In the Name of God

Neuroscience of Learning, Memory, Cognition

Homework_2 Report

- Phase Planes & Limit Cycles
- Integrate & Fire Model
- Spike−Triggered Average
- Raster Plot & PETH
- SR Latch
- Limit Cycles

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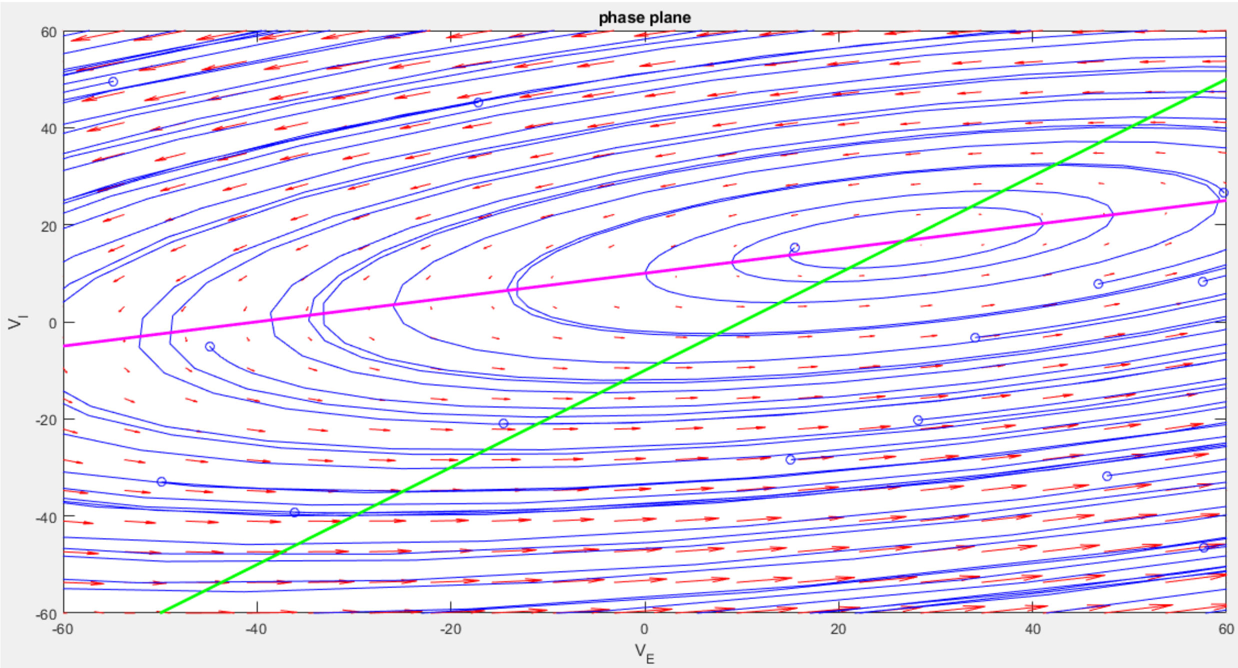
1- Phase Planes & Limit Cycles: First Neuron model

1.1,1 –



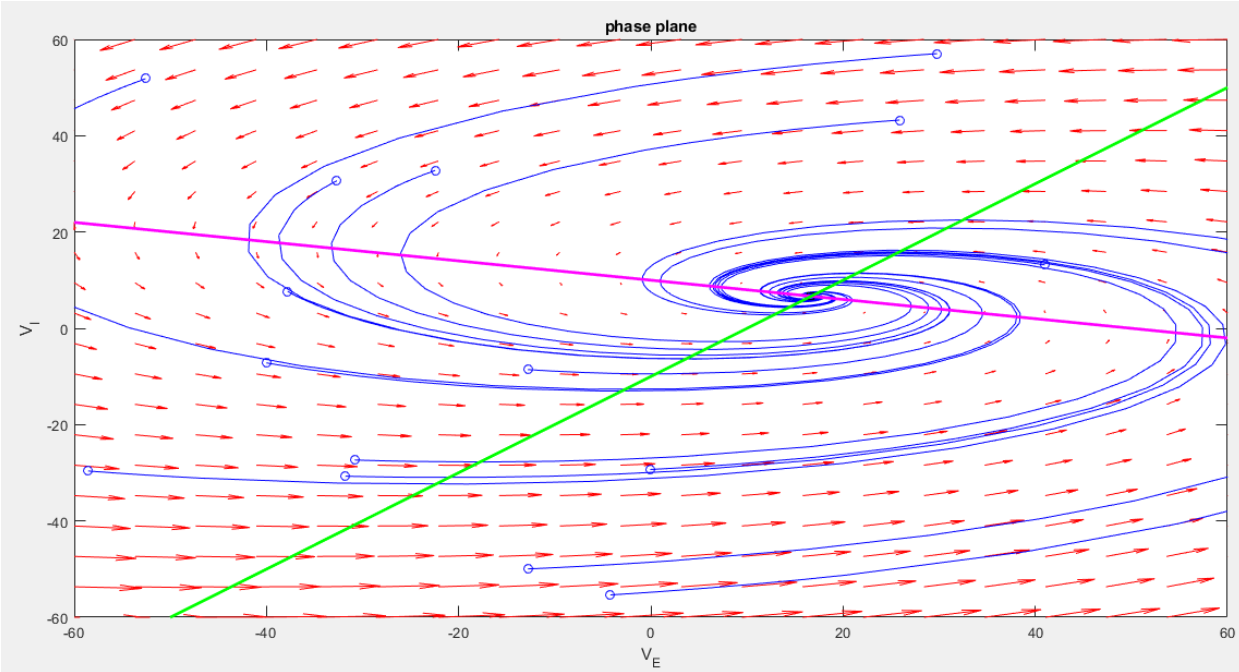
It`s an **unstable focus**. As we can see, if we slightly move away from the fixed point, we won`t return there. This means real part of the eigenvalues are positive. (the reason that the behavior near the fixed point is oscillatory, is because of sign of under the radical which is negative)

1.1.2 –

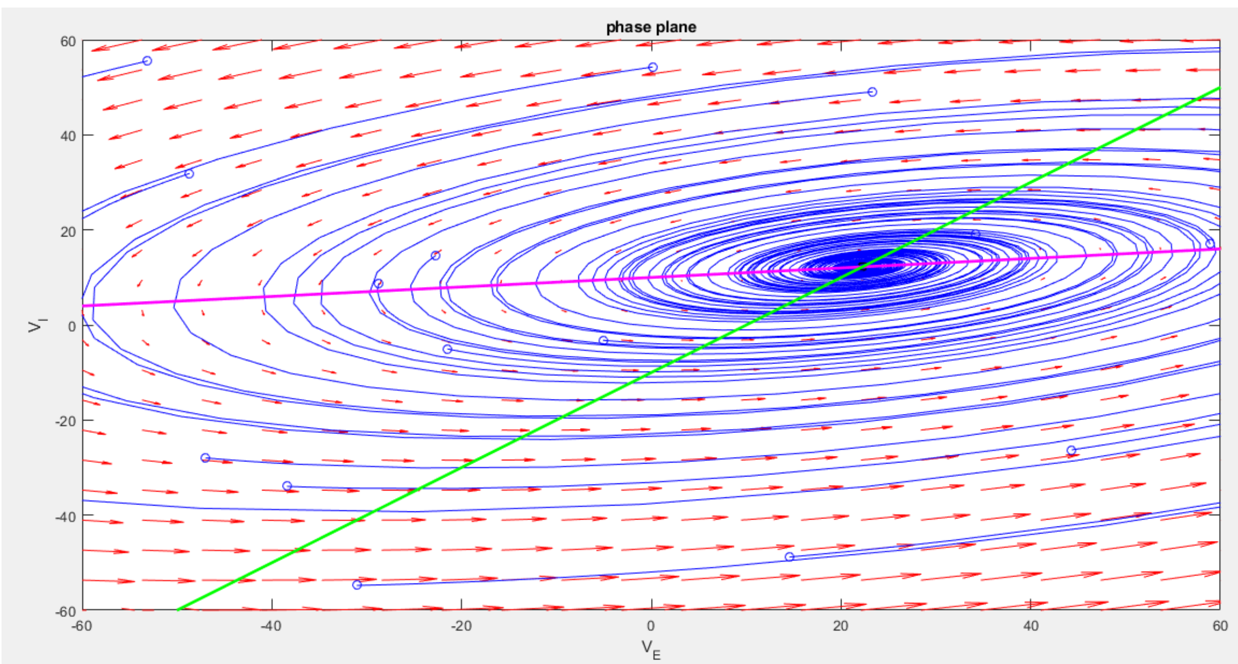
$$\frac{dv_E}{dt} = \frac{-v_E + [M_{EE}\, v_E + M_{EI}\, v_I - \gamma_E]_+}{\tau_E} \;,\; \frac{dv_I}{dt} = \frac{-v_E + [(M_{IE}\, v_E + M_{II}\, v_I - \gamma_E)]_+}{\tau_E}$$

As we can see, if we change M_{EE}, $\boldsymbol{v_I}$ nullcline won`t change because it isn`t related to M_{EE}, but the slope of $\boldsymbol{v_E}$ nullcline will change when we are changing the M_{EE}. Another thing we see, is change of the **stability and type** of the fixed point. Let`s take a closer look.

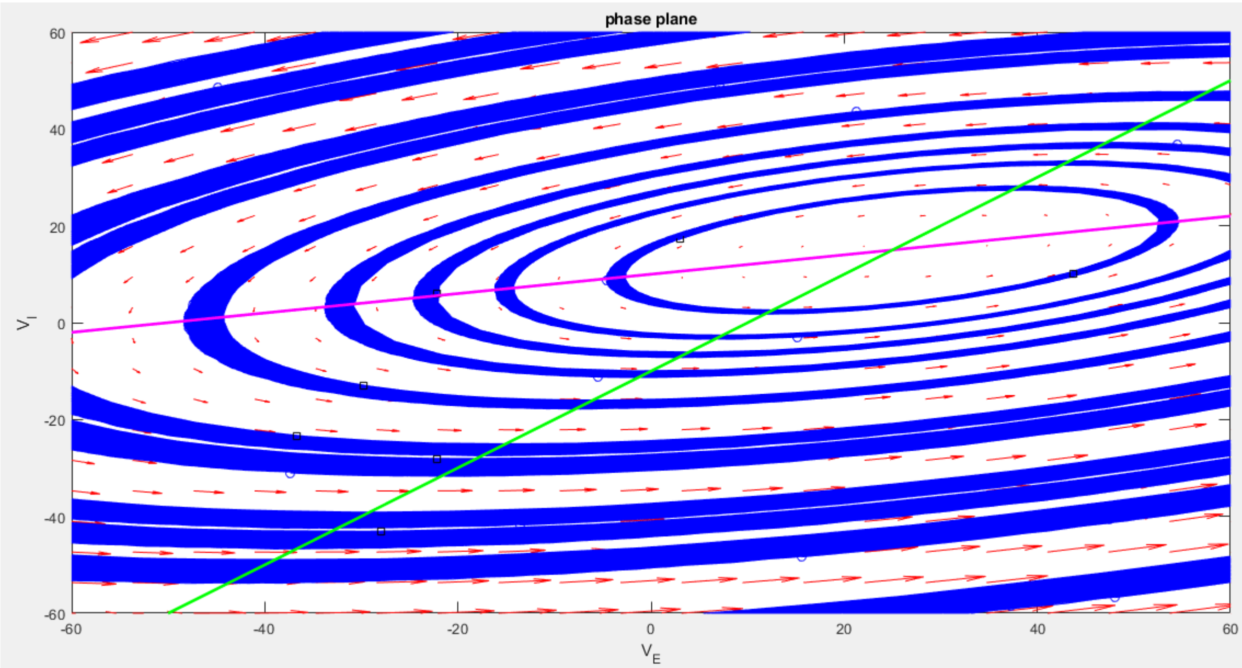
$M_{EE} = ۰.۸$: (stable focus)



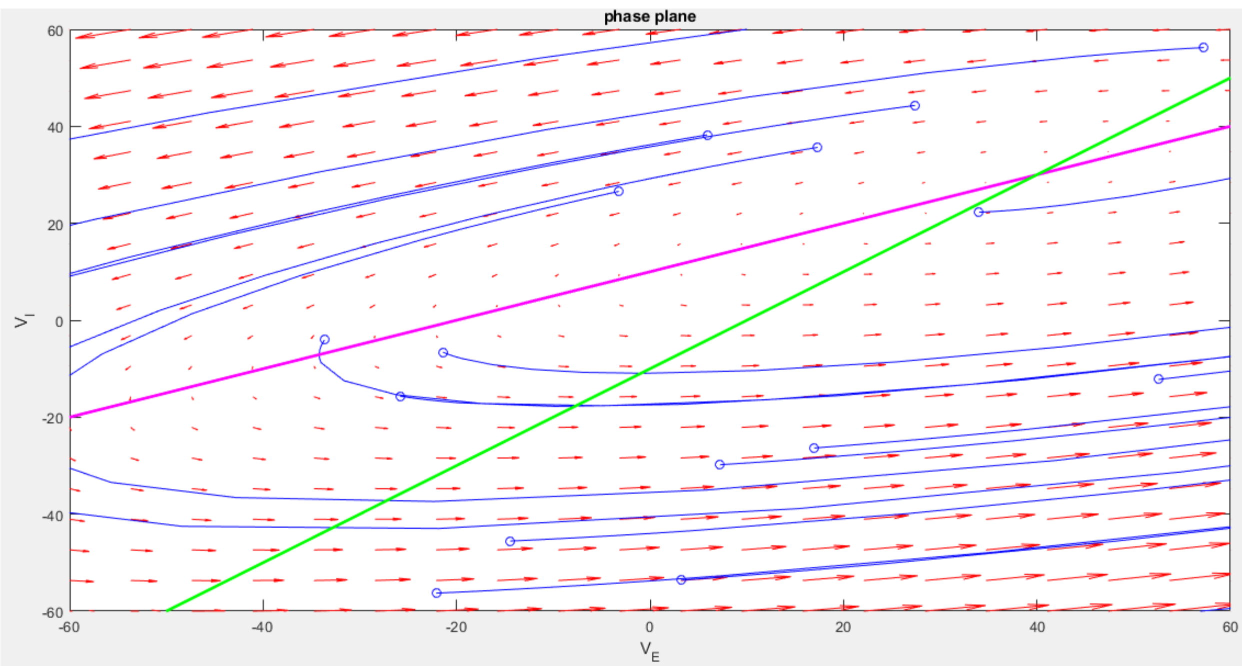
$M_{EE} = ۱.۱$: (stable focus)



M_{EE} = ١.٢ :(bifurcation point)



M_{EE} = ١.٥ : (unstable focus)



As we can see, by passing 1.2 our **stable focus** changes to an **unstable focus**. So, $M_{EE} = 1.2$ is like a bifurcation point.

But what is the mathematical reason that makes 1.2 a bifurcation point?

As we can calculate, our eigenvalues are:

$$\lambda = \frac{1}{2}\left(\frac{M_{EE}-1}{\tau_E}+\frac{M_{II}-1}{\tau_I} \pm \sqrt{\left(\frac{M_{EE}-1}{\tau_E}+\frac{M_{II}-1}{\tau_I}\right)^2+\frac{4M_{EI}M_{IE}}{\tau_E\tau_I}}\right)$$

So if the real part of eigenvalues gets positive, $\frac{M_{EE}-1}{\tau_E}+\frac{M_{II}-1}{\tau_I}$: our stable focus will change to an unstable one.

Because of our initial values, if M_{EE} passes 1.2, the real part will get positive and the focus would be unstable.

The reason why the plot of M_{EE} = ١.٢ looks like this is that the real parts are zero. So, we won`t get returned to the fix point or moved away from it. Neither diverge nor converge.

When we compare the plots of $M_{EE} = 0.8$ and $M_{EE} = 1.1$, we can see when M_{EE} is 0.8, we get to the fixed point faster that when M_{EE} is 1.1. The reason is obvious because if we get further from 1.2, the real part will be more negative and so we return to the fixed point faster.

So, When M_{EE} is below 1.2, we have a stable focus and the less M_{EE} is, the faster it returns to the fixed point and if M_{EE} gets more than 1.2 we have an unstable fixed point.

Now if we increase or decrease M_{EE} a lot, what will happen?

We know if the value of M_{EE} is more than 1.2 the real part is positive and we have an unstable focus and when the values is less than 1.2 we have a stable focus. Up to this moment, all the plots I`ve put here are showing oscillatory behavior around the fixed point and so they`re **focus**. Why?

Because if we take a look at the equation which describes the eigenvalues, the sign of under the radical is negative and because of that we have **focus**. So if we increase or decrease M_{EE} A lot, can the sign under radical change to positive?

$$\lambda = \frac{1}{2}\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I} \pm \sqrt{\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}\right)^2 + \frac{4M_{EI}M_{IE}}{\tau_E\tau_I}}\right)$$

Absolutely it can. Let`s take a look at our initial values:

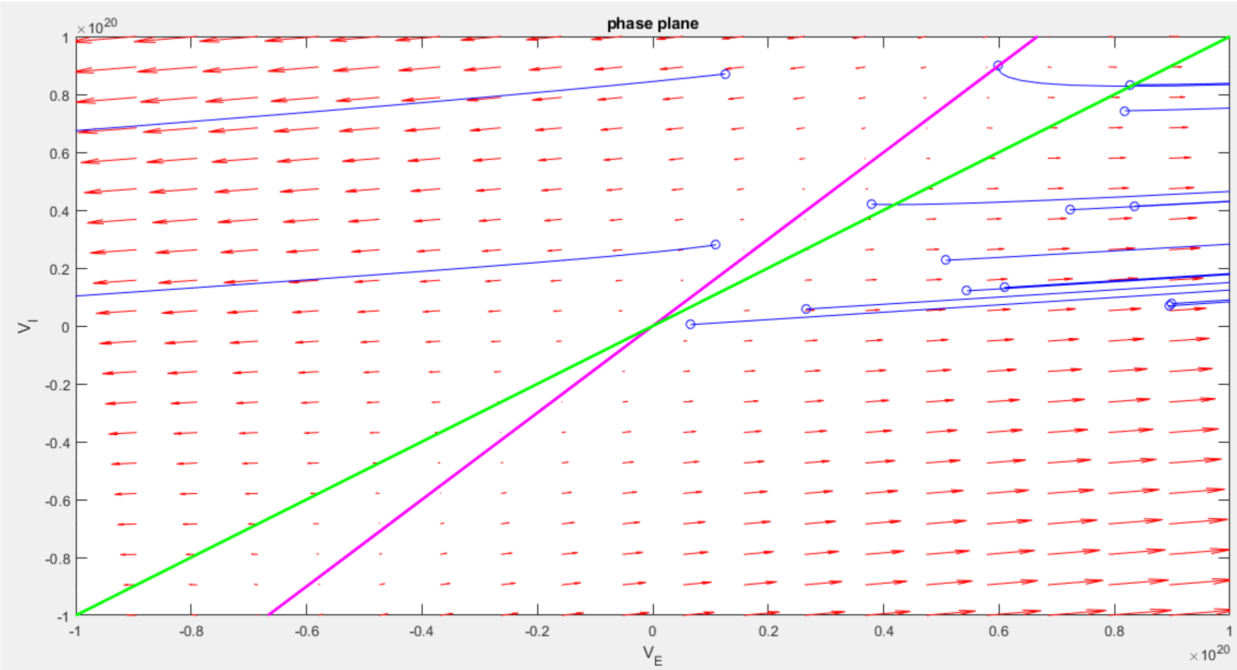
```
Mee = % changing
Mei = -1;
Mie = 1;
Mii = 0;
Ye = -10;
Yi = 10;
Te = 0.01;
Ti = 0.05;
```

$+\frac{4M_{EI}M_{IE}}{\tau_E\tau_I}\Bigg)$ The value of this part of the radical is -8000.

$\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}\right)^2$ So if the value of goes above 8000, we won`t have a focus anymore because sign of under the radical has changed to positive and we have **node or saddle** now. So, when this part gets more than 8000? If we put initial values in to it, we get if $M_{EE}$ goes near 2.09 or more, we`ll have a **bifurcation from an unstable focus to an unstable node**(because real part of eigenvalues are still positive and the value of radical is less than the other part out of the radical and so real part is always positive and we won`t have a saddle) and if $M_{EE}$ goes to nearly 0.3 or less, again we`ll have a **bifurcation from a stable focus to a stable node**. (because real part of eigenvalues are still negative and the value of radical is less than the other part out of the radical and so real part is always negative and we won`t have a saddle) .

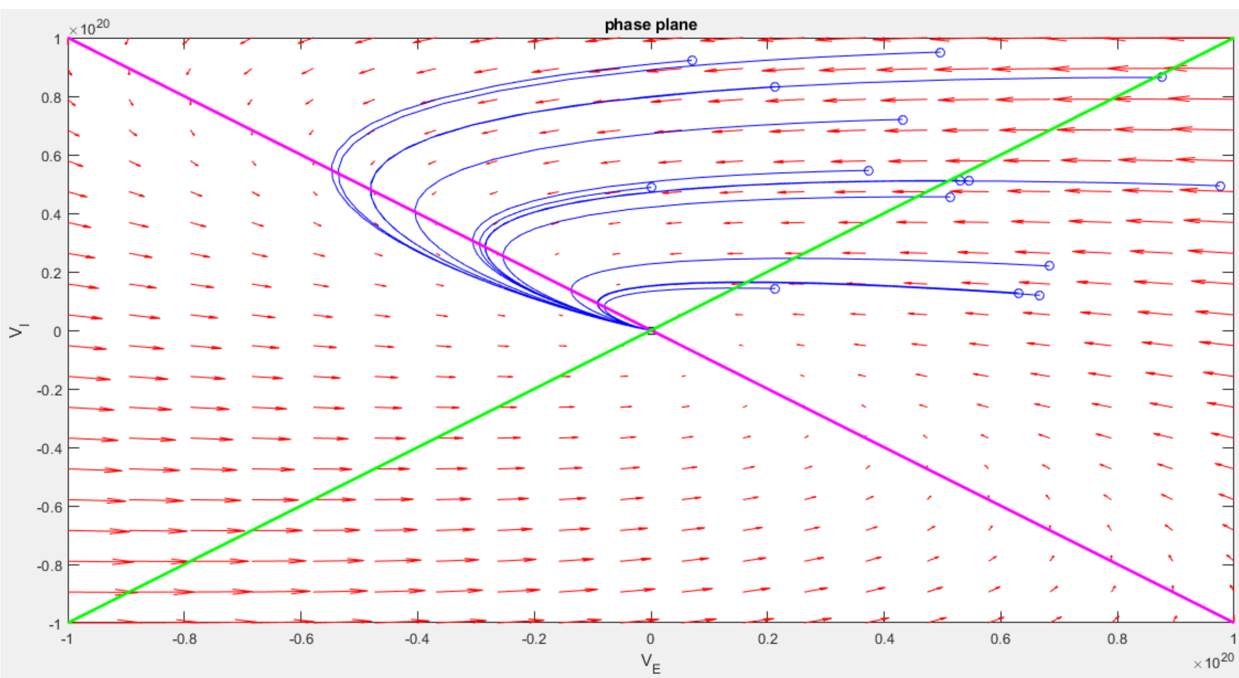
$+\frac{4M_{EI}M_{IE}}{\tau_E\tau_I}\Bigg)$ If this part was positive we could have saddle after bifurcation instead of node but it`s negative.

$M_{EE} =$ ٢.٥:



Unstable node.

$M_{EE} =$ ٠.١:



Stable node.

So, we can have different bifurcations by changing M_{EE}. In a summary by changing M_{EE} the phase plan and our fixed point behaves like the chart next page:

$M_{EE} < 0.3$ → $0.3 < M_{EE} < 1.2$ → $1.2 < M_{EE} < 2.1$ → $2.1 < M_{EE}$

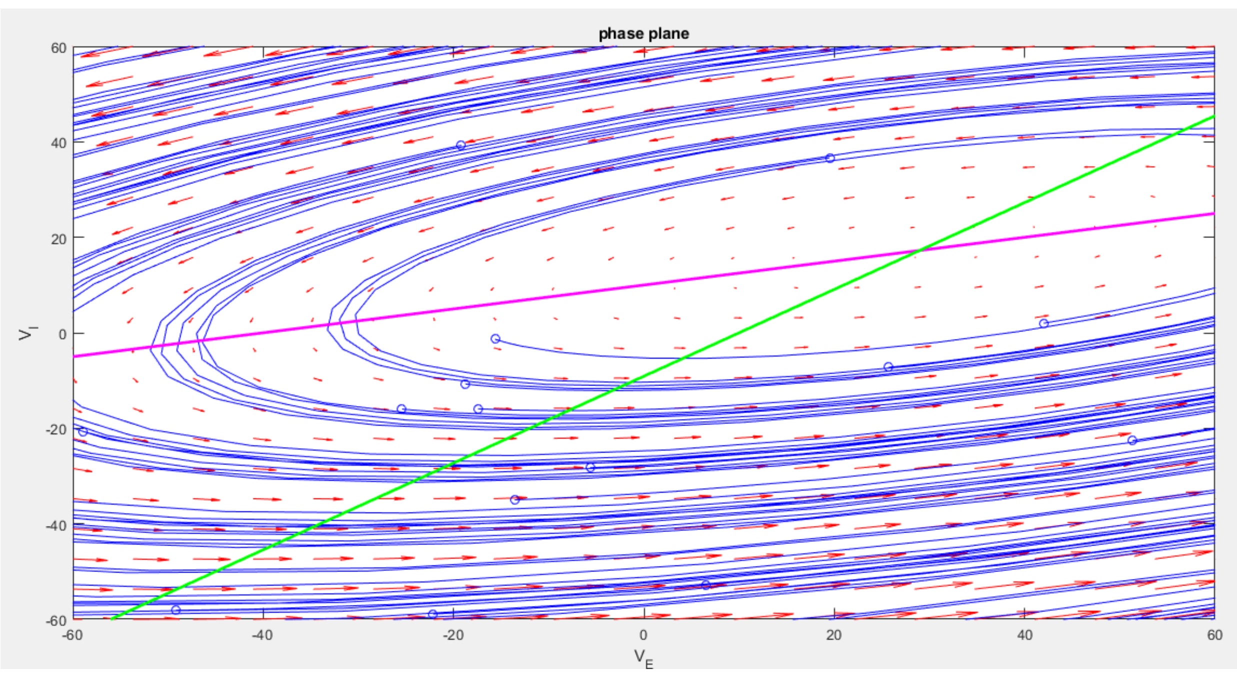
Stable node → **Stable focus** → **unstable focus** → **unstable node**

۱,۱.۳ –

- Changing M_{II}:(other values are constant)

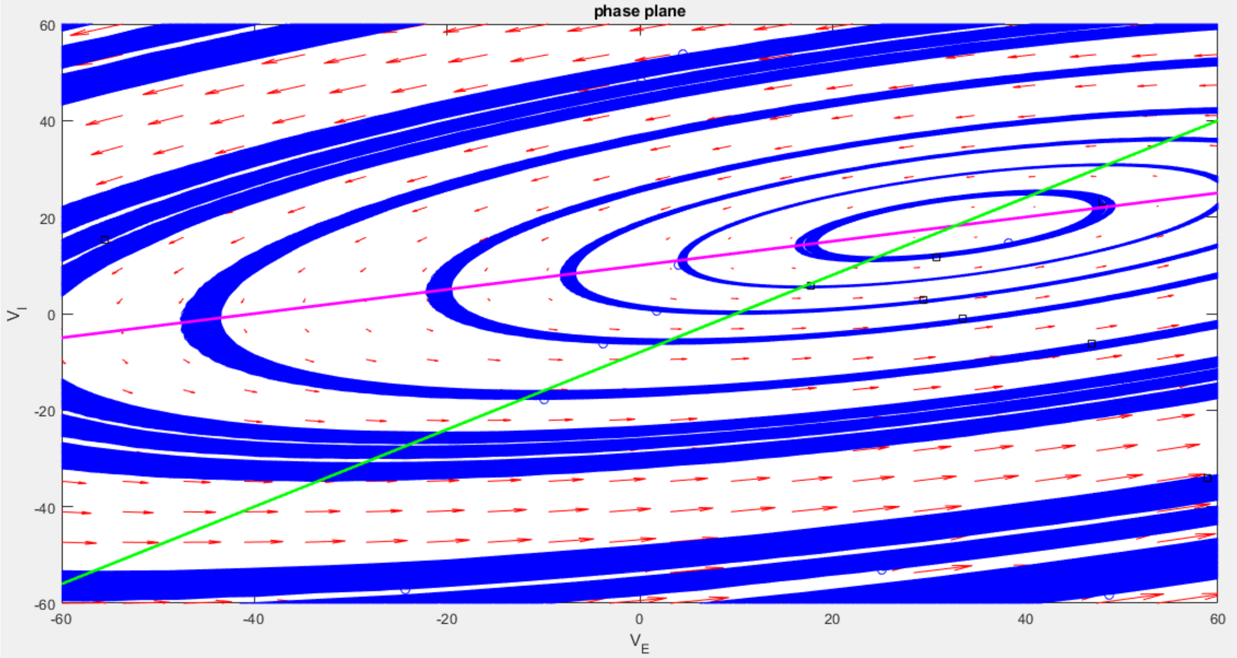
This time $\boldsymbol{v_E}$ nullcline won`t change because it isn`t related to M_{II}, but the slope of $\boldsymbol{v_I}$ nullcline will change when we are changing the M_{II}. Another thing we see, is change of the **stability and type** of the fixed point.

$M_{II} = -$۰.۱ :



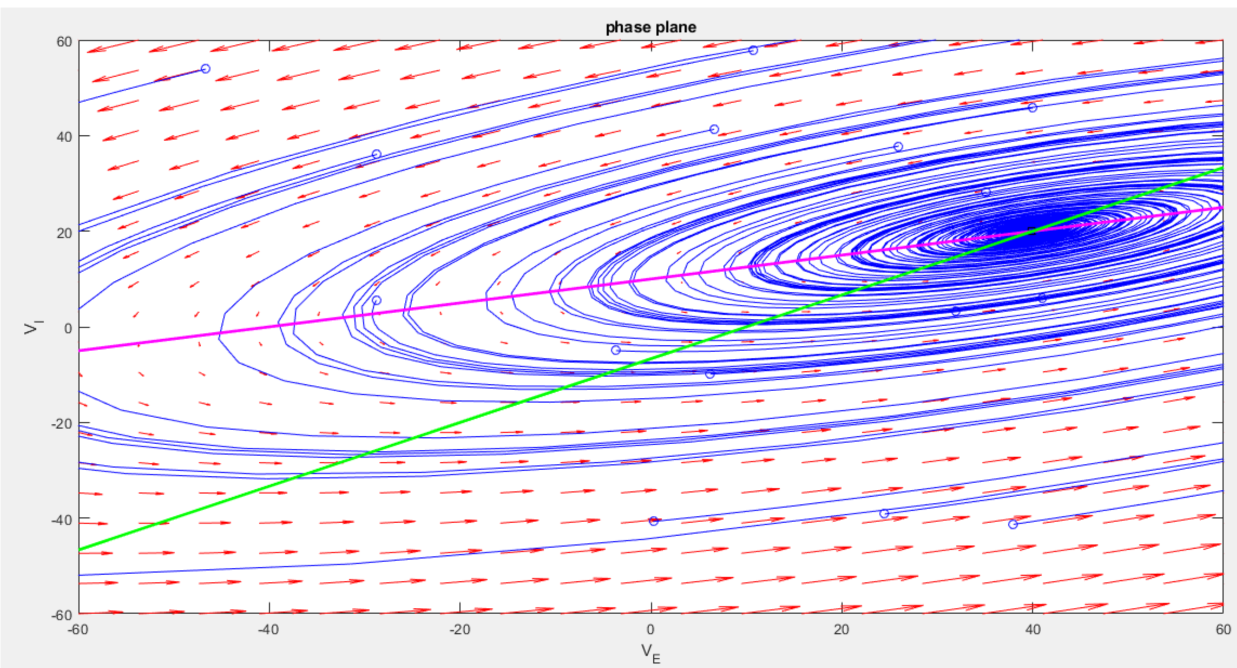
Unstable focus

$M_{II} = -۰.۲۵$:



Bifurcation point

$M_{II} = -۰.۵$:



Stable focus

as we can see, -0.25 is our **bifurcation point** which when M_{II} is lower than this, we`ll have a stable focus because real values of eigenvalues are negative but when $M_{II}$ is more than -0.25 we`ll have an unstable focus. (note that here M_{EE} is equal to 1.25 and if we look at the eigenvalues equation we can calculate -0.25 from there too.) When we are below -0.25 which we have a stable focus, if we get further from -0.25, we will return to the fixed point faster. For example, when M_{II} = -1.5, we return to fixed point faster than when it`s equal to -1 because size of the real parts is bigger.

$$\lambda = \frac{1}{2}\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I} \pm \sqrt{\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}\right)^2 + \frac{4M_{EI}M_{IE}}{\tau_E\tau_I}}\right)$$

Real part

$$\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}$$

$\boldsymbol{\tau_E}$ **= 0.01,** $\boldsymbol{\tau_I}$ **= 0.05,** $\boldsymbol{M_{EE}}$ **= 1.25**

Now what will happen if we increase or decrease M_{II} a lot?

We know if the value of M_{II} is more than -0.25 the real part is positive and we have an unstable focus and when the values is less than -0.25 we have a stable focus. Up to this moment, all the plots I`ve put here are showing oscillatory behavior around the fixed point and so they`re **focus**. Why?

$$\lambda = \frac{1}{2}\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I} \pm \sqrt{\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}\right)^2 + \frac{4M_{EI}M_{IE}}{\tau_E\tau_I}}\right)$$

Because if we take a look at the equation which describes the eigenvalues, the sign of under the radical is negative and because of that we have **focus**. So if we increase or decrease M_{EE} a lot, can the sign under radical change to positive?

Absolutely it can. Let`s take a look at our initial values:

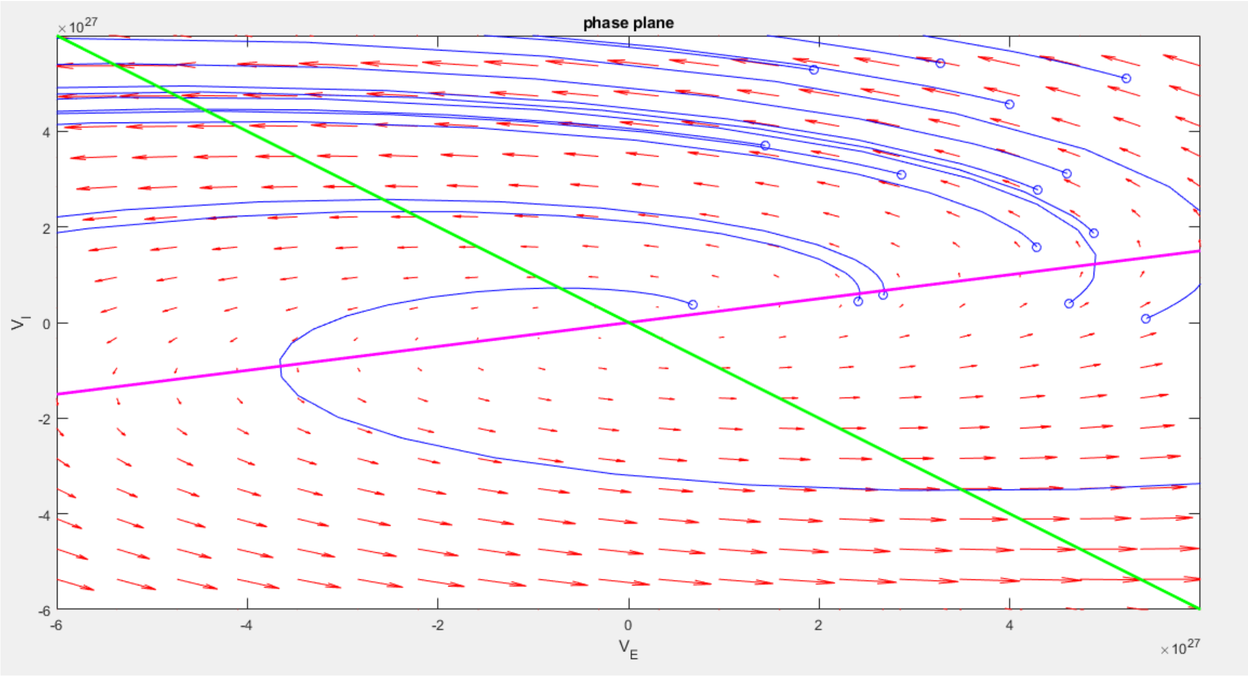
```
Mee = 1.25;
Mei = -1;
Mie = 1;
Mii = % changing
Ye = -10;
Yi = 10;
Te = 0.01;
Ti = 0.05;
```

$+\frac{4M_{EI}M_{IE}}{\tau_E\tau_I}\Big)$ The value of this part of the radical is -8000.

$\left(\frac{M_{EE}-1}{\tau_E}+\frac{M_{II}-1}{\tau_I}\right)^2$ So if the value of goes above 8000, we won`t have a focus anymore because sign of under the radical has changed to positive and we have **node or saddle** now. So, when this part gets more than 8000? If we put initial values in to it, we get if $M_{II}$ goes near 4.2 or more, we`ll have a **bifurcation from an unstable focus to an unstable node**(because real part of eigenvalues are still positive and the value of radical is less than the other part out of the radical and so real part is always positive and we won`t have a saddle) and if $M_{II}$ goes to nearly -4.72 or less, again we`ll have a **bifurcation from a stable focus to a stable node.** (because real part of eigenvalues are still negative and the value of radical is less than the other part out of the radical and so real part is always negative and we won`t have a saddle) .

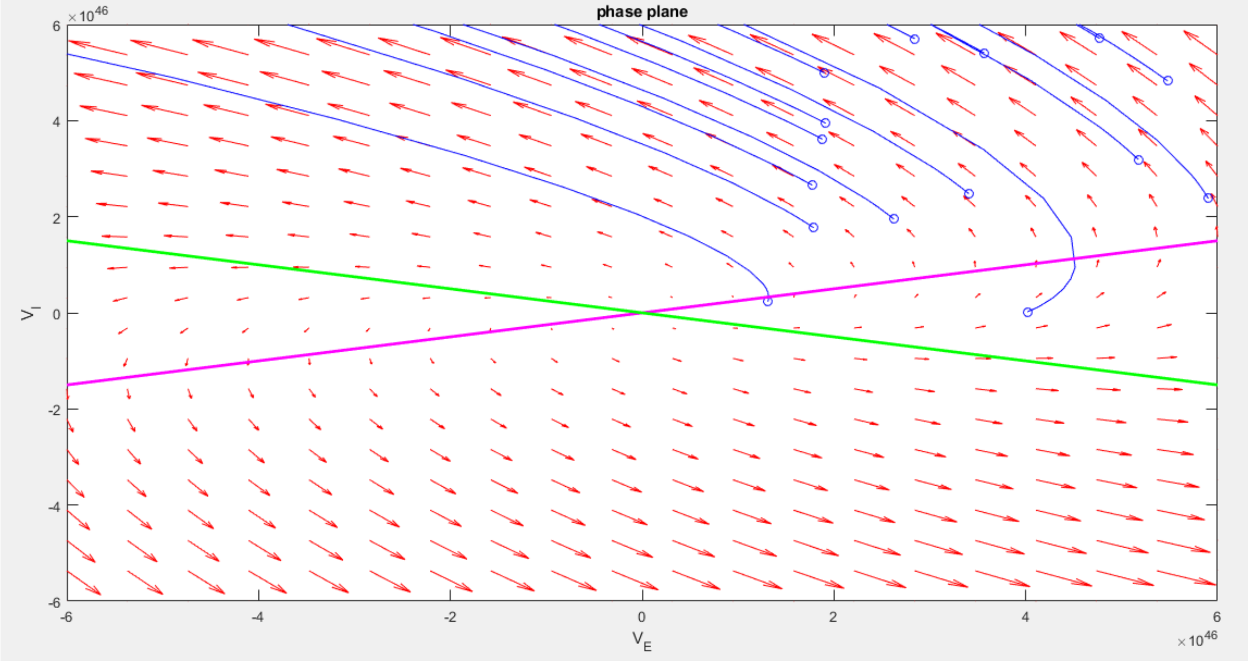
$+\frac{4M_{EI}M_{IE}}{\tau_E\tau_I}\Big)$ If this part was positive we could have saddle after bifurcation instead of node but it`s negative.

$M_{II} = ٢$:



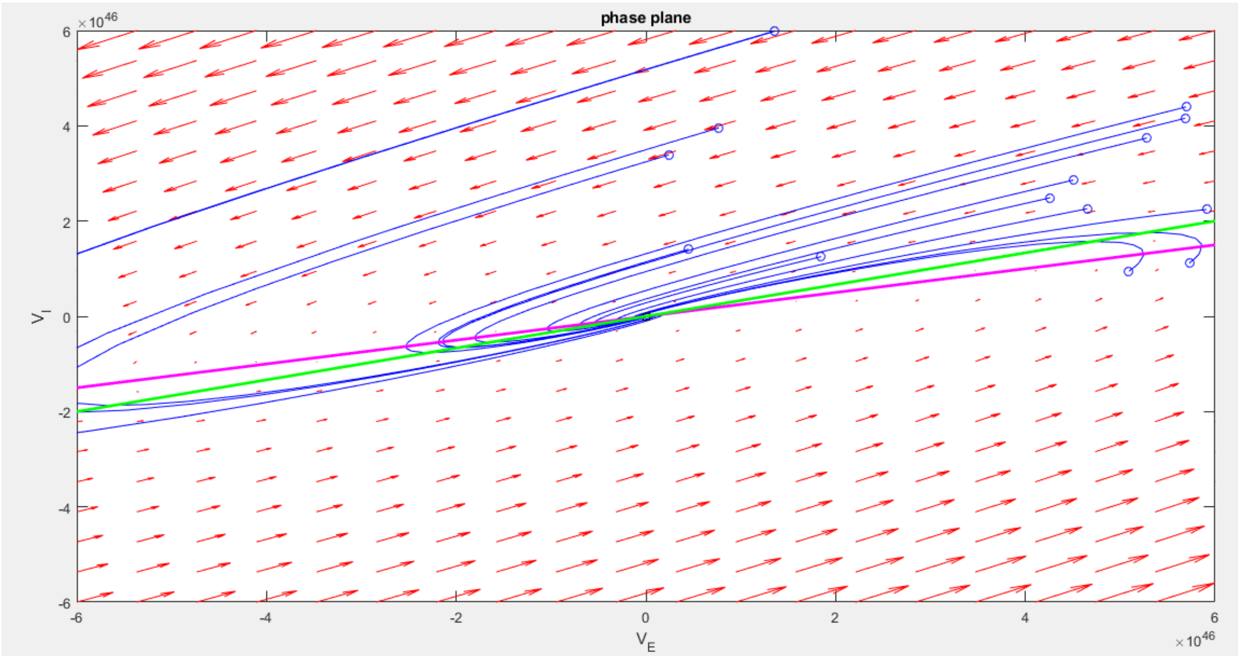
Unstable focus

$M_{II} = ٥$:



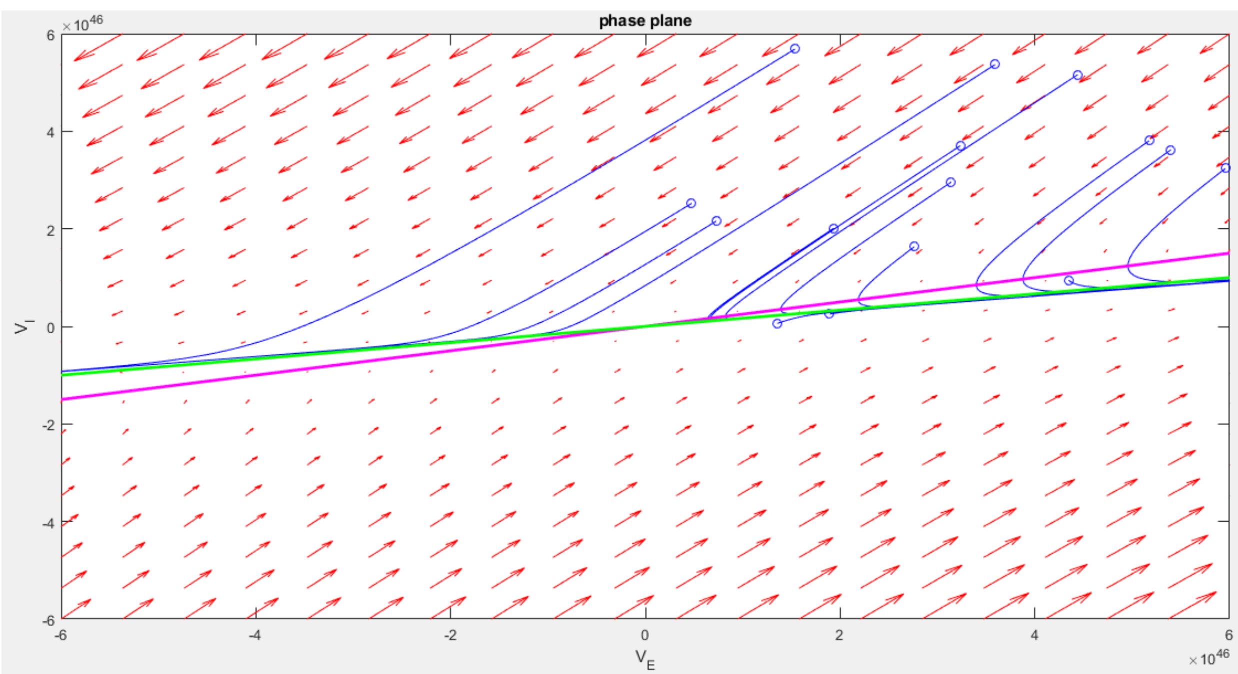
Unstable node

$M_{II} = -$۲:



Stable focus

$M_{II} = -$۵:



Stable node

M_{II} < -4.72	-4.72 < M_{II} < -0.25	-0.25 < M_{II} < 4.2	4.2 < M_{II}
Stable node →	**Stable focus** →	**unstable focus** →	**unstable node**

- Changing M_{EI}:(other values are constant)

By changing M_{EI}, $\boldsymbol{v_I}$ nullcline won`t change because it isn`t related to M_{EI}, but the slope of $\boldsymbol{v_E}$ nullcline will change when we are changing the M_{EI}. Another thing we see, is change of the **stability and type** of the fixed point.

$$\lambda = \frac{1}{2}\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I} \pm \sqrt{\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}\right)^2 + \frac{4M_{EI}M_{IE}}{\tau_E\tau_I}}\right)$$

$$\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}$$

This part is equal to 5 and so if under the radical is negative we`ll have an unstable focus and if under the radical is positive we can have a saddle or an unstable node. If the value of radical was bigger than the other part, we`ll have a saddle and if not, we`ll have an unstable node.

Let`s take a look at our initial values:

```
Mee = 1.25;
Mei = %changing
Mie = 1;
Mii = 0;
Ye = -10;
Yi = 10;
Te = 0.01;
Ti = 0.05;
```

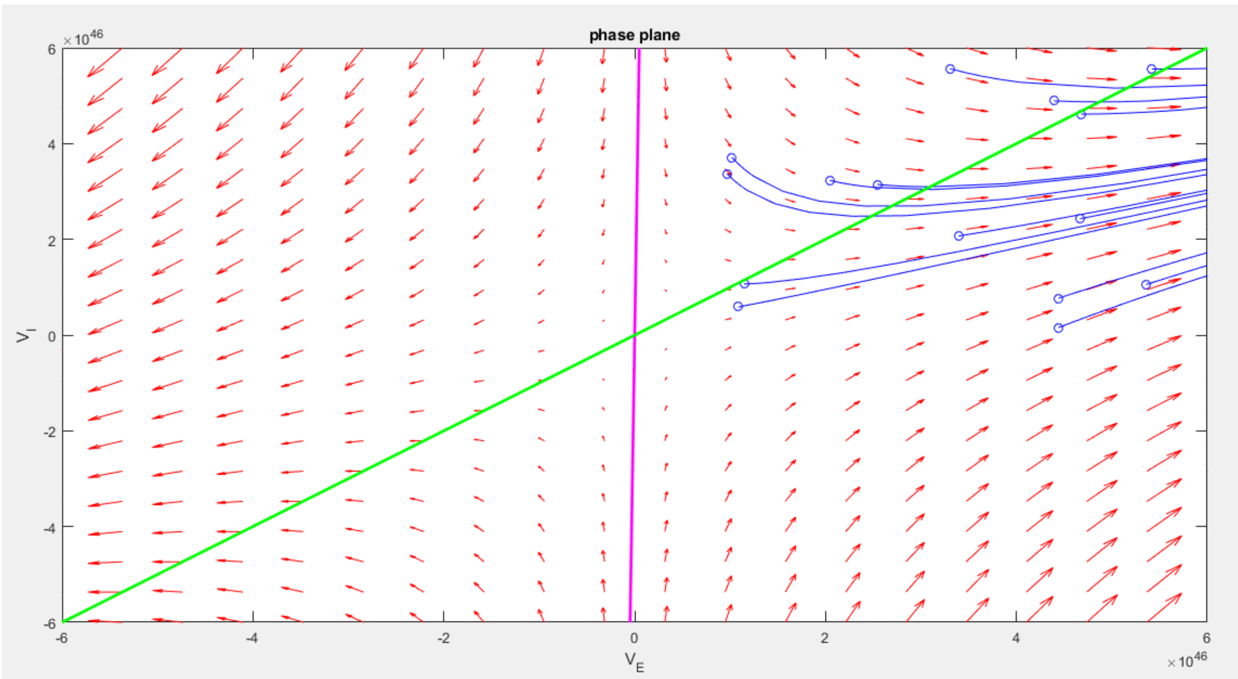
$$\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}\right)^2 = 25,$$

$$\left.\frac{4M_{EI}M_{IE}}{\tau_E\tau_I}\right) = 8000M_{EI},$$

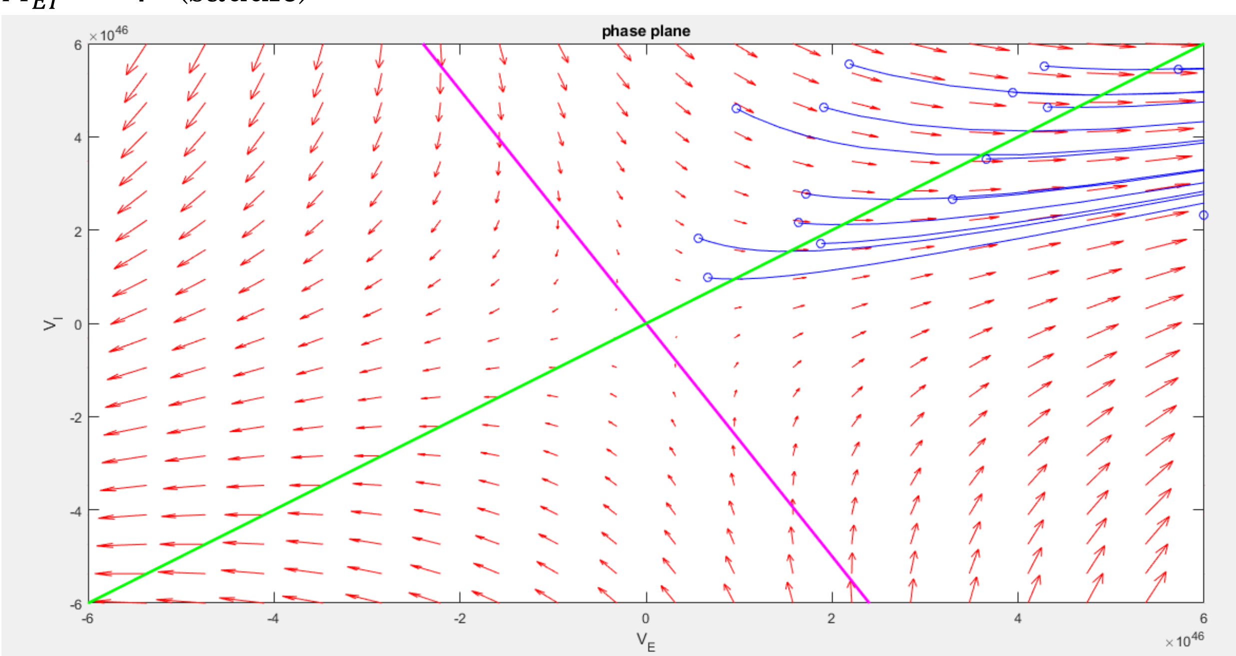
So if M_{EI} is more than $\frac{-١}{٣٢٠}$, under the radical would be positive and we would have a saddle or an unstable node.

If M_{EI} is positive we would have a **saddle** and if M_{EI} is between $\frac{-١}{٣٢٠}$ and zero, we would have an **unstable node**. And if M_{EI} is less than $\frac{-١}{٣٢٠}$ we would have an unstable focus because under the radical is negative. Let`s check what we said with simulation:

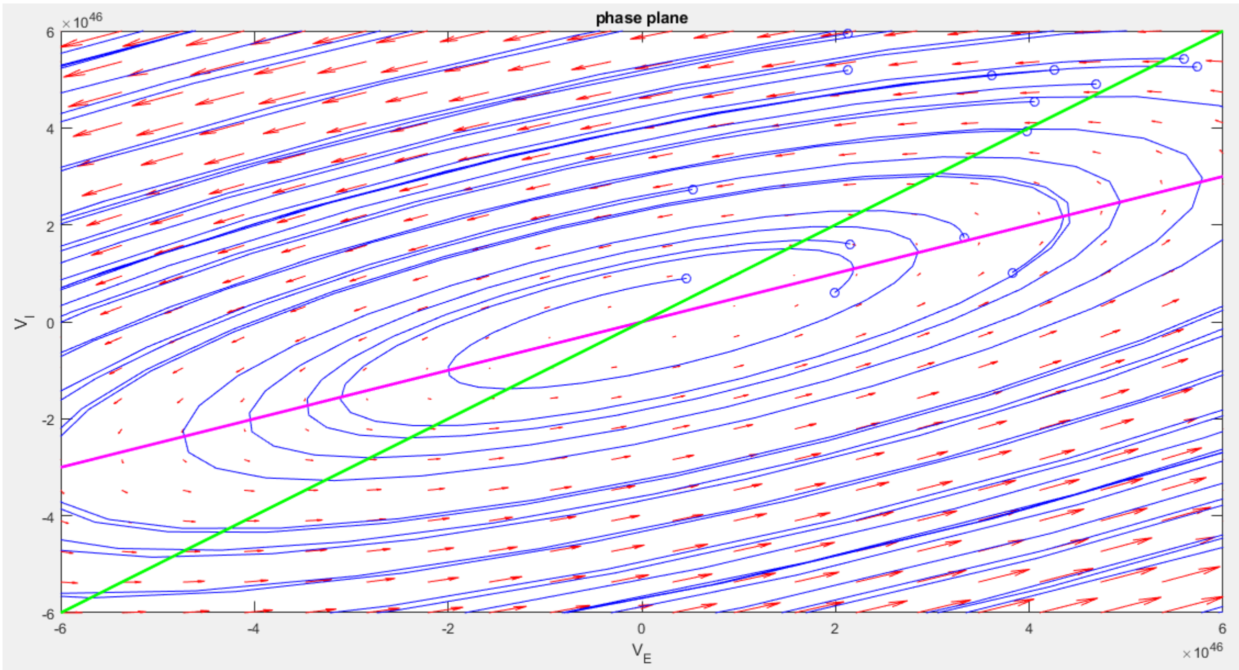
$M_{EI} = -$٠.٠٠٢:(unstable node)



$M_{EI} =$ ٠.١:(saddle)



$M_{EI} = -$۰.۵:(unstable focus)



M_{EI} < -1/320 **Unstable focus** → -1/320 < M_{EI} < 0 **Unstable node** → 0 < M_{EI} **Saddle**

- Changing M_{IE}:(other values are constant)

By changing M_{IE}, $\boldsymbol{v_E}$ nullcline won`t change because it isn`t related to M_{IE}, but the slope of $\boldsymbol{v_I}$ nullcline will change when we are changing the M_{IE}. Another thing we see, is change of the **stability and type** of the fixed point.

$$\lambda = \frac{1}{2}\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I} \pm \sqrt{\left(\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}\right)^2 + \frac{4M_{EI}M_{IE}}{\tau_E\tau_I}}\right)$$

$$\frac{M_{EE}-1}{\tau_E} + \frac{M_{II}-1}{\tau_I}$$

This part is equal to 5 and so if under the radical is negative we`ll have an unstable focus and if under the radical is positive we can have a saddle or an unstable node. If the value of radical was bigger than the other part, we`ll have a saddle and if not, we`ll have an unstable node.

Let`s take a look at our initial values:

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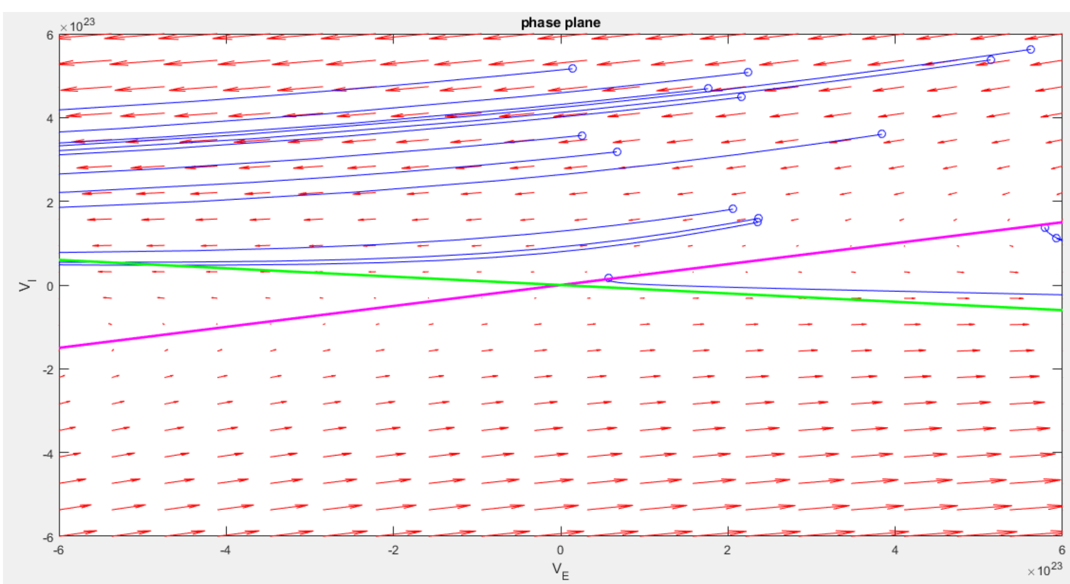
Mee = 1.25;
Mei = -1;
Mie = %changing
Mii = 0;
Ye = -10;
Yi = 10;
Te = 0.01;
Ti = 0.05;
```

$$\left(\frac{M_{EE}-1}{\tau_E}+\frac{M_{II}-1}{\tau_I}\right)^2 = 25,$$

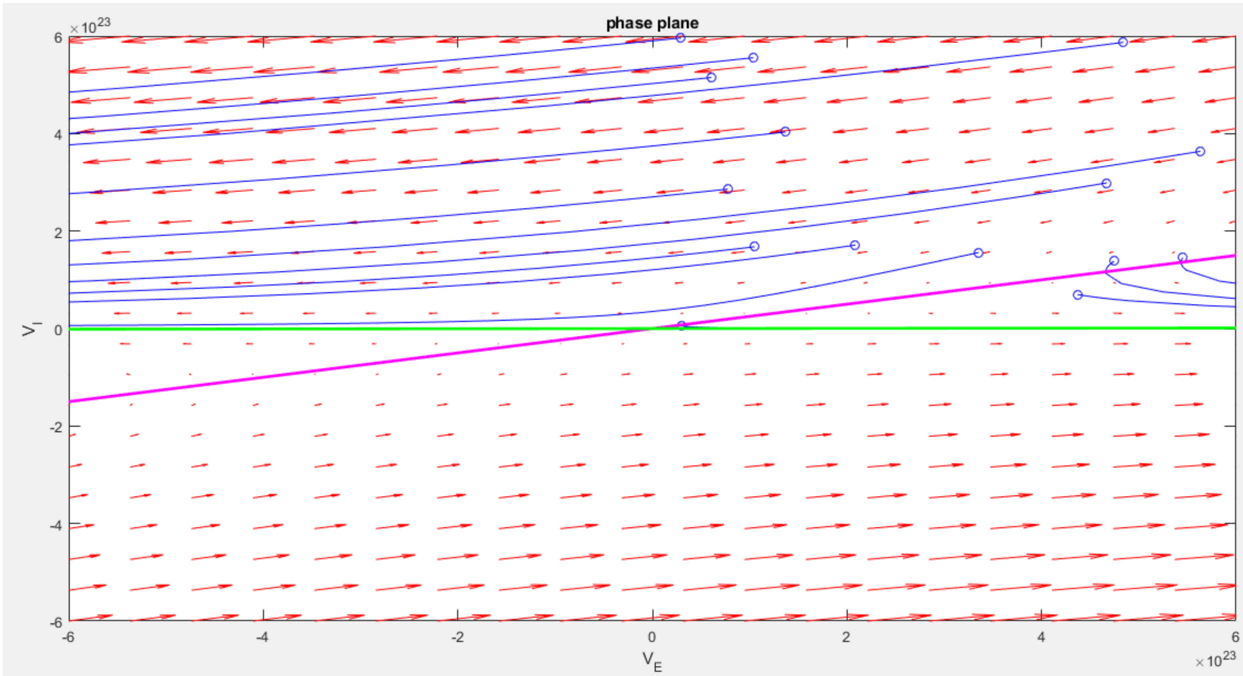
$$\left.\frac{4M_{EI}M_{IE}}{\tau_E\tau_I}\right) = -8000M_{IE},$$

So if M_{IE} is more than $\frac{١}{٣٢٠}$, under the radical would be negative and we would have a **unstable focus.** If M_{IE} is between $\frac{١}{٣٢٠}$ and zero we would have a **unstable node** and if M_{IE} is negative, we would have an **saddle**. Let`s check what we said with simulation

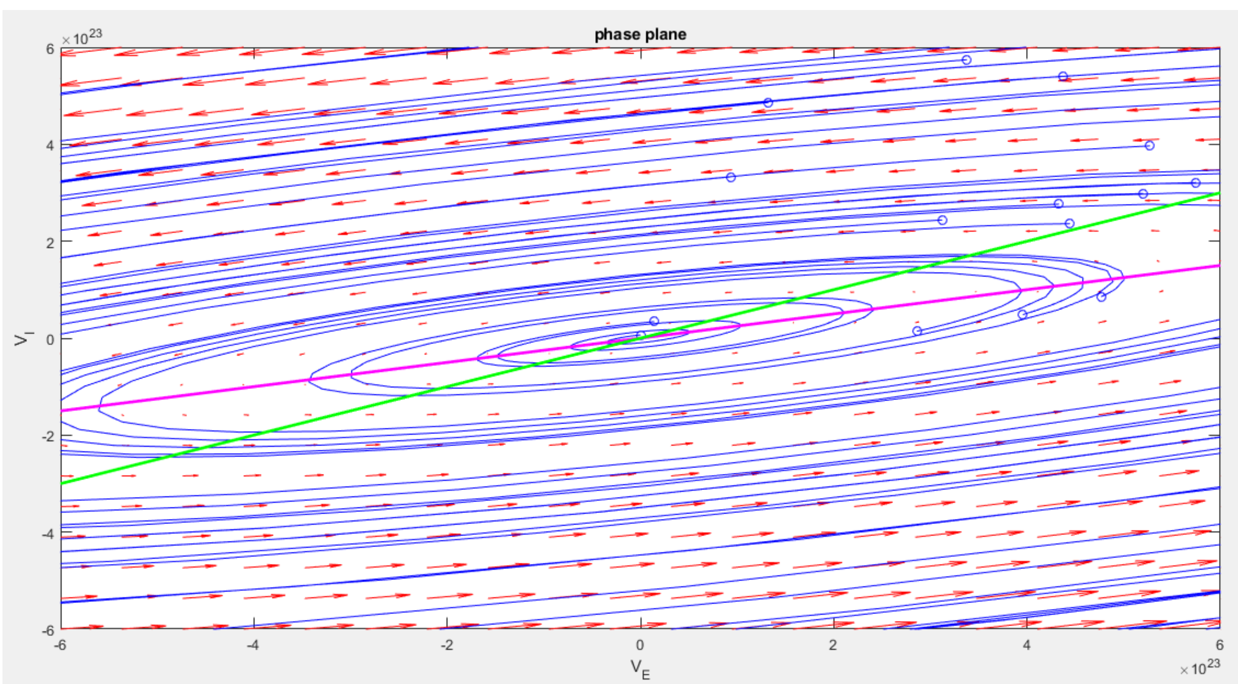
$M_{IE} = -$٠.١:(saddle)



$M_{IE} =$ ٠.٠٠٠٢:(unstable node)



$M_{IE} =$ ٠.٥:(unstable focus)



$M_{IE} <$ ٠		$0 < M_{IE} < 1/320$		$1/320 < M_{IE}$
Saddle	→	**Unstable node**	→	**Unstable focus**

1.1.4 –

Yes. If the initial conditions of M_{EE}, M_{EI}, M_{IE}, M_{II} are something that makes our fixed point, an unstable node or saddle, we can have noip that diverges and it`s diverge is not on a limit cycle and it goes. As I`ve shown in the previous parts we can have unstable node or saddle. Click here to see a example of an unstable node that a noip goes to infinity.

If a noip position is on the fixed point, it can stay there and don`t move.

1.1.5 –

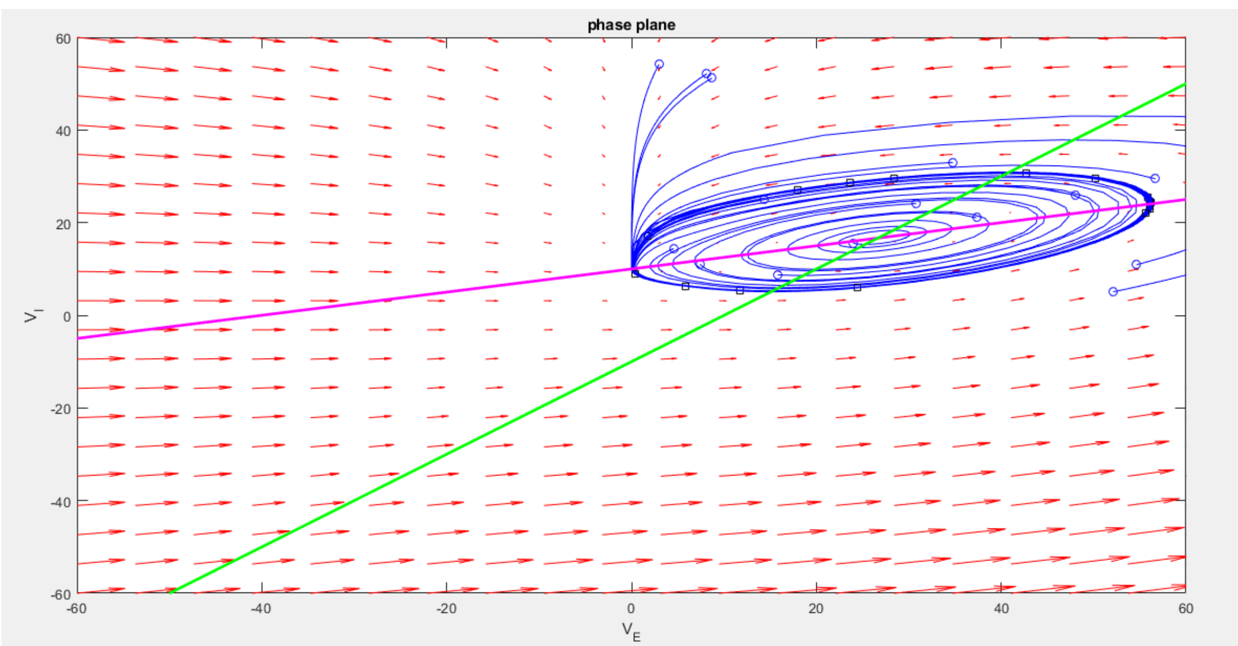
My rectification function is :

$$F(x) = [x]_+ = \begin{cases} x & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

To apply this change to my function handler, I`ve add x to its absolute and divide it to 2 and it will give us F(x).

```
f = @(t,Y) [(-Y(1)+((Mee*Y(1)+Mei*Y(2)-Ye)+abs(Mee*Y(1)+Mei*Y(2)-
Ye))/2)/Te;(-Y(2)+((Mie*Y(1)+Mii*Y(2)-Yi)+abs(Mie*Y(1)+Mii*Y(2)-Yi))/2)/Ti];
```

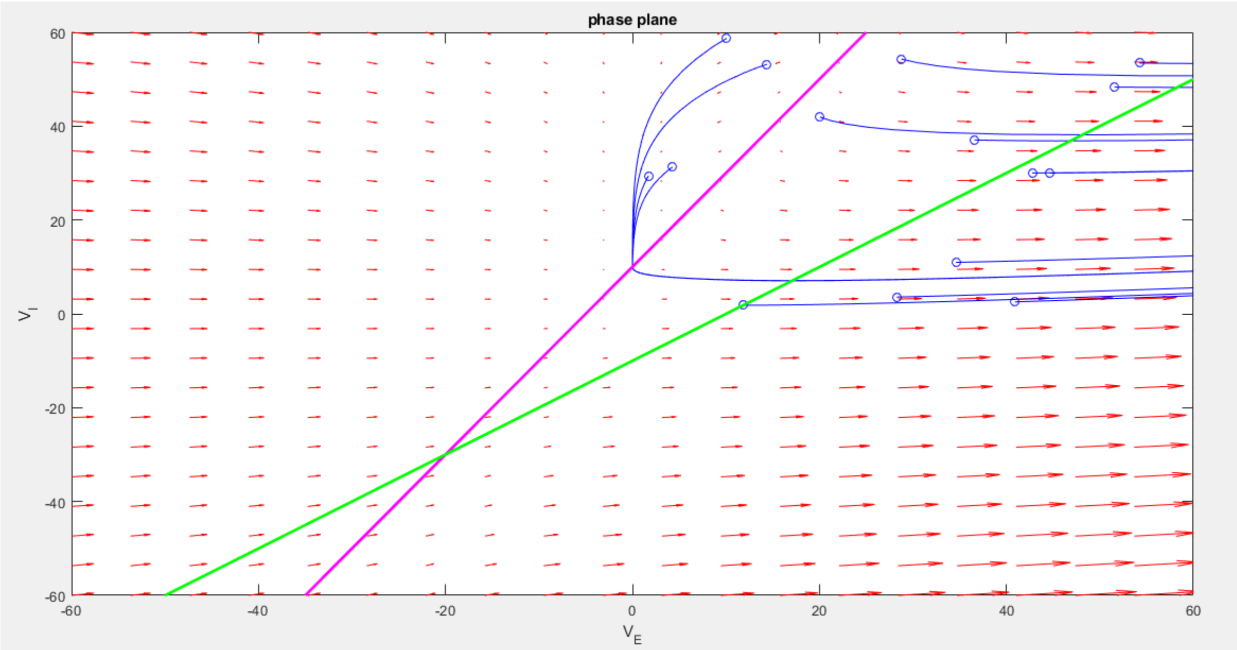
for the initial values in the question: (unstable focus)



We can see here, that our rectifier has stopped the firing rate V_E from getting negative and we have a limit cycle. (M_{EE} = ١.٢٥)

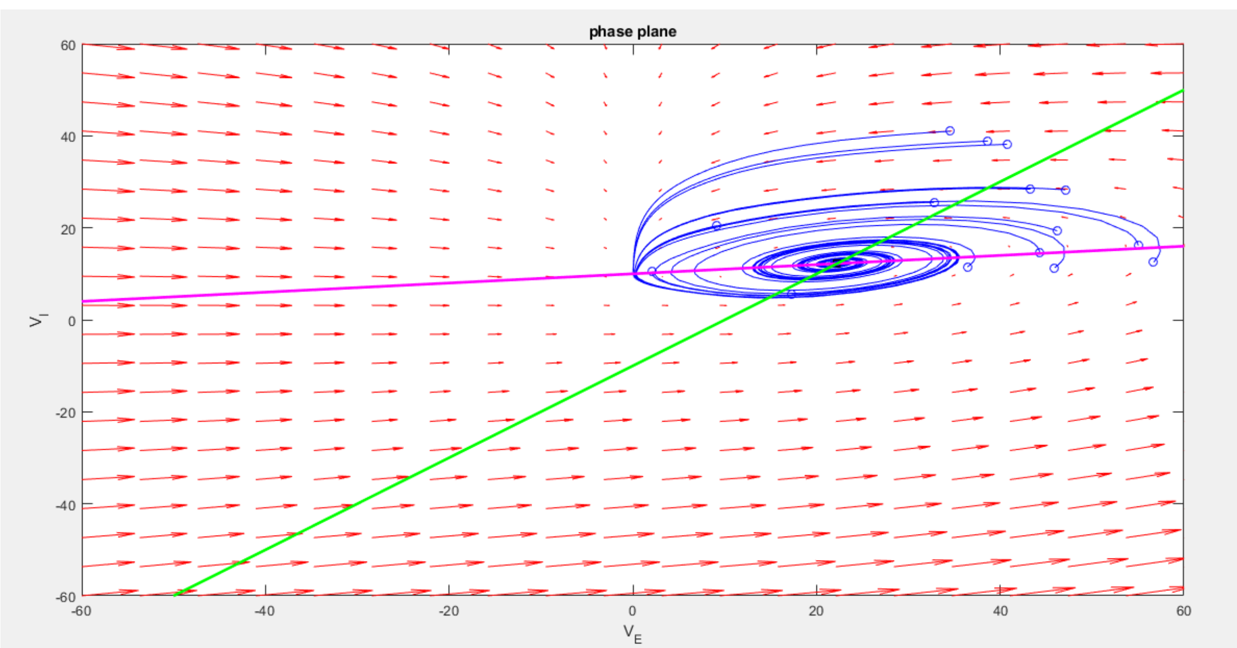
```
Mee = 1.25; Mei = -1; Mie = 1; Mii = 0;
```

- By increasing M_{EE}, we can see that the fixed point will be unstable:

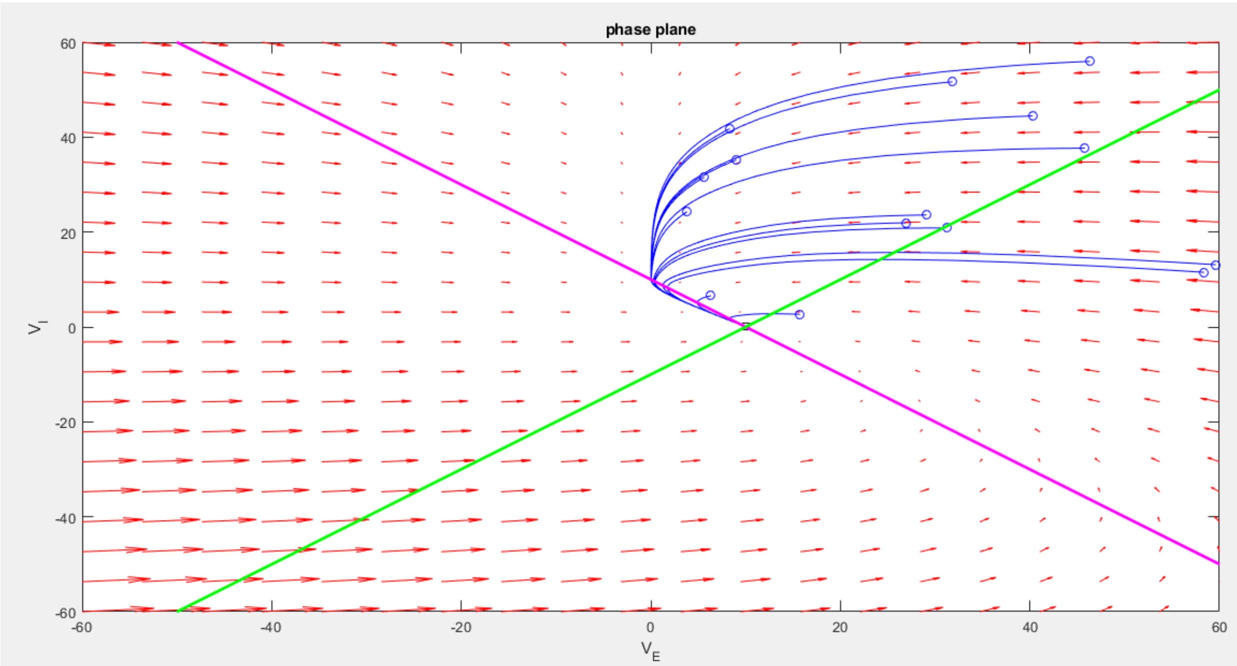


Unstable node(M_{EE} = ٣)

By decreasing M_{EE}, we can have stable node or stable focus:

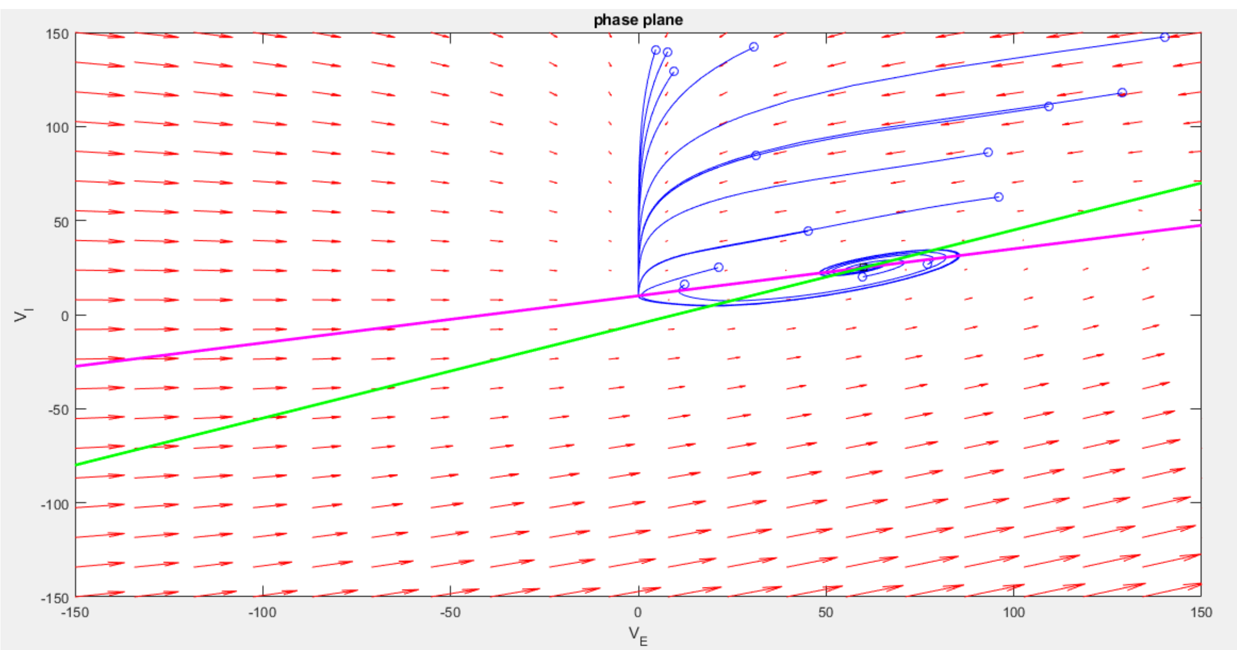


Stable focus(M_{EE} = ١.١)

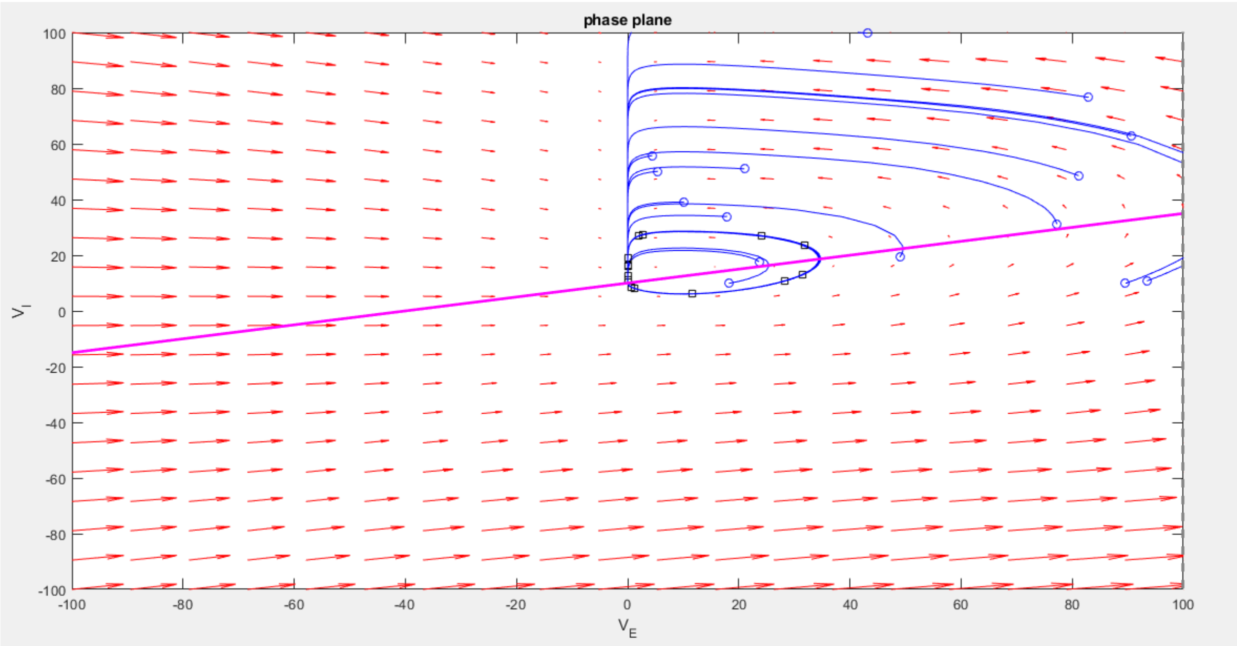


Stable node($M_{EE} = 0$)

- By changing M_{II}:

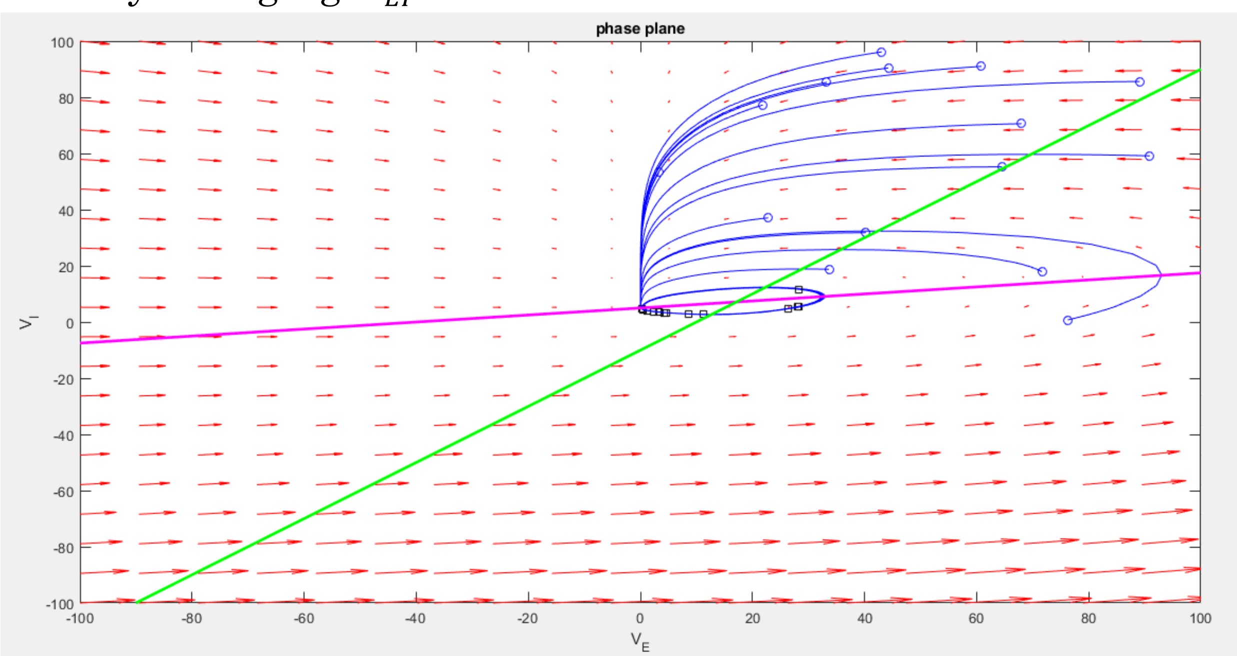


Stable focus($M_{II} = -١$)



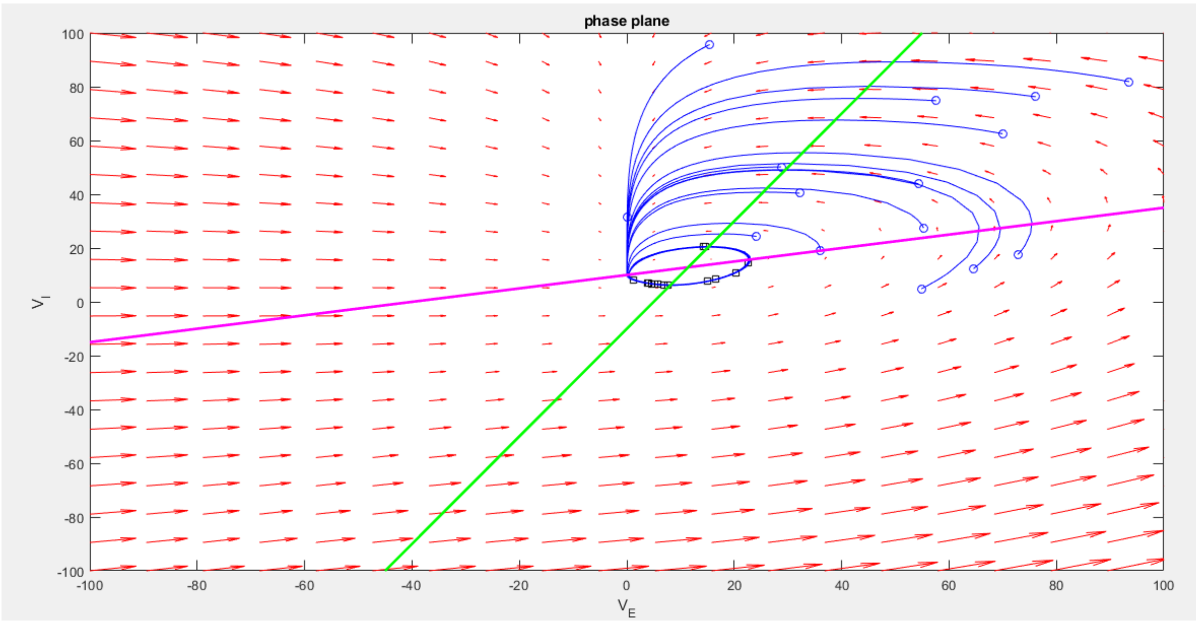
Untable focus(M_{II} = ١)

- By changing M_{EI}:



Untable focus(M_{EI} = −٢)

By changing M_{IE}:



Unstable focus(M_{IE} = ٢)

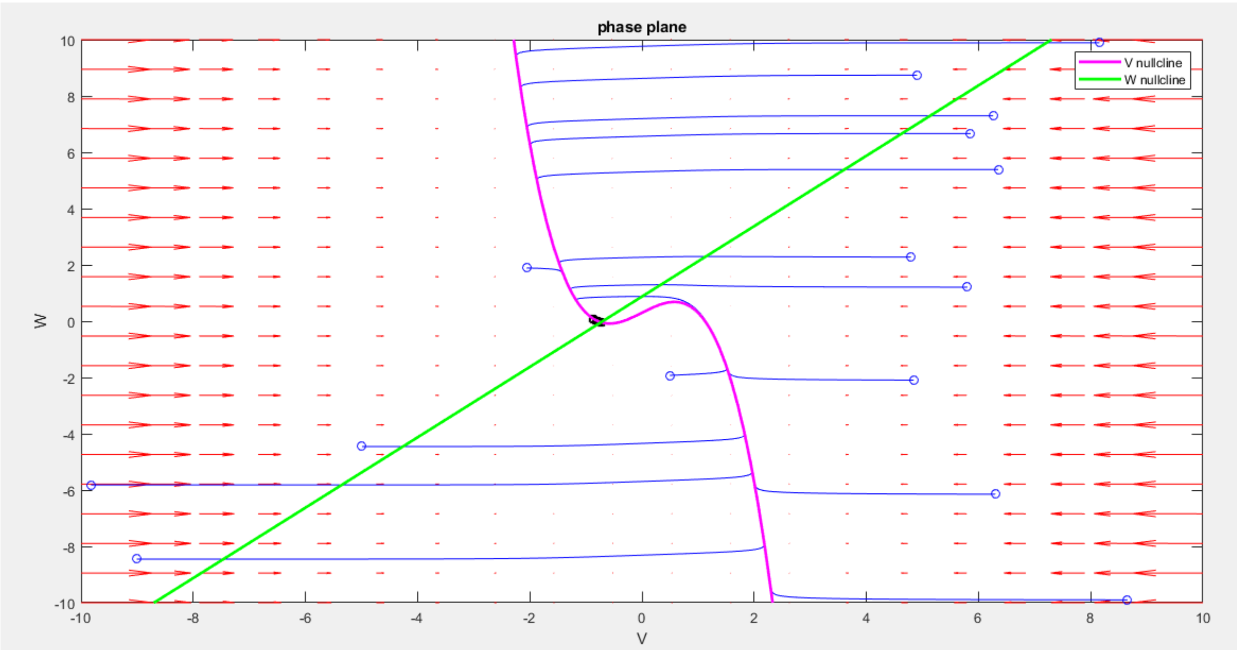
As we can see in all the plots, rectifier has stopped the fire rate from getting negative.

We can have unstable node and saddle by changing M_{IE} and M_{EI} in here too but I don`t bring them because it`s the same as part2 and 3 with just a rectifier applied to it. We can have an unstable and a stable node, too.

1- Phase Planes & Limit Cycles: Second Neuron model

1.2.1 –

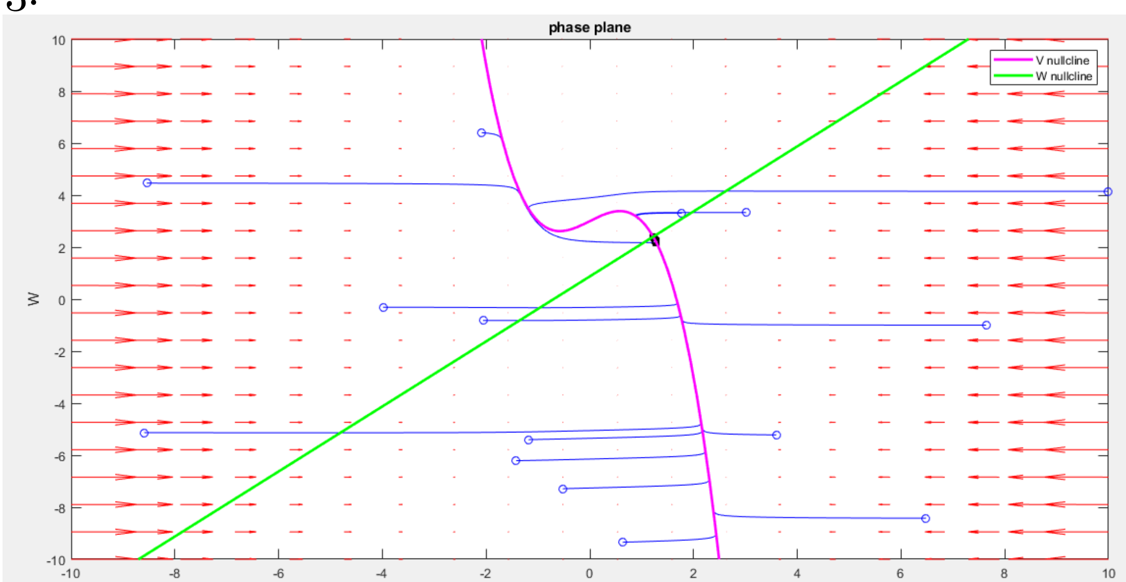
By changing the f function and nullclines our phase plane would be like this:(stable)



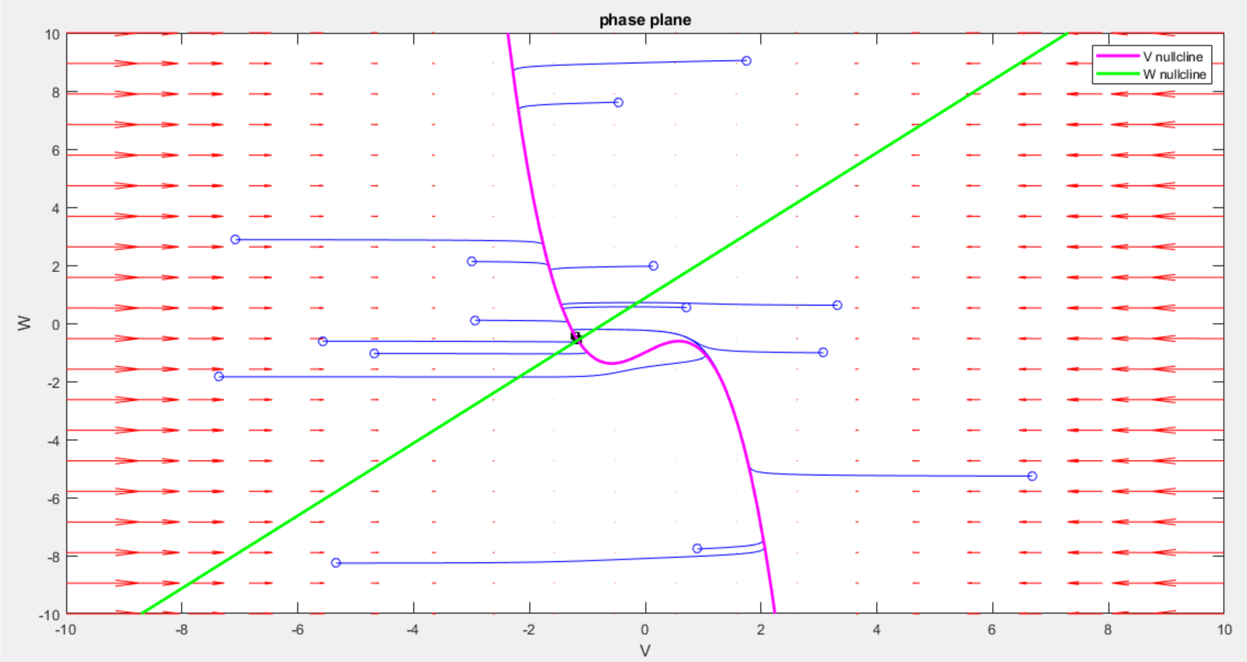
1.2.2 –

By increasing/decreasing the value of I, v nullcline shifts up/down! But remains stable.

I = 3:



I = -1:



1.2.3 –

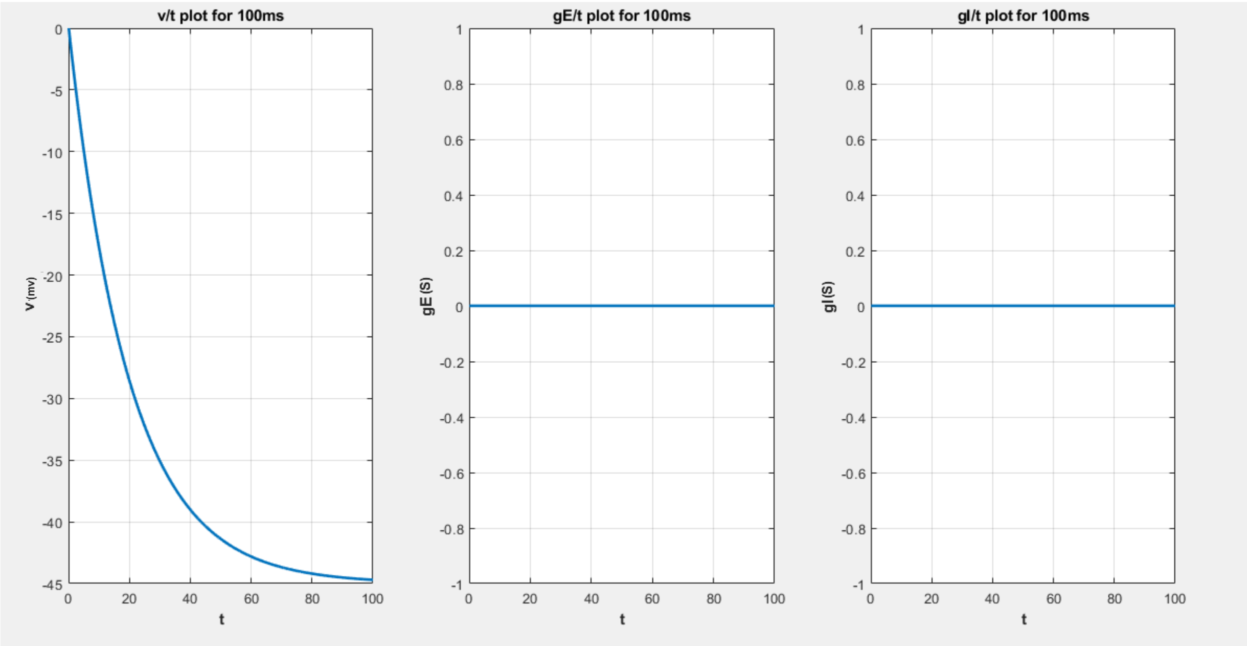
No, this model doesn't have an oscillatory behavior as we can see from the previous plots.

2- Integrate and Fire model:

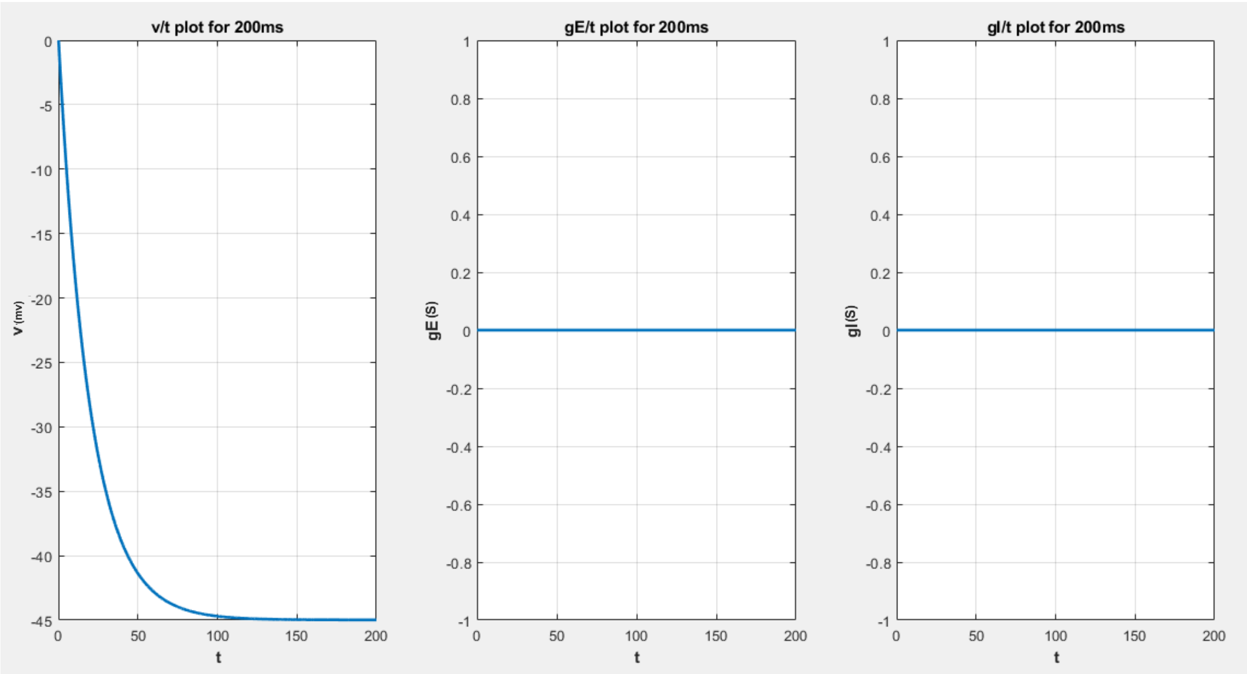
2.1.1 –

We solve the differential equations by **Euler Method**.

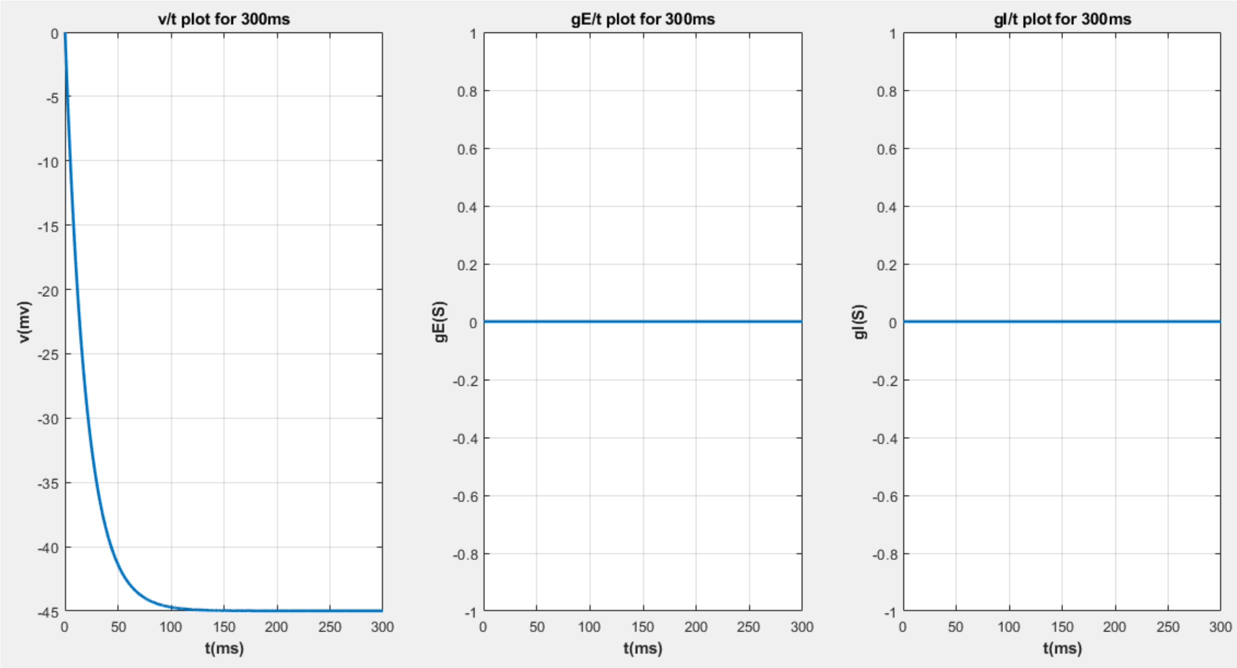
v(t)/g_e(t)/ g_i(t) for 100ms:



v(t)/g_e(t)/ g_i(t) for 200ms:



v(t)/g_e(t)/ g_i(t) for 300ms:



$$\frac{dv}{dt} = (g_l(E_l - v) + g_e(E_e - v) + g_i(E_i - v) + I_{ex})/C_m$$

$$\frac{dg_e}{dt} = -g_e/tau_e$$

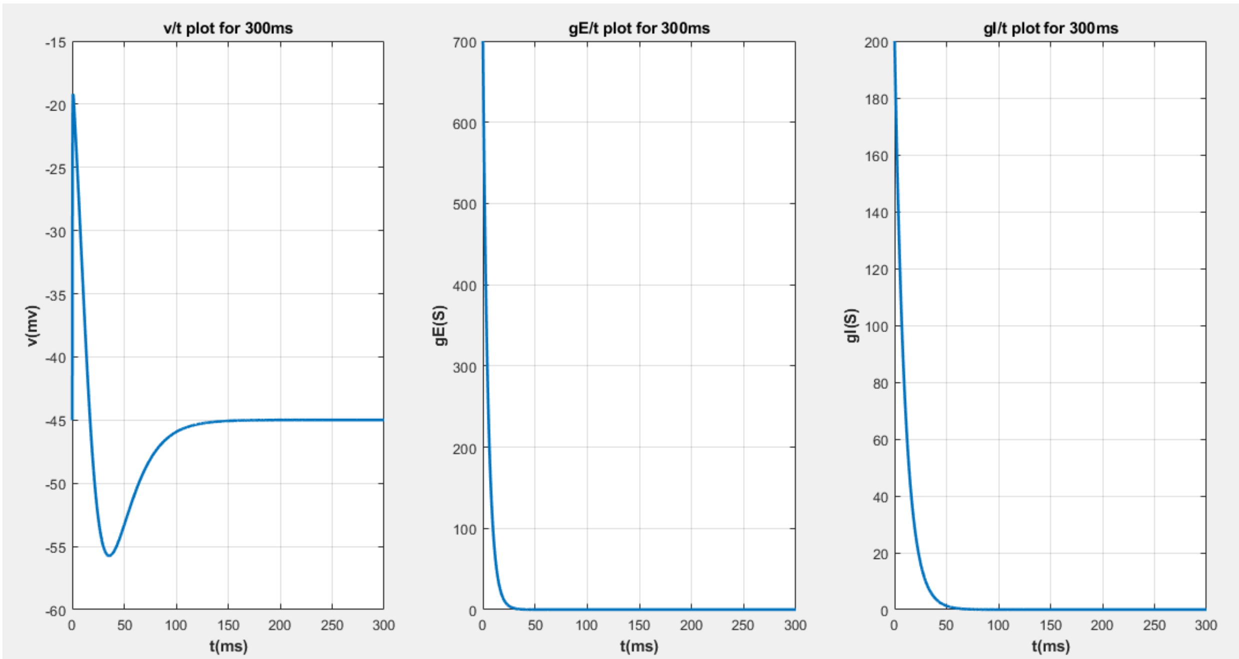
$$\frac{dg_i}{dt} = -g_i/tau_i$$

Because of the initial states that are zero, g_e and g_i will be always zero. So, the equation of v would be $\frac{dv}{dt} = \frac{(g_l(E_l - v) + I_{ex})}{C_m}$. Answer of this equation is some exponential function with a negative power and so it will converges to zero as we can see in the simulations too. $\boldsymbol{g_i}$ and $\boldsymbol{g_e}$ are zero so they behave linearly.

2.1.2 –

g_i and g_e`s dimension is **mho or S**, so they`re conductance and when they`re zero, it means the channels are closed and so the neuron won`t spike.

2.1.3 –

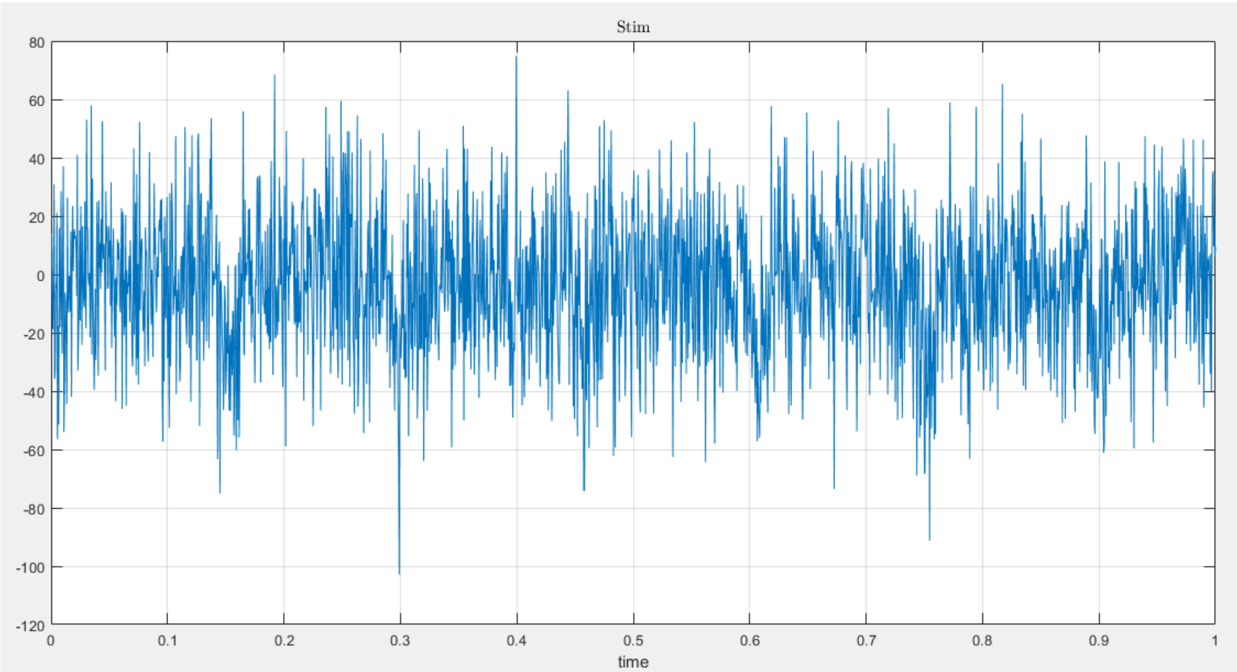


Here initial value of g_e is 800 nS, for g_i is 200nS and for voltage is -45mV. We can see g_i and g_e are decreasing exponentially as we expected from their equations. We can see the voltage plot showing a **spike**, because at first g_i and g_e have a initial high value and it will makes the channels open and the neuron will spike and after that, neuron will go to rest state and the channels will be closed as we`re seeing in the plots.

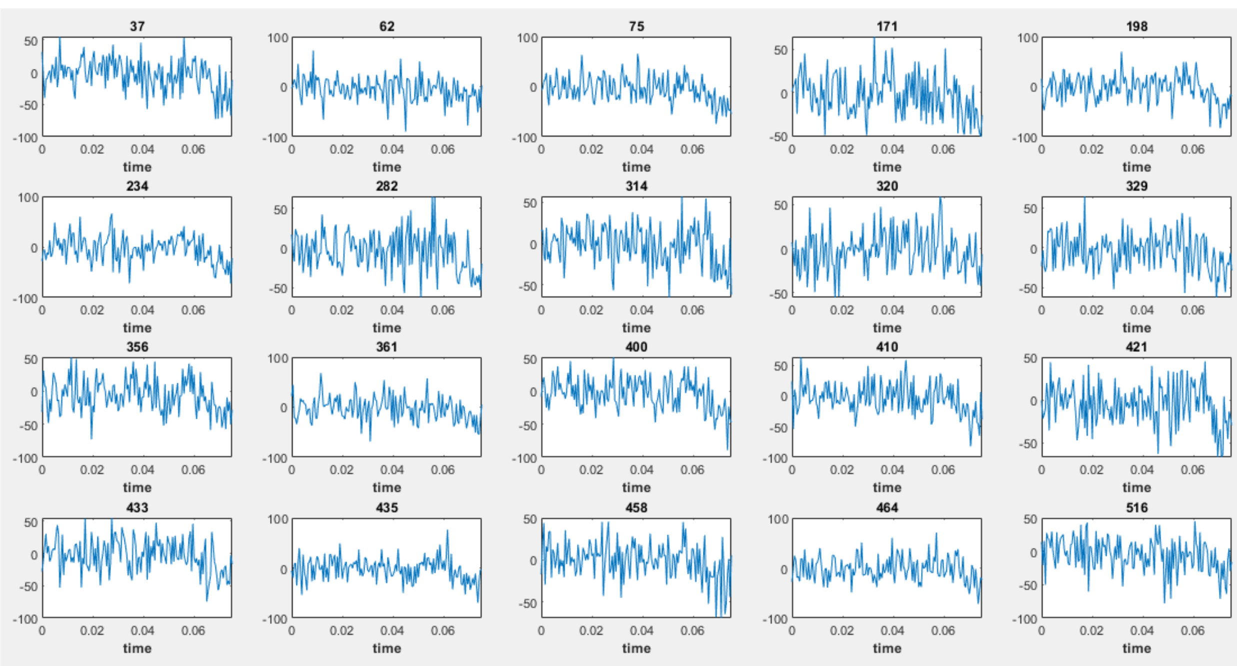
3- Spike-Triggered Average:

3.1 –

first second of Stimulus signal:



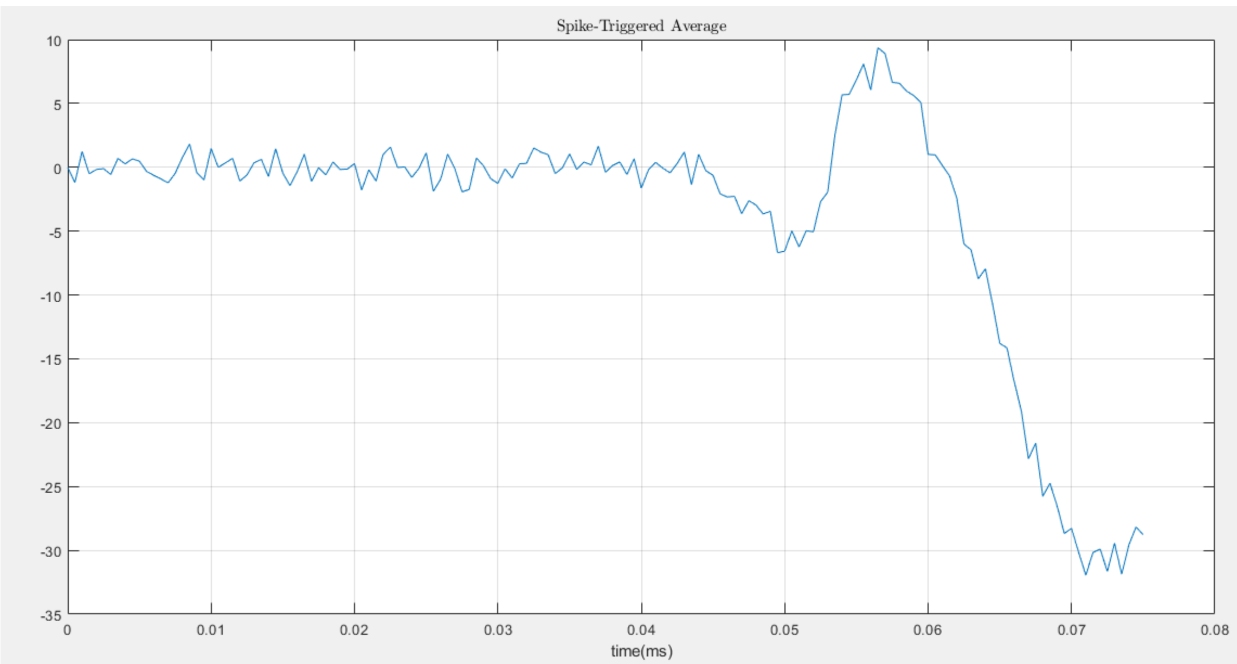
3.2 –



We have 75ms of stimulus signal before 20 times of neuron spiking that has been randomly chosen throw out 598 spikes.

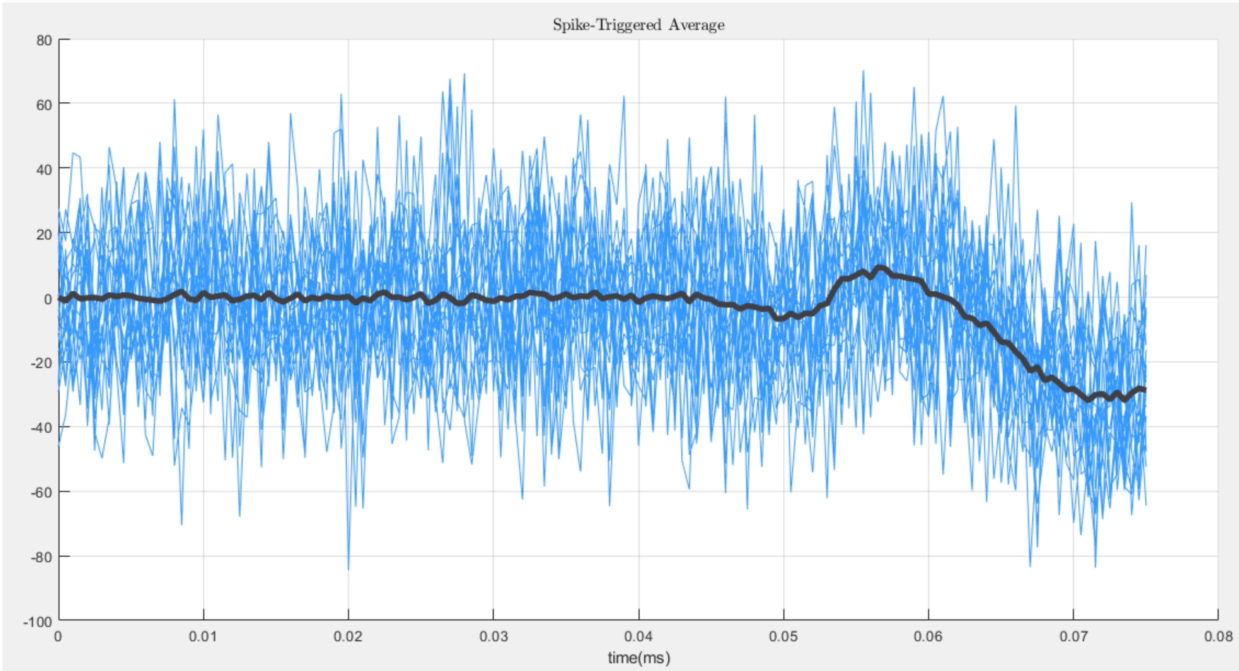
It can be concluded from the plots that during the 75ms before the neuron spikes, the stimulus signal is at first around zero about 50ms. It goes up and down but it doesn't have the value to make the neuron spikes. But after that its value will increase and makes the neuron spike and after the spike it`s value will decrease as we can see and it goes to negative.

3.3 –



As we can see, the Spike-Triggered average is showing us the pattern that we were seeing in 20 spikes in the previous part. At first the average of stimulus signal is around zero and after that it will reach a threshold and the neuron spikes and then it decreases.

3.4 and 3.5 –

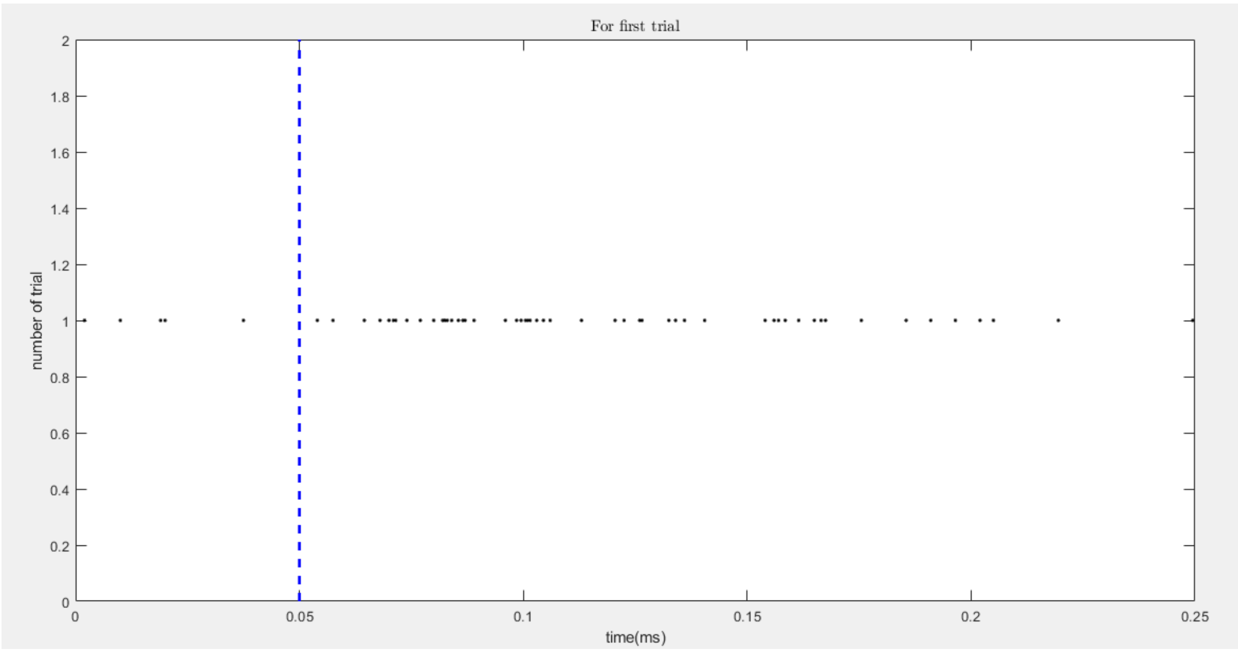


Here we can see the spike of the neuron and the stimulus signal before 20 spikes and we can see the average of them that is changing like the spike-triggered average. So the estimated time for the neuron to spike is about 50ms. Actually, the neuron will spike when all the pre-synaptic neurons spike and it takes about 50ms. In the first 50ms, all neurons won`t spike together and that`s why we don`t have the Spike triggered average sooner.

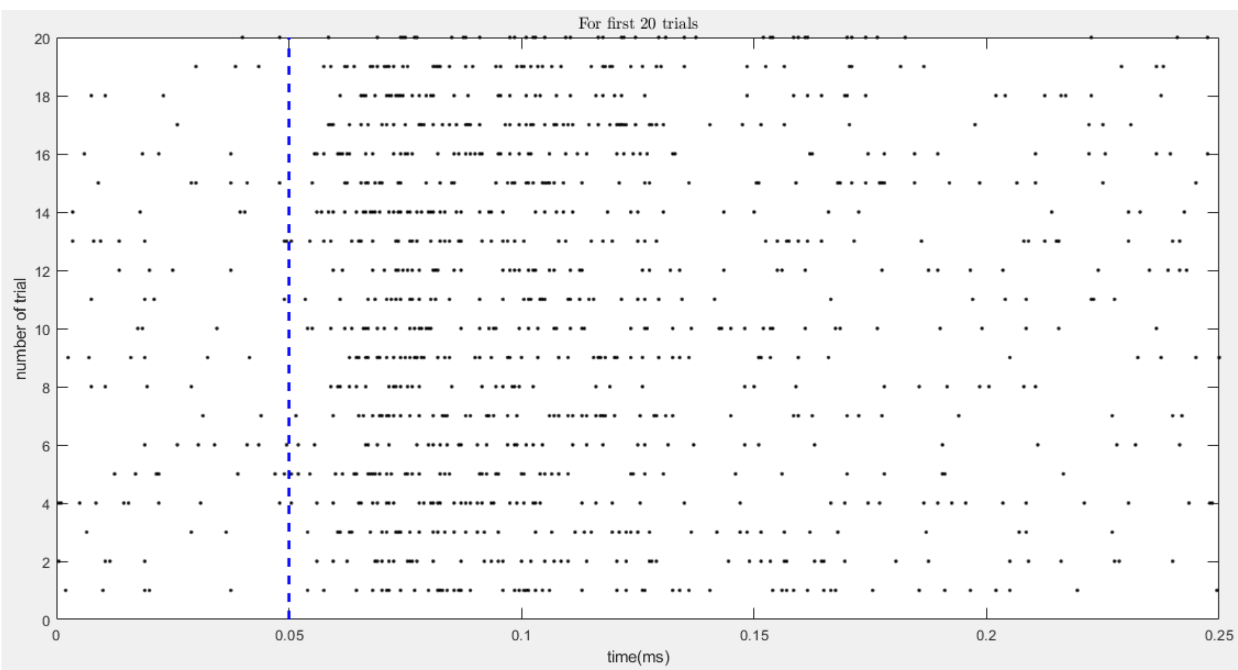
4- Raster Plot and PETH:

4,1 –

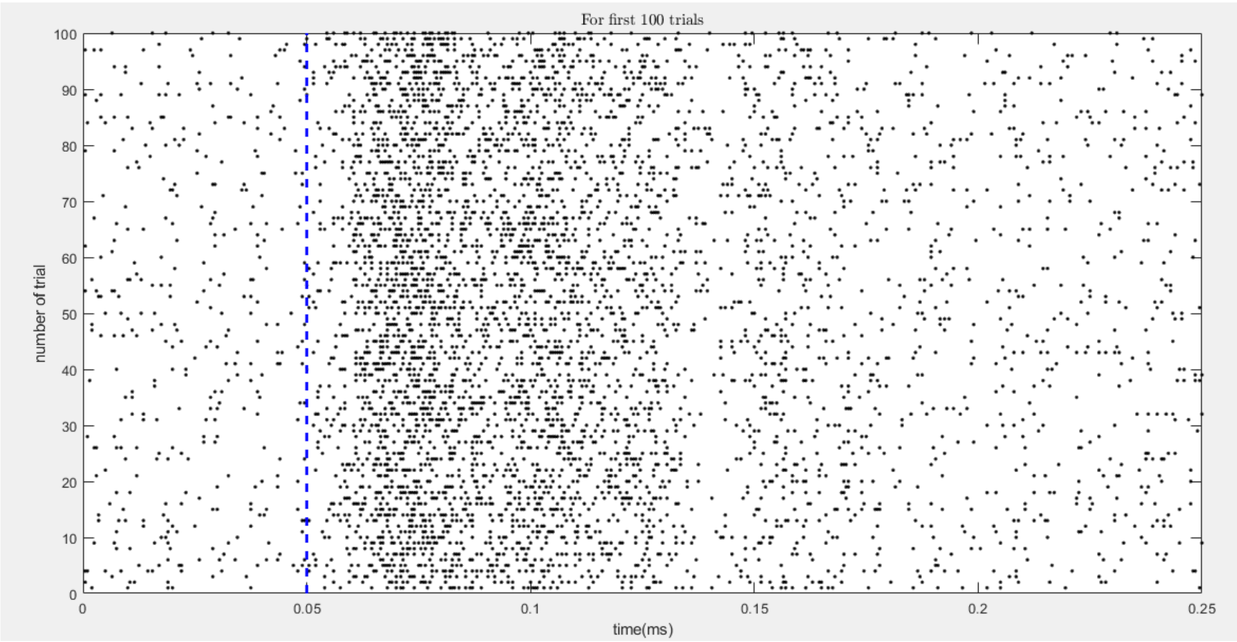
-for first trial:



-for first 20 trials:



-for all trials:



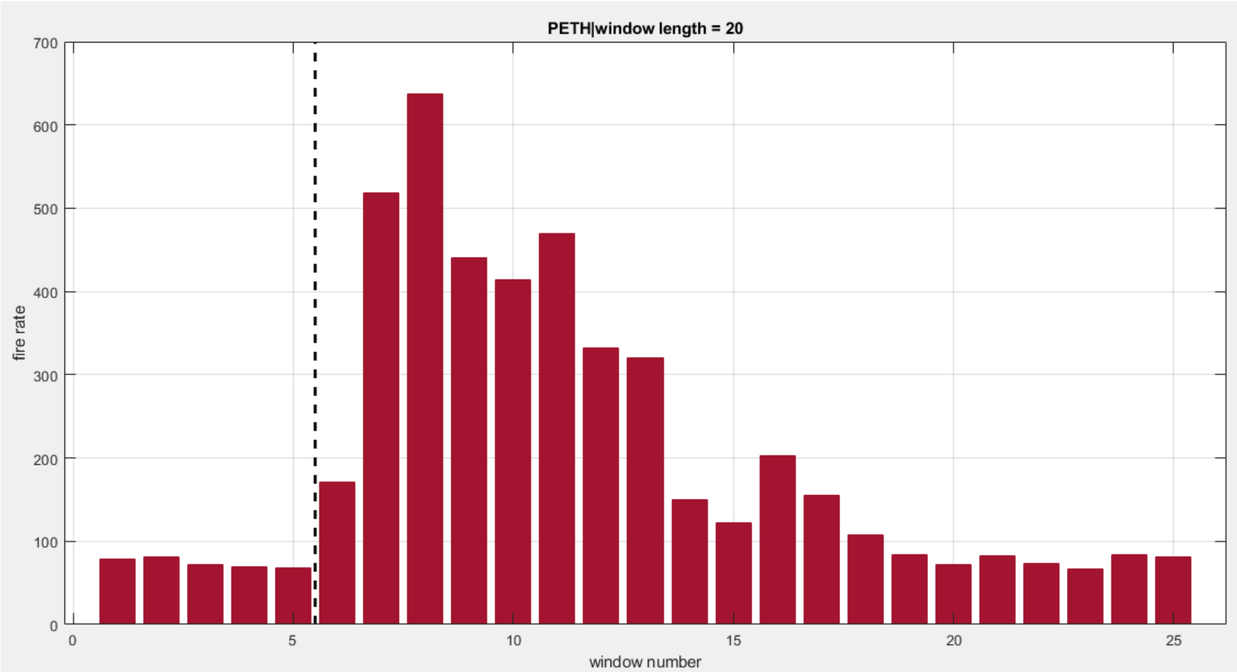
As the question is saying we have 50ms before the stimulation and 200ms after it. So we expect to have a lot of spikes in all the trials around 50ms. As the plot is showing, we have a lot of spikes around 50ms which is around the stimulation time. This high density of spikes remains up to 130ms and after that our neurons will go to rest state and number of spikes decreases.

4,2 –

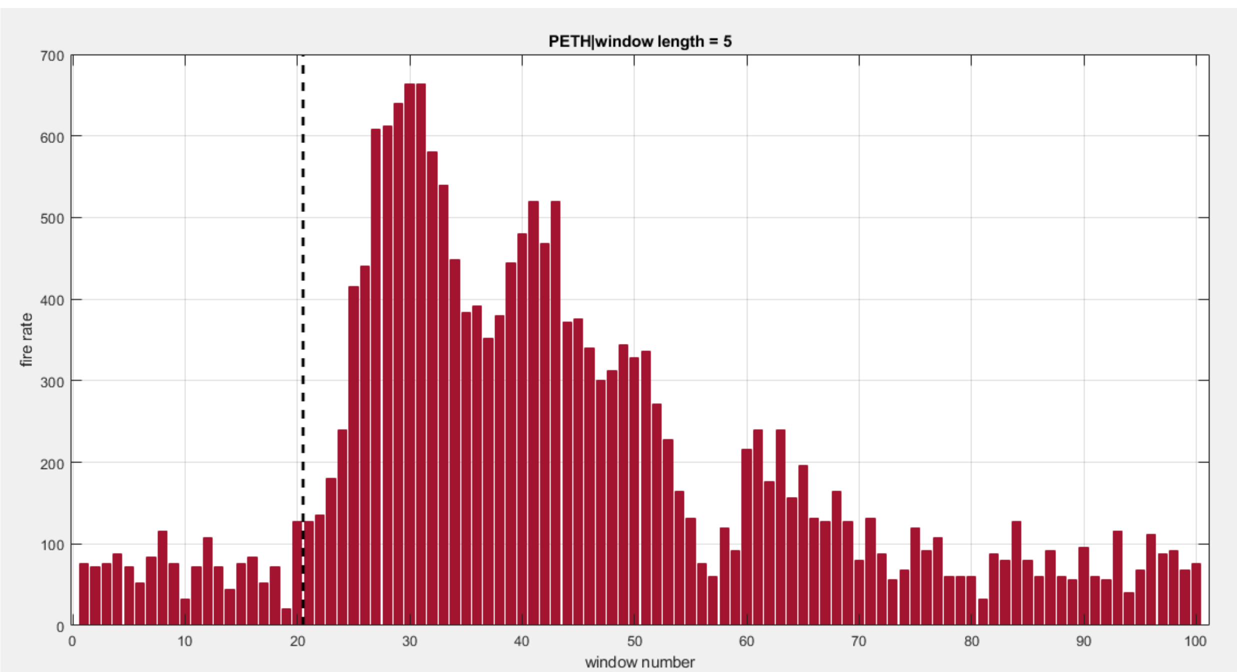
We want to model neurons using their firing rate instead of modeling them with their spikes` number. This method has a greater efficiency and scales well to large networks. Also, modeling networks using spikes will make equations, so hard to solve. In order to model neurons using their firing rate, we use a rectangular moving window and count number of spikes and then calculate the fire rate base on the window length. This is actually a low-pass filter on our spike trains.

4,3_4,4 –

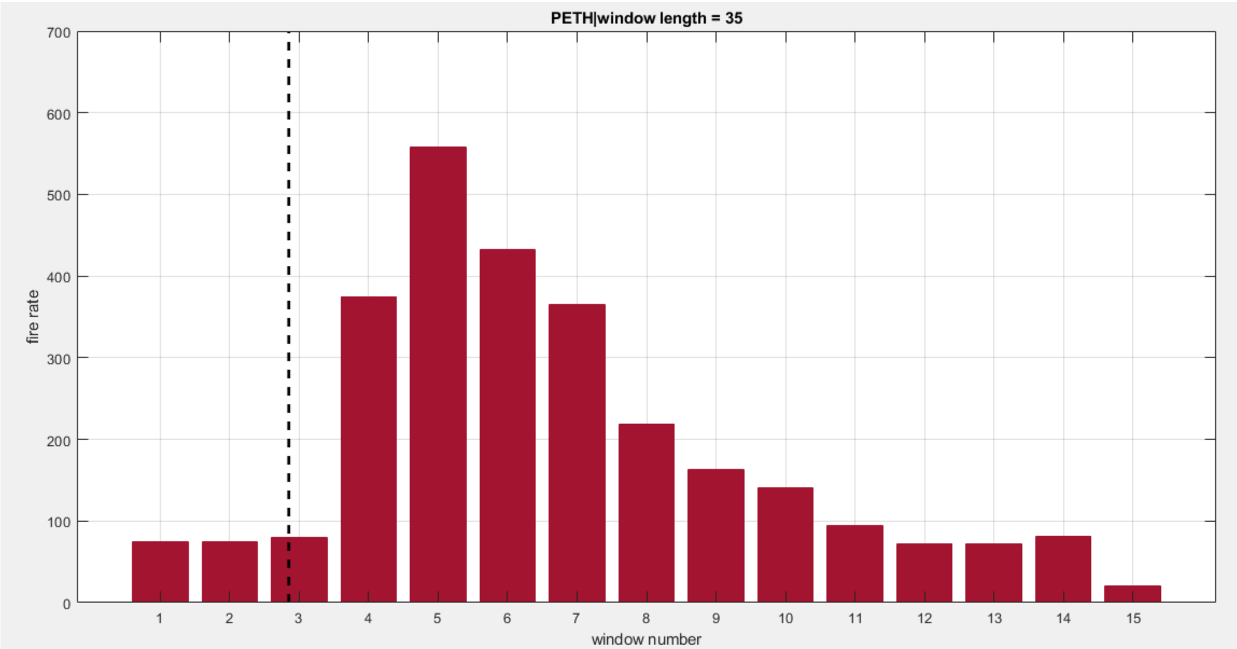
window length = 20 samples or 10ms



window length = 5 samples or 2.5ms



window length = 35 samples or 17.5ms



As we can see by increasing the window length, will decrease the accuracy of fire rates. Maximum fire rate for window_L 35 is about 559 and for window_L 20 is 637 and for 5 is 644. When we have a smaller window, we can have a better visualization of the firing rates. The window length shouldn`t be very small, because for example, the window will just contain one spike and the firing rate would be 1000 and maybe the firing rate around there is not this much. So window length shouldn`t be very small or very big.

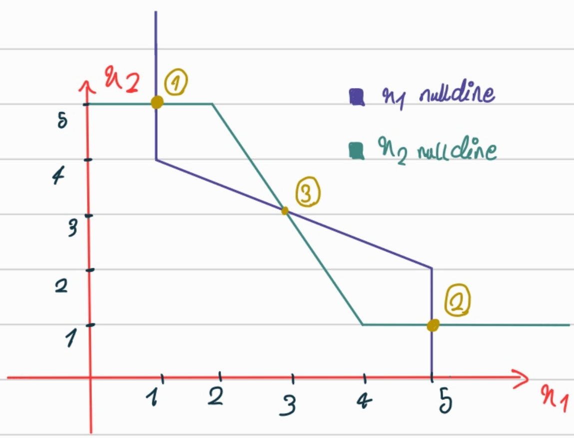
٥- Dynamic of SR Latch system:

5,1 –

Question_5_SR Latch

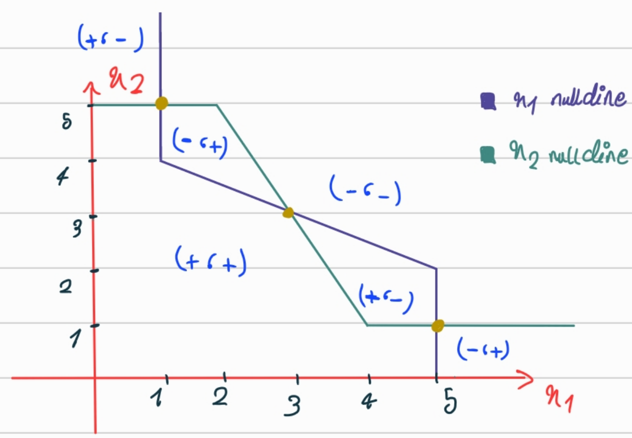
5.1: $\dot{x}_1 = 0 \rightsquigarrow$ nullcline @ x_1 : $NOT(x_2) - x_1 = 0$

$\dot{x}_2 = 0 \rightsquigarrow$ nullcline @ x_2 : $NOT(x_1) - x_2 = 0$



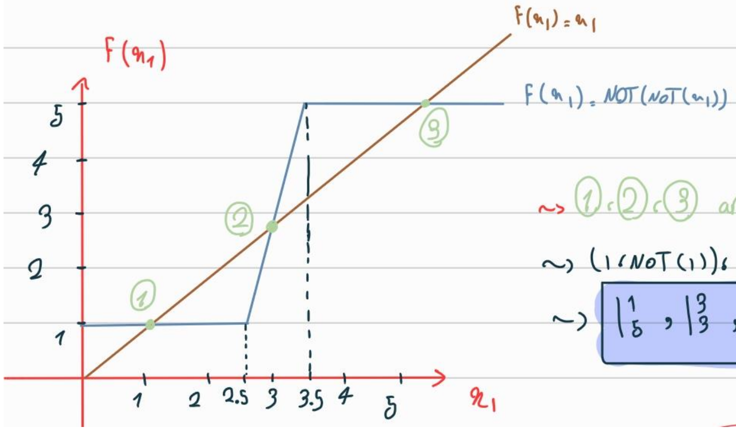
* ① $\Big|^{1}_{5}$ and ② $\Big|^{3}_{3}$ and ③ $\Big|^{5}_{1}$

are the fixed points of this system.



۵,۲

$$\left.\begin{aligned} &\text{nullcline @ } x_1 : NOT(x_2) - x_1 = 0 \\ &\text{nullcline @ } x_2 : NOT(x_1) - x_2 = 0 \end{aligned}\right\} \Rightarrow x_1 = NOT(NOT(x_1)) = F(x_1)$$

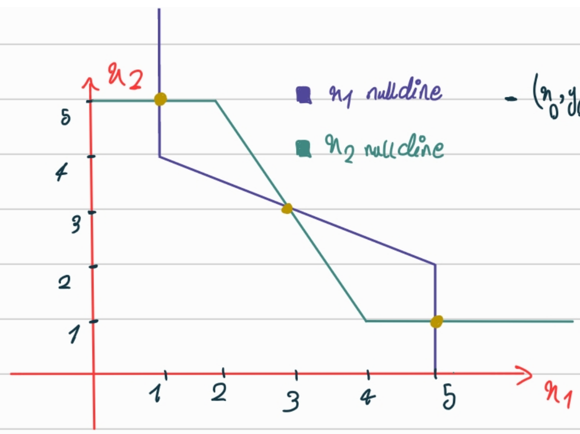


⇝ ①, ②, ③ are the roots of this equation.

⇝ $(1, NOT(1)), (3, NOT(3)), (5, NOT(5))$

⇝ $\boxed{\begin{vmatrix}1\\5\end{vmatrix}, \begin{vmatrix}3\\3\end{vmatrix}, \begin{vmatrix}5\\1\end{vmatrix}}$ ⇝ As we calculated in 5.1 ☑

۵,۳ –



- $(x_0, y_0) \begin{vmatrix}5\\1\end{vmatrix}, (x_1, y_1) \begin{vmatrix}3\\3\end{vmatrix}, (x_2, y_2) \begin{vmatrix}1\\5\end{vmatrix}$

- $\dot{x}_1 = NOT(x_2) - x_1 = f(x_1, x_2)$
- $\dot{x}_2 = NOT(x_1) - x_2 = g(x_1, x_2)$

①

$$f(x_1, x_2) \simeq a(x_1 - 5) + b(x_2 - 1), \quad a = \frac{\partial f}{\partial x_1}\Big|_{(x_0, y_0)}, \quad b = \frac{\partial f}{\partial x_2}\Big|_{(x_0, y_0)}$$

$$g(x_1, x_2) \simeq c(x_1 - 5) + d(x_2 - 1), \quad c = \frac{\partial g}{\partial x_1}\Big|_{(x_0, y_0)}, \quad d = \frac{\partial g}{\partial x_2}\Big|_{(x_0, y_0)}$$

$$\xrightarrow[x_2 - 1 = w]{x_1 - 5 = u} \begin{cases} \dot{u} = au + bw \\ \dot{w} = cu + dw \end{cases} \rightsquigarrow \begin{bmatrix}\dot{u}\\ \dot{w}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}u\\ w\end{bmatrix}$$

$$L = \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \rightsquigarrow \det(L - \lambda I) = 0 \rightsquigarrow (-1-\lambda)(-1-\lambda) = 0$$

$$\rightsquigarrow \boxed{\lambda_{1,2} = -1} \rightsquigarrow \underline{\text{stable}}$$

۲

$$f(x_1, x_2) \simeq a(x_1 - 3) + b(x_2 - 3) \quad , \quad a = \frac{\partial f}{\partial x_1}\Big|_{(x_0, y_0)} , \; b = \frac{\partial f}{\partial x_2}\Big|_{(x_0, y_0)}$$

$$g(x_1, x_2) \simeq c(x_1 - 3) + d(x_2 - 3) \qquad c = \frac{\partial g}{\partial x_1}\Big|_{(x_0, y_0)} , \; d = \frac{\partial g}{\partial x_2}\Big|_{(x_0, y_0)}$$

$$\xrightarrow[x_2 - 3 = w]{x_1 - 3 = u} \begin{cases} \dot{u} = au + bw \\ \dot{w} = cu + dw \end{cases} \rightsquigarrow \begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}$$

$$L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} \rightsquigarrow \det(L - \lambda I) = 0 \rightsquigarrow (-1-\lambda)(-1-\lambda) - 4 = 0$$

$$\rightsquigarrow (-1-\lambda) = \pm 2 \rightsquigarrow \boxed{\lambda_1 = 1} , \boxed{\lambda_2 = -3}$$

unstable ← Saddle

۳

$$f(x_1, x_2) \simeq a(x_1 - 1) + b(x_2 - 5) \quad , \quad a = \frac{\partial f}{\partial x_1}\Big|_{(x_0, y_0)} , \; b = \frac{\partial f}{\partial x_2}\Big|_{(x_0, y_0)}$$

$$g(x_1, x_2) \simeq c(x_1 - 1) + d(x_2 - 5) \qquad c = \frac{\partial g}{\partial x_1}\Big|_{(x_0, y_0)} , \; d = \frac{\partial g}{\partial x_2}\Big|_{(x_0, y_0)}$$

$$\xrightarrow[x_2 - 5 = w]{x_1 - 1 = u} \begin{cases} \dot{u} = au + bw \\ \dot{w} = cu + dw \end{cases} \rightsquigarrow \begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}$$

$$L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \rightsquigarrow \det(L - \lambda I) = 0 \rightsquigarrow (-1-\lambda)(-1-\lambda) = 0$$

$$\rightsquigarrow \boxed{\lambda_{1,2} = -1} \rightsquigarrow \boxed{\text{Stable}}$$

٦- Limit Cycles: $\begin{cases}\dot{x} = x + y - x(x^2+y^2)\\ \dot{y} = -(x-y) - y(x^2+y^2)\end{cases}$

- Fixed points and their stability

$\dot{x}=0= x+y-x(x^2+y^2)\ ,\quad \dot{y}=0=-(x-y)-y(x^2+y^2)$

$\rightsquigarrow x^2+y^2=\frac{x-y}{-y} \rightsquigarrow 0=x+y+x\left(\frac{x-y}{y}\right) \rightsquigarrow 0=xy+y^2+x^2-xy \rightsquigarrow \boxed{y^2=-x^2}$

$\longrightarrow$ (0,0) is our only fixed point

$L=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} \rightsquigarrow (1-\lambda)^2+1=0 \rightsquigarrow \lambda^2-2\lambda+2=0$

$\rightsquigarrow \lambda_{1,2}=\frac{2\pm 2i}{2} \rightsquigarrow \boxed{\begin{matrix}\lambda_1=1+i\\ \lambda_2=1-i\end{matrix}} \longrightarrow$ unstable focus

- Limit cycle:

We use polar coordinates:

$\rightsquigarrow \dot{r}\cos(\theta)-r\dot{\theta}\sin(\theta)=r\cos\theta+r\sin\theta-r\cos\theta\,(r^2) \xrightarrow{\times\cos\theta}$

$\xrightarrow{+}$

$\rightsquigarrow \dot{r}\sin\theta+r\dot{\theta}\cos(\theta)=-r\cos\theta+r\sin\theta-r\sin\theta\,(r^2) \xrightarrow{\times\sin\theta}$

$\rightsquigarrow \dot{r}(\cos^2\theta+\sin^2\theta)=r(\cos^2\theta+\sin^2\theta)-r^3(\cos^2\theta+\sin^2\theta) \rightsquigarrow \boxed{\dot{r}=r-r^3}$

$\xrightarrow{r>0}$

	-1		0		+1	
+		-		+		-

$\rightsquigarrow$ So we have a limit cycle with $r=1$

- Movement on limit cycle:

$$\rightsquigarrow \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta) = r\cos\theta + r\sin\theta - r\cos\theta(r^2) \xrightarrow{\times \sin\theta}$$

$$\xrightarrow{-}$$

$$\rightsquigarrow \dot{r}\sin\theta + r\dot{\theta}\cos(\theta) = -r\cos\theta + r\sin\theta - r\sin\theta(r^2) \xrightarrow{\times \cos\theta}$$

$$-r\dot{\theta}\sin^2\theta - r\dot{\theta}\cos^2\theta = r\sin^2\theta + r\cos^2\theta \rightsquigarrow \boxed{\dot{\theta} = -1}$$

$\rightsquigarrow$ we move with the speed of 1 Rad/s in clockwise.