Department of Mathematics

Math 126: Algebra and Geometry

Exercise 4

TK/KD/ALM/EAYA/GAB

22/7/22

1. If $\mathbf{A}$ is the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{pmatrix}$ and $\mathbf{I}$, the unit matrix of order 3, show that $\mathbf{A}^3 = p\mathbf{I} + q\mathbf{A} + r\mathbf{A}^2$.

2. A matrix $\mathbf{A}$ has x rows and $x+5$ columns. $\mathbf{B}$ has y rows and $11-y$ columns. Both $\mathbf{AB}$ and $\mathbf{BA}$ exist. Find x and y.

3. If $\mathbf{A}_{mn}$, $\mathbf{B}_{pq}$ are two matrices, state the conditions when they are conformable for (i) addition, (ii) multiplication (iii) addition and multiplication.

4. Let $\mathbf{A}$ be a square matrix and $\mathbf{A}^T$ denote the transpose of $\mathbf{A}$. Show that $\mathbf{A}+\mathbf{A}^T$ is a symmetric matrix and $\mathbf{A}-\mathbf{A}^T$ is a skew symmetric matrix. Hence express the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ -3 & 1 & 4 \end{pmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

5. $\mathbf{P}$ is a 3×3 matrix such that $\mathbf{P}^2 = \mathbf{P} - \mathbf{I}$.

   (a) Find $\mathbf{P}^{-1}$ in terms of $\mathbf{P}$ and $\mathbf{I}$.

   (b) Show that $\mathbf{P}^3 + \mathbf{I} = \mathbf{0}$.

   (c) If $\mathbf{PQ} = 2\mathbf{I} - \mathbf{P}$, find $\mathbf{Q}$ in the form $\lambda\mathbf{I} + \mu\mathbf{P}$, where λ and μ are constants to be determined.

6. Use the method of row reduction to solve the simultaneous equations: $2x-y+z=1$, $2x-2y+3z=-1$, $x+y-2z=3$.

7. Apply Cramer's rule to solve the equations: $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$.

8. (a) For what value of k will the system $4x+y+4z=0$, $2x+ky+z=0$, $(k-1)x-y+2z=0$ have non-trivial solutions?

   (b) Obtain non-trivial solutions when they exist:

      i. $3x-2y+z=0$, $x+2y-2z=0$, $2x-y+2z=0$,

      ii. $x+y-3z=0$, $3x-y-z=0$, $2x+y-4z=0$.

9. Find the values of λ for which the equations $(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$, $(\lambda-1)x+(4\lambda-2)y+(\lambda+)z=0$, $2x+(3\lambda+1)y+3(\lambda-1)z=0$ are consistent and find the ratios of $x:y:z$ when λ has the smallest of these values. What happens when λ has the greatest of these values?

10. Find the adjoint of the matrix $\mathbf{A}=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ and verify that

    $\mathbf{A}(adj\mathbf{A})=(adj\mathbf{A})\mathbf{A}=|\mathbf{A}|\mathbf{I}$.

11. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 3 & 4 & -2 \\ -1 & 1 & 4 \end{pmatrix}$ by elementary row operations. Hence solve the simultaneous equations $x+y-z=1$, $3x+4y-2z=3$, $-x+y+4z=2$, and find the matrix $\mathbf{X}$ such that $\mathbf{XA}=\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$.

12. Determine the values of λ and μ so that the following systems of equations have (i) Unique solution (ii) No solution (iii) Infinite number of solutions

    (a) $x-2y+3z=6$, $3x+4y-10z=1$, $3x+4y+\lambda z=\mu$,

    (b) $x+y+z=1$, $x+2y+3z=2$, $2x+3y+\lambda y=\mu$,

    (c) $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$.

13. Show that the following sets of equations are consistent and solve them.

    (a) $4x-2y+6z=8$, $x+y-3z=-1$, $15x-3y+9z=21$,

    (b) $x+y+2z=4$, $2x-y+3z=9$, $3x-y-z=2$,

    (c) $5x+3y+7z=4$, $3x+26y+2z=9$, $7x+2y+10x=5$.

14. Find the values of α for which the equations $x+2y+3z=1$, $5x+y+3z=\alpha$, $3x+9y+13z=\alpha^2$ have solutions. Find all the solutions for each value of α.

15. For the matrix equation $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 6 \\ 4+k \end{pmatrix}$, where $\mathbf{M}=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & k \end{pmatrix}$, find the value of k for which the equation does not have unique solution. For this value of k, solve the equations and interpret the solution geometrically in relation to the linear transformation with matrix $\mathbf{M}$.

16. Show that the linear transformation with matrix $\begin{pmatrix} 1 & -3 & 0 \\ 2 & -4 & 3 \\ 1 & 1 & 6 \end{pmatrix}$ maps every point of the three-dimensional space onto a single plane and give the Cartesian equation of this plane. Describe the set of points mapped onto the origin under this transformation.

17. Write down the 2×2 matrices representing the following transformations of the plane. (i) Reflection in the y-axis, (ii) Reflection in the line $y = x$ (iii) Rotation through $180°$ about the origin (iv) Enlargement from the origin with scale factor λ.

18. (a) A matrix $\mathbf{A}$ of a linear transformation $T(x, y) = (px + qy, rx + sy)$ is given by $\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$.

    i. Determine the values of the constant coefficients p, q, r and s.

    ii. Find the matrix of the inverse of the transformation T and use it to find the point whose image is $(2, 1)$ under T.

    (b) Write down the matrix $\mathbf{B}$ of the linear transformation $S(x, y) = (x+3y, 2x+y)$. Find the image of the point $(2, 1)$ under the transformation given by $\mathbf{AB}$.

19. The linear transformation $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where $\mathbf{M}$ is 3×3 matrix, maps the points with position vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to the points with position vectors $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ respectively. Write down the matrix $\mathbf{M}$ and find the inverse matrix $\mathbf{M}^{-1}$. Show that the transformation with the matrix $\mathbf{M}$ maps points of the plane $x + y + z = 0$ to points of the plane $x = y$ and verify that the inverse transformation with matrix $\mathbf{M}^{-1}$ maps points of the plane $x = y$ to points of the plane $x + y + z = 0$.

20. (a) Show that for all values of θ, the determinant $\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$ lies between 2 and 4 inclusive. State the value of θ for which the determinant has the value 2, and one for which it has the value 4.

    (b) Expand the determinant $y = \begin{vmatrix} x & x^2 & x^3 \\ a & b & c \\ p & q & r \end{vmatrix}$ by the first row, and from the expansion, find $\dfrac{dy}{dx}$. Express $\dfrac{dy}{dx}$ in the form of a determinant, and hence or otherwise, find the two values of x for which the determinant $y = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix}$ has stationary values.