ENG 122 LECTURE 15 FALL 2016 Nov 9, 2016

Square Wave Homework

$$\underset{\S}{x(t)} = \underbrace{e^{-\zeta\omega_n t} \underset{\S}{A} \sin(\omega_d t + \underset{\S}{\phi})} + \underbrace{\Sigma}_{i}$$

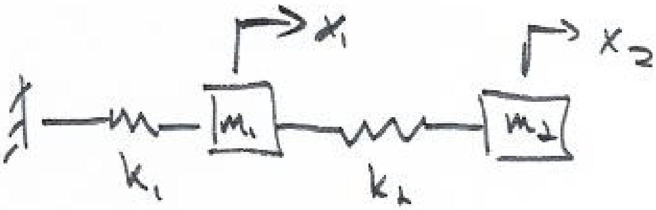
$A = \ldots$

$\phi = \ldots$

$\S$

$X(0) = X_0 = \boxed{}$

Solve for A, ϕ



EOM

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\substack{M \\ \text{mass matrix}}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\substack{K \\ \text{stiffness matrix}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Steps

① Multiplied both sides by $M^{-1} \Rightarrow \ddot{x} + K'x = 0$

$$K' = M^{-1}K$$

② assume a solution $\tilde{X} = \tilde{X}_0 \sin(\omega t)$

substitute into the EoM

$$(-\omega^2 I + K')\tilde{X}_0 \sin \omega t = 0$$

③ only possible solution $(-\omega^2 I + k')\tilde{X}_0 = 0$

ω^2 is the eigen value in the above equation

↑ scalar

$\tilde{X}_0$: eigenvector

$$K'\tilde{X}_0 = \omega^2 \tilde{X}_0$$

④ Solve for ω^2. Two DoF $\Rightarrow$ Two Eigenvalues

Two natural frequencies for the system.

$m = m_1 = m_2$, $k = k_1 = k_2$

$\omega_1 \neq \omega_2$

⑤ Solve for the corresponding eigenvectors: $\tilde{X}_{01}, \tilde{X}_{02}$

⑥ Construct a solution for the trajectories of $\tilde{X}$

$$\boxed{\tilde{X}(t) = \tilde{A} \sin(\underset{\xi}{\omega_1} t + \phi_1) + \tilde{B} \sin(\underset{\xi}{\omega_2} t + \phi_2)}$$

Matrix k' was $\underline{\underline{\text{not}}}$ symmetric!

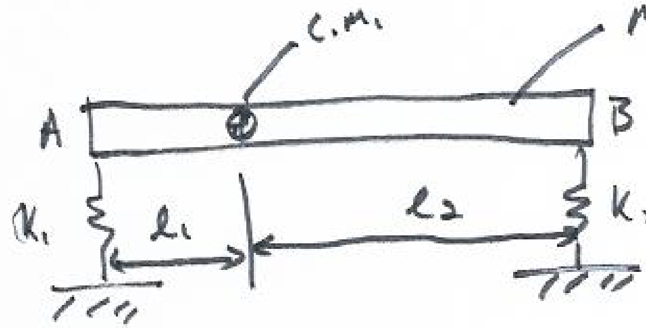
$$k' = \begin{bmatrix} \frac{k_1 + k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{bmatrix} \Rightarrow$$ "elastic" or "static" coupling

M : in general alway symmetric and also positive definite.

Solving for the ~~eq~~ eigenvalues is computationally inefficient!

We have the power to make k' symmetric with careful choices of coordinates.

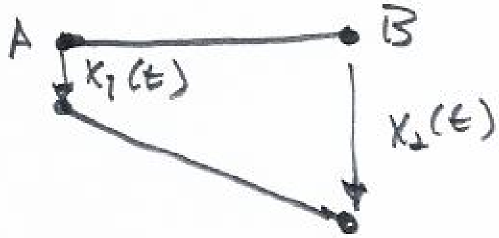
Example



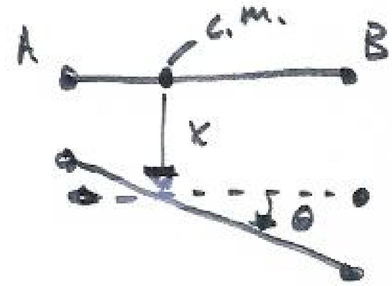
2 DoF

Possible of possible sets of coordinates.

i)

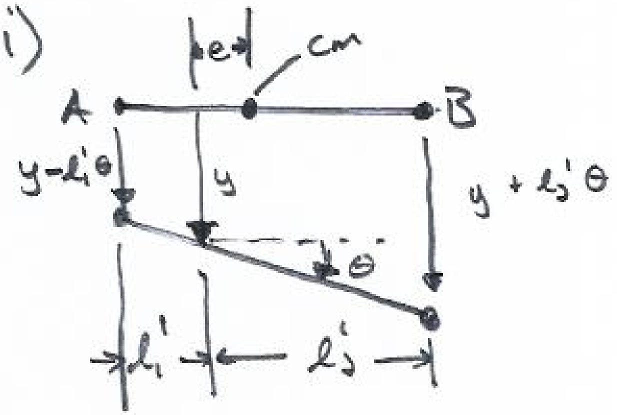


ii)



X and θ

iii)



infinite # of combinations of coordinates

Derive the EoM

ii)
$$m\ddot{x} = -k_1(x-l_1\theta) - k_2(x+l_2\theta)$$
$$J_o\ddot{\theta} = k_1(x-l_1\theta)l_1 - k_2(x+l_2\theta)l_2$$

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -(l_1k_1-l_2k_2) \\ -(k_1l_1-k_2l_2) & k_1l_1^2+k_2l_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

iii)

$$m\ddot{y} = -k_1(y - l_1'\theta) - k_2(y + l_2'\theta) - me\ddot{\theta}$$

$$J_p\ddot{\theta} = k_1(y - l_1'\theta)l_1' - k_2(y + l_2'\theta)l_2' - me\ddot{y}$$

$$J_p = J_0 + me^2$$

$$\begin{bmatrix} m & me \\ me & J_p \end{bmatrix}\begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & \cancelto{0}{k_2 l_2' - k_1 l_1'} \\ -\cancelto{0}{(k_1 l_1' - k_2 l_2')} & k_1 l_1'^2 + k_2 l_2'^2 \end{bmatrix}\begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we choose y such the $k_1 l_1' = k_2 l_2'$, the K matrix will become uncoupled.

The system is going to vibrate exactly the same way regardless of your choice of coordinates!

If you choose coordinates to ensure an uncoupled K', then you have chosen a special set of coordinates: "natural coordinates" or "principal coordinates"

$$\text{Generalized coordinates} \Rightarrow \underbrace{\text{principal coordinates}}_{\displaystyle\downarrow}$$

modal analysis

① Factorize M such that $M = L L^T$ where L is a lower triangular matrix $\left(L = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix} \right)$

This is Cholesky Factorization if M is positive definite and symmetric.

If M is diagonal $\left(\text{e.g. } M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right)$

$L = M^{1/2}, \quad L^T = M^{1/2}, \quad L^{-1} = M^{-1/2}$

eg. $M^{1/2} = \begin{bmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{bmatrix} \qquad L^{-1} = \begin{bmatrix} 1/\sqrt{m_1} & 0 \\ 0 & \frac{1}{\sqrt{m_2}} \end{bmatrix}$

② Transform coordinates:

$\tilde{q} = L^T \tilde{x} \qquad \tilde{x} = (L^T)^{-1} \tilde{q} \qquad L^{-1} M = L^T$

EOM

$M\left[(L^T)^{-1} \ddot{\tilde{q}} \right] + K\left((L^T)^{-1} \tilde{q} \right) = 0$

$\underbrace{L^{-1} M (L^T)^{-1}} \ddot{\tilde{q}} + \underbrace{\left(L^{-1} K (L^T)^{-1} \right)} q = 0$

$$\boxed{I \ddot{\tilde{q}} + \tilde{K} \tilde{q} = 0}$$

$\tilde{K} = L^{-1} K (L^T)^{-1}$: mass normalized stiffness matrix

akin $\frac{k}{m}$

$\tilde{K}$: if M and K are symmetric, $\tilde{K}$ is guaranteed to be symmetric!

(3) Solve eigenvalue problem

$q(t) = \tilde{V} \sin \omega t$, $\dot{q}(t) = \omega \tilde{V} \cos \omega t$, $\ddot{q}(t) = -\omega^2 \tilde{V} \sin \omega t$

Subs. to EOM:

$$-\omega^2 \tilde{V} \sin \omega t + \tilde{K} \tilde{V} \sin \omega t = 0$$

$$\underline{\tilde{K}\tilde{V} = \omega^2 \tilde{V}}$$

get: ω's and the $\tilde{V}$'s

(4) If $\tilde{V}$ is an eigenvector so is $\alpha \tilde{V}$ if ~~...~~ α is an arbitrary constant.

Normalize the $\tilde{V}$'s such that $\underline{|\tilde{V}| = 1}$.

Eigenvectors for a symmetrix $\tilde{K}$ are always orthogonal to each other. $(\tilde{V}_i^T \cdot V_j = 0, \ i \neq j)$

dot product

orthonormal vectors

$P = [\tilde{V}_1, \tilde{V}_2, \dots] \Rightarrow$ orthogonal matrix

(5) use matrix P to diagonalize $\tilde{k}$

since $\tilde{k}\,\tilde{v}_i = \omega_i^2\,\tilde{v}_i$

$$P^T \tilde{k} P = P^T \left[\omega_1^2 \tilde{v}_1, \omega_2^2 \tilde{v}_2, \ldots \right]$$

$$= \begin{bmatrix} \omega_1^2 & & & 0 \\ & \omega_2^2 & & \\ & & \omega_3^2 & \\ 0 & & & \ddots \end{bmatrix}$$

(6) Consider a second coordinate transform:

$\tilde{q} \to \tilde{r}$ $\qquad$ $\tilde{q} = P\,\tilde{r}$

$$\dot{\tilde{q}} = P\,\dot{\tilde{r}}$$

$$\ddot{\tilde{q}} = P\,\ddot{\tilde{r}}$$

<u>EOM</u> in $\tilde{r}$

$$P\ddot{\tilde{r}} + \tilde{k} P \tilde{r} = 0$$

Premultiply

$$\underbrace{P^T P}_{I}\,\ddot{\tilde{r}} + \underbrace{P^T \tilde{k} P}_{\Lambda}\,\tilde{r} = 0$$

$$I\,\ddot{\tilde{r}} + \Lambda\,\tilde{r} = 0$$

lambda matrix

This completely decouples the equations in this system.

Modal Equations of Motion!

modal coordinates: $\tilde{r}$

$$\ddot{r}_i + \omega_i^2 r_i = 0 \quad (i = 1, 2, \ldots n)$$

7 The initial conditions:

$$\tilde{r}(0) = P^T \tilde{q}(0) = P^T L^T \tilde{x}(0)$$

$$\dot{\tilde{r}}(0) = P^T \dot{\tilde{q}}(0) = P^T L^T \dot{\tilde{x}}(0)$$

Solution to system: eigen frequency

$$r_i(t) = \sqrt{r_{i0}^2 + \frac{\dot{r}_{i0}^2}{\omega_i^2}} \sin(\omega_i t + \phi_i)$$

$$\phi_i = \arctan\left(\frac{\omega_i r_{i0}}{\dot{r}_{i0}}\right)$$

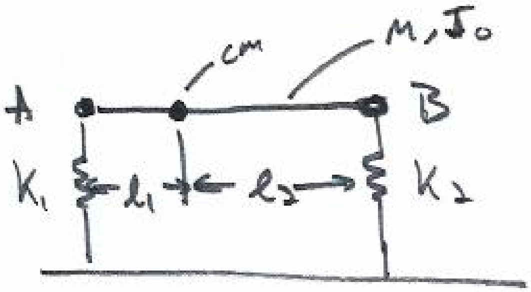
$$\tilde{x}(t) = (L^T)^{-1} \tilde{q}(t) = (L^T)^{-1} P \tilde{r}(t)$$

Matrix $S = (L^T)^{-1} P$. This matrix is the matrix of <u>mode shapes</u> in $\tilde{x}$.

Each column is mode shape

$$S = [S_1, S_2, S_3, \ldots]$$

$$\boxed{\tilde{x}(t) = \sum_{i=1}^{n} S_i r_i(t)}$$



$$\begin{bmatrix} M & 0 \\ 0 & J_0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -(\ell_1 k - \ell_2 k_2) \\ -(k_1 \ell_1 - k_2 \ell_2) & k_1 \ell_1^2 + k_2 \ell_2^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$k_1 = k_2 = k, \quad \ell_1 = \tfrac{1}{2} \ell_2 = \ell$$

① Cholesky factorization of M

$$M = LL^T \qquad L = M^{1/2} = \begin{bmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{J_0} \end{bmatrix} = L^T$$

② coordinate transform $x \to q$

$$M (L^T)^{-1} \ddot{q} + K (L^T)^{-1} q = 0$$

$$\ddot{q} + \hat{k} q = 0 \qquad \hat{k} = L^{-1} K (L^T)^{-1}$$

$$(L^{T^{-1}}) = L^{-1} = \begin{bmatrix} \frac{1}{\sqrt{m}} & 0 \\ 0 & \frac{1}{\sqrt{J_0}} \end{bmatrix} \qquad k = \begin{bmatrix} 2k & -\ell k \\ -\ell k & 5k\ell^2 \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} \frac{1}{\sqrt{m}} & 0 \\ 0 & \frac{1}{\sqrt{J_0}} \end{bmatrix} \begin{bmatrix} 2k & -\ell k \\ -\ell k & 5k\ell^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{m}} & 0 \\ 0 & \frac{1}{\sqrt{J_0}} \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} \frac{2k}{m} & -\frac{\ell k}{\sqrt{m J_0}} \\ -\frac{\ell k}{\sqrt{m J_0}} & \frac{5\ell^2 k}{J_0} \end{bmatrix}$$ symmetric!

③ Solve for eigenvalues

$$\tilde{K} v = \omega^2 V$$

$$\det\left(I\omega^2 - \tilde{K}\right) = 0 = \begin{vmatrix} \omega^2 - \frac{2k}{m} & \frac{lk}{\sqrt{mJ_0}} \\ \frac{lk}{\sqrt{mJ_0}} & \omega^2 - \frac{5l^2k}{J_0} \end{vmatrix}$$

$$\underbrace{(\omega^2)^2 - \left(\frac{2k}{m} + \frac{5l^2K}{J_0}\right)\omega^2 + \frac{10l^2k^2}{mJ_0} - \frac{l^2k^2}{mJ_0} = 0}_{\text{characteristic equation}}$$

let $m = k = l = J_0 = 1$

$$\omega^4 - 7\omega^2 + 9 = 0 \Rightarrow \omega^2 = \frac{7 \pm \sqrt{15}}{2}$$

$$\begin{bmatrix} \omega_1 \approx 2.3 \text{ rad/s} \\ \omega_2 \approx 1.3 \text{ rad/s} \end{bmatrix}$$

eigen frequencies

modal frequencies

solve for eigenvectors

$$\begin{bmatrix} \omega_1^2 - 2 & 1 \\ 1 & \omega_1^2 - 5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} -1 \\ \frac{3+\sqrt{13}}{2} \end{bmatrix} \approx \begin{bmatrix} -1 \\ 3.3027 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 1 \\ 0.3027 \end{bmatrix}$$

eigenvectors

④ Normalize the eigen vector (convert to unit vector i.e. divide by mag)

$$V_1 = \begin{bmatrix} -0.2897 \\ 0.9571 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 0.9571 \\ 0.2897 \end{bmatrix}$$

$$|V_1| = |V_2| = 1$$

$$P = [V_1, V_2] = \begin{bmatrix} -0.2897 & 0.9571 \\ 0.9571 & 0.2897 \end{bmatrix}$$

⑤ compute $\Lambda = P^T \tilde{K} P$

$$\Lambda = \begin{bmatrix} -0.2897 & 0.9571 \\ 0.9571 & 0.2897 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -0.2897 & 0.9571 \\ 0.9571 & 0.2897 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 5.3026 & 0 \\ 0 & 1.6972 \end{bmatrix} = \begin{bmatrix} \frac{7+\sqrt{13}}{2} & 0 \\ 0 & \frac{7-\sqrt{13}}{2} \end{bmatrix}$$

⑥ decoupled EOM in modal coordinates r:

$$\ddot{r} + \begin{bmatrix} \frac{7+\sqrt{13}}{2} & 0 \\ 0 & \frac{7-\sqrt{13}}{2} \end{bmatrix} r = 0$$

$$X = (L^T)^{-1} P\, r = S\, r$$

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underbrace{\begin{matrix} -0.2897 \\ 0.9571 \end{matrix}}_{S_1} & \underbrace{\begin{matrix} 0.9571 \\ 0.2897 \end{matrix}}_{S_2} \end{bmatrix} r$$

$$X(t) = \sum_{i=1}^{n} S_i + r_i(t)$$

$$= S_1 r_1(t) + S_2 r_2(t)$$

$$= \begin{bmatrix} -0.2897 \\ 0.9571 \end{bmatrix} \sqrt{r_1(0)^2 + \frac{\dot{r}_1(0)^2}{\omega_1^2}} \sin(\omega_1 t + \phi_1)$$

$$+ \begin{bmatrix} 0.9571 \\ 0.2897 \end{bmatrix} \sqrt{r_2(0)^2 + \frac{\dot{r}_2(0)^2}{\omega_2^2}} \sin(\omega_2 t + \phi_2)$$

$$\phi_1 = \arctan\left(\frac{\omega_1 r_1(0)}{\dot{r}_1(0)}\right) \qquad \phi_2 = \arctan\left(\frac{\omega_2 r_2(0)}{\dot{r}_2(0)}\right)$$

$$r(0) = (P^T L^T) X_0 = S^{-1} X_0$$

$$\dot{r}(0) = (P^T L^T) \dot{X}_0 = S^{-1} \dot{X}_0$$

If $\underline{X_0 = S_1}$ and $\dot{X}(0) = 0$

$$X(t) = S_1 r_1(t) + S_2 r_2(t) = S_1 r_1(0) \cos\omega_1 t + S_2 r_2(0) \cos\omega_2 t$$

$$r(0) = S^{-1} S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{matrix} r_1(0) = 1 \\ r_2(0) = 0 \end{matrix}$$

$$X(t) = S_1 \cdot 1 \cdot \cos\omega_1 t + \cancelto{0}{S_2 \cdot 0 \cdot \cos\omega_2 t}$$

$$= S_1 \cos\omega_1 t$$

~~The~~ If the system is initially displaced by an amount that equals one of the eigenvectors (mode shapes), it will vibrate <u>only</u> at the frequency and modeshape!

$$r = S^{-1}x = \begin{bmatrix} -0.8021 & 2.6998 \\ 0.8021 & 2.8021 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix}$$

$$r_1 = -0.8021x + 0.26498\theta$$

$$r_2 = 0.8021x + 0.8021\theta$$