CS7.302: Computer Graphics

Final Exam on Feb 26, 2024. Total: **100 points**
(Answer *any 5 out of 6* questions)

1. **[20 points]** Given $p(\omega) = k \cdot \cos\theta \cdot e^{\phi}$, which is a PDF to sample a direction vector on the upper hemisphere. Recall also that $p(\omega) = \sin\theta \cdot p(\theta, \phi)$. Solve the following sub-questions in order:

   (a) Derive the normalization constant k of the PDF $p(\omega)$. **[5 points]**

   (b) Write expression for $p(\theta)$ & $p(\phi)$. **[5 points]**

   (c) Write expression for the CDFs $P(\theta)$ & $P(\phi)$. **[5 points]**

   (d) Given two random numbers $\xi_1 \in [0, 1)$ & $\xi_2 \in [0, 1)$, give the expressions that sample θ and ϕ proportional to $p(\theta, \phi)$. **[5 points]**

   Hint: $\sin 2x = 2 \sin x \cos x$

2. **[20 points]** Given an integral $I = \int_D f(x)dx$ over a domain D, where $|D| = 2\pi$, solve the following:

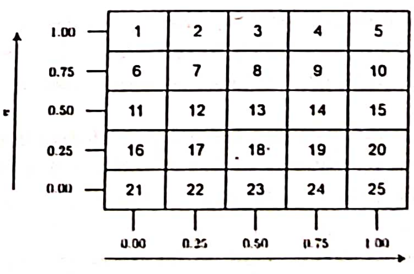
   (a) Write the Monte Carlo Estimator $\langle I \rangle$, assuming X_i's are sampled uniformly over the domain. **[5 points]**

   (b) Write the Monte Carlo Estimator $\langle I \rangle$, assuming X_i's are sampled according to some PDF $p(X_i)$. **[5 points]**

   (c) Prove that the Monte Carlo Estimator from the previous question computes the right answer on average. **[10 points]**

   Hint: For a continuous random variable X sampled with probability $p(X)$, $E[f(X)] = \int_D f(x)p(x)dx$.

3. **[20 points]** Given this 5×5 monochromatic image, answer the following questions:



(a) What will be the value at $(u, v) = (0.4, 0.24)$ using nearest neighbour interpolation. [5 points]

(b) What will be the value at $(u, v) = (0.4, 0.24)$ using bi-linear interpolation. [5 points]

(c) A object is using spherical mapping. Assume the following spherical mapping:

$$x = r \cos u \sin v$$
$$y = r \sin u \sin v$$
$$z = r \cos v$$

What will be the value at the point on the surface $x = (1, 1, \sqrt{2})$ assuming bi-linear interpolation? [10 points]

4. [20 points] You are given the following function and an integral:

$$f(x) = (x + 2)^2, \qquad I = \int_0^2 f(x)dx.$$

You are also given four uniform random numbers in $[0, 1)$:

$$\xi_1 = 0.58 \qquad \xi_2 = 0.99 \qquad \xi_3 = 0.27 \qquad \xi_4 = 0.63.$$

(a) Analytically integrate to find I. [2 points]

(b) Use Monte Carlo (MC) integration, and sample X_i's using the random numbers as:

$$X_i = 2 \cdot \xi_i$$

The PDF to be used in MC is $p(X_i) = \frac{1}{2}$.
Show the steps for each of the four Monte Carlo samples and write the final answer. [7 points]

(c) Use Monte Carlo integration and sample X_i's using the random numbers as:

$$X_i = \sqrt[3]{56 \cdot \xi_i + 8} - 2$$

The PDF to be used in MC is $p(X_i) = \frac{3}{56}(x + 2)^2$.
Show the steps for each of the four Monte Carlo samples and write the final answer. [7 points]

(d) Plot a rough graph of number of MC samples on the x-axis, and for each sample plot the value of the MC estimate on the y-axis for (b) and (c). Also draw a line of $y = A$ where A is the analytic answer from (a). Which converges faster, (b) or (c)? [4 points]

5. [20 points] Derive the expression to determine if a ray intersects with a sphere and the location of the intersection. A ray can be defined as $\bar{o} + t\bar{d}$, where $\bar{o}$ is the start point of the ray and $\bar{d}$ is the unit vector in the direction of the ray. A point $\bar{p}$ lying on the sphere with center $\bar{c}$ and radius r satisfies the property $|\bar{p} - \bar{c}| = r$. Find the expression for t in terms of $\bar{o}, \bar{p}, \bar{c}$ and r, which would give the point of intersection with the sphere and the condition when the ray would intersect the sphere.
Hint: The roots of a quadratic equation $ax^2 + bx + c$ are given by $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

6. [20 points] The rendering equation is as follows:

$$L_o(x, \omega_o) = \int_{\Omega} f(x, \omega_o, \omega_i) L_i(x, \omega_i) \cos\theta d\omega_i, \tag{1}$$

where Ω is the upper hemisphere. The scene also contains an area light A in the scene, which is located at p with normal vector n_l. Answer the following questions:

(a) Derive $d\omega$ in terms of dA, where dA is a differential area on A and $d\omega$ is a differential solid angle subtended by dA on Ω. [8 points]

(b) Given the previous derivation, write the rendering equation over A instead of over Ω. [4 points]

(c) Write a Monte Carlo Estimator for the modified rendering equation from the previous question using a uniform sampling PDF over the area light A. [8 points]