Department of Mathematics
Math 126: Algebra and Geometry
Exercise 4

TK/KD/ALM/EAYA/GAB
22/7/22

1. If $\mathbf{A}$ is the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{pmatrix}$ and $\mathbf{I}$, the unit matrix of order 3, show that $\mathbf{A}^3 = p\mathbf{I} + q\mathbf{A} + r\mathbf{A}^2$.

2. A matrix $\mathbf{A}$ has x rows and $x + 5$ columns. $\mathbf{B}$ has y rows and $11 - y$ columns. Both $\mathbf{AB}$ and $\mathbf{BA}$ exist. Find x and y.

3. If $\mathbf{A}_{mn}$, $\mathbf{B}_{pq}$ are two matrices, state the conditions when they are conformable for (i) addition, (ii) multiplication (iii) addition and multiplication.

4. Let $\mathbf{A}$ be a square matrix and $\mathbf{A}^T$ denote the transpose of $\mathbf{A}$. Show that $\mathbf{A} + \mathbf{A}^T$ is a symmetric matrix and $\mathbf{A} - \mathbf{A}^T$ is a skew symmetric matrix. Hence express the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ -3 & 1 & 4 \end{pmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

5. $\mathbf{P}$ is a 3×3 matrix such that $\mathbf{P}^2 = \mathbf{P} - \mathbf{I}$.
   (a) Find $\mathbf{P}^{-1}$ in terms of $\mathbf{P}$ and $\mathbf{I}$.
   (b) Show that $\mathbf{P}^3 + \mathbf{I} = \mathbf{0}$.
   (c) If $\mathbf{PQ} = 2\mathbf{I} - \mathbf{P}$, find $\mathbf{Q}$ in the form $\lambda\mathbf{I} + \mu\mathbf{P}$, where λ and μ are constants to be determined.

6. Use the method of row reduction to solve the simultaneous equations: $2x - y + z = 1$, $2x - 2y + 3z = -1$, $x + y - 2z = 3$.

7. Apply Cramer's rule to solve the equations: $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$.

8. For what value of k will the system $4x + y + 4z = 0$, $2x + ky + z = 0$, $(k-1)x - y + 2z = 0$.

9. Find the values of λ for which the equations $(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$, $(\lambda - 1)x + (4\lambda - 2)y + (\lambda +)z = 0$, $2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$ are consistent and find the ratios of $x : y : z$ when λ has the smallest of these values. What happens when λ has the greatest of these values?

10. Find the adjoint of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ and verify that

$\mathbf{A}(adj\,\mathbf{A}) = (adj\,\mathbf{A})\mathbf{A} = |\mathbf{A}|\mathbf{I}$.

11. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 3 & 4 & -2 \\ -1 & 1 & 4 \end{pmatrix}$ by elementary row operations. Hence solve the simultaneous equations $x + y - z = 1$, $3x + 4y - 2z = 3$, $-x + y + 4z = 2$, and find the matrix $\mathbf{X}$ such that $\mathbf{XA} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$.

12. Determine the values of λ and μ so that the following systems of equations have (i) Unique solution (ii) No solution (iii) Infinite number of solutions

    (a) $x - 2y + 3z = 6$, $3x + 4y - 10z = 1$, $3x + 4y + \lambda z = \mu$,

    (b) $x + y + z = 1$, $x + 2y + 3z = 2$, $2x + 3y + \lambda y = \mu$,

    (c) $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$.

13. Show that the following sets of equations are consistent and solve them.

    (a) $4x - 2y + 6z = 8$, $x + y - 3z = -1$, $15x - 3y + 9z = 21$,

    (b) $x + y + 2z = 4$, $2x - y + 3z = 9$, $3x - y - z = 2$,

    (c) $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10x = 5$.

14. Find the values of α for which the equations $x + 2y + 3z = 1$, $5x + y + 3z = \alpha$, $3x + 9y + 13z = \alpha^2$ have solutions. Find all the solutions for each value of α.

15. For the matrix equation $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 4+k \end{pmatrix}$, where $\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & k \end{pmatrix}$, find the value of k for which the equation does not have unique solution. For this value of k, solve the equations and interpret the solution geometrically in relation to the linear transformation with matrix $\mathbf{M}$.

16. Show that the linear transformation with matrix $\begin{pmatrix} 1 & -3 & 0 \\ 2 & -4 & 3 \\ 1 & 1 & 6 \end{pmatrix}$ maps every point of the three-dimensional space onto a single plane and give the Cartesian equation of this plane. Describe the set of points mapped onto the origin under this transformation.

17. Write down the 2×2 matrices representing the following transformations of the plane. (i) Reflection in the y-axis, (ii) Reflection in the line $y = x$ (iii) Rotation through $180°$ about the origin (iv) Enlargement from the origin with scale factor λ.

18. (a) A matrix $\mathbf{A}$ of a linear transformation $T(x, y) = (px + qy, rx + sy)$ is given by $\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$.

i. Determine the values of the constant coefficients p, q, r and s.

ii. Find the matrix of the inverse of the transformation T and use it to find the point whose image is $(2, 1)$ under T.

(b) Write down the matrix $\mathbf{B}$ of the linear transformation $S(x, y) = (x+3y, 2x+y)$. Find the image of the point $(2, 1)$ under the transformation given by $\mathbf{AB}$.

19. The linear transformation $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where $\mathbf{M}$ is 3×3 matrix, maps the points with position vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to the points with position vectors $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ respectively. Write down the matrix $\mathbf{M}$ and find the inverse matrix $\mathbf{M}^{-1}$. Show that the transformation with the matrix $\mathbf{M}$ maps points of the plane $x + y + z = 0$ to points of the plane $x = y$ and verify that the inverse transformation with matrix $\mathbf{M}^{-1}$ maps points of the plane $x = y$ to points of the plane $x + y + z = 0$.

20. (a) Show that for all values of θ, the determinant $\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$ lies between 2 and 4 inclusive. State the value of θ for which the determinant has the value 2, and one for which it has the value 4.

(b) Expand the determinant $y = \begin{vmatrix} x & x^2 & x^3 \\ a & b & c \\ p & q & r \end{vmatrix}$ by the first row, and from the expansion, find $\dfrac{dy}{dx}$. Express $\dfrac{dy}{dx}$ in the form of a determinant, and hence or otherwise, find the two values of x for which the determinant $y = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix}$ has stationary values.