Chapters to review

1.1 - 1.6, 1.8 - 1.10

2.1 - 2.5, 2.7 - 2.9

3.1

Main Topics

- deriving equations of motion for 1 DoF systems using Lagrange's method. We've linearized about equilibrium points.
- Stability of linear form
- numerically simulated non-linear & linear EoMs
- unforced (free) response with and without damping
- harmonically forced systems: with & without damping
- non-linear: coulomb, aero, etc
- equivalent mass, stiffness, and damping
- Specific models: base excitation and unbalanced rotating masses
- impulse response

Votes

7 deriving EOM
3 unbalanced mass
2 base excitation
2 what eqs?
2 impulse
2 compound pendulum

Equations

$$m\ddot{x} + c\dot{x} + kx = F$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \underline{\underline{F/m = f}}$$

$$\omega_n^2 = \frac{k}{m} \qquad \zeta = \frac{c}{2\omega_n m} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$T = \frac{2\pi}{\omega_n} \qquad f_n = \frac{\omega_n}{2\pi}$$

Lagrange's

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial \dot{q}_i} = Q_i$$

generalized force

$$L = T - U$$

kinetic energ

total potential energy

* Rayleigh's dissapation function *

Axial

$$k = \frac{EA}{l}$$

torsion

$$k = \frac{GJ_p}{l}$$

Helical

$$k = \frac{Gd^4}{64nR^3}$$

cantilever

$$k = \frac{3EI}{l^3}$$

seiries

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

parallel

$$k = k_1 + k_2$$

Solutions to ODEs

Equivalent damping : table of c_{eq}

$\delta(t-\tau)$
Dirac Delta
unit impulse

$H(t-\tau)$
Heaviside
unit step

Unbalanced Mass

offset distance

$$m\ddot{x}+c\dot{x}+kx = m_o e\,\underbrace{\omega_r^2 \sin\omega_r}_{\text{driving frequency}} t$$

m: total mass of the system

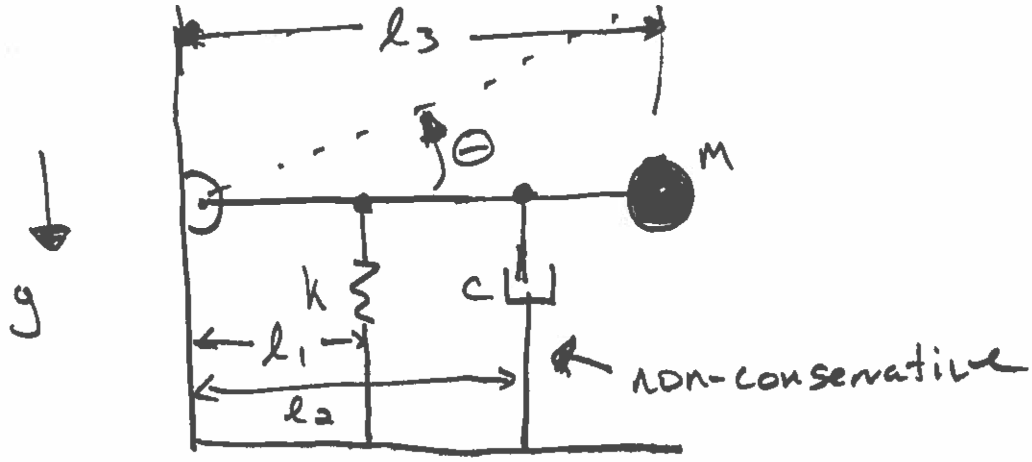
m_o: mass of offset particle

$$x_p(t) = X\sin(\omega_r t - \theta)$$

$$X = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \qquad r = \frac{\omega_r}{\omega_n}$$

$$\theta = \arctan\left(\frac{2\zeta r}{1-r^2}\right)$$

$$\frac{mX}{m_o e} \Rightarrow$$ non-dimensional



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = Q_i$$

$$L = T - U$$

$$Q_i = -\frac{\partial R}{\partial \dot\theta}$$

$$T = \frac{1}{2}m\left(l_3\dot\theta\right)^2$$

$$U = \frac{1}{2}k\left(l_1 \underbrace{\sin\theta}_{\sin\theta \approx \theta}\right)^2$$

$$L = T - U$$

$$L = \frac{1}{2}m(l\dot\theta)^2$$

$$-\frac{1}{2}k(l_1\theta)^2$$

$$-mg\, l_3\theta$$

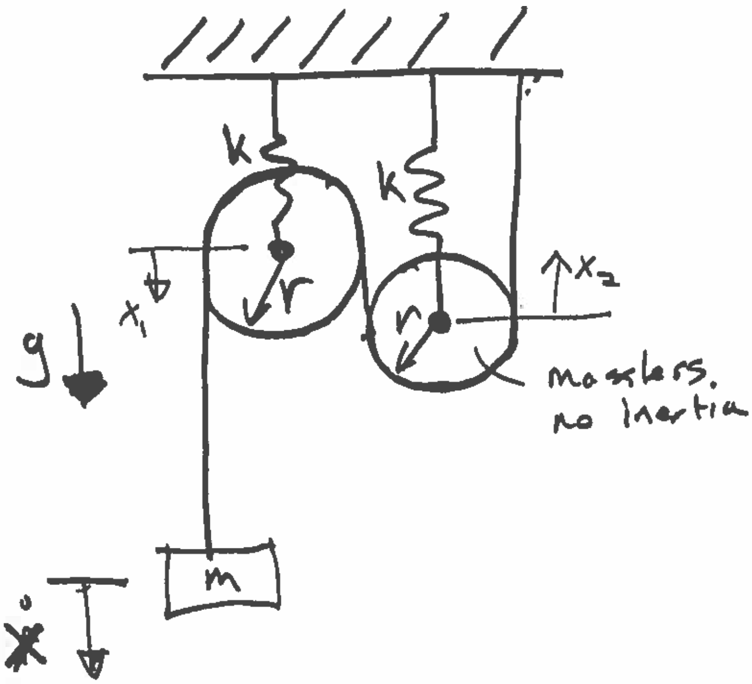
$$U = \frac{1}{2}k(l_1\theta)^2 + mg\, l_3 \underbrace{\sin\theta}_{\theta}$$

$$U = \frac{1}{2}k(l_1\theta)^2 + mgl_3\theta \longleftarrow \text{linear}$$

$$R = \frac{1}{2}c\left(l_2\dot\theta\right)^2$$

$$\frac{d}{dt}\left(ml_3^2\dot\theta\right) - \left(-kl_1^2\theta - mgl\right) = -cl_2^2\dot\theta$$

$$\boxed{ml_3^2\ddot\theta + kl_1^2\theta + mgl_3 + cl_2^2\dot\theta = 0}$$



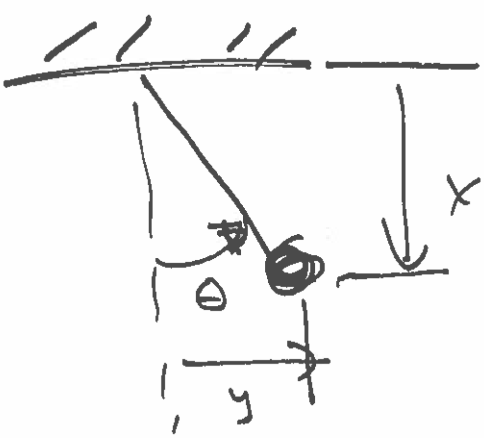
$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2}k\left(\frac{1}{4}x\right)^2 + \frac{1}{2}k\left(\frac{1}{4}x\right)^2 - mgx$$

$$X = 2x_2 + 2x_1$$

$$X_1 = X_2$$

$$X = 2x_1 + 2x_1 = \underline{\underline{4x_1}}$$

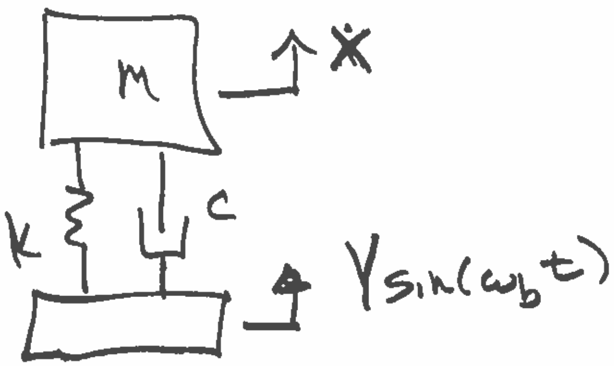


Base Excitation

$$m\ddot{x} + c\dot{x} + Kx = cY\omega_b \cos\omega_b t + KY\sin\omega_b t$$

magnitude of input

frequency of excitation



$m = 100$ kg

$c = 50$ kg/s

$K = 1000$ N/m

$Y = 0.03$ m

$\omega_b = 3$ rad/s

What is the magnitude displacement of the mass?

$$x_p(t) = X\sin(\omega_b t - \theta)$$

$$X = Y\left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 3.16 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{50 \text{ kg/s}}{2(100\text{kg})\cdot(3.16 \text{ rad/s})} = 0.079$$

underdamped $0 < \zeta < 1$

$$r = \frac{\omega_b}{\omega_n} = 0.949$$

$$X = 0.03\text{m}\left[\frac{1 + [2(0.079)(0.949)]^2}{(1-0.949^2)^2 + [2(0.079)(0.949)]^2}\right]^{1/2} = 5.62 \text{ m}$$

Extra Notes

Chapters to review

1.1 - 1.6, 1.8 - 1.10

2.1 - 2.5, 2.7 - 2.9

3.1

Main Topics

- deriving the equations of motion for 1 DoF systems using Lagrange's method.
- unforced (free) response with and without damping
- Stability of linear systems
- numerical simulation of non-lin systems
- non-linear damping : coulomb, aero, etc
- harmonically forced systems : damped + undamped
- specific models: base excitation, unbalanced mass in rotating machines
- equivalent mass, stiffness, damping
- impulse response

Equations

$$m\ddot{x} + c\dot{x} + kx = F$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F/m$$

$$\omega_n^2 = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2\omega_n m} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$T = \frac{2\pi}{\omega_n} \qquad f_n = \frac{\omega_n}{2\pi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad L = T - U$$

Axial

$$k = \frac{EA}{l}$$

Torsional

$$k = \frac{GJ_p}{I}$$

Helical Tension/Comp Spring

$$k = \frac{Gd^4}{64nR^3}$$

Cantilever

$$k = \frac{3EI}{l^3}$$

series

$$K = \frac{k_1 k_2}{k_1 + k_2}$$

parallel

$$k = k_1 + k_2$$

ODE solution forms

Damping models table

$\delta(t-\tau)$ Dirac Delta

$H(t-\tau)$ Heaviside

Undamped Free Harmonic Motion

$$m\ddot{x} + kx = 0$$

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t} \qquad \omega_n = \sqrt{\frac{k}{m}},\ j = \sqrt{-1}$$

or

$$x(t) = A \sin(\omega_n t + \phi)$$

or

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \qquad \phi = \arctan \frac{\omega_n x_0}{v_0}$$

Key points

- oscillates @ ω_n : natural frequency
- amplitude and phase shift depend on initial conditions

Damped Free Harmonic Motion

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{or} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2\omega_n m}$$

Under damped : $0 < \zeta < 1$

$$x(t) = A e^{-\zeta\omega_n t} \sin(\omega_d + \phi)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2}{\omega_d^2}} \qquad \phi = \arctan \frac{x_0 \omega_d}{v_0 + \zeta\omega_n x_0}$$

Overdamped $\zeta > 1$

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_n\sqrt{\zeta^2-1}\,t} + a_2 e^{+\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

$$a_1 = \frac{-v_0 \mp (-\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2-1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

Critically damped $\zeta = 1$

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t}$$

$$a_1 = x_0 \qquad a_2 = v_0 + \omega_n x_0$$

Stability

If effective stiffness is negative for undamped system $\Rightarrow$ instability.

effective damping < 0 could lead to instability

First order explicit form

$$\dot{x} = v$$

$$\dot{v} = -\frac{c}{m}\cancel{v} - \frac{k}{m}x$$

statespace

$$\dot{s} = F \qquad \dot{s} = \begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} \qquad s = \begin{bmatrix}x\\ v\end{bmatrix}$$

linear state space $F = \begin{bmatrix} v \\ -\frac{c}{m}v - \frac{k}{m}x\end{bmatrix}$

$$\dot{s} = A s \qquad A = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{bmatrix}$$

Coulomb Friction

$$m\ddot{x} + \mu m g\,\mathrm{sign}(\dot{x}) + kx = 0$$

Linear decay $N = mg$

slope $= \pm \dfrac{2\mu N \omega_n}{\pi k}$

$X_0 > \dfrac{\mu_s N}{k}$

Harmonic Excitation Undamped

$$m\ddot{x} + kx = F_0 \cos \omega t \qquad f_0 = \frac{F_0}{m}$$

$$\ddot{x} + \omega_n^2 x = f_0 \cos \omega t$$

$$x = x_h + x_p$$

$$x_p = \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

$$x = \frac{v_0}{\omega_n} \sin \omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

Freq response: undefined @ $r = 1$ $\quad r = \dfrac{\omega}{\omega_n}$

near ω_n: beating $\omega_{beat} = |\omega_n - \omega|$

resonance $\Rightarrow$ output grows (for undamped, without bounds)

Harmonic Ext. damped

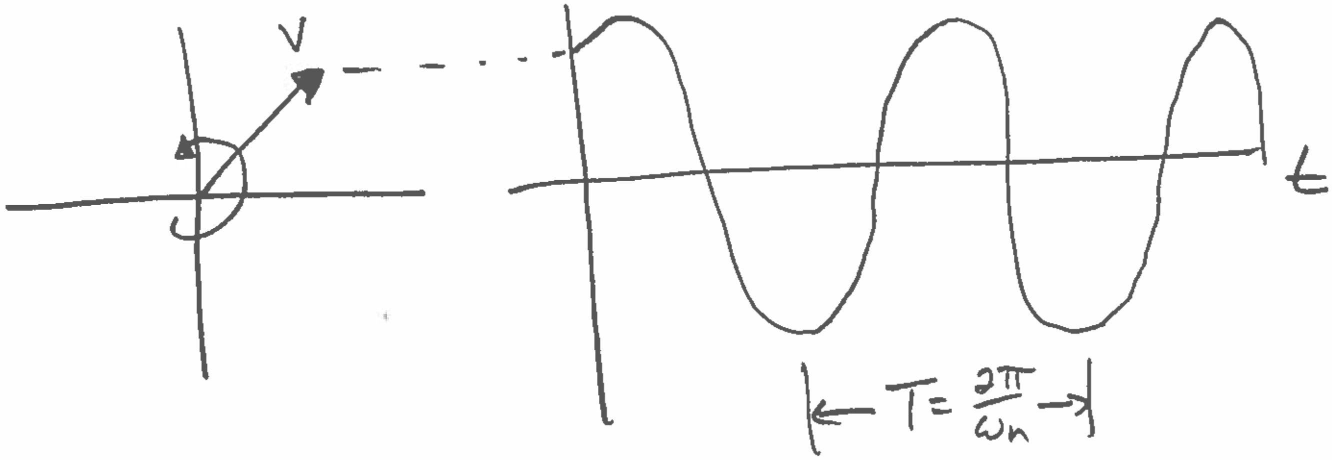
$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 \cos \omega t$$

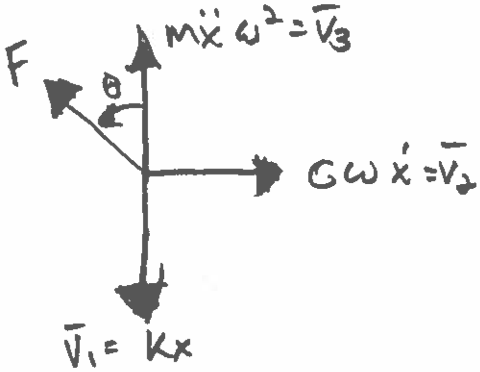
$$x_p = X \cos \omega t - \theta \leftarrow \text{Steady state}$$

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \qquad \theta = \arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \right)$$

Phasor

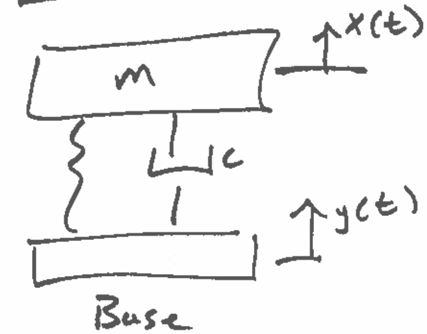


Sum of phasors is just vector addition



$$\bar{F} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3$$

Base Excitation



$$m\ddot{x} + c\dot{x} + kx = cY\omega_b \cos\omega_b t + kY\sin\omega_b t$$

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

$$y(t) = Y\sin\omega_b t$$

Unbalanced mass

$$m\ddot{x} + c\dot{x} + kx = m_0 e \omega_r^2 \sin \omega_r t$$

$$x_p(t) = X \sin(\omega_r t - \theta)$$

$$X = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\theta = \arctan \frac{2\zeta r}{1-r^2}$$

Equivalent mass, stiffness

$$T_{eq} = T_{act} \qquad U_{eq} = U_{act}$$

$\Downarrow$ mass $\qquad\qquad$ $\Downarrow$ stiffness

Equiv damping

$$\Delta E = \oint F_d \, dx = \int_0^{2\pi/\omega} c\dot{x}\dot{x} \, dt$$

$$\Delta E = \pi c \omega X^2 \qquad \text{viscous}$$

$$\Delta E = \Delta E_{act}$$

aero: $c_{eq} = \frac{8}{3\pi} \alpha \omega X$

Coulomb: $c_{eq} = \frac{4\mu mg}{\pi\omega X}$

hystertic

ΔE : area in loop

$$c_{eq} = \frac{k\beta}{\omega} = \frac{h}{\omega}$$

Impulse Response

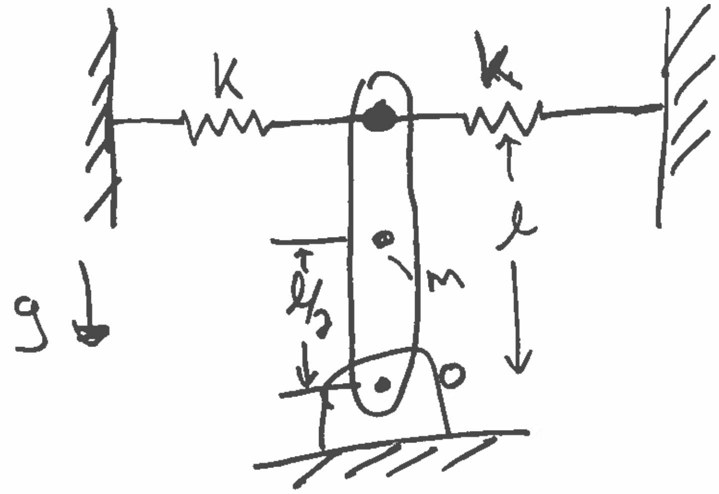
Underdamped:

$$m\ddot{x} + c\dot{x} + kx = \hat{F}(t) \qquad \hat{F}(t) = F\Delta t$$

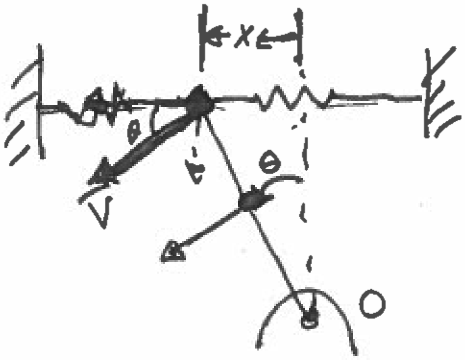
$$x(t) = \frac{\hat{F} e^{-\zeta\omega_n t}}{m\omega_d} \sin\omega_d t$$

$$x(t) = \hat{F}\, h(t)$$

$$h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$$



$$I_0 = \frac{ml^2}{3}$$



$$l \sin\theta = x$$

θ ~~is~~ small

$$|\bar{v}| = l\dot{\theta}$$

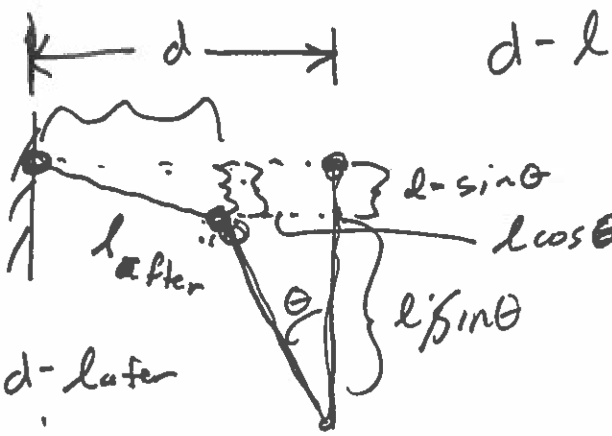
$$V_x = l\dot{\theta}\cos\theta$$

$$|V_x| = |\bar{V}|\cos\theta \approx |\bar{V}|$$

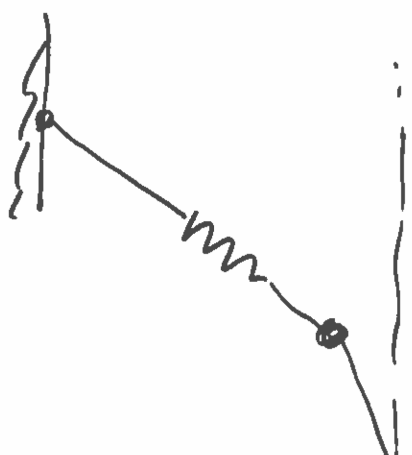
$$l_{after}^2 = d^2 + l^2 - 2dl$$

$$T = \frac{1}{2} m\left(\frac{l}{2}\dot{\theta}\right)^2 + \frac{1}{2} I_0 \dot{\theta}^2$$

$$U = \frac{1}{2} k x^2 + \frac{1}{2} k x^2 - mg\left(\frac{l}{2} - \frac{l}{2}\cos\theta\right)$$



$$d - l\sin\theta$$



$$\Delta = d - l_{after}$$

$$l_{after}^2 = (d - l\cos\theta)^2 + (l - l\sin\theta)^2$$

$$\ldots = (d-l)^2 + (l)^2 = d^2 + l^2 - 2dl$$

2.58

$$\frac{X}{Y} < 0.55 \qquad \frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} < 0.55$$

$r = 1.8$ $\qquad \zeta = ?$

Force trans?

$$1+(2\zeta r)^2 < (0.55)^2\left[(1-r^2)^2+(2\zeta r)^2\right]$$

$$1+(2\zeta r)^2 < 0.55^2(1-r^2)^2 + 0.55^2(2\zeta r)^2$$

$$(2\zeta r)^2 - 0.55^2(2\zeta r)^2 < 0.55^2(1-r^2)^2 - 1$$

$$\cancel{\zeta^2\left[2r^2 - 0.55^2\right]}$$

$$(2\zeta r)^2\left[1-0.55^2\right] < 0.55^2(1-r^2)^2 - 1$$

$$\zeta^2 < \frac{0.55^2(1-r^2)^2-1}{2^2 r^2\left[1-0.55^2\right]}$$

$$\zeta < \sqrt{\frac{0.55^2(1-1.8^2)^2-1}{2^2\, 1.8^2\left[1-0.55^2\right]}} = 0.239$$

$$\frac{F_T}{kY} = r^2\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} = \underline{\underline{1.78}}$$

twice the force!