Lecture Notes, Topic-3

Review from previous class

1. Solved the ODE (EOM) through observing a vibrating system
2. Obtain an equation for the free response of a system

Objectives for today's class

1. Review Euler's (pronounced oy-ler) formula
2. Obtain a general solution for vibrating systems

Lecture

Review of complex numbers.

Vibration analysis uses complex numbers to solve the EOM's differential equation. In this class the imaginary number is termed j (sometimes referred to as i): such that:

$$j = \sqrt{-1} \tag{1}$$

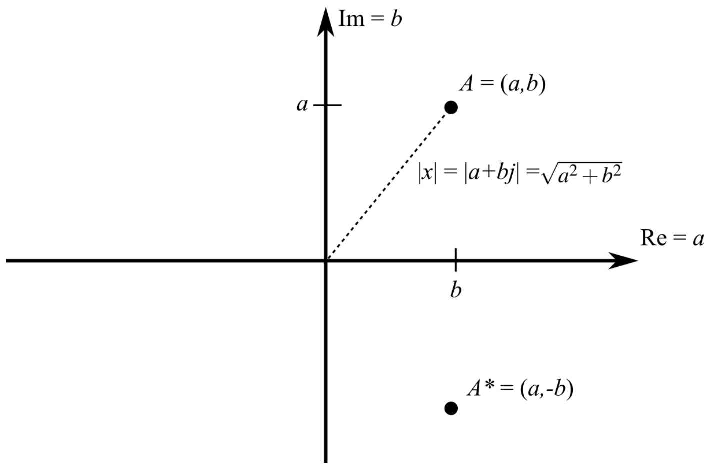
and:

$$j^2 = -1 \tag{2}$$

a general complex number. x can be expressed as

$$x = a + bj \tag{3}$$

here, a is referred to as the real number and b is the imaginary part of the number x. Such complex numbers can be represented in the complex plane, also called a Argand plot. The absolute value or modules is defined as $|x|$ presented on the Argand plot.



A and A^* prime are complex conjugate pairs. In mathematics, the complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. In other words, a conjugate pair is $a+bj$ and $a-bj$.

Definition $\rightarrow$ **con·ju·gate** (adjective): Coupled, connected, or related.

Review of Euler's formula.

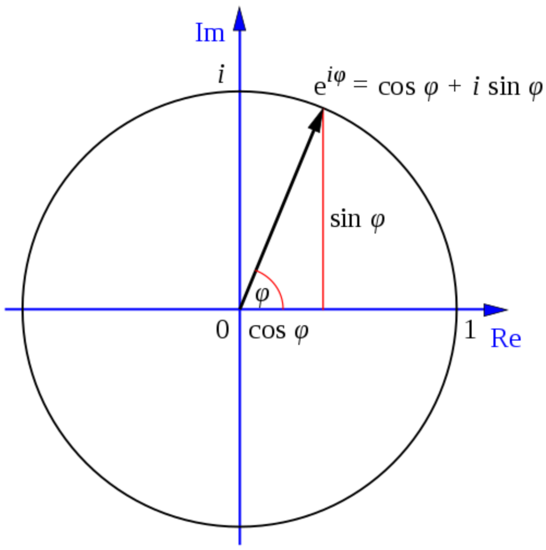
Euler's formula, named after Swiss engineer and mathematician Leonhard Euler (1707-1783), is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number x,

$$e^{j\psi} = \cos(\psi) + j\sin(\psi) \tag{4}$$

where $j = \sqrt{-1}$. This equation can also be expressed as:

$$e^{-j\psi} = \cos(\psi) - j\sin(\psi) \tag{5}$$

This can be expressed in terms of polar coordinates as:



Theory of elementary differential equations.

We can also solve the following EOM:

$$m\ddot{x} + kx = 0 \tag{6}$$

in a more analytical manner using the theory of elementary differential equations. To do this we have to assume a solution, in the form of

$$x(t) = ae^{\lambda t} \tag{7}$$

here, a and t are nonzeros constants that need to be determined. Using successive differentiation, we get:

$$\dot{x}(t) = \lambda ae^{\lambda t} \tag{8}$$

and

$$\ddot{x}(t) = \lambda^2 ae^{\lambda t} \tag{9}$$

therefore, $m\ddot{x}(t) + kx(t) = 0$ becomes:

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \tag{10}$$

Now we divide by $ae^{\lambda t}$ to obtain the **characteristic equation**:

$$m\lambda^2 + k = 0 \tag{11}$$

We can do this because $ae^{\lambda t}$ is never zero, therefore, we never divide by zero. The quadratic formula gives us:

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \tag{12}$$

remember that $\omega_n = \sqrt{\frac{k}{m}}$. Notice that the $\pm$ tells us there are two solutions to this problem. So, putting λ back into our assumed solution, we get two solutions:

$$x(t) = a_1 e^{+\omega_n jt} \tag{13}$$

and

$$x(t) = a_2 e^{-\omega_n jt} \tag{14}$$

As we deal with linear systems, we know that the sum of the solutions is also a solution, resulting in:

$$x(t) = a_1 e^{+\omega_n jt} + a_2 e^{-\omega_n jt} \tag{15}$$

where a_1 and a_2 are complex valued constants of integration. This equation derived using Euler's formula is equivalent to the $A\sin(\omega_n + \phi)$. To recover the previously assumed solution we apply the knowledge that a_1 and a_2 are complex congregate pairs and as such the magnitude can be expressed as $a_1 = a_2$. Using Euler's polar notation, a_1 and a_2 can be expressed as

$$a_1 = a_2 = ae^{j\psi} \tag{16}$$

where a and ψ are real numbers, the equation becomes:

$$x(t) = ae^{j(\omega_n t + \psi)} + ae^{-j(\omega_n t + \psi)} \tag{17}$$

this becomes:

$$x(t) = a(e^{j(\omega_n t + \psi)} + e^{-j(\omega_n t + \psi)}) \tag{18}$$

Remembering Euler's equations from before, this becomes:

$$x(t) = a\big(\cos(\omega_n t + \psi) + j\sin(\omega_n t + \psi) + \cos(\omega_n t + \psi) - j\sin(\omega_n t + \psi)\big) \tag{19}$$

combining the "cos" terms and canceling out the "sin" terms this becomes:

$$x(t) = 2a \cdot \cos(\omega_n t + \psi) \tag{20}$$

This is equivalent to $x(t) = A\sin(\omega_n t + \phi)$ if we take $A = 2a$ and knowing $\sin(\phi) = \cos(\phi + \psi)$.

Formulate the general solution from Euler's expression.

From Euler's equation we saw that:

$$x(t) = a_1 e^{+\omega_n jt} + a_2 e^{-\omega_n jt} \tag{21}$$

we can expand this into the form:

$$x(t) = a_1\big(\cos(\omega_n t) + j\sin(\omega_n t)\big) + a_2\big(\cos(\omega_n t) - j\sin(\omega_n t)\big) \tag{22}$$

using trigonometric functions. This equates to:

$$x(t) = (a_1 + a_2) \cdot \cos(\omega_n t) + (a_1 - a_2) j \cdot \sin(\omega_n t) \tag{23}$$

as $x(t)$ is always real, we can define:

$$A_1 = (a_1 + a_2) \tag{24}$$

and

$$A_2 = (a_1 - a_2) j \tag{25}$$

lastly, as the **general solution** is written as:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{26}$$

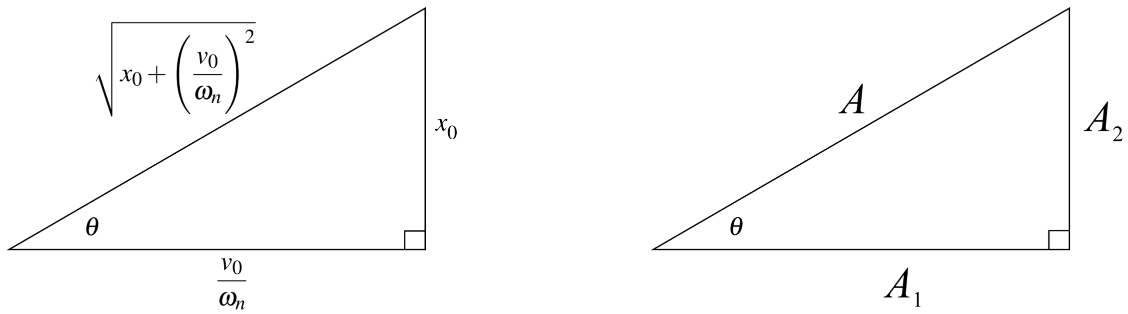
This is the general solution for the EOM ($m\ddot{x} + kx = 0$) of the considered oscillating system where A_1 and A_2 are defined as:

$$A = \sqrt{A_1^2 + A_2^2} \tag{27}$$

and

$$\phi = \tan^{-}1\left(\frac{A_1}{A_2}\right) \tag{28}$$

These are obtained from a trigonometric relationship, similar to that used before:



again, A and ϕ are:

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \tag{29}$$

$$\phi = \tan^{-1}\left(\frac{x_0 \omega_n}{v_0}\right) \tag{30}$$

One Equation in three forms:

Form one, for $m\ddot{x} + kx = 0$ subject to nonzero initial conditions can be written as:

$$x(t) = a_1 e^{+\omega_n jt} + a_2 e^{-\omega_n jt} \tag{31}$$

where a_1 and a_2 are complex terms. Form two is:

$$x(t) = A\sin(\omega_n t + \phi) \tag{32}$$

while form three is:

$$x(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) \tag{33}$$

where A, ϕ, A_1, and A_2, are all real-valued constants. Each set of constants can be related to each other by:

$$A = \sqrt{A_1^2 + A_2^2} \qquad \phi = \tan^{-}1\left(\frac{A_1}{A_2}\right) \tag{34}$$

$$A_1 = (a_1 + a_2) \qquad A_2 = (a_1 - a_2)j \tag{35}$$

$$a_1 = \frac{A_1 - A_2 j}{2} \qquad a_2 = \frac{A_1 + A_2 j}{2} \tag{36}$$

Which follow from trigonometric identities and the Euler's formulas.

Example 1

Using the general solution:

$$x(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) \tag{37}$$

Calculate the values of A_1 and A_2 in terms of their initial conditions x_0 and v_0.

Solution: Knowing the following for x and $\dot{x}$:

$$x(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) \tag{38}$$

$$\dot{x}(t) = -A_1\omega_n\sin(\omega_n t) + A_2\omega_n\cos(\omega_n t) \tag{39}$$

Now apply the initial conditions, $x(0) = 0$ and $v(0) = 0$, this yields:

$$x(0) = x_0 = A_1 \tag{40}$$

$$\dot{x}(0) = v_0 = A_2\omega_n \tag{41}$$

Solving for A_1 and A_2 shows us:

$$A_1 = x_0, \text{ and } A_2 = \frac{v_0}{\omega_n} \tag{42}$$

thus:

$$x(t) = x_0\cos(\omega_n t) + \frac{v_0}{\omega_n}\sin(\omega_n t) \tag{43}$$

Important items from today

- The solution for a vibrating system can be expressed in various forms
- These forms relate to each other through Euler's equations