Quiz2: Probability and Statistics (30 Marks)

Instruction:

- Please state reasons wherever applicable.
- Use precise mathematical arguments, no speeches.
- **Universal Hint:** Often, checking for almost sure convergence using the definition is going to be difficult, in which case use the following lemma.
  Lemma: Consider a sequence $X_1, X_2, \ldots$. If for every $\epsilon > 0$ we have

  $$\sum_{n=1}^{\infty} P(|X_n - X| > \epsilon) < \infty,$$

  then it implies that $X_n \to X$ almost surely.

Each question: 6 marks

1. Show that convergence in mean square implies convergence in probability. (Hint: Use Markov Inequality)

2. Consider a sequence of random variables $\{X_n, n = 1, 2, 3, \ldots\}$ such that

   $$X_n = \begin{cases} \frac{-1}{n^2} \text{ with probability } 0.3 \\ \frac{1}{n^2} \text{ with probability } 0.7. \end{cases}$$

   Show that X_n converges to 0 almost surely.

3. Suppose X_n are i.i.d Binomial$(n, \frac{\lambda}{n})$. Show that X_n converges in distribution to Poisson(λ)

4. (a) Suppose you have access to samples from $U[0,1]$ random variable. Now consider a random variable X with $F_X(x) = 1 - e^{-\sqrt{x}}$. How would you use samples from U to generate samples of X? (3marks).
   (b) Now suppose you have samples of X (you just generated them!). How would you use them to generate samples of $U[0,1]$. Give justification (2 marks)

5. Suppose $\{X_n, n = 1, 2, 3, ...\}$ are i.i.d unifrom $U[0,1]$ and let $Y_n = min\{X_1, \ldots, X_n\}$. Show that Y_n converges to 0 in probability. (4marks) Does it also converge in almost sure sense? Justify your answer. (2)marks