Mass, Centers of Mass, and Double Integrals

Suppose a 2-D region R has density $\rho(x, y)$ at each point (x, y). We can partition R into subrectangles, with m of them in the x-direction, and n in the y-direction. Suppose each subrectangle has width Δx and height Δy. Then a subrectangle containing the point $(\hat{x}, \hat{y})$ has approximate mass

$$\rho(\hat{x}, \hat{y})\Delta x \Delta y$$

and the mass of R is approximately

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_i, y_i)\Delta x \Delta y$$

where (x_i, y_i) is a point in the i, j-th subrectangle. Letting m and n go to infinity, we have

$$M = \text{mass of } R = \iint_R \rho(x, y)\, dA.$$

Similary, the moment with respect to the x axis can be calculated as

$$M_x = \iint_R y\rho(x, y)\, dA$$

and the moment with respect to the y axis can be calculated as

$$M_y = \iint_R x\rho(x, y)\, dA.$$

The we may calculate the center of mass of R via

$$\text{center of mass of } R = (\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right).$$

Example 1

Let R be the unit square, $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. Suppose the density of R is given by the function

$$\rho(x, y) = \frac{1}{y + 1}$$

so that R is denser near the x-axis. As a result, we would expect the center of mass to be below the geometric center, $(1/2, 1/2)$. However, since the density does not depend on x, we do expect $\bar{x} = 1/2$.

We have:

$$M = \iint\limits_R \frac{1}{y+1}\,dA = \int_0^1 \int_0^1 \frac{1}{y+1}\,dy\,dx = \int_0^1 \ln(y+1)\,\big|_0^1\,dx = \int_0^1 \ln 2\,dx = \ln 2 = 0.693147....$$

$$M_x = \iint\limits_R \frac{y}{y+1}\,dA = \int_0^1 \int_0^1 \left(1 - \frac{1}{y+1}\right)\,dy\,dx = \int_0^1 (y - \ln(y+1))\,\big|_0^1\,dx$$
$$= \int_0^1 (1 - \ln 2)\,dx = 1 - \ln 2 = 0.306852819....$$
$$M_y = \iint\limits_R \frac{x}{y+1}\,dA = \int_0^1 \int_0^1 \frac{x}{y+1}\,dy\,dx = \int_0^1 x\ln 2\,dx = \frac{1}{2}x^2 \ln 2\,\big|_0^1 = \frac{1}{2}\ln 2 = 0.346573590....$$

Thus the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\frac{1}{2}\ln 2}{\ln 2}, \frac{1-\ln 2}{\ln 2}\right) = \left(\frac{1}{2}, 0.442095...\right).$$

Example 2 (Polar)

Let $0 \leq z \leq 1$. Let R be the polar region

$$R = \{(r,\theta) : z \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}.$$

Suppose R has constant density ρ. Then:

$$M = \iint\limits_R \rho\,dA = \rho \iint\limits_R\,dA = \rho \cdot \text{ area of } R\ = \rho\left(\frac{\pi}{4} - \frac{\pi z^2}{4}\right) = \frac{\pi}{4}\rho\left(1 - z^2\right).$$

$$M_x = \iint\limits_R \rho y\,dA = \rho \int_z^1 \int_0^{\pi/4} r^2 \sin\theta\,d\theta\,dr = \rho \int_z^1 -r^2\cos\theta\,\big|_0^{\pi/2}\,dr = \rho \int_z^1 r^2\,dr = \frac{1}{3}\rho(1 - z^3).$$

$$M_y = \iint\limits_R \rho x\,dA = \rho \int_z^1 \int_0^{\pi/2} r^2 \cos\theta\,d\theta\,dr = \rho \int_z^1 r^2\sin\theta\,\big|_0^{\pi/2}\,dr = \rho \int_z^1 r^2\,dr = \frac{1}{3}\rho(1 - z^3).$$

Thus, the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{\frac{1}{3}(1-z^3)}{\frac{\pi}{4}(1-z^2)}, \frac{\frac{1}{3}(1-z^3)}{\frac{\pi}{4}(1-z^2)}\right).$$

An interesting feature of this region is that if z is sufficiently large, the center of mass will be outside the region. This happens when the distance from the center of mass to $(0,0)$ is less than z. That is, when

$$\sqrt{2}\,\frac{\frac{1}{3}(1-z^3)}{\frac{\pi}{4}(1-z^2)} < z.$$

By factoring, we see that this is equivalent to

$$\frac{\frac{\sqrt{2}}{3}(1+z+z^2)}{\frac{\pi}{4}(1+z)} < z.$$

The critical z value is the positive solution to

$$0 = z^2 + z - \frac{\frac{\sqrt{2}}{3}}{\frac{\pi}{4} - \frac{\sqrt{2}}{3}}$$

which is about $0.82337397....$ Thus, if $z > 0.82337397...$, the region is very thin, and the center of mass lies outside of the region.