Finite Element Formulation of the Extended Euler-Bernoulli Beam with Axial Forces

Governing Equation

Nomenclature

E	[Pa]	Young's modulus (steel: 2.1×10^{11}).
A	[m^2]	Cross-sectional area (e.g. 0.01).
I	[m^4]	Second moment of area (e.g. 8.333×10^{-6}).
ρ	[kg/m^3]	Mass density (e.g. 7850).
L	[m]	Total beam length (e.g. 2).
L_e	[m]	Element length: $L_e = L/n_{\text{elem}}$.
x	[m]	Spatial coordinate.
t	[s]	Time coordinate.
$u(x,t)$	[m]	Axial displacement.
$w(x,t)$	[m]	Transverse displacement.
$\delta u,\ \delta w$	–	Virtual (test) displacements.
k_e	–	Full element stiffness matrix (6×6).
m_e	–	Full element mass matrix (6×6).

1 Governing Equations

The classical Euler–Bernoulli beam equation for transverse deflection $w(x,t)$ is

$$EI\,w''''(x,t) + \rho A\,\ddot{w}(x,t) = q(x,t). \tag{1}$$

With an added axial degree of freedom, the beam's strong form separates into two equations:

$$\rho A\,\ddot{u}(x,t) - \big(EA\,u'(x,t)\big)' = q(x,t), \tag{2}$$

$$\rho A\,\ddot{w}(x,t) - \big(EI\,w''(x,t)\big)'' = q(x,t). \tag{3}$$

An example of beam under analysis is shown in Figure 1. The beam is fixed at both ends and loaded transversely at its center by a point force P, resulting in internal axial forces N_A, N_B, bending moments M_A, M_B, and vertical reaction forces F_A, F_B.
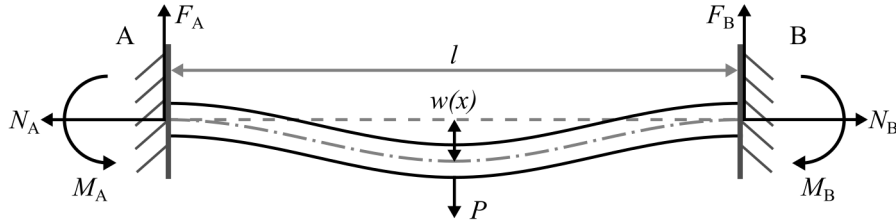

Figure 1: Free body diagram of a fixed-fixed beam subjected to a transverse point load.

Explanation for Trotter: This is the main rule that tells us how the beam bends and moves when a force is applied.

2 Weak Form

To reduce the continuity requirements on trial and test spaces and to impose natural boundary conditions, we derive the variational form by multiplying each strong equation by its virtual displacement and integrating by parts.

2.1 Axial

Multiply Eq. (2) by δu, integrate by parts:

$$\int_0^L EA\,u'\,\delta u'\,dx - \left[EA\,u'\,\delta u\right]_0^L = 0 \implies \int_0^L EA\,u'\,\delta u'\,dx = 0. \tag{4}$$

2.2 Bending

Multiply Eq. (3) by δw, integrate by parts twice:

$$\int_0^L EI\,w''\,\delta w''\,dx - \left[EI\,w''\,\delta w' - (EI\,w'')'\,\delta w\right]_0^L = 0 \implies \int_0^L EI\,w''\,\delta w''\,dx = 0. \tag{5}$$

3 Finite-Element Interpolations

3.1 Axial (linear)

We approximate $u(x)$ over each element of length L_e by

$$N^{(u)}(x) = \begin{pmatrix} 1 - \frac{x}{L_e} \\ \frac{x}{L_e} \end{pmatrix}, \qquad u(x) = \left[N^{(u)}(x)\right]^T (u_1, u_2)^T. \tag{6}$$

3.2 Transverse (Hermite cubic)

To capture bending curvature with C^1 continuity, we use four cubic polynomials:

$$\begin{aligned}
N_1(x) &= 1 - 3(\tfrac{x}{L_e})^2 + 2(\tfrac{x}{L_e})^3, \\
N_2(x) &= x[1 - 2(\tfrac{x}{L_e}) + (\tfrac{x}{L_e})^2], \\
N_3(x) &= 3(\tfrac{x}{L_e})^2 - 2(\tfrac{x}{L_e})^3, \\
N_4(x) &= x[-(\tfrac{x}{L_e}) + (\tfrac{x}{L_e})^2].
\end{aligned} \tag{7}$$

Mapping $x \to \xi = x/L_e \in [0,1]$ for integration:

$$\begin{aligned}
N_1(\xi) &= 1 - 3\xi^2 + 2\xi^3, \quad N_2(\xi) = L_e\,\xi(1 - 2\xi + \xi^2), \\
N_3(\xi) &= 3\xi^2 - 2\xi^3, \quad N_4(\xi) = L_e\,\xi(-\xi + \xi^2).
\end{aligned} \tag{8}$$

4 Element Stiffness Matrices

Substituting the above interpolations into the weak forms (4)–(5) and evaluating the resulting integrals yields:

4.1 Axial

By inserting $u(x)$ from (6) into (4) and noting the constant derivative,

$$k_e^{(\text{axial})} = \int_0^{L_e} EA\,\frac{dN^{(u)}}{dx}\frac{dN^{(u)}}{dx}^T dx = \frac{EA}{L_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{9}$$

4.2 Bending

From the bending weak form (5), define the curvature–shape vector in ξ:

$$B(\xi) = \frac{d^2 N^{(w)}}{dx^2} = \frac{1}{L_e^2}[6\xi - 6, \; 3\xi - 4, \; -6\xi + 6, \; 3\xi - 2], \tag{10}$$

then

$$k_e^{(\text{bending})} = EI \int_0^{L_e} B^T B \, dx = \frac{EI}{L_e^3} \begin{pmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{pmatrix}. \tag{11}$$

4.3 Assembled stiffness

Finally, the full 6×6 element stiffness matrix is

$$k_e = \begin{pmatrix} k_e^{(\text{axial})} & \mathbf{0} \\ \mathbf{0} & k_e^{(\text{bending})} \end{pmatrix}. \tag{12}$$

5 Element Mass Matrices

The element mass matrix comes from the beam's kinetic energy:

$$T = \tfrac{1}{2} \int_0^{L_e} \rho A \, \dot{u}^2(x,t) \, dx + \tfrac{1}{2} \int_0^{L_e} \rho A \, \dot{w}^2(x,t) \, dx. \tag{13}$$

5.1 Axial

We start with the axial kinetic energy

$$T_{\text{axial}} = \tfrac{1}{2} \int_0^{L_e} \rho A \, \dot{u}^2 \, dx. \tag{14}$$

Substitute the linear interpolation (6):

$$\dot{u}(x,t) = \left[N^{(u)}(x) \right]^T \dot{\mathbf{u}}, \tag{15}$$

which gives

$$T_{\text{axial}} = \tfrac{1}{2} \dot{\mathbf{u}}^T \left[\int_0^{L_e} \rho A \, N^{(u)} N^{(u)T} \, dx \right] \dot{\mathbf{u}} = \tfrac{1}{2} \dot{\mathbf{u}}^T m_e^{(\text{axial})} \dot{\mathbf{u}}. \tag{16}$$

Hence the consistent axial mass block is

$$m_e^{(\text{axial})} = \int_0^{L_e} \rho A \, N^{(u)} N^{(u)T} \, dx = \frac{\rho A L_e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{17}$$

5.2 Bending

Likewise, for bending

$$T_{\text{bend}} = \tfrac{1}{2} \int_0^{L_e} \rho A \, \dot{w}^2 \, dx. \tag{18}$$

Using the Hermite-cubic interpolation (7):

$$\dot{w}(x,t) = \left[N^{(w)}(\xi) \right]^T \dot{\mathbf{w}}, \tag{19}$$

we obtain

$$T_{\text{bend}} = \tfrac{1}{2} \dot{\mathbf{w}}^T \left[\int_0^{L_e} \rho A \, N^{(w)}(\xi) N^{(w)T}(\xi) \, dx \right] \dot{\mathbf{w}} = \tfrac{1}{2} \dot{\mathbf{w}}^T m_e^{(\text{bending})} \dot{\mathbf{w}}. \tag{20}$$

Thus the bending mass block is

$$m_e^{(\text{bending})} = \int_0^{L_e} \rho A \, N^{(w)}(\xi) N^{(w)T}(\xi) \, dx = \frac{\rho A L_e}{420} \begin{pmatrix} 156 & 22L_e & 54 & -13L_e \\ 22L_e & 4L_e^2 & 13L_e & -3L_e^2 \\ 54 & 13L_e & 156 & -22L_e \\ -13L_e & -3L_e^2 & -22L_e & 4L_e^2 \end{pmatrix}. \tag{21}$$

5.3 Assembled Mass

Putting axial and bending together gives the full element mass matrix:

$$m_e = \begin{pmatrix} m_e^{(\text{axial})} & \mathbf{0} \\ \mathbf{0} & m_e^{(\text{bending})} \end{pmatrix}. \tag{22}$$

6 Equation of Motion and Time Integration

The global equation of motion for the discretized beam—assembled from all element mass, damping, and stiffness contributions—can be written in matrix form as

$$\underbrace{\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{u}_1 \\ \ddot{w}_1 \\ \ddot{\theta}_1 \\ \vdots \\ \ddot{\theta}_N \end{bmatrix}}_{\ddot{W}} + \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dot{u}_1 \\ \dot{w}_1 \\ \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_N \end{bmatrix}}_{\dot{W}} + \underbrace{\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} u_1 \\ w_1 \\ \theta_1 \\ \vdots \\ \theta_N \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \\ \vdots \\ F_n(t) \end{bmatrix}}_{F(t)}. \tag{23}$$

Here:

- $W = [\,u_1,\ w_1,\ \theta_1,\ \ldots,\ u_N,\ w_N,\ \theta_N\,]^T$ collects all $3N$ nodal degrees of freedom: u_i (axial displacement), w_i (transverse displacement), and θ_i (slope) at node i.

- $\ddot{W}$ and $\dot{W}$ are the corresponding acceleration and velocity vectors.

- $M = [m_{ij}]$ is the global mass matrix, assembled from each element's consistent mass blocks (Eqs. (17) and (21)).

- $C = [c_{ij}]$ is the global damping matrix (e.g. Rayleigh damping $C = \alpha M + \beta K$).

- $K = [k_{ij}]$ is the global stiffness matrix, assembled from each element's axial and bending stiffness (Eqs. (9)–(11)).

- $F(t) = [F_1(t), F_2(t), \ldots, F_n(t)]^T$ is the vector of external (or control) forces and moments applied at each DOF.

Equation (23) succinctly represents the dynamic equilibrium of the entire beam structure in its finite-element discretization.

Newmark–Beta Implicit Time Integration (Step-by-Step)

We solve the second-order equation of motion:

$$\mathbf{M\ddot{w}} + \mathbf{C\dot{w}} + \mathbf{Kw} = \mathbf{f}(t), \tag{24}$$

where:

- $\mathbf{M}$: Mass matrix (inertia),

- $\mathbf{C} = \alpha_R \mathbf{M} + \beta_R \mathbf{K}$: Rayleigh damping matrix,

- $\mathbf{K}$: Stiffness matrix,

- $\mathbf{f}(t)$: External force vector,

- $\mathbf{w}, \dot{\mathbf{w}}, \ddot{\mathbf{w}}$: Displacement, velocity, and acceleration vectors.

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Step 1: Initial Conditions

We assume known:

$$\mathbf{w}_0, \quad \dot{\mathbf{w}}_0$$

Then compute the initial acceleration $\ddot{\mathbf{w}}_0$ using the equation of motion:

$$\ddot{\mathbf{w}}_0 = \mathbf{M}^{-1}\left[\mathbf{f}_0 - \mathbf{C}\dot{\mathbf{w}}_0 - \mathbf{K}\mathbf{w}_0\right] \tag{25}$$

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Step 2: Define Effective Stiffness Matrix

The Newmark parameters β and γ (typically $\beta = \frac{1}{4}, \gamma = \frac{1}{2}$) control stability and accuracy. We define:

$$\mathbf{K}_{\text{eff}} = \mathbf{K} + \frac{\gamma}{\beta\Delta t}\mathbf{C} + \frac{1}{\beta\Delta t^2}\mathbf{M} \tag{26}$$

This matrix stays constant if $\mathbf{M}, \mathbf{C}, \mathbf{K}$ are time-invariant.

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Step 3: Construct Effective Force

The force vector on the right-hand side includes known values from the previous time step:

$$\mathbf{F}_{\text{eff}} = \mathbf{f}_{n+1} + \mathbf{M}\left(\frac{1}{\beta\Delta t^2}\mathbf{w}_n + \frac{1}{\beta\Delta t}\dot{\mathbf{w}}_n + \frac{1-2\beta}{2\beta}\ddot{\mathbf{w}}_n\right) \tag{27}$$

$$+ \mathbf{C}\left(\frac{\gamma}{\beta\Delta t}\mathbf{w}_n + \left(\frac{\gamma}{\beta}-1\right)\dot{\mathbf{w}}_n + \Delta t\left(\frac{\gamma}{2\beta}-1\right)\ddot{\mathbf{w}}_n\right) \tag{28}$$

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Step 4: Solve for Displacement at Next Step

$$\mathbf{w}_{n+1} = \mathbf{K}_{\text{eff}}^{-1}\mathbf{F}_{\text{eff}} \tag{29}$$

This is the main linear solve for each time step.

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Step 5: Update Acceleration

Once $\mathbf{w}_{n+1}$ is known, compute:

$$\ddot{\mathbf{w}}_{n+1} = \frac{1}{\beta\Delta t^2}\left(\mathbf{w}_{n+1} - \mathbf{w}_n - \Delta t\dot{\mathbf{w}}_n\right) - \frac{1-2\beta}{2\beta}\ddot{\mathbf{w}}_n \tag{30}$$

This gives the actual physical acceleration at t_{n+1}.

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Step 6: Update Velocity

Now update velocity:

$$\dot{\mathbf{w}}_{n+1} = \dot{\mathbf{w}}_n + \Delta t\left[(1-\gamma)\ddot{\mathbf{w}}_n + \gamma\ddot{\mathbf{w}}_{n+1}\right] \tag{31}$$

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Interpretation:

- The system first predicts displacement using known values. - The acceleration and velocity are then *corrected* based on the actual displacement obtained. - Damping is included directly in the matrix system (via Rayleigh terms), which affects how the system resists motion — especially at high frequency.

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Typical Parameter Choice:

$$\beta = \frac{1}{4}, \quad \gamma = \frac{1}{2} \quad \Rightarrow \quad \text{Unconditionally stable and second-order accurate}$$

Control Force Calculation

Control forces are introduced at selected nodes along the beam to actively influence both axial and bending responses. As shown in Figure 2, each actuator applies a concentrated axial force F_{control} along the beam axis and induces a moment M_{control} by acting off-center relative to the beam's neutral axis. These control actions are designed to counteract undesirable vibrations or deflections and are computed based on real-time deviations from a desired trajectory or shape.

The control force is defined as

$$F_{\text{control}} = K_{\text{ctrl}} \cdot e(t), \tag{32}$$

where K_{ctrl} represents a control gain and $e(t)$ is the error between the measured and desired beam states. The corresponding moment is defined as

$$M_{\text{control}} = F_{\text{control}} \cdot \frac{h}{2}, \tag{33}$$

where h is the beam thickness, and the factor $h/2$ accounts for the distance from the neutral axis to the surface where the actuator applies the force.

In the finite element framework, F_{control} is applied to the axial degree of freedom u at the control node, while M_{control} contributes to the rotational degree of freedom θ at the same node. These contributions are inserted directly into the global force vector in the positions corresponding to those degrees of freedom.

The modified equation of motion, including the control forces and moments, becomes

$$M\ddot{W} + C\dot{W} + KW = F_{\text{control}}(t), \tag{34}$$

where M, C, and K are the assembled global mass, damping, and stiffness matrices, and $F_{\text{control}}(t)$ contains the time-varying control inputs mapped to the appropriate DOFs.
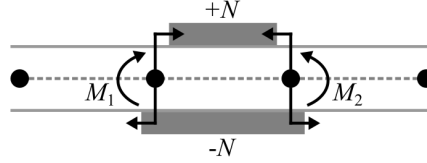

Figure 2: Control force F_{control} and generated moment M_{control} applied at the control node.

Explanation for Trotter: The control force is calculated based on how much the beam is off from where you want it to be. This force is applied at a specific point on the beam, called the control node. Along with this force, a moment is created to control bending. Both are included in the beam's equations of motion to guide its behavior.