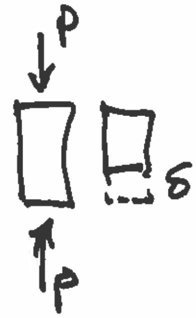
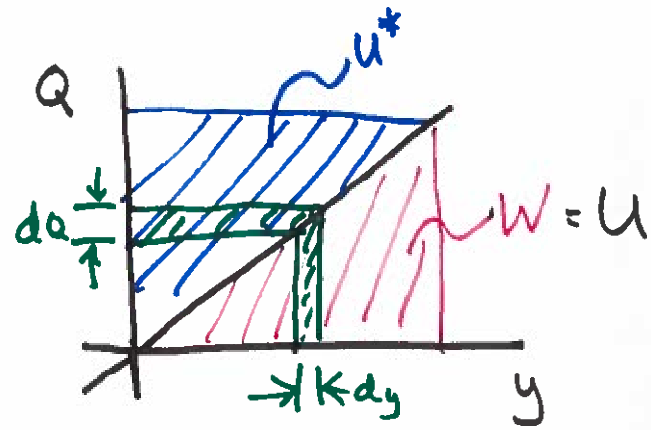
## Strain Energy




W: work done on the material by a load Q to achieve a deflection y.

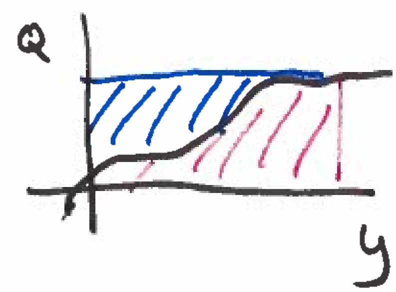
W= PS




$$W = \int Q(y)\,dy$$

Work cause a potentral energy to be stored in the material.

"Strain energy" ⇒ U




$$W = U$$

$U^*$ ⇒ complementary energy

For linear only
$$dU^* = dU$$

$$U^* = \int y(Q)\,dQ \Rightarrow dU^* = y\,dQ$$

$$dU^* = dU = y\,dQ = Q\,dy \Rightarrow \boxed{y = \frac{dU}{dQ}}$$

- Valid for single applied loads on structural element

If element is subjected to multiple loads, $Q_i$, all within the elastic range:

The the deflection, $y_i$, associated with the point of application of $Q_i$ is:

$$\boxed{y_i = \frac{\partial U}{\partial Q_i}} \Rightarrow$$ ~~Castiliano~~ Castigliano's Theorem

$U$: total strain energy

$Q_i$: Single applied load

$y_i$: deflection at the point of load

: When an element is elastically deflected by an combination of loads, the deflection at any point, in any direction is equal to the partial derivative of the total strain energy wrt load at that point acting in the direction.

- The applied load may or may not exist.

L, A, E $\Rightarrow$ constant wrt to P

$$u^* = \int y(Q)\, dQ = \int \frac{PL}{AE}\, dP$$

$$\delta = \frac{PL}{AE} \Rightarrow y = \frac{QL}{AE}$$ as func of displacement

$$U = u^* = \frac{P^2 L}{2AE}$$

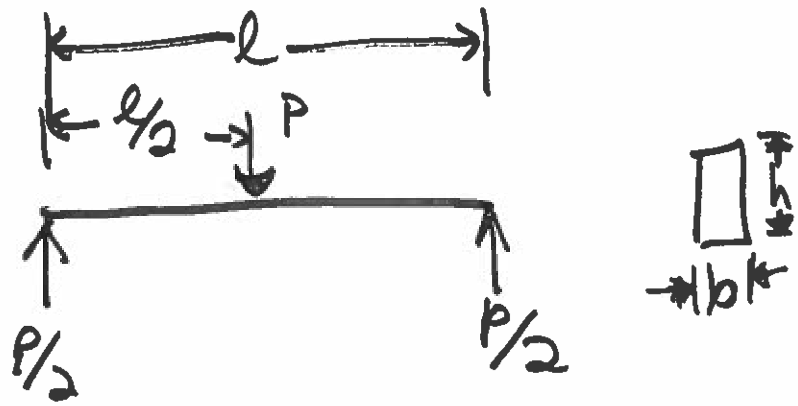
| Load | Factors | Strain Energy Constant | Strain Energy, variable factors |
|---|---|---|---|
| Axial | A, E, P | $U = \frac{P^2 L}{2AE}$ | $U = \int_0^L \frac{P^2}{2EA}\, dx$ |
| Bending | I, E, M | $U = \frac{M^2 L}{2EI}$ | $U = \int_0^L \frac{M^2}{2EI}\, dx$ |
| Torsion | J, G, T | $U = \frac{T^2 L}{2GJ}$ | $U = \int_0^L \frac{T^2}{2GJ}\, dx$ |
| Shear | A, G, V | $U = \frac{CV^2 L}{2AG}$ | $U = \int_0^L \frac{CV^2}{2AG}\, dx$ |

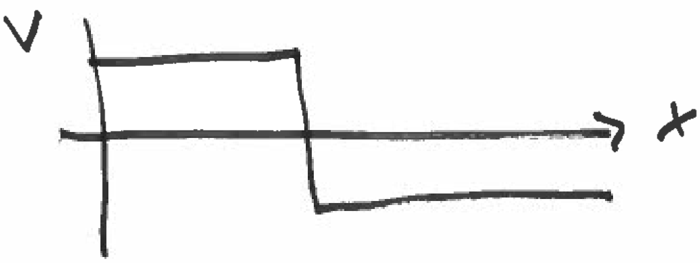
Table 4-1

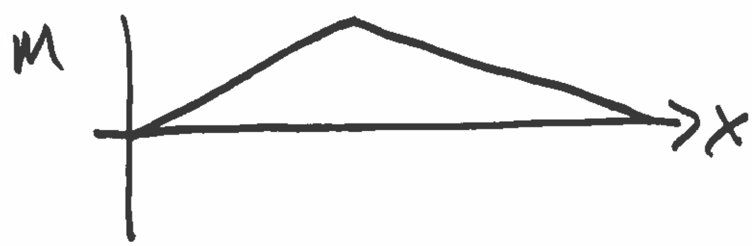
Cross sections

| Cross sec | C |
|---|---|
| Rectangular | 1.2 |
| Circular | 1.11 |

**Example** Find the deflection at the midpoint.




**Solution**: Beam has bending and shear.

$$V = \begin{cases} P/2 & 0 \le x \le \frac{l}{2} \\ -P/2 & \frac{l}{2} \le x \le l \end{cases}$$

$$M = \begin{cases} \frac{P}{2}x & 0 \le x \le \frac{l}{2} \\ \frac{Pl}{2} - \frac{Px}{2} & \frac{l}{2} \le x \le l \end{cases}$$

## Bending Strain Energy

$$U_{1} = \int_0^{\ell/2} \frac{M^2}{2EI}\,dx$$

$$U_{1} = U_{2} = \int_{\ell/2}^{\ell} \frac{M^2}{2EI}\,dx$$

right

left

## Shear

$$U_{3} = \int_0^{\ell} \frac{cV^2}{2AG}\,dx$$

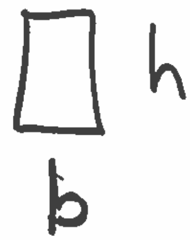
## Total

$$U = U_{1} + U_{2} + U_{3}$$

$$U = 2\int_0^{\ell/2} \frac{M^2}{2EI}\,dx + \int_0^{\ell} \frac{cV^2}{2AG}\,dx$$

$$U = \frac{P^2L^3}{96EI} + \frac{15P^2L}{AG}$$

$$\boxed{y\left(\tfrac{l}{3}\right) = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \underbrace{\frac{3PL}{AG}}_{\text{not present in table A-9}}}$$

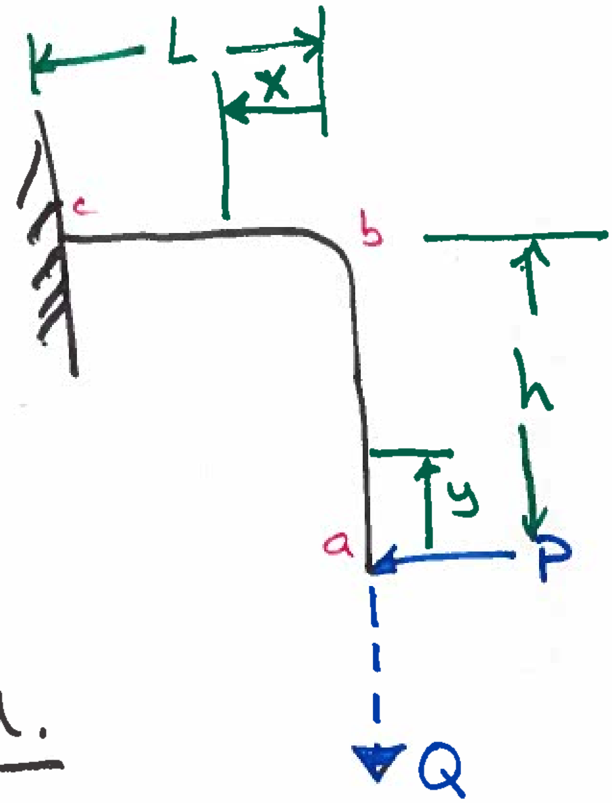



$C = 1.2$

$$\boxed{\text{For } l/h > 10 \Rightarrow \text{shear typically negligible}}$$

## Example

Find the vertical deflection at the free end.




## Solution

Dummy load Q is needed.

1) Bending ab, $M_{ab} = Py$ varying

2) Bending in bc, $M_{bc} = Qx + Ph$ varying

3) Tension in ab, $Q$ constant

4) compression cb, $P$ constant

neglecting transverse shear

$$U = \int_0^h \frac{M_{ab}^2}{2EI}\,dy + \int_0^L \frac{M_{bc}^2}{2EI}\,dx + \frac{Q^2 h}{2EA} + \frac{P^2 L}{2EA}$$

$$U = \frac{P^2 h^3}{6EI} + \frac{Q^2 L^3}{6EI} + \frac{PQhL^2}{2EI} + \frac{P^2 h^2 L}{2EI} + \frac{Q^2 h}{2EA} + \frac{P^2 L}{2EA}$$

$$\delta_y = \frac{\partial U}{\partial Q}\bigg|_{Q=0} = 0 + 0 + \frac{PhL^2}{2EI} + 0 + 0 + 0$$

$$\boxed{\delta_y = \frac{PhL^2}{2EI}}$$

Often simpler to bring derivative inside integral!

See page 180 in book

eg. Bending

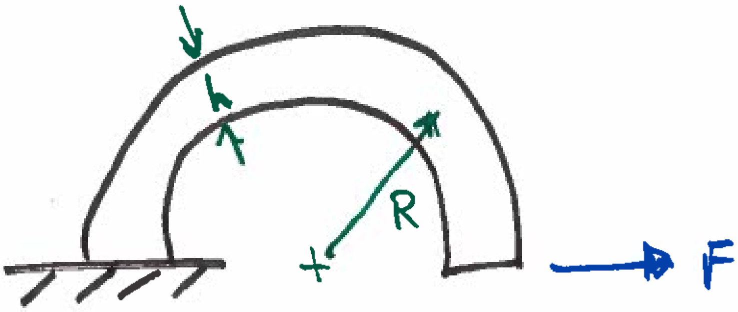
$$y_i = \frac{\partial U}{\partial Q_i} = \frac{\partial}{\partial Q_i}\left(\int \frac{M^2}{2EI}\,dx\right)$$

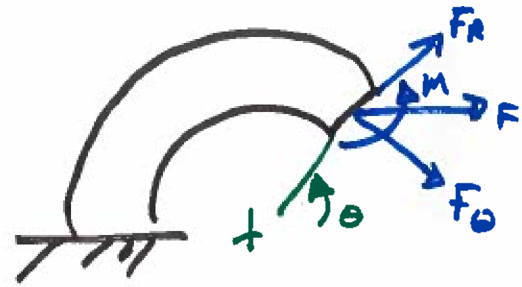
$$= \int \frac{\partial}{\partial Q_i}\left(\frac{M^2}{2EI}\right)dx$$

$$y_i = \int \frac{1}{EI}\left(M\frac{\partial M}{\partial Q_i}\right)dx$$

Ex 4-10 Study

## Circular




Moment alone

$$U_1 = \int \frac{M^2 \, d\theta}{2AeE} \qquad e = R - r_n$$

### axial

$$U_2 = \int \frac{F_\theta^2 \, R \, d\theta}{2AE}$$

### Coupling

$$U_3 = -\int \frac{M F_\theta \, d\theta}{AE}$$

### Transverse Shear

$$U_4 = \int \frac{C \, F_r^2 \, R \, d\theta}{2AG}$$