Volume using Double Integrals

Example 1: Find the volume of the region in $\mathbb{R}^3$ that is under the paraboloid $z = x^2 + 2y^2$, above the xy - plane and below $z = 3$.

Solution: The region is shown below. The planes $z = 0$ and $z = 3$ are colored blue..

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ContourPlot3D[{z == x^2 + 2 y^2, z == 0, z == 3}, {x, -3, 3}, {y, -3, 3}, {z, 0, 3.5}]
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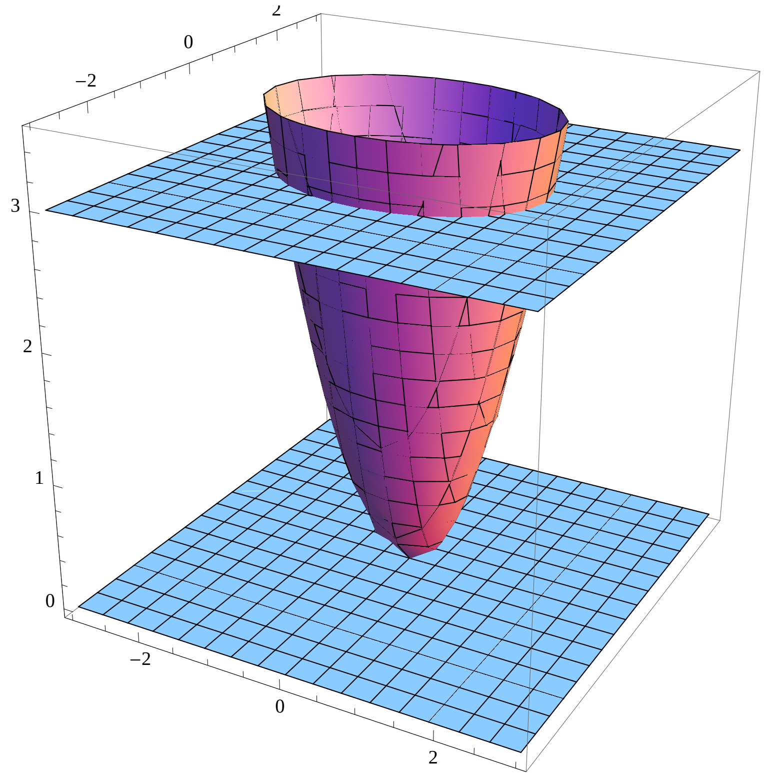

The integral to set up is, informally, $\int_R (\text{Top} - \text{Bottom})\, dA$. The region R is the region in the plane over which the integration takes place. How do we determine the region of integration? We note that the integral has to be evaluated over the region in the xy - plane which "supports" the volume. It's easy to see that the region to integration over is the region under the elliptical "disk" that lies under the intersection of the paraboloid with the plane at $z = 3$. The level curve is $x^2 + 2y^2 = 3$ and the graph of the region bounded by this ellipse is

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ContourPlot[x^2 + 2 y^2 == 3, {x, -2, 2}, {y, -2, 2}]
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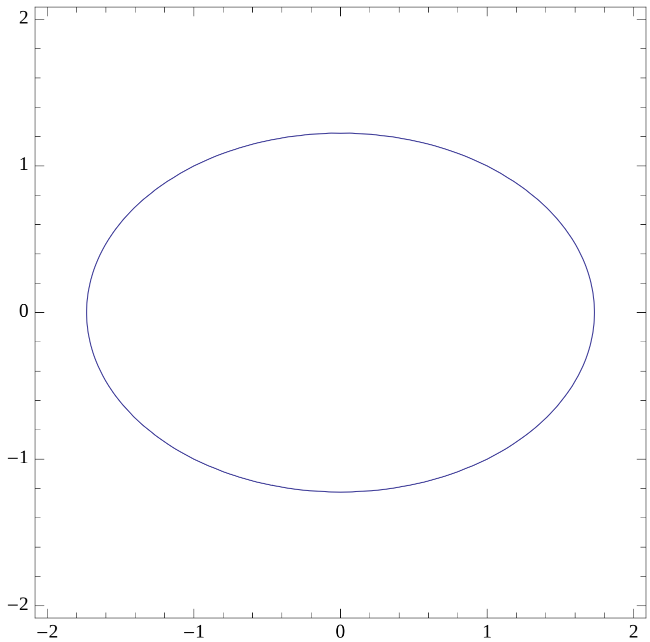


We can set up the integral over this elliptical region as a *y*-simple region:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{\frac{3-x^2}{2}}}^{\sqrt{\frac{3-x^2}{2}}} x^2 + 2\,y^2 \, dy \, dx$$

This integral can be evaluated as follows. The inner integral is

$$\int_{-\sqrt{\frac{3-x^2}{2}}}^{\sqrt{\frac{3-x^2}{2}}} \left(x^2 + 2\,y^2\right) dy$$

$$\frac{1}{3} \sqrt{6 - 2\,x^2}\,\left(3 + 2\,x^2\right)$$

Hence, we have to integrate this function over *x* and the limits for *x*:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{3} \sqrt{6 - 2\,x^2}\,\left(3 + 2\,x^2\right) dx$$

$$\frac{9\,\pi}{2\,\sqrt{2}}$$

This number, then, is the volume of the region.

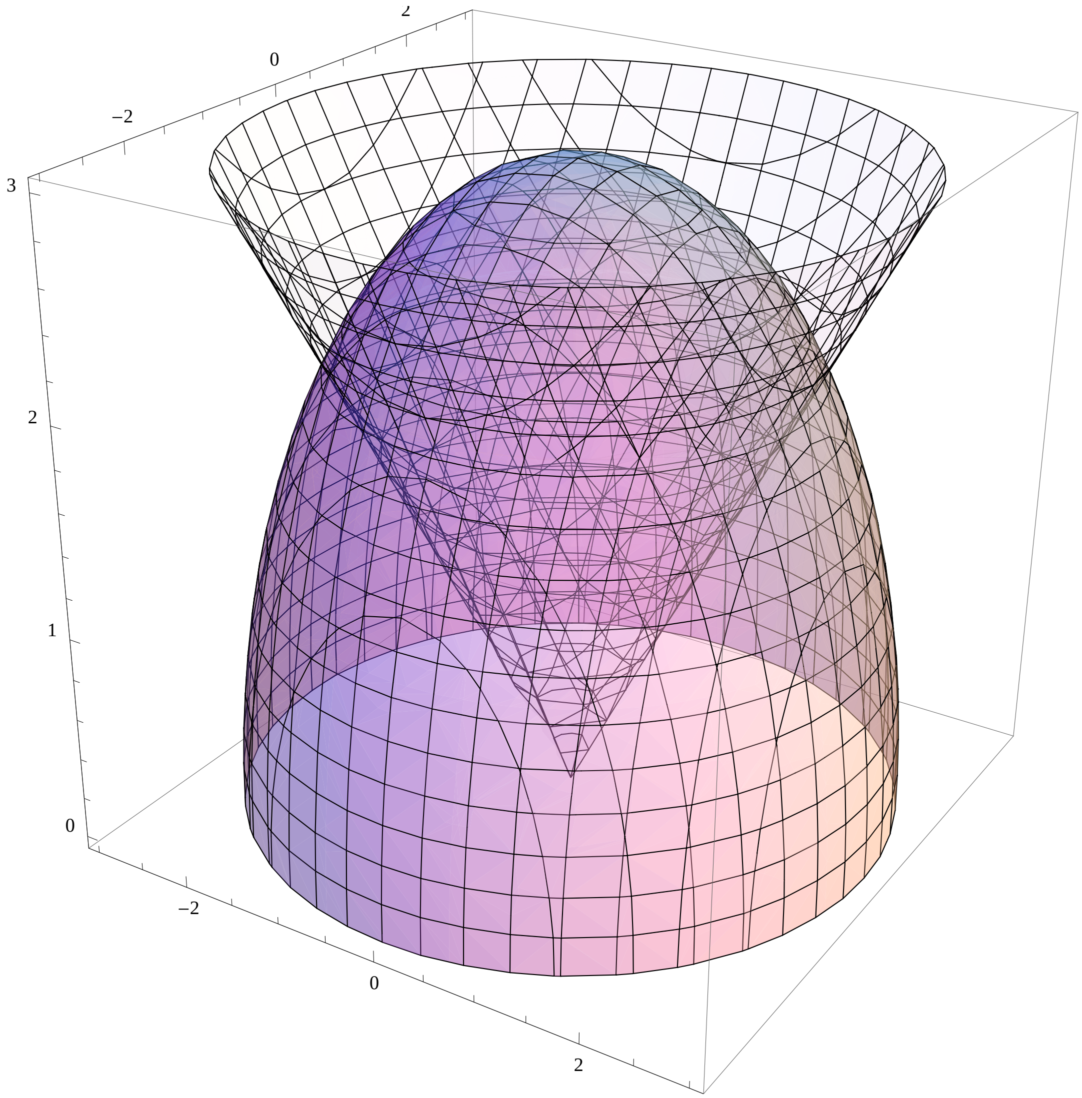
NOTE: We could have exploited the symmetry of the 3-dimensional region to write the volume as

$$4 \int_0^{\sqrt{3}} \int_0^{\sqrt{\frac{3-x^2}{2}}} x^2 + 2\,y^2 \, dy \, dx$$

Example 2: Set up the integral for the volume of the region inside the sphere of radius 3 centered at the origin and above the upper nappe of the cone $z^2 = x^2 + y^2$.

Solution: The region is shown below:

```
ContourPlot3D[{z^2 == x^2 + y^2, x^2 + y^2 + z^2 == 9}, {x, -3, 3},
 {y, -3, 3}, {z, 0, 3}, ContourStyle → {Opacity[0.2], Opacity[0.7]}]
```



We again set up an integral of the form $\int_R (\text{Top} - \text{Bottom})\, dA$. The top surface for our is the sphere and the bottom surface for our region is the cone. Hence Top − Bottom = $\sqrt{9 - x^2 - y^2} - \sqrt{x^2 + y^2}$. The region of integration is again determined by looking at the intersecion of the surfaces. Here, we have an intersection along the circle $x^2 + y^2 = \frac{9}{2}$. The region of integration in the *xy*-plane is this disk of radius $\frac{3}{\sqrt{2}}$. The double integral is

$$\int_{\frac{-3}{\sqrt{2}}}^{\frac{3}{\sqrt{2}}} \int_{-\sqrt{\frac{9}{2}-x^2}}^{\sqrt{\frac{9}{2}-x^2}} \left(\sqrt{9 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dy\, dx$$

As you can see, the integral is nearly impossible to evaluate. In fact, the inner integral is:

$$\int_{-\sqrt{\frac{9}{2}-x^2}}^{\sqrt{\frac{9}{2}-x^2}} \left(\sqrt{9 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dy$$

$$\text{If}\Bigg[\left(\sqrt{\frac{-9+2x^2}{-9+x^2}} \notin \text{Reals} \,||\, \text{Re}\left[\sqrt{\frac{9-2x^2}{18-2x^2}}\right] \le 1\right) \&\&$$

$$\left(\text{Im}\left[\frac{x}{\sqrt{\frac{9}{2}-x^2}}\right] \le -1 \,||\, \text{Im}\left[\frac{x}{\sqrt{\frac{9}{2}-x^2}}\right] \ge 1 \,||\, \text{Re}\left[\frac{x}{\sqrt{18-4x^2}}\right] \ne 0\right),$$

$$-\left(-9+x^2\right) \text{ArcCot}\left[\frac{3}{\sqrt{9-2x^2}}\right] - x^2\, \text{ArcCoth}\left[\frac{3}{\sqrt{9-2x^2}}\right],$$

$$\text{Integrate}\left[2\sqrt{\frac{9}{2}-x^2}\left(\sqrt{9-x^2-\left(-\sqrt{\frac{9}{2}-x^2}+2y\sqrt{\frac{9}{2}-x^2}\right)^2} - \sqrt{x^2+\left(-\sqrt{\frac{9}{2}-x^2}+2y\sqrt{\frac{9}{2}-x^2}\right)^2}\right),\right.$$

$$\{y, 0, 1\}, \text{Assumptions} \to \,!\left(\left(\sqrt{\frac{9-2x^2}{18-2x^2}} \notin \text{Reals} \,||\, \text{Re}\left[\sqrt{\frac{9-2x^2}{18-2x^2}}\right] \le 1\right) \&\&\right.$$

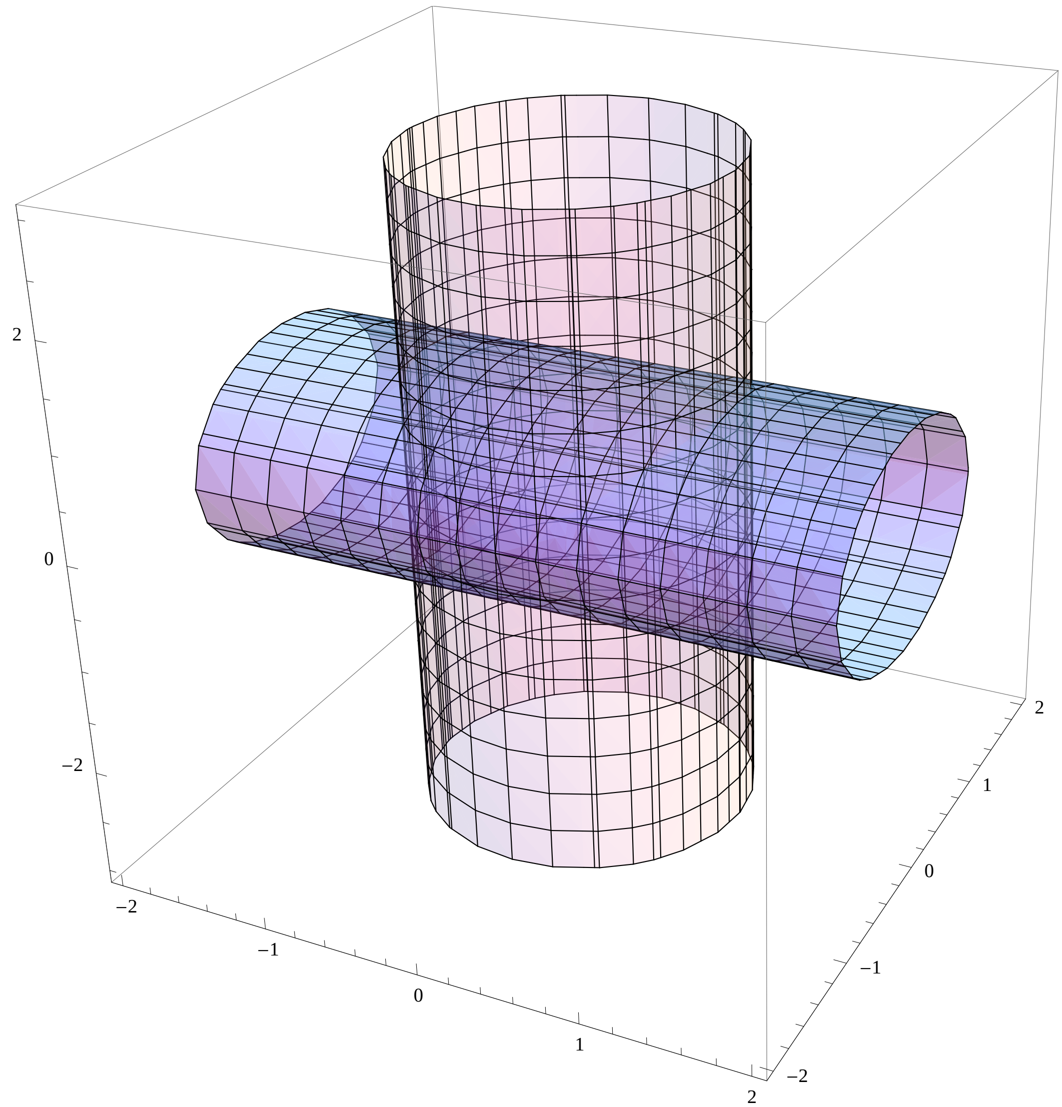
$$\left.\left.\left(\text{Im}\left[\frac{x}{\sqrt{\frac{9}{2}-x^2}}\right] \le -1 \,||\, \text{Im}\left[\frac{x}{\sqrt{\frac{9}{2}-x^2}}\right] \ge 1 \,||\, \text{Re}\left[\frac{x}{\sqrt{\frac{9}{2}-x^2}}\right] \ne 0\right)\right)\right]\Bigg]$$

We'd have to evaluate the integral of hte appropriate part of this output to get the volume. It turns out that there is an easy way to evaluate this integral once we know polar coordinates (or spherical coordinates).

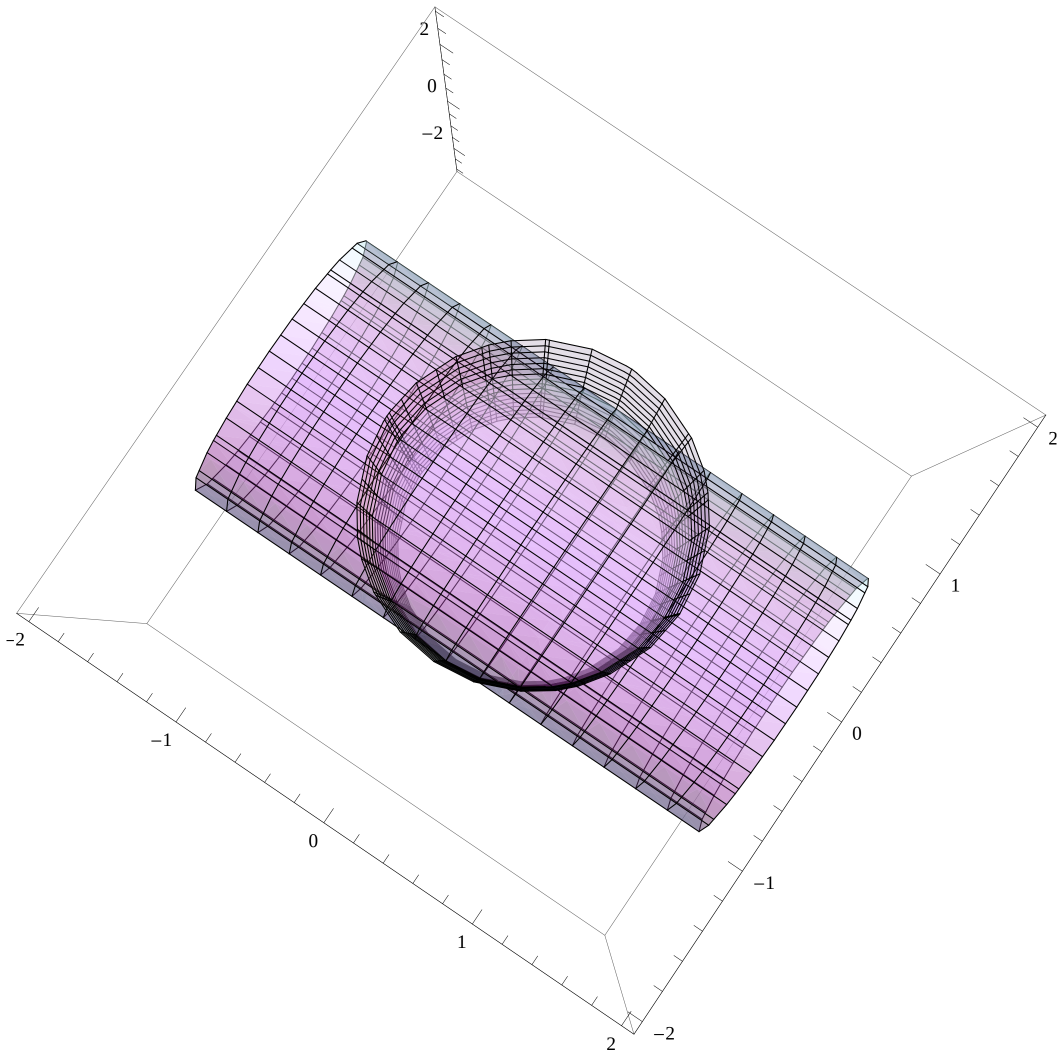
Example 3: Find the volume of the region of intersection of the cylinder $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

Solution: The region of integration is shown below

```
ContourPlot3D[{x^2 + y^2 == 1, y^2 + z^2 == 1}, {x, -2, 2},
 {y, -2, 2}, {z, -3, 3}, ContourStyle → {Opacity[0.4], Opacity[0.7]}]
```



The top and bottom of the region is formed by the cylinder $y^2 + z^2 = 1$. Hence the function that describes the top of the region is $z = \sqrt{1 - y^2}$ and the bottom is $z = -\sqrt{1 - y^2}$. The region of integration is the unit circle in the xy-plane; look straight down on the intersection of the cylinders:



We therefore have, for our integral,

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(2\sqrt{1-y^2}\right)dy\,dx=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}2\sqrt{1-y^2}\;dx\,dy$$

Which of these integrals is easier? Clearly, the second one seems to be. Let's evaluate it from inside out, as is always the case for an iterated integral. We have, for the *x*-intetgral,

$$\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}2\sqrt{1-y^2}\;dx$$

$$4\left(1-y^2\right)$$

The *y*-integral is easy:

$$\int_{-1}^{1} 4\left(1-y^2\right) dy$$

$$\frac{16}{3}$$

Note that the other integral is, on the inside,

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(2\sqrt{1-y^2}\right) dy$$

$$2\,\text{If}\left[\text{Re}\left[\sqrt{1-x^2}\right] \le 1 \;||\; \sqrt{1-x^2} \notin \text{Reals},\ \sqrt{x^2-x^4} + \text{ArcSin}\left[\sqrt{1-x^2}\right],\ \text{Integrate}\left[\sqrt{1-y^2},\ \left\{y, -\sqrt{1-x^2}, \sqrt{1-x^2}\right\},\ \text{Assumptions} \to\ !\left(\text{Re}\left[\sqrt{1-x^2}\right] \le 1 \;||\; \sqrt{1-x^2} \notin \text{Reals}\right)\right]\right]$$

This is not so easy. We can make *Mathematica* evaluate the double integral directly, however, and the result is:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(2\sqrt{1-y^2}\right) dy\, dx$$

$$\frac{16}{3}$$

It's good to see that we get the same answer as above.