Review

Unbalanced mass

offset dist.

$$m\ddot{x} + c\dot{x} + kx = m_o e \omega_r^2 \underbrace{\sin \omega_r t}_{\text{driven freq}}$$

total mass

offset mass

steady state $X_p(t) = X \sin(\omega_r t - \theta)$

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\theta = \arctan \frac{2\zeta r}{1-r^2}$$

If $\zeta > 1$: $\frac{mX}{m_o e} < 1$ – increase damping alot and solve any issues

as $r \to \infty$: $\frac{mX}{m_o e} \to 1$ – at high freq mass imbalance doesn't matter so much

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$\underbrace{\text{low damping} + r \approx 1}$ – problematic!

↳ car speed

Equivalent mass, stiffness, damping

mass: equate Kinetic energy

stiffness: equate potentia energy

damping: equate "energy loss per cycle" [in the context of forced vibrations]

$$\Delta E = \oint F_d \, dx$$

$$= \int_0^{\frac{2\pi}{\omega}} F_d \frac{dx}{dt} \, dt$$

Coulomb

$$c_{eq} = \frac{4\mu m g}{\pi \omega X}$$

Aero Drag

$$c_{eq} = \frac{8}{3\pi} \alpha \omega X$$

$$\alpha = \frac{C_D A}{2}$$

C_D

$$F_D = V^2$$

Hysteretic Damping

$$c_{eq} = \frac{k\beta}{\omega} \text{ or } \frac{h}{\omega}$$

area inside a hysteris loop is ΔE

$$\pi c \omega X^2$$

General Force Response

- forces applied to system are rarely modeled well by sinusoids (harmonic excitation)
- develop methods to handle generic forces

Superposition

For linear systems the response is the sum of the responses from additive forcing functions.

e.g.

$$\ddot{x} + \omega_n^2 x = 0$$

if $x_1(t)$ and $x_2(t)$ are both solutions

then

$x = a_1 x_1 + a_2 x_2$ is also a solution

similarly

$\ddot{x} + \omega_n^2 x = f_1$ $\quad x_p = x_1$

$\ddot{x} + \omega_n^2 x = f_2$ $\quad x_p = x_2$

the solution to $\ddot{x} + \omega_n^2 x = f_1 + f_2$

is simply $x_1 + x_2$

Periodic : repeats in time

- harmonic : sinusodial

Nonperiodic : doesn't repeat

- impulses, step, ramp, etc
- random

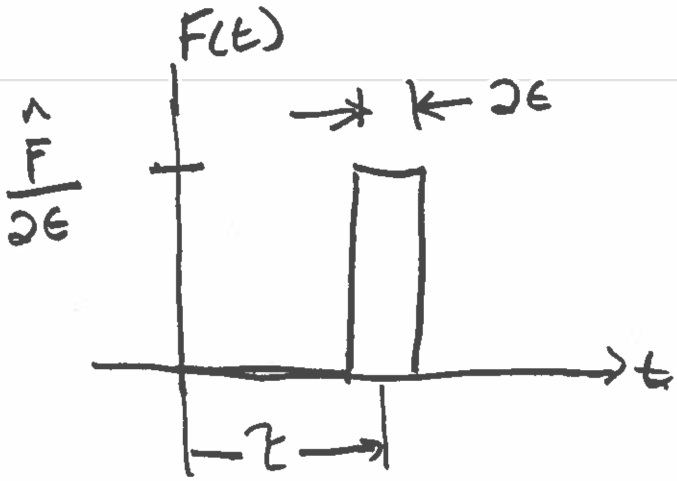
Transient : goes to zero in finite amount of time.

Impulse Response Function

Impulse : large force applied for a very short duration (non-periodic)

impulses cause "shock loading"

Key Idea : response from an impulse is equivalent to the response from some initial conditions



$$F(t) = \begin{cases} 0 & t \leq \tau - \epsilon \\ \frac{\hat{F}}{2\epsilon} & \tau - \epsilon \leq \tau \leq \tau + \epsilon \\ 0 & t \geq \tau + \epsilon \end{cases}$$

ϵ is small and $\epsilon > 0$

Impulse of Force: measure of the forcing functions "strength"

$$I(\epsilon) = \int_{\tau-\epsilon}^{\tau+\epsilon} F(t)\,dt = \int_{-\infty}^{\infty} F(t)\,dt$$

→ units N·s

$$I(\epsilon) = \int_{-\infty}^{\infty} F(t)\,dt = \frac{\hat{F}}{2\epsilon}\,2\epsilon = \hat{F}$$

$I(\epsilon)$ is independent of ϵ if $\epsilon \neq 0$

Define "impulse function" with two properties:

$$F(t-\tau) = 0 \qquad t \neq \tau$$

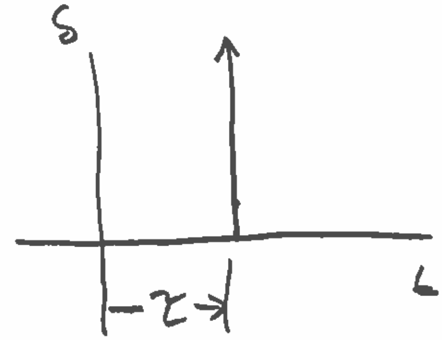
and

$$\int_{-\infty}^{\infty} F(t-\tau)\,dt = \hat{F}$$

<u>Unit</u> <u>Impulse</u> <u>function</u> (Dirac Delta Function)

$$\delta(t-\tau) = 0 \qquad t = \tau$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\,dt = 1$$



response to $\hat{F}(t)$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \hat{F}(t)$$

Recall that an impulse imparts a change in momentum.

$$\Delta p = \hat{F}$$

applying $\hat{F}$ impulse is equivalent to $x_0 = 0$ and $v_0 = ?$

$$m v_0 = F \Delta t$$

$$\boxed{v_0 = \frac{F \Delta t}{m}}$$

If $x_0 = 0$ and $v_0 = \frac{F\Delta t}{m}$ and $0 < \zeta < 1$

$$x(t) = \frac{v_0}{\omega_d} e^{-\zeta \omega_n t} \sin \omega_d t$$

$$x(t) = \frac{\hat{F} e^{-\zeta \omega_n t}}{m \omega_d} \sin \omega_d t$$

$$x(t) = \hat{F}\,\underbrace{h(t)}_{\text{response to unit impulse}}$$

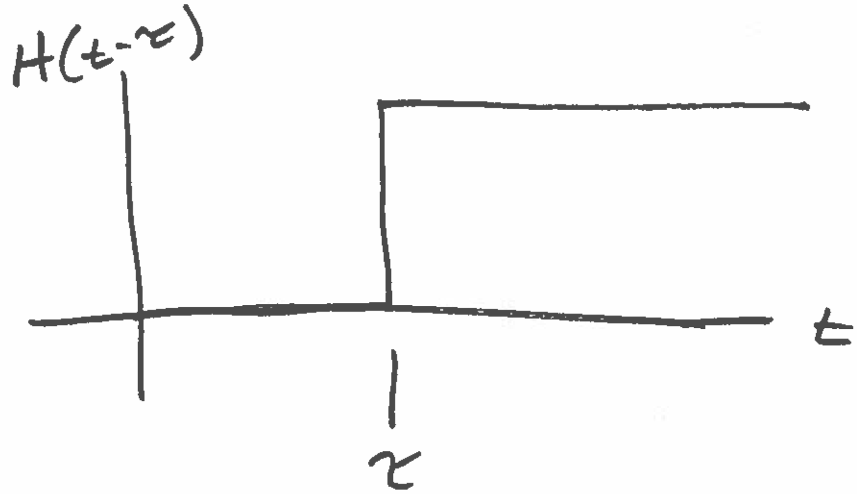
$$h(t-\tau) = \frac{1}{m\omega_d} e^{-\zeta \omega_n (t-\tau)} \sin \omega_d (t-\tau)$$

<u>valid</u> $t > \tau$

$t < \tau \quad h(t-\tau) = 0$

Heaviside Function

$$H(t-\tau) = \begin{cases} 0 & t < \tau \\ 1 & t \geq \tau \end{cases}$$



Model is only valid for large forces and short durations.

if $\Delta t \ll T = \frac{2\pi}{\omega_n}$

Example

Consider a linear system with

$m = 0.1$ kg

$c = 0.25$ Ns/m

$k = 0.5$ N/m

A $t = 2$ s the system is struck by a force of 10 N for a duration of 0.01 s. 1) What is the response function?

2) Sketch the response from $t = 0$ to $t = 10$ s.

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{0.5\,\frac{N}{m}}{0.1\,kg}} = 2.24\ \frac{rad}{s}$$

$$\hat{F} = (10\,N)(0.01\,s)$$

$$\hat{F} = 0.1\ Ns$$

$$\zeta = \frac{c}{2\,m\cdot\omega_n} = \frac{0.25\,\frac{Ns}{m}}{2(0.1\,kg)(2.24\,\frac{rad}{s})} \rightarrow \zeta = 0.559$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = (2.24\,\tfrac{rad}{s})\sqrt{1-(0.559)^2} = 1.857\ \frac{rad}{s}$$

$$x(t) = \frac{\hat{F}\,e^{-\zeta\omega_n(t-\tau)}}{m\,\omega_d}\sin\omega_d(t-\tau) \qquad \tau = 2\,s$$

$$= \frac{(0.1\,Ns)\,e^{-(0.558\cdot 2.24\,\frac{rad}{s}(t-\tau))}}{(0.1\,kg)(1.86\,rad/s)}\sin 1.86(t-\tau)$$

$$= 0.538\,e^{-1.25(t-2s)}\sin 1.86(t-2s)$$

$$T = \frac{2\pi}{\omega_d} = 3.38\ s$$

e

$x(0)=0$
$x(2)=0$
$x(3.7)=0$