ENG 122 FALL 2016 LECTURE 17 Monday, Nov. 21, 2016

- How eigenvalues relate back to the system?

- Practical implications of 2 DoF systems? What does math mean!

* - Stability for 2 DoF (interpreting eigenvalue plots)

- How to model and Derive EoMs in N-DoF (choosing coordinates)?

- How does damping apply to 2DoF?

* - Physically, what is a mode?

Review

Matrix Form of Linear EOM

$$M\ddot{\bar{x}} + C\dot{\bar{x}} + K\bar{x} = 0$$

n : degrees of freedom

M, C, K : $n \times n$ matrices

$\bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}$: $n \times 1$ vecters

↑
coordinate vector

M, K ⇒ symmetric and $C = 0$

$$\bar{x} = L\bar{q}$$

$M = LL^T$ ↳ cholesky decomposition (triangular matrix)

$$\ddot{\bar{q}} + \tilde{K}\bar{q} = 0$$

$$\tilde{K} = L^{-1} K L^{-1}$$

if M is diagonal

$$\tilde{K} = M^{-1/2} K M^{-1/2}$$

e.g.

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad M^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 \\ 0 & \frac{1}{\sqrt{m_2}} \end{bmatrix}$$

$\tilde{K}$ is guaranteed to be symmetric!

Solve Eigenvalue problem

$$\tilde{K}\bar{v} = \lambda \bar{v} \quad \text{or} \quad \tilde{K}\bar{v} = \omega^2 \bar{v}$$

λ ↳ eigenvalue

ω ↳ eigenfrequencies

$\det(\tilde{K} - \lambda I) = 0 \longrightarrow$ characteristic equation

char. eq. → polynomial in λ (or ω)
where one λ per DoF or two ω's per DoF.

Since $\tilde{K}$ is symmetric ⟹ - all eigenvalues and eigenvectors are real!

- eigenvectors are orthogonal

Each eigenvalue and eigenvector pair corresponds to one mode shape of the system.

- Normalize the eigenvectors ⟹ magnitude to be equal to 1!

Othonormal eigenvectors

$$P = [\bar{V}_1, \ldots, \bar{V}_n]$$ matrix of orthonormal eigenvectors

$$P^T = P^{-1}$$

Decouple equations of motion

$$\bar{q} = P\bar{r}$$ $\bar{r}$: modal coordinates

$$\dot{\bar{q}} = P\dot{\bar{r}}$$

EOM

$$\ddot{\bar{r}} + \underline{\Lambda}\,\bar{r} = 0$$ modal form of EOMs

$\underline{\Lambda}$: spectral matrix

$$\underline{\Lambda} = \text{diag}(\lambda_i) = P^T\tilde{K}P$$

$$\underline{\Lambda} = \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots & \\ & & & \omega_n^2 \end{bmatrix}$$

undamped

$$r_i(t) = \frac{\sqrt{\omega_i^2 r_{i0}^2 + \dot{r}_{i0}^2}}{\omega_i} \sin\left[\omega_i t + \arctan\left(\frac{\omega_i r_{i0}}{\dot{r}_{i0}}\right)\right]$$

$$r_i(t) = A_i \sin(\omega_i t + \phi_i)$$

Convert the modal coordinates to your gen. coords:

$$\bar{X} = S\bar{r} \quad \text{where } S = L^{-1}P$$

$$S^{-1} = P^T L$$

S: matrix of mode shapes

$$\bar{r} = S^{-1}\bar{x}, \quad \bar{x} = S\bar{r}$$

<u>Mode Summation</u>

$$\bar{X} = \sum_{i=1}^{n} d_i \sin(\omega_i t + \phi_i)\,\bar{u}_i$$

d_i → modal participation factor

$\bar{u}_i = L^{-1}\bar{V}_i$ → ith mode shape

$$S = \left[\bar{u}_1, \ldots, \bar{u}_n\right]$$

If $\bar{x}_0$ is $\bar{u}_i$ then only that mode will be excited.

e.g. $\bar{x}_0 = \bar{u}_1$ then $d_2, \ldots, d_n = 0$

Rigid Body Modes

if $\lambda_i = 0 \Rightarrow$ corresponds to translation or rotation away equilibrium

$$d_i = \frac{\bar{V}_i^T \bar{q}_0}{\sin \phi_i} \qquad \phi_i = \arctan \frac{\omega_i \bar{V}_i^T \bar{q}(0)}{\bar{V}_i^T \dot{\bar{q}}_0}$$

If $C \neq 0$ in $M\ddot{\bar{x}} + C\dot{\bar{x}} + K\bar{x} = 0$

If $C = \underbrace{\alpha M + \beta K}_{\text{Scalars}}$

$$\ddot{\bar{r}} + (\alpha I + \beta \Lambda)\dot{\bar{r}} + \Lambda \bar{r} = 0 \quad \text{modal form}$$

$$\ddot{r}_i + 2\zeta_i \omega_i \dot{r}_i + \omega^2 r_i = 0$$

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta \omega_i}{2} \quad \text{modal damping}$$

$$r_i(t) = A_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} + \phi_i)$$

$$\bar{X}(t) = \sum_{i=1}^{n} d_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} + \phi_i)\, \bar{u}_i$$

If C is more general

First order form

$$\underset{\text{state matrix}}{A} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \qquad 2n \times 2n$$

$$\dot{\bar{X}} = A\bar{X}$$

$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} \quad 2n \times 1$$

$2n$ eigenvalues : $\lambda_i \Rightarrow$ these can be complex!!

$$\bar{X}(t) = \sum_{i=1}^{2n} c_i \bar{u}_i e^{\lambda_i t}$$

c_i: from initial conds.

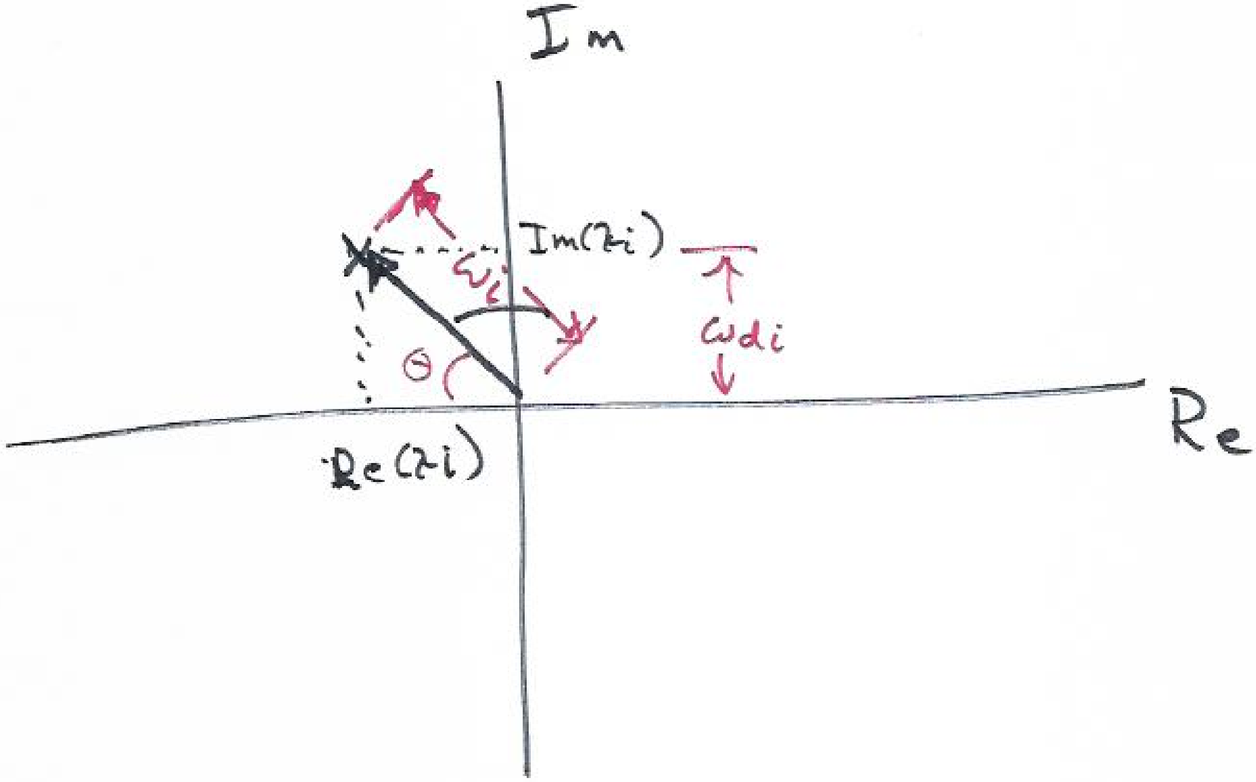
$\bar{u}_i$: mode shape vector

Complex conjugate pairs will arise if a mode is oscillatory: $j = \sqrt{-1}$

$$\text{pair} \begin{cases} \lambda_i = -\zeta_i \omega_i - \omega_i \sqrt{1-\zeta_i^2}\, j = -\zeta_i \omega_i - \omega_{di} j \\ \lambda_{i+1} = \quad " \quad + \quad " \quad = \quad " \quad + \quad " \end{cases}$$

$$\omega_i = \sqrt{Re(\lambda_i)^2 + Im(\lambda_i)^2}$$

$$\zeta_i = \frac{-Re(\lambda_i)}{\sqrt{Re(\lambda_i)^2 + Im(\lambda_i)^2}} = \cos\theta$$



If critically damped or overdamped mode the λ_i is <u>real</u>.