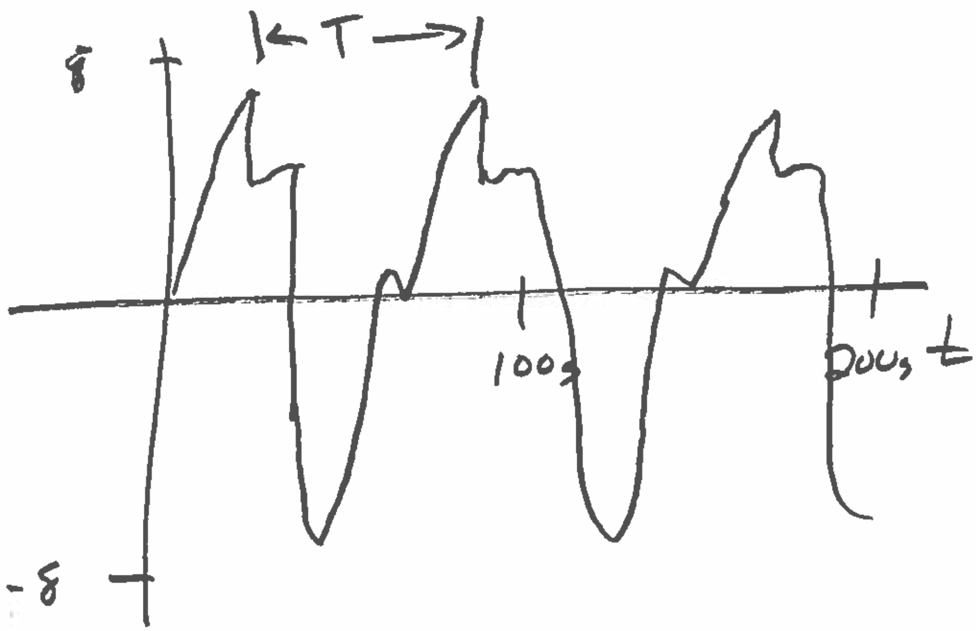
Response to Arbitrary Periodic Inputs

periodic: repeats in time

periodic function: $f(t) = f(t+T)$

e.g. $f(t) = A \sin \omega_1 t + B \sin \omega_2 t$



$A = 5$
$B = 2$
$\omega_1 = 0.2$
$\omega_2 = 0.5$

Fourier Series

Any periodic function $F(t)$ with a period T can be represented by an infinite series:

F

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_T t + b_n \sin n\omega_T t)$$

$$\omega_T = \frac{2\pi}{T}$$

Fourier Coefficients

$$\left\{ \begin{aligned} a_0 &= \frac{2}{T}\int_0^T F(t)\,dt \\ a_n &= \frac{2}{T}\int_0^T F(t)\cos n\omega_T t\,dt \quad n=1,2,\ldots \\ b_n &= \frac{2}{T}\int_0^T F(t)\sin n\omega_T t\,dt \end{aligned} \right.$$

Fourier Series have the property "orthogonality":

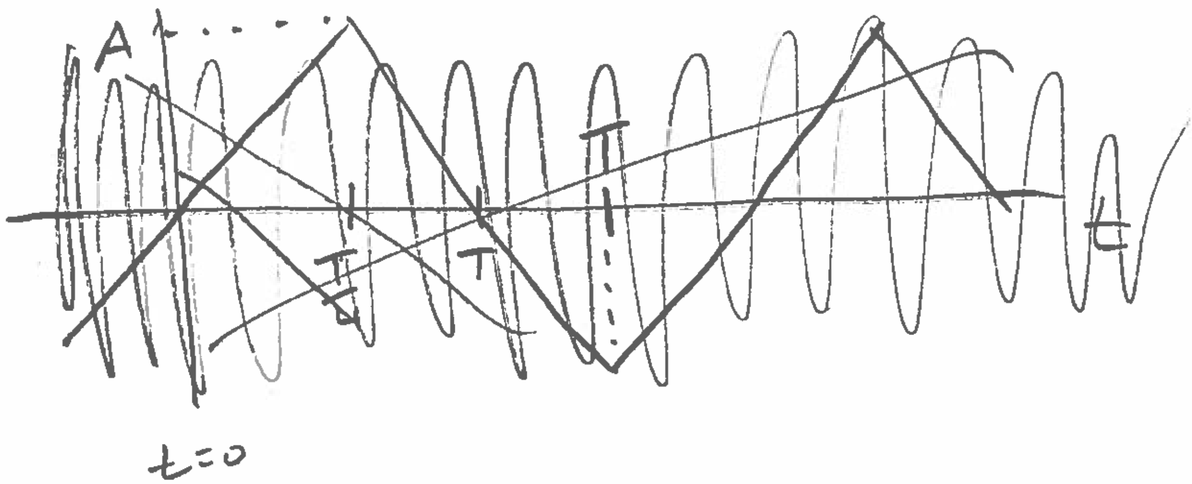
$$\int_0^T \sin n\omega_T t \sin m\omega_T t\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

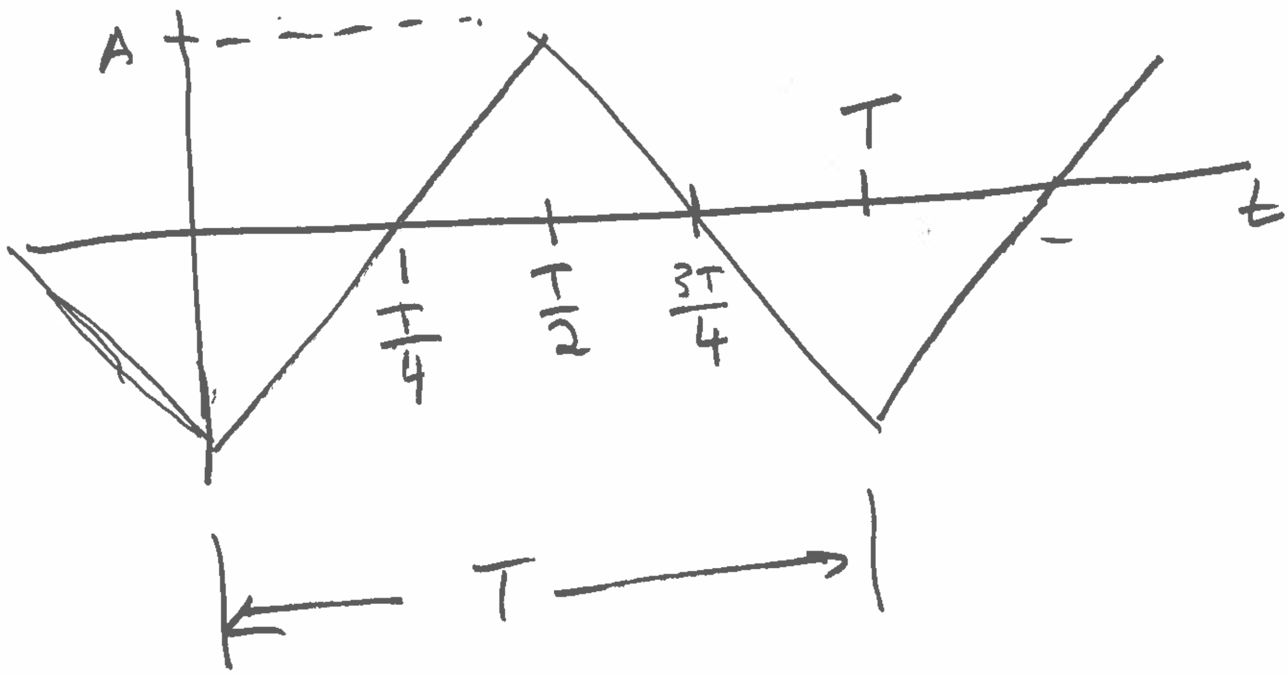
$$\int_0^T \cos n\omega_T t \cos m\omega_T t\,dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_0^T \cos n\omega_T t \sin m\omega_T\,dt = 0$$

$m, n \Rightarrow$ integers

Ex Saw tooth function





$$F(t) = \begin{cases} A\left(\frac{4}{T}t - 1\right) & 0 \le t < \frac{T}{2} \\ A\left(3 - \frac{4t}{T}\right) & \frac{T}{2} \le t < T \end{cases}$$

Solution to arbitrary periodic forcing

$$m\ddot{x} + c\dot{x} + kx = \underline{F(t)}$$

If $F(t)$ is periodic:

$$x_p(t) = x_1(t) + \sum_{n=1}^{\infty} [\underset{\substack{\uparrow \\ \cos}}{x_{cn}(t)} + \underset{\substack{\uparrow \\ \sin}}{x_{sn}(t)}]$$

Superposition of the sines and cosines

Start with $x_1(t)$

$$m\ddot{x}_1 + c\dot{x}_1(t) + k\,x_1(t) = \frac{a_0}{2}$$

$$x_1(t) = \frac{a_0}{2k}$$

then cos

$$m\ddot{x}_{cn}(t) + c\dot{x}_{cn}(t) + k\,x_{cn}(t) = a_n \cos n\omega_T t$$

$$x_{cn}(t) = \frac{a_n/m}{\left[[\omega_n^2 - (n\omega_T)^2]^2 + (2\zeta\omega_n n\omega_T)^2\right]^{1/2}} \cos(n\omega_T - \theta_n)$$

$$\theta_n = \arctan\left(\frac{2\zeta\omega_n n\omega_T}{\omega_n^2 - (n\omega_T)^2}\right)$$

$$X_{sn}(t) = \frac{b_n/m}{\left[\left[\omega_n^2 - (n\omega_T)^2\right]^2 + (2\zeta\omega_n n\omega_T)^2\right]^{1/2}} \sin(n\omega_T - \theta_n$$

$$X(t) = \underbrace{A e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}_{\substack{\text{homogeneous} \\ \text{transient}}} + \underbrace{\frac{a_0}{2k} + \sum_{n=1}^{\infty}\left[X_{cn}(t) + X_{sn}(t)\right]}_{\substack{\text{steady} \\ \text{state} \\ \text{particular} \\ \text{solution}}}$$

A, ϕ : depend initial conditions *and* periodic forcing

$M =$

$C =$

$K =$

$A =$

$T =$

$\omega_n = 10\ rad/s$

$T = \dfrac{2\pi}{5\ rad/s}$

$\zeta = 0.01$

$A = 1000\ N$

$m = 100\ kg$