Department of Mathematics

Math 126: Algebra and Geometry

Exercise 9

TK/PKO/JB/JKA/GAB

16/4/20

1. If $\mathbf{A} = \begin{pmatrix} 1 & 4 & 0 & 3 \\ -2 & 0 & 2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & -1 & -3 & 1 \\ 3 & 5 & 1 & -2 \end{pmatrix}$ find

   (a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A} - \mathbf{B}$ (c) $2\mathbf{A} - 3\mathbf{B}$.

2. If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -3 & 1 & -2 \\ 0 & 4 & -1 \\ 2 & 0 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & -4 \\ 2 & 0 \\ -3 & 1 \end{pmatrix}$,

   (a) calculate where possible the products $\mathbf{A}^2$, $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{AD}$, $\mathbf{BA}$, $\mathbf{B}^2$, $\mathbf{BC}$, $\mathbf{BD}$, $\mathbf{CA}$, $\mathbf{CB}$, $\mathbf{C}^2$, $\mathbf{CD}$, $\mathbf{DA}$, $\mathbf{DB}$, $\mathbf{DC}$ and $\mathbf{D}^2$.

   (b) Verify that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ and that $(\mathbf{BC})\mathbf{D} = \mathbf{B}(\mathbf{CD})$.

3. If $\mathbf{A} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, where $i = \sqrt{-1}$, find $\mathbf{A}^2$ and $\mathbf{A}^4$.

4. Find the product $\begin{pmatrix} 3-i & 4 \\ -4 & 3+i \end{pmatrix} \begin{pmatrix} 2 & 5-2i \\ -5-2i & 2 \end{pmatrix}$.

5. If $\mathbf{X} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix}$, find the values of a and b such that $\mathbf{XY} = \mathbf{YX}$.

6. Given that $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -4 & -1 \end{pmatrix}$, show that $\mathbf{A} - \mathbf{A}^2 = 10\mathbf{I}$.

7. Given that $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ and that $\mathbf{M}^2 - 6\mathbf{M} + k\mathbf{I} = \mathbf{0}$, find k.

8. Given that $\mathbf{M}$ is a 2×2 matrix such that $\mathbf{M}^2 = \mathbf{M} + \mathbf{I}$, show that $\mathbf{M}^4 - 3\mathbf{M} = 2\mathbf{I}$.

9. Given that $\mathbf{A}$ is a 2×2 matrix such that $\mathbf{A}^2 = \mathbf{A} - 2\mathbf{I}$, show that $\mathbf{A}^4 + 3\mathbf{A}^2 + 4\mathbf{I} = \mathbf{0}$.

10. By letting $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, prove that for all 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$,

    (a) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$, (b) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$, (c) $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$, (d) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

11. Given that $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same order, simplify

(a) $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})-(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})$ (b) $(\mathbf{A}+\mathbf{B})^3+(\mathbf{A}-\mathbf{B})^3-2\mathbf{A}(\mathbf{A}^2+\mathbf{B}^2)$

12. If $\mathbf{r} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 1 & 0 & g \\ 0 & 1 & f \\ g & f & c \end{pmatrix}$ show that $\mathbf{r}^T\mathbf{A}\mathbf{r} = 0$ is the equation of the circle in $\mathbb{R}^2$ with centre $(-g,-f)$ and radius $\sqrt{g^2+f^2-c}$.

13. $\mathbf{A}$ and $\mathbf{X}$ are the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} x & y \\ u & v \end{pmatrix}$ respectively, where b is not equal to zero. Prove that if $\mathbf{AX} = \mathbf{XA}$ then $u = cy/b$ and $v = x+(d-a)y/b$. Hence prove that if $\mathbf{AX} = \mathbf{XA}$ then there are numbers p and q such that $\mathbf{X} = p\mathbf{A}+q\mathbf{I}$, and find p and q in terms of a, b, x, y.