## Section 13.3: Area and Volume by Double Integration

Problems 1-10: Use double integration to find the area of the region in the $xy$-plane bounded by the given curves.

$y = x \qquad y = x^4$

These two curves intersect at $x = 0$ and $x = 1$. Thus, our integral is

$$\int_0^1 \int_{x^4}^{x} 1 \, \mathrm{d}y \mathrm{d}x = \int_0^1 \left( y \Big|_{x^4}^{x} \right) \mathrm{d}x = \int_0^1 x - x^4 \, \mathrm{d}x$$

$$= \left( \frac{x^2}{2} - \frac{x^5}{5} \right) \Bigg|_0^1 = \frac{3}{10}.$$

Problems 11-26: Find the volume of the solid that lies below the surface $z = f(x, y)$ and above the region in the $xy$-plane bounded by the given curves.

$z = y + e^x \qquad x = 0 \quad x = 1 \quad y = 0 \quad y = 2$

Our integral is

$$\int_0^1 \int_0^2 y + e^x \, \mathrm{d}y \mathrm{d}x = \int_0^1 \left( \frac{y^2}{2} + ye^x \right) \Bigg|_0^2 \mathrm{d}x = \int_0^1 2 + 2e^x \, \mathrm{d}x$$

$$= (2x + 2e^x) \Big|_0^1 = 2e.$$

$z = x^2 \qquad y = x^2 \quad y = 1$

The curves $y = x^2$ and $y = 1$ intersect at $x = \pm 1$, so we can set up our integral as

$$\int_{-1}^1 \int_{x^2}^1 x^2 \, \mathrm{d}y \mathrm{d}x = \int_{-1}^1 yx^2 \Big|_{y=x^2}^{y=1} \mathrm{d}x$$

$$= \int_{-1}^{1} x^2 - x^4 \, \mathrm{d}x = \left( \frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_{-1}^{1} = \frac{4}{15}.$$

Alternatively, we could have set up the integral as

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} x^2 \, \mathrm{d}x\mathrm{d}y = \int_0^1 \frac{x^3}{3} \Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}y$$

$$= \int_0^1 \frac{2}{3} y^{3/2} \, \mathrm{d}y = \frac{4}{15} y^{5/2} \Big|_0^1 = \frac{4}{15}.$$

<u>$z = 10 + y - x^2 \qquad y = x^2 \quad x = y^2$</u>

The two bounding curves intersect at $(0,0)$ and $(1,1)$. Therefore our integral is

$$\int_0^1 \int_{x^2}^{\sqrt{x}} 10 + y - x^2 \, \mathrm{d}y\mathrm{d}x = \int_0^1 \left( 10y + \frac{y^2}{2} - yex^2 \right) \Big|_{x^2}^{\sqrt{x}} \mathrm{d}x$$

$$\int_0^1 10\sqrt{x} + \frac{x}{2} - x^{5/2} - 10x^2 - \frac{x^4}{2} + x^4 \, \mathrm{d}x$$

$$= \left( \frac{20}{3} x^{3/2} + \frac{x^2}{4} - \frac{2}{7} x^{7/2} - \frac{10}{3} x^3 - \frac{x^5}{10} + \frac{1}{5} x^5 \right) \Big|_0^1$$

$$= \frac{20}{3} + \frac{1}{4} - \frac{2}{7} - \frac{10}{3} - \frac{1}{10} + \frac{1}{5} = \frac{2851}{840}.$$

<u>Problems 27-30:</u> Find the volume of the given solid.

<u>The solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $3x + 2y + z = 6$</u>

This region is simple in all directions, so we can integrate first in either direction. Since all coordinates are nonnegative in this region, $x \leq 2$. Therefore our integral is

$$\int_0^2 \int_0^{(6-3x)/2} 6 - 3x - 2y \, \mathrm{d}y\mathrm{d}x = \int_0^2 (6y - 3xy - y^2) \Big|_0^{(6-3x)/2} \mathrm{d}x$$

$$= \int_0^2 18-9x-\frac{3}{2}(6x-3x^2)-(6-3x)^2/4\,\mathrm{d}x = 18x-9x^2+\frac{3}{2}x^3+(6-3x)^3/36\Big|_0^2 = 6.$$

<u>Problems 31-34:</u> Set up an iterated integral that gives the volume of the given solid.

<u>The solid above the $xy$-plane and below $z = 9 - x^2 - y^2$.</u>

The region of integration is where $z$ is nonnegative, that is, on the disc $x^2 + y^2 \leq 9$. Since $x$ can range from $-3$ to 3 in this region, our integral is

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 9 - x^2 - y^2 \,\mathrm{d}y\mathrm{d}x.$$

<u>Problem 39:</u> Find the volume of a sphere of radius $a$ by double integration.

On a sphere of radius $a$, the $x$ value can range from $-a$ to $a$. With a given value of $x$, $y$ can only range from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$. Having fixed an $x$ and $y$ value, $z$ can be anywhere in $(-\sqrt{a^2 - x^2 - y^2}, \sqrt{a^2 - x^2 - y^2})$. Therefore the integral we want to evaluate is

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2 - y^2} \,\mathrm{d}y\mathrm{d}x.$$

Using Integral 54 on page TA-2, we get this integral is

$$\int_{-a}^{a} \left(y\sqrt{a^2 - x^2 - y^2} + (a^2 - x^2)\sin^{-1}(y/\sqrt{a^2 - x^2})\right)\Big|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \mathrm{d}x$$

$$\int_{-a}^{a} \frac{\pi}{2}(a^2 - x^2)\,\mathrm{d}x = \pi\left(2a^3 - \frac{2a^3}{3}\right) = \frac{4}{3}\pi a^3.$$

## Section 13.4: Double Integrals in Polar Coordinates

Problems 1-7: Find the indicated area by double integration in polar coordinates.

The area bounded by the circle $r = 1$.

Since $\theta$ can range from 0 to $2\pi$ and $r$ ranges from 0 to 1, the area is

$$\int_0^{2\pi} \int_0^1 r \, \mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2} \, \mathrm{d}\theta = \pi.$$

The area bounded by the cardioid $r = 1 + \cos(\theta)$.

Again, $\theta$ ranges from 0 to $2\pi$, but $r$ goes from 0 to $1 + \cos(\theta)$. Therefore the area is

$$\int_0^{2\pi} \int_0^{1+\cos\theta} r \, \mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \frac{(1+\cos(\theta))^2}{2} \, \mathrm{d}\theta = \left. \frac{\sin(\theta)\cos(\theta) + 4\sin(\theta) + 3\theta}{4} \right|_0^{2\pi}$$

$$= \frac{3\pi}{2}.$$

Problems 8-12: Use double integration in polar coordinates to find the volume of the solid that lies below the given surface and above the given plane region.

$z = x^2 + y^2 \qquad r = 3$

In polar form, $z = (r\cos(\theta))^2 + (r\sin(\theta))^2 = r^2$, so our integral is

$$\int_0^{2\pi} \int_0^3 r^3 \, \mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \frac{81}{4} \, \mathrm{d}\theta = \frac{81\pi}{2}.$$

$z = 10 + 2x + 3y \qquad r = \sin(\theta)$

In polar form, $z = 10 + 2r\cos(\theta) + 3r\sin(\theta)$. Also, $r = \sin(\theta)$ is a circle traced out completely by $\theta$ between 0 and $\pi$. Therefore, the volume is

$$\int_0^{\pi}\int_0^{\sin(\theta)} 10r + 2r^2\cos(\theta) + 3r^2\sin(\theta)\,\mathrm{d}r\mathrm{d}\theta$$

$$= \int_0^{\pi} 5\sin^2(\theta) + \frac{2}{3}\sin^3(\theta)\cos(\theta) + \sin^4(\theta)\,\mathrm{d}\theta$$

$$= \left(\frac{23}{8}\theta + \frac{\sin^4(\theta)}{6} - \frac{\sin(\theta)\cos(\theta)(2\sin^2(\theta)+23)}{8}\right)\Bigg|_0^{\pi} = \frac{23\pi}{8}.$$

Problems 13-18: Evaluate the integral by first converting to polar coordinates.

$\int_0^1\int_0^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2}\,\mathrm{d}x\mathrm{d}y$

We can see from the bounds of integration that the region of integration is the quarter of the circle $x^2 + y^2 = 1$ lying in the first quadrant. Therefore, in polar form, we have $0 \le \theta \le \pi/2$ and $0 \le r \le 1$. Also, $\frac{1}{1+x^2+y^2} = \frac{1}{1+r^2}$. Therefore

$$\int_0^1\int_0^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2}\,\mathrm{d}x\mathrm{d}y = \int_0^{\pi}\int_0^1 \frac{r}{1+r^2}\,\mathrm{d}r\mathrm{d}\theta$$

$$= \int_0^{\pi} \frac{1}{2}\ln(1+r^2)\Big|_0^1\,\mathrm{d}\theta = \int_0^{\pi}\frac{\ln 2}{2}\,\mathrm{d}\theta = \frac{\pi\ln 2}{4}.$$

$\int_0^1\int_x^1 x^2\,\mathrm{d}y\mathrm{d}x$

From the bounds of integration, we are integrating over the triangle formed by the lines $x = 0$, $y = 1$ and $y = x$. The line $y = 1$ in polar form is $r = \csc(\theta)$, and we only need $\theta$ to range from $\pi/4$ to $\pi/2$ to cover the region. Since $x^2 = r^2\cos^2(\theta)$, we have

$$\int_0^1\int_x^1 x^2\,\mathrm{d}y\mathrm{d}x = \int_{\pi/2}^{\pi/4}\int_0^{\csc(\theta)} r^3\cos^2(\theta)\,\mathrm{d}r\mathrm{d}\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{4} \csc^4(\theta)\cos^2(\theta)\,\mathrm{d}r\mathrm{d}\theta = \frac{\sin^2(\theta)\cos(\theta) + 2\cos(\theta) + 3\theta\sin(\theta)}{-2\sin(\theta)}\Bigg|_{\pi/4}^{\pi/2}$$

$$= \frac{5}{2} - \frac{3\pi}{8}.$$

Problems 19-22: Find the volume of the given solid.

$z = 1 \qquad z = 3 + x + y \qquad r = 1$

We integrate $r$ times the difference of the given functions over the region $r = 1$:

$$\int_0^{2\pi}\int_0^1 r(2 + r\cos(\theta) + r\sin(\theta))\,\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} 1 + \frac{\cos(\theta)}{3} + \frac{\sin(\theta)}{3}\,\mathrm{d}\theta$$

$$= (\theta - \frac{\sin(\theta)}{3} + \frac{\cos(\theta)}{3})\Bigg|_0^{2\pi} = 2\pi.$$

Problem 23: Find the volume of a sphere of radius $a$ by double integration.

We will integrate to get the area of a half-sphere. This is the volume under the curve $z = \sqrt{a^2 - r^2}$ over the circle $r = a$. Therefore the volume is

$$\int_0^{2\pi}\int_0^a r\sqrt{a^2 - r^2}\,\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \frac{-1}{3}(a^2 - r^2)^{3/2}\Bigg|_0^a \mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{a^3}{3}\,\mathrm{d}\theta = \frac{2\pi}{3}a^3.$$

Doubling this gives the familiar formula $V = \frac{4\pi}{3}a^3$.

## Section 13.5: Applications of Double Integrals

Problems 1-10: Find the centroid of the given plane region with constant density.

$x = 0 \qquad x = 4 \qquad y = 0 \qquad y = 6$

The centroid of a rectangle is its center, so the centroid of this region is $(2, 3)$.

To use calculus, though, the mass of this region is its area, or 24. Then

$$\overline{x} = \frac{1}{24}\int_0^4 \int_0^6 x \, \mathrm{d}y\mathrm{d}x = \frac{1}{24}\int_0^4 6x \, \mathrm{d}x = 2$$

$$\overline{y} = \frac{1}{24}\int_0^4 \int_0^6 y \, \mathrm{d}y\mathrm{d}x = \int_0^4 18 \, \mathrm{d}x = 3,$$

so the centroid is $(2, 3)$.

$x = 0 \qquad y = 0 \qquad x + y = 3$

We start by calculating the mass:

$$m = \int_0^3 \int_0^{3-x} \mathrm{d}y\mathrm{d}x = \int_0^3 3 - x \, \mathrm{d}x = \frac{9}{2}.$$

Then we know

$$\overline{x} = \frac{2}{9}\int_0^3 \int_0^{3-x} x \, \mathrm{d}y\mathrm{d}x = \frac{2}{9}\int_0^3 3x - x^2 \, \mathrm{d}x$$

$$= \frac{2}{9}\left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^3 = 1$$

$$\overline{y} = \frac{2}{9}\int_0^3 \int_0^{3-x} y \, \mathrm{d}y\mathrm{d}x = \frac{2}{9}\int_0^3 \frac{(3-x)^2}{2} \, \mathrm{d}x$$

$$= \frac{2}{9}\left(\frac{-(3-x)^3}{6}\right)\Bigg|_0^3 = 1,$$

so the centroid is $(1, 1)$.

Problems 11-30: Find the mass and centroid of the plane lamina

Triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$, $\delta(x, y) = xy$

We start by integrating $\delta$ to get the mass:

$$\int_0^1 \int_0^{1-x} xy \,\mathrm{d}y\mathrm{d}x = \frac{1}{2} \int_0^1 x - 2x^2 + x^3 \,\mathrm{d}x = \frac{1}{24}.$$

Now we can find $\overline{x}$:

$$\overline{x} = \frac{1}{m} \int_0^1 \int_0^{1-x} x\delta \,\mathrm{d}y\mathrm{d}x = 24 \int_0^1 \int_0^{1-x} x^2 y, \mathrm{d}y\mathrm{d}x$$

$$= 12 \int_0^1 x^2 - 2x^3 + x^4 \,\mathrm{d}x = \frac{2}{5}.$$

A similar calculation shows that $\overline{y} = \frac{2}{5}$, although this also follows since the region and the density function are symmetric in $x$ and $y$. Hence the mass of the region is $\frac{1}{24}$ and its centroid is $(2/5, 2/5)$.

Region bounded by $y = x^2$ and $y = 2 - x^2$, $\delta(x, y) = y$

Note that these two curves intersect at $x = -1$ and $x = 1$. We start by calculating the mass of the region.

$$m = \int_{-1}^1 \int_{x^2}^{2-x^2} y \,\mathrm{d}y\mathrm{d}x = \int_{-1}^1 2 - 2x^2 \,\mathrm{d}x = \frac{8}{3}.$$

Now we find that

$$\overline{x} = \frac{1}{m} \int_{-1}^1 \int_{x^2}^{2-x^2} xy \,\mathrm{d}y\mathrm{d}x = \frac{3}{8} \int_{-1}^1 2x - 2x^3 \,\mathrm{d}x = 0$$

$$\overline{y} = \frac{1}{m} \int_{-1}^1 \int_{x^2}^{2-x^2} y^2 \,\mathrm{d}y\mathrm{d}x = \frac{3}{8}, \frac{1}{3} \int_{-1}^1 8 - 12x^2 + 6x^4 - 2x^6 \,\mathrm{d}x = \frac{43}{35}.$$

Therefore the centroid of the region is $(0, 43/35)$.

Region bounded by cardioid $r = 1 + \cos(\theta)$, $\delta(r, \theta) = r$

We integrate $\delta$ to find the mass of the region:

$$m = \int_0^{2\pi} \int_0^{1+\cos(\theta)} r \cdot r \, \mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \frac{(1+\cos(\theta))^3}{3} \, \mathrm{d}x$$

$$= \left( \frac{2\sin(\theta)\cos^2(\theta) + 9\sin(\theta)\cos(\theta) + 22\sin(\theta) + 15\theta}{18} \right) \Bigg|_0^{2\pi} = \frac{5\pi}{3}.$$

We can now find $\overline{x}$ and $\overline{y}$:

$$\overline{x} = \frac{1}{m} \int_0^{2\pi} \int_0^{1+\cos(\theta)} r \cdot \delta(x,y)(r\cos(\theta)) \, \mathrm{d}r\mathrm{d}\theta = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos(\theta)} r^3 \cos(\theta) \, \mathrm{d}r\mathrm{d}\theta$$

$$= \frac{3}{5\pi} \int_0^{2\pi} \frac{(1+\cos(\theta))^4 \cos(\theta)}{4} \, \mathrm{d}\theta = \frac{3}{5\pi} \frac{105\pi}{60} = \frac{21}{20}$$

$$\overline{y} = \frac{1}{m} \int_0^{2\pi} \int_0^{1+\cos(\theta)} r \cdot \delta(x,y)(r\sin(\theta)) \, \mathrm{d}r\mathrm{d}\theta = \frac{3}{5\pi} \int_0^{2\pi} \int_0^{1+\cos(\theta)} r^3 \sin(\theta) \, \mathrm{d}r\mathrm{d}\theta$$

$$= \frac{3}{5\pi} \int_0^{2\pi} \frac{(1+\cos(\theta))^4 \sin(\theta)}{4} \, \mathrm{d}\theta = \frac{3}{5\pi} \left( \frac{-(1+\cos(\theta))^5}{20} \right) \Bigg|_0^{2\pi} = 0.$$

Therefore the centroid of the cardioid is $\left(\frac{21}{20}, 0\right)$.

<u>Problems 31-35: Find the polar moment of inertia $I_0$ of the indicated lamina.</u>

<u>Disc bounded by $r = a$, $\delta(x,y) = r^n$ $(n \neq -1)$</u>

We will integrate $r^2\delta(x,y) \cdot r = r^{n+3}$ in polar form:

$$I_0 = \int_0^{2\pi} \int_0^a r^{n+3} \, \mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \frac{a^{n+4}}{n+4} \, \mathrm{d}\theta = \frac{a^{n+4}}{n+4} 2\pi.$$

<u>Right-hand leaf of lemniscate $r^2 = \cos(2\theta)$, $\delta(x,y) = r^2$</u>

In polar form, we integrate $r^2\delta(x,y) \cdot r = r^5$. The right-hand leaf of the lemniscate is traced out by $\theta$ between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, so our integral is

$$I_0 = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos(2\theta)}} r^5 \, \mathrm{d}r\mathrm{d}\theta = \int_{-\pi/4}^{\pi/4} \frac{\cos^3(\theta)}{6} \, \mathrm{d}\theta$$

$$= \left( \frac{\sin(2\theta)\cos^3(2\theta)}{36} + \frac{\sin(2\theta)}{18} \right) \Bigg|_{-\pi/4}^{\pi/4} = \frac{1}{9}.$$

Problems 41-56

Use Pappus's Theorem to find the centroid of the first quadrant of $x^2 + y^2 \leq r^2$

We know the area of the first quadrant of this disc is $A = \frac{\pi}{4} r^2$. Also, if we revolve this region around the $y$-axis, the resulting solid is a hemisphere with volume $V = \frac{2\pi}{3} r^3$. The distance traveled by the centroid is $2\pi\overline{x}$. Pappus's Theorem tells us

$$V = 2\pi\overline{x}A,$$

so $\overline{x} = \frac{4}{3\pi} r$. By symmetry, $\overline{y} = \overline{x}$, so the centroid is $\left(\frac{4}{3\pi} r, \frac{4}{3\pi} r\right)$.

Use Pappus's Theorem to find the centroid of the first quadrant arc of $x^2 + y^2 = r^2$

We know the length of the given arc is $s = \frac{\pi}{2} r$. Also, if we revolve this arc around the $y$-axis, the resulting surface will be a hemisphere with surface area $S = 2\pi r^2$. The distance traveled by the centroid is $2\pi\overline{x}$. Pappus's Theorem tells us

$$S = 2\pi\overline{x}s,$$

so $\overline{x} = \frac{2}{\pi} r$. By symmetry $\overline{x} = \overline{y}$, so the centroid of the arc is $\left(\frac{2}{\pi} r, \frac{2}{\pi} r\right)$.

## Section 13.6: Triple Integrals

Problems 1-10: Evaluate the given triple integrals.

$f(x, y, z) = xy\sin(z)$, $0 \leq x, y, z \leq \pi$

Our integral is clearly

$$\int_0^\pi \int_0^\pi \int_0^\pi xy\sin(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^\pi \int_0^\pi \frac{\pi^2}{2} y\sin(z)\,\mathrm{d}y\mathrm{d}z$$
$$= \int_0^\pi \frac{\pi^4}{4}\sin(z)\,\mathrm{d}z = \left(-\frac{\pi^4}{4}\cos(z)\right)\Bigg|_0^\pi = \frac{\pi^4}{2}.$$

$f(x, y, z) = x^2$, bounded by coordinates planes and $x + y + z = 1$

We see that $x$ can range from 0 to 1. Given an $x$ between 0 and 1, $y$ ranges from 0 to $1 - x$. Given admissible $x$ and $y$, $z$ ranges from 0 to $1 - x - y$. Therefore our integral is

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1 \int_0^{1-x} x^2 - x^2y - x^3\,\mathrm{d}y\mathrm{d}x$$
$$= \int_0^1 \left(x^2y - \frac{x^2y^2}{2} - x^3y\right)\Bigg|_0^{1-x} \mathrm{d}x = \int_0^1 \frac{x^2}{2} - x^3 + \frac{x^4}{2}\,\mathrm{d}x$$
$$= \left(\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10}\right)\Bigg|_0^1 = \frac{1}{60}.$$

$f(x, y, z) = z$, region bounded by $z = 8 - y^2$ and $z = y^2$, $-1 \leq x \leq 1$

The curves $8 - y^2$ and $y^2$ intersect at $y = -2$ and $y = 2$. Therefore our integral is

$$\int_{-1}^1 \int_{-2}^2 \int_{y^2}^{8-y^2} z\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \frac{1}{2}\int_{-1}^1 \int_{-2}^2 64 - 16y^2\,\mathrm{d}y\mathrm{d}x$$

$$= \frac{1}{2}\int_{-1}^{1}\left(64y - \frac{16}{3}y^3\right)\Big|_{-2}^{2}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} = \frac{1}{2}\int_{-1}^{1}\frac{512}{3}\,\mathrm{d}x = \frac{512}{3}.$$

Problems 11-20: Find the volume of the given solids by triple integration.

Between $z = 0$ and $z = x^2 + y^2$ over region bounded by $x = 0$, $y = 0$ and $x + y = 1$

Clearly $x$ ranges from 0 to 1, $y$ ranges from 0 to $1-x$, and $z$ ranges from 0 to $x^2 + y^2$. Therefore the volume is

$$\int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} \mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1 \int_0^{1-x} x^2 + y^2\,\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 (x^2 y + y^3/3)\Big|_0^{1-x}\,\mathrm{d}x = \int_0^1 x^2 - x^3 + \frac{1}{3}(1-x)^3\,\mathrm{d}x$$

$$= \left(\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}.$$

Bounded by $y = z^2$, $z = y^2$, $x + y + z = 2$, $x = 0$

The curves $y = z^2$ and $z = y^2$ intersect at $y = 0$ and $y = 1$. Given an admissible value of $y$, $z$ ranges from $y^2$ to $\sqrt{y}$. Clearly $x$ ranges from 0 to $2 - y - z$. Therefore the volume is

$$\int_0^1 \int_{y^2}^{\sqrt{y}} \int_0^{2-y-z} \mathrm{d}x\mathrm{d}z\mathrm{d}y = \int_0^1 \int_{y^2}^{\sqrt{y}} 2 - y - z\,\mathrm{d}z\mathrm{d}y$$

$$= \int_0^1 (2z - yz - z^2/2)\Big|_{y^2}^{\sqrt{y}}\,\mathrm{d}y = \int_0^1 2\sqrt{y} - y^{3/2} - y/2 - 2y^2 + y^3 + y^4/2\,\mathrm{d}y$$

$$= \left(\frac{4y^{3/2}}{3} - \frac{2y^{5/2}}{5} - \frac{y^2}{4} - \frac{2y^3}{3} + \frac{y^4}{4} + \frac{y^5}{10}\right)\Big|_0^1 = \frac{11}{30}.$$

Problems 21-32: Assume the indicated solid has constant density.

Find the centroid of the hemisphere $x^2 + y^2 + z^2 \leq R^2$, $z \geq 0$

We know the mass of the hemisphere is simply its volume $V = \frac{2}{3}\pi R^3$. Also, since the hemisphere is symmetric about the $x$- and $y$-axes, $\overline{x} = \overline{y} = 0$. Then

$$\overline{z} = \frac{1}{V}\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{0}^{\sqrt{R^2-x^2-y^2}} z\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \frac{1}{2V}\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} R^2-x^2-y^2\,\mathrm{d}y\mathrm{d}x$$

$$= \frac{1}{2V}\int_{-R}^{R}(R^2y - x^2y - y^3/3)\Big|_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\mathrm{d}x = \frac{2}{3V}\int_{-R}^{R}(R^2-x^2)^{3/2}\,\mathrm{d}x$$

$$= \frac{2}{3V}\left(\frac{3}{8}R^4\tan^{-1}\left(\frac{x}{\sqrt{R^2-x^2}}\right) + \frac{x(R^2-x^2)^{3/2}}{4} + \frac{3R^2x\sqrt{R^2-x^2}}{8}\right)\Bigg|_{-R}^{R}$$

$$= \frac{2}{3V}\left(\frac{3}{8}\pi R^4\right) = \frac{3R}{8}.$$

Therefore the centroid is $(0, 0, 3R/8)$.

Problems 33-40: Assume the indicated solid has uniform density

For a cube with edge $a$, find the moment of inertia about an edge

Picture the cube lying with an edge each on the $x$-, $y$-, and $z$-axis. We find the moment of inertia about the edge on the $x$-axis. The distance from a point $(x, y, z)$ to this edge is $\sqrt{y^2+z^2}$; we integrate the square of this distance over the whole cube to get the moment of inertia $I$.

$$I = \int_0^a\int_0^a\int_0^a y^2 + z^2\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^a\int_0^a y^2a + a^3/3\,\mathrm{d}y\mathrm{d}x$$

$$= \int_0^a \frac{2a^4}{3}\,\mathrm{d}x = \frac{2a^5}{3}.$$

Problems 41-44: Find the volume of the given solids.

Bounded by $z = 2x^2 + y^2$ and $z = 12 - x^2 - 2y^2$

The intersection of these two curves projects onto the $xy$-plane as the circle $x^2 + y^2 = 4$. To find the volume we just integrate:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int 2x^2 + y^{2\,12-x^2-2y^2} \mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 12 - 3x^2 - 3y^2 \,\mathrm{d}y\mathrm{d}x$$

$$= \int_{-2}^{2} (12y - 3x^2y - y^3)\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} = 4 \int_{-2}^{2} (4 - x^2)^{3/2} \,\mathrm{d}x$$

$$= 4\left(6\sin^{-1}(x/2) + \frac{x(4-x^2)^{3/2}}{4} + \frac{3x\sqrt{4-x^2}}{2}\right)\Bigg|_{-2}^{2} = 24\pi.$$

**Section 13.7: Integration in Cylindrical and Spherical Coordinates**

Problems 1-20: Solve the following problems using integration in cylindrical coordinates.

Find the volume and centroid of solid bounded by $z = 4$ and $z = r^2$.

There are no restrictions on $\theta$, so $\theta$ can range from 0 to $2\pi$. On our region, $r \leq 2$, so $0 \leq r \leq 2$. Therefore the volume of this solid is

$$V = \int_0^2 \int_0^{2\pi} \int_{r^2}^4 r \,\mathrm{d}z\mathrm{d}\theta\mathrm{d}r = \int_0^2 \int_0^{2\pi} 4r - r^2 \,\mathrm{d}\theta\mathrm{d}r$$

$$= \int_0^2 8\pi r - 2\pi r^2 \,\mathrm{d}r = 8\pi.$$

Since this region is symmetric around the $z$-axis, we conclude that $\overline{x} = \overline{y} = 0$. We compute

$$\overline{z} = \frac{1}{V} \int_0^2 \int_0^{2\pi} \int_{r^2}^4 rz \,\mathrm{d}z\mathrm{d}\theta\mathrm{d}r = \frac{1}{8\pi} \int_0^2 \int_0^{2\pi} 8r - \frac{r^5}{2} \,\mathrm{d}\theta\mathrm{d}r$$

$$= \frac{1}{8\pi} \int_0^2 16\pi r - \pi r^5 \,\mathrm{d}r = \frac{8}{3}.$$

Therefore the volume is $8\pi$ and the centroid is $(0, 0, 8/3)$.

Find volume and centroid of region bounded by $z = 0$ and $z = 9 - x^2 - y^2$.

Over this region, $0 \leq r \leq 3$, and $\theta$ can range from 0 to $2\pi$. Since $9 - x^2 - y^2 = 9 - r^2$, the volume is

$$V = \int_0^3 \int_0^{2\pi} \int_0^{9-r^2} r \,\mathrm{d}z\mathrm{d}\theta\mathrm{d}r = \int_0^3 \int_0^{2\pi} 9r - r^3 \,\mathrm{d}\theta\mathrm{d}r$$

$$= \int_0^3 18\pi r - 2\pi r^3 \,\mathrm{d}r = \frac{81\pi}{2}.$$

Since this region is symmetric around the $z$-axis, $\overline{x} = \overline{y} = 0$. We compute

$$\overline{z} = \frac{1}{V} \int_0^3 \int_0^{2\pi} \int_0^{9-r^2} rz \,\mathrm{d}z\mathrm{d}\theta\mathrm{d}r = \frac{1}{81\pi} \int_0^3 \int_0^{2\pi} 81r - 18r^3 + r^5 \,\mathrm{d}\theta\mathrm{d}r$$

$$= \frac{2\pi}{81\pi}\int_0^3 81r - 18r^3 + r^5 \,\mathrm{d}r = \frac{2}{81}\left(\frac{81r^2}{2} - \frac{9r^4}{2} + \frac{r^6}{6}\right)\Big|_0^3 = 3.$$

Therefore the volume is $81\pi/2$ and the centroid is $(0, 0, 3)$.

<u>Problems 21-40:</u> Solve the following problems using integration in spherical coordinates.

<u>Find the centroid of a homogeneous solid hemisphere of radius $a$.</u>

Consider the hemisphere to be $\rho \leq a$, $\phi \leq \pi/2$. We know the volume of the hemisphere is $V = \frac{2}{3}\pi a^3$. Since the hemisphere is symmetric around the $z$-axis, $\overline{x} = \overline{y} = 0$. Therefore we must only calculate $\overline{z}$:

$$\overline{z} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos(\phi))(\rho^2\sin(\phi))\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/2}\frac{a^4}{4}\sin(\phi)\cos(\phi)\,\mathrm{d}\phi\mathrm{d}\theta = \frac{1}{V}\int_0^{2\pi}\left(\frac{a^4}{8}\sin^2(\phi)\right)\Big|_0^{\pi/2}\mathrm{d}\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\frac{a^4}{8}\,\mathrm{d}\theta = \frac{1}{V}\frac{\pi a^4}{4} = \frac{3a}{8}.$$

Therefore the centroid is $(0, 0, 3/8)$.

<u>Find volume and centroid of solid inside $\rho = a$ and above $r = z$.</u>

The equation $r = z$ is equivalent to $\rho\sin(\phi) = \rho\cos(\phi)$, or $\phi = \pi/4$. Therefore our volume is

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^a \rho^2\sin(\phi)\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{a^3}{3}\sin(\phi)\,\mathrm{d}\phi\mathrm{d}\theta$$

$$= \int_0^{2\pi}\frac{-a^3\cos(\phi)}{3}\Big|_0^{\pi/4}\mathrm{d}\theta = \int_0^{2\pi}\left(1 - \frac{\sqrt{2}}{2}\right)\frac{a^3\pi}{3}\,\mathrm{d}\theta = \left(2 - \sqrt{2}\right)\frac{a^3\pi}{3}.$$

Now that we know the volume, we can compute the coordinates of the centroid.

$$\overline{x} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\int_0^a (\rho\sin(\phi)\cos(\theta))\rho^2\sin(\phi)\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\frac{a^4}{4}\sin^2(\phi)\cos(\theta)\,\mathrm{d}\phi\mathrm{d}\theta = \frac{1}{V}\int_0^{2\pi}\frac{a^4}{4}\cos(\theta)\left(\frac{\phi}{2}-\frac{\sin^2(\phi)}{4}\right)\Bigg|_0^{\pi/4}\mathrm{d}\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\frac{a^4}{4}\left(\frac{\pi}{8}-\frac{1}{4}\right)\cos(\theta)\,\mathrm{d}\theta = \frac{1}{V}\frac{a^4}{4}\left(\frac{\pi}{8}-\frac{1}{4}\right)\sin(\theta)\Bigg|_0^{2\pi} = 0.$$

A similar computation shows that

$$\overline{y} = \frac{1}{V}\frac{a^4}{4}\left(\frac{\pi}{8}-\frac{1}{4}\right)(-\cos(\theta))\Bigg|_0^{2\pi} = 0.$$

Finally,

$$\overline{z} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a}(\rho\cos(\phi))\rho^2\sin(\phi)\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\frac{a^4}{4}\sin(\phi)\cos(\phi)\,\mathrm{d}\phi\mathrm{d}\theta = \frac{1}{V}\int_0^{2\pi}\frac{-a^4\cos^2(\phi)}{8}\Bigg|_0^{\pi/4}\mathrm{d}\theta$$

$$= \frac{1}{V}\int_0^{2\pi}\frac{a^4}{16}\,\mathrm{d}\theta = \frac{a^4\pi}{8V} = \frac{3a}{8(2-\sqrt{2})} = \frac{3a}{16}(2+\sqrt{2}).$$

Therefore the volume is $V = \frac{(2-\sqrt{2})\pi a^3}{3}$ and the centroid is

$$(0, 0, 3a(2+\sqrt{2})/16).$$

**Section 13.9: Change of Variables in Multiple Integrals**

Problems 1-6: Solve for $x$ and $y$ in terms of $u$ and $v$. Then compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

$u = x + y \qquad v = x - y$

It is easy to see that $x = (u+v)/2$ and $y = (u-v)/2$. Therefore the partial derivatives of $x$ and $y$ are $x_u = x_v = y_u = 1/2$ and $y_v = -1/2$. Therefore the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2}.$$

$u = xy \qquad v = y/x$

It is easy to see that $x^2 = u/v$ and $y^2 = uv$, so $x = \sqrt{u/v}$ and $y = \sqrt{uv}$. Then the partial derivatives are $x_u = \frac{1}{2\sqrt{uv}}$, $x_v = -\frac{\sqrt{u}}{2v^{3/2}}$, $y_u = \frac{\sqrt{v}}{2\sqrt{u}}$, and $y_v = \frac{\sqrt{u}}{2\sqrt{v}}$. Then the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v^{3/2}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}.$$

Problems 7-18: Find the area of the following regions by change of variables.

$R$ bounded by $x + y = 1$, $x + y = 2$, $2x - 3y = 2$, and $2x - 3y = 5$

Let $u = x + y$ and $v = 2x - 3y$, so $x = (3u + v)/5$ and $y = (2u - v)/5$. Then the partial derivatives of $x$ and $y$ are $x_u = 3/5$, $x_v = 1/5$, $y_u = 2/5$, and $y_v = -1/5$, so

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{3}{5} \cdot \frac{-1}{5} - \frac{1}{5} \cdot \frac{2}{5} = \frac{-1}{5}.$$

Because of the way we defined $u$ and $v$, to get the area, we integrate the absolute value of the Jacobian over $1 \leq u \leq 2$ and $2 \leq v \leq 5$. Therefore the area is

$$A = \int_1^2 \int_2^5 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}v\mathrm{d}u = \int_1^2 \int_2^5 \frac{1}{5} \,\mathrm{d}v\mathrm{d}u = \frac{3}{5}.$$

<u>$R$ bounded by $x = y$, $2x = y$, $xy = 1$, and $xy = 2$</u>

Let $u = xy$ and $v = y/x$, so that, as we have already seen, $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$. To find the area of this region, we must integrate over $1 \leq u \leq 2$ and $1 \leq v \leq 2$. Therefore the area is

$$A = \int_1^2 \int_1^2 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v = \int_1^2 \frac{1}{2v} \,\mathrm{d}v = \frac{\ln 2}{2}.$$

<u>Problem 19:</u> Change to spherical coordinates to show that, for $k > 0$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2} \cdot e^{-k(x^2+y^2+z^2)} \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{2\pi}{k^2}.$$

Our region of integration is all of three-dimensional space, so if we integrate over this region in spherical coordinates, we will let $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $0 \leq \rho \leq \infty$. We already know from the book that $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin(\phi)$. Also, the integrand above becomes $\rho e^{-k\rho^2}$ in spherical. Therefore the integral is

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \rho^3 \sin(\phi) e^{-k\rho^2} \,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin(\phi) \left. \frac{-(k\rho^2+1)e^{-k\rho^2}}{2k^2} \right|_0^{\infty} \mathrm{d}\phi\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{\sin(\phi)}{2k^2} \,\mathrm{d}\phi\mathrm{d}\theta = \int_0^{2\pi} \frac{1}{k^2} \,\mathrm{d}\theta = \frac{2\pi}{k^2}.$$