Modified Euler–Bernoulli Beam Theory with Axial Displacement: Linear axial interpolation

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1 Introduction

The classical Euler–Bernoulli beam theory models only transverse displacement $w(x,t)$ and rotation

$$\theta(x,t) \;=\; \frac{\partial w}{\partial x}.$$

To extend this model, we add an axial displacement $u(x,t)$ as an additional degree of freedom. This allows the beam to deform both in bending and in axial stretch simultaneously.

2 Degrees of freedom and physical meaning

Each node in the extended model carries three DOFs:

$$u \quad \text{(axial)}, \qquad w \quad \text{(transverse)}, \qquad \theta \quad \text{(rotation)}.$$

A two-node element therefore has the local vector $\{u_1,\, w_1,\, \theta_1,\, u_2,\, w_2,\, \theta_2\}^T$.

3 Shape functions

3.1 Axial displacement $u(x)$

We use linear interpolation over an element of length L_e:

$$u(x) \;=\; N_1^{(u)}(x)\,u_1 \;+\; N_2^{(u)}(x)\,u_2, \quad N_1^{(u)} = 1 - \frac{x}{L_e}, \quad N_2^{(u)} = \frac{x}{L_e}.$$

3.2 Transverse displacement $w(x)$

We use Hermite cubic polynomials:

$$w(x) = N_1^{(w)}(x)\,w_1 + N_2^{(w)}(x)\,\theta_1 + N_3^{(w)}(x)\,w_2 + N_4^{(w)}(x)\,\theta_2,$$

where

$$\begin{aligned}
N_1^{(w)}(x) &= 1 - 3\Big(\tfrac{x}{L_e}\Big)^2 + 2\Big(\tfrac{x}{L_e}\Big)^3,\\
N_2^{(w)}(x) &= x\Big(1 - 2\tfrac{x}{L_e} + \big(\tfrac{x}{L_e}\big)^2\Big),\\
N_3^{(w)}(x) &= 3\Big(\tfrac{x}{L_e}\Big)^2 - 2\Big(\tfrac{x}{L_e}\Big)^3,\\
N_4^{(w)}(x) &= x\Big(-\tfrac{x}{L_e} + \big(\tfrac{x}{L_e}\big)^2\Big).
\end{aligned}$$

4 Strain energy of the element

The total strain energy is the sum of an axial part and a bending part:

$$U = \frac{1}{2}\int_0^{L_e} EA\Big(\tfrac{du}{dx}\Big)^2\,dx + \frac{1}{2}\int_0^{L_e} EI\Big(\tfrac{d^2w}{dx^2}\Big)^2\,dx.$$

5 Derivation of element stiffness matrix

5.1 Axial Part

Because du/dx is constant,

$$\varepsilon = \frac{du}{dx} = -\frac{u_1}{L_e} + \frac{u_2}{L_e},$$

one obtains

$$U_{\text{axial}} = \frac{EA}{2L_e}(u_1^2 - 2u_1u_2 + u_2^2), \quad \Longrightarrow \quad k^{(\text{axial})} = \frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

5.2 Bending part

The standard Hermite–Euler–Bernoulli bending stiffness is

$$k^{(\text{bending})} = \frac{EI}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix}.$$

6 Full element stiffness matrix

Placing the 2×2 axial block in the $\{u_1, u_2\}$ rows and cols, and the 4×4 bending block in the $\{w_1, \theta_1, w_2, \theta_2\}$ rows and cols, gives the full 6×6 matrix

$$k_e = \begin{bmatrix} \frac{EA}{L_e} & -\frac{EA}{L_e} & 0 & 0 & 0 & 0 \\ -\frac{EA}{L_e} & \frac{EA}{L_e} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12EI}{L_e^3} & \frac{6EI}{L_e^2} & -\frac{12EI}{L_e^3} & \frac{6EI}{L_e^2} \\ 0 & 0 & \frac{6EI}{L_e^2} & \frac{4EI}{L_e} & -\frac{6EI}{L_e^2} & \frac{2EI}{L_e} \\ 0 & 0 & -\frac{12EI}{L_e^3} & -\frac{6EI}{L_e^2} & \frac{12EI}{L_e^3} & -\frac{6EI}{L_e^2} \\ 0 & 0 & \frac{6EI}{L_e^2} & \frac{2EI}{L_e} & -\frac{6EI}{L_e^2} & \frac{4EI}{L_e^3} \end{bmatrix}.$$

7 Dimensional analysis and units

- Entries like $\frac{EA}{L_e}$ carry units N/m.
- Mixed terms like $\frac{6EI}{L_e^2}$ carry units N.
- Pure rotation–rotation terms like $\frac{4EI}{L_e}$ carry units N·m.

8 Note about radians

In SI, radians are dimensionless. Thus a term such as $(EI/L_e)\,\theta$ has units N·m when θ is in radians.

llustration of the beam element

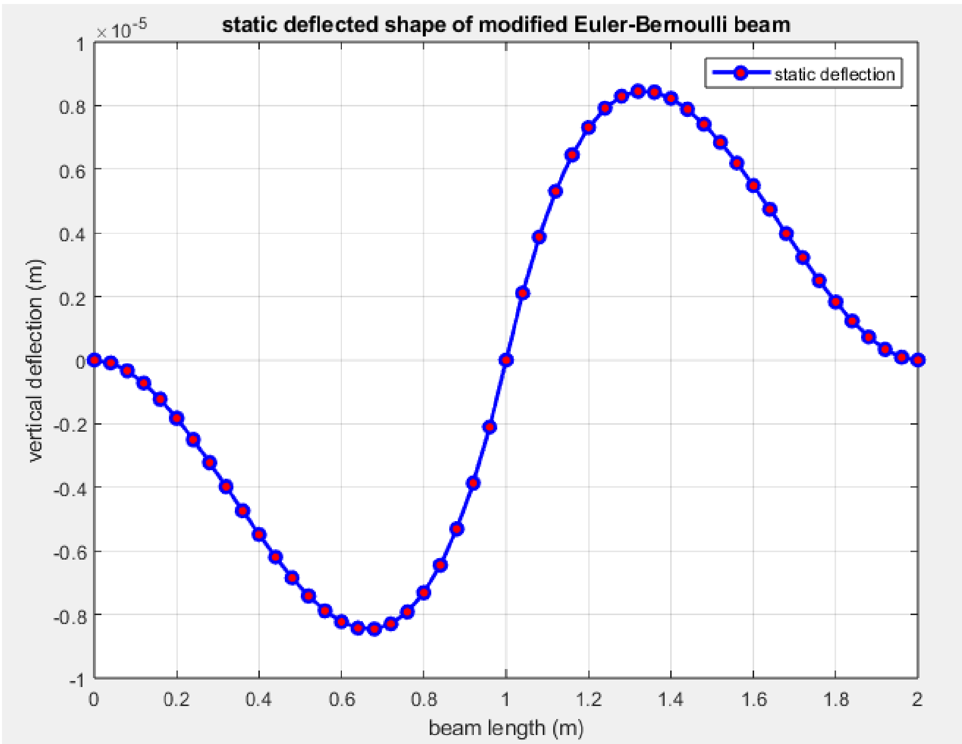

Figure 1: Schematic of the deflection extended Euler–Bernoulli beam element by applying moment at the middle of the beam, using linear interpolating function.