Homework 5

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Problem 1 (Theoretical)

Part 1: Carry out von Neumann stability analysis to show that the BTCS method is unconditionally stable

Part 2: Carry out Taylor expansions to show that the local truncation error of the Crank-Nicolson method is

$$e_i^n(\Delta x, \Delta t) = \Delta t\, O\left((\Delta t)^2 + (\Delta x)^2\right)$$

In the final expression, be sure to convert r back to $\Delta t/(\Delta x)^2$.

1)

Part 1

$$U_i^n = \rho^n \exp(\sqrt{-1}\,\xi i \Delta x)$$

sub into numerical method

$$\rho^{n+1}\exp(\sqrt{-1}\,\xi\, i\,\Delta x) = \rho^n \exp(\sqrt{-1}\,\xi i \Delta x) + r\rho^{n+1}\exp(\sqrt{-1}\,\xi i \Delta x)\left(\exp(\sqrt{-1}\,\xi\Delta x) - 2 + \exp(-\sqrt{-1}\,\xi\Delta x)\right)$$

$$\rho = 1 + r\rho\left(\cos(\xi\Delta x) + \sqrt{-1}\sin(\xi\Delta x) - 2 + \cos(-\xi\Delta x) + \sqrt{-1}\sin(-\xi\Delta x)\right)$$

$$\rho = 1 + r\rho\left(2\cos(\xi\Delta x) - 2\right)$$

$$\rho - r\rho\left(2\cos(\xi\Delta x) - 2\right) = 1$$

$$\rho\left(1 - r(2\cos(\xi\Delta x) - 2)\right) = 1$$

$$\rho = \frac{1}{1 - 2r(\cos(\xi\Delta x) - 1)}$$

For stability we need $|\rho| \le 1 + C\Delta t$

$$|\rho| = \frac{1}{1 + 4r\sin^2\left(\frac{\xi\Delta x}{2}\right)}$$

since the denom is always positive and

greater than 1 $|\rho| \le 1 + C\Delta t$ $\blacksquare$

Part 2

C-N method

$$U_i^{n+1} = U_i^n + \frac{r}{2}\left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}\right) + \frac{r}{2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)$$

NTS

$$e_i^n(\Delta x, \Delta t) = \Delta t\, O\left((\Delta t)^2 + (\Delta x)^2\right)$$

$$e_i^n(\Delta x, \Delta t) = \{U(x_i, t_{n+1})\} - L_{num}\{U(x_i, t_n)\}$$

for CN

$$U(x_i, t_{n+1}) - U(x_i, t_n) - \frac{r}{2}\Big[U(x_{i+1}, t_n) - 2U(x_i, t_n) + U(x_{i-1}, t_n)$$
$$+ U(x_{i+1}, t_{n+1}) - 2(x_i, t_{n+1}) + U(x_{i-1}, t_{n+1})\Big]$$

Expand around (x_i, t_n)

$$U(x_i, t_{n+1}) - U(x_i, t_n) = U_t \Delta t + \frac{1}{2} U_{tt} (\Delta t)^2 + O(\Delta t^3)$$

$$U(x_{i+1}, t_n) - 2U(x_i, t_n) + U(x_{i-1}, t_n) =$$

$$\cancel{U(x_i, t_n)} + \cancel{U_x}\Delta x + \frac{U_{xx}}{2}\Delta x^2 + O(\Delta x^4) - \cancel{2U(x_i, t_n)}$$
$$+ \cancel{U(x_i, t_n)} - \cancel{U_x}\Delta x + \frac{U_{xx}}{2}\Delta x^2 + O(\Delta x^4)$$
$$= U_{xx}\Delta x^2 + O(\Delta x^4)$$

$$U(x_{i+1}, t_{n+1}) - 2U(x_i, t_{n+1}) + U(x_{i-1}, t_{n+1}) = U(x_i + \Delta x, t_n + \Delta t) - 2U(x_i, t_n + \Delta t)$$

$+ U(x_i - \Delta x, t_n + \Delta t)$

$$= \underbrace{(\Delta x)^2 U_{xx}\Big|_{(x_i, t_{n+1})}}_{} + O(\Delta x)^4$$

$$\Delta x^2 U_{xx} + \Delta x^2 \Delta t\, U_{xxt} + O((\Delta x)^4 + (\Delta t)^2)$$

$$U_{xxt} = U_{xxxx} \qquad r = \frac{\Delta t}{(\Delta x)^2}$$

$$U_{tt} = U_{xxxx}$$

$$e_i^n(\Delta x, \Delta t) = U_t \Delta t + \frac{1}{2} U_{tt} (\Delta t)^2 + O(\Delta t^3) - \frac{\Delta t}{2(\Delta x)^2}\left[U_{xx} (\Delta x)^2 + U_{xx}(\Delta x)^2 + U_{xxt}(\Delta x^2)(\Delta t) + O(\Delta x^4) + O(\Delta t^2) \right]$$

$$e_i^n(\Delta x, \Delta t) = \cancel{U_{xx}\Delta t} + \cancel{\frac{1}{2} U_{xxxx} (\Delta t)^2} - \cancel{U_{xx}(\Delta t)} - \cancel{\frac{U_{xxxx}(\Delta t)^2}{2}} + \Delta t\, O(\Delta x^2 + \Delta t^2) + O((\Delta t)^3)$$

$$= \Delta t\, O(\Delta x^2 + \Delta t^2)$$

Problem 2 (Theoretical)

Consider matrix

$$A = \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}_{(N-1)\times(N-1)}, \qquad \Delta x = \frac{1}{N}$$

Part 1: Verify that the set below are eigenvalues and eigenvectors of matrix A.

$$\left.\begin{aligned} \lambda^{(k)} &= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right) \\ w^{(k)} &= \left\{\sin(k\pi i\Delta x),\ i = 1,2,\ldots,N-1\right\} \end{aligned}\right\}, \quad k = 1,2,\ldots,N-1$$

Part 2: Verify that

$$\frac{1}{(\Delta x)^2}\left(\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i\Delta x) + \cos(k\pi(i+1)\Delta x)\right)$$
$$= \frac{2}{(\Delta x)^2}\left(\cos(k\pi\Delta x) - 1\right)\cdot\cos(k\pi i\Delta x), \qquad k = 1,2,\ldots,N-1$$

Part 3: Explain why $u^{(k)} = \left\{\cos(k\pi i\Delta x),\ i = 1,2,\ldots,N-1\right\}$ is NOT an eigenvector of A.

Hint: What boundary conditions did we use in defining matrix A?

2) $\underline{Au}$

$$A \longleftrightarrow D_x^2$$

$$D_x^2\{u\}_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{(\Delta x)^2} = \lambda U_i$$

$U_0 = 0 \quad U_N = 0$

W.T.S.

$$\frac{w_{i+1}^{(k)} - 2w_i^{(k)} + w_{i-1}^{(k)}}{(\Delta x)^2} = \lambda^{(k)} w_i^{(k)}$$

$w_0^{(k)} = 0, \; w_N^{(k)} = 0$

$$\frac{\sin(k\pi(i+1)\Delta x) - 2\sin(k\pi(i)\Delta x) + \sin(k\pi(i-1)\Delta x)}{(\Delta x)^2} \stackrel{?}{=} \frac{2}{(\Delta x)^2}(\cos(k\pi\Delta x - 1) \cdot \sin(k\pi i\Delta x)$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

LHS

$$= \sin(k\pi i\Delta x)\cos(k\pi\Delta x) + \cancel{\cos(k\pi i\Delta x)\sin(k\pi\Delta x)} - 2\sin(k\pi i\Delta x)$$
$$+ \sin(k\pi i\Delta x)\cos(k\pi\Delta x) - \cancel{\cos(k\pi i\Delta x)\sin(k\pi\Delta x)}$$

$$= \frac{2\sin(k\pi i\Delta x)\cos(k\pi\Delta x) - 2\sin(k\pi i\Delta x)}{(\Delta x)^2}$$

$$= \frac{2\sin(k\pi i\Delta x)}{\Delta x^2}(\cos(k\pi\Delta x) - 1) = \frac{2}{\Delta x^2}\sin(k\pi i\Delta x)(\cos(k\pi\Delta x) - 1)$$ RHS ✓

$i = 0 \Rightarrow \sin(k\pi(0)\Delta x) = \sin(0) = 0$ ✓

$i = N \Rightarrow \sin(k\pi N(1/N)) = \sin(k\pi) = 0$ ✓

We have verified that $\lambda^{(k)}$ and $w^{(k)}$ are

We have verified that $\lambda^{(k)}$ and $w^{(k)}$ are eigenvalues and eigen vectors of matrix A respectively

Part 2

$$\frac{1}{(\Delta x)^2}\left[\cos(k\pi(i-1)\Delta x) - 2\cos(k\pi i\Delta x) + \cos(k\pi(i-1)\Delta x)\right]$$

$$= \frac{1}{(\Delta x)^2}\left[\begin{array}{l}\cos(k\pi i\Delta x)\cos(k\pi\Delta x) + \cancel{\sin(k\pi i\Delta x)}\sin(k\pi\Delta x) - 2\cos(k\pi i\Delta x) \\ \qquad + \cos(k\pi i\Delta x)\cos(k\pi\Delta x) - \cancel{\sin(k\pi i\Delta x)}\sin(k\pi\Delta x)\end{array}\right]$$

$$= \frac{1}{(\Delta x)^2}\left[2\cos(k\pi i\Delta x)\cos(k\pi\Delta x) - 2\cos(k\pi i\Delta x)\right]$$

$$= \frac{1}{(\Delta x)^2}\, 2\cos(k\pi i\Delta x)\left[\cos(k\pi\Delta x) - 1\right] \checkmark$$

Part 3 If we take the natural extention of $U^{(k)}$ we see that $\cos(k\pi(0)\Delta x) \neq 0$ and $\cos(k\pi) \neq 0$ thus $U^{(k)}$ dose not satisfy the Zero boundry conditions.