(CS1.301) Algorithm Analysis and Design (Monsoon 2023) Date: 23.09.2023

Mid-Semester Examination

Alloted time: 90 minutes Total marks: 30

Instructions:

- **There are a total of 6 questions with varying credit printed on pages 1 to 3.**
- **Discussions amongst the students are not allowed. No electronic devices (including smart watches) nor notes/books of any kind are allowed.**
- **Any dishonesty shall be penalized heavily.**
- **Any theorem/lemma/claim/fact that was proved in the class can be used without proof in the exam, only by *explicitly* writing its statement, and a clear remark that it was chosen from the class notes.**
- **Questions have been framed to be disambiguous, and queries will not be answered during the examination. In case you find any ambuguity, please mention that in your answer scripts and work with it. Answers got by misreading of questions may not be given credit.**
- **Be clear in your arguments. Partial marking is available for every question but vague arguments shall not be given any credit.**
- **Analysis of running times, and proofs of correctness need to be done unless explicitly asked not to.**

Question 1 [2 + 3 marks]

A directed graph $G = (V, E)$ is strongly connected if and only if every pair of vertices is strongly connected.

(a) Give an algorithm that computes all strong components of a graph in time at most $O(|V| \cdot |E|)$.

(b) Can this be improved to $O(|V|+|E|)$ by considering $G_{\textbf{reverse}}$ (which is the graph obtained by reversing edge directions)?

Question 2 [3 marks]

In a city there are N houses, each of which is in need of a water supply. It costs w_i dollars to build a well at house i, and it costs $c_{i,j}$ to build a pipe in between houses i and j. A house can receive water if either there is a well built there or there is some path of pipes to a house with a well. Design an algorithm to find the minimum amount of money needed to supply every house with water. Proof of correctness is not required.

Question 3 [4 marks]

Given a list of n jobs $J_1, J_2, ..., J_n$ with processing times $p_1, p_2, ..., p_n$ and weights $w_1, w_2, ..., w_n$. Starting from $t = 0$, the cost for completing a job J_i is given by w_i*(*total time from* $t = 0$ *to the time at which job* J_i *finishes*). Give an algorithm to find the order to perform the jobs with minimum total cost. Give proof of correctness.

Question 4 [6 marks]

Given an array of $n \geqslant 2$ integers, say $[x(1), \ldots, x(n)]$, we want to find the largest step d, which is defined to be the max of $x(j) - x(i)$ over all $j > i$. For example, for $x = [22, 5, 8, 10, -3, 1]$ and in this case, $d = x(4) - x(2) = 10 - 5 = 5$.

Question 5 [2 + 4 marks]

Let $G = (V, E)$ be a connected graph on n vertices and m edges such that their edge weights are all distinct. Algorithm 1 presents a different algorithm than what we studied in our classes. Your task is to

(a) analyse the running time, and

(b) prove the correctness of this algorithm.

Algorithm 1: Boruvka's algorithm for Minimum Spanning Tree

Input: A weighted connected graph $G = (V, E)$ with unique edge weights.
Output: The minimum spanning tree T for G.

Let T be a subgraph of G initially containing just the vertices in V (with no edges, just isolated vertices);
while T *has fewer than* $|V| - 1$ *edges* **do**
    for *each connected component,* C_i, *of* T **do**
        Let $e = (u, v)$ be the smallest-weight edge in E with $u \in C_i$ and $v \notin C_i$;
        Add e to T (unless e is already in T);
return T

Question 6 [2 + 2 + 2 marks]

Let there be n medical schools $\{M_1, \ldots, M_n\}$ each with exactly 1 vacancy for a medical residency program and let there be n medical students $\{S_1, \ldots, S_n\}$ applying for those positions. Each medical school has a prior knowledge of all the students who are applying and thus has a (strict) preference[1] amongst them. Similarly, each student has a strict preference for the medical schools. A matching in this case is the assignment of these n medical students to the n vacancies across all the medical schools. A matching is said to be unstable if there is a medical school M_i, and a student S_j such that

- S_j is not matched with M_i,
- S_j prefers M_i to their current match, and

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[1] A preference in this case means that each medical school rates the students in a strict decreasing order of desirability for the position.

- M_i prefers S_j to their current match.

A matching is said to be stable if it is not unstable. Gale and Shapley proposed Algorithm 2.

Algorithm 2: Gale-Shapley Proposal Algorithm

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while *there is an unmatched student* **do**
    Let S_i be an unmatched student;
    S_i approaches a medical school M_j that they most prefer which they were not already rejected from;
    if *M_j's vacancy is open* **then**
        M_j selects student S_i;
    else
        if *S_i is more prefered by M_j than their current selection* **then**
            M_j selects S_i and unmatches their current selection;
            $S_{i'}$ adds M_j into the list of schools they were rejected from;
        else
            S_i remains unmatched;
            S_i adds M_j into the list of schools they were rejected from;
        end
    end
end

Given this information, prove the following three statements.

(a) Whenever there is an unmatched student, there is a medical school that they have not approached.

(b) Algorithm 2 terminates in $O(n^2)$ steps.

(c) The matching thus found by Algorithm 2 is stable.