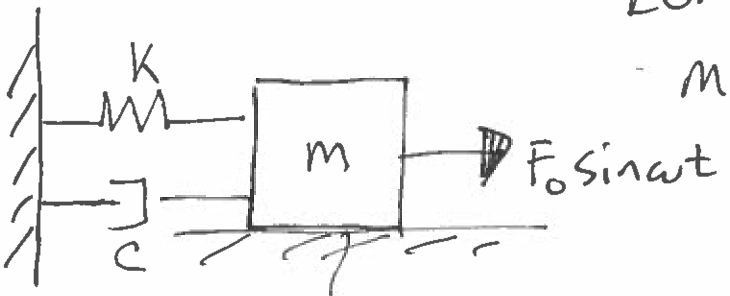
Harmonic Excitation with Damping



frictionless surface

EOM:

$$m\ddot{x} + c\dot{x} + kx = F_o \sin\omega t$$

recall

Homogeneous solution: underdamped $\zeta < 1$

$$x_h(t) = A e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$A = \sqrt{x_o^2 + \left(\frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d}\right)^2} \qquad \phi = \arctan\left(\frac{\omega_d x_o}{\dot{x}_o + \zeta\omega_n x_o}\right)$$

particular solution

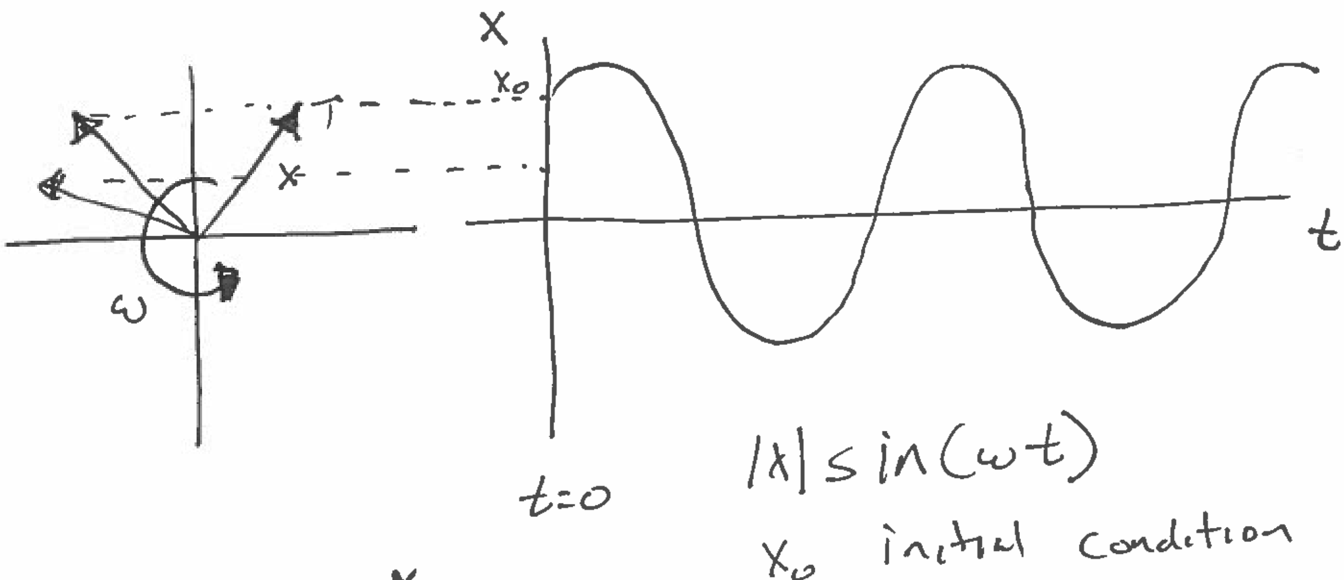
$$x_p(t) = X \sin(\omega t - \theta)$$

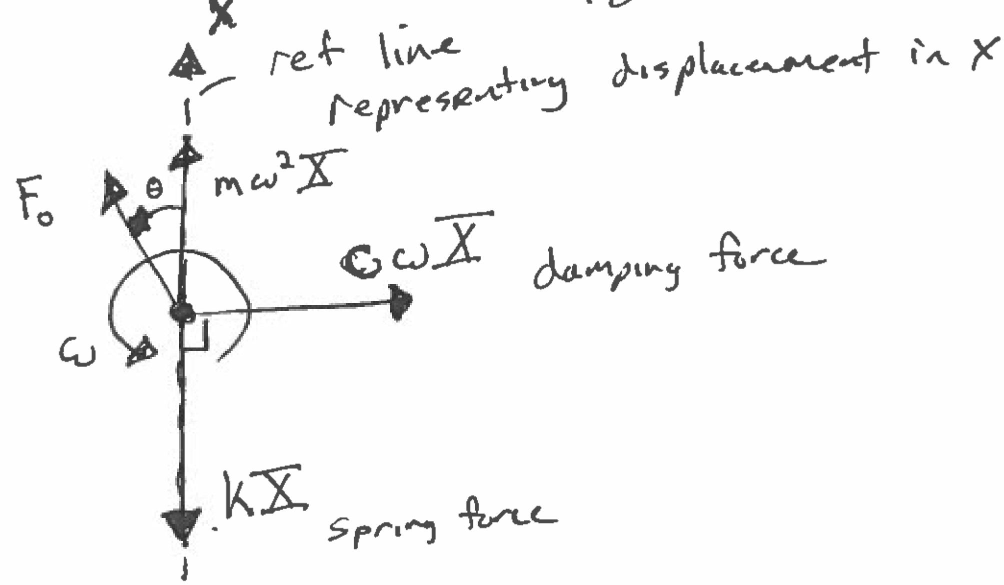
$$\dot{x}_p(t) = \omega X \cos(\omega t - \theta)$$

$$\ddot{x}_p(t) = -\omega^2 X \sin(\omega t - \theta)$$

$$-m\omega^2 X \sin(\omega t - \theta) + c\omega X \cos(\omega t - \theta) + kX \sin(\omega t - \theta) = F_o \sin\omega t$$

Phasors : a graphical representation of oscillating values





Sum forces in x/y directions :

$$\left.\begin{aligned} c\omega \bar{X} - F_0 \sin\theta &= 0 \\ F_0 \cos\theta + m\omega^2 \bar{X} - k\bar{X} &= 0 \end{aligned}\right\} \text{ solve for } \bar{X}, \theta$$

$$X = \frac{F_o}{\sqrt{(c\omega)^2 + (k - m\omega^2)^2}} = \frac{F_o/k}{\sqrt{\left(2\zeta \frac{\omega}{\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{\omega_n^2}\right)}}$$

$$r = \frac{\omega}{\omega_n} \qquad X = \frac{F_o/k}{\sqrt{(2\zeta r)^2 + (1 - r^2)}}$$

no initial conditions in eq.

$$\tan(\theta) = \frac{c\omega}{k - m\omega^2} = \frac{2\zeta r}{1 - r^2}$$

$$x(t) = x_h(t) + x_p(t)$$

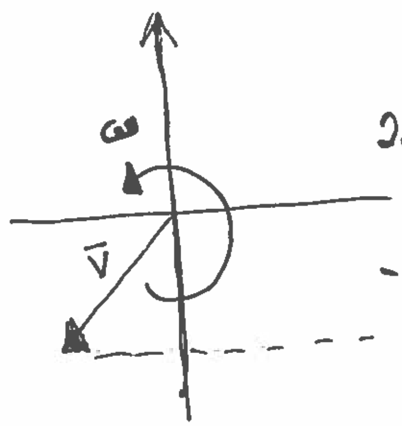
$$x(t) = \underbrace{A e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)}_{\text{transient response}} + \underbrace{X \sin(\omega t - \theta)}_{\text{steady state response}}$$

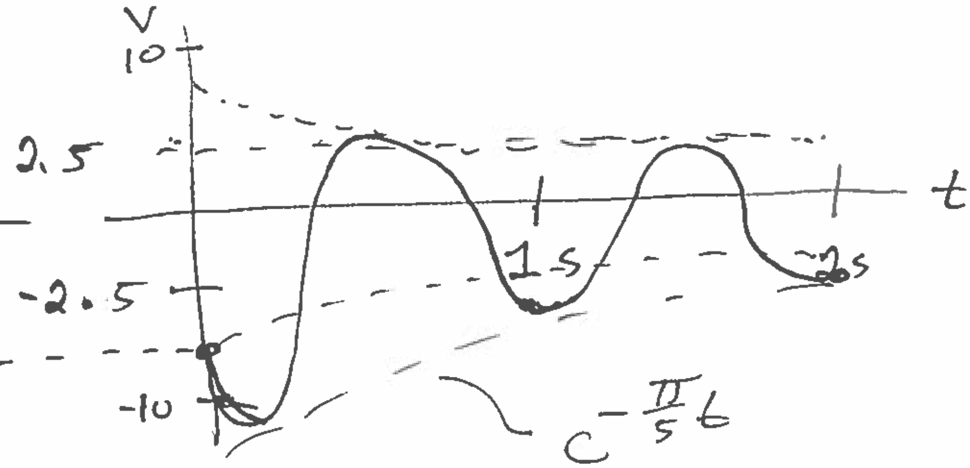
homogeneous

particular

Mini-quiz

Sketch the response of the phasor.





$\omega = 2\pi$ rad/s

$$|\bar{V}| = 10\,e^{-\frac{\pi}{5}t}$$

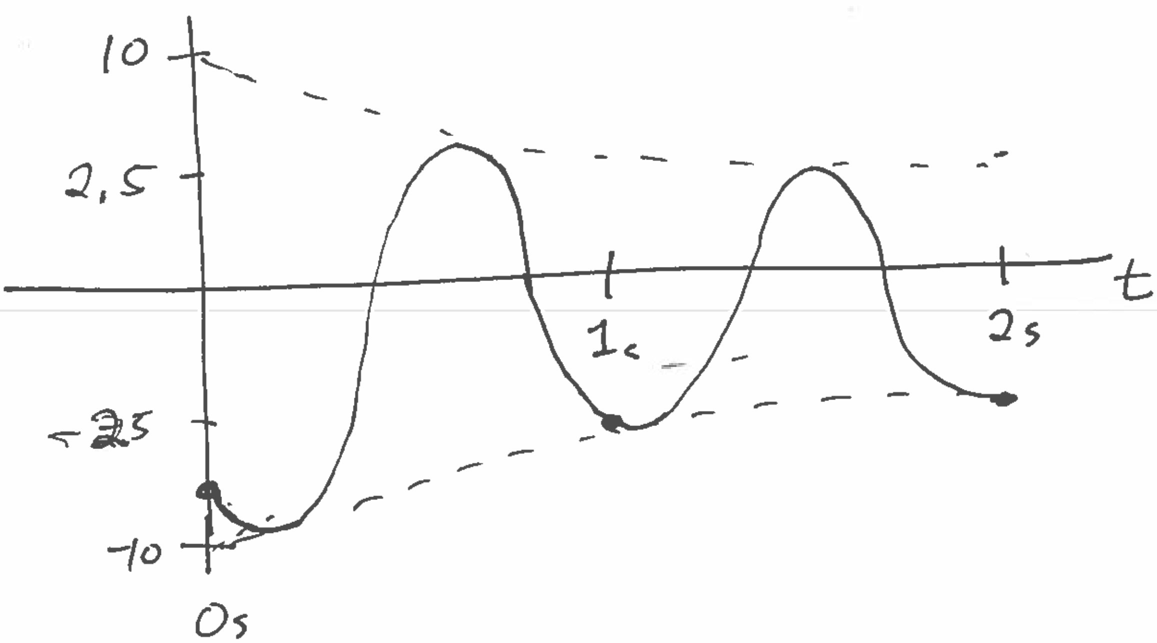
$|\bar{V}|(0) = 10$

$|\bar{V}|(2s) = 2.84$

$|\bar{V}|(1.0s) = 5.33$

$$-\frac{\pi}{5} = 2\zeta\omega_n = -4\pi\zeta$$

$$\zeta = +\frac{1}{20} = 0.05 \qquad T = \frac{\omega_n}{2\pi} = 1\,s$$



Look steady state is most interesting

$$\frac{X}{F_o/k} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

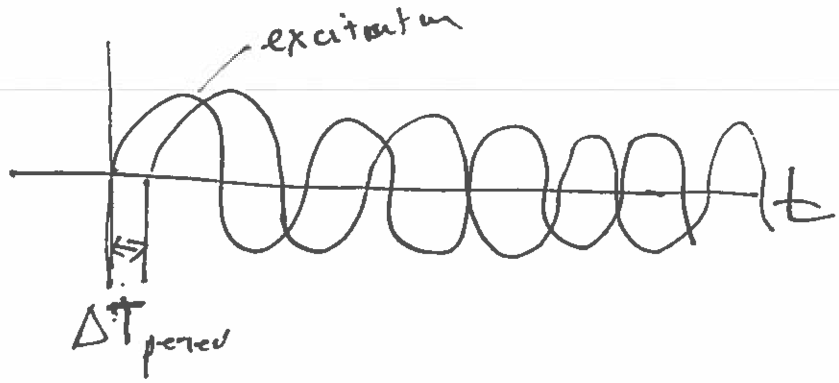
characteristic length

Max output, X, to max static input, F_o/k,

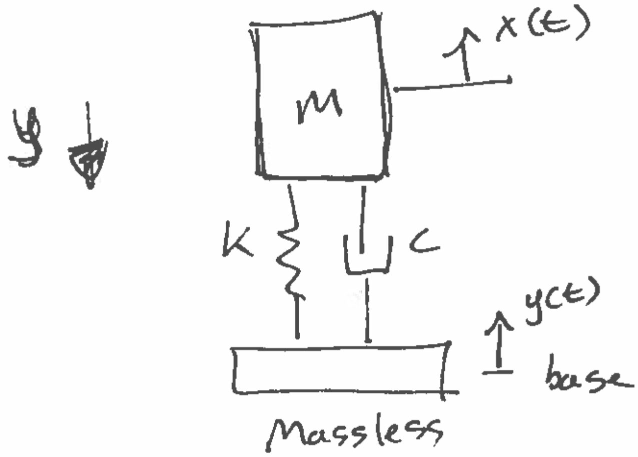
$$\frac{d}{dr}\left(\frac{Xk}{F_o}\right) = 0 \Rightarrow r_{peak} = \sqrt{1-2\zeta^2} = \frac{\omega_{peak}}{\omega_n}$$

only true for underdamped system $\zeta < \frac{1}{\sqrt{2}}$

$$\frac{Xk}{F_o}(r_{peak}) = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$



Base Excitation



EOM:

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

$$y(t) = Y\sin\omega_{b(ase)} t$$

$$\underbrace{m\ddot{x} + c\dot{x} + kx}_{} = cY\omega_b\cos\omega_b t + kY\sin\omega_b t$$

linear comb of forcing
solve for 2 particular
solutions

X_{p1} X_{p2}

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n Y\cos\omega_b t + \omega_n^2 Y\sin\omega_b t$$

$$X_{p1} = \frac{2\zeta\omega_n\omega_b Y}{\sqrt{(\omega_n^2-\omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}\cos(\omega_b t - \theta_1)$$

$$\theta_1 = \arctan\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2-\omega_b^2}\right)$$

$$x_{p2} = \frac{\omega_n^2 Y}{\sqrt{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}} \sin(\omega_b t - \theta_1)$$

$$x_p = \omega_n Y \left[\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}\right]^{1/2} \cos(\omega_b t - \theta_1 - \theta_2)$$

$$\theta_2 = \arctan\left(\frac{\omega_n}{2\zeta\omega_b}\right)$$

$$\frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

ratio of max response amplitude to input displacemet

"displacement transmissibility"

$$F(t) = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x}(t)$$

$$F(t) = F_T \cos(\omega_b t - \theta_1 - \theta_2)$$

$$\frac{F_T}{kY} = r^2 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

"Force transmissibility ratio"