CARLISI Nolan DM 1

Exercice 1: Suites

1a)

$$\begin{aligned} v_n &= \frac{u_{n+1}}{u_n} \\ &= \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \\ &= \frac{n^2+2n+1}{2^n \times 2} \times \frac{2^n}{n^2} \\ &= \frac{(n^2+2n+1) \times 2^n}{2^n \times 2 \times n^2} \end{aligned}$$

$$\boxed{v_n = \frac{(n+1)^2}{2n^2}}$$

1b)

$$\begin{aligned} v_n &> \frac{1}{2} \\ \frac{(n+1)^2}{2n^2} - \frac{1}{2} &> 0 \\ \frac{(n+1)^2 - n^2}{2n^2} &> 0 \\ \frac{n^2+2n+1-n^2}{2n^2} &> 0 \\ \boxed{\frac{2n+1}{2n^2} > 0} \end{aligned}$$

c)
p égal 5, car $v_5 < \frac{3}{4} < v_6$

d)
pour $n \geq 5$
$v_n \leq \frac{3}{4}$

$$v_n \leq \frac{3}{4}$$
$$\frac{u_{n+1}}{u_n} \leq \frac{3}{4}$$
$$\boxed{u_{n+1} \leq \frac{3}{4}u_n}$$

2a)

$$P(n) = \text{«}\, u_n \leq (\frac{3}{4})^{n-5} \times u_5 \,\text{»}$$

<u>Initialisation:</u>

$$u_5 \leq (\frac{3}{4})^{5-5} \times u_5$$
$$\leq 1 \times u_5$$
$$\boxed{u_5 \leq u_5}$$

$P(5)$ est Vrai

<u>Hérédité:</u>
<u>HR:</u> $u_n \leq (\frac{3}{4})^{n-5} \times u_5$
<u>CCL:</u> $u_{n+1} \leq (\frac{3}{4})^{n+1-5} \times u_5 = (\frac{3}{4})^{n-4} \times u_5$

$$u_n \leq (\frac{3}{4})^{n-5} \times u_5$$
$$\frac{3}{4}u_n \leq (\frac{3}{4})(\frac{3}{4})^{n-5} \times u_5$$
$$\boxed{u_{n+1} \leq (\frac{3}{4})^{n-4} \times u_5}$$

$P(n)$ est Vrai, $\forall n \geq 5$

2b)

$$u_5 \leq (\frac{3}{4})^{5-5} \times u_5$$
$$u_5 + u_6 \leq (\frac{3}{4})^{0} \times u_5 + (\frac{3}{4})^{6-5} \times u_5$$
$$u_5 + u_6 + u_7 \leq 1 \times u_5 + (\frac{3}{4})^{1} \times u_5 + (\frac{3}{4})^{7-5} \times u_5$$
$$u_5 + u_6 + u_7 + \cdots + u_n \leq 1 \times u_5 + \frac{3}{4} \times u_5 + (\frac{3}{4})^{2} \times u_5 + \cdots + (\frac{3}{4})^{n-5} \times u_5$$
$$u_5 + u_6 + u_7 + \cdots + u_n \leq [1 + \frac{3}{4} + (\frac{3}{4})^{2} + \cdots + (\frac{3}{4})^{n-5}] \times u_5$$

$$\boxed{S_n \leq [1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5}] \times u_5}$$

c)
$1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5}$ est la somme d'une suite géométrique de raison $\frac{3}{4}$

$$\begin{aligned} 1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5} &= \frac{1 - (\frac{3}{4})^{n+1-5}}{1 - \frac{3}{4}} \\ &= \frac{1 - (\frac{3}{4})^{n-4}}{\frac{1}{4}} \\ &= (1 - (\frac{3}{4})^{n-4}) \times 4 \\ &\boxed{= 4 - 4(\frac{3}{4})^{n-4}} \end{aligned}$$

Or $-4(\frac{3}{4})^{n-4}$ est négatif

Donc $1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5} \leq 4$
$[1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5}]u_n \leq 4u_n$
$\boxed{S_n \leq 4u_n}$

3.)

$$S_{n+1} - S_n = [1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5} + (\frac{3}{4})^{n-4}] \times u_5 - [1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5}] \times u_5$$

$$\begin{aligned} S_{n+1} - S_n &= \left([1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5} + (\frac{3}{4})^{n-4}] - [1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5}]\right) \times u_5 \\ S_{n+1} - S_n &= \left(\cancel{[1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5}} + (\frac{3}{4})^{n-4}] - \cancel{[1 + \frac{3}{4} + (\frac{3}{4})^2 + \cdots + (\frac{3}{4})^{n-5}]}\right) \times u_5 \end{aligned}$$

$$\boxed{S_{n+1} - S_n = (\frac{3}{4})^{n-4} \times u_5}$$

Or $(\frac{3}{4})^{n-4} \times u_5$ est positif, donc la suite est croissante

Exercice 2:

Partie A)

1.)
$g(x) = x + 2 - e^x$
$g'(x) = 1 - e^x$

x	0		$+\infty$
$g'(x)$	0	$-$	
$g(x)$	1	$\searrow$	

2.)

Si $g(x) = 0$, pour $\alpha \in [0; +\infty[$

$\boxed{1,146 \leq \alpha \leq 1,147}$

3.)

x	0		α		$+\infty$
$g(x)$		$+$	0	$-$	

Partie B)

1a)

$f(x) = \frac{e^x - 1}{1 + xe^x} = \frac{u}{v}$

Avec

$$\begin{aligned} u &= e^x - 1 \\ v &= 1 + xe^x \\ u' &= e^x \\ v' &= e^x x + e^x \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{u'v - uv'}{v^2} \\ &= \frac{e^x(1 + xe^x) - (e^x - 1)(e^x x + e^x)}{(1 + xe^x)^2} \\ &= \frac{e^x[1 + xe^x - (x - 1)(e^x - 1)]}{(1 + xe^x)^2} \\ &= \frac{e^x[1 + \cancel{xe^x - xe^x} + x - e^x + 1]}{(1 + xe^x)^2} \\ &= \frac{e^x[2 + x - e^x]}{(1 + xe^x)^2} \\ &\boxed{= \frac{e^x \times g(x)}{(1 + xe^x)^2}} \end{aligned}$$

1b)

$f(0) = \frac{e^0-1}{1+0e^0} = \frac{0}{1} = 0$

x	0		α		$+\infty$
$g(x)$		$+$	0	$-$	
e^x			$+$		
$(1+xe^x)^2$			$+$		
$f'(x)$		$+$	0	$-$	
$f(x)$	$f(0)$	$\nearrow$	$f(\alpha)$	$\searrow$	

2a)

<u>CCL:</u>$f(\alpha) = \frac{1}{\alpha+1}$

$$\begin{aligned} \alpha + 2 - e^\alpha &= 0 \\ \alpha &= e^\alpha - 2 \\ e^\alpha &= \alpha + 2 \end{aligned}$$

$$\begin{aligned} f(\alpha) &= \frac{e^\alpha - 1}{1 + \alpha e^\alpha} \\ &= \frac{\alpha + 2 - 1}{1 + \alpha(\alpha + 2)} \\ &= \frac{\alpha + 2 - 1}{1 + \alpha^2 + 2\alpha} \\ &= \frac{\alpha + 1}{(\alpha + 1)^2} \end{aligned}$$

$$\boxed{f(\alpha) = \frac{1}{\alpha + 1}}$$

2b)

$\boxed{1,146 \leq \alpha \leq 1,147}$

3.)

$y : f'(a)(x-a) + f(a)$

$(T)\ y : f'(0)(x-0) + f(0)$

$f'(0) = \frac{e^0 \times g(0)}{(e^0+1)^2} = \frac{1\times(0+2-e^0)}{1^2} = \frac{1}{1} = 1$

$f(0) = \frac{e^0-1}{1+0e^0} = \frac{1-1}{1+0} = 0$

$(T)y : 1(x-0) + 0 = x$

$\boxed{(T)y = x}$

4a)

$$
\begin{aligned}
f(x) - x &= \frac{e^x - 1}{1 + xe^x} - x \\
&= \frac{e^x - 1}{1 + xe^x} - \frac{x(1 + xe^x)}{1 + xe^x} \\
&= \frac{e^x + 1 - x - x^2e^x}{1 + xe^x} \\
&= \frac{xe^x + e^x - x^2e^x - xe^x - x - 1}{1 + xe^x} \\
&= \frac{(e^x - xe^x - 1)(x + 1)}{1 + xe^x} \\
&\boxed{f(x) - x = \frac{(x+1) \times u(x)}{xe^x + 1}}
\end{aligned}
$$

4b)

$u'(x) = e^x - xe^x + e^x = 2e^x - xe^x = \boxed{e^x(2 - x) = u'(x)}$

Signe de $u'(x)$ est le même que $2 - x$

$$
\begin{aligned}
2 - x &> 0 \\
x &< 2
\end{aligned}
$$

x	0		2		$+\infty$
$u'(x)$		$+$	0	$-$	
$u(x)$		$\nearrow$	$u(2)$	$\searrow$	
$u(x)$			$-$		

4c)

$u(2) = e^2 - 2e^2 - 1 = -e^2 - 1 = \boxed{-(e^2 + 1) = u(2)}$

Or $e^2 + 1$ positif donc $-(e^2 + 1)$ négatif donc $u(2)$ négatif

4d)

x	0		$+\infty$
(xe^x+1)		$+$	
$x+1$		$+$	
$u(x)$		$-$	
$f(x)-x$		$-$	

$$
\begin{aligned}
&f(x) - y < 0 \\
&f(x) < y, \forall x \in [0; +\infty[
\end{aligned}
$$

Donc Cf est en dessous de T, pour tout x dans $[0; +\infty[$