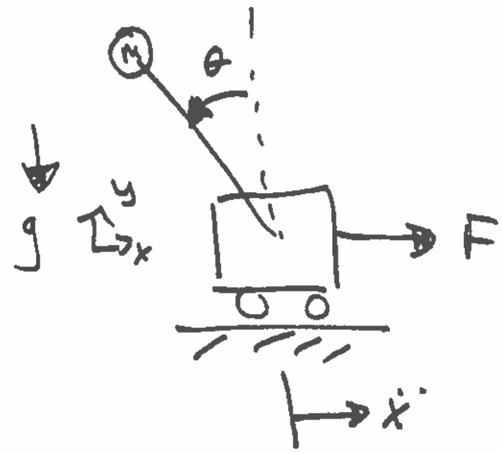
First order form of ODEs (state space)

1) $(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$

2) $\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$



2 DoF

Step 1: Create new state variables

4 first order

$$\left.\begin{aligned}\dot{x} &= v\\ \dot{\theta} &= \omega\end{aligned}\right\} \text{defined}$$

$$\left.\begin{aligned}(m+M)\underline{\dot{v}} - m\ell\underline{\dot{\omega}}\cos\theta + m\ell\omega^2\sin\theta &= F\\ \ell\underline{\dot{\omega}} - g\sin\theta &= \underline{\dot{v}}\cos\theta\end{aligned}\right\} \text{linear in accelerations}$$

$$\begin{bmatrix} M+m & -m\ell\cos\theta \\ -\cos\theta & \ell \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} F - m\ell\omega^2\sin\theta \\ g\sin\theta \end{bmatrix}$$

$$A \quad x = b$$

In general, solve with Gaussian Elimination.

For 2x2: Cramer's Rule

$$|A| = (M+m)\ell - m\ell\cos^2\theta$$

$$\dot{v} = \frac{\begin{vmatrix} b_1 & -m\ell\cos\theta \\ b_2 & \ell \end{vmatrix}}{|A|} = \frac{F\ell - m\ell^2\omega^2\sin\theta + m\ell g\cos\theta\sin\theta}{|A|}$$

$$\dot{\omega} = \frac{\begin{vmatrix} M+m & F - m\ell\omega^2\sin\theta \\ -\cos\theta & g\sin\theta \end{vmatrix}}{|A|} = \frac{(M+m)g\sin\theta - \left[-F\cos\theta + m\ell\omega^2\sin\theta\cos\theta\right]}{|A|}$$

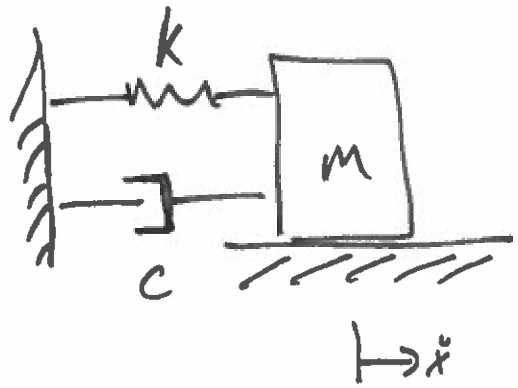
$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v \\ \omega \\ \\ \\ \end{bmatrix}$$

↑ derivatives of the states

states: $\begin{bmatrix} x \\ \theta \\ v \\ \omega \end{bmatrix}$

4 states for 2DoF

Lagrange's Methods For Systems with non-conservative forces



$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} K x^2$$

$$L = T - U = \frac{1}{2}(m\dot{x}^2 - kx^2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

Q_i ← generalized forces (anything that is non-conservative)

$$Q_i = \frac{\delta W}{\delta V}$$

δW ← virtual work

δV ← virtual displacement

Rayleigh's Dissapation Function

$$R = \frac{1}{2} \sum_{i=1}^{N} c_i \dot{q}_i^2$$

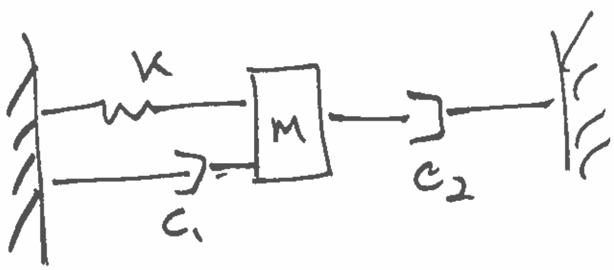
linear damping (non-conservative force)

$$Q_i = -\frac{\partial R_i}{\partial \dot{q}_i}$$

$$R = \frac{1}{2} c \dot{x}^2$$

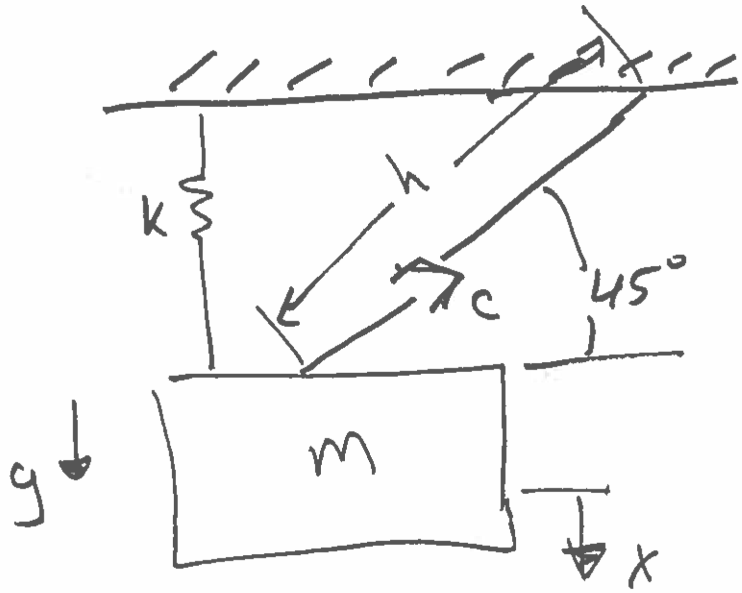
$$Q_i = -\frac{\partial R}{\partial \dot{x}} = -c\dot{x}$$

$$\boxed{m\ddot{x} + kx + c\dot{x} = 0}$$



$$R = \frac{1}{2}\left(c_1 \dot{x}^2 + c_2 \dot{x}^2\right)$$

$$-\frac{\partial R}{\partial \dot{x}} = -(c_1 + c_2)\dot{x}$$



$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k \dot{x}^2 - mg\,x$$

$$R = \frac{1}{2} c \left(\frac{2}{\sqrt{2}} \dot{x} \right)^2$$

$$\dot{h} = \frac{\dot{x}}{\sin(45^\circ)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = - \frac{\partial R}{\partial \dot{x}}$$

$$\boxed{m \ddot{x} + kx + c \frac{2}{\sqrt{2}} \dot{x} - mg\,x = 0}$$

Two Impulses

$$\ddot{x}(t) + 2\dot{x}(t) + 4x(t) = \underbrace{\delta(t) - \delta(t-4)}_{\text{two unit impulses}}$$

$X_0 = 1$ mm

$V_0 = 1$ mm

$\omega_n = 2$ rad/s

$\zeta = 0.5 \Rightarrow$ underdamped system

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 2\sqrt{1-(\tfrac{1}{2})^2} = \sqrt{3}$$

From $0 \leq t \leq 4$ only first impulse has occurred

$$X_{p1}(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t = \frac{1}{\sqrt{3}} e^{-t} \sin\sqrt{3}t$$

We have to include homogenous solution:

$$X_h(t) = e^{-t}(A\sin\omega_d t + B\cos\omega_d t)$$

$$\dot{X}_h(t) = -e^{-t}(A\sin\sqrt{3}t + B\cos\sqrt{3}t) + e^{-t}(\sqrt{3}A\cos\sqrt{3}t - \sqrt{3}B\sin\sqrt{3}t)$$

with $X_0 = 1$ mm, $V_0 = -1$ mm/s

$A = 0$, $B = 1$ $\leftarrow$ from solving above equations

$$x_h(t) = e^{-t}\cos\sqrt{3}\,t \qquad 0 \le t \le 4$$

add $x_{p_1} + x_h$

$$x(t) = e^{-t}\left(\cos\sqrt{3}\,t + \frac{1}{\sqrt{3}}\sin\sqrt{3t}\right) \quad 0 \le t \le 4$$

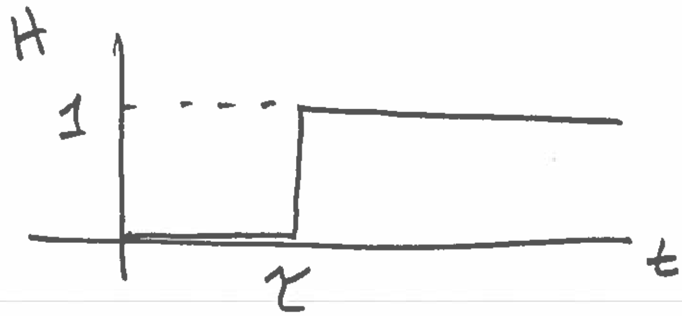
$t > 4 \qquad \tau = 4\text{ s}$

$$x_{p_2}(t) = \frac{\hat{F}}{m\omega_d}\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)$$

$$x_{p_2}(t) = -\frac{1}{\sqrt{3}}\, e^{-t+4} \sin\sqrt{3}\,(t-4)$$

Heaviside Step Function

$$H(t-\tau) = \Phi(t-\tau) = \begin{cases} 0 & t < \tau \\ 1 & t \geq \tau \end{cases}$$

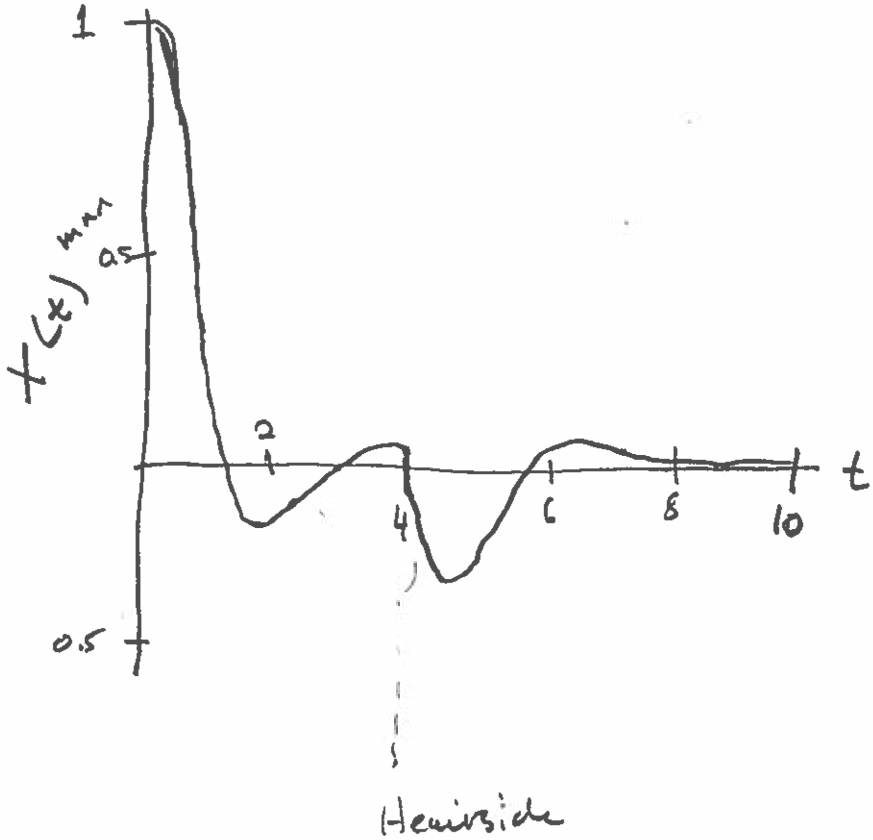


function that "turns on" at τ

Using superposition to add the particular solutions and use Heaviside function for compact notation.

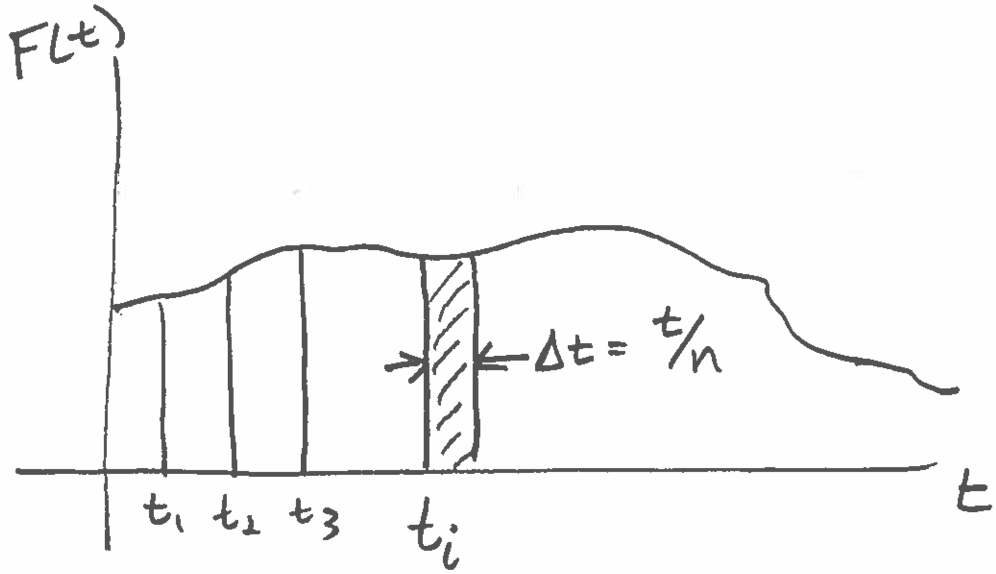
$$\boxed{X(t) = e^{-t}\left(\cos\sqrt{3}\,t + \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t\right) - \left[\frac{e^{-(t-4)}}{\sqrt{3}}\sin\sqrt{3}\,(t-4)\right]\underline{H}(t-4)}$$

total Solution


1
0.5
x(t) mm
0.5
2
4
6
8
10
t
Heaviside

Response to Arbitrary Inputs

Arbitrary inputs can be thought of as a summation of impulses.



n: # of impulses

$i = 1, \dots, n$

Response due to an impulse on t_i to t_{i+1}

$$\Delta X(t_i) = F(t_i)\, h(t - t_i)\, \Delta t$$

Total response (use superposition)

$$X(t_n) = \sum_{i=1}^{n} F(t_i) \cdot h(t - t_i) \Delta t$$

As $\Delta t \to 0$ $(n \to \infty)$

$$X(t) = \underbrace{\int_0^t F(\tau)\, h(t - \tau)\, d\tau}_{\text{convolution integral}}$$

convolution integral: integral of a product to two functions, one which is shifted by the variable of integration

For $X_0, V_0 = 0$ and 1 DoF system:

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right] d\tau$$

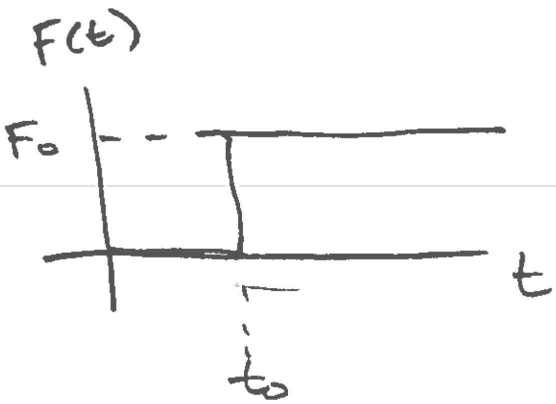
Property of convolution integral:

$$\int_0^t F(\tau) h(t-\tau) d\tau = \int_0^t F(t-\tau) h(\tau) d\tau$$

So

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(t-\tau) e^{-\zeta\omega_n \tau} \sin\omega_d \tau \, d\tau$$

<u>Response to Step Function</u>



$$F(t) = \begin{cases} 0 & t_0 > t > 0 \\ F_0 & t \geq t_0 \end{cases}$$

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t}\left[\int_0^{t_0} (0)\, e^{\zeta\omega_n t} \sin\omega_d(t-\tau)\,d\tau + \int_{t_0}^{t} F_0\, e^{\zeta\omega_n t} \sin\omega_d(t-\tau)\,d\tau\right]$$

$$= \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t}\underbrace{\int_{t_0}^{t} e^{\zeta\omega_n t} \sin\omega_d(t-\tau)\,d\tau}$$

Table of Integrals

$$x(t) = \frac{F_0}{K} - \frac{F_0}{K\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n(t-t_0)} \cos\left[\omega_d(t-t_0)-\theta\right]$$

$$\theta = \arctan\frac{\zeta}{\sqrt{1-\zeta^2}}$$

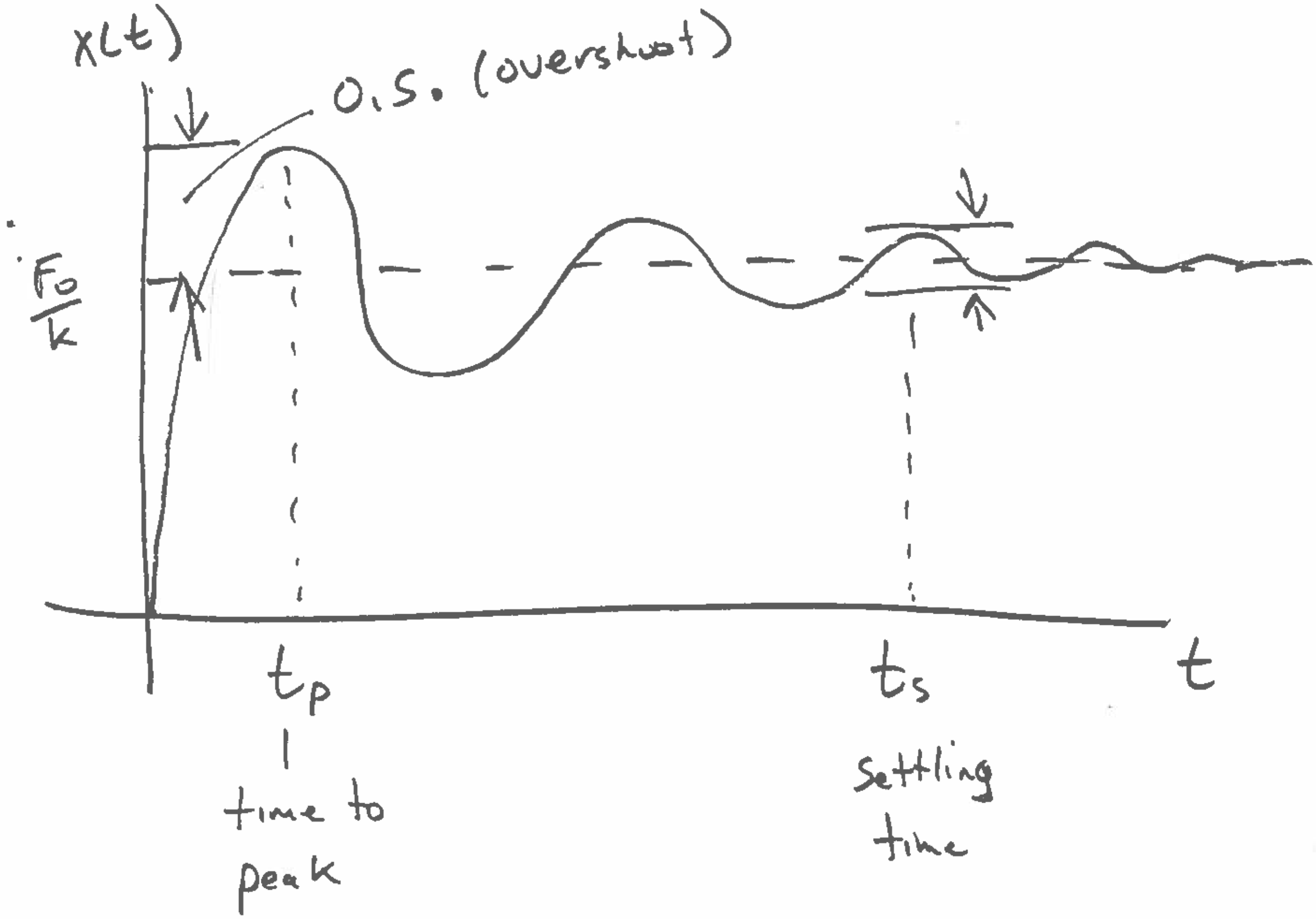
as $t \Rightarrow \infty$ $\quad x_{ss}(t) \Rightarrow \frac{F_0}{K}$

if $t_0 = 0$

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \cos(\omega_d t - \theta)$$

$\zeta = 0$

$$x(t) = \frac{F_0}{K}(1 - \cos\omega_n t)$$



$$t_0 = 0 \Rightarrow t_p = \pi/\omega_d, \quad t_s = \frac{3.5}{\zeta\omega_n} \text{ for } 3\% \text{ of } F_0/k$$