

Quiz Solutions

Q1

$$P(X_3 < X_1 \text{ and } X_3 < X_2)$$

$$= P(X_1 > X_3 \text{ and } X_2 > X_3)$$

1 mark $\left\{ \begin{aligned} &= \int_0^{\infty} \underbrace{P(X_1 > x \text{ and } X_2 > x)}_{(\because \text{Independent})} f_{X_3}(x) dx \end{aligned} \right.$

1 mark $\left\{ \begin{aligned} &= \int_0^{\infty} P(X_1 > x) P(X_2 > x) (\lambda_3 e^{-\lambda_3 x}) dx \end{aligned} \right.$

$$= \lambda_3 \int_0^{\infty} e^{-\lambda_1 x} e^{-\lambda_2 x} e^{-\lambda_3 x} dx$$

3 marks $\left\{ \begin{aligned} &= \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \quad \underline{\text{Ans}} \end{aligned} \right.$

$\Rightarrow \left\{ \begin{aligned} &2 \text{ for correct substitution of PDF \& CDF} \\ &1 \text{ for correct working \& final answer} \end{aligned} \right.$

Marking Scheme

(Total scaled by 2)

Q2 $N(t)$: Poisson Process with rate ' λ '

Check Markov Property 2 marks

$$\begin{aligned} P(N(t) = k \mid N(t_1) = k_1, N(t_2) = k_2, \dots, N(t_m) = k_m) \\ = P(N(t - t_m) = k - k_m) \\ = P(N(t) = k \mid N(t_m) = k_m) \end{aligned}$$

TPM $P_{ij}(t) = P(N(t) = j \mid N(0) = i)$

4 marks $\left[\begin{aligned} &= \begin{cases} 0 & \text{if } j < i \\ P(N(t) = j - i) & \text{otherwise} \end{cases} \end{aligned} \right. \rightarrow e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} \text{ (By Definition)}$

2 marks $\left\{ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} \right. \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} e^{-\lambda t} \\ 0 \\ 0 \\ \vdots \end{array} \begin{array}{c} e^{-\lambda t}(\lambda t) \\ e^{-\lambda t} \\ 0 \\ \vdots \end{array} \begin{array}{c} e^{-\lambda t} \frac{(\lambda t)^2}{2!} \\ e^{-\lambda t}(\lambda t) \\ e^{-\lambda t} \\ \vdots \end{array} \dots$

Dimensions $\Rightarrow |s_{0,1,2,\dots}| \times |s_{0,1,2,\dots}|$
2 marks

Q3. Binomial Process $\{S_n, n \geq 1\}$

we know $S_n = \sum_{i=1}^n X_i$ where X_i are
iid Bernoulli r.v. let $X_i \sim \text{Bernoulli}(p)$

So, 2 marks
$$= \begin{cases} P(S_{n+1} = j \mid S_n = i) \\ P(X_{n+1} = j - i) \end{cases} \quad (\text{By Bernoulli r.v. PMF})$$

$$= \begin{cases} (1-p) & \text{if } j = i \\ p & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \quad \left. \vphantom{\begin{cases} (1-p) \\ p \\ 0 \end{cases}} \right\} \text{3 marks}$$

$$\therefore P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1-p & p & 0 & 0 & \dots \\ 0 & 1-p & p & 0 & \dots \\ 0 & 0 & 1-p & p & 0 & \dots \\ 0 & 0 & 0 & 1-p & p & \dots \end{bmatrix} \end{matrix}$$

\vdots

Marking Scheme

Total Scaled by 2

Q4

Stationary Distribution

1 mark $\Rightarrow \pi Q = 0$

$$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} -\lambda_H & \lambda_H \\ \lambda_T & -\lambda_T \end{bmatrix} = 0$$

2 marks $\Rightarrow \lambda_H \pi_1 = \lambda_T \pi_2 \quad \text{or} \quad \pi_1 = \frac{\lambda_T}{\lambda_H} \pi_2$

1 mark [and $\pi_1 + \pi_2 = 1$

$$\Rightarrow \left(\frac{1}{2} + \frac{1}{2} \right) \left\{ \pi_1 = \frac{\lambda_T}{\lambda_H + \lambda_T}, \quad \pi_2 = \frac{\lambda_H}{\lambda_H + \lambda_T} \right\}$$

For Embedded DTMC, we only consider jumps from one state to a different state] 2 marks

So $P_{11} = P_{22} = 0$] 1 mark

and $P_{ij}^* = \frac{q_{ij}}{|q_{ii}|}$] 1 mark

\therefore TPM $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$] 1 mark