TUTORIAL 1.6 SOLUTIONS

1. Table where $*$ is multiplication.

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Check $e^2 = a^2 = b^2 = c^2 = e$. This is a K-4 group.

2. Let $T = \{hk \mid h \in H, k \in K\}$ and G abelian.
$T \neq \emptyset$ since $e = ee \in T$ and $T \subseteq G$ since G is closed and $h, k \in G \to hk \in G$.

$$\begin{aligned} h_1k_1, h_2k_2 \in T \ &\Rightarrow \ (h_1k_1)(h_2k_2)^{-1} = (h_1k_1)(k_2^{-1}h_2^{-1}) \\ &= \ h_1h_2^{-1}k_1k_2^{-1} \in T \quad \text{since } G \text{ is abelian.} \\ & \quad \ \therefore T \leq G. \end{aligned}$$

3. $H = \{x \in G \mid x^2 = e\}$ and G abelian.
$H \neq \emptyset$ since $e^2 = e \in H$.
By definition of H, it is clear that $H \subseteq G$.

$$\begin{aligned} x_1, x_2 \in H \ &\Rightarrow \ x_1^2 = x_2^2 = e \\ &\Rightarrow \ (x_1x_2^{-1})^2 = (x_1x_2^{-1})(x_1x_2^{-1}) \\ &= \ (x_1x_2^{-1})(x_2x_1^{-1}) = e \qquad (\text{since } x^2 = e \Rightarrow x^{-1} = x \text{ and } G \text{ abelian}) \\ &\Rightarrow \ x_1x_2^{-1} \in H \quad \therefore H \leq G. \end{aligned}$$

4. $H \neq \emptyset$, $\quad |H| = k$, $\quad H \subseteq G$ such that H is closed under $*$. Say $h \in H$ then
$\langle h \rangle = \{h^r \mid r \in \mathbb{Z}\} \subseteq H$, H closed under $*$.
But $|H| = k$ so $h^{r_1} = h^{r_2}$ for some $r_1, r_2 \in \mathbb{Z}$ and $h^{r_1 - r_2} = e \in H$.
Further, say $r_1 - r_2 = n$ then $h^n = e = h(h^{n-1}) = (h^{n-1})h$.
So $h^{-1} = h^{n-1}$ for all $h \in H \quad \therefore H \leq G$.

5. $H_a = \{x \in G \mid xa = ax\} = \{x \in G \mid x = axa^{-1}\}$, so $H_a \subseteq G$.
$aea^{-1} = aa^{-1} = e$ so $e \in H_a$, so $H_a \neq \emptyset$.

$$\begin{aligned} x, y \in H_a &\Rightarrow x = axa^{-1}, y = aya^{-1} \\ &\Rightarrow xy^{-1} = (axa^{-1})(aya^{-1})^{-1} \\ &= axy^{-1}a^{-1} \quad \Rightarrow xy^{-1} \in H_a \\ &\therefore H_a \leq G. \end{aligned}$$

6. (a) $S \subseteq G$ and
   $H_S = \{x \in G \mid xs = sx \ \forall s \in S\} = \{x \in G \mid x = sxs^{-1}, \ \forall s \in S\}$
   so $H_S \subseteq G$.
   $e = ses^{-1} \quad \forall s \in S$ so $e \in H_S$ and $H_S \neq \emptyset$.

$$\begin{aligned} x, y \in H_S &\Rightarrow sxs^{-1} = x, \ sys^{-1} = y \quad \forall s \in S \\ &\Rightarrow xy^{-1} = (sxs^{-1})(sys^{-1})^{-1} = sxy^{-1}s^{-1} \quad \forall s \in S \\ &\Rightarrow xy^{-1} \in H_S. \\ &\therefore H_S \leq G. \end{aligned}$$

   (b) $G \subseteq G$ so $H_G = \{x \in G \mid xy = yx \quad \forall y \in G\} = Z(G)$ centre of G is a subgroup of G by part (a).
   Hence H_G is an belian group.

   For any $x, z \in H_G \ \Rightarrow xz = zx \ \Rightarrow H_G$ abelian.

7. (a) $a \equiv a$ since $aa^{-1} = e \in H \leq G$.
   (b)

$$\begin{aligned} a \equiv b &\Rightarrow ab^{-1} \in H \\ &\Rightarrow (ab^{-1})^{-1} \in H \leq G \\ &\Rightarrow ba^{-1} \in H \\ &\Rightarrow b \equiv a. \end{aligned}$$

   (c)

$$\begin{aligned} a \equiv b \text{ and } b \equiv c &\Rightarrow ab^{-1} \in H \text{ and } bc^{-1} \in H \\ &\Rightarrow (ab^{-1})(bc^{-1}) \ \in H \leq G. \\ &\Rightarrow a(b^{-1}b)c^{-1} = ac^{-1} \in H \leq G. \end{aligned}$$

8. Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Assume (i) $HK \leq G$.

$$\begin{aligned} hk \in HK &\Rightarrow (hk)^{-1} \Rightarrow k^{-1}h^{-1} \in HK \\ &\Rightarrow k^{-1}h^{-1} = h_1k_1 \in HK \\ &\Rightarrow hk = k_1^{-1}h_1^{-1} \in KH \\ &\Rightarrow HK \subseteq KH. \end{aligned}$$

$$\begin{aligned} kh \in KH &\Rightarrow k \in K \text{ and } h \in H \Rightarrow k^{-1} \in K \text{ and } h^{-1} \in H \\ &\Rightarrow h^{-1}k^{-1} \in HK \Rightarrow (kh)^{-1} \in HK \\ &\Rightarrow kh \in HK \end{aligned}$$

As assumed HK is a group $\Rightarrow KH \subseteq HK$.
Therefore $KH = HK$.

Assume (ii) $HK = KH$
$e \in KH$ so $KH \neq \emptyset$.

$$\begin{aligned} k_1h_1, k_2h_2 \in KH &\Rightarrow (k_1h_1)(k_2h_2)^{-1} \\ &\Rightarrow (k_1h_1)(h_2^{-1}k_2^{-1}) \\ &\Rightarrow k_1h_3k_2^{-1} = k_3h_3 \in KH. \end{aligned}$$

Therefore $KH \leq G$.

By similar argument $KH \leq G \Rightarrow KH = HK$ and $KH = HK \Rightarrow HK \leq G$.

9. $G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{R}, a \neq 0 \right\}$.
$G \neq \emptyset$ since $I_2 \in G$.

$$A, B \in G \Rightarrow A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \text{ where } a \neq 0 \text{ and } B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \text{ where } c \neq 0.$$

Then $B^{-1} = \dfrac{1}{c^2}\begin{pmatrix} c & -d \\ 0 & c \end{pmatrix}$. Hence

$$\begin{aligned} AB^{-1} &= \frac{1}{c^2}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} c & -d \\ 0 & c \end{pmatrix} \\ &= \frac{1}{c^2}\begin{pmatrix} ac & -ad+bc \\ 0 & ac \end{pmatrix}. \end{aligned}$$

But $ac \neq 0$ since $a \neq 0$ and $c \neq 0 \;\Rightarrow\; AB^{-1} \in G$. Therefore $G \leq GL_2(\mathbb{R})$.

10. (a) (1 2 3)(9 8 5)(4 7 6).

    (b) (7 2 3 1 9 8 4)(5 6).

    (c) (1 2)(3 7 8 5 6)(9 4).

11. $S_3 = \{e, (1\ 2\ 3), (1\ 3\ 2), (1\ 2)(1\ 3)(2\ 3)\}$.
    But $(1\ 2\ 3) = (1\ 3)(1\ 2)$ and $(1\ 3\ 2) = (1\ 2)(1\ 3)$.
    Therefore, e, (1 2 3), (1 3 2) are even permutations in S_3.

12. e, (1 2 3), (1 2 4), (1 3 2), (1 3 4), (1 4 2), (1 4 3),
    (2 3 4), (2 4 3), (1 3)(2 4), (1 4)(2 3), (1 2)(3 4)
    are even permutations in S_4.