SC1.110a: Science 1 (Monsoon 2023)

Final

Total marks: 100

Duration: 3 hrs

1. (6 points) Use linear stability analysis to classify the fixed points of the following system. If the linear analysis is inconclusive, draw the phase diagram to conclude the stability.

$$\dot{x} = 1 - e^{-x^2} \tag{1}$$

2. Consider the differential equation $\dot{x} = \beta x - x^3$, where β is a real number.
   (a) (5 points) Find and classify the fixed points of the above dynamical system.
   (b) (1 point) Find the value of β at which bifurcation occurs.
   (c) (4 points) Draw the bifurcation diagram for the above dynamical system and identify the type of bifurcation.

3. Consider the following dynamical system

$$\begin{aligned} \dot{x} &= y - xy^2 \\ \dot{y} &= -x + yx^2 \end{aligned} \tag{2}$$

   (a) (3 points) Find all fixed points for this dynamical system.
   (b) (5 points) Linearize around each fixed point and classify its type.
   (c) (4 points) Plot the phase portrait in the region $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.

4. (8 points) Consider the following dynamical system,

$$\begin{aligned} \dot{x} &= \zeta x - x^2 \\ \dot{y} &= -y \end{aligned} \tag{3}$$

Find the eigenvalues at the stable fixed point as a function of ζ, and show that one of the eigenvalues tends to zero as $\zeta \to 0$.

5. Consider the system

$$\begin{aligned} \dot{x} &= x[x(1-x) - y], \\ \dot{y} &= y(x - a), \end{aligned}$$

where $x \geq 0$ is the dimensionless population of the rabbit, $y \geq 0$ is the dimensionless population of the fox, and $a \geq 0$ is a control parameter.

   (a) (3 points) Sketch the nullclines (i.e., the curve $\dot{x} = 0$ and the curve $\dot{y} = 0$) in the first quadrant $x, y \geq 0$.
   (b) (5 points) Show that the fixed points are $(0,0), (1,0), (a, a - a^2)$. Classify the fixed points $(0,0)$ and $(1,0)$.

6. Consider the following dynamical system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 \end{aligned}$$

(a) (5 points) Show that the fixed points $(-1, 0)$ and $(+1, 0)$ are true centers for the dynamical system

(b) (4 points) Sketch the phase portrait for the above dynamical system.

7. Consider the initial value problem $\frac{dy}{dx} = f(x, y) = 3y^{2/3}$, $y(0) = y_0$. Determine y_0 such that the equation has

   (a) (3 points) No real solution (Explain).

   (b) (4 points) A local unique solution (Explain).

   (c) (5 points) Suppose $y_0 = 0$. Explain why a unique solution is not guaranteed and build a family of (infinitely many) solutions for this initial value problem.

8. (6 points) Write the total differential for the Gibbs free energy and Enthalpy of a system.

9. Consider a box of volume V containing photons, whose internal energy at temperature T is given by

$$U = bVT^4 \tag{4}$$

where b is a constant.

   (a) (3 points) Write an expression for the heat capacity C_v in terms of T, V and b.

   (b) (6 points) Write an expression for entropy S in terms of T, V and b by assuming $S(T = 0) = 0$.

10. Consider the following dynamical system

$$\ddot{x} + f(\dot{x}) + g(x) = 0,$$

where f is an even function.

   (a) (5 points) Show that the dynamical system is reversible.

   (b) (7 points) Show that the fixed points cannot be stable nodes or stable spirals.

11. (8 points) Consider the following dynamical system.

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= x + y \end{aligned} \tag{6}$$

Locate the fixed points and calculate the index of the fixed point.