Coulomb Damping

- multiple equilibrium points

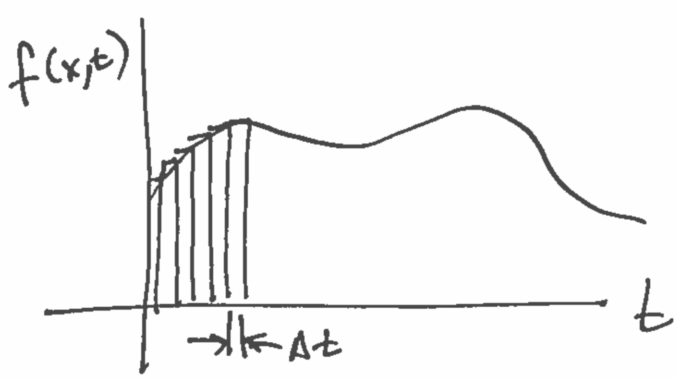
$$-\frac{\mu_s mg}{k} \leq X_0 \leq \frac{\mu_s mg}{k}$$

- the solution is bounded by intervals between $\dot{X}=0$ points.

- The amplitude of oscillation decreases linearly.

$$\text{slope} = \pm \frac{2\mu N \omega_n}{\pi K}$$

Numerial Integration



$$\underset{\approx}{X} = \int_{t_0}^{t_1} f(x,t)\, dt$$

$$\dot{x} = f(x,t)$$

$$X \approx \sum_{i=1}^{N} f(x_i, t_i)\, \Delta t$$

$$\Delta t \Rightarrow 0$$

Euler's Method

$$\frac{dx(t_i)}{dt} = \frac{x(t_{i+1}) - x(t_i)}{\Delta t} \quad \text{as } \Delta t \to 0$$

$$f(x_i, t_i) = \frac{x_{i+1} - x_i}{\Delta t} \qquad \Delta t = t_{i+1} - t_i$$

Δt ~~was~~ $t_{i+1} - t_i$

$$x_{i+1} = x_i + \Delta t \, f(x_i, t_i)$$

numerical integration error

- approximation (how poorly the integration routine matches reality)
- truncation error (computer machine precision)

State Space Form

1st order form required

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x$$

$$\dot{x} = v$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{c}{m}v - \frac{k}{m}x \end{aligned}}$$

state space form

$$\dot{S} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -\frac{c}{m}v - \frac{k}{m}x \end{bmatrix}$$

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x[i+1] = x[i] + del_t * coulomb_eqns(x[i], ti)
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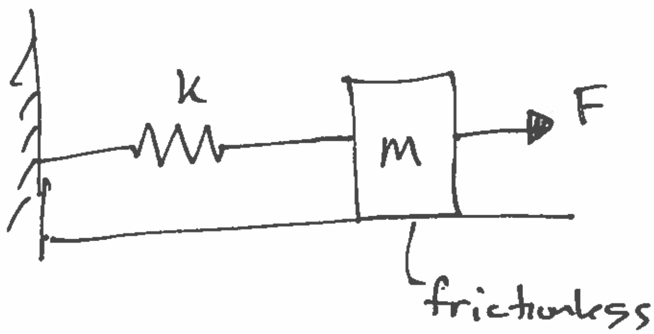
<u>Single</u> <u>DoF</u> <u>Forced</u> <u>Vibrations</u>

A system that vibrates under a *periodic* excitation force, i.e. $F(t+T) = F(t)$



If $F = F_0 \cos\omega t$

$\left.\begin{matrix} F_0 \\ \omega \end{matrix}\right\}$ constants

we now have a <u>harmonic</u> excitation.

<u>EOM</u>: $m\ddot{x} + kx = F_0 \cos\omega t$ (1) non-homogeneous linear ODE in $x(t)$

Solution:

$$x(t) = x_h(t) + x_p(t)$$

$x_h(t)$: homogeneous solution

$x_p(t)$: particular solution

$$x_h(t) = C_1 \cos\omega_n t + C_2 \sin\omega_n t \qquad \omega_n = \sqrt{\frac{k}{m}}$$

$C_1, C_2 \Rightarrow$ from I.C.s

$$X_p(t) = \bar{X} \cos \omega t$$

$$\dot{X}_p(t) = \bar{X} \omega \sin \omega t$$

$$\ddot{X}_p(t) = -\bar{X} \omega^2 \cos \omega t$$

Plug it in to the EOM.

$$m(-\bar{X}\omega^2 \cos \omega t) + k\bar{X} \cos \omega t = F_o \cos \omega t$$

solve for $\bar{X}$

$$\bar{X} = \frac{F_o}{k - m\omega^2}$$

define $r = \dfrac{\omega}{\omega_n}$ frequency ratio

$\sigma_{st} = \dfrac{F_o}{k}$ characteristic distance

$$\frac{\bar{X}}{\sigma_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} = \frac{1}{1-r^2}$$

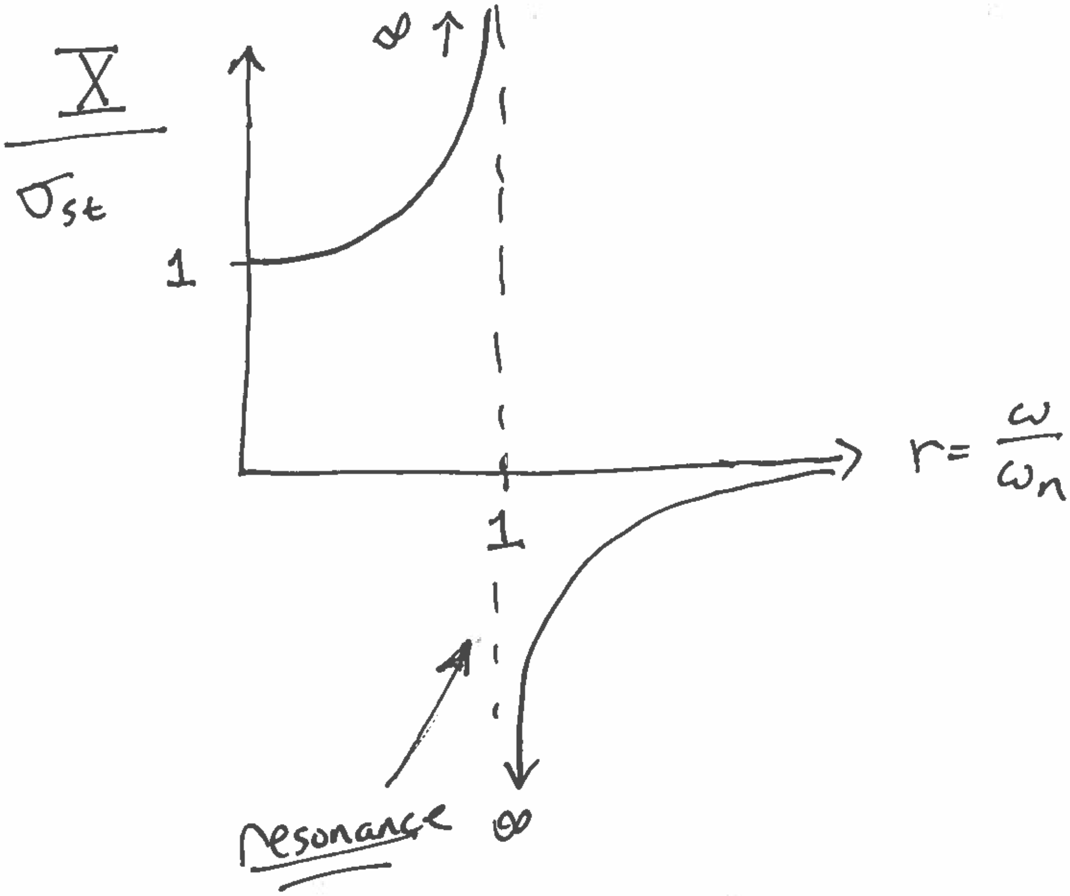
dimensionless ratio

<u>Total Solution</u>:

$$X(t) = C_1 \cos \omega_n t + C \sin \omega_n t + \frac{F_o}{k - m\omega^2} \cos \omega t$$

$x_o, \dot{x}_o$ $\quad C_1 = X_o - \dfrac{F_o}{k - m\omega^2} \quad C_2 = \dfrac{\dot{X}_o}{\omega_n}$

Plot the frequency response



If ω is close to ω_n and $x_0 = \dot{x}_0 = 0$

$$X(t) = \left(0 - \frac{F_0}{K - m\omega^2}\right)\cos\omega_n t + 0 + \frac{F_0}{K_0 - m\omega^2}\cos\omega t$$

$$= \frac{F_0/m}{\omega_n^2 - \omega^2}\left(\cos\omega t - \cos\omega_n t\right)$$

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$$= \frac{F_0/m}{\omega_n^2 - \omega^2}\left[-2\sin\left(\frac{\omega+\omega_n}{2}\right)t \cdot \sin\left(\frac{\omega-\omega_n}{2}\right)t\right]$$

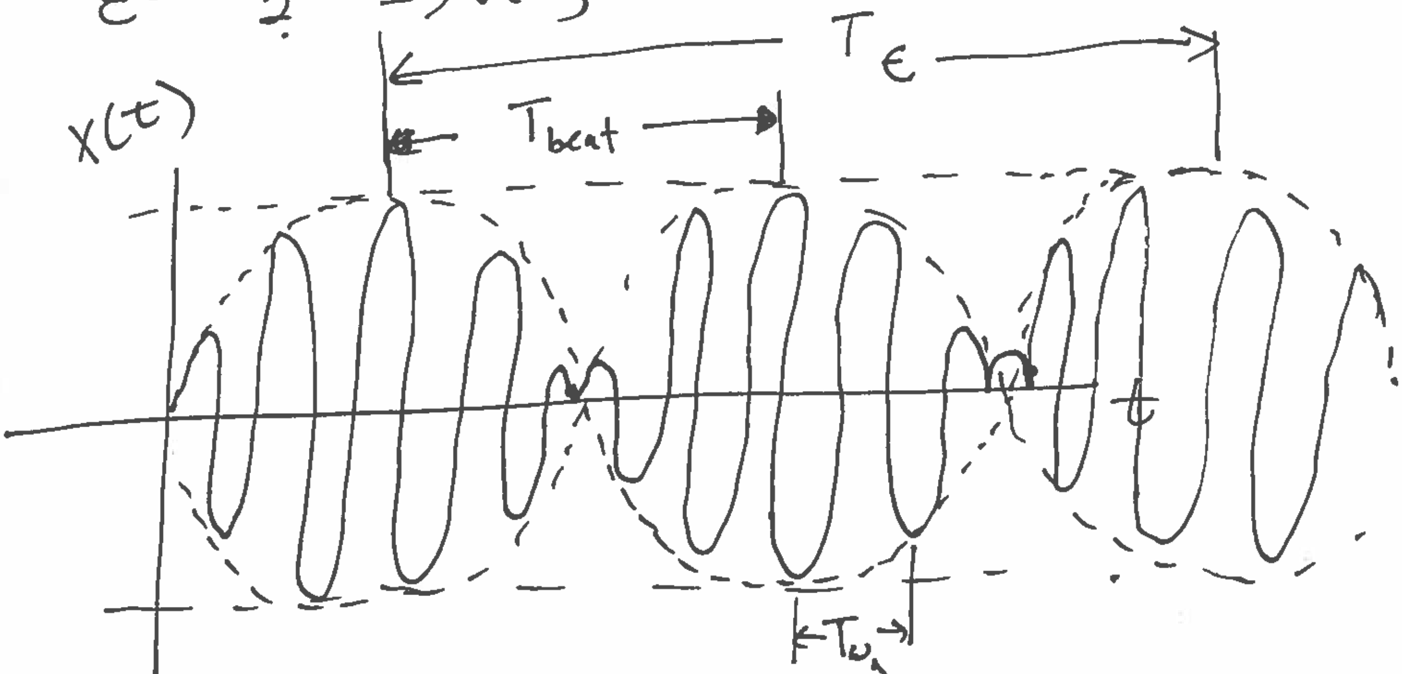
If ω is close to ω_n $\quad \omega + \omega_n \approx 2\omega_n$

$$\epsilon = \frac{\omega_n - \omega}{2}$$

$$X(t) = \frac{F_o/m}{(\omega_n+\omega)(\omega_n-\omega)}\left[-2\sin\left(\frac{\omega+\omega_n}{2}\right)t \cdot \sin\left(\frac{\omega-\omega_n}{2}\right)t\right]$$

$$X(t) = \underbrace{\frac{F_o/m}{2\omega_n\epsilon}\sin\epsilon t}_{\text{amplitude}} \cdot \sin\omega_n t$$

$\epsilon = \frac{\omega_n-\omega}{2} \Rightarrow$ very small compared to ω_n



Beating

$\omega_{beat} = |\omega - \omega_n| = 2|\epsilon|$

frequency between maximum peaks

Now if $\omega = \omega_n$ (at resonance):

$$x_p(t) = t\bar{X}\sin\omega t$$

$$\dot{x}_p(t) = t\bar{X}\omega\cos\omega t + \bar{X}\sin\omega t$$

$$\ddot{x}(t) = \bar{X}\omega\cos\omega t - t\bar{X}\omega^2\sin\omega t + \bar{X}\omega\cos\omega t$$

Plug into EOM

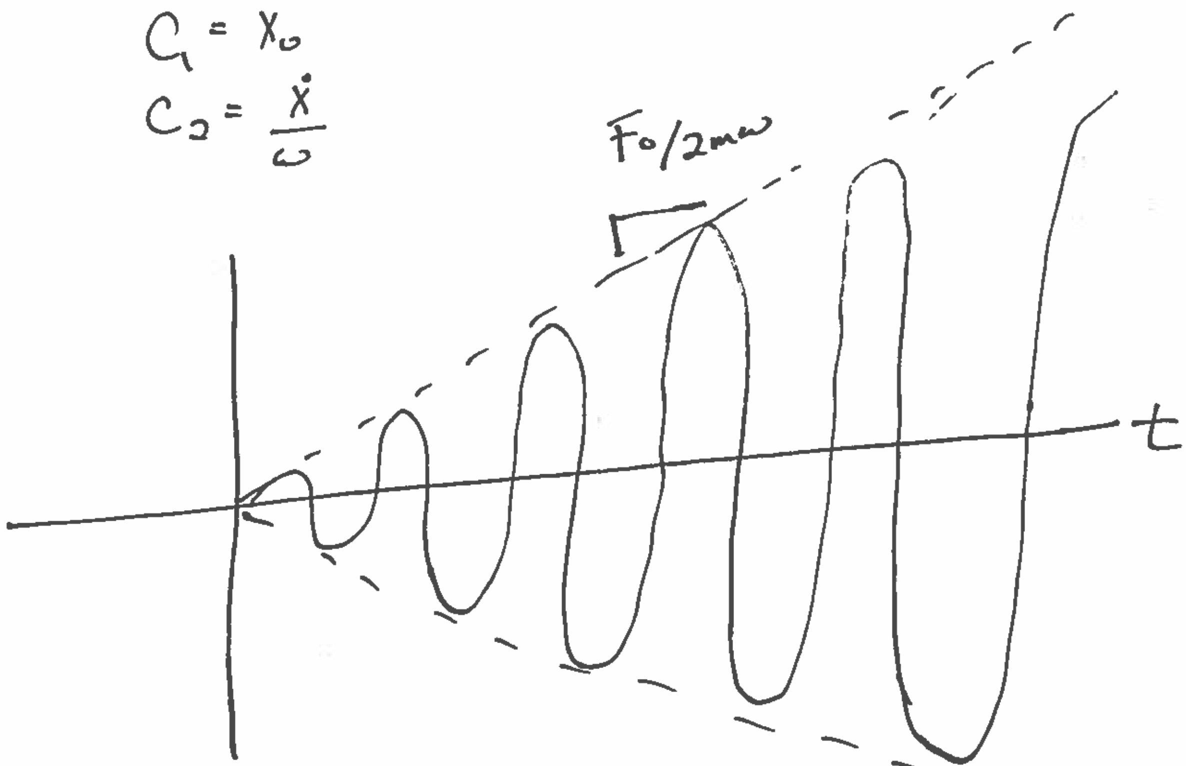
$$2\bar{X}\omega\cos\omega t = \frac{F_0}{m}\cos\omega t \quad \text{for all } t$$

$$\therefore \bar{X} = \frac{F_0}{2m\omega}$$

$$x(t) = C_1\cos\omega_n t + c_2\sin\omega_n t + \frac{F_0}{2m\omega}t\sin\omega t,$$

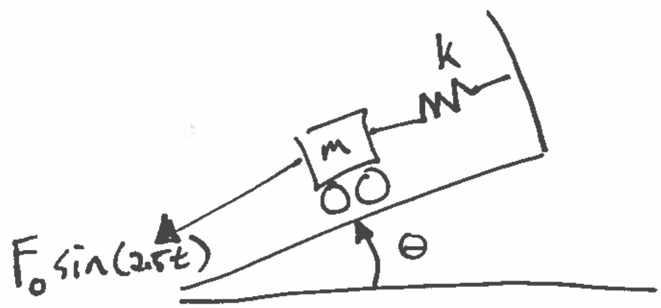
$C_1 = x_0$

$C_2 = \frac{\dot{x}}{\omega}$



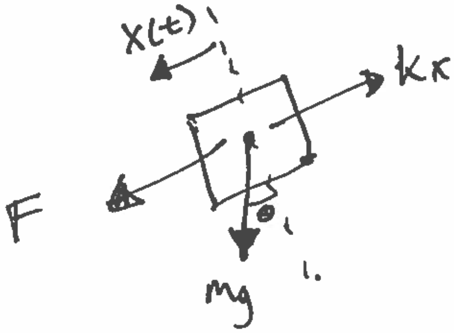
<u>unstable</u>

Problem 2.10



$m = 50\ \text{kg}$
$\theta = 30^\circ$
$k = 1000\ \text{N/m}$

Find the response of the system.



$$\sum F_x = m\ddot{x} = -kx + F + mg\sin\theta$$

$$m\ddot{x} + kx = F + mg\sin\theta$$

$$m\ddot{x} + kx = F_0 \sin(\omega t) + mg\sin\theta$$

$$x_p(t) = X\sin(\omega t) + A$$
$$\dot{x}_p(t) = \omega X\cos(\omega t)$$
$$\ddot{x}_p(t) = -\omega^2 X\sin(\omega t)$$

$$m\left[-\omega^2 X\sin(\omega t)\right] + k\left[X\sin\omega t + A\right] = F_0\sin\omega t + mg\sin\theta$$

$$kA = mg\sin\theta \qquad F_0 = (-m\omega^2 + k)X$$

$$A = \frac{mg}{k}\sin\theta \qquad X = \frac{F_0}{k - m\omega^2}$$

$$x_p(t) = \frac{F_0}{k - m\omega^2}\sin(\omega t) + \frac{mg}{k}\sin\theta$$

$$X(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t) + \frac{F_0}{k-m\omega^2}\sin\omega t + \frac{mg\sin\theta}{k}$$

$$X(0)=0 \Rightarrow C_1 = -\frac{mg}{k}\sin\theta$$

$$\dot{X}(0)=0 \Rightarrow C_2 = -\frac{F_0}{k-m\omega^2}$$

$m = 50\ kg$

$\theta = 30°$

$k = 1000\ N/m$

$F_0 = 90\ N$

$\omega = 2.5\ rad/s$

$$C_1 = \frac{-(50kg)(9.81\ m/s^2)\sin(30°)}{1000\ N/m}$$

$$C_2 = -\frac{90N}{1000\frac{N}{m} - (50kg)(2.5\frac{rad}{s})^2}$$

$C_1 = -0.245\ m$

$C_2 = -0.131\ m$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{50}}$$

$X(t) =$

$$\frac{mg}{k}\sin\theta = \quad 9.8$$

$$X(t) = (-0.245\,m)\cos(4.5t) - (0.131\,m)\sin(4.5t) + 0.131\sin(2.5\omega t) + 0.245\,m$$

plot curve with computer!