Matrix Chain Multiplication

Problem

We are given n matrices, we have to multiply them in such a way that the total number of operations are minimum.

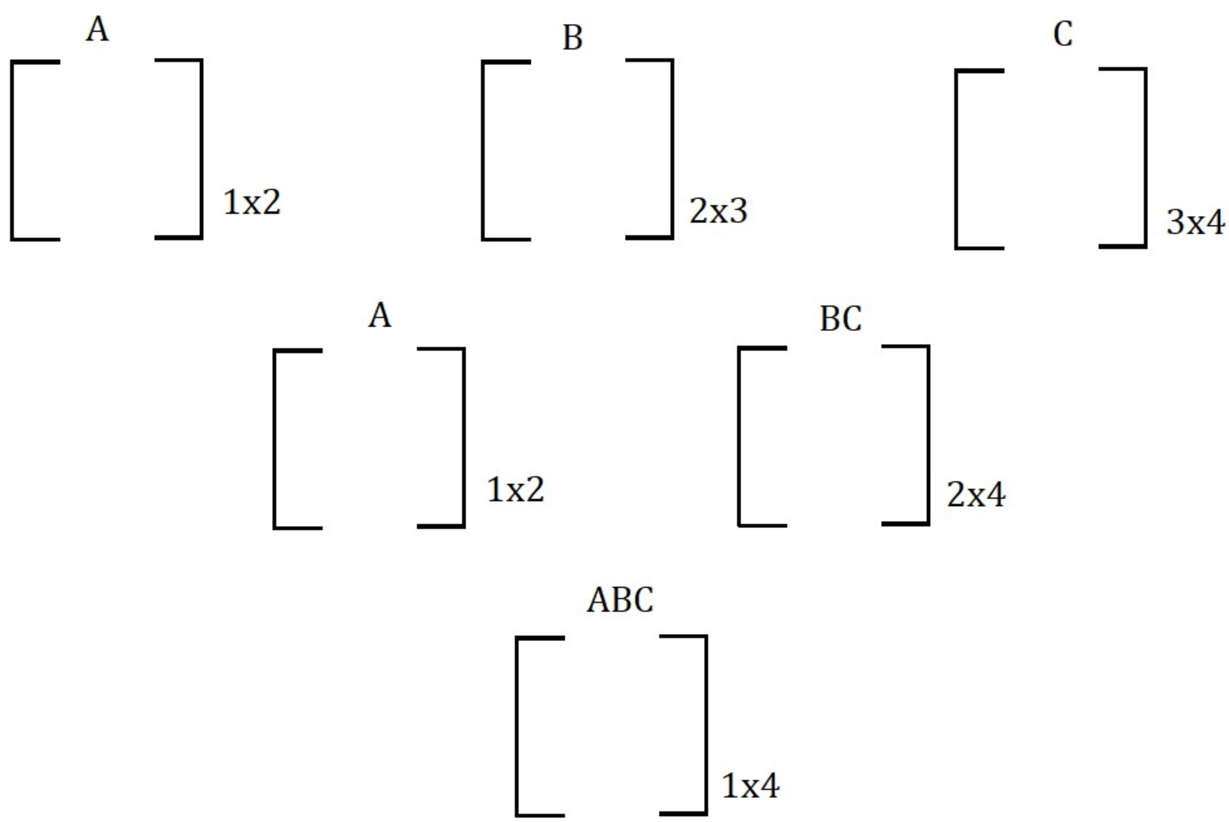
Example

$$[\,]_{1x2} \quad [\,]_{2x3} \quad [\,]_{3x4}$$

A B C

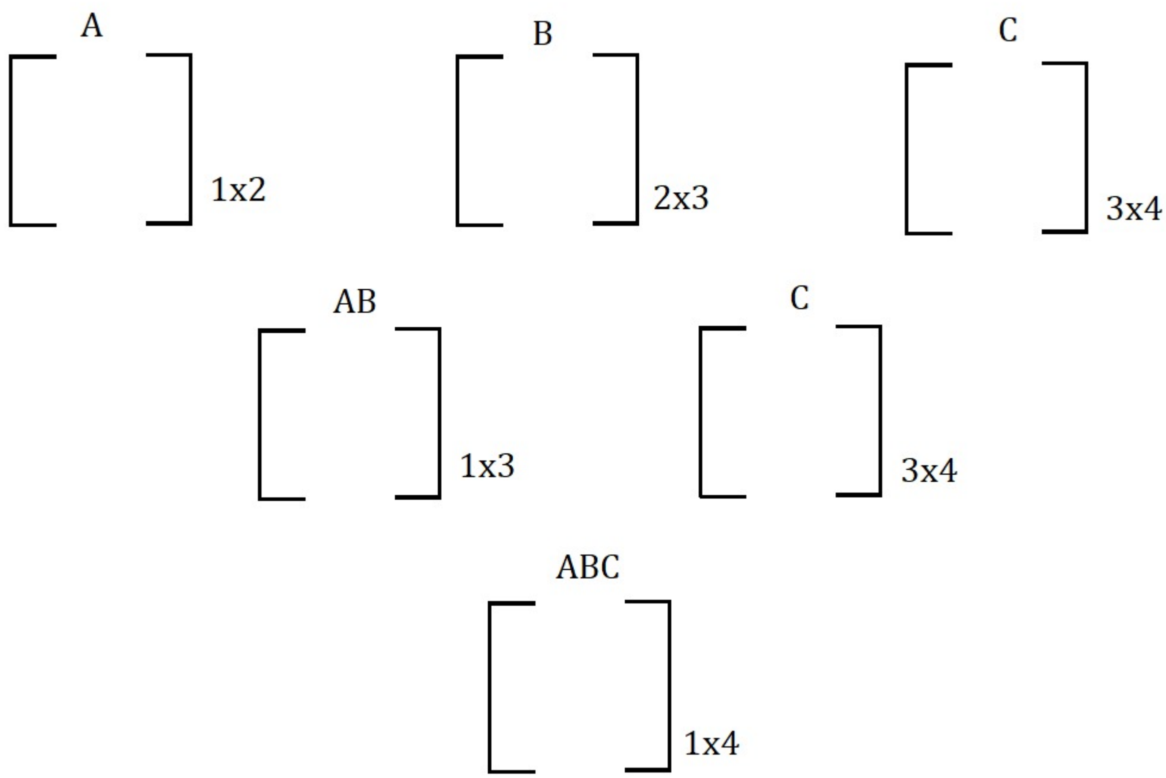
Since we know multiplication of matrices is associative, hence
A(BC) = (AB)C

Operations in A(BC)



(Total operations)$_{A(BC)}$ = 2x3x4 + 1x2x4 = 32 operations

<u>Operations in (AB)C</u>



(Total Operations)$_{A(BC)}$ = 1X2X3 + 1X3X4 = 18 operations

Therefore, (AB)C is more efficient than A(BC).

Dimensions of matrices will be given in the form of an array.

Example

10	20	30	20	30

The Dimension of i^{th} matrix is a[i-1] x a[i].
Example

$$M_1 \text{ -> } a[0] \times a[1] = 10 \times 20$$
$$M_2 \text{ -> } a[1] \times a[2] = 20 \times 30$$
$$M_3 \text{ -> } a[2] \times a[3] = 30 \times 20$$
$$M_4 \text{ -> } a[3] \times a[4] = 20 \times 30$$

Therefore dimension of matrix multiplication from

$$M_i \text{ to } M_j \text{ -> } a[i-1] \times a[j]$$
$$\text{Example: } M_1\ M_2\ M_3 \text{ -> } a[0] \times a[3] = 10 \times 20$$

Our Recurrence Relation becomes

$$f(M_1M_2....M_N) = min(f(M_1....M_k) + f(M_{k+1}....M_N) + a[0] \times a[1] \times a[N]\ )$$
$$\text{where } 1 <= k <= N-1$$

Let us take 4 matrices A, B, C, D.

We can see that answer of ABCD depends on

1. (A)(BCD)
2. (AB)(CD)
3. (ABC)(D)

Whichever from 1. , 2. Or 3 gives minimum operations, that is the answer.

In other words, we can say that 3 cuts are possible,
(i) A | B C D
(ii) A B | C D
(iii) A B C | D

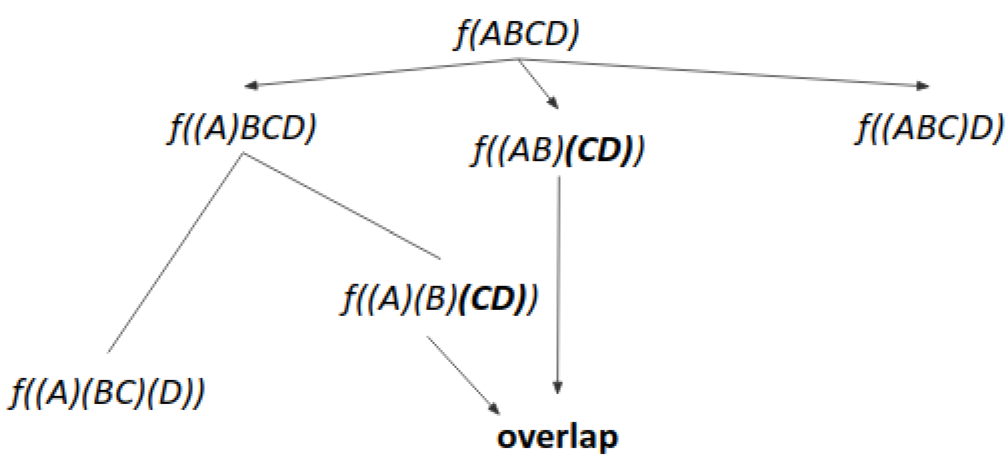
We can write its recurrence as

$$f(ABCD) = min(f(A|BCD), f(AB|CD), f(ABC|D))$$

Since it has a recurrence relation, therefore it follows <u>optimal substructure property</u>.

Checking whether it has overlapping subproblem property also?

<u>Making recursion tree</u>



We can see that computation of *f(CD)* is repeated, hence it possesses <u>overlapping subproblem property</u>.

Hence it can be solved using <u>dynamic programming</u>.

<u>Approach 1</u>

1. Write the recursive solution.
2. Memoize it.

Approach 2 (Tabulation (Bottom Up))

1. Build from base.
2. For each gap=0 to gap=n-2, compute all submatrix multiplication and their results.
3. Build the answer using,

for every k=i to k=j-1

dp[i][j] = min(dp[i][j], dp[i][k] + dp[k+1][j] + a[i-1] x a[k] x a[j])

Time complexity: $O(n^3)$

Code (Recursive)

```
// matrix chain multiplication
int dp[N][N];

int fun(vi &a, int i, int j)
{
    if(j-i == 1)
    {
        dp[i][j] = a[i-1]*a[i]*a[j];
        return dp[i][j];
    }

    if(j == i)
        return 0;

    if(dp[i][j] != MOD)
        return dp[i][j];

    for(int k=i; k<j; k++)
    {
        int temp = fun(a,i,k) + fun(a,k+1,j) + a[i-1]*a[k]*a[j];
        dp[i][j] = min(temp, dp[i][j]);


    }

    return dp[i][j];
}
```

```cpp
void solve()
{
    rep(i,0,N)
    {
        rep(j,0,N)
            dp[i][j] = MOD;
    }

    int n;
    cin >> n;

    vi a(n);
    rep(i,0,n)
        cin >> a[i];

    cout << fun(a, 1, n-1) << endl;
}
```

<u>Code (Iterative)</u>

```cpp
int dp[N][N];

void solve()
{
    rep(i,0,N)
    {
        rep(j,0,N)
        dp[i][j] = MOD;
    }

    int n;
    cin >> n;

    vi a(n);

    rep(i,0,n)
        cin >> a[i];
```

```
    for(int gap=0; gap <= n-2; gap++)
    {
        for(int i=1; i<=n-gap-1; i++)
        {
            int j = i+gap;
            if(i == j)
                dp[i][j] = 0;
            else if(j-i == 1)
                dp[i][j] = a[i-1]*a[i]*a[j];
            else
            {
                for(int k=i; k<j; k++)
                {
                    dp[i][j] = min(dp[i][j], dp[i][k] + dp[k+1][j] + a[
                }
            }
        }
    }

    cout << dp[1][n-1] << endl;
}
```