LDPC codes
The sum-product algorithm

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1 Hyperbolic tangent

1.1 Definition

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2\cdot x}}{1 + e^{-2\cdot x}} \tag{1}$$

1.2 Inverse function

$$\frac{1+\tanh(x)}{1-\tanh(x)} = \frac{1 + \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}{1 - \dfrac{e^x - e^{-x}}{e^x + e^{-x}}} = \frac{e^x}{e^{-x}} = e^{2\cdot x} \tag{2}$$

$$x = \operatorname{artanh}\big(\tanh(x)\big) = \frac{1}{2}\cdot\ln\left(\frac{1+\tanh(x)}{1-\tanh(x)}\right) \tag{3}$$

2 Log-likelihood ratio

2.1 Definitions

$$\mathrm{LLR}^{(0)}(x) = \ln\left(\frac{P(x=0)}{P(x=1)}\right) \tag{4}$$

$$\mathrm{LLR}^{(1)}(x) = \ln\left(\frac{P(x=1)}{P(x=0)}\right) \tag{5}$$

2.2 Moving back and forth between probabilities and log-likelihood ratio

$$P(x=0) = \frac{e^{\,\mathrm{LLR}^{(0)}(x)}}{1+e^{\,\mathrm{LLR}^{(0)}(x)}} = \frac{1}{1+e^{\,-\,\mathrm{LLR}^{(0)}(x)}} \tag{6}$$

$$P(x=1) = \frac{1}{1+e^{\,\mathrm{LLR}^{(0)}(x)}} = \frac{e^{\,-\,\mathrm{LLR}^{(0)}(x)}}{1+e^{\,-\,\mathrm{LLR}^{(0)}(x)}}$$

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$$
\begin{aligned}
P(x=0) &= \frac{1}{1+e^{\,\mathrm{LLR}^{(1)}(x)}} = \frac{e^{\,-\mathrm{LLR}^{(1)}(x)}}{1+e^{\,-\mathrm{LLR}^{(1)}(x)}} \\
P(x=1) &= \frac{e^{\,\mathrm{LLR}^{(1)}(x)}}{1+e^{\,\mathrm{LLR}^{(1)}(x)}} = \frac{1}{1+e^{\,-\mathrm{LLR}^{(1)}(x)}}
\end{aligned} \tag{7}
$$

2.3 Property

From equation (1)

$$
\begin{aligned}
1-2\cdot P(x=1) &= 1-2\cdot\frac{e^{\,-\mathrm{LLR}^{(0)}(x)}}{1+e^{\,-\mathrm{LLR}^{(0)}(x)}} = \frac{1-e^{\,-\mathrm{LLR}^{(0)}(x)}}{1+e^{\,-\mathrm{LLR}^{(0)}(x)}} \\
1-2\cdot P(x=1) &= \tanh\left(\frac{\mathrm{LLR}^{(0)}(x)}{2}\right)
\end{aligned} \tag{8}
$$

3 Sum-product algorithm. Low Density Parity Check codes

3.1 Definitions

The sets of nodes are defined as follows:

- $A(c_j)$ is the set of variable nodes connected to check node c_j.
- $A(v_i)$ is the set of check nodes connected to variable node v_i.

The estimation of probabilities by check nodes is given by:

$$
P(v_i=0, c_j\to v_i) = \frac{1}{2}+\frac{1}{2}\cdot\prod_{v\in A(c_j)\setminus v_i}\left(1-2\cdot P(v=1, v\to c_j)\right) \tag{9}
$$

3.2 Message from variable node to check node. Estimation of probabilities by variable nodes, LLR($v_i \to c_j$)

$$
\mathrm{LLR}(v_i, v_i\to c_j) = \mathrm{LLR}^{(0)}(v_i, v_i\to c_j) = \ln\left(\frac{P(v_i=0, v_i\to c_j)}{P(v_i=1, v_i\to c_j)}\right) \tag{10}
$$

From equations (4) and (6),

$$
\begin{aligned}
P(v_i=0, v_i\to c_j) &= \frac{e^{\,\mathrm{LLR}(v_i,v_i\to c_j)}}{1+e^{\,\mathrm{LLR}(v_i,v_i\to c_j)}} = \frac{1}{1+e^{\,-\mathrm{LLR}(v_i,v_i\to c_j)}} \\
P(v_i=1, v_i\to c_j) &= \frac{1}{1+e^{\,\mathrm{LLR}(v_i,v_i\to c_j)}} = \frac{e^{\,-\mathrm{LLR}(v_i,v_i\to c_j)}}{1+e^{\,-\mathrm{LLR}(v_i,v_i\to c_j)}}
\end{aligned} \tag{11}
$$

3.3 Message from chek node to variable node. Estimation of probabilities by check nodes, LLR($c_j \to v_i$)

$$
\mathrm{LLR}(v_i, c_j\to v_i) = \mathrm{LLR}^{(0)}(v_i, c_j\to v_i) = \ln\left(\frac{P(v_i=0, c_j\to v_i)}{P(v_i=1, c_j\to v_i)}\right) \tag{12}
$$

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From equation (9)

$$
\begin{aligned}
\mathrm{LLR}(v_i, c_j \to v_i) &= \ln\left(\frac{\frac{1}{2} + \frac{1}{2} \cdot \prod_{v \in A(c_j)\setminus v_i} (1 - 2 \cdot P(v = 1, v \to c_j))}{1 - \frac{1}{2} - \frac{1}{2} \cdot \prod_{v \in A(c_j)\setminus v_i} (1 - 2 \cdot P(v = 1, v \to c_j))} \right) \\
\mathrm{LLR}(v_i, c_j \to v_i) &= \ln\left(\frac{1 + \prod_{v \in A(c_j)\setminus v_i} (1 - 2 \cdot P(v = 1, v \to c_j))}{1 - \prod_{v \in A(c_j)\setminus v_i} (1 - 2 \cdot P(v = 1, v \to c_j))} \right) \qquad (13)
\end{aligned}
$$

Using equation(8)

$$
\mathrm{LLR}(v_i, c_j \to v_i) = \ln\left(\frac{1 + \prod_{v \in A(c_j)\setminus v_i} \tanh\left(\frac{\mathrm{LLR}(v, v \to c_j)}{2}\right)}{1 - \prod_{v \in A(c_j)\setminus v_i} \tanh\left(\frac{\mathrm{LLR}(v, v \to c_j)}{2}\right)} \right) \qquad (14)
$$

From equations (3) and (14) we get

$$
\mathrm{LLR}(v_i, c_j \to v_i) = 2 \cdot \operatorname{artanh}\left(\prod_{v \in A(c_j)\setminus v_i} \tanh\left(\frac{\mathrm{LLR}(v, v \to c_j)}{2}\right) \right) \qquad (15)
$$