

## 2 Free Vibration of Single-Degree-of-Freedom Systems

Vibrations (i.e. the exchange of potential and kinetic energy) requires oscillatory motion that may repeat itself regularly or irregularly. A motion that is repeated on time intervals is called periodic motion. If this motion has a single frequency and amplitude it is called simple harmonic motion and represents the most basic form of oscillatory motion as depicted in figure 1. For a 1-DOF system simple harmonic motion is defined as a periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

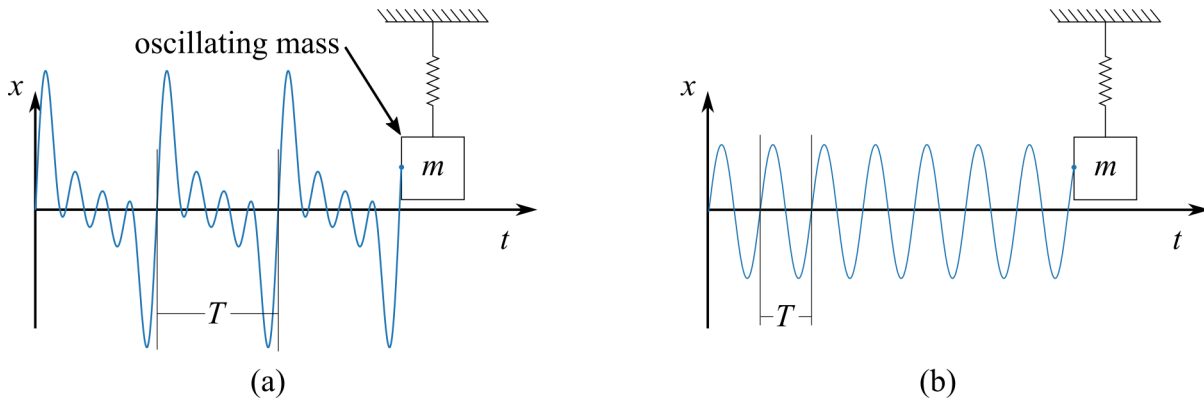


Figure 1: Oscillatory motion for a single degree of freedom system showing (a) periodic motion; and (b) simple harmonic motion.

Given the nature of simple harmonic motion, constant amplitude and frequency, the wave starting at the origin  $O$  can be modeled at a point on the end of a vector with length  $A$  rotating at a constant angular velocity  $\omega_n$  where the angle from the origin of the vector is  $\phi$ , defined as  $\phi = \omega t$ . Where  $\omega$  is the lowercase Greek letter Omega and  $\phi$  is the lowercase Coptic letter phi. This is similar to a Greek phi ( $\phi$ ) and either can be used in this context. The subscript  $n$  on  $\omega$  denotes that this frequency relates to the natural frequency of the system, the only frequency in simple harmonic motion. A visualization of the harmonic motion obtained from projecting the point on the edge of a vector onto the  $\omega_n t$  space is presented in figure 2.

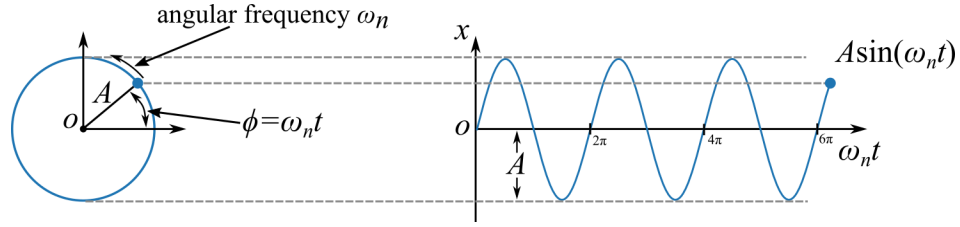


Figure 2: Harmonic motion represented as the projection of a point on the end of a vector moving on a circle. Note the axis  $\omega_n t$ .

## 2.1 Mathematical Modeling of Free Vibration

The Development of a mathematical model for a system under free vibration would enable the practitioner to predict, or model, the vibrating system of interest. Therefore, considering that the following system,

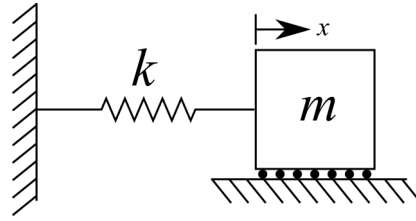


Figure 3: 1-DOF spring-mass system.

can be modeled expressed with the following EOM

$$m\ddot{x}(t) + kx(t) = 0 \quad (1)$$

it becomes prudent to solve this homogeneous ordinary differential equation (ODE) to obtain a model of the vibrating system. The simplest method for solving an ODE is to propose a solution based on observations of a vibrating physical system. Figure 4 reports and annotates the key components from an observation of a vibrating system.

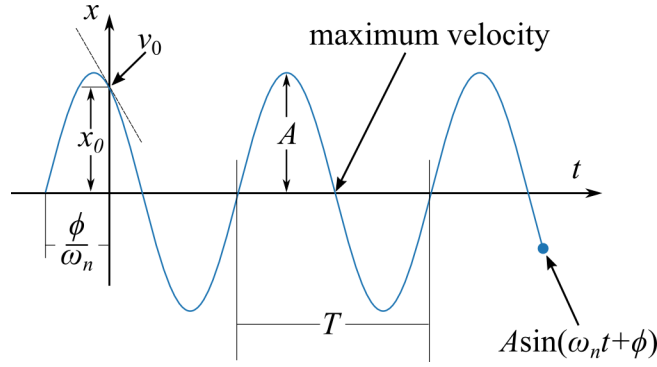


Figure 4: Summary of the temporal response for a 1-DOF system.

where  $x_0$  and  $v_0$  are the displacement and velocity at  $t=0$  (i.e. the initial displacement).

A mathematical expression can now be formulated to represent the observed simple harmonic motion. This expression can be based on the projection of a point on a vector (transposed into the time domain) or assembled from constituent parts as done in what follows. Solving for a location  $x$ , at a time  $t$ ;  $x(t)$ , the various characteristics of the expression can be identified:

### 2.1.1 Solve for the Natural Frequency ( $\omega_n$ ) of the System

- System oscillates  $\rightarrow$  a sin function models this
- System oscillates at different speed  $\rightarrow$  use a parameter to adjust  $\omega_n$  in rad/s.
- Systems have different amplitudes  $\rightarrow$  use a parameter to adjust  $A$  in meters.
- System has different starting points  $\rightarrow$  use a parameter to adjust  $\phi$  in rad.

Using these four constituent components, an equation can be proposed:

$$x(t) = A \sin(\omega_n t + \phi) \quad (2)$$

Take the derivative to get velocity:

$$\dot{x}(t) = A \omega_n \cos(\omega_n t + \phi) \quad (3)$$

Take the derivative again to get acceleration:

$$\ddot{x}(t) = -A \omega_n^2 \sin(\omega_n t + \phi) \quad (4)$$

Substituting  $x$  and  $\ddot{x}$  into the EOM for the considered 1-DOF system ( $m\ddot{x}(t) + kx(t) = 0$ ) yields:

$$m(-A\omega_n^2 \sin(\omega_n t + \phi)) + k(A\omega_n \sin(\omega_n t + \phi)) = 0 \quad (5)$$

Thereafter, dividing both sides by  $A\sin(\omega_n t + \phi)$  results in the expression:

$$-m\omega_n^2 + k = 0 \quad (6)$$

This expression can be rearranged into the more useful standard form:

$$\omega_n = \sqrt{\frac{k}{m}} \quad (7)$$

This form is not an ODE so the equation is solvable and testable. This equation relates the frequency of a system to stiffness and mass. This equation leads to:

$$T = \frac{2\pi}{\omega_n} \quad (8)$$

where  $T$  is the period of oscillations and

$$f_n = \frac{\omega_n}{2\pi} \quad (9)$$

where  $f_n$  is the frequency of the oscillations.

### 2.1.2 Solve for Initial Phase ( $\phi$ ) of the System

The EOM is a second-order ODE so there needs to exist two initial conditions (constants) to solve it. For the systems under considerations, the displacement ( $x$ ) and velocity ( $\dot{x}$  or  $v$ ) at  $t = 0$  are the initial conditions. For simplicity, these are written as

$$x(0) = x_0 \quad (10)$$

$$\dot{x}(0) = v(0) = v_0 \quad (11)$$

Setting the equation to its initial state  $t = 0$ , the equations for displacement and velocity can be simplified to:

$$x(0) = x_0 = A\sin(\omega_n 0 + \phi) = A\sin(\phi) \quad (12)$$

$$\dot{x}(0) = v_0 = A\omega_n \cos(\omega_n 0 + \phi) = A\omega_n \cos(\phi) \quad (13)$$

Thereafter, mathematical meanings for  $\phi$  and  $A$  can be derived. To do this,  $\phi$  can be solved for by rearranging equations 12 and 13 for  $A$ :

$$A = \frac{x_0}{\sin(\phi)} \quad (14)$$

and:

$$A = \frac{v_0}{\omega_n \cos(\phi)} \quad (15)$$

Setting these two equations equal to each other cancels out  $A$  and creates:

$$\frac{x_0 \omega_n}{\sin(\phi)} = \frac{v_0}{\cos(\phi)} \quad (16)$$

therefore:

$$\frac{x_0 \omega_n}{v_0} = \frac{\sin(\phi)}{\cos(\phi)} \quad (17)$$

finally:

$$\phi = \tan^{-1} \left( \frac{x_0 \omega_n}{v_0} \right) \quad (18)$$

### 2.1.3 Solve for Amplitude ( $A$ ) of the System

The amplitude of the vibrating system ( $A$ ) is solved for in a similar manner to  $\phi$  where the expressions for  $x$  and  $\dot{x}$  are solved for at  $t = 0$  and rearranged as to isolate  $\phi$ . This operations results in the the equations:

$$\sin(\phi) = \frac{x_0}{A} \quad (19)$$

and:

$$\cos(\phi) = \frac{v_0}{\omega_n A} \quad (20)$$

From these equations a value for  $\phi$  can be obtained knowing that  $\sin(\phi)^2 + \cos(\phi)^2 = 1$ . Therefore:

$$\left( \frac{x_0}{A} \right)^2 + \left( \frac{v_0}{\omega_n A} \right)^2 = 1 \quad (21)$$

multiplying each expression by 1 (also expressed as  $\frac{\omega_n}{\omega_n}$ ), gives the equation:

$$\left( \frac{\omega_n}{\omega_n} \right)^2 \left( \frac{x_0}{A} \right)^2 + 1 \left( \frac{v_0}{\omega_n A} \right)^2 = 1 \times 1 \quad (22)$$

which becomes:

$$\left(\frac{\omega_n x_0}{\omega_n A}\right)^2 + \left(\frac{v_0}{\omega_n A}\right)^2 = 1 \quad (23)$$

Further simplification is obtained by multiplying each side by  $(\omega_n A)^2$  to obtain:

$$\omega_n^2 x_0^2 + v_0^2 = A^2 \omega_n^2 \quad (24)$$

Solving for A, this expression rearranges to:

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \quad (25)$$

#### 2.1.4 Response for Simple Harmonic Motion

The time-varying displacement of a 1-DOF vibrating system under free response is expressed by the equation  $x(t) = A \sin(\omega_n t + \phi)$ . Substituting in the expressions for A and  $\phi$  results in:

$$x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \sin\left(\omega_n t + \left(\tan^{-1}\left(\frac{x_0 \omega_n}{v_0}\right)\right)\right) \quad (26)$$

This equation provides a mathematical solutions that relates displacement of the mass to the initial conditions  $x_0$  and  $v_0$ . The solution is considered a free response because no input is applied after  $t=0$ . The relationship between the initial conditions ( $x_0$  and  $v_0$ ) and the amplitude and phase of the response can be expressed using the Pythagorean theorem,  $a^2 + b^2 = c^2$ , as annotated in figure 5.

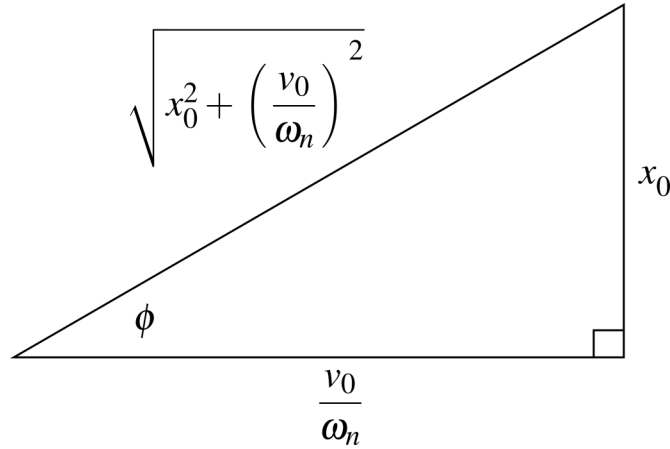


Figure 5: Trigonometric relationship between the initial conditions ( $x_0$  and  $v_0$ ), amplitude A, and phase  $\phi$  for free vibration of a 1-DOF system.

**Example 2.1** A vehicle wheel, tire, and suspension can be modeled as a SDOF spring and mass. The mass is measured to be 300 kg, its frequency of oscillation is observed to be 10 rad/sec. What is the stiffness of the wheel assembly?

**Solution:** Knowing  $k = m\omega_n^2 = (300 \text{ kg})(10 \text{ rad/s})^2 = 30 \text{ KN/m}$

**Example 2.2** Consider the following 1-DOF system, where  $k = 857.8 \text{ N/m}$  and  $m = 49.2 \times 10^{-3}$ , calculate the natural frequency in rad/s and Hz. Also find the period of oscillations and the maximum displacement if the spring is initially displaced 10 mm with no initial velocity.

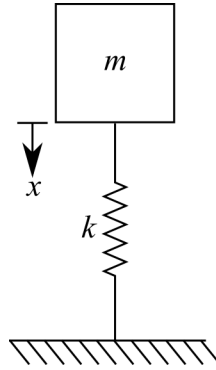


Figure 6: 1-DOF spring-mass system.

**Solution:**

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132 \text{ rad/sec} \quad (27)$$

In Hz, this is:

$$f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz} \quad (28)$$

The period is:

$$T = \frac{2\pi}{\omega_n} = 0.0476 \text{ s} \quad (29)$$

The maximum displacement will happen when  $\sin(\omega_n t + \phi) = 0$ , therefore, the value of  $A$  is the maximum displacement. For an undamped system,  $A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}$ ,

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \frac{\sqrt{132^2 0.01^2 + 0^2}}{132} = 0.01 \text{ m} \quad (30)$$

## 2.2 General Solution for Vibrating Systems

The EOM for a vibrating system has many solutions and can be expressed in various forms including a general solution. These forms offer different mathematical approaches to solve the same 1-DOF spring-mass system and relate to each other through Euler's equations.

**Review 2.1** Vibration analysis uses complex numbers to solve the EOM's differential equation. In this text the imaginary number is termed  $j$  (sometimes referred to as  $i$ ): such that:

$$j = \sqrt{-1} \quad (31)$$

and:

$$j^2 = -1 \quad (32)$$

a general complex number.  $x$  can be expressed as

$$x = a + bj \quad (33)$$

here,  $a$  is referred to as the real number and  $b$  is the imaginary part of the number  $x$ . Such complex numbers can be represented in the complex plane, also called a Argand plot. The absolute value or modules is defined as  $|x|$  presented on the complex plot.



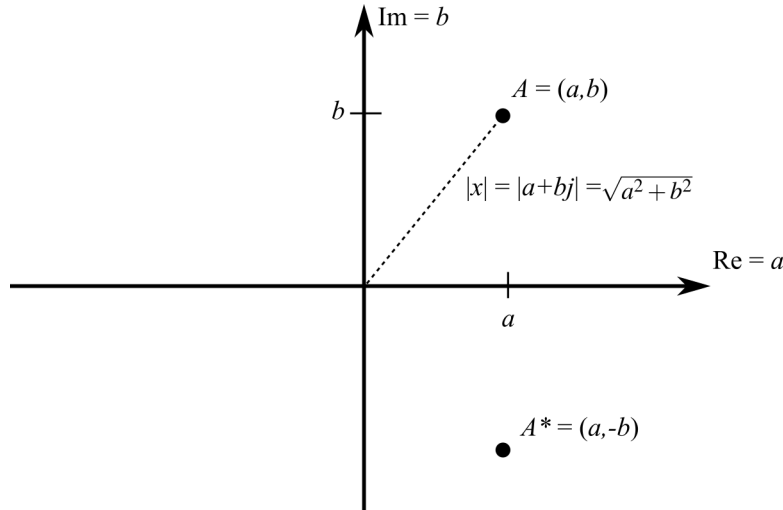


Figure 7: A conjugate pair of numbers ( $A$  and  $A^*$ ) represented on the complex plane.

$A$  and  $A^*$  prime are complex conjugate pairs. In mathematics, the complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. In other words, a conjugate pair is  $a + bj$  and  $a - bj$ .

**Definition** - con·ju·gate (adjective): Coupled, connected, or related.

**Review 2.2** Euler's (pronounced oy-ler) formula, named after Swiss engineer and mathematician Leonhard Euler (1707-1783), is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number  $x$ ,

$$e^{j\psi} = \cos(\psi) + j\sin(\psi) \quad (34)$$

where  $j = \sqrt{-1}$ . This equation can also be expressed as:

$$e^{-j\psi} = \cos(\psi) - j\sin(\psi) \quad (35)$$

This can be expressed in terms of polar coordinates as:

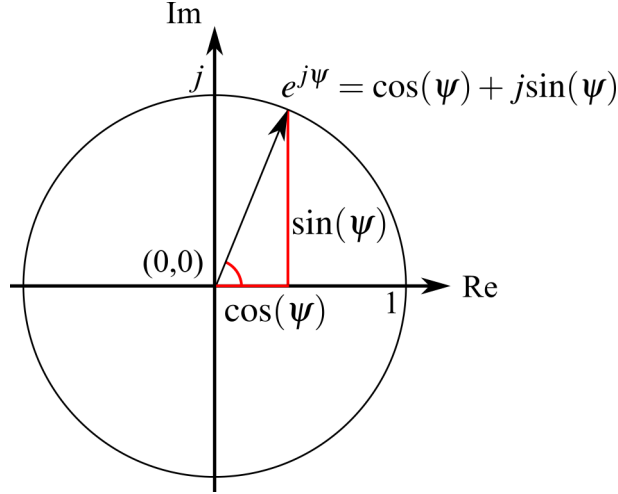


Figure 8: Euler's formula illustrated on the unit circle in the complex plane.

### 2.2.1 Formulating the General solution for a 1-DOF Spring-Mass System

We can also solve the following EOM as an elementary differential equation:

$$m\ddot{x} + kx = 0 \quad (36)$$

in a more analytical manner using the theory of elementary differential equations. To do this is the form:

$$x(t) = ae^{\lambda t} \quad (37)$$

is assumed where,  $a$  and  $t$  are nonzeros constants that need to be determined. Using successive differentiation, the assumed solution becomes:

$$\dot{x}(t) = \lambda ae^{\lambda t} \quad (38)$$

and

$$\ddot{x}(t) = \lambda^2 ae^{\lambda t} \quad (39)$$

therefore,  $m\ddot{x}(t) + kx(t) = 0$  becomes:

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \quad (40)$$

Next, the above expressions is divide by  $ae^{\lambda t}$  to obtain the characteristic equation:

$$m\lambda^2 + k = 0 \quad (41)$$

This can be done because  $ae^{\lambda t}$  is never zero, therefore, the expressions is never divide by zero. The quadratic formula gives us:

$$\lambda = \pm \sqrt{-\frac{k}{m}} = \pm \sqrt{\frac{k}{m}}j = \pm \omega_n j \quad (42)$$

remember that  $\omega_n = \sqrt{\frac{k}{m}}$ . Notice that the  $\pm$  tells us there are two solutions to this problem. So, putting  $\lambda$  back into the assumed solution results in two solutions (one positive, one negative):

$$x(t) = a_1 e^{+\omega_n j t} \quad (43)$$

and

$$x(t) = a_2 e^{-\omega_n j t} \quad (44)$$

As these solutions only consider, and are only valid for, linear systems, the sum of the solutions is also a solution. This simplification results in:

$$x(t) = a_1 e^{+\omega_n j t} + a_2 e^{-\omega_n j t} \quad (45)$$

where  $a_1$  and  $a_2$  are complex valued constants of integration. This equation derived using Euler's formula is equivalent to the  $A \sin(\omega_n t + \phi)$ . To recover the previously assumed solution, the knowledge that  $a_1$  and  $a_2$  are complex conjugate pairs and as such the magnitude can be expressed as  $a_1 = a_2$  is leveraged. Using Euler's polar notation,  $a_1$  and  $a_2$  can be expressed as

$$a_1 = a_2 = a e^{j\psi} \quad (46)$$

where  $a$  and  $\psi$  are real numbers, the equation becomes:

$$x(t) = a e^{j(\omega_n t + \psi)} + a e^{-j(\omega_n t + \psi)} \quad (47)$$

this becomes:

$$x(t) = a(e^{j(\omega_n t + \psi)} + e^{-j(\omega_n t + \psi)}) \quad (48)$$

Remembering Euler's equations from before, this becomes:

$$x(t) = a(\cos(\omega_n t + \psi) + j\sin(\omega_n t + \psi) + \cos(\omega_n t + \psi) - j\sin(\omega_n t + \psi)) \quad (49)$$

combining the "cos" terms and canceling out the "sin" terms this becomes:

$$x(t) = 2a \cdot \cos(\omega_n t + \psi) \quad (50)$$

This is equivalent to  $x(t) = A \sin(\omega_n t + \phi)$  considering that  $A = 2a$  and knowing  $\sin(\phi) = \cos(\phi + \psi)$ . Euler's equation states:

$$x(t) = a_1 e^{+\omega_n j t} + a_2 e^{-\omega_n j t} \quad (51)$$

we can expand this into the form:

$$x(t) = a_1(\cos(\omega_n t) + j\sin(\omega_n t)) + a_2(\cos(\omega_n t) - j\sin(\omega_n t)) \quad (52)$$

using trigonometric functions. This equates to:

$$x(t) = (a_1 + a_2) \cdot \cos(\omega_n t) + (a_1 - a_2)j \cdot \sin(\omega_n t) \quad (53)$$

As  $x(t)$  is always real,  $A_1$  and  $A_2$  can be defined as:

$$A_1 = (a_1 + a_2) \quad (54)$$

and

$$A_2 = (a_1 - a_2)j \quad (55)$$

Lastly, as the general solution is written as:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \quad (56)$$

This is the general solution for the EOM ( $m\ddot{x} + kx = 0$ ) of the considered oscillating system where  $A_1$  and  $A_2$  are defined as:

$$A = \sqrt{A_1^2 + A_2^2} \quad (57)$$

and

$$\phi = \tan^{-1} \left( \frac{A_1}{A_2} \right) \quad (58)$$

These are obtained from a trigonometric relationship, similar to that used before:

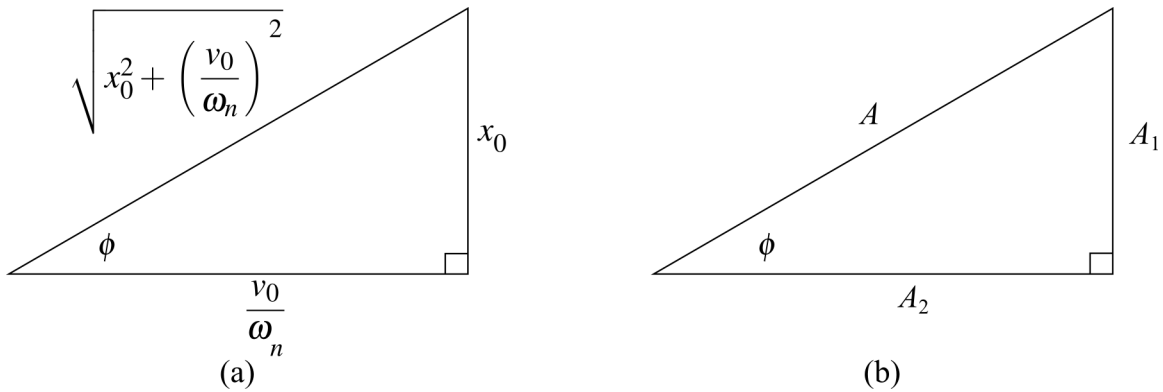


Figure 9: Trigonometric relationship between the initial conditions, amplitude, and phase, for free vibration of a 1-DOF system expressed with: (a) variables for initial conditions; and (b) generic variables  $A_1$  and  $A_2$ .

again,  $A$  and  $\phi$  are:

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \quad (59)$$

$$\phi = \tan^{-1}\left(\frac{x_0 \omega_n}{v_0}\right) \quad (60)$$

### 2.2.2 Solution of 1-DOF System in Three Forms

Form one, for  $m\ddot{x} + kx = 0$  subject to nonzero initial conditions can be written as:

$$x(t) = a_1 e^{+\omega_n j t} + a_2 e^{-\omega_n j t} \quad (61)$$

where  $a_1$  and  $a_2$  are complex terms. Form two is:

$$x(t) = A \sin(\omega_n t + \phi) \quad (62)$$

while form three is:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \quad (63)$$

where  $A$ ,  $\phi$ ,  $A_1$ , and  $A_2$ , are all real-valued constants. Each set of constants can be related to each other by:

$$A = \sqrt{A_1^2 + A_2^2} \quad \phi = \tan^{-1}\left(\frac{A_1}{A_2}\right) \quad (64)$$

$$A_1 = (a_1 + a_2) \quad A_2 = (a_1 - a_2)j \quad (65)$$

$$a_1 = \frac{A_1 - A_2 j}{2} \quad a_2 = \frac{A_1 + A_2 j}{2} \quad (66)$$

Which follow from trigonometric identities and the Euler's formulas.

**Example 2.3** Using the general solution:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \quad (67)$$

Calculate the values of  $A_1$  and  $A_2$  in terms of their initial conditions  $x_0$  and  $v_0$ .

**Solution:** Knowing the following for  $x$  and  $\dot{x}$ :

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \quad (68)$$

$$\dot{x}(t) = -A_1 \omega_n \sin(\omega_n t) + A_2 \omega_n \cos(\omega_n t) \quad (69)$$

Now apply the initial conditions,  $x(0) = 0$  and  $v(0) = 0$ , this yields:

$$x(0) = x_0 = A_1 \quad (70)$$

$$\dot{x}(0) = v_0 = A_2 \omega_n \quad (71)$$

Solving for  $A_1$  and  $A_2$  shows us:

$$A_1 = x_0, \text{ and } A_2 = \frac{v_0}{\omega_n} \quad (72)$$

thus:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \quad (73)$$

## 2.3 Damping

The response of a spring-mass system predicts that a system will oscillate indefinitely. However, we know that this is not true from observing real-world solutions. So based on real-world observations and mathematical conveniences, we need to add a term that will remove “energy” from the system with time. To do this the idea of the ideal dashpot is introduced. A linear dashpot is diagrammed in figure 10 and is a mechanical device that resists motion via viscous friction and therefore converts the mechanical energy of the system into thermal energy that is dissipated.

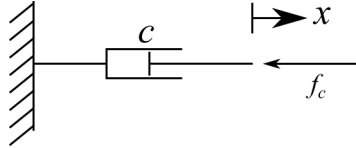


Figure 10: Schematic of a linear dashpot showing the damping force ( $f_c$ ) acting in the opposite direction of the displacement ( $x$ ).

Just as spring forms a physical model of the cause vibration, through its storage and release of energy, a dashpot (sometimes called a damper) forms a physical model for dissipating energy. Dashpots create a resisting or damping force that acts opposite to the direction of travel (as annotated in figure 10) and is proportional to the velocity. Therefore, the damping forces  $f_c$  can be computed as:

$$f_c = c\dot{x} \quad (74)$$

the constant  $c$ , called the damping coefficient, has the units of kg/s. Dashpots are a mathematical representation of viscous dampers installed in automobiles, aircraft, structures, and other mechanical devices. However, all systems have inherent damping not just systems with physical dampers. The spring-mass system can be used as a representation of real-world systems with inherent damping as demonstrated by the rubber engine mount depicted in figure 11.

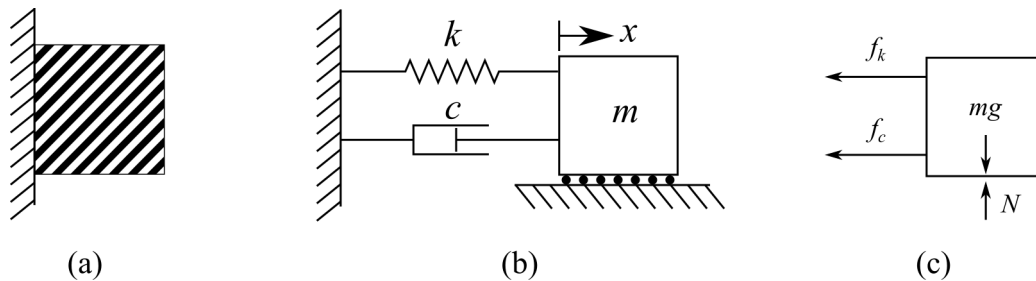


Figure 11: Modeling of a rubber engine mount as a spring-dashpot-mass model showing (a) the rubber engine mount; (b) idealized model of the rubber month; and (c) the FBD of the idealized model

Depending on the amount of damping present in a system, the temporal response of the system will represent itself in various way, as represented in figure 12. To reiterate, an undamped case will oscillates around the equilibrium and does not decay. If a limited amount of damping is present in a system it will oscillates around the equilibrium and slowly decay with time to the equilibrium position, this is termed underdamped. If an excessive amount of damping is present, the system will not oscillate but decay directly to the equilibrium position, this is termed the overdamped case. Lastly, there exists a special case that results in the system converging as quickly to possible to the equilibrium without oscillations; this cases is termed the critically damped case. Furthermore, the amount of damping required to obtain a critically damped system is the damping value that the underdamped and overdamped cases for a specific system. To recap, the key types of damping are:

- **Undamped** - Oscillates around the equilibrium and does not decay.
- **Underdamped** - Oscillates around the equilibrium and slowly decays and is the most common case.
- **Overdamped** - Does not pass the equilibrium position and is a simple a decay with no oscillation.

- **Critically damped** - provides the quickest approach to zero amplitude for a damped oscillator.

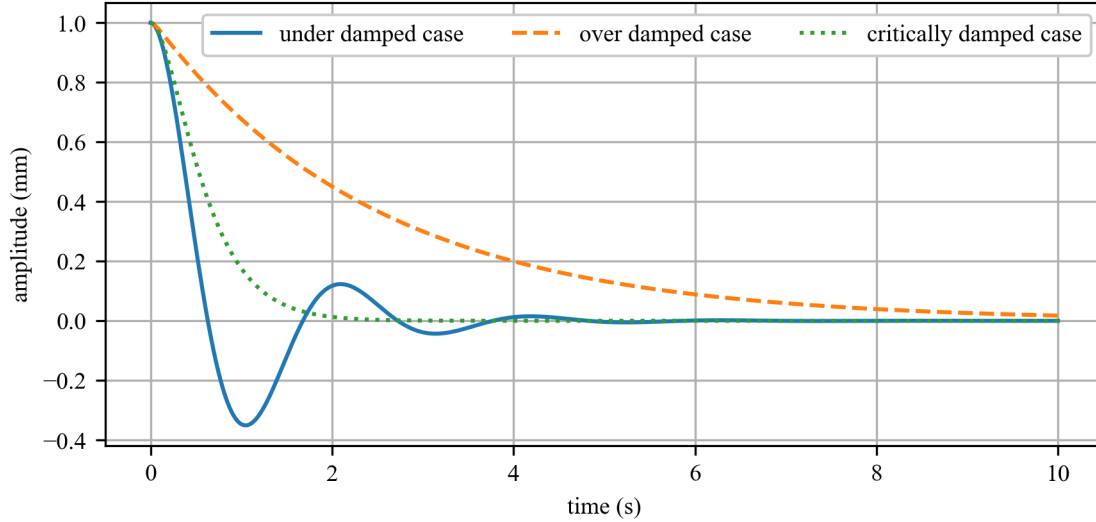


Figure 12: Temporal responses for the three types of damping: underdamped, over damped, and critically damped.

### 2.3.1 Modeling Vibrating Systems with Damping

The spring-mass system of chapter 1 can be expanded to a spring-dashpot-mass system that considers the damping component of the system. A mathematical model of the spring-dashpot-mass system can be developed for the case present in figure 13. Using the FBD for the system, it can conclude that the EOM for this system:

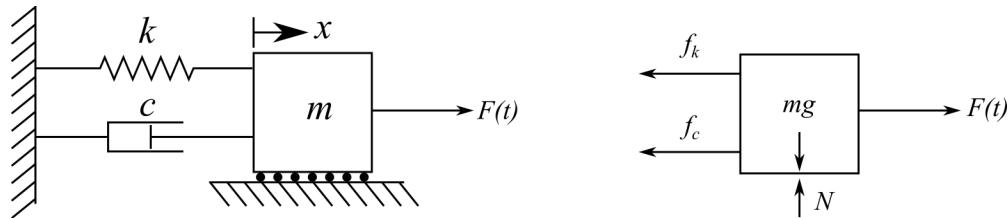


Figure 13: Spring-dashpot-mass model showing: (a) a schematic of the system; and (b) the FBD of the system.

is:

$$m\ddot{x}(t) = -f_c - f_k \quad (75)$$



Rearranging into standard form and concerting forces into parameters  $c$  and  $k$  results in:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \quad (76)$$

This system is subject to the same initial conditions as before,  $x(0) = x_0$  and  $\dot{x}(0) = v_0$ . Again, choosing to model it this way for convinces, so let's solve it in a similar manner to the EOM without damping. Again, assume the solution:

$$x(t) = ae^{\lambda t} \quad (77)$$

here,  $a$  and  $t$  are nonzeros constants that need to be determined. Using successive differentiation, we get:

$$\dot{x}(t) = \lambda ae^{\lambda t} \quad (78)$$

and

$$\ddot{x}(t) = \lambda^2 ae^{\lambda t} \quad (79)$$

therefore,  $m\ddot{x} + c\dot{x} + kx = 0$  becomes:

$$m\lambda^2 ae^{\lambda t} + c\lambda ae^{\lambda t} + kae^{\lambda t} = 0 \quad (80)$$

Now we divide by  $ae^{\lambda t}$  to obtain the **characteristic equation**:

$$m\lambda^2 + c\lambda + k = 0 \quad (81)$$

We can do this because  $ae^{\lambda t}$  is never zero, therefore, we never divide by zero. The quadratic formula gives us:

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{-c}{2m} \pm \frac{1}{2m} \sqrt{c^2 - 4km} \quad (82)$$

Some key points from this equation:

- The  $\pm$  tells us there are two solutions to this problem
- if  $c^2 - 4km < 0$ , system is Underdamped, solutions are complex conjugate pairs
- if  $c^2 - 4km = 0$ , system is critically damped, solutions are equal negative real numbers
- if  $c^2 - 4km > 0$ , system is Overdamped, solutions are distinct negative real numbers

From this, we can see that  $c^2 - 4km = 0$  is a special value, let us define a value for  $c$  that will give us this critical damping number. We will call it the **critical damping coefficient** ( $c_{cr}$ ). So setting the equation as:

$$c_{cr}^2 - 4km = 0 \quad (83)$$

giving us:

$$c_{cr}^2 = 4km \quad (84)$$

next we can derive the function:

$$c_{cr} = 2\sqrt{km} = 2\left(\frac{\sqrt{m}}{\sqrt{m}}\right)\sqrt{km} = 2m\omega_n \quad (85)$$

remember that  $\omega_n = \sqrt{\frac{k}{m}}$  for an undamped system. Next, we generate a non-dimensional number ( $\zeta$ ), pronounced ‘zeta’ that will allow us to distinguish between different types of damping.  $\zeta$  is called the **critical damping ratio**.

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n} \quad (86)$$

Now if we put the  $\zeta$  back into the characteristic equation and resolve using the quadratic equation we get:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \quad (87)$$

From this equation it become clear that  $\zeta$  determines whether the roots are complex or real, this in turn determines the nature of the response of the structure. Listing our possible responses we get: For each damping case, we will have a

damping case	critical damping ratio	radicand	solutions
under damped	$0 < \zeta < 1$	$c^2 - 4km < 0$	complex conjugate pairs
critically damped	$\zeta = 1$	$c^2 - 4km = 0$	equal negative real numbers
over damped	$1 < \zeta$	$c^2 - 4km > 0$	distinct negative real numbers

different solution to the problem.

### 2.3.2 Modeling Underdamped Motion

In the case that  $0 < \zeta < 1$ , a complex conjugate pair of roots are the solutions to the characteristic equation after pulling out a  $\sqrt{-1}$ :

$$\lambda_1 = -\zeta\omega_n + \omega_n\sqrt{1 - \zeta^2}j \quad (88)$$

and:

$$\lambda_2 = -\zeta \omega_n - \omega_n \sqrt{1 - \zeta^2} j \quad (89)$$

Where the  $j$  is pulled out because:

$$\sqrt{1 - \zeta^2} j = \sqrt{(1 - \zeta^2)(-1)} = \sqrt{\zeta^2 - 1} \quad (90)$$

Next, let us “arbitrarily” define:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (91)$$

where  $\omega_d$  is the **damped natural frequency**. Therefore, the equations become:

$$\lambda_1 = -\zeta \omega_n + \omega_d j \quad (92)$$

and:

$$\lambda_2 = -\zeta \omega_n - \omega_d j \quad (93)$$

Again, we have two solutions to a linear problem, so we can combine these into one solution and insert  $\lambda$  into the assumed solution  $ae^{\lambda t}$  to obtain:

$$x(t) = a_1 e^{-\zeta \omega_n t + \omega_d t j} + a_2 e^{-\zeta \omega_n t - \omega_d t j} \quad (94)$$

where  $a_1$  and  $a_2$  are complex valued constants. This can now be simplified into:

$$x(t) = e^{-\zeta \omega_n t} (a_1 e^{\omega_d t j} + a_2 e^{-\omega_d t j}) \quad (95)$$

Using Euler’s equations, (same as before) and choosing:

$$A_1 = (a_1 - a_2) j \quad (96)$$

and

$$A_2 = (a_1 + a_2) \quad (97)$$

The **general form** of this solution is then:

$$x(t) = e^{-\zeta \omega_n t} (A_1 \sin(\omega_d t) + A_2 \cos(\omega_d t)) \quad (98)$$

Recall that for undamped 1-DOF systems we showed

$$x(t) = A \sin(\omega_d t + \phi) = A_1 \sin(\omega_d t) + A_2 \cos(\omega_d t) \quad (99)$$

As  $e^{-\zeta \omega_n t}$  accounts for the damping, our current solution becomes:

$$x(t) = A e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) \quad (100)$$

Now that we have  $x$  and  $\dot{x}$ , we can solve for the boundary conditions  $x_0$  and  $v_0$  by setting  $t = 0$ , we get:

$$x(0) = x_0 = A \sin(\phi) \quad (101)$$

and taking the derivative of  $x(t)$  using the product rule  $(fg)' = f'g + fg'$ , we get:

$$\dot{x}(t) = -\zeta \omega_n A e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) + A e^{-\zeta \omega_n t} \omega_d \cos(\omega_d t + \phi) \quad (102)$$

$$\dot{x}(0) = v_0 = -\zeta \omega_n A \sin(\phi) + A \omega_d \cos(\phi) \quad (103)$$

letting  $A = x_0 / \sin(\phi)$  gives us the equation:

$$\dot{x}(0) = v_0 = -\zeta \omega_n \left( \frac{x_0}{\sin(\phi)} \right) \sin(\phi) + \left( \frac{x_0}{\sin(\phi)} \right) \omega_d \cos(\phi) \quad (104)$$

This simplifies to:

$$\dot{x}(0) = v_0 = -\zeta \omega_n x_0 + x_0 \omega_d \cot(\phi) \quad (105)$$

solving for  $\phi$ :

$$\cot(\phi) = \frac{v_0 + \zeta \omega_n x_0}{x_0 \omega_d} \quad (106)$$

and as the  $\tan(\phi) = 1 / \cot(\phi)$ :

$$\phi = \tan^{-1} \left( \frac{x_0 \omega_d}{v_0 + \zeta \omega_n x_0} \right) \quad (107)$$

Using the trigonometric relationship:

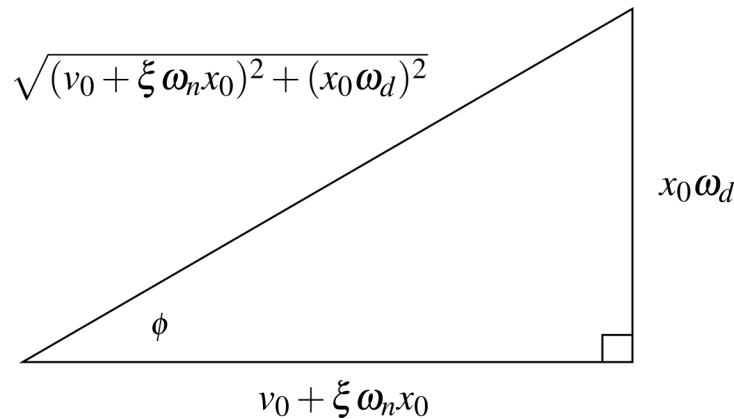


Figure 14: Trigonometric relationship between the initial conditions ( $x_0$  and  $v_0$ ), amplitude  $A$ , and phase  $\phi$  for underdamped motion of a 1-DOF system.

we get:

$$\sin(\phi) = \frac{x_0 \omega_d}{\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}} \quad (108)$$

and applying  $A = x_0 / \sin(\phi)$  we get:

$$A = \frac{\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}}{\omega_d} = \sqrt{x_0^2 + \left( \frac{v_0 + \zeta \omega_n x_0}{\omega_d} \right)^2} \quad (109)$$

Finally, collecting all of our important equations:

- Critical damping coefficient:  $c_{cr} = 2\sqrt{km} = 2m\omega_n$
- Damping ratio:  $\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$
- Damped natural frequency:  $\omega_d = \omega_n \sqrt{1 - \zeta^2}$
- Solution for underdamped system:  $x(t) = Ae^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$ , where:

$$A = \frac{\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}}{\omega_d} \quad \phi = \tan^{-1} \left( \frac{x_0 \omega_d}{v_0 + \zeta \omega_n x_0} \right)$$

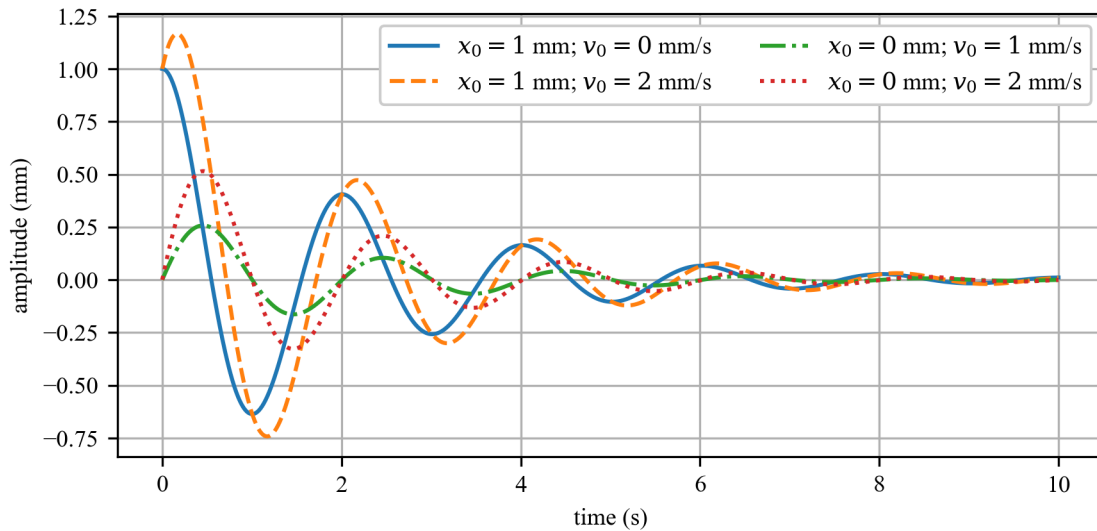


Figure 15: Four example responses for an under damped 1-DOF system ( $\zeta = 0.142$ ) with various initial conditions.

### 2.3.3 Modeling Overdamped Motion

In the case of overdamped systems,  $1 < \zeta$ , the solutions for  $\lambda$  are distinct real roots that are written as:

$$\lambda_1 = -\zeta \omega_n - \omega_n \sqrt{\zeta^2 - 1} \quad (110)$$

and:

$$\lambda_2 = -\zeta \omega_n + \omega_n \sqrt{\zeta^2 - 1} \quad (111)$$

The solution for the EOM using the assumed solution then becomes:

$$x(t) = e^{-\zeta \omega_n t} (a_1 e^{-\omega_n \sqrt{\zeta^2 - 1} t} + a_2 e^{+\omega_n \sqrt{\zeta^2 - 1} t}) \quad (112)$$

This equation represents a non-oscillating response of the system. Again,  $a_1$  and  $a_2$  are solved for using known boundary conditions  $x_0$  and  $v_0$  such that:

$$a_1 = \frac{-v_0 + \left(-\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n x_0}{2\omega_n \sqrt{\zeta^2 - 1}} \quad (113)$$

$$a_2 = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right) \omega_n x_0}{2\omega_n \sqrt{\zeta^2 - 1}} \quad (114)$$

Typical responses for a overdamped system with various initial conditions are shown below:

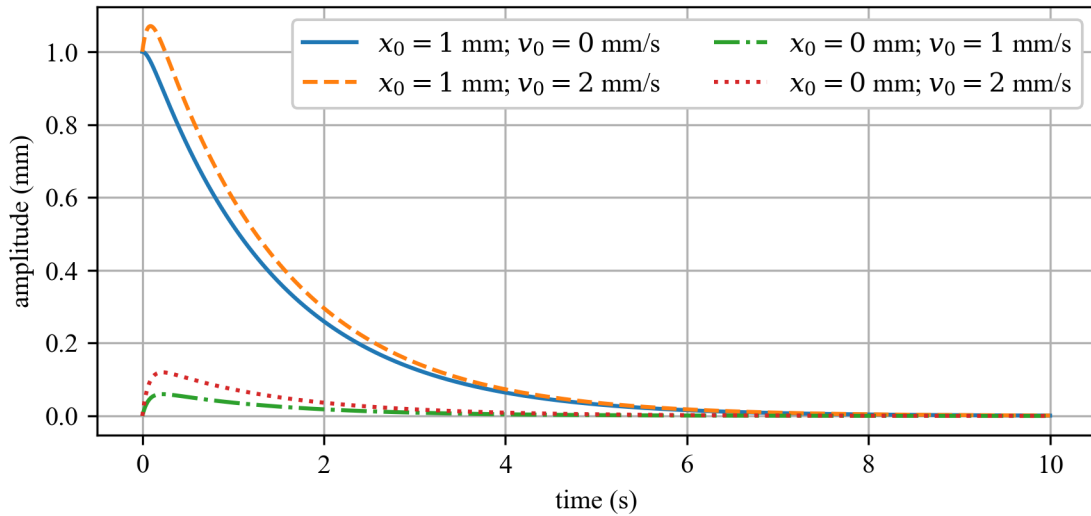


Figure 16: Four example responses for an over damped 1-DOF system ( $\zeta = 2.371$ ) with various initial conditions.

### 2.3.4 Modeling critically damped motion

In the case of critically damped systems,  $\zeta = 1$ , the solutions for  $\lambda$  will be equal negative real numbers, therefore from before:

$$\lambda_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \quad (115)$$

We get:

$$\lambda_1 = \lambda_2 = -\omega_n \quad (116)$$

Because both solutions ( $a_1$  and  $a_2$ ) are the same, we multiply the second solution by  $t$  so the solution for a critically damped system is in the same form as before. The solution for the EOM using the assumed solution then becomes:

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t} \quad (117)$$

This simplifies into:

$$x(t) = (a_1 + a_2 t) e^{-\omega_n t} \quad (118)$$

This equation represents a non-oscillating response of the system. Again,  $a_1$  and  $a_2$  are solved for using known boundary conditions  $x_0$  and  $v_0$  such that:

$$a_1 = x_0 \quad (119)$$

$$a_2 = v_0 + \omega_n x_0 \quad (120)$$

### 2.3.5 Standard Form of the EOM

The EOM for a damped 1-DOF system is written in the form:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \quad (121)$$

In what is known as standard form. First, every term is divided by  $m$  such that:

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = 0 \quad (122)$$

This can be rearranged to the **standard form**:

$$\ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) = 0 \quad (123)$$

### Example 2.4 Example 1

Consider the following 1-DOF system, where  $k = 857.8$  N/m,  $c = 7.8$  kg/s, and  $m = 49.2 \times 10^{-3}$  kg, calculate the damped frequency in rad/s and Hz. What damping case is this system?

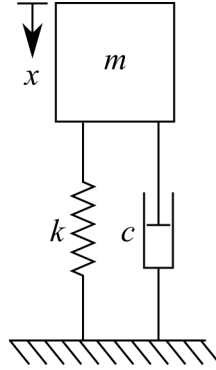


Figure 17: 1-DOF spring-dashpot-mass system.

Calculate the undamped frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132 \text{ rad/s} \quad (124)$$

The systems critical damping value:

$$c_{cr} = 2\sqrt{km} = 2\sqrt{k} = 857.8 \cdot 49.2 \times 10^{-3} = 12.993 \text{ kg/s} \quad (125)$$

And the critical damping ratio:

$$\zeta = \frac{c}{c_{cr}} = \frac{7.8}{12.993} = 0.600 \quad (126)$$

This can also be expressed as 60% damped, this is a underdamped system, and the system will oscillate. Now we can calculate the damped frequency:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \omega_n \sqrt{1 - 0.600^2} = 105.6 \text{ rad/s} \quad (127)$$

Therefore, the system oscillates at 105.6 rad/sec or 16.81 Hz



**Example 2.5** For a damped one DOF system where  $m$ ,  $c$ , and  $k$  are known to be  $m = 1$  kg,  $c = 2$  kg/s, and  $k = 10$  N/m. Calculate the value of  $\zeta$  and  $\omega_n$ . Is the system overdamped, underdamped, or critically damped?

The natural frequency is calculated as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16 \text{ rad/s} \quad (128)$$

The damping can be calculated as:

$$\zeta = \frac{c}{2\omega_n m} = \frac{2}{2\left(\sqrt{\frac{10}{1}}\right)(1)} = \frac{1}{\sqrt{10}} = 0.316 \text{ rad/s} \quad (129)$$

So the damped natural frequency is equal to:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{10} \sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2} = 0.3 \text{ rad/s} \quad (130)$$

As  $0 < \zeta < 1$  the system is underdamped.

**Example 2.6** For the following industrial device consisting of a mass isolated from its fixtures by rubber dampers provide an estimate of the systems oscillating frequency.

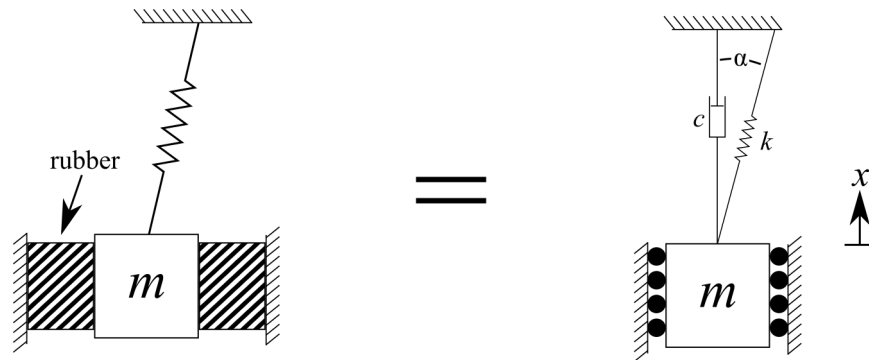
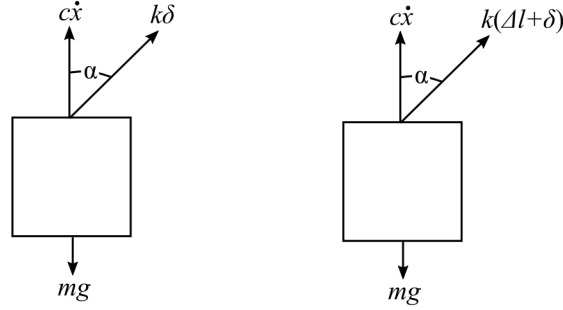


Figure 18: Representing a industrial damper as a spring-dashpot-mass system.

As we only want an estimate of the frequency, we can assume the displacement is small and as such  $\alpha$  of the displaced mass is equal  $\alpha$  of the resting mass. This leads to the FBD for the resting and displaced masses.



$$\sum F_x = mg - k\delta \cos(\alpha) = 0$$

$$\sum F_x = mg - c\dot{x} - k\cos(\alpha)(\Delta l + \delta)$$

Applying newtons second law and combining these equations yields:

$$m\ddot{x} + c\dot{x} + k\Delta l \cos(\alpha) = 0 \quad (131)$$

As we assumed the displacement is small and  $\alpha$  remains unchanged, the assumption that  $\cos(\alpha) = h/l$  is equivalent to  $\cos(\alpha) = x/\Delta l$  is logical (draw out the triangles if needed), therefore the prior equation becomes:

$$m\ddot{x} + c\dot{x} + k\Delta l \frac{x}{\Delta l} = m\ddot{x} + c\dot{x} + kx = 0 \quad (132)$$

as  $\alpha$  and  $x$  are small. Therefore, the frequency can be estimated as:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (133)$$

once the values for the system are known or measured.

## 2.4 Logarithmic decrement

Obtaining the damping coefficient of a system using the logarithmic decrement method

For a vibrating system, the mass ( $m$ ) and stiffness ( $k$ ) can be measured using scales and static deflection tests. However, the damping coefficient ( $c$ ) is a more difficult quantity to determine. From  $k$  and  $m$  we can compute the natural frequency ( $\omega_n$ ) and the critical damping coefficient ( $c_{cr}$ ). Therefore, knowing that the critical damping ratio ( $\zeta$ ) is defined as:

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n} \quad (134)$$

if we calculate  $\zeta$ , we can obtain  $c$  for the system of interest. This is made possible

because  $c_{cr}$  can be calculated from  $k$  and  $m$ . Observing the temporal response for the underdamped system,

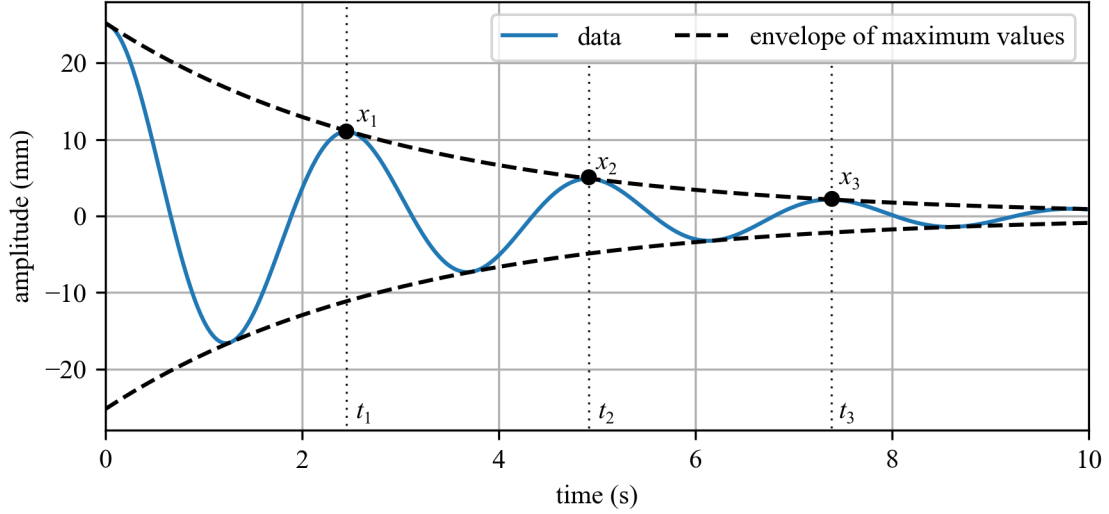


Figure 19: Measuring the peak displacements points in an experimental system with decay caused by damping.

we mark three points of maximum amplitude,  $x_1$ ,  $x_2$ , and  $x_3$  that happen at  $t_1$ ,  $t_2$ , and  $t_3$ , respectively. Considering displacement values for the first two points  $x_1$  and  $x_2$ , separated by a complete period ( $T$ ). Knowing that one cycle is  $2\pi$ , the time period for this complete cycle is given by:

$$t_2 - t_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1 - \zeta^2} \omega_n} \quad (135)$$

where  $\omega_d$  is the damped natural frequency. This is the time period ( $T$ ) of damped oscillations. If derive an equation for the values of the peaks, also called the envelope of maximum values, we get:

$$x_{\text{peaks}} = A e^{-\zeta \omega_n t} \quad (136)$$

Knowing that the system is underdamped,  $A$  can be solved for using the initial conditions  $x_0$  and  $v_0$ , therefore:

$$A = \frac{\sqrt{(v_0 + \zeta \omega_n x_0)^2 + (x_0 \omega_d)^2}}{\omega_d} \quad (137)$$

In terms of  $t_1$  and  $t_2$ , we can express the displacement at these times as:

$$x_1 = A e^{-\zeta \omega_n t_1} \quad (138)$$

and

$$x_2 = Ae^{-\zeta \omega_n t_2} \quad (139)$$

therefore:

$$\frac{x_1}{x_2} = e^{\zeta \omega_n (t_2 - t_1)} \quad (140)$$

However, from before we know that  $t_2 - t_1 = \frac{2\pi}{\sqrt{1-\zeta^2}\omega_n}$ . Therefore, we can express this last equation as:

$$\frac{x_1}{x_2} = e^{\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \quad (141)$$

Next, we take the natural log of both sides to get the logarithmic decrement, denoted by  $\delta$ :

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{x(t_1)}{x(t_1 + T)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad (142)$$

This shows us that the ratio of any two successive amplitudes for an underdamped system, vibrating freely, is constant and is a function of the damping only. Sometimes, in experiments, it is more convenient/accurate to measure the amplitudes after say “ $n$ ” peaks rather than two successive peaks (because if the damping is very small, the difference between the successive peaks may not be significant). The logarithmic decrement can then be given by the equation

$$\delta = \frac{1}{n} \ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{1}{n} \ln\left(\frac{x(t_1)}{x(t_1 + nT)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad (143)$$

Once we use the experimental data to obtain  $\delta$ , and knowing that:

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad (144)$$

we can calculate the value of  $\zeta$ :

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \quad (145)$$

Therefore, having  $\zeta$  we can solve for the coefficient of damping,  $c$ , as:

$$c = \zeta 2\sqrt{km} \quad (146)$$

**Example 2.7** Calculate the damping coefficient of the problem expressed above given that  $m = 3$  kg and  $k = 20$  N/m,  $x_1 = 11$  mm, and  $x_3 = 2$  mm.

First, we solve for  $\delta$ , for  $n = 2$ :

$$\delta = \frac{1}{2} \ln \left( \frac{x_1}{x_3} \right) = \frac{1}{2} \ln \left( \frac{11.03}{2.15} \right) = 0.852 \quad (147)$$

Next, we can calculate  $\zeta$ , as:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.817}{\sqrt{4\pi^2 + 0.817^2}} = 0.134 \quad (148)$$

And lastly:

$$c = \zeta 2\sqrt{km} = 0.134 \cdot 2\sqrt{20 \cdot 3} = 2.08 \text{ kg/s} \quad (149)$$

**Example 2.8** The free response of a 1000-kg automobile with a stiffness of  $k = 400,000$  N/m is observed to be underdamped. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, determine the damping coefficient if the displacement at  $t_1$  is measured to be 2 cm and 0.22 cm at  $t_2$ .

Knowing  $x_1 = 2$  cm and  $x_2 = 0.22$  cm and  $t_2 = T + t_1$ , therefore:

$$\delta = \ln \frac{x_1}{x_2} = \ln \frac{2}{0.22} = 2.207 \quad (150)$$

and:

$$\zeta = \left( \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \right) = \left( \frac{2.207}{\sqrt{4\pi^2 + 2.207^2}} \right) = 0.331 \quad (151)$$

therefore, we can obtain the damping coefficient as

$$c = 2\zeta\sqrt{km} = 2(0.331)\sqrt{400,000 \cdot 1,000} = 13,256 \text{ kg/s} \quad (152)$$