derivation of FEM for extended Euler–Bernoulli beam (Quadratic axial + Cubic bending)

Step 1: quadratic axial element (mapping from x-domain to ξ-domain)

We define the element over $x \in [x_1, x_2]$. To simplify integrals, map to $\xi \in [-1, 1]$:

$$x = \frac{L_e}{2}\,\xi + \frac{x_1 + x_2}{2} \qquad \Longrightarrow \qquad \frac{d}{dx} = \frac{2}{L_e}\,\frac{d}{d\xi}.$$

Interpolate axial displacement as

$$u(\xi) = N_1(\xi)\,u_1 + N_2(\xi)\,u_m + N_3(\xi)\,u_2,$$

with

$$N_1(\xi) = \tfrac{1}{2}\,\xi(\xi - 1), \quad N_2(\xi) = 1 - \xi^2, \quad N_3(\xi) = \tfrac{1}{2}\,\xi(\xi + 1).$$

Compute strain:

$$\varepsilon(\xi) = \frac{du}{dx} = \frac{2}{L_e}\,\frac{du}{d\xi} = \frac{2}{L_e}\,B^{(u)}(\xi)\begin{bmatrix} u_1 \\ u_m \\ u_2 \end{bmatrix},$$

where

$$B^{(u)}(\xi) = \begin{bmatrix} \xi - \frac{1}{2} & -2\xi & \xi + \frac{1}{2} \end{bmatrix}.$$

Thus the axial stiffness matrix is

$$k_e^{(\text{axial})} = EA\int_{-1}^{1} (B^{(u)})^T B^{(u)}\,\frac{L_e}{2}\,d\xi = \frac{2EA}{L_e}\begin{bmatrix} \frac{7}{6} & -\frac{4}{3} & \frac{1}{6} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ \frac{1}{6} & -\frac{4}{3} & \frac{7}{6} \end{bmatrix}.$$

Step 2: Cubic bending element derivation

With Hermite cubic shape functions $N_1^{(w)}, \ldots, N_4^{(w)}$:

$$w(\xi) = N_1^{(w)}\,w_1 + N_2^{(w)}\,\theta_1 + N_3^{(w)}\,w_2 + N_4^{(w)}\,\theta_2,$$

and curvature

$$\kappa = -\frac{d^2 w}{dx^2} = -\left(\tfrac{2}{L_e}\right)^2 \frac{d^2 w}{d\xi^2}.$$

The bending stiffness becomes

$$k_e^{(\text{bending})} = \int_{-1}^{1} EI\left(\tfrac{d^2 N}{dx^2}\right)^T \left(\tfrac{d^2 N}{dx^2}\right)\frac{L_e}{2}\,d\xi = \frac{EI}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix}.$$

Step 3: Merge axial and bending

Using DOFs $\{u_1, w_1, \theta_1, u_m, u_2, w_2, \theta_2\}$,

$$k_e = \begin{bmatrix} k_{3\times3}^{(\text{axial})} & \mathbf{0} \\ \mathbf{0} & k_{4\times4}^{(\text{bending})} \end{bmatrix}.$$

Step 4: Apply boundary conditions

For a fixed–fixed element:

$$u_1 = w_1 = \theta_1 = 0, \quad u_2 = w_2 = \theta_2 = 0.$$

Remove those rows and columns from the global K and F.

Step 5: Solve for displacement

After reduction:

$$K_{\text{red}}\, U = F_{\text{red}} \quad \Longrightarrow \quad U = K^{-1}F.$$

llustration of the beam element

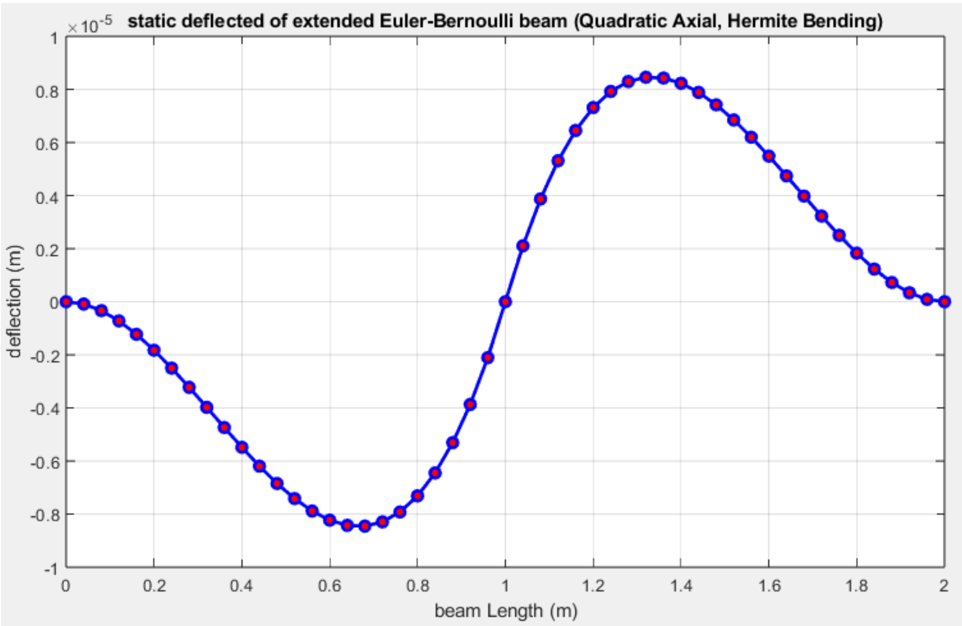

Figure 1: Schematic of the deflection extended Euler–Bernoulli beam element by applying moment at the middle of the beam