to solve the folllowing IBVP of the heat equation

$$u_t = u_{xx},\ x \in (0,2),\ t > 0$$

$$u(x,0) = f(x),\ x \in (0,2)$$

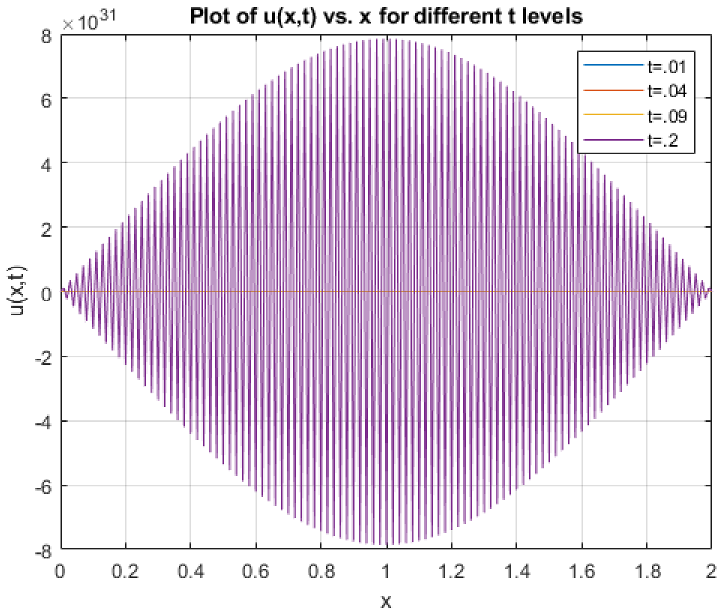
$$u(0,t) = g_L(t),\ u(2,t) = g_R(t)$$

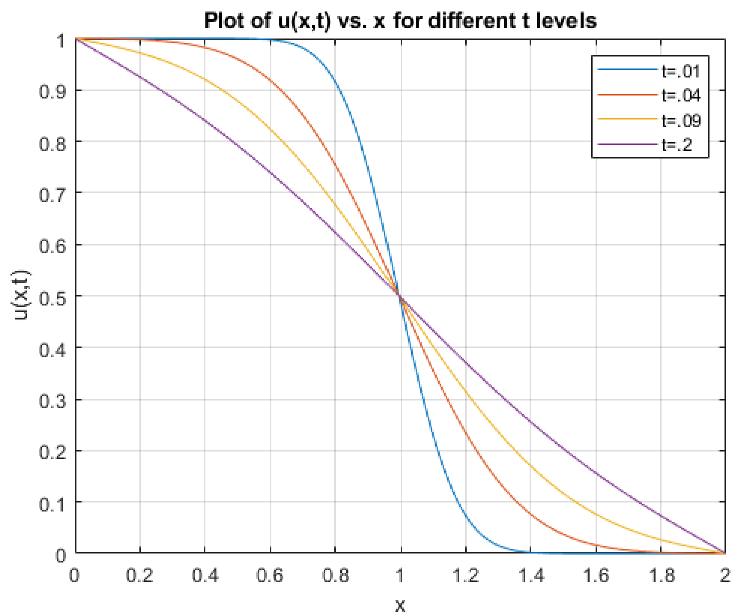
Where $g_L(t) = 1$, $g_R(t) = 0$, $f(x) = \{1,\ 0 < x < 1, \text{ and } 0,\ x \geq 1\ \}$. We are to implement FTCS to solve the IBVP to $T = 0.2$ using $\Delta x = 0.01$ and for 2 different values of Δt.

$$\Delta t = \frac{(\Delta x)^2}{2(.99)}$$

$$\Delta t = \frac{(\Delta x)^2}{2(1.01)}.$$

We are then asked to plot $u(x,t)$ vs x for $t = 0.01, 0.04, 0.09, 0.2$ for both Δt. Those figure are shown below.

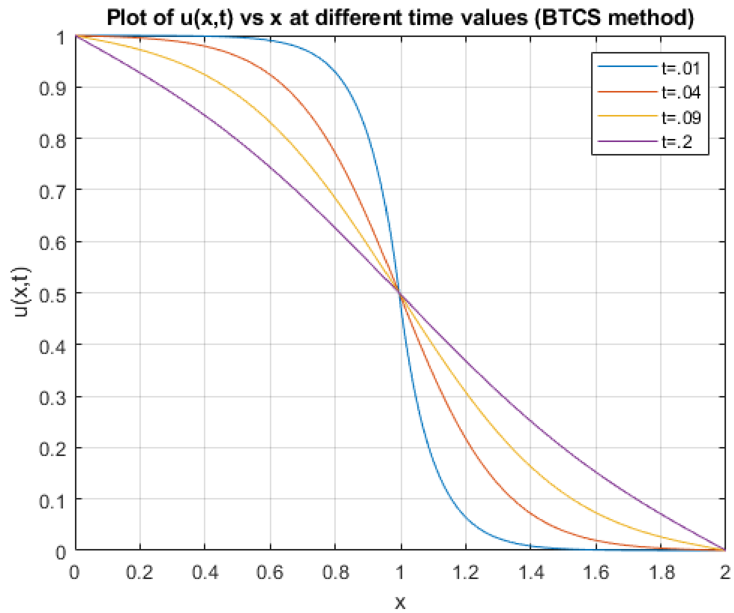


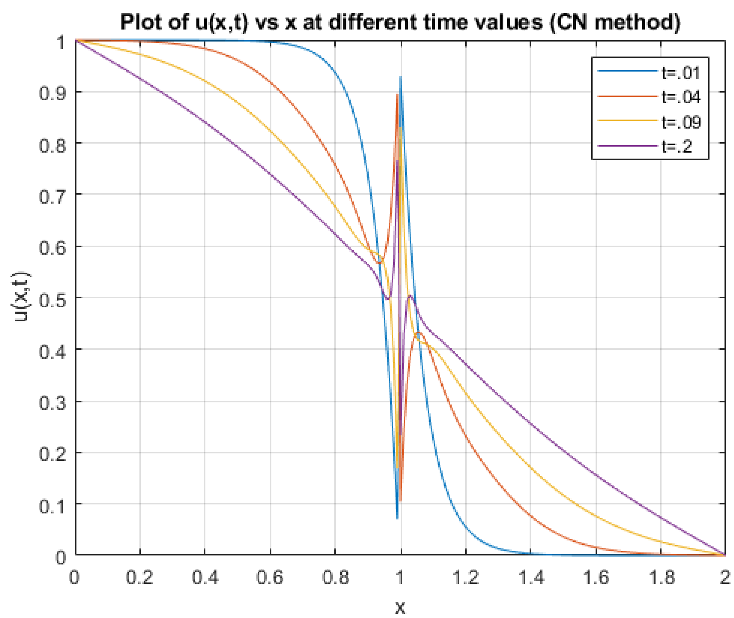


Problem 4

In problem 4 we implemented the BTCS method and C-N method to solve the same IBVP as in problem 3. We again plot $u(x,t)$ vs x at $t = 0.01, 0.04, 0.09, 0.2$

for both methods. From the figures below we can see that the C-N method becomes unstable around x=1.



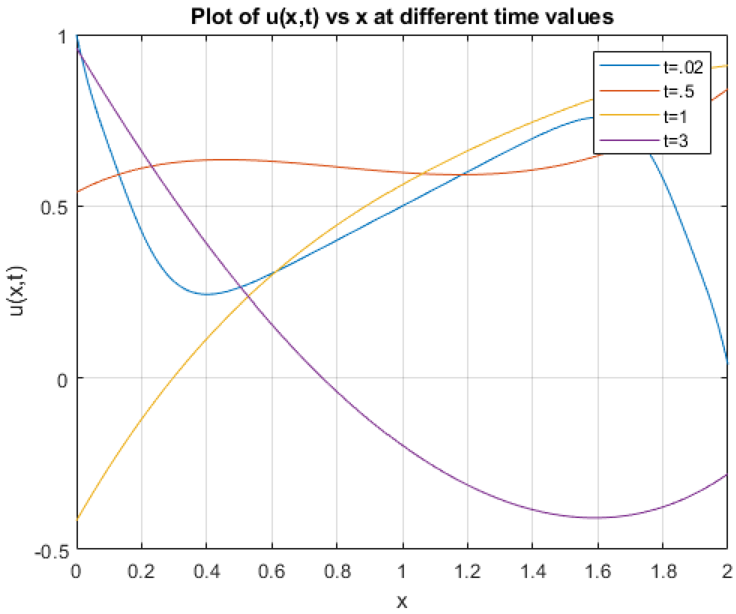


Problem 5

In Problem 5 we use the same IBVP as problem but this time we change the initial and boundary conditions to $f(x) = 0.5x$, $g_L(t) = cos(2t)$, $g_R(t) = sin(2t)$. We solve this IBVP using 2s-DIRK with $\alpha = 1-1/\sqrt{2}$. Where $T = 3$, $\Delta x = 0.01$ and $\Delta t = 0.01$

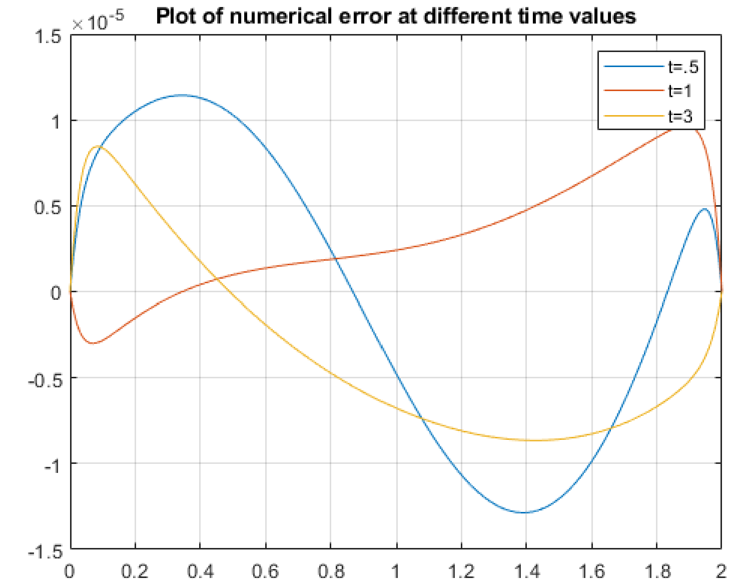
Part 1

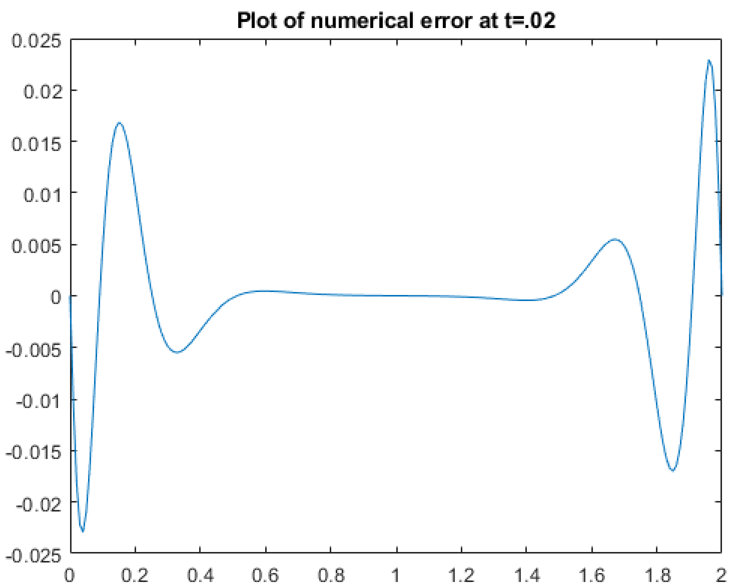
For part one we were simply asked to plot the numerical solution $u(x,t)$ vs x at $t = 0.02, 0.5, 1, 3$ shown below.



Part 2

In part 2 we estimated the error of the numerical solution from part 1 and plot it when $t = 0.5, 1, 3$ and then $t = 0.02$ in a separate figure.


Plot of numerical error at different time values
×10^-5
t=.5
t=1
t=3
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

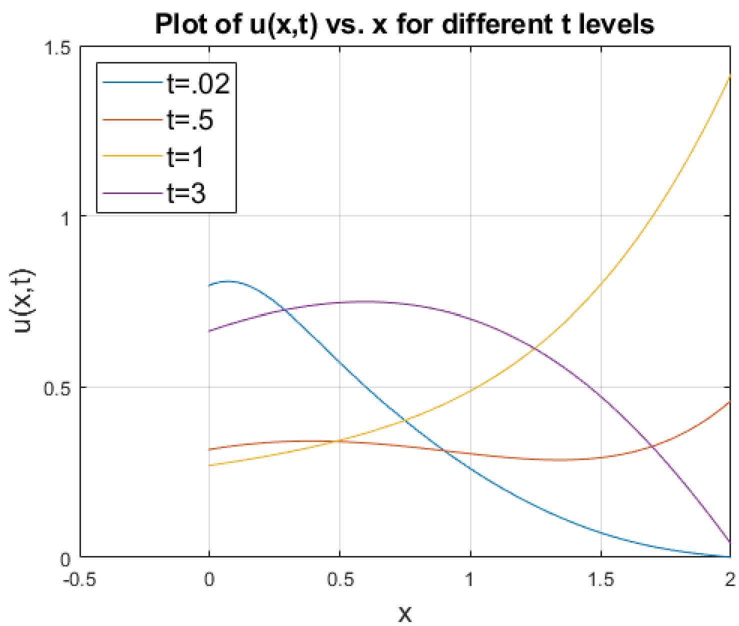

Plot of numerical error at t=.02
0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Problem 6

Problem 6 we has to solve the following IBVP to $T = 3$ using the FTCS where $\Delta t = 4 \times 10^{-5}$ and $\Delta x = \frac{L}{N-0.5}$ $N = 200$. We then plot $u(x,t)$ vs x at $t = 0.02, 0.5, 1, 3$.

$$\begin{cases} u_t = u_{xx}, & x \in (0, L), \quad t > 0 \\ u(x,0) = p(x), & x \in (0, L) \\ u_x(0,t) - \alpha u(0,t) = 0, & u(L,t) = q(t) \end{cases}$$

where $L = 2$, $\alpha = 0.4$, $p(x) = (1-0.5x)^2$, $q(t) = 2\sin^2(t)$.



Homework 5

Thursday, April 29, 2021 5:03 PM

Problem 1 (Theoretical)

Part 1: Carry out von Neumann stability analysis to show that the BTCS method is unconditionally stable

Part 2: Carry out Taylor expansions to show that the local truncation error of the Crank-Nicolson method is

$$e_i^n(\Delta x, \Delta t) = \Delta t\, O\left((\Delta t)^2 + (\Delta x)^2\right)$$

In the final expression, be sure to convert r back to $\Delta t/(\Delta x)^2$.

1)

Part 1

$$U_i^n = \rho^n \exp(\sqrt{-1}\,\xi i \Delta x)$$

sub into numerical method

$$\rho^{n+1} \exp(\sqrt{-1}\,\xi\, i\, \Delta x) = \rho^n \exp(\sqrt{-1}\,\xi i \Delta x) + r\rho^{n+1} \exp(\sqrt{-1}\,\xi i \Delta x)\left(\exp(\sqrt{-1}\,\xi\Delta x) - 2 + \exp(-\sqrt{-1}\,\xi\Delta x)\right)$$

$$\rho = 1 + r\rho\left(\cos(\xi\Delta x) + \sqrt{-1}\sin(\xi\Delta x) - 2 + \cos(-\xi\Delta x) + \sqrt{-1}\sin(-\xi\Delta x)\right)$$

$$\rho = 1 + r\rho\left(2\cos(\xi\Delta x) - 2\right)$$

$$\rho - r\rho(2\cos(\xi\Delta x) - 2) = 1$$

$$\rho(1 - r(2\cos(\xi\Delta x) - 2)) = 1$$

$$\rho = \frac{1}{1 - 2r(\cos(\xi\Delta x) - 1)}$$

For stability we need $|\rho| \leq 1 + C\Delta t$

$$|\rho| = \frac{1}{1 + 4r\sin^2\left(\frac{\xi\Delta x}{2}\right)}$$

since the denom is always positive and

greater than 1 $|\rho| \le 1 + C\Delta t$ ▨

Part 2

C-N method

$$U_i^{n+1} = U_i^n + \frac{r}{2}\left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}\right) + \frac{r}{2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)$$

NTS

$$e_i^n(\Delta x, \Delta t) = \Delta t\, O((\Delta t)^2 + (\Delta x)^2)$$

$$e_i^n(\Delta x, \Delta t) = \{U(x_i, t_{n+1})\} - L_{num}\{U(x_i, t_n)\}$$

for CN

$$U(x_i, t_{n+1}) - U(x_i, t_n) - \frac{r}{2}\Big[U(x_{i+1}, t_n) - 2U(x_i, t_n) + U(x_{i-1}, t_n)$$
$$+ U(x_{i+1}, t_{n+1}) - 2(x_i, t_{n+1}) + U(x_{i-1}, t_{n+1})\Big]$$

Expand around (x_i, t_n)

$$U(x_i, t_{n+1}) - U(x_i, t_n) = U_t \Delta t + \frac{1}{2}U_{tt}(\Delta t)^2 + O(\Delta t^3)$$

$$U(x_{i+1}, t_n) - 2U(x_i, t_n) + U(x_{i-1}, t_n) =$$

$$\cancel{U(x_i, t_n)} + \cancel{U_x \Delta x} + \frac{U_{xx}}{2}\Delta x^2 + O(\Delta x^4) \cancel{- 2U(x_i, t_n)}$$
$$+ \cancel{U(x_i, t_n)} - \cancel{U_x \Delta x} + \frac{U_{xx}}{2}\Delta x^2 + O(\Delta x^4)$$
$$= U_{xx}\Delta x^2 + O(\Delta x^4)$$

$$U(x_{i+1}, t_{n+1}) - 2U(x_i, t_{n+1}) + U(x_{i-1}, t_{n+1}) = U(x_i + \Delta x, t_n + \Delta t) - 2U(x_i, t_n + \Delta t)$$

$$+ U(x_i - \Delta x, t_n + \Delta t)$$

$$= \underbrace{(\Delta x)^2 U_{xx}\Big|_{(x_i,\, t_{n+1})}}_{\Delta x^2 U_{xx} + \Delta x^2 \Delta t\, U_{xxt} + O((\Delta x)^4 + (\Delta t)^2)} + O(\Delta x)^4$$

$$U_{xxt} = U_{xxxx} \qquad r = \frac{\Delta t}{(\Delta x)^2}$$

$$U_{tt} = U_{xxxx}$$

$$e_i^n(\Delta x, \Delta t) = U_t \Delta t + \frac{1}{2} U_{tt} (\Delta t)^2 + O(\Delta t^3) - \frac{\Delta t}{2(\Delta x)^2}\left[U_{xx} (\Delta x)^2 + U_{xx} (\Delta x)^2 + U_{xxt} (\Delta x^2)(\Delta t) + O(\Delta x^4) + O(\Delta t^2) \right]$$

$$e_i^n(\Delta x, \Delta t) = \cancel{U_{xx} \Delta t} + \cancel{\frac{1}{2} U_{xxxx} (\Delta t)^2} - \cancel{U_{xx} (\Delta t)} - \cancel{\frac{U_{xxxx} (\Delta t)^2}{2}} + \Delta t\, O(\Delta x^2 + \Delta t^2) + O((\Delta t)^3)$$

$$= \Delta t\, O(\Delta x^2 + \Delta t^2) \blacksquare$$

Problem 2 (Theoretical)

Consider matrix

$$A = \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}_{(N-1)\times(N-1)}, \qquad \Delta x = \frac{1}{N}$$

Part 1: Verify that the set below are eigenvalues and eigenvectors of matrix A.

$$\left.\begin{aligned} \lambda^{(k)} &= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right) \\ w^{(k)} &= \left\{\sin(k\pi i \Delta x),\ i = 1, 2, \ldots, N-1\right\} \end{aligned}\right\}, \quad k = 1, 2, \ldots, N-1$$

Part 2: Verify that

$$\frac{1}{(\Delta x)^2}\left(\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i \Delta x) + \cos(k\pi(i+1)\Delta x)\right)$$

$$= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right)\cdot\cos(k\pi i\Delta x), \qquad k = 1, 2, \ldots, N-1$$

Part 3: Explain why $u^{(k)} = \left\{\cos(k\pi i\Delta x),\ i = 1, 2, \ldots, N-1\right\}$ is NOT an eigenvector of A.

Hint: What boundary conditions did we use in defining matrix A?

2) Au

$$A \longleftrightarrow D_x^2$$

$$D_x^2\{u\}_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{(\Delta x)^2} = \lambda U_i$$

$V_0 = 0 \qquad V_N = 0$

W.T.S.

$$\frac{W^{(k)}_{i+1} - 2W^{(k)}_i + W^{(k)}_{i-1}}{(\Delta x)^2} = \lambda^{(k)} w^{(k)}_i$$

$$w^{(k)}_0 = 0, \quad w^{(k)}_N = 0$$

$$\frac{\sin(k\pi(i+1)\Delta x) - 2\sin(k\pi(i)\Delta x) + \sin(k\pi(i-1)\Delta x)}{(\Delta x)^2} \stackrel{?}{=} \frac{2}{(\Delta x)^2}(\cos(k\pi\Delta x - 1) \cdot \sin(k\pi i \Delta x)$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$\underline{\text{LHS}}$

$$= \sin(k\pi i\Delta x)\cos(k\pi\Delta x) + \cancel{\cos(k\pi i\Delta x)\sin(k\pi\Delta x)} - 2\sin(k\pi i\Delta x)$$
$$+ \sin(k\pi i\Delta x)\cos(k\pi\Delta x) - \cancel{\cos(k\pi i\Delta x)\sin(k\pi\Delta x)}$$

$$= \frac{2\sin(k\pi i\Delta x)\cos(k\pi\Delta x) - 2\sin(k\pi i\Delta x)}{(\Delta x)^2}$$

$$= \frac{2\sin(k\pi i\Delta x)}{\Delta x^2}(\cos(k\pi\Delta x) - 1) = \overset{\text{RHS}}{\frac{2}{\Delta x^2}\sin(k\pi i\Delta x)(\cos(k\pi\Delta x) - 1)} \checkmark$$

$$i = 0 \Rightarrow \sin(k\pi(0)\Delta x) = \sin(0) = 0 \checkmark$$

$$i = N \Rightarrow \sin(k\pi N(1/N)) = \sin(k\pi) = 0 \checkmark$$

We have verified that $\lambda^{(k)}$ and $w^{(k)}$ are

We have verified that $\lambda^{(k)}$ and $w^{(k)}$ are eigenvalues and eigen vectors of matrix A respectively

Part 2 $$\frac{1}{(\Delta x)^2}\left[\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i\Delta x) + \cos(k\pi(i-1)\Delta x)\right]$$

$$= \frac{1}{(\Delta x)^2}\left[\begin{array}{l}\cos(k\pi i\Delta x)\cos(k\pi\Delta x) + \cancel{\sin(k\pi i\Delta x)\sin(k\pi\Delta x)} - 2\cos(k\pi i\Delta x) \\ \quad + \cos(k\pi i\Delta x)\cos(k\pi\Delta x) - \cancel{\sin(k\pi i\Delta x)\sin(k\pi\Delta x)}\end{array}\right]$$

$$= \frac{1}{(\Delta x)^2}\left[2\cos(k\pi i\Delta x)\cos(k\pi\Delta x) - 2\cos(k\pi i\Delta x)\right]$$

$$= \frac{1}{(\Delta x)^2}\, 2\cos(k\pi i\Delta x)\left[\cos(k\pi\Delta x) - 1\right] \checkmark$$

Part 3 If we take the natural extention of $U^{(k)}$ we see that $\cos(k\pi(0)\Delta x) \neq 0$ and $\cos(k\pi) \neq 0$ thus $U^{(k)}$ dose not satisfy the zero boundry conditions.