Engineering Mathematics 233

Solutions: Double and triple integrals

Double Integrals

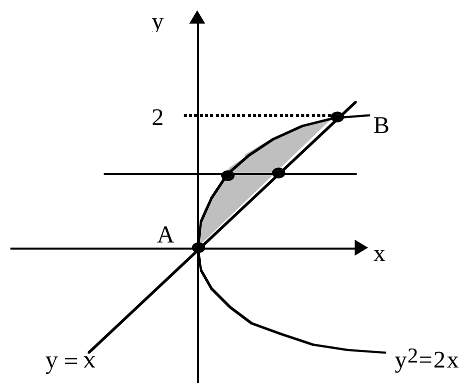
1. Sketch the region R in the xy-plane bounded by the curves $y^2 = 2x$ and $y = x$, and find its area.

Solution

The region R is bounded by the parabola $x = \frac{1}{2}y^2$ and the straight line $y = x$. The points of intersection of the two curves are given by

$$y = \frac{y^2}{2} \iff y^2 - 2y = 0 \iff y(y-2) = 0 \iff y = 0, 2.$$

This gives the two points $A = (0,0)$ and $B = (2,2)$.



The region R is a Type II region, and can be described by

$$R: \quad \begin{array}{l} 0 \leq y \leq 2 \\ \dfrac{y^2}{2} \leq x \leq y. \end{array}$$

Then,

$$\begin{aligned} \text{area}(R) &= \int\int_R 1\, dA = \int_0^2 \int_{y^2/2}^{y} 1\, dx\, dy \\ &= \int_0^2 y - \frac{y^2}{2}\, dy = \left(\frac{y^2}{2} - \frac{y^3}{6}\right)\Big|_0^2 \\ &= 2 - \frac{8}{6} = \frac{2}{3}. \end{aligned}$$

2. Evaluate the integral

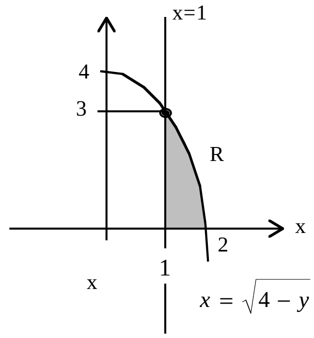
$$\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\, dx\, dy$$

by interchanging the order of integration.

Solution

The region of integration is the Type II region R

$$R: \quad \begin{array}{l} 0 \le y \le 3 \\ 1 \le x \le \sqrt{4-y}. \end{array}$$



We have

$$x = \sqrt{4-y} \quad \Rightarrow \quad x^2 = 4 - y \quad \Leftrightarrow \quad y = 4 - x^2.$$

Then, from the drawing above, we can rewrite the region R as the Type I region

$$R: \quad \begin{array}{l} 1 \le x \le 2 \\ 0 \le y \le 4 - x^2. \end{array}$$

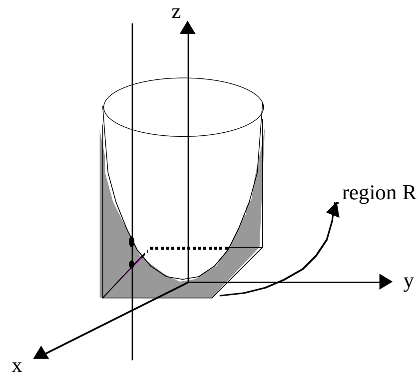
Then,

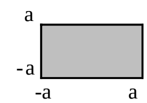
$$\begin{aligned}
\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\, dx\, dy &= \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\, dy\, dx \\
&= \int_{1}^{2} \left(xy + \frac{y^2}{2} \right) \Big|_{0}^{4-x^2} \\
&= \int_{1}^{2} 4x - x^3 + \frac{16 - 8x^2 + x^4}{2}\, dx \\
&= \left(2x^2 - \frac{x^4}{4} + 8x - \frac{4x^3}{3} + \frac{x^5}{10} \right) \Big|_{1}^{2} = \frac{241}{60}.
\end{aligned}$$

3. Find the volume of the region of $\mathbb{R}^3$ bounded by the paraboloid $z = x^2 + y^2$ and by the planes $z = 0$, $x = -a$, $x = a$, $y = -a$ and $y = a$.

<u>Solution</u>

Let S be the 3D region bounded by the paraboloid and the planes.





Then,

$$\text{volume}(S) = \int\int_R \left(x^2 + y^2 - 0\right)\ dA,$$

where R is the projection of S in the xy-plane, i.e.

$$R: \qquad \begin{array}{l} -a \le x \le a \\ -a \le y \le a. \end{array}$$

Then,

$$\begin{aligned}
\text{volume}(S) &= \int\int_R x^2 + y^2\ dA = \int_{-a}^{a}\int_{-a}^{a} x^2 + y^2\ dy\ dx \\
&= 4\int_0^a\int_0^a x^2 + y^2\ dy\ dx = 4\int_0^a \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^a\ dx \\
&= 4\int_0^a x^2 a + \frac{a^3}{3}\ dx = 4\left(\frac{x^3 a}{3} + \frac{a^3 x}{3}\right)\Big|_0^a \\
&= 4\left(\frac{2a^4}{3}\right) = \frac{8a^4}{3}.
\end{aligned}$$

4. Evaluate $\displaystyle\int\int_R \sqrt{x^2 + y^2}\ dA$, where R is the region of the plane given by $x^2 + y^2 \le a^2$.

<u>Solution</u>

The region R and the integrand $\sqrt{x^2 + y^2}$ are best described with polar coordinates (r, θ). In those coordinates, the region R, which is the region inside the circle $x^2 + y^2 = a^2$, becomes

$$R: \qquad \begin{array}{l} 0 \le r \le a \\ 0 \le \theta \le 2\pi. \end{array}$$

Then,

$$\begin{aligned}
\int\int_R \sqrt{x^2 + y^2}\ dA &= \int_0^{2\pi}\int_0^a r\ r\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^a r^2\ dr\ d\theta \\
&= \int_0^{2\pi} \frac{a^3}{3}\ d\theta = \frac{2\pi a^3}{3}.
\end{aligned}$$

5. Evaluate $\displaystyle\int\int_R e^{-(x^2+y^2)}\ dA$, where R is the region of **4.** above.

Solution

Using polar coordinates again, we write

$$\begin{aligned}\int\int_R e^{-(x^2+y^2)}\ dA &= \int_0^{2\pi}\int_0^a e^{-r^2}\ r\ dr\ d\theta \\ &= \int_0^{2\pi} \frac{e^{-r^2}}{-2}\Big|_0^a\ d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(1-e^{-a^2})\ d\theta = \pi(1-e^{-a^2}).\end{aligned}$$

6. Evaluate the integral $\int_0^1\int_{y^2}^1 ye^{x^2}\ dx\ dy$. **Hint:** First reverse the order of integration.

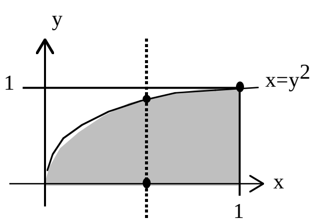
Solution

If we try to evaluate the integral as written above, then the first step is to compute the indefinite integral

$$\int e^{x^2}\ dx.$$

But e^{x^2} does not have an indefinite integral that can be written in terms of elementary functions. Then, we will fist reverse the order of integration. The region of integration is the Type II region

$$\begin{aligned} R: \quad & 0 \le y \le 1 \\ & y^2 \le x \le 1.\end{aligned}$$



Then, R can also be described as the Type I region

$$\begin{aligned} R: \quad & 0 \le x \le 1 \\ & 0 \le y \le \sqrt{x}.\end{aligned}$$

This gives

$$\begin{aligned}\int_0^1\int_{y^2}^1 ye^{x^2}\ dx\ dy &= \int_0^1\int_0^{\sqrt{x}} ye^{x^2}\ dy\ dx \\ &= \int_0^1 \left(\frac{y^2}{2}e^{x^2}\right)\Big|_0^{\sqrt{x}}\ dx \\ &= \frac{1}{2}\int_0^1 xe^{x^2}\ dx \\ &= \frac{1}{2}\left(\frac{e^{x^2}}{2}\right)\Big|_0^1 = \frac{1}{4}(e-1).\end{aligned}$$

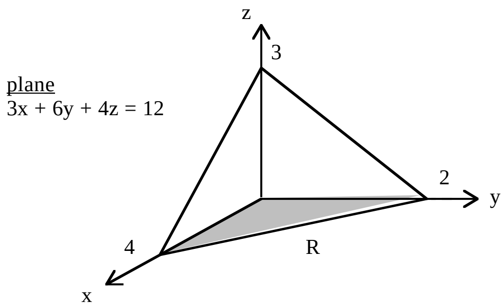
7. Find the volume of the tetrahedron bounded by the coordinate axes and the plane $3x + 6y + 4z = 12$.

Solution

We have to find the volume of the tetrahedron S bounded by the plane

$$3x + 6y + 4z = 12 \iff z = \frac{12 - 3x - 6y}{4}$$

and the coordinate axes. This is the portion of the plane in the first octant, as one can see from Picture (1).



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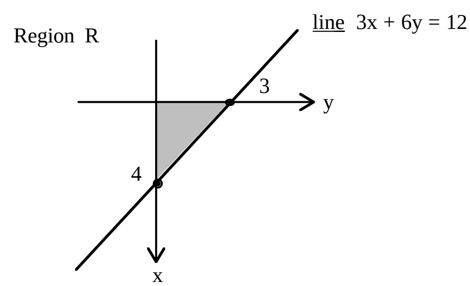


(2)

Then, we have

$$\text{volume}(S) = \int\int_R \frac{12 - 3x - 6y}{4}\, dA,$$

where R is the projection of the tetrahedron in the xy-plane. Then, R is the Type I region (see Picture (2))

$$R: \qquad \begin{array}{l} 0 \le x \le 4 \\ 0 \le y \le \dfrac{12 - 3x}{6}. \end{array}$$

Finally, this gives

$$\begin{aligned}
\text{volume}(S) &= \frac{1}{4}\int_0^4 \int_0^{12-3x/6} (12 - 3x - 6y)\, dy\, dx \\
&= \frac{1}{4}\int_0^4 \left((12 - 3x)y - 3y^2\right)\Big|_0^{2-\frac{x}{2}}\, dx \\
&= \frac{1}{4}\int_0^4 (12 - 3x)\left(2 - \frac{x}{2}\right) - 3\left(2 - \frac{x}{2}\right)^2\, dx \\
&= \frac{1}{4}\int_0^4 \frac{3}{4}x^2 - 6x + 12\, dx \\
&= \frac{1}{4}\left(\frac{x^3}{4} - 3x^2 + 12x\right)\Big|_0^4 = 4.
\end{aligned}$$

8. Evaluate the integral

$$\int_0^1 \int_{\sqrt{3}\,y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\; dx\; dy.$$

Hint: Use polar coordinates.

<u>Solution</u>

The region R of integration is the Type II region

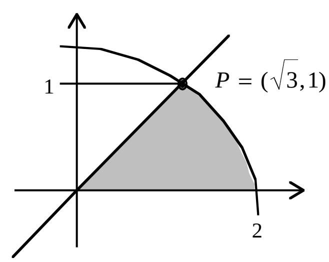
$$R: \qquad \begin{array}{l} 0 \le y \le 1 \\ \sqrt{3}\, y \le x \le \sqrt{4-y^2} \end{array}$$

We have

$$x = \sqrt{4-y^2} \quad \Rightarrow \quad x^2 = 4 - y^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4.$$

Then, x varies between the straight line $x = \sqrt{3}\, y$ and the circle $x^2 + y^2 = 4$.

The region R is



In polar coordinates, the region R is

$$R: \qquad \begin{array}{l} 0 \le r \le 2 \\ 0 \le \theta \le \alpha, \end{array}$$

where α is the angle made by the straight line $x = \sqrt{3}\, y$. The straight line and the cicle meet at the points

$$\left(\sqrt{3}y\right)^2 + y^2 = 4 \iff 4y^2 = 4 \iff y^2 = 1 \iff y = \pm 1.$$

The intersection point in the first quadrant is then $P = (\sqrt{3}, 1) = (2\cos\alpha, 2\sin\alpha)$. Then,

$$\alpha = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

Finally, the integrand $\sqrt{x^2+y^2}$ is r in polar coordinates. This gives

$$\begin{aligned} \int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\; dx\; dy &= \int_0^{\pi/6} \int_0^2 r\; r\; dr\; d\theta \\ &= \int_0^{\pi/6} \frac{r^3}{3}\Big|_0^2\; d\theta \\ &= \frac{\pi}{6}\,\frac{8}{3} = \frac{4\pi}{9}. \end{aligned}$$

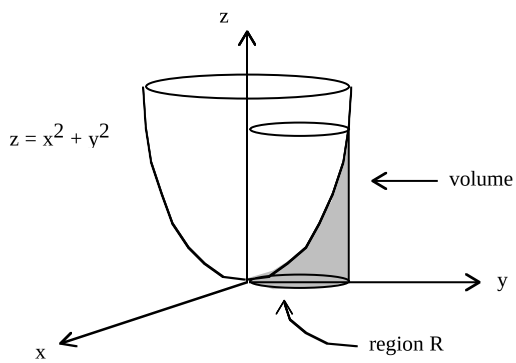
9. Find the volume below the surface $z = x^2 + y^2$, above the plane $z = 0$, and inside the cylinder $x^2 + y^2 = 2y$.

Solution

Completing the squares, we rewrite the equation of the cylinder as

$$x^2 + y^2 = 2y \iff x^2 + (y^2 - 2y) = 0 \iff x^2 + (y-1)^2 = 1.$$

The base of the cylinder is then the circle of radius 1 centered at $(0, 1)$. Then, we have to find the volume of the 3D region:



From the picture above, we write

$$V = \int\int_R x^2 + y^2 \; dA,$$

where R is the projection of the 3D region in the plane, i.e. the circle $x^2 + y^2 = 2y$. Using polar coordinates, this gives

$$V = \int\int_R r^2 r \; dr \; d\theta.$$

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, the circle writes as

$$x^2 + y^2 = 2y \iff r^2 = 2r\sin\theta \iff r = 2\sin\theta,$$

and R is the region

$$R: \quad \begin{array}{l} 0 \le \theta \le \pi \\ 0 \le r \le 2\sin\theta. \end{array}$$

Then,

$$\begin{aligned} V &= \int_0^{\pi} \int_0^{2\sin\theta} r^3 \; dr \; d\theta = \int_0^{\pi} \frac{r^4}{4} \Big|_0^{2\sin\theta} d\theta \\ &= 4\int_0^{\pi} \sin^4\theta \; d\theta = 4\left(\frac{3\theta}{8} - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta \Big|_0^{\pi}\right) \\ &= 4\left(\frac{3\pi}{8}\right) = \frac{3\pi}{2}. \end{aligned}$$

Triple Integrals

10. Evaluate

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=\sqrt{x^2+y^2}}^{2} xyz \; dz \; dy \; dx.$$

Solution

$$\begin{aligned}
\int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^2 xyz\;dz\;dy\;dx &= \int_0^1\int_0^1 \frac{xyz^2}{2}\Big|_{z=\sqrt{x^2+y^2}}^2 dy\;dx\\
&= \int_0^1\int_0^1 2xy-\frac{xy(x^2+y^2)}{2}\;dy\;dx\\
&= \int_0^1\int_0^1 2xy-\frac{x^3y}{2}-\frac{y^3x}{2}\;dy\;dx\\
&= \int_0^1 \left(xy^2-\frac{x^3y^2}{4}-\frac{y^4x}{8}\right)\Big|_0^1 dx\\
&= \int_0^1 x-\frac{x^3}{4}-\frac{x}{8}\;dx\\
&= \int_0^1 \frac{7x}{8}-\frac{x^3}{4}\;dx\\
&= \left(\frac{7x^2}{16}-\frac{x^4}{16}\right)\Big|_0^1\\
&= \frac{7}{16}-\frac{1}{16}=\frac{3}{8}.
\end{aligned}$$

11. Find the mass of the 3D region B given by

$$x^2+y^2+z^2\leq 4,\; x\geq 0,\; y\geq 0,\; z\geq 0,$$

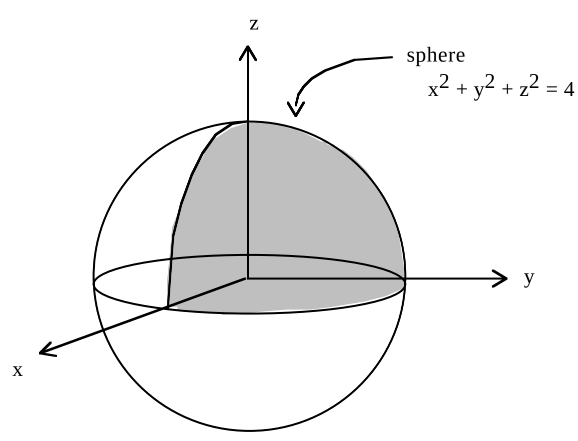
if the density is equal to xyz.

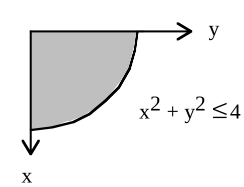
Solution

We have

$$\text{mass}(B)=\int\int\int_B xyz\;dV,$$

and the region B is the portion of the sphere of radius 2 in the first octant.





3D region B 2D region R

Then, B can be described as

$$0 \leq z \leq \sqrt{4-x^2-y^2}, \quad \text{for all } (x,y) \in R,$$

where R is the projection of B in the xy-plane. Describing R as a Type I region, this gives

$$\begin{aligned} B: \quad & 0 \leq x \leq 2 \\ & 0 \leq y \leq \sqrt{4-x^2} \\ & 0 \leq z \leq \sqrt{4-x^2-y^2}. \end{aligned}$$

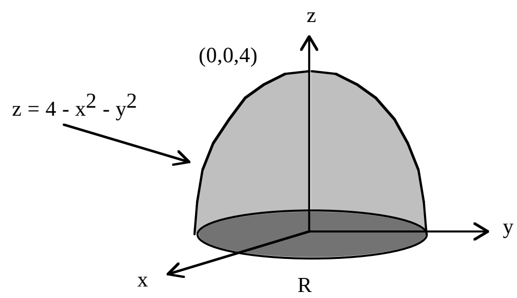
Then,

$$\begin{aligned}
\text{mass}(B) &= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz \; dz \; dy \; dx \\
&= \int_0^2 \int_0^{\sqrt{4-x^2}} \left(xy\frac{z^2}{2} \right) \Big|_0^{\sqrt{4-x^2-y^2}} \; dy \; dx \\
&= \frac{1}{2} \int_0^2 \int_0^{\sqrt{4-x^2}} xy(4-x^2-y^2) \; dy \; dx \\
&= \frac{1}{2} \int_0^2 \int_0^{\sqrt{4-x^2}} 4xy - x^3y - y^3x \; dy \; dx \\
&= \frac{1}{2} \int_0^2 \left(2xy^2 - \frac{x^3y^2}{2} - \frac{y^4x}{4} \right) \Big|_0^{\sqrt{4-x^2}} \; dx \\
&= \frac{1}{2} \int_0^2 2x(4-x^2) - \frac{x^3(4-x^2)}{2} - \frac{(4-x^2)^2x}{4} \; dx \\
&= \frac{1}{2} \int_0^2 \frac{x^5}{4} - 2x^3 + 4x \; dx \\
&= \frac{1}{2} \left(\frac{x^6}{24} - \frac{x^4}{2} + \frac{4x^2}{2} \right) \Big|_0^2 \\
&= \frac{1}{2} \left(\frac{64}{24} - 8 + 8 \right) = \frac{4}{3}.
\end{aligned}$$

12. Find the volume of the region B bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

<u>**Solution**</u>

We have volume$(B) = \int\int\int_B 1\ dV$.



Then, B can be described as

$$0 \leq z \leq 4 - x^2 - y^2, \quad \text{for all } (x, y) \in R,$$

where R is the projection of B in the xy-plane. Then, R is the interior of the circle $x^2 + y^2 = 4$. In polar coordinates, the region R is

$$R: \qquad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2, \end{array}$$

and in cylindrical coordinates, the region B is

$$B: \qquad \begin{array}{l} 0 \leq z \leq (4 - r^2) \\ 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2. \end{array}$$

Then,

$$\begin{aligned} \text{volume}(B) &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\ dz\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^2 (4-r^2) r\ dr\ d\theta \\ &= 2\pi \int_0^2 4r - r^3\ dr \\ &= 2\pi\left(2r^2 - \frac{r^4}{4}\Big|_0^2\right) = 2\pi(4) = 8\pi. \end{aligned}$$

13. Find the center of gravity of the region in **12.**, assuming constant density σ.

<u>**Solution**</u>

By symmetry, $\overline{x} = \overline{y} = 0$. Also, as the density is $d(x, y, z) = \sigma$,

$$\begin{aligned} \overline{z} &= \frac{\int\int\int_B d(x,y,z) z\ dV}{\int\int\int_B d(x,y,z)\ dV} = \frac{\sigma \int\int\int_B z\ dV}{\sigma \int\int\int_B 1\ dV} \\ &= \frac{\int\int\int_B z\ dV}{\text{volume}(B)} = \frac{1}{8\pi}\int\int\int_B z\ dV. \end{aligned}$$

Using the description of the region B in cylindrical coordinates of **12.**, we get

$$\begin{aligned}\int\int\int_B z\, dV &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} z\ r\ dz\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^2 \frac{(4-r^2)^2}{2} r\ dr\ d\theta \\ &= \frac{2\pi}{2}\int_0^2 16r - 8r^3 + r^5\ dr \\ &= \pi\left(8r^2 - 2r^4 + \frac{r^6}{6}\Big|_0^2\right) = \frac{32}{3}\pi.\end{aligned}$$

Then,

$$\overline{z} = \frac{1}{8\pi}\left(\frac{32}{3}\pi\right) = \frac{4}{3}.$$

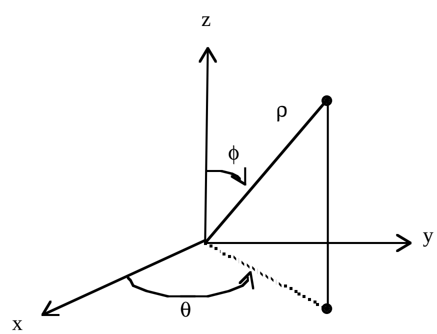
14. Evaluate

$$\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV,$$

where B is the region bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2+y^2}$.

<u>Solution</u>

We will use the spherical coordinates



to describe the region B. In those coordinates,

$$\begin{aligned} x &= \rho\ \sin\phi\ \cos\theta \\ y &= \rho\ \sin\phi\ \sin\theta \\ z &= \rho\ \cos\phi. \end{aligned}$$

Then, the cone $z = \sqrt{x^2+y^2}$ writes as

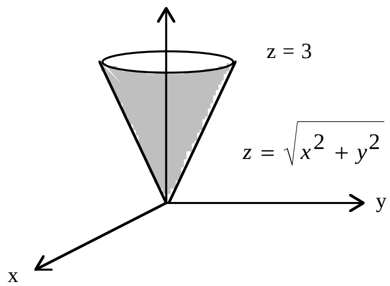
$$\rho\cos\phi = \rho\sin\phi \iff \frac{\sin\phi}{\cos\phi} = \tan\phi = 1 \iff \phi = \frac{\pi}{4},$$

the plane $z = 3$ as

$$3 = \rho\cos\phi \iff \rho = \frac{3}{\cos\phi},$$

and the region B can be described as

$$B: \qquad \begin{aligned} &0 \leq \phi \leq \frac{\pi}{4} \\ &0 \leq \phi \leq 2\pi \\ &0 \leq \rho \leq \frac{3}{\cos\phi}. \end{aligned}$$



Finally, in spherical coordinates, $\sqrt{x^2+y^2+z^2} = \rho$.
Then,

$$\begin{aligned} \int\int\int_B \sqrt{x^2+y^2+z^2}\, dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3/\cos\phi} \rho\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \\ &= 2\pi \int_0^{\pi/4} \sin\phi \int_0^{3/\cos\phi} \rho^3 \, d\rho\, d\phi \\ &= 2\pi \int_0^{\pi/4} \sin\phi \left(\frac{\rho^4}{4}\Big|_0^{3/\cos\phi} \right) \, d\phi \\ &= 2\pi \frac{3^4}{4} \int_0^{\pi/4} \frac{\sin\phi}{\cos^4\phi}\, d\phi \\ &= \frac{81\pi}{2}\left(\frac{(\cos\phi)^{-3}}{3}\Big|_0^{\pi/4} \right) \\ &= \frac{81\pi}{6}\left(\left(\frac{\sqrt{2}}{2}\right)^{-3} - 1 \right) = \frac{27\pi}{2}\left(2\sqrt{2}-1\right). \end{aligned}$$

15. Evaluate

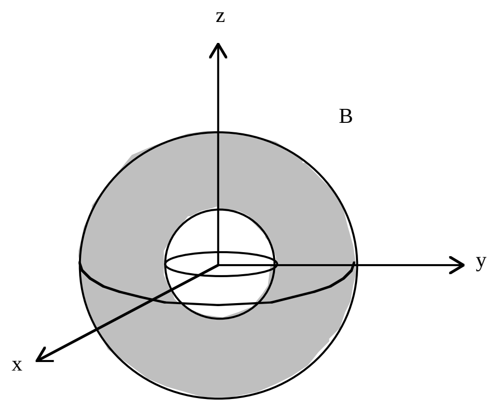
$$\int\int\int_B \left(x^2+y^2+z^2\right)^{-3/2}\, dV,$$

where B is the region bounded by the spheres $x^2+y^2+z^2 = a^2$ and $x^2+y^2+z^2 = b^2$, where $a > b > 0$.

Solution

Using spherical coordinates, the region B between the 2 spheres can be described as

$$B: \qquad \begin{aligned} &0 \leq \theta \leq 2\pi \\ &0 \leq \phi \leq \pi \\ &a \leq \rho \leq b. \end{aligned}$$



Then,

$$
\begin{aligned}
\int\int\int_B (x^2+y^2+z^2)^{-3/2}\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_a^b \rho^{-3}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 2\pi\int_0^{\pi}\int_a^b \frac{1}{\rho}\sin\phi\,d\rho\,d\phi \\
&= 2\pi\int_0^{\pi}\left(\ln\rho\Big|_a^b\right)\sin\phi\,d\phi \\
&= 2\pi\ln\left(\frac{b}{a}\right)\int_0^{\pi}\sin\phi\,d\phi \\
&= 2\pi\ln\left(\frac{b}{a}\right)\left(-\cos\phi\Big|_0^{\pi}\right) = 4\pi\ln\left(\frac{b}{a}\right).
\end{aligned}
$$

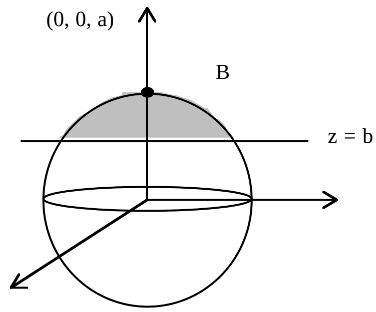
16. Find the volume of the region B bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by the plane $z=b$, where $a>b>0$.

Solution

The region B can be described as the set of $(x,y,z)\in\mathbb{R}^3$ such that

$$b \le z \le \sqrt{a^2-x^2-y^2}$$

for all (x, y) in the plane region bounded by the circle $x^2 + y^2 + b^2 = a^2 \iff x^2 + y^2 = a^2 - b^2$.



sphere $x^2 + y^2 + z^2 = a^2$

The region B is best described in cylindrical coordinates, and this gives

$$B: \quad \begin{aligned} &b \le z \le \sqrt{a^2 - r^2} \\ &0 \le r \le \sqrt{a^2 - b^2} \\ &0 \le \theta \le 2\pi. \end{aligned}$$

Then,

$$\begin{aligned}
\text{volume}(B) &= \iiint_B 1\, dV \\
&= \int_0^{2\pi} \int_0^{\sqrt{a^2-b^2}} \int_b^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{a^2-b^2}} (\sqrt{a^2-r^2} - b) r\, dr\, d\theta \\
&= 2\pi \int_0^{\sqrt{a^2-b^2}} \sqrt{a^2-r^2}\, r - br\, dr \\
&= 2\pi \left(\frac{(a^2-r^2)^{3/2}}{(3/2)(-2)} - b\frac{r^2}{2} \Big|_0^{\sqrt{a^2-b^2}} \right) \\
&= 2\pi \left(-\frac{1}{3}\left((b^2)^{3/2} - (a^2)^{3/2} \right) - \frac{b}{2}(a^2 - b^2) \right) \\
&= 2\pi \left(\frac{a^3 - b^3}{3} - \frac{a^2 b}{2} + \frac{b^3}{2} \right) = 2\pi \left(\frac{a^3}{3} - \frac{a^2 b}{2} + \frac{b^3}{6} \right).
\end{aligned}$$

17. Sketch the region B whose volume is given by the triple integral

$$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-6y)/4} 1\, dz\, dy\, dx.$$

Rewrite the triple integral using the order of integration $dV = dy\, dx\, dz$.

Solution

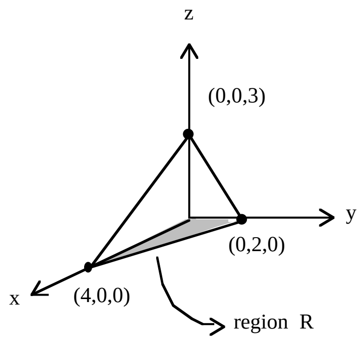
From the triple integral, the region B is described by

$$0 \le z \le \frac{12 - 3x - 6y}{4},$$

i.e. z varies between the planes $z = 0$ and $z = \dfrac{12 - 3x - 6y}{4} \iff 3x + 6y + 4z = 12$. Furthermore, (x, y) are in the region R described by

$$R: \qquad \begin{aligned} &0 \leq y \leq \frac{4 - x}{2} \\ &0 \leq x \leq 4, \end{aligned}$$

which is the projection of the plane $3x + 6y + 4z = 12$ in the xy-plane.



We now use the ordre of integration $dV = dy\ dx\ dz$. The region B be can be described as

$$0 \leq y \leq \frac{12 - 3x - 4x}{6}$$

for all (x, z) in the region R which is the projection of B in the xz-plane. Then, R can be described as

$$R: \qquad \begin{aligned} &0 \leq z \leq 3 \\ &0 \leq x \leq \frac{(12 - 4z)}{3}. \end{aligned}$$

This gives

$$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-6y)/4} 1\ dz\ dy\ dx = \int_0^3 \int_0^{(12-4z)/3} \int_0^{(12-3x-4z)/6} 1\ dy\ dx\ dz.$$

18. Evaluate

$$\int\int\int_B \sqrt{x^2 + y^2}\ dV,$$

where B is the region lying above the xy-plane, and below cone $z = 4 - \sqrt{x^2 + y^2}$.

<u>Solution</u>

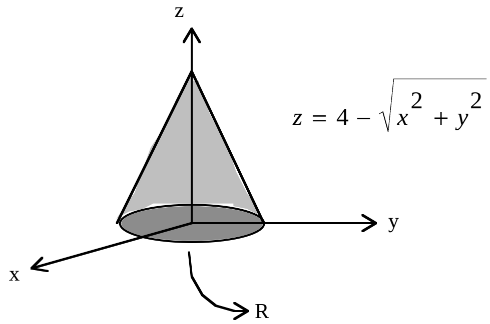
The region B be can be described as

$$0 \leq z \leq 4 - \sqrt{x^2 - y^2}, \quad \text{for all } (x, y) \in R,$$

where R is the projection of B in the xy-plane. Then, R is the region inside the curve

$$0 = 4 - \sqrt{x^2 + y^2} \iff x^2 + y^2 = 16,$$

which is a circle of radius 4.



We then use cylindrical coordinates to describe the region B. This gives

$$B: \quad \begin{aligned} &0 \le z \le 4 - r \\ &0 \le r \le 4 \\ &0 \le \theta \le 2\pi. \end{aligned}$$

Then,

$$\begin{aligned} \int\int\int_B \sqrt{x^2+y^2}\, dV &= \int_0^{2\pi}\int_0^4\int_0^{4-r} r\ r\ dz\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^4 r^2(4-r)\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^4 4r^2 - r^3\ dr \\ &= \int_0^{2\pi} \frac{4r^3}{3} - \frac{r^4}{4}\bigg|_0^4\ d\theta \\ &= \frac{64}{3}\int_0^{2\pi}\ d\theta = \frac{64}{3}(2\pi). \end{aligned}$$

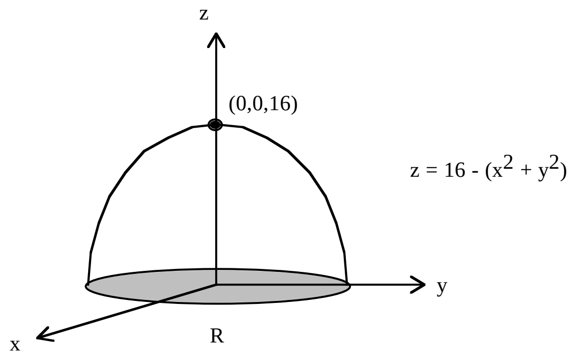
19. Evaluate the integral

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{(16-x^2-y^2)} \sqrt{x^2+y^2}\ dz\ dy\ dx.$$

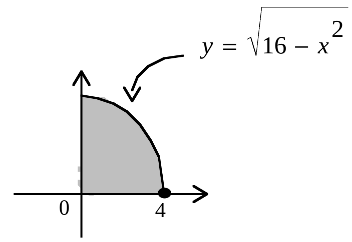
Hint: First convert to cylindrical coordinates.

Solution

The region B described by the integral is the region given by $0 \le z \le (16 - x^2 - y^2)$, i.e. bounded below by the plane $z = 0$ and above by the paraboloid $z = 16 - x^2 - y^2$, for all $(x, y) \in R$. For the region R, we have $0 \le y \le \sqrt{16 - x^2}$, i.e. y varies between the straight line $y = 0$ and the top part of the circle $x^2 + y^2 = 16$. Similarly, x varies between the straight lines $x = 0$ and $x = 4$. Then, R is the portion of the circle $x^2 + y^2 = 16$ in the first quadrant.



3D region B



2D region R

The region B is best described in cylindrical coordinates as

$$B: \quad \begin{aligned} &0 \le z \le 16 - r^2 \\ &0 \le \theta \le \pi/2 \\ &0 \le r \le 4. \end{aligned}$$

Then,

$$\begin{aligned} \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{16-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx &= \int_0^{\pi/2} \int_0^4 \int_0^{16-r^2} r\, r\, dz\, dr\, d\theta \\ &= \frac{\pi}{2} \int_0^4 r^2(16 - r^2)\, dr \\ &= \frac{\pi}{2} \int_0^4 16r^2 - r^4\, dr \\ &= \frac{\pi}{2} \left(\frac{16r^3}{3} - \frac{r^5}{5} \Big|_0^4 \right) \\ &= \frac{\pi}{2} \left(\frac{2048}{15} \right) = \frac{1024\pi}{15}. \end{aligned}$$

20. Evaluate

$$\int\int\int_B \sqrt{x^2+y^2+z^2}\, dV,$$

where B is the region above the xy-plane bounded by the cone $z^2 = 3(x^2 + y^2)$ and by the sphere $x^2 + y^2 + z^2 = 1$.

Solution

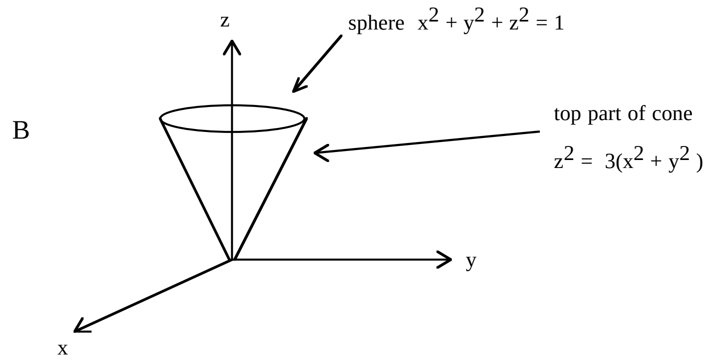
In spherical coordinates (ρ, θ, ϕ), the sphere is

$$x^2 + y^2 + z^2 = 1 \iff \rho^2 = 1 \iff \rho = 1,$$

and the cone is

$$z^2 = 3(x^2+y^2) \iff \rho^2 \cos^2\phi = 3\rho^2 \sin^2\phi \iff \tan^2\phi = \frac{1}{3} \iff \tan\phi = \pm\frac{1}{\sqrt{3}} \iff \phi = \frac{\pi}{6} \text{ or } \frac{4\pi}{6}.$$

The part of the cone above the xy-plane corresponds to $\phi = \dfrac{4\pi}{6}$.



Then, in spherical coordinates, the region B is

$$B: \quad \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \phi \le \dfrac{\pi}{6} \\ 0 \le \theta \le 2\pi. \end{array}$$

Then,

$$\begin{aligned}
\int\int\int \sqrt{x^2+y^2+z^2}\, dV &= \int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \sin\phi \left(\frac{\rho^4}{4}\bigg|_0^1 \right) d\phi\, d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/6} \sin\phi\, d\phi\, d\theta \\
&= \frac{1}{4}\int_0^{2\pi} \left(-\cos\phi \bigg|_0^{\pi/6} \right) d\theta \\
&= \frac{1}{4}\int_0^{2\pi} 1 - \frac{\sqrt{3}}{2}\, d\theta \\
&= \left(1 - \frac{\sqrt{3}}{2}\right)\frac{\pi}{2}.
\end{aligned}$$