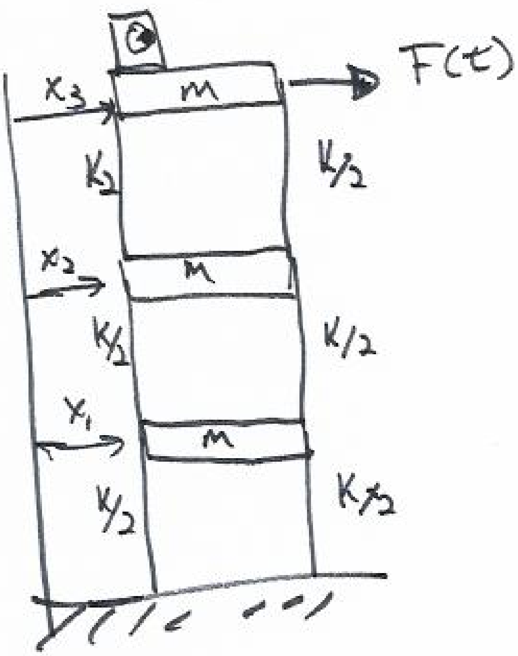
$P^T M$

$P^T \cancel{M} P = I$ Typo

$P^T \bar{K} P = \Lambda$ Typo

$\left.\right\}$ 4.22

Forced responsed using Modal Analysis



Suppose a rotating machine is on top floor causing $F(t)$.

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

$$M\ddot{\bar{x}} + C\dot{\bar{x}} + K\bar{x} = B\bar{F}$$

assume this proportional damping $\rightarrow C = \alpha M + \beta K$

$$\bar{F} = \begin{bmatrix} 0 \\ 0 \\ m_0 e \omega^2 \sin \omega t \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\bar{x} \to \bar{r}$

$$\ddot{\bar{r}} + \mathrm{diag}[2\zeta_i \omega_i]\dot{\bar{r}} + \Lambda \bar{r} = P^T L^{-1} B \bar{F}$$

$$P^T L^{-1} B \bar{F} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

f_1, f_2, f_3 are linear combinations of the elements of $\bar{F}$

Decoupled form:

$$\ddot{r}_1 + 2\zeta_1 \omega_1 \dot{r}_1 + \omega_1^2 r_1 = f_1$$

$\vdots$

Can be treated as three single dof non-homogeneous systems.

$\bar{x} = S \bar{r}$ ⇒ maps back to generalized coordinates

Resonance exists in multi-dof systems.

A harmonic driving force will excite the system at its natural frequency and cause unbounded vibration (undamped) or max amplitude (damped case).

However:

#1) How the force is applied matters.
If force is applied orthogonally to the mode of the excitation frequency, the system will not resonate.

#2) Resonance in one mode can lead to resonance in more than one generalized coordinate. (Any generalized coordinate is a linear combination of the modal coordinates).

Example: Which system will resonate?

a) $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.642 \\ 0.761 \end{bmatrix} \sin 2t$

b) $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.235 \\ 2.971 \end{bmatrix} \sin 2.75t$

let $\underbrace{m_1 = 4,\ m_2 = 9}_{kg}$ $\quad k_1 = 25,\ k_2 = 5\ \left[\frac{N}{m}\right]$

$\tilde{K} = L^{-1} K L^{-1}$ $\quad L^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

$\tilde{K} = \begin{bmatrix} 7.5 & -0.833 \\ -0.833 & 0.556 \end{bmatrix}$

Solve the ~~eigenvalues~~ eigenvector problem.

eigenvalues: $\lambda_1 = 0.4569$ $\quad \lambda_2 = 7.5986$

frequencies: $\omega_1 = 0.676$ rad/s $\quad \omega_2 = 2.75$ rad/s

eigenvectors: $\bar{V}_1 = \begin{bmatrix} 0.1175 \\ 0.9931 \end{bmatrix}$ $\quad \bar{V}_2 = \begin{bmatrix} 0.9931 \\ -0.1175 \end{bmatrix}$

$P = [\bar{V}_1, \bar{V}_2]$

a) will not resonate

b) might resonate

$P^T L^{-1} B \cdot F$

$$\begin{bmatrix} 0.1175 & 0.9931 \\ 0.9931 & -0.1175 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.235 \\ 2.979 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\ddot{\bar{r}} + \mathrm{diag}[2\zeta_i \omega_i]\dot{\bar{r}} + \Lambda \bar{r} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 2.75t$$

$$BF = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.235 \\ 2.979 \end{bmatrix}}_{b}$$

$b^T \cdot S_i = b^T \bar{u}_i$

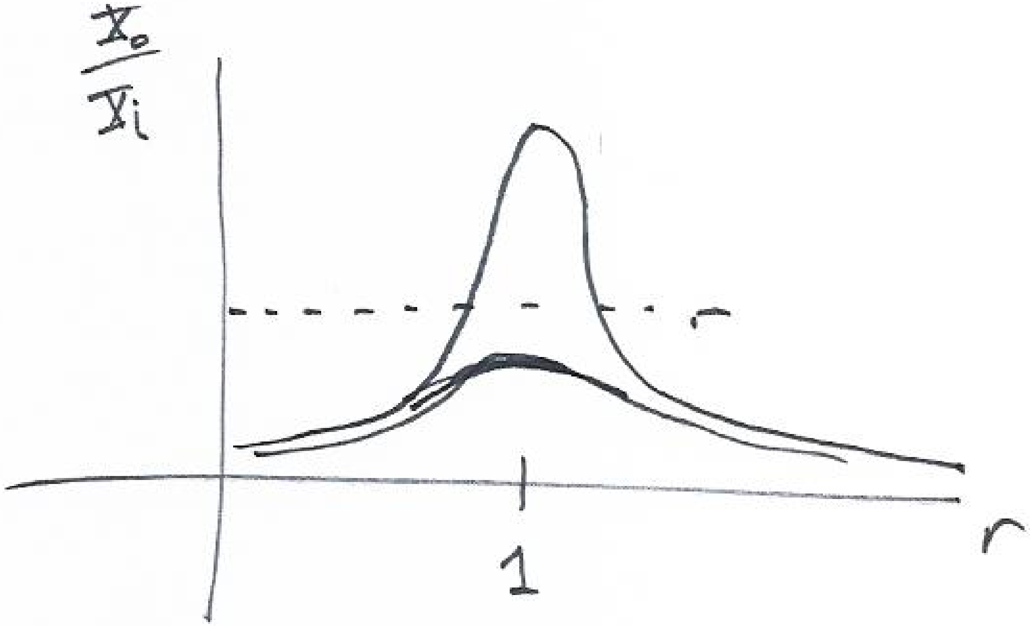
$b^T \cdot S_i = 0 \Rightarrow$ no resonance

$$S = \begin{array}{c} \begin{bmatrix} 0.59 & 0.496 \\ 0.331 & -0.039 \end{bmatrix} \\ \quad S_1 \qquad\quad S_2 \end{array}$$

$b = \begin{bmatrix} 0.235 \\ 2.979 \end{bmatrix}$ $\quad b^T \cdot S_1 = 1 \quad b^T \cdot S_2 = 0$

b is orthogonal to S_i

This rarely happens by chance but may often by design.



Design of Vibratory Systems

1) Good and Bad vibrations are found in daily life.

Good: calm baby, sleep strip (rumblestrip) on freeways, stereo, electric toothbrush

Bad: Earthquake, unbalanced rotor (washing machine, car rim), in most machines you don't want vibration

Why bad?, - structural and mechanical failures
- cost
- pain and discomfort

Acceptable vibration

1) freq?,
2) amplitude of displacement
3) amplitude of velocity
4) Amplitude of acceleration

Internationals Organization for Standards (ISO) has lots of guidelines for acceptable vibrations. Mil-Specs also have extensive details on acceptable vibrations.

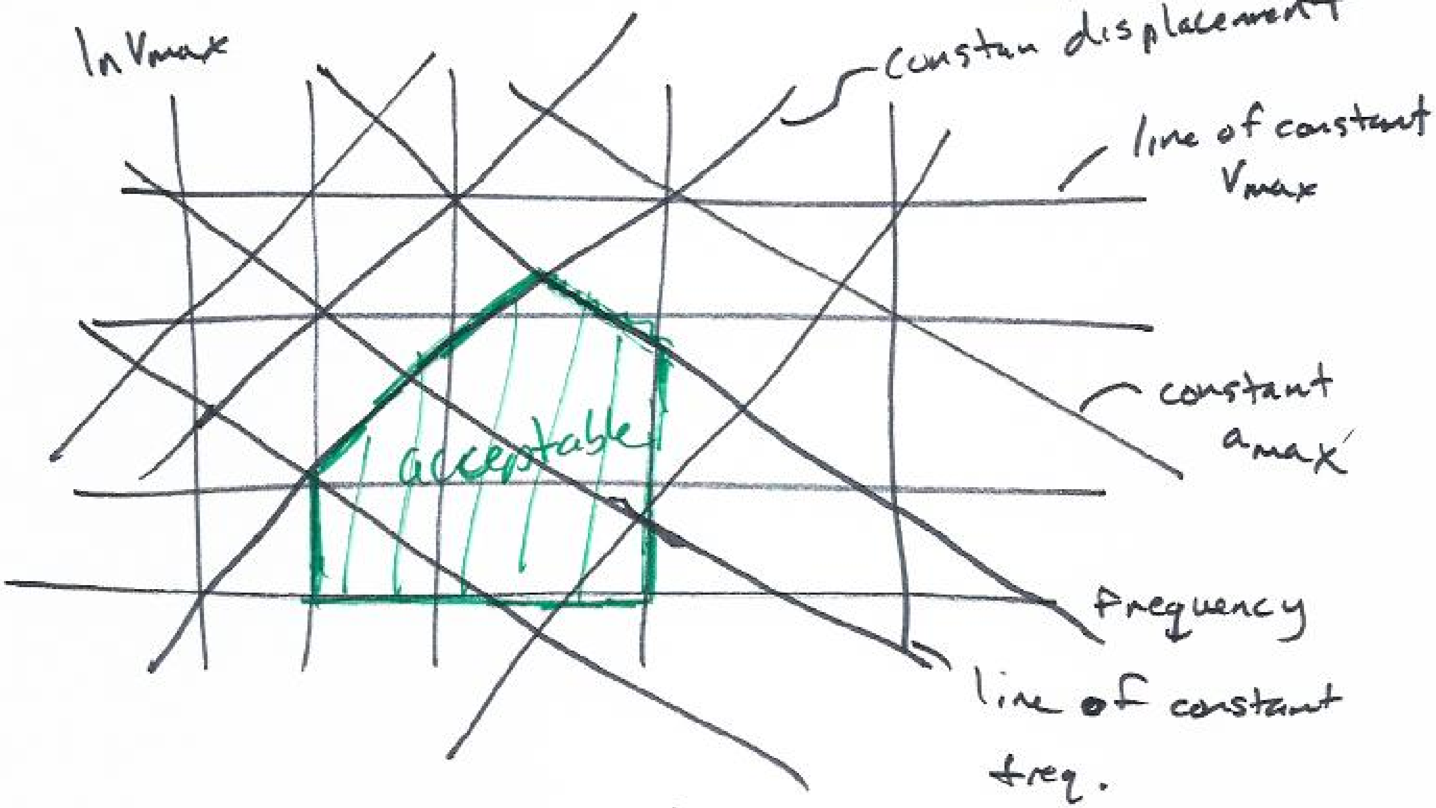
For a simple undamped harmonic system:

$$x(t) = A \sin \omega t$$
$$v = \dot{x}(t) = A\omega \cos \omega t$$
$$a = \dot{v}(t) = -A\omega^2 \sin \omega t$$

$\ln X_{max} = \ln A$

$\ln V_{max} = \ln X + \ln \omega = \ln(A\omega)$

$\ln a_{max} = -\ln(A\omega^2) = -\ln V_{max} - \ln \omega$



Vibration nomograph

For the human body (sensitive to vibrations)

common standards:

- motion sickness (0.1–1 Hz)
- blurry vision (2–20 Hz)
- speech disturbance (1–20 Hz)

To control vibration can use either:

a) active systems (inject energy into system)

b) passive systems (no extra energy needed)

Active systems: cost a lot, and can have physical limitations

Passive: can't always deal with vibrations

3 ways to suppress vibration:

a) remove the source of vibration

b) isolate the vibration

c) absorption

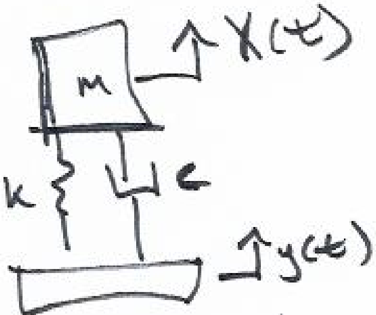
Removal

Try to alter source so it produces less vibration. (Not always feasible, especially if the source is a disturbance from nature). Dealing with offset masses in rotating machines is very common.

Isolation

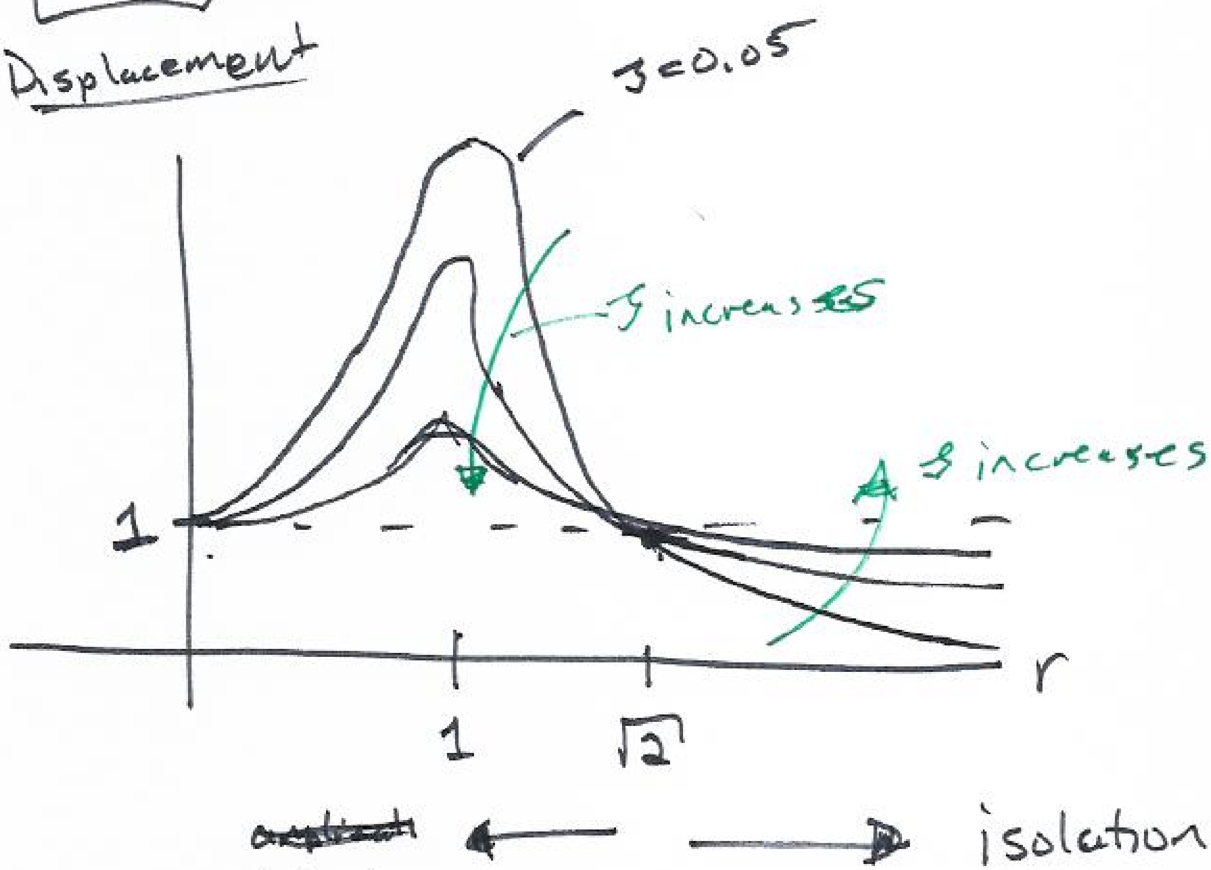
If vibration from source cannot be removed, one can design an isolation system to reduce the vibration transmitted.

recall
Base excitation problem

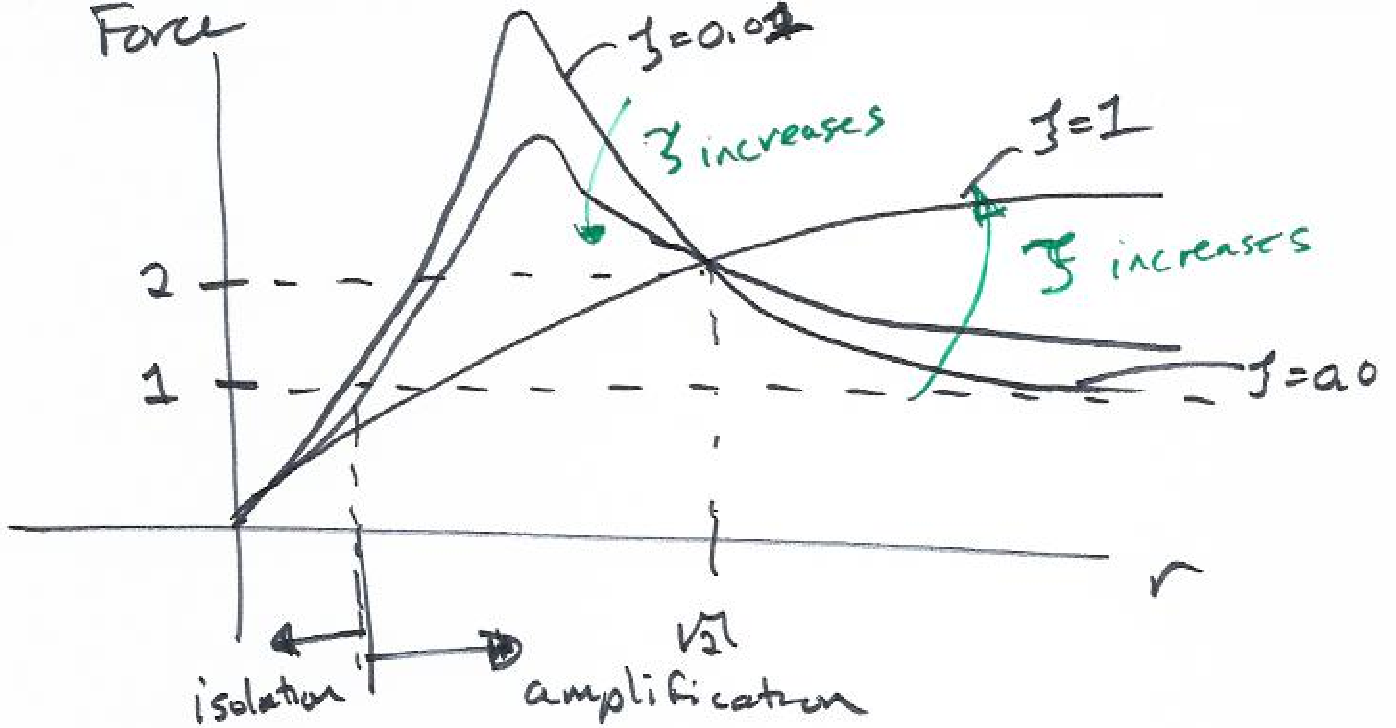


- Displacement Transmissibility
- Force Transmissibility.

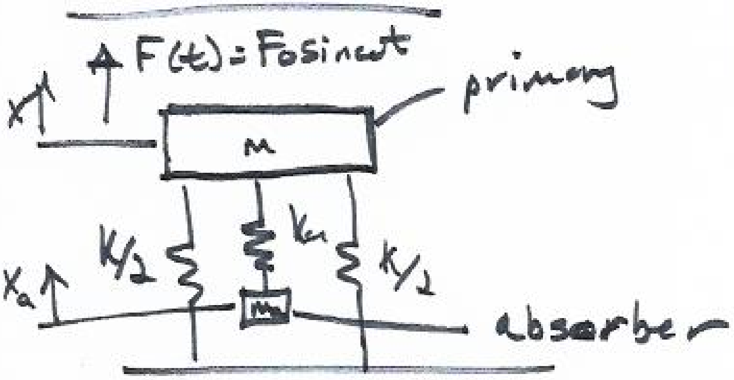
Displacement



~~amplificat~~ amplification ← → isolation



Absorbtion:



$$\begin{bmatrix} M & 0 \\ 0 & m_a \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} k+k_a & -k_a \\ -k_a & k_a \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} F_0 \sin\omega t \\ 0 \end{bmatrix}$$

Goal: choose k_a, m_a such that $x(t) = 0$

Sub:

$$x(t) = X \sin\omega t$$

$$x_a(t) = X_a \sin\omega t$$

$$\begin{bmatrix} k + k_a - M\omega^2 & -k_a \\ -k_a & k_a - m_a\omega^2 \end{bmatrix} \begin{bmatrix} X \\ X_a \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin\omega t$$

Solve for X and X_a

$$\begin{bmatrix} X \\ X_a \end{bmatrix} = \frac{1}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2}\begin{bmatrix} k_a-m_a\omega^2 & k_a \\ k_a & k+k_a-m\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$X = \frac{(k_a-m_a\omega^2)F_0}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2}$$ primary amplitude

$$X_a = \frac{k_a F_0}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2}$$

Choose k_a and m_a such that $X=0$.

$$\frac{(k_a-m_a\omega^2)}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2} = 0$$

$$\hookrightarrow \quad \underbrace{\omega^2 = \frac{k_a}{m_a}}_{\text{tuning condition}}$$

Sub $k_a = m_a \omega^2$ into X_a

$$X_a = \frac{m_a \omega^2 F_0}{(k + m_a \omega^2 - m\omega^2)\cancelto{0}{(m_a\omega^2 - m_a\omega^2)} - m_a^2 \omega^4}$$

$$X_a = \frac{F_0}{-m_a \omega^2} = -\frac{F_0}{k_a}$$

$$x_a(t) = -\frac{F_0}{k_a} \sin \omega t$$

Force on the absorber

$$k_a x_a(t) \Rightarrow k_a\left[-\frac{F_0}{k_a}\right] \Rightarrow \underbrace{-F_0 \sin \omega t}_{\text{force on absorber}}$$

all force is transmitted to the absorber

Check for resonance:

$$\mu = \frac{m_a}{m} \quad \text{mass ratio}$$

$$\omega_p = \sqrt{\frac{k}{m}} \quad \text{nat. freq of system w/o absorber}$$

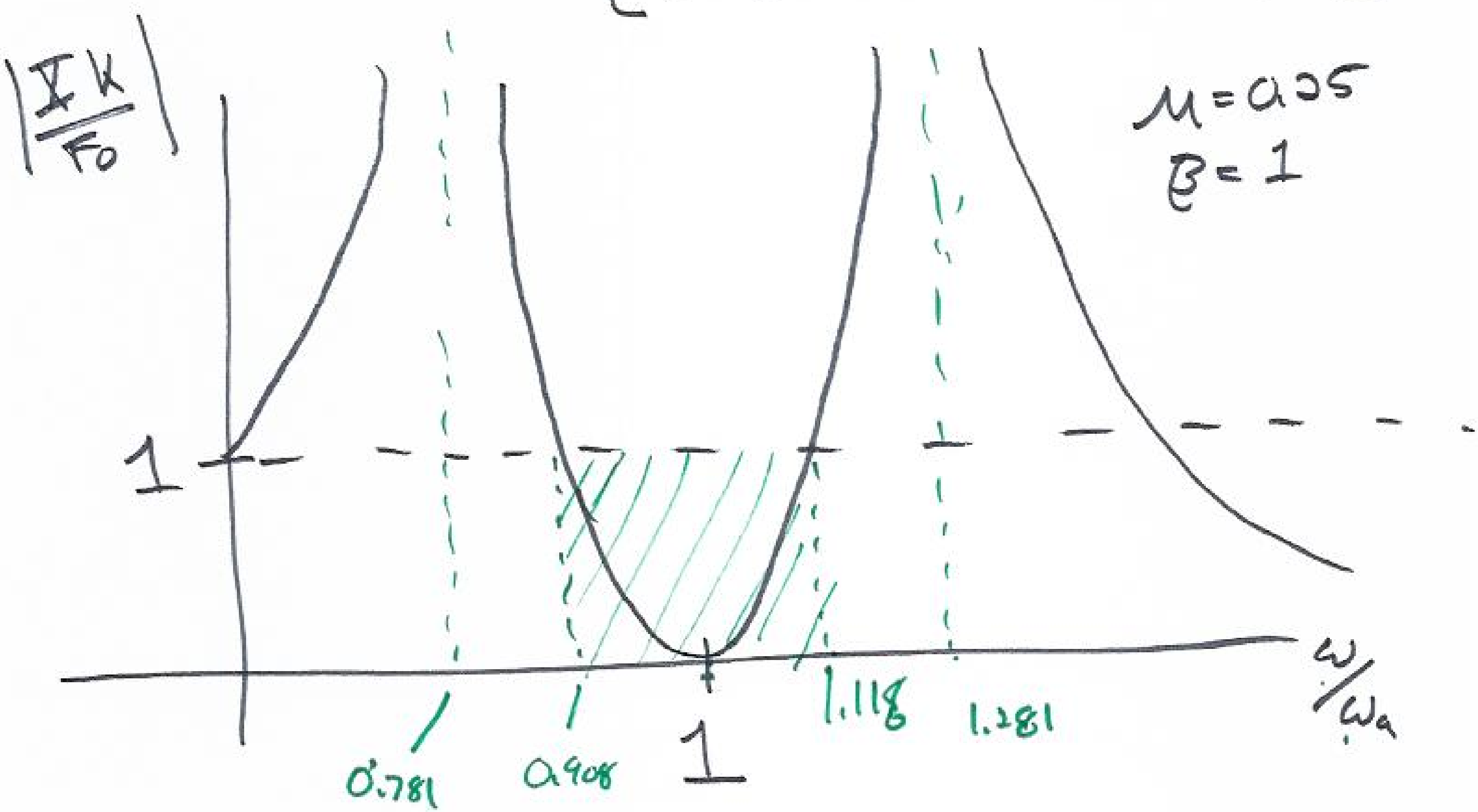
$$\omega_a = \sqrt{\frac{k_a}{m_a}} \quad \text{nat. freq of absorber alone}$$

Note that

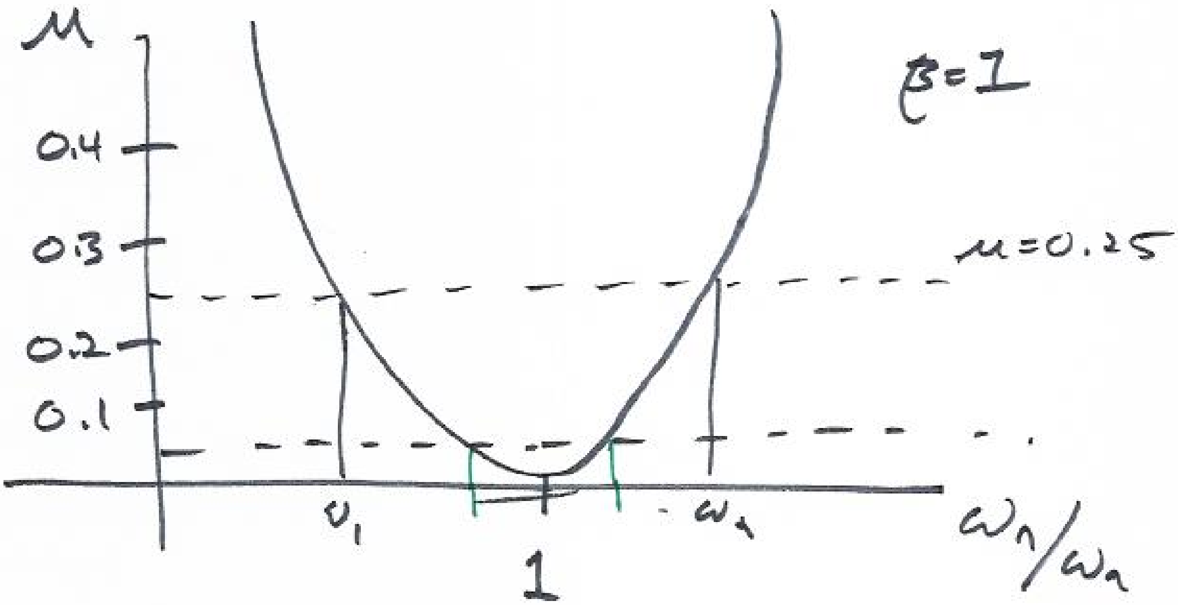
$$\frac{k_a}{k} = \mu \frac{\omega_a^2}{\omega_p^2} = \mu \beta^2 \qquad \beta = \frac{\omega_a}{\omega_p}$$

Sub into $\bar{X}$

$$\frac{\bar{X} k}{F_0} = \frac{1 - \omega^2/\omega_n^2}{\left[1 + \mu(\omega_a/\omega_p)^2 - (\omega/\omega_p)^2\right]\left[1 - (\omega/\omega_a)^2 - \mu(\omega_a/\omega_p)^2\right]}$$



How does the system's ω_n change with μ and β?



Rule of thumb: $0.05 \leq \mu \leq 0.25$

Large $\mu \Rightarrow$ poor design wrt large absorber mass

smaller than $0.05 \Rightarrow$ too easy to get resonance

Also check max $X_a \Rightarrow$ spring could fail in fatigue