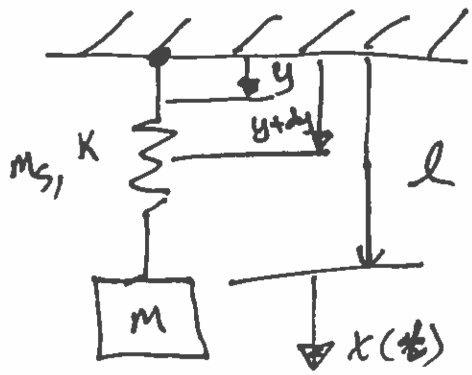
What if the mass of the spring is not negligible?



M_s: mass of the spring

m: suspended mass

ℓ: nominal length of spring

y: distance along the spring

dy: differential length element of the spring

mass of the dy element?

$\underbrace{\frac{M_s}{\ell}}_{\text{mass per unit length}} dy$: mass dy

assume that $V(y)$ varies linearly from zero to $\dot{x}(t)$

$$V(y) = \frac{y}{\ell}\dot{x}(t)$$

$$V(0) = 0$$

$$V(\ell) = \dot{x}(t)$$

Kinetic energy of spring

$$T_s = \frac{1}{2} \int_0^{l} \underbrace{\frac{m_s}{l} dy}_{\text{mass}} \underbrace{\left[\frac{y}{l} \dot{x}(t)\right]^2}_{\text{energy}}$$

$$= \frac{1}{2} \underbrace{\left(\frac{m_s}{3}\right)}_{\text{effective mass of spring}} \dot{x}^2$$

$$\omega_n = \sqrt{\frac{k}{m + m_s/3}}$$

$$m >>> m_s/3$$

Measurement in Vibrating Systems

need : m, c, k, I etc

- inertial terms: mass, moment of inertia
- damping terms: viscous friction coeff
  air drag
  coulomb friction coeff.
- stiffness: spring constants, modulus of elasticity
- geometry: lengths, loc of CoM
- other: forces

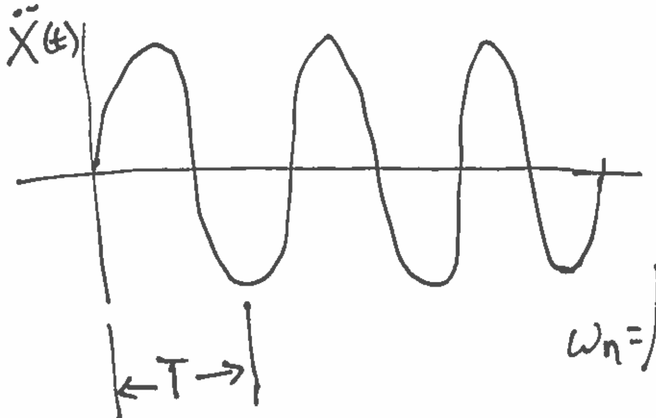
measuring:

Mass: use a scale, easy and accurate

geometry: use ruler

inertia: not always easy
1) draw everything in 3D detail and estimate inertia from simple shapes
2) vibrate it!

damping: hard to measure
dynamic measurement



$\omega_n = \sqrt{\dfrac{k}{m}}$

$T = \dfrac{2\pi}{\omega_n}$

measure $\ddot{X}$ with an accelerometer

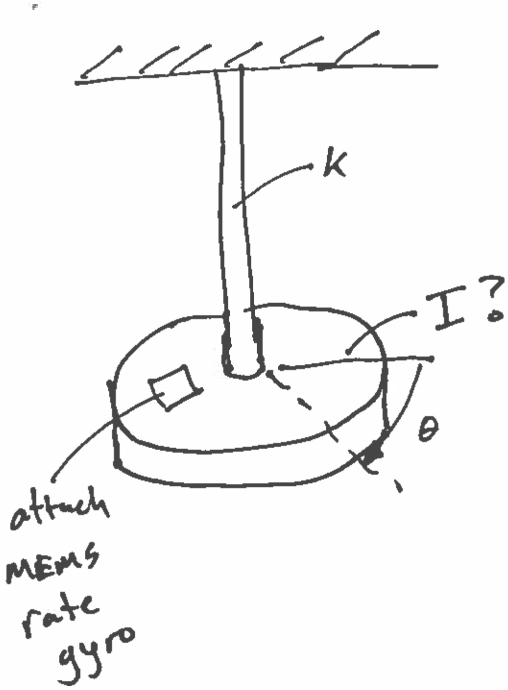
estimate m from period

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$m \frac{4\pi^2}{T^2} = k$$

$$\boxed{m = \frac{kT^2}{4\pi^2}}$$

Inertia

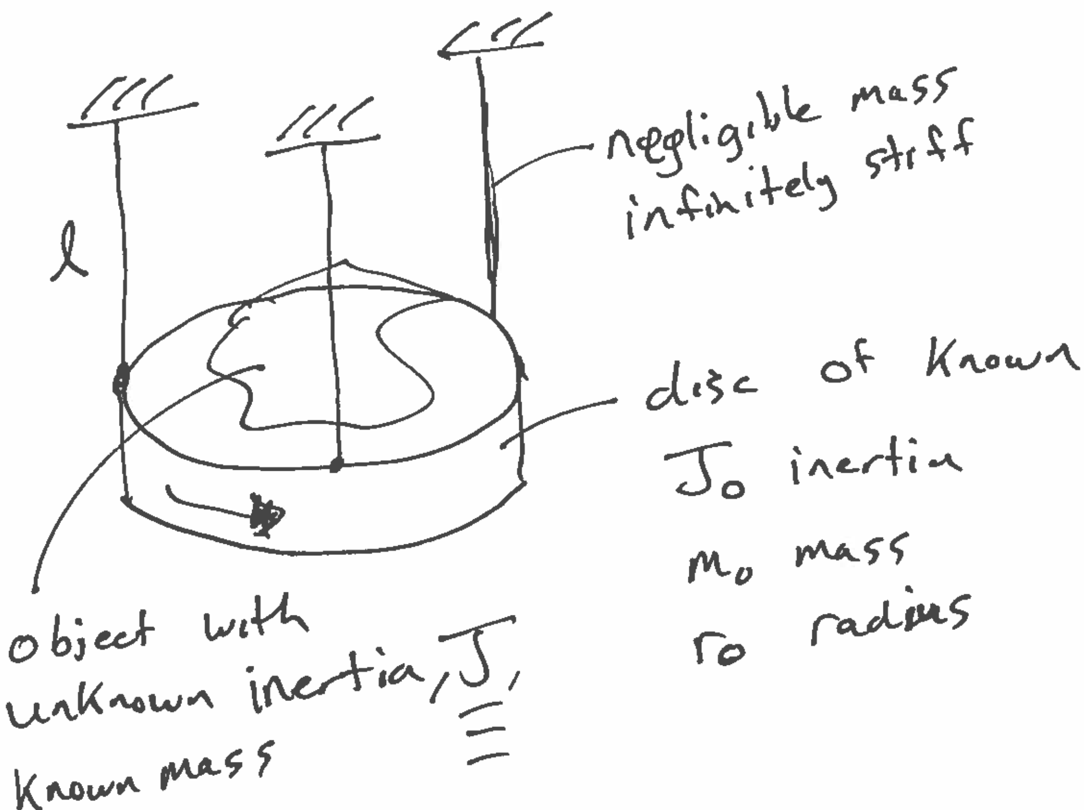


$$I\ddot{\theta} + k\theta = 0$$

$$\omega_n = \sqrt{\frac{k}{I}}$$

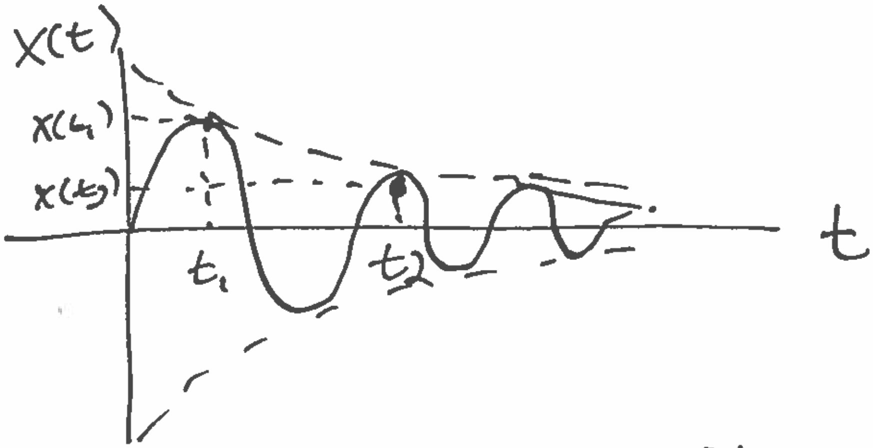
$$\boxed{I \simeq \frac{kT^2}{4\pi^2}}$$

Common setup: trifilar pendulum



$$J = \frac{g T^2 r_0^2 (m_0 + m)}{4\pi^2 l} - J_0$$

Damping is hard to measure



$$x(t) = A e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$$

5 parameters

Simple method:

log decrement

$$\delta = \ln \frac{x(t)}{x(t+T)}$$

↑ damped period

$$\delta = \ln \frac{A e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)}{A e^{-\zeta \omega_n (t+T)} \sin(\omega_d t + \omega_d T + \phi)}$$

$$\omega_d T = 2\pi$$

$$\delta = \ln e^{\zeta \omega_n T} = \zeta \omega_n T$$

$$T = \frac{2\pi}{\omega_d}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

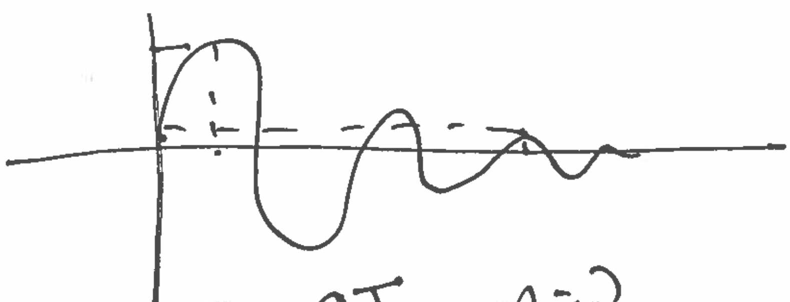
solve for ζ

$$\delta = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}} \Longrightarrow \boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}}$$

$$\delta = \frac{1}{n} \ln\left(\frac{x(t)}{x(t+nT)}\right)$$

n: # of periods

bit more accurate estimate

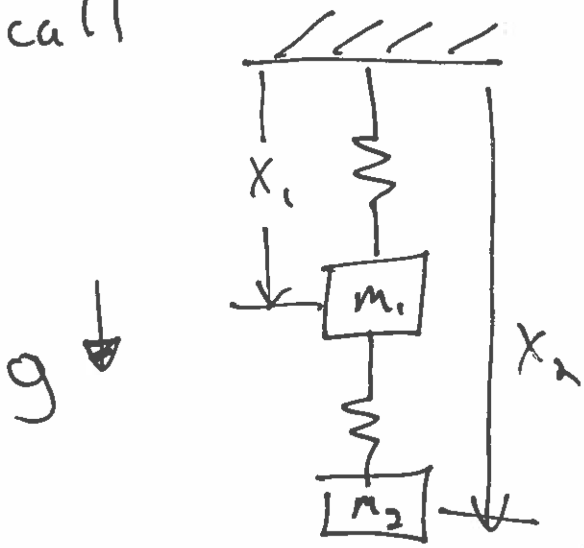


$2T$ $n=2$

ζ

Equilibrium Points

Recall



$$m_1\ddot{x}_1 + 2kx_1 - kx_2 - gm_1 = 0$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) - gm_2 = 0$$

$$\dot{x}_1,\ \dot{x}_2,\ \ddot{x}_1,\ \ddot{x}_2 = 0$$

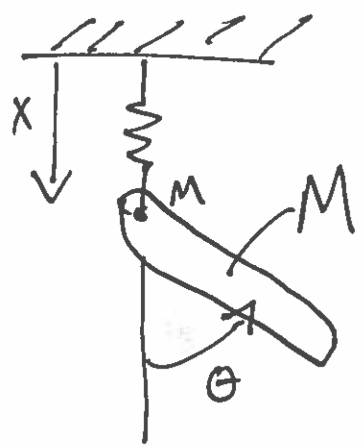
$$2kx_1 - 2x_2 - gm_1 = 0$$

$$-k(x_1 - x_2) - gm_2 = 0$$

$\Downarrow$ solve for x_1, x_2

$$x_1 = \frac{g}{k}(m_1 + m_2) \qquad x_2 = \frac{g}{k}(m_1 + 2m_2)$$

Recall



$$(I + Ml^2)\ddot{\theta} + Mgl\sin\theta - Ml\sin\theta\,\ddot{x} = 0$$

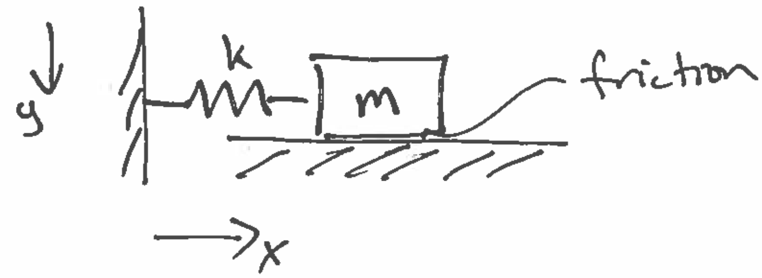
$$(m+M)\ddot{x} - M(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta) - (M+m)g + kx = 0$$

$$\left.\begin{aligned} mgl\sin\theta &= 0 \\ -(m+M)g + kx &= 0 \end{aligned}\right\} \begin{aligned} &\theta = 0, \pi, 2\pi, \ldots \\ &x = \frac{(M+m)g}{k} \end{aligned}$$

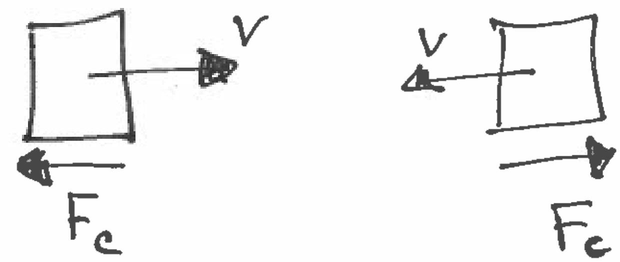
$\theta = 0$ stable eq pt

$\theta = \pi$ unstable eq pt

Coulomb Friction



$$|F_c| = \mu mg$$



$$\sum F = ma$$

$$m\ddot{x} + kx + F_c = 0$$

$$m\ddot{x} + kx = -F_c = -\underbrace{\begin{cases} +\mu mg & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ -\mu mg & \dot{x} < 0 \end{cases}}_{\text{non linear term}}$$

$$\boxed{m\ddot{x} + kx + \operatorname{sgn}(\dot{x})\mu mg = 0}$$