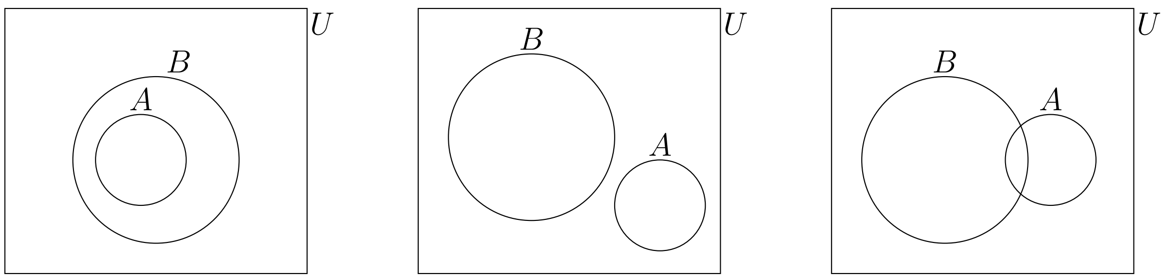
4 SETS II

Definition 4.1. (Subset) Let A and B be sets. The set A is a subset of the set B, denoted $A \subseteq B$, if every element of A is also an element of B. In other words, The set A is a subset of the set B if $x \in A$ implies $x \in B$.

If A is not a subset of B, we write $A \nsubseteq B$. Note that if A and B are sets and if $A \nsubseteq B$, then it is still possible that some of the elements of A are in B, just not all. We show this in Figure 4.1. The sets A and B are said to be **disjoint** in Figure 4.1 (b).



(a) $A \subseteq B$ (b) $A \nsubseteq B$ (c) $A \nsubseteq B$

Figure 4.1:

The notion of an object being an element of a set, and the notion of a set being a subset of another set are not the same and should be clearly understood.

Example 4.1. Let the sets A and B be given by $A = \{a, b, c\}$ and $B = \{\{a\}, b, c\}$. Which of the following statements are true and which are false?

(1)	$a \in A$	(2)	$\{a\} \subseteq A$	(3)	$a \subseteq A$	(4)	$\{a\} \in A$
(5)	$\{a\} \in B$	(6)	$\{\{a\}\} \subseteq B$	(7)	$\{a\} \subseteq B$	(8)	$a \in B$

Solution. Let the sets A and B be given by $A = \{a, b, c\}$ and $B = \{\{a\}, b, c\}$. Then the statements "$a \in A$" and "$\{a\} \subseteq A$" are true, whereas the statements "$a \subseteq A$" and "$\{a\} \in A$" are false. We see that the set B is not the same as the set A because $a \neq \{a\}$. We have that "$\{a\} \in B$" and "$\{\{a\}\} \subseteq B$" are true, but "$\{a\} \subseteq B$" and "$a \in B$" are false. ■

There is a standard strategy for proving a statement of the form "$A \subseteq B$." The proof has the following form.

Proof. Let $a \in A$.

$\vdots$

(argumentation)

$\vdots$

Then $a \in B$. Hence $A \subseteq B$. □

However, to prove a statement of the form "$A \nsubseteq B$," we must find some $a \in A$ such that $A \notin B$.

Lemma 4.1. Let X, Y and Z be sets.

1. $X \subseteq X$.
2. $\emptyset \subseteq X$
3. If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

Proof. Let X, Y and Z be sets.

1. Let $x \in X$(on the LHS of "$X \subseteq X$"). Then $x \in X$(on the RHS of "$X \subseteq X$"). Hence $X \subseteq X$.

2. We observe that $x \in \emptyset$ is always false. Thus, the logical implication "if $x \in \emptyset$, then $x \in X$" is always true. Therefore, $\emptyset \subseteq X$.

3. Suppose that $X \subseteq Y$ and $Y \subseteq Z$. Let $x \in X$. Then $x \in Y$, since $X \subseteq Y$. Consequently, we deduce that $x \in Z$ as $Y \subseteq Z$. Hence $X \subseteq Z$. □

Lemma 4.1 says that every set S is a subset of itself. We call S the **improper subset** of S.

Definition 4.2. (The Power Set) Let S be a set. The power set of S, denoted $P(S)$, is the set defined by

$$P(S) = \{H \mid H \subseteq S\}.$$

That is, the power set of S is the set of all subsets of the set S.

Example 4.2. Find the power set of the set Y given by $Y = \{a, b, c\}$.

Solution. Let $Y = \{a, b, c\}$. Then

$$P(Y) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\}.$$

■

Definition 4.3. (Equality of Sets) Let A and B be sets. The two sets A and B are equal if they have the same elements. That is, the set A equals the set B if $A \subseteq B$ and $B \subseteq A$. We write $A = B$ if A and B are equal sets.

To indicate that a set A is a subset of the set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B. For $A \subset B$ to be true, we must have that $A \subseteq B$ and B contains an element x such that $x \notin A$.

To prove the statement "$A = B$" we proceed as follows.

Proof. Let $a \in A$.

$\vdots$

(argumentation)

$\vdots$

Then $a \in B$. Hence $A \subseteq B$.
Conversely, let $b \in B$.

$\vdots$

(argumentation)

$\vdots$

Then $b \in A$. Hence $B \subseteq A$.
Therefore, $A = B$. □

Lemma 4.2. Let A, B and C be sets.

1. $A = A$.
2. If $A = B$ then $B = A$.
3. If $A = B$ and $B = C$ then $A = C$.

Proof. Let A, B and C be sets.

1. We see that $A \subseteq A$, by Lemma 4.1. It follows that $A = A$.

2. Suppose that $A = B$. Then, by the definition of equality of sets, we have that $B \subseteq A$ and $A \subseteq B$. This gives us $B = A$. Therefore, $A = B$ implies $B = A$.

3. Suppose that $A = B$ and $B = C$. Then $A \subseteq B$ and $B \subseteq C$, from which we get $A \subseteq C$ on applying Lemma 4.1. Furthermore, $C \subseteq B$ and $B \subseteq A$. Application of Lemma 4.1 yields $C \subseteq A$. Hence, by the definition of equality of sets, we conclude that $A = C$. Therefore, $A = B$ and $B = C$ implies $A = C$. □

Definition 4.4. (The Union of Two Sets) Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both. That is, the union $A \cup B$ of A and B is the set defined by

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

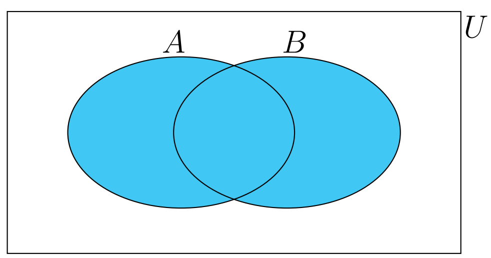


Figure 4.2: The union of the sets A and B.

Definition 4.5. (The Intersection of Two Sets) Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B. That is, the intersection $A \cap B$ of A and B is the set defined by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Example 4.3. Let $S = \{u, w, x, y, z\}$ and $\{u, v, z\}$. Find $S \cup T$ and $S \cap T$.

Solution. Let $S = \{u, w, x, y, z\}$ and $\{u, v, z\}$. Then

$$S \cup T = \{u, v, w, x, y, z\} \quad \text{and} \quad S \cap T = \{u, z\}.$$

■

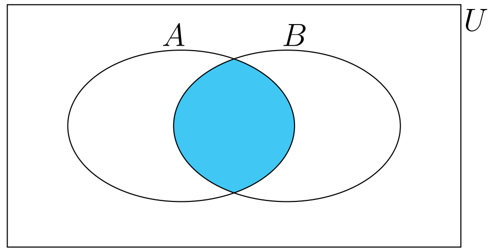


Figure 4.3: The intersection of the sets A and B.

Definition 4.6. (Disjoint Sets) Let A and B be sets. The sets A and B are disjoint if $A \cap B = \emptyset$.

Theorem 4.1. Let A, B and C be sets.

1. $A \cap B \subseteq A$ and $A \cap B \subseteq B$. If X is a set such that $X \subseteq A$ and $X \subseteq B$ then $X \subseteq A \cap B$.
2. $A \subseteq A \cup B$ and $B \subseteq A \cup B$. If Y is a set such that $A \subseteq Y$ and $B \subseteq Y$ then $A \cup B \subseteq Y$.
3. $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (Commutative Laws).
4. $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative Laws).
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Laws).
6. $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ (Identity Laws).
7. $A \cup A = A$ and $A \cap A = A$ (Idempotent Laws).
8. $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$ (Absorption Laws).
9. If $A \subseteq B$ then $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$.

Proof. Let A, B and C be sets.

1. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus, $x \in A$. Hence $A \cap B \subseteq A$. Let $y \in A \cap B$. Then $y \in A$ and $y \in B$. This means that $y \in B$. Hence $A \cap B \subseteq B$.

Next, suppose that $X \subseteq A$ and $X \subseteq B$. Now, let $x \in X$. Then $x \in A$ and $x \in B$, which leads to $x \in A \cap B$. Consequently, $X \subseteq A \cap B$. Therefore, $X \subseteq A$ and $X \subseteq B$ implies $X \subseteq A \cap B$.

2. Let $z \in A$. By definition, we have that $z \in A \cup B$. Hence, $A \subseteq A \cup B$. Let $w \in B$. By definition, we get $w \in A \cup B$. Therefore, $B \subseteq A \cup B$.

Now, suppose that $A \subseteq Y$ and $B \subseteq Y$. Let $y \in A \cup B$. Then $y \in A$ or $y \in B$. If $y \in A$ then $y \in Y$, and so $A \cup B \subseteq Y$. If $y \in B$ then $y \in Y$, and so $A \cup B \subseteq Y$. Hence $A \subseteq Y$ and $B \subseteq Y$ implies $A \cup B \subseteq Y$.

3. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$ then $x \in B \cup A$, by definition, and so $A \cup B \subseteq B \cup A$. If $x \in B$ then $x \in B \cup A$, by definition, and so $A \cup B \subseteq B \cup A$. Thus, we

obtain $A \cup B \subseteq B \cup A$. Conversely, let $x \in B \cup A$. Then $x \in B$ or $x \in A$. If $x \in B$ then $x \in B \cup A$, by definition, and so $B \cup A \subseteq A \cup B$. If $x \in A$ then $x \in B \cup A$, by definition, and so $A \cup B \subseteq B \cup A$. Consequently, we have that $B \cup A \subseteq A \cup B$. Therefore, $B \cup A = A \cup B$.

Let $z \in A \cap B$. Then $z \in A$ and $z \in B$. Thus, $z \in B \cap A$ by definition, meaning $A \cap B \subseteq B \cap A$. Let $u \in B \cap A$. Then $u \in B$ and $u \in A$. It follows that $u \in A \cap B$ by definition, and so $B \cap A \subseteq A \cap B$. Therefore, $A \cap B = B \cap A$.

4. Let $z \in (A \cup B) \cup C$. Then $z \in A \cup B$ or $z \in C$. First, suppose that $z \in A \cup B$. Then $z \in A$ or $z \in B$. If $z \in A$ then $z \in A \cup (B \cup C)$ by Part (2) of Theorem 4.1, and if $x \in B$ then $x \in B \cup C$, and hence $z \in A \cup (B \cup C)$. Second, suppose that $x \in C$. It follows from Part (2) of Theorem 4.1 that $x \in B \cup C$, and hence $z \in A \cup (B \cup C)$. Putting the two cases together, we deduce that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Conversely, let $y \in A \cup (B \cup C)$. Then $y \in A$ or $y \in B \cup C$. Firstly, suppose that $y \in A$. It follows that $y \in A \cup B$, and hence $y \in (A \cup B) \cup C$, where we used Part (2) of Theorem 4.1. Secondly, suppose that $y \in B \cup C$. Then $y \in B$ or $y \in C$. If $y \in B$ then $y \in A \cup B$, and so by Part (2) of Theorem 4.1, $y \in (A \cup B) \cup C$. If $y \in C$ then $y \in (A \cup B) \cup C$. Putting the two cases together, we deduce that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. Therefore, $(A \cup B) \cup C = A \cup (B \cup C)$.

5. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Hence $x \in B$ or $x \in C$. If $x \in B$ we deduce that $x \in A \cap B$, and if $x \in C$ we deduce that $x \in A \cap C$. In either case, we use Part (2) of this theorem to see that $x \in (A \cap B) \cup (A \cap C)$. Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Conversely, let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. First, suppose that $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in B \cup C$ by Part (2) of Theorem 4.1, and therefore $x \in A \cap (B \cup C)$. Second, suppose that $x \in A \cap C$. Then $x \in A$ and $x \in C$. By part 2 of Theorem 4.1, it follows that $x \in B \cup C$, and so $x \in A \cap (B \cup C)$. Combining the two cases we deduce that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Therefore $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

We leave the rest of the proof of Theorem 4.1 to the exercises.

Definition 4.7. (Set Difference) Let A and B be sets. The difference of A and B, denoted by $A \setminus B$, is the set containing those elements that are in A but not in B. That is, the difference of A and B is the set defined by

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

The difference of A and B is also called the **complement of B with respect to A**.

The set $A \setminus B$ is defined for any two sets A and B; it is not necessary to have $B \subseteq A$.

Example 4.4. Let U, A and B be given by

$$U = \{0, 1, a, 2, b, 3, c, 4, d\}, \quad A = \{1, a, 2, b\} \quad \text{and} \quad B = \{2, b, 3, c\}$$

respectively, where U is the universal set. Find the difference of A and B.

Solution. Let U, A and B be given by

$$U = \{0, 1, a, 2, b, 3, c, 4, d\}, \quad A = \{1, a, 2, b\} \quad \text{and} \quad B = \{2, b, 3, c\}$$

respectively, where U is the universal set. Then the difference $A \setminus B$ of A and B is given by

$$A \setminus B = \{1, a\}.$$

■

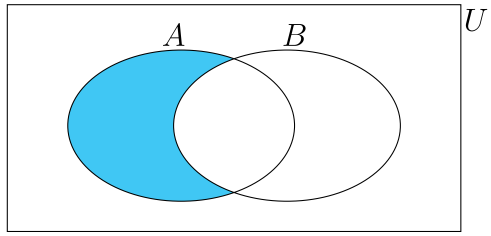


Figure 4.4: The difference of A and B.

Theorem 4.2. Let A, B and C be sets.

1. $A \setminus B \subseteq A$.
2. $(A \setminus B) \cap B = \emptyset$.
3. $A \setminus B = \emptyset$ if and only if $A \subseteq B$.
4. $B \setminus (B \setminus A) = A$ if and only if $A \subseteq B$.
5. If $A \subseteq B$ then $A \setminus C = A \cap (B \setminus C)$.
6. If $A \subseteq B$ then $C \setminus A \supseteq C \setminus B$.
7. $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$ and $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$ (De Morgan's Laws).

Proof. Let A, B and C be sets.

1. Let $u \in A \setminus B$. Then $u \in A$ and $u \notin B$. Thus $u \in A$. Therefore $A \setminus B \subseteq A$.

2. Let $x \in (A \setminus B) \cap B$. Then $x \in A \setminus B$ and $x \in B$. Thus $x \in A$ and $x \notin B$, since $x \in A \setminus B$. This means that $x \in B$ and $x \notin B$, which is a contradiction. As a result, the statement $x \in (A \setminus B) \cap B$ is false. Therefore $x \in (A \setminus B) \cap B \implies x \in \emptyset$. Hence $(A \setminus B) \cap B \subseteq \emptyset$. Using part 2 of Lemma 4.1, we see that $\emptyset \subseteq (A \setminus B) \cap B$. We conclude that $(A \setminus B) \cap B = \emptyset$.

3. Suppose that $A \setminus B = \emptyset$. Now, let $y \in A$. Assume that $y \notin B$. Then we have that $y \in A \setminus B$, which is a contradiction since $A \setminus B = \emptyset$. Thus $y \in B$. Therefore $A \subseteq B$. Hence we have shown that $A \setminus B = \emptyset$ implies $A \subseteq B$.

Conversely, suppose that $A \subseteq B$. Next, assume that there exists x such that $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. This means that $x \in B$ because of the supposition that $A \subseteq B$. So we have that $x \in B$ and $x \notin B$, which is a contradiction. This implies that there is no x such that $x \in A \setminus B$. Hence $A \setminus B = \emptyset$. Therefore $A \subseteq B$ is necessary and sufficient for $A \setminus B = \emptyset$.

4. Suppose that $B \setminus (B \setminus A) = A$. Let $z \in A$. Then $z \in B \setminus (B \setminus A)$. It follows that $z \in B$ and $z \notin B \setminus A$. Thus $z \in A \cap B$, which means that $z \in B$. Hence $A \subseteq B$.

Conversely, suppose that $A \subseteq B$. Next, let $w \in B \setminus (B \setminus A)$. Then $w \in B$ and $w \notin B \setminus A$, which means that $w \in A \cap B = A$. Thus $w \in A$, and so $B \setminus (B \setminus A) \subseteq A$. Now, let $y \in A$.

Then $y \in B$. We see that $y \in A \cap B$, which mean that $y \notin B \setminus A$. That is, $y \in B \setminus (B \setminus A)$. We obtain $A \subseteq B \setminus (B \setminus A)$. Hence $B \setminus (B \setminus A) = A$. Therefore, we have shown that $B \setminus (B \setminus A) = A$ if and only if $A \subseteq B$.

7. Let $x \in C \setminus (A \cup B)$. Then $x \in C$ and $x \notin A \cup B$. It follows that $x \notin A$ and $x \notin B$. Since $x \in C$ and $x \notin A$, we see that $x \in C \setminus A$. Since $x \in C$ and $x \notin B$, then $x \in C \setminus B$. Hence $x \in (C \setminus A) \cap (C \setminus B)$. Therefore $C \setminus (A \cup B) \subseteq (C \setminus A) \cap (C \setminus B)$.

Now let $y \in (C \setminus A) \cap (C \setminus B)$. Hence $y \in C \setminus A$ and $y \in C \setminus B$. Because $y \in C \setminus A$, it follows that $y \in C$ and $y \notin A$. Because $y \in C \setminus B$, it follows that $y \in C$ and $y \notin B$. We have that $y \notin A$ and $y \notin B$, which yields $y \notin A \cup B$. Thus $y \in C \setminus (A \cup B)$. Hence $(C \setminus A) \cap (C \setminus B) \subseteq C \setminus (A \cup B)$. We conclude that $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$. $\square$

We leave the rest of the proof of Theorem 4.2 to the exercises.

Definition 4.8. (The complement of a set) Let U be the universal set. The complement of the set B, denoted by B', is the complement of B with respect to U. In other words, the complement of the set B is $U \setminus B$.

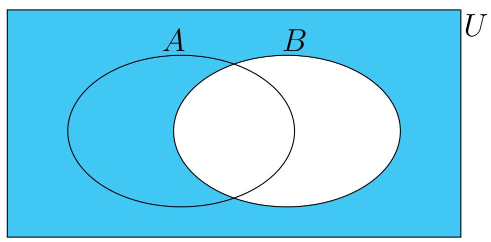


Figure 4.5: The complement of B.

Example 4.5. Let U, A and B be given by

$$U = \{0, 1, a, 2, b, 3, c, 4, d\}, \quad A = \{1, a, 2, b\} \quad \text{and} \quad B = \{2, b, 3, c\}$$

respectively, where U is the universal set. Find the complement of B.

Solution. Let U, A and B be given by

$$U = \{0, 1, a, 2, b, 3, c, 4, d\}, \quad A = \{1, a, 2, b\} \quad \text{and} \quad B = \{2, b, 3, c\}$$

respectively, where U is the universal set. The complement B' of B is given by

$$B' = \{0, 1, a, 4, d\}.$$

$\blacksquare$

Theorem 4.3. Let A and B be sets.

1. $(A')' = A$ (Complementation law).
2. $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ (De Morgan's laws).
3. $A \cup A' = U$ and $A \cap A' = \emptyset$ (Complement laws).

Definition 4.9. (Symmetric Difference) Let A and B be sets. The symmetric difference of A and B, denoted $A \triangle B$, is the set defined by

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

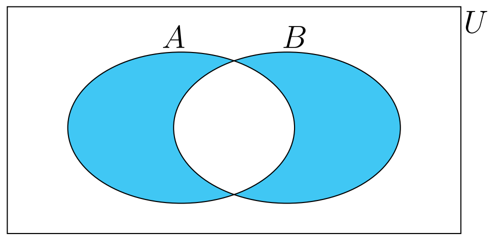


Figure 4.6: The symmetric difference of A and B.

Example 4.6. Let U, A and B be given by

$$U = \{0, 1, a, 2, b, 3, c, 4, d\}, \quad A = \{1, a, 2, b\} \quad \text{and} \quad B = \{2, b, 3, c\}$$

respectively, where U is the universal set. Find the symmetric difference $A \triangle B$ of A and B.

Solution. Let U, A and B be given by

$$U = \{0, 1, a, 2, b, 3, c, 4, d\}, \quad A = \{1, a, 2, b\} \quad \text{and} \quad B = \{2, b, 3, c\}$$

respectively, where U is the universal set. The symmetric difference $A \triangle B$ of A and B is given by

$$A \triangle B = \{1, a\} \cup \{3, c\} = \{1, a, 3, c\}.$$

■

Theorem 4.4. Let X, Y and Z be sets.

1. $X \triangle \emptyset = X$.
2. $X \triangle X = \emptyset$.
3. $X \triangle Y = Y \triangle X$.
4. $X \triangle (Y \triangle Z) = (X \triangle Y) \triangle Z$.
5. $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$.
6. $X \triangle Y = (X \cup Y) \setminus (X \cap Y)$.

Proof. Let X, Y and Z be sets.

1. Let $h \in X \triangle \emptyset$. Then $h \in X \setminus \emptyset$ or $h \in \emptyset \setminus X$. If $h \in X \setminus \emptyset$, then $h \in X$, which means that $X \triangle \emptyset \subseteq X$. If $h \in \emptyset \setminus X$, then $h \in X$ because $h \in \emptyset \setminus X$ is false. Thus $X \triangle \emptyset \subseteq X$. Both cases lead to $X \triangle \emptyset \subseteq X$. Next, let $g \in X$. Then $g \in X \setminus \emptyset$, and so $g \in X \setminus \emptyset \cup \emptyset \setminus X$. Hence $X \subseteq X \triangle \emptyset$. Therefore $X \triangle \emptyset = X$.

2. Let $z \in X \triangle X$. Then $z \in X \setminus X$. That is $z \in X$ and $z \notin X$, which is false. Hence $X \triangle X \subseteq \emptyset$. Next, we observe that $\emptyset \subseteq X \triangle X$. Therefore, $X \triangle X = \emptyset$. □

Definition 4.10. (Cartesian Product) Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. That is, the Cartesian product $A \times B$ of A and B is the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 4.7. Let the sets A and B be given by $A = \{x, y, z\}$ and $B = \{a, b\}$. Find $A \times B$.

Solution. Let $A = \{x, y, z\}$ and $B = \{a, b\}$. Then

$$A \times B = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}.$$

■

Theorem 4.5. Let A, B, C and D be sets.

1. If $A \subseteq B$ and $C \subseteq D$ then $A \times C \subseteq B \times D$.
2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $(B \cup C) \times A = (B \times A) \cup (C \times A)$.
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$ and $(B \cap C) \times A = (B \times A) \cap (C \times A)$.
4. $A \times \emptyset = \emptyset$ and $\emptyset \times A = \emptyset$.
5. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Proof. Let A, B, C and D be sets.

1. Suppose that $A \subseteq B$ and $C \subseteq D$. Let $(u, v) \in A \times C$. Then $u \in A$ and $v \in C$. It follows that $u \in B$ and $v \in D$. Thus $(u, v) \in B \times D$. Therefore, $A \times C \subseteq B \times D$.

2. Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. Thus $y \in B$ or $y \in C$. If $y \in B$, then $(x, y) \in A \times B$, in which case $(x, y) \in (A \times B) \cup (A \times C)$. If $y \in C$, then $(x, y) \in A \times C$, which means that $(x, y) \in (A \times B) \cup (A \times C)$. We deduce that $(x, y) \in (A \times B) \cup (A \times C)$. Hence $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Next, let $(a, b) \in (A \times B) \cup (A \times C)$. Then $(a, b) \in (A \times B)$ or $(a, b) \in (A \times C)$. If $(a, b) \in (A \times B)$, then $a \in A$ and $b \in B$. Since $b \in B$, we have that $b \in B \cup C$. We see that $(a, b) \in A \times (B \cup C)$. If $(a, b) \in (A \times C)$, then $a \in A$ and $b \in C$. We get $b \in B \cup C$, since $b \in C$. It follows that $(a, b) \in A \times (B \cup C)$. We see that $(a, b) \in A \times (B \cup C)$, and so $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Therefore, we have established that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. □

Definition 4.11. (The Union of a Family of Sets) Let $\mathcal{F}$ be a family of sets. The union of the sets in $\mathcal{F}$, denoted $\bigcup_{X \in \mathcal{F}} X$, is defined as follows. If $\mathcal{F} \neq \emptyset$, then

$$\bigcup_{X \in \mathcal{F}} X = \{x \mid x \in A \text{ for some } A \in \mathcal{F}\};$$

if $\mathcal{F} = \emptyset$, then $\bigcup_{X \in \mathcal{F}} X = \emptyset$. If $\mathcal{F} = \{A_i\}_{i \in I}$ is indexed by a set I, then we write

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}.$$

Furthermore, if $I = \mathbb{N}$ or $I = \{1, 2, 3, \ldots, n\}$, where $n \in \mathbb{N}$, then we write

$$\bigcup_{i=1}^{\infty} A_i \quad \text{or} \quad \bigcup_{i=1}^{n} A_i$$

respectively.

Definition 4.12. (The Intersection of a Family of Sets) Let $\mathcal{F}$ be a family of sets. The intersection of the sets in $\mathcal{F}$, denoted $\bigcap_{X\in\mathcal{F}} X$, is defined as follows. If $\mathcal{F} \neq \emptyset$, then

$$\bigcap_{X\in\mathcal{F}} X = \{x \mid x \in A \text{ for all } A \in \mathcal{F}\};$$

if $\mathcal{F} = \emptyset$, then $\bigcup_{X\in\mathcal{F}} X$ is not defined. If $\mathcal{F} = \{A_i\}_{i\in I}$ is indexed by a set I, then we write

$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}.$$

Furthermore, if $I = \mathbb{N}$ or $I = \{1, 2, 3, \ldots, n\}$, where $n \in \mathbb{N}$, then we write

$$\bigcap_{i=1}^{\infty} A_i \quad \text{or} \quad \bigcap_{i=1}^{n} A_i$$

respectively.

Theorem 4.6.
Non-Indexed Version: Let $\mathcal{F}$ be a non-empty family of sets and let B be a set.

1. $\bigcap_{X\in\mathcal{F}} X \subseteq A$ for all $A \in \mathcal{F}$. If $B \subseteq X$ for all $X \in \mathcal{F}$ then $B \subseteq \bigcap_{X\in\mathcal{F}} X$.
2. $A \subseteq \bigcup_{X\in\mathcal{F}} X$ for all $A \in \mathcal{F}$. If $X \subseteq B$ for all $X \in \mathcal{F}$ then $\bigcup_{X\in\mathcal{F}} X \subseteq B$.
3. $B \bigcap \left(\bigcup_{X\in\mathcal{F}} X\right) = \bigcup_{X\in\mathcal{F}} (B \bigcap X)$ (Distributive Law).
4. $B \bigcup \left(\bigcap_{X\in\mathcal{F}} X\right) = \bigcap_{X\in\mathcal{F}} (B \bigcup X)$ (Distributive Law).
5. $B \setminus \left(\bigcup_{X\in\mathcal{F}} X\right) = \bigcap_{X\in\mathcal{F}} B \setminus X$ (De Morgan's law).
6. $B \setminus \left(\bigcap_{X\in\mathcal{F}} X\right) = \bigcup_{X\in\mathcal{F}} B \setminus X$ (De Morgan's law).
7. $B \times \left(\bigcup_{X\in\mathcal{F}} X\right) = \bigcup_{X\in\mathcal{F}} (B \times X)$ (Distributive Law).
8. $B \times \left(\bigcap_{X\in\mathcal{F}} X\right) = \bigcap_{X\in\mathcal{F}} (B \times X)$ (Distributive Law).

Indexed Version: Let I be a non-empty set, let $\{A_i\}_{i\in I}$ be a family of sets indexed by I and let B be a set.

1. $\bigcap_{i\in I} A_i \subseteq A_k$ for all $k \in I$. If $B \subseteq A_k$ for all $k \in I$ then $B \subseteq \bigcap_{i\in I} A_i$.
2. $A_k \subseteq \bigcup_{i\in I} A_i$ for all $k \in I$. If $A_k \subseteq B$ for all $k \in I$ then $\bigcup_{i\in I} A_i \subseteq B$.
3. $B \bigcap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} (B \bigcap A_i)$ (Distributive Law).
4. $B \bigcup \left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} (B \bigcup A_i)$ (Distributive Law).

5. $B \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} B \setminus A_i$ (De Morgan's law).
6. $B \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I} B \setminus A_i$ (De Morgan's law).
7. $B \times \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} (B \times A_i)$ (Distributive Law).
8. $B \times \left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} (B \times A_i)$ (Distributive Law).

Proof.

1. **Non-Indexed**: Let $z \in \bigcap_{X\in\mathcal{F}} X$. Then $z \in A$ for all $A \in \mathcal{F}$. Therefore $\bigcap_{X\in\mathcal{F}} X \subseteq A$ for all $A \in \mathcal{F}$.
Indexed: Let $z \in \bigcap_{i\in I} A_i$. Then $z \in A_k$ for all $k \in I$. Therefore $\bigcap_{i\in I} A_i \subseteq A_k$ for all $k \in I$.

3. **Indexed**: Let $u \in B \bigcap \left(\bigcup_{i\in I} A_i\right)$. Then $u \in B$ and $u \in \bigcup_{i\in I} A_i$. It follows that $u \in A_k$ for some $k \in I$. Hence $u \in B \bigcap A_k$ for some $k \in I$. Therefore $u \in \bigcup_{i\in I} (B \bigcap A_i)$ by Part (2) of Theorem 4.6. Hence $B \bigcap \left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I} (B \bigcap A_i)$.
Now let $v \in \bigcup_{i\in I} (B \bigcap A_i)$. Then $v \in B \bigcap A_j$ for some $j \in I$. Hence $v \in B$ and $v \in A_j$ for some $j \in I$. Therefore $v \in \bigcup_{i\in I} A_i$ by Part (2) of Theorem 4.6. It follows that $v \in B \bigcap \left(\bigcup_{i\in I} A_i\right)$. Hence $\bigcup_{i\in I} (B \bigcap A_i) \subseteq B \bigcap \left(\bigcup_{i\in I} A_i\right)$. We conclude that $B \bigcap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} (B \bigcap A_i)$.

6. Let $x \in B \setminus \left(\bigcap_{X\in\mathcal{F}} X\right)$. Then $x \in B$ and $x \notin \bigcap_{X\in\mathcal{F}} X$. Then $x \notin Y$ for some $Y \in \mathcal{F}$. Thus, $x \in B \setminus Y$ for some $Y \in \mathcal{F}$. Hence $x \in \bigcup_{X\in\mathcal{F}} (B \setminus X)$ by Part (2) of this theorem. It follows that $B \setminus \left(\bigcap_{X\in\mathcal{F}} X\right) \subseteq \bigcup_{X\in\mathcal{F}} (B \setminus X)$.
Now let $y \in \bigcup_{X\in\mathcal{F}} (B \setminus X)$. Then $y \in B \setminus Z$ for some $Z \in \mathcal{F}$. Then $y \in B$ and $y \notin Z$. Hence $y \notin \bigcap_{X\in\mathcal{F}} X$. It follows that $y \in B \setminus \left(\bigcap_{X\in\mathcal{F}} X\right)$. Therefore $\bigcup_{X\in\mathcal{F}} (B \setminus X) \subseteq B \setminus \left(\bigcap_{X\in\mathcal{F}} X\right)$.
We conclude that $B \setminus \left(\bigcap_{X\in\mathcal{F}} X\right) = \bigcup_{X\in\mathcal{F}} (B \setminus X)$. $\square$