1. ELS NOMBRES

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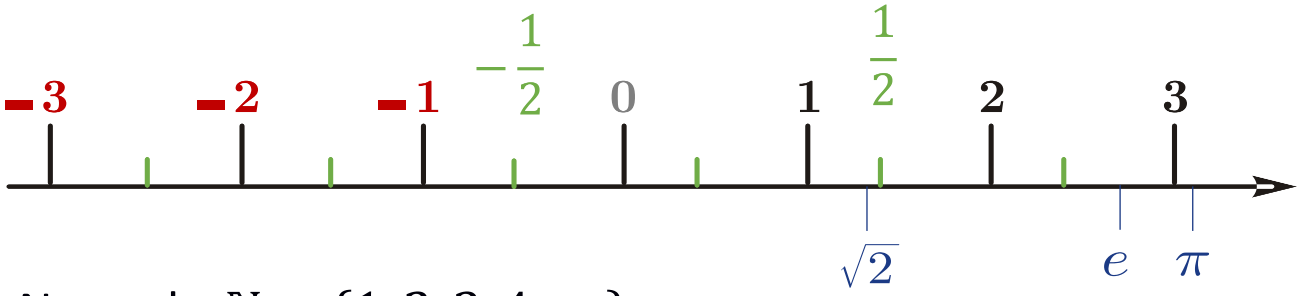
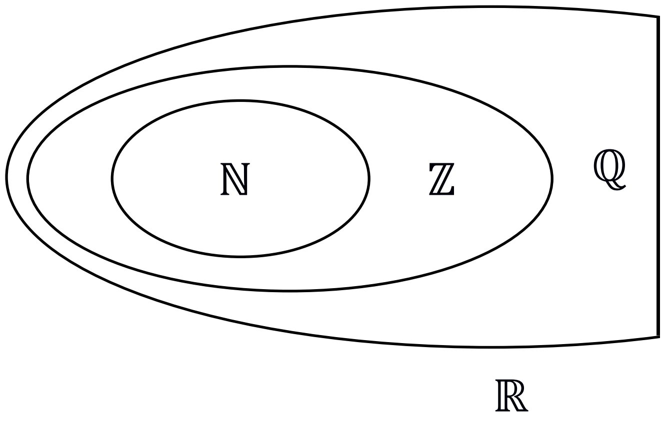
Setembre 2022

Continguts

1.1 TIPUS DE NOMBRES





- Naturals $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$
- Enters $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, \underbrace{1, 2, 3, 4, \cdots}_{\mathbb{N}}\}$ $\quad \mathbb{N} \subset \mathbb{Z}$ Els nombres naturals són un subconjunt dels enters
- Racionals $\mathbb{Q} = \left\{\frac{a}{b} \text{ tal que } a, b \in \mathbb{Z}, b \neq 0\right\}$ $\quad$ Ex: $\frac{1}{2}, \frac{1}{3}, \frac{-1}{2}, \underbrace{\frac{4}{2}}_{\mathbb{N}}, \underbrace{\frac{-6}{3}}_{\mathbb{Z}}$ $\quad \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$

poden expressar-se en forma decimal: $\begin{cases} \text{limitada } 1'6 = \frac{16}{10} = \frac{8}{5} \\ \text{il·limitada i periòdica: } \begin{cases} \text{periòdics purs} & 2'\widehat{31} = 2{,}313131 \\ \text{periòdics mixtes} & 5'2\widehat{41} = 5'24141 \end{cases} \end{cases}$

- Irracionals: aquells nombres amb expressió decimal il·limitada i no periòdica

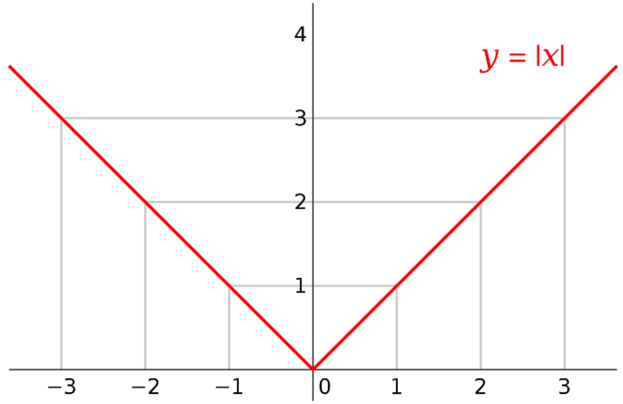
Ex: $\sqrt{2} = 1.41421356237\cdots$ $\quad \pi = 3'141592654\cdots$ $\quad e = 2'7182818\cdots$

- Reals $\mathbb{R}$: el conjunt format per tots els nombres racionals i irracionals
- Complexos $\mathbb{C}$

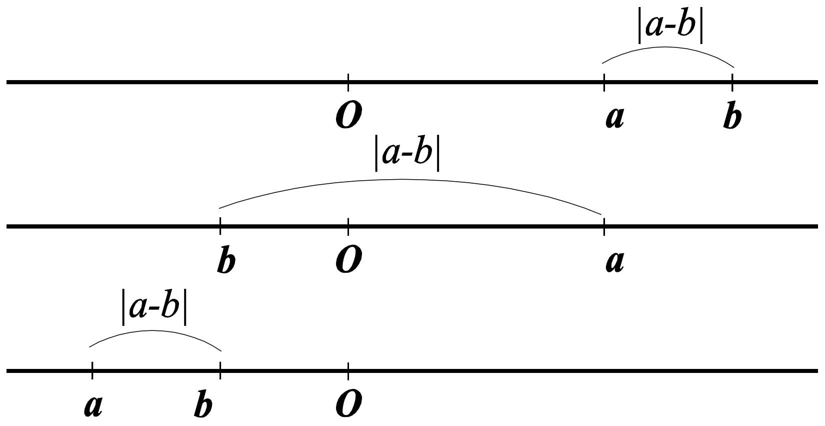
1.2 VALOR ABSOLUT D'UN NOMBRE REAL

Donat un nombre real x, definim el seu valor absolut, denominat per |x|, com:

$$|x| = \begin{cases} x & si\ x \geq 0, \\ -x & si\ x < 0. \end{cases}$$



$|a - b|$ representa la distancia que separa dos punts a i b



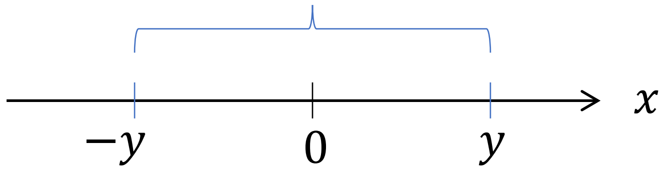
Propietats

∀ per qualsevol
∈ pertany
ℝ reals

1) $|x| \geq 0 \quad \forall x \in \mathbb{R}$

a més, $|x| = 0 \iff x = 0$

2) $|x| \leq y \Leftrightarrow -y \leq x \leq y$



3) $|x + y| \leq |x| + |y|$;

$|x - y| \geq \big||x| - |y|\big|$

4) $|x \cdot y| = |x| \cdot |y|$;

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \quad si\ \ y \neq 0$$

Intervals

a) Intervals acotats

Donats $a, b \in \mathbb{R}$

$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ Interval obert

$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ Interval tancat

$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

b) Intervals no acotats

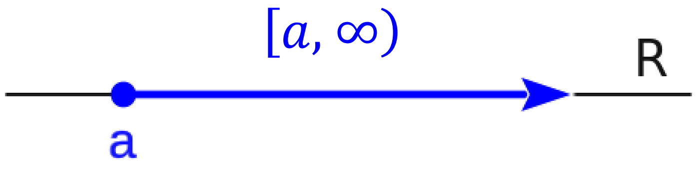
Donat $a \in \mathbb{R}$

$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$

$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$

$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$

$(-\infty,a] = \{x \in \mathbb{R} \mid x \leq a\}$



$\in$ pertany
$\mathbb{R}$ reals
$|$ tal que

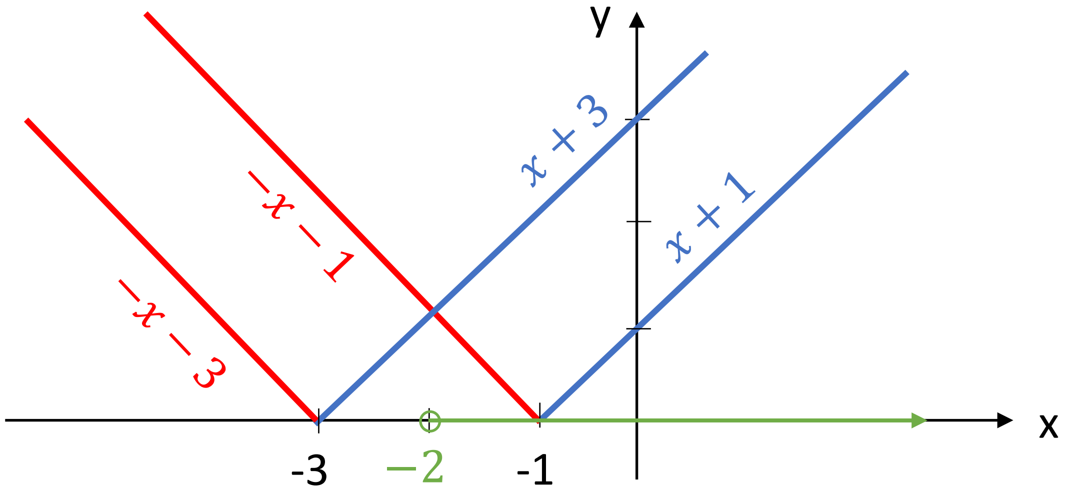
Inequacions amb valors absoluts.

Exemple $|x+3| > |x+1|$

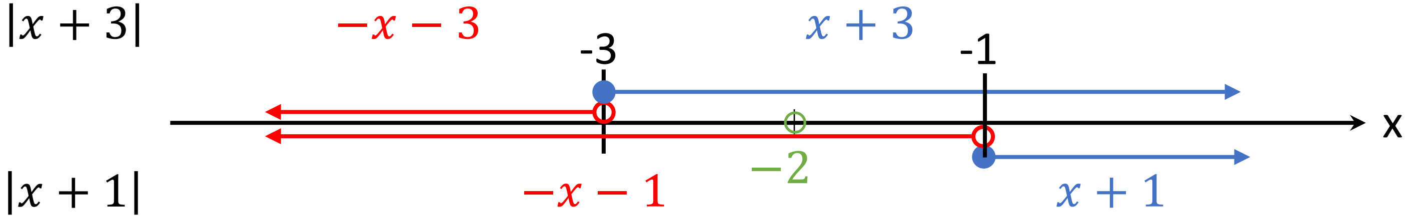
Apliquem la definició $|x| = \begin{cases} x & si\ \ x \geq 0 \\ -x & si\ \ x < 0 \end{cases}$

$$|x+3| = \begin{cases} x+3 & si\ \ x+3 \geq 0 \ \Rightarrow x \geq -3 \\ -(x+3) & si\ \ x+3 < 0 \Rightarrow x < -3 \end{cases}$$

$$|x+1| = \begin{cases} x+1 & si\ \ x+1 \geq 0 \ \Rightarrow x \geq -1 \\ -(x+1) & si\ \ x+1 < 0 \Rightarrow x < -1 \end{cases}$$



Reescrivim i avaluem la inequació per cadascun dels trams: i) $x < -3$ ii) $-3 \leq x < -1$ iii) $x \geq -1$



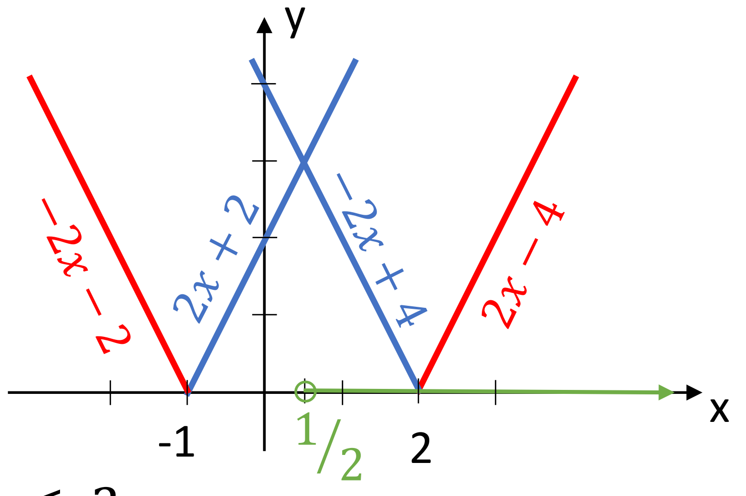
Tram i	Tram ii	Tram iii
$-x-3 > -x-1$	$x+3 > -x-1$	$x+3 > x+1$
$-3 > -1$ FALS	$2x > -4$	$3 > 1$ CERT
No existeix solució en aquest tram	$x > -2$	Tots els valors d'aquest tram compleixen la condició
	sol: $x \in (-2,-1)$	sol: $x \in [-1,\infty)$

Solució:
$x \in (-2,-1) \cup [-1,\infty)$

Ho podem simplificar com:
$x \in (-2,\infty)$ o $x > -2$

- **Exercici 1.** $|2x+2| > |-2x+4|$

Apliquem la definició $|x| = \begin{cases} x & si\ \ x \geq 0 \\ -x & si\ \ x < 0 \end{cases}$

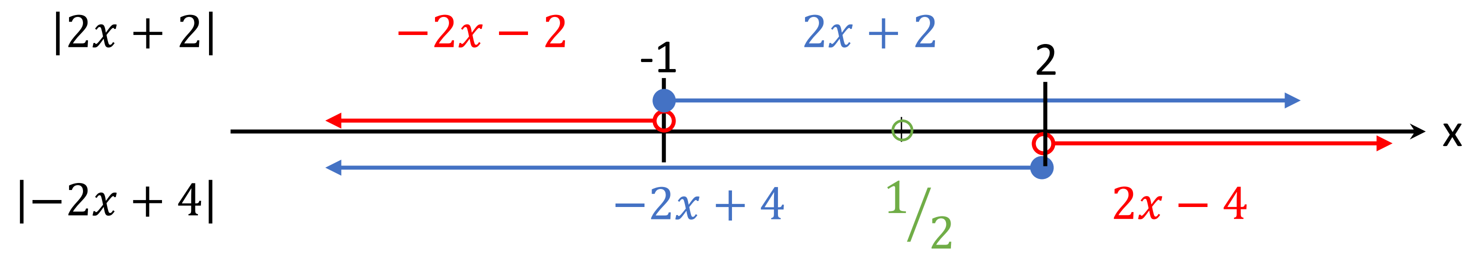


$$|2x+2| = \begin{cases} 2x+2 & si\ \ 2x+2 \geq 0 \ \Rightarrow 2x \geq -2 \Rightarrow x \geq -1 \\ -2x-2 & si\ \ 2x+2 < 0 \Rightarrow 2x < -2 \Rightarrow x < -1 \end{cases}$$

$$|-2x+4| = \begin{cases} -2x+4 & si\ \ -2x+4 \geq 0 \ \Rightarrow \boxed{-2}x \geq -4 \Rightarrow x \boxed{\leq} -\dfrac{4}{\boxed{-2}} \Rightarrow x \leq\ 2 \\ 2x-4 & si\ -2x+4 < 0 \ \Rightarrow \boxed{-2}x < -4 \Rightarrow x \boxed{>} -\dfrac{4}{\boxed{-2}} \Rightarrow x >\ 2 \end{cases}$$

El signe de la inequació canvia quan un nombre negatiu que multiplica un costat de la inequació passa dividint a l’altre costat

Reescrivim i avaluem la inequació per cadascun dels trams:



Tram i

$-2x-2 > -2x+4$

$-2 > 4$ FALS

No existeix solució en aquest tram

Tram ii

$2x+2 > -2x+4$

$4x > 2$

$x > {}^1/_2$

sol: $x \in ({}^1/_2, 2]$

Tram iii

$2x+2 > 2x-4$

$2 > -4$ CERT

Tots els valors d’aquest tram compleixen la condició

sol: $x \in (2, \infty)$

Solució:

$x \in ({}^1/_2, 2] \cup (2, \infty)$

Ho podem simplificar com:

$x \in ({}^1/_2, \infty)$ o $x > {}^1/_2$

1.3 ELS NOMBRES COMPLEXOS

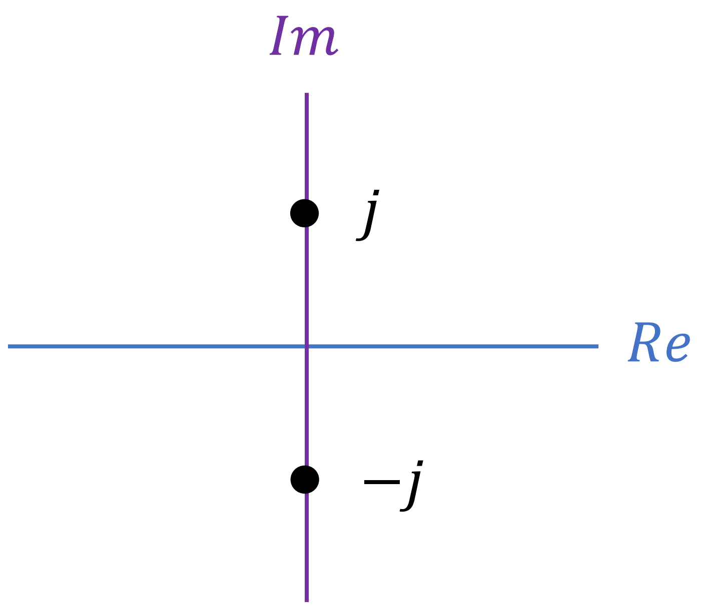
1.3.1 Definició

Ens permeten resoldre equacions que no tenen solucions reals:

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \pm\boxed{\sqrt{-1}}$$

Definim: $i = \sqrt{-1} = j$ unitat imaginària

La solució de l'equació serà per tant $x = \pm j$

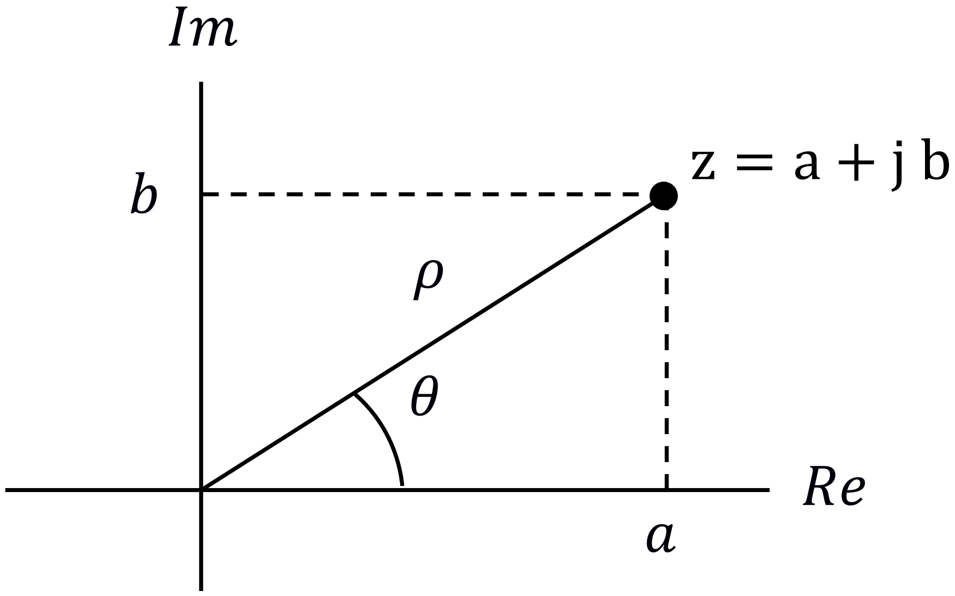


Passem de la recta del reals (1D)
al pla complex (2D)

Són de la forma $z = \boxed{a} + j\,\boxed{b}\,,\quad a, b \in \mathbb{R}$

part real $\quad$ part imaginària

$Re(z) = a \qquad Im(z) = b$



Per representar un nombre en el pla complex (2D) podem utilitzar:

(a, b) coordenades cartesianes

a part real b part imaginària

(ρ, θ) coordenades polars

ρ mòdul θ argument

A partir d'aquestes coordenades tenim vàries formes de representar un nombre complex:

a) Forma cartesiana (rectangular o binòmica)

$$z = a + j\,b$$

b) Forma polar

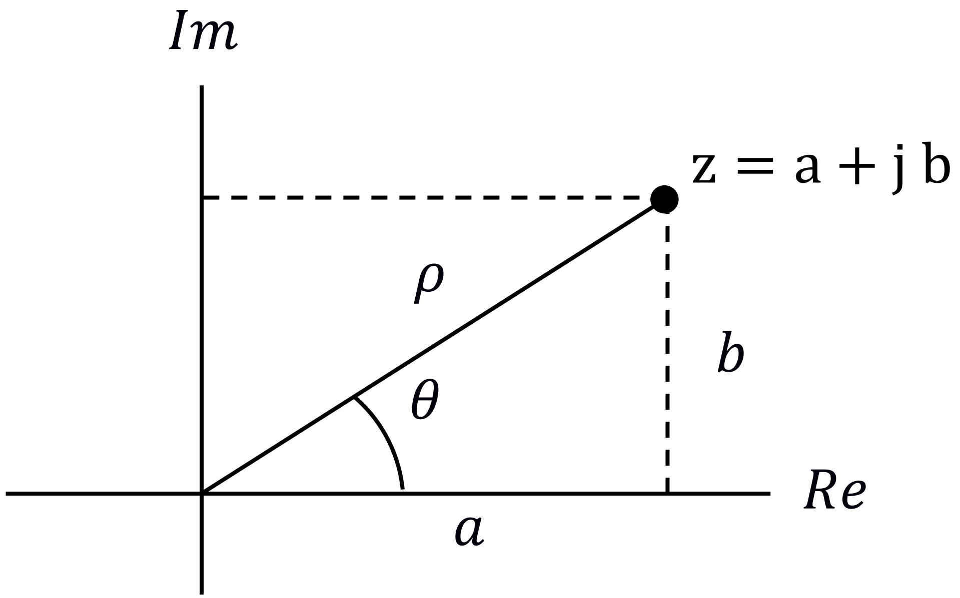
$$\mathrm{z} = \rho_{\underline{|\theta}}$$

c) Forma mòdul-argument

$$\mathrm{z} = (\rho, \theta), \qquad \rho = |z|, \qquad \theta = arg\{z\}$$

d) Forma exponencial

$$\mathrm{z} = \rho e^{j\theta}$$



cartesianes $\rightarrow$ polars

$$(a, b) \rightarrow (\rho, \theta)$$

$$\rho^2 = a^2 + b^2$$

$$\boxed{\rho = \sqrt{a^2 + b^2}}$$ mòdul

$$tg\theta = {}^{b}/_{a}$$

$$\boxed{\theta = arctg\left(\frac{b}{a}\right)}$$ argument

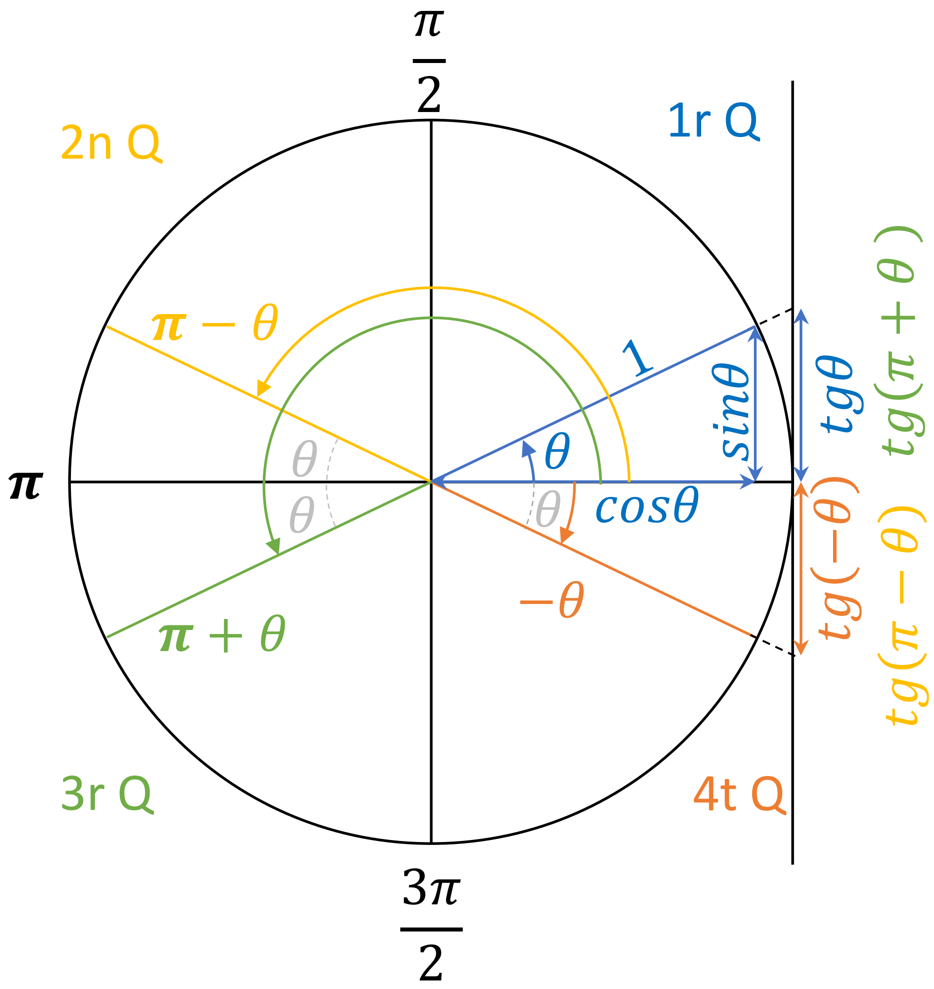
polars $\rightarrow$ cartesianes

$$(\rho, \theta) \rightarrow (a, b)$$

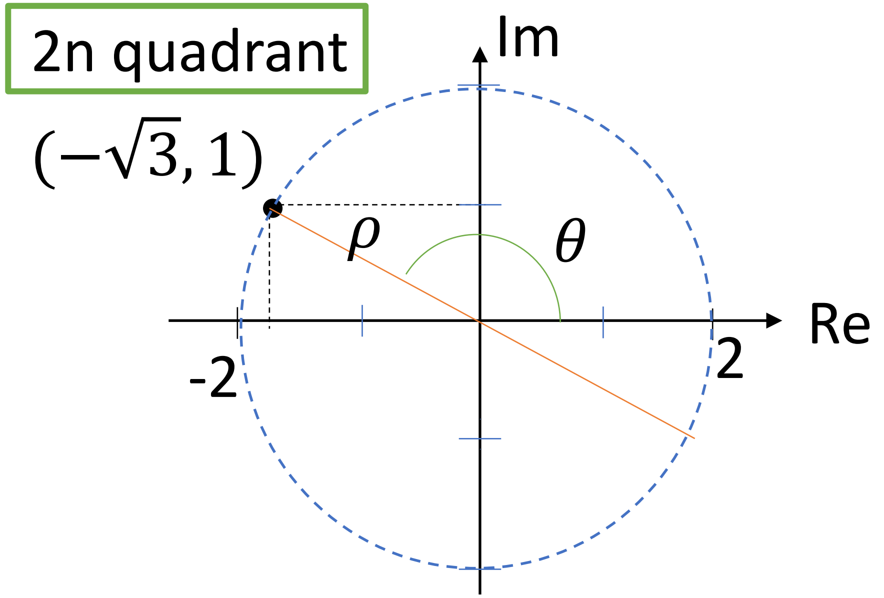
$$cos\theta = \frac{a}{\rho}, \qquad sin\theta = \frac{b}{\rho}$$

$$\boxed{a = \rho\, cos\theta,\ \ b = \rho sin\theta}$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	0	$30°$	$45°$	$60°$	$90°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$



- **Exemple 1, passar de forma cartesiana a polar.**



$$z = -\sqrt{3} + j$$

$$\rho = \sqrt{a^2 + b^2} = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\operatorname{tg} \theta = \frac{b}{a} = \frac{1}{-\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

4t quadrant | 2n quadrant

$$\theta = \operatorname{arctg}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6} \quad \text{o} \quad \boxed{\pi - \frac{\pi}{6} = \frac{5\pi}{6}}$$

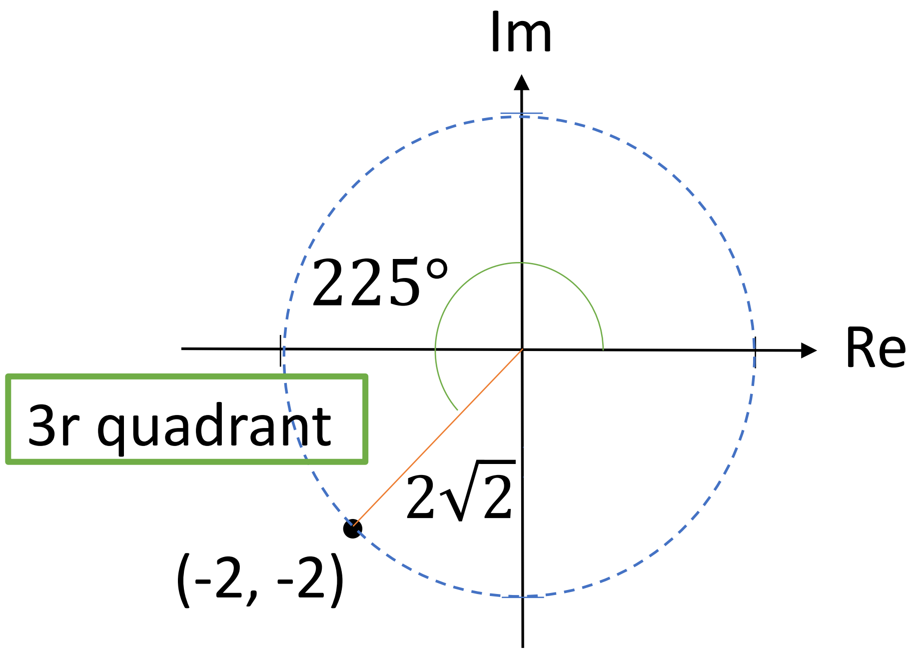
$$-30° \quad \text{o} \quad \boxed{180° - 30° = 150°}$$

$$z = 2_{\frac{5\pi}{6}} \qquad z = 2_{150°}$$

θ	0 0	$\frac{\pi}{6}$ $30°$	$\frac{\pi}{4}$ $45°$	$\frac{\pi}{3}$ $60°$	$\frac{\pi}{2}$ $90°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$

- **Exemple 2, passar de forma polar a cartesiana.**

$$z = 2\sqrt{2}_{225^\circ} \qquad z = 2\sqrt{2}_{180^\circ+\boxed{45^\circ}} \qquad z = 2\sqrt{2}_{\pi+\boxed{\frac{\pi}{4}}}$$



$$a = \rho\cos\theta = 2\sqrt{2}\cos\left(\frac{5\pi}{4}\right) = 2\sqrt{2}\left(\boxed{-}\frac{\sqrt{2}}{2}\right) = -2$$

$$b = \rho\sin\theta = 2\sqrt{2}\sin\left(\frac{5\pi}{4}\right) = 2\sqrt{2}\left(\boxed{-}\frac{\sqrt{2}}{2}\right) = -2$$

$$z = -2 - j2$$

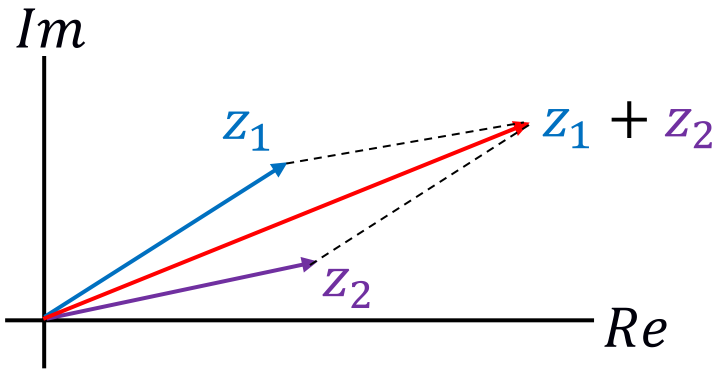
θ	0 0	$\frac{\pi}{6}$ 30°	$\frac{\pi}{4}$ 45°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$

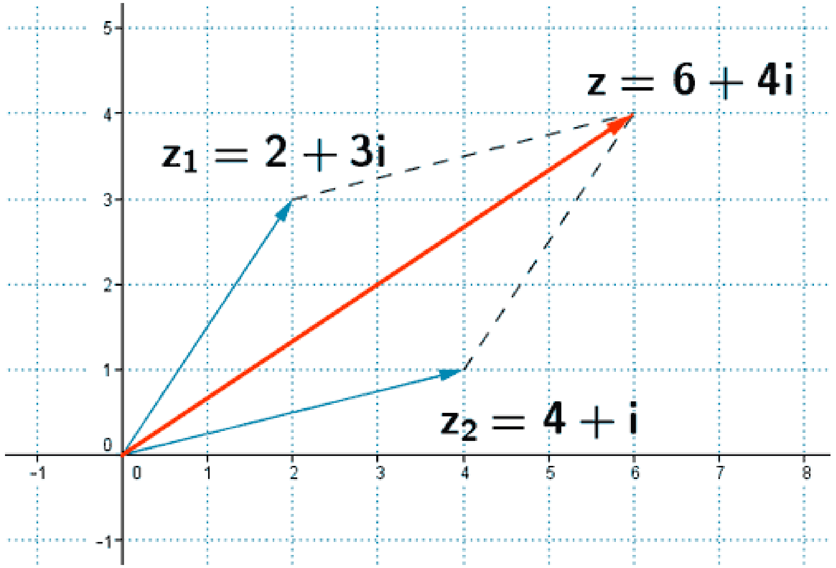
1.3.2 Operacions

$$z_1 = a_1 + jb_1$$
$$z_2 = a_2 + jb_2$$

a) Suma

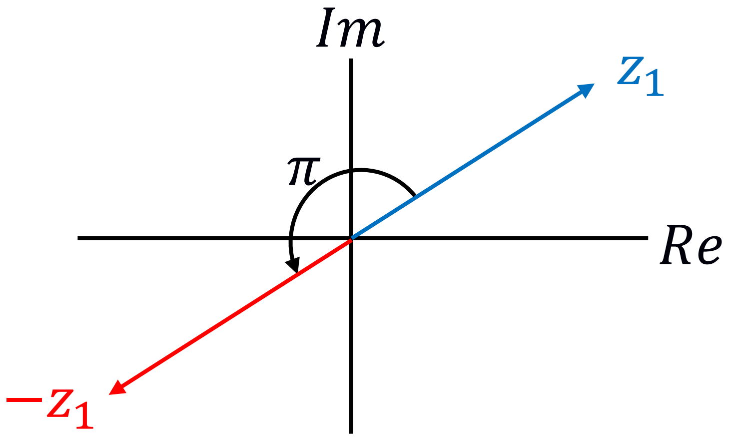
$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$$





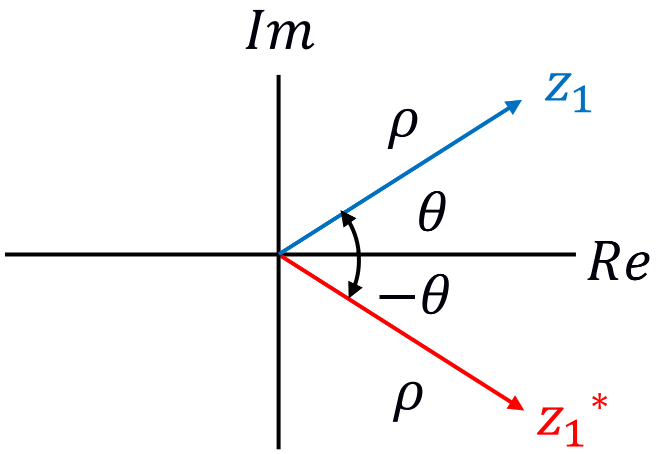
b) Complex oposat a z_1

$$-z_1 = -(a_1 + jb_1)$$



c) Complex conjugat de z_1

$$z_1{}^* = a_1 - jb_1$$



d) Producte

$$z_1 z_2 = (a_1 + jb_1)(a_2 + jb_2) =$$

$$= a_1 a_2 + j a_1 b_2 + j b_1 a_2 + j^2 b_1 b_2 =$$

$$= (a_1 a_2 \underset{j^2 = -1}{-} b_1 b_2) + j(a_1 b_2 + b_1 a_2)$$

$$z_1 {z_1}^* = (a_1 + jb_1)(a_1 - jb_1) = {a_1}^2 + {b_1}^2$$

e) Quocient

Multipliquem i dividim pel conjugat del denominador

$$\frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \boxed{\frac{a_2 - jb_2}{a_2 - jb_2}} =$$

$$= \frac{a_1 a_2 - j a_1 b_2 + j b_1 a_2 - j^2 b_1 b_2}{{a_2}^2 + {b_2}^2} =$$

$$= \frac{a_1 a_2 + b_1 b_2}{{a_2}^2 + {b_2}^2} + j \frac{b_1 a_2 - a_1 b_2}{{a_2}^2 + {b_2}^2}$$

f) Potència de nombres complexos

$$\begin{tikzcd} j^0 = 1,\ j^4 = 1 \arrow[r, "\times j"] & j^1 = j \arrow[d, "\times j"] \\ j^3 = -j \arrow[u, "\times j"] & j^2 = -1 \arrow[l, "\times j"] \end{tikzcd}$$

Reduïm l’exponent: residu de la divisió entre 4

$$\begin{array}{l|l} 26 & 4 \\ \hline 24 & \boxed{6} \\ \boxed{2} & \end{array}$$

$$j^{26} = j^{4 \cdot \boxed{6} + \boxed{2}} = \underset{j^4 = 1}{(j^4)^6} j^2 = j^{\boxed{2}} = -1$$

$z^n = (\mathrm{a} + \mathrm{j}\ \mathrm{b})^n$

Ho veurem amb un exemple

$(2 - \mathrm{j}\ )^4$

<u>Forma 1</u>

$(2 - \mathrm{j}\ )^4 = \underbrace{(2 - \mathrm{j}\ )^2}(2 - \mathrm{j}\ )^2 =$

$(2 - \mathrm{j}\ )^2 = 2^2 - 4j + j^2 = 3 - 4j$

$= (3 - 4j)(3 - 4j) =$

$= 9 - j12 - j12 + j^2 16$

$= 9 - j24 - 16 = -7 - j24$

<u>Forma 2</u>: Binomi de Newton

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \,, \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$(2 - j)^4 = \binom{4}{0} 2^{\boxed{4}}(-j)^{\boxed{0}} + \binom{4}{1} 2^{\boxed{3}}(-j)^{\boxed{1}} +$$

$$+ \binom{4}{2} 2^{\boxed{2}}(-j)^{\boxed{2}} + \binom{4}{3} 2^{\boxed{1}}(-j)^{\boxed{3}} + \binom{4}{4} 2^{\boxed{0}}(-j)^{\boxed{4}} =$$

$$= \boxed{1} \cdot 2^4 + \boxed{4} \cdot 2^3(-j) + \boxed{6} \cdot 2^2(1) + \boxed{4} \cdot 2(j) + \boxed{1} \cdot 1(1) =$$

$$= 16 - j32 - 24 + j8 + 1 = -7 - j24$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1.3.3 Representació d'un nombre complex en forma exponencial

Tot nombre complex $\mathrm{z} = a + j\,b$ es pot expressar de la forma $\mathrm{z} = \rho e^{j\theta}$

on $\rho = |z|$ i $\theta = arg\{z\}$

<u>Fórmula d'Euler</u>

Donada $\varphi \in \mathbb{R}$, $e^{j\varphi} = cos\varphi + j\,sin\varphi$

$$a = \rho\cos\theta$$
$$b = \rho sin\theta$$
$$\downarrow$$

$$z = a + j\,b = \rho\cos\theta + j\rho sin\theta = \rho\,\underbrace{(\cos\theta + jsin\theta)}_{\text{segons Euler: } e^{j\theta}} = \rho e^{j\theta}$$

Operacions en forma exponencial

Sigui $z_1 = \rho_1 e^{j\theta_1}$ i $z_2 = \rho_2 e^{j\theta_2}$

a) Producte

$$z_1 z_2 = \rho_1 e^{j\theta_1} \rho_2 e^{j\theta_2} = \rho_1 \rho_2 \, e^{j(\theta_1 + \theta_2)}$$

b) Divisió

$$\frac{z_1}{z_2} = \frac{\rho_1 e^{j\theta_1}}{\rho_2 e^{j\theta_2}} = \frac{\rho_1}{\rho_2} e^{j(\theta_1 - \theta_2)}$$

c) Potencia

$$z^n = \left(\rho e^{j\theta}\right)^n = \rho^n \, e^{j\theta n}$$

Fórmula de Moivre

$$z^n = \rho^n \, e^{jn\theta} = \rho^n [\cos(\mathrm{n}\theta) + j\sin(n\theta)]$$

$$e^{j(n\theta)} = cos(n\theta) + j \, sin(n\theta)$$ Euler

d) Arrels d’un nombre complex

Volem calcular per exemple $\sqrt[3]{8}$

Per fer el càlcul passem 8 a forma exponencial complexa: $8 = 8\,e^{j0}$

$$\sqrt[3]{8} = \sqrt[3]{8e^{j0}} = \left(8e^{j0}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}} e^{j\frac{\boxed{0}}{3}} = \sqrt[3]{8}\, e^{j0} = 2$$

Hem trobat una solució, però una arrel cúbica té 3 solucions.

sumem voltes senceres a l’angle:

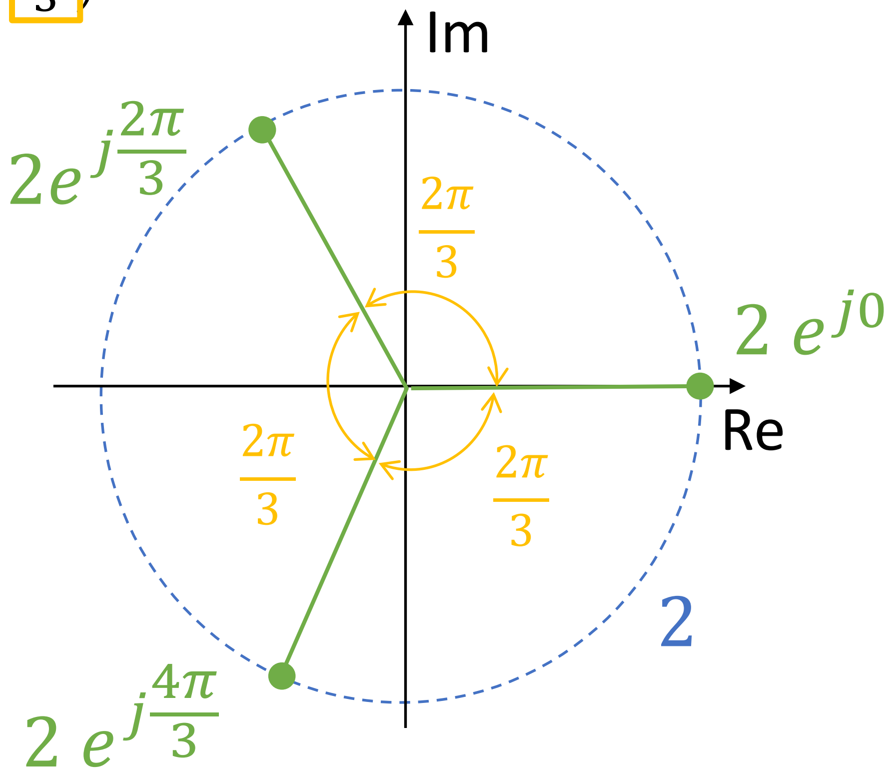
$$\boxed{0 = 0 + k2\pi} \qquad \sqrt[3]{8} e^{j\frac{\boxed{0+k2\pi}}{3}} = 2\, e^{j\left(\frac{0}{3}+k\boxed{\frac{2\pi}{3}}\right)}$$

Donant valors a k trobarem les 3 solucions:

$$k = 0 \rightarrow \omega_0 = 2e^{j\left(\frac{0}{3}+0\frac{2\pi}{3}\right)} = 2$$

$$k = 1 \rightarrow \omega_1 = 2e^{j\left(\frac{0}{3}+1\frac{2\pi}{3}\right)} = 2e^{j\frac{2\pi}{3}}$$

$$k = 2 \rightarrow \omega_2 = 2e^{j\left(\frac{0}{3}+2\frac{2\pi}{3}\right)} = 2e^{j\frac{4\pi}{3}}$$



Si ens demanen expressar els resultats en forma binòmica:

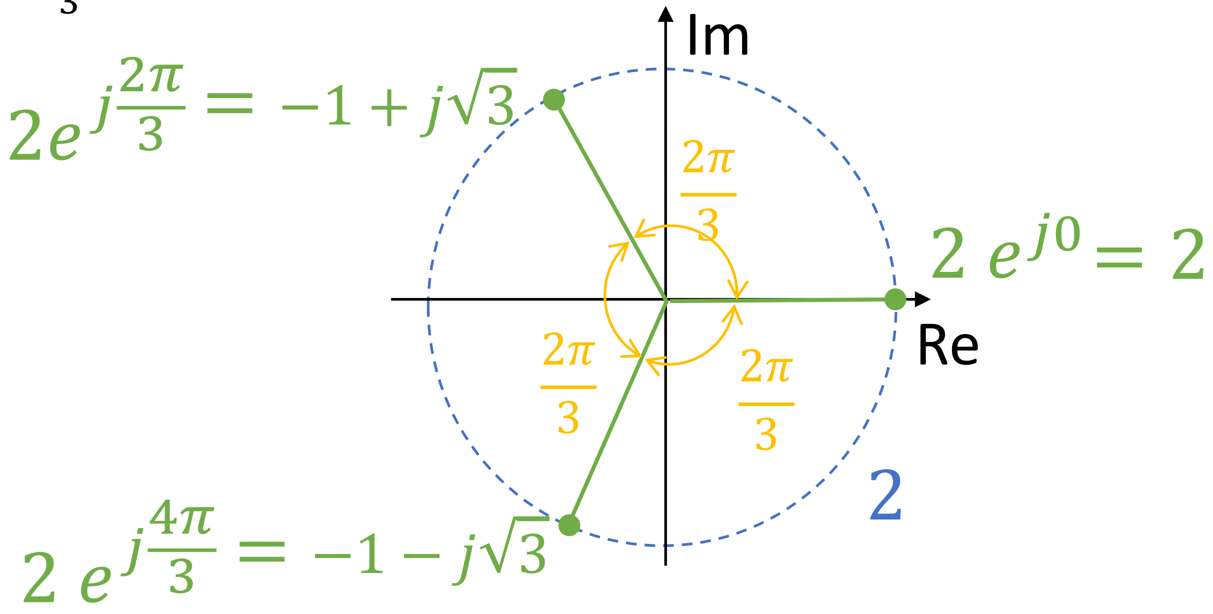
$$\omega_0 = 2$$

$$\omega_1 = 2e^{j\frac{2\pi}{3}} = 2\cos\frac{2\pi}{3} + j2\sin\frac{2\pi}{3} = -2\cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = -2\frac{1}{2} + j\frac{2\sqrt{3}}{2} = -1 + j\sqrt{3}$$

$\frac{2\pi}{3} = \pi - \frac{\pi}{3}$ (2n quadrant)

$$\omega_2 = 2e^{j\frac{4\pi}{3}} = 2\cos\frac{4\pi}{3} + j2\sin\frac{4\pi}{3} = -2\cos\frac{\pi}{3} - j\sin\frac{\pi}{3} = -2\frac{1}{2} - j\frac{2\sqrt{3}}{2} = -1 - j\sqrt{3}$$

$\frac{4\pi}{3} = \pi + \frac{\pi}{3}$ (3r quadrant)



Donat un nombre complex $z = \rho e^{j\theta} \neq 0$,

existeixen n arrels n-èssimes de z, $\omega_k = \sqrt[n]{z}$, és a dir,

existeixen n nombres ω_k, $k = 0, 1, \cdots, n-1$, que verifiquen que $(\omega_k)^n = z$

$$\omega_k = \sqrt[n]{z} = \sqrt[n]{\rho e^{j\theta}} = \left(\rho e^{j\theta}\right)^{\frac{1}{n}} = \rho^{\frac{1}{n}} e^{j\frac{\theta}{n}} \underset{\substack{\uparrow \\ \theta = \theta + k2\pi}}{=} \rho^{\frac{1}{n}} e^{j\frac{\theta + k2\pi}{n}} = \boxed{\sqrt[n]{\rho}}\, e^{j\left(\boxed{\frac{\theta}{n}} + k\boxed{\frac{2\pi}{n}}\right)}$$

Per tant, hi haurà n solucions:

- Totes tenen el mateix mòdul: $\boxed{\sqrt[n]{\rho}}$
- Els seus arguments s'obtenen començant en $\boxed{\frac{\theta}{n}}$ i incrementant successivament $\boxed{\frac{2\pi}{n}}$ radians:

$$\boxed{\frac{\theta}{n}} + k\boxed{\frac{2\pi}{n}}, \qquad k = 0, 1, \cdots, n-1$$

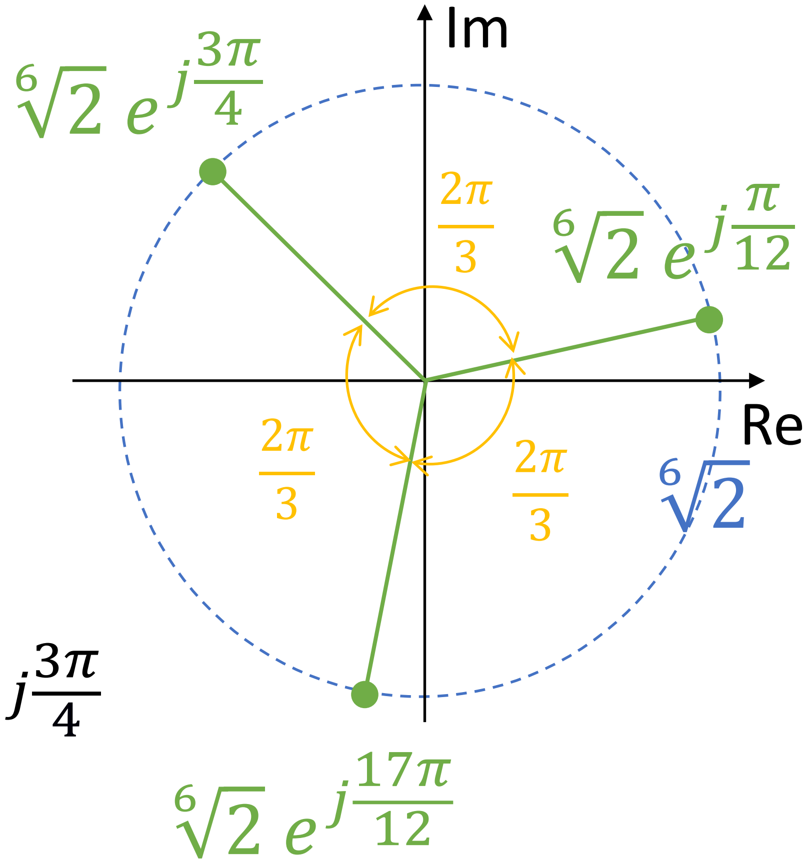
Exemple. Troba les arrels cúbiques de $z = 1 + \mathrm{j}$

Passem a forma exponencial:

$$\rho = \sqrt{1^2 + 1^2} = \sqrt{2} \qquad \theta = \operatorname{arctg}\left(\frac{1}{1}\right) = \frac{\pi}{4} \quad \text{1r quadrant} \qquad z = 1+\mathrm{j} = \sqrt{2}\, e^{j\frac{\pi}{4}}$$

$$\omega_k = \sqrt[3]{z} = \sqrt[3]{\rho}\, e^{j\left(\frac{\theta}{3}+k\frac{2\pi}{3}\right)} = \sqrt[3]{\sqrt{2}\, e^{j\left(\frac{\pi/4}{3}+k\frac{2\pi}{3}\right)}} =$$

$$= \sqrt[6]{2}\, e^{j\left(\frac{\pi}{12}+k\frac{2\pi}{3}\right)}$$



Per tant, les solucions són:

$$\omega_0 = \sqrt[6]{2}\, e^{j\left(\frac{\pi}{12}+0\frac{2\pi}{3}\right)} = \sqrt[6]{2}\, e^{j\frac{\pi}{12}}$$

$$\omega_1 = \sqrt[6]{2}\, e^{j\left(\frac{\pi}{12}+1\frac{2\pi}{3}\right)} = \sqrt[6]{2}\, e^{j\frac{\pi+8\pi}{12}} = \sqrt[6]{2}\, e^{j\frac{9\pi}{12}} = \sqrt[6]{2}\, e^{j\frac{3\pi}{4}}$$

$$\omega_2 = \sqrt[6]{2}\, e^{j\left(\frac{\pi}{12}+2\frac{2\pi}{3}\right)} = \sqrt[6]{2}\, e^{j\frac{\pi+16\pi}{12}} = \sqrt[6]{2}\, e^{j\frac{17\pi}{12}}$$

Exemple. Troba les arrels cúbiques de $z = 1 + \mathrm{j}$

Les solucions són:

$$\omega_0 = \sqrt[6]{2}\, e^{j\frac{\pi}{12}}$$

$$\omega_1 = \sqrt[6]{2}\, e^{j\frac{3\pi}{4}}$$

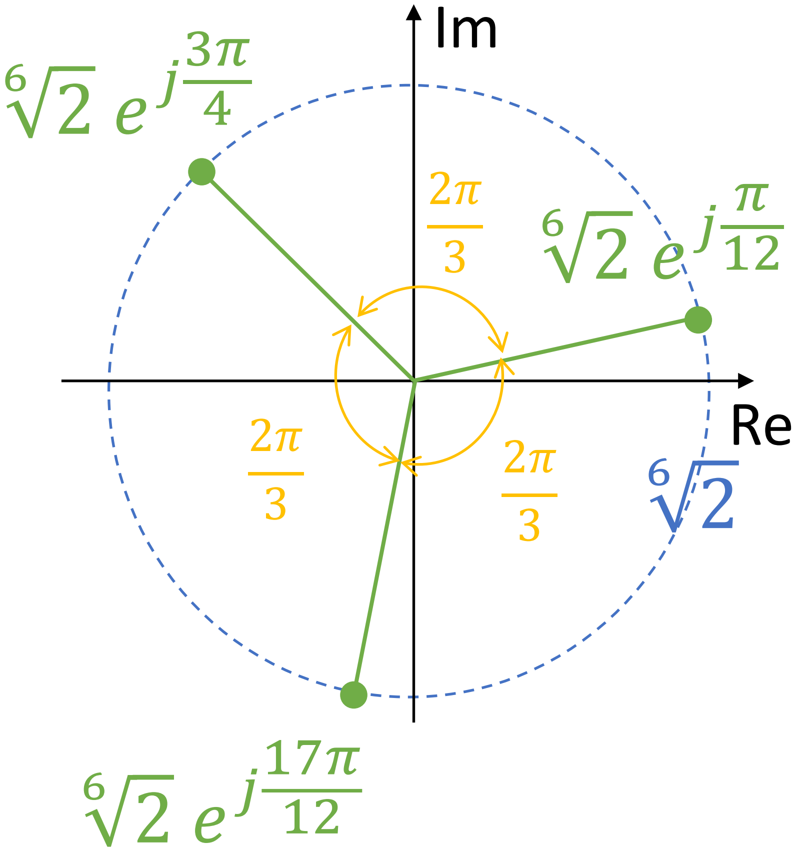
$$\omega_2 = \sqrt[6]{2}\, e^{j\frac{17\pi}{12}}$$

Comprovació:

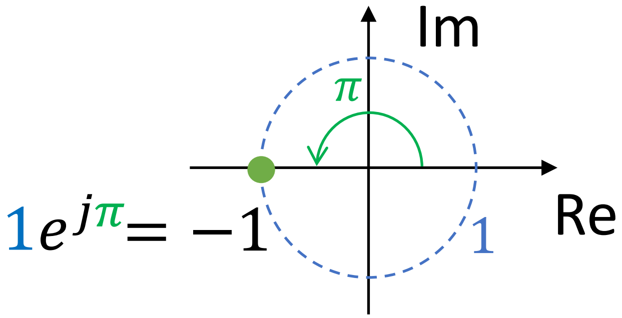
$$\omega_0^3 = \left(\sqrt[6]{2}\right)^3 e^{j3\frac{\pi}{12}} = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$\omega_1^3 = \left(\sqrt[6]{2}\right)^3 e^{j3\frac{3\pi}{4}} = \sqrt{2}e^{j\frac{9\pi}{4}} = \sqrt{2}e^{j2\pi+\frac{\pi}{4}} = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$\omega_2^3 = \left(\sqrt[6]{2}\right)^3 e^{j3\frac{17\pi}{12}} = \sqrt{2}e^{j\frac{17\pi}{4}} = \sqrt{2}e^{j4\pi+\frac{\pi}{4}} = \sqrt{2}e^{j\frac{\pi}{4}}$$



Fórmules d'Euler $\quad e^{j\varphi} = cos\varphi + j\ sin\varphi$



a) Identitat d'Euler $\quad e^{j\pi} + 1 = 0$

Demo: $e^{j\pi} = cos\pi + j\ sin\pi = -1 + j\,0 = -1$

b) Logaritme d'un nombre negatiu

$$e^{j\pi} = -1 \Rightarrow \ln e^{j\pi} = \ln(-1) \Rightarrow j\pi = \ln(-1)$$

apliquem ln als 2 costats de l'equació

$$\ln(-k) = \ln(k(-1)) = \ln(k) + \ln(-1) = \ln(k) + j\pi$$

$\ln(AB) = \ln A + \ln B$

c) cosinus i sinus

$$e^{jx} = \cos x + j\sin x \quad (1)$$
$$e^{-jx} = \cos x - j\sin x \quad (2)$$

$e^{-jx} = \cos(-x) + j\sin(-x)$

(1)+(2) $\quad e^{jx} + e^{-jx} = 2\cos x \quad \Rightarrow \cos x = \dfrac{e^{jx} + e^{-jx}}{2}$

(1)-(2) $\quad e^{jx} - e^{-jx} = 2j\sin x \quad \Rightarrow \sin x = \dfrac{e^{jx} - e^{-jx}}{2j}$

Identitats trigonomètriques

$\sin^2\alpha + \cos^2\alpha = 1$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$\xrightarrow{\beta = \alpha}$

Angle doble

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = (1-\sin^2\alpha) - \sin^2\alpha = 1 - 2\sin^2\alpha \longrightarrow \sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$

$\cos(2\alpha) = \cos^2\alpha - (1-\cos^2\alpha) = -1 + 2\cos^2\alpha \longrightarrow \cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$

Angle meitat

(1) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(2) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(1)+(2) $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

(1)−(2) $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$

(3) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

(4) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

(3)+(4) $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

Producte → Suma