Assignment-Equations

Course: Dynamics & Control of Aerial Robots

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Lecture Timings

Monday & Wednesday: 2.00 pm -3.00pm

Tutorial Timings

Friday: 2.00 pm -3.00 pm

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$$
\begin{aligned}
\dot{p}_n &= (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w \\
\dot{p}_e &= (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w \\
\dot{h} &= u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta \\
\dot{u} &= rv - qw - g\sin\theta + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e\right] + \frac{\rho S_{\mathrm{prop}} C_{\mathrm{prop}}}{2\mathrm{m}}\left[(k_{\mathrm{motor}}\delta_t)^2 - V_a^2\right] \\
\dot{v} &= pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right] \\
\dot{w} &= qu - pv + g\cos\theta\cos\phi + \frac{\rho V_a^2 S}{2\mathrm{m}}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{cq}{2V_a} + C_{Z_{\delta_e}}(\alpha)\delta_e\right] \\
\dot{\phi} &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\
\dot{\theta} &= q\cos\phi - r\sin\phi \\
\dot{\psi} &= q\sin\phi\sec\theta + r\cos\phi\sec\theta \\
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] \\
\dot{q} &= \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{cq}{2V_a} + C_{m_{\delta_e}}\delta_e\right] \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr + \frac{1}{2}\rho V_a^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right]
\end{aligned}
$$

$$C_{p_0} = \Gamma_3 C_{l_0} + \Gamma_4 C_{n_0}$$

$$C_{p_\beta} = \Gamma_3 C_{l_\beta} + \Gamma_4 C_{n_\beta}$$

$$C_{p_p} = \Gamma_3 C_{l_p} + \Gamma_4 C_{n_p}$$

$$C_{p_r} = \Gamma_3 C_{l_r} + \Gamma_4 C_{n_r}$$

$$C_{p_{\delta_a}} = \Gamma_3 C_{l_{\delta_a}} + \Gamma_4 C_{n_{\delta_a}}$$

$$C_{p_{\delta_r}} = \Gamma_3 C_{l_{\delta_r}} + \Gamma_4 C_{n_{\delta_r}}$$

$$C_{r_0} = \Gamma_4 C_{l_0} + \Gamma_8 C_{n_0}$$

$$C_{r_\beta} = \Gamma_4 C_{l_\beta} + \Gamma_8 C_{n_\beta}$$

$$C_{r_p} = \Gamma_4 C_{l_p} + \Gamma_8 C_{n_p}$$

$$C_{r_r} = \Gamma_4 C_{l_r} + \Gamma_8 C_{n_r}$$

$$C_{r_{\delta_a}} = \Gamma_4 C_{l_{\delta_a}} + \Gamma_8 C_{n_{\delta_a}}$$

$$C_{r_{\delta_r}} = \Gamma_4 C_{l_{\delta_r}} + \Gamma_8 C_{n_{\delta_r}}.$$

Objective is to compute trim states and inputs when aircraft simultaneously satisfies three conditions:

- Traveling at constant speed V_a^*,
- Climbing at constant flight path angle γ^*,
- In constant orbit of radius R^*.

V_a^*, γ^*, and R^*, are inputs to the trim calculations.

$$\begin{aligned} \text{States:} \qquad & x \triangleq (p_n, p_e, p_d, u, v, w, \phi, \theta, \psi, p, q, r)^\top \\ \text{Inputs:} \qquad & u \triangleq (\delta_e, \delta_t, \delta_a, \delta_r)^\top \end{aligned}$$

For constant-climb orbit:
$\rightarrow$ speed of aircraft not changing $\rightarrow \dot{u}^* = \dot{v}^* = \dot{w}^* = 0$
$\rightarrow$ roll and pitch angles constant $\rightarrow \dot{\phi}^* = \dot{\theta}^* = \dot{p}^* = \dot{q}^* = 0$

Turn rate constant and given by

$$\dot{\psi}^* = \frac{V_a^*}{R^*}\cos\gamma^* \quad \rightarrow \quad \dot{r}^* = 0$$

Climb rate constant, and given by

$$\dot{h}^* = V_a^* \sin\gamma^*$$

Given parameters V_a^*, γ^*, and R^*, can specify $\dot{x}^*$ as

$$\dot{x}^* = (\dot{p}_n^* \;\; \dot{p}_e^* \;\; \dot{h}^* \;\; \dot{u}^* \;\; \dot{v}^* \;\; \dot{w}^* \;\; \dot{\phi}^* \;\; \dot{\theta}^* \;\; \dot{\psi}^* \;\; \dot{p}^* \;\; \dot{q}^* \;\; \dot{r}^* \;)^\top$$

Given parameters V_a^*, γ^*, and R^*, can specify $\dot{x}^*$ as

$$\dot{x}^* = \begin{pmatrix} \dot{p}_n^* \\ \dot{p}_e^* \\ \dot{h}^* \\ \dot{u}^* \\ \dot{v}^* \\ \dot{w}^* \\ \dot{\phi}^* \\ \dot{\theta}^* \\ \dot{\psi}^* \\ \dot{p}^* \\ \dot{q}^* \\ \dot{r}^* \end{pmatrix} = \begin{pmatrix} [\text{don't care}] \\ [\text{don't care}] \\ V_a^* \sin\gamma^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{V_a^*}{R^*}\cos\gamma^* \\ 0 \\ 0 \\ 0 \end{pmatrix} = f(x^*, u^*)$$

Problem of finding x^* and u^* such that $\dot{x}^* = f(x^*, u^*)$, reduces to solving nonlinear algebraic systems of equations.

Linearized Lateral Model

$$
\begin{pmatrix} \dot{\bar{v}} \\ \dot{\bar{p}} \\ \dot{\bar{r}} \\ \dot{\bar{\phi}} \\ \dot{\bar{\psi}} \end{pmatrix} = \begin{pmatrix} Y_v & Y_p & Y_r & g\cos\theta^*\cos\phi^* & 0 \\ L_v & L_p & L_r & 0 & 0 \\ N_v & N_p & N_r & 0 & 0 \\ 0 & 1 & \cos\phi^*\tan\theta^* & q^*\cos\phi^*\tan\theta^* - r^*\sin\phi^*\tan\theta^* & 0 \\ 0 & 0 & \cos\phi^*\sec\theta^* & p^*\cos\phi^*\sec\theta^* - r^*\sin\phi^*\sec\theta^* & 0 \end{pmatrix} \begin{pmatrix} \bar{v} \\ \bar{p} \\ \bar{r} \\ \bar{\phi} \\ \bar{\psi} \end{pmatrix} + \begin{pmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_a \\ \bar{\delta}_r \end{pmatrix}
$$

Lateral	Formula
Y_v	$\frac{\rho S b v^*}{4mV_a^*}\left[C_{Y_p}p^* + C_{Y_r}r^*\right] + \frac{\rho S v^*}{m}\left[C_{Y_0} + C_{Y_\beta}\beta^* + C_{Y_{\delta_a}}\delta_a^* + C_{Y_{\delta_r}}\delta_r^*\right] + \frac{\rho S C_{Y_\beta}}{2m}\sqrt{u^{*2} + w^{*2}}$
Y_p	$w^* + \frac{\rho V_a^* S b}{4m} C_{Y_p}$
Y_r	$-u^* + \frac{\rho V_a^* S b}{4m} C_{Y_r}$
Y_{δ_a}	$\frac{\rho V_a^{*2} S}{2m} C_{Y_{\delta_a}}$
Y_{δ_r}	$\frac{\rho V_a^{*2} S}{2m} C_{Y_{\delta_r}}$
L_v	$\frac{\rho S b^2 v^*}{4V_a^*}\left[C_{p_p}p^* + C_{p_r}r^*\right] + \rho S b v^*\left[C_{p_0} + C_{p_\beta}\beta^* + C_{p_{\delta_a}}\delta_a^* + C_{p_{\delta_r}}\delta_r^*\right] + \frac{\rho S b C_{p_\beta}}{2}\sqrt{u^{*2} + w^{*2}}$
L_p	$\Gamma_1 q^* + \frac{\rho V_a^* S b^2}{4} C_{p_p}$
L_r	$-\Gamma_2 q^* + \frac{\rho V_a^* S b^2}{4} C_{p_r}$
L_{δ_a}	$\frac{\rho V_a^{*2} S b}{2} C_{p_{\delta_a}}$
L_{δ_r}	$\frac{\rho V_a^{*2} S b}{2} C_{p_{\delta_r}}$
N_v	$\frac{\rho S b^2 v^*}{4V_a^*}\left[C_{r_p}p^* + C_{r_r}r^*\right] + \rho S b v^*\left[C_{r_0} + C_{r_\beta}\beta^* + C_{r_{\delta_a}}\delta_a^* + C_{r_{\delta_r}}\delta_r^*\right] + \frac{\rho S b C_{r_\beta}}{2}\sqrt{u^{*2} + w^{*2}}$
N_p	$\Gamma_7 q^* + \frac{\rho V_a^* S b^2}{4} C_{r_p}$
N_r	$-\Gamma_1 q^* + \frac{\rho V_a^* S b^2}{4} C_{r_r}$
N_{δ_a}	$\frac{\rho V_a^{*2} S b}{2} C_{r_{\delta_a}}$
N_{δ_r}	$\frac{\rho V_a^{*2} S b}{2} C_{r_{\delta_r}}$

$$C_X(\alpha) \triangleq -C_D(\alpha)\cos\alpha + C_L(\alpha)\sin\alpha$$
$$C_{X_q}(\alpha) \triangleq -C_{D_q}\cos\alpha + C_{L_q}\sin\alpha$$
$$C_{X_{\delta_e}}(\alpha) \triangleq -C_{D_{\delta_e}}\cos\alpha + C_{L_{\delta_e}}\sin\alpha$$
$$C_Z(\alpha) \triangleq -C_D(\alpha)\sin\alpha - C_L(\alpha)\cos\alpha$$
$$C_{Z_q}(\alpha) \triangleq -C_{D_q}\sin\alpha - C_{L_q}\cos\alpha$$
$$C_{Z_{\delta_e}}(\alpha) \triangleq -C_{D_{\delta_e}}\sin\alpha - C_{L_{\delta_e}}\cos\alpha.$$

$$C_L(\alpha) = C_{L_0} + C_{L_\alpha}\alpha$$
$$C_D(\alpha) = C_{D_0} + C_{D_\alpha}\alpha.$$

If we assume a constant altitude and a constant thrust input, and longitudinal velocity $u = \dot{u} = 0$ is constant.

$$\dot{\bar{\alpha}} = Z_w \bar{\alpha} + \frac{Z_q}{V_a^* \cos \alpha^*}\dot{\bar{\theta}} - \frac{g \sin \theta^*}{V_a^* \cos \alpha^*}\bar{\theta} + \frac{Z_{\delta_e}}{V_a^* \cos \alpha^*}\bar{\delta}_e$$

$$\ddot{\bar{\theta}} = M_w V_a^* \cos \alpha^* \bar{\alpha} + M_q \dot{\bar{\theta}},$$

$$\lambda_{\text{short}} = \frac{Z_w + M_q}{2} \pm \sqrt{\left(\frac{Z_w + M_q}{2}\right)^2 - M_q Z_w + M_w Z_q}.$$

If we assume a constant altitude and a constant thrust input, and longitudinal velocity $u = \dot{u} = 0$ is constant.

$$\begin{pmatrix} \dot{\bar{u}} \\ 0 \\ \dot{\bar{q}} \\ \dot{\bar{\theta}} \end{pmatrix} = \begin{pmatrix} X_u & X_w V_a^* \sin\alpha^* & X_q & -g\cos\theta^* \\ \frac{Z_u}{V_a^*\cos\alpha^*} & Z_w & \frac{Z_q}{V_a^*\cos\alpha^*} & \frac{-g\sin\theta^*}{V_a^*\cos\alpha^*} \\ M_u & M_w V_a^* \cos\alpha^* & M_q & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta^* & -V_a^*\cos\theta^*\cos\alpha^* & 0 & u^*\cos\theta^* + w^*\sin\theta^* \end{pmatrix} \begin{pmatrix} \bar{u} \\ 0 \\ \bar{q} \\ \bar{\theta} \end{pmatrix} + \begin{pmatrix} X_{\delta_e} \\ \frac{Z_{\delta_e}}{V_a^*\cos\alpha^*} \\ M_{\delta_e} \\ 0 \\ 0 \end{pmatrix} \bar{\delta}_e.$$

$$s^2 + \left(\frac{Z_u X_q - X_u Z_q}{Z_q}\right) s - \frac{g Z_u}{Z_q} = 0.$$

$$\lambda_{\text{phugoid}} = -\frac{Z_u X_q - X_u Z_q}{2Z_q} \pm \sqrt{\left(\frac{Z_u X_q - X_u Z_q}{2Z_q}\right)^2 + \frac{g Z_u}{Z_q}}.$$

Reduced order Modes- Roll Mode

If we assume a constant altitude and a constant thrust input, and longitudinal velocity $u = \dot{u} = 0$ is constant.

The rolling mode is obtained by assuming that $\bar{\beta} = \bar{r} = \bar{\delta}_r = 0$:

$$\dot{\bar{p}} = +L_p \bar{p} + L_{\delta_a} \bar{\delta}_a.$$

The transfer function is therefore

$$\bar{p}(s) = \frac{L_{\delta_a}}{s - L_p} \bar{\delta}_a(s).$$

An approximation of the eigenvalue for the rolling mode is therefore given by

$$\lambda_{\text{rolling}} = L_p.$$

If we assume a constant altitude and a constant thrust input, and longitudinal velocity $u = \dot{u} = 0$ is constant.

For the spiral-divergence mode we assume that $\dot{\bar{p}} = \bar{p} = 0$, and that the rudder command is negligible.

$$0 = L_v V_a^* \cos \beta^* \bar{\beta} + L_r \bar{r} + L_{\delta_a} \bar{\delta}_a$$

$$\dot{\bar{r}} = N_v V_a^* \cos \beta^* \bar{\beta} + N_r \bar{r} + N_{\delta_a} \bar{\delta}_a.$$

$$\bar{r}(s) = \frac{\left(\frac{N_{\delta_a} L_v - N_v L_{\delta_a}}{L_v}\right)}{s - \left(\frac{N_r L_v - N_v L_r}{L_v}\right)} \bar{\delta}_a(s)$$

$$\lambda_{\text{spiral}} = \frac{N_r L_v - N_v L_r}{L_v}$$

Reduced order Modes- Dutch Roll Mode

If we assume a constant altitude and a constant thrust input, and longitudinal velocity $u = \dot{u} = 0$ is constant.

For the dutch-roll mode, we neglect the rolling motions and focus on the equations for sideslip and yaw.

$$\begin{pmatrix} \dot{\bar{\beta}} \\ \dot{\bar{r}} \end{pmatrix} = \begin{pmatrix} Y_v & \frac{Y_r}{V_a^* \cos \beta^*} \\ N_v V_a^* \cos \beta^* & N_r \end{pmatrix} \begin{pmatrix} \bar{\beta} \\ \bar{r} \end{pmatrix} + \begin{pmatrix} \frac{Y_{\delta_r}}{V_a^* \cos \beta^*} \\ N_{\delta_r} \end{pmatrix} \bar{\delta}_r .$$

The characteristic equation is given by

$$\det \left(sI - \begin{pmatrix} Y_v & \frac{Y_r}{V_a^* \cos \beta^*} \\ N_v V_a^* \cos \beta^* & N_r \end{pmatrix} \right) = s^2 + (-Y_v - N_r)s + (Y_v N_r - N_v Y_r) = 0.$$

Therefore, the poles of the dutch-roll mode are approximated by

$$\lambda_{\text{dutch roll}} = \frac{Y_v + N_r}{2} \pm \sqrt{\left(\frac{Y_v + N_r}{2}\right)^2 - (Y_v N_r - N_v Y_r)}.$$