COMS 3003A
HW 6

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1. Translate the following statements into the language $\mathcal{L}_{\mathcal{A}}$ of arithmetic:
   (a) Multiplication is commutative.
   (b) There exists a multiplicative identity.
   (c) The does not exist largest natural number.
   (d) The number 0 is the smallest natural number.
   (e) There are infinitely many primes.
   (f) Every even number greater than 2 is a sum of two primes (this is Goldbach's conjecture).
   (g) There are infinitely many pairs of prime numbers n and m such that $n - m = 2$ (this is a twin primes conjecture).
2. To define a (in this case, binary) predicate letter $<$ in the language $\mathcal{L}_{\mathcal{A}}$ of arithmetic means to write a formula with the matching number (in these case, two, say x and y) of free variables that is true if and only if $x < y$ is true. Define the following predicate letters in the language of arithmetic:
   (a) $<$ (binary);
   (b) $\leqslant$ (binary);
   (c) $>$ (binary);
   (d) $\geqslant$ (binary);
   (e) *Prime* (unary).
3. Translate into the language $\mathcal{L}_{\mathcal{S}}$ of set theory the following statements:
   (a) If every element of x is an element of y and vice versa, then x and y are the same set.
   (b) The set $x \cap y$ contains precisely those elements that are both in x and in y.
   (c) For every set x, there exists a set whose elements are precisely subsets of x.
4. Consider a first-order language whose only non-logical symbol is a binary function symbol f. Write formulas that state the following:
   (a) f is an injection;
   (b) f is a surjection;
   (c) f is a bijection.

5. You have been put in charge of drawing up the schedule for a basketball league. The league involves eight teams, each of which has to play each of the other seven teams exactly two times: one at home and once away. Think of a reasonable language for this situation. What constants would you need? Do you need any predicate symbols? Do you need any function symbols? Using your language, express the following statements:

   (a) Two teams in the league have the same hometown.

   (b) No team plays two games on the same day.

   (c) Each team plays every other team exactly twice: once at home and once at the other team's hometown.

6. For each of the following formulas, find a model where the formula is true and a model where the formula is false:

   (a) $\forall x\, R(x, x)$;

   (b) $\forall x \forall y\, (R(x, y) \to R(y, x))$;

   (c) $\forall x \forall y \forall z\, (R(x, y) \land R(y, z) \to R(x, z))$;

   (d) $\forall x \forall y\, (R(x, y) \to \exists z\, (R(x, z) \land R(z, y)))$;

   (e) $\exists x\, P(x) \to \forall x\, P(x)$;

   (f) $\forall x \exists y\, R(x, y)$;

   (g) $\exists x \forall y R(y, x)$;

   (h) $\forall x \exists y R(x, y) \to \exists x \forall y R(y, x)$;

   (i) $\forall x \forall y \forall z\, (R(x, y) \land R(y, z) \to R(x, z)) \land \forall x \exists y R(x, y) \land \neg R(x, x)$.