ASEN2012 - Project 2 - Water Bottle Rocket Flight, Fall 2024
Experimental and Computational Methods in Aerospace Engineering Sciences

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Key Dates:

- Assigned: Monday November 4th
- Individual Component Due: Sunday November 24th 11:59 pm uploaded to Canvas
- Group Component Due: Thursday December 12th 11:59 pm uploaded to Canvas

1 Background and Introduction

1.1 Project Motivation

The goal of this project is to model the trajectory of the bottle rocket launch, using the numerical integration of a system of ordinary differential equations. Numerical, also known as computational or mathematical, modeling is a common tool used by engineers to understand or predict the behavior of a physical system when analytical solutions do not exist or are too complicated to solve for. Instead of a "plug-and-chug" analytical solution, a computer model approximates a solution by performing a large number of calculations. Further, engineers can use the results of their numerical simulations to determine how to best design the physical system. A bottle rocket is a very simple rocket and consists of a plastic bottle (typically a two-liter soda bottle) filled partially with a liquid (usually water) and pressurized by air. When the stopper is removed, the water is forced out leading to a reactionary force that propels the bottle according to Newton's laws of motion. The photograph below shows a typical bottle rocket launch. Even though the water bottle rocket system is easy to state, the underlying governing equations must be solved using numerical integration as no analytical solution exists.

To prepare you for the Bottle Rocket Design and Performance Analysis Lab in ASEN 2704/2804 in spring, you will create a numerical simulation of a bottle rocket flight trajectory to understand the functional dependence of bottle rocket performance on the design parameters, such as the volumetric fraction of water in the bottle, the initial pressure of air, drag of the rocket, and the launch angle. Your task is to use the knowledge you have gained to date to develop a MATLAB code to determine the bottle rocket thrust as a function of time, and predict the resulting height and range of the rocket. You are then asked to use the code to explore the parameter space in order to determine how each of the parameters affect the height and the range of the rocket, and what combination of parameters will allow the rocket to land within 1 meter of a 85 meter target.

1.2 Trajectory of a Bottle Rocket

To determine the bottle rocket trajectory, we will apply Newton's laws of motion, using a free body diagram. We will simplify this problem to only look at motion in 2 dimensions, in the horizontal (x) and vertical (z) directions:
The sum of the forces acting on this rocket can be written in vector notation:

$$\Sigma\ \vec{\text{Forces}} = m_r\vec{a} = m_r \begin{bmatrix} a_x \\ a_z \end{bmatrix} = m_r\dot{\vec{v}} = \vec{F} - \vec{D} + m_r\vec{g}\ , \tag{1}$$

where m_r is the mass of the rocket, $\vec{a}$ is the acceleration vector of the rocket, $\vec{F}$ is the thrust vector of the rocket (noting that F is used instead of T to avoid confusion with temperature), $\vec{g}$ is the gravity vector that acts only in the z direction equal to 9.81 m/s^2, and $\vec{v}$ is the velocity vector of the rocket. Note that

Figure 1: Image of students in ASEN 2804 launching a water bottle rocket!

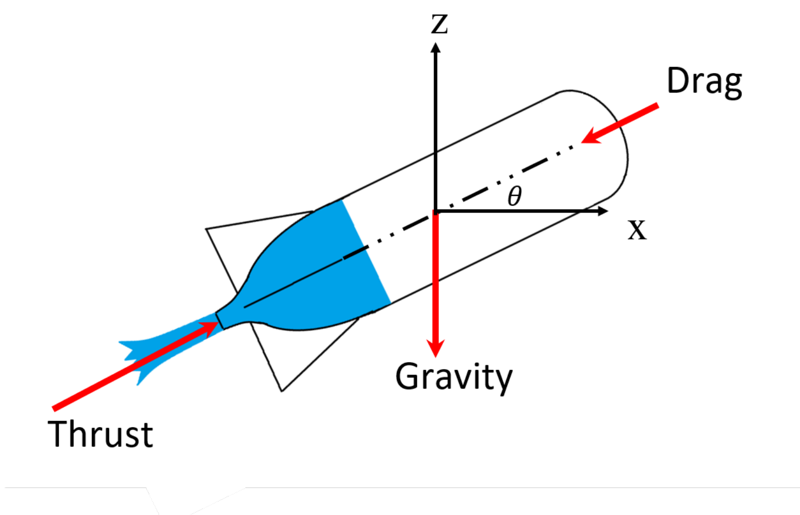

Figure 2: Free body diagram of the rocket

m_r, v, F, and D are all functions of time, t.

To find the components of the thrust and drag force vectors in the x and z directions, which is needed to understand the rocket trajectory based on acceleration in each direction, you can multiply the magnitude of each force by the velocity unit vector, $\hat{h}$; this is equivalent to the heading. The heading of the rocket at any instant can be found by taking the velocity components in the x and z direction over the normalized magnitude of velocity at that instant:

$$\hat{h} = \frac{\vec{v}}{|\vec{v}|} = \frac{v_x}{\sqrt{v_x^2 + v_z^2}}\hat{i} + \frac{v_z}{\sqrt{v_x^2 + v_z^2}}\hat{k}. \tag{2}$$

These components are equivalent to $\cos\theta$ and $\sin\theta$, respectively, by trigonometry and can be used to determine the respective components of the forces acting on the rocket.

Let's start by understanding the drag force on the rocket. The magnitude of the drag force is a function of dynamic pressure, q, the drag coefficient, C_D, and the cross-sectional area of the bottle, A_B. The dynamic pressure is a function of the air density, ρ, and the magnitude of the rocket's velocity, $|\vec{v}|$ as

$$q = \frac{1}{2}\rho|\vec{v}|^2, \tag{3}$$

and therefore, the drag force can then be written as

$$D = qC_D A_B = \frac{1}{2}\rho|\vec{v}|^2 C_D A_B. \tag{4}$$

The drag coefficient depends on a variety of factors, including the Reynolds number of the flow, any flow separation toward the rear of the bottle, the drag on the stabilizing surfaces and the shape of the nose cone, and it is hard to determine theoretically. It is usually measured in a wind tunnel, we can expect C_D=0.3 to 0.5.

The most important term in Eq. 1 is the thrust F. It can be estimated by applying the laws of thermodynamics and aerodynamics to the expansion of air in the bottle (see Section 1.3). We also need to keep track of the mass of the rocket, as water and air is expelled through the bottle's mouth (we will call it the throat in conformity with rocket terminology). The following sections explain the principles behind how thrust is generated in 2 phases followed by a ballistic trajectory. Note that at the end, the final equations needed for thrust are summarized.

1.3 Bottle Rocket Thermodynamics

Let p^i_{air}, $\forall^i_{air}$ and T^i_{air} be the initial pressure, volume and temperature of air inside the bottle, respectively. Then the initial mass of air would be represented by the ideal gas law as $m^i_{air} = p^i_{air}\forall^i_{air}/(R_{air}T^i_{air})$, where $R_{air} = 287J/(kg \cdot K)$. The bottle rocket thrust phase can be divided into two phases:

1. Before the water is fully exhausted
2. After the water is fully exhausted

1.3.1 Before the water is fully exhausted

During this phase, the mass of air m_{air} remains constant but the air volume $\forall_{air}$ increases as water is expelled, and therefore the air density is inversely proportional to its volume. We will assume the expansion of air during the rocket operation is isentropic, meaning the process is adiabatic (no heat transfer to or from the air mass) and reversible (there are no frictional losses). This is a good approximation and the air pressure p at any future time t is then given by:

$$\frac{p}{p^i_{air}} = \left(\frac{\forall^i_{air}}{\forall}\right)^\gamma , \tag{5}$$

where γ is the ratio of specific heats and $\gamma = 1.4$ for air.

The mass flow rate of the water out of the throat of the bottle is:

$$\dot{m}_w = c_{dis}\rho_w A_t v_e \ , \tag{6}$$

where ρ_w is the water density, v_e is the velocity of the exhaust, A_t is the throat area, and c_{dis} is the discharge coefficient, typically less than 1.

The thrust of the rocket, F is given by:

$$F = \dot{m}_w v_e + (p_e - p_a)A_t \ , \tag{7}$$

where p_a is the ambient pressure (atmospheric) and p_e is the pressure at the exit. Since the water is incompressible, we can apply the Bernoulli equation for incompressible flows using the air pressure p,

$$(p - p_a) = \frac{\rho_w}{2} v_e^2 \rightarrow v_e = \sqrt{\frac{2(p - p_a)}{\rho_w}}. \tag{8}$$

Since $p_e = p_a$ (the exit pressure equals the ambient air pressure), the equation for F can be simplified to the following:

$$F = \dot{m}_w v_e = 2c_{dis} A_t (p - p_a). \tag{9}$$

Note that the thrust is independent of the liquid density! The air pressure p decreases with time as the air volume expands, and therefore the thrust decreases with time. The air pressure at time t can be computed if the volume is known from Eq. 5. The rate of change of volume of air with time can be written as:

$$\frac{d\cancel{V}}{dt} = c_{dis} A_t v_e = c_{dis} A_t \sqrt{\frac{2(p - p_a)}{\rho_w}} = c_{dis} A_t \sqrt{\frac{2}{\rho_w}\left[p_0 \left(\frac{\cancel{V}_o}{\cancel{V}}\right)^{\gamma} - p_a\right]}. \tag{10}$$

This formulation involves first inserting Eq. 8 and then the relationship from Eq. 5, but now using initial condition nomenclature for the volume ($\cancel{V}_0 = \cancel{V}^i_{air}$) and air pressure ($p_0 = p^i_{air}$). From Eq. 10 the volume $\cancel{V}(t)$ of air can be determined. This equation needs to be solved with the initial conditions $\cancel{V} = \cancel{V}^i_{air}$ at time $t = 0$. Because of the nonlinear nature, this has to be done numerically using an ODE solver such as the 4th order Runge-Kutta or `ode45`. The integration stops when $\cancel{V} = \cancel{V}_B$, where $\cancel{V}_B$ is the volume of the bottle. This is physically when all the water has been expelled.

Eq. 5 can be used to determine $p(t)$, and from this, $F(t)$ can be determined by Eq. 9.
The rocket mass changes with time as the water mass flows out of the bottle according to:

$$\dot{m}_r = -\dot{m}_w = -c_{dis} \rho_w A_t v_e = -c_{dis} A_t \sqrt{2\rho_w (p - p_a)}. \tag{11}$$

The initial mass of the rocket is the sum of the masses of the bottle, the initial pressurized air, and the initial water:

$$m^i_r = m_B + m^i_{water} + m^i_{air}. \tag{12}$$

Substituting in, this can be formulated in terms of the density of water, the initial volume of the water (which is the difference between the total volume of the bottle and that of the air, initially), and finally the pressure and temperature of the air:

$$m^i_r = m_B + \rho_w \left(\cancel{V}_B - \cancel{V}^i_{air}\right) + \frac{p^i_{air} \cancel{V}^i_{air}}{R_{air} T^i_{air}} \tag{13}$$

1.3.2 After the water is fully exhausted

Let p_{end} be the pressure and T_{end} be the temperature of air in the bottle at the time all the water is expelled, such that

$$p_{end} = p^i_{air} \left(\frac{\cancel{V}^i_{air}}{\cancel{V}_B}\right)^{\gamma} \text{ and} \tag{14}$$

$$T_{end} = T^i_{air} \left(\frac{\cancel{V}^i_{air}}{\cancel{V}_B}\right)^{\gamma - 1}. \tag{15}$$

Once the water is exhausted, the volume of air remains constant but its mass decreases, and therefore the density is proportional to the mass. Again, we assume air expands isentropically, until the air pressure p drops to the ambient pressure, p_a. Then the pressure at any time t is given by:

$$\frac{p}{p_{end}} = \left(\frac{m_{air}}{m^i_{air}}\right)^{\gamma}, \tag{16}$$

and the corresponding density is $\rho = m_{air}/\forall_B$ and the corresponding temperature is $T = p/(\rho \cdot R_{air})$.

To determine the exit velocity of the air, define a critical pressure, p_*:

$$p_* = p\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{(\gamma-1)}} , \tag{17}$$

where the following are true:

1. If $p_* > p_a$, the follow is choked (exit Mach number, $M_e = 1$ and the exit velocity is

$$v_e = \sqrt{\gamma R_{air} T_e} \text{ where } T_e = \left(\frac{2}{\gamma+1}\right)T \ , \ \rho_e = \frac{p_e}{R_{air}T_e} \text{ and } p_e = p_*. \tag{18}$$

2. If $p_* < p_a$, the flow is not choked and the exit Mach number is obtained from:

$$\frac{p}{p_a} = \left(1 + \frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma}{\gamma-1}} \text{ where } \frac{T}{T_e} = \left(1 + \frac{\gamma-1}{2}M_e^2\right) \ , \ \rho_e = \frac{p_a}{R_{air}T_e} \text{ and } p_e = p_a. \tag{19}$$

   And the exit velocity is given by $v_e = M_e\sqrt{\gamma R_{air} T_e}$

For both cases, the thrust is given by

$$F = \dot{m}_{air} v_e + (p_e - p_a)A_t \ , \tag{20}$$

where $\dot{m}_{air} = c_{dis}\rho_e A_t v_e$. During this phase, the rocket mass decreases according to

$$\dot{m}_r = -\dot{m}_{air} = -c_{dis}\rho_e A_t v_e. \tag{21}$$

1.4 Ballistic Phase

Thrust is generated by the bottle rocket until the air pressure in the bottle, p falls to the ambient pressure, p_a. After that the thrust is zero and the rocket enters its free ballistic phase under the influence of gravity,

$$F = 0 \text{ and } m_r \sim m_B \tag{22}$$

The rocket is on a ballistic trajectory for the rest of the flight (until the rocket hits the ground), with initial conditions corresponding to those at the end of the thrust phase.

2 Your Assignment: Determine the Rocket Trajectory

As you can see, the bottle rocket flight consists of three distinct phases:

1. From the moment the stopper is removed until the water is exhausted,
2. After the water is exhausted until the air pressure drops to the ambient value and the thrust phase ends,
3. Ballistic phase

The first two comprise the thrust phase, which is usually a small fraction of the total flight time.
Your assignment is to model the trajectory of the water bottle rocket using the equations of motion described above using `ode45`. The rocket trajectory in the first phase of flight can be estimated using **Eqs. 1 and 9**, during the second phase of flight using **Eqs. 1 and 20**, and **Eqs. 1 and 22** in the ballistic phase of flight. To convert force to acceleration, you must also keep track of rocket mass using **Eqs. 11 and 21**.

To launch, the rocket is set on a test stand, which guides the rocket during the initial portion of flight to ensure a straight trajectory. The initial angle of the test stand is the launch angle, θ^i and the initial velocity is $\vec{v}^i = 0$. You may assume the test stand is 0.5m in length, and after the rocket clears the test stand it is free to change heading by calculating the unit vector. Putting the rocket on the test stand elevates it off the ground by 0.25 m. As in Project 1, there are both individual and group components of this project.

Flow Chart: You are **not** required to submit a flow chart for this project. However, creating one will be **very** helpful!

Coding Requirements: within your code, you **may not** do any of the following:

- use global variables
- adjust the provided constants found in Sec. 4.1
- adjust the tolerances or set a maximum/minimum step size for `ode45`
- use "events" option for `ode45`

Individual Deliverable: You will model the flight of the rocket and compare to a verification test case, which we provide. The test case will show the range, height and parameters needed to achieve this trajectory. You should aim to replicate the flight trajectory (z vs. x position) and thrust profile (F vs. t). Once you have reproduced this result, you will know your code is working and you can be confident completing the group portion of the project. You must submit your code and a published PDF containing your code and figures you produced with your code to demonstrate it matches the verification case. To complete verification, you will need to replicate data in Sec. 4.2: height versus distance and thrust versus time; provided are values for maximum thrust, maximum height (with corresponding distance traveled) and maximum distance. You will find the .m file containing the verification data on Canvas, it consists of a structure called `verification` with fields consisting of `time`, `thrust`, `velocity_x`, `velocity_z`, `distance`, `height`, and `volume_air`. You should plot your computed trajectories and thrust profile ***with*** the verification data provided. **Please follow the detailed instructions for exporting your code as a PDF.**

In order to assist you, the following "state" vector (the vector of quantities that d/dt acts on) has been provided with equation number references for each phase of flight. You do **not** need to use this particular setup, as there are others. Here x and z are the current horizontal and vertical displacement from x_0 and z_0, respectively. v_x and v_z are the current horizontal and vertical velocity components, respectively. a_x and a_z are the current horizontal and vertical acceleration components, respectively. m_r, $\cancel{V}_{air}$, and m_{air} are the rocket's current total mass, current volume of air, and current mass of air, respectively. $\dot{m}_w$ and $\dot{m}_{air}$ are the current mass flow rates of water and air, respectively.

State Vector — Phases of flight: 1 | 2 | 3

$$\frac{d}{dt}\begin{bmatrix} x \\ v_x \\ z \\ v_z \\ m_r \\ \cancel{V}_{air} \\ m_{air} \end{bmatrix} = \left[\begin{array}{c|c|c} v_x & v_x & v_x \\ a_x & a_x & a_x \\ v_z & v_z & v_z \\ a_z & a_z & a_z \\ -\dot{m}_w \text{ (Eqn. 6)} & -\dot{m}_{air} \text{ (Eqn. 21)} & 0 \\ \text{Eqn. 10} & 0 & 0 \\ 0 & -\dot{m}_{air} \text{ (Eqn. 21)} & 0 \end{array}\right]$$

Group Deliverable: From code developed in the individual component, you and your partner must determine the flight parameters that will allow the bottle rocket to land within 1 m of a target 92 m away. You cannot use the same partner you had for Project 1. Please include a plot of the trajectory and a plot of your thrust profile with time over the flight, including markers on the plot to indicate where the transition between the three phases of flight occur. The performance of the bottle rocket depends on four parameters:

1. p^i_{air}: the initial pressure of the air (the limit being the burst pressure of the bottle, with some factory of safety)
2. the initial volume fraction of water (or equivalently initial mass)
3. C_D: drag coefficient
4. θ^i: the launch angle

You and your partner should explore the impact of *each* parameter and discuss its influence on the range and height of the rocket trajectory. You should use figures to demonstrate these results.

For submission, you and your partner are required to give a **6-minute** presentation discussing your project outcomes using the principles discussed in class. Record the presentation and submit the video on Canvas. The video must be a single take and may not be edited. It should be filmed from the perspective of an audience member. Each person must speak approximately 50% of the time. Your slides are not required to be in the video, but you must be standing and speaking as though you are giving the talk to aerospace customers. There are penalties for being outside of $\pm$ 30 seconds of the time limit.

In addition to the video, please submit your slides (PowerPoint or PDF), and your MATLAB code as a zip file. The MATLAB code must be ready and able to be ran as submitted. You must submit your code and a PDF containing your code and figures you produced with your code. **Please follow the detailed instructions for exporting your code as a PDF.**
We suggest the following format for your presentation. You may use an outline slide, based on preference.

- Introduction - Objective of study and relevant background
- Methodology - Approach to complete the objective
- Results - Tables, graphs, and charts
- Discussion - Explanation of what was observed and why
- Conclusion - Summary of findings
- References - List of sources that provide information that helped you carry out the study
- Backup Slides - Any remaining material you feel would be important to answer questions. You should include your answers to the 10-step problem solving method as backup slides (note that you may also discuss some of the 10 steps in your presentation, but all 10 steps should be presented in backup slides).

Please note: All projects will be graded exactly as you submitted them. There will be no exceptions to this policy, so please ensure your submission is accurate and completed on time. No late work will be accepted.

3 References

Anderson, J. D., Jr., **Introduction to Flight, 7th Ed.**, McGraw-Hill (2009).
Sutton, G. and Biblarz, O., **Rocket Propulsion Elements, 8th Ed.**, Wiley (2010).

4 Verification Case

The following values are the constants and initial conditions used to verify your code; **do not round them or add additional significant figures**.

4.1 Constants

- $g = 9.81$ m/s^2: acceleration due to gravity
- $c_{dis} = 0.78$: discharge coefficient
- $\rho_{air} = 0.961$ kg/m^3: ambient air density
- $\forall_B = 0.002$ m^3: volume of empty bottle
- $p_a = 12.1$ psia: atmospheric pressure (you will need to convert this to Pa)
- $\gamma = 1.4$: ratio of specific heats for air
- $\rho_w = 1000$ kg/m^3: water density
- $d_e = 2.1$ cm: diameter of the throat (exit)
- $d_B = 10.5$ cm: diameter of the bottle
- $R_{air} = 287$ J/(kg$\cdot$ K): specific gas constant of air
- $m_B = 0.15$ kg: mass of empty 2-liter bottle with cone and fins
- $C_D = 0.425$: drag coefficient
- $p_0 = 48$ psig: initial **gauge** pressure of air in bottle (you will need to convert this to Pa)
- $\forall^i_{water} = 0.0005$ m^3: initial volume of water inside bottle
- $T_0 = 310$ K: initial temperature of air
- $v_0 = 0.0$ m/s: initial velocity of rocket
- $\theta^i = 40°$: initial angle of rocket
- $x_0 = 0.0$ m: initial horizontal distance
- $z_0 = 0.25$ m: initial vertical distance
- $l_s = 0.5$ m: length of test stand

Integration time is 0 to 5 seconds for the *individual* component. During the group component, you may find that you need more time for integration to guarantee the rocket lands.

4.2 Outputs

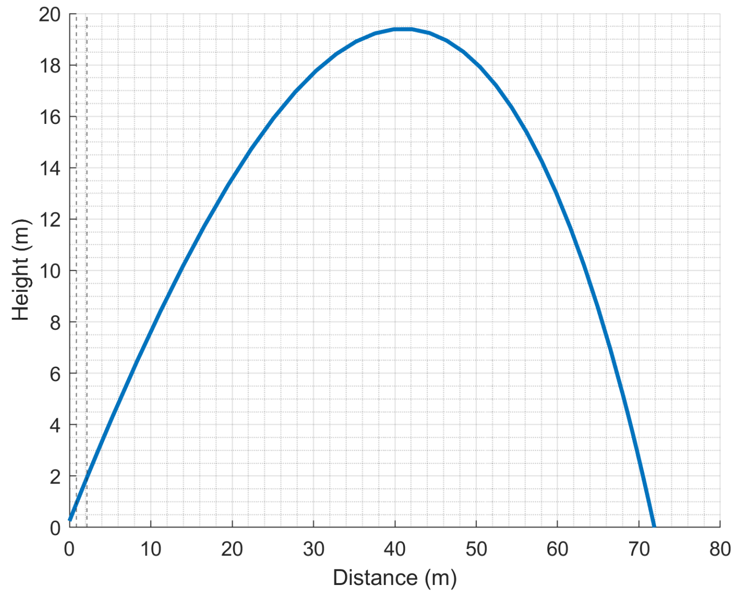

Figure 3: Flight trajectory for verification case

Max Height: 19.4 ± 0.1 m occurring at 42.1 ± 0.3 m
Max Distance: 71.9 ± 0.4 m

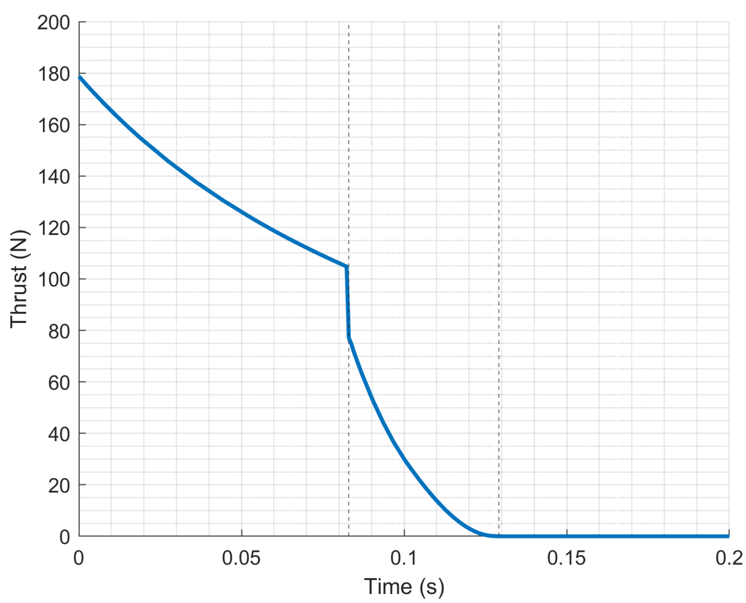

Figure 4: Thrust time-history for verification case

Peak Thrust: 178.8 ± 1.0 N

4.2.1 Additional Debugging Plots

These are from the same case as above, and will be helpful reference as you work through and debug your code!

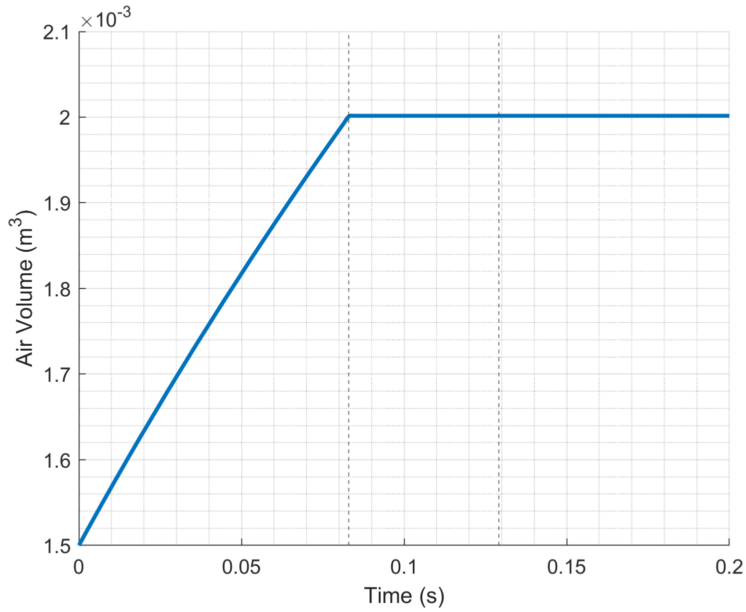

Figure 5: A plot of the volume of air inside the bottle rocket for phase 1 of the rocket's flight.

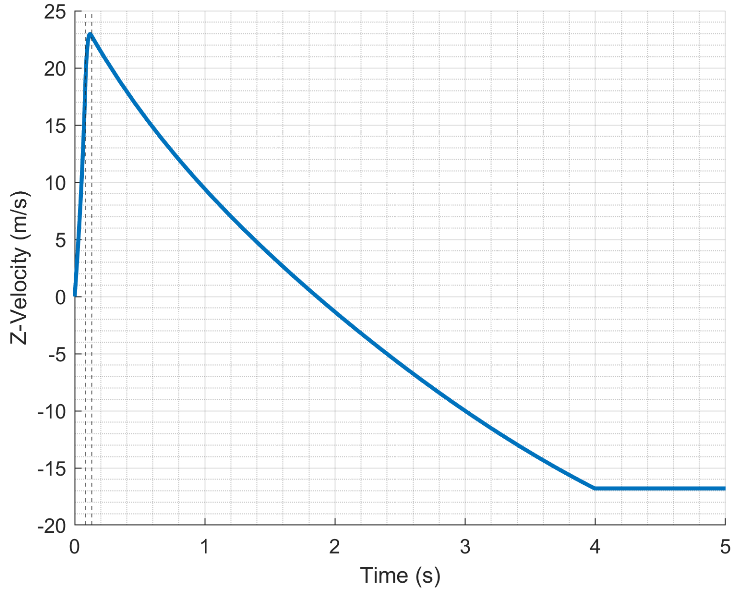

Figure 6: A plot of the vertical (up/down) velocity (v_z) of the bottle rocket, from the time it launches to the time it hits the ground.

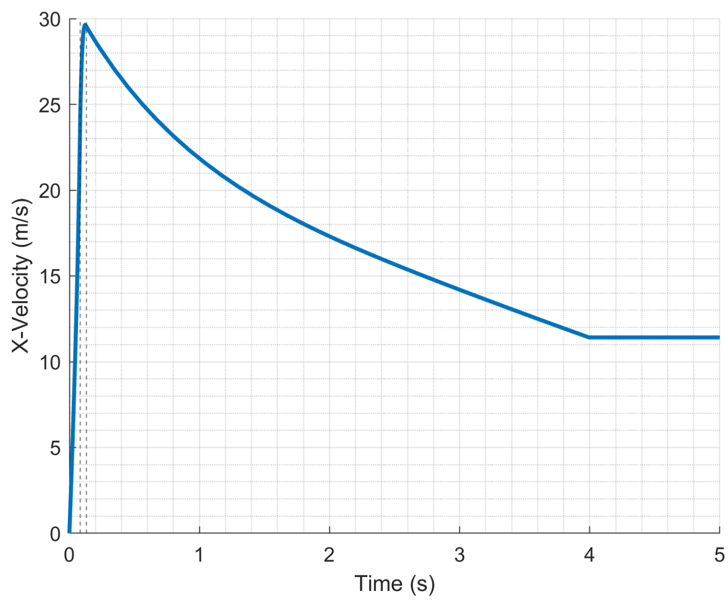

Figure 7: A plot of the horizontal (left/right on a graph of trajectory) velocity (v_x) of the bottle rocket, from the time it launches to the time it hits the ground.