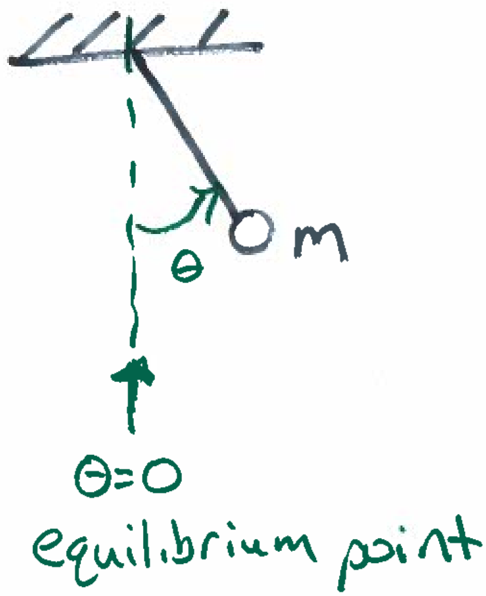
Lecture 3

Wednesday, September 28, 2016

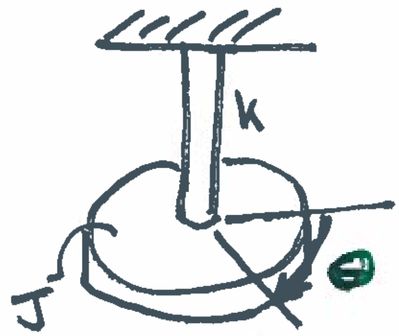
Linearized EoM of a simple pendulum



$$\ddot{\theta} + \frac{g}{l}\theta = 0 \quad 1)$$

equivalent to:

$$\ddot{x} + \frac{k}{m}x = 0$$



$$\ddot{\theta} + \frac{k}{J}\theta = 0$$

inertial force
restoring force

1 DoF
second order linear ODE

Solution to ODE

Sol of 1)

$$\Theta(t) = A\sin(\omega_n t + \phi)$$

?

A: amplitude

ω_n: natural freq

ϕ: phase shift

$$\dot{\Theta}(t) = \omega_n A \cos(\omega_n t + \phi)$$

$$\ddot{\Theta}(t) = -\omega_n^2 A \sin(\omega_n t + \phi)$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$-\omega_n^2 A \sin(\omega_n t + \phi) + \frac{g}{l} A \sin(\omega_n t + \phi) = 0$$

$$\omega_n^2 = \frac{g}{l} \Rightarrow \boxed{\omega_n = \sqrt{\frac{g}{l}}}$$ natural frequency

A and ϕ depends initial conditions.

1) $\Theta_0 = \Theta(0) = A \sin(\omega_n \cdot 0 + \phi) = A \sin\phi$

2) $\omega_0 = \dot{\Theta}(0) = \omega_n A \cos(\omega_n \cdot 0 + \phi) = \omega_n A \cos\phi$

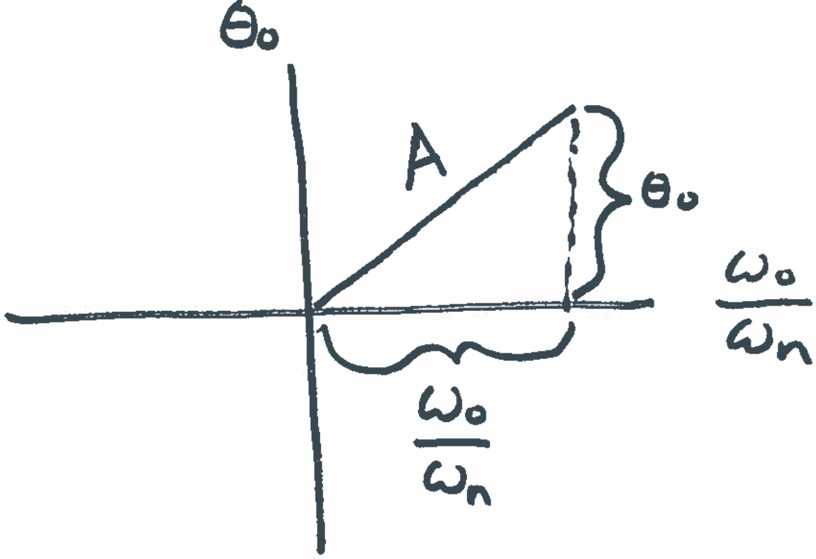
Solve for A and ϕ:

$$A = \sqrt{\frac{\omega_n^2 \theta_0^2 + \omega_0^2}{\omega_n}} \qquad \phi = \arctan\left(\frac{\omega_n \theta_0}{\omega_0}\right)$$

Full equation

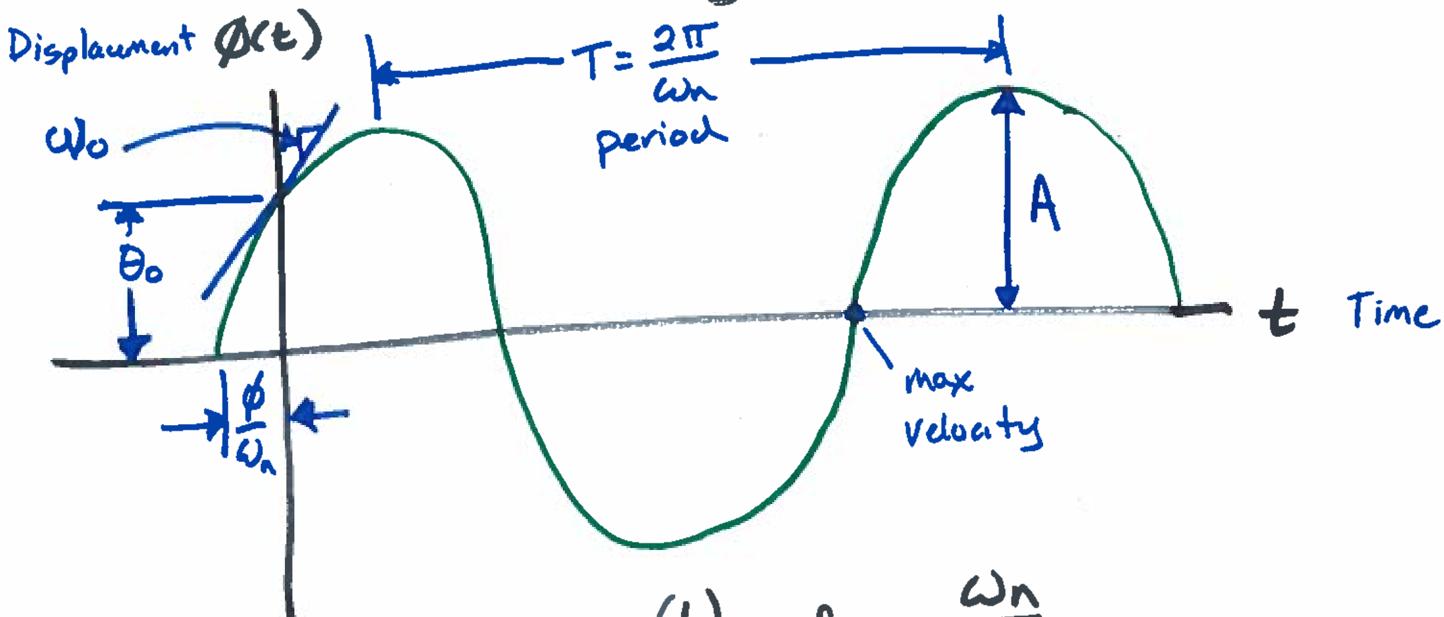
$$\Theta(t) = \sqrt{\frac{\omega_n^2 \theta_0^2 + \omega_0^2}{\omega_n}} \sin\left[\omega_n t + \arctan\left(\frac{\omega_n \theta_0}{\omega_0}\right)\right]$$

Geometric relationship among solution unknowns.



$$A = \sqrt{\theta_0^2 + \left(\frac{\omega_0}{\omega_n}\right)^2}$$

Beware of the quadrant this falls in when computing the arctan!!



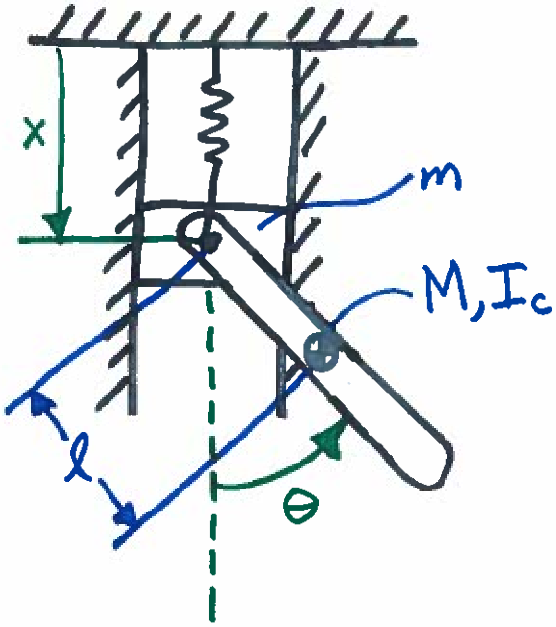
f_n: frequency in Hertz $\left(\frac{1}{s}\right)$ $\quad f_n = \frac{\omega_n}{2\pi}$

T: period in seconds $\quad T = \frac{1}{f_n}$

I.C.s contain info about initial energy in system

* Corrected wrt lecture *

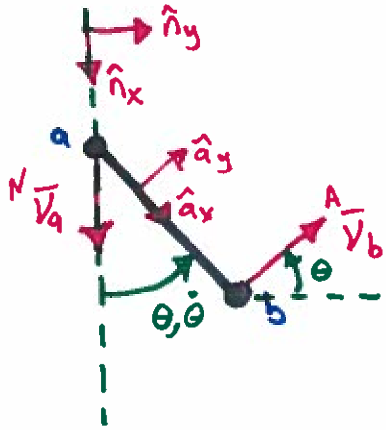
Example: Compound pendulum with base on vertical spring



Goal: Find the equations of motion for this model using Lagrange's method.

Model has 2 DoF with generalized coordinates x and θ.

Step 1: Form an expression for the Kinetic energy



N: inertial ref. frame

A: ref. frame attached to bar ab

$$ {}^N\bar{V}_b = {}^N\bar{V}_a + {}^A\bar{V}_b $$

$$ {}^A\bar{V}_b = {}^N\bar{\omega}^A \times \bar{r}^{\,b/a} $$

* Note the corrected magnitude of the velocity at point b! *

$$ \left|{}^N\bar{V}_b\right| = \left|{}^N\bar{V}_a + {}^A\bar{V}_b\right| $$

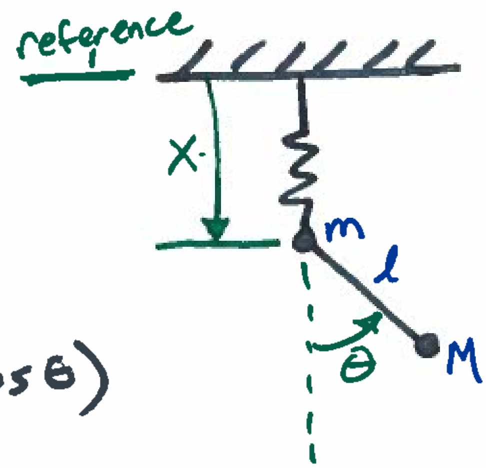
$$ = \sqrt{(\dot{x} - \dot{\theta}\ell\sin\theta)^2 + (\dot{\theta}\ell\cos\theta)^2} $$

$$ \left|{}^N\bar{V}_b\right| = \sqrt{\dot{x}^2 + \ell^2\dot{\theta}^2 - 2\dot{x}\ell\dot{\theta}\sin\theta} $$

$$ T = \frac{1}{2}m\left|{}^N\bar{V}_a\right|^2 + \frac{1}{2}M\left|{}^N\bar{V}_b\right|^2 + \frac{1}{2}I_c\dot{\theta}^2 $$

$$ T = \underbrace{\frac{m}{2}\dot{x}^2 + \frac{M}{2}(\dot{x}^2 + \ell^2\dot{\theta}^2 - 2\dot{x}\ell\dot{\theta}\sin\theta)}_{\text{linear K.E.}} + \underbrace{\frac{I_c}{2}\dot{\theta}^2}_{\text{rotational Kinetic energy}} $$

$$U = U_{spring} + \underbrace{U_{mg} + U_{Mg}}_{\text{P.E. due to gravity}}$$



$$U = \frac{1}{2}kx^2 - mgx - Mg(x + l\cos\theta)$$

Step 2: Form Lagrangian

$$L = T - U$$

$$L = \frac{m}{2}\dot{x}^2 + \frac{M}{2}(\dot{x}^2 + l^2\dot{\theta}^2 - 2\dot{x}l\dot{\theta}\sin\theta) + \frac{I_c}{2}\dot{\theta}^2$$

$$- \frac{k}{2}x^2 + mgx + Mg(x + l\cos\theta)$$

Step 3: Form Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$x_1 = x$, $x_2 = \theta$
two DoF
two G.C.s
two second order coupled non-linear ODEs

① x

$$\frac{d}{dt}\left(m\dot{x} + M\dot{x} - Ml\dot{\theta}\sin\theta\right) + kx - mg - Mg = 0$$

$$\boxed{m\ddot{x} + M\ddot{x} - M(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta) + kx - mg - Mg = 0}$$

② θ

$$\frac{d}{dt}\left(M\ell^2\dot{\theta} - M\dot{x}\ell\sin\theta + I_c\dot{\theta}\right) + M\dot{x}\ell\dot{\theta}\cos\theta + Mg\ell\sin\theta = 0$$

$$M\ell^2\ddot{\theta} - M\ddot{x}\ell\sin\theta - M\dot{x}\ell\dot{\theta}\cos\theta + I_c\ddot{\theta} + M\dot{x}\ell\dot{\theta}\cos\theta + Mg\ell\sin\theta = 0$$

$$\boxed{M\ell^2\ddot{\theta} - M\ddot{x}\ell\sin\theta + I_c\ddot{\theta} + Mg\ell\sin\theta = 0}$$

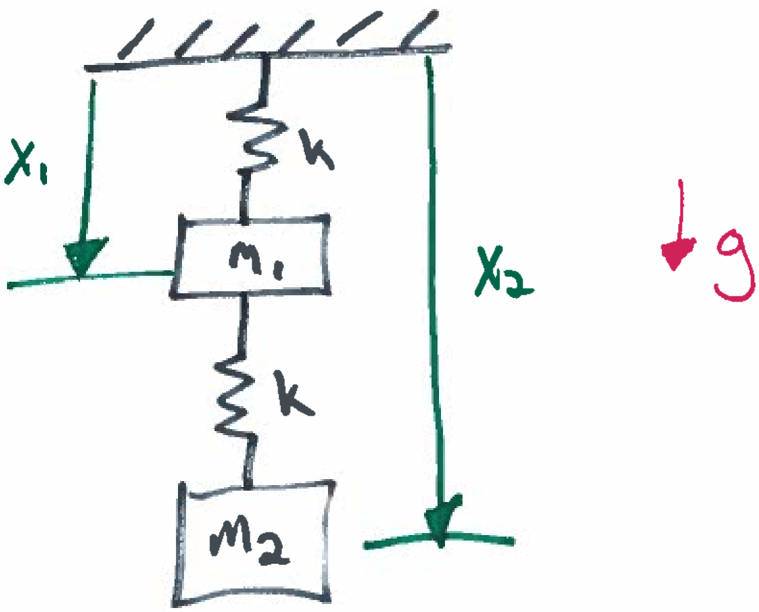
We now have two coupled second order non-linear ordinary differential equations. With some simplification we have:

$$\underbrace{(m+M)\ddot{x}}_{\text{force required to linearly accel. both masses}} - \underbrace{M\ell(\ddot{\theta}\sin\theta +}_{\text{force req. to angularly accel bar}} \underbrace{\dot{\theta}^2\cos\theta)}_{\text{force req to centripetally accel bar}} + \underbrace{Kx}_{\text{force due to spring}} - \underbrace{(m+M)g}_{\text{force due to gravity}} = 0$$

$$\underbrace{(m\ell^2 + I_c)\ddot{\theta}}_{\text{torque required to angularly accel bar}} \underbrace{- M\ddot{x}\ell\sin\theta}_{\text{torque req. to linearly accel bar}} \underbrace{+ Mg\ell\sin\theta}_{\text{torque required for bar to overcome gravity}} = 0$$

Exercise

Write out the potential energy, U, term for the following system:



Do not assume springs are at equilibrium

Both systems are conservative $\Rightarrow$ no loss of energy simply transforms kinetic to potential and vice versa.

So $\underbrace{T+U}_{\text{total energy}}$ is constant wrt to time

$$U = U_{s1} + U_{s2} + U_{g1} + U_{g1}$$ S: spring
g: gravity

$$U = \frac{K}{2} x_1^2 + \frac{K}{2}(x_2 - x_1)^2 - m_1 g x_1 - m_2 g x_2$$

What is the equilibrium point?

Spring 1 supports weight of m_1 and m_2 so:

$$x_1^{eq} = \frac{(m_1 + m_2) g}{K}$$

Spring 2 only supports weight of m_2 so:

$$x_2^{eq} = x_1^{eq} + \frac{m_2 g}{K} = \frac{(m_1 + 2m_2) g}{K}$$

The potential energy stored at equilibrium can be found by substituting the equilibrium point into U.