CS9.312 Introduction to Quantum Information and Computation

Friday the 3rd, February 2023

Answer any two questions out of five. Provide unambiguous justifications for your solutions.

1. Let $\vec{v}$ be any real, three-dimensional unit vector and θ a real number. Prove that

$$\exp\left(i\theta\vec{v}\cdot\vec{\sigma}\right) = \cos(\theta)\mathbb{1} + i\sin(\theta)\vec{v}\cdot\vec{\sigma}$$

where $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ represents the Pauli matrices and $\vec{v}\cdot\vec{\sigma} = v_x\sigma_x + v_y\sigma_y + v_z\sigma_z$. Given,

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. A qubit is subjected to a magnetic field in the z direction, so it experiences the following Hamiltonian, $H = \gamma B\sigma_z/2$ What is the state of the qubit as a function of time if its initial state is described by the density operator,

$$\rho_0 = \frac{1}{2}\left(\mathbb{1} + \frac{1}{2}\sigma_y + \frac{1}{2}\sigma_z\right).$$

3. Consider a maximally entangled state $|\Phi\rangle\langle\Phi|_{AB}$ where

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{d}}\sum_{i,j=0}^{d-1} |i\rangle_A \otimes |i\rangle_B\,,$$

where $\{|i\rangle\}_i$ forms an orthonormal basis and $|A| = |B| = d$. Prove that for any operator M (a $d \times d$ matrix), we have

$$M_A \otimes \mathbb{1}_B\,|\Phi\rangle_{AB} = \mathbb{1}_A \otimes M_B^T\,|\Phi\rangle_{AB}\,, \tag{1}$$

where M^T is transpose of M with respect to orthonormal basis $\{|i\rangle_B\}_i$.

4. Consider a qubit channel $\mathcal{E}_{A\to B} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$, i.e, A and B are qubit systems. The action of the channel $\mathcal{E}_{A\to B}$ is given as

$$\mathcal{E}_{A\to B}(\rho) = p\rho + (1-p)\left[\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z\right].$$

Find the Choi state $\mathcal{E}_{A\to B}(\Phi_{RA})$ of the channel, where Φ_{RA} is a two-qubit maximally entangled state.

5. Consider Alice and Bob hold qubit systems A and B, respectively. Let the two-qubit system AB be in the state $|\psi\rangle\langle\psi|_{AB}$, where $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$. Let Alice and Bob perform measurements $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ in their respective systems. Then calculate the probabilities
$p(a=0, b=0)$,
$p(a=0, b=1)$,
$p(b=0)$,
$p(a=1, b=1)$,
where $p(a,b)$ represents joint probability of Alice's outcome being a while Bob's outcome being b, and $p(b)$ represents probability of Bob's outcome being b. Consider $a, b \in \{0, 1\}$ such that 0 and 1 are outcomes corresponding to $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, respectively.