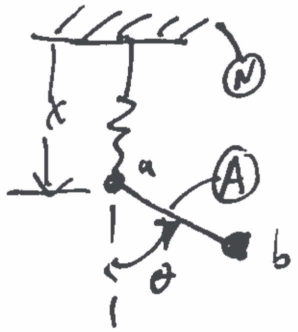
ENG122 Lecture #4

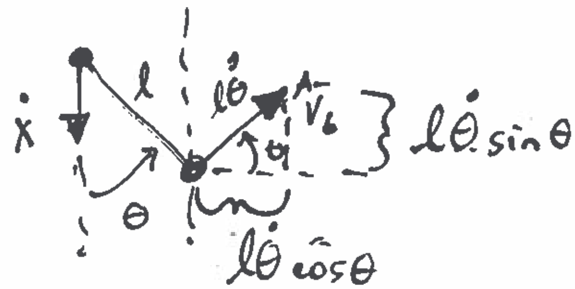
Monday, October 3, 2016

Corrected Velocity From Bouncy Compound Pendulum



$$ {}^{N}\bar{V}_b = {}^{N}\bar{V}_a + \underbrace{{}^{N}\omega^{A} \times \bar{r}^{b/a}}_{{}^{A}\bar{V}_b} $$

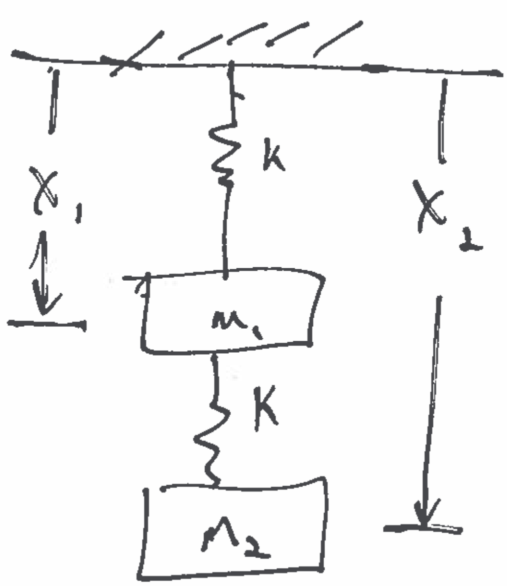
$$ |{}^{A}\bar{V}_b| = l\dot{\theta} $$



$$ |{}^{N}\bar{V}_b| = \sqrt{(\dot{x} - l\dot{\theta}\sin\theta)^2 + (l\dot{\theta}\cos\theta)^2} $$

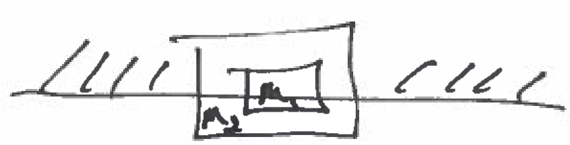
$$ T = \frac{1}{2} m \left(|{}^{N}\bar{V}_b| \right)^2 + \ldots $$

Potential Energy



$$ U = \frac{1}{2} k (x_1)^2 + \frac{1}{2} K (x_2 - x_1)^2 $$

$$ - m_1 g x_1 - M_2 g x_2 $$



$x_1 = 0, \; x_2 = 0$

<u>Viscous Damping</u>

$$m\ddot{x} + c\dot{x} + kx = 0$$

or

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$\zeta = \frac{c}{2\sqrt{km}}$

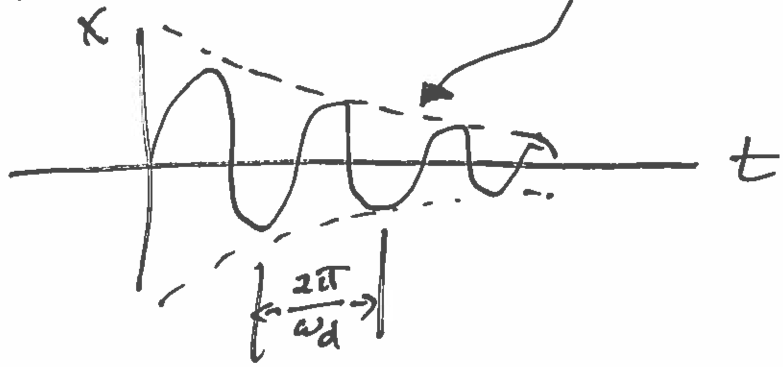
$\omega_d = \omega_n\sqrt{1-\zeta^2}$

3 solutions!!

Underdamped $(0 < \zeta < 1)$ (complex roots)

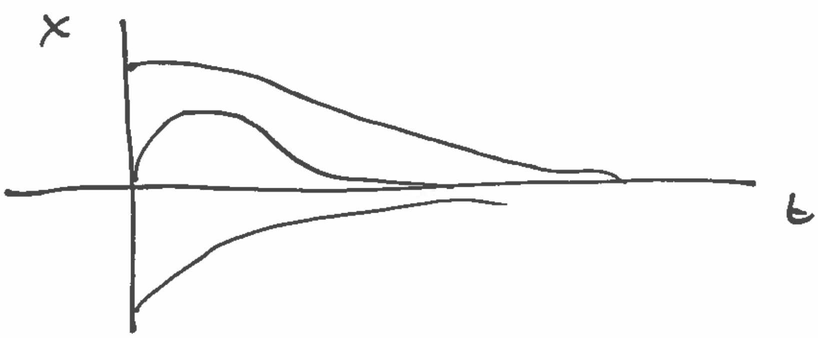
$$x(t) = A\underline{e^{-\zeta\omega_n t}}\sin(\omega_d t + \phi)$$

Damped oscillation



Over-damped $(\zeta > 1)$ pair of distinct real roots

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_n\sqrt{\zeta^2-1}\,t} + a_2 e^{\omega_n\sqrt{\zeta^2-1}\,t}\right)$$



no oscillation

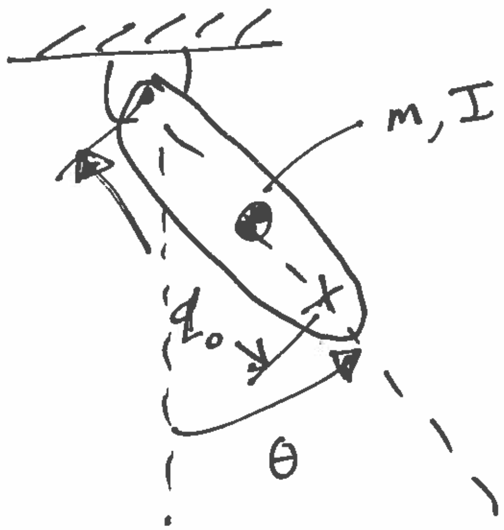
Critically damped $(\zeta = 1)$ pair of repeated roots

$$x(t) = (a_1 + a_2 t)\,e^{-\omega_n t}$$

no oscillation

fastest return to zero without oscillation

Compound Pendulum



There exists a simple pendulum that has the same period of oscillation as the compound pendulum. The length of this simple pendulum is refered as the <u>center of percussion</u>.

compound $\omega_n^2 = \dfrac{mgl}{I}$

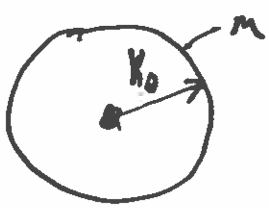
simple $\omega_n^2 = \dfrac{g}{q_o}$

$$q_o = \frac{I}{ml} \qquad \text{C.o.P}$$

↳ length

Radius of gyration

radius of ring that has same moment of inertia as the object in question.



$$mk_o^2 = I \Rightarrow K_o = \sqrt{\frac{I}{m}}$$

↑ length

$$q_o l = k_o^2 \qquad q_o = \frac{\cancel{m}k_o^2}{\cancel{m}l}$$

↑ length to CoM of compound pendulum

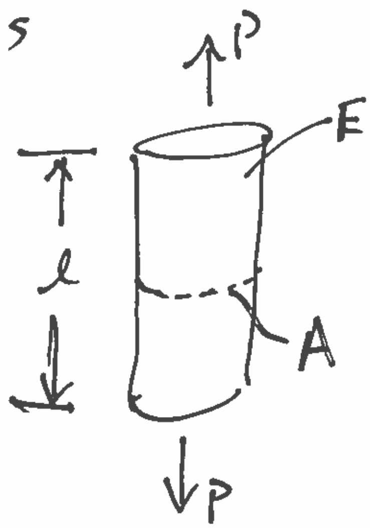
Stiffness

Function describing how much load is required to produce a unit of deflection in a mechanical structural element.

Stiffness is a function of both the geometry and material properties.

For examples

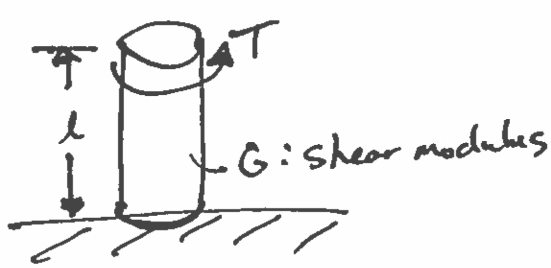
slender elastic rod



modulus of elasticity

$$k = \frac{EA}{l}$$

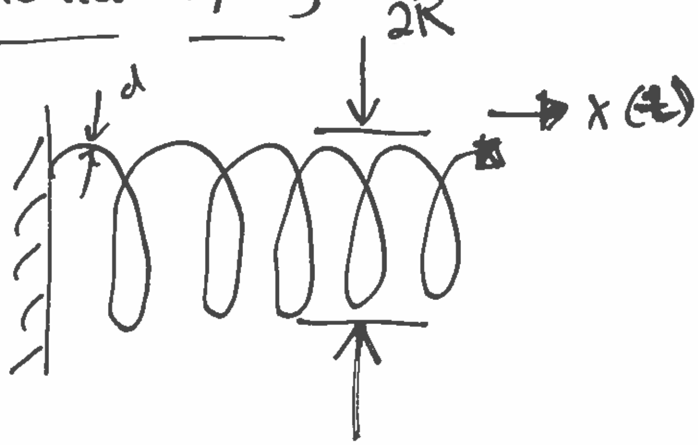
slender torsion bar



$$k = \frac{GJ}{l}$$

J: second polar moment of area of the cross section

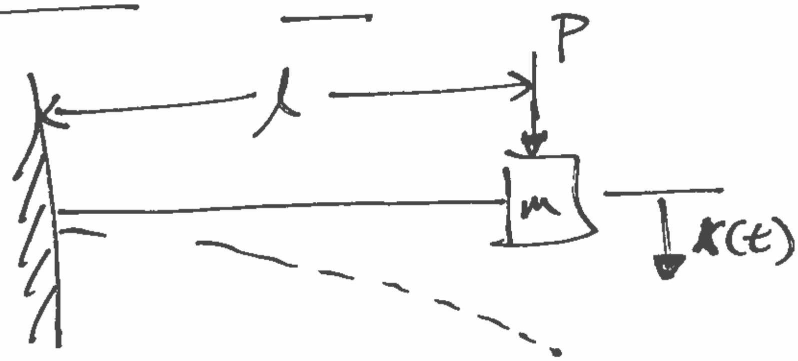
helical spring



$$k = \frac{G d^4}{64 n R^3}$$

n: number of coils

massless beam

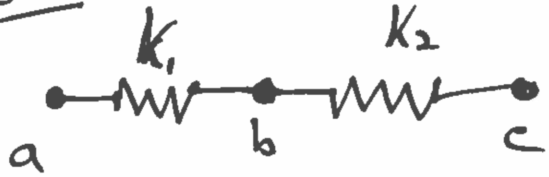


$$k = \frac{3EI}{l^3}$$

I: second moment of area of the beam's cross section

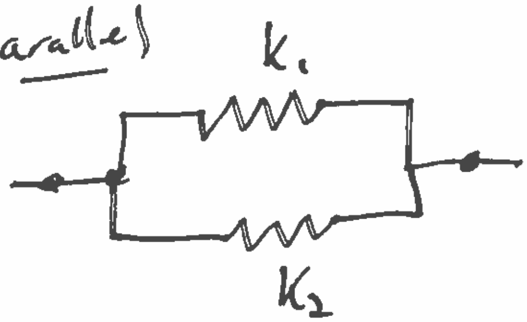
Combinations of springs (stiffnesses)

series



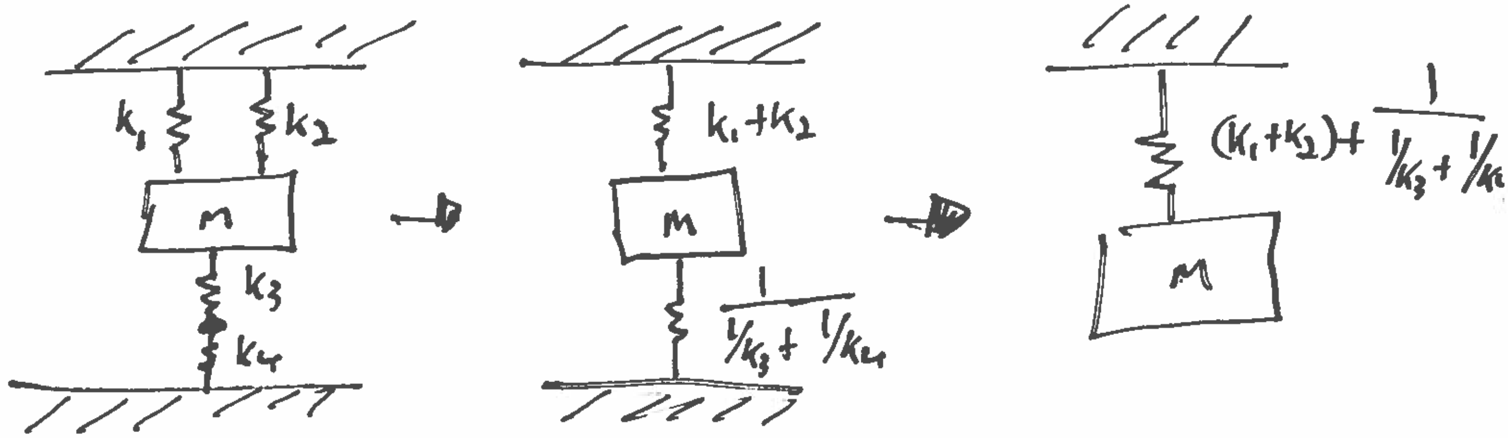
$$K_{total} = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2}$$

parallel



$$K_{total} = k_1 + k_2$$

Example



Stability

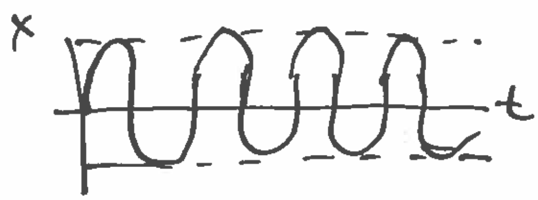
Describes behavior of a system when $t \to \infty$. All systems so far have been stable. A system can be classified as stable or unstable.

Stable system

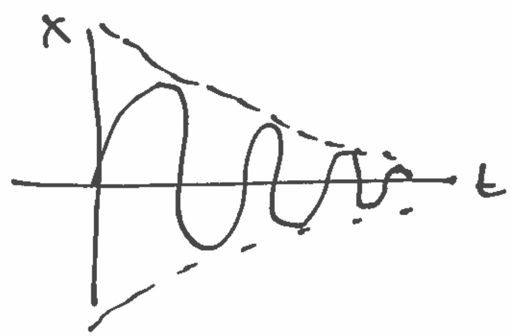
the behavior is such that as $t \to \infty$ it is always bounded, $\| X_{t \to \infty} \| \leq X_{max}$.

For unstable system the behavior is not bounded.

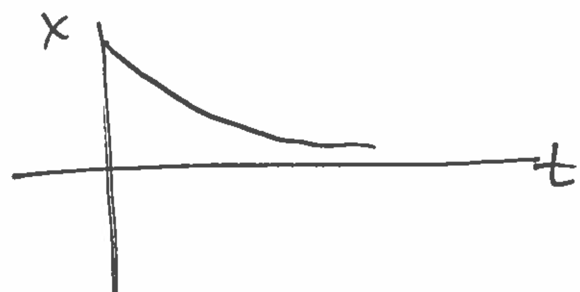
Stable



marginally stable

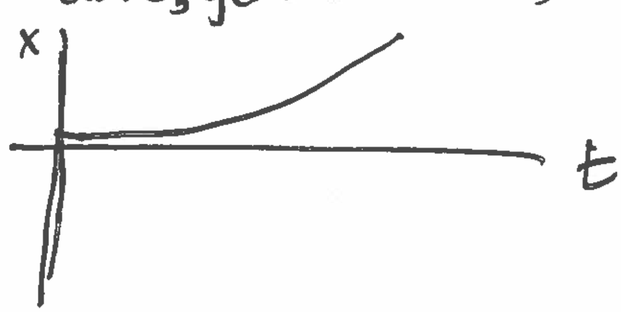


asymptotic stability

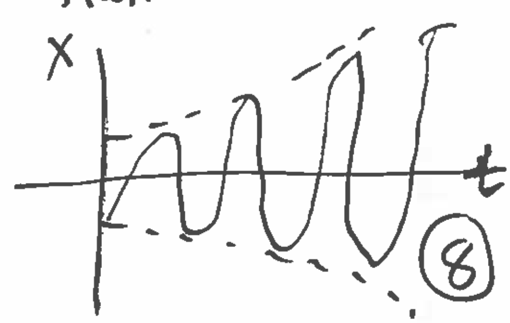


Unstable

divergent instability



flutter instability



Stability is a function of the system parameters : m, c, k

Examples

$$m\ddot{x} + kx = 0 \qquad (+\ k, m) \qquad x(t) = A\sin(\omega t + \phi)$$

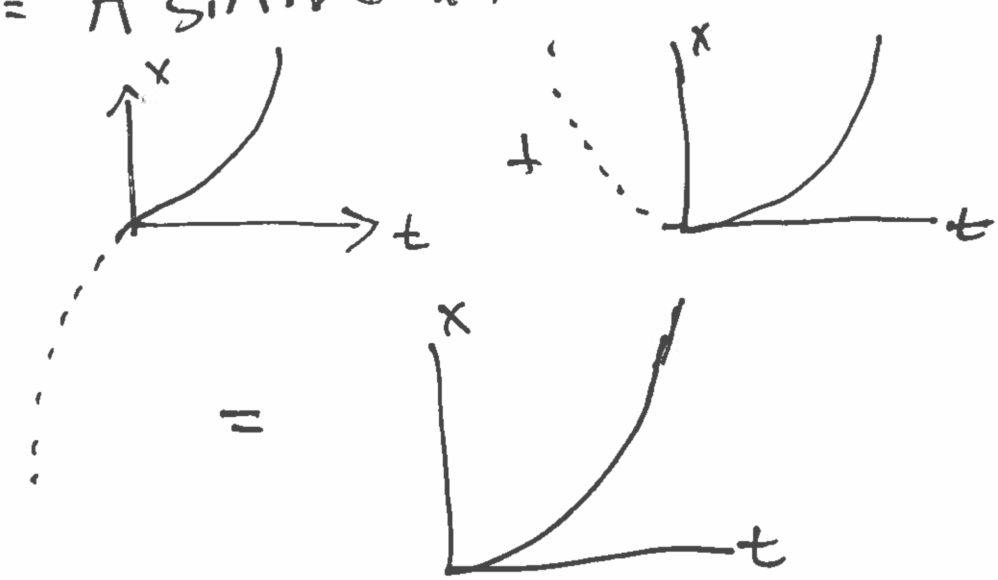
$$|x(t)| \leq |A|$$

(marginally) stable!

$$m\ddot{x} - kx = 0 \qquad (k, m\ +)$$

$$x(t) = a_1 e^{-\sqrt{\frac{k}{m}}\,t} + a_2 e^{-\sqrt{\frac{k}{m}}\,t}$$

$$= A\sinh(\omega_n t) + B\cosh(\omega_n t)$$

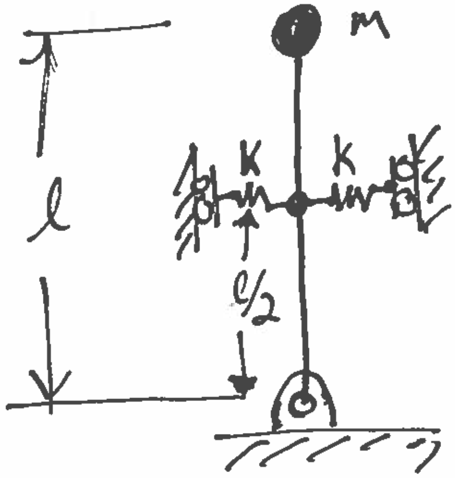


unstable!

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad m, c, k > 0$$

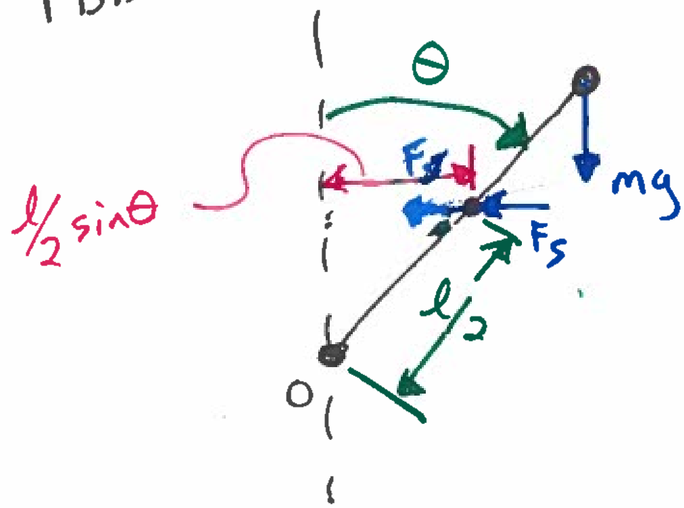
stable solution

Example



Newton and Euler's equations

FBD



$$I_o \ddot{\theta} = \Sigma M_o$$

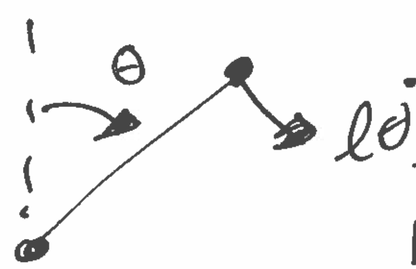
$$I_o = m\ell^2$$

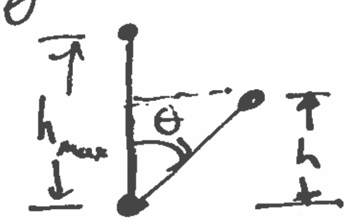
$$F_s = k\Delta x_1 + k\Delta x_2$$
$$= 2k\left(\frac{\ell}{2}\sin\theta\right)$$
$$= k\ell\sin\theta$$

$$F_g = mg$$

$$\ddot{\theta} m\ell^2 = -k\ell\sin\theta\,\frac{\ell}{2}\cos\theta + mg\ell\sin\theta$$

Lagranges Method





$$T = \frac{1}{2}m(\ell\dot{\theta})^2$$

$$U = U_{s_1} + U_{s_2} + U_g$$
$$= \frac{1}{2}k\left(\frac{\ell}{2}\sin\theta\right)^2 + \frac{1}{2}k\left(\frac{\ell}{2}\sin\theta\right)^2 + mg\ell\cos\theta$$
$$= k\left(\frac{\ell}{2}\sin\theta\right)^2 + mg\ell\cos\theta$$

$L = T - U$

$L = \frac{1}{2} m(l\dot{\theta})^2 - K\left(\frac{l}{2}\sin\theta\right)^2 - mgl\cos\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt}\left(ml^2\dot{\theta}\right) + 2K\frac{l}{2}\sin\theta\,\frac{l}{2}\cos\theta - mgl\sin\theta = 0$

$\frac{Kl^2}{2}\sin\theta\cos\theta$

$ml^2\ddot{\theta} + \frac{Kl^2}{2}\sin\theta\cos\theta - mgl\sin\theta = 0$

Assume $\sin\theta = \theta$, $\cos\theta = 1$

$ml^2\ddot{\theta} + \frac{Kl^2}{2}\theta - mgl\theta = 0$

$\ddot{\theta} + \underbrace{\frac{K}{2m}\theta - \frac{g}{l}}\theta = 0$

$\ddot{\theta} + \underbrace{\left(\frac{K}{2m} - \frac{g}{l}\right)}_{\text{negative?}}\theta = 0$

$\frac{K}{2m} - \frac{g}{l} < 0$

$\boxed{\frac{K}{2m} < \frac{g}{l}} \rightarrow$ lead unstable behavior!