

Quiz 1 : Probability and Statistics

September 3, 2023

Question 1

Let X be a random variable with cdf $F_X(x)$. The CDF of a random variable X is defined as

$F_X(x_1) = P(\omega \in \Omega : X(\omega) \leq x_1) = \sum_{x < x_1} p_x(x)$ where p_x is the PMF.

For $a < b$, we can consider the following events:

- $C = X \leq a$
- $D = a < X \leq b$
- $E = X \leq b$

Then C and D are mutually exclusive and their union is the event E .

By the third axiom of probability, we know that

$$P(E) = P(D) + P(C)$$

$$P(X \leq b) = P(a < X \leq b) + P(X \leq a)$$

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

Marking Scheme:

1 mark for cdf definition

2 marks for writing disjoint sets and invoking probability axiom

2 marks for logic

Question 2

Mean:

$$E[x] = \sum_{x=1}^{\infty} x \cdot p(x) \quad (0.5 \text{ mark})$$

$$= \sum_{x=1}^{\infty} x \cdot p \cdot (1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} (x) \cdot (1-p)^{x-1} \quad (0.5 \text{ mark})$$

$$= p (1 \cdot (1-p)^0 + 2 \cdot (1-p) + 3 \cdot (1-p)^2 + \dots)$$

$$= p (1 + 2(1-p) + 3(1-p)^2 + \dots)$$

$$= p \left(\frac{1}{(1 - (1-p))^2} \right) \quad (1 \text{ mark})$$

$$= p \left(\frac{1}{p^2} \right)$$

$$= \frac{1}{p} \quad (0.5 \text{ mark})$$

Variance:

$$\text{Var}(x) = E(X^2) - [E(X)]^2 \quad (0.5 \text{ mark})$$

$$= E(X(X-1) + X) - \frac{1}{p^2}$$

$$= E(X(X-1)) + E(X) - \frac{1}{p^2} \quad (0.5 \text{ mark})$$

$$= \sum_{x=1}^{\infty} x(x-1)p(1-p)^{x-1} + \frac{1}{p} - \frac{1}{p^2}$$

$$= [2p(1-p) + 6p(1-p)^2 + 12p(1-p)^3 + \dots] + \frac{1}{p} - \frac{1}{p^2}$$

$$= 2p(1-p) [1 + 3(1-p) + 6(1-p)^2 + \dots] + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{2p(1-p)}{p^3} + \frac{1}{p} - \frac{1}{p^2} \quad (1 \text{ mark})$$

$$= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1-p}{p^2} \quad (0.5 \text{ mark})$$

Question 3

Suppose you choose Door 1. Let us say presenter chooses door 3. Let C_i denote the event that door i conceals a car is, and G denote the event that a goat is shown at door 3. (1/2 mark for mentioning bayes' law or conditional probability and 1/2 mark for proper assumptions of events) Using Conditional probability we get,

$$P(C_2|G) = \frac{P(C_2 \cap G)}{P(G)}$$

which we can further write as,

$$P(C_2|G) = \frac{P(C_2 \cap G|C_1) \cdot P(C_1) + P(C_2 \cap G|\neg C_1) \cdot P(\neg C_1)}{P(G)}$$

Now, let's find the required probabilities:

1. $P(C_2 \cap G|C_1) = 0$ because car cant be behind both the doors 1 and 2.

2. $P(C_2 \cap G | \neg C_1) = 1$ the probability that car is behind door 2 and goat is behind door 3 given that car is not behind door 1 can happen with probability 1.
3. $P(G|C_2) = 1$ probability that behind door 3 there is a goat, given that behind door 1 there is a car, these two events are independent, and the probability that behind door 3 there is a goat is equal to 1, as presenter always opens a door with a goat behind it.
4. $P(C_2) = P(C_1) = \frac{1}{3}$ because initially, the probability of the car being behind any door is equal ($\frac{1}{3}$ for each door).

5. $P(G)$ This total probability can be written as:

$$P(G) = P(G|C_1) \cdot P(C_1) + P(G|\neg C_1) \cdot P(\neg C_1)$$

$$P(G) = 1 \cdot \left(\frac{1}{3}\right) + (1) \cdot \left(\frac{2}{3}\right)$$

Reason : Using 3rd point and 4th point we have made the above substitutions

$$P(G) = \frac{1}{3} + \frac{2}{3}$$

$$P(G) = 1$$

(2 marks for finding the probabilities)

So:

$$P(C_2|G) = \frac{1 \cdot \frac{2}{3}}{1} = \frac{2}{3}$$

(1/2 mark)

$$P(C_3|G) = 0$$

By axioms of probability,

$$P(C_1|G) = 1 - \frac{2}{3} - 0 = \frac{1}{3}.$$

(1/2 mark)

Therefore,

$$P(C_2|G) > P(C_1|G)$$

(1/2 mark)

So we should switch the doors because the probability of finding the car in other door is higher than door we chose first. (1/2 mark)

Note : Any other method with proper mathematical proof will be graded in a similar fashion for the approach.

Question 4

For Part 1

We have

$$\begin{aligned} P\left(\bigcap_{i=1}^n A_i\right) &= P\left(\left(\bigcup_{i=1}^n A_i^c\right)^c\right) \text{ [de Morgan's law]} \\ &= 1 - P\left(\bigcup_{i=1}^n A_i^c\right) \end{aligned}$$

We have

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \Rightarrow P(A \cup B) &\leq P(A) + P(B) \end{aligned}$$

By using induction on the above inequality for n events, we have

$$P\left(\bigcup_{i=1}^n A_i^c\right) \leq \sum_{i=1}^n P(A_i^c)$$

So, by using the equation, we get

$$\Rightarrow 1 - P\left(\bigcap_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i^c)$$

$$\Rightarrow P\left(\bigcap_{i=1}^n A_i\right) \geq 1 - \sum_{i=1}^n P(A_i^c)$$

MARKING SCHEME:

+0.5 for using de Morgan's law

+2.0 for stating and proving above inequality

+0.5 if stating Boole's inequality (the inequality above) without proof

For Part 2

The given problem implies that either (i) A happens and B does not or (ii) A does not happen and B happens. So, we can write the desired probability as :-

$$P(A \text{ or } B) = P(A \cap B^c) + P(A^c \cap B) \text{ (1 mark)}$$

By law of total probability applied on event A , we have

$$P(A) = P(A \cap B^c) + P(A \cap B) \dots (1) \text{ (0.5 marks)}$$

By law of total probability applied on even B , we have

$$P(B) = P(B \cap A^c) + P(B \cap A) \dots (2) \text{ (0.5 marks)}$$

Adding equations (1) and (2), we get

$$\Rightarrow P(A) + P(B) = 2P(A \cap B) + P(A^c \cap B) + P(A \cap B^c)$$

$$\Rightarrow P(A^c \cap B) + P(A \cap B^c) = P(A) + P(B) - 2P(A \cap B) \quad (0.5 \text{ marks})$$

MARKING SCHEME :

+1 mark for writing correct expression for the problem

+0.5 mark for applying law of total probability on A

+0.5 mark for applying law of total probability on B

+0.5 mark for solving the two equations

Question 5

Definition : Event space or sigma-algebra \mathcal{F} is a collection of measurable sets (1/2 marks for Definition) that satisfy:

1. $\emptyset \in \mathcal{F}$ - (1/2 marks)
2. $A \in \mathcal{F} \implies A^C \in \mathcal{F}$ (closed under complementation) - (1/2 marks)
3. $A_k \in \mathcal{F} \implies \bigcup_{k \in I} A_k \in \mathcal{F} \quad k \in I$ (closed under countable union) - (1/2 marks)

Now, we will prove that sigma-algebra \mathcal{F} is closed under countable intersections:

$$\begin{aligned} A_k \in \mathcal{F} &\implies A_k^C \in \mathcal{F} && \text{(closed under complementation) - 1/2 marks} \\ &\implies \left(\bigcup_{k \in I} A_k^C \right) \in \mathcal{F} && \text{(closed under union) - 1/2 marks} \\ &\implies \left(\bigcup_{k \in I} A_k^C \right)^C \in \mathcal{F} && \text{(closed under complementation) - 1/2 marks} \\ &\implies \left(\bigcap_{k \in I} (A_k^C)^C \right) \in \mathcal{F} && \text{(de Morgan's Law) - 1 marks} \\ &\implies \bigcap_{k \in I} A_k \in \mathcal{F} \\ &\implies \mathcal{F} \text{ is closed under intersection.} && \text{- 1/2 marks} \end{aligned}$$

Question 6

The plane which has the higher probability of more people showing up than the number of seats is more overbooked. (1 Mark)

Probability that k people show up for a flight with l seats is

$$\binom{k}{l} \left(\frac{9}{10}\right)^k \left(\frac{1}{10}\right)^{l-k} \quad (1Mark)$$

Here $\binom{k}{l}$ stands for $\frac{k!}{(l!)(k-l)!}$ and $k \geq l$

Let N people come to the indigo flight and M people for Air india. The probability that more people show up for the indigo flight is than number of seats is-

$$\begin{aligned} P(N > 9) &= P(N = 10) \\ &= \binom{10}{10} \left(\frac{9}{10}\right)^{10} \left(\frac{1}{10}\right)^0 \quad (1Mark) \\ &= \left(\frac{9}{10}\right)^{10} \end{aligned}$$

Probability that more people show up for Air india flight than number of seats is-

$$\begin{aligned} P(M > 18) &= P(M = 19) + P(M = 20) \\ &= \binom{20}{19} \left(\frac{9}{10}\right)^{19} \left(\frac{1}{10}\right)^1 + \binom{20}{20} \left(\frac{9}{10}\right)^{20} \left(\frac{1}{10}\right)^0 \quad (1.5Marks) \end{aligned}$$

Thus $P(M > 18) > P(N > 9)$. Hence, Air India is more over booked (0.5 Marks)