Analytical Expressions and Notations Used in BER Analysis

This topic covers the analytical expressions and notations for the theoretical analysis used in the BER functions (`berawgn`, `bercoding`, `berconfint`, `berfadingberfit`, `bersync`), **Bit Error Rate Analysis** app, and Bit Error Rate Analysis Techniques topic.

Common Notation

This table defines the notations used in the analytical expressions in this topic.

Description	Notation
Size of modulation constellation	M
Number of bits per symbol	$k = \log_2 M$
Energy per bit-to-noise power-spectral-density ratio	$\frac{E_b}{N_0}$
Energy per symbol-to-noise power-spectral-density ratio	$\frac{E_s}{N_0} = k\frac{E_b}{N_0}$
Bit error rate (BER)	P_b
Symbol error rate (SER)	P_s
Real part	$\mathrm{Re}[\cdot]$
Floor, largest integer smaller than the value contained in braces	$\lfloor \cdot \rfloor$

This table describes the terms used for mathematical expressions in this topic.

Function	Mathematical Expression
Q function	$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-t^2/2)dt$
Marcum Q function	$Q(a,b) = \int_b^\infty t \exp\left(-\frac{t^2+a^2}{2}\right) I_0(at)dt$
Modified Bessel function of the first kind of order ν	$I_\nu(z) = \sum_{k=0}^\infty \frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$ where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.
Confluent hypergeometric function	${}_1F_1(a,c;x) = \sum_{k=0}^\infty \frac{(a)_k}{(c)_k}\frac{x^k}{k!}$ where the Pochhammer symbol, $(\lambda)_k$, is defined as $(\lambda)_0 = 1$, $(\lambda)_k = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1)$.

This table defines the acronyms used in this topic.

Acronym	Definition
M-PSK	*M*-ary phase-shift keying
DE-M-PSK	Differentially encoded *M*-ary phase-shift keying
BPSK	Binary phase-shift keying
DE-BPSK	Differentially encoded binary phase-shift keying
QPSK	Quaternary phase-shift keying
DE-QPSK	Differentially encoded quadrature phase-shift keying
OQPSK	Offset quadrature phase-shift keying
DE-OQPSK	Differentially encoded offset quadrature phase-shift keying
M-DPSK	*M*-ary differential phase-shift keying
M-PAM	*M*-ary pulse amplitude modulation
M-QAM	*M*-ary quadrature amplitude modulation
M-FSK	*M*-ary frequency-shift keying
MSK	Minimum shift keying
M-CPFSK	*M*-ary continuous-phase frequency-shift keying

Analytical Expressions Used in `berawgn` Function and Bit Error Rate Analysis App

- M-PSK
- DE-M-PSK
- OQPSK
- DE-OQPSK
- M-DPSK
- M-PAM
- M-QAM
- Orthogonal M-FSK with Coherent Detection
- Nonorthogonal 2-FSK with Coherent Detection
- Orthogonal M-FSK with Noncoherent Detection
- Nonorthogonal 2-FSK with Noncoherent Detection
- Precoded MSK with Coherent Detection
- Differentially Encoded MSK with Coherent Detection
- MSK with Noncoherent Detection (Optimum Block-by-Block)
- CPFSK Coherent Detection (Optimum Block-by-Block)

These sections cover the main analytical expressions used in the `berawgn` function and **Bit Error Rate Analysis** app.

M-PSK

From equation 8.22 in [2],

$$P_s = \frac{1}{\pi} \int_0^{(M-1)\pi/M} \exp\left(-\frac{kE_b}{N_0} \frac{\sin^2[\pi/M]}{\sin^2\theta}\right) d\theta$$

This expression is similar, but not strictly equal, to the exact BER (from [4] and equation 8.29 from [2]):

$$P_b = \frac{1}{k}\left(\sum_{i=1}^{M/2} (w'_i) P_i\right)$$

where $w'_i = w_i + w_{M-i}$, $w'_{M/2} = w_{M/2}$, w_i is the Hamming weight of bits assigned to symbol i,

$$P_i = \frac{1}{2\pi} \int_0^{\pi(1-(2i-1)/M)} \exp\left(-\frac{kE_b}{N_0} \frac{\sin^2[(2i-1)\pi/M]}{\sin^2\theta}\right) d\theta$$
$$-\frac{1}{2\pi} \int_0^{\pi(1-(2i+1)/M)} \exp\left(-\frac{kE_b}{N_0} \frac{\sin^2[(2i+1)\pi/M]}{\sin^2\theta}\right) d\theta$$

For M-PSK with M = 2, specifically BPSK, this equation 5.2-57 from [1] applies:

$$P_s = P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

For M-PSK with M = 4, specifically QPSK, these equations 5.2-59 and 5.2-62 from [1] apply:

$$P_s = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]$$
$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

DE-M-PSK

For DE-M-PSK with M = 2, specifically DE-BPSK, this equation 8.36 from [2] applies:

$$P_s = P_b = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - 2Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

For DE-M-PSK with M = 4, specifically DE-QPSK, this equation 8.38 from [2] applies:

$$P_s = 4Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - 8Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) + 8Q^3\left(\sqrt{\frac{2E_b}{N_0}}\right) - 4Q^4\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

From equation 5 in [3],

$$P_b = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1 - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]$$

OQPSK

For OQPSK, use the same BER and SER computations as for QPSK in [2].

DE-OQPSK

For OQPSK, use the same BER and SER computations as for DE-QPSK in [3].

M-DPSK

For M-DPSK, this equation 8.84 from [2] applies:

$$P_s = \frac{\sin(\pi/M)}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{\exp(-(kE_b/N_0)(1-\cos(\pi/M)\cos\theta))}{1-\cos(\pi/M)\cos\theta} d\theta$$

This expression is similar, but not strictly equal, to the exact BER (from [4]):

$$P_b = \frac{1}{k}\left(\sum_{i=1}^{M/2}(w'_i)A_i\right)$$

where $w'_i = w_i + w_{M-i}$, $w'_{M/2} = w_{M/2}$, w_i is the Hamming weight of bits assigned to symbol i,

$$A_i = F\left((2i+1)\frac{\pi}{M}\right) - F\left((2i-1)\frac{\pi}{M}\right)$$

$$F(\psi) = -\frac{\sin\psi}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{\exp(-kE_b/N_0(1-\cos\psi\cos t))}{1-\cos\psi\cos t}dt$$

For M-DPSK with M = 2, this equation 8.85 from [2] applies:

$$P_b = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$$

M-PAM

From equations 8.3 and 8.7 in [2] and equation 5.2-46 in [1],

$$P_s = 2\left(\frac{M-1}{M}\right)Q\left(\sqrt{\frac{6}{M^2-1}\frac{kE_b}{N_0}}\right)$$

From [5],

$$P_b = \frac{2}{M\log_2 M}\times$$

$$\sum_{k=1}^{\log_2 M}\sum_{i=0}^{(1-2^{-k})M-1}\left\{(-1)^{\left\lfloor\frac{i2^{k-1}}{M}\right\rfloor}\left(2^{k-1}-\left\lfloor\frac{i2^{k-1}}{M}+\frac{1}{2}\right\rfloor\right)Q\left((2i+1)\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right)\right\}$$

M-QAM

For square M-QAM, $k = \log_2 M$ is even, so equation 8.10 from [2] and equations 5.2-78 and 5.2-79 from [1] apply:

$$P_s = 4\frac{\sqrt{M}-1}{\sqrt{M}}Q\left(\sqrt{\frac{3}{M-1}\frac{kE_b}{N_0}}\right) - 4\left(\frac{\sqrt{M}-1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{3}{M-1}\frac{kE_b}{N_0}}\right)$$

From [5],

$$P_b = \frac{2}{\sqrt{M}\log_2\sqrt{M}}$$

$$\times\sum_{k=1}^{\log_2\sqrt{M}}\sum_{i=0}^{(1-2^{-k})\sqrt{M}-1}\left\{(-1)^{\left\lfloor\frac{i2^{k-1}}{\sqrt{M}}\right\rfloor}\left(2^{k-1}-\left\lfloor\frac{i2^{k-1}}{\sqrt{M}}+\frac{1}{2}\right\rfloor\right)Q\left((2i+1)\sqrt{\frac{6\log_2 M}{2(M-1)}\frac{E_b}{N_0}}\right)\right\}$$

For rectangular (non-square) M-QAM, $k = \log_2 M$ is odd, $M = I \times J$, $I = 2^{\frac{k-1}{2}}$, and $J = 2^{\frac{k+1}{2}}$. So that,

$$P_s = \frac{4IJ - 2I - 2J}{M}$$
$$\times Q\left(\sqrt{\frac{6\log_2(IJ)}{(I^2+J^2-2)}\frac{E_b}{N_0}}\right) - \frac{4}{M}(1 + IJ - I - J)Q^2\left(\sqrt{\frac{6\log_2(IJ)}{(I^2+J^2-2)}\frac{E_b}{N_0}}\right)$$

From [5],

$$P_b = \frac{1}{\log_2(IJ)}\left(\sum_{k=1}^{\log_2 I} P_I(k) + \sum_{l=1}^{\log_2 J} P_J(l)\right)$$

where

$$P_I(k) = \frac{2}{I}\sum_{i=0}^{(1-2^{-k})I-1}\left\{(-1)^{\left\lfloor\frac{i2^{k-1}}{I}\right\rfloor}\left(2^{k-1} - \left\lfloor\frac{i2^{k-1}}{I} + \frac{1}{2}\right\rfloor\right)Q\left((2i+1)\sqrt{\frac{6\log_2(IJ)}{I^2+J^2-2}\frac{E_b}{N_0}}\right)\right\}$$

and

$$P_J(k) = \frac{2}{J}\sum_{j=0}^{(1-2^{-l})J-1}\left\{(-1)^{\left\lfloor\frac{j2^{l-1}}{J}\right\rfloor}\left(2^{l-1} - \left\lfloor\frac{j2^{l-1}}{J} + \frac{1}{2}\right\rfloor\right)Q\left((2j+1)\sqrt{\frac{6\log_2(IJ)}{I^2+J^2-2}\frac{E_b}{N_0}}\right)\right\}$$

Orthogonal M-FSK with Coherent Detection

From equation 8.40 in [2] and equation 5.2-21 in [1],

$$P_s = 1 - \int_{-\infty}^{\infty}\left[Q\left(-q - \sqrt{\frac{2kE_b}{N_0}}\right)\right]^{M-1}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{q^2}{2}\right)dq$$

$$P_b = \frac{2^{k-1}}{2^k - 1}P_s$$

Nonorthogonal 2-FSK with Coherent Detection

For $M = 2$, equation 5.2-21 in [1] and equation 8.44 in [2] apply:

$$P_s = P_b = Q\left(\sqrt{\frac{E_b(1 - \mathrm{Re}[\rho])}{N_0}}\right)$$

ρ is the complex correlation coefficient, such that:

$$\rho = \frac{1}{2E_b}\int_0^{T_b}\tilde{s}_1(t)\tilde{s}_2^*(t)dt$$

where $\tilde{s}_1(t)$ and $\tilde{s}_2(t)$ are complex lowpass signals, and

$$E_b = \frac{1}{2}\int_0^{T_b}\left|\tilde{s}_1(t)\right|^2 dt = \frac{1}{2}\int_0^{T_b}\left|\tilde{s}_2(t)\right|^2 dt$$

For example, with

$$\tilde{s}_1(t) = \sqrt{\frac{2E_b}{T_b}}e^{j2\pi f_1 t},\ \tilde{s}_2(t) = \sqrt{\frac{2E_b}{T_b}}e^{j2\pi f_2 t}$$

then

$$\rho = \frac{1}{2E_b}\int_0^{T_b}\sqrt{\frac{2E_b}{T_b}}e^{j2\pi f_1 t}\sqrt{\frac{2E_b}{T_b}}e^{-j2\pi f_2 t}dt = \frac{1}{T_b}\int_0^{T_b}e^{j2\pi(f_1-f_2)t}dt$$
$$= \frac{\sin(\pi\Delta f T_b)}{\pi\Delta f T_b}e^{j\pi\Delta ft}$$

where $\Delta f = f_1 - f_2$.

From equation 8.44 in [2],

$$\mathrm{Re}[\rho] = \mathrm{Re}\left[\frac{\sin(\pi\Delta f T_b)}{\pi\Delta f T_b}e^{j\pi\Delta ft}\right] = \frac{\sin(\pi\Delta f T_b)}{\pi\Delta f T_b}\cos(\pi\Delta f T_b) = \frac{\sin(2\pi\Delta f T_b)}{2\pi\Delta f T_b}$$
$$\Rightarrow P_b = Q\left(\sqrt{\frac{E_b(1-\sin(2\pi\Delta f T_b)/(2\pi\Delta f T_b))}{N_0}}\right)$$

where $h = \Delta f T_b$.

Orthogonal M-FSK with Noncoherent Detection

From equation 5.4-46 in [1] and equation 8.66 in [2],

$$P_s = \sum_{m=1}^{M-1}(-1)^{m+1}\binom{M-1}{m}\frac{1}{m+1}\exp\left[-\frac{m}{m+1}\frac{kE_b}{N_0}\right]$$
$$P_b = \frac{1}{2}\frac{M}{M-1}P_s$$

Nonorthogonal 2-FSK with Noncoherent Detection

For $M = 2$, this equation 5.4-53 from [1] and this equation 8.69 from [2] apply:

$$P_s = P_b = Q(\sqrt{a}, \sqrt{b}) - \frac{1}{2}\exp\left(-\frac{a+b}{2}\right)I_0(\sqrt{ab})$$

where

$$a = \frac{E_b}{2N_0}(1-\sqrt{1-|\rho|^2}),\ b = \frac{E_b}{2N_0}(1+\sqrt{1-|\rho|^2})$$

Precoded MSK with Coherent Detection

Use the same BER and SER computations as for BPSK.

Differentially Encoded MSK with Coherent Detection

Use the same BER and SER computations as for DE-BPSK.

MSK with Noncoherent Detection (Optimum Block-by-Block)

The upper bound on error rate from equations 10.166 and 10.164 in [6]) is

$$P_s = P_b$$

$$\leq \frac{1}{2}\left[1 - Q\left(\sqrt{b_1}, \sqrt{a_1}\right) + Q\left(\sqrt{a_1}, \sqrt{b_1}\right)\right] + \frac{1}{4}\left[1 - Q\left(\sqrt{b_4}, \sqrt{a_4}\right) + Q\left(\sqrt{a_4}, \sqrt{b_4}\right)\right] + \frac{1}{2}e^{-\frac{E_b}{N_0}}$$

where

$$a_1 = \frac{E_b}{N_0}\left(1 - \sqrt{\frac{3 - 4/\pi^2}{4}}\right), \quad b_1 = \frac{E_b}{N_0}\left(1 + \sqrt{\frac{3 - 4/\pi^2}{4}}\right)$$

$$a_4 = \frac{E_b}{N_0}\left(1 - \sqrt{1 - 4/\pi^2}\right), \quad b_4 = \frac{E_b}{N_0}\left(1 + \sqrt{1 - 4/\pi^2}\right)$$

CPFSK Coherent Detection (Optimum Block-by-Block)

The lower bound on error rate (from equation 5.3-17 in [1]) is

$$P_s > K_{\delta_{\min}} Q\left(\sqrt{\frac{E_b}{N_0}\delta_{\min}^2}\right)$$

The upper bound on error rate is

$$\delta_{\min}^2 > \min_{1 \leq i \leq M-1}\{2i(1 - \mathrm{sinc}(2ih))\}$$

where h is the modulation index, and $K_{\delta_{\min}}$ is the number of paths with the minimum distance.

$$P_b \cong \frac{P_s}{k}$$

Analytical Expressions Used in `berfading` Function and Bit Error Rate Analysis App

- Notation
- M-PSK with MRC
- DE-M-PSK with MRC
- M-PAM with MRC
- M-QAM with MRC
- M-DPSK with Postdetection EGC
- Orthogonal 2-FSK, Coherent Detection with MRC
- Nonorthogonal 2-FSK, Coherent Detection with MRC
- Orthogonal M-FSK, Noncoherent Detection with EGC
- Nonorthogonal 2-FSK, Noncoherent Detection with No Diversity

This section covers the main analytical expressions used in the `berfading` function and the **Bit Error Rate Analysis** app.

Notation

This table describes the additional notations used in analytical expressions in this section.

Description	Notation
Power of the fading amplitude r	$\Omega = E[r^2]$, where $E[\cdot]$ denotes statistical expectation
Number of diversity branches	L

Signal to Noise Ratio (SNR) per symbol per branch	$\overline{\gamma}_l = \left(\Omega_l \frac{E_s}{N_0}\right)/L = \left(\Omega_l \frac{kE_b}{N_0}\right)/L$ For identically-distributed diversity branches, $\overline{\gamma} = \left(\Omega \frac{kE_b}{N_0}\right)/L$
Moment generating functions for each diversity branch	For Rayleigh fading channels: $M_{\gamma_l}(s) = \frac{1}{1 - s\overline{\gamma}_l}$ For Rician fading channels: $M_{\gamma_l}(s) = \frac{1+K}{1+K-s\overline{\gamma}_l} e^{\left[\frac{Ks\overline{\gamma}_l}{(1+K)-s\overline{\gamma}_l}\right]}$ K is the ratio of the energy in the specular component to the energy in the diffuse component (linear scale). For identically-distributed diversity branches, $M_{\gamma_l}(s) = M_\gamma(s)$ for all l.

This table defines the additional acronyms used in this section.

Acronym	**Definition**
MRC	Maximal-ratio combining
EGC	Equal-gain combining

M-PSK with MRC

From equation 9.15 in [2],

$$P_s = \frac{1}{\pi} \int_0^{(M-1)\pi/M} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{\sin^2(\pi/M)}{\sin^2\theta}\right) d\theta$$

From [4] and [2],

$$P_b = \frac{1}{k}\left(\sum_{i=1}^{M/2} (w_i')\overline{P}_i\right)$$

where $w_i' = w_i + w_{M-i}$, $w_{M/2}' = w_{M/2}$, w_i is the Hamming weight of bits assigned to symbol i,

$$\overline{P}_i = \frac{1}{2\pi} \int_0^{\pi(1-(2i-1)/M)} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{1}{\sin^2\theta}\sin^2\frac{(2i-1)\pi}{M}\right) d\theta$$
$$- \frac{1}{2\pi} \int_0^{\pi(1-(2i+1)/M)} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{1}{\sin^2\theta}\sin^2\frac{(2i+1)\pi}{M}\right) d\theta$$

For the special case of Rayleigh fading with $M = 2$ (from equations C-18 and C-21 and Table C-1 in [6]),

$$P_b = \frac{1}{2}\left[1 - \mu \sum_{i=0}^{L-1} \binom{2i}{i} \left(\frac{1-\mu^2}{4}\right)^i\right]$$

where

$$\mu = \sqrt{\frac{\overline{\gamma}}{\overline{\gamma} + 1}}$$

If $L = 1$, then:

$$P_b = \frac{1}{2}\left[1 - \sqrt{\frac{\overline{\gamma}}{\overline{\gamma} + 1}}\right]$$

DE-M-PSK with MRC

For $M = 2$ (from equations 8.37 and 9.8-9.11 in [2]),

$$P_s = P_b = \frac{2}{\pi}\int_0^{\pi/2}\prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{1}{\sin^2\theta}\right)d\theta - \frac{2}{\pi}\int_0^{\pi/4}\prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{1}{\sin^2\theta}\right)d\theta$$

M-PAM with MRC

From equation 9.19 in [2],

$$P_s = \frac{2(M-1)}{M\pi}\int_0^{\pi/2}\prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{3/(M^2-1)}{\sin^2\theta}\right)d\theta$$

From [5] and [2],

$$P_b = \frac{2}{\pi M\log_2 M}$$

$$\times\sum_{k=1}^{\log_2 M}\sum_{i=0}^{(1-2^{-k})M-1}\left\{(-1)^{\left\lfloor\frac{i2^{k-1}}{M}\right\rfloor}\left(2^{k-1} - \left\lfloor\frac{i2^{k-1}}{M} + \frac{1}{2}\right\rfloor\right)\int_0^{\pi/2}\prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{(2i+1)^2 3/(M^2-1)}{\sin^2\theta}\right)d\theta\right\}$$

M-QAM with MRC

For square M-QAM, $k = \log_2 M$ is even (equation 9.21 in [2]),

$$P_s = \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{M}}\right)\int_0^{\pi/2}\prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{3/(2(M-1))}{\sin^2\theta}\right)d\theta$$

$$-\frac{4}{\pi}\left(1 - \frac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4}\prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{3/(2(M-1))}{\sin^2\theta}\right)d\theta$$

From [5] and [2]:

$$P_b = \frac{2}{\pi\sqrt{M}\log_2\sqrt{M}}$$

$$\times\sum_{k=1}^{\log_2\sqrt{M}}\sum_{i=0}^{(1-2^{-k})\sqrt{M}-1}\left\{(-1)^{\left\lfloor\frac{i2^{k-1}}{\sqrt{M}}\right\rfloor}\left(2^{k-1} - \left\lfloor\frac{i2^{k-1}}{\sqrt{M}} + \frac{1}{2}\right\rfloor\right)\int_0^{\pi/2}\prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{(2i+1)^2 3/(2(M-1))}{\sin^2\theta}\right)d\theta\right\}$$

For rectangular (nonsquare) M-QAM, $k = \log_2 M$ is odd, $M = I \times J$, $I = 2^{\frac{k-1}{2}}$, $J = 2^{\frac{k+1}{2}}$, $\overline{\gamma}_l = \Omega_l \log_2(IJ)\frac{E_b}{N_0}$,

$$P_s = \frac{4IJ - 2I - 2J}{M\pi} \int_0^{\pi/2} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{3/(I^2 + J^2 - 2)}{\sin^2\theta}\right) d\theta$$

$$-\frac{4}{M\pi}(1 + IJ - I - J) \int_0^{\pi/4} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{3/(I^2 + J^2 - 2)}{\sin^2\theta}\right) d\theta$$

From [5] and [2],

$$P_b = \frac{1}{\log_2(IJ)} \left(\sum_{k=1}^{\log_2 I} P_I(k) + \sum_{l=1}^{\log_2 J} P_J(l) \right)$$

$$P_I(k) = \frac{2}{I\pi} \sum_{i=0}^{(1-2^{-k})I-1} \left\{ (-1)^{\left\lfloor \frac{i2^{k-1}}{I} \right\rfloor} \left(2^{k-1} - \left\lfloor \frac{i2^{k-1}}{I} + \frac{1}{2} \right\rfloor \right) \int_0^{\pi/2} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{(2i+1)^2 3/(I^2 + J^2 - 2)}{\sin^2\theta} \right) d\theta \right\}$$

$$P_J(k) = \frac{2}{J\pi} \sum_{j=0}^{(1-2^{-l})J-1} \left\{ (-1)^{\left\lfloor \frac{j2^{l-1}}{J} \right\rfloor} \left(2^{l-1} - \left\lfloor \frac{j2^{l-1}}{J} + \frac{1}{2} \right\rfloor \right) \int_0^{\pi/2} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{(2j+1)^2 3/(I^2 + J^2 - 2)}{\sin^2\theta} \right) d\theta \right\}$$

M-DPSK with Postdetection EGC

From equation 8.165 in [2],

$$P_s = \frac{\sin(\pi/M)}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{[1 - \cos(\pi/M)\cos\theta]} \prod_{l=1}^{L} M_{\gamma_l}(-[1 - \cos(\pi/M)\cos\theta]) d\theta$$

From [4] and [2],

$$P_b = \frac{1}{k} \left(\sum_{i=1}^{M/2} (w_i') \overline{A}_i \right)$$

where $w_i' = w_i + w_{M-i}$, $w'_{M/2} = w_{M/2}$, w_i is the Hamming weight of bits assigned to symbol i,

$$\overline{A}_i = \overline{F}\left((2i+1)\frac{\pi}{M} \right) - \overline{F}\left((2i-1)\frac{\pi}{M} \right)$$

$$\overline{F}(\psi) = -\frac{\sin\psi}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{(1 - \cos\psi\cos t)} \prod_{l=1}^{L} M_{\gamma_l}(-(1 - \cos\psi\cos t)) dt$$

For the special case of Rayleigh fading with $M = 2$ and $L = 1$ (equation 8.173 from [2]),

$$P_b = \frac{1}{2(1 + \overline{\gamma})}$$

Orthogonal 2-FSK, Coherent Detection with MRC

From equation 9.11 in [2],

$$P_s = P_b = \frac{1}{\pi} \int_0^{\pi/2} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{1/2}{\sin^2\theta} \right) d\theta$$

For the special case of Rayleigh fading (equations 14.4-15 and 14.4-21 in [1]),

$$P_s = P_b = \frac{1}{2^L}\left(1 - \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \frac{1}{2^k}\left(1 + \sqrt{\frac{\overline{\gamma}}{2+\overline{\gamma}}}\right)^k$$

Nonorthogonal 2-FSK, Coherent Detection with MRC

From equations 9.11 and 8.44 in [2],

$$P_s = P_b = \frac{1}{\pi}\int_0^{\pi/2} \prod_{l=1}^{L} M_{\gamma_l}\left(-\frac{(1-\mathrm{Re}[\rho])/2}{\sin^2\theta}\right) d\theta$$

For the special case of Rayleigh fading with $L = 1$ (equations 20 in [8] and 8.130 in [2]),

$$P_s = P_b = \frac{1}{2}\left[1 - \sqrt{\frac{\overline{\gamma}(1-\mathrm{Re}[\rho])}{2+\overline{\gamma}(1-\mathrm{Re}[\rho])}}\right]$$

Orthogonal M-FSK, Noncoherent Detection with EGC

For Rayleigh fading, from equation 14.4-47 in [1],

$$P_s = 1 - \int_0^{\infty} \frac{1}{(1+\overline{\gamma})^L (L-1)!} U^{L-1} e^{-\frac{U}{1+\overline{\gamma}}} \left(1 - e^{-U}\sum_{k=0}^{L-1} \frac{U^k}{k!}\right)^{M-1} dU$$

$$P_b = \frac{1}{2}\frac{M}{M-1} P_s$$

For Rician fading from equation 41 in [8],

$$P_s = \sum_{r=1}^{M-1} \frac{(-1)^{r+1} e^{-LK\overline{\gamma}_r/(1+\overline{\gamma}_r)}}{(r(1+\overline{\gamma}_r)+1)^L} \binom{M-1}{r} \sum_{n=0}^{r(L-1)} \beta_{nr} \frac{\Gamma(L+n)}{\Gamma(L)} \left[\frac{1+\overline{\gamma}_r}{r+1+r\overline{\gamma}_r}\right]^n {}_1F_1\left(L+n, L; \frac{LK\overline{\gamma}_r/(1+\overline{\gamma}_r)}{r(1+\overline{\gamma}_r)+1}\right)$$

$$P_b = \frac{1}{2}\frac{M}{M-1} P_s$$

where

$$\overline{\gamma}_r = \frac{1}{1+K}\overline{\gamma}$$

$$\beta_{nr} = \sum_{i=n-(L-1)}^{n} \frac{\beta_{i(r-1)}}{(n-i)!} I_{[0,(r-1)(L-1)]}(i)$$

$$\beta_{00} = \beta_{0r} = 1$$

$$\beta_{n1} = 1/n!$$

$$\beta_{1r} = r$$

and $I_{[a,b]}(i) = 1$ if $a \le i \le b$ and 0 otherwise.

Nonorthogonal 2-FSK, Noncoherent Detection with No Diversity

From equation 8.163 in [2],

$$P_s = P_b = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1-\varsigma^2}{1+2\varsigma\sin\theta+\varsigma^2} M_{\gamma}\left(-\frac{1}{4}(1+\sqrt{1-\rho^2})(1+2\varsigma\sin\theta+\varsigma^2)\right) d\theta$$

where

$$\varsigma = \sqrt{\frac{1-\sqrt{1-\rho^2}}{1+\sqrt{1-\rho^2}}}$$

Analytical Expressions Used in `bercoding` Function and Bit Error Rate Analysis App

This section covers the main analytical expressions used in the `bercoding` function and the **Bit Error Rate Analysis** app.

- Common Notation
- Block Coding
- Convolutional Coding

Common Notation

This table describes the additional notations used in analytical expressions in this section.

Description	Notation
Energy-per-information bit-to-noise power-spectral-density ratio	$\gamma_b = \frac{E_b}{N_0}$
Message length	K
Code length	N
Code rate	$R_c = \frac{K}{N}$

Block Coding

This section describes the specific notation for block coding expressions, where $d_{\min}$ is the minimum distance of the code.

Soft Decision

For BPSK, QPSK, OQPSK, 2-PAM, 4-QAM, and precoded MSK, equation 8.1-52 in [1]) applies,

$$P_b \le \frac{1}{2}(2^K - 1)Q\left(\sqrt{2\gamma_b R_c d_{\min}}\right)$$

For DE-BPSK, DE-QPSK, DE-OQPSK, and DE-MSK,

$$P_b \le \frac{1}{2}(2^K - 1)\left[2Q\left(\sqrt{2\gamma_b R_c d_{\min}}\right)\left[1 - Q\left(\sqrt{2\gamma_b R_c d_{\min}}\right)\right]\right]$$

For BFSK coherent detection, equations 8.1-50 and 8.1-58 in [1] apply,

$$P_b \le \frac{1}{2}(2^K - 1)Q\left(\sqrt{\gamma_b R_c d_{\min}}\right)$$

For BFSK noncoherent square-law detection, equations 8.1-65 and 8.1-64 in [1] apply,

$$P_b \le \frac{1}{2}\frac{2^K - 1}{2^{2d_{\min}-1}}\exp\left(-\frac{1}{2}\gamma_b R_c d_{\min}\right)\sum_{i=0}^{d_{\min}-1}\left(\frac{1}{2}\gamma_b R_c d_{\min}\right)^i \frac{1}{i!}\sum_{r=0}^{d_{\min}-1-i}\binom{2d_{\min}-1}{r}$$

For DPSK,

$$P_b \le \frac{1}{2}\frac{2^K - 1}{2^{2d_{\min}-1}}\exp(-\gamma_b R_c d_{\min})\sum_{i=0}^{d_{\min}-1}(\gamma_b R_c d_{\min})^i \frac{1}{i!}\sum_{r=0}^{d_{\min}-1-i}\binom{2d_{\min}-1}{r}$$

Hard Decision

For general linear block code, equations 4.3 and 4.4 in [9], and 12.136 in [6] apply,

$$P_b \le \frac{1}{N} \sum_{m=t+1}^{N} (m+t) \binom{N}{m} p^m (1-p)^{N-m}$$

$$t = \left\lfloor \frac{1}{2}(d_{\min} - 1) \right\rfloor$$

For Hamming code, equations 4.11 and 4.12 in [9] and 6.72 and 6.73 in [7] apply

$$P_b \approx \frac{1}{N} \sum_{m=2}^{N} m \binom{N}{m} p^m (1-p)^{N-m} = p - p(1-p)^{N-1}$$

For rate (24,12) extended Golay code, equations 4.17 in [9] and 12.139 in [6] apply:

$$P_b \le \frac{1}{24} \sum_{m=4}^{24} \beta_m \binom{24}{m} p^m (1-p)^{24-m}$$

where β_m is the average number of channel symbol errors that remain in corrected N-tuple format when the channel caused m symbol errors (see table 4.2 in [9]).

For Reed-Solomon code with $N = Q - 1 = 2^q - 1$,

$$P_b \approx \frac{2^{q-1}}{2^q - 1} \frac{1}{N} \sum_{m=t+1}^{N} m \binom{N}{m} (P_s)^m (1-P_s)^{N-m}$$

For FSK, equations 4.25 and 4.27 in [9], 8.1-115 and 8.1-116 in [1], 8.7 and 8.8 in [7], and 12.142 and 12.143 in [6] apply,

$$P_b \approx \frac{1}{q} \frac{1}{N} \sum_{m=t+1}^{N} m \binom{N}{m} (P_s)^m (1-P_s)^{N-m}$$

otherwise, if $\log_2 Q / \log_2 M = q/k = h$, where h is an integer (equation 1 in [10]) applies,

$$P_s = 1 - (1-s)^h$$

where s is the SER in an uncoded AWGN channel.

For example, for BPSK, $M = 2$ and $P_s = 1 - (1-s)^q$, otherwise P_s is given by table 1 and equation 2 in [10].

Convolutional Coding

This section describes the specific notation for convolutional coding expressions, where d_{free} is the free distance of the code, and a_d is the number of paths of distance d from the all-zero path that merges with the all-zero path for the first time.

Soft Decision

From equations 8.2-26, 8.2-24, and 8.2-25 in [1] and 13.28 and 13.27 in [6] apply,

$$P_b < \sum_{d=d_{free}}^{\infty} a_d f(d) P_2(d)$$

The transfer function is given by

$$T(D,N) = \sum_{d=d_{free}}^{\infty} a_d D^d N^{f(d)}$$

$$\left.\frac{dT(D,N)}{dN}\right|_{N=1} = \sum_{d=d_{free}}^{\infty} a_d f(d) D^d$$

where $f(d)$ is the exponent of N as a function of d.

This equation gives the results for BPSK, QPSK, OQPSK, 2-PAM, 4-QAM, precoded MSK, DE-BPSK, DE-QPSK, DE-OQPSK, DE-MSK, DPSK, and BFSK:

$$P_2(d) = P_b|_{\frac{E_b}{N_0} = \gamma_b R_c d}$$

where P_b is the BER in the corresponding uncoded AWGN channel. For example, for BPSK (equation 8.2-20 in [1]),

$$P_2(d) = Q\left(\sqrt{2\gamma_b R_c d}\right)$$

Hard Decision

From equations 8.2-33, 8.2-28, and 8.2-29 in [1] and 13.28, 13.24, and 13.25 in [6] apply,

$$P_b < \sum_{d=d_{free}}^{\infty} a_d f(d) P_2(d)$$

When d is odd,

$$P_2(d) = \sum_{k=(d+1)/2}^{d} \binom{d}{k} p^k (1-p)^{d-k}$$

and when d is even,

$$P_2(d) = \sum_{k=d/2+1}^{d} \binom{d}{k} p^k (1-p)^{d-k} + \frac{1}{2}\binom{d}{d/2} p^{d/2}(1-p)^{d/2}$$

where p is the bit error rate (BER) in an uncoded AWGN channel.

Analytical Expressions Used in `bersync` Function and Bit Error Rate Analysis App

- Timing Synchronization Error
- Timing Synchronization Error

This section covers the main analytical expressions used in the `bersync` function and the **Bit Error Rate Analysis** app.

Timing Synchronization Error

To compute the BER for a communications system with a timing synchronization error, the `bersync` function uses this formula from [13]:

$$\frac{1}{4\pi\sigma}\int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{2\sigma^2}\right)\int_{\sqrt{2R}(1-2|\xi|)}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx\, d\xi + \frac{1}{2\sqrt{2\pi}}\int_{\sqrt{2R}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx$$

where σ is the timing error, and R is the linear E_b/N_0 value.

Timing Synchronization Error

To compute the BER for a communications system with a carrier synchronization error, the `bersync` function uses this formula from [13]:

$$\frac{1}{\pi\sigma}\int_{0}^{\infty}\exp(-\frac{\phi^2}{2\sigma^2})\int_{\sqrt{2R}\cos\phi}^{\infty}\exp(-\frac{y^2}{2})dyd\phi$$

where σ is the phase error R is the linear E_b/N_0 value.

See Also

Apps

Bit Error Rate Analysis

Functions

`berawgn` | `bercoding` | `berconfint` | `berfading` | `berfit` | `bersync`

Related Topics

- Bit Error Rate Analysis Techniques
- Use Bit Error Rate Analysis App