

End Semester
Science I (Classical and Quantum Mechanics)
Total Marks:75 , Time: 3 hrs

Q1. The time dependent Lagrangian of a particle moving in one dimension is given by

$$L = e^{\lambda t} \left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 \right)$$

- (1) Write down the Lagrange equation of motion
- (2) Obtain the expression for generalized momentum and the Hamiltonian $H(p, x)$.
- (3) Write the Hamilton's equation of motion.

4+4+4

Q2. The matrix representations of two quantum operators are given by

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

- (1) Calculate the commutation relation $[a_1, a_2]$.
- (2) Evaluate eigenvalues and normalized eigenstates of a_1 and a_2
- (3) What are the measured values of a_1 ? Calculate their probabilities in one of the eigenstates of a_2 .
- (4) Calculate the uncertainty of measuring a_1 in the above case, i.e. Δa_1 .

4+4+4+3

Q3. The normalized eigenstate of the Hamiltonian of an electron in harmonic oscillator at ground state is given by

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} x^2 \right)$$

- (1) Calculate $\langle x \rangle$ and $\langle x^2 \rangle$ for position operator
- (2) Calculate the uncertainty in measuring X , i.e. Δx and $\langle p \rangle$ for momentum operator
- (4) Calculate the probability of momentum value p

3×4

Q4. An electron is moving inside a infinite well potential

$$\begin{aligned} V(x) &= \infty \text{ at } x < 0, x > a \\ V(x) &= 0 \text{ at } 0 < x < a \end{aligned}$$

The wave-function for the electron is given by $|\psi\rangle = \sqrt{\frac{3}{5}}|\phi_1\rangle + \sqrt{\frac{2}{5}}|\phi_2\rangle$

$|\phi_1\rangle$ and $|\phi_2\rangle$ are two lowest energy eigenstates.

- (1) What are the energy values their probabilities you will get if energy is measured.

- (2) Calculate the average energy $\langle E \rangle$ and energy uncertainty ΔE .
- (3) What is the probability of the particle to remain within 0 to $a/2$.
- (4) How will the probability change at time t , Calculate the minimum probability value achieved at some time t .

3+4+4+4

Q5. An electron of charge e is moving under a central potential $\frac{-e^2}{r}$

- (1) Write down the Lagrangian and Hamiltonian of the electron if angular momentum is L in polar coordinate.
- (2) Write the classical Hamilton's equation of motion
- (3) Calculate the maximum radius r_{max} in the classical bound state, if total energy is $E < 0$ and $L = 0$
- (4) If the electron is at the ground state energy eigenstate, evaluate the probability that the electron will be found within r_{max} .

4×4

The ground state eigenfunction, $\phi_0(r) = \frac{1}{\sqrt{\pi}a_0^{3/2}} \exp\left(\frac{-r}{a_0}\right)$ and energy $E = \frac{-e^2}{2a_0}$

Q6. In a region of space, a particle with mass m and with zero energy has a energy eigenfunction where A and L are constants.

$$\psi(x) = a \exp\left(\frac{-x^2}{L^2}\right)$$

- (1) Determine the potential energy $U(x)$ of the particle using Schrodinger equation
- (2) Determine the ground state energy eigenvalue.

3+2

Useful Integrals

$$\begin{aligned} \int_0^{\frac{a}{2}} \sin^2\left(\frac{\pi x}{a}\right) dx &= \frac{a}{4} & \int_0^{\frac{a}{2}} \sin^2\left(\frac{2\pi x}{a}\right) dx &= \frac{a}{4} & \int_0^{\frac{a}{2}} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx &= \frac{a}{4} \\ \int_0^{\frac{a}{2}} \cos^2\left(\frac{\pi x}{a}\right) dx &= \frac{a}{4} & \int_0^{\frac{a}{2}} \cos^2\left(\frac{2\pi x}{a}\right) dx &= \frac{a}{4} & \int_0^{\frac{a}{2}} \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) dx &= \frac{a}{3\pi} \\ \int x^2 \exp(-x/a) dx &= -a \exp(-x/a) (2a^2 + 2ax + x^2) \end{aligned}$$