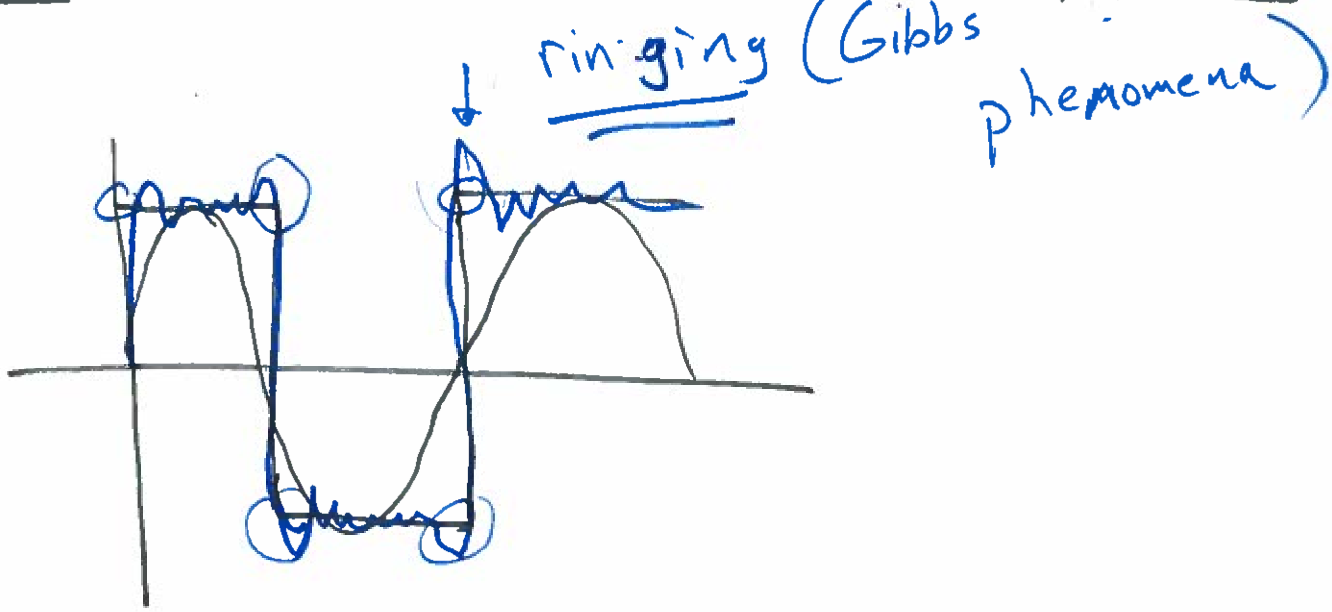
ENG122 FALL 2016 LECTURE 13 November 7, 2016



$$x(t) = A \sin$$

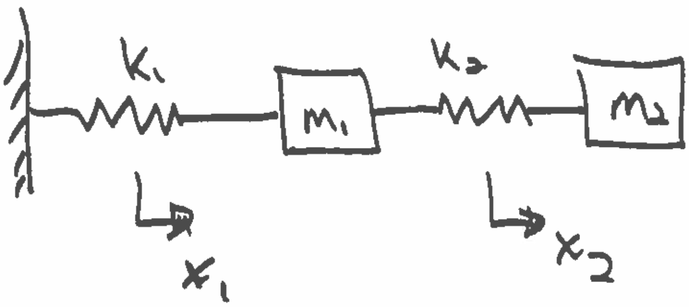
$$x(t) = A e^{-\zeta \omega_n t} \sin(\omega_d t - \phi) + \underbrace{\Sigma}_{c_1}$$

$$x(0) = x_0$$

$$\underbrace{x_0 - c_1}_{x^*} = A e^{-\zeta \omega_n t} \sin(\omega_d t - \phi)$$

$$x^* = A$$

2 DoF Systems



EOM

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) \Rightarrow m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) \Rightarrow m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

2 coupled second order ordinary differential equations, homogenous, linear

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \dot{\tilde{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \qquad \ddot{\tilde{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \overbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}^{\ddot{\tilde{x}}} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

mass matrix — stiffness matrix

$$M\ddot{\tilde{x}} + K\tilde{x} = 0$$

$$\underset{\substack{\text{has} \\ \text{to } \cancel{\text{be}} \\ \text{invertable}}}{M^{-1}} M \ddot{\tilde{x}} + \underbrace{M^{-1} k}_{k'} \tilde{x} = 0$$

$$\ddot{\tilde{x}} + k' x = 0$$

Assume a solution

$$\tilde{x} = \tilde{x}_0 \sin \omega t$$

$$\dot{\tilde{x}} = \omega \tilde{x}_0 \cos \omega t$$

$$\ddot{\tilde{x}} = -\omega^2 \tilde{x}_0 \sin \omega t$$

$$-\omega^2 \tilde{x}_0 \sin \omega t + k \tilde{x}_0 \sin \omega t = 0$$

$$\tilde{x}_0 = I \tilde{x}_0$$

$$(-\omega^2 I + k') \tilde{x}_0 \sin \omega t = 0$$

how can this equation be true?

$\sin \omega t = 0$ (not for any t)

$\tilde{x}_0 = 0$ ($x_1 = x_2 = 0$ trivial solution)

$(-\omega^2 I + k') = 0$ (impossible)

$$\underbrace{\left(-\omega^2 I + k'\right)\tilde{X}_0 = 0}$$

$$K'\tilde{X}_0 = \omega^2 \tilde{X}_0$$

$$\underset{\sim}{\mathbf{A}}\,\tilde{X}_0 = \lambda\,\tilde{X}_0$$

eigenvectors

eigenvalue problem to solve

eigenvalues

ω^2 is the eigenvalue

$\tilde{X}_0$ is the eigenvector

K' 2x2 matrix (the "A" matrix)

$$\underbrace{\det(A - \lambda I) = 0}$$

finds the eigenvalues

$$K' = M^{-1}K \qquad M^{-1} = \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix}$$

$$K' = \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix}\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{bmatrix}$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - K'\right) = 0$$

$$\det\begin{bmatrix} \lambda - \frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \lambda - \frac{k_2}{m_2} \end{bmatrix} = 0$$

$$\left(\lambda - \frac{k_1+k_2}{m_1}\right)\left(\lambda - \frac{k_2}{m_2}\right) - \frac{k_2}{m_1}\frac{k_2}{m_2} = 0$$

$$\underbrace{\lambda^2 - \frac{(k_1+k_2)m_2 + k_2 m_1}{m_1 m_2}\lambda + \frac{k_1 k_2}{m_1 m_2} = 0}_{\text{characteristic equation}}$$

Second order polynomial for two DoF system

For $k_1 = k_2 = k$, $m_1 = m_2 = m$

$$\lambda^2 - \frac{3k}{m}\lambda + \frac{k^2}{m^2} = 0$$

$$\lambda = \frac{\frac{3k}{m} \pm \sqrt{\left(\frac{3k}{m}\right)^2 - 4\frac{k^2}{m^2}}}{2} = \left(\frac{3k}{m} \pm \sqrt{5}\,\frac{k}{m}\right)/2$$

$$\lambda = \frac{3 \pm \sqrt{5}}{2}\,\frac{k}{m} = \omega^2$$ two eigenvalues

$$\pm\omega_1 = \pm\sqrt{\frac{k}{m}}\sqrt{\frac{3+\sqrt{5}}{2}} \approx \pm 1.6\sqrt{\frac{k}{m}}$$

$$\pm\omega_2 = \pm\sqrt{\frac{k}{m}}\sqrt{\frac{3-\sqrt{5}}{2}} \approx 0.62\sqrt{\frac{k}{m}}$$

not equal to the natural frequency

Find eigenvectors

$$K'\tilde{x}_0 = \lambda\tilde{x}_0$$

$$(-K' + \lambda^2 I)\tilde{x}_0 = 0$$

$$\begin{bmatrix} \omega_1^2 - \frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & \omega_1^2 - \frac{k}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{3+\sqrt{5}}{2} - 2\right)\frac{k}{m}x_1 + \frac{k}{m}x_2 = 0 \quad (1)$$

$$\frac{k}{m}x_1 + \left(\frac{3+\sqrt{5}}{2} - 1\right)\frac{k}{m}x_2 = 0 \quad (2)$$

$$\tilde{x}_{01} = \begin{bmatrix} -\frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$ associated with ω_1

plug in ω_2

$$\tilde{x}_{02} = \begin{bmatrix} -\frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$ associated with ω_2

Now we construct the full solution.
for any initial condition $\tilde{X}(0)$ and $\dot{\tilde{x}}(0)=0$.

$$\tilde{X}(t) = C_1 \tilde{X}_{01} \sin\omega_1 t + C_2 \tilde{X}_{02} \sin\omega_2 t + C_3 \tilde{X}_{01} \cos\omega_1 t + C_4 \tilde{X}_{02} \cos\omega_2 t$$

Solve for C_1, C_2, C_3, C_4 by substituting initial conditions.

Midterm review

$$e^{-\zeta\omega_n t} \qquad -\zeta\omega_n t > 0$$

$$-\frac{c}{2m_{eff}\,\cancel{\omega_n}}\,\cancel{\omega_n} > 0$$

$$-\frac{\alpha u h m n}{2 I_p} > 0$$

$-u > 0$ or $u < 0$ to be unstable

$\boxed{u > 0 \text{ stable}}$

static deflection

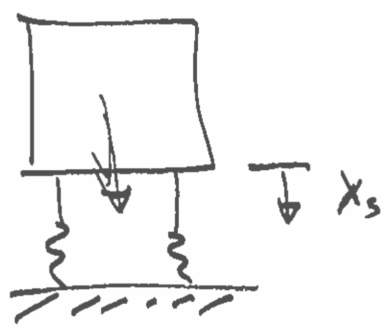
$$X_s = \frac{mg}{2K_T} = 76.6 \text{ mm} \qquad X_d = (0.25\%)X_s$$

$$0.0025 \cdot X_s$$

153 mm

$k_T = 2 * K$

$mg = K_T X$



$$X_d = \frac{M_0 e r^2}{m\sqrt{(1+r^2)^2 + (2\zeta r)^2}}$$

polynomial in r $\qquad r^4 + \dots$