Finite Element Formulation of the Extended Euler-Bernoulli Beam with Axial Forces

Governing Equation

The governing equation for an Euler-Bernoulli beam with an axial force N is given by:

$$EIw''''(x) + Nw''(x) + \rho A w_{tt}(x,t) = 0, \tag{1}$$

where:

- $w(x,t)$ is the transverse displacement (how much the beam moves up and down),
- E is Young's modulus (stiffness of the material),
- I is the second moment of area (how resistant the beam is to bending),
- N is the tensile axial force (positive in tension, negative in compression),
- ρ is the material density (how heavy the beam is),
- A is the cross-sectional area (thickness of the beam).

An example of beam under analysis is shown in Figure 1. The beam is fixed at both ends and loaded transversely at its center by a point force P, resulting in internal axial forces N_A, N_B, bending moments M_A, M_B, and vertical reaction forces F_A, F_B.

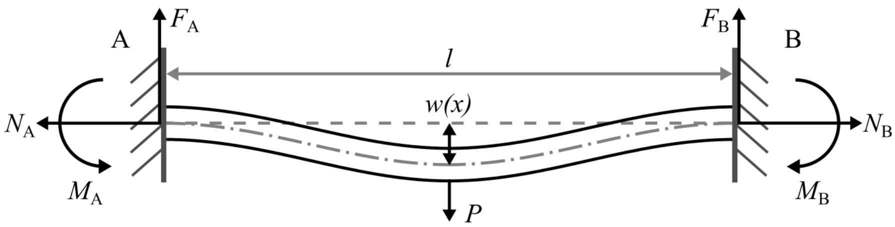

Figure 1: Free body diagram of a fixed-fixed beam subjected to a transverse point load.

Explanation for Trotter: This is the main rule that tells us how the beam bends and moves when a force is applied.

Weak Formulation

Multiplying by a test function $v(x)$ and integrating over the beam length L:

$$\int_0^L \left(EIw''''(x)v(x) + Nw''(x)v(x) + \rho A w_{tt}(x,t)v(x)\right) dx = 0. \tag{2}$$

Applying integration by parts twice,

$$\int_0^L EIw''(x)v''(x)dx + \int_0^L Nw''(x)v(x)dx + \int_0^L \rho A w_{tt}(x,t)v(x)dx = 0. \tag{3}$$

Explanation for Trotter: We multiply the equation by another function and integrate it. This helps us break the beam into smaller pieces for easier calculations.

Finite Element Approximation

We approximate the displacement field as

$$w(x,t) = \sum_{j=1}^{N} w_j(t)\phi_j(x), \tag{4}$$

where $\phi_j(x)$ are the shape functions. For an element of length h, the shape functions $\Phi(x)$ and corresponding nodal degrees of freedom are:

$$W_e = \begin{bmatrix} u_1 & w_1 & \theta_1 & u_2 & w_2 & \theta_2 \end{bmatrix}^\intercal. \tag{5}$$

Thus,

$$w_e(x,t) = \Phi(x)W_e(t). \tag{6}$$

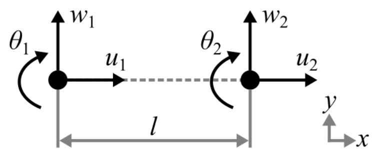

Figure 2: Degrees of freedom and internal forces for a 1D beam finite element.

Explanation for Trotter: We chop the beam into smaller sections and use math functions (shape functions) to describe how each section moves.

Element Stiffness and Mass Matrices

The element stiffness matrix is given by

$$K_{ij}^{(e)} = \int_0^h EI\phi_i''(x)\phi_j''(x)dx + \int_0^h N\phi_i'(x)\phi_j'(x)dx. \tag{7}$$

The integrals are evaluated using cubic Hermite shape functions:

$$\phi(x) = [1 - 3\xi^2 + 2\xi^3,\ h(\xi - 2\xi^2 + \xi^3),\ 3\xi^2 - 2\xi^3,\ h(-\xi^2 + \xi^3)] \tag{8}$$

with $\xi = x/h$. Taking first and second derivatives, and integrating term by term over $\xi \in [0,1]$, we obtain

$$K_e = \frac{EI}{h^3}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} + \frac{N}{h}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{9}$$

The element mass matrix is given by

$$M_{ij}^{(e)} = \int_0^h \rho A\phi_i(x)\phi_j(x)dx, \tag{10}$$

Using the same Hermite basis functions as for stiffness, and integrating symbolically, we obtain

$$M_e = \frac{\rho A h}{420}\begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix}. \tag{11}$$

Explanation for Trotter: The stiffness matrix tells us how resistant each small piece of the beam is to bending. The mass matrix accounts for the weight of the beam.

Global Assembly

The global system matrices are assembled as:

$$K = \sum_e K_e, \quad M = \sum_e M_e. \tag{12}$$

For a fixed-fixed beam, we apply boundary conditions:

$$K_{0,:} = K_{:,0} = 0, \quad K_{1,:} = K_{:,1} = 0, \quad K_{-1,:} = K_{:,-1} = 0, \quad K_{-2,:} = K_{:,-2} = 0, \tag{13}$$

with enforced constraints:

$$K_{0,0} = K_{1,1} = K_{-1,-1} = K_{-2,-2} = 1. \tag{14}$$

Explanation for Trotter: We put together all the small beam sections into one big system that represents the full beam.

Equation of Motion and Time Integration

The discretized equation of motion is:

$$M\ddot{W} + C\dot{W} + KW = F(t), \tag{15}$$

where the damping matrix is given by

$$C = \alpha M + \beta K. \tag{16}$$

Using Newmark-Beta time integration,

$$W_{n+1} = W_n + \dot{W}_n \Delta t + \frac{1}{2}\ddot{W}_n \Delta t^2, \tag{17}$$

we solve:

$$(K + \frac{\gamma}{\beta \Delta t} C + \frac{1}{\beta \Delta t^2} M) W_{n+1} = F_{n+1} + \text{previous terms}. \tag{18}$$

Explanation for Trotter: The beam moves over time, so we use math formulas to predict how it bends at each moment.

Control Force Calculation

Control forces are calculated and applied at specified control nodes, where they influence both axial and bending behaviors of the beam. The control force is computed as:

$$F_{\text{control}} = \text{Control Force Coefficients} \cdot \text{Error Terms}, \tag{19}$$

where the *Control Force Coefficients* represent parameters that define the magnitude of the force, and the *Error Terms* are the deviations from the desired system state (such as displacement, velocity, or acceleration). The generated moment from the control force is computed as:

$$M_{\text{control}} = F_{\text{control}} \cdot \frac{h}{2}, \tag{20}$$

where h is the thickness of the beam.

The control forces and moments are then applied to the system's equation of motion, specifically to the control node. The control forces are applied as axial forces at the control node, and moments are generated at the node to simulate the effect of actuators (such as piezoelectric actuators).

For a control node, the axial force F_{control} influences the beam's axial stiffness. Additionally, the moment M_{control} modifies the bending behavior of the beam. The axial force and moment are applied to the corresponding degrees of freedom (DOFs) at the control node, affecting the overall system dynamics. The control forces are incorporated into the equation of motion for the beam, influencing both axial and bending behaviors, as described by:

$$M\ddot{W} + C\dot{W} + KW = F_{\text{control}}(t), \tag{21}$$

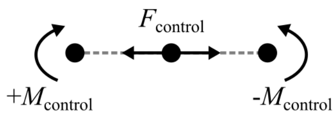

Figure 3: Control force F_{control} and generated moments M_{control} applied at the control node.

where M is the mass matrix, C is the damping matrix, K is the stiffness matrix, and $F_{\text{control}}(t)$ represents the control forces applied at the control node, similar to the discretized equation of motion.

Explanation for Trotter: The control force is calculated based on how much the beam is off from where you want it to be (for example, how much it's bending or moving). This force is applied at a specific point on the beam, called the control node, which could be the middle or another part. The control force is an axial force, meaning it pushes or pulls along the length of the beam. Along with this, a moment (or turning force) is created by the control force, which helps control the bending of the beam. Both the pushing/pulling force and the moment are used to adjust the beam's movement, and these changes are included in the equations that describe how the beam behaves, making sure it bends the way you want it to.