Transform Methods

Express linear differential equations as algebraic relationships.

Laplace Tranform $\Rightarrow$ maps time domain to complex domain

complex number

$$L[f(t)] = F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

$f(t)$ has integrable and $f(t)=0$ for $t<0$

$$L^{-1}[F(s)] = f(t)$$

Ex

$$L[\dot{f}(t)] = \int_0^\infty \dot{f}(t) e^{-st}\, dt = \int_0^\infty e^{-st} \frac{d[f(t)]}{\cancel{dt}} \cancel{dt}$$

$$L[\dot{f}(t)] = \int_0^\infty e^{-st}\, d f(t)$$

by parts

$$L[\dot{f}(t)] = e^{-st} f(t) \Big|_0^\infty + \underbrace{s \int_0^\infty e^{-st} f(t)\, dt}$$

$$\boxed{L[\dot{f}(t)] = -f(0) - sF(s)}$$

$$\boxed{L[\ddot{f}(t)] = s^2 F(s) - s f(0) - \dot{f}(0)}$$

Ex Unit step

$$L[\Phi(t)] = \int_0^\infty e^{-st} \cdot 1 \, dt = -\frac{e^{-st}}{s}\Big|_0^\infty$$

$$= -\cancelto{0}{\frac{e^{-\infty}}{s}} + \frac{e^{-0}}{s} = \frac{1}{s}$$

EoM

$x_0 = \dot{x}_0 = 0$

$$m\ddot{x} + kx = \Phi(t)$$

$$ms^2 X(s) + kX(s) = \frac{1}{s}$$ algebraic equation

$$X(s) = \frac{1}{s(ms^2+k)} = \frac{1/m}{s(s^2+\omega_n^2)}$$

Look at Table 3.1 ⇒ entry 7

inverse Laplace Transform

$$x(t) = \frac{1/m}{\omega_n^2}(1 + \cos \omega_n t) = \frac{1}{k}(1 - \cos\omega_n t)$$

$m\ddot{x} + cx + kx = \delta(t)$ impulse response

$$(ms^2 + cs + k)\,X(s) = 1$$

$$X(s) = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

inverse $L^{-1}[\;]$

$x_0 = \dot{x}_0 = 0$

if $\zeta < 1$ underdamped case

Table 3.1 entry 8

$$x(t) = \frac{1/m}{\omega_n\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t\right)$$

$$x(t) = \frac{1}{m\omega_d}\, e^{-\zeta\omega_n t} \sin\omega_d t \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

<u>Ex</u> $\ddot{x}(t) + 2\dot{x}(t) + 2x(t) = \delta(t-\pi)$ $\quad x_0 = \dot{x}_0 = 0$

$\omega_n = \sqrt{2}\ \frac{\text{rad}}{\text{s}} \qquad \zeta = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \omega_d = \sqrt{2}\sqrt{1-(1/\sqrt{2})^2}$

$\omega_d = 1$ rad/s

Transform the EoM

$$(s^2 + 2s + 2)\,X(s) = e^{-\pi s}$$

$$X(s) = \frac{e^{-\pi s}}{s^2+2s+2} = e^{-\pi s} \cdot \frac{1}{s^2+2s+2}$$

$\omega_d = 1$ rad/s
$\omega_n = \sqrt{2}$ rad/s
$\zeta = \frac{1}{\sqrt{3}}$

$$L^{-1}\left[\frac{1}{s^2+2s+2}\right] = e^{-t}\sin t$$

$$\boxed{x(t) = e^{-(t-\pi)}\sin(t-\pi)\,\Phi(t-\pi)}$$

entry 12 in table 3.1

Fourier Transform

maps time domain to the frequency domain

recall solution to arbitrary forcing

$$x(t) = \int_0^t F(\tau)\,h(t-\tau)\,dt$$

Laplace Transform of the convolution integral.

$x_0 = \dot{x}_0 = 0$

$$X(s) = F(s)\,\underbrace{H(s)}_{\text{laplace transform of impulse response function}}$$

response

driving force

$$\frac{X(s)}{F(s)} = H(s)$$

transfer function

$$\boxed{H(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

If you examine the Fourier series of non-periodic function, you derive the Fourier Transform:

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$\uparrow$ frequency

$$\mathcal{L}[x(t)] = \int_0^{\infty} f(t)\, e^{-st}$$

$$s = \underline{\underline{j}}\omega \qquad j = \sqrt{-1}$$

$$H(\omega) = \frac{1}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{1}{-\omega^2 + 2\zeta j\omega\omega_n + \omega_n^2}$$

Response to Random Inputs

random: non-deterministic

$X(t)$: all possible trajectories

Stationary signal: statistical properties of the signal (e.g. mean, std) do not change with time

Average of signal $x(t)$

$$\bar{X} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\, dt$$

Mean square (variance)

magnitude of signal

$$\bar{X}^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t)\, dt$$

root mean square

$$X_{rms} = \sqrt{\bar{X}^2}$$

Auto correlation Function

: how fast a signal is changing

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\, x(t+\tau)\, d\tau$$

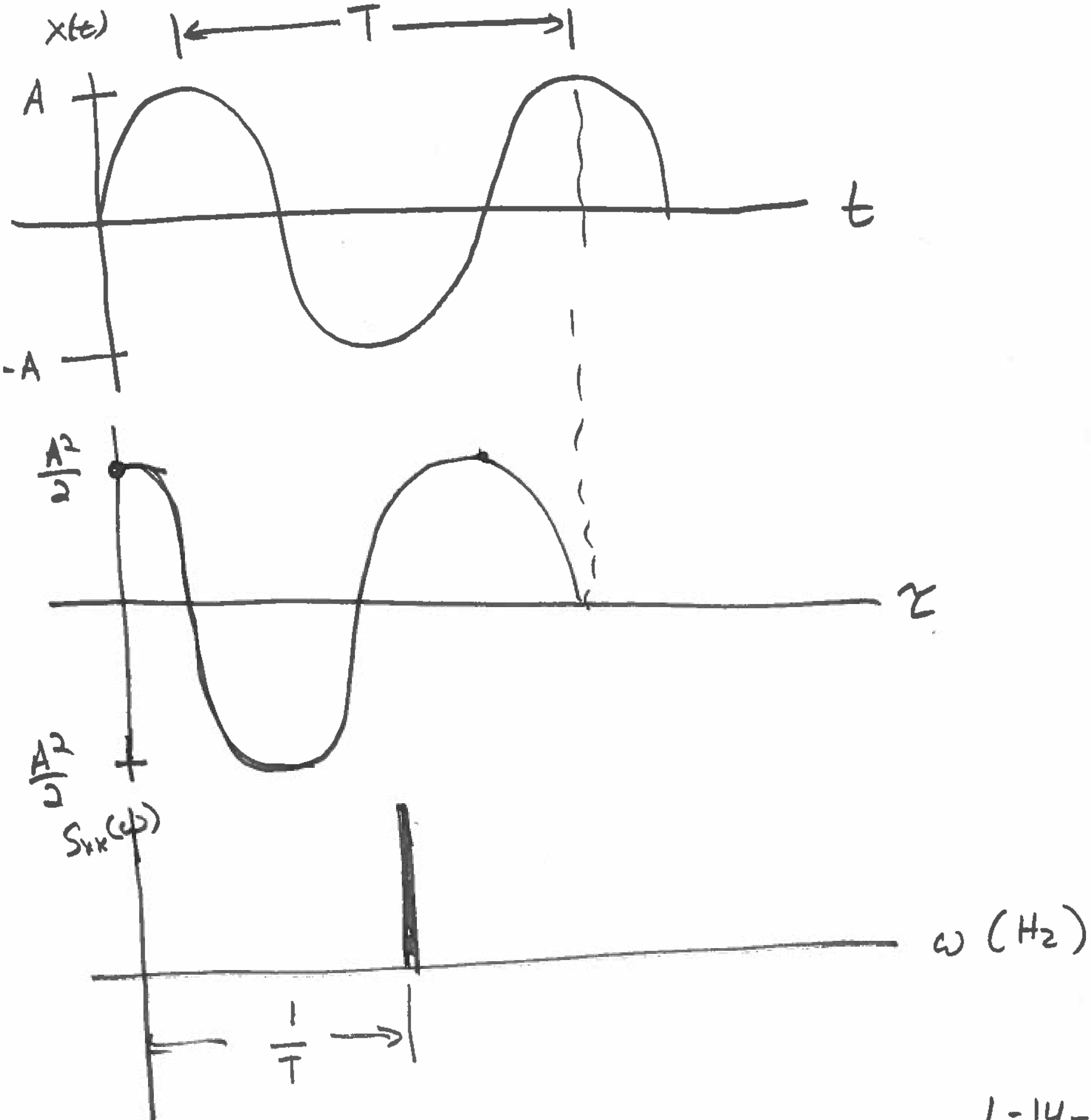
τ: time difference between samples of $x(t)$

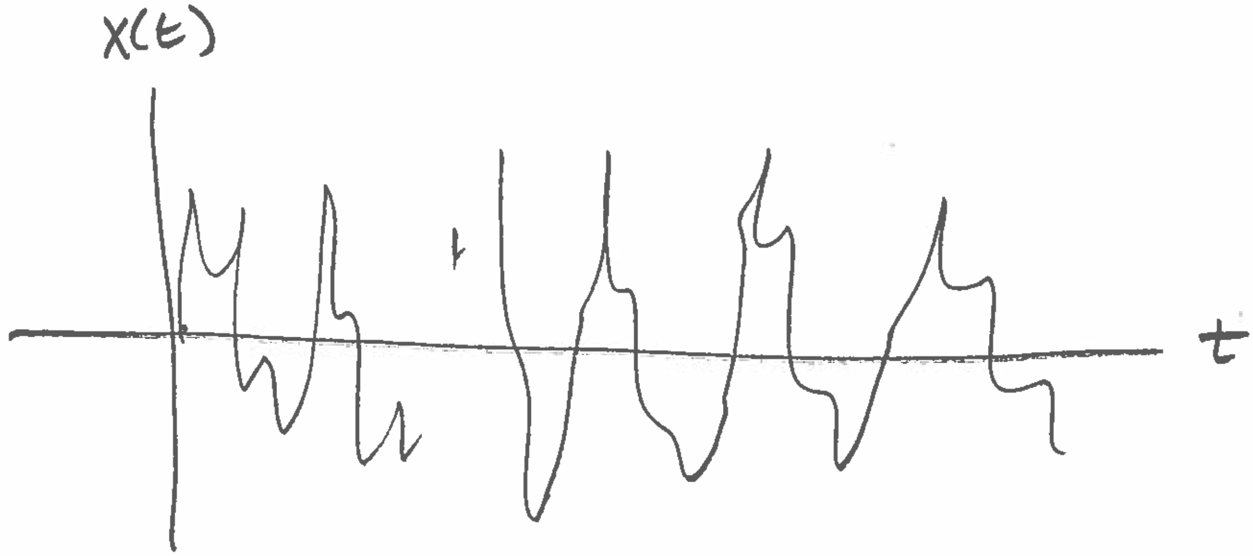
Power Spectral Density

Fourier Transform of $R_{xx}(\tau)$

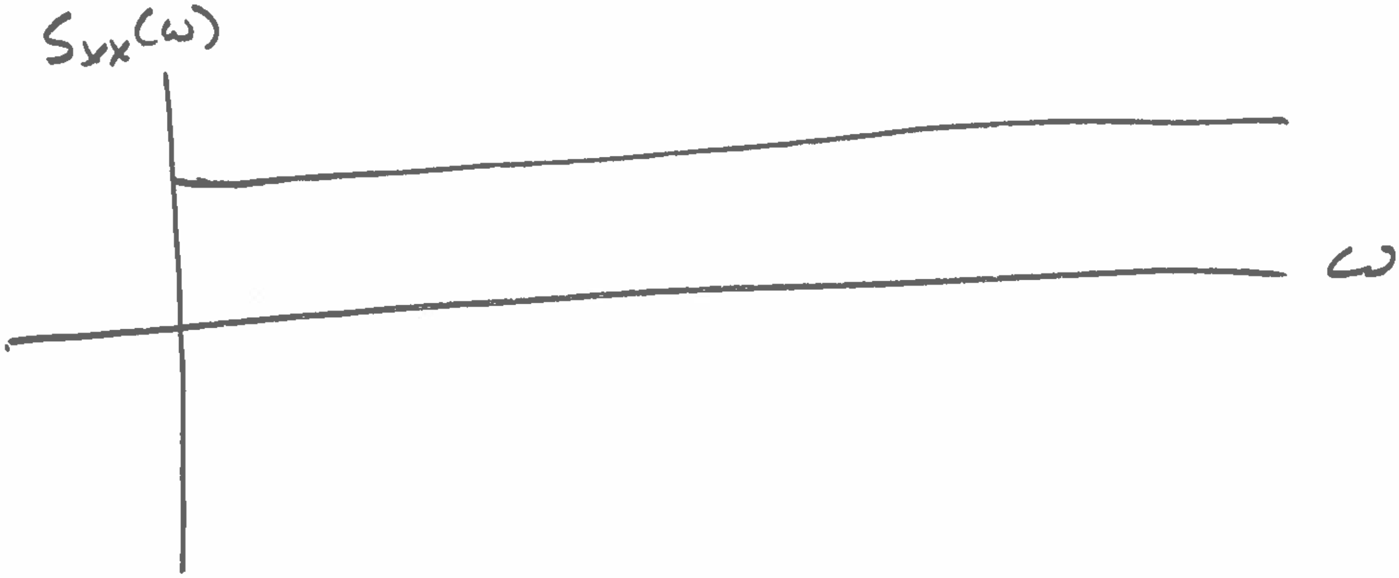
$\tau \rightarrow \omega$

$$S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\omega)\, e^{-j\omega\tau}\, d\tau$$




X(t)
t


A^2_{rms}
τ


$S_{xx}(\omega)$
ω

$$S_{xx}(\omega) = |\underbrace{H(\omega)}_{\text{System dynamics}}|^2 \, S_{ff}(\omega)$$

response PSD

input PSD

$$\left.\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right|_{s=j\omega}$$