2 Free Vibration of Single-Degree-of-Freedom Systems

Vibrations (i.e. the exchange of potential and kinetic energy) requires oscillatory motion that may repeat itself regularly or irregularly. A motion that is repeated on time intervals is called periodic motion. If this motion has a single frequency and amplitude it is called simple harmonic motion are represents the most basic form of oscillatory motion as depicted in figure 1. For a 1-DOF system simple harmonic motion is defined as a periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

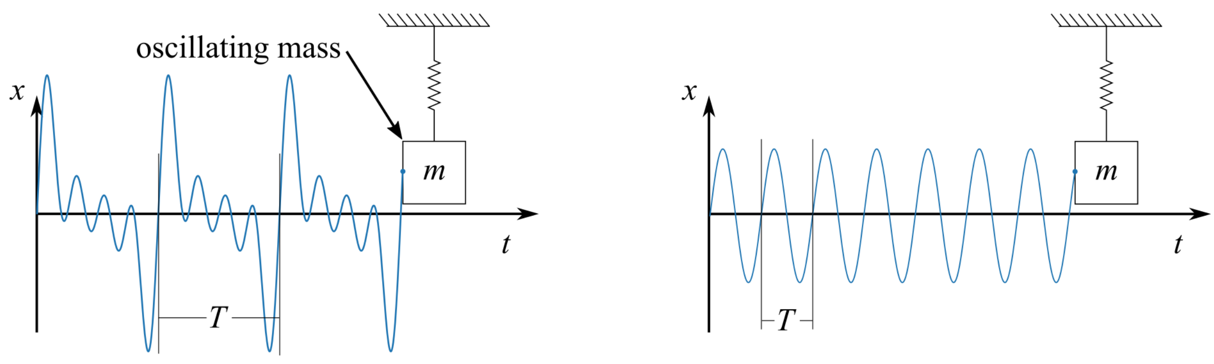


Figure 1: Oscillatory motion for a single degree of freedom system showing (a) periodic motion; and (b) simple harmonic motion.

Given the nature of simple harmonic motion, constant amplitude and frequency, the wave starting at the origin O can be modeled at a point on the end of a vector with length A rotating rotating at a constant angular velocity ω_n where the angle from the origin of the vector is ϕ, defined as $\phi = \omega t$. Where ω is the lowercase Greek letter Omega and ϕ is the lowercase Coptic letter phi. This is similar to a Greek phi (φ) and either can be used in this context. The subscript n on ω denotes that this frequency relates to the natural frequency of the system, the only frequency in simple harmonic motion. A visualization of the harmonic motion obtained from projecting the point on the edge of a vector onto the $\omega_n t$ space is presented in figure 2.

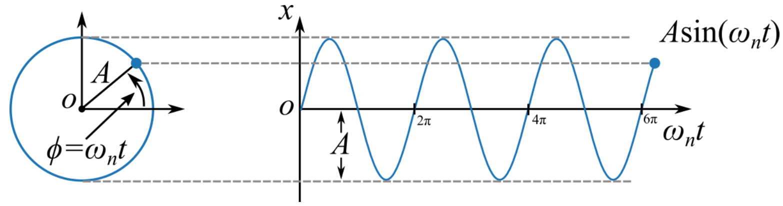


Figure 2: Harmonic motion represented at the projection of a point on the end of a vector moving on a circle. Note the axis $\omega_n t$.

2.1 Mathematical Modeling of Free Vibration

The Development of a mathematical model for a system under free vibration would enable the practitioner to predict, or model, the vibrating system of interest. Therefore, considering that the following system,

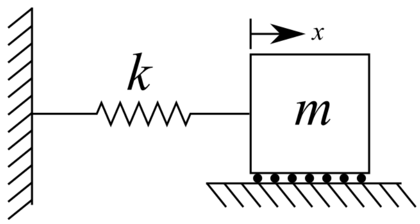


Figure 3: 1-DOF spring-mass system.

can be modeled expressed with the following EOM

$$m\ddot{x}(t) + kx(t) = 0 \tag{1}$$

it becomes prudent to solve this homogeneous ordinary differential equation (ODE) to obtain a model of the vibrating system. The simplest method for solving an ODE is to propose a solution based on observations of a vibrating physical system. Figure 4 reports and annotates the key components from an observation of a vibrating system.

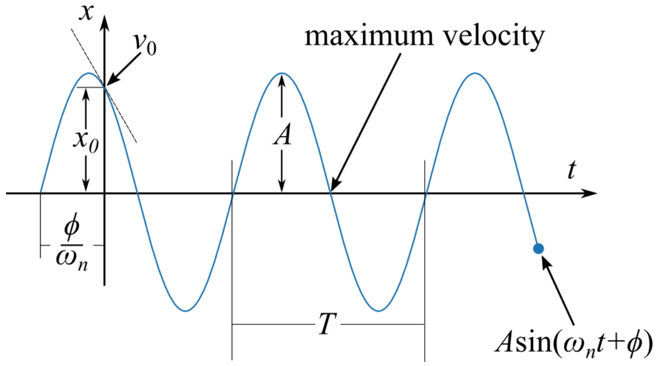


Figure 4: Summary of the temporal response for a 1-DOF system.

where x_0 and v_0 are the is the displacement and velocity at t=0 (i.e. the initial displacement).

A mathematical expression can now be formulate to represent the observed simple harmonic motion. This expression can be based on the projection of a point on a vector (transposed into the time domain) or assembled from constituent parts as done in what follows. Solving for a location x, at a time t; $x(t)$, the various characteristics of the expression can be identified:

2.1.1 Solve for the natural frequency (ω_n) of the system

- System oscillates $\rightarrow$ a sin function models this
- System oscillates at different speed $\rightarrow$ use a parameter to adjust ω_n in rad/s.
- Systems have different amplitudes $\rightarrow$ use a parameter to adjust A in meters.
- System has different starting points $\rightarrow$ use a parameter to adjust ϕ in rad.

Using these four constituent components, an equation can be proposed:

$$x(t) = A\sin(\omega_n t + \phi) \tag{2}$$

Take the derivative to get velocity:

$$\dot{x}(t) = A\omega_n\cos(\omega_n t + \phi) \tag{3}$$

Take the derivative again to get acceleration:

$$\ddot{x}(t) = -A\omega_n^2\sin(\omega_n t + \phi) \tag{4}$$

Substituting x and $\ddot{x}$ into the EOM for the considered 1-DOF system ($m\ddot{x}(t) + kx(t) = 0$) yields:

$$m\big(-A\omega_n^2 \sin(\omega_n t + \phi)\big) + k\big(A\omega_n \sin(\omega_n t + \phi)\big) = 0 \tag{5}$$

Thereafter, dividing both sides by $A\sin(\omega_n t + \phi)$ results in the expression:

$$-m\omega_n^2 + k = 0 \tag{6}$$

This expression can be rearranged into the more useful standard form:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{7}$$

This form is not an ODE so the equation is solvable and testable. This equation relates the frequency of a system to stiffness and mass. This equation leads to:

$$T = \frac{2\pi}{\omega_n} \tag{8}$$

where T is the period of oscillations and

$$f_n = \frac{\omega_n}{2\pi} \tag{9}$$

where f_n is the frequency of the oscillations.

2.1.2 Solve for initial phase (ϕ) of the system

The EOM is a second-order ODE so we need two constants (initial conditions) to solve it. Here we will use the displacement (x) and velocity ($\dot{x}$ or v) at $t = 0$. For simplicity, these are written as

$$x(0) = x_0 \tag{10}$$

$$\dot{x}(0) = v(0) = v_0 \tag{11}$$

Setting the equation to its initial state $t = 0$, the equations for displacement and velocity can be simplified to:

$$x(0) = x_0 = A\sin(\omega_n 0 + \phi) = A\sin(\phi) \tag{12}$$

$$\dot{x}(0) = v_0 = A\omega_n \cos(\omega_n 0 + \phi) = A\omega_n \cos(\phi) \tag{13}$$

Now that we can derive mathematical meanings for ϕ and A. To do this, ϕ can be solved for by rearranging equations 12 and 13 for A:

$$A = \frac{x_0}{\sin(\phi)} \tag{14}$$

and:

$$A = \frac{v_0}{\omega_n \cos(\phi)} \tag{15}$$

Setting these two equations equal to each other cancels out A and creates:

$$\frac{x_0 \omega_n}{\sin(\phi)} = \frac{v_0}{\cos(\phi)} \tag{16}$$

therefore:

$$\frac{x_0 \omega_n}{v_0} = \frac{\sin(\phi)}{\cos(\phi)} \tag{17}$$

finally:

$$\phi = \tan^{-1}\left(\frac{x_0 \omega_n}{v_0}\right) \tag{18}$$

2.1.3 Solve for amplitude (A) of the system

The amplitude of the vibrating system (A) is solved for in a similar manner to ϕ where the expressions for x and $\dot{x}$ are solved for at $t = 0$ and rearranged as to isolate ϕ. This operations results in the the equations:

$$\sin(\phi) = \frac{x_0}{A} \tag{19}$$

and:

$$\cos(\phi) = \frac{v_0}{\omega_n A} \tag{20}$$

From these equations a value for ϕ can be obtained knowing that $\sin(\phi)^2 + \cos(\phi)^2 = 1$. Therefore:

$$\left(\frac{x_0}{A}\right)^2 + \left(\frac{v_0}{\omega_n A}\right)^2 = 1 \tag{21}$$

multiplying each expression by 1 (also expressed as $\frac{\omega_n}{\omega_n}$), gives the equation:

$$\left(\frac{\omega_n}{\omega_n}\right)^2 \left(\frac{x_0}{A}\right)^2 + 1\left(\frac{v_0}{\omega_n A}\right)^2 = 1 \times 1 \tag{22}$$

which becomes:

$$\left(\frac{\omega_n x_0}{\omega_n A}\right)^2 + \left(\frac{v_0}{\omega_n A}\right)^2 = 1 \tag{23}$$

Further simplification is obtained by multiplying each side by $(\omega_n A)^2$ to obtain:

$$\omega_n^2 x_0^2 + v_0^2 = A^2 \omega_n^2 \tag{24}$$

Solving for A, this expression rearranges to:

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \tag{25}$$

2.1.4 Response for simple harmonic motion

The time-varying displacement of a 1-DOF vibrating system under free response is expressed by the equation $x(t) = A\sin(\omega_n t + \phi)$. Substituting in the expressions for A and ϕ results in:

$$x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \sin\left(\omega_n t + \left(\tan^{-1}\left(\frac{x_0 \omega_n}{v_0}\right)\right)\right) \tag{26}$$

This equation provides a mathematical solutions that relates displacement of the mass to the initial conditions x_0 and v_0. The solution is considered a free response because no input is applied after t=0. The relationship between the initial conditions (x_0 and v_0) and the amplitude and phase of the response can be expressed using the pathagreom theorem, $a^2 + b^2 = c^2$, as annotated in figure 5.

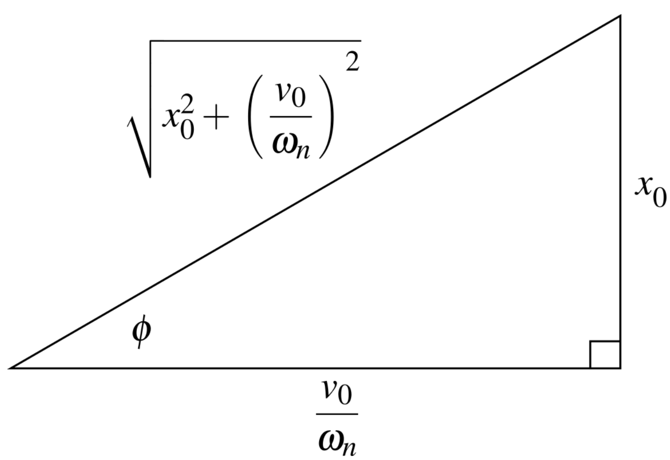


Figure 5: Trigonometric relationship between the initial conditions (x_0 and v_0) and the amplitude A and phase ϕ for free vibration of a 1-DOF system.

Example 2.1 A vehicle wheel, tire, and suspension can be modeled as a SDOF spring and mass. The mass is measured to be 300 kg, its frequency of oscillation is observed to be 10 rad/sec. What is the stiffness of the wheel assembly?

Solution: Knowing $k = m\omega_n^2 = (300 \text{ kg})(10 \text{ rad/s})^2 = 30 \text{ KN/m}$

Example 2.2 Consider the following 1-DOF system, where $k = 857.8$ N/m and $m = 49.2 \times 10^{-3}$, calculate the natural frequency in rad/s and Hz. Also find the period of oscillations and the maximum displacement if the spring is initially displaced 10 mm with no initial velocity.

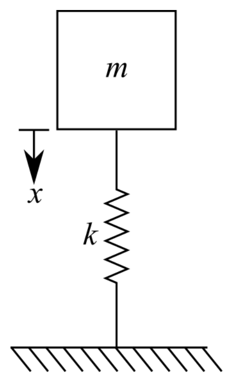


Figure 6: 1-DOF spring-mass system

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132 \text{ rad/sec} \tag{27}$$

In Hz, this is:

$$f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz} \tag{28}$$

The period is:

$$T = \frac{2\pi}{\omega_n} = 0.0476 \text{ s} \tag{29}$$

To determine the maximum displacement, we can see that this will happen when $\sin(\omega_n t + \phi)$= 0, therefore, the value of A is the maximum displacement. For an undamped system, $A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}$,

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \frac{\sqrt{132^2 0.01^2 + 0^2}}{132} = 0.01 \text{ m} \tag{30}$$

2.2 General solution for vibrating systems

The solution for a vibrating system can be expressed in various forms and these forms relate to each other through Euler's equations.

2.2.1 Review: complex numbers

Vibration analysis uses complex numbers to solve the EOM's differential equation. In this class the imaginary number is termed j (sometimes referred to as i): such that:

$$j = \sqrt{-1} \tag{31}$$

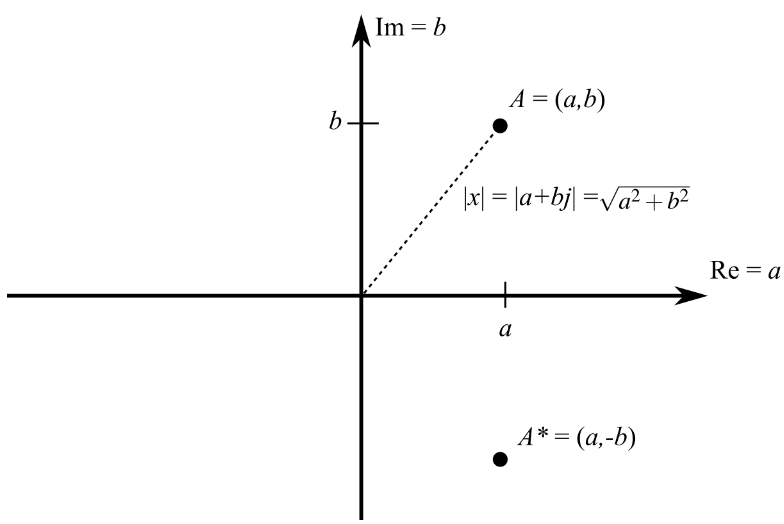
and:

$$j^2 = -1 \tag{32}$$

a general complex number. x can be expressed as

$$x = a + bj \tag{33}$$

here, a is referred to as the real number and b is the imaginary part of the number x. Such complex numbers can be represented in the complex plane, also called a Argand plot. The absolute value or modules is defined as $|x|$ presented on the complex plot.



A and A^* prime are complex conjugate pairs. In mathematics, the complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. In other words, a conjugate pair is $a+bj$ and $a-bj$.

Definition → **con·ju·gate** (adjective): Coupled, connected, or related.

2.2.2 Review: Euler's formula.

Euler's (pronounced oy-ler) formula, named after Swiss engineer and mathematician Leonhard Euler (1707-1783), is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number x,

$$e^{j\psi} = \cos(\psi) + j\sin(\psi) \tag{34}$$

where $j = \sqrt{-1}$. This equation can also be expressed as:

$$e^{-j\psi} = \cos(\psi) - j\sin(\psi) \tag{35}$$

This can be expressed in terms of polar coordinates as:

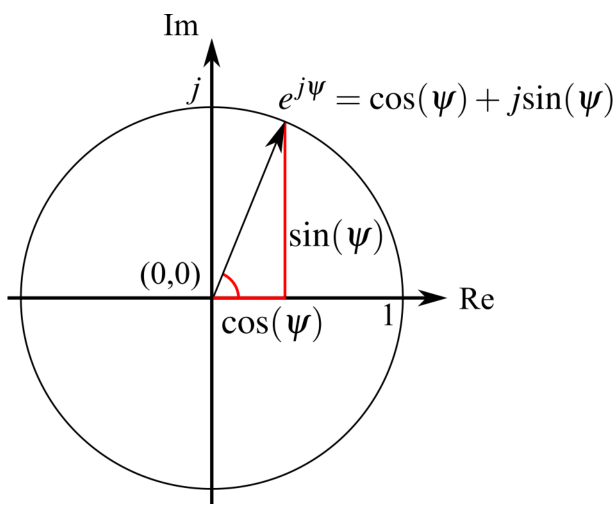


Figure 7: Euler's formula illustrated on the unit circle in the complex plane.

2.2.3 Solving the EOM

We can also solve the following EOM as an elementary differential equation:

$$m\ddot{x} + kx = 0 \tag{36}$$

in a more analytical manner using the theory of elementary differential equations. To do this we have to assume a solution, in the form of

$$x(t) = ae^{\lambda t} \tag{37}$$

here, a and t are nonzeros constants that need to be determined. Using successive differentiation, we get:

$$\dot{x}(t) = \lambda ae^{\lambda t} \tag{38}$$

and

$$\ddot{x}(t) = \lambda^2 ae^{\lambda t} \tag{39}$$

therefore, $m\ddot{x}(t) + kx(t) = 0$ becomes:

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \tag{40}$$

Now we divide by $ae^{\lambda t}$ to obtain the **characteristic equation**:

$$m\lambda^2 + k = 0 \tag{41}$$

We can do this because $ae^{\lambda t}$ is never zero, therefore, we never divide by zero. The quadratic formula gives us:

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \tag{42}$$

remember that $\omega_n = \sqrt{\frac{k}{m}}$. Notice that the $\pm$ tells us there are two solutions to this problem. So, putting λ back into our assumed solution, we get two solutions:

$$x(t) = a_1 e^{+\omega_n jt} \tag{43}$$

and

$$x(t) = a_2 e^{-\omega_n jt} \tag{44}$$

As we deal with linear systems, we know that the sum of the solutions is also a solution, resulting in:

$$x(t) = a_1 e^{+\omega_n jt} + a_2 e^{-\omega_n jt} \tag{45}$$

where a_1 and a_2 are complex valued constants of integration. This equation derived using Euler's formula is equivalent to the $A\sin(\omega_n + \phi)$. To recover the previously assumed solution we apply the knowledge that a_1 and a_2 are complex congregate pairs and as such the magnitude can be expressed as $a_1 = a_2$. Using Euler's polar notation, a_1 and a_2 can be expressed as

$$a_1 = a_2 = ae^{j\psi} \tag{46}$$

where a and ψ are real numbers, the equation becomes:

$$x(t) = ae^{j(\omega_n t+\psi)} + ae^{-j(\omega_n t+\psi)} \tag{47}$$

this becomes:

$$x(t) = a(e^{j(\omega_n t+\psi)} + e^{-j(\omega_n t+\psi)}) \tag{48}$$

Remembering Euler's equations from before, this becomes:

$$x(t) = a\big(\cos(\omega_n t+\psi) + j\sin(\omega_n t+\psi) + \cos(\omega_n t+\psi) - j\sin(\omega_n t+\psi)\big) \tag{49}$$

combining the "cos" terms and canceling out the "sin" terms this becomes:

$$x(t) = 2a \cdot \cos(\omega_n t+\psi) \tag{50}$$

This is equivalent to $x(t) = A\sin(\omega_n t+\phi)$ if we take $A = 2a$ and knowing sin(ϕ) = cos($\phi + \psi$).

Formulate the general solution from Euler's expression.

From Euler's equation we saw that:

$$x(t) = a_1 e^{+\omega_n jt} + a_2 e^{-\omega_n jt} \tag{51}$$

we can expand this into the form:

$$x(t) = a_1\big(\cos(\omega_n t) + j\sin(\omega_n t)\big) + a_2\big(\cos(\omega_n t) - j\sin(\omega_n t)\big) \tag{52}$$

using trigonometric functions. This equates to:

$$x(t) = (a_1 + a_2) \cdot \cos(\omega_n t) + (a_1 - a_2) j \cdot \sin(\omega_n t) \tag{53}$$

as $x(t)$ is always real, we can define:

$$A_1 = (a_1 + a_2) \tag{54}$$

and

$$A_2 = (a_1 - a_2) j \tag{55}$$

lastly, as the **general solution** is written as:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{56}$$

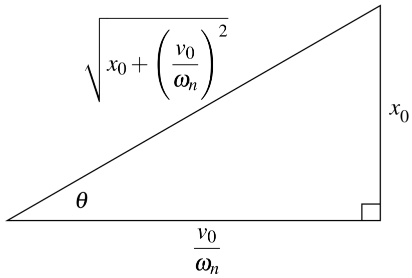
This is the general solution for the EOM ($m\ddot{x} + kx = 0$) of the considered oscillating system where A_1 and A_2 are defined as:

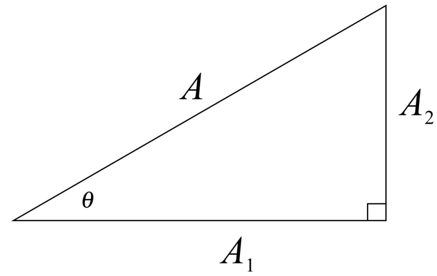
$$A = \sqrt{A_1^2 + A_2^2} \tag{57}$$

and

$$\phi = \tan^{-}1 \left(\frac{A_1}{A_2} \right) \tag{58}$$

These are obtained from a trigonometric relationship, similar to that used before:





again, A and ϕ are:

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n} \right)^2} \tag{59}$$

$$\phi = \tan^{-1} \left(\frac{x_0 \omega_n}{v_0} \right) \tag{60}$$

One Equation in three forms:

Form one, for $m\ddot{x} + kx = 0$ subject to nonzero initial conditions can be written as:

$$x(t) = a_1 e^{+\omega_n jt} + a_2 e^{-\omega_n jt} \tag{61}$$

where a_1 and a_2 are complex terms. Form two is:

$$x(t) = A\sin(\omega_n t + \phi) \tag{62}$$

while form three is:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{63}$$

where A, ϕ, A_1, and A_2, are all real-valued constants. Each set of constants can be related to each other by:

$$A = \sqrt{A_1^2 + A_2^2} \qquad \phi = \tan^{-}1\left(\frac{A_1}{A_2}\right) \tag{64}$$

$$A_1 = (a_1 + a_2) \qquad A_2 = (a_1 - a_2)j \tag{65}$$

$$a_1 = \frac{A_1 - A_2 j}{2} \qquad a_2 = \frac{A_1 + A_2 j}{2} \tag{66}$$

Which follow from trigonometric identities and the Euler's formulas.

Example 2.3 Using the general solution:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{67}$$

Calculate the values of A_1 and A_2 in terms of their initial conditions x_0 and v_0.
Solution: Knowing the following for x and $\dot{x}$:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{68}$$

$$\dot{x}(t) = -A_1 \omega_n \sin(\omega_n t) + A_2 \omega_n \cos(\omega_n t) \tag{69}$$

Now apply the initial conditions, $x(0) = 0$ and $v(0) = 0$, this yields:

$$x(0) = x_0 = A_1 \tag{70}$$

$$\dot{x}(0) = v_0 = A_2 \omega_n \tag{71}$$

Solving for A_1 and A_2 shows us:

$$A_1 = x_0, \text{ and } A_2 = \frac{v_0}{\omega_n} \tag{72}$$

thus:

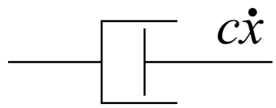
$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{73}$$

2.3 Damping

What is damping?

The response of a spring-mass system predicts that a system will oscillate indefinitely. However, we know that this is not true from observing real-world solutions.

So based on real-world observations and mathematical conveniences, we need to add a term that will remove "energy" from the system with time. To do this we introduce the ideal dashpot, modeled with the terms $c\dot{x}$.



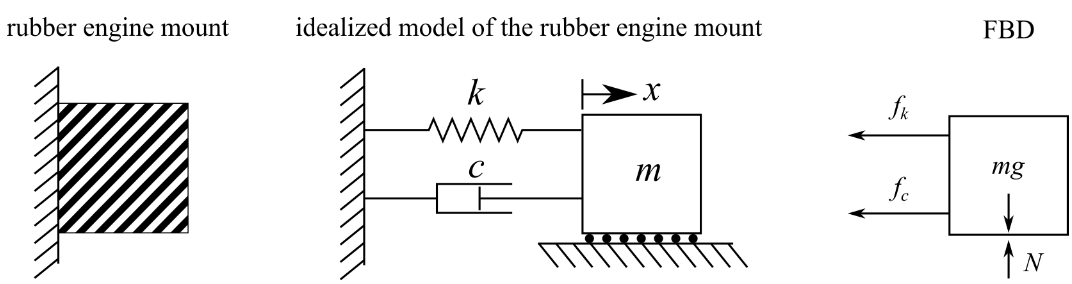
A spring forms a physical model of the cause vibration, through its storage and release of energy, a damper forms a physical model for dissipating energy We typically model dampers as viscous dampers, and as such the force is proportional to the velocity of the damper. The damping force f_c has the form

$$f_c = c\dot{x}(t) \tag{74}$$

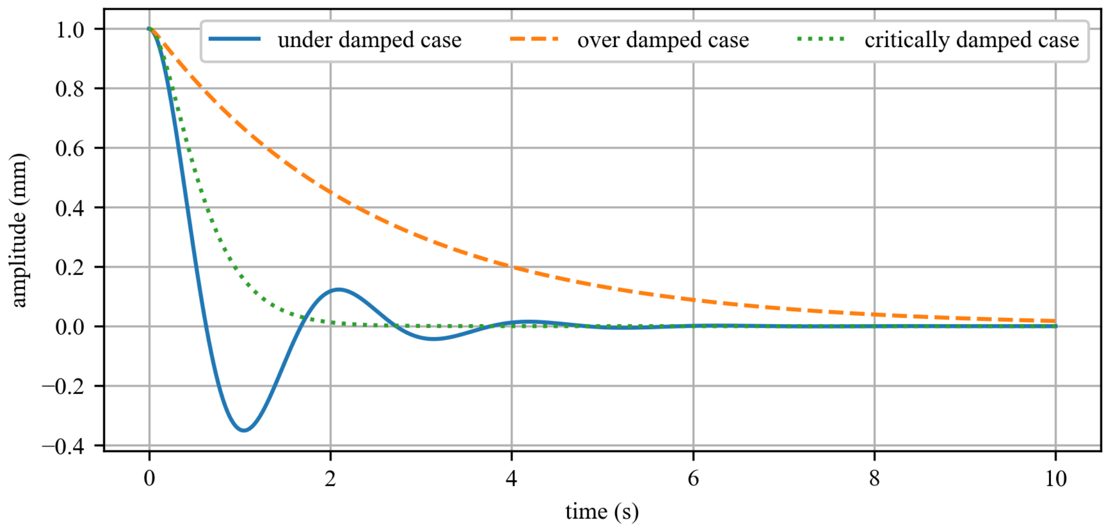
the constant c, called the damping coefficient, has the units of kg/s.

What does it mean in real life?

Systems have damping, but not with just physical dampers. Our spring-mass systems are just representations of real-world systems.



What does damping look like in terms of displacement vs time



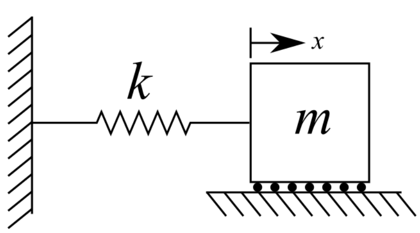
The key types of damping are:

- **Undamped** - Oscillates around the equilibrium and does not decay.
- **Under damped** - Oscillates around the equilibrium and slowly decays and is the most common case.
- **Critically damped** - provides the quickest approach to zero amplitude for a damped oscillator.
- **Overdamped** - Does not pass the equilibrium position and is a simple a decay with no oscillation.

In this class we are only considering positive damping, mass, and stiffness values.

Solve the EOM with damping?

Using the FBD for the system, we can conclude that the EOM for this system:



is:

$$m\ddot{x}(t) = -f_c - f_k \tag{75}$$

Rearranging into standard form and concerting forces into parameters c and k results in:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{76}$$

This system is subject to the same initial conditions as before, $x(0) = x_0$ and $\dot{x}(0) = v_0$. Again, we chose to model it this way for convinces, so let's solve it in a similar manner to the EOM without damping. Again, assume the solution:

$$x(t) = ae^{\lambda t} \tag{77}$$

here, a and t are nonzeros constants that need to be determined. Using successive differentiation, we get:

$$\dot{x}(t) = \lambda ae^{\lambda t} \tag{78}$$

and

$$\ddot{x}(t) = \lambda^2 ae^{\lambda t} \tag{79}$$

therefore, $m\ddot{x} + c\dot{x} + kx = 0$ becomes:

$$m\lambda^2 ae^{\lambda t} + c\lambda ae^{\lambda t} + kae^{\lambda t} = 0 \tag{80}$$

Now we divide by $ae^{\lambda t}$ to obtain the **characteristic equation**:

$$m\lambda^2 + c\lambda + k = 0 \tag{81}$$

We can do this because $ae^{\lambda t}$ is never zero, therefore, we never divide by zero. The quadratic formula gives us:

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{-c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \tag{82}$$

Some key points from this equation:

- The $\pm$ tells us there are two solutions to this problem
- if $c^2 - 4km < 0$, system is Underdamped, solutions are complex conjugate pairs
- if $c^2 - 4km = 0$, system is critically damped, solutions are equal negative real numbers

- if $c^2 - 4km > 0$, system is Overdamped, solutions are distinct negative real numbers

From this, we can see that $c^2 - 4km = 0$ is a special value, let us define a value for c that will give us this critical damping number. We will call it the **critical damping coefficient** (c_{cr}). So setting the equation as:

$$c_{\text{cr}}^2 - 4km = 0 \tag{83}$$

giving us:

$$c_{\text{cr}}^2 = 4km \tag{84}$$

next we can derive the function:

$$c_{\text{cr}} = 2\sqrt{km} = 2\left(\frac{\sqrt{m}}{\sqrt{m}}\right)\sqrt{km} = 2m\omega_n \tag{85}$$

remember that $\omega_n = \sqrt{\frac{k}{m}}$ for an undamped system. Next, we generate a non-dimensional number (ζ), pounced 'zeta' that will allow us to distinguish between different types of damping. ζ is called the **critical damping ratio**.

$$\zeta = \frac{c}{c_{\text{cr}}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n} \tag{86}$$

Now if we put the ζ back into the characteristic equation and resolve using the quadratic equation we get:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{87}$$

From this equation it become clear that ζ determines whether the roots are complex or real, this in turn determines the nature of the response of the structure. Listing our possible responses we get: For each damping case, we will have a

damping case	critical damping ratio	radicand	solutions
under damped	$0 < \zeta < 1$	$c^2 - 4km < 0$	complex conjugate pairs
critically damped	$\zeta = 1$	$c^2 - 4km = 0$	equal negative real numbers
over damped	$1 < \zeta$	$c^2 - 4km > 0$	distinct negative real numbers

different solution to the problem.

Under damped motion

In the case that $0 < \zeta < 1$, a complex conjugate pair of roots are the solutions to the characteristic equation after pulling out a $\sqrt{-1}$:

$$\lambda_1 = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}\, j \tag{88}$$

and:

$$\lambda_2 = -\zeta\omega_n - \omega_n\sqrt{1-\zeta^2}\, j \tag{89}$$

Where the j is pulled out because:

$$\sqrt{1-\zeta^2}\, j = \sqrt{(1-\zeta^2)(-1)} = \sqrt{\zeta^2 - 1} \tag{90}$$

Next, let us "arbitrarily" define:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{91}$$

where ω_d is the **damped natural frequency**. Therefore, the equations become:

$$\lambda_1 = -\zeta\omega_n + \omega_d j \tag{92}$$

and:

$$\lambda_2 = -\zeta\omega_n - \omega_d j \tag{93}$$

Again, we have two solutions to a linear problem, so we can combine these into one solution and insert λ into the assumed solution $ae^{\lambda t}$ to obtain:

$$x(t) = a_1 e^{-\zeta\omega_n t + \omega_d t j} + a_2 e^{-\zeta\omega_n t - \omega_d t j} \tag{94}$$

where a_1 and a_2 are complex valued constants. This can now be simplified into:

$$x(t) = e^{-\zeta\omega_n t}(a_1 e^{\omega_d t j} + a_2 e^{-\omega_d t j}) \tag{95}$$

Using Euler's equations, (same as before) and choosing:

$$A_1 = (a_1 - a_2) j \tag{96}$$

and

$$A_2 = (a_1 + a_2) \tag{97}$$

The **general form** of this solution is then:

$$x(t) = e^{-\zeta\omega_n t}\big(A_1 \sin(\omega_d t) + A_2 \cos(\omega_d t)\big) \tag{98}$$

Recall that for undamped 1-DOF systems we showed

$$x(t) = A\sin(\omega_d t + \phi) = A_1\sin(\omega_d t) + A_2\cos(\omega_d t) \tag{99}$$

As $e^{-\zeta\omega_n t}$ accounts for the damping, our current solution becomes:

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \tag{100}$$

Now that we have x and $\dot{x}$, we can solve for the boundary conditions x_0 and v_0 by setting $t = 0$, we get:

$$x(0) = x_0 = A\sin(\phi) \tag{101}$$

and taking the directive of $x(t)$ using the product rule (fg)'= f'g+fg', we get:

$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + Ae^{-\zeta\omega_n t}\omega_d\cos(\omega_d t + \phi) \tag{102}$$

$$\dot{x}(0) = v_0 = -\zeta\omega_n A\sin(\phi) + A\omega_d\cos(\phi) \tag{103}$$

letting $A = x_0/\sin(\phi)$ gives us the equation:

$$\dot{x}(0) = v_0 = -\zeta\omega_n\left(\frac{x_0}{\sin(\phi)}\right)\sin(\phi) + \left(\frac{x_0}{\sin(\phi)}\right)\omega_d\cos(\phi) \tag{104}$$

This simplifies to:

$$\dot{x}(0) = v_0 = -\zeta\omega_n x_0 + x_0\omega_d\cot(\phi) \tag{105}$$

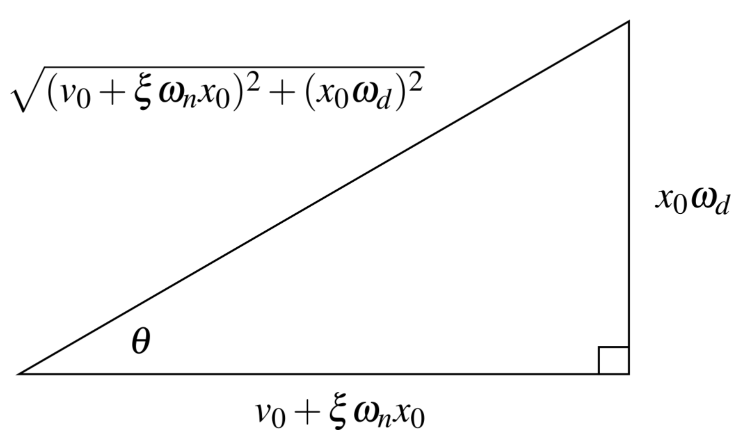
solving for ϕ:

$$\cot(\phi) = \frac{v_0 + \zeta\omega_n x_0}{x_0\omega_d} \tag{106}$$

and as the $\tan(\phi) = 1/\cot(\phi)$:

$$\phi = \tan^{-}1\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right) \tag{107}$$

Using the trigonometric relationship:



we get:

$$\sin(\phi) = \frac{x_0\omega_d}{\sqrt{(v_0+\zeta\omega_n x_0)^2+(x_0\omega_d)^2}} \tag{108}$$

and applying $A = x_0/\sin(\phi)$ we get:

$$A = \frac{\sqrt{(v_0+\zeta\omega_n x_0)^2+(x_0\omega_d)^2}}{\omega_d} = \sqrt{x_0^2+\left(\frac{v_0+\zeta\omega_n x_0}{\omega_d}\right)^2} \tag{109}$$

Finally, collecting all of our important equations:

- Critical damping coefficient: $c_{\text{cr}} = 2\sqrt{km} = 2m\omega_n$
- Damping ratio: $\zeta = \frac{c}{c_{\text{cr}}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$
- Damped natural frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- Solution for underdamped system: $x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$, where:

$$A = \frac{\sqrt{(v_0+\zeta\omega_n x_0)^2+(x_0\omega_d)^2}}{\omega_d} \qquad \phi = \tan^-1\left(\frac{x_0\omega_d}{v_0+\zeta\omega_n x_0}\right)$$

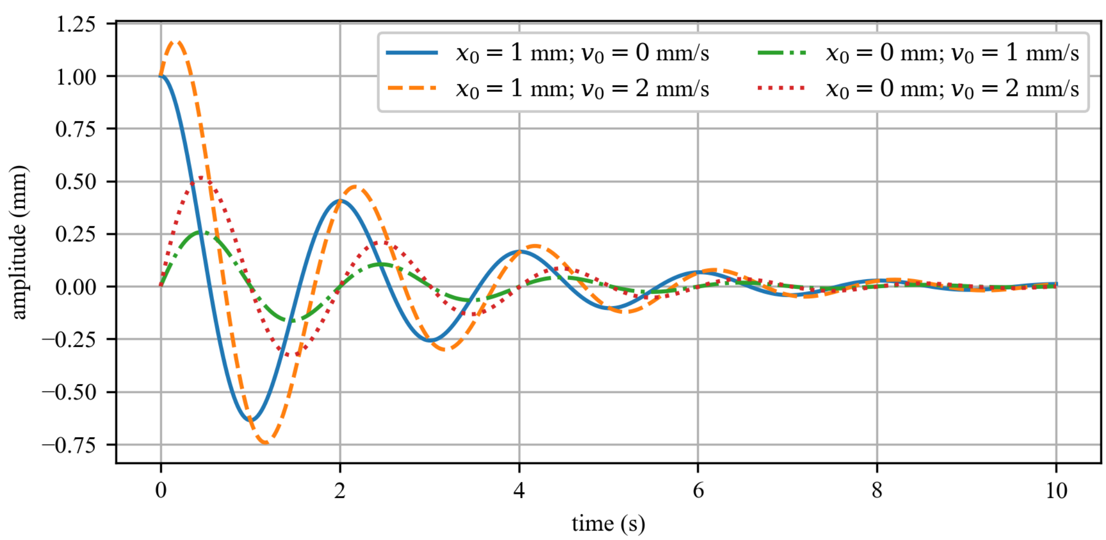


Figure 8: Four example responses for an under damped 1-DOF system ($\zeta = 0.142$) with various initial conditions.

Over damped motion

In the case of overdamped systems, $1 < \zeta$, the solutions for λ are distinct real roots that are written as:

$$\lambda_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \tag{110}$$

and:

$$\lambda_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \tag{111}$$

The solution for the EOM using the assumed solution then becomes:

$$x(t) = e^{-\zeta\omega_n t}(a_1 e^{-\omega_n\sqrt{\zeta^2-1}t} + a_2 e^{+\omega_n\sqrt{\zeta^2-1}t}) \tag{112}$$

This equation represents a non-oscillating response of the system. Again, a_1 and a_2 are solved for using known boundary conditions x_0 and v_0 such that:

$$a_1 = \frac{-v_0 + \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{113}$$

$$a_2 = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{114}$$

Typical responses for a overdamped system with various initial conditions are shown below:

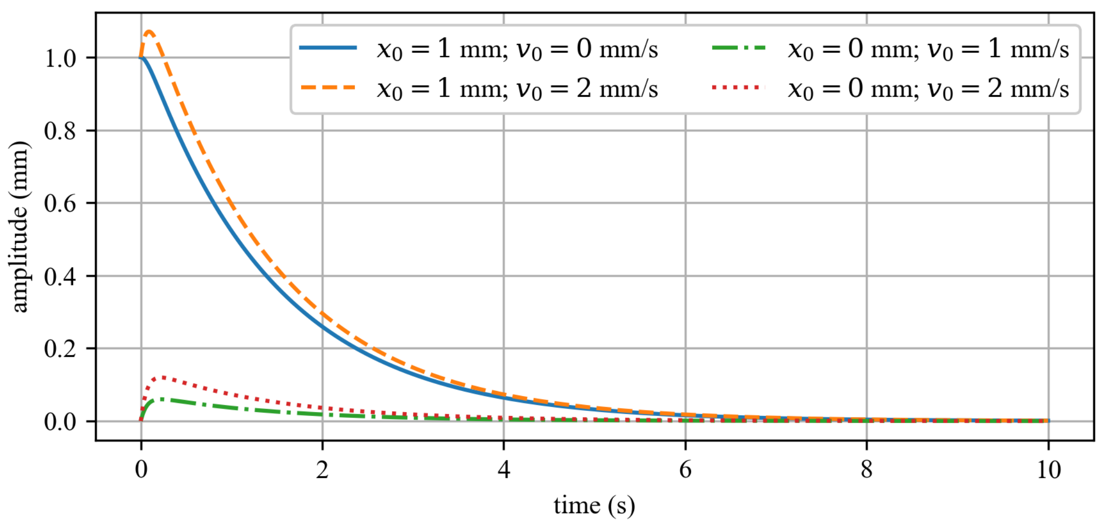


Figure 9: Four example responses for an over damped 1-DOF system ($\zeta = 2.371$) with various initial conditions.

Critically damped motion

In the case of critically damped systems, $\zeta = 1$, the solutions for λ will be equal negative real numbers, therefore from before:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{115}$$

We get:

$$\lambda_1 = \lambda_2 = -\omega_n \tag{116}$$

Because both solutions (a_1 and a_2) are the same, we multiply the second solution by t so the solution for a critically damped system is in the same form as before. The solution for the EOM using the assumed solution then becomes:

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t} \tag{117}$$

This simplifies into:

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t} \tag{118}$$

This equation represents a non-oscillating response of the system. Again, a_1 and a_2 are solved for using known boundary conditions x_0 and v_0 such that:

$$a_1 = x_0 \tag{119}$$

$$a_2 = v_0 + \omega_n x_0 \tag{120}$$

Standard Form of the EOM

Typically, we write the EOM:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{121}$$

In what is known as standard form. First, every term is divided by m such that:

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = 0 \tag{122}$$

This can be rearranged to the **standard form**:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0 \tag{123}$$

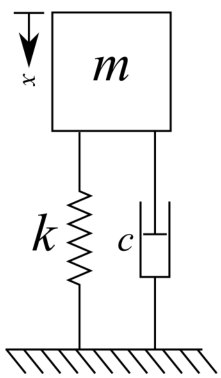
damping case	critical damping ratio	radicand	solutions
under damped	$0 < \zeta < 1$	$c^2 - 4km < 0$	complex conjugate pairs
critically damped	$\zeta = 1$	$c^2 - 4km = 0$	equal negative real numbers
over damped	$1 < \zeta$	$c^2 - 4km > 0$	distinct negative real numbers

Important items from today

- There are three types of damping cases, under, critical, and over.
- The under damped system is the only one that oscillates, and as such is of the most importance to this class.

Example 2.4 Example 1

Consider the following 1-DOF system, where $k = 857.8$ N/m, $c = 7.8$ kg/s, and $m = 49.2 \times 10^{-3}$ kg, calculate the damped frequency in rad/s and Hz. What damping case is this system?



Calculate the undamped frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132 \ \text{rad/s} \tag{124}$$

The systems critical damping value:

$$c_{\text{cr}} = 2\sqrt{km} = 2\sqrt{k = 857.8 \cdot 49.2 \times 10^{-3}} = 12.993 \ \text{kg/s} \tag{125}$$

And the critical damping ratio:

$$\zeta = \frac{c}{c_{\text{cr}}} = \frac{7.8}{12.993} = 0.600 \tag{126}$$

This can also be expressed as 60% damped, this is a underdamped system, and the system will oscillate. Now we can calculate the damped frequency:

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = \omega_n \sqrt{1-0.600^2} = 105.6 \ \text{rad/s} \tag{127}$$

Therefore, the system oscillates at 105.6 rad/sec or 16.81 Hz

Example 2.5 For a damped one DOF system where m, c, and k are known to be m = 1 kg, c = 2 kg/s, and k = 10 N/m. Calculate the value of ζ and ω_n. Is the system overdamped, underdamped, or critically damped?

The natural frequency is calculated as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16 \ \text{rad/s} \tag{128}$$

The damping can be calculated as:

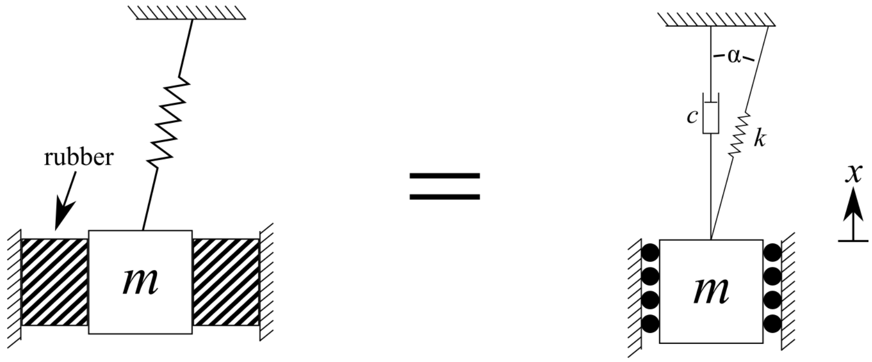
$$\zeta = \frac{c}{2\omega_n m} = \frac{2}{2\left(\sqrt{\frac{10}{1}}\right)(1)} = \frac{1}{\sqrt{10}} = 0.316 \ \text{rad/s} \tag{129}$$

So the damped natural frequency is equal to:

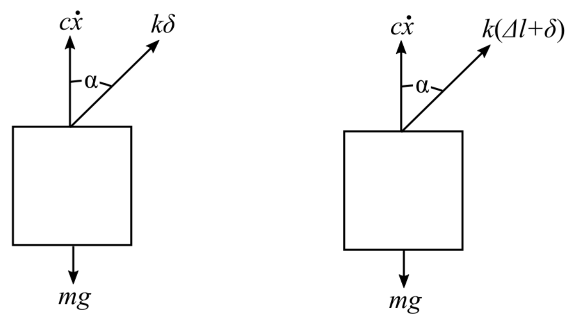
$$\omega_d = \omega_n \sqrt{1-\zeta^2} = \sqrt{10}\sqrt{1-\left(\frac{1}{\sqrt{10}}\right)^2} = 0.3 \ \text{rad/s} \tag{130}$$

As $0 < \zeta < 1$ the system is underdamped.

Example 2.6 For the following industrial device consisting of a mass isolated from its fixtures by rubber dampers provide an estimate of the systems oscillating frequency.



As we only want an estimate of the frequency, we can assume the displacement is small and as such α of the displaced mass is equal α of the resting mass. This leads to the FBD for the resting and displaced masses.



$\sum F_x = mg - k\delta\cos(\alpha) = 0$ $\qquad$ $\sum F_x = mg - c\dot{x} - k\cos(\alpha)(\Delta l + \delta)$

Applying newtons second law and combining these equations yields:

$$m\ddot{x} + c\dot{x} + k\Delta l\cos(\alpha) = 0 \tag{131}$$

As we assumed the displacement is small and α remains unchanged, the assumption that $\cos(\alpha) = h/l$ is equivalent to $\cos(\alpha) = x/\Delta l$ is logical (draw out the triangles if needed), therefore the prior equation becomes:

$$m\ddot{x} + c\dot{x} + k\Delta l\frac{x}{\Delta l} = m\ddot{x} + c\dot{x} + kx = 0 \tag{132}$$

as α and x are small. Therefore, the frequency can be estimated as:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{133}$$

once the values for the system are known or measured.

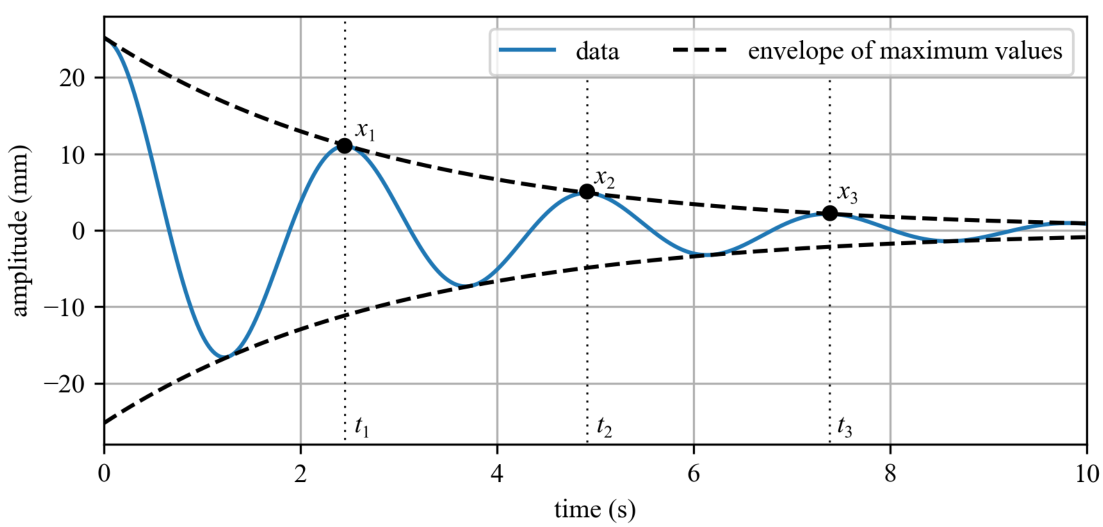
2.4 Logarithmic decrement

Obtaining the damping coefficient of a system using the logarithmic decrement method

For a vibrating system, the mass (m) and stiffness (k) can be measured using scales and static deflection tests. However, the damping coefficient (c) is a more difficult quantity to determine. From k and m we can compute the natural frequency (ω_n) and the critical damping coefficient (c_{cr}). Therefore, knowing that the critical damping ratio (ζ) is defined as:

$$\zeta = \frac{c}{c_{\text{cr}}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n} \tag{134}$$

if we calculate ζ, we can obtain c for the system of interest. This is made possible because c_{cr} can be calculated from k and m. Observing the temporal response for the underdamped system,



we mark three points of maximum amplitude, x_1, x_2, and x_3 that happen at t_1, t_2, and t_3, respectively. Considering displacement values for the first two points x_1 and x_2. separated by a complete period (T). Knowing that one cycle is 2π, the time period for this complete cycle is given by:

$$t_2 - t_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}\omega_n} \tag{135}$$

where ω_d is the damped natural frequency. This is the time period (T) of damped oscillations. If derive an equation for the values of the peaks, also called the envelope of maximum values, we get:

$$x_{\text{peaks}} = Ae^{-\zeta\omega_n t} \tag{136}$$

Knowing that the system is underdamped, A can be solved for using the initial conditions x_0 and v_0, therefore:

$$A = \frac{\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}{\omega_d} \tag{137}$$

In terms of t_1 and t_2, we can express the displacement at these times as:

$$x_1 = Ae^{-\zeta\omega_n t_1} \tag{138}$$

and

$$x_2 = Ae^{-\zeta\omega_n t_2} \tag{139}$$

therefore:

$$\frac{x_1}{x_2} = e^{\zeta\omega_n(t_2-t_1)} \tag{140}$$

However, from before we know that $t_2 - t_1 = \frac{2\pi}{\sqrt{1-\zeta^2}\omega_n}$. Therefore, we can express this last equation as:

$$\frac{x_1}{x_2} = e^{\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \tag{141}$$

Next, we take the natural log of both sides to get the **logarithmic decrement**, denoted by δ:

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{x(t_1)}{x(t_1+T)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{142}$$

This shows us that the ratio of any two successive amplitudes for an underdamped system, vibrating freely, is constant and is a function of the damping only. Sometimes, in experiments, it is more convenient/accurate to measure the amplitudes after say "n" peaks rather than two successive peaks (because if the damping is very small, the difference between the successive peaks may not be significant). The logarithmic decrement can then be given by the equation

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{\text{n+1}}}\right) = \frac{1}{n}\ln\left(\frac{x(t_1)}{x(t_1+nT)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{143}$$

Once we use the experimental data to obtain δ, and knowing that:

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{144}$$

we can calculate the value of ζ:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{145}$$

Therefore, having ζ we can solve for the coefficient of damping, c, as:

$$c = \zeta 2\sqrt{km} \tag{146}$$

Example 2.7 Calculate the damping coefficient of the problem expressed above given that m = 3 kg and k = 20 N/m, x_1 = 11 mm, and x_3 = 2 mm.

First, we solve for δ, for n = 2:

$$\delta = \frac{1}{2}\ln\left(\frac{x_1}{x_3}\right) = \frac{1}{2}\ln\left(\frac{11.03}{2.15}\right) = 0.852 \tag{147}$$

Next, we can calculate ζ, as:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.817}{\sqrt{4\pi^2 + 0.817^2}} = 0.134 \tag{148}$$

And lastly:

$$c = \zeta 2\sqrt{km} = 0.134 \cdot 2\sqrt{20 \cdot 3} = 2.08 \text{ kg/s} \tag{149}$$

Example 2.8 The free response of a 1000-kg automobile with a stiffness of k = 400,000 N/m is observed to be underdamped. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, determine the damping coefficient if the displacement at t_1 is measured to be 2 cm and 0.22 cm at t_2.

Knowing x_1 = 2 cm and x_2 = 0.22 cm and $t_2 = T + t_1$, therefore:

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{2}{0.22} = 2.207 \tag{150}$$

and:

$$\zeta = \left(\frac{\delta}{\sqrt{4\pi^2 + \delta^2}}\right) = \left(\frac{2.207}{\sqrt{4\pi^2 + 2.207^2}}\right) = 0.331 \tag{151}$$

therefore, we can obtain the damping coefficient as

$$c = 2\zeta\sqrt{km} = 2(0.331)\sqrt{400,000 \cdot 1,000} = 13,256 \text{ kg/s} \qquad (152)$$