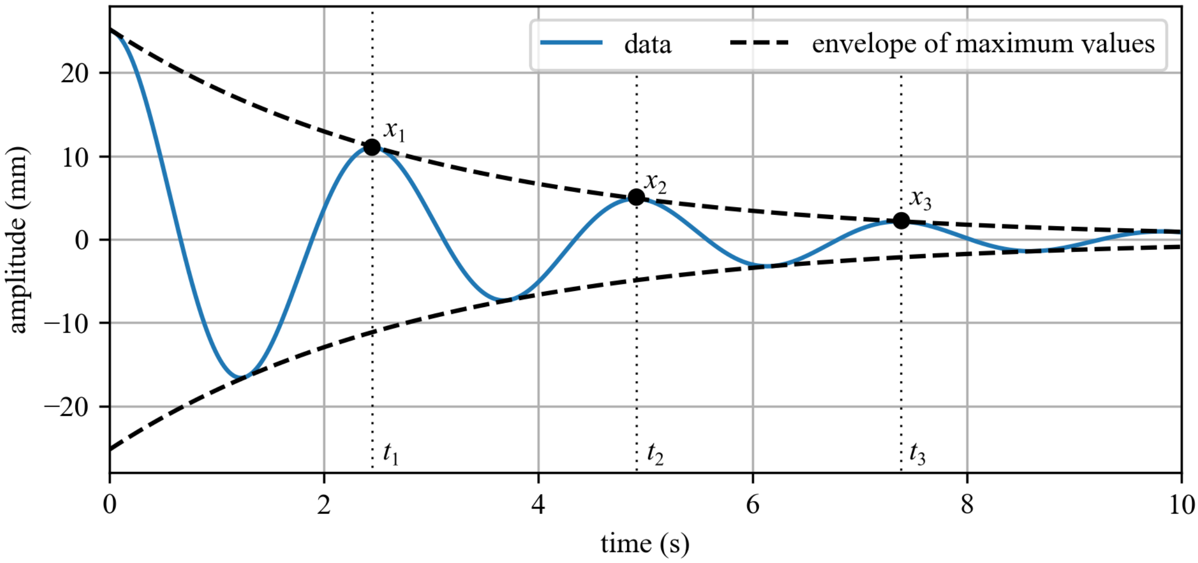
Obtaining the damping coefficient for a system using the logarithmic decrement method

For a vibrating system, the mass (m) and stiffness (k) can be measured using scales and static deflection tests. However, the damping coefficient (c) is a more difficult quantity to determine. From k and m we can compute the natural frequency (ω_n) and the critical damping coefficient (c_{cr}). Therefore, knowing that the critical damping ratio (ξ) is defined as:

$$\xi = \frac{c}{c_{\mathrm{cr}}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n} \tag{1}$$

if we calculate ξ, we can obtain c for the system of interest. This is made possible because c_{cr} can be calculated from k and m. Observing the temporal response for the underdamped system,



we mark three points of maximum amplitude, x_1, x_2, and x_3 that happen at t_1, t_2, and t_3, respectively. Considering displacement values for the first two points x_1 and x_2. separated by a complete period (T). Knowing that one cycle is 2π, the time period for this complete cycle is given by:

$$t_2 - t_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\xi^2}\omega_n} \tag{2}$$

where ω_d is the damped natural frequency. This is the time period (T) of damped oscillations. If derive an equation for the values of the peaks, also called the envelope of maximum values, we get:

$$x_{\mathrm{peaks}} = Ae^{-\xi\omega_n t} \tag{3}$$

Knowing that the system is underdamped, A can be solved for using the initial conditions x_0 and v_0, therefore:

$$A = \frac{\sqrt{(v_0 + \xi\omega_n x_0)^2 + (x_0\omega_d)^2}}{\omega_d} \tag{4}$$

In terms of t_1 and t_2, we can express the displacement at these times as:

$$x_1 = Ae^{-\xi\omega_n t_1} \tag{5}$$

and

$$x_2 = Ae^{-\xi\omega_n t_2} \tag{6}$$

therefore:

$$\frac{x_1}{x_2} = e^{\xi\omega_n(t_2 - t_1)} \tag{7}$$

However, from before we know that $t_2 - t_1 = \frac{2\pi}{\sqrt{1-\xi^2}\omega_n}$. Therefore, we can express this last equation as:

$$\frac{x_1}{x_2} = e^{\left(\frac{2\pi\xi}{\sqrt{1-\xi^2}}\right)} \tag{8}$$

Next, we take the natural log of both sides to get the **logarithmic decrement**, denoted by δ:

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{x(t_1)}{x(t_1+T)}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \tag{9}$$

This shows us that the ratio of any two successive amplitudes for an underdamped system, vibrating freely, is constant and is a function of the damping only. Sometimes, in experiments, it is more convenient/accurate to measure the amplitudes after say "n" peaks rather than two successive peaks (because if the damping is very small, the difference between the successive peaks may not be significant). The logarithmic decrement can then be given by the equation

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{\mathrm{n+1}}}\right) = \frac{1}{n}\ln\left(\frac{x(t_1)}{x(t_1+nT)}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \tag{10}$$

Once we use the experimental data to obtain δ, and knowing that:

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \tag{11}$$

we can calculate the value of ξ:

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{12}$$

Therefore, having ξ we can solve for the coefficient of damping, c, as:

$$c = \xi 2\sqrt{km} \tag{13}$$

Example 1

Calculate the damping coefficient of the problem expressed above given that m = 3 kg and k = 20 N/m, x_1 = 11 mm, and x_3 = 2 mm.

First, we solve for δ, for n = 2:

$$\delta = \frac{1}{2}\ln\left(\frac{x_1}{x_3}\right) = \frac{1}{2}\ln\left(\frac{11.03}{2.15}\right) = 0.852 \tag{14}$$

Next, we can calculate ξ, as:

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.817}{\sqrt{4\pi^2 + 0.817^2}} = 0.134 \tag{15}$$

And lastly:

$$c = \xi 2\sqrt{km} = 0.134 \cdot 2\sqrt{20 \cdot 3} = 2.08 \text{ kg/s} \tag{16}$$

Example 2

The free response of a 1000-kg automobile with stiffness of k = 400,000 N/m is observed to be underdamped. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, determine the damping coefficient if the displacement at t_1 is measured to be 2 cm and 0.22 cm at t_2.

Knowing x_1 = 2 cm and x_2 = 0.22 cm and $t_2 = T + t_1$, therefore:

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{2}{0.22} = 2.207 \tag{17}$$

and:

$$\xi = \left(\frac{\delta}{\sqrt{4\pi^2 + \delta^2}}\right) = \left(\frac{2.207}{\sqrt{4\pi^2 + 2.207^2}}\right) = 0.331 \tag{18}$$

therefore, we can obtain the damping coefficient as

$$c = 2\xi\sqrt{km} = 2(0.331)\sqrt{400,000 \cdot 1,000} = 13256 \;\text{ kg/s} \tag{19}$$