INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2006; **16**:485–503
Published online 20 April 2006 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc.1075

Nonsingular path following control of a unicycle in the presence of parametric modelling uncertainties

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SUMMARY

A new type of control law is derived to steer the dynamic model of a wheeled robot of unicycle type along a desired path. The methodology adopted for path following control deals explicitly with vehicle dynamics and plant parameter uncertainty. Furthermore, it overcomes stringent initial condition constraints that are present in a number of path following control strategies described in the literature. This is done by controlling explicitly the rate of progression of a 'virtual target' to be tracked along the path, thus bypassing the problems that arise when the position of the virtual target is simply defined by the projection of the actual vehicle on that path. In the paper, a nonlinear adaptive control law is derived that yields convergence of the (closed-loop system) path following error trajectories to zero. Controller design relies on Lyapunov theory and backstepping techniques. Simulation results illustrate the performance of the control system proposed. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: path following; wheeled robots; adaptive control; backstepping

1. INTRODUCTION

The problem of motion control of autonomous vehicles (including air, land, and marine robots) has received considerable attention during the last few years. The problems addressed in the literature can be roughly classified in three groups:

1. *point stabilization*—the goal is to stabilize the vehicle at a given point, with a given orientation;

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Contract/grant sponsor: EC
Contract/grant sponsor: Portuguese Foundation for Science and Technology

Received 4 October 2005
Revised 5 January 2006
Accepted 5 March 2006

2. *trajectory tracking*—the vehicle is required to track a time parameterized reference, and
3. *path following*—the vehicle is required to converge to and follow a path, without explicit temporal specifications.

Point stabilization presents a true challenge to control system designers when the vehicle has nonholonomic (or nonintegrable) constraints, since there is no smooth (or even continuous) state-feedback law that will yield stability, as pointed out by Brockett [1]. To overcome this difficulty three main approaches have been proposed: smooth time-varying control laws [2–4] and discontinuous as well as hybrid feedback laws [5–9].

The trajectory tracking problem for fully actuated systems is now well understood and satisfactory solutions can be found in advanced nonlinear control textbooks. However, in the case of underactuated vehicles, that is, when the vehicle has less actuators than state variables to be tracked, the problem is still a very interesting topic of research. Linearization and feedback linearization methods [10, 11], as well as Lyapunov-based control laws [2,12] have been proposed.

Path following control has received relatively less attention than the other two problems. See the publications of Samson and Ait-Abderrahim [13] and Micaelli and Samson [14] for pioneering work in the area as well as Canudas de Wit *et al.* [2] and Jiang and Nijmeijer [15] and the references therein. Path following systems for marine vehicles have been reported by Encarnação *et al.* [16, 17]. The underlying assumption in path following control is that the vehicle's forward speed tracks a desired speed profile, while the controller acts on the vehicle orientation to drive it to the path. Typically, smoother convergence to a path is achieved, in comparison with the performance obtained with trajectory tracking controllers, and the control signals are less likely pushed to saturation.

This paper addresses the problem of steering the dynamic model of a wheeled robot of unicycle type along a desired path. Its main contribution is twofold: (i) it extends the results obtained in Reference [14]—for kinematic models of wheeled robots—to a more general setting, in order to deal with vehicle dynamics and parameter uncertainty, and (ii) it overcomes stringent initial condition constraints that are present in a number of path following control strategies described in the literature. This is done by controlling explicitly the rate of progression of a 'virtual target' to be tracked along the path, thus bypassing the problems that arise when the position of the virtual target is simply defined by the projection of the actual vehicle on that path. This procedure avoids the *singularities* that occur when the distance to path is not well defined and allows for a proof of global convergence of the actual path of the vehicle to the desired path. This is in striking contrast with the results described in Reference [14] for example, where only local convergence has been proven. To the best of the authors' knowledge, the idea of exploring the extra degree of freedom that comes from controlling the motion of a virtual target along a path appeared for the first time in the work of Aicardi *et al.* [18] for the control of wheeled robots. The circle of ideas exploited in Reference [18] was extended to deal with marine craft control in Reference [19]. However, none of these references addresses the issues of vehicle dynamics and parameter uncertainty. Furthermore, the methodologies adopted in References [18, 19] for control system design build on an entirely different technique that requires the introduction of a nonsingular transformation in the original error space. Interestingly enough, a recent publication, by del Rio *et al.* [20] explores the same concept of a virtual target for path following of wheeled robots.

In this paper, controller design builds on the work reported in Reference [14] on path following control and relies heavily on Lyapunov-based backstepping techniques established by Krstić *et al.* [21] in order to extend kinematic control laws to a dynamic setting. Parameter uncertainties are dealt with in an adaptive framework by augmenting Lyapunov candidate functions with terms that are quadratic in the parameter errors. See Reference [5], where identical techniques were used in the design of an adaptive control law to steer the dynamic model of a wheeled robot to a point, with a desired orientation, in the presence of parameter uncertainty.

The paper is organized as follows. Section 2 introduces the problem of path following control for a wheeled robot of unicycle type. Section 3 develops a nonlinear, adaptive, path following control law to deal with vehicle dynamics and parameter uncertainty. The performance of the control system proposed is illustrated in simulation in Section 4. Finally, Section 5 contains the conclusions and describes some problems that warrant further research.

2. PATH FOLLOWING CONTROL. PROBLEM FORMULATION

This section reviews the dynamic model of a wheeled robot and provides a rigorous formulation of the problem of steering it along a desired path. The reader is referred to References [5, 14] for background material.

The following assumptions are made regarding the robot, see Figure 1. The vehicle has two identical parallel, nondeformable rear wheels which are controlled by two independent motors, and a steering front wheel. It is assumed that the plane of each wheel is perpendicular to the

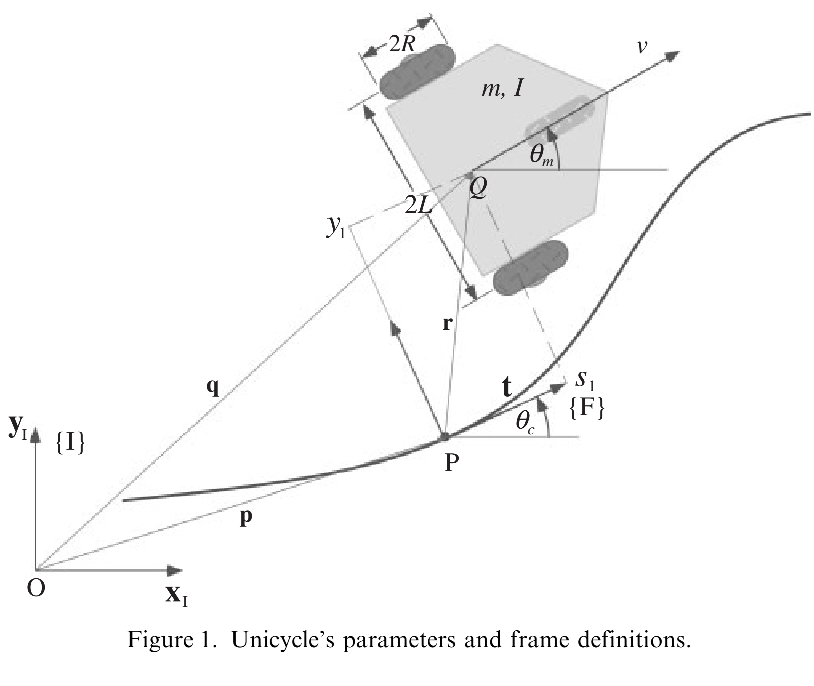

Figure 1. Unicycle's parameters and frame definitions.

ground and that the contact between the wheels and the ground is pure rolling and nonslipping, i.e. the velocity of the centre of mass of the robot is orthogonal to the rear wheels axis. It is further assumed that the masses and inertias of the wheels are negligible and that the centre of mass of the mobile robot is located in the middle of the axis connecting the rear wheels. Each rear wheel is powered by a motor which generates a control torque τ_i; $i = 1, 2$.

2.1. Kinematic equations of motion. The Serret–Frenet frame

The solution to the problem of path following derived in Reference [14] admits an intuitive explanation: a path following controller should look at (i) the distance from the vehicle to the path and (ii) the angle between the vehicle velocity vector and the tangent to the path, and reduce both to zero. This motivates the development of the kinematic model of the vehicle in terms of a Serret–Frenet frame $\{F\}$ that moves along the path; $\{F\}$ plays the role of the body axis of a 'virtual target vehicle' that should be tracked by the 'real vehicle'. Using this set-up, the abovementioned distance and angle become the coordinates of the error space where the control problem is formulated and solved. In this paper, motivated by the work in Reference [14], a Frenet frame $\{F\}$ that moves along the path to be followed is used with a significant difference: *the Frenet frame is not attached to the point on the path that is closest to the vehicle*. Instead, the origin of $\{F\}$ along the path is made to evolve according to a conveniently defined function of time, effectively yielding an extra controller design parameter. As it will be seen, this seemingly simple procedure allows to lift the stringent initial condition constraints that arise with the path following controlled described in References [14]. The notation that follows is by now standard. See for example References [13, 14].

Consider Figure 1, where P is an arbitrary point on the path to be followed and Q is the centre of mass of the moving vehicle. Associated with P, consider the corresponding Serret–Frenet frame $\{F\}$. The signed curvilinear abscissa of P along the path is denoted s. Clearly, Q can either be expressed as $\mathbf{q} = (X, Y, 0)$ in a selected inertial reference frame $\{I\}$ or as $(s_1, y_1, 0)$ in $\{F\}$. Stated equivalently, Q can be given in (X, Y) or (s_1, y_1) coordinates (see Figure 1). Let the position of point P in $\{I\}$ be vector $\mathbf{p}$. Let

$$\mathbf{R} = \begin{bmatrix} \cos\theta_c & \sin\theta_c & 0 \\ -\sin\theta_c & \cos\theta_c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be the rotation matrix from $\{I\}$ to $\{F\}$, parameterized locally by the angle θ_c. Define $\omega_c = \dot{\theta}_c$. Then,

$$\begin{aligned} \omega_c &= \dot{\theta}_c = c_c(s)\dot{s} \\ \dot{c}_c(s) &= g_c(s)\dot{s} \end{aligned} \tag{1}$$

where $c_c(s)$ and $g_c(s) = \mathrm{d}c_c(s)/\mathrm{d}s$ denote the path curvature and its derivative, respectively. The velocity of P in $\{I\}$ can be expressed in $\{F\}$ to yield

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_F = \begin{bmatrix} \dot{s} \\ 0 \\ 0 \end{bmatrix}$$

It is also straightforward to compute the velocity of Q in $\{I\}$ as

$$\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_I + \mathbf{R}^{-1}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_F + \mathbf{R}^{-1}(\omega_c \times \mathbf{r})$$

where $\mathbf{r}$ is the vector from P to Q. Multiplying the above equation on the left by $\mathbf{R}$ gives the velocity of Q in $\{I\}$ expressed in $\{F\}$ as

$$\mathbf{R}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)_F + \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_F + \omega_c \times \mathbf{r}$$

Using the relations

$$\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_I = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ 0 \end{bmatrix}$$

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_F = \begin{bmatrix} \dot{s}_1 \\ \dot{y}_1 \\ 0 \end{bmatrix}$$

and

$$\begin{aligned} \omega_c \times \mathbf{r} &= \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} = c_c(s)\dot{s} \end{bmatrix} \times \begin{bmatrix} s_1 \\ y_1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -c_c(s)\dot{s}y_1 \\ c_c(s)\dot{s}s_1 \\ 0 \end{bmatrix} \end{aligned}$$

Equation (2) can be rewritten as

$$\mathbf{R}\begin{bmatrix} \dot{X} \\ \dot{Y} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{s}(1 - c_c(s)y_1) + \dot{s}_1 \\ \dot{y}_1 + c_c(s)\dot{s}s_1 \\ 0 \end{bmatrix}$$

Solving for $\dot{s}_1$ and $\dot{y}_1$ yields

$$\begin{aligned} \dot{s}_1 &= [\cos\theta_c \quad \sin\theta_c]\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} - \dot{s}(1 - c_c y_1) \\ \dot{y}_1 &= [-\sin\theta_c \quad \cos\theta_c]\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} - c_c \dot{s} s_1 \end{aligned} \tag{2}$$

At this point it is important to notice that in Reference [14] $s_1 = 0$ for all t, since the location of point P is defined by the projection of Q on the path, assuming the projection is well defined. One is then forced to solve for $\dot{s}$ in the equation above. However, by doing so $1 - c_c y_1$ appears in the denominator, thus creating a singularity at $y_1 = 1/c_c$. As a result, the control law derived in Reference [14] requires that the initial position of Q be restricted to a tube around the path, the radius of which must be less than $1/c_{c,\max}$, where $c_{c,\max}$ denotes the maximum curvature of the path. Clearly, this constraint is very conservative since the occurrence of a large $c_{c,\max}$ in even a very small section of the path only will impose a rather strict constraint on the initial vehicle's position, no matter where it starts with respect to that path.

By making s_1 not necessarily equal to zero, a virtual target that is not coincident with the projection of the vehicle on the path is created, thus introducing an extra degree of freedom for controller design. By specifying how fast the newly defined target moves, the occurrent of a singularity at $y_1 = 1/c_c$ is removed . The velocity of the unicycle in the $\{I\}$ frame satisfies the equation

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = v \begin{bmatrix} \cos\theta_m \\ \sin\theta_m \end{bmatrix} \tag{3}$$

where θ_m and v denote the yaw angle of the vehicle and its body-axis speed, respectively. Substituting (3) in (2) and introducing the variable $\theta = \theta_m - \theta_c$ gives the kinematic model of the unicycle in the (s, y) coordinates is given by

$$\begin{aligned} \dot{s}_1 &= -\dot{s}(1 - c_c y_1) + v\cos\theta \\ \dot{y}_1 &= -c_c \dot{s} s_1 + v \sin\theta \\ \dot{\theta} &= \omega_m - c_c \dot{s} \end{aligned} \tag{4}$$

where $\omega_m = \dot{\theta}_m$.

2.2. Dynamics. Problem formulation

The complete dynamic model of the unicycle is obtained by augmenting (4) with the equations

$$\begin{aligned} \dot{v} &= \frac{F}{m} \\ \dot{\omega} &= \dot{\omega}_m - c_c \ddot{s} - g_c \dot{s}^2 \end{aligned} \tag{5}$$

where $\dot{\omega}_m = N/I$ and m and I are the mass and the mass moment of inertia of the unicycle, respectively. Notice how equation $\dot{v} = F/m$ captures the fact that the motion of the unicycle is along its longitudinal axis, in reaction to the longitudinal force F. This a consequence of the common assumption that there is no slippage of the vehicle along its lateral axis (nonholonomic constraint).

Finally, F and N can be rewritten in terms of the control inputs τ_1 and τ_2 as

$$\begin{aligned} F &= \frac{(\tau_1 + \tau_2)}{R} \\ N &= \frac{L(\tau_1 - \tau_2)}{R} \end{aligned} \tag{6}$$

where R is the radius of the rear wheels and $2L$ is the distance between them.

With the above notation, the problem under study can be formulated as follows:

Given a desired speed profile $v_d(t) > v_{\min} > 0$ for the vehicle speed v, and a path to be followed parameterized in terms of its length, derive a feedback control law for τ_1 and τ_2 to drive y_1, θ, and $v - v_d$ asymptotically to zero in the presence of plant parameter uncertainties in m, I, R, and L.

3. NONLINEAR CONTROL DESIGN

This section introduces a nonlinear closed-loop control law to steer the dynamic model of a wheeled robot described by (4)–(5) along a desired path, in the presence of parametric uncertainties by resorting to backstepping techniques. The reader will find in Reference [21] a lucid exposition of interesting theoretical and practical issues involved in backstepping. See also Reference [5] for related work on wheeled robot control in the face of parameter uncertainty. Controller design unfolds in three basic steps where a sequence of Lyapunov functions are built to deal with kinematics, dynamics, and parameter uncertainty in succession.

3.1. Nonlinear controller design using the kinematic model

The analysis that follows in inspired by the work in Reference [13,14] on path following control for kinematic models of wheeled robots. Recall from the problem definition in Section 2 that the main objective of the path following control law is to drive y_1 and θ to zero. Starting at the kinematic level, these objectives can be embodied in the Lyapunov function candidate, see Reference [14].

$$V_1 = \frac{1}{2}(s_1^2 + y_1^2) + \frac{1}{2\gamma}(\theta - \delta(y_1, v))^2 \tag{7}$$

where it is assumed that

A.1. $\delta(0, v) = 0$.
A.2. $y_1 v \sin \delta(y_1, v) \leqslant 0 \ \forall y \ \forall v$.
A.3. $\lim_{t\to\infty} v(t) \neq 0$.

In the V_1 Lyapunov function adopted, the first term $\frac{1}{2}(s_1^2 + y_1^2)$ captures the distance between the vehicle and the path, which must be reduced to 0. The second term aims to shape the approach angle $\theta = \theta_m - \theta_c$ of the vehicle to the path as a function of the 'lateral' distance y_1 and speed v, by forcing it to follow a desired orientation profile embodied in the function δ. See Reference [13], where the use of a δ function of this kind was first proposed.

Assumption A.1 specifies that the desired relative heading vanishes as y_1 goes to zero, thus imposing the condition that the vehicles main axis must be tangent to the path when the lateral distance y_1 is 0. Assumption A.2 provides an adequate reference sign definition in order to drive the vehicle to the path (turn left when the vehicle is on the right side of the path, and turn right in the other situation). Finally, Assumption A.3 stated that the vehicle does not tend to a state of rest. The need for these conditions will become apparent in the development that follows. The derivative of V_1 can be easily computed to give

$$\begin{aligned}
\dot{V}_1 &= s_1\dot{s}_1 + y_1\dot{y}_1 + \frac{1}{\gamma}(\theta - \delta)(\dot{\theta} - \dot{\delta}) \\
&= s_1(v\cos\theta - \dot{s}(1 - c_c y_1) - \dot{s}c_c y_1) + y_1 v \sin\theta + \frac{1}{\gamma}(\theta - \delta)(\dot{\theta} - \dot{\delta}) \\
&= s_1(v\cos\theta - \dot{s}) + y_1 v \sin\delta + \frac{1}{\gamma}(\theta - \delta)\left(\dot{\theta} - \dot{\delta} + \gamma y_1 v \frac{\sin\theta - \sin\delta}{\theta - \delta}\right)
\end{aligned}$$

Let the ideal (also called virtual) 'kinematic control laws' for s and θ be defined as

$$\begin{aligned} \dot{s} &= v\cos\theta + k_1 s_1 \\ \dot{\theta} &= \dot{\delta} - \gamma y_1 v \frac{\sin\theta - \sin\delta}{\theta - \delta} - k_2(\theta - \delta) \end{aligned} \tag{8}$$

where k_1 and k_2 are positive gains. Then,

$$\dot{V}_1 = -k_1 s_1^2 + y_1 v \sin\delta - \frac{(\theta-\delta)^2}{\gamma} \leqslant 0 \tag{9}$$

Note the presence of the term $y_1 v \sin\delta$ in the previous equation and how assumption A.2 is justified.

3.2. *Backstepping the dynamics*

The above feedback control law applies to the kinematic model of the wheeled robot only. In what follows, using backstepping techniques, that control law is extended to deal with the vehicle dynamics. Notice how in the kinematic design the speed of the robot $v(t)$ was assumed to follow a *desired speed profile*, say $v_d(t)$. In the dynamic design this assumption is dropped, and a feedback control law must be designed so that the tracking error $v(t) - v_d(t)$ approaches zero. Notice also that the robot's angular speed ω_m (or equivalently the variable $\dot{\theta}$) was assumed to be a control input. This assumption will be lifted by taking into account the vehicle dynamics. Following Reference [21] define the virtual control law for $\dot{\theta}$ (desired behaviour of $\dot{\theta}$ in (8)) as

$$\zeta = \dot{\delta} - \gamma y_1 v \frac{\sin\theta - \sin\delta}{\theta - \delta} - k_2(\theta - \delta) \tag{10}$$

and let $\varepsilon = \dot{\theta} - \zeta$ be the difference between actual and desired values of $\dot{\theta}$. Replacing $\dot{\theta}$ by $\varepsilon + \zeta$ in the computation of $\dot{V}_1$ gives

$$\dot{V}_1 = -k_1 s_1^2 + y_1 v \sin\delta - \frac{(\theta-\delta)^2}{\gamma} + \frac{(\theta-\delta)}{\gamma}\varepsilon \tag{11}$$

Augment now the candidate Lyapunov function V_1 with the terms $\varepsilon^2/2$ and $(v - v_d)^2/2$ to obtain

$$V_2 = V_1 + \tfrac{1}{2}[\varepsilon^2 + (v - v_d)^2]$$

with derivative

$$\dot{V}_2 = -k_1 s_1^2 + y_1 v \sin\delta - \frac{1}{\gamma}(\theta-\delta)^2 + \varepsilon\left(\frac{1}{\gamma}(\theta-\delta) + \dot{\varepsilon}\right) + (v - v_d)(\dot{v} - \dot{v}_d)$$

Simple computations show that if

$$\begin{aligned} \dot{\varepsilon} &= -\frac{1}{\gamma}(\theta - \delta) - k_3\varepsilon \\ \dot{v} &= \dot{v}_d - k_4(v - v_d) \end{aligned} \tag{12}$$

where k_3 and k_4 are positive gains. Then

$$\dot{V}_2 = -k_1 s_1^2 + y_1 v \sin\delta - \frac{1}{\gamma}(\theta-\delta)^2 - k_3\varepsilon^2 - k_4(v - v_d)^2 \leqslant 0 \tag{13}$$

It is now straightforward to compute the control inputs F and N (and thus τ_1 and τ_2) by solving the dynamics equation (5) to obtain

$$\begin{aligned} N &= I(f_1(\cdot) - k_3\varepsilon) \\ F &= m(f_2(\cdot) - k_4(v - v_d)) \end{aligned} \tag{14}$$

where

$$f_1(\cdot) = \dot{\zeta} - \frac{1}{\gamma}(\theta - \delta) + c_c\ddot{s} + g_c\dot{s}^2$$

$$f_2(\cdot) = \dot{v}_d$$

The above control law makes $\dot{V}_2$ negative semidefinite. This fact plays an important role in the proof of convergence of the robot to the path.

3.3. *Choice of the approach angle $\delta(y_1, v)$ and control computation*

As explained before, the choice of the $\delta(y_1, v)$ function is instrumental in shaping the transient maneuvers during the path approach phase. In Reference [14], the authors propose to use $\delta(y_1, v) = -\text{sign}(v)\theta_a \tanh(y_1)$. This choice is natural, but raises some subtle mathematical difficulties because $\delta(y_1, v)$ is not differentiable with respect to v at $v = 0$.

We propose instead the approach function

$$\delta(y_1, v) = -\theta_a \tanh(k_\delta y_1 v) \tag{15}$$

where $0 < \theta_a < \pi/2$ and k_δ is an arbitrary positive gain.

Clearly, this function satisfies the assumptions A.1 and A.2. The candidate controller proposed so far requires the computation of $\dot{\delta}$ and $\ddot{\delta}$ as

$$\begin{aligned} \dot{\delta} &= \delta'_y \dot{y}_1 + \delta'_v \dot{v} \\ \ddot{\delta} &= \delta''_y \dot{y}_1^2 + \delta''_v \dot{v}^2 + \delta'_y \ddot{y}_1 + \delta''_v \ddot{v} + (\delta'_{vy} + \delta'_{yv1})\dot{y}_1 \dot{v} + \delta'_{yv2} \dot{v} \end{aligned} \tag{16}$$

where

$$\begin{aligned} \delta'_y &= -\theta_a k_\delta v(1 - \tanh^2(k_\delta y_1 v)) \\ \delta'_v &= -\theta_a k_\delta y_1(1 - \tanh^2(k_\delta y_1 v)) \\ \delta''_y &= \theta_a (k_\delta v)^2 2\tanh(k_\delta y_1 v)(1 - \tanh^2(k_\delta y_1 v)) \\ \delta''_v &= \theta_a (k_\delta y_1)^2 2\tanh(k_\delta y_1 v)(1 - \tanh^2(k_\delta y_1 v)) \\ \delta'_{yv1} &= -\theta_a k_\delta + \theta_a k_\delta \tanh^2(k_\delta y_1 v) + \theta_a k_\delta v 2\tanh(k_\delta y_1 v) k_\delta y_1 (1 - \tanh^2(k_\delta y_1 v)) \\ \delta'_{yv2} &= \delta'_y(\sin\theta - c_c s_1 \cos\theta) \\ \delta'_{vy} &= -\theta_a k_\delta + \theta_a k_\delta \tanh^2(k_\delta y_1 v) + \theta_a k_\delta y_1 2\tanh(k_\delta y_1 v) k_\delta v (1 - \tanh^2(k_\delta y_1 v)) \end{aligned} \tag{17}$$

Equation (16) show that it is necessary to compute the forward acceleration $\dot{v}$ and jerk $\ddot{v}$. Because measuring these quantities is hard at best, one must resort to the dynamic model of the vehicle. Assume for the time being that the unicycle's physical parameters m, I, L, and R are known exactly. At this point it is convenient to redefine the control inputs u_i; $i = 1, 2$ as

$u_1 = \tau_1 - \tau_2$ and $u_2 = \tau_1 + \tau_2$. This yields the dynamic equations (see (6))

$$\begin{aligned} \dot{\omega}_m &= \frac{u_1}{c_1} \\ \dot{v} &= \frac{u_2}{c_2} \end{aligned} \tag{18}$$

where $c_1 = IR/L$ and $c_2 = mR$ are positive parameters. With this input transformation the controller can be re-written as

$$\begin{aligned} u_1 &= c_1(f_{u1}^1 + f_{u1}^{\dot{v}}\dot{v} + f_{u1}^{\dot{v}^2}\dot{v}^2 + f_{u1}^{\ddot{v}}\ddot{v}) \\ u_2 &= c_2 f_{u2}^1 \end{aligned} \tag{19}$$

where

$$\begin{aligned}
f_{u1}^1 &= f_{f1}^1 - k_3\left(\dot{\theta} - \delta'_y \dot{y}_1 + \gamma y_1 v \frac{\sin\theta - \sin\delta}{\theta - \delta} + k_2(\theta - \delta)\right) \\
f_{u1}^{\dot{v}} &= f_{f1}^{\dot{v}} + k_3 \delta'_v \\
f_{u1}^{\dot{v}^2} &= \delta''_v \\
f_{u1}^{\ddot{v}} &= \delta'_v \\
f_{u2}^1 &= \dot{v}_d - k_4(v - v_d) \\
f_{f1}^1 &= f_{\dot{\zeta}}^1 - \frac{1}{\gamma}(\theta - \delta) + g_c \dot{s}^2 + c_c(v\dot{\theta}\sin\theta + k_1 \dot{s}_1) \\
f_{f1}^{\dot{v}} &= f_{\dot{\zeta}}^{\dot{v}} + c_c \cos\theta \\
f_{\dot{\zeta}}^1 &= f_{\ddot{\delta}}^1 - \gamma v \frac{\sin\theta - \sin\delta}{\theta - \delta}\dot{y}_1 + \dot{\theta}\left[-k_2 - \frac{\gamma v y_1}{(\theta - \delta)^2}(\cos\theta(\theta - \delta) - (\sin\theta - \sin\delta))\right] \\
&\quad + \delta'_y \dot{y}_1 \left[k_2 + \frac{\gamma v y_1}{(\theta - \delta)^2}(\cos\delta(\theta - \delta) - (\sin\theta - \sin\delta))\right] \\
f_{\dot{\zeta}}^{\dot{v}} &= f_{\ddot{\delta}}^{\dot{v}} - \gamma y_1 \frac{\sin\theta - \sin\delta}{\theta - \delta} + \delta'_v\left[k_2 + \frac{\gamma v y_1}{(\theta - \delta)^2}(\cos\delta(\theta - \delta) - (\sin\theta - \sin\delta))\right] \\
f_{\ddot{\delta}}^1 &= \delta''_y \dot{y}_1^2 + \delta'_y f_{\ddot{y}1}^1 \\
f_{\ddot{\delta}}^{\dot{v}} &= (\delta'_{vy} + \delta'_{yv1})\dot{y}_1 + \delta'_{yv2} + \delta'_y f_{\ddot{y}1}^{\dot{v}} \\
f_{\ddot{y}1}^1 &= -c_c \dot{s}\dot{s}_1 - g_c \dot{s}^2 s_1 + \dot{\theta} v \cos\theta - c_c s_1(k_1 \dot{s}_1 - \dot{\theta} v \sin\theta) \\
f_{\ddot{y}1}^{\dot{v}} &= \sin\theta - c_c s_1 \cos\theta
\end{aligned} \tag{20}$$

At this stage, assuming that the vehicle parameters are known exactly, the acceleration $\dot{v}$ and jerk $\ddot{v}$ are obtained from the physical model of the vehicle as

$$\dot{v} = \frac{u_2}{c_2} = \dot{v}_d - k_4(v - v_d), \quad \ddot{v} = \frac{\dot{u}_2}{c_2} = \ddot{v}_d - k_4(\dot{v} - \dot{v}_d) \tag{21}$$

3.4. Parameter adaptation

We now tackle the case where the unicycle's physical parameters m, I, L, and R are not known accurately. To this effect, we start by re-writing the control law obtained in (19). Let $\bar{c}_i$ be the estimated values of the parameters, and c_i their actual unknown values. Define

$$\begin{aligned} u_1^{\text{opt}} &= c_1 f_{u1,1} + c_3 f_{u1,2} + c_4 f_{u1,3} \\ u_2^{\text{opt}} &= c_2 f_{u2} \end{aligned} \tag{22}$$

with

$$\begin{aligned} f_{u1,1} &= f_{u1}^{1} \\ c_3 &= \frac{c_1}{c_2} \\ f_{u1,2} &= f_{u1}^{\dot{v}} u_2 + f_{u1}^{\ddot{v}} \bar{c}_2 (\ddot{v}_d - k_4 \dot{v}_d) \\ c_4 &= \frac{c_1}{c_2^2} \\ f_{u1,3} &= f_{u1}^{\dot{v}^2} u_2^2 - f_{u1}^{\ddot{v}} \bar{c}_2 k_4 u_2 \\ f_{u2} &= \dot{v}_d - k_4 (v - v_d) \end{aligned} \tag{23}$$

where the notation u_i^{opt} was introduced to stress the fact that this (ideal) control law is computed with the true values of the parameters. Notice how the control law depends on four new parameters c_i; $i = 1, \ldots, 4$ that are determined by the true values of the physical parameters defined above. In practice, the control law defined above must be implemented using the estimates of $\bar{c}_i$ of c_i; $i = 1, \ldots, 4$ yielding the actual control law

$$\begin{aligned} u_1 &= \bar{c}_1 f_{u1,1} + \bar{c}_3 f_{u1,2} + \bar{c}_4 f_{u1,3} \\ u_2 &= \bar{c}_2 f_{u2} \end{aligned} \tag{24}$$

Let $\Delta c_i = \bar{c}_i - c_i$, be the errors between true and estimated parameters. Notice that $\Delta \dot{c}_i = \dot{\bar{c}}_i - \dot{c}_i = \dot{\bar{c}}_i$; $i = 1, 2$, since the true parameters are constant. Using (24),

$$\begin{aligned} \dot{V}_2 = & -k_1 s_1^2 + y_1 v \sin\delta - \frac{1}{\gamma}(\theta - \delta)^2 - k_3 \varepsilon^2 - k_4 (v - v_d)^2 \\ & + \frac{\varepsilon}{c_1}(\Delta c_1 f_{u1,1} + \Delta c_3 f_{u1,2} + \Delta c_4 f_{u1,3}) + \frac{(v - v_d)}{c_2} \Delta c_2 f_{u2} \end{aligned} \tag{25}$$

which is no longer guaranteed to be negative semidefinite. In order to deal with this problem, augment V_3 with the parameter error terms to obtain the final candidate Lyapunov function

$$V_3 = V_2 + \frac{k_5}{2} \frac{\Delta c_1^2 + \Delta c_3^2 + \Delta c_4^2}{c_1} + \frac{k_6}{2} \frac{\Delta c_2^2}{c_2} \tag{26}$$

and compute its derivative

$$\dot{V}_3 = -k_1 s_1^2 + y_1 v \sin\delta - \frac{1}{\gamma}(\theta-\delta)^2 - k_3\varepsilon^2 - k_4(v-v_d)^2 + \frac{\Delta c_1}{c_1}(\varepsilon f_{u1,1} + k_5\dot{\bar{c}}_1)$$
$$+ \frac{\Delta c_2}{c_2}((v-v_d)f_{u2} + k_6\dot{\bar{c}}_2) + \frac{\Delta c_3}{c_1}(\varepsilon f_{u1,3} + k_5\dot{\bar{c}}_3) + \frac{\Delta c_4}{c_1}(\varepsilon f_{u1,4} + k_5\dot{\bar{c}}_4) \quad (27)$$

Let a parameter update law be defined as

$$\dot{\bar{c}}_1 = -\frac{\varepsilon f_{u1,1}}{k_5}$$
$$\dot{\bar{c}}_2 = -\frac{(v-v_d)f_{u2}}{k_6}$$
$$\dot{\bar{c}}_3 = -\frac{\varepsilon f_{u1,3}}{k_5}$$
$$\dot{\bar{c}}_4 = -\frac{\varepsilon f_{u1,4}}{k_5}$$

This choice of adaptation law cancels the nonnegative terms of $\dot{V}_3$, yielding

$$\dot{V}_3 = -k_1 s_1^2 + y_1 v \sin\delta - \frac{1}{\gamma}(\theta-\delta)^2 - \frac{k_3}{I}\varepsilon^2 - \frac{k_4}{m}(v-v_d)^2 \leqslant 0 \quad (28)$$

that is, $\dot{V}_3$ becomes negative semidefinite.

3.5. *Complete system. Convergence analysis*

From the presentation above the complete adaptive, path following control system is described by the state equations

$$\begin{aligned}
\dot{s}_1 &= -\dot{s}(1 - c_c y_1) + v\cos\theta \\
\dot{y}_1 &= -c_c \dot{s} s_1 + v\sin\theta \\
\dot{\theta} &= \omega \\
\dot{\omega} &= \frac{u_1}{c_1} - c_c\ddot{s} - g_c\dot{s}^2 \\
\dot{v} &= \frac{u_2}{c_2}
\end{aligned} \quad (29)$$

together with the control law

$$\begin{aligned}
u_1 &= \bar{c}_1 f_{u1,1} + \bar{c}_3 f_{u1,2} + \bar{c}_4 f_{u1,3} \\
u_2 &= \bar{c}_2 f_{u2}
\end{aligned} \quad (30)$$

the parameter adaptation law

$$\begin{aligned}\dot{\hat{c}}_1 &= -\frac{\varepsilon f_{u1,1}}{k_5}\\ \dot{\hat{c}}_2 &= -\frac{(v-v_d)f_{u2}}{k_6}\\ \dot{\hat{c}}_3 &= -\frac{\varepsilon f_{u1,3}}{k_5}\\ \dot{\hat{c}}_4 &= -\frac{\varepsilon f_{u1,4}}{k_5}\end{aligned} \tag{31}$$

and the virtual target dynamics

$$\dot{s} = v\cos\theta + k_1 s_1 \tag{32}$$

In the above equations, all functions are replaced by their estimated values. The main result of the paper is stated next.

Proposition 1

Consider the closed-loop adaptive control system given by the dynamic model of unicycle (29) driven by the control law (30), together with the adaptation scheme (31) and the virtual target dynamics (32). Let v_d be a desired velocity profile. Assume that v_d and its first- and second-order derivatives are bounded. Further assume that v_d does not tend to zero as t tends to infinity. Then y_1, θ, and $v - v_d$ tend asymptotically to zero in the presence of bounded plant parameter uncertainties in m, I, R, and L. Stated differently, the robot converges to the prescribed path asymptotically, in the presence of parametric modelling uncertainties, and tracks the desired speed profile v_d asymptotically.

Proof

In what follows, we describe the main steps involved in the proof of Proposition 1. The proof builds on the partial results obtained so far and makes ample use of Barbalat's lemma stated below.

Barbalat's lemma

Let the function $f(t)$ admit a second-order derivative with respect to t and assume that the limit of $f(t)$ when t tends to infinity is well defined. Further assume that $\dot{f}(t)$ is uniformly continuous. Then $\dot{f}(t)$ tends to 0 as t tends to ∞. □

For details about the Barbalat's lemma, its proof, and application, please refer to Reference [22]. The key steps in the proof can be described as follows:

- Start by considering the positive definite Lyapunov candidate V_3 in (26). Straightforward computations show that the choice of the control expression (30), together with the adaptation scheme (31) and the virtual target dynamics (32) makes $\dot{V}_3$ is negative semidefinite. Hence V_3 is a positive and monotonically decreasing function, and therefore $\lim_{t\to\infty} V_3$ exists and is finite.

- It is now possible to show that s_1, y_1, θ, ε, and v are bounded because δ and v_d are assumed to be bounded. From the equations above it then follows that $\dot{s}_1$, $\dot{y}_1$, $\dot{\varepsilon}$, and $\dot{v}$ are bounded as well. Based on this result, a straightforward differentiation of $\dot{V}_3$ will show that $\ddot{V}_3$ is bounded and therefore $\dot{V}_3$ is uniformly continuous.
- Finally, a simple application of Barbalat's lemma shows that $\dot{V}_3 \to 0$ as $t \to \infty$. As a consequence, the variables s_1, y_1, δ, θ, ε and $v - v_d$ vanish as t tends to infinity, thus proving that the vehicle converges to the path and tracks the desired speed profile asymptotically.

Remark

The errors Δc_i, $i = 1, 2, 3, 4$ of the parameter estimation process do not show up in the final derivative of the Lyapunov function V_3. Thus, convergence of the errors to zero cannot be guaranteed. It can be shown, however that Δc_i, $i = 1, 2, 3, 4$ converge to some limit values that are not necessarily zero. Nevertheless, the control law still drives the required error states to zero. This behaviour for the parameter estimates can be observed in a large class of adaptive control systems.

4. SIMULATION RESULTS

This section illustrates the performance of the path following control law derived through simulations with a dynamic model of a wheeled robot. The reference and actual robot paths are shown in Figure 2. The desired speed $v_d(t)$ was set to $1\ \mathrm{m\,s^{-1}}$. The values of the design parameters were $\gamma = 1$ and $k_i = 1$; $i = 1, \ldots, 6$. The following initial conditions were adopted in

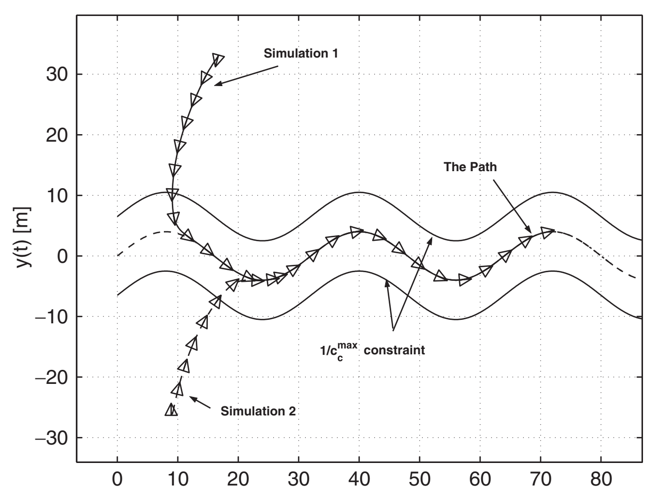

Figure 2. Desired and actual robot paths.

the simulations:

$$x_0 = \begin{bmatrix} s_1(0)\ (\mathrm{m}) \\ y_1(0)\ (\mathrm{m}) \\ \theta(0)\ (\mathrm{rad}) \\ v(0)\ (\mathrm{m/s}) \\ \dot{\theta}(0)\ (\mathrm{rad/s}) \\ s(0)\ (\mathrm{m}) \\ \bar{c}_1(0)\ (\mathrm{kg}^{-1}) \\ \bar{c}_2(0)\ (\mathrm{m}) \\ \bar{c}_3(0)\ (\mathrm{kg\ m}) \\ \bar{c}_4(0)\ (\mathrm{kg\ m}^2) \end{bmatrix} = \begin{bmatrix} 4 \\ 30 \\ -1.7 \\ 0.1 \\ 0 \\ 10 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{\text{Simulation 1}} = \begin{bmatrix} 4 \\ -30 \\ 1.7 \\ 0.1 \\ 0 \\ 10 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{\text{Simulation 2}}$$

The robot parameters used in the simulation are those of the Pioneer P3-DX robot: $m = 9$ kg, $I = 0.1\ \mathrm{kg\,m}^2$, $R = 0.1$ m and $L = 0.15$ m. The corresponding values for c_1, c_2, c_3 and c_4 are $0.06, 0.9, 0.07$ and 0.08, respectively.

Figures 2–6 shows the results of the simulations. Notice how the lateral distance to the path is driven to zero and how the actual speed $v(t)$ converges to $v_d(t)$. Notice that $\dot{s}$ tends to $v_d(t)$ as well. Finally, it is important to remark that the errors in the parameter estimates do not go to zero. The objective of following the desired path is achieved nonetheless.

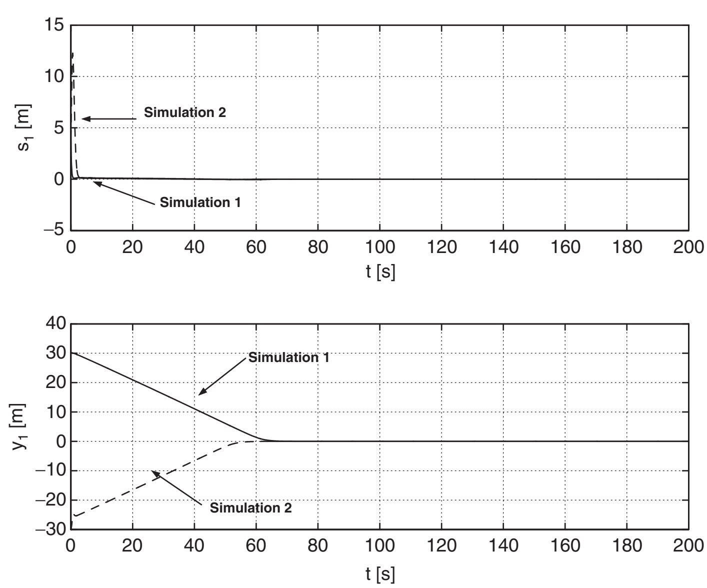

Figure 3. Time history of $s_1(t)$ and $y_1(t)$.

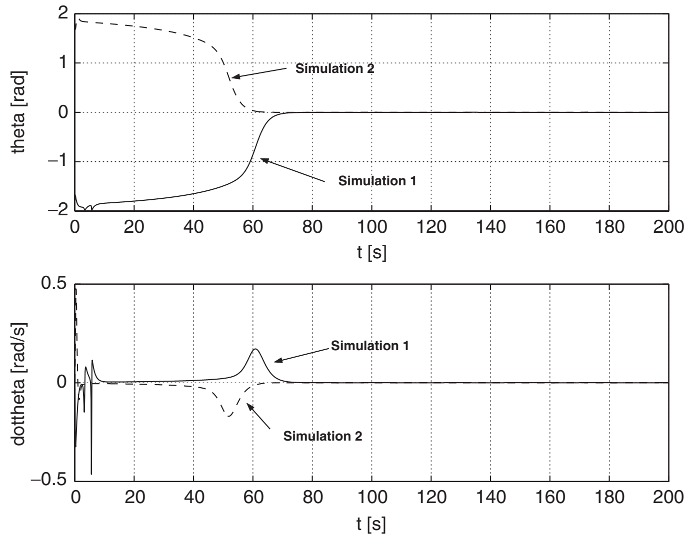

Figure 4. Time history of $\theta(t)$ and $\dot{\theta}(t)$.

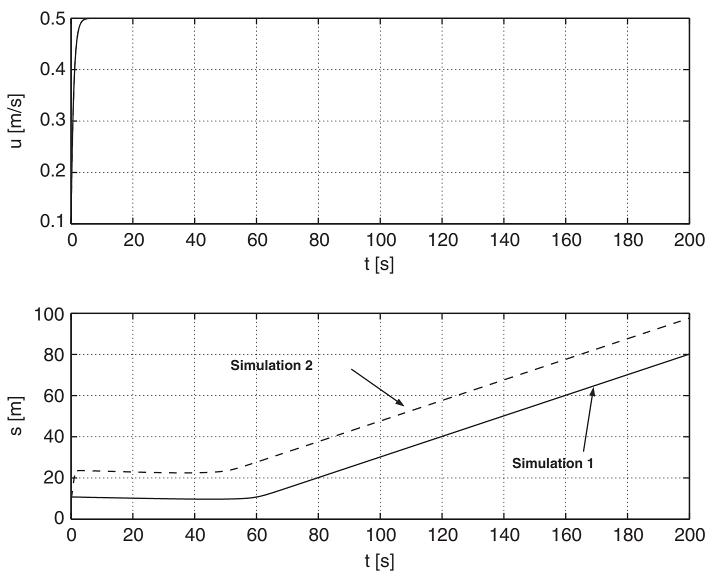

Figure 5. Time history of $v(t)$ and $s(t)$.

5. CONCLUSIONS AND FUTURE WORK

The paper derived a nonlinear, adaptive control law for accurate path following of a dynamic-wheeled robot in the presence of parameter uncertainty. The key idea behind the new control law developed was to control explicitly the rate of progression of a 'virtual target' to be tracked

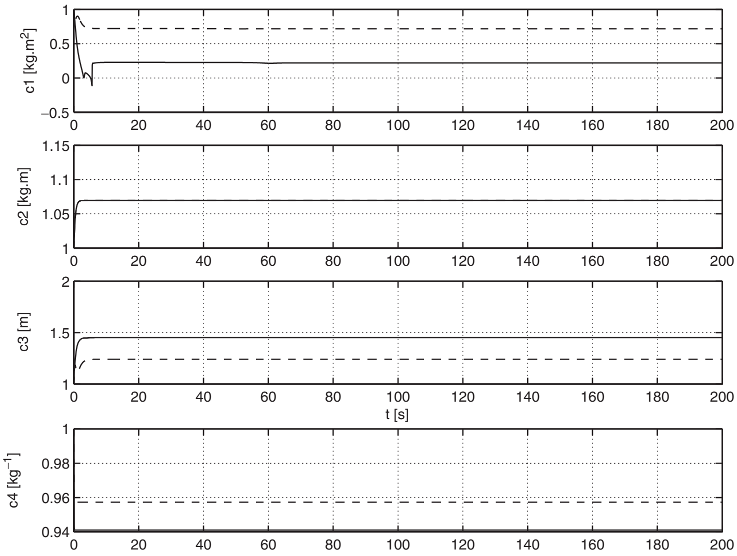

Figure 6. Time history of $c_1(t)$, $c_2(t)$, $c_3(t)$ and $c_4(t)$ (simulation 1: full line; Simulation 2: dashed line).

along the path, thus bypassing the 'singularity' problems that arise when the position of the virtual target is simply defined by the projection of the actual vehicle position on that path. Controller design relied on backstepping techniques. The paper offered a formal proof of convergence of the robot to the path. Simulation results illustrated the performance of the control system proposed. The research done in the scope of the present work unveiled some important issues that warrant further research. Among them, the following are worth emphasizing.

The idea of using a virtual target for path following is present, albeit in an implicit manner, in the work of Aicardi *et al.* [19], Skjetne *et al.* [23] and del Rio *et al.* [20], where a virtual target control law was 'intuitively chosen' even before a path following control law was developed. The originality of the present work lies in the fact that the virtual target control law falls naturally from the type of path following control law proposed, thus yielding superior performance and avoiding singularity conditions even in the presence of parameter uncertainty. This method seems to be very promising. Currently, we are investigating the benefits of using the virtual target vehicle principle to solve the combined problem of path following and obstacle avoidance for a unicycle-type robot.

In the study adopted, access to vehicle acceleration (and jerk) was done indirectly by resorting to the vehicle dynamics. The possible criticism that such a strategy may lead to poor performance or even instability (because of model parameter uncertainty) was partially addressed by incorporating an adaptive scheme. However, in the development that followed, we tacitly assumed that external disturbances, measurement noise, and unmodelled dynamics were absent. Measuring the accelerations directly via a dedicated sensor suite and using that information in a newly designed control law has the potential to yield better performance and to simplify the complexity of the path following control scheme. However, this would certainly introduce a path for noise through the acceleration measurements. Clearly, this introduces a tradeoff in system design that must be explicitly studied.

Finally, and in order to fully complete this study, the problem of path following in the presence of actuator saturation must also be addressed.

ACKNOWLEDGEMENTS

The first author benefited from a grant of the EC, under project FREESUB. The work of the second author was supported by a post-doctoral grant from the Portuguese Foundation for Science and Technology.

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