

Homework 6

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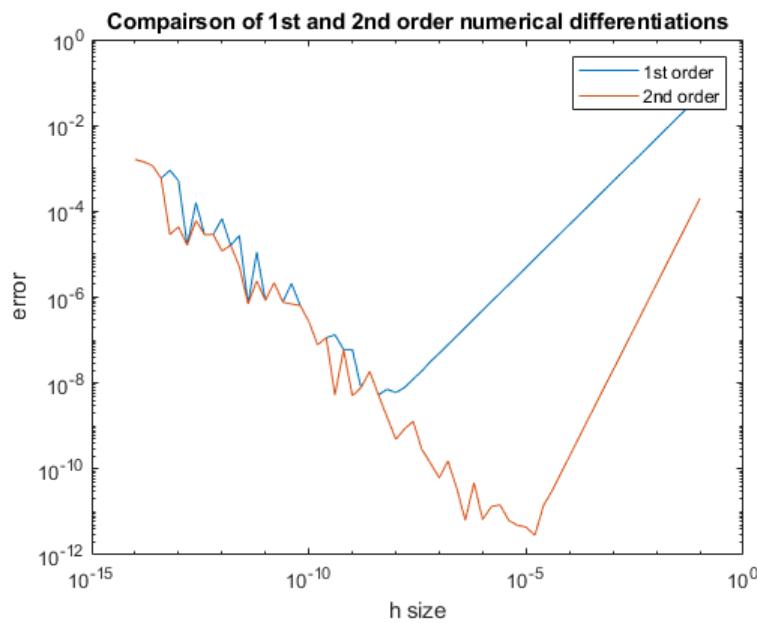
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Problem 1-2

Submitted as a hand written pdf attached at the end of this report.

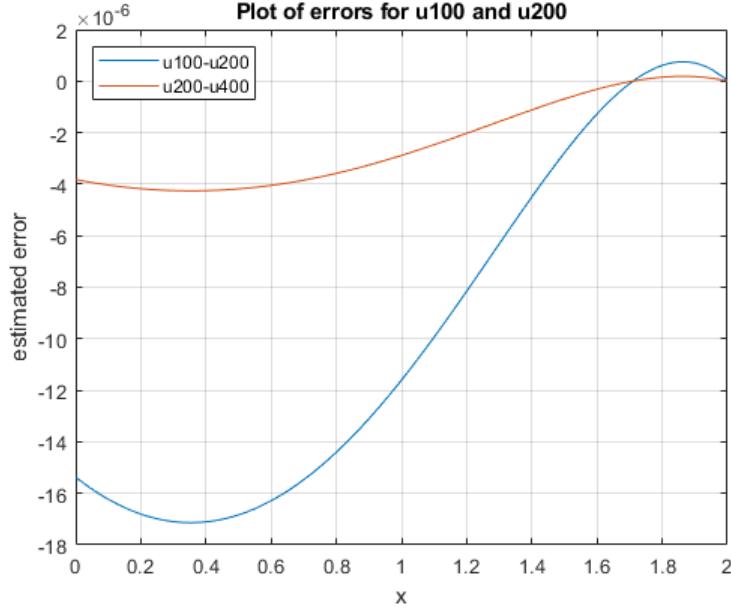
Problem 3

In Problem 3 we were given the simple task of approximating $\frac{d}{dx} \sin(x)$ when $x = 1.45$. From there we found the exact error of each method for different values of h . From the graph below we see that the second order method is more accurate at a larger h value.



Problem 4

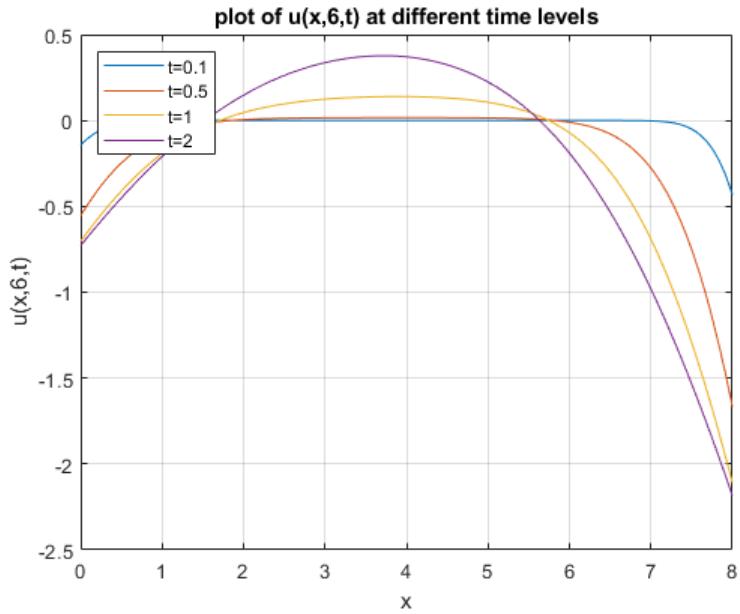
In problem 4 we continue with problem 6 homework 5. We program FTCS to solve the problem to $T=3$ and $N=100,200,400$. Then using the spline function in matlab we plotted $u_{(N=100)} - u_{(N=200)}$ and $u_{(N=200)} - u_{(N=400)}$ vs x . At $T = 3$.



Problem 5

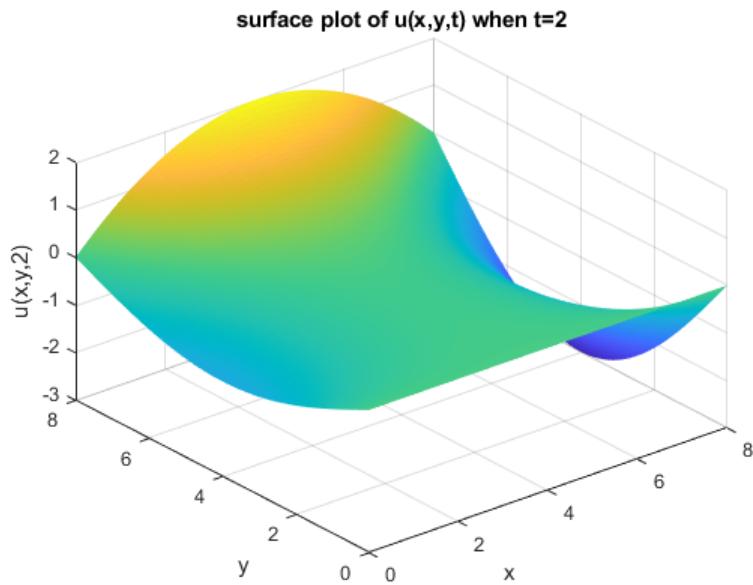
Part 1

Problem 5 was a 2D IBVP of the heat equation we needed to solve the equation to $T=2$ using FTCS. Then plot $u(x, 6)$ vs. x at $t = 0.1, 0.5, 1, 2$. Pictured below



Part 2

In part 2 we did a surface plot u vs (x, y) at $T = 2$. shown below.

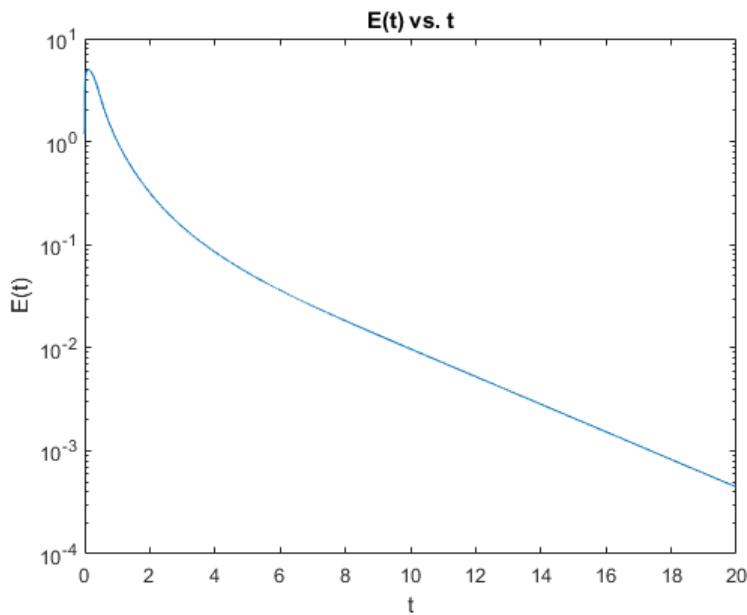


Problem 6

Using the same IBVP from Problem 5 problem 6 asks us to solve the IBVP to $T = 20$.

Part 1

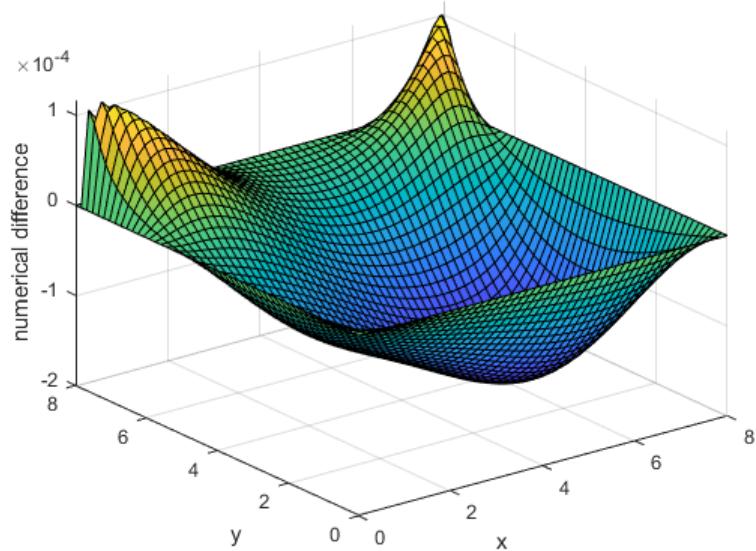
Part 1 asks us to plot the max difference between the current time and the next time step of the internal points the plot is shown below.



Part 2

Part 2 has us rerun the simulation using twice the step size for x and y then we calculate the difference at $T=20$ between the two grids on their common grid points. The surface it creates is shown below.

Numerical difference between $(\Delta x, \Delta y)$ and $(2^* \Delta x, 2^* \Delta y)$



Problem 1 (Theoretical)Consider the Lax-Friedrichs method for the general case ($a > 0$ or $a < 0$)

$$u_j^{n+1} = \frac{u_{j-1}^n + u_{j+1}^n}{2} - \frac{\alpha r}{2} (u_{j+1}^n - u_{j-1}^n), \quad r = \frac{\Delta t}{\Delta x}$$

Part 1: Carry out von Neumann stability analysis.

Part 2: Use Taylor expansion to find the local truncation error ϵ_j^n .Find coefficients of $(\Delta t)^2$, $(\Delta t)(\Delta x)$ and $(\Delta x)^2$ terms in ϵ_j^n .

Part 3: Answer the two questions below.

For fixed $\frac{\Delta t}{\Delta x} = r$, write $\frac{\epsilon_j^n}{\Delta t}$ in terms of Δx only. Do we have $\lim_{\Delta x \rightarrow 0} \frac{\epsilon_j^n}{\Delta t} = 0$?For fixed $\frac{\Delta t}{(\Delta x)^2} = c$, write $\frac{\epsilon_j^n}{\Delta t}$ in terms of Δx only. Do we have $\lim_{\Delta x \rightarrow 0} \frac{\epsilon_j^n}{\Delta t} = 0$?Part 1

$$U_j^n = \rho^n \exp(\sqrt{-1} \xi_j^n i \Delta x)$$

$$\int_{-r}^{r} \exp(\sqrt{-1} \xi_j^n i \Delta x) = \underbrace{\int_{-\rho}^{\rho} \exp(\sqrt{-1} \xi_{(i+1)}^n i \Delta x)}_2 + \underbrace{\int_{\rho}^{-\rho} \exp(\sqrt{-1} \xi_{(i-1)}^n i \Delta x)}_2$$

$$= \frac{\alpha r}{2} \left(\int_{-\rho}^{\rho} \exp(\sqrt{-1} \xi_{(i+1)}^n i \Delta x) - \int_{\rho}^{-\rho} \exp(\sqrt{-1} \xi_{(i-1)}^n i \Delta x) \right)$$

$$\xi_j^n = \underbrace{\exp(\sqrt{-1} \xi_{\Delta x}) + \exp(-\sqrt{-1} \xi_{\Delta x})}_2$$

$$= \frac{\alpha r}{2} \left(\exp(\sqrt{-1} \xi_{\Delta x}) - \exp(-\sqrt{-1} \xi_{\Delta x}) \right)$$

$$\xi_j^n = \cos(\xi_{\Delta x}) - \alpha r \sin(\xi_{\Delta x})$$

For stability we need $|\rho| \leq 1 + C \Delta t$

$$|\rho| = \sqrt{\cos^2(\xi_{\Delta x}) + \alpha^2 r^2 \sin^2(\xi_{\Delta x})}$$

$|\beta| > 0$ for any αr

$$|\beta| = \sqrt{1 - \sin^2(\beta_{\Delta x}) + \alpha^2 r^2 \sin^2(\beta_{\Delta x})}$$

$$= \sqrt{1 + (\alpha^2 r^2 - 1)(\sin^2(\beta_{\Delta x}))}$$

let $\alpha r > 1$ then $\alpha^2 r^2 > 1$

$$\text{let } \omega = \alpha^2 r^2 - 1 > 0$$

thus

$$\sqrt{1 + \omega \sin^2(\beta_{\Delta x})} > 1$$

and also let

$$\alpha r < -1 \text{ then } \alpha^2 r^2 > 1$$

$$U = \alpha^2 r^2 - 1 > 0$$

thus

$$\sqrt{1 + U \sin^2(\beta_{\Delta x})} > 1$$

Finally $|\alpha r| < 1$ then $\alpha^2 r^2 < 1$

$$V = \alpha^2 r^2 - 1 < 0$$

$$\sqrt{1 + V \sin^2(\beta_{\Delta x})} \leq 1$$

We see for stability $|s| \leq 1$ it
must be that $-1 \leq \alpha r < 1$

Part 2

$$u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} - \frac{\alpha r}{2}(u_{j+1}^n - u_{j-1}^n), \quad r = \frac{\Delta t}{\Delta x}$$

expand around (x_i, t_n)

$$e_i^n = U(x_i, t_{n+1}) - \frac{1}{2} [U(x_{i+1}, t_n) + U(x_{i-1}, t_n)] + \frac{\alpha r}{2} [U(x_{i+1}, t_n) - U(x_{i-1}, t_n)]$$

Expand

$$U(x_i, t_{n+1}) = U + V_t \Delta t + \frac{V_{tt} \Delta t^2}{2} + O(\Delta t^3)$$

Expand

$$U(x_{i+1}, t_n) + U(x_{i-1}, t_n) = 2U + U_{xx} \Delta x^2 + O(\Delta x^4)$$

Expand

$$U(x_{i+1}, t_n) - U(x_{i-1}, t_n) = 2V_x \Delta x + O(\Delta x^3)$$

Sub in

$$\cancel{U + V_t \Delta t + \frac{V_{tt} \Delta t^2}{2}} - \cancel{U - \frac{U_{xx} \Delta x^2}{2} + \alpha \Delta t V_x + \Delta t O(\Delta x^2) + O(\Delta x^4)} + O(\Delta t^3)$$

$$\underbrace{V_t \Delta t + \alpha \Delta t V_x}_{=0} + \frac{V_{tt} \Delta t^2}{2} - \frac{U_{xx} \Delta x^2}{2} + \Delta t O(\Delta x^2) + O(\Delta x^4) + O(\Delta t^3)$$

from problem

$$e_i^n = \frac{1}{2} \left(U_{t+} \Delta t - U_{xx} \Delta x^2 \right) + \Delta t O(\Delta x^2) + O(\Delta x^4) + O(\Delta t^3)$$

Part 3

$$\frac{e_i^n}{\Delta t} = \frac{1}{2} \left(U_{t+} \Delta t - U_{xx} \frac{\Delta x^2}{\Delta t} \right) + O(\Delta x^2) + \frac{O(\Delta x^3)}{\Delta t} + O(\Delta t^2)$$

when

$$\Delta t = r \Delta x \Rightarrow \frac{1}{2} \left(U_{t+} r \Delta x - U_{xx} \frac{\Delta x}{r} \right) + O(\Delta x^2) + \frac{O(\Delta x^3)}{r} + O(r^2 \Delta x^2)$$

$$\lim_{\Delta x \rightarrow 0} \frac{e_i^n}{\Delta t} = 0$$

when

$$\Delta t = c \Delta x^2$$

$$\Rightarrow \frac{1}{2} \left(U_{t+} c \Delta x^2 - U_{xx} c^{-1} \right) + O(\Delta x^2) + \frac{O(\Delta x^3)}{c} + O(c^2 \Delta x^4)$$

$$\lim_{\Delta x \rightarrow 0} \frac{e_i^n}{\Delta t} = - \frac{U_{xx} c^{-1}}{2} \neq 0$$

Problem 2 (Theoretical)

Carry out von Neumann stability analysis on each of the methods below

- i) the BTCS method for the general case ($a > 0$ or $a < 0$)

$$u_i^{n+1} = u_i^n - \frac{ar}{2} (u_{i+1}^{n+1} - u_{i-1}^{n+1}), \quad r = \frac{\Delta t}{\Delta x}$$

- ii) the implicit upwind method for the case of $a > 0$

$$u_i^{n+1} = u_i^n - ar(u_i^{n+1} - u_{i-1}^{n+1}), \quad r = \frac{\Delta t}{\Delta x}$$

Hint: Examine $|1/\rho|^2$.

2 i)

$$\int_{-1}^{n+1} \exp(\sqrt{-1} \zeta_i \Delta x) = \int_{-1}^n \exp(\sqrt{-1} \zeta_i \Delta x) - \frac{ar}{2} \int_{-1}^{n+1} \left(\exp(\sqrt{-1}(i+1)\zeta \Delta x) - \exp(\sqrt{-1}(i-1)\zeta \Delta x) \right)$$

$$f = 1 - \frac{\alpha r \beta}{2} (\exp(\sqrt{-1}\beta \Delta x) - \exp(-\sqrt{-1}\beta \Delta x))$$

$$f = 1 - \alpha r \beta i \sin(\beta \Delta x)$$

$$f + \alpha r \beta i \sin(\beta \Delta x) = 1$$

$$f = \frac{1}{1 + \alpha r i \sin(\beta \Delta x)}$$

$$|f| = \frac{1}{\sqrt{1 + \alpha^2 r^2 \sin^2(\beta \Delta x)}} \quad |$$

Since the denominator is always greater than or equal to 1
 this implies $|f| \leq 1$ so we have stability \oplus

ii)

$$\sum_{n=1}^{n+1} \exp(\sqrt{-1}\beta i \Delta x) = \sum_{n=1}^n \exp(\sqrt{-1}\beta i \Delta x) + \alpha r \beta i \exp(\sqrt{-1}\beta i \Delta x) (1 - \exp(\sqrt{-1}\beta \Delta x))$$

$$f = 1 - \alpha r \beta (1 - \exp(-\sqrt{-1}\beta \Delta x))$$

$$f = \frac{1}{1 + \alpha r (1 - \cos(\beta \Delta x) + i \sin(\beta \Delta x))}$$

$$\begin{aligned}
 |\beta|^2 &= \frac{1}{(1+\alpha r - \alpha \cos(\xi \Delta x))^2 + (\alpha r \sin(\xi \Delta x))^2} \\
 &= \frac{1}{1 + 2\alpha r - 2\alpha r \cos(\xi \Delta x) + (\alpha r)^2 - 2(\alpha r)^2 \cos(\xi \Delta x) + (\alpha r)^2 \cancel{\cos^2(\xi \Delta x)} + (\alpha r)^2 \cancel{\sin^2(\xi \Delta x)}} \\
 &= \frac{1}{1 + 2\alpha r - 2\alpha r \cos(\xi \Delta x) + 2\alpha^2 r^2 - 2\alpha^2 r^2 \cos(\xi \Delta x)} \\
 &= \frac{1}{1 + 2\alpha r (1 - \cos(\xi \Delta x)) + \alpha r - \alpha r \cos(\xi \Delta x)} \\
 \frac{1}{|\beta|^2} &= 1 + 2\alpha r (1 - \cos(\xi \Delta x)) + \alpha r - \alpha r \cos(\xi \Delta x) \\
 &= 1 + 2\alpha r (1 - \cos(\xi \Delta x)) + \alpha r (1 - \cos(\xi \Delta x)) \\
 &= 1 + 2\alpha r (1 - \cos(\xi \Delta x))(1 + \alpha r)
 \end{aligned}$$

Since $\alpha > 0$

$$\Rightarrow \frac{1}{|\beta|^2} \geq 1 \Rightarrow |\beta| \leq 1$$

Thus we have shown that the

Implicit upwind method is unconditionally stable. 