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# DIGITAL SAT® MATH ORANGE BOOK

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J. Ernest Gotta, Daniel Kirchheimer, and George Rimakis

# 1600.io Digital SAT® Math Orange Book

J. Ernest Gotta
Daniel Kirchheimer
George Rimakis

A DIFFICULT SUM.

HARRY is pulling his hair, as if that would help him out of the difficulty. The figures won't come right, do what he will, and he has worked at it an hour. Many little boys would give it up; but Harry is persevering, and intends to go over the sum until he finds out his mistake.

*My Pretty Present*. (New York: Thomas Nelson & Sons, 1789)

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J. Ernest Gotta, Daniel Kirchheimer, and George Rimakis

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Joe, you made this possible, and your unwavering loyalty and commitment cannot be forgotten. Megan, the only reason I've done anything is for you. John and Paige, I couldn't wish for more fun and intelligent kids. Mom and Dad, you inspire me to be a good person, a good father, and a good teacher. Nettie, Billy, Steffie, Keith, Danny, Jared, Matt, Sean, and Brandon, you shaped the humor needed to survive writing a book this long. Mustafa, Radhika, Mahoney, Timo, Elijah, Carolina, Arun, Lenny, Corbett, Will, McKayla, Sarah, Annesha, Selena, Lana, and Tanay, I'm honored to have worked alongside you. Prof. Goloubeva and Mr. Speidel, you showed me what it was to be a math educator. My former students, you inspired many of the weirdos found in the problems.

J. Ernest

My contributions to this book are dedicated to my amazing parents, who celebrated learning every day and who made me believe that I could do anything; to my beautiful, patient, and indulgent wife Jill and my brilliant son Matthew, who both endured a ceaseless barrage of terrible dad jokes (but I repeat myself) while I worked on this text; and to my extraordinary daughter Danielle, the world's best 3rd-grade teacher, who knew long before I did what a joy it is to help educate young people.

Dan

This project is dedicated to all the students reading this book right now who dare to become smarter. You've made it this far: remember that the pursuit of knowledge, with its ups and downs, is always a worthy adventure. I would also like to dedicate this book to my parents, who have supported me in everything I have ever done; to my sister Katherine, who shared her love of math with me at an early age; and to all of the teachers and professors I have had the good fortune to learn from. Without their support and inspiration, I don't think that this book would have been possible.

George

*We could use up two Eternities in learning all that is to be learned about our own world and the thousands of nations that have arisen and flourished and vanished from it. Mathematics alone would occupy me eight million years.*

–Mark Twain

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# Preface

Why did we decide to write an SAT math book? Amazon is groaning under the weight of such works. Does the world really need another? Our answer was an emphatic "yes." We felt that existing books either relied too much on tricks and tactics for trying to game the test, or the topics they taught weren't well-aligned with the current style of SAT, or the instruction was cursory, confusing, and scattershot, and these realizations crystallized our shared determination to wade into this crowded field and craft a contribution based on our collective wisdom.

Though the notion of creating a math book had been bandied about among us for some time, credit for initiating the project goes to Joe Zirkel, founder of Clarion Prep, who pushed the boat away from shore. Using our amazing math skills, we computed that it would take no more than six months to complete, and we only missed that estimate by a factor of four, which is close enough for jazz. Despite (or, perhaps, because of) the intensity and length of this process, the three authors forged an extraordinary working relationship that exceeded our expectations, and though the hours (and days, and months) have been long, the gratification we received along the way has been immense, and we could not be prouder of each other's efforts, contributions, willingness to accept constructive criticism, and good humor along the way. We set out to create the finest SAT math book possible, and we hope the reader finds that we have indeed succeeded.

Finally, a word about us. Collectively, we've helped thousands of students of all abilities, backgrounds, and educational levels prepare for the SAT. We operate under the conceit that we see things that others don't, and with the humility that drives us to believe there are always better ways to teach. This book, in its structure, tone, and approach, is a distillation of everything we've observed, learned, and implemented over the years with great success, and we think it's the best SAT math book ever written.

# Acknowledgements

We must offer our profound thanks to the thousands of wonderful students who contributed to this book through their innumerable interactions with us as we helped them educate themselves. This encompasses the legions of people who have entrusted to us a part of their futures whether through our years of private tutoring, as members of 1600.io, as participants in our rich and diverse Discord server, via their posts and comments on Reddit, in the form of participation in our live classes, through their thousands of constructive comments on our explanatory videos, via countless DM conversations, and in their emails. This book is, in many ways, a distillation of everything we've learned from all those interactions, and our gratitude is deep and heartfelt. We couldn't have done it without you.

# Introduction

This is not an SAT book.

That is, it's not a traditional "test prep" book that discourages students from actually becoming better-educated but instead touts a "secret method" or an array of "tricks" that are purported to help the student "game the test." We think that's insulting and fundamentally counter to the central reason students take the SAT, which is to further their education by gaining admission to the best possible college, and the reason students want to go to college is to become better-educated still. Education isn't merely an annoying chore; education is the point, the larger goal, and a good SAT score helps students along the path to a great education.

One underlying assumption that appears to be built into many books and test-prep programs, and that pervades the thoughts and discussions among students studying for the test, is that the SAT is like a complicated safe, and if you can just figure out the combination, you can unlock it and ace the test—that even though the SAT purports to assess students' skills, if you know the "secrets," you don't have to actually *have* those skills in order to score well.

This is complete nonsense. It's insulting and demeaning to students, and it serves to unjustifiably undermine their self-image with respect to their intelligence and potential. The reality is that the SAT is an exquisitely well-crafted test, and it does an extremely good job of assessing exactly what it tries to assess. And what it tries to assess is a thoughtfully chosen and appropriately broad set of basic academic skills. Therefore, the best way to have your score indicate that you have mastery of all those skills is *to actually have those skills*. And it sends a terrible message: that without tricks and a gaming-the-test approach, you can't get a good score, because you can't *really* learn—you have to resort to clever tactics in hopes you can dupe the test into indicating you're smart when you're really not.

We think better of you. We know for a certainty that given the right guidance, you can be a great reader, you can learn all the grammatical and writing structure principles necessary, and, most relevantly for this work, you can really learn the math—all of it. You're not dumb, you're not un-educable, and you're not so desperate that you'll grasp for any cultish trick to try to avoid actually learning what's being tested. We believe in you, and we don't want you to accept the message that all these side-door methods subtly send you—that you're just not smart enough to learn the material, so you have to find some way to fool the test (and colleges) about your abilities. Don't be lured into the "tricks" and "secrets" cult; think more of yourself. You're better than that.

Our philosophy is radically different from that of most test preparation resources. As alluded to earlier, we view the SAT not as a *goal*, but rather as a *tool* that provides colleges with an assessment of students' level of proficiency in three core academic areas. It follows that what students should be focused on isn't getting any particular score on the test, but rather on becoming as well-educated as possible in the areas the SAT assesses. The test will then do its job and reflect that level of achievement as a natural consequence. This, in our view, is by far the best way to approach the test: go right through it rather than attempting to sneak around it. The test is far too good; it can't be tricked into reporting that you're better at math, reading, or writing than you really are, and any plan to "beat the test" is doomed, as well as being a cop-out, a surrender, an acceptance of defeat. If, instead, you acquire the skills assessed by the test, you can free yourself of all the confusing static about strategies and tricks, and you can be confident that if you put in a smart, honest effort, the test will reward you with a fair result.

Lest the reader think that we've conjured up our philosophy merely to be contrarian, we'd like to make it clear that we, too, once believed that it was a winning mindset to view the test as a challenge that could be gamed through a set of strategies, but our long experience has proven that to be a fool's errand. We now know that we were not seeing the deeper truth about the test, and it is absolutely clear to us that the approach we are now preaching, as embodied in this work, is without question the better one.

## The Essence of the SAT

Many students have a core misconception about the SAT, due no doubt to their scholastic experiences with tests. It is essential to understand that the SAT in general, and the math section in particular, is not a test of *facts*; it's a test of *processes and principles*. This distinguishes it from many of the exams students take in school: it doesn't demand *memorization of information*; it requires mental muscle memory gained through *performing processes*. It's about *doing* things, not *knowing* things. It's not a history test that demands the recitation of names and dates; it's not an English test that relies on remembering the hidden meanings in an assigned work of literature; it's not a vocabulary test that's susceptible to memorizing word lists; and it's not a math or science formula regurgitation exam. Many students don't realize these essential facts about the test, and their studying proceeds along all sorts of tangents and detours that don't make any sense. In short, you should not feel tormented by the fear of forgetting something when you open the test booklet; just focus on *doing*, and the rest will take care of itself.

The math section of the SAT does have a certain format, and the problems that appear tend to fit into certain general templates, so familiarity with the format and templates is valuable. This is not at all a trick or a secret; the College Board™ itself specifically encourages students to learn their way around the test by taking practice exams and by using Khan Academy for practice. But, importantly, there aren't any very difficult math problems on the SAT (the reading section is another story!). That's right: There are no problems that take ten minutes and dozens of steps to solve, that require some weird math skill you've never been taught (if you use this text, of course), or that demand some special "math smarts" insight that either you have or you don't. Indeed, the problems that seem to be the hardest often turn out to have very simple solutions; don't mistake a failure to recognize how to solve a particular problem for an assessment that the problem is therefore extremely difficult.

We're going to repeat this, because it's so important: *There are no very hard math problems on the SAT*. The difficulty is *intentionally* capped, and the complete set of skills needed to solve *any problem that has ever appeared on an administered SAT* is well-defined by the College Board itself in a public document. So, you might not know how to approach a specific problem; you may be completely baffled by it. You might be flummoxed by several problems, or by nearly *all* the problems, if you're just starting your studying and you've had terrible math teachers. But you must understand that this does not mean that the problems are inherently *difficult*; it just means you don't *yet* know how to solve them. That's all. And you *can* learn how to do them—all of them. We will teach you everything you need to know.

## What's in This Text, and Why

The structure and content of this work is unlike that of any other math text or SAT prep book you've ever seen. On the one hand, it's not a collection of "tricks," intended to add a few points here and there while avoiding the matter of actually educating you. On the other, it's not just a general-purpose math textbook, though it covers every one of the topics that are at the core of the American high school mathematics curriculum.

Instead, this text teaches every math skill that has appeared on the SAT, and it is intimately linked to that test. We've been guided by the College Board's official specifications for the math section of the SAT and we've exhaustively analyzed every single released test problem—nearly two thousand of them—to ensure that the skills needed to solve them are thoroughly explained herein, and we've crafted over 900 example and practice problems that exercise those skills. That means that students can have the confidence that they are learning *all the skills needed* to solve the problems they're likely to see on future tests.

The size of this work might be intimidating, and that's understandable. However, it is crucial for the reader to know that the reason this text is so hefty is because it is complete, thorough, and provides multi-level instruction throughout. It doesn't just cover some of the topics on the SAT; it covers every single one of them. It has hundreds of example problems so that students aren't just thrown into practice sets after an explanation; these examples are very carefully worked through with detailed comments along the way (and, often, with alternative solutions, too). There are a dozen tests' worth of practice problems, each carefully crafted to maximize their value, with a pop-up written explanation and an explanatory video for every one, along with hundreds of interactive graphs. EEvery concept is explained in great detail, complete with illustrative diagrams, summary boxes to highlight essential principles, interactive graphs, and a concise but complete topic summary that concludes each chapter. So, the size of this text is a *good* thing, because it means that no corners have been cut and no compromises have been made, but there are multiple levels of summarization available, too, so if you want the shorter version, or are looking for a quick refresher, we've got that covered. Once you dive in, we





think you'll realize that the heft of this tome is a consequence of the depth and quality of the instruction we've provided.

As noted, this isn't an SAT strategy guide. More specifically, we don't engage in digressions about test-taking tactics or time management. This text is intended to teach you the skills you need to do the math, so we've focused our attention on clear, in-depth explanations and extensive, high-quality example and practice questions. However, we decided that while having a wealth of examples and practice problems is necessary, it's not enough. Most math problems on the SAT center around one topic—finding the solutions to a quadratic equation, for example—but they necessarily also require the use of several constituent skills. Furthermore, just as math expressions are like molecules assembled from atoms such as numbers and variables (a parallel we draw herein), many SAT math topics are also like molecules; they're composed of elemental skills (the "atoms" of the topic) that are combined to form the higher-level skill. This means that trying to solve complete SAT-style problems before mastering the underlying atomic skills can be frustrating and inefficient.

In response to this realization, we invented the SkillDrill™. SkillDrills are sets of micro-problems that focus on just one atomic skill, such as finding the two numbers that sum to the $b$ coefficient and that multiply to the $c$ coefficient in a quadratic expression, which is an essential atomic skill for the larger skill of factoring (which itself is a component of solving a quadratic). Another such SkillDrill just requires the student to decompose an improper fraction into a mixed number; this is never tested on its own, but it's an atomic skill essential for solving certain exponent problems. And for every SkillDrill, there?s an explanatory video that walks the student through the solutions. By working through these highly-specific drills, students engage in the essential repetitive practice needed to become comfortable with the "atomic toolkit" that serves as the foundation for proficiency with the larger topic—solving quadratics, manipulating exponential expressions, and so on. That this approach is highly effective is not mere speculation: we've tested it extensively, and the results confirm the value of this methodology. As with so much of education, a firm and complete foundation provides the solid support needed for building upon.

Every topic here not only has SAT-style example and practice problems and SkillDrills; we've also provided a button at the end of each topic section that opens our innovative SAT Math Topic Explorer (SMTEx), which displays a listing of *every SAT problem related to that topic that has ever appeared on a released test*. This provides the student with an additional resource—real test problems—for practicing the skills they just learned.

The linkage between the text and the real tests doesn't only go in one direction. To close the loop, SMTEx also allows a student to look up any math problem from any released SAT and to find and jump to the topic or topics in the Orange Book that relate to that problem. This means that after a student takes a practice test, they can look up our detailed, patient, thorough, and empathetic guide for acquiring the mathematics techniques needed to solve the questions they struggled with without having to hunt through the text for relevant information.

Driving all of the ideas detailed above is a simple set of principles that we hold and that we kept foremost in our minds as we were crafting this text:

- We want you to understand concepts and techniques no one was ever able to explain to you before
- We want you to realize that math topics you thought were really hard are actually easy
- We want you to truly understand aspects of math that you had blindly accepted on faith

These principles have a common root: the goal of instilling in students the confidence that mastery of a subject produces, because mastery plus confidence equals 800.

A final word about the structure of the text: There is an expectation that students have a foundation in elementary algebra and that they are familiar with the various symbols that are used at the level of high school mathematics, and there are a few questions on each test that probe those elementary skills. In support of that, we start off the instruction with what we came to call "Chapter 0," now named Foundations, and it is a concise collection of the most basic math skills that are necessary for all the more-focused topics that follow. It is *essential* that students feel comfortable with these building-block skills before moving into the topic instruction, so give the Foundations chapter a read-through to be sure you're on solid footing before proceeding further.

## How to Use This Text

There are many ways to leverage the instruction in this work. Here are a few of the primary scenarios we anticipate, though we expect that nearly all students will end up using a hybrid approach:

1. **Read and work through the text cover-to-cover:**

   If your math skills are particularly shaky or you never even learned the high school math topics assessed by the SAT, or you just want to be absolutely sure you've covered all the bases, you can simply go through the entire text, reading all the instructional material, doing all the SkillDrills, and solving all the practice problems. Practically speaking, most students will end up skipping or skimming portions of many topics when they realize they are already knowledgable in the subject being discussed, but there are explanations for every aspect of every topic in case you need help.

2. **Use the summaries as a refresher:**

   If you have a reasonably solid foundation but are rusty, you can focus on the condensed topic and subtopic explanations that are broken out in the boxed text and chapter recaps, referring to the full explanations as needed, and then you can work the practice problems to check your skills.

3. **Focus on your weaknesses:**

   Students can identify the major topics in which they are less proficient and then work through just the chapters that cover those areas.

4. **Use the text as an instructional reference:**

   Some students prefer to use math problems from a variety of sources for practice; they can use the text as a reference tool when they discover a topic or a concept with which they are unfamiliar. This work makes that mode of use particularly easy when students work problems from released SATs, because SMTEx lets students look up any problem on any released test and jump directly to the location in the text that explains the relevant topics, so when you're confused by a practice problem, you can get right to the explanation of the associated math principles with one click.

5. **Let practice tests drive your learning:**

   If your studying centers on taking sections from practice SATs, you can use our Autoscoring Answer Forms (AAFs) to score your answers; for each problem on the score report, there's a link that opens the Orange Book to the exact page where the relevant topic is explained. This allows a student to go seamlessly from assessment and practice to perfectly-focused learning with the best SAT math resource available.

6. **Work the problems for practice:**

   If you feel that your concept knowledge is basically sound, but you need more practice, you can use the hundreds of practice problems to solidify your skills. You can concentrate on the topics that need the most work, and any needed instructional support is available right where the practice problems are presented. For help, pop-up written explanations, expert video explanations, and interactive graphs are just one click away.

7. **As a sleep aid:**

   See scenario 1.





## Beyond the Book

As thorough as it is, this text does not stand alone; it's part of the 1600.io ecosystem of educational resources. We have extensive expertise in video-based standardized test instruction, and in addition to the embedded explanatory videos for every practice problem and SkillDrill in this book, we also offer video explanations for every reading, writing, and math question on every released SAT for those who want to supplement their self-studying. Other supporting resources are also available at our website, providing a rich set of tools that support students? efforts to master the skills assessed by the SAT.

## Wait, What's a "Wormhole?"

At the start of some of the example problem solutions, you'll see this curious symbol:

Sprinkled throughout the SAT, hiding in plain sight, are members of a class of math problems that have a non-obvious, accelerated solution that we've dubbed a **wormhole**. A real wormhole is a warp in the fabric of space that brings two distant locations close together; the wormhole allows travel from one such location to the other very quickly, as the normal distance between the points does not have to be traversed. So it is too with wormhole solutions: they let the student go from the question to the answer with just a few easy steps, rather than requiring them to travel through the usual lengthier solving procedures for that general type of problem.

Importantly, **nearly all wormhole problems can also be solved with more generalized methods**. What this means is that wormhole problems are really like several little tests, all packed into one. Though from a scoring standpoint, a right answer is a right answer, no matter how you get there, the solving procedure that takes advantage of a wormhole will proceed much more rapidly (and with a smaller probability of error due to the simpler procedure) than will the longer, more generalized procedure. That rewards the student with more than just the points for the answer; in effect, it grants the student a bit of bonus time that can be used on some other problems, or the time can be used for double-checking answers once the first pass through the section is done. Also, as alluded to earlier, the simpler procedure is less susceptible to error; the more steps there are in a process, the greater the likelihood an error will creep in. Wormhole problems are thus very powerful in sifting the test-takers by ability, because one problem can classify students into those who spot the wormhole, those who don't but who nonetheless solve the problem correctly with a generalized method, and those who fail to solve the problem.

Taking advantage of wormholes is not a test-taking trick. It doesn't attempt to evade the test's purpose in assessing the student's ability to efficiently solve math problems—quite the contrary, in fact: one of the test's many facets is that it allows students who have the aptitude and mental acuity to see things that others might not to demonstrate that ability and to have it pay off in their score.

Because this underappreciated aspect of the test is important, we've got a chapter dedicated to it, where example problems that are inspired by real SAT problems are presented and explained. Don't overlook it; a few precious seconds saved, or errors avoided, could be the difference between a good score and a great one. Also, it (rightly) makes you feel clever, and that builds confidence, which can fuel your efforts throughout the test.

# A Few Words of Advice

While this text is not going to delve into test-taking tactics, we do have a few words of advice (and caution) that students should heed.

## Calculators: Blessing or Curse?

We strongly discourage the use of calculators while learning the math skills assessed by the SAT for anything other than time-consuming arithmetic. We're fond of declaring that calculators are tools of the devil because they produce a reliance on the device that causes math skills to atrophy (if they ever developed at all).

The skillful student will have no difficulty completing the math sections on the test without resorting to the use of a calculator for solving or graphing, and there is hardly a worse feeling than finding out during the test that your calculator won't turn on, and you're so reliant on it that you can't solve some problems without it. Don't be that person.

When studying, if you find yourself tempted to turn to your calculator to do something you don't know how to do—or don't *want* to do—just say no. You'll be far better off.

However, the Digital SAT presents a rather different situation, because the Desmos® graphing calculator is available for use on any problem in the math sections; it can't run out of power, and you can't forget to bring it to the exam. Furthermore, because Desmos provides a math visualization tool, it can actually help impart understanding during the learning process, and so it is not just a solving tool, but a learning tool. In recognition of that power, this resource includes Desmos graphs that support the instruction being provided, and we also demonstrate how to use Desmos efficiently for solving SAT-style problems in a way that adds insight rather than obscuring the underlying mathematical principles, as can happen with mindless calculator solving.

## Plugging-in Is Surrender

There is an alarmingly widespread belief that the best way to solve certain types of problems is to pick some value and substitute it for the variable in the provided expression or equation to see if a unique match with an answer choice is obtained. This is a terrible solving method, because it's nondeterministic (it's not guaranteed to produce the answer) and it's a cop-out—a way of avoiding actually learning and understanding the math needed to simply solve the problem directly. We strongly urge the reader not to resort to this technique. Note, importantly, that we are *not* referring to the practice of testing the provided answer choices to see which one(s) produce a valid equation; in some cases, that is a perfectly appropriate method for solving a multiple-choice math problem. We are instead directing our scorn at the very different procedure whereby the student cooks up some value—0, 1, 2, for example—and plugs it in. Please think better of yourself than to employ this method.

## Writing Out Your Work Is Smart; Mental Math Is Dumb

One of the chief score killers in the math section of the SAT is sloppy arithmetic mistakes: dropping a sign, adding instead of subtracting, failure to distribute across all terms in an equation, and so on. One of the leading causes of these errors is students' unhealthy tendency to use mental math to compress solving steps rather than taking the effort to carefully (and legibly) write out their work as they go. Students perceive it as some sort of show of prowess to be able to do math in their heads rather than write it out, and, on the other side, they view it as some sort of failing to rely on written-out steps when doing a math problem that they feel they could probably handle without that process. This is dumb, and you don't want to do dumb things. **The best way to eliminate sloppy mistakes is to write out your work.**





## A Note about Student-Produced Responses ("Grid-ins")

Many of the practice problems in this text and on the test require the student to write their answer rather than select it from a set of choices. Keep in mind the constraints on these grid-in answers; they must be enterable into an answer form like this:

Here are the restrictions that apply:

- You cannot enter a negative number (so if you get a negative answer, you made a mistake!)
- You cannot enter a zero in the first column (that is, as the first digit in a four-character answer); you are *never* required to enter a leading zero, so this should never cause any confusion
- You have a maximum of four character positions including a decimal point or a fraction slash
- Unless the problem specifies otherwise, you can either round or truncate decimal answers that don't fit into the grid, so $\frac{5}{3}$ could be entered as 5/3, 1.66, or 1.67, but you must use all four character positions for such values
- You cannot enter a mixed number such as $2\frac{1}{2}$; you must either enter the decimal equivalent (2.5) or an improper fraction (5/2)
- It doesn't matter in which position you start entering your answer as long as you enter the complete value (unless it's a value with more decimal digits than would fit into the four positions, in which case you must start with the first position)

The Digital SAT allows the direct entry of positive and negative answers with up to six characters including a possible minus sign and a decimal point.

# Help Us Help You

We've made this work as good as we can make it, but all the same, we know it can be made better. If you have ideas, suggestions, criticisms, or if you spot errors, we very much would like to hear what you have to say. Email us at MathBook@1600.io, use the member contact form at our website, chat with us through our site, or DM us in our Discord server or through Reddit. This text could not have been written without your input, and that feedback is also essential to improving it. To ensure that all readers get the benefit of any corrections as soon as they are recognized, we will maintain an updates page on our website to log these changes.

# Get Started!

You're viewing this powerful resource on your screen. Take the next step forward: get a sharp pencil and some scrap paper, get focused, and turn the (virtual) page. You're about to start getting smarter.

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# Foundations 0

This chapter covers the foundational arithmetic and algebraic knowledge that is required in order to get the most out of this book. The instruction in this text progresses in a methodical manner, working its way from writing simple expressions all the way through most of Algebra II and covering the necessary topics in statistics and geometry. However, you need to be proficient with skills such as manipulating fractions, simple factoring, and isolating variables in order to get started. Some skills in this chapter will be covered in more depth in later chapters when necessary, but others are going to be assumed at all points going forward. Even if you consider yourself reasonably well-grounded in basic techniques, look over this chapter to be sure you're well-equipped to move ahead.

## 0.1 Definitions and Fundamentals

First, we need to differentiate between **expressions** and **equations**. **Expressions**, strictly speaking, consist only of numerical and variable terms being added, subtracted, multiplied, and so on. For example, $40 + 2x$ is an expression. When we set this value equal to another expression, we can make an equation. **Equations** consist of two expressions that are set equal to each other using an equals sign (=). For example, $40 + 2x = 80 - 4x$ is an equation.

The expressions on both sides of the equals sign are exactly equivalent to each other (that's what equals means), even if they don't *look* exactly the same. Therefore, the expressions can switch sides of the equals sign; the equation $40 + 2x = 80 - 4x$ could also be written as $80 - 4x = 40 + 2x$.

One of the most essential foundational principles of mathematics is that **because the expressions on both sides of an equation are equivalent, we can perform the same operation on both of those expressions and the resulting equation will still be valid**. This can be analogized to a scale that has an assortment of weights on each side that produce a perfect balance; if we then add the same amount of weight to each side of the scale, the balance will be maintained. It is *absolutely crucial* that you understand and accept this principle so that you to have confidence in all that follows.

Obviously, you need to be familiar with how to add, subtract, multiply, and divide numbers (positive and negative) in order to solve any of the problems on this test.

A few basic arithmetic terms that will be used frequently herein:

**Integers** are all of the **natural numbers** (positive counting numbers), the negatives of these numbers, and zero, e.g. $\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$.

**Non-negative** numbers are all positive numbers (not just integers) and 0.

**Non-positive** numbers are all negative numbers (not just integers) and 0.

The result of an addition is called the **sum**. The result of a subtraction is called the **difference**. The result of a multiplication is called the **product**. The result of a division is called the **quotient**.

### Important Symbols

The following table shows some of the symbols you will encounter on this test and in this text, some of which are discussed in greater detail in chapters in the book when needed. Consider this a quick primer on what to expect.

| Name | Symbol | Usage |
|---|---|---|
| Equals | $=$ | This symbol is used in equations to show that two values or expressions are equal. The equation $a = b$ is read as "$a$ equals $b$" or "$a$ is equal to $b$" |
| Not Equals | $\neq$ | This symbol is used to show that two values or expressions are NOT equal. For example, $3 \neq 5$ because 3 is not equal to 5. |
| Approximately Equals | $\approx$ | Used to show that a value is approximate. For example, $a \approx 3.14$ means that $a$ is approximately—but not exactly—3.14. |
| Infinity | $\infty$ | An indefinitely large number. |
| Less Than | $<$ | The value on the left (on the smaller side of the symbol) is less than the value on the right (on the larger side of the symbol).<br>Mathematical statements involving less-than signs (and the next three symbols as well) are called **inequalities**. The inequality $x < 5$ means that $x$ can be any value from $-\infty$ up to, but **not** including, 5. |
| Less Than Or Equal To | $\leq$ | The value on the left is less than or equal to (no more than) the value on the right. For example, $x \leq 5$, means that $x$ can be any value from $-\infty$ up to 5, including 5 itself. |
| Greater Than | $>$ | The value on the left (on the larger side of the symbol) is greater than the value on the right (on the smaller side of the symbol).<br>The inequality $x > 5$ means that $x$ can be any value greater than 5 up to $\infty$. |
| Greater Than Or Equal To | $\geq$ | The value on the left is greater than or equal to (no less than) the value on the right. For example, $x \geq 5$ means that $x$ can be any value from 5 (including 5 itself) up to $\infty$. |
| Absolute Value | $\lvert a \rvert$ | Produces a non-negative value indicating how far a number is from 0. ($\lvert -5 \rvert = 5$) |
| Root | $\sqrt[n]{x}$ | Indicates the $n$-th root of the operand $x$; when $n$ is absent, its implied value is 2, so $\sqrt{x}$ indicates the second or square root of $x$.<br>(See Chapter 6.) |
| Braces | $\{m, n, \ldots\}$ | Braces indicate that elements $m$, $n$, etc. belong to a set. |
| Plus-Minus | $\pm$ | Indicates that both the positive and negative values should be considered. |
| Ratio | $a : b$ | Shows the relative value of $a$ compared to $b$.<br>(See Chapter 12.) |
| Percent | $\%$ | A relative measure expressed in hundredths of some reference value.<br>(See Chapter 13.) |
| Line Segment | $\overline{AB}$ | Used to refer to a line segment between points $A$ and $B$. |
| Parallel | $\ell \parallel k$ | Used to show that lines $\ell$ and $k$ are parallel to each other (spaced a constant distance apart). |
| Perpendicular | $\ell \perp k$ | Used to show that lines $\ell$ and $k$ are perpendicular to each other (intersecting at a right angle). |
| Angle | $\angle A$ or $\angle ABC$ | Used to refer to an angle in a figure, such as a triangle. |
| Pi | $\pi$ | This Greek letter is used to represent the ratio of a circle's circumference to its diameter. $\pi \approx 3.14$ (See Chapter 20) |
| Theta | $\theta$ | This Greek letter is often used to represent the measurement of an angle. |
| Arc | $\widehat{AB}$ or $\widehat{ACB}$ | Used to denote an arc on a circle. |





## What You Should Already Know About Fractions

The **numerator** is the expression on the top of a fraction. The **denominator** is the expression on the bottom of a fraction.

$$\frac{\text{numerator}}{\text{denominator}}$$

Any number can be represented as a fraction where the numerator is the number itself and the denominator is 1 because any number divided by 1 is equal to itself. For example, 2 can be written as $\frac{2}{1}$.

Fractions can be **reduced** if the numerator and denominator share a common divisor (factor) other than 1. For example, the fraction $\frac{4}{6}$ can be reduced to $\frac{2}{3}$ because the numerator, 4, and the denominator, 6, share a common factor of 2. Dividing both the numerator and denominator by 2 produces the reduced fraction. Other examples are $\frac{3}{9} = \frac{1}{3}$ and $\frac{7}{7} = \frac{1}{1} = 1$.

When a fraction cannot be reduced any further, it is said to be **in lowest terms**. Note that reducing is **not** the same as simplifying, which refers to making expressions simpler (a somewhat subjective concept); try to use the correct terms for these totally different procedures.

To multiply a fraction by a single value, **multiply just the numerator** of the fraction by the value. For example, $6\left(\frac{4}{5}\right) = \frac{6(4)}{5} = \frac{24}{5}$. Note that this is equivalent to multiplying $\frac{6}{1}$ and $\frac{4}{5}$ (see the next principle).

To multiply two fractions, multiply the numerators to form the numerator of the product and multiply the denominators to form the denominator of the product. For example, $\frac{1}{4}\left(\frac{3}{5}\right) = \frac{1(3)}{4(5)} = \frac{3}{20}$.

Dividing by a fraction is the same as multiplying by the **reciprocal** of the fraction (formed by "flipping" the fraction, swapping the positions of the numerator and denominator).

$$\frac{6}{\frac{1}{2}} = 6\left(\frac{2}{1}\right) = \frac{12}{1} = 12$$

When dividing one fraction by another, the same principle applies; multiply the fraction in the numerator by the reciprocal of the fraction in the denominator.

$$\frac{\frac{2}{3}}{\frac{1}{2}} = \frac{2}{3}\left(\frac{2}{1}\right) = \frac{4}{3}$$

You can **add or subtract fractions *only* when they have the same denominator**. When doing so, the addition or subtraction operation is applied to the numerators, while the denominator stays the same. For example, $\frac{1}{5} + \frac{2}{5} = \frac{1+2}{5} = \frac{3}{5}$.

When the denominators are *not* the same, you can make a **common denominator** (make the denominators match) in order to then be able to add or subtract the fractions. To create the common denominator, you have to multiply both the numerator and denominator of one or both fractions by values that will make the denominators of both fractions equal; that is, you will multiply one or both fractions by 1 expressed as a fraction (such as $\frac{2}{2}$) so the original fractions' *values* don't change, but the way they're *written* does. The simplest case arises when the denominator of one fraction is a multiple of the denominator of the other fraction as in the example below.

$$\frac{2}{3} + \frac{1}{6} = \frac{2}{3}\left(\frac{2}{2}\right) + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$$

If neither denominator is a multiple of the other, then you can try to make the common denominator of the fractions the **lowest (or least) common multiple** of the two denominators. Often the lowest common multiple is just the product of the two denominators. For example, in the example below, the denominators are 3 and 4, whose lowest common multiple is 12, which is equal to 3 times 4.

$$\frac{1}{3}-\frac{1}{4}=\frac{1}{3}\left(\frac{4}{4}\right)-\frac{1}{4}\left(\frac{3}{3}\right)=\frac{4}{12}-\frac{3}{12}=\frac{1}{12}$$

In the following example, the denominators, 8 and 12, share a lowest common multiple of 24 because both can go into 24 evenly (finding this value will result in smaller numbers than would using the product of 8 and 12, which is 96, as the common denominator).

$$\frac{3}{8}+\frac{5}{12}=\frac{3}{8}\left(\frac{3}{3}\right)+\frac{5}{12}\left(\frac{2}{2}\right)=\frac{9}{24}+\frac{10}{24}=\frac{19}{24}$$

When fractions appear in an equation rather than just in a lone expression, you can avoid making common denominators by eliminating fractions entirely (multiplying both sides of the equation by the denominators of any fractions so the denominators all get canceled).

**Mixed numbers** consist of a whole number part and a fraction part. For example, "two and a half" can be represented by the mixed number $2\frac{1}{2}$; however, mixed numbers are confusing when working in a mathematical context, and you can't fill them in as answers on the test anyway, so avoid them except in certain situations involving exponents as explained in Chapter 6.

If you are presented with a mixed number, you should immediately convert it into a fraction; because the absolute value of the fraction will be greater than or equal to 1 due to the presence of the whole number in the original mixed number, this represents a particular type of fraction called an **improper fraction** (a fraction in which the numerator has a larger absolute value than does the denominator). Do so by converting the whole number part into a fraction that shares a common denominator with the fraction part of the mixed number and then adding the two parts together.

$$3\frac{1}{4}=3+\frac{1}{4}=3\left(\frac{4}{4}\right)+\frac{1}{4}=\frac{12}{4}+\frac{1}{4}=\frac{13}{4}$$

## 0.2 Solving Equations/Isolating Variables

When we want to solve equations for a variable, symbolic constant, or expression, such as $x$, we need to isolate the term or expression whose value we want to find. To do so, we simply need to undo any mathematical operations that are being performed on the term we want to isolate. To undo an operation, we perform the inverse operation; addition and subtraction are inverse operations, and multiplication and division are inverse operations. Exponentiation and taking a root are also inverse operations, but there are some special considerations when performing those operations as explained in Chapter 9.

Let's start with the very basic equation $2x = 8$ and solve for $x$. The variable $x$ is being multiplied by the **coefficient** 2, so we should perform the inverse operation to that multiplication and divide both sides of the equation by 2 to determine that $x = 4$.

Let's go one step further and solve the equation $-2x - 3 = 7$. We need to "peel away" the other parts of the expression on the left side of the equation to isolate $x$. The coefficient of $-2$ is most closely grouped with the $x$ term, so we will come to that last. The outermost operation is the subtraction between the two terms, so first, we need to add 3 to both sides of the equation in order to cancel the $-3$. The equation becomes $-2x = 10$, and then we can divide both sides of the equation by $-2$ to get $x = -5$.





$$\begin{aligned}
-2x-3&=7\\
-2x-3+\mathbf{3}&=7+\mathbf{3}\\
-2x\cancel{-3}\cancel{+3}&=10\\
-2x&=10\\
\frac{-2x}{\mathbf{-2}}&=\frac{10}{\mathbf{-2}}\\
\frac{\cancel{-2}x}{\cancel{-2}}&=\frac{10}{-2}\\
x&=-5
\end{aligned}$$

**Example 0.2-1**

**1** 1600.io 0.2

What value of $x$ satisfies the equation $4x + 4 = 24$ ?

A) 4

B) 5

C) 7

D) 24

**Solution**

1. Subtract 4 from both sides of the equation to cancel the +4, which is the outermost operation.

$$\begin{aligned}
4x+4&=24\\
4x+4-\mathbf{4}&=24-\mathbf{4}\\
4x&=20
\end{aligned}$$

2. Divide both sides of the equation by 4 to cancel the multiplication of $x$ by 4 in order to solve for $x$.

$$\begin{aligned}
4x&=20\\
\frac{4x}{\mathbf{4}}&=\frac{20}{\mathbf{4}}\\
x&=5
\end{aligned}$$

3. The answer is B.

If the **coefficient** of the variable (the constant that is multiplying the variable) you need to isolate is in the form of a fraction, you can first multiply both sides of the equation by the coefficient's denominator and then divide both sides of the equation by the numerator. However, the best approach is to multiply both sides of the equation by the **reciprocal** of the fraction, which isolates the variable in one step, as it combines the multiplication and division operations. Note that this procedure is equivalent to dividing both sides of the equation by the fraction; division by a fraction can be implemented as multiplication by the reciprocal of the fraction.

**Example 0.2-2**

**2** 1600.io 0.2

$$\frac{3}{5}t = \frac{7}{3}$$

What value of $t$ is the solution of the equation above?

**Solution 1**

1. To solve for $t$, we want to divide both sides of the equation by the coefficient of $t$, which is $\frac{3}{5}$. We can accomplish this by multiplying both sides by the reciprocal of $\frac{3}{5}$, which is $\frac{5}{3}$.

$$\frac{3}{5}t = \frac{7}{3}$$
$$\frac{5}{3}\left(\frac{3}{5}\right)t = \frac{5}{3}\left(\frac{7}{3}\right)$$
$$\frac{\cancel{15}}{\cancel{15}}t = \frac{35}{9}$$
$$t = \frac{35}{9}$$

2. The answer is $\frac{35}{9}$.

**Solution 2**

1. Multiply both sides of the equation by 5, the denominator of the fraction on the left side of the equation.

$$\frac{3}{5}t = \frac{7}{3}$$
$$\cancel{5}\left(\frac{3}{\cancel{5}}\right)t = 5\left(\frac{7}{3}\right)$$
$$3t = \frac{35}{3}$$

2. Divide both sides of the equation by 3 (or multiply by $\frac{1}{3}$) to isolate and solve for $t$.

$$3t = \frac{35}{3}$$
$$\frac{1}{\cancel{3}}(\cancel{3}t) = \frac{1}{3}\left(\frac{35}{3}\right)$$
$$t = \frac{35}{9}$$

3. The answer is $\frac{35}{9}$.





**Section 0.2 Suggested P**

- Test 3-C-7
- Test 6-NC-17
- Test 9-C-1

2021-NC-18

View related real-test problems in SMTEx

SMTEx

## 0.3 Substitution

If you are given the value of a constant or a particular value of a variable, you can replace that constant or variable with the given value. For example, if we are given the equation $3x + a = 13$ and told that $a = 1$, we can **substitute** 1 for $a$ to make the equation $3x + 1 = 13$. From there, we can solve for $x$ by subtracting 1 from both sides of the equation to get $3x = 12$ and then dividing both sides by 3 to arrive at $x = 4$.

**Example 0.3-1**

**1** 1600.io 0.3

If $\frac{x-2}{4} = k$ and $k = 3$, what is the value of $x$?

A) 3
B) 7
C) 12
D) 14

**Solution**

1. Substitute 3 for $k$.

$$\frac{x-2}{4} = k$$
$$\frac{x-2}{4} = 3$$

2. Multiply both sides of the equation by 4.

$$\frac{x-2}{4} = 3$$
$$\cancel{4}\left(\frac{x-2}{\cancel{4}}\right) = 4(3)$$
$$x - 2 = 12$$

3. Add 2 to both sides of the equation to solve for $x$.

$$x - 2 = 12$$
$$x - \cancel{2} + \cancel{2} = 12 + 2$$
$$x = 14$$

4. The answer is D.

Section 0.3 Suggested P

- Test 1-NC-1
- Test 4-C-5

View related real-test problems in SMTEx

SMTEx

## 0.4 Solving for Expressions; Distributing and Factoring

Many questions on the test ask you to solve for an expression rather than for a lone variable, and usually this is a clue that you might be able to take a more direct route to the answer if you are thoughtful. However, in this section we are just interested in solving for the lone variables and substituting their values into the solve-for expression (we'll discuss the optimizations in later chapters). Most importantly, don't solve for a variable and assume that is the answer; **always double check what the problem is asking you for**.

**Example 0.4-1**

**1** 1600.io 0.4

If $3x + 5 = 7$, what is the value of $6x + 8$ ?

A) 2

B) 5

C) 12

D) 14

**Solution**

1. Subtract 5 from both sides of the equation.

$$3x + 5 = 7$$
$$3x + \cancel{5} - \cancel{5} = 7 - 5$$
$$3x = 2$$

2. Divide both sides of the equation by 3 to solve for $x$, whose value needs to be substituted into the solve-for expression.

$$3x = 2$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{2}{3}$$
$$x = \frac{2}{3}$$

3. Now that we have the value of $x$, we can substitute $\frac{2}{3}$ for $x$ in the solve-for expression, which is $6x + 8$.

$$6x + 8 = 6\left(\frac{2}{3}\right) + 8$$
$$6x + 8 = \frac{12}{3} + 8$$
$$6x + 8 = 4 + 8$$
$$6x + 8 = 12$$

4. The answer is C.





In the following example, we must first translate a sentence into an equation and then solve for the desired expression. When translating a sentence, handle one phrase at a time. Place the equals sign immediately when you see a phrase like "is equal to" or "the result is," and anything that comes after that phrase should go on the other side of the equals sign.

**Example 0.4-2**

**2** 1600.io 0.4

When 6 times the number $x$ is added to 8, the result is 2.
What number results when 3 times $x$ is added to 4 ?

A) $-1$

B) 1

C) 2

D) 7

**Solution**

1. The phrase "6 times the number $x$" translates to $6x$, and that is added to 8, so the first expression we should write is $8+6x$ (note that because you can perform addition in any order—it's **commutative**—this could also be written as $6x+8$, which you might find more natural).

2. The phrase "the result is 2" means that we should set the first expression we wrote equal to 2.

$$8 + 6x = 2$$

3. Subtract 8 from both sides of the equation.

$$8 + 6x = 2$$
$$\cancel{8} + 6x - \cancel{8} = 2 - 8$$
$$6x = -6$$

4. Divide both sides of the equation by 6 to solve for $x$.

$$6x = -6$$
$$\frac{\cancel{6}x}{\cancel{6}} = \frac{-6}{6}$$
$$x = -1$$

5. Don't forget that we need to construct the solve-for expression, which is specified in the second sentence of the question. Based on the phrase "3 times $x$ is added to 4," the solve-for expression is $4 + 3x$ (you could also write this as $3x + 4$).

6. Substitute $-1$ for $x$ in the solve-for expression.

$$4 + 3x = 4 + 3(-1)$$
$$4 + 3x = 4 - 3$$
$$4 + 3x = 1$$

7. The answer is B.

In the following example, the equation contains two unknown quantities represented by letters, so we cannot solve for the value of either one by itself. Instead, we must take advantage of an existing grouping and isolate the grouped expression to solve.

**Example 0.4-3**

**3** 1600.io 0.4

If $5(a+b)=7$, what is the value of $a+b$ ?

A) $\frac{5}{7}$

B) $\frac{7}{5}$

C) 5

D) 7

**Solution**

1. The solve-for expression consists of two unknown terms added to each other. Because we have only one equation, but two unknown values, $a$ and $b$, we cannot find the value of either one individually. However, because the solve-for expression $a+b$ appears in the original equation, we can isolate the solve-for expression by simply dividing both sides of the equation by 5.

$$5(a+b)=7$$
$$\frac{\cancel{5}(a+b)}{\cancel{5}}=\frac{7}{5}$$
$$a+b=\frac{7}{5}$$

2. The answer is B.

## Distributing and Factoring Constants

When an expression involving addition or subtraction is multiplied by a constant (the expression is often enclosed in parentheses as in the previous example) or when an expression is divided by a constant (which is the same as multiplying by a fraction), to perform the multiplication or division the constant must be **distributed** to all the terms in the expression, meaning every term in the expression must be multiplied by the constant, and then all those products (results of multiplication) must be added together. For example, given the expression $3(x+2)$, we can distribute the 3 to both terms in the parenthetical expression (multiply both $x$ and 2 by 3) to rewrite the expression as $3x+3(2)=3x+6$.

If you have to distribute a negative number, just **make sure the negative part is also distributed** (using extra parentheses around negative terms helps reinforce this). For example, $-2(x-4)=-2x-2(-4)=-2x+8$; notice how we placed $-4$ in parentheses to remind us to be careful about multiplying two negative numbers.





**Example 0.4-4**

**4** 1600.io 0.4

Which of the following is equivalent to $4(x+3)-8$ ?

A) $4x-5$

B) $4x-4$

C) $4x+4$

D) $12x-8$

**Solution**

1. Distribute the 4 to both terms in the parentheses, then combine the numerical constants.

$$4(x+3)-8$$
$$4x+4(3)-8$$
$$4x+12-8$$
$$4x+4$$

2. The answer is C.

Distribution also occurs if we divide an expression by a constant: all the terms in the expression must be divided by the constant. For example, if we want to divide the expression $8x+4$ by 4, we simply have to divide both of the terms in the expression by 4 (split the numerator into two fractions to visualize this).

$$\frac{8x+4}{4}=\frac{8x}{4}+\frac{4}{4}=2x+1$$

Alternatively, you can think about this as multiplying the expression in the numerator by $\frac{1}{4}$ (the reciprocal of 4) and then distributing the multiplication.

$$\frac{8x+4}{4}=\frac{1}{4}(8x+4)=\frac{1}{4}(8x)+\frac{1}{4}(4)=\frac{8x}{4}+\frac{4}{4}=2x+1$$

Another way to think about and handle the operation is to realize that since both terms in the expression $8x+4$ include multiples of 4, we can **factor** 4 out of both terms.

$$8x+4=4(2x)+4(1)=4(2x+1)$$

Note, importantly, that **factoring is the inverse operation of distributing**:

$$8x+4 \underset{\text{Distributing}}{\overset{\text{Factoring}}{=}} 4(2x+1)$$

Factoring constants out and then dividing by the factor to isolate the remaining expression is often used to produced wormhole-like shortcut solutions to problems like the following example (see the Wormholes chapter for a thorough definition of that term and an in-depth discussion of this principle). Realizing that we can factor as the first step leads to an optimization that helps us find the solve-for expression immediately. These processes are covered in later chapters, but the clue that factoring is the best approach is the need to make the coefficient of the variable term in the given equation match the coefficient of the variable term in the solve-for expression (this can be used as a mini optimization on some of the real-test problems that are suggested in the previous section but is more applicable to the ones suggested in this section). Note that the following problem has two unknown values, so we cannot solve for the individual values but must instead solve directly for an expression containing both of those unknowns.

**Example 0.4-5**

**5** 1600.io 0.4

$$6ax - 3 = 30$$

Based on the equation above, what is the value of $2ax - 1$ ?

A) 5

B) 6

C) 10

D) 15

**Solution 1**

1. We can factor 3 out of both terms on the left side of the equation, which will produce the solve-for expression as the other factor.

$$6ax - 3 = 30$$
$$3(2ax - 1) = 30$$

2. Divide both sides of the equation by 3 to isolate and solve for the solve-for expression.

$$3(2ax - 1) = 30$$
$$\frac{\cancel{3}(2ax - 1)}{\cancel{3}} = \frac{30}{3}$$
$$2ax - 1 = 10$$

3. The answer is C.





**Solution 2**

1. Add 3 to both sides of the equation.

$$6ax - 3 = 30$$
$$6ax = 33$$

2. Divide both sides of the equation by 6 to solve for $ax$ (we cannot solve for either unknown value individually, but the term $ax$ appears in the solve-for expression, so solving for $ax$ is sufficient) and reduce the resulting fraction.

$$6ax = 33$$
$$\frac{\cancel{6}ax}{\cancel{6}} = \frac{33}{6}$$
$$ax = \frac{11}{2}$$

3. Substitute $\frac{11}{2}$ for $ax$ in the solve-for expression.

$$2ax - 1 = 2\left(\frac{11}{2}\right) - 1$$
$$2ax - 1 = 11 - 1$$
$$2ax - 1 = 10$$

4. The answer is C.

**Section 0.4 Suggested Problems from Real Tests**

- Test 1-C-4
- Test 2-NC-1
- Test 2-C-6
- Test 3-NC-2
- Test 3-NC-4
- Test 5-C-4
- Test 9-C-3
- May 2018-C-3
- May 2019 (US)-C-6
- Oct 2019-C-5
- Mar 2021-C-17
- Apr 2021-C-14
- May 2021 (US)-C-2

View related real-test problems in SMTEx

## 0.5 Combining Like Terms

We already know that numbers can be combined with addition or subtraction, but the same goes for variable terms as well as constants and expressions. For example, $x + x = 2x$ because all we've done is add two $x$ terms together. Similarly, we can add $2x$ and $3x$ to make $5x$, and we can subtract $6x$ from $4x$: $4x - 6x = -2x$. Constants can be treated the same way, so $4a + 3a = 7a$, and like expressions can also be combined: $9(x+4) - 7(x+4) = 2(x+4)$.

Note that **two different variables, constants, or expressions cannot be combined in any way like this**. For example, $2x + 5y \neq 7xy$ (or any other nonsense way people can think of to combine two incompatible things).

**Example 0.5-1**

**1** 1600.io 0.5

If $3x + 5 = 2x - 3$, what is the value of $x + 9$ ?

A) $-8$

B) $-1$

C) $1$

D) $8$

**Solution**

1. Subtract 5 from both sides of the equation in order to eliminate constants on the left side of the equation. Combine constants on the right side of the equation.

$$3x + 5 = 2x - 3$$
$$3x + \cancel{5} - \mathbf{5} = 2x - 3 - \mathbf{5}$$
$$3x = 2x - 8$$

2. Subtract $2x$ from both sides of the equation in order to combine the $x$-terms on the left side of the equation.

$$3x = 2x - 8$$
$$3x - \mathbf{2x} = \cancel{2x} - 8 - \mathbf{2x}$$
$$x = -8$$

3. Substitute $-8$ for $x$ in the solve-for expression.

$$x + 9 = -8 + 9$$
$$x + 9 = 1$$

4. The answer is C.

In the following example, in order to combine the variable terms, we also have to either make use of common denominators or multiply both sides of the equation by a value that will eliminate the fractions entirely.





**Example 0.5-2**

**2** 1600.io 0.5

$$\frac{1}{2}x - 1 = 2 - x$$

What value of $x$ satisfies the equation above?

A) $\frac{1}{2}$

B) $2$

C) $3$

D) $\frac{9}{2}$

**Solution 1**

1. Multiply both sides of the equation by 2 in order to eliminate any fractions, and then distribute.

$$\frac{1}{2}x - 1 = 2 - x$$
$$\mathbf{2}\left(\frac{1}{2}x - 1\right) = \mathbf{2}(2 - x)$$
$$\cancel{2}\left(\frac{1}{\cancel{2}}x\right) - 2(1) = 2(2) - 2(x)$$
$$x - 2 = 4 - 2x$$

2. Add 2 to both sides of the equation in order to eliminate constants on the left side of the equation. Combine constants on the right side of the equation.

$$x - 2 = 4 - 2x$$
$$x - \cancel{2} + \mathbf{2} = 4 - 2x + \mathbf{2}$$
$$x = 6 - 2x$$

3. Add $2x$ to both sides of the equation in order to combine $x$-terms on the left side of the equation.

$$x = 6 - 2x$$
$$x + \mathbf{2x} = 6 - \cancel{2x} + \mathbf{2x}$$
$$3x = 6$$

4. Divide both sides of the equation by 3 in order to solve for $x$.

$$3x = 6$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{6}{\mathbf{3}}$$
$$x = 2$$

5. The answer is B.

**Solution 2**

1. Add 1 to both sides of the equation in order to eliminate constants on the left side of the equation. Combine constant terms on the right side of the equation.

$$\frac{1}{2}x - 1 = 2 - x$$
$$\frac{1}{2}x \cancel{-1} \cancel{+1} = 2 - x + \mathbf{1}$$
$$\frac{1}{2}x = 3 - x$$

2. Add $x$ to both sides of the equation in order to combine the $x$-terms on the left side of the equation. Make a common denominator (write the second $x$-term's implied coefficient of 1 as $\frac{2}{2}$) in order to allow the adding of the coefficients.

$$\frac{1}{2}x = 3 - x$$
$$\frac{1}{2}x + \mathbf{x} = 3 \cancel{-x} \cancel{+x}$$
$$\frac{1}{2}x + \frac{\mathbf{2}}{\mathbf{2}}x = 3$$
$$\frac{1+2}{2}x = 3$$
$$\frac{3}{2}x = 3$$

3. To solve for $x$, divide both sides of the equation by $\frac{3}{2}$; this is accomplished by multiplying by the reciprocal of $\frac{3}{2}$, which is $\frac{2}{3}$ (this method is much neater than writing out division by a fraction).

$$\frac{3}{2}x = 3$$
$$\frac{\cancel{2}}{\cancel{3}}\left(\frac{\cancel{3}}{\cancel{2}}\right)x = \frac{\mathbf{2}}{\cancel{3}}(\cancel{3})$$
$$x = 2$$

4. The answer is B.

The next problem has us combining many like terms, which follows exactly the same principles as are used in the simpler examples above. For example, $x + 2x + 3x + 4x = 10x$ because the coefficients add to 10 ($1 + 2 + 3 + 4 = 10$).

**Example 0.5-3**

**3** 1600.io 0.5

$$5(p+1) + 9(p-1) = 11p$$

What value of $p$ is the solution of the equation above?





**Solution**

1. Distribute the constants 5 and 9 through their respective parenthetical expressions on the left side of the equation. Combine like terms on the left side of the equation.

$$5(p+1) + 9(p-1) = 11p$$
$$5p + 5 + 9p - 9 = 11p$$
$$14p - 4 = 11p$$

2. Add 4 to both sides of the equation in order to eliminate constants on the left side of the equation (f you wanted to, you could instead subtract $14p$ from both sides of the equation in order to combine $p$-terms on the right side of the equation, but it is good practice to maintain a positive coefficient for the variable that you want to solve for, and following the steps shown below will allow us to do that).

$$14p - 4 = 11p$$
$$14p \cancel{-4} \cancel{+4} = 11p + \mathbf{4}$$
$$14p = 11p + 4$$

3. Subtract $11p$ from both sides of the equation in order to combine $p$-terms on the left side of the equation.

$$14p = 11p + 4$$
$$14p - \mathbf{11}p = \cancel{11p} + 4 \cancel{-11p}$$
$$3p = 4$$

4. Divide both sides of the equation by 3 in order to solve for $p$.

$$3p = 4$$
$$\frac{\cancel{3}p}{\cancel{3}} = \frac{4}{\mathbf{3}}$$
$$p = \frac{4}{3}$$

5. The answer is $\frac{4}{3}$.

**Section 0.5 Suggested Problems from Real Tests**

- Test 3-NC-17
- Test 5-NC-17
- Test 5-C-33
- Test 6-C-32
- Test 8-NC-1
- Test 10-NC-1
- Test 10-C-27
- Apr 2017-C-1
- May 2017-NC-18
- May 2017-C-24
- Mar 2018-NC-6
- May 2018-NC-14
- Mar 2019-NC-4
- Mar 2019-NC-16
- Mar 2019-C-3
- Apr 2019-NC-6
- May 2019 (Int)-NC-18
- May 2019 (Int)-C-4
- Oct 2019-C-22
- Mar 2020-C-1
- Oct 2020-NC-17
- Mar 2021-NC-3
- May 2021 (US)-NC-4
- May 2021 (Int)-NC-17
- May 2021 (Int)-C-9

View related real-test problems in SMTEx

SMTEx

## 0.6 Unknown Values in Denominators

If an unknown value is in the denominator of a fraction, you should eliminate the fraction by multiplying both sides of the equation by the denominator (just as you would when numbers are in the denominator). For example, if you are given the equation $\frac{8}{x} = 160$ and are asked to solve for $x$, simply multiply both sides of the equation by $x$ to get $8 = 160x$, and then divide both sides of the equation by 160 to solve for $x$: $x = \frac{8}{160} = \frac{1}{20}$.

Note that sometimes, having an unknown in the denominator can lead to extraneous solutions (which are covered in Chapter 9) because the denominator might end up being equal to 0, but those problems are easily spotted on the test, and none of those are included in the scope of this section. Therefore, in the cases covered here, simply multiplying by the denominator to eliminate the fractions works well.

**When an expression appears in the denominator, we cannot simply split the fraction into parts** as we could when the numerator is an expression and the denominator is just a number. For example, $\frac{2x+2}{2} = \frac{2x}{2} + \frac{2}{2} = x + 1$, but $\frac{2}{x+2} \neq \frac{2}{x} + \frac{2}{2}$, so be careful not to make that mistake moving forward.

In the following example, the unknown value appears in the denominator of two fractions (and the denominators aren't the same so we can't just combine the fractions). Instead of making a common denominator, we just need to multiply both sides of the equation by *both* of those denominators to eliminate both fractions.

**Example 0.6-1**

**1** 1600.io 0.6

$$\frac{1}{x} + \frac{2}{x-2} = 0$$

What value of $x$ satisfies the equation above?

**Solution**

1. Our first goal is to eliminate fractions, so first multiply both sides of the equation by $x$, which is the denominator of the first fraction.

$$\frac{1}{x} + \frac{2}{x-2} = 0$$
$$x\left(\frac{1}{x} + \frac{2}{x-2}\right) = x(0)$$
$$\cancel{x}\left(\frac{1}{\cancel{x}}\right) + x\left(\frac{2}{x-2}\right) = 0$$
$$1 + \frac{2x}{x-2} = 0$$





2. Multiply both sides of the equation by $x - 2$, which is the denominator of the remaining fraction. Combine like terms.

$$1 + \frac{2x}{x-2} = 0$$
$$(x-2)\left(1 + \frac{2x}{x-2}\right) = (x-2)(0)$$
$$(x-2)(1) + \cancel{(x-2)}\left(\frac{2x}{\cancel{x-2}}\right) = 0$$
$$x - 2 + 2x = 0$$
$$3x - 2 = 0$$

3. Add 2 to both sides of the equation in order to isolate the $x$-term.

$$3x - 2 = 0$$
$$3x \cancel{-2} \cancel{+2} = 0 + 2$$
$$3x = 2$$

4. Divide both sides of the equation by 3 to solve for $x$.

$$3x = 2$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{2}{3}$$
$$x = \frac{2}{3}$$

5. The answer is $\frac{2}{3}$.

Occasionally, you will see a fraction in which both the numerator and denominator are unknown. If you know the value of a fraction $\frac{x}{y}$, then the value of the reciprocal, $\frac{y}{x}$, is equal to the reciprocal of the original value (this should come as no surprise). For example, if $\frac{x}{y} = \frac{2}{3}$, then $\frac{y}{x} = \frac{3}{2}$; we've just taken the reciprocal of both sides of an equation. Looked at another way (literally), if you stand on your head, the equation is still true.

In the following example, we cannot find the individual values of $a$ and $b$, but we can solve for an expression that is useful to us by taking advantage of the reciprocal.

**Example 0.6-2**

**2** 1600.io 0.6

If $\frac{3a}{b} = \frac{1}{3}$, what is the value of $\frac{b}{a}$ ?

A) $\frac{1}{9}$

B) 1

C) 3

D) 9

**Solution 1**

1. Multiply both sides of the equation by $\frac{1}{3}$ (this is the same as dividing by 3) in order to isolate the expression $\frac{a}{b}$, which is the reciprocal of the solve-for expression.

$$\frac{3a}{b} = \frac{1}{3}$$
$$\frac{1}{\cancel{3}}\left(\frac{\cancel{3}a}{b}\right) = \frac{1}{3}\left(\frac{1}{3}\right)$$
$$\frac{a}{b} = \frac{1}{9}$$

2. Since $\frac{a}{b} = \frac{1}{9}$, then the reciprocals of both sides are also equal.

$$\frac{b}{a} = \frac{9}{1} = 9$$

3. The answer is D.

**Solution 2**

1. Multiply both sides of the equation by $b$ to eliminate the fraction on the left side of the equation.

$$\frac{3a}{b} = \frac{1}{3}$$
$$\cancel{b}\left(\frac{3a}{\cancel{b}}\right) = b\left(\frac{1}{3}\right)$$
$$3a = \frac{b}{3}$$

2. Multiply both sides of the equation by 3 to eliminate the fraction on the right side of the equation.

$$3a = \frac{b}{3}$$
$$3(3a) = \cancel{3}\left(\frac{b}{\cancel{3}}\right)$$
$$9a = b$$

3. Divide both sides of the equation by $a$ to create and solve for the solve-for expression.

$$9a = b$$
$$\frac{9\cancel{a}}{\cancel{a}} = \frac{b}{a}$$
$$9 = \frac{b}{a}$$

4. The answer is D.





**Section 0.6 Suggested Problems from Real Tests**

- Test 1-NC-8
- Test 3-NC-5
- Test 4-NC-10
- Test 8-NC-12

**View related real-test problems in SMTEx**

SMTEx

# Linear Relationships 1

## 1.1 Demystifying Linear Relationships

You might believe that you don't understand some of the math concepts on the SAT, but if I had a dollar for every student who wrongly thought they didn't understand linear relationships...

Let's say it's been a thousand students; then, you'd probably be able to figure out pretty quickly that I would have $1,000. If it had been twice as many students, I'd have twice as much money. If this makes sense, then you already get the essence of **linear relationships**. We often encounter real-life situations in which two quantities are related in such a way that a change in one of them results in a corresponding change in the other. If you get paid hourly, then working twice as many hours will double your pay. If your new school is only half as far away as your old one, it will take you half as long to get there if you travel at the same speed.

The phrase "linear relationships" doesn't necessarily make sense as a result of this knowledge because the phrase comes from the shapes of the graphs of these relationships, and we will get to those eventually. To get started though, we just need to be able to model these relationships by writing mathematical expressions that allow us to calculate values without simply listing a long series of numbers. You don't have to understand lines in an abstract way in order to express these relationships. Before we get into an algebraic understanding of lines, we can use common sense to work our way through some questions based on this type of relationship.

For example, Mindy lives within walking distance of three buffet restaurants that have different price structures.

The first buffet charges $20 to enter and dine, no matter how much food she eats. Whether Mindy eats 4 plates of food or 20 plates of food, she will pay a fixed amount of $20 to eat there.

$$\text{Total Cost} = \text{Entrance Charge}$$

The second buffet only charges by the plate (there is no entrance fee). Each plate of food Mindy eats costs $4. Here, Mindy will pay $4 times the number of plates of food she eats. If she eats one plate of food, her bill will be $4; if she eats 4 plates, her bill will be $\$4 \times 4 = \$16$. If she eats $p$ plates of food, her bill will be $4p$ dollars.

$$\text{Total Cost} = \text{Price Per Plate} \times \text{Number of Plates}$$

The third buffet combines the approaches of both of the other buffets. This third buffet imposes a smaller entrance charge than does the first buffet: entrance to the third buffet only costs $5. The third buffet also charges less per plate than does the second buffet: each plate of food costs $2. Here, Mindy will pay $5 plus $2 per plate of food. In order to enter the restaurant, she pays $5; if she has a heart attack while waiting in line and gets 0 plates of food, the whole trip will still cost her only $5 (aside from medical bills). If she eats one plate of food, the trip will cost $2 for the one plate of food ($2 times 1 plate) plus the $5 entrance fee (a total of $7). If she eats two plates of food, the cost will be $4 for the two plates of food ($2 times 2 plates) plus the $5 entrance charge for a total of $9. If she eats $p$ plates of food, her bill will be $2p$ ($2 times $p$ plates) plus the $5 entrance charge for a total of $2p + 5$ dollars.

$$\text{Total Cost} = (\text{Price Per Plate} \times \text{Number of Plates}) + \text{Entrance Charge}$$

The different styles of buffet demonstrate different facets of linear relationships. In the simplest case, there can just be a constant amount that never changes (the entrance charge). More commonly, the value produced can be changed by some amount (here, the price per plate) every time a variable (the number of plates) is changed. And finally, in the most general situation, there can be a combination of both of these attributes.

## 1.2 Writing Linear Expressions

On the test, you will be responsible for writing mathematical expressions to represent linear relationships.

**Example 1.2-1**

**1** 1600.io 1.2

One pound of peanuts costs \$3. At this price, how much will $p$ pounds of peanuts cost?

A) $3p$

B) $3 + p$

C) $\frac{3}{p}$

D) $\frac{p}{3}$

**Solution**

1. Since each pound costs \$3, multiply 3 by the number of pounds, $p$, to find the total cost, which is $3p$.
2. The answer is A.

**Notes**

If you need to verify that this works, just list out a few values to see that this is true.

| Pounds of Peanuts | Cost | Pattern Representation |
|---|---|---|
| 1 | \$3 | 3(1) |
| 2 | \$6 | 3(2) |
| 3 | \$9 | 3(3) |

Notice in each row that the value in the Cost column is three times the value in the Pounds of Peanuts column. If $p$ is the number of pounds of peanuts, then three times $p$, or $3p$, is the cost of those $p$ pounds of peanuts.

Now suppose that George starts a pen collection with 20 pens that were gifted to him by his grandfather. He plans on buying 3 new pens every month to add to his collection. How could we write an expression to show the total number of pens he would have after $m$ months?





The number of new pens George would have after $m$ months follows the same pattern we have already talked about. If he adds 3 pens a month for $m$ months, he would have $3m$ *new* pens. All we have to do is add the original 20 pens to find out the total number of pens he has after $m$ months: $3m + 20$.

Let's look at this situation as represented by a table to see how this expression makes sense as a way of representing the pattern of how his collection grows.

| Number of Months | Total Number of Pens | Pattern Representation |
|---|---|---|
| 0 | 20 | $20 + 3(0)$ |
| 1 | 23 | $20 + 3(1)$ |
| 2 | 26 | $20 + 3(2)$ |
| $m$ | $20 + 3m$ | $20 + 3(m)$ |

**Example 1.2-2**

**2** 1600.io 1.2

One of the requirements for becoming a journalist was the ability to write notes at a rate of 100 words per minute. Lewis was able to write only 40 words per minute when he started reporting, but through practice, he was able to increase his speed by 2 words per minute each week. Which of the following represents the number of words per minute that Lewis could write after $w$ weeks of practice?

A) $2 + 40w$

B) $40 + 2w$

C) $100 + 2w$

D) $100 - 2w$

**Solution**

1. Since Lewis's speed increases by 2 words per minute every week, the amount his speed increases is equal to 2 times the number of weeks, $w$. His speed increases by $2w$ overall.
2. Since Lewis started at a speed of 40 words per minute, to find his overall speed, we need to add his speed increase to his original starting speed. Lewis's overall speed after $w$ weeks of practice is $40 + 2w$.
3. The answer is B.

**Notes**

You may have noticed that in this example, the constant term (40) appears before the variable term ($2w$), but it's important to understand that the ordering of the terms does not matter: $40 + 2w$ is equivalent to $2w + 40$. Linear expressions on the test can appear in either arrangement. By convention, variable terms appear first when writing expressions, but the ordering has no mathematical consequences.

Note that in this problem, you're given some information that you do not need: the requirement that a journalist be able to write at 100 words per minute. This happens occasionally on word problems (though very rarely on strictly mathematical problems).

You should understand that just because you don't use a given quantity or other piece of information, that does not mean that you are not properly solving the problem.

**SkillDrill 1.2-1**

**Directions:** Based on the situation described, write a mathematical expression that can be used to answer the question.

1. Emily earns \$12 for every pizza she sells. How much money will she have if she sells $p$ pizzas?
2. The Tooth Fairy gains one tooth for every house she visits. How many teeth will she have after visiting $h$ houses?
3. Brandon started the day with \$50 in his wallet, and he will get another \$20 from each of his aunts. How much money will he have after he visits $a$ of his aunts?
4. Charlie has 84 baseball cards and he buys 3 more every weekend. How many cards will he have after $w$ weekends?
5. Priscilla has to eat 20 hamburgers to win a prize. It takes her 3 minutes to eat each hamburger. If she has already eaten $h$ hamburgers, how much *longer* will it take her to eat the remaining hamburgers?

**Section 1.2 Suggested Problems from Real Tests**

- Test 4-C-2
- Test 8-C-1
- Apr 2017-C-21
- Mar 2018-NC-2
- May 2021 (US)-C-19

**View related real-test problems in SMTEx**

SMTEx

## Section 1.2 Practice Problems

**1** 1600.io 1.2

Saquon spent a total of \$450 on his hotel room during his vacation. Each day of his vacation, he bought a dinner at a restaurant for \$28. If Saquon bought $d$ dinners, how much money, in dollars, did Saquon spend during his vacation on his hotel room and dinners?

A) $450 + 56d$

B) $450 + 28d$

C) $450 + 14d$

D) $28d$

**2** 1600.io 1.2

An artisan is crafting a custom chess set by carving each of the 32 pieces in the set by hand. It takes the artisan 40 minutes to carve each piece. If $p$ of the pieces are already carved, which of the following represents the number of additional minutes needed to finish carving the set of chess pieces?

A) $32(40 - p)$

B) $32(p - 40)$

C) $40(32 - p)$

D) $40(p - 32)$





## 1.3 Writing and Solving Linear Equations

We have seen a few problems in which we wrote mathematical expressions that represent a certain type of simple pattern or relationship. **Expressions**, strictly speaking, consist only of numerical and variable terms being added, subtracted, multiplied, etc. For example, $40 + 2w$ is an expression. When we set this value equal to another expression, we can make an equation. **Equations** consist of two expressions that are set equal to each other using an equals sign (=). For example, if we wanted to make an equation to represent Lewis's writing speed, we could set $s$, the speed that Lewis can write words, equal to the expression $40 + 2w$, which tells us Lewis's writing speed after $w$ weeks of practice. In equation form, the relationship is written $s = 40 + 2w$. Note that the linear expression could appear on either side of the equals sign; the equation $s = 40 + 2w$ could also be written as $40 + 2w = s$.

### Choose Variable Letters that Make Sense

When writing your own equations, you should always choose letters that tell you what a variable stands for. For example, in the case of Lewis's writing speed, we used $s$ to represent the *speed* that he writes (we were told to use $w$ for the number of weeks).

Avoid always using $x$ and $y$ when writing your own equations for real world word problems, because—although $x$ and $y$ are usually the most familiar variable letters to most students—it is very easy to forget what each letter represents (unless you're working with xylophones and yodelers).

In the case of Lewis, if we write $y = 40 + 2x$ it is not apparent what $x$ and $y$ represent, but when we write $s = 40 + 2w$ it is much easier to see that $s$ represents the speed and $w$ represents the number of weeks he practiced.

To take the next step into writing linear *equations*, all we need to do is set the kinds of *expressions* we have been building equal to some other quantity or variable that represents the total.

**Example 1.3-1**

**1** 1600.io 1.3

A drone, initially hovering 80 feet above the ground, begins to lose altitude at a rate of 3 feet per second. Which of the following equations represents the drone's altitude above the ground $a$, in feet, $t$ seconds after the drone begins to lose altitude?

A) $a = 80 - 3$

B) $a = 80 + 3t$

C) $a = 80 - 3t$

D) $a = 80t - 3$

**Solution**

1. The altitude, or height, goes down by 3 feet every second (that is, the change in height is $-3$ feet every second), so to find the total change in the height, we have to multiply $-3$ by the number of seconds, $t$. The change in height after $t$ seconds is $-3t$.

2. Since the drone's altitude started at 80 feet and *decreased* by $3t$ feet after $t$ seconds, we start with 80 and *subtract* $3t$, then set it equal to $a$, the altitude of the drone after $t$ seconds. The final equation used to find the height after $t$ seconds is $a = 80 - 3t$.

3. The answer is C.

**Notes**

This is the first time we've seen a negative change in the value in a linear relationship. If you are unsure whether the change should be added or subtracted in this example, think about the way the height changes. A negative change in the height should be reflected in the equation by subtracting the change in height from the initial height.

Here, we have an ordinary linear relationship written as a constant first, then a variable term: $a = 80 - 3t$. You might already be familiar with Slope-Intercept Form for a linear equation, which, by convention, places the variable term first, followed by the constant (the "$y$-intercept", which we will explain thoroughly a bit later); here, that would be $a = -3t + 80$. Note, importantly, that this has *no effect whatsoever on the relationship*; it's just a matter of writing the terms in a different order which happens to make sense for the situation in the problem.

Having equations to represent this type of situation is useful for when we need to calculate values that we would not easily be able to count to. For example, it's easy to add the cost of a single item two or three times, but when we are buying hundreds or thousands of items, we can use the equations to easily find particular values in a way that involves no repetition.

**Example 1.3-2**

**2** 1600.io 1.3

A partially filled aquarium contains 20 gallons of water. A hose is turned on, and water flows into the tank at a rate of 4 gallons per minute. How many gallons of water will be in the aquarium after 30 minutes?

**Solution**

1. We have to start by writing the equation to represent the situation. Let $w$ be the total amount of water in the aquarium after being filled for $m$ minutes. The amount of water starts at 20 gallons and then increases by 4 gallons every minute, so the equation that represents the situation is $w = 4m + 20$ (again, note that we could have written this as $20 + 4m$, which you might feel matches the real-world situation more naturally; it makes no difference).

2. We want to know how much water there will be in the aquarium after 30 minutes, so we should replace $m$ with 30 and find the value of $w$.

$$\begin{aligned} w &= 4m + 20 \\ w &= 4(30) + 20 \\ w &= 120 + 20 \\ w &= 140 \end{aligned}$$

3. The answer is 140.





We now know how to find final values for situations where we know what to substitute into the equations: we can find the cost of a buffet when we know how many plates of food are eaten, and we can find the number of gallons of water in an aquarium after filling it for a certain amount of time. But what if we were told the total amount Mindy spent at a buffet, and we were then asked to figure out how many plates of food she ate? What if we were told how much water was in the aquarium after a certain amount of time, and we were then asked how many minutes the hose must have been filling the aquarium in order for there to be that much water? Essentially, we are being asked to substitute a different value for a different variable in an equation that models a linear relationship. The following example illustrates this concept.

**Example 1.3-3**

**3** 1600.io 1.3

The monthly membership fee for a food delivery service is \$22.50. The cost of dinners is included in the membership fee, but there is an additional fee of \$2.50 for each dessert. In April, Gus's membership and dessert fees were \$32.50. How many desserts did Gus have delivered that month?

A) 2

B) 3

C) 4

D) 5

**Solution**

1. We have to start by writing the equation that represents the total cost. Let's use $t$ for the total amount of money Gus pays each month and $d$ for the number of desserts. There is a constant charge of \$22.50, and the cost goes up by \$2.50 for every dessert Gus orders that month, so the equation is $t = 2.50d + 22.50$.

2. We know that his total charge in April was \$32.50, so we should replace $t$ with 32.50.

$$\begin{aligned} t &= 2.50d + 22.50 \\ 32.50 &= 2.50d + 22.50 \end{aligned}$$

3. Now that we have substituted the total cost for April in the correct place, we can solve for $d$, the number of desserts Gus ordered that month.

$$\begin{aligned} 32.50 &= 2.50d + 22.50 \\ 10 &= 2.50d \\ \frac{10}{2.50} &= d \\ 4 &= d \end{aligned}$$

4. The answer is C.

**Notes**

A problem like this could appear on either the Calculator or No Calculator section of the test. Don't panic when you see decimals and fractions. You know how many times \$2.50 goes into 10 dollars, just like you know how many times 25 cents goes into \$1.

When we're working with a linear equation of the form $y = mx + b$, we can be given any three of the four components (the value of $x$, the value of $y$, the coefficient $m$ of $x$, and the value of the constant $b$) that comprise these equations and use them to determine the value of the remaining component. As long as you understand what type of information is represented by each component, you can easily work through problems that involve these types of relationships regardless of which values are given and which are unknown. If you write a nice equation, with well-chosen letters, it should be easy to substitute values for the information that you know in order to solve for the variable whose value you would like to find.

**SkillDrill 1.3-1**

**Directions:** Based on the situation described, write a mathematical equation that can be used to answer the question. If you have to choose your own variable, make sure to choose a letter that makes sense. Solve the equation to find the unknown value.

1. Emily earns \$12 for every pizza she sells. How much money, in dollars, will she have if she sells 5 pizzas?
2. The Tooth Fairy gains one tooth for every house she visits. How many teeth will she have after visiting 31 houses?
3. Brandon started the day with \$50 in his wallet, and he received another \$20 from each of his aunts during the day. If he has \$110 at the end of the day, how many aunts did he visit?
4. Charlie has 84 baseball cards and he buys 3 more every weekend. How many cards will he have after 2 weekends?
5. Priscilla has to eat 20 hamburgers to win a prize. It takes her 3 minutes to eat each hamburger. If she has been eating for 15 minutes, how many more hamburgers does she still have to eat?

**Section 1.3 Suggested Problems from Real Tests**

- Test 1-C-10
- Test 2-C-3
- Test 2-C-8
- Test 4-C-1
- Test 4-C-31
- Test 5-C-7
- Test 5-C-16
- Test 5-C-18
- Test 6-C-4
- Test 7-NC-6
- Test 8-NC-19
- May 2017-C-5
- May 2017-C-15
- May 2017-C-20
- Mar 2018-C-9
- Apr 2019-C-4
- Apr 2019-C-31
- May 2019 (US)-C-1
- May 2019 (Int)-C-2
- May 2019 (Int)-C-3
- May 2019 (Int)-C-36
- Oct 2019-C-32
- Mar 2020-NC-5
- Oct 2020-NC-1
- May 2021 (Int)-C-4

View related real-test problems in SMTEx

SMTEx





## Section 1.3 Practice Problems

**1** 1600.io 1.3

The Finer Things Club (FTC) in a certain high school plans to increase its membership by a total of $n$ members per year. There were $m$ members in the FTC at the beginning of this year. Which function best models the total number of members, $y$, the FTC plans to have as members $x$ years from now?

A) $y = nx + m$

B) $y = nx - m$

C) $y = mx + n$

D) $y = mx - n$

**2** 1600.io 1.3

A home security company charges homeowners a onetime setup fee of \$220 plus $d$ dollars for each month of home monitoring. If a homeowner paid \$640 for the first 12 months, including the setup fee, what is the value of $d$ ?

A) 25

B) 35

C) 40

D) 50

**3** 1600.io 1.3

An eagle takes off from a perch that is 10 feet above the ground. If the eagle rises at a constant rate of 23 feet per second, which of the following equations gives the height, $h$, in feet, of the eagle $s$ seconds after it starts flying upwards?

A) $h = 10s + 23$

B) $h = 10s + \frac{23}{10}$

C) $h = 23s + 10$

D) $h = 23s + \frac{23}{10}$

**4** 1600.io 1.3

Charles is planning to ride his bike 200 miles to prove something to himself. If he plans to ride 15 miles per hour, which of the following represents the remaining distance $d$, in miles, that Charles will have to ride his bike to achieve his goal after riding for $n$ hours?

A) $d = 200 + 15n$

B) $d = 200n - 15$

C) $d = 15n - 200$

D) $d = 200 - 15n$

## 1.4 Function Notation/Inputs and Outputs

Let's look at an alternative way of writing equations that uses a different notation to represent the total amount. We call equations written in this new form **functions**. With functions, an input value is plugged into the function, some operations are performed on the input value, and then an output value is produced. We can represent the relationships we have talked about in this chapter using function notation, so let's write an equation as a function to see what functions look like.

For example, if 30 cubic feet of water is being pumped into a pool every minute, we could use $v$ to represent the volume of water and $t$ to represent the amount of time the pool is being filled. We could say that the volume of water in the pool is a **function** of time, $t$, because the volume of water is dependent only on the amount of time that has passed. Instead of just using $v$ to represent the volume, we can use the notation $v(t)$ (pronounced "$v$ of $t$") to show that $v$ is a function of time. The equation representing this relationship could be written as $v(t) = 30t$ (the volume of water in the pool is 30 gallons per minute times the number of minutes the water flows into the pool).

The $t$ in the parentheses is a placeholder for an input value just as variables are always placeholders. If we wanted to know how much water there would be after 3 minutes, we would use 3 for the value of the function's input variable $t$. Using function notation, we may be asked to find the value of $v(3)$ (pronounced "$v$ of 3"), which means the value of $v$ when $t$ is 3. Whatever value replaces the variable in the parentheses gets substituted in for the variable everywhere it appears in the expression that defines the function, so in this case we replace each instance of $t$ with 3.

$$v(t) = 30t$$
$$v(3) = 30(3)$$
$$v(3) = 90$$

Realize that **the function notation $v(3)$ does not mean "$v$ times 3."** Many students get this confused because this is the first time they have seen parentheses used for a purpose other than multiplication. The test will let you know that $v$ is a function by using the word "function" in the question. Even if the word "function" is not explicitly stated, it is more likely that $v(3)$ is function notation because $v$ times 3 (or 3 times $v$) would normally be written as $3v$.

If a function called $f(x)$ is defined by the equation $f(x) = 3x + 2$, we can find $f(4)$, the output value when the input value is 4, by substituting 4 for $x$ regardless of whether the function represents some real-world situation or it's a purely abstract mathematical construct; we simply substitute the input value and evaluate the expression to get the output value.

**SkillDrill 1.4-1**

**Directions:** Given the three functions below, $f(x)$, $g(x)$, and $h(x)$, find the value you are asked to find.

$$f(x) = 2x + 1 \qquad g(x) = -3x + 4 \qquad h(x) = \frac{1}{2}x - 2$$

Note that answers here can be negative, which is not the case with the test's Student Produced Response ("grid-in") questions.

**1.** $f(1)$ **2.** $g(0)$ **3.** $h(4)$ **4.** $f(-2)$

**5.** $g(4)$ **6.** $h(5)$ **7.** $f(10)$ **8.** $g(-2)$

**9.** $h(2)$ **10.** $f(0)$ **11.** $g(-3)$ **12.** $h(12)$

**Directions:** Using the same three functions listed above, find the input value $x$ that will result in the given output value of the function.

**13.** $f(x) = 5$ **14.** $g(x) = 10$ **15.** $h(x) = 4$ **16.** $f(x) = 14$

**17.** $g(x) = -20$ **18.** $h(x) = -31$ **19.** $f(x) = 0$ **20.** $g(x) = 4$

**21.** $h(x) = -4$ **22.** $f(x) = 11$ **23.** $g(x) = 1$ **24.** $h(x) = \frac{1}{2}$





**Example 1.4-1**

**1** 1600.io 1.4

$$f(x) = 4x + 1$$

The function $f$ is defined above. What is the value of $f(-2)$?

A) $-8$
B) $-7$
C) $7$
D) $9$

**Solution**

1. Replace each $x$ with $-2$.

$$f(x) = 4x + 1$$
$$f(-2) = 4(-2) + 1$$
$$f(-2) = -8 + 1$$
$$f(-2) = -7$$

2. The answer is B.

If you are told the value of a function for a particular $x$-value, you should substitute the $x$-value, but also replace the $f(x)$ term with the value of the function.

**Example 1.4-2**

**2** 1600.io 1.4

$$f(x) = \frac{1}{2}x + b$$

In the function above, $b$ is a constant. If $f(4) = 5$, what is the value of $f(-4)$?

A) $-5$
B) $-2$
C) $1$
D) $3$

**Solution**

1. When we are told that $f(4) = 5$, we know that the value of the function is 5 when $x$ is 4. Replace each $x$ with 4.

$$f(x) = \frac{1}{2}x + b$$

$$f(4) = \frac{1}{2}(4) + b$$

$$f(4) = 2 + b$$

2. Since we know that $f(4) = 5$, we can replace $f(4)$ with 5. Remember that $f(4) = 5$ does NOT mean "$f$ times 4 equals 5," but instead tells us that the value of the function is 5 when $x$ is 4. Making this replacement enables us to solve for $b$.

$$f(4) = 2 + b$$
$$5 = 2 + b$$
$$3 = b$$

3. Now that we have the value of $b$, we can replace $b$ with 3 in the given function.

$$f(x) = \frac{1}{2}x + b$$

$$f(x) = \frac{1}{2}x + 3$$

4. Since we want to find $f(-4)$, which denotes the value of the function when $x$ is $-4$, we should replace each instance of $x$ with $-4$.

$$f(x) = \frac{1}{2}x + 3$$

$$f(-4) = \frac{1}{2}(-4) + 3$$

$$f(-4) = -2 + 3$$

$$f(-4) = 1$$

5. The answer is C.

Function notation is used in a variety of problem types on the test, so it's essential that you become familiar and comfortable with it. In general, when dealing with the graphs of functions in the $xy$-plane (which begins in the next chapter), the expression $f(x)$ refers to the $y$-value for a given $x$-value. For example, $f(4)$ is the $y$-value of the function when the $x$-value is 4 (this may be a hint that plugging in 4 for $x$ is a useful step in that hypothetical problem).

### Section 1.4 Suggested Problems from Real Tests

- Test 4-NC-2
- Test 10-C-5
- Apr 2017-NC-13
- Apr 2017-C-7
- May 2017-NC-2
- Apr 2019-NC-3
- Apr 2021-C-3
- Apr 2021-C-28
- May 2021 (Int)-NC-18

View related real-test problems in SMTEx

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## Section 1.4 Practice Problems

**1** 1600.io 1.4

$$f(x) = 3(x - 4) + 1$$

For the function $f$ defined above, what is the value of $f(4)$?

A) $-2$

B) $0$

C) $1$

D) $4$

**2** 1600.io 1.4

A tank initially contains 4,678 cubic inches of orange juice. A pump begins emptying the orange juice at a constant rate of 63 cubic inches per minute. Which of the following functions best approximates the volume $v(t)$, in cubic inches, of orange juice in the tank $t$ minutes after pumping begins, for $0 \le t \le 74$?

A) $v(t) = 4{,}678 - 63t$

B) $v(t) = 4{,}678 - 74t$

C) $v(t) = 4{,}678 + 63t$

D) $v(t) = 4{,}678 + 74t$

**3** 1600.io 1.4

$$b(t) = \frac{3(6t - 10k)}{2} - 120$$

The number of people who bike on a public trail can be modeled by the function $b$ above, where $k$ is a constant and $t$ is the air temperature in degrees Fahrenheit (°F) for $70 < t < 100$. If 480 people are predicted to bike on the trail when the temperature is 80°F, what is the value of $k$?

A) 8

B) 10

C) 28

D) 30

**4** 1600.io 1.4

$$f(x) = bx + 4$$

In the function above, $b$ is a constant. If $f(1) = 3$, what is the value of $f(-3)$?

A) $-3$

B) $-1$

C) $1$

D) $7$

## CHAPTER 1 RECAP

- Total Amount = (Amount per Thing × Number of Things) + Starting Amount
- **Expressions** consist only of numerical and variable terms being added, subtracted, multiplied, etc. For example, $40+2w$ is an expression. When we set this value equal to another expression, we can make an equation. **Equations** consist of two expressions that are set equal to each other using an equals sign (=).
- When writing your own expressions and equations, use letters that make sense for the variables so that you don't mistake what the letters stand for.
- The four components of a linear equation in the form $y = mx + b$ are the $y$-value (also called an output value when an equation is represented as a function), the $x$-value (the input value when working with function notation), the coefficient $m$ of the $x$-term, and the constant $b$. We can solve for any one of these when we know the other three. Make sure you substitute given values for the the correct component.
- Functions are a type of equation for which an input value, often $x$, is plugged into an equation, producing an output value $f(x)$, which is read as "$f$ of $x$."

  Note that $f(x)$ means the output value of a function when $x$ is plugged into the function. It *does not* mean "$f$ times $x$." For example, $f(3)$ means the value of the function when 3 is substituted for $x$.
- Whatever goes into the parentheses in a function is the input value. Replace all instances of the input variable with whatever is in the parentheses in order to find the function's value.

  If you are given the value of $f(x)$ and you're asked to find the corresponding value of $x$, you should replace the expression $f(x)$ with that value and then solve for the $x$-value that produced that output.





## Additional Problems

**1** 1600.io 1.4

$$f(x) = \frac{(x+8)}{5}$$

For the function $f$ defined above, what is the value of $f(7) - f(-3)$?

A) 3

B) $\frac{12}{5}$

C) 2

D) $\frac{4}{5}$

**2** 1600.io 1.3

A magazine debuted with 1,329 subscribers. The magazine's growth plan assumes that 270 new subscribers will be added each quarter (every 3 months) for the first 10 years. If an equation is written in the form $y = ax + b$ to represent the number of subscribers, $y$, subscribed to the magazine $x$ quarters after the magazine debuted, what is the value of $b$?

**3** 1600.io 1.2

John earns \$14.70 per hour at his part-time job. When he works $z$ hours, he earns $14.70z$ dollars. Which of the following expressions gives the amount, in dollars, John will earn if he works $5z$ hours?

A) $5 + 14.70z$

B) $14.70(z + 5)$

C) $5z + 14.70z$

D) $5(14.70z)$

**4** 1600.io 1.3

Cranjis rented a motorcycle for one day from a company that charges \$100 per day plus \$0.35 per mile driven. If she was charged a total of \$149 for the rental and mileage, for how many miles of driving was Cranjis charged? (Assume there is no tax.)

A) 35

B) 140

C) 285

D) 425

**5** 1600.io 1.3

The acceleration due to gravity, in feet per second per second (ft/s$^2$), on Jupiter is 7 ft/s$^2$ less than 3 times the acceleration due to gravity on Venus. If the acceleration due to gravity on Jupiter is 80 ft/s$^2$, what is the acceleration due to gravity, in ft/s$^2$, on Venus?

**6** 1600.io 1.3

The equation $y = 0.2x$ models the relationship between the number of different exercises a certain fitness model performs, $y$, during an $x$-minute workout session. How many exercises did the fitness model perform if the session lasted 40 minutes?

A) 2

B) 5

C) 8

D) 20

**7** 1600.io 1.3

$$p = 17f - (6f + t)$$

The profit, $p$, in dollars, from producing and selling $f$ frisbees is given by the equation above, where $t$ is a constant. If 300 frisbees are produced and sold for a profit of \$2,150, what is the value of $t$?

**8** 1600.io 1.4

$$c(x) = mx + 360$$

A company's total cost $c(x)$, in euros (€), to produce $x$ pantaloons is given by the function above, where $m$ is a constant and $x > 0$. The total cost to produce 50 pantaloons is €660. What is the total cost, in euros, to produce 500 pantaloons? (Disregard the € sign when writing your answer.)

**9** 1600.io 1.4

$$f(x) = \frac{x+2}{3}$$

For the function $f$ defined above, what is the value of $f(-2)$?

A) $-\frac{2}{3}$

B) 0

C) $\frac{1}{3}$

D) 1

**10** 1600.io 1.3

$$T = 2{,}000 + 20a$$

In the equation above, $T$ represents Penny's total points in a video game, where $a$ represents the number of apples she has caught in the game and 2,000 represents a starting bonus for choosing to play on a harder difficulty. If Penny scored a total of 4,400 points, how many apples did she catch in the game?

A) 100

B) 120

C) 220

D) 440





**11** 1600.io 1.2

The top floor of a trapezoidal building has 4 rooms. There are 45 floors in total. If each floor has 3 more rooms than the floor above it, which expression gives the total number of rooms on the bottom floor?

A) $4 + 3(45 - 1)$

B) $4 + 3(45)$

C) $45(4 + 3)$

D) $4 + 3^{45}$

**12** 1600.io 1.3

Paige was 37 inches tall the day she turned 4 years old, and she was 41 inches tall the day she turned 5 years old. If Paige's height increases by the same amount each year between the ages of 4 and 9, how many inches tall will she be the day she turns 8 years old?

**Questions 13 and 14 refer to the following information.**

| Species of tree | Growth factor |
|---|---|
| Hackberry | 3.1 |
| Cottonwood | 1.6 |
| Silver maple | 2.0 |
| Norway maple | 3.0 |
| Blue spruce | 2.1 |
| Ponderosa pine | 2.3 |
| Crabapple | 2.6 |
| Honeylocust | 2.7 |

One method of calculating the approximate age, in years, of a tree of a particular species is to multiply the diameter of the tree, in inches, by a constant called the growth factor for that species. The table above gives the growth factors for eight species of trees.

**13** 1600.io 1.3

According to the information in the table, what is the approximate age of a Norway maple tree with a diameter of 18 inches?

A) 36 years

B) 42 years

C) 54 years

D) 58 years

**14** 1600.io 1.3

If a ponderosa pine tree and a hackberry tree each now have a diameter of 10 inches, which of the following will be closest to the difference, in inches, of their diameters 10 years from now?

A) 1.0

B) 1.1

C) 1.2

D) 1.3

**15** 1600.io 1.3

A water company charges Franklin \$0.04 per gallon of water he uses in his house. If Franklin was charged \$40 by the water company, how many gallons of water did Franklin use?

A) 0.01

B) 100

C) 160

D) 1,000

**16** 1600.io 1.4

The function $g$ is defined as $g(x) = \frac{4x}{5} + 4$. What is the value of $g(-50)$ ?

A) $-46$

B) $-44$

C) $-36$

D) $-6$

**17** 1600.io 1.3

On its opening day, a bicycle shop had an inventory of 42 bikes. During the first 3 months, 21 additional bikes were purchased by the shop each week, and the sales team sold an average of 14 bikes per week. During the first three months, which of the following equations best models the bike inventory, $b$, at the shop $t$ weeks after opening?

A) $b = -7t + 42$

B) $b = \frac{2}{3}t + 42$

C) $b = 7t + 42$

D) $b = 14t + 42$

**18** 1600.io 1.3

A company that makes plastic toys purchases a 3D printer for \$4,360. The printer depreciates in value at a constant rate for 8 years, after which it is considered to have no monetary value. How much is the printer worth 2 years after it is purchased?

A) \$1,090

B) \$2,180

C) \$3,270

D) \$3,815





# Answer Key

### SkillDrill 1.2-1

1. $12p$
2. $h$
3. $50 + 20a$
4. $84 + 3w$
5. $3(20 - h)$ or $60 - 3h$

### Section 1.2 Practice Problems

1. B
2. C

### SkillDrill 1.3-1

1. 60
2. 31
3. 3
4. 90
5. 15

### Section 1.3 Practice Problems

1. A
2. B
3. C
4. D

### SkillDrill 1.4-1

1. 3
2. 4
3. 0
4. $-3$
5. $-8$
6. $\frac{1}{2}$
7. 21
8. 10
9. $-1$
10. 1
11. 13
12. 4
13. 2
14. $-2$
15. 12
16. $\frac{13}{2}$ or 6.5
17. 8
18. $-58$
19. $\frac{-1}{2}$ or $-0.5$
20. 0
21. $-4$
22. 5
23. 1
24. 5

### Section 1.4 Practice Problems

1. C
2. A
3. A
4. D

### Additional Problems

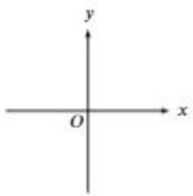
1. C
2. 1329
3. D
4. B
5. 29
6. C
7. 1150
8. 3360
9. B
10. B
11. A
12. 53
13. C
14. B
15. D
16. C
17. C
18. C

# Slope-Intercept Form 2

## 2.1 Why They're Called Linear Relationships

Now it's time to see where linear relationships get their name. To start, you need to be familiar with the ***xy*-plane**.

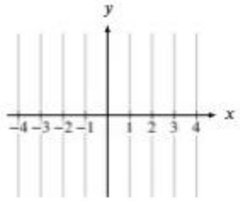
The $xy$-plane consists of two lines called **axes**. The ***x*-axis** goes from left to right (horizontal). The ***y*-axis** goes from bottom to top (vertical).



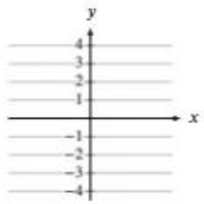

Typically, the point where the two axes cross is called the **origin** (usually marked with an $O$), and it is the point where both the $x$-value and $y$-value are equal to 0. When working with graphs of mathematical functions, the $x$- and $y$-axes will comply with this arrangement, but it's important to note that data sets represented as scatter plots, lines of best fit, and other statistical information are sometimes shown on graphs where the axes do **not** cross at their 0 values, and you must be alert for this situation, as it comes up often on the test.

There are often additional vertical and horizontal lines drawn on the plane that form a grid; these are evenly spaced lines that mark successive $x$- or $y$-values.

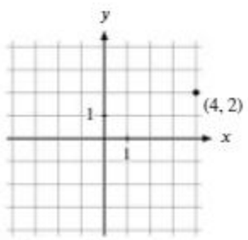
Because the $y$-axis goes straight up through the origin, which has an $x$-value of 0, all the points on the $y$-axis also have an $x$-value of 0. To the right of the $y$-axis are positive values of $x$, and to the left of the $y$-axis are negative values of $x$. The other vertical gridlines are usually (but not always!) used to denote the successive whole number values of $x$: 1, 2, 3, etc. To find the $x$-value of a point from a graph in that common arrangement, just count how many gridlines there are to the left or right of the $y$-axis.



The $x$-axis goes straight through the origin from left to right, and because the origin has a $y$-value of 0, all the points on the $x$-axis also have a $y$-value of 0. Above the $x$-axis are positive values of $y$, and below the $x$-axis are negative values of $y$. The other horizontal gridlines are usually (but not always!) used to denote the successive whole number values of $y$: 1, 2, 3, etc. To find the $y$-value of a point from a graph like that, count how many gridlines there are above or below the $x$-axis.

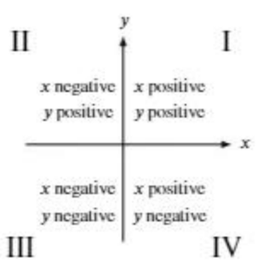



Points on this $xy$-plane will often be provided using the notation $(x, y)$ where the first number tells us the $x$-value, and the second number tells us the corresponding $y$-value. For example, the point $(4, 2)$ is 4 lines to the right of the $y$-axis because the $x$-value is 4, and it's 2 lines above the $x$-axis because the $y$-value is 2. Because the *order* in which the two elements appear is important—the first value is the $x$-value, and the second, the $y$-value—this notation represents what is called an **ordered pair**. The ordered pair that represents the origin is $(0, 0)$ because both the $x$-and $y$-values are 0.




Another way you may be given the coordinates of points is based on function notation. For example, if you are told that $f(4) = 2$, then you know that $(4, 2)$ is a point on the graph of the function because the function's $y$-value is equal to 2 when the $x$-value is equal to 4. In other words, when plotting a function $f$ on the $xy$-plane, $y = f(x)$.

The $x$- and $y$-axes divide the $xy$-plane into four **quadrants** that are numbered with Roman numerals starting in the upper right hand corner and going counter-clockwise.

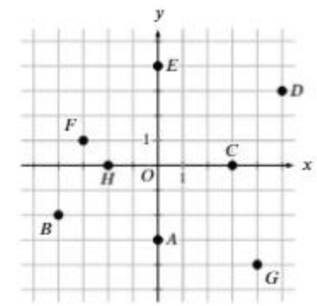







**SkillDrill 2.1-1**

**Directions:** Write the ordered pair of $x$- and $y$-coordinates for each labeled point in the figure below.

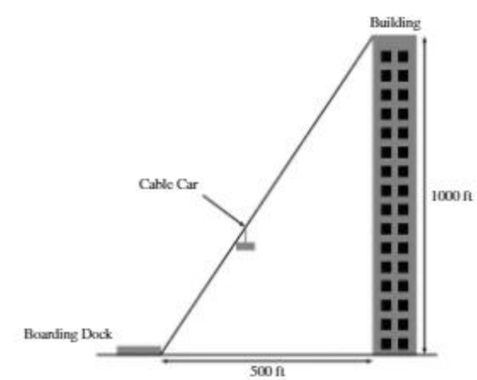



1. Point $A$
2. Point $B$
3. Point $C$
4. Point $D$
5. Point $E$
6. Point $F$
7. Point $G$
8. Point $H$

## Slopes

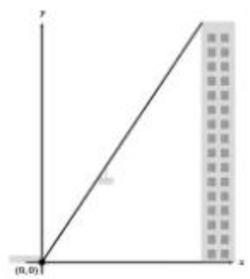
Now that we know about how points are represented on the $xy$-plane, let's relate this idea to a real world situation. We will plot some data points based on a scenario and see how they form a **line**, and linear relationships, when graphed in the $xy$-plane, produce **lines**—that's precisely why these relationships are "linear" (line-ar).

Molly is taking a ride on a cable car to the top of a tall building in a big city. The boarding platform is at ground level, and the building, which is located 500 feet away from the boarding dock, is 1,000 feet tall.

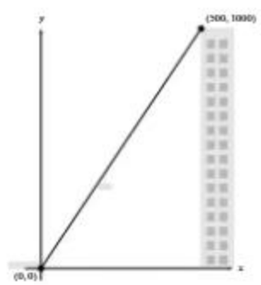


If Molly wanted to get to the top of the building without riding the cable car, she would have to walk horizontally 500 ft along the ground and then ride an elevator vertically 1,000 ft. The cable car, on the other hand, travels a straight line path from the dock to the top of the building that covers both the horizontal distance and vertical distance at the same time.

We can represent the cable on the $xy$-plane and use it to find the equation of the linear relationship between the height of the cable car and its horizontal distance from the boarding dock. If we use $x$ to represent Molly's horizontal distance from the dock and $y$ to represent her height above the ground, then when Molly is first starting her ride, we can say that 0 is an $x$-value and 0 is the corresponding $y$-value (she is at the boarding dock). Therefore, if we were to graph this point, we would graph the point (0, 0), which is the origin.



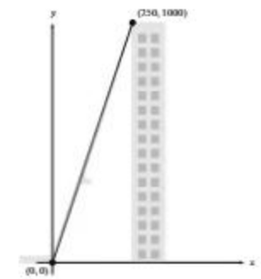

In order to define a specific line on a graph, we need at least two points so we can draw a line through both of them. When Molly is at the top of the building, she is 500 feet away horizontally (the $x$-value is 500) from the boarding dock, and she is 1,000 ft above the boarding dock (the $y$-value is 1000), so another point on the graph is (500, 1000). When we graph this point and draw a line connecting the two points we have labelled, we can see a representation of the vertical distance from the boarding dock as a function of the horizontal distance from the boarding dock.




If the boarding dock were built closer to the base of the building, this would affect how slanted the line graph would be. The cable car would have to cover the same vertical distance in a smaller horizontal space. For example, if the boarding dock were only 250 ft away from the building, the tow line would be twice as steep.



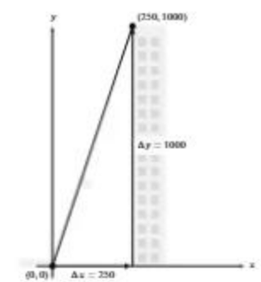





We can write an equation to represent Molly's height as a function of the distance from the boarding dock based on the new, steeper configuration. Her starting height is 0, but we also need to find the amount that her height changes as she moves away from the boarding dock. When we look at any two points on the line, we can find how much Molly moves upward for every foot she moves forward. Let's use the points (0, 0) and (250, 1000).

We can draw two arrows that show how far away the end point at the top of the building lies from the starting point horizontally (in the $x$-direction) and vertically (in the $y$-direction). How far across we go represents the *change* in $x$-value; we can use the Greek letter delta ($\Delta$) to mean "change in," so we write the change in $x$-value as $\Delta x$, pronounced "delta $x$." Similarly, how far up or down we go (that is, the movement parallel to the $y$-axis) represents the *change* in $y$-value; we write that change in $y$-value as $\Delta y$, pronounced "delta $y$."

Between the two points in our picture, (0, 0) and (250, 1000), we can draw in two arrows (the over-and-up arrows) showing the change in the $x$- and $y$-values.



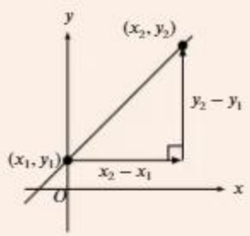

The $y$-value goes up by 1,000, so $\Delta y = 1000$. The $x$-value increases by 250, so $\Delta x = 250$. The word that we use to describe the steepness of such a graphed line is **slope**, and we define slope as the amount $y$ changes relative to a change in $x$, so slope is calculated simply by dividing $\Delta y$ (the change in $y$) by $\Delta x$ (the change in $x$). For this line, the slope is therefore $\frac{1000}{250}$ or 4.

## Slope

Slope is a quantity that shows the "steepness" of a line. It is commonly denoted by the variable $m$, and it is calculated using the following formula, where $\Delta y$ is the change in $y$-values and $\Delta x$ is the change in $x$-values.

$$m = \frac{\Delta y}{\Delta x}$$

Alternatively, the change in $y$-values can be more formally written as $y_2 - y_1$, where $y_1$ is the $y$-value of the first point and $y_2$ is the $y$-value of the second point. The change in $x$-values can be written as $x_2 - x_1$, where $x_1$ is the $x$-value of the first point and $x_2$ is the $x$-value of the second point. This equation is referred to as the **Slope Formula**.




$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

While this second form of the equation is not incorrect and is sometimes useful, in general, a quicker and less error-prone method of finding the slope is by traveling from one point to the other by drawing the over-and-up arrows (as in the example above) and determining their lengths to find $\Delta y$ and $\Delta x$ based on the graph alone. It is very easy to mess up signs and values when substituting the coordinates of two points into the Slope Formula. However, if the points are far apart, it could be very time-consuming to do all the incremental counting required using the traveling method, and errors can creep in, so in these circumstances, the Slope Formula is the better choice.

**When given points on a line, you should always start by finding the slope whenever possible.** It is very rare that the slope is not important to solving a problem involving lines.

For completeness, we will demonstrate using the Slope Formula to find the same information, though we recommend drawing the over-and-up arrows on the graph to find the slope directly. The first point we will choose is $(0, 0)$, which means that $x_1 = 0$ and $y_1 = 0$. The second point is $(250, 1000)$, which means that $x_2 = 250$ and $y_2 = 1000$. Plug these values into the slope formula and be careful that you are consistent about which point is $(x_1, y_1)$ and which point is $(x_2, y_2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{1000 - 0}{250 - 0}$$

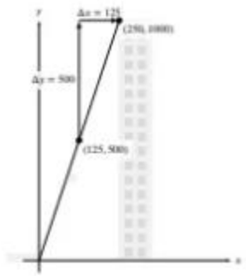
$$m = \frac{1000}{250}$$

$$m = 4$$





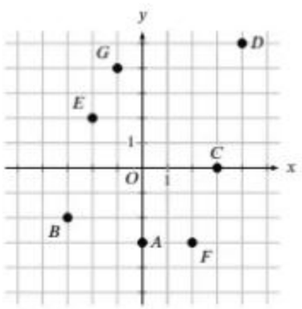
**The slope of a particular line will be the same no matter which two points are chosen or which point is $(x_1, y_1)$ and which point is $(x_2, y_2)$.** What if, instead of $(0, 0)$ and $(250, 1000)$, we chose the points $(250, 1000)$ and $(125, 500)$ (this is the midpoint of the tow line because it is half of the total $x$ distance and half of the total $y$-distance between the boarding dock and the top of the building)? Note also that it doesn't matter whether we draw the over-and-up arrows above or below the line in question, as long as we make sure to go from left to right (it is not wrong to go from right to left, but you have to be careful to take negative signs into account because $\Delta x$ would be negative if we go from right to left).




The $y$-value goes up by 500, so $\Delta y$ is 500. The $x$-value goes up by 125, so $\Delta x$ is 125. The slope is $\frac{500}{125}$, which still simplifies to 4.

**SkillDrill 2.1-2**

**Directions:** Find the slope of the line containing the two points found in the figure below.




1. Points $A$ and $C$
2. Points $A$ and $D$
3. Points $C$ and $D$
4. Points $B$ and $E$
5. Points $C$ and $F$
6. Points $D$ and $F$
7. Points $B$ and $D$
8. Points $B$ and $G$

## Slope-Intercept Form

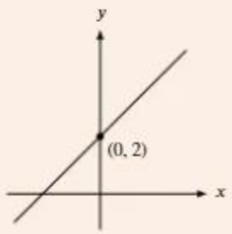
The next thing to determine when you are trying to find the equation of a line based on a graph is the points where the line crosses the axes. These crossing points are called **intercepts**.

Specifically, we are interested in the point where the line crosses the $y$-axis. For any linear graph, the point where the line crosses the $y$-axis is called the **$y$-intercept**.

### $y$-intercept

**The $y$-intercept is the point where a line crosses the $y$-axis.** Another way of thinking about the $y$-intercept is that it's **the value of $y$ when the $x$-value is 0** (the $y$-intercept is the value of $f(0)$). This quantity is often represented by the variable $b$.

In the graph below, the $y$-intercept is 2 because the line passes through the point $(0, 2)$, which is on the $y$-axis.



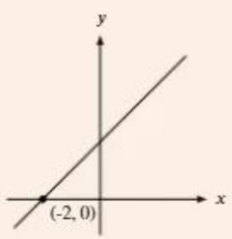

In real world problems, the $y$-intercept represents a starting or base amount when the $x$-value is 0. For example, if your cell phone plan charges you a fixed fee for the line each month plus an additional amount depending on how much data you use, the $y$-intercept of the function representing this linear relationship will be that base fee for the line; that's what you'd pay even if you use 0 gigabytes of data in a given month. As another example, in the last chapter, we talked about George's numerous pens, and the number of pens he started his collection with is 20, so 20 is the $y$-intercept.

**When you need to solve for the $y$-intercept of the graph of an equation or function, substitute 0 for $x$ and solve for $y$.**

### $x$-intercept

**The $x$-intercept is the point where a line crosses the $x$-axis, and it can also be thought of as the value of $x$ when the $y$-value is 0.**

In the graph below, the $x$-intercept is -2 because the line passes through the point $(-2, 0)$, which is on the $x$-axis.




**When you need to solve for the $x$-intercept of the graph of an equation or function, substitute 0 for $y$ and solve for $x$.**





All we have done is given new names to concepts we were already familiar with. The starting amount is the $y$-intercept, and the amount the values change is the slope. We can use this information to define a general linear equation that can be used even when we don't have a real world context to work from. It's called the **Slope-Intercept Form** of a line, and it's one of the most useful representations of lines on this test.

### Slope-Intercept Form

For a line with slope $m$ and $y$-intercept $b$, the Slope-Intercept Form of linear equations is the following:

$$y = mx + b$$

Note that any of the letters in the Slope-Intercept Form can change; the letters used above are the ones most commonly used by convention, but sometimes different letters that are more meaningful in the context of a specific problem are used instead for clarity. The arrangement and meaning of the terms are what really matter.

In the case of Molly on the cable car, we can find the equation that represents the linear relationship between her height above the ground and her distance (horizontally) from the boarding dock based on just two points on the line. We used the points $(0, 0)$ and $(250, 1000)$ to find that the slope of the line is 4. Also, we know the $y$-intercept of the line because the cable passes through the point $(0, 0)$; since the $x$-value of the point is 0, this point is the $y$-intercept of the line. Since we know the slope is 4 and the $y$-intercept is 0, the equation that represents the linear relationship in this example is $y = 4x + 0$, or just $y = 4x$.

**SkillDrill 2.1-3**

**Directions:** Given a linear equation in Slope-Intercept Form, identify the value of the slope, $m$, and the $y$-intercept, $b$.

**1.** $y = 2x + 1$ **2.** $y = 3x + 6$ **3.** $y = x - 7$ **4.** $y = 5x$

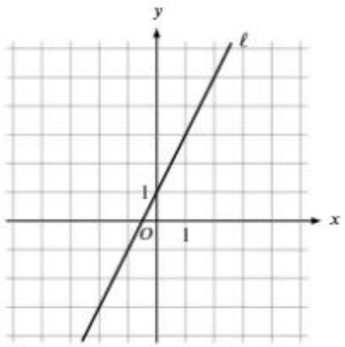
**Directions:** Write the equation of the line in Slope-Intercept Form given the value of the slope, $m$, and the $y$-intercept, $b$.

**5.** $m = 5, b = -4$ **6.** $m = 8, b = 1$ **7.** $m = 1, b = 0$ **8.** $m = 2, b = -2$

Let's walk through an example problem that requires us to write the equation of the line without any real world context. To do so, we will need to pull points from the graph in order to find the equation of the line. Sometimes points on a line are marked by black dots that are labelled with the $x$- and $y$-value of the point. Often, however, you will have to determine points on a line by using the gridlines to read the $x$- and $y$-values of the point. **Make sure to use points where the line crosses through the intersection of two gridlines whenever possible so that you know the exact coordinates of the point on the line.**

**Example 2.1-1**

**1** 1600.io 2.1




Which of the following is an equation of line $\ell$ in the $xy$-plane above?

A) $x = 2$

B) $y = 1$

C) $y = x + 2$

D) $y = 2x + 1$

**Solution**

1. We know the equation of a line is of the form $y = mx + b$, and we need to find $m$, which represents the slope, and $b$, which represents the $y$-intercept (you can find them in any order depending on which is easier).

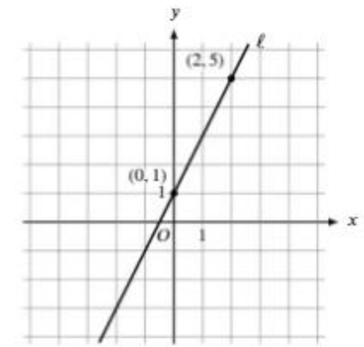
2. It is clear that the line passes through the point (0, 1), which tells us that the $y$-intercept of the line is 1. We can replace $b$ in the general Slope-Intercept Form with 1.

$$y = mx + b$$
$$y = mx + 1$$

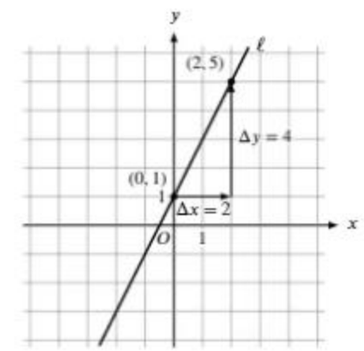
3. We can find the slope by picking any two points on the line and finding the changes in $x$- and $y$-values between those points. Make sure to pick points that are located at the crossing of two gridlines. The best one to start with is the point (0, 1) because this is the $y$-intercept and this tells us that the $y$-intercept is 1. We can pick any other point on the line that is also where two grid lines meet. Let's use (2, 5) (the point (1, 3) is also a good choice).








4. Draw the over-and-up arrows to find $\Delta y$ and $\Delta x$.




5. The $y$-value increases by 4, so $\Delta y = 4$, and the $x$-value increases by 2, so $\Delta x = 2$. Therefore, the slope, which is equal to $\frac{\Delta y}{\Delta x}$, is $\frac{4}{2}$ or 2.

6. Replace $m$ with 2 in the general Slope-Intercept Form to find the equation of the line.

$$y = 2x + 1$$

7. The answer is D.

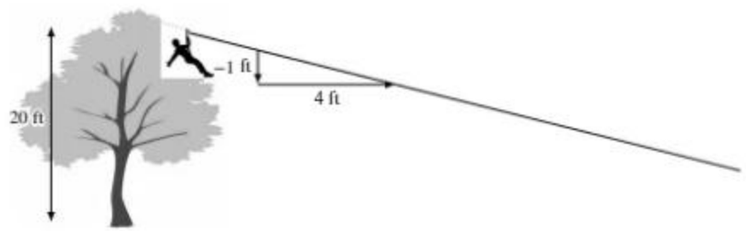
**Notes**

You can also use the Slope Formula to find the slope of the line, but usually drawing the over-and-up arrows is the preferred method, especially when the picture is already halfway drawn for you.

## Slopes Aren't Always Positive Integers

The slopes of lines are not always positive; they can be positive, negative, or zero. Also, a slope does not need to be an integer.

David owns an adventure course with a zipline on which the rider starts 20 ft in the air, and the zipline is sloped such that the height drops by 1 ft for every 4 ft that a rider travels horizontally.

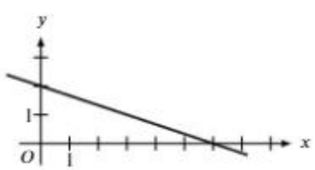



In this example, we can say that the $y$-intercept is 20 because the zipliner starts at a height of 20 feet off the ground when he has moved 0 ft away from the tree. The slope is $\frac{-1}{4}$ because the rider's height goes down by 1 ft (that is, it changes by −1 ft) for every 4 ft he moves away from the tree. The equation of the line is $y = \frac{-1}{4}x + 20$.

Note that a negative fractional slope might be written in the form $\frac{-1}{4}$ or $-\frac{1}{4}$, but the value is the same; in the example below, the second form is used, so don't let that confuse you. Also note that when we draw the over-and-down arrows, we are still working from left to right so that $\Delta x$ (the "over") will always be positive and $\Delta y$ can be positive or negative depending on how the $y$-values change between the two points.

**Example 2.1-2**

**2** 1600.io 2.1




Which of the following could be an equation for the graph shown in the $xy$-plane above?

A) $y = -\frac{1}{3}x + 6$

B) $y = -3x + 2$

C) $y = -\frac{1}{3}x + 2$

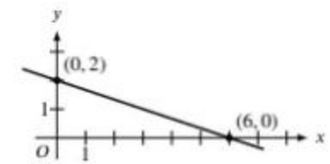
D) $y = -3x + 6$



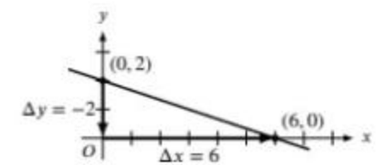


**Solution**

1. We need to pick points to find the equation of the line. In this case, since there are no gridlines, the only points we can use are the $y$-intercept, which is the point (0, 2), and the $x$-intercept, which is the point (6, 0). Since we know from the graph that the $y$-intercept is 2, we can eliminate choices A and D, which show 6 as the $y$-intercept.




2. Draw the over-and-down arrows to show the change in the $y$-values and the change in the $x$-values.




The $y$-value decreases by 2, so $\Delta y = -2$. The $x$-value increases by 6, so $\Delta x = 6$. Therefore, the slope is $\frac{-2}{6}$, or $\frac{-1}{3}$ when we reduce the fraction.

3. Since the slope is $\frac{-1}{3}$ and the $y$-intercept is 2, the equation of the line is $y = \frac{-1}{3}x + 2$.

4. The answer is C.

**Notes**

You could have used the Slope Formula to find the slope; we'll use (0, 2) as point 1 and (6, 0) as point 2, but you could have chosen the reverse assignment, as it has no effect on the result.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{0 - 2}{6 - 0}$$

$$m = \frac{-2}{6}$$

$$m = \frac{-1}{3}$$

Once again, this method is not incorrect, but it is easy to mess up. If you are going to choose to use the Slope Formula, make sure your work is neatly written out, with all steps shown.

As you can see, it's possible to encounter linear relationships that do not have positive integer slopes, but finding the slope of these lines is just as easy when you are given a graph on which to find the slope. If you are given a real world situation in which the $y$-variable decreases as the $x$-variable increases, you do not necessarily need to graph the points to find the slope or equation of the line. For example, if the average temperature in a certain country drops by 1°C every 10 days during the winter, then in a linear equation representing the average temperature in this country during winter, we can say that the slope must be $\frac{-1}{10}$ because the temperature decreases by 1 when the number of days increases by 10.

## Vertical and Horizontal Lines

One last topic that is related to slopes of lines is the slopes of vertical and horizontal lines.

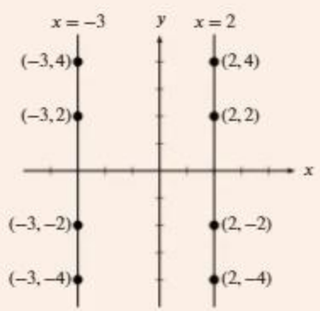
### Vertical Lines

**Vertical lines are lines where the $x$-value never changes**. They have equations like $x = 5$, which tells us that the $x$-value is always 5 and the $y$-value does not depend on the $x$-value at all.

The slope of these lines is undefined, and looking at the Slope Formula will tell us why. Since the $x$-value does not change, $\Delta x = 0$. When we substitute 0 for $\Delta x$ in the Slope Formula, we have a problem because it is impossible to divide by 0.

$$m = \frac{\Delta y}{0} = \text{undefined}$$

**From this, it should be clear that vertical lines have equations written in the form $x = c$, where $c$ is a constant value**. Note, importantly, that this equation is not in Slope-Intercept form, because that form cannot represent a vertical line. For any point on vertical lines, the $x$-value is fixed, but each point will have a different $y$-value.




We can use the Slope Formula to verify that the slope between any two of these points is undefined. From the line $x = 2$, let's use $(2, 4)$ as $(x_1, y_1)$ and $(2, 2)$ as $(x_2, y_2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-4}{2-2} = \frac{-2}{0} = \text{undefined}$$





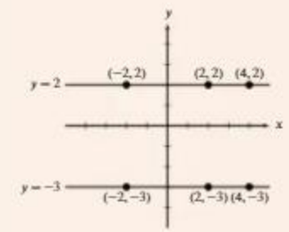
### Horizontal Lines

**Horizontal lines are lines where the $y$-value never changes**. They have equations like $y = -2$, which tells us that the $y$-value is always $-2$ no matter what the $x$-value is.

The slope of these lines is 0. Since the $y$-value does not change, $\Delta y = 0$. When we substitute 0 for $\Delta y$ in the Slope Formula, the numerator is 0, and 0 divided by a number is still 0.

$$m = \frac{0}{\Delta x}$$

**From this, it should be clear that horizontal lines have equations written in the form $y = c$, where $c$ is a constant value.** Note that, unlike the equation for a vertical line, the horizontal line equation is in Slope-Intercept form, as it is equivalent to $y = 0x + c$.

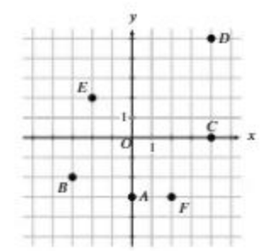



We can use the Slope Formula to verify that the slope between any two of these points is equal to 0. From the line $y = 2$, let's use $(4, 2)$ as $(x_1, y_1)$ and $(-2, 2)$ as $(x_2, y_2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-2}{4-(-2)} = \frac{0}{6} = 0$$

**SkillDrill 2.1-4**

**Directions:** Based on the points in the figure below, find the equation of the line in Slope-Intercept Form (or $x = c$ form for vertical lines) that contains the two listed points, and find the coordinates of the $x$-intercept (if there is an $x$-intercept).




1. Points $A$ and $B$
2. Points $B$ and $D$
3. Points $E$ and $D$
4. Points $E$ and $F$
5. Points $A$ and $F$
6. Points $C$ and $D$
7. Points $B$ and $C$
8. Points $E$ and $C$

### Section 2.1 Suggested Problems from Real Tests

- Test 1-NC-12
- Test 1-C-16
- Test 2-C-25
- Test 3-NC-8
- Test 3-C-17
- Test 3-C-26
- Test 4-NC-7
- Test 4-NC-20
- Test 4-C-8
- Test 5-NC-1
- Test 6-C-14
- Test 7-C-28
- Test 7-C-32
- Test 8-NC-2
- Test 10-C-8
- May 2017-NC-1
- Apr 2018-NC-5
- May 2018-NC-3
- May 2018-C-13
- Mar 2019-NC-12
- Mar 2019-C-8
- May 2019 (Int)-NC-10
- Oct 2019-NC-6
- Oct 2019-C-7
- Oct 2019-C-33
- Mar 2020-NC-16
- Oct 2020-C-33
- Mar 2021-NC-4
- Mar 2021-C-16
- Mar 2021-C-35
- May 2021 (US)-C-13
- May 2021 (Int)-NC-5
- May 2021 (Int)-C-10
- May 2021 (Int)-C-11

**View related real-test problems in SMTEx**

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## Section 2.1 Practice Problems

**1** 1600.io 2.1

In the $xy$-plane, what is the $y$-intercept of the line with the equation $y = 3x - 2$?

A) $3$

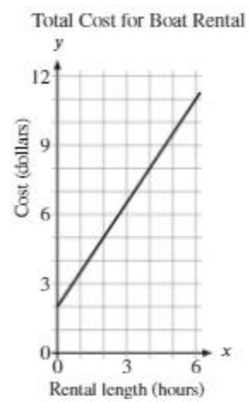
B) $\frac{1}{3}$

C) $-\frac{1}{3}$

D) $-2$

**2** 1600.io 2.1

The line graphed in the $xy$-plane below models the total cost, in dollars, for a paddleboat rental, $y$, in a certain city park based on the number of hours the paddleboat is rented, $x$.




According to the graph, what is the cost for each additional hour, in dollars, of the rental?

A) \$0.50

B) \$1.00

C) \$1.50

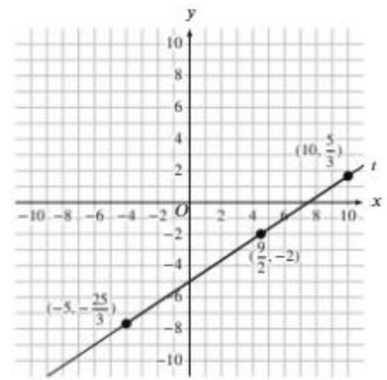
D) \$2.00





**3** 1600.io 2.1

Line $t$ is shown in the $xy$-plane below.




What is the slope of line $t$?

**4** 1600.io 2.1

The graph of $y = f(x)$ is a line in the $xy$-plane that passes through the point $(0, 7)$ and has a slope of 3. Which of the following equations could define the function $f$?

A) $f(x) = -\frac{1}{7}x + 3$

B) $f(x) = -\frac{1}{3}x + 7$

C) $f(x) = 3x + 7$

D) $f(x) = 7x + 3$

## 2.2 Analyzing Graphs of Linear Equations

In a way, we have already walked through the process of making a graphical representation of a linear relationship, but we used real world examples like hills and ziplines, which both have natural slopes and automatically conjure up a mental image of a sloped line. However, we need to be able to identify the graphs of linear equations based on the equations themselves, without the aid of any real world context.

Luckily, the Slope-Intercept Form of lines provides us with enough useful information to identify the graph of a linear equation based on the slope and $y$-intercept, which are the two key visual features of line graphs.

### Graphing Slope and $y$-intercept

When you have to identify the graph that corresponds to a linear equation, the key features to look out for are the slope and $y$-intercept.

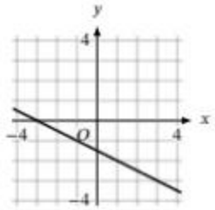
Start by eliminating any choices that do not pass through the correct $y$-intercept value. Then check that the line slopes in the correct direction (positive or negative slope). Finally, if there are still multiple remaining choices, make sure that the slope is the correct value (i.e. if the slope is supposed to be $-2$, make sure that the graph goes down by 2 units for every 1 unit it goes to the right).

**Example 2.2-1**

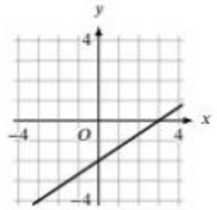
**1** 1600.io 2.2

Which of the following is the graph of the equation $y = -2x + 3$ in the $xy$-plane?

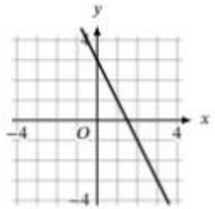
A)



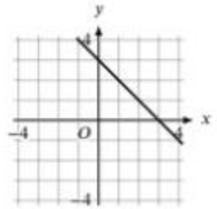
B)



C)



D)

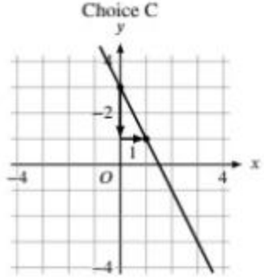


**Solution**

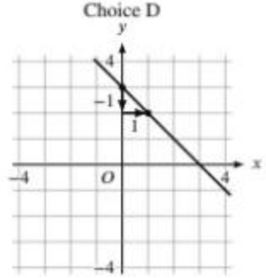
1. The $y$-intercept of the given equation is 3, so eliminate choices A and B, which do not pass through the $y$-axis at $y = 3$ (that is, at the point $(0, 3)$).

2. The slope of the line is negative (it's $-2$), but the remaining choices (C and D) have negative slopes, so we cannot eliminate either of them immediately. We need to choose the graph that has a slope of $-2$.
   Looking at the two remaining choices, we can start from the $y$-intercept and see how much the graphs go down and over to see which has a slope of $-2$. The correct answer will be the graph that goes down by 2 units for every 1 unit it goes over.

Choice C



Choice D



3. The graph of choice C is the one that has a slope of $-2$ because the graph goes down by 2 units for every 1 unit it goes over; the graph of choice D goes down by 1 unit for every 1 unit it goes over, so its slope is $-1$, and that does not match the slope we're looking for.

4. The answer is C.





**Section 2.2 Suggested P**

- Test 6-NC-5
- Apr 2017-NC-8
- Test 9-C-20
- 2020-C-12
- 2021-NC-1
- 2021 (Jan)-C-5

View related real-test problems in SMTEx

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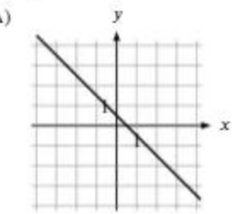
## Section 2.2 Practice Problems

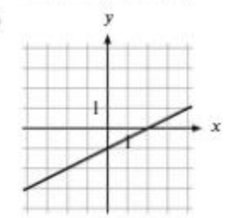
**1** 1600.io 2.2

Which of the following is the graph of the equation $y = \frac{1}{2}x - 1$ in the $xy$-plane?

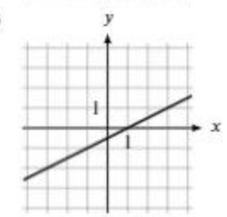
A)



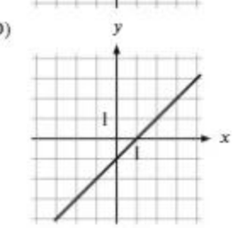
B)



C)



D)



**2** 1600.io 2.2

The function $f$ is defined for all real numbers, and the graph of $y = f(x)$ in the $xy$-plane is a line with a positive slope. Which of the following must be true?

I. If $a < b$, then $f(a) > f(b)$

II. If $a < 0$, then $f(a) < 0$

III. If $a > 0$, then $f(a) > 0$

A) None

B) I only

C) II and III only

D) I, II, and III

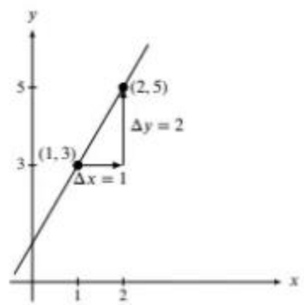
## 2.3 Using Points and Tables

You don't need an actual graph to find the equation of a line. Sometimes you are just given two or more points and are told to find the equation of the line that goes through those points. **Any two points are sufficient to find the equation of a line**.

For example, let's find the slope of the line that goes through $(1, 3)$ and $(2, 5)$ using the Slope Formula by replacing $y_2$ with 5 and $x_2$ with 2 (from the point $(2, 5)$) and replacing $y_1$ with 3 and $x_1$ with 1 (from the point $(1, 3)$).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{5-3}{2-1}$$
$$m = \frac{2}{1}$$
$$m = 2$$

Alternatively, we can draw a quick picture, plotting the points $(1, 3)$ and $(2, 5)$, and then using the over-and-up arrows to find the slope from $\Delta y$ and $\Delta x$.




The $x$-values of the points increase by 1 (from 1 to 2) so $\Delta x = 1$. The $y$-values of the points increase by 2 (from 3 to 5) so $\Delta y = 2$. The slope of the line is $\frac{2}{1} = 2$. If we want to find the equation of this line in Slope-Intercept Form, we can replace $m$ with 2 because the slope is 2.

$$y = mx + b$$
$$y = 2x + b$$

We can plug in the $x$ and $y$-values of either given point, and once we have those three values ($m$, $x$, and $y$) substituted into the equation, we can solve for $b$, the $y$-intercept of the line. Let's use the point $(1, 3)$. **Note that any point on the line can be used**, but you can make smarter, easier choices if you choose points that have 0 or 1 as a value of the $x$- or $y$-coordinates because those values result in the simplest arithmetic.

$$y = 2x + b$$
$$3 = 2(1) + b$$
$$3 = 2 + b$$
$$1 = b$$

Since the slope is 2 and the $y$-intercept is 1, the equation of the line is $y = 2x + 1$.





### Plugging In Points to Find Unknown Values

Once you have the slope (or $y$-intercept) of a line and you know a point on the line, the next logical step is to substitute the coordinates of the point into the Slope-Intercept Form. Replace the $x$ variable with the $x$-value of the point you know, and replace the $y$ variable with the $y$-value of the point you know.

You will have three of the four unknown values filled in and you can then solve for the remaining unknown value. In the previous example, we replaced $m$ with the slope that we calculated, leaving us with the equation $y = 2x + b$.

Next comes the step where many students freeze up because they do not realize that they should substitute the coordinates of one of the points they know in order to figure out the $y$-intercept and complete the equation.

This tip does not just apply to lines but for all equations that will come up on the test: **if you are trying to fill in unknown values in an equation, you should substitute the coordinates of any points that are *given* to you by the text, graph, or table** (tables of points are discussed below).

**SkillDrill 2.3-1**

**Directions:** Based on the points given, find the equation of the line in Slope-Intercept Form.

**1.** $(-2, 3)$ and $(4, 1)$ **2.** $(0, -5)$ and $(2, 3)$ **3.** $(5, 14)$ and $(1, 11)$ **4.** $(-8, 0)$ and $(2, 0)$

**5.** $(-3, 4)$ and $(3, -4)$ **6.** $(5, -3)$ and $(3, 1)$ **7.** $(10, 2)$ and $(20, 3)$ **8.** $(8, 4)$ and $(-1, 5)$

**Example 2.3-1**

**1** 1600.io 2.3

The graph of a line in the $xy$-plane passes through the point $(5, 2)$ and crosses the $x$-axis at the point $(7, 0)$. The line crosses the $y$-axis at the point $(0, b)$. What is the value of $b$?

**Solution 1**

1. Recognize that this problem asks us to find the $y$-value of the point $(0, b)$, which is when the $x$-value is 0. **This is just another way of asking for the $y$-intercept of the line**. This means that the symbol $b$ in the problem represents the $y$-value of the $y$-intercept, just as it does in the template $y = mx + b$ in the Slope-Intercept Form of a linear equation.

2. First we will find the slope of the line using the Slope Formula. The point $(x_1, y_1)$ is $(5, 2)$, and the point $(x_2, y_2)$ is $(7, 0)$.

$$m = \frac{0-2}{7-5}$$
$$m = \frac{-2}{2}$$
$$m = -1$$

3. Since we know the slope and a point on the line, we can substitute these values into the general Slope-Intercept Formula to find the $y$-intercept of the line. Replace $m$ with $-1$, then, based on the point $(7, 0)$, substitute 7 for $x$ and 0 for $y$.

$$\begin{aligned} y &= mx + b \\ y &= (-1)x + b \\ y &= -x + b \\ 0 &= -(7) + b \\ 0 &= -7 + b \\ 7 &= b \end{aligned}$$

4. Since the slope is $-1$ and the $y$-intercept is 7, the equation of the line is $y = -x + 7$. Since the $y$-intercept is equal to 7, you can stop here and finalize 7 as your answer.

5. If you did not notice at the beginning of the problem that you were being asked for the $y$-intercept, which is the $y$-value of the point where the $x$-value is 0, you can find the $y$-value of the point $(0, b)$ by substituting 0 for $x$ and $b$ for $y$ in the equation of this line (which is $y = -x + 7$).

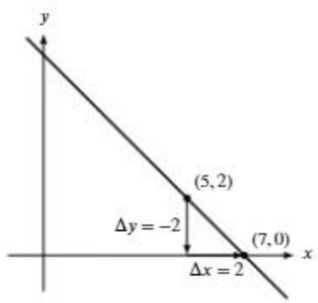
$$\begin{aligned} y &= -x + 7 \\ (b) &= -(0) + 7 \\ b &= 7 \end{aligned}$$

6. The answer is 7.

**Solution 2**

1. Recognize that this problem asks us to find the $y$-value of the point $(0, b)$, which is when the $x$-value is 0. This is just another way of asking for the $y$-intercept of the line.

2. First we will graph the points $(5, 2)$ and $(7, 0)$, and then use the picture to find the slope by drawing in the over-and-down arrows showing the change in $y$ and the change in $x$.




3. The $y$-value decreases by 2, so $\Delta y = -2$, and the $x$-value increases by 2, so $\Delta x = 2$. Therefore, the slope, which is $\Delta y$ divided by $\Delta x$, is $\frac{-2}{2}$ or $-1$.





4. Since we know the slope and a point on the line, we can substitute these values into the general Slope-Intercept Formula to find the $y$-intercept of the line. Replace $m$ with $-1$, then, based on the point $(7, 0)$, substitute 0 for $y$ and 7 for $x$.

$$\begin{aligned} y &= mx + b \\ y &= (-1)x + b \\ y &= -x + b \\ 0 &= -(7) + b \\ 0 &= -7 + b \\ 7 &= b \end{aligned}$$

5. Since the slope is $-1$ and the $y$-intercept is 7, the equation of the line is $y = -x + 7$. Since the $y$-intercept is equal to 7, you can stop here and finalize 7 as your answer.

6. If you did not notice at the beginning of the problem that you were being asked for the $y$-intercept, you can find the $y$-value of the point $(0, b)$ by substituting 0 for $x$ and $b$ for $y$ in the equation of this line (which is $y = -x + 7$).

$$\begin{aligned} y &= -x + 7 \\ (b) &= -(0) + 7 \\ b &= 7 \end{aligned}$$

7. The answer is 7.

Since we know how to find the equation of a line when we are given two points, we can also find the equation of a line when we are given a table of values that represent the $x$- and $y$- coordinates of a set of points on a line. The points will be represented as a column (or row) of $x$-values and a column (or row) of $y$-values that correspond to those $x$-values. Let's look at the following table of values.

| $x$ | $y$ |
|---|---|
| $-1$ | $-2$ |
| 0 | 2 |
| 1 | 6 |
| 2 | 10 |

Each row is a point, with the $x$-value in the left column and the $y$-value in the right column. For example, the first row shows us that $(-1, -2)$ is one point on the line. Similarly, the second row shows us that $(0, 2)$ is another point on the line. Also notice that $(0, 2)$ is the $y$-intercept of the line because the $x$-value of the point is 0. To find the equation of the line, we would start by using any two points (rows) from the table to find the slope of the line.

Remember that we found the slope from a graph by drawing a right triangle and finding $\Delta y$ and $\Delta x$ directly from the picture. We can similarly avoid formally plugging points into the Slope Formula on these problems. First, choose any two points from the table. By subtraction or counting, see how much the $x$- and $y$-values increase or decrease between those two points (finding the value of $\Delta x$ and $\Delta y$, respectively). You can easily notate this by marking the change in $x$ and $y$-values next to the table with brackets of some kind. Divide $\Delta y$ by $\Delta x$ to find the slope.

Let's use the first two points (rows). The $y$-value increases by 4 (from $-2$ to 2), so $\Delta y = 4$. The $x$-value increases by 1, so $\Delta x = 1$.

$\Delta x = 1$

| $x$ | $y$ |
|---|---|
| −1 | −2 |
| 0 | 2 |
| 1 | 6 |
| 2 | 10 |

$\Delta y = 4$

Since $\Delta y$ is 4 and $\Delta x$ is 1, the slope of this line is $\frac{4}{1}$ or 4. As it happens, we also know that the $y$-intercept is 2 because (0, 2) is a point given to us in the table, so the equation of the line is $y = 4x + 2$.

Note that **you can always find the equation of a line when you are given any two points on the line (or the slope and any single point)**, so we could have used any two of the points in the table above to find the same equation of the line. If the $y$-intercept is not shown in the table, then you can substitute the slope (for $m$) and the $x$- and $y$ coordinates of any point on the line (for $x$, and $y$, respectively) into the Slope-Intercept Form, $y = mx + b$, in order to solve for the $y$-intercept ($b$).

## Using Tables of Points on Lines

Each row (or column) in a table is a point on a line. **Many tables show you the $y$-intercept of the line. Look out for the point where the $x$-value is 0.**

There is no need to use the Slope Formula when you are given a table of points on a line. **Subtract or count to see how much the $x$- and $y$-values increase or decrease between any two points to find $\Delta x$ and $\Delta y$, repectively.**

### Example 2.3-2

2 1600.io 2.3

Shipping Charges

| Book Weight (pounds) | Shipping Charge ($) |
|---|---|
| 2 | 2 |
| 4 | 3 |
| 8 | 5 |
| 16 | 9 |

The table above shows the shipping charges for an online retailer that sells books. There is a linear relationship between the shipping charge and the weight of the books. Which equation can be used to determine the total shipping charge $y$, in dollars, for an order with a book weight of $x$ pounds?

A) $y = \frac{1}{2}x$

B) $y = \frac{1}{2}x + 1$

C) $y = x$

D) $y = x + 2$





### Solution

1. The left column contains $x$-values and the right column contains $y$-values. Each row in the table is a point on the line. The first row tells us that (2, 2) is a point, and the second row tells us that (4, 3) is a point. We are told that the relationship between the cost and weight is linear, so we can use these points to find the slope of the line that describes the relationship.

2. Between the first and second row, the $x$-value increases by 2 (from 2 to 4), so $\Delta x = 2$, and the $y$-value increases by 1 (from 2 to 3), so $\Delta y = 1$.

$\Delta x = 2$

| Book Weight (pounds) | Shipping Charge ($) |
|---|---|
| 2 | 2 |
| 4 | 3 |
| 8 | 5 |
| 16 | 9 |

$\Delta y = 1$

Since $\Delta y = 1$ and $\Delta x = 2$, the slope, $m$, which is defined as $\frac{\Delta y}{\Delta x}$, is $\frac{1}{2}$.

Alternatively, substitute the coordinates of the two points into Slope Formula to find the slope of the line.

$$m = \frac{y_2 - y_1}{y_2 - y_1}$$

$$m = \frac{3-2}{4-2}$$

$$m = \frac{1}{2}$$

3. Since we know that the slope is $\frac{1}{2}$, we can replace $m$ with $\frac{1}{2}$ in the general Slope-Intercept equation.

$$y = \frac{1}{2}x + b$$

4. Choose any point from the table to substitute into the equation so we can solve for $b$. Let's use (2, 2). Replace the $x$-value with 2, and replace the $y$-value with 2.

$$y = \frac{1}{2}x + b$$

$$2 = \frac{1}{\cancel{2}}(\cancel{2}) + b$$

$$2 = 1 + b$$

5. Subtract 1 from both sides of the equation to solve for $b$, the $y$-intercept.

$$1 = b$$

6. Replace $b$ with 1 in the general Slope-Intercept equation to find the equation of the line.

$$y = \frac{1}{2}x + 1$$

7. The answer is B.

**Notes**

If you are really comfortable working directly on the table, you can add an extra row to the table to work out what the $y$-value would be when the $x$-value is 0 in order to find the $y$-intercept more quickly.

| | Book Weight (pounds) | Shipping Charge ($) | |
|---|---|---|---|
| | 0 | $b$ | |
| $\Delta x = 2$ | 2 | 2 | $\Delta y = 1$ |
| $\Delta x = 2$ | 4 | 3 | $\Delta y = 1$ |
| $\Delta x = 4$ | 8 | 5 | $\Delta y = 2$ |
| $\Delta x = 8$ | 16 | 9 | $\Delta y = 4$ |

The value of $b$ has to be 1 because between the first row and the second row, the $y$-value has to increase by 1 because the $x$-value increases by 2 (the slope still has to be $\frac{1}{2}$ ).

Note also that the slope can be calculated using any two rows in the table. Even when $\Delta x$ is different, the corresponding $\Delta y$ will also change in such a way that the ratio of $\Delta y$ to $\Delta x$ (the slope) will always be constant because the slope of a line is the same everywhere by definition.

**SkillDrill 2.3-2**

**Directions:** Based on the points given in the tables below, find the equation of each line in Slope-Intercept Form and find the value of $k$ either by counting the change in values between rows or by substituting into your equation the value of the known coordinate for the row (point) in which $k$ appears.

**1.**

| $x$ | $y$ |
|---|---|
| 0 | −5 |
| 1 | −3 |
| 2 | −1 |
| 3 | $k$ |

**2.**

| $x$ | $y$ |
|---|---|
| −2 | 10 |
| −1 | 6 |
| 0 | $k$ |
| 1 | −2 |

**3.**

| $x$ | $y$ |
|---|---|
| 2 | 3 |
| 4 | 6 |
| $k$ | 15 |
| 14 | 21 |

**4.**

| $x$ | $y$ |
|---|---|
| −5 | 2 |
| −1 | 1 |
| 3 | 0 |
| $k$ | −3 |

**Section 2.3 Suggested Problems from Real Tests**

- Test 1-C-22
- Test 3-C-4
- Test 3-C-10
- Test 3-C-11
- Test 5-C-2
- Test 5-C-23
- Test 5-C-28
- Test 6-C-15
- Test 6-C-25
- Test 7-NC-19
- Test 8-C-25
- Test 9-NC-5
- Test 9-C-14
- Test 10-NC-3
- Mar 2019-C-12
- Mar 2019-C-36
- May 2019 (US)-NC-3
- May 2019 (US)-NC-10
- May 2019 (US)-C-19
- May 2019 (Int)-NC-6
- May 2019 (Int)-C-24
- Mar 2020-NC-9
- Mar 2020-C-8
- Mar 2020-C-36
- Oct 2020-NC-16
- Mar 2021-C-13
- Mar 2021-C-29
- Mar 2021-C-38
- Apr 2021-NC-5
- Apr 2021-NC-9
- Apr 2021-C-20
- Apr 2021-C-27
- May 2021 (US)-NC-16
- May 2021 (US)-C-16
- May 2021 (Int)-C-16

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SMTEx





## Section 2.3 Practice Problems

**1** 2.3

For a function $f$, $f(3) = 18$ and $f(5) = 26$. If the graph of $y = f(x)$ is a line in the $xy$-plane, what is the slope of the line?

**2** 2.3

| $x$ | $h(x)$ |
|---|---|
| 1 | 1 |
| 3 | 5 |
| 5 | 9 |
| 7 | 13 |

For the linear function $h$, the table above shows several values of $x$ and their corresponding values of $h(x)$. Which of the following defines $h$?

A) $h(x) = 2x - 3$

B) $h(x) = 2x - 1$

C) $h(x) = 4x - 3$

D) $h(x) = 4x - 7$

**3** 2.3

In the $xy$-plane, line $k$ intersects the $y$-axis at the point $(0, -4)$ and passes through the point $(3, 2)$. If the point $(15, w)$ lies on line $k$, what is the value of $w$?

**4** 2.3

| $x$ | $f(x)$ |
|---|---|
| 1 | 3 |
| 3 | 17 |
| 4 | 24 |

Some values of the linear function $f$ are shown in the table above. Which of the following defines $f$?

A) $f(x) = x + 3$

B) $f(x) = 3x + 7$

C) $f(x) = 7x + 3$

D) $f(x) = 7x - 4$

## 2.4 Interpreting Linear Equations

Now that we know about slope and $y$-intercept and we can read and create graphs of linear equations, we can come back to real world problems to round out the topic of lines in Slope-Intercept Form. Some of the questions people struggle with the most are based on interpreting the *meaning* of the slope and $y$-intercept of linear equations that represent real world situations.

Remember the four parts of the equation that we talked about in the last chapter, and relate each part to a real world situation. The $y$-intercept (the value of $b$ in the Slope-Intercept Form template) is the starting amount. The slope ($m$ in the template) is the rate of change (how the $y$-value changes when the $x$-value changes) and is usually indicate by words like "each" or "per." Recall that slopes can be positive (to show an increase in a real world quantity) or negative (to show a decrease).

**Example 2.4-1**

**1** 1600.io 2.4

$$1500 - 32t = 220$$

In 1954, the population of a frog species was 1500. The population $t$ years after 1954 was 220, and $t$ satisfies the equation above. Which of the following is the best interpretation of the number 32 in this context?

A) The population $t$ years after 1954

B) The value of $t$ when the population was 220

C) The difference between the population in 1954 and the population $t$ years after 1954

D) The average decrease in the population per year from 1954 to $t$ years after 1954

**Solution**

1. The equation given to us is a linear equation. The terms are on the opposite side of the equals sign from what we are used to, but the pattern still fits: instead of being in the form $y = mx + b$, this line is in the form $b + mx = y$. The letter $t$ is being used to represent the time, in years, instead of the generic $x$, in order to add meaning to the variable.

2. The coefficient of the $x$-, or in this case $t$-, variable is the slope, so, for this problem, we know that the slope is $-32$.

3. The slope is the change in $y$ divided by the change in $x$. Since the $y$-value represents the population and the $x$-value (or $t$-value, in this case) is the number of years, **the slope of this line is the change in the population each year**. We choose to represent $-32$ as $\frac{-32}{1}$ in order to show the slope as a fraction that represents the change in $y$ divided by the change in $x$.

$$\frac{\Delta y}{\Delta t} = \frac{\text{change in population}}{\text{change in years}} = \frac{-32}{1}$$

As you can see the population decreases by 32 every time the year increases by 1.

4. The answer is D.





**Notes**

For this example, let's identify each of the four parts of the equation and what those mean in the real world context.

The value 1500 is the $y$-intercept, or the starting population when the variable $t$ is 0, which corresponds to the year 1954. The population in 1954 was 1500 and has been decreasing every year since then, until it finally reached a value of 220 after $t$ years. The value 220 is like a particular $y$-value: it is the population that still exists after $t$ years, so there exists a point $(t, 220)$, though note that we are never asked to find the value of $t$ in this problem (spoiler alert: it's 40, which you should be able to figure out by solving the original equation for $t$). The variable $t$ is acting as the $x$-variable: $t$ represents the number of years since 1954, and there is a change in the population every year. If we want to know the remaining population after $t$ years, we would have to subtract the population loss from the starting amount. The total loss would be found by multiplying the yearly decrease in population by the number of years since 1954. Since $t$ is the number of years since 1954, then 32 must be the amount that the population decreases each year. Also recall that based on the general Slope-Intercept Form, the slope of the line is $-32$, which represents how much the population changes each year.

**Example 2.4-2**

**2** 1600.io 2.4

Corinne is a computer technician. Each week, she receives a batch of computers that need repairs. The number of computers that she has left to fix at the end of each day can be estimated with the equation $C = 30 - 6d$, where $C$ is the number of computers left and $d$ is the number of days she has worked that week. What is the meaning of the value 30 in this equation?

A) Corinne will complete the repairs within 30 days.

B) Corinne starts each week with 30 computers to fix.

C) Corinne repairs computers at a rate of 30 per hour.

D) Corinne repairs computers at a rate of 30 per day.

**Solution**

1. We can see that this is a linear equation. We know that the $y$-variable is $C$, the number of computers left to fix after $d$ days of work. We know that the $x$-variable is $d$, the number of days Corinne has worked.

2. The number of computers she has left to fix at the beginning of the week would be how many computers she has left to fix at the start of the week, or when $d = 0$. One way to contextualize the number 30 is to substitute 0 for $d$ and see that when she has not worked any days yet, she still has 30 computers left to fix.

$$C = 30 - 6d$$
$$C = 30 - 6(0)$$
$$C = 30$$

3. Alternatively, you can just realize that the $y$-intercept (which is 30 based on the given linear equation) can be interpreted as the starting amount and immediately realize that she starts the week with 30 computers to fix.

4. The answer is B.

Some of the most difficult real-world linear equation interpretation questions involve complicated-looking fractional or decimal slopes. Remember that the slope is the change in $y$-values ($\Delta y$) divided by the change in $x$-values ($\Delta x$), so the change in $y$-values is measured in the same units as the $y$-value and the change in the $x$-values is measured in the same units as the $x$-value. Therefore, the slope will tell you how many units the $y$-value increases (or decreases) as the $x$-value increases by a certain number of units.

For example, if the weight, $w$, in pounds (or lbs), of a certain cactus that is currently in a pot of soil weighing 15 lbs increases by 13 lbs as the height, $h$, in inches, of the cactus increases by 2 in, then the linear equation representing the weight of the cactus and pot of soil (acting as the $y$-value) as a function of the height of the cactus (acting as the $x$-value) is $w = \frac{13}{2}h + 15$, where the slope is measured in units of lbs/in.

Because we know that the slope must be constant, we can use a simple proportion to see what change in the $x$-values (the change in height in this case) would produce a certain change in the $y$-values (the weight in this case). See the chapter on ratios and proportions if you're fuzzy on this type of setup.

$$\frac{\text{Change in Weight}}{\text{Change in Height}} = \frac{\Delta y}{\Delta x}$$

$$\frac{13}{2} = \frac{\Delta y}{\Delta x}$$

So, for example, if we wanted to know how much taller the cactus would have to grow in order to increase the weight by 1 lb, we would just substitute 1 for the change in weight ($\Delta y$), and solve for the corresponding change in height ($\Delta x$) by cross multiplying and then solving for $\Delta x$.

$$\frac{13}{2} = \frac{\Delta y}{\Delta x}$$

$$\frac{13}{2} = \frac{1}{\Delta x}$$

$$13(\Delta x) = 2(1)$$

$$13(\Delta x) = 2$$

$$\Delta x = \frac{2}{13}$$

In this example, if the height of the cactus increases by $\frac{2}{13}$ in, the weight of the cactus will increase by 1 lb.

We can verify this if we choose to input any two weights that are 1 lb different from each other into the equation $w = \frac{13}{2}h + 15$, and see that the heights corresponding to those weights are $\frac{2}{13}$ inches different from each other. Let's use the weights 15 lbs (the $y$-intercept, which makes our job easier because it corresponds to a cactus height of 0 in) and 16 lbs (which is 1 lb heavier than the other weight).

For the first plant and pot weighing 15 lbs, substitute 15 for $w$ and solve for $h$ (which will end up being 0).

$$w = \frac{13}{2}h + 15$$

$$15 = \frac{13}{2}h + 15$$

$$0 = \frac{13}{2}h$$





It should be obvious what $h$ will be, but for completeness, let's multiply both sides of the equation by $\frac{2}{13}$ in order to solve for $h$.

$$\frac{2}{13}(0) = \frac{\cancel{2}}{\cancel{13}}\left(\frac{\cancel{13}}{\cancel{2}}h\right)$$

$$0 = h$$

The first plant and pot weighs 15 lbs when the height of the cactus is 0 in. For the second plant and pot, which weighs 16 lbs (1 lb more than the other plant and pot), substitute 16 for $w$ and solve for $h$ (which we predict should be $\frac{2}{13}$ in, which is $\frac{2}{13}$ in more than 0 in, the height of the other potted cactus).

$$w = \frac{13}{2}h + 15$$

$$16 = \frac{13}{2}h + 15$$

$$1 = \frac{13}{2}h$$

Multiply both sides of the equation by $\frac{2}{13}$ in order to solve for $h$.

$$\frac{2}{13}(1) = \frac{\cancel{2}}{\cancel{13}}\left(\frac{\cancel{13}}{\cancel{2}}h\right)$$

$$\frac{2}{13} = h$$

As we can see, in comparison to substituting $x$- and $y$-values into the linear equation, using the proportion based on the slope (which is constant for lines) was a much quicker and easier way to find corresponding changes in the $x$- and $y$-values. Problems that ask you to find how much one quantity changes in response to a specific change in the other are routine on the SAT, so it's important to become comfortable with using the proportion technique to rapidly solve such problems.

You don't always have to use the proportion in interpretation questions, but **you should be aware of the units of measurement for the $x$-and $y$-values so that you can easily interpret what the slope means in a real-world linear equation problem**. If we were just told that the relationship between the weight, $w$, of a potted cactus (measured in lbs) and the height, $h$, of the potted cactus (measured in in) is represented by the equation $w = \frac{13}{2}h + 15$, then we know that the meaning of the fraction $\frac{13}{2}$ is that the weight increases by 13 lbs for every 2 in increase in height.

Be aware that this slope could also be interpreted as a $\frac{13}{2}$-pound increase in weight for every 1-in increase in height, a 130-pound increase in weight for every 20-in increase in height, or even a 1-lb increase in weight for every $\frac{2}{13}$-in increase in height because all of the changes would still produce the same slope.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\text{Change in Weight}}{\text{Change in Height}} = \frac{13}{2} = \frac{\frac{13}{2}}{1} = \frac{130}{20} = \frac{1}{\frac{2}{13}}$$

**Example 2.4-3**

**3** 1600.io 2.4

$$d(t) = \frac{1}{248}t + 1{,}857$$

A geologist estimates that, as a result of erosion, the depth of the Grand Canyon has been increasing at a constant rate since it was first formed. The function above is used by the geologist to model the depth $d(t)$, in meters, of the canyon $t$ years after the year 2000. According the the function, which of the following statements is true?

A) Every 248 years the depth of the canyon increases by 0.1 meters.

B) Every 2,480 years the depth of the canyon increases by 10 meters.

C) Every 100 years the depth of the canyon increases by 2.48 meters.

D) Every year the depth of the canyon increases by 248 meters.

**Solution**

1. In this equation, the $d$-value (acting as the $y$-value) is the depth of the canyon, measured in meters, and the $t$-value (acting as the $x$-value) is the number of years since the year 2000. Therefore, the slope, which is $\frac{\Delta d}{\Delta t}$, is measured in units of m/yr. As such, the fraction $\frac{1}{248}$ tells us that the depth of the canyon increases by 1 m every 248 yr.
2. Choice A is clearly wrong because we know that the depth increase by 1 full meter and not by 0.1 m every 248 yr. Eliminate choice A.
3. Choice D is frankly outlandish if you think about it, but more directly to the point, it inverts the units of the numerator and denominator of the slope, so eliminate choice D.
4. Choice C is not nearly as freakish as choice D; however, if we know that it takes 248 years for the depth to increase by 1 m, then the depth could not increase by more than 1 m in a smaller amount of time (only 100 years versus 248 years). Eliminate choice C.
5. The only remaining choice is B, which isn't immediately recognizable with our correct interpretation of the slope; however, if we set up the fraction representing the slope, $\frac{\Delta d}{\Delta t}$, we will see that a change in depth of 10 m ($\Delta d$) for a change in years of 2,480 yr still reduces to the correct slope.

$$\frac{\Delta d}{\Delta t} = \frac{10}{2{,}480}$$

$$\frac{\Delta d}{\Delta t} = \frac{1}{248}$$

6. The answer is B.





**Notes**

You could do this problem slowly by substituting values into the equation to check each answer choice. For example, to check choice A, you could input two values of $t$ that are 248 years different from one another and see if the corresponding depths are 0.1 m different from one another (they will not be because this answer is not correct). To check choice B, you could input two values of $t$ that are 2,480 yr different from one another and see if the corresponding depths are 10 m different from one another (they will be because this answer is correct). And so on…

**Section 2.4 Suggested Problems from Real Tests**

- Test 1-NC-4
- Test 1-NC-6
- Test 1-C-15
- Test 2-C-35
- Test 3-NC-15
- Test 3-C-8
- Test 4-C-17
- Test 4-C-32
- Test 5-NC-8
- Test 5-NC-13
- Test 6-NC-1
- Apr 2017-NC-14
- May 2017-NC-11
- May 2017-C-19
- May 2017-C-29
- May 2018-C-15
- Test 10-C-29
- Apr 2019-NC-13
- Apr 2019-C-7
- May 2019 (US)-C-10
- Oct 2019-NC-10
- Oct 2019-C-9
- Oct 2020-C-20
- Mar 2021-NC-9
- Mar 2021-C-21
- May 2021 (Int)-C-15

**View related real-test problems in SMTEx**

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## Section 2.4 Practice Problems

**1** 1600.io 2.4

$$a = 20t + 130$$

Gregory made an initial deposit into his vacation fund jar. Each week thereafter he deposited a fixed amount to the jar. The equation above models the amount $a$, in dollars, that Gregory has deposited after $t$ weekly deposits. According to the model, how many dollars was Gregory's initial deposit? (Disregard the $ sign when writing your answer.)

**2** 1600.io 2.4

$$4{,}200 - 35k = 175$$

In 1806, the population of a fish species was 4,200. The population $k$ years after 1806 was 175, and $k$ satisfies the equation above. Which of the following is the best interpretation of the number 35 in this context?

A) The difference between the population in 1806 and the population $k$ years after 1806

B) The average decrease in the population per year from 1806 to $k$ years after 1806

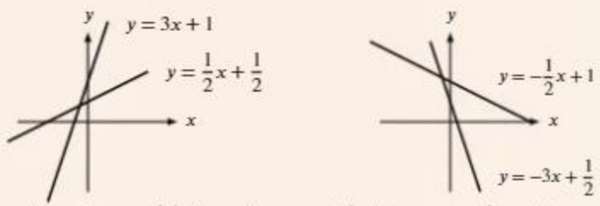
C) The population $k$ years after 1806

D) The value of $k$ when the population was 175

## CHAPTER 2 RECAP

- Slope-Intercept Form: $y = mx + b$
- Slope Formula: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$
- A positive slope means that the $y$-value increases as $x$-value increases. Steeper lines have higher slopes because the $y$-value goes up more as the $x$-value increases. A line with a slope of 3 is steeper than a line with a slope of $\frac{1}{2}$.

  A negative slope means that the $y$-value decreases as the $x$-value increases. Steeper lines have lower (more negative) slopes because the $y$-value decreases more as the $x$-value increases. A line with a slope of $-3$ is steeper than a line with a slope of $-\frac{1}{2}$.




- Horizontal lines have a slope of zero and their equations are of the form $y = c$, where $c$ is a constant (the $y$-value is the same for all points on the line).

  Vertical lines have an undefined slope and their equations are of the form $x = c$, where $c$ is constant (the $x$-value is the same for all points on the line). Note that this is **not** a Slope-Intercept equation, because a vertical line cannot be represented in that form because the slope is undefined.
- The $y$-intercept is where a line crosses the $y$-axis. One useful way of thinking about this (especially for real world problems) is that the $y$-intercept is the starting amount or the value of $y$ when the $x$-value is 0. To find the $y$-intercept when it is not clearly marked on a graph, substitute the slope and $x$- and $y$-coordinates of a point into the Slope-Intercept Form and solve for $b$.

  The $x$-intercept is where a line crosses the $x$-axis. Another useful way of thinking about this (especially for real world problems) is that this is the value of $x$ when the $y$-value is equal to 0. Unlike the $y$-intercept, this value is not visible in any common representation of linear equations, but to find it when it is not clearly marked on a graph, substitute the slope and $y$-intercept into the Slope-Intercept Form, then set the $y$-value equal to 0 and solve for $x$.
- You can find the equation of any line as long as you know any two points on the line or know the slope and any one point. If you are given the slope and $y$-intercept, then you already have everything you need to make the equation.
- Using using up-and-over arrows with graphical representations to find $\Delta y$ and $\Delta x$, or counting differences between coordinate values between rows of tables, is often less error prone than substituting the coordinates of points into the long from of the Slope Formula. There is no mathematical difference between methods, but graphical methods are harder to mess up.
- When asked to interpret the meaning of a coefficient or constant in a Slope-Intercept Form linear equation, first determine which one of the four components of a linear equation it is (the $y$-value, the slope, the $x$-value, or the $y$-intercept).

  The $y$-intercept is the "starting amount" and is measured in the same units as the $y$-value.

  The slope is the rate of change of the $y$-value as the $x$-value changes and is measured in the units of the $y$-value divided by the units of the $x$-value (look for keywords like "per" which indicate division).





## Additional Problems

**1** 1600.io 2.1

A line in the $xy$-plane has a slope of 0. Which of the following could be the equation of the line?

A) $x = 0$

B) $y = x$

C) $y = 0$

D) $y = -x$

**2** 1600.io 2.1

The line $y = kx + 5$, where $k$ is a constant, is graphed in the $xy$-plane. If the line contains the point $(c, d)$, where $c \neq 0$ and $d \neq 0$, what is the slope of the line in terms of $c$ and $d$?

A) $\frac{d-5}{c}$

B) $\frac{c-5}{d}$

C) $\frac{4-d}{c}$

D) $\frac{4-c}{d}$

**3** 1600.io 2.1

Which of the following is an equation of the line in the $xy$-plane that has slope $-5$ and passes through the point $(0, 2)$?

A) $y = -5x - 2$

B) $y = -5x + 2$

C) $y = -5(x + 2)$

D) $y = -5(x - 3)$

**4** 1600.io 2.3

$$C = 85h + 100$$

The equation above gives the amount $C$, in dollars, an HVAC technician charges for a job that takes $h$ hours. Ms. Porter and Mr. Green each hired this technician. The technician worked 3 hours longer on Ms. Porter's job than on Mr. Green's job. How much more did the technician charge Ms. Porter than Mr. Green?

A) \$85

B) \$100

C) \$170

D) \$255

**5** 1600.io 2.3

The graph in the $xy$-plane of the linear function $f$ contains the point $(1, 6)$. For every increase of 3 units in $x$, $f(x)$ increases by 4 units. Which of the following equations defines the function?

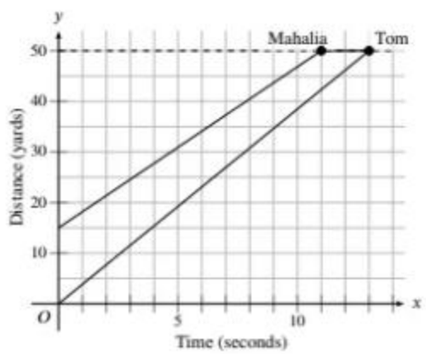
A) $f(x) = -\frac{4}{3}x + 10$

B) $f(x) = -\frac{3}{4}x + \frac{21}{4}$

C) $f(x) = \frac{3}{4}x + 2$

D) $f(x) = \frac{4}{3}x + \frac{14}{3}$

**6** 1600.io 2.4




The graph above shows the positions of Tom and Mahalia during a race. Tom and Mahalia each rollerbladed at a constant rate, and Mahalia was given a head start to shorten the distance she needed to skate. Tom finished the race in 13 seconds, and Mahalia finished the race in 11 seconds. According to the graph, Mahalia was given a head start of how many yards?

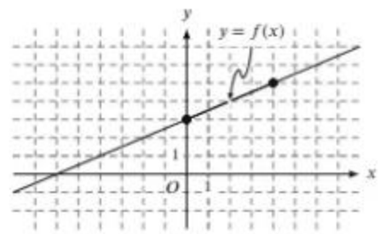
A) 3

B) 10

C) 15

D) 20

**7** 1600.io 2.1




The graph of the function $f$ is shown in the $xy$-plane above. The function $f$ is defined by the equation $f(x) = \frac{a}{b}x + c$ for positive constants $a$, $b$, and $c$, where $\frac{a}{b}$ is a fraction in lowest terms. Which of the following orders $a$, $b$, and $c$ from least to greatest?

A) $a < b < c$

B) $a < c < b$

C) $b < c < a$

D) $c < a < b$

**8** 1600.io 2.3

In the $xy$-plane, the points $(-1, 4)$ and $(2, -1)$ lie on the graph of which of the following linear functions?

A) $f(x) = -\frac{5}{3}x + \frac{7}{3}$

B) $f(x) = -\frac{3}{5}x + 1$

C) $f(x) = \frac{1}{2}x - 2$

D) $f(x) = 5x + 9$





**9** 1600.io 2.4

The equation $p = 14.7 + 15.4d$ approximates the pressure $p$, in pounds per square inch, exerted on a peanut at a depth of $d$ inches (in) below the surface of a chocolate syrup vat. What is the increase in depth that is necessary to increase the pressure by 1 pound per square inch?

A) $\frac{1}{14.7}$ in

B) $\frac{1}{15.4}$ in

C) 14.7 in

D) 15.4 in

**10** 1600.io 2.3

| $x$ | $f(x)$ |
|---|---|
| 4 | 15 |
| 8 | 7 |

The table above shows two pairs of values for the linear function $f$. The function can be written in the form $f(x) = ax + b$, where $a$ and $b$ are constants. What is the value of $a + b$?

**11** 1600.io 2.4

$$A = 205 + 24.14m$$

The equation above can be used to estimate the body surface area $A$, in square inches, of a child with mass $m$, in pounds mass, where $7 \le m \le 67$. Which of the following statements is consistent with the equation?

A) For each increase of 205 pounds in mass, $A$ increases by approximately 1 square inch.

B) For each increase of 1 pound in mass, $A$ increases by approximately 205 square inches.

C) For each increase of 1 pound in mass, $A$ increases by approximately 24.14 square inches.

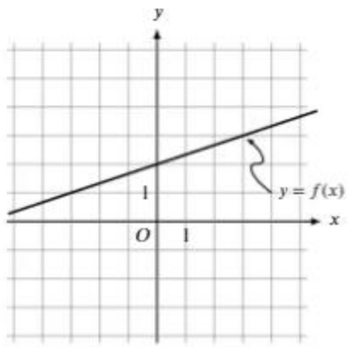
D) For each increase of 24.14 pounds in mass, $A$ increases by approximately 1 square inch.

**12** 1600.io 2.3

| $x$ | $y$ |
|---|---|
| 1 | $-2$ |
| $k$ | 10 |
| 5 | $n$ |

The table above shows the coordinates of three points on a line in the $xy$-plane, where $k$ and $n$ are constants. If the slope of the line is 6, what is the value of $n - k$ ?

**13** 1600.io 2.3




The graph of the linear function $f$ is shown in the $xy$-plane above. The slope of the graph of the linear function $g$ is 3 times the slope of the graph of $f$. If the graph of $g$ passes through the point $(0, -8)$, what is the value of $g(6)$?

A) $-8$

B) $-6$

C) $-2$

D) $10$

**14** 1600.io 2.1

A line in the $xy$-plane passes through the origin and has a slope $\frac{1}{10}$. Which of the following points lies on the line?

A) $(20, 2)$

B) $(10, 10)$

C) $(1, 10)$

D) $(0, 10)$

**15** 1600.io 2.3

The function $f$ is linear, $f(1) = 20$, and $f(4) = 11$. If $f(x) = mx + b$, where $m$ and $b$ are constants, what is the value of $b$?

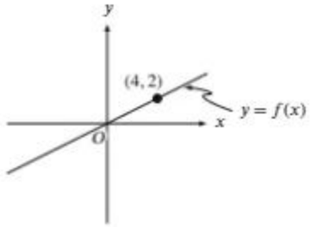
A) $\frac{61}{3}$

B) $23$

C) $17$

D) $\frac{34}{3}$

**16** 1600.io 2.1




In the $xy$-plane above, a point (not shown) with coordinates $(s, t)$ lies on the graph of the linear function $f$. If $s$ and $t$ are positive integers, what is the ratio of $t$ to $s$?

A) 1 to 2

B) 2 to 3

C) 2 to 1

D) 3 to 2





**17** 1600.io 2.4

$$s = 32.2t$$

The equation above can be used to approximate the speed $s$, in feet per second (ft/s), of an object $t$ seconds after being dropped into a free fall. Which of the following is the best interpretation of the number 32.2 in this context?

A) The speed, in ft/s, of the object when it hits the ground

B) The initial speed, in ft/s, of the object when it is dropped

C) The speed, in ft/s, of the object $t$ seconds after it is dropped

D) The increase in speed, in ft/s, of the object for each second after it is dropped

**18** 1600.io 2.3

| $x$ | $f(x)$ |
|---|---|
| 0 | $-2$ |
| 3 | 7 |
| 7 | 19 |

Some values of the linear function $f$ are shown in the table above. What is the value of $f(5)$?

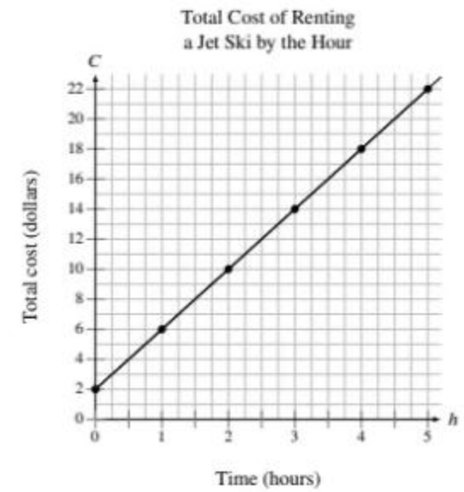
A) 10

B) 13

C) 14

D) 15

**Questions 19 and 20 refer to the following information.**




The graph above displays the total cost $C$, in dollars, of renting a jet ski for $h$ hours.

**19** 1600.io 2.4

What does the $C$-intercept represent in the graph?

A) The total number of jet skis rented

B) The total number of hours the jet ski is rented

C) The increase in cost to rent the jet ski for each additional hour

D) The initial cost of renting the jet ski

**20** 1600.io 2.1

Which of the following represents the relationship between $h$ and $C$?

A) $h = 4C$

B) $C = h + 2$

C) $C = 2h$

D) $C = 4h + 2$

**21** 2.3

A business owner purchased a vehicle valued at \$90,000. The values of the vehicle depreciates by the same amount each year so that after 5 years the value will be \$50,000. Which of the following equations gives the value, $v$, of the machine, in dollars, $t$ years after it was purchased, for $0 \le t \le 5$ ?

A) $v = 50{,}000 - 8{,}000t$

B) $v = 90{,}000 - 50{,}000t$

C) $v = 90{,}000 - 8{,}000t$

D) $v = 90{,}000 + 8{,}000t$

**22** 2.4

$$3n + 10 = 46$$

A snake had a length of 10 inches when it was hatched. The equation above can be used to find how many months $n$ it took the snake to reach a length of 46 inches. Which of the following is the best interpretation of the number 3 in this context?

A) The number of months it took the snake to triple its length

B) The length, in inches, of the snake when the snake was 1 month old

C) The average number of months it takes similar snakes to grow 46 inches

D) The average number of inches that the snake grew per month

**23** 2.3

| $x$ | $h(x)$ |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 3 | 7 |

The table above shows selected values for the function $h$. In the $xy$-plane, the graph of $y = h(x)$ is a line. What is the value of $h(6)$ ?

A) 9

B) 11

C) 13

D) 14

**24** 2.3

| $x$ | $f(x)$ |
|---|---|
| 0 | $c$ |
| 1 | $-2c$ |
| 2 | $-5c$ |

For the linear function $f$, the table above gives some values of $x$ and their corresponding values $f(x)$, where $c$ is a constant. Which of the following equations defines $f$ ?

A) $f(x) = -5cx - 5c$

B) $f(x) = -3cx + c$

C) $f(x) = -cx + c$

D) $f(x) = cx + c$





**25** 2.3

Which of the following is an equation of the line in the $xy$-plane that contains the points (1, 4) and (4, 16) ?

A) $y = 4x$

B) $y = x + 3$

C) $y = 3x + 4$

D) $y = \frac{1}{4}x$

**26** 2.3

Population of Hardlocke, Texas

| Year | Population |
|---|---|
| 1900 | 711 |
| 1910 | 687 |

The table above shows the population of Hardlocke, Texas, for the years 1900 and 1910. If the relationship between population and year is linear, which of the following functions $P$ models the population of Hardlocke $t$ years after 1900?

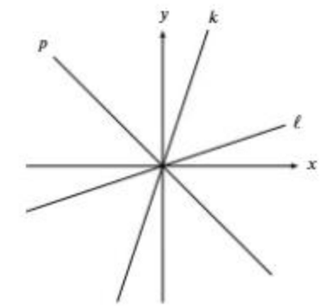
A) $P(t) = 711 - 2.4t$

B) $P(t) = 711 - 24t$

C) $P(t) = 711 - 2.4(t - 1900)$

D) $P(t) = 711 - 24(t - 1900)$

**27** 2.1




In the $xy$-plane, lines $k$, $\ell$, and $p$ are shown. Which of the following lists the slopes from least to greatest?

A) The slope of $p$, the slope of $\ell$, the slope of $k$

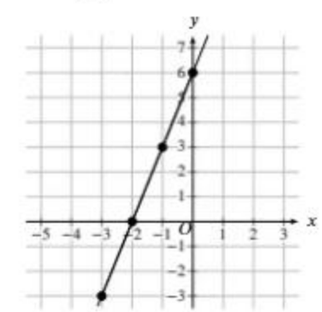
B) The slope of $\ell$, the slope of $p$, the slope of $k$

C) The slope of $k$, the slope of $\ell$, the slope of $p$

D) The slope of $p$, the slope of $k$, the slope of $\ell$

**28** 2.1

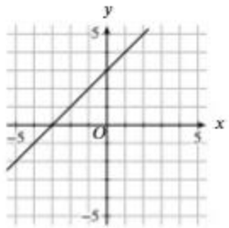
The graph of $y = mx + b$, where $m$ and $b$ are constants, is shown in the $xy$-plane.

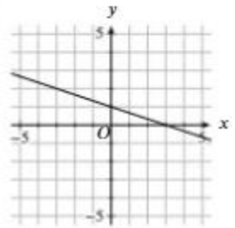


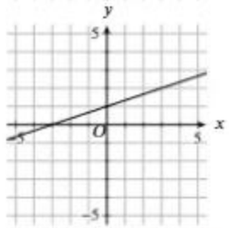

What is the value of $m$ ?

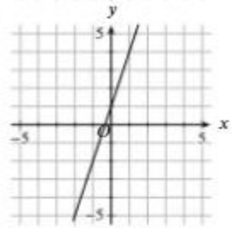
**29** 1600.io 2.2

Which of the following is the graph of the equation $y = 3x + 1$ in the $xy$-plane?

A)



B)



C)



D)



**30** 1600.io 2.3

| Speed (mph) | Pulse (bpm) |
|---|---|
| 4 | 65 |
| 8 | 77 |
| 12 | 89 |

The table lists selected values of Gina's rollerblading speed, in miles per hour (mph), and her corresponding pulse, in beats per minute (bpm). There is a linear relationship between Gina's speed, $x$, and her pulse, $f(x)$. Which of the following equations describes $f(x)$ ?

A) $f(x) = 12x + 65$

B) $f(x) = 12x + 53$

C) $f(x) = 3x + 65$

D) $f(x) = 3x + 53$

**31** 1600.io 2.4

Milk production in a certain area dropped from 10 million cartons in 2000 to 2.7 millions cartons in 2018. Assuming that the milk production decreased at a constant rate, which of the following linear functions $f$ best models the production, in millions of cartons, $t$ years after the year 2000?

A) $f(t) = \frac{73}{180}t + 10$

B) $f(t) = \frac{27}{180}t + 10$

C) $f(t) = -\frac{73}{180}t + 10$

D) $f(t) = -\frac{27}{180}t + 10$





# Answer Key

### SkillDrill 2.1-1

1. $(0, -3)$
2. $(-4, -2)$
3. $(3, 0)$
4. $(5, 3)$
5. $(0, 4)$
6. $(-3, 1)$
7. $(4, -4)$
8. $(-2, 0)$

### SkillDrill 2.1-2

1. 1
2. 2
3. 5
4. 4
5. 3
6. 4
7. 1
8. 3

### SkillDrill 2.1-3

1. $m = 2, b = 1$
2. $m = 3, b = 6$
3. $m = 1, b = -7$
4. $m = 5, b = 0$
5. $y = 5x - 4$
6. $y = 8x + 1$
7. $y = x$
8. $y = 2x - 2$

### SkillDrill 2.1-4

1. $y = \frac{-1}{3}x - 3, x_{int} = (-9, 0)$
2. $y = x + 1, x_{int} = (-1, 0)$
3. $y = \frac{1}{2}x + 3, x_{int} = (-6, 0)$
4. $y = \frac{-5}{4}x - \frac{1}{2}, x_{int} = \left(\frac{-2}{5}, 0\right)$
5. $y = -3, x_{int}$ does not exist
6. $x = 4, x_{int} = (4, 0)$
7. $y = \frac{2}{7}x - \frac{8}{7}, x_{int} = (4, 0)$
8. $y = \frac{-1}{3}x + \frac{4}{3}, x_{int} = (4, 0)$

### Section 2.1 Practice Problems

1. D
2. C
3. $\frac{2}{3}$, .666, or .667
4. C

### Section 2.2 Practice Problems

1. B
2. A

### SkillDrill 2.3-1

1. $y = \frac{-1}{3}x + \frac{7}{3}$
2. $y = 4x - 5$
3. $y = \frac{3}{4}x + \frac{41}{4}$
4. $y = 0$
5. $y = \frac{-4}{3}x$
6. $y = -2x + 7$
7. $y = \frac{1}{10}x + 1$
8. $y = \frac{-1}{9}x + \frac{44}{9}$

### SkillDrill 2.3-2

1. $y = 2x - 5, k = 1$
2. $y = -4x + 2, k = 2$
3. $y = \frac{3}{2}x, k = 10$
4. $y = \frac{-1}{4}x + \frac{3}{4}, k = 15$

### Section 2.3 Practice Problems

1. 4
2. B
3. 26
4. D

### Section 2.4 Practice Problems

1. 130
2. B

**Additional Problems**

1. C
2. A
3. B
4. D
5. D
6. C
7. A
8. A
9. B
10. 21
11. C
12. 19
13. C
14. A
15. B
16. A
17. D
18. B
19. D
20. D
21. C
22. D
23. C
24. B
25. A
26. A
27. A
28. 3
29. D
30. D
31. C



# Standard Form/Parallel and Perpendicular Lines

# 3

## 3.1 Standard Form

Jimmy wants to buy notebooks that cost \$4 each. We've learned already how we could calculate the total amount that Jimmy would spend on notebooks by multiplying the cost of each notebook by the number of notebooks he buys. If Jimmy buys $n$ notebooks, then he will spend $4n$ dollars.

Suppose he also needs to buy pens that cost \$2 each. The amount Jimmy would spend on $p$ pens would be $2p$ dollars. Even though he is buying two different items, we can make one equation that represents the total amount he spends on both items by adding the total cost of the notebooks and pens together. If $t$ is the total cost of the supplies, then the equation representing Jimmy's total cost is $2p + 4n = t$.

Let's say that Jimmy can only spend a total of \$40 on supplies. In the equation, the value of $t$ would be 40:

$$2p + 4n = 40$$

Jimmy wants to use this equation to figure out how many pens and notebooks he can buy.

The first thing to note is that the more he buys of one thing, the fewer he could buy of the other. If he buys 0 notebooks, he can buy 20 pens because $2(\mathbf{20}) + 4(\mathbf{0}) = 40$. If he buys 1 notebook, he still has enough money for 18 pens ($2(\mathbf{18}) + 4(\mathbf{1}) = 40$), but if he buys 2 notebooks, he can only buy 16 pens ($2(\mathbf{16}) + 4(\mathbf{2}) = 40$).

As soon as we limit the total amount of money he is going to spend, the number of pens and notebooks has to strike a balance: the two values are related, much in the same way $x$ and $y$-values are related to each other for the points on a line. That is because the equation we just wrote is a linear equation, but it's not in Slope-Intercept Form; it's in what we call **Standard Form**.

### Standard Form Linear Equation

The following is the equation of a line in Standard Form, where $A$, $B$, and $C$ are constants:

$$Ax + By = C$$

In a lot of word problems, it is easier to write the equation in Standard Form than in Slope-Intercept Form due to how the information is given to you. Once you have the equation, you can use it to solve for any values that are unknown.

**SkillDrill 3.1-1**

**Directions:** Write a Standard Form linear equation to represent the given situation. In order to make sure that you use variable letters that make sense (this is an important habit), we will provide them for you.

1. Bella gets paid \$2 for every backflip she does and \$3 for every front flip she does. In one day, she makes \$56 doing backflips and front flips. (Use $b$ for the number of backflips and $f$ for the number of front flips.)
2. In an eating contest, Ferdinand takes 1 minute to eat each hot dog and 3 minutes to eat each cake. It took Ferdinand 60 minutes to eat $h$ hot dogs and $c$ cakes.
3. It takes Paula 8 minutes to drink a small water bottle and 20 minutes to drink a big water bottle. In one week, it took her a total of 112 minutes to drink $s$ small water bottles and $b$ big water bottles.
4. A company loses \$100 for every inappropriate comment they make on social media and loses \$2.50 every time one of their employees exposes the hazardous working conditions in the company's overseas sweatshops. In one year, the company loses \$197.50 from $c$ inappropriate comments and $h$ reports of hazardous working conditions.
5. Jeremetrius gains 5 subscribers for every reaction video he posts and loses 10 subscribers for every prank video he posts. In one week, Jeremetrius loses a total of 42 subscribers. (Use $r$ for the number of reaction videos and $p$ for the number of prank videos.)

**Example 3.1-1**

**1** 1600.io 3.1

Nick spends \$57 per month on chips. A 10-bag pack of chips costs \$8.50, and a single bag of chips costs \$1.20. If $p$ represents the number of 10-bag packs Nick buys in a month, and $s$ represents the number of single bags Nick buys in a month, which of the following equations best represents the relationship between $p$ and $s$?

A) $p + s = 8.50 + 1.20$

B) $p + s = 57$

C) $1.20p + 8.50s = 57$

D) $8.50p + 1.20s = 57$

**Solution**

1. We need to write an equation that represents the total amount of money Nick spends on chips in a month. To do this, we should start by finding the amount he spends on $p$ 10-bag packs and $s$ single bags of chips.
2. The cost of $p$ 10-bag packs is equal to the cost per 10-bag pack (\$8.50) times the number of 10-bag packs ($p$), which is $8.50p$.
3. The cost of $s$ single bags is equal to the cost per single bag (\$1.20) times the number of single bags ($s$), which is $1.20s$.
4. We know the total cost was \$57, so we can add the costs of buying $p$ 10-bag packs ($8.50p$) and $s$ single bags ($1.20s$) and set the sum equal to 57.

$$8.50p + 1.20s = 57$$

5. The answer is D.





For many word problems, it is easier to write linear equations in Standard Form; however, when lines are graphed, it is usually easier to write the equations first in Slope-Intercept Form and then convert the equations into Standard Form if necessary to match the format of the answer choices.

## Converting from Slope-Intercept Form to Standard Form

We can convert from Slope-Intercept Form to Standard Form through simple algebraic steps by moving the $x$-variable to the same side of the equation as the $y$-variable.

We want to convert from a line in Slope-Intercept Form ($y = mx + b$) to a line in Standard Form ($Ax + By = C$), so we should subtract $mx$ from both sides of the equation.

$$\begin{aligned} y &= mx + b \\ -mx + y &= b \end{aligned}$$

The equation above is now in Standard Form ($Ax + By = C$) where $A = -m$, $B = 1$, and $C = b$.

For example, the line $y = -2x + 3$ is in Slope-Intercept Form, but by adding $2x$ to both sides of the equation (subtracting $-2x$ from both sides of the equation) we arrive at the same linear equation in Standard Form: $2x + y = 3$.

**When the slope of a line in Slope-Intercept Form is a fraction**, you may also wish to (or have to in order to match answer choices) **multiply both sides of the equation by the denominator of the $x$-coefficient to eliminate the fraction**.

For example, we can rewrite the linear equation $y = \frac{2}{3}x - 4$ in Standard Form in multiple equivalent ways. Start by subtracting $\frac{2}{3}x$ from both sides of the equation: $-\frac{2}{3}x + y = -4$. This is already in Standard Form, but it is not necessarily the stopping point (the answer choices for many questions asking you for the Standard Form equation of a line based on a graph will generally not have any fractional coefficients). We can multiply both sides of the equation by 3 (the denominator of the $x$-coefficient) in order to eliminate the fractional coefficient.

$$\begin{aligned} -\frac{2}{3}x + y &= -4 \\ 3\left(-\frac{2}{3}x + y\right) &= 3(-4) \\ \cancel{3}\left(-\frac{2}{\cancel{3}}x\right) + 3y &= -12 \\ -2x + 3y &= -12 \end{aligned}$$

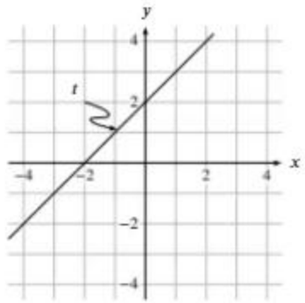
Similarly, we can (if necessary to match answer choices) multiply both sides of the equation by $-1$ in order to rewrite the equation into yet another equivalent representation.

$$\begin{aligned} -2x + 3y &= -12 \\ -1(-2x + 3y) &= -1(-12) \\ -1(-2x) - 1(3y) &= 12 \\ 2x - 3y &= 12 \end{aligned}$$

This highlights the fact that while there is only **one** Slope-Intercept Form of a linear equation because one side of the equation must be just $y$, there are **infinite** equivalent Standard Form versions of the same equation because there are no constraints on the form or value of the variables' coefficients.

**Example 3.1-2**

2 1600.io 3.1




Which of the following is an equation of line $t$ in the $xy$-plane above?

A) $x + y = 2$

B) $x + y = -2$

C) $x - y = 2$

D) $x - y = -2$

**Solution 1**

1. All of the answer choices are in Standard Form, $Ax + By = C$, where the value of $A$ is 1 (because the coefficient of $x$ is 1 in every choice). Therefore, we can simplify the equation of this line to $x + By = C$.

2. We can plug in points on the line to find the value of $B$ and $C$. If we start with the point $(-2, 0)$, the $y$-term will drop out of the equation, helping us find the value of $C$ easily.

$$\begin{aligned} x + By &= C \\ -2 + B(0) &= C \\ -2 &= C \end{aligned}$$

3. Since the value of $C$ is $-2$, the equation of this line is $x + By = -2$ (limiting us to choices B and D), and we can solve for $B$ by plugging in the point $(0, 2)$.

$$\begin{aligned} x + By &= -2 \\ 0 + B(2) &= -2 \\ 2B &= -2 \\ B &= -1 \end{aligned}$$

4. Since the value of $B$ is $-1$, the equation of this line is $x - y = -2$.

5. The answer is D.



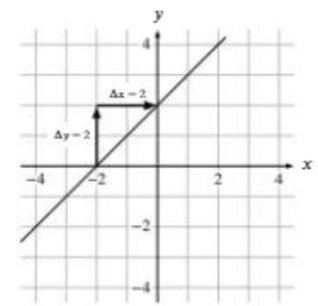


**Solution 2**

1. Start by finding the equation of the line in Slope-Intercept Form, which can then be converted into Standard Form in order to match the answer choices.

2. Since the graph goes through the point $(0, 2)$, we know that the $y$-intercept of the line is 2.

3. We can draw in over-and-up arrows that show the changes in the $y$- and $x$-values so that we can find the slope of the line. Let's use the points $(0, 2)$ and $(-2, 0)$.




Since the slope is equal to $\frac{\Delta y}{\Delta x}$, the slope of this line is $\frac{2}{2}$ or 1.

4. We can write the equation of the line in Slope-Intercept Form ($y = mx + b$). We know that the slope is 1 and the $y$-intercept is 2, so we can plug in 1 for $m$ (remember when the coefficient of a variable is 1, you do not have to write the coefficient) and 2 for $b$.

$$\begin{aligned} y &= mx + b \\ y &= (1)x + (2) \\ y &= x + 2 \end{aligned}$$

5. In order to match the form of the answer choices, subtract $x$ from both sides of the equation in order to rewrite the equation in Standard Form.

$$\begin{aligned} y &= x + 2 \\ -x + y &= 2 \end{aligned}$$

6. None of the answer choices match this form. Since all of the answer choices start with a positive $x$ instead of $-x$, multiply both sides of the equation by $-1$.

$$\begin{aligned} -x + y &= 2 \\ -1(-x + y) &= -1(2) \\ -1(-x) - 1(y) &= -2 \\ x - y &= -2 \end{aligned}$$

7. The answer is D.

Just as lines written in Slope-Intercept Form can be rewritten in Standard Form, lines written in Standard Form can be converted into Slope-Intercept Form.

## Converting from Standard Form to Slope-Intercept Form

Sometimes we are given lines in Standard Form, but it is more useful for the problem to have them in Slope-Intercept Form. We can convert from Standard Form to Slope-Intercept Form through simple algebraic steps by isolating the $y$-variable.

We want to convert from a line in Standard Form ($Ax + By = C$) to a line in Slope-Intercept Form ($y = mx + b$), so we will complete the two following steps:

1. Subtract $Ax$ from both sides of the equation: $By = -Ax + C$
2. Divide both sides of the equation by $B$: $y = \frac{-A}{B}x + \frac{C}{B}$

Remember, the slope of a line expressed in Slope-Intercept Form is the coefficient of $x$, so looking at the equation above, you can see that the **slope of a line in Standard Form is $\frac{-A}{B}$**.

Similarly, the $y$-intercept of a line expressed in Slope-Intercept Form is the constant term, so the **$y$-intercept of a line in Standard Form is $\frac{C}{B}$**.

You do not need to memorize this conversion formula; what's essential is that you understand the *procedure* involved, which is simply isolating the $y$ term with no coefficient on one side of the equation. As long as you know that principle (and your algebra is solid), you can perform this conversion.

**SkillDrill 3.1-2**

**Directions:** Convert each linear equation from Standard Form to Slope-Intercept Form.

**1.** $4x - 2y = 8$ **2.** $9x - 3y = -21$ **3.** $-3x - 2y = 7$ **4.** $-5x - 6y = -18$

**Directions:** Convert each linear equation from Slope-Intercept Form to Standard Form, eliminating fractional coefficients if any. There are multiple correct answers; the answer key gives simple versions, with lowest non-fractional coefficients and non-negative $x$-coefficients.

**5.** $y = -5x - 10$ **6.** $y = 2x + 4$ **7.** $y = \frac{-1}{3}x - \frac{2}{3}$ **8.** $y = \frac{5}{2}x + 7$

Let's look back at the example of Jimmy buying pens and notebooks. We said if he buys $p$ pens for \$2 each and $n$ notebooks for \$4 each and spends a total of $t$ dollars, then the equation that represents the relationship is $2p + 4n = t$. While this is a linear equation in Standard Form, it doesn't represent a specific line because $t$, the total amount that can be spent, is a variable, not a constant. As soon as we have a specific total amount that Jimmy can spend, the equation will represent a specific line that reflects the relationship.

For example, when we said that Jimmy has to spend exactly \$40, we have the Standard Form linear equation $2p + 4n = 40$, and now we have a linear equation for a specific line (which can be graphed) for which the number of pens and notebooks must be in balance so that regardless of how many of each are bought, the total cost will come to \$40.



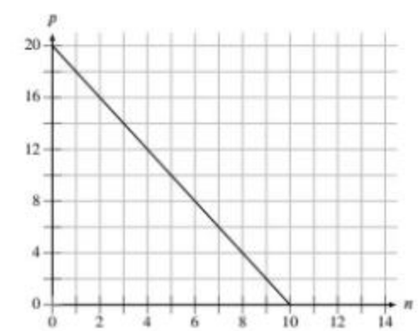


It might be easier to visualize this if we write the equation in Slope-Intercept Form, which makes it easy to graph the line so we can select some points and plug them into the Standard Form equation; converting the equation to Slope-Intercept Form lets us find the value of $p$ (which will be on the vertical axis) for any value of $n$ (the horizontal axis), which lets us plot points and draw the line through them. Let's find the number of pens he buys as a function of the number of notebooks he buys by solving for $p$ in terms of $n$, converting the equation into Slope-Intercept Form (where $p$ is like the $y$-value and $n$ is like the $x$-value). That is, we'd like an equation into which we can input the number of notebooks, and the equation will then produce the number of pens; to accomplish that, we will isolate the number of pens $p$ on one side so when we evaluate the other side, we'll know the number of pens.

$$\begin{aligned} 2p + 4n &= 40 \\ 2p &= -4n + 40 \\ \frac{1}{\cancel{2}}(\cancel{2}p) &= \frac{1}{2}(-4n + 40) \\ p &= \frac{-4}{2}n + \frac{40}{2} \\ p &= -2n + 20 \end{aligned}$$

Now that the line is in Slope-Intercept Form, we can verify what we said before about how the number of pens is affected by the number of notebooks. The $y$-intercept is 20, because if Michael bought 0 notebooks, he could buy 20 pens (that is, $n$ would equal zero, so the equation would be $p = 20$). The slope, given as the coefficient of the variable $n$, is $-2$, which means that the number of pens he can buy decreases by 2 every time he buys another notebook. This makes perfect sense, because a notebook costs \$2 and a pen costs \$1.

Let's now graph the line and pick a couple of points, so we can show how all of the points on the line also satisfy the Standard Form equation and provide ordered pairs of the number of pens and notebooks Jimmy can buy that will result in a total cost of \$40.




Remember that the cost of the supplies can be represented with the Standard Form equation $2p + 4n = t$. If we plug in any points from the graph into the Standard Form equation, we should see that the cost will add up to \$40. The two obvious points to look at are the $p$- and $n$-intercepts, $(0, 20)$ and $(10, 0)$, respectively. If we plug in the point $(0, 20)$, where $n = 0$ and $p = 20$, we will see that the cost of the supplies, $t$, is \$40.

$$\begin{aligned} 2p + 4n &= t \\ 2(20) + 4(0) &= t \\ 40 &= t \end{aligned}$$

If we plug in the point (10, 0), where $n = 10$ and $p = 0$, we will see agin that the cost of the supplies, $t$, is \$40.

$$\begin{aligned} 2p + 4n &= t \\ 2(0) + 4(10) &= t \\ 40 &= t \end{aligned}$$

Any point that we pick on the line will show that the total cost of supplies is \$40. Let's try (5, 10), where $n = 5$ and $p = 10$.

$$\begin{aligned} 2p + 4n &= t \\ 2(10) + 4(5) &= t \\ 20 + 20 &= t \\ 40 &= t \end{aligned}$$

Now let's check the point (8, 4), where $n = 8$ and $p = 4$.

$$\begin{aligned} 2p + 4n &= t \\ 2(4) + 4(8) &= t \\ 8 + 32 &= t \\ 40 &= t \end{aligned}$$

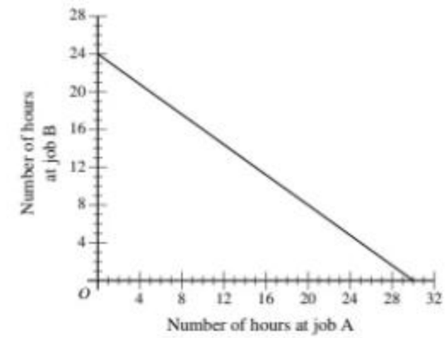
As soon as the total cost of the supplies was set at \$40, a single line was established, which could be graphed with the aid of a conversion to Slope-Intercept Form. Every point on the line could be plugged in to the Standard Form equation to show that any combination of notebooks and pens that appears as a point on the line would result in a total cost of \$40 (note that a point with a non-counting number coordinate value such as (1.5, 17) is on the line and satisfies the equation because $2(17) + 4(1.5) = 34 + 6 = 40$, but Jimmy probably can't buy one and half notebooks).





**Example 3.1-3**

3 1600.io 3.1




To earn money for piano lessons, Carlita works two part-time jobs: A and B. She earns \$8 an hour working at job A and \$10 an hour working at job B. In one week, Carlita earned a total of $s$ dollars for working at the two part-time jobs. The graph above represents all possible combinations of numbers of hours Carlita could have worked at the two jobs to earn $s$ dollars. What is the value of $s$?

A) 192

B) 240

C) 300

D) 720

**Solution**

1. Normally, when there is a linear graph presented above a question, it is useful to find the equation of the line in Slope-Intercept Form. However, for this problem, that step would probably not be very useful to understanding the problem (it is not necessarily a waste of time or useless, and it should still generally be considered a good first step for many line problems).
   We can see that the horizontal axis (usually the $x$-axis) tells us the number of hours that Carlita worked at job A, which we will denote as $a$, and the vertical axis (usually the $y$-axis) tells us the number of hours that Carlita worked at job B, which we will denote as $b$. We are told that the line in the graph represents all possible combinations of hours that Carlita could have worked in order to earn a specific (but unknown) amount of money: $s$ dollars.
   If we want to figure out the value of $s$, we need only pick any point $(a, b)$ on this line and figure out how much money Carlita would make for working $a$ hours at job A and $b$ hours at job B (any and every point on the line tells us a combination of hours she could have worked to make $s$ dollars). First, we need to write the Standard Form equation representing the line, which is easier to do based on the question's text than based on the Slope-Intercept Form of the line.

2. If Carlita works $a$ hours at job A for \$8 an hour, then the total money she makes from job A is $8a$ dollars. If she works $b$ hours at job B for \$10 an hour, then the total money she makes from job B is $10b$ dollars. In total, the amount of money she makes from working $a$ hours at job A and $b$ hours at job B, which is said to be $s$ dollars, is equal to $8a + 10b$. Therefore, we can write the Standard Form linear equation $8a + 10b = s$ to represent the total amount of money she makes in a week.

3. Now that we have the Standard Form equation of the line (where $s$ is still unknown), we can find the value of $s$ by plugging in the $a$- and $b$-values of any point shown on the line. The easiest points to use would be either the $a$-intercept $(30, 0)$ (where the line hits the horizontal axis), or the $b$-intercept $(0, 24)$ (where the line hits the vertical axis). Note that any points on the line would work, but the lack of gridlines in the graph makes it harder to accurately choose any other points, and by choosing a point where one of the coordinate values is 0, we simplify the arithmetic we're about to perform. Let's check both intercepts just to verify that the total will be the same. First, let's use the point $(30, 0)$ by plugging in 30 for $a$ and 0 for $b$.

$$\begin{aligned} 8a + 10b &= s \\ 8(30) + 10(0) &= s \\ 240 &= s \end{aligned}$$

We can comfortably choose choice B as the correct answer at this point, but for completeness, let's also check that the point $(0, 24)$ also produces the same value of $s$ by plugging in 0 for $a$ and 24 for $b$.

$$\begin{aligned} 8a + 10b &= s \\ 8(0) + 10(24) &= s \\ 240 &= s \end{aligned}$$

4. The answer is B.

## Section 3.1 Suggested Problems from Real Tests

- Test 1-NC-3
- Test 2-NC-16
- Test 2-C-1
- Test 6-C-35
- Test 7-NC-1
- Test 7-C-25
- Test 7-C-26
- Test 9-NC-3
- Test 10-NC-9
- Test 10-NC-16
- Test 10-C-23
- Mar 2019-NC-15
- Apr 2019-NC-19
- Apr 2019-C-27
- May 2019 (US)-C-21
- Mar 2020-C-13
- Mar 2020-C-21
- Oct 2020-NC-7
- Mar 2021-NC-5
- Apr 2021-NC-6
- Apr 2021-C-1
- Apr 2021-C-26
- May 2021 (US)-C-28
- May 2021 (Int)-NC-1
- May 2021 (Int)-NC-2

View related real-test problems in SMTEx

SMTEx





## Section 3.1 Practice Problems

**1** 1600.io 3.1

Kenzie burns 3 calories per minute lifting weights and 15 calories per minute swimming. Which of the following equations represents the total number of calories, $T$, Kenzie has burned after lifting weights for 30 minutes and swimming for $m$ minutes?

A) $T = 45m + 30$

B) $T = 15m + 90$

C) $T = 3m + 450$

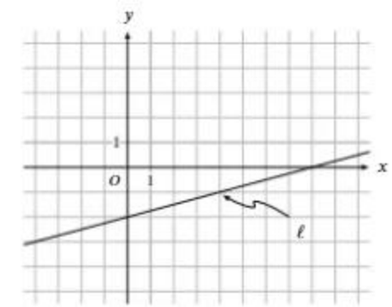
D) $T = 18m + 30$

**2** 1600.io 3.1

$$T = 4b + 10s$$

A manufacturer produces units of two different products. The equation above shows the total manufacturing cost $T$, in dollars, for manufacturing $b$ units of the big product and manufacturing $s$ units of the small product. If the total manufacturing cost was \$5,670 and 800 units of the big product were manufactured, how many units of the small product were manufactured?

**3** 1600.io 3.1




Line $\ell$ is shown in the $xy$-plane above. Which of the following is an equation of line $\ell$?

A) $8x + 2y = 0$

B) $8x - 2y = 0$

C) $4x + y = 8$

D) $x - 4y = 8$

**4** 1600.io 3.1

What are the slope and the $y$-intercept of the graph in the $xy$-plane of the equation $2x - 5y - 7 = 0$?

A) The slope is $-\frac{2}{5}$, and the $y$-intercept is $(0, -\frac{7}{5})$.

B) The slope is $-\frac{2}{5}$, and the $y$-intercept is $(0, \frac{7}{5})$.

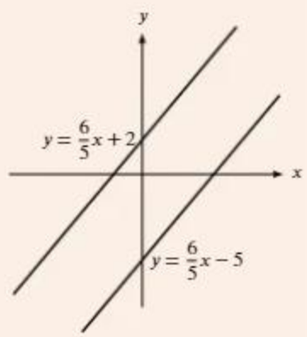
C) The slope is $\frac{2}{5}$, and the $y$-intercept is $(0, -\frac{7}{5})$.

D) The slope is $\frac{2}{5}$, and the $y$-intercept is $(0, \frac{7}{5})$.

## 3.2 Parallel and Perpendicular Lines

One of the main reasons we may need to convert from Standard Form to Slope-Intercept Form is to solve problems that ask us about parallel or perpendicular lines. These problems usually need you to make use of the lines' slopes, which are most easily found when the lines are written in Slope-Intercept Form.

### Parallel Lines




**Parallel lines are the same distance apart everywhere, and thus they never intersect. Parallel Lines have the same slope.** The line $y = \frac{6}{5}x + 2$ is parallel to the line $y = \frac{6}{5}x - 5$ because they have the same slope.

So we can see what parallel lines look like in Standard Form, convert both lines into Standard Form.

$$y = \frac{6}{5}x + 2 \qquad\qquad y = \frac{6}{5}x - 5$$

$$\frac{-6}{5}x + y = 2 \qquad\qquad \frac{-6}{5}x + y = -5$$

$$5\left(\frac{-6}{5}x + y\right) = 5(2) \qquad\qquad 5\left(\frac{-6}{5}x + y\right) = 5(-5)$$

$$\cancel{5}\left(\frac{-6}{\cancel{5}}x\right) + 5y = 10 \qquad\qquad \cancel{5}\left(\frac{-6}{\cancel{5}}x\right) + 5y = -25$$

$$-6x + 5y = 10 \qquad\qquad -6x + 5y = -25$$

We now have both equations in Standard Form, and we should notice that the $x$- and $y$-coefficients match in both equations.

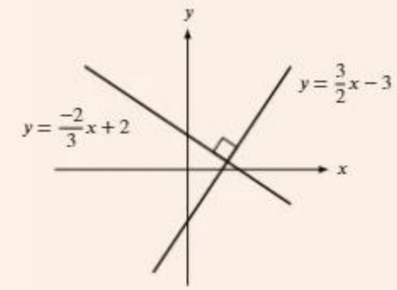
$$\mathbf{-6}x + \mathbf{5}y = 10$$
$$\mathbf{-6}x + \mathbf{5}y = -25$$

If we were given these two lines in Standard Form, we would know that the lines are parallel because they $x$- and $y$-coefficients match. You can also verify this by remembering that the slope of a line in Standard Form, $Ax + By = C$, is equal to $\frac{-A}{B}$, so the slopes will match as long as the $x$- and $y$-coefficients are the same (or are in the same ratio to each other, which is covered more in the next chapter).





### Perpendicular Lines




**Perpendicular lines cross at a right angle. The slope of each line in a pair of perpendicular lines is the negative reciprocal of the slope of the other line.** The **negative reciprocal** is found by swapping the values in the numerator and denominator of a slope and multiplying by $-1$. For example, a line with a slope of $\frac{-2}{3}$ is perpendicular to a line with a slope of $\frac{3}{2}$, which was found by flipping the fraction and multiplying by $-1$. Similarly, a line with a slope of 4 (or $\frac{4}{1}$) is perpendicular to a line with a slope of $\frac{-1}{4}$.

To consider what perpendicular lines look like when written in Standard Form, let's convert the perpendicular lines $y = 4x + 2$ and $y = \frac{-1}{4}x - 3$ into Standard Form.

$$y = 4x + 2$$
$$-4x + y = 2$$

$$y = \frac{-1}{4}x - 3$$
$$\frac{1}{4}x + y = -3$$
$$4\left(\frac{1}{4}x + y\right) = 4(-3)$$
$$\cancel{4}\left(\frac{1}{\cancel{4}}x\right) + 4y = -12$$
$$x + 4y = -12$$

These two lines in Standard Form are the following.

$$-4x + y = 2$$
$$x + 4y = -12$$

If we look at the two perpendicular lines in Standard Form, it looks like the $x$- and $y$-coefficients have swapped positions, and one of them has been multiplied by $-1$. In a way, you can think of this as a "horizontal negative reciprocal"—just as the negative reciprocal is found by swapping the numerator and denominator and multiplying by $-1$, the "horizontal negative reciprocal" here is found by swapping the $x$- and $y$-coefficients and multiplying one of them by $-1$. Remember again that the slope of a line in Standard Form, $Ax + By = C$, is equal to $\frac{-A}{B}$, so swapping the coefficients and multiplying one of them by $-1$ (to get the line $Bx - Ay = C$) effectively makes the slope equal to $\frac{B}{A}$, which is the negative reciprocal of $\frac{-A}{B}$.

**SkillDrill 3.2-1**

**Directions:** Given a linear equation in Standard Form, write the equation in both Standard Form and Slope-Intercept Form of the parallel or perpendicular line that contains the given point. The answers in Standard Form are written with lowest non-fractional coefficients and a non-negative $x$-coefficient.

**1.** parallel to $3x - 2y = 7$ contains $(0, 4)$

**2.** perpendicular to $3x - 2y = 7$ contains $(0, 4)$

**3.** parallel to $4x + 2y = 10$ contains $(2, -6)$

**4.** perpendicular to $4x + 2y = 10$ contains $(-1, 10)$

**5.** parallel to $-5x - 4y = 90$ contains $(5, 4)$

**6.** perpendicular to $-5x - 4y = 90$ contains $(10, 2)$

**7.** parallel to $-8x + 5y = 35$ contains $(1, 1)$

**8.** perpendicular to $-8x + 5y = 35$ contains $(-4, 0)$

**Example 3.2-1**

**1** 1600.io 3.2

Lines $t$ and $w$ are parallel in the $xy$-plane. The equation of line $t$ is $4x + 7y = 14$, and line $w$ passes through $(7, 6)$. What is the value of the $y$-intercept of line $w$?

**Solution 1**

1. Remember that parallel lines in Slope-Intercept Form should have the same $x$- and $y$-coefficients (or be in the same ratio), so we know that for line $w$ in Standard Form, $Ax + By = C$, the value of $A$ should be 4 and the value of $B$ should be 7 (we do not know $C$, but that is part of what is needed to solve the problem).

$$Ax + By = C$$
$$4x + 7y = C$$

2. In order to solve for $C$, we should plug in the only point on line $w$ that we know (which we are given), $(7, 6)$. Plug in 7 for $x$, and plug in 6 for $y$.

$$4x + 7y = C$$
$$4(7) + 7(6) = C$$
$$28 + 42 = C$$
$$70 = C$$

3. Since $C = 70$, we can substitute 70 in for $C$ to find the Standard Form equation of line $w$.

$$4x + 7y = C$$
$$4x + 7y = 70$$





4. The $y$-intercept is the $y$-value of the point on the line where $x = 0$, so plug in 0 for $x$ in order to find the $y$-intercept of line $w$.

$$4x + 7y = 70$$
$$4(0) + 7y = 70$$
$$7y = 70$$
$$y = 10$$

5. The answer is 10.

**Solution 2**

1. First rewrite the equation of line $t$ in Slope-Intercept Form, so we can find its slope (note that we don't actually need the entire equation including the constant that represents the $y$-intercept, because all we need to find is the slope, but we'll perform the conversion for completeness).

$$4x + 7y = 14$$
$$7y = -4x + 14$$
$$y = \frac{-4}{7}x + 2$$

2. The slope of line $t$ is $\frac{-4}{7}$, and since line $w$ is parallel to line $t$, the slope of line $w$ is also $\frac{-4}{7}$. Replace $m$ with $\frac{-4}{7}$ in the general Slope-Intercept equation to begin writing the equation of line $w$.

$$y = \frac{-4}{7}x + b$$

3. Since we know that $(7, 6)$ is a point on line $w$, we can plug in 7 for $x$ and 6 for $y$ in order to solve for $b$, the $y$-intercept of line $w$.

$$y = \frac{-4}{7}x + 2$$
$$6 = \frac{-4}{7}(7) + b$$
$$6 = -4 + b$$
$$10 = b$$

4. The answer is 10.

**Notes**

In Step 4 of Solution 1, instead of plugging in 0 for $x$ to find the $y$-intercept, you could remember that the $y$-intercept of a line in Standard Form $Ax + By = C$ is equal to $\frac{C}{B}$ (but this doesn't save much time, is easy to screw up, and doesn't reinforce the fact the $y$-intercept of an equation is the $y$-value of the equation when the $x$-value is 0).

**Example 3.2-2**

**2** 1600.io 3.2

$$-4x + 3y = 8$$

In the $xy$-plane, the graph of which of the following equations is perpendicular to the graph of the equation above?

A) $3x + 4y = 8$

B) $3x - 4y = 8$

C) $4x + 3y = 8$

D) $4x - 3y = 8$

**Solution 1**

1. First we need to write the original equation in Slope-Intercept Form to find its slope.

$$-4x + 3y = 8$$
$$3y = 4x + 8$$
$$y = \frac{4}{3}x + \frac{8}{3}$$

2. The slope of the given line is $\frac{4}{3}$, so the slope of a perpendicular line is the negative reciprocal of $\frac{4}{3}$. Flip the fraction and multiply by $-1$ to determine that the slope of a perpendicular line is $\frac{-3}{4}$.

3. Start converting each answer choice from Standard Form to Slope-Intercept Form until you find one with a slope of $\frac{-3}{4}$. Let's begin with choice A: $3x + 4y = 8$.

$$4y = -3x + 8$$
$$y = \frac{-3}{4}x + 2$$

4. Luckily, the first choice was the correct choice, but if you want to double check, you could put all of the other lines in Slope-Intercept Form. None of the other lines will have a slope of $\frac{-3}{4}$.

5. The answer is A.





**Solution 2**

1. Instead of converting to Slope-Intercept Form, use the fact that the slope of a line in Standard Form, $Ax + By = C$, is equal to $\frac{-A}{B}$. In the given equation, $-4x + 3y = 8$, we know that $A = -4$ and $B = 3$, so the slope is equal to $\frac{-(-4)}{3}$ or $\frac{4}{3}$.

2. The slope of the given line is $\frac{4}{3}$, so the slope of a perpendicular line is the negative reciprocal of $\frac{4}{3}$. Flip the fraction and multiply by $-1$ to determine that the slope of a perpendicular line is $\frac{-3}{4}$.

3. Find the slope of the lines in the answer choices using the same method in order to see which choice has a slope of $\frac{-3}{4}$. Let's begin with choice A: $3x + 4y = 8$.

4. For this equation, $A = 3$ and $B = 4$, so the slope of the line is equal to $\frac{-3}{4}$, which is the slope of a line perpendicular to the line $-4x + 3y = 8$, which has a slope of $\frac{4}{3}$.

5. The answer is A.

**Notes**

Once you become familiar with finding the equation of a line perpendicular to a given line with both lines in Standard Form, you'll start to think about looking for (or creating) an equation where the coefficients of $x$ and $y$ are swapped and one of them is negated; inspect the coefficients in the given line equation and in answer choice A to see this relationship. This procedure streamlines the solving process, because you never actually need to determine the slopes explicitly.

**Section 3.2 Suggested Problems from Real Tests**

- Test 2-NC-6
- Test 2-C-28
- Test 4-NC-8
- Test 5-C-11
- Test 9-NC-20
- Test 9-C-32
- May 2018-NC-16
- May 2018-C-35
- 2019-C-24
- 2020-C-28
- 2021-NC-10
- 2021 (US)-NC-7
- May 2021 (US)-C-20
- May 2021 (Int)-C-38

View related real-test problems in SMTEx

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## Section 3.2 Practice Problems

**1** 1600.io 3.1

Which of the following statements is true about the graph of the equation $3y + 5x = -12$ in the $xy$-plane?

A) It has a negative slope and a positive $y$-intercept.

B) It has a negative slope and a negative $y$-intercept.

C) It has a positive slope and a positive $y$-intercept.

D) It has a positive slope and a negative $y$-intercept.

**2** 1600.io 3.2

In the $xy$-plane, line $\ell$ has a slope of $-3$. If line $k$ is perpedicular to line $\ell$, which of the following could be an equation of line $k$ ?

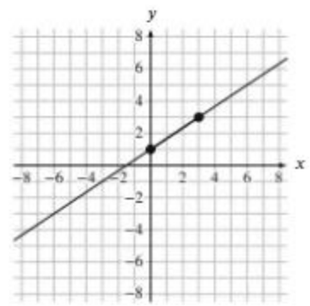
A) $-9x + 3y = 10$

B) $-2x + 6y = 12$

C) $-x - 3y = 4$

D) $12x + 4y = 7$

**3** 1600.io 3.2




A line is shown in the $xy$-plane above. A second line (not shown) is parallel to the line shown and passes through the points $(0, 3)$ and $(4, c)$, where $c$ is a constant. What is the value of $c$ ?





## CHAPTER 3 RECAP

- The Standard Form of Linear Equations is $Ax + By = C$.
- Standard Form can be converted into Slope-Intercept Form algebraically or by using the formula template $y = \frac{-A}{B}x + \frac{C}{B}$.
- The slope of a line in Standard Form is $\frac{-A}{B}$.
- The $y$-intercept of a line in Standard Form is $\frac{C}{B}$.
- Parallel lines are the same distance apart everywhere and thus they never intersect. Parallel Lines have the same slope.
- Perpendicular lines cross at a right angle. The slope of each line in a pair of perpendicular lines is the **negative reciprocal** of the slope of the other line. The negative reciprocal is found by swapping the values in the numerator and denominator of a slope and multiplying by $-1$.
- Parallel lines in Standard Form will have the same $x$- and $y$-coefficients (or the ratio of $x$- and $y$-coefficients will be the same).
- Perpendicular lines in Standard Form will have the $x$- and $y$-coefficients swapped, and one of the coefficients will be multiplied by $-1$ (or the ratio of the $x$- and $y$-coefficients will be flipped and multiplied by $-1$).

## Additional Problems

**1** 1600.io 3.1

The line with the equation $\frac{3}{4}x + \frac{1}{5}y = 1$ is graphed in the $xy$-plane. What is the $x$-coordinate of the $x$-intercept of the line?

**2** 1600.io 3.1

$$x + y = 20$$

The equation above relates the number of minutes, $x$, Kevin spends showering each day and the number of minutes, $y$, he spends eating breakfast each day. In the equation, what does the number 20 represent?

A) The number of minutes spent showering each day

B) The number of minutes spent eating breakfast each day

C) The number of minutes spent eating breakfast for each minute spent showering

D) The total number of minutes spent showering and eating breakfast each day

**3** 1600.io 3.1

Gaston bought two types of candies: red candies that cost \$0.60 each and green candies that each cost $z$ times as much as a red candy. If the cost of 3 red candies and 1 green candy was \$3, what is the value of $z$ ?

**4** 1600.io 3.2

In the $xy$-plane, line $k$ passes through the point $(5, 7)$ and is parallel to the line with the equation $y = \frac{7}{5}x - \frac{11}{5}$. What is the slope of line $k$ ?

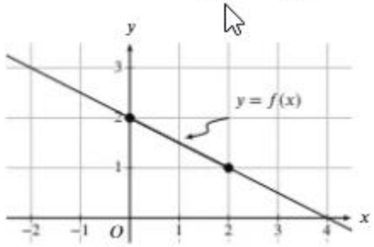




**5** 1600.io 3.1

The headmaster of a wizarding school awarded a total of 1200 bonus points to the most upstanding students. The points were awarded in amounts of 50 points or 200 points. If at least one 50-point bonus and at least one 200-point bonus were awarded, what is one possible number of 50-point bonuses awarded?

**6** 1600.io 3.2




The graph of the linear function $f$ is shown in the $xy$-plane above. The graph of the linear function $g$ (not shown) is perpendicular to the graph of $f$ and passes through the point $(2, 6)$. What is the value of $g(0)$ ?

**7** 1600.io 3.1

The graph of the equation $ax + ky = -4$ is a line in the $xy$-plane, where $a$ and $k$ are constants. If the line contains the points $(2, 0)$ and $(0, -2)$, what is the value of $k$ ?

A) $-4$

B) $-2$

C) $2$

D) $4$

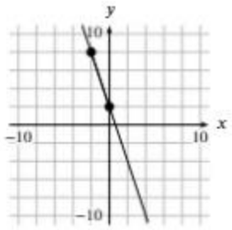
**8** 1600.io 3.2

Lines $z$ and $w$ are parallel in the $xy$-plane. The equation of line $z$ is $2x + 5y = 10$, and line $w$ passes through $(-4, 6)$. What is the value of the $y$-intercept of line $w$ ?

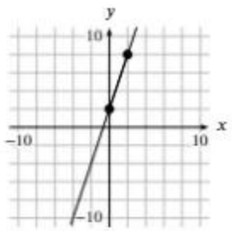
**9** 1600.io 3.1

Which of the following is the graph of $y + 3x = 2$ in the $xy$-plane?

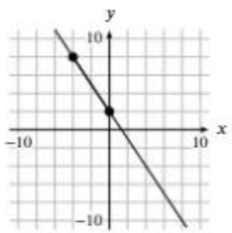
A)



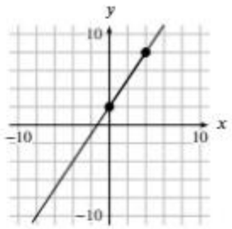

B)




C)




D)




**10** 1600.io 3.1

$$-3x + 5y = -2$$

The graph of the equation above in the $xy$-plane is a line. What is the $x$-coordinate of the $x$-intercept of the line?

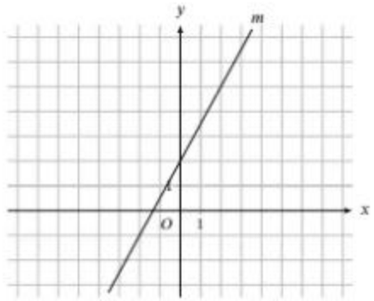
**11** 1600.io 3.2

In the $xy$-plane, line $\ell$ has a $y$-intercept of $-2$ and is perpendicular to the line with the equation $y = -\frac{3}{5}x$. If the point $(6, b)$ is on line $\ell$, what is the value of $b$ ?





**12** 1600.io 3.2




In the $xy$-plane above, line $m$ is perpendicular to line $\ell$ (not shown). Which of the following could be an equation of line $\ell$ ?

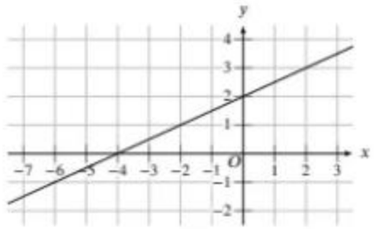
A) $2x + 3y + 6 = 0$

B) $2x - 3y + 6 = 0$

C) $3x - 2y + 2 = 0$

D) $3x + 2y + 2 = 0$

**13** 1600.io 3.1




An equation of the graph shown is $ax + by = -8$, where $a$ and $b$ are constants. What is the value of $b$ ?

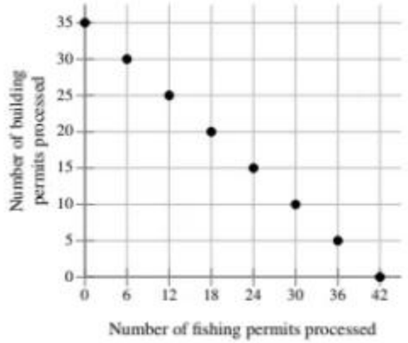
A) $-4$

B) $-2$

C) $2$

D) $4$

**14** 1600.io 3.1




For his job, Kelvin spent a total of $n$ minutes processing building permits and fishing permits. It takes Kelvin 30 minutes to process a building permit and 25 minutes to process a fishing permit. The graph above represents all possible combinations for the number of building permits and the number of fishing permits that Kelvin could have processed in the $n$ minutes. What is the value of $n$ ?

A) 1,470

B) 1,260

C) 1,050

D) 875

**15** 1600.io 3.1

The graph of a line in the $xy$-plane has a negative slope and intersects the $y$-axis at a point that has a positive $y$-coordinate. Which of the following could be an equation of the line?

A) $-7x + 4y = -10$

B) $-7x + 4y = 10$

C) $7x + 4y = -10$

D) $7x + 4y = 10$

**16** 1600.io 3.1

| $x$ | $y$ |
|---|---|
| $2b$ | 0 |
| $4b$ | $-b$ |
| $6b$ | $-2b$ |

Some values of $x$ and their corresponding values of $y$ are shown in the table above, where $b$ is a constant and $b \neq 0$. If there is a linear relationship between $x$ and $y$, which of the following equations represents the relationship?

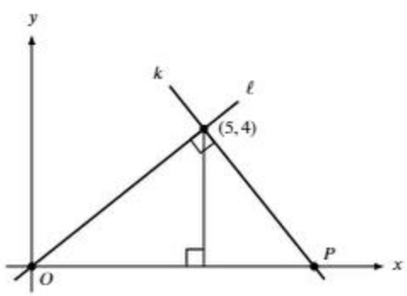
A) $x + 2y = b$

B) $x + 2y = 2b$

C) $2x - y = -6b$

D) $2x - y = -b$

**17** 1600.io 3.2




In the $xy$-plane above, lines $k$ and $\ell$ are perpendicular. What is the $x$-coordinate of point $P$ ?

A) 8.2

B) 8.25

C) 8.5

D) 8.8

**Questions 18 and 19 refer to the following information.**

Cost per Pound of Nuts

| Nut | Dollars | Euros |
|---|---|---|
| Cashew | 4.00 | 4.40 |
| Brazil | 3.00 | 3.30 |
| Macadamia | 4.00 | 4.40 |

The table above gives the typical costs per pound of nut, expressed in both dollars and euros, of the three nuts that a vendor sells.

**18** 1600.io 3.1

If $x$ dollars is equivalent to $k$ euros, of the following, which best represents the relationship between $x$ and $k$ ?

A) $k = 0.91x$

B) $xk = 1.1$

C) $k = 1.1x$

D) $x = 1.1k$

**19** 1600.io 3.1

If a client pays \$300 for a mixture of nuts consisting entirely of $c$ pounds of cashews, $b$ pounds of brazil nuts, and $m$ pounds of macadamia nuts, which of the following expresses $b$ in terms of $c$ and $m$ ?

A) $b = 100 - \frac{4}{3}(c + m)$

B) $b = 100 - \frac{4}{3}(c - m)$

C) $b = 100 + \frac{4}{3}(c + m)$

D) $b = 100 + \frac{3}{4}(c + m)$





# Answer Key

### SkillDrill 3.1-1

1. $2b + 3f = 56$
2. $h + 3c = 60$
3. $8s + 20b = 112$
4. $-100c - 2.50h = -197.50$
5. $5r - 10p = -42$

### SkillDrill 3.1-2

1. $y = 2x - 4$
2. $y = 3x + 7$
3. $y = \frac{-3}{2}x - \frac{7}{2}$
4. $y = \frac{-5}{6}x + 3$
5. $5x + y = -10$
6. $2x - y = -4$
7. $x + 3y = -2$
8. $5x - 2y = -14$

### Section 3.1 Practice Problems

1. B
2. 247
3. D
4. C

### SkillDrill 3.2-1

1. $y = \frac{3}{2}x + 4,\ 3x - 2y = -8$
2. $y = \frac{-2}{3}x + 4,\ 2x + 3y = 12$
3. $y = -2x - 2,\ 2x + y = -2$
4. $y = \frac{1}{2}x + \frac{21}{2},\ x - 2y = -21$
5. $y = \frac{-5}{4}x + \frac{41}{4},\ 5x + 4y = 41$
6. $y = \frac{4}{5}x - 6,\ 4x - 5y = 30$
7. $y = \frac{8}{5}x - \frac{3}{5},\ 8x - 5y = 3$
8. $y = \frac{-5}{8}x - \frac{5}{2},\ 5x + 8y = -20$

### Section 3.2 Practice Problems

1. B
2. B
3. $\frac{17}{3}$, 5.66, or 5.67

### Additional Problems

1. $\frac{4}{3}$ or 1.33
2. D
3. 2
4. $\frac{7}{5}$ or 1.4
5. 4, 8, 12, 16, or 20
6. 2
7. C
8. $\frac{22}{5}$ or 4.4
9. A
10. $\frac{2}{3}$, .666, or .667
11. 8
12. A
13. A
14. C
15. D
16. B
17. A
18. C
19. A

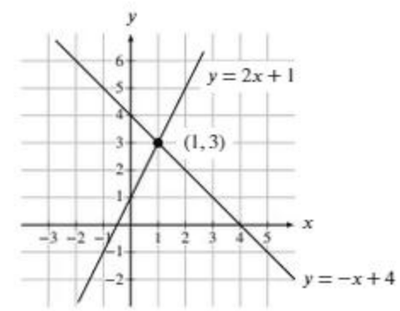
# Systems of Linear Equations

# 4

## 4.1 Introduction to Systems of Linear Equations

A **system of linear equations** is a collection of two linear equations with the same set of variables. In previous chapters, we talked about the information that can be gleaned from a single linear equation and its graph, but what happens when we have two lines in the same system?

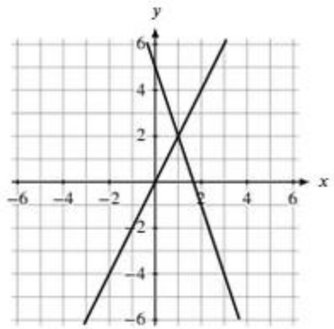
Let's look at a graph of two distinct lines: $y = 2x + 1$ and $y = -x + 4$.




Notice that the two lines intersect at the point (1, 3). This intersection point exists as a point on the graphs of both functions. Remember that the graph of a linear equation represents all of the points that are solutions to that linear equation. When two intersecting lines are graphed together to represent a system of linear equations, their intersection point gives one pair of $x$- and $y$-values that satisfies both equations. The intersection point is called a **solution to the system of equations** because it is a solution to both of the equations in the system.

**Example 4.1-1**

**1** 1600.io 4.1




The lines in the $xy$-plane above are the graphs of two linear equations. What is the solution $(x, y)$ to the system formed by the equations?

A) $(0, 5)$

B) $(0, 0)$

C) $(1, 2)$

D) $(2, 1)$

**Solution**

1. Since the solution point for a system of two linear equations is the point where the two lines intersect, we should check the gridlines to see if the intersection point is easily found.
2. Because the lines intersect at a meeting of gridlines, we can clearly see that the lines intersect at the point $(1, 2)$, so the $x$- and $y$-value pair that constitutes the solution to the system of linear equations is represented by the point $(1, 2)$.
3. The answer is C.

**Section 4.1 Suggested P**

- Mar 2018-NC-1
- Mar 2019-C-5

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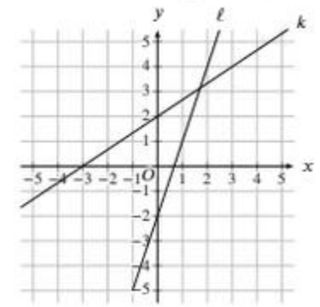
## Section 4.1 Practice Problems

**1** 1600.io 4.1




Lines $\ell$ and $k$ in the $xy$-plane above are the graphs of the equations in a system. How many solutions does the system of equations have?

A) None

B) One

C) Two

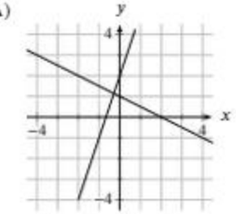
D) More than two

**2** 1600.io 4.1

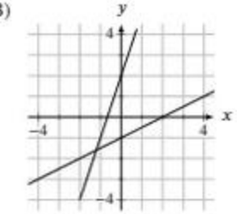
$$-x + 2y = -2$$
$$3x - y = -2$$

Which of the following graphs in the $xy$-plane could be used to solve the system of equations above?

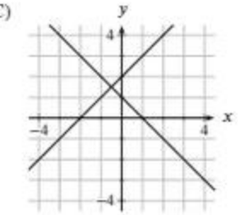
A)



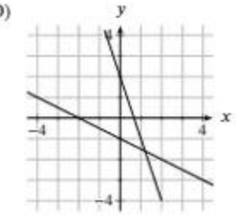

B)




C)




D)



## 4.2 Using Substitution to Solve Systems of Linear Equations

Let's come back to our opening example with the lines $y = 2x + 1$ and $y = -x + 4$. We determined from the graph that the point $(1, 3)$ existed on both lines. But suppose the lines were not graphed on a carefully laid out grid, and we were asked to find the solution to the system if we only had the equations themselves. Perhaps there is a more mathematical approach to solving this system than sloppily drawing the lines ourselves and hoping we could draw a perfectly proportioned grid that would enable us to correctly identify the solution to the system (**NEVER** attempt to solve system of equations problem this way).

Instead of simply considering the solution to a system of linear equations as an intersection point on a graph, we can also consider that (as mentioned earlier) a solution to the system is also a pair of $x$- and $y$-values that satisfies both equations. Algebraically, we can solve the system using substitution. Since we are looking for an $x$- and $y$-coordinate pair that is a solution to both lines, the same $x$- and $y$-values that satisfy one of the equations also satisfy the other equation. When we are looking for this solution, we know that the $y$-value is the same in both equations (and so are the $x$-values), so we can plug one in for the other, which is very easy when both equations are written in Slope-Intercept Form.

When both lines are in Slope-Intercept Form, as in our current example, they are both solved for $y$ in terms of $x$. For example, we know from one of our equations that $y$ must be equal to $2x + 1$, and we know from the other equation that $y$ must also be equal to $-x + 4$.

Here's what we're given:

$$\begin{aligned} y &= 2x + 1 \\ y &= -x + 4 \end{aligned}$$

Let's just rewrite the first equation by switching the sides of the terms:

$$2x + 1 = y$$

Arranging our two equations:

$$\begin{aligned} 2x + 1 &= y \\ y &= -x + 4 \end{aligned}$$

Now, it should be apparent that

$$2x + 1 = -x + 4$$

because equality is transitive: generally, if we're given

$$\begin{aligned} y &= u \\ y &= v \end{aligned}$$

we know that

$$u = v$$

Sometimes, this procedure is casually referred to as "setting the equations equal to each other," but that's not really right: we're not setting the two *equations* equal; we're setting the two *expressions* that are on the opposite side of the equals sign from $y$ equal to each other, so be sure you understand that.





This procedure can equivalently be thought of as **substituting** an expression in terms of $x$ for $y$ in an equation, which has the effect of eliminating $y$ from the equation, leaving $x$ as the only variable. So, we have

$$y = 2x + 1$$

We also have

$$y = -x + 4$$

so we can substitute $-x + 4$ for $y$ in the first equation:

$$\begin{aligned} y &= 2x + 1 \\ -x + 4 &= 2x + 1 \end{aligned}$$

Of course, we could do the same thing but switching the equations, so we could substitute $2x + 1$ for $y$ in the second equation:

$$2x + 1 = -x + 4$$

The result is the same, except the sides of the expressions are swapped, which has no mathematical importance. Let's proceed with the solving using the first unified equation we made.

Now that the two expressions are set equal to each other, we can collect and combine all the $x$-terms on one side of the equation and all the constant terms on the other side of the equation in order to solve for $x$.

$$\begin{aligned} -x + 4 &= 2x + 1 \\ 4 &= 3x + 1 \\ 3 &= 3x \\ 1 &= x \end{aligned}$$

By setting the two expressions that represent $y$ equal to each other or, equivalently, by substituting the expression that gives the value of $y$ from one equation in for $y$ in the other equation, we have determined the $x$-value of the solution point. Now that we know that $x = 1$ is *part* of the solution, we can **substitute this value for $x$ in the equation for either line** (remember that the solution to the system is the coordinate pair that exists on **both** lines in the system) in order to find the $y$-value of the solution.

$$\begin{aligned} y &= 2x + 1 \\ y &= 2(1) + 1 \\ y &= 2 + 1 \\ y &= 3 \end{aligned}$$

By using substitution (instead of locating the intersection point on a graph), we have found the pair of $x$- and $y$-variables that form the solution to the system of equations.

**SkillDrill 4.2-1**

**Directions:** Use substitution to find the intersection point that is the solution to the system of linear equations.

**1.** $y = 8x + 2$, $y = -2x + 12$

**2.** $y = -3x$, $y = 2x + 25$

**3.** $y = x - 7$, $y = 3x - 14$

**4.** $y = 9x + 6$, $y = -3x + 5$

In the following example, we will solve for the solution to a system of linear equations by setting the expressions representing $y$ equal to each other (we will have to write the second equation based on the question prompt before we can solve).

**Example 4.2-1**

**1** 1600.io 4.2

$$y = x + 4$$

An equation of line $\ell$ in the $xy$-plane is shown above. Another line, $k$, has a slope equal to triple the slope of $\ell$ and a $y$-intercept equal to triple the $y$-intercept of $\ell$. At which point $(x, y)$ do lines $\ell$ and $k$ intersect?

A) $(-4, 0)$

B) $\left(-\frac{1}{4}, 0\right)$

C) $\left(0, \frac{1}{4}\right)$

D) $(0, 4)$

**Solution**

1. Remember, the point of intersection of two lines represents the solution to the system the lines represent. Write the equation of the second line in the system based on the information given in the question. The slope of line $k$ is triple (three times) the slope of line $\ell$, and as the slope of line $\ell$ is 1, the slope of line $k$ is 3. The $y$-intercept of line $k$ is triple the $y$-intercept of line $\ell$, and as the $y$-intercept of line $\ell$ is 4, the $y$-intercept of line $k$ is 12. Since its slope is 3 and its $y$-intercept is 12, the equation of line $k$ is $y = 3x + 12$.

2. Both equations are solved for $y$ in terms of $x$. Since the $y$-values have to be equal at the intersection point, we can use substitution and set the right sides of both equations equal to each other.

$$x + 4 = 3x + 12$$

3. Combine the $x$-terms on one side of the equation and the constant terms on the other side of the equation, then solve for $x$.

$$\begin{aligned} x + 4 &= 3x + 12 \\ 4 &= 2x + 12 \\ -8 &= 2x \\ -4 &= x \end{aligned}$$

4. There is only one choice in which the $x$-value of the point is $-4$, so there is no need to solve for the $y$-value of the intersection point (by plugging the $x$-value into either of the original equations in the system), but you can if you like to play it safe.

$$\begin{aligned} y &= x + 4 \\ y &= -4 + 4 \\ y &= 0 \end{aligned}$$

5. The answer is A.





It is not necessary for both equations to be in Slope-Intercept Form (or, more generally, to be solved in terms of the same variable, whether it's $x$ or $y$) in order to use substitution. **Whenever at least one of the equations in a system is solved for one variable in terms of another (one variable is isolated on one side of the equation), substitution will probably be the best method of solution** (we will discuss other cases later in the chapter). In the next example, neither linear equation is written in Slope-Intercept Form. DON'T PANIC; both of the equations are still linear, and we can still use substitution even when the linear equations are not written in either of the traditional forms of linear equations that we have discussed.

**Example 4.2-2**

**2** 1600.io 4.2

$$\begin{aligned} \frac{1}{4}(x + 3y) &= \frac{9}{4} \\ x &= 3y \end{aligned}$$

The system of equations above has solution $(x, y)$. What is the value of $y$?

**Solution**

1. Start by getting rid of the fractions in the first equation (a smart first step in most situations). Multiply both sides of the first equation by 4.

$$\begin{aligned} \cancel{4}\left(\frac{1}{\cancel{4}}\right)(x + 3y) &= \cancel{4}\left(\frac{9}{\cancel{4}}\right) \\ x + 3y &= 9 \end{aligned}$$

2. Since the second equation tells us that $x$ is equal to $3y$, substitute $3y$ in for $x$ in the simplified version of the first equation $(x + 3y = 9)$, then solve for $y$.

$$\begin{aligned} x + 3y &= 9 \\ (3y) + 3y &= 9 \\ 6y &= 9 \\ y &= \frac{9}{6} \\ y &= \frac{3}{2} \end{aligned}$$

3. The answer is $\frac{3}{2}$.

**Notes**

It was not essential that the fractions be eliminated as the first step; you could have started by substituting $3y$ for $x$ in the first equation, which would result in $\frac{1}{4}(6y) = \frac{9}{4}$. At that point, you could multiply through by 4 to get $6y = 9$ and then proceed with solving for $y$ from there.

**Section 4.2 Suggested Problems from Real Tests**

- Test 1-NC-11
- Test 2-NC-9
- Test 2-C-34
- Test 3-C-18
- Test 3-C-24
- Test 4-NC-3
- Test 4-C-6
- Test 5-NC-18
- May 2017-NC-4
- May 2017-C-34
- May 2019 (US)-NC-16
- Oct 2019-C-11
- Mar 2020-NC-17
- Oct 2020-NC-11
- Oct 2020-C-30
- Apr 2021-C-2

View related real-test problems in SMTEx

SMTEx

## Section 4.2 Practice Problems

**1** 1600.io 4.2

$$7x + y = 53$$
$$x = 6$$

If $(x, y)$ is the solution to the given system of equations, what is the value of $y$ ?

**2** 1600.io 4.4

$$x - 4y = 16$$
$$4y = 12$$

If $(x, y)$ is the solution to the system of equations above, what is the value of $x$ ?

A) $-4$

B) $4$

C) $28$

D) $64$

**3** 1600.io 4.2

$$x + 5y = 16$$
$$y = 3x$$

If the ordered pair $(x, y)$ satisfies the system of equations above, what is the value of $x$ ?

**4** 1600.io 4.2

In the $xy$-plane, the graph of $y = 3x - 2$ intersects the graph of $y = 2x + 4$ at the point $(a, b)$. What is the value of $a$ ?

A) $-2$

B) $2$

C) $6$

D) $16$

**5** 1600.io 4.2

The graph of a line in the $xy$-plane has slope 4 and contains the point $(1, 10)$. The graph of a second line passes through the points $(1, 4)$ and $(2, 5)$. If the two lines intersect at the point $(a, b)$, what is the value of $b - a$ ?

A) $-3$

B) $-1$

C) $2$

D) $3$

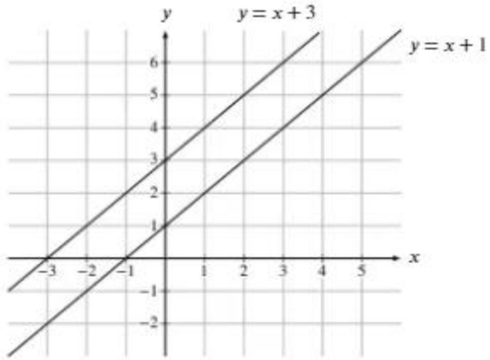




## 4.3 Number of Solutions to Systems of Linear Equations Part I

We know that a solution to a system of two linear equations is an $x$- and $y$-value pair that satisfies both linear equations, and the graphical representation of that solution is a point with that $x$- and $y$-value that lies on both lines. **When the lines intersect at a point, a single solution to the system exists**. But what happens if the lines in a system never intersect? There can be no solution to the system because there is no common point that exists on both lines.

**The only way for two lines to never intersect (that is, for the system of equations to have no solution) is for them to be parallel**: the two lines must have **the same slope and different $y$-intercepts**. For example, let's consider the lines $y = x + 1$ and $y = x + 3$.




There are no $x$- and $y$-coordinate pairs that exist on both lines (there is no intersection point). Therefore, there is no solution to the system of equations.

If we tried to solve this system using substitution to create the equation $x + 1 = x + 3$, we would arrive at an impossible situation (which indicates that there is no solution) when we attempt to solve for $x$. When we subtract $x$ from both sides of the equation (in an attempt to group the $x$-terms on one side of the equation), we end up with no $x$ terms and have the irreconcilable, untrue equation $1 = 3$. Since 1 is obviously never equal to 3, there is no value of $x$ for which $x + 1$ equals $x + 3$; there is no solution to the system of equations (no matter what value of $x$ we plug into the equations).

Most of the time, problems about systems of linear equations with no solutions are disguised in the form of a single equation with only one variable. For example, if we are given the equation

$$ax + 5 = 2x + 3$$

and told that there are no solutions, and we are asked to find the value of $a$, then we need to understand that **this is actually a system of equations problem, but the first step of substitution is already done for us** (the expressions representing $y$ for two separate linear equations have already been set equal to each other).

Essentially, it is as if we have been given a system of linear equations in which one equation is $y = ax + 5$ (based on the left side of the equation) and the other equation is $y = 2x + 3$ (based on the right side of the equation). If these two lines have no solution, then they must be parallel. **In order for the two lines to be parallel, their slopes must be the same, and their $y$-intercepts must be different.**

Since the slope of the first line is $a$, and the slope of the second line is 2, it must be true that $a = 2$ in order for the lines to have the same slope.

In essence, you do not have to actually split the equation into two separate equations; you simply need to **match the $x$-coefficients**

**(the slopes) on both sides of the equation** (and verify that the constants, or $y$-intercepts, are not the same).

Indeed, if $a = 2$, we can see that plugging 2 in for $a$ in the original single equation given to us, $ax + 5 = 2x + 3$, will once again produce an impossible situation.

$$ax + 5 = 2x + 3$$
$$(2)x + 5 = 2x + 3$$
$$5 = 3????$$

**SkillDrill 4.3-1**

**Directions:** All of the following equations have no solution. Find the value of $a$.
Challenge questions are boxed.

**1.** $ax + 2 = 4x - 7$

**2.** $x + 2 = 2a(x - 7)$

**3.** $a(3x + 2) + 2x = 7x - 7$

**4.** $\boxed{ax + 2 = -7}$

**Example 4.3-1**

**1** 1600.io 4.3

$$a(-4x + 2) + 3x = 4x - 1$$

The equation above has no solutions, and $a$ is a constant.
What is the value of $a$ ?

A) $-1$

B) $-\frac{1}{2}$

C) $-\frac{1}{4}$

D) $1$

**Solution**

1. Our goal is to get both sides of the equation into the form $mx + b$ (as if the $y$-expressions for two lines have already been set equal to each other) so we can simply match the $x$-coefficients (and verify that the constants are not the same).

2. Distribute $a$ on the left side of the equation.

$$a(-4x + 2) + 3x = 4x - 1$$
$$-4ax + 2a + 3x = 4x - 1$$





3. Subtract $3x$ from both sides of the equation to group the $x$-terms with numerical coefficients on the right side of the equation.

$$-4ax + 2a + 3x = 4x - 1$$
$$-4ax + 2a = x - 1$$

4. In order for there to be no solutions, the $x$-coefficients (which represent the slopes) must be the same (and the constants must not be equal). Set the $x$-coefficients ($-4a$ and 1) equal to each other to solve for $a$.

$$-4a = 1$$
$$a = -\frac{1}{4}$$

5. The answer is C.

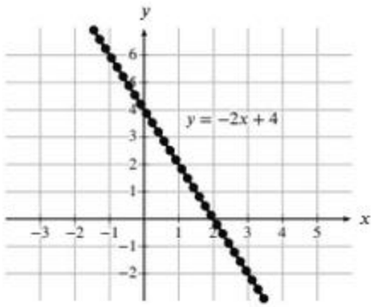
**Notes**

You can double check the value of $a$ by making sure that the constant terms (the $y$-intercepts) DO NOT match (if the slopes were the same and the $y$-intercepts were the same, the lines would be the same line instead of parallel lines). If $a = \frac{-1}{4}$, then the constant on the left side of the equation is $2\left(\frac{-1}{4}\right)$ or $\frac{-1}{2}$, which is not equal to the constant term on the other side of the equation ($-1$). The value we found for $a$ passes this test, so we know that the two lines are parallel and are not the same line.

In Step 3, we subtracted $3x$ from both sides in order to avoid combining the $x$-coefficients ($-4a$ and 3) and factoring out $x$. This would not be wrong but is probably confusing for some students. In general, you want to try to keep unknown constants such as $a$, $b$, or $c$ on one side of the equation (separate from numerical constants) when you have to match coefficients.

We've seen that a system of equations can have one solution where there's a single pair of values $(x, y)$ that satisfies both equations, and we showed that the graphical representation of this is the point of intersection of the two lines that represent the equations. If the lines do not cross anywhere—they are parallel—there's no point that lies on both lines, and thus there is no solution.

**There is only one other relationship between two linear equations: they're actually the same, and thus they produce the same line when graphed; such lines are called coincident lines. In that case, every solution to one of the equations is also a solution to the other equation; graphically, this is evident because the two lines are in fact the same, so every point lies on both, and is thus a solution to both.** Because there are an infinite number of points on a line, this means that **there are infinitely many solutions to the system of equations in which the lines are exactly the same**. In the graph below, the line $y = -2x + 4$ is extra thick to show that the same line has been graphed twice (one on top of the other) and several of the infinite solution points are shown along the line.



If we were given a system of linear equations in which both equations were the same, say $y = -2x + 4$ and $y = -2x + 4$ (it should be obvious the lines are the same if they are given in the same form, but let's continue anyway), we could try to solve the system with substitution.

$$-2x + 4 = -2x + 4$$

When we try to group the $x$-terms, they will once again cancel each other (as with a system of parallel lines), but this time we will be left with the undeniably true equation $4 = 4$. Since 4 is obviously always equal to 4 no matter what the value of $x$ is, there are infinitely many solutions to the system of equations—no matter what value of $x$ or $y$ we plug into the equations, the two lines will always be equal because they are the same line.

Most of the time, problems about systems of linear equations with infinitely many solutions are disguised in the form of a single equation with only one variable. For example, if we are given the equation $ax + b = -3x + 5$ and told that there are infinitely many solutions, and we are asked to find the value of $a$, we need to understand that this is actually a systems of equations problem, but the first step of substitution is already done for us (the expressions that equal $y$ for two separate linear equations have already been set equal to each other).

Essentially, it is as if we have been given a system of linear equations in which one equation is $y = ax + b$ (based on the left side of the equation) and the other equation is $y = -3x + 5$ (based on the right side of the equation). If these two lines have infinitely many solutions, then they must be exactly the same line. **In order for the two lines to be the same, their slopes *and* their $y$-intercepts must be the same.**

Since the slope of the first line is $a$, and the slope of the second line is $-3$, it must be true that $a = -3$ in order for the lines to have the same slope. Since the $y$-intercept of the first line is $b$, and the $y$-intercept of the second line is 5, it must be true that $b = 5$ for the lines to have the same $y$-intercept.

In essence, you do not have to actually split the equation into two separate equations; you simply need to **match the $x$-coefficients (the slopes) and the constant terms (the $y$-intercepts) on both sides of the equation when dealing with a system that has infinitely many solutions**.

Indeed, if $a = -3$ and $b = 5$, we can see that plugging $-3$ in for $a$ and 5 in for $b$ in the original single equation given to us, $ax + b = -3x + 5$, will once again produce a statement that is always true for any value of $x$.

$$ax + b = -3x + 5$$
$$-3x + 5 = -3x + 5$$
$$5 = 5$$

**SkillDrill 4.3-2**

**Directions:** Each of the following equations has infinite solutions. Find the values of $a$ and $b$, reducing any fractions to lowest terms.

**1.** $ax - b = 4x - 7$ **2.** $a(x + 2) = 6x + b$ **3.** $ax + 2 = b(3x - 1)$ **4.** $\boxed{abx + b = 3(5x + 3)}$





**Example 4.3-2**

**2** 1600.io 4.3

$$ax - 5(4 - 3x) = -20$$

In the equation above, $a$ is a constant. For what value of $a$ does the equation have infinitely many solutions?

A) $-15$

B) $-3$

C) $3$

D) $5$

**Solution**

1. Our goal is to get both sides of the equation into the form $mx + b$ (as if the expressions representing $y$ for two lines have already been set equal to each other) so we can simply match the $x$-coefficients and constant terms on both sides of the equation (satisfying the conditions for infinitely many solutions).
2. Distribute $-5$ on the left side of the equation.

$$ax - 5(4 - 3x) = -20$$
$$ax - 20 + 15x = -20$$

3. Subtract $15x$ from both sides of the equation in order to have an $x$-term on both sides of the equation.

$$ax - 20 + 15x = -20$$
$$ax - 20 = -15x - 20$$

4. In order for there to be infinitely many solutions, the $x$-coefficients ($a$ and $-15$), which represent the slopes, must be the same (so too must the constants, which, as expected, already match). Therefore $a = -15$.
5. The answer is A.

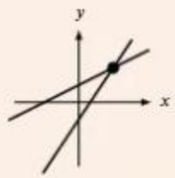
**Notes**

In Step 3, we subtracted $15x$ from both sides in order to avoid combining the $x$-coefficients ($a$ and 15) and factoring out $x$. This would not be wrong but is probably confusing for some students. In general, you want to try to keep unknown constants such as $a$, $b$, or $c$ on one side of the equation (separate from numerical constants) when you have to match coefficients.

### Systems of Linear Equations with One Solution

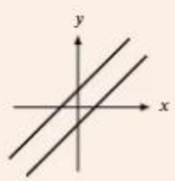
**In a system of equations with one solution, the lines intersect at one point**, with the coordinates of the intersection point representing the $x$- and $y$-values of the solution. Remember, a solution represents a pair of $x$- and $y$-values that satisfies both equations, which, in graphical terms, means these values define a single point that lies on both lines; a point that's on both lines is a point of intersection.

***Any* two lines that have different slopes will intersect once.** Note that the lines do not have to be perpendicular.




### Systems of Linear Equations with No Solution

**In a system of equations with no solutions, the lines are parallel, so they do not intersect.** Because a solution is an $x$-value and a $y$-value that satisfies both linear equations, and the graphical representation of that solution is a point with that $x$- and $y$-value that lies on both lines, for a single solution to exist, the lines must intersect, with the point of intersection representing that solution. The only way two lines can never intersect is for the two lines to be parallel; **to be parallel, two lines must have the *same* slope and they must have *different* $y$-intercepts** (otherwise they'd be the same lines).




When the lines representing two linear equations are parallel, you will see when inspecting the equations in Slope-Intercept Form that the lines will have the same slope, but different $y$-intercepts. For example, the lines $y = x + 1$ and $y = x - 1$ are parallel and thus will never intersect because they have the same slope but different $y$-intercepts.

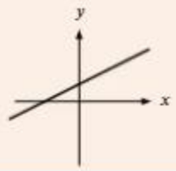




### Systems of Linear Equations with Infinite Solutions

**In a system of equations with infinitely many solutions, the lines must be exactly the same**, and thus they produce the same line when graphed. Every solution to one of the equations is also a solution to the other equation; graphically, this is evident because the two lines are in fact the same (they're coincident), so every point on one line is common to the other line and is thus a solution to both. Because there are an infinite number of points on a line, this means that there are infinitely many solutions to the system of equations in this situation.

When two linear equations in Slope-Intercept Form are the same, you will see that **the lines will have the *same* slope and *same* $y$-intercept**. In the following picture, it may look like only one line is graphed, but in reality, the same line is graphed twice, and the two lines are actually drawn on top of each other.




### Section 4.3 Suggested Problems from Real Tests

- Test 7-C-9
- Test 9-NC-8
- Test 10-C-35
- Apr 2018-NC-10
- May 2018-NC-4
- May 2018-C-28
- Mar 2019-C-27
- May 2019 (US)-NC-1
- May 2019 (Int)-NC-17
- Mar 2020-C-29
- Oct 2020-NC-13
- Mar 2021-NC-20
- Apr 2021-NC-13
- 2021-C-34
- 2021 (US)-NC-14
- 2021 (US)-C-33
- 2021 (Int)-NC-12
- May 2021 (Int)-C-27

View related real-test problems in SMTEx

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## Section 4.3 Practice Problems

**1** 1600.io 4.3

$$y = -\frac{1}{3}x + 1$$
$$y = ax - 4$$

In the system of equations above, $a$ is a constant. If the system of equations has no solution, what is the value of $a$?

A) $-3$

B) $-\frac{1}{3}$

C) $0$

D) $3$

**2** 1600.io 4.3

$$4k(x - 6) = x - 6$$

In the equation above, $k$ is a constant. If the equation has infinitely many solutions, what is the value of $k$?

**3** 1600.io 4.3

$$3ax - 12 = 2(x + 4) + 4(x - 3)$$

In the equation above, $a$ is a constant. If no value of $x$ satisfies the equation, what is the value of $a$?

A) 0

B) 1

C) 2

D) 3

**4** 1600.io 4.3

$$5(x + b) = ax + 2c$$

In the equation above, $a$, $b$, and $c$ are constants. If the equation has infinitely many solutions, which of the following must be equal to $c$?

A) $5a$

B) $5b$

C) $\frac{5a}{2}$

D) $\frac{5b}{2}$





# 4.4 Solving Systems of Linear Equations by Elimination/Combination

The intersection of two graphs is one way to think about the solution to a system of linear equations. However, it is probably more useful to think about the solution to a system of equations as an ordered pair $(x, y)$ of values that satisfies both equations because most systems of linear equations on the test are based on the Standard Form of lines and do not lend themselves to graphing and finding intersections (unlike lines in Slope-Intercept Form).

We will discuss the methods of solving systems of linear equations shortly, but for now let's briefly focus on writing our own system of linear equations in Standard Form (rather than in Slope-Intercept Form as in the previous example).

For many of the problems on the test, you need to write your own system of linear equations in Standard Form. Most of these are found in the form of word problems about how much money is made from selling a certain amount of two different items. The following example reflects something you are sure to see on every test. You will have to write one equation that represents the total number of items sold and one equation that represents the amount of money made from selling those things. It is easiest to write these equations in Standard Form based on how they are presented (not to mention that the answer choices will almost always be in Standard Form).

**Example 4.4-1**

**1** 1600.io 4.4

A butcher sold 89 pounds of meat that consisted of $b$ pounds of beef and $p$ pounds of pork. The butcher sold the beef for \$2.39 per pound and the pork for \$1.39 per pound and collected a total of \$172.71. Which of the following systems of equations can be used to find the number of pounds of beef that were sold?

A) $b + p = 172.71$
$2.39b + 1.39p = 89$

B) $b + p = 89$
$2.39b + 1.39p = 172.71$

C) $b + p = 89$
$1.39b + 2.39p = 172.71$

D) $b + p = 172.71$
$1.39b + 2.39p = 89$

**Solution**

1. We need to write one equation that describes the total number of pounds of meat sold. We know that $b$ is the number of pounds of beef, $p$ is the number of pounds of pork, and 89 is the total number of pounds of meat sold. So one of the equations is the following:

$$b + p = 89$$

2. We also need to write an equation that describes the amount of money the butcher made. Since beef is sold for \$2.39 per pound and $b$ pounds of beef are sold, the total amount of money made from beef is $2.39b$. Since pork is sold for \$1.39 per pound and $p$ pounds of pork are sold, the total amount of money made from pork is $1.39p$. Add these two amounts and set them equal to 172.71, which is the total amount of money the butcher made from selling this meat.

$$2.39b + 1.39p = 172.71$$

3. The two equations that describe the number of pounds of meat sold and the amount of money made from selling the meat are the following:

$$\begin{aligned} b + p &= 89 \\ 2.39b + 1.39p &= 172.71 \end{aligned}$$

4. The answer is B.

In the previous example, we only had to write the system of linear equations, but notice that the question indicated that we were writing these equations in order to hypothetically solve for the number of pounds of beef that were sold.

How would we actually solve one of these systems of linear equations (find the values that satisfy both of the equations in the system)? In general, the process of solving systems of equations having two variables is performed by reducing the system of equations to a single equation with only one variable (and temporarily eliminating the other variable). Once we know the value of one of the variables, we can substitute the value back into either of the original equations (remember that the solution values will satisfy both of the original equations) and solve for the value of the remaining variable.

There are several ways to isolate the single variable and reduce the system to a single equation (namely, Elimination and Substitution), but the majority of the system of linear equations questions on the test (especially those in which the equations are written in Standard Form) can and should be solved using the **Elimination** method, which is based on the procedure of adding and subtracting entire linear equations; this technique can be leveraged to eliminate one of the variables. Before we outline the method, let's first understand why we can combine linear equations with addition or subtraction.





## Combining Equations

Equations can be added together or subtracted from each other because both sides of an equation are equal. If we add the left sides of two equations and we add the right sides of two equations, the resulting equation will still be valid.

$$\begin{array}{rr} & a + b = 4 \\ + & c + d = 3 \\ \hline & (a+b) + (c+d) = 7 \end{array}$$

Since $a + b$ is 4 and $c + d$ is 3, then we can substitute 4 for $a + b$ and 3 for $c + d$ and verify that the result is 7.

$$\begin{array}{ccccc} (a+b) & + & (c+d) & = & 7 \\ 4 & + & 3 & = & 7 \\ & 7 & & = & 7 \end{array}$$

Note also that we can multiply both sides of any equation by a constant and the resulting equation is still the same equation as before, just with scaled coefficients and constant. Multiplying equations by constants is sometimes a necessary step in solving by elimination.

In order to solve linear equations by Elimination, we need to build off of the notion that we can combine linear equations by addition and subtraction. **Our goal is to add the equations in such a way that one of the variables will be eliminated**.

## Solving Systems of Linear Equations by Elimination

Consider the following system of linear equations.

$$\begin{aligned} x + 2y &= 5 \\ x - 2y &= 3 \end{aligned}$$

Notice that the coefficients of $y$ in the two equations are negatives of each other (the coefficient is 2 in the top equation and $-2$ in the bottom equation). If we add the two equations, then the $y$-terms will cancel each other out (their coefficients will sum to 0), eliminating the $y$-variables and leaving us with a single equation with only one variable remaining (the $x$-variable). We can use this new equation to solve for $x$.

$$\begin{array}{rr} & x + 2y = 5 \\ + & x - 2y = 3 \\ \hline & 2x + 0y = 8 \\ & \Downarrow \\ & 2x = 8 \\ & x = 4 \end{array}$$

Now that we have the value of $x$, we can plug it into either of the original two equations and solve for $y$. Let's use the first equation, $x + 2y = 5$.

$$\begin{aligned} x + 2y &= 5 \\ 4 + 2y &= 5 \\ 2y &= 1 \\ y &= \frac{1}{2} \end{aligned}$$

---

We sometimes need to multiply one or both equations in a system by a constant so that one of the variables will be canceled out when we add the the two equations to each other.

Consider the following system of linear equations.

$$\begin{aligned} 2x + 3y &= 6 \\ x - y &= 3 \end{aligned}$$

Let's solve for $x$ by eliminating the $y$-variables. In order to eliminate the $y$-variable and solve for $x$, we need the $y$-coefficients to cancel each other. The easiest way to do that in this case is to multiply the bottom equation by 3 so that the top equation has the term $3y$ and the bottom equation has the term $-3y$. When we add the equations together, the $y$-terms will cancel.

$$\begin{aligned} 2x + 3y &= 6 \\ 3(x - y) &= 3(3) \\ &\Downarrow \\ 2x + 3y &= 6 \\ 3x - 3y &= 9 \end{aligned}$$

Now, we can add the two equations, and the $y$-variables will cancel, allowing us to solve for $x$.

$$\begin{array}{rl} & 2x + 3y = 6 \\ + & 3x - 3y = 9 \\ \hline & 5x + 0y = 15 \\ & \Downarrow \\ & 5x = 15 \\ & x = 3 \end{array}$$

We could have also eliminated the $y$ terms by multiplying the top equation by $\frac{1}{3}$, but we chose the easier route of working with whole numbers instead of fractions. When it comes to systems of equations, working with whole-number coefficients is almost always the better option.

We could also have solved directly for $y$ by multiplying both sides of the second equation by $-2$ in order for the $x$-coefficients to cancel when the equations are added, thus using the elimination strategy in order to solve for each variable separately instead of solving for one using elimination and then using substitution to find the other as above.





In the following example, we will find the solution of a system using Elimination, but we will have to multiply *both* of the equations by constants in order to eliminate a variable. As a generalized procedure that will allow you to eliminate a variable, multiply the first equation by the coefficient of that variable in the second equation, and multiply the second equation by the coefficient of that variable in the first equation; this will result in that variable having the same coefficient in both equations, allowing that variable to be eliminated (if one coefficient is $a$ and the other is $b$, you'll end up with coefficients $ab$ and $ba$, which have the same value because multiplication is commutative). If both of the coefficients have the same sign, multiply one of the equations by $-1$ as well before adding the two equations so the variable terms will cancel each other when summed.

**Example 4.4-2**

**2** 1600.io 4.4

$$\begin{aligned} -5x + 2y &= 10 \\ 4x + 5y &= 25 \end{aligned}$$

If $(x, y)$ is the solution to the system of equations above, what is the value of $x$?

**Solution**

1. Since we want to solve for $x$, we need to eliminate the $y$-terms. To do that while avoiding fractions as much as possible, we need to multiply both equations by constants that will cause the $y$-coefficients to be opposites that will cancel when we add the two equations. Multiply the **top** equation by 5 (which is the $y$-coefficient in the **bottom** equation) and multiply the **bottom** equation by 2 (which is the $y$-coefficient in the **top** equation) so that the $y$-coefficients will both be 10.

$$\begin{aligned} 5(-5x + 2y) &= 5(10) \\ 2(4x + 5y) &= 2(25) \\ &\Downarrow \\ -25x + 10y &= 50 \\ 8x + 10y &= 50 \end{aligned}$$

2. We need to eliminate the $y$-variables, so multiply the top equation by $-1$ so the coefficients of the $y$-terms will cancel when they are added (we could have multiplied the bottom equation by $-1$, but we notice that if we use the top equation, the sign of the resulting $x$ term will be positive, which is always preferred).

$$\begin{aligned} -1(-25x + 10y) &= -1(50) \\ 8x + 10y &= 50 \\ &\Downarrow \\ 25x - 10y &= -50 \\ 8x + 10y &= 50 \end{aligned}$$

3. Combine the equations and solve for $x$.

$$\begin{array}{rl} & 25x - 10y = -50 \\ + & 8x + 10y = 50 \\ \hline & 33x + 0y = 0 \\ & \Downarrow \\ & 33x = 0 \\ & x = 0 \end{array}$$

4. The answer is 0.

**Notes**

Steps 1 and 2 could have been combined. If you notice that both of the coefficients of the variable you want to eliminate have the same sign (both positive or both negative), include a factor of $-1$ when multiplying one of the equations by the coefficient of the variable in the other equation. For example, in this problem, we could have multiplied the top equation by $-5$ instead of 5 if we kept in mind that we want the $y$-coefficients to be negatives of each other.

Alternatively, you could forego the multiplication by $-1$ and simply subtract the equations rather than adding them; just take care that you keep all the signs straight during the operation.

**SkillDrill 4.4-1**

**Directions:** Use elimination to find the intersection point that is the solution to the system of equations.

**1.** $2x + y = 6$
$x + y = 5$

**2.** $x - 3y = 4$
$x + 6y = 13$

**3.** $-2x - 3y = 13$
$4x + 9y = -35$

**4.** $4x + 5y = 2$
$2x - 3y = 1$

**Example 4.4-3**

**3** 1600.io 4.4

A group of 140 people are going to a wedding, where there are 30 tables. Some of the tables seat 4 people each, and the rest seat 6 people each. Assuming all the seats are filled and every person gets to sit in a seat, exactly how many of the tables were 4-person tables?

A) 10

B) 15

C) 20

D) 24

**Solution**

1. We need to write an equation that represents the total number of tables. Use the letter $f$ to represent the number of 4-person tables and $s$ to represent the number of 6-person tables. Since $f + s$ is the total number of tables, and there are 30 total tables, our first equation is $f + s = 30$.

2. We need to write an equation that represents the total number of seats. Since there are $f$ tables that seat 4 people, the number of seats from those $f$ tables is $4f$. Since there are $s$ tables that seat 6 people, the number of seats from those $s$ tables is $6s$. The total number of seats is $4f + 6s$, and there are 140 seats, so our second equation is $4f + 6s = 140$.

3. The system of equations is the following:

$$\begin{aligned} f + s &= 30 \\ 4f + 6s &= 140 \end{aligned}$$





4. Since we want to solve for $f$, we should multiply both sides of the top equation by $-6$ so the $y$-coefficients will cancel when we combine the equations.

$$\begin{aligned} -6(f + s) &= -6(30) \\ 4f + 6s &= 140 \\ &\Downarrow \\ -6f - 6s &= -180 \\ 4f + 6s &= 140 \end{aligned}$$

5. Combine the equations and solve for $f$.

$$\begin{array}{r} -6f - 6s = -180 \\ +\quad 4f + 6s = 140 \\ \hline -2f + 0s = -40 \end{array}$$
$$\Downarrow$$
$$\begin{aligned} -2f &= -40 \\ f &= 20 \end{aligned}$$

6. The answer is C.

**Notes**

Take note of the fact that we did not have to find the value of $s$ at any point in order to find the value of $f$. The true advantage of elimination over other methods of solving systems of equations is that there is no confusion about which variable you are actually solving for first. The answer is in the name "elimination": you eliminate the variable you don't want to solve for, leaving only the variable whose value you're seeking.

Occasionally, there is a test problem that asks you for the value of an expression rather than for the value of one variable. Most of the time, the expression has terms with both $x$- and $y$-variables. In almost all of these cases, it is much simpler to find the value of that expression directly rather than to first find the value of one variable, substitute, find the value of the other variable, then construct the desired expression and evaluate it. An example will illustrate this clearly. Usually, simply adding or subtracting the equations from each other will make the coefficients in the combined equation match up perfectly with the expression that you are asked to find.

For a deeper dive into the subject of problems for which there is a rapid, direct route to a solution that bypasses the more general approach, see the "Wormholes" chapter.

**Example 4.4-4**

**4** 1600.io 4.4

$$\begin{aligned} 2x + 3y &= 100 \\ 4x + 3y &= 360 \end{aligned}$$

In the system of equations above, what is the value of $6x + 6y$?

**Solution 1**

1. Notice that you are asked to solve for $6x + 6y$. You could solve this by finding the values of $x$ and $y$ independently and then plugging those values into the expression $6x + 6y$ and evaluating it, but notice that if we just combine the equations by adding them, the two $x$-terms will add up to $6x$, and the two $y$-terms will add up to $6y$.

$$\begin{array}{rr} & 2x + 3y = 100 \\ + & 4x + 3y = 360 \\ \hline & 6x + 6y = 460 \end{array}$$

2. The answer is 460.

**Solution 2**

1. We can choose to solve for either variable first if we decide to solve for them individually. Based on the starting conditions of the problem, it makes sense to eliminate $y$ first because its coefficients already match. Multiply the top equation by $-1$, and then add the two equations in order to eliminate $y$ and solve for $x$.

$$\begin{array}{r} -1(2x + 3y) = -1(100) \\ 4x + 3y = 360 \\ \Downarrow \\ -2x - 3y = -100 \\ + \quad 4x + 3y = 360 \\ \hline 2x + 0y = 260 \\ \Downarrow \\ 2x = 260 \\ x = 130 \end{array}$$

2. Now that we know the value of $x$, we can plug the value into either equation in order to solve for $y$. Let's use the top equation (because the coefficients are smaller).

$$\begin{aligned} 2x + 3y &= 100 \\ 2(130) + 3y &= 100 \\ 260 + 3y &= 100 \\ 3y &= -160 \\ y &= \frac{-160}{3} \end{aligned}$$





3. Now that we have the values of both $x$ and $y$, we can substitute the values into the expression $6x + 6y$ in order to find its value. Plug in 130 for $x$ and $\frac{-160}{3}$ for $y$.

$$\begin{aligned} 6x + 6y &= 6(130) + 6\left(\frac{-160}{3}\right) \\ 6x + 6y &= 780 - 320 \\ 6x + 6y &= 460 \end{aligned}$$

4. The answer is 460.

**Notes**

Solution 2 is not horribly long, but it involves messy fractions and large numbers that can be completely avoided if you know the test well and look for the shortcut provided by Solution 1.

**Section 4.4 Suggested Problems from Real Tests**

- Test 1-NC-9
- Test 1-NC-18
- Test 1-C-19
- Test 2-NC-2
- Test 3-NC-6
- Test 3-NC-19
- Test 3-C-30
- Test 4-NC-19
- Test 5-C-6
- Test 5-C-12
- Test 7-NC-3
- Test 7-C-11
- Test 7-C-33
- Test 8-NC-10
- Test 8-NC-18
- Test 8-C-6
- Test 8-C-27
- Apr 2017-NC-1
- Apr 2017-NC-7
- Apr 2017-C-36
- May 2017-NC-17
- May 2017-C-9
- Mar 2018-C-26
- Apr 2018-NC-1
- Apr 2018-C-14
- May 2018-C-11
- May 2019 (Int)-NC-2
- May 2019 (Int)-NC-19
- May 2019 (Int)-C-11
- Oct 2019-NC-4
- Oct 2019-NC-19
- Oct 2019-C-24
- Mar 2020-C-18
- Mar 2020-C-33
- Mar 2020-C-36
- Oct 2020-C-38
- Mar 2021-NC-19
- Mar 2021-C-18
- Apr 2021-NC-20
- May 2021 (US)-NC-13
- May 2021 (US)-C-11
- May 2021 (Int)-NC-16
- May 2021 (Int)-C-33

**View related real-test problems in SMTEx**

SMTEx

## Section 4.4 Practice Problems

**1** 1600.io 4.4

The Hartford ferry charges \$9 for an adult and \$1 for a child to ride one way. During a certain 8-hour shift, a ticket booth attendant collects \$1,051 from 155 passengers. Which of the following systems of equations could be used to determine the number of adult riders, $A$, and the number of child riders, $C$, during this 8-hour shift?

A) $9A + C = 8(1{,}051)$
$A + C = 8(155)$

B) $8(9A) + 8(C) = 1{,}051$
$8(A + C) = 155$

C) $9A + C = 155$
$A + C = 1{,}051$

D) $9A + C = 1{,}051$
$A + C = 155$

**2** 1600.io 4.4

$$5m + 3p = 24$$
$$m + p = 8$$

If $(m_1, p_1)$ is the solution to the system of equations above, what is the value of $p_1$ ?

**3** 1600.io 4.4

The sum of two different numbers $x$ and $y$ is 80, and the difference when the smaller number is subtracted from the larger number is 40. What is the value of $xy$ ?

A) 120

B) 320

C) 1,200

D) 3,200

**4** 1600.io 4.4

$$5x + 6y = 6$$
$$3x + 8y = 9$$

If $(x, y)$ is the solution to the system of equations above, what is the value of $2x - 2y$ ?

A) $-15$

B) $-3$

C) 3

D) 15

**5** 1600.io 4.4

$$x + y = 13$$
$$x - 3y = -3$$

According to the system of equations above, what is the value of $x$ ?

A) 2

B) 4

C) 9

D) 11





## 4.5 Number of Solutions to a System of Linear Equations Part II

We already know that the number of solutions to a system of linear equations can be thought of in terms of the slopes and $y$-intercepts of the lines in the system and the number of intersections of those lines: nonparallel lines will intersect once (one solution), parallel lines (lines with the same slope but different $y$-intercepts) will never intersect (0 solutions), and coincident lines (two equal lines with the same slope and $y$-intercept) share every point in common (infinitely many solutions). However, it is not so readily apparent (based on slope and $y$-intercept) how many solutions there will be to a system of linear equations in Standard Form because the slope and $y$-intercept do not exist as individual coefficients or constants in the equations.

Remember that lines in Standard Form can be easily rewritten in Slope-Intercept Form (though this is not a necessary step in the problems we are going to discuss), and that the slope and $y$-intercept are directly related to the constants and coefficients of the line in Standard Form.

For example, let's look at the symbolic Standard Form line $A_1x + B_1y = C_1$ and convert it into Slope-Intercept Form.

$$A_1x + B_1y = C_1$$
$$B_1y = -A_1x + C_1$$
$$y = \frac{-A_1}{B_1}x + \frac{C_1}{B_1}$$

The slope of a Standard Form line $A_1x + B_1y = C_1$ is equal to $\frac{-A_1}{B_1}$, and the $y$-intercept is equal to $\frac{C_1}{B_1}$.

Let's form a system of linear equations by also considering another line in Standard Form, $A_2x + B_2y = C_2$, whose slope is equal to $\frac{-A_2}{B_2}$ and whose $y$-intercept is equal to $\frac{C_2}{B_2}$ (based on the same conversion that was done for the line $A_1x + B_1y = C_1$).

We can use what we know about the number of solutions based on slope and $y$-intercept to draw some conclusions about how we can handle questions about the number of solutions to a system of linear equations in Standard Form.

We know that any two lines with **different slopes** (lines that are not parallel) will have **one solution**. In order for the system of equations consisting of the two lines mentioned above ($A_1x + B_1y = C_1$ and $A_2x + B_2y = C_2$) to have one solution, the ratios $\frac{-A_1}{B_1}$ and $\frac{-A_2}{B_2}$, which represent the slopes of their respective lines, must NOT be equal (we can actually omit the negative signs because they will cancel out anyway when the two ratios are set equal).

$$\frac{-A_1}{B_1} \neq \frac{-A_2}{B_2}$$
$$\frac{A_1}{B_1} \neq \frac{A_2}{B_2}$$

**As long as the ratio of the $x$-coefficient to the $y$-coefficient is different for both equations, the system will have one solution** (remember that if these ratios are not the same, it indicates that the slopes are not the same, despite the missing negative signs). For example, given the lines $4x + 2y = 5$ and $-3x + y = -5$, we can tell that the system will have one solution because the ratio of $x$- to $y$-coefficients is not the same for both equations: $\frac{4}{2} \neq \frac{-3}{1}$.

In order for the system to have **no solutions**, the lines would have to have the **same slope and different $y$-intercepts**, so we would know that **the ratio of $x$- to $y$-coefficients would have to be the same for both equations** (note again that the negative signs can be omitted): $\frac{A_1}{B_1} = \frac{A_2}{B_2}$ We also know that the $y$-intercepts of the lines must not be equal, and because the $y$-intercepts are equal to the ratio of the constant terms to the $y$-coefficients, these ratios must be different for both equations $\left(\frac{C_1}{B_1} \neq \frac{C_2}{B_2}\right)$.

For example, given the lines $4x + 2y = 5$ and $10x + 5y = 3$, we can tell that the system will have no solutions because not only is the ratio of the $x$- to $y$-coefficients the same for both equations $\left(\frac{4}{2} = \frac{10}{5} \text{ which reduces to } 2 = 2\right)$, which means that the slopes of these two line are the same, but the ratio of constant terms to $y$-coefficients is also not the same $\left(\frac{5}{2} \neq \frac{3}{5}\right)$, which means that the $y$-intercepts of these two lines are different. This means the lines are parallel and therefore have no intersection point, and thus there is no solution to the system.

Finally, in order for the system to have **infinitely many solutions**, the lines have to be **exactly the same, with the same slope and $y$-intercept**. Even if the coefficients in the equations are different, the coefficients and constant in one equation may have the same *ratio* among them as do the coefficients and constant in the other equation.

If the ratios of the $x$- and $y$-coefficients are equal $\left(\frac{A_1}{B_1} = \frac{A_2}{B_2}\right)$, which means that the slopes of these two lines are the same for both equations, and the ratios of the constant terms and $y$-coefficients are equal $\left(\frac{C_1}{B_1} = \frac{C_2}{B_2}\right)$, which means that the $y$-intercepts of these two lines are equal for both equations, then the lines are the same, and the system will have infinitely many solutions.

For example, given the lines $4x + 2y = 5$ and $10x + 5y = \frac{25}{2}$, we can see that the system will have infinitely many solutions because not only is the ratio of the $x$- to $y$-coefficients is the same for both equations $\left(\frac{4}{2} = \frac{10}{5}\right)$, but the ratio of the constant terms to the $y$-coefficients is also the same for both equations.

$$\frac{5}{2} = \frac{\frac{25}{2}}{5}$$
$$\frac{5}{2} = \frac{25}{10}$$
$$\frac{5}{2} = \frac{5}{2}$$

## Systems of Linear Equations with One Solution

There will be **one solution** to a system of equations in Standard Form when the **ratio of $x$- to $y$-coefficients is different**.

For example, the lines $2x + 3y = 4$ and $-4x + 6y = 4$ will have one solution because the ratio of $x$- to $y$-coefficients for the first equation is $\frac{2}{3}$, but the ratio of $x$- to $y$-coefficients for the second equation is $\frac{-4}{6}$, which reduces to $\frac{-2}{3}$, which is not equal to $\frac{2}{3}$.

The $y$-intercept doesn't matter in this situation.





## Systems of Linear Equations with No Solution

There will be **no solution** to a system of linear equations in Standard Form when the **ratio of $x$- to $y$-coefficients is the same for both equations, but the ratio of constant terms to $y$-coefficients is different for both equations**.

For example, the lines $2x + 3y = 4$ and $2x + 3y = 6$ will be parallel because the left sides of the equations are the same (so the ratio of the $x$- to $y$-coefficients is the same) but the right sides are different. If you want, you can convert the lines into Slope-Intercept Form to verify that they are parallel (they will have the same slope and different $y$-intercepts).

$$2x + 3y = 4 \quad \Rightarrow \quad y = \frac{-2}{3}x + \frac{4}{3}$$

$$2x + 3y = 6 \quad \Rightarrow \quad y = \frac{-2}{3}x + 2$$

Similarly, the lines $2x+3y = 4$ and $4x+6y = 10$ are also parallel lines (with the same slope), but how is this possible? Though the coefficients might be different in the two equations, the *ratio* of the coefficients is the same.

For the lines $2x + 3y = 4$ and $4x + 6y = 10$, the ratio of $x$-coefficient to $y$-coefficient for the first equation is 2 to 3 or $\frac{2}{3}$. The ratio of the $x$-coefficient to $y$-coefficient for the second equation is 4 to 6 or $\frac{4}{6}$, which reduces to $\frac{2}{3}$. Since these ratios match, these lines have the same slope.

Notice also that the ratio of constant to $y$-coefficient is 4 to 3 for the first equation but 10 to 6, which reduces to 5 to 3, for the second equation. These ratios aren't equal, so the $y$-intercepts are different.

## Systems of Linear Equations with Infinite Solutions

There will be **infinite solutions** to a system of linear equations in Standard Form when **the ratio among the two variables' coefficients and the constant are the same in both equations**. In the simplest case, these values will simply be equal in both equations, but things will rarely be that simple, and you will usually have to test the ratios among the corresponding components of the equations to see if they are all identical.

For example, the lines $2x + 3y = 4$ and $2x + 3y = 4$ are very obviously the same line. However, the lines $2x + 3y = 4$ and $4x + 6y = 8$ are also the same line. In this example, it's easy to realize this; all the terms in the second equation are just double those in the first, so you can multiply both sides of the first equation by 2 to get the line $4x + 6y = 8$, which is exactly the same as the second equation.

In the general case, when you can't tell whether one equation is a scaled version of the other because the numbers are larger and/or are in non-integral ratios, you can scale one or both equations so as to match one of the terms, and then you can see if the other two terms also end up matching, which would indicate that the equations have the same solution set and thus that there are infinite solutions to the system.

You *could* choose any pair of corresponding coefficients, or the constants, and form a fraction that represents the scaling factor of one of the equations relative to the other, and then multiply the other two values by this scaling factor to see if the results match the corresponding elements in the other equation, which would indicate that the equations are the same. However, in some situations you will have to deal with fractions if you use this method, and fractions can make things messy. Alternatively, you can use one of the techniques used when performing elimination: you can pick any pair of corresponding coefficients, or the constants, and perform the complementary multiplication operations so the selected terms end up with exactly the same coefficients (be sure to preserve the signs of the coefficients when performing this procedure). Then, you can inspect the other terms to see if they, too, are now identical in both equations as a result of this equalizing procedure; if they are, the equations are the same and there are infinite solutions.

We need to determine if the ratio among these three components is the same in both equations; in the first equation in the example above, it's 2:3:4, and in the second, it's 4:6:8. Though the answer might be obvious here due to the simple numbers, let's work through it more rigorously to show the procedure so you're ready if you're faced with a less-pleasant situation.

Let's select the $x$-coefficients as our reference. We're going to perform what we call complementary intermultiplication, which is an appropriate term because we said so. We'll multiply the first equation by the $x$-coefficient of the second equation (which is 4), and we'll multiply the second equation by the $x$-coefficient of the first equation (which is 2). This will, of course, result in matching coefficients for the $x$ terms in both equations (because $ab = ba$), but we need to determine if the *other* two terms end up matching as well.

$$\begin{gathered} 4(2x + 3y = 4) \\ 2(4x + 6y = 8) \\ \Downarrow \\ 8x + 12y = 16 \\ 8x + 12y = 16 \end{gathered}$$

All the terms match, which means the equations represent the exact same relationship (the lines represented by the equations are coincident), and therefore there are infinite solutions to the system.

You can also realize that **the ratios of the $x$- to $y$-coefficients and the ratios of the constant terms to the $y$-coefficients is the same for the two equations**.

The ratio of the $x$- to $y$-coefficients for the first equation is $\frac{2}{3}$. The ratio of the $x$- to $y$-coefficients for the second equation is $\frac{4}{6}$, which reduces to $\frac{2}{3}$, so this ratio matches in the two equations, indicating that the slopes are the same. Similarly, the ratio of the constant terms to $y$-coefficient is $\frac{4}{2}$, which reduces to 2, and in the second equation, this ratio is equal to $\frac{8}{4}$, which also reduces to 2; this means the $y$-intercepts are the same as well. Since both of these ratios are the same for both equations, the lines are the same (they have the same slope and $y$-intercept) and will therefore have infinitely many solutions.

Finally, yet another option to check for infinitely many solutions is to convert both equations to Slope-Intercept Form and see if the slope and $y$-intercept are the same.

If you are told that a system has no solutions, but you need to determine the value of one of the coefficients, you can use the fact that the ratio of $x$- to $y$-coefficients must be equal to set up a simple proportion that will allow you to solve for the unknown quantity. For example, if you are given the following system with no solutions,

$$\begin{aligned} 12x - 4y &= 2 \\ ax + 2y &= 7 \end{aligned}$$

you can set up the following proportion which leverages the known ratio between the $y$-coefficients (we like to place the unknown in the numerator of the fraction on the left to facilitate solving):

$$\frac{a}{12} = \frac{2}{-4}$$

Solving for $a$,

$$\begin{aligned} a &= 12\left(\frac{2}{-4}\right) \\ a &= 2(-3) \\ a &= -6 \end{aligned}$$





The use of a proportion keeps things organized, and more generally, writing down your work instead of relying on mental math is the better habit to maintain.

If you are told that a system has infinite solutions, not only will the $x$- and $y$-coefficients be in the same ratio, but the constants will be, too, so you can use the same procedure described above to determine the value of an unknown constant. If we have a system with infinite solutions such as this,

$$\begin{aligned} 4x + 6y &= 12 \\ ax - 3y &= b \end{aligned}$$

you can set up a proportion to find either $a$ or $b$ by making use of the known ratio of the $y$-coefficients. We showed the procedure for finding $a$, the $x$-coefficient, above; here's what the proportion involving $b$ would look like:

$$\frac{b}{12} = \frac{-3}{6}$$

Solving for $b$,

$$\begin{aligned} b &= 12\left(\frac{-3}{6}\right) \\ b &= -3(2) \\ b &= -6 \end{aligned}$$

Generally, you will use a known ratio (whether of a pair of coefficients or of the constants) to set up a proportion to solve for an unknown value (coefficient or constant).

**SkillDrill 4.5-1**

**Directions:** All of the following systems of linear equations have no solution. Find the value of $a$.

**1.** $\begin{aligned} 8x - 2y &= 1 \\ ax + y &= 3 \end{aligned}$

**2.** $\begin{aligned} 5x - 15y &= 45 \\ ax - 3y &= 8 \end{aligned}$

**3.** $\begin{aligned} 2x + 3y &= 6 \\ ax - y &= 1 \end{aligned}$

**4.** $\begin{aligned} 5x + 2y &= 10 \\ x + ay &= 4 \end{aligned}$

**Directions:** All of the following systems of linear equations have infinite solutions. Find the values of $a$ and $b$.

**5.** $\begin{aligned} 3x - 9y &= 12 \\ ax + by &= 4 \end{aligned}$

**6.** $\begin{aligned} 2x + 3y &= 6 \\ ax - 3y &= b \end{aligned}$

**7.** $\begin{aligned} x + 5y &= 8 \\ ax - by &= 1 \end{aligned}$

**8.** $\begin{aligned} -6x + 5y &= 6 \\ 4ax + 2y &= b \end{aligned}$

**Example 4.5-1**

**1** 1600.io 4.5

$$-5x + y = 12$$
$$ax + 3y = 32$$

In the system of equations above, $a$ is a constant. For which of the following values of $a$ does the system have no solution?

A) $-15$

B) $-10$

C) $-5$

D) $0$

**Solution 1**

1. In order for a system of linear equations to have no solution, the two lines must be parallel but have different $y$-intercepts. When the lines are in Standard Form, we can tell that they are parallel when the ratio of $x$- to $y$-coefficients is the same for both equations.

2. The ratio of $x$- to $y$-coefficients for the first equation is $-5$ to 1, or $\frac{-5}{1}$. The ratio of $x$- to $y$-coefficients for the second equation is a to 3, or $\frac{a}{3}$. Create a proportion by setting the two fractions equal to each other and solve for $a$. Remember to cross-multiply.

$$\frac{-5}{1} = \frac{a}{3}$$
$$-5(3) = a$$
$$-15 = a$$

3. The answer is A.

**Solution 2**

1. In order for a system of linear equations to have no solution, the two lines must be parallel but have different $y$-intercepts. When the lines are in Standard Form, we can take the approach of making $x$- and $y$-coefficients match in both equations. We just have to verify the constants on the right side of the equations are different from each other (technically, we know this will be true because the problem is set up so that there will be no solutions, and thus the constant terms will definitely not match).

2. We do not know how to make the $x$-coefficients match, since we do not know the value of $a$, but we can make the $y$-coefficients match by multiplying both sides of the first equation by 3. This is the first step in trying to get the left sides of the equations to look identical.

$$3(-5x + y) = 3(12)$$
$$ax + 3y = 32$$
$$\Downarrow$$
$$-15x + 3y = 36$$
$$ax + 3y = 32$$





3. The $y$-coefficients match, and the constants on the right side of the equation are different (again, this should be true given that the problem is set up to ensure there will be no solutions to the system of equations for a certain value of $a$), so there will be no solution when the $x$-coefficients are the same (making the lines parallel), which means that the value of $a$ is $-15$.

4. The answer is A.

**Solution 3**

1. In order for a system of linear equations to have no solution, the two lines must be parallel but have different $y$-intercepts. We can convert both lines into Slope-Intercept Form and then check what value of of $a$ would make the slopes equal.

2. Convert the first equation into Slope-Intercept Form.

$$-5x + y = 12$$
$$y = 5x + 12$$

3. Convert the second equation into Slope-Intercept Form.

$$ax + 3y = 32$$
$$3y = -ax + 32$$
$$y = \frac{-a}{3}x + \frac{32}{2}$$

4. The $y$-intercepts do not match, which is expected since the problem is about two lines that never intersect. In order for the lines to be parallel, the slopes must be the same. Set the two slopes equal to each other to find what value of $a$ would make the slopes equal. The slope of the first line is 5. The slope of the second line is $\frac{-a}{3}$.

$$5 = \frac{-a}{3}$$
$$15 = -a$$
$$-15 = a$$

5. The answer is A.

**Example 4.5-2**

**2** 1600.io 4.5

$$ax + by = 4$$
$$3x + 6y = 60$$

In the system of equations above, $a$ and $b$ are constants. If the system has infinitely many solutions, what is the value of $\frac{a}{b}$?

**Solution 1**

1. In order for a system of linear equations to have infinitely many solutions, the two lines must be exactly the same. The shortest way to handle this one is to remember that if the lines are in Standard Form, then the ratio of $x$ and $y$-coefficients will be the same in both equations, so $\frac{a}{b} = \frac{3}{6}$, or (reducing the fraction) $\frac{a}{b} = \frac{1}{2}$.

2. The answer is $\frac{1}{2}$.

**Solution 2**

1. Since the problem states that there are infinitely many solutions, the two equations must be equal. If you are confused by the fact that the constants 4 and 60 don't match, then you can multiply both sides of the first equation by 15 (because $\frac{60}{4} = 15$) so that the 4 will match the 60 on the right side of the equation.

$$15(ax + by) = 15(4)$$
$$3x + 6y = 60$$
$$\Downarrow$$
$$15ax + 15by = 60$$
$$3x + 6y = 60$$

2. Now that the constants match, the next step is setting the ratios of $x$ and $y$-coefficients for both equations equal to each other because the equations must be the same. Set $\frac{15a}{15b}$ equal to $\frac{3}{6}$, and reduce the fractions to solve for $\frac{a}{b}$.

$$\frac{\cancel{15}a}{\cancel{15}b} = \frac{3}{6}$$
$$\frac{a}{b} = \frac{1}{2}$$

3. The answer is $\frac{1}{2}$.

**Notes**

In Solution 1, we ignored the fact that the constant terms didn't match in the original forms of the equation. If the problem states that there are infinitely many solutions, you don't have to worry about making the constant terms match first before you start comparing the ratios of the coefficients.

The problem can also be done by converting both equations into Slope-Intercept Form and setting the two slopes equal to each other, then solving for $\frac{a}{b}$.





**Section 4.5 Suggested Problems from Real Tests**

- Test 2-NC-20
- Test 3-NC-9
- Test 6-C-11
- Test 8-C-33
- May 2019 (US)-C-29
- 2020-NC-19
- 2020-C-21
- 2021-C-28
- May 2021 (Int)-NC-3

**View related real-test problems in SMTEx**

SMTEx

## Section 4.5 Practice Problems

**1** 1600.io 4.5

In the system of equations below, $a$ and $c$ are constants.

$$\frac{1}{3}x + \frac{1}{5}y = \frac{1}{15}$$
$$ax + y = c$$

If the system of equations has an infinite number of solutions $(x, y)$, what is the value of $a$?

A) $-\frac{5}{3}$

B) $-\frac{1}{3}$

C) $\frac{1}{3}$

D) $\frac{5}{3}$

**2** 1600.io 4.5

In the $xy$-plane, the equations $x + 5y = 8$ and $4x + 20y = c$ represent the same line for some constant $c$. What is the value of $c$?

**3** 1600.io 4.5

$$kx - 2y = 3$$
$$7x + 4y = 4$$

In the system of equations above, $k$ is a constant and $x$ and $y$ are variables. For what value of $k$ will the system of equations have no solution?

A) $-\frac{21}{4}$

B) $-\frac{7}{2}$

C) $\frac{7}{2}$

D) $\frac{21}{4}$

**4** 1600.io 4.5

$$3x + 2y = 10$$
$$6x + cy = 21$$

In the system of equations above, $c$ is a constant. For what value of $c$ will there be no solution $(x, y)$ to the system of equations?

A) 2

B) 3

C) 4

D) 6

## CHAPTER 4 RECAP

- Solutions to systems of linear equations can be thought of as points where the graphs of the lines intersect. At these intersection points, both lines have the same $x$- and $y$- values.
- A system of linear equations has **one solution when the two lines are *not* parallel and cross at one point**.
- A system of linear equations has **no solutions when the lines are parallel**, because the lines do not intersect. If the equations of the lines are in Slope-Intercept Form, the equations of parallel lines will display the ***same* slope and *different* $y$-intercepts**.
- A system of linear equations has **infinite solutions when the two lines are exactly the same**. The **slopes and $y$-intercepts must be the same**.
- Set expressions for $y$ equal to each other (substitution) to find the solution point when both equations are already in Slope-Intercept Form.
- Use substitution when the equations are in different forms, particularly when one variable is already solved for in terms of the other. Substitution is RARELY preferable to elimination for problems on the test.
- Use elimination to solve most systems of linear equations, particularly when both equations are in Standard Form. Multiply one or both equations by numbers that will cause the coefficients of the variable you want to eliminate to cancel out when the equations are combined through addition or subtraction.
- Use combination without eliminating either variable if possible when the problem asks for an expression involving terms with both variables.
- A system of linear equations in Standard Form has **one solution when the ratio of $x$- to $y$-coefficients is NOT the same for both equations** (this correlates to the fact that the slopes are not the same).
- A system of linear equations in Standard Form has **no solutions when the ratio of $x$- to $y$-coefficients is the same for both equations** (which correlates to the slopes being the same) **but the ratio of constant terms to $y$-coefficients is NOT the same for both equations** (which correlates to the $y$-intercepts being different).
- A system of linear equations in Standard Form has **infinitely many solutions when the ratio of $x$- to $y$-coefficients is the same for both equations** (which correlates to the slopes being the same) **and when the ratio of constant terms to $y$-coefficients is the same for both equations** (which correlates to the $y$-intercepts being the same).
- When asked to find unknown coefficient values in systems of linear equations in Standard Form that have no solutions or infinitely many solutions, use proportions of the ratios of the coefficients and constants to solve for unknown values.

  Alternatively, multiply one or both equations by scale factors so that the any known coefficients or constants in the same position can be matched between the two equations.
- When you are asked about the number of solutions to a system of linear equations in Standard Form, you can always convert Standard Form linear equations into Slope-Intercept Form to make it easier to match the slope and $y$-intercept.



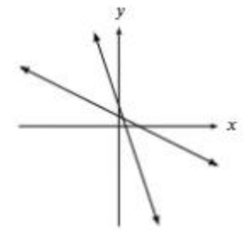


## Additional Problems

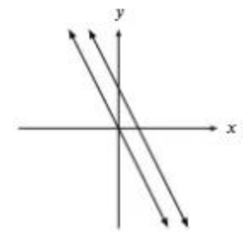
**1** 1600.io 4.1

Which of the following is a graph of a system of equations with no solution?

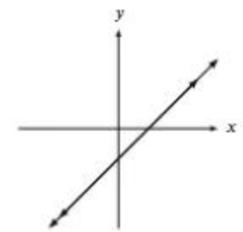
A)



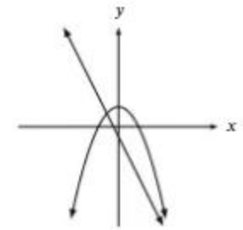

B)




C)




D)




**2** 1600.io 4.4

An aquarium sells two types of tickets. The standard ticket, for admission only, costs \$20. The premium ticket, which includes admission and a swim with angelfish, costs \$40. One Tuesday, the aquarium sold a total of 400 tickets and collected a total of \$9,000 from ticket sales. Which of the following systems of equations can be used to find the number of standard tickets, $s$, and premium tickets, $p$, sold on that Tuesday?

A) $s + p = 400$
$20s + 40p = 9{,}000$

B) $s + p = 400$
$40s + 20p = 9{,}000$

C) $20s + 40p = 400$
$s + p = 9{,}000$

D) $40s + 20p = 400$
$s + p = 9{,}000$

**3** 1600.io 4.3

$$ax = 3$$

In the equation above, $a$ is a constant. For which of the following values of $a$ will the equation have no solution?

A) 0

B) 1

C) 3

D) 6

**4** 1600.io 4.4

$$\frac{1}{3}y = 6$$
$$x - \frac{1}{3}y = 2$$

The system of equations above has solution $(x, y)$. What is the value of $x$?

A) 4
B) $\frac{17}{3}$
C) 8
D) 20

**5** 1600.io 4.4

Marcus sells only hats and scarves on his website. Hats sell for \$15 each, and scarves sell for \$10 each. If Marcus sold 30 pieces of clothing and his sales totaled \$400, how many scarves did Marcus sell?

**6** 1600.io 4.3

$$12x - 4x(c - 2) = 3x$$

In the equation above, $c$ is a constant. If the equation has infinitely many solutions, what is the value of $c$?

A) $\frac{11}{4}$
B) $\frac{13}{4}$
C) $\frac{15}{4}$
D) $\frac{17}{4}$

**7** 1600.io 4.4

$$x = y - 5$$
$$\frac{x}{2} + 4y = 2$$

Which ordered pair $(x, y)$ satisfies the system of equations shown above?

A) $(-5, 0)$
B) $(-4, 1)$
C) $(4, 0)$
D) $(5, 10)$





**8** 1600.io 4.2

$$\frac{x}{y} = 4$$
$$3(y + 2) = x$$

If $(x, y)$ is the solution of the system of equations above, what is the value of $y$?

A) 2
B) 6
C) 12
D) 24

**9** 1600.io 4.4

$$2x + 5y = 800$$
$$5x + 2y = 1500$$

Based on the system of equations above, what is the value of $7x + 7y$?

**10** 1600.io 4.2

$$4(x + y) = 56$$
$$\frac{x}{2} = 5$$

If $(x, y)$ is a solution to the system of equations above, what is the value of $y$?

A) 16
B) 4
C) $-4$
D) $-16$

**11** 1600.io 4.4

If $3w + 5t = 21$ and $5w - t = 7$, what is the value of $4w + 2t$?

A) 14
B) 18
C) 24
D) 28

**12** 1600.io 4.4

At a food truck, each empanada has 40 more calories than each taco. If 1 empanada and 8 tacos have a total of 1,570 calories, how many calories does an empanada have?

**13** 1600.io 4.4

$$x - 4y = 24$$
$$0.25x + y = 4$$

The solution to the system of equations above is $(x, y)$. What is the value of $x$?

A) $-4$

B) $4$

C) $20$

D) $40$

**14** 1600.io 4.4

$$-x + y = -4.75$$
$$x + 5y = 11.75$$

If $(x, y)$ satisfies the system of equations above, what is the value of $y$?

**15** 1600.io 4.4

In the $xy$-plane, the point $(p, r)$ lies on the line with equation $y = 2x + b$, where $b$ is a constant. The point with coordinates $(4p, 3r)$ lies on the line with the equation $y = 3x + b$. If $p \neq 0$, what is the value of $\frac{r}{p}$?

A) $5$

B) $2$

C) $\frac{1}{2}$

D) $\frac{1}{5}$





**16** 1600.io 4.3

The equation $7x - 4 = a(x - b)$, where $a$ and $b$ are constants, has no solutions. Which of the following must be true?

I. $a = 7$

II. $b = 4$

III. $b \neq \frac{4}{7}$

A) None

B) I only

C) I and II only

D) I and III only

**17** 1600.io 4.2

In a wedding hall, during a reception, guests are seated at tables. If 6 guests are assigned to each table, 3 additional tables will be needed to seat all of the guests. If 12 guests are assigned to each table, 3 tables will not be used. How many guests will be attending the wedding reception?

**18** 1600.io 4.5

$$\frac{2}{3}x - \frac{4}{5}y = 10$$
$$ax - by = 15$$

The system of equations above has no solutions. If $a$ and $b$ are constants, what is the value of $\frac{a}{b}$?

**19** 1600.io 4.2

$$\frac{1}{4}y = \frac{21}{20} - \frac{1}{5}x$$
$$8y = 9x$$

In the $xy$-plane, the lines that correspond to the system of equations above intersect at the point $(a, b)$. What is the value of $\frac{a}{b}$?

**20** 1600.io 4.4

The score on a history test is obtained by subtracting the number of incorrect answers from 4 times the number of correct answers. If a student answered all 25 questions and obtained a score of 65, how many questions did the student answer correctly?

**21** 1600.io 4.2

In one month, Dan and John spent a combined 500 hours writing a math book. If Dan spent 50 more hours writing the book than John did, how many hours did John spend writing the book?

**22** 1600.io 4.4

$$2x + y = a$$
$$-x + 2y = 4$$

In the system of equations above, $a$ is a constant. What is the $y$-value of the solution to the system in terms of $a$?

A) $\frac{a+8}{5}$

B) $\frac{2a-4}{5}$

C) $\frac{4a+4}{5}$

D) $\frac{-3a-1}{5}$

**23** 1600.io 4.2

Ms. Morrissey has a bag containing $n$ pieces of chocolate to distribute to the students in her second grade class. If she gives each student 2 pieces of chocolate, she will have 6 pieces of chocolate left over. In order to give each student 3 pieces of chocolate, she will need an additional 16 pieces of chocolate. How many students are in the class?

A) 10

B) 16

C) 19

D) 22





# Answer Key

### Section 4.1 Practice Problems

1. B
2. B

### SkillDrill 4.2-1

1. (1, 10)
2. (−5, 15)
3. $\left(\frac{7}{2}, \frac{-7}{2}\right)$ or (3.5, −3.5)
4. $\left(\frac{-1}{12}, \frac{21}{4}\right)$ or $(-0.08\overline{33}, 5.25)$

### Section 4.2 Practice Problems

1. 11
2. C
3. 1
4. C
5. D

### SkillDrill 4.3-1

1. 4
2. $\frac{1}{2}$ or 0.5
3. $\frac{5}{3}$ or $1.\overline{66}$
4. 0

### SkillDrill 4.3-2

1. $a = 4, b = 7$
2. $a = 6, b = 12$
3. $a = -6, b = -2$
4. $a = \frac{5}{3}, b = 9$

### Section 4.3 Practice Problems

1. B
2. $\frac{1}{4}$ or .25
3. C
4. D

### SkillDrill 4.4-1

1. (1, 4)
2. (7, 1)
3. (−2, −3)
4. $\left(\frac{1}{2}, 0\right)$ or (0.5, 0)

### Section 4.4 Practice Problems

1. D
2. 8
3. C
4. B
5. C

### SkillDrill 4.5-1

1. −4
2. 1
3. $\frac{-2}{3}$ or $-0.\overline{66}$
4. $\frac{2}{5}$ or 0.4
5. $a = 1, b = -3$
6. $a = -2, b = -6$
7. $a = \frac{1}{8}, b = \frac{-5}{8}$ or $a = 0.125, b = -0.625$
8. $a = \frac{-3}{5}, b = \frac{12}{5}$ or $a = -0.6, b = 2.4$

### Section 4.5 Practice Problems

1. D
2. 32
3. B
4. C

### Additional Problems

1. B
2. A
3. A
4. C
5. 10
6. D
7. B
8. B
9. 2300
10. B
11. A
12. 210
13. C
14. $\frac{7}{6}$, 1.16, or 1.17
15. A
16. D
17. 72
18. $\frac{5}{6}$ or .833
19. $\frac{8}{9}$, .888, or .889
20. 18
21. 225
22. A
23. D

# Linear Inequalities and Absolute Value

# 5

## 5.1 Writing and Solving Linear Inequalities

Developing children are known to test their boundaries as a way of learning their limits. How many cookies can you sneak out of the kitchen before your stomach aches? How loud can you be before your dad loses his mind? Though the breaking point is important, the goal of this testing is to figure out what the safe zone is—as long as you eat fewer cookies than your limit, your mission can be considered a success.

When we work with linear expressions, sometimes we need to figure out boundaries, maximums, and minimums instead of just finding one particular value: we are interested in **inequalities** rather than equations. Writing and solving linear inequalities is extremely similar to writing and solving linear equations (in either Slope-Intercept Form or Standard Form). There are just a few key differences, the first being that we use inequality signs rather than equals signs.

### Inequality Signs

Linear inequalities are written using one of four signs.

- **Less Than**
  The less-than sign is the following:

  $$<$$

  It signifies that the left side of the inequality is less than the right side. For example, $x < 8$ means that $x$ is less than 8 (note that $x$ cannot be equal to 8).

- **Less Than or Equal To**
  The less-than-or-equal-to sign is the following:

  $$\leq$$

  It signifies that the left side of the inequality is less than or equal to the right side. Another way the relationship might be phrased is that the value of the left side must be "at most" a certain number (or "no greater than" a certain number). The left side can match the right side, but cannot exceed its value. For example, $x \leq 8$ means that $x$ is less than or equal to 8 (the maximum value of $x$ is 8, but $x$ can be any number that is no more than 8, including 8 itself).

- **Greater Than**
  The greater-than sign is the following:

  $$>$$

  It signifies that the left side of the inequality is greater than the right side. For example, $x > 8$ means that $x$ is greater than 8 (note that $x$ cannot be equal to 8).

- **Greater Than or Equal To**
  The greater-than-or-equal-to sign is the following:

  $$\geq$$

  It signifies that the left side of the inequality is greater than or equal to the right side. Another way the relationship might be phrased is that the value of the left side must be "at least" a certain number (or "no less than" a certain number). The left side can match the right side, but cannot be less than the right side. For example, $x \geq 8$ means that $x$ is greater than or equal to 8 (the minimum value of $x$ is 8, but $x$ can be any number that is no less than 8, including 8 itself).

People often get confused which sign to use, but you only need to know two things. First, the small side of the inequality sign (the pointy end) is pointing to the smaller value side of the inequality, and the big side of the inequality (the open end) is on the same side as the greater value side of the inequality. Second, if the sign has a line underneath it, it means the two sides can also be equal.

**SkillDrill 5.1-1**

**Directions:** Write an inequality to represent the information given.

1. Bella gets paid \$2 for every backflip she does and \$3 for every front flip she does. In one day, she makes less than \$56 doing backflips and front flips. (Use $b$ for the number of backflips and $f$ for the number of front flips.)

2. In an eating contest, Ferdinand takes 1 minute to eat each hot dog and 3 minutes to eat each cake. It took Ferdinand at least 60 minutes to eat $h$ hot dogs and $c$ cakes.

3. A number $x$ is at least three more than twice a number $y$.

4. Five less than three times a number $n$ is no more than 35.

5. It takes Paula 8 minutes to drink a small water bottle and 20 minutes to drink a big water bottle. This week, she wants to spend fewer than 112 minutes drinking $s$ small water bottles and $b$ big water bottles.





**Example 5.1-1**

**1** 1600.io 5.1

A cookware store is having a sale on knives and pans. During the sale, the cost of each knife is \$20, and the cost of each pan is \$30. Gordon can spend at most \$240 at the store. If Gordon buys $k$ knives and $p$ pans, which of the following must be true?

A) $20k + 30p \leq 240$

B) $20k + 30p \geq 240$

C) $30k + 20p \leq 240$

D) $30k + 20p \geq 240$

**Solution**

1. Start by writing an expression for the total cost of Gordon's items, but instead of writing an equation, we will form an inequality.

2. Since he is buying $k$ knives for \$20 each, the total cost of $k$ knives is $20k$. Since he is buying $p$ pans for \$30 each, the cost of $p$ pans is $30p$. Therefore, Gordon's total cost for knives and pans is $20k + 30p$.

3. The total amount he spends has an upper limit; he can spend at most \$240, so the total cost of buying $k$ knives and $p$ pans must be less than or equal to \$240. Set the expression $20k + 30p$ less than or equal to 240 in the inequality.

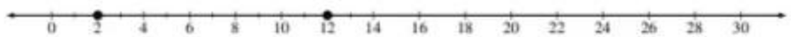
   $$20k + 30p \leq 240$$

4. The answer is A.

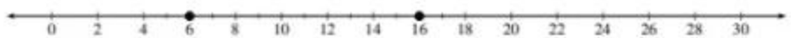
## Solving Linear Inequalities

When you have to solve inequalities, most operations can be applied just as you would apply them when solving equations, though there are a few notable exceptions.

Let's consider the true inequality $2 < 12$ and look at how performing operations on this inequality can result in other true inequalities. We will also graph the points 2 and 12 on a number line and verify that the inequality makes sense.




It is clear on the number line that 2 is further to the left than 12 (2 is less than 12). If we add 4 to both sides of the inequality, we have the inequality $6 < 16$, which is still true (the two original points are simply shifted, but maintain their positions relative to each other).



If we subtract 1 from both sides of this inequality, we have the inequality $5 < 15$, which is still true (again, the points have merely shifted).

If we multiply both sides by 2, we have the inequality $10 < 30$, which is still true. Unlike when we added and subtracted, the points have not only shifted, but the distance between the points has grown due to the scale factor of 2 (they are now twice as far apart).

If we divide both sides of this inequality by 5, we have the inequality $2 < 6$, which is still true. Once again, the points have not only shifted, but the distance between them has shrunk (by a scale factor of 5).

All of the operations we applied so far have produced valid inequalities. But what if we had multiplied both sides of the inequality by 0? The inequality breaks down because $0 \not< 0$ (0 is not less than 0).

Perhaps even more interesting is what happens when we multiply (or divide) an inequality by a negative number. Let's start back at the inequality $2 < 12$. If we multiply (or divide) both sides of the inequality by $-1$ (essentially mirroring both points, 2 and 12, about 0), we will see once again that the resulting inequality would not be true: $-2 \not< -12$ (notice that $-2$ is further to the right than $-12$).

**When multiplying (or dividing) an inequality by a negative number, you need to reverse the inequality sign in order to produce the correct inequality**: $-2 > -12$. For a moment, let's consider what happens when we deal with some unknown constants, $a$ and $b$, which for now, we will assume are both positive (though this is not important). If we know that $a < b$, then we can draw the following number line graph of the two values $a$ and $b$.

If we multiply both sides of the inequality by $-1$, we mirror both points, and we can see that $-a > -b$.

We can also prove this algebraically using subtraction instead of multiplication. Starting with the inequality $a < b$, let's subtract $a$ from both sides.

$$a < b$$
$$0 < b - a$$

Now, subtract $b$ from both sides of the inequality.

$$0 < b - a$$
$$-b < -a$$

Once again, we see that $-b$ is less than $-a$. Essentially, we have not "flipped the inequality sign," but reversed the sides of the terms by normal algebraic manipulation. Regardless, it is sufficient to say that **when we multiply (or divide) an inequality by a negative number, we need to reverse the inequality sign**.

Solving inequalities that contain variables is just like solving equations, but **if one of your algebraic steps involves multiplying or dividing both sides of the inequality by a negative number, you must remember to reverse the inequality sign**.

For example, if we are given the inequality $-8x + 3 \leq 59$ and are asked to solve for $x$, we can start by subtracting 3 from both sides of the equation.

$$-8x + 3 \leq 59$$
$$-8x \leq 56$$

Divide both sides of the inequality by $-8$. When we do that, we have to reverse the less-than-or-equal-to sign into a greater-than-or-equal-to sign to get the correct inequality: $x \geq -7$.

Any value of $x$ that is greater than or equal to $-7$ is a solution to this inequality.

**SkillDrill 5.1-2**

**Directions:** Solve the given inequality for $x$ (with $x$ on the left side of the inequality) and find the maximum or minimum *integer* value of $x$ that satisfies the inequality.

**1.** $3x + 2 \leq 8$

**2.** $5x - 3 \geq 4x + 5$

**3.** $10(x - 2) < 3x$

**4.** $3x - 5 < 4x - 7$

**5.** $3x + 2 \leq 8x$

**6.** $2x + 10 \geq 4x + 30$

**7.** $2x - 4 < 3x - 4$

**8.** $6x + 3 < 4x + 15$

**Example 5.1-2**

**2** 1600.io 5.1

Which of the following numbers is NOT a solution of the inequality $6x - 9 \leq 7x - 6$ ?

A) $-1$

B) $-2$

C) $-3$

D) $-4$

**Solution**

1. Group the $x$-terms on the right side of the inequality by subtracting $6x$ from both sides.

$$6x - 9 \leq 7x - 6$$
$$-9 \leq x - 6$$

2. Group the constant terms on the left side of the inequality by adding 6 to both sides in order to isolate and solve for $x$.

$$-9 \leq x - 6$$
$$-3 \leq x$$

3. Any value of $x$ that *IS* greater than or equal to $-3$ is a solution of the inequality. Since the question asks us to pick the choice that is *NOT* a solution, we need to pick the only choice that is less than $-3$ (that is, not greater than or equal to $-3$), which is $-4$.

4. The answer is D.

**Notes**

You could do this problem by plugging in each choice and looking for the one that does not produce a valid inequality, but this method wastes time and is potentially confusing, especially since this inequality is very easy to solve.

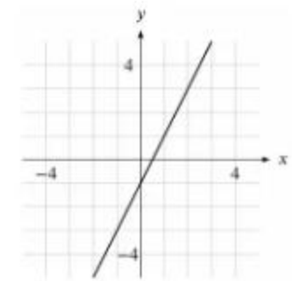




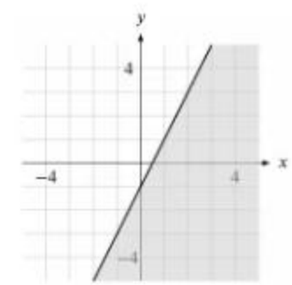
## Graphing Linear Inequalities

Let's extend the concept of inequalities into two dimensions (with two variables, $x$ and $y$) by considering what it means to have a linear inequality, and what it means to be a solution to the linear inequality.

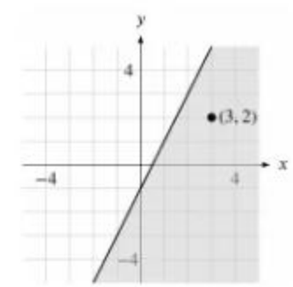
Let's consider the linear inequality (in Slope-Intercept Form) $y \leq 2x - 1$. We already know that if this were the linear equation $y = 2x - 1$, then the solution set could be represented with a line on the $xy$-plane that shows all possible pairs of $x$- and $y$-values that satisfy the equation.




In the inequality, all of these points where the $y$-value is equal to $2x - 1$ are still solutions to the inequality because it includes the values of $y$ that are **equal** to $2x - 1$, but they are not the only solutions to the inequality. Any point that is **below** a point on this line is also a solution to the system because the $y$-values at these points are **less than** the value of $2x - 1$ for any given $x$-value, so the solution region is represented by the entire shaded region under the line including the line itself.



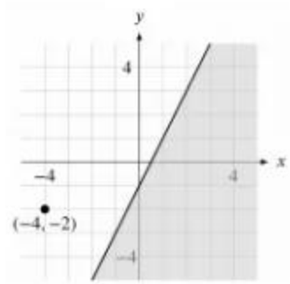

If we pick any point on the line or in the shaded region, we can plug the $x$- and $y$-values into the inequality to verify that it is still true. Let's use the point $(3, 2)$, which is located in the shaded region.



Let's plug in 3 for $x$ and 2 for $y$ in the linear inequality.

$$\begin{aligned} y &\le 2x - 1 \\ (2) &\le 2(3) - 1 \\ 2 &\le 6 - 1 \\ \checkmark \quad 2 &\le 5 \end{aligned}$$

It is true that $2 \le 5$, so the point $(3, 2)$ is a solution to the inequality. But what if we choose a point outside of the shaded region (and not on the line)? Let's use the point $(-4, -2)$.

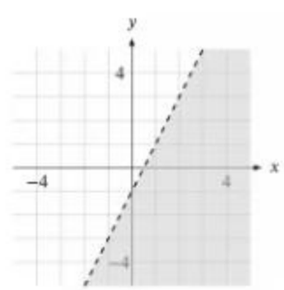



Let's plug in $-4$ for $x$ and $-2$ for $y$ in the linear inequality (note that we are simply plugging in negative values and not multiplying both sides by a negative, so we don't have to touch the inequality sign).

$$\begin{aligned} y &\le 2x - 1 \\ (-2) &\le 2(-4) - 1 \\ -2 &\le -8 - 1 \\ \times \quad -2 &\le -9 \\ -2 &\not\le -9 \end{aligned}$$

Since $-2$ is not less than or equal to $-9$, this point is not a solution to the system of inequalities.

If the inequality were not a "less than or equal to" inequality ($y \le 2x - 1$), but instead were just a "less than" inequality ($y < 2x - 1$), the only difference would be that the line $y = 2x - 1$ would no longer be included in the solution set (because the $y$-values on the line are not strictly less than $2x - 1$ for any given value of $x$), so a dashed line instead of a solid line would be used to indicate the border of the region.





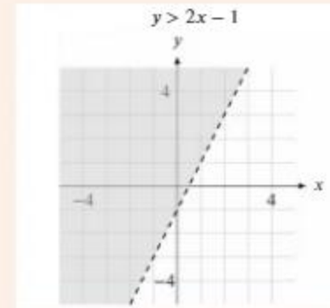



## Graphing Linear Inequalities

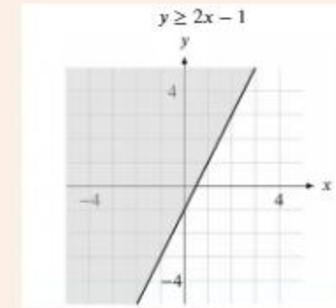
When the sign in the inequality is $\le$ or $\ge$, you will graph the line as a solid line because the points on the line are included in the solutions to the inequality, but if the symbol is $<$ or $>$, you will use a dashed line to graph the line because the points on the line will not satisfy the inequality and therefore are excluded from the solution set.

When we graph any linear inequality in Slope-Intercept Form with a greater-than or greater-than-or-equal-to sign, we shade above the line because any value of $y$ that is above the $y$-values on the line is greater than the value of the $y$-coordinate of the point on the line below it, and thus will satisfy the inequality.

$y > 2x - 1$

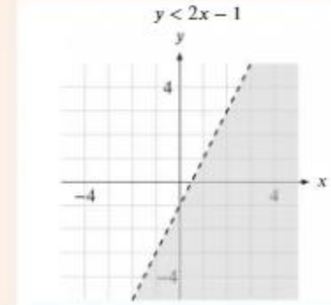



$y \ge 2x - 1$

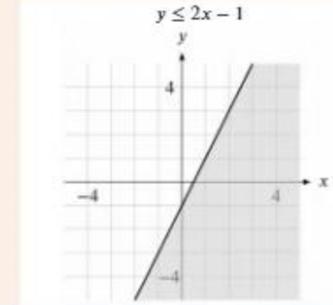



When we graph any linear inequality in Slope-Intercept Form with a less-than or less-than-or-equal-to sign, we shade below the line because any value of $y$ that is below the $y$-values on the line will satisfy the inequality.

$y < 2x - 1$




$y \le 2x - 1$




Most of the time, graphing is reserved for Systems of Linear Inequalities problems, which will be discussed later in the chapter. For now, it is sufficient that we know what it means for a point to be a solution to a linear inequality, and for many of the inequality problems, all we have to do is plug in values to check if the $x$- and $y$-values satisfy the inequality.

**SkillDrill 5.1-3**

**Directions:** For each inequality, see if each of the three following points is a solution to the inequality. List the Roman numerals of the points that are solutions to each inequality.

I. $(0, -5)$

II. $(1, 1)$

III. $(2, 4)$

**1.** $y < x$ **2.** $y \leq 2x$ **3.** $y < 2x - 5$ **4.** $y \geq 3x - 2$

**Example 5.1-3**

**3** 1600.io 5.1

Gretel rented an electric scooter. The scooter rental cost \$4 per hour, and she had to also pay for a helmet that costs \$8. Gretel spent more than \$42 for the rental and the helmet. If the scooter was available for only a whole number of hours, what was the minimum number of hours that Gretel could have rented the scooter?

**Solution**

1. Start by writing a linear expression for the total cost of Gretel's rental in Slope-Intercept Form, but instead of writing an equation, we will form an inequality.
2. There is a cost of \$4 per hour, so if she rents the scooter for $h$ hours, she will spend $4h$ dollars plus the initial one-time fee of \$8 for the helmet. Therefore, the total cost of her rental was $4h + 8$.
3. The total amount she spends has a boundary; she spent more than \$42, so her total cost must be greater than \$42. Set the expression $4h + 8$ greater than 42 in the inequality.

$$4h + 8 > 42$$

4. Subtract 8 from both sides of the inequality.

$$4h > 34$$





5. Divide both sides of the inequality by 4 in order to solve for $h$.

$$4h > 34$$
$$h > \frac{34}{4}$$
$$h > \frac{17}{2}$$

6. Since $\frac{17}{2}$ is 8.5 and Gretel had to have rented the scooter for a whole number of hours, the least number of hours she could have rented the scooter that is still greater than 8.5 is 9.
7. The answer is 9.

Linear Inequalities can be written in Standard Form as well, and the math works the same (though graphing would require converting the inequality into Slope-Intercept Form, but graphing is rarely necessary). Simply plug in any given values and check if the points satisfy the inequality.

**Example 5.1-4**

**4** 1600.io 5.1

Which of the following ordered pairs $(x, y)$ satisfies the inequality $6x - 2y < 8$ ?

I. $(0, -4)$

II. $(1, 1)$

III. $(2, 2)$

A) I only

B) II only

C) I and II only

D) II and III only

**Solution**

1. Divide both sides of the inequality by 2 (because 2 is the greatest common factor of all of the coefficients) in order to simplify the inequality and make it easier to plug in points and evaluate the results (this step is not essential, but it simplifies the arithmetic to come, and why shouldn't you make life a little easier for yourself?).

$$6x - 2y < 8$$
$$\frac{1}{2}(6x - 2y) < \frac{1}{2}(8)$$
$$3x - y < 4$$

2. For each of the given points, plug in the $x$-coordinate for $x$ and the $y$-coordinate for $y$ in the simplified form of the inequality in order to check if the resulting inequality is true (which means the point is in the solution set).

3. Start with point I, $(0, -4)$: plug in 0 for $x$ and $-4$ for $y$.

$$\begin{aligned} 3x - y &< 4 \\ 3(0) - (-4) &< 4 \\ 0 + 4 &< 4 \\ \text{✗} \quad 4 &< 4 \\ 4 &\not< 4 \end{aligned}$$

Since 4 is not less than 4, this point does not satisfy the inequality. Eliminate choices A and C, which include point I.

4. By process of elimination, point II, $(1, 1)$ will satisfy the inequality because it is included in both of the remaining answers, but if you want to be thorough, plug in 1 for $x$ and 1 for $y$.

$$\begin{aligned} 3x - y &< 4 \\ 3(1) - (1) &< 4 \\ 3 - 1 &< 4 \\ \text{✓} \quad 2 &< 4 \end{aligned}$$

Point II satisfies the inequality because 2 is less than 4.

5. Check if point III, $(2, 2)$ satisfies the inequality by plugging in 2 for $x$ and 2 for $y$.

$$\begin{aligned} 3x - y &< 4 \\ 3(2) - (2) &< 8 \\ 6 - 2 &< 4 \\ \text{✗} \quad 4 &< 4 \\ 4 &\not< 4 \end{aligned}$$

Since 4 is not less than 4, this point does not satisfy the inequality, so eliminate choice D, which includes point III.

6. The answer is B.

**Notes**

The points can simply be plugged into the inequality as provided (skipping Step 1), but here, as is sometimes the case with substitute-and-test problems, the inequality (or an equality, if that's what's being evaluated) can be simplified. While this adds a step up front (increasing the time needed and adding an opportunity for an error to be made), this investment is repaid with interest as it simplifies the substitution/evaluation procedure due to smaller (or no) coefficients and constants, making the method shown in the solution above more rapid and less error-prone.

When dealing with linear inequalities in Standard Form, sometimes you have to handle maximization or minimization problems. Remember a few chapters ago when we said that Standard Form linear equations can be though of as striking a balance between two variable quantities? The same holds true here: when two variable terms add up to a constant value, if one of those terms increases in value, the other term must decrease to compensate so the sum remains the same. In the following example, this means that in order to *maximize* one variable, we must *minimize* the other variable.





**Example 5.1-5**

**5** 1600.io 5.1

$$C = 20x + 6y$$

The formula above gives the monthly cost $C$, in dollars, of operating a printer when the technician works a total of $x$ hours and when $y$ reams of paper are used. If, in a particular month, it cost no more than \$3,800 to operate the printer and at least 100 reams of paper were used, what is the maximum number of hours the technician could have worked?

A) 160

B) 300

C) 1,800

D) 3,200

**Solution**

1. Start by rewriting the given equation as an inequality. The total cost is no more than (less than or equal to) \$3,800, so set the expression $20x + 6y$ less than or equal to 3,800.

$$20x + 6y \leq 3{,}800$$

2. Since we want to maximize the number of hours the technician worked, we need to minimize the number of reams of paper used (so that the cost of paper will be as low as possible, meaning that the maximum portion of the budget will go towards the technician's pay, which indicates that the technician worked as many hours as possible). Since they used at least 100 reams, then 100 is the lowest possible number of reams of paper in order to maximize the number of hours. Substitute 100 for $y$.

$$\begin{aligned} 20x + 6y &\leq 3{,}800 \\ 20x + 6(100) &\leq 3{,}800 \\ 20x + 600 &\leq 3{,}800 \\ 20x &\leq 3{,}200 \\ x &\leq 160 \end{aligned}$$

3. If $x \leq 160$, the maximum value $x$ can have is 160, so the answer is A.

Be aware that in real world problems, the limit of an inequality is not necessarily the answer due to additional constraints, such as a requirement that the result must be a counting number (a non-negative integer). For example, if someone is buying $b$ books that cost \$20 each, and they can spend at most \$75, then the inequality that represents the relationship is $20b \leq 75$. Solving for $b$, we see that $b \leq 3.75$, so as long as this person buys less than 3.75 books they will come in under budget. However, this person cannot buy three quarters of a book, so this person is limited to only buying 3 books. Be on the alert for problems that have an additional (sometimes implied but not directly stated) constraint that informs us that the answer must be an integer.

**Section 5.1 Suggested Problems from Real Tests**

- Test 1-C-11
- Test 1-C-28
- Test 1-C-32
- Test 2-NC-8
- Test 2-C-12
- Test 4-C-16
- Test 4-C-19
- Test 5-NC-16
- Test 5-C-21
- Apr 2017-NC-2
- Apr 2017-C-32
- May 2017-NC-8
- Mar 2019-C-35
- Apr 2019-C-26
- Mar 2020-NC-4
- Mar 2020-C-15
- Oct 2020-C-13
- Apr 2021-NC-7
- Apr 2021-C-24
- Apr 2021-C-29
- May 2021 (US)-C-14

**View related real-test problems in SMTEx**

SMTEx

## Section 5.1 Practice Problems

**1** 1600.io 5.1

Fred harvests corn at a constant rate of 1.2 acres per day. He harvested 3 acres so far this month and plans to spend $d$ days harvesting for the rest of the month. If Fred wants to harvest at least 10 acres of corn this month, which of the following inequalities best represents this situation?

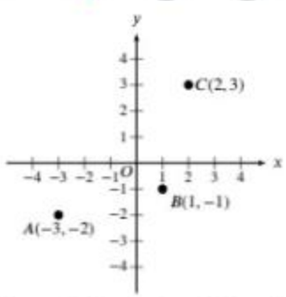
A) $3d + 1.2 \geq 10$

B) $1.2d + 3 \geq 10$

C) $1.2d - 3 \geq 10$

D) $1.2d \geq 10$

**2** 1600.io 5.1




The coordinates of points $A$, $B$, and $C$ are shown in the $xy$-plane above. For which of the following inequalities will each of the points $A$, $B$, and $C$ be contained in the solution region?

A) $x > -3$

B) $y > x - 2$

C) $y < x + 2$

D) $y < x + 1$

**3** 1600.io 5.1

$$C = 5x + y$$

The formula above gives the weekly cost $C$, in dollars, of operating a lemon press when the lemonade maker works a total of $x$ hours and when $y$ lemons are used. If, in a particular week, it cost at least \$400 to operate the lemon press and no more than 100 lemons were used, what is the least number of hours the lemonade maker could have worked?

A) 40

B) 60

C) 100

D) 400

**4** 1600.io 5.1

If $5p - 7 \geq 4$, what is the least possible value of $5p + 1$?

A) $\frac{11}{5}$

B) 5

C) 11

D) 12





## 5.2 Compound Linear Inequalities

A **Compound Linear Inequality** is when a linear expression is bounded on both sides by both an upper and lower limit. The value of the expression must be between two values. For example, the compound inequality $3 \leq x < 7$ means that $x$ is greater than or equal to 3, but $x$ is also less than 7. By convention, the lower bound is written on the left, and the upper bound is written on the right, so only the less-than and less-than-or-equal-to signs are used.

When working with compound inequalities, it can be tempting to split the inequality up into two separate inequalities, but most of the time, it is easier to just work with what you are given. Doing algebraic work on compound inequalities is simple: just apply the same operations to all three parts of the inequality. Extending the principle that applies to simpler linear inequalities that we discussed earlier, if you multiply or divide by a negative number, you have to remember to flip **both** inequality signs; when you do this, re-write the inequality in the conventional orientation, with the lower bound on the left, thus using only less-than and less-than-or-equal-to signs.

For example, let's solve for $x$ in the compound inequality $2 < -3x + 2 < 8$.

| | |
|---|---|
| $2 < -3x + 2 < 8$ | |
| $0 < -3x < 6$ | Subtract 2 from all three parts of the compound inequality. |
| $\frac{0}{-3} > x > \frac{6}{-3}$ | Divide all three parts by $-3$ and flip the inequality signs. |
| $0 > x > -2$ | |
| $-2 < x < 0$ | Rewrite the compound inequality in traditional orientation with only less-than signs. |

**SkillDrill 5.2-1**

**Directions:** Solve each compound inequality for $x$.

**1.** $2 < 5x + 2 < 10$ **2.** $-6 < 4x - 3 < 9$ **3.** $0 < -7x - 1 < 6$ **4.** $-4 < 3 - 2x < 7$

**Example 5.2-1**

**1** 1600.io 5.2

Tickets for a school's winter concert cost \$2.50 for students and \$5 for adults. If Harold spends at least \$15 but no more than \$24 on $x$ students tickets and 2 adult tickets, what is one possible value of $x$?

**Solution**

1. Start by writing an expression to represent the total cost of tickets in Standard Form. Let's use $s$ for the number of student tickets (even though they tell us to use $x$ later), and let's use $a$ for the number of adult tickets. If student tickets cost \$2.50 each, and Harold buys $s$ student tickets, then the cost of those tickets is $2.5s$. Since adult tickets are \$5 each and Harold buys $a$ of them, the price of those tickets is $5a$. Therefore, the total amount that Harold spends on tickets is $2.5s + 5a$.

2. The amount Harold spends must be between \$15 and \$24, inclusive, so we should set the expression $2.5s + 5a$ greater than or equal to 15, but also less than or equal to 24.

$$15 \le 2.5s + 5a \le 24$$

3. Since the problem tells us that Harold buys $x$ student tickets, replace $s$ with $x$. Since we know that Harold buys 2 adult tickets, replace $a$ with 2.

$$\begin{aligned} 15 \le 2.5s + 5a &\le 24 \\ 15 \le 2.5x + 5(2) &\le 24 \\ 15 \le 2.5x + 10 &\le 24 \end{aligned}$$

4. Subtract 10 from all three parts of the inequality to isolate the $x$ term.

$$\begin{aligned} 15 \le 2.5x + 10 &\le 24 \\ 5 \le 2.5x &\le 14 \end{aligned}$$

5. Divide all three parts of the inequality by 2.5 (if you like working without a calculator, you can convert 2.5 into the fraction $\frac{5}{2}$, then multiply all three parts by $\frac{2}{5}$ in order to isolate and solve for $x$).

$$\begin{aligned} 5 \le 2.5x &\le 14 \\ 2 \le x &\le 5.6 \end{aligned}$$

or

$$\begin{aligned} 5 \le 2.5x &\le 14 \\ 5 \le \frac{5}{2}x &\le 14 \\ \frac{2}{\cancel{5}}(\cancel{5}) \le \frac{\cancel{2}}{\cancel{5}}\left(\frac{\cancel{5}}{\cancel{2}}x\right) &\le \frac{2}{5}(14) \\ 2 \le x &\le \frac{28}{5} \end{aligned}$$

6. For your answer, you can fill in any whole number (integer) value between 2 and 5.6, inclusive (the valid answers must be whole numbers because $x$ represents a number of tickets, and you can't buy part of a ticket). A number like 3 is a safe choice if you are unsure whether the boundaries should be included or not, though 2 is still a valid answer because the inequality indicates that $x$ can be greater than *or equal to* 2.

**Section 5.2 Suggested Problems from Real Tests**

- Test 1-C-31
- Test 3-C-31
- Test 6-C-18
- Test 6-C-19
- Test 7-NC-14

**View related real-test problems in SMTEx**

SMTEx





## Section 5.2 Practice Problems

**1** 1600.io 5.2

Jenny can shuck at least 20 dozen oysters per hour and at most 24 dozen oysters per hour. Based on this information, what is a possible amount of time, in hours, that it could take Jenny to shuck 120 dozen oysters?

**2** 1600.io 5.2

A certain rhinoceros weighs 100 pounds at birth and gains more than 3 but less than 5 pounds per day during its first year. Which of the following inequalities represents all possible weights $w$, in pounds, for the rhinoceros 180 days after birth?

A) $640 < w < 1{,}000$

B) $540 < w < 900$

C) $300 < w < 500$

D) $280 < w < 900$

## 5.3 Systems of Linear Inequalities

We can write systems of linear inequalities just like we write systems of linear equations. The only difference is we also have to make sure that we use the correct inequality signs in all of the inequalities that we write.

**Example 5.3-1**

**1** 1600.io 5.3

A maid service is buying window cleaner and floor polish from its supplier. The supplier will deliver no more than 450 pounds in a shipment. Each container of window cleaner weighs 5.65 pounds, and each container of floor polish weighs 8.4 pounds. The service wants to buy at least three times as many containers of floor polish as containers of window cleaner. Let $w$ represent the number of containers of window cleaner, and let $f$ represent the number of containers of floor polish, where $w$ and $f$ are non-negative integers. Which of the following systems of inequalities best represents this situation?

A) $8.4w + 5.65f \le 450$
$f \ge 3w$

B) $8.4w + 5.65f \le 450$
$3f \ge w$

C) $5.65w + 8.4f \le 450$
$f \ge 3w$

D) $5.65w + 8.4f \le 450$
$3f \ge w$

**Solution**

1. We need to write one inequality that describes the total number of pounds of product bought. We know that $w$ is the number of containers of window cleaner, and each one weighs 5.65 pounds, so the total weight of window cleaner is $5.65w$. We know that $f$ is the number of containers of floor polish, and each one weighs 8.4 pounds, so the total weight of floor polish is $8.4f$. Therefore, the total weight of the supplies is $5.65w + 8.4f$. The total weight must be no more than (less than or equal to) 450 pounds, so this expression should be set less than or equal to 450 to form the first linear inequality in the system.

$$5.65w + 8.4f \leq 450$$

Eliminate choices A and B, leaving only choices C and D.

2. This step is the crucial step in this problem, and it requires careful translation. We are told that the service wants to buy at least three times as many containers of floor polish as containers of window cleaner. The most common mistake made on questions like this is mistranslating this part into the wrong inequality. **Don't just try to write this relationship based on the order that the information appears in the sentence**. Because the number 3 is presented close to the words "floor polish," you might think that $f$ should be multiplied by 3. However, this is incorrect.

Visualize the situation before you write the inequality. For example, if you had one container of window cleaner, you would need at least 3 containers of floor polish. The number of floor polishes is larger, and it is three times the number of window cleaners, so therefore we would have to multiply the number of window cleaners by 3 in order to match the number of floor polishes. We have now rephrased the statement in a way that reflects its true meaning and helps us write the inequality: *the number of floor polish containers* ***is at least*** *(greater than or equal to) 3 times the number of window cleaner containers.*

$$f \geq 3w$$

3. The set of inequalities that describe the given circumstances is the following:

$$5.65w + 8.4f \leq 450$$
$$f \geq 3w$$

4. The answer is C.

**Notes**

One straightforward way to approach the construction of inequalities (or equations) in scenarios like in Step 2 of the solution above is to start by setting up a proportion (they are covered in a later chapter); using proportions is really easy and you don't have to rely on intuition or plugging in actual values to figure out the solution. The given constraint that there must be at least three times as many containers of floor polish as containers of window cleaner simply means the ratio of the number of floor polish containers to the number of window cleaner containers must be at least 3 to 1. The phrase "at least" tells us to use a greater-than-or-equal-to sign.

$$\frac{\text{floor polish}}{\text{window cleaner}} \geq \frac{3}{1}$$

$$\frac{f}{w} \geq 3$$





Now, we just need to do some simple algebra to get the equation into the form of the answer choices. Multiply both sides of the equation by $w$.

$$f \geq 3w$$

Solving systems of linear inequalities can be confusing. Luckily, some problems gives you points as choices, and a lot of times, the easiest way to handle those problems is by plugging the answer choices into the two inequalities and seeing which points satisfy both inequalities. If a point only makes one of the inequalities true, then it is NOT a solution to the system.

**Example 5.3-2**

**2** 1600.io 5.3

$$y \geq -x + 3$$
$$y > 2x - 3$$

In the $xy$-plane, point $A$ is contained in the graph of the solution set of the system of inequalities above. Which of the following could be the coordinates of point $A$ ?

A) (2, 1)

B) (3, 0)

C) (3, 3)

D) (3, 5)

**Solution**

1. Try plugging the $x$-coordinate and $y$-coordinate of each answer choice into both inequalities to find the point that satisfies both inequalities.

2. Start with choice A, (2, 1); plug in 2 for $x$ and 1 for $y$.

$$y \geq -x + 3$$
$$y > 2x - 3$$
$$\Downarrow$$
$$1 \geq -(2) + 3$$
$$1 > 2(2) - 3$$
$$\Downarrow$$
$$1 \geq -2 + 3$$
$$1 > 4 - 3$$
$$\Downarrow$$
$$1 \geq 1$$
$$1 > 1$$
$$\Downarrow$$
$$\checkmark \quad \mathbf{1 \geq 1}$$
$$\times \quad \mathbf{1 \not> 1}$$

Even though the first inequality is true, the second inequality is not because 1 is not greater than 1. Eliminate choice A.

3. Try point B, (3, 0); plug in 3 for $x$ and 0 for $y$.

$$\begin{array}{ll} & y \geq -x + 3 \\ & y > 2x - 3 \\ & \Downarrow \\ & 0 \geq -(3) + 3 \\ & 0 > 2(3) - 3 \\ & \Downarrow \\ & 0 \geq -3 + 3 \\ & 0 > 6 - 3 \\ & \Downarrow \\ & 0 \geq 0 \\ & 0 > 3 \\ & \Downarrow \\ \checkmark & \mathbf{0 \geq 0} \\ \times & \mathbf{0 \not> 3} \end{array}$$

Even though the first inequality is true, the second inequality is not because 0 is not greater than 3. Eliminate choice B.

4. Try point C, (3, 3); plug in 3 for $x$ and 3 for $y$.

$$\begin{array}{ll} & y \geq -x + 3 \\ & y > 2x - 3 \\ & \Downarrow \\ & 3 \geq -(3) + 3 \\ & 3 > 2(3) - 3 \\ & \Downarrow \\ & 3 \geq -3 + 3 \\ & 3 > 6 - 3 \\ & \Downarrow \\ & 3 \geq 0 \\ & 3 > 3 \\ & \Downarrow \\ \checkmark & \mathbf{3 \geq 0} \\ \times & \mathbf{3 \not> 3} \end{array}$$

Even though the first inequality is true, the second inequality is not because 3 is not greater than 3. Eliminate choice C.

5. By process of elimination D is the answer, but you should check all the choices when you use plugging in as your solving method. Try point D, (3, 5); plug in 3 for $x$ and 5 for $y$.

$$\begin{array}{ll} & y \geq -x + 3 \\ & y > 2x - 3 \\ & \Downarrow \\ & 5 \geq -(3) + 3 \\ & 5 > 2(3) - 3 \\ & \Downarrow \\ & 5 \geq -3 + 3 \\ & 5 > 6 - 3 \\ & \Downarrow \\ \checkmark & \mathbf{5 \geq 0} \\ \checkmark & \mathbf{5 > 3} \end{array}$$

Both inequalities are true, so point D satisfies the system of inequalities.

6. The answer is D.





**Notes**

Here, after we eliminated choice A, we could have observed that the $x$-value for all of the remaining three answer choices is 3, so we wouldn't need to recalculate the right side of the inequalities over and over—we could just do it once, so we'd be left with the following simplified inequalities:

$$\begin{array}{l} y \geq -x + 3 \\ y > 2x - 3 \\ \Downarrow \\ y \geq -(3) + 3 \\ y > 2(3) - 3 \\ \Downarrow \\ y \geq -3 + 3 \\ y > 6 - 3 \\ \Downarrow \\ y \geq 0 \\ y > 3 \end{array}$$

Now, we just need to substitute the $y$-values for each of the choices and determine if the inequalities are valid.

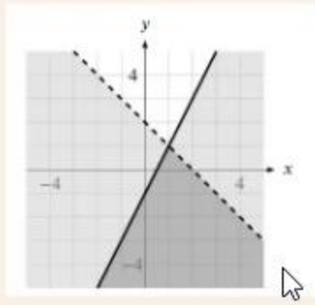
While systems of linear inequalities questions are rare, you do occasionally need to graph systems of linear inequalities and shade the region that contains the solution set. A good starting point is to write the inequalities in Slope-Intercept Form by isolating the $y$-variable. Once you have the inequalities in Slope-Intercept Form, you can begin graphing them.

When you graph a system of linear inequalities, you need to graph both linear inequalities. The region where the two shaded areas overlap is the solution set for the system of inequalities because it represents the set of all points that satisfy *both* inequalities in the system. When you need to pick a point that satisfies the system, you should limit your search to points that appear in this overlap region (or on any solid—not dashed—lines that bound the overlap region).

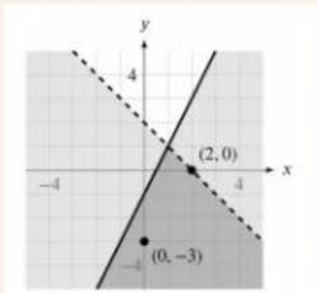
## Graphing Systems of Linear Inequalities

For this demonstration of the graphical solution to a system of linear inequalities, let's look at the following system of linear inequalities and graph them (make sure the inequalities are written in Slope-Intercept Form if you have to graph them):

$$\begin{array}{l} y \leq 2x - 1 \\ y < -x + 2 \end{array}$$



Notice the overlapping, darker region of shading. A point in this region will satisfy both inequalities, but points that are outside of the overlapping region will not. Look at the points $(0, -3)$, which is in the overlapping shaded region, and $(2, 0)$, which is in *a* shaded region, but on the dashed line and *not in the region of overlapping shading*.




If we plug these points into both inequalities in the system, the point $(0, -3)$ will satisfy both inequalities, but the point $(2, 0)$ will only satisfy the first inequality.

$$\begin{array}{ll} y \le 2x - 1 & y \le 2x - 1 \\ y < -x + 2 & y < -x + 2 \\ \Downarrow & \Downarrow \\ -3 \le 2(0) - 1 & 0 \le 2(2) - 1 \\ -3 < -(0) + 2 & 0 < -(2) + 2 \\ \Downarrow & \Downarrow \\ \checkmark \quad \mathbf{-3 \le -1} & 0 \le 4 - 1 \\ \checkmark \quad \mathbf{-3 < 2} & 0 < -2 + 2 \\ & \Downarrow \\ & \checkmark \quad \mathbf{0 \le 3} \\ & \times \quad \mathbf{0 \not< 0} \end{array}$$

Similar to systems of linear *equations*, the only time there will be no solutions to a system of linear *inequalities* is when the boundary lines are parallel (or the same line, with one or both not being solid), and no shaded regions or solid lines overlap each other at all.





**SkillDrill 5.3-1**

**Directions:** For each system of inequalities, check if each of the following points is a solution to the system (remember that a point must satisfy both inequalities in order to be a solution).

I. $(0, 6)$
II. $(1, 4)$
III. $(-2, -3)$

**1.** $2x + y \le 6$, $x + y \ge 5$

**2.** $4x + 5y < 2$, $2x - 3y \ge 1$

**3.** $x + 6y > 10$, $2x + 3y < 2$

**4.** $7x - 2y \le 4$, $y > -3$

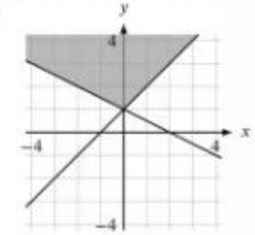
**Example 5.3-3**

**3** 1600.io 5.3

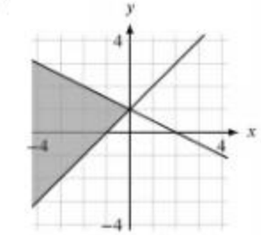
$$y \ge x + 1$$
$$x + 2y \le 2$$

In which of the following does the shaded region represent the solution set in the $xy$-plane to the system of inequalities above?

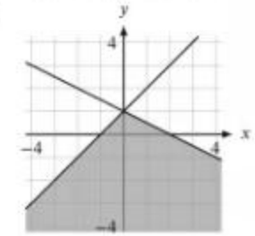
A)



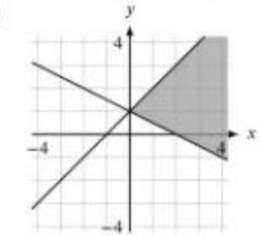

B)




C)




D)




**Solution 1**

1. If you notice immediately that, in each of the answer choices, the same two lines are graphed, then you can cut out a lot of time on this problem. The first inequality is already given to us in Slope-Intercept Form, so we can identify the line $y = x + 1$ (the line going up from left to right), so we can narrow the possible choices to those that have shading above this line. That leaves choices A and B and eliminates choices C and D.

2. For the second inequality, the line is not in Slope-Intercept Form, but since we are not going to divide by a negative number, we know that the inequality will tell us that $y$ is less than or equal to an expression, and therefore we will be shading below the second line (the line that goes down from left to right). Of the remaining choices, only choice B has shading below the second line, and we can eliminate choice A.

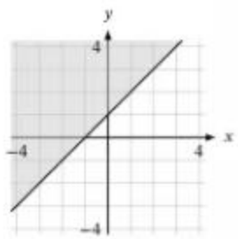
3. The answer is B.

**Solution 2**

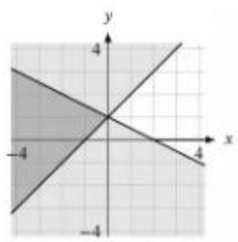
1. The first inequality is already in Slope-Intercept Form. Write the second inequality, $x + 2y \leq 2$, in Slope-Intercept Form.

$$\begin{aligned} x + 2y &\leq 2 \\ 2y &\leq -x + 2 \\ y &\leq \frac{-1}{2}x + 1 \end{aligned}$$

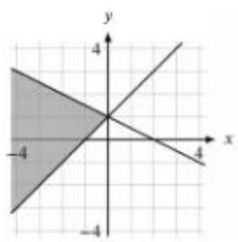
2. Graph the first inequality $y \geq x + 1$. Use a solid line to represent the line $y = x + 1$, then shade above the line.




3. Add to the graph the second inequality, $y \leq \frac{-1}{2}x + 1$. Use a solid line to represent the line $y = \frac{-1}{2}x + 1$, then shade below the line.




4. The darker shaded region (where the two shaded regions overlap) is the solution set of the system of inequalities.




5. The answer is B.

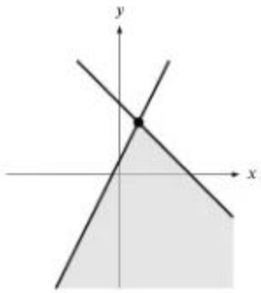




Occasionally, the exact point where the two boundary lines intersect is actually the most important point to find when solving a system of linear inequalities problem, and to determine the coordinates of that point, you simply use the equations of those boundary lines (which are just the inequalities converted to equations) and then solve the resulting system of linear equations to find the intersection point. Specifically, if a question asks about the greatest (or least) possible value of $x$ or $y$, then the intersection point of the boundary lines in the system of linear inequalities *must* be at an extreme point on the graph of the overlap region such that there is actually a maximum or minimum value of $x$ or $y$, so you don't need to figure out whether that is the case or not. Therefore, **the intersection point of the boundary lines of the inequalities in a system will always represent the maximum or minimum coordinate value you are asked to find**; you will never be asked to determine whether or not such a finite extreme value exists, so you can always assume that it does and just convert the system to one of equations and solve it. Problems of this type rarely appear on the test, but there have been a few sightings of them over the years, so we will cover the needed concepts here.

If we graph the system of linear inequalities below, we will see that the overlap region has a maximum $y$-value (there is, however, no maximum or minimum $x$-value).

$$\begin{aligned} y &\leq 2x + 1 \\ y &\leq -x + 5 \end{aligned}$$




If we look at the bounded region, we can see that the point where the two lines meet has the greatest $y$-value of any point in the bounded region (which includes both of the graphed lines) because all of the other points in the shaded region lie below the intersection point.

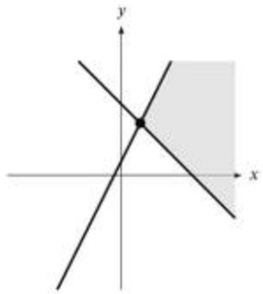
If we treat the system of linear inequalities as a system of linear equations which represent the boundary lines, we can find the intersection point with substitution (setting $2x + 1$ equal to $-x + 5$) in order to find the $x$-value of the intersection point and then we can plug that $x$-value back into either boundary line equation to find the maximum value of $y$ (alternatively, we could use elimination to solve directly for the $y$-value of the intersection point).

$$\begin{aligned} 2x + 1 &= -x + 5 \\ 3x + 1 &= 5 \\ 3x &= 4 \\ x &= \frac{4}{3} \end{aligned} \quad \Rightarrow \text{Substitute } \frac{4}{3} \text{ for } x \Rightarrow \quad \begin{aligned} y &= 2x + 1 \\ y &= 2\left(\frac{4}{3}\right) + 1 \\ y &= \frac{8}{3} + \frac{3}{3} \\ y &= \frac{11}{3} \end{aligned}$$

The intersection point of the two boundary lines is $\left(\frac{4}{3}, \frac{11}{3}\right)$, and every $y$-value in the bounded region must be less than or equal to $\frac{11}{3}$.

Of course, the $x$-value can also have a maximum or minimum in a system of inequalities (without the $y$-value having a maximum or minimum). If we graph the system of inequalities below (which has the same two boundary lines as in the previous example, but the second inequality has a different sign), we will see that the bounded region has a minimum $x$-value (while the $y$-value has no maximum or minimum).

$$y \leq 2x + 1$$
$$y \geq -x + 5$$

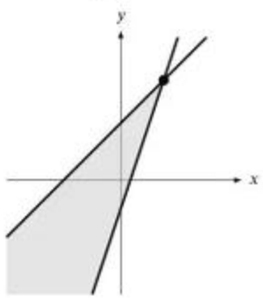



Since the two boundary lines are the same as those in the previous example, we know that the intersection point is $\left(\frac{4}{3}, \frac{11}{3}\right)$, and, in this case, the $x$-value of all points in the solution area must always be greater than or equal to $\frac{4}{3}$.

Finally, it is possible for two linear inequalities to meet in such a way that the intersection point's $x$-coordinate and $y$-coordinate *both* represent extreme values—that is, both the $x$- and $y$-coordinate of the intersection point are a maximum or minimum.

Let's graph the system of linear inequalities below.

$$y \leq x + 4$$
$$y \geq 3x - 2$$




Notice that the intersection point is both the highest up (greatest $y$-value) and furthest right (greatest $x$-value) of any of the points in the shaded region. Let's use the equations of the boundary lines, $y = x + 4$ and $y = 3x - 2$, set the expressions $x + 4$ and $3x - 2$ equal to each other to solve for the $x$-value of the intersection point, and then substitute the $x$-value into either equation to find the $y$-value of the intersection point.





$$\begin{aligned} x + 4 &= 3x - 2 \\ 4 &= 2x - 2 \\ 6 &= 2x \\ 3 &= x \end{aligned} \qquad \Rightarrow \text{Substitute 3 for } x \Rightarrow \qquad \begin{aligned} y &= x + 4 \\ y &= (3) + 4 \\ y &= 7 \end{aligned}$$

The intersection point is $(3, 7)$, and for every point in the solution region, the $x$-coordinate must be less than or equal to 3, and the $y$-coordinate must be less than or equal to 7.

One more note: because, in this last case, there is a maximum $x$-value and a maximum $y$-value, you could be asked for the maximum value of $x + y$; you couldn't answer such a question if either value did not have a maximum in the system of inequalities.

To reiterate the central principle here: **if you're asked for a maximum or minimum coordinate value in a system of linear inequalities, that coordinate *must* be at the intersection point of the boundary lines, and that's found by converting the system to one of equations and solving that.**

**Example 5.3-4**

**4** 1600.io 5.3

$$y \leq -15x + 2000$$
$$y \leq 5x$$

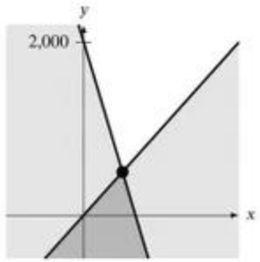
In the $xy$-plane, if a point with coordinates $(a, b)$ lies in the solution set of the system of inequalities above, what is the maximum possible value of $b$?

**Solution**

1. We need to find the point in the solution set that has the greatest $y$-value because $b$ is the $y$-value of a point in the solution set. This maximum value of $y$ *must* be the $y$-coordinate of the point of intersection of the boundary lines of the system, so we could forge ahead with converting the system to one of equations and solving. However, both of the linear inequalities in the system are in Slope-Intercept Form, so visualizing the graph of the systems is a good and simple start, even if it doesn't directly provide the answer to the question.

   Start by roughly drawing the line $y = -15x + 2000$, which has a steep negative slope of $-15$ and goes through the point $(0, 2000)$ (because the $y$-intercept is 2,000). Shade below this line.

   Next, roughly draw the line $y = 5x$, which has a much less steep positive slope of 5 (the axes do not have to be on the same scale as each other, as this is just a rough sketch to get an idea of what the graph looks like) and goes through the origin (because the $y$-intercept is 0). Shade below this line as well, then look at the area of overlapping shading.



2. We can see that the maximum $y$-value for any points in the solution area exists at the point where the two boundary lines intersect. Therefore, we can find this value by treating the two inequalities like equations and using elimination to solve for $y$.

$$\begin{aligned} y &= -15x + 2000 \\ y &= 5x \\ &\Downarrow \\ y &= -15x + 2000 \\ 3(y) &= 3(5x) \\ &\Downarrow \\ y &= -15x + 2000 \\ + \quad 3y &= 15x \\ \hline 4y &= 2000 \\ &\Downarrow \\ y &= 500 \end{aligned}$$

3. The answer is 500.

**Notes**

Of course, you can use substitution to solve for the $x$-value of the intersection point, but you have to remember to plug the value back into either equation in order to find the $y$-value that is the correct answer to the question.

**Section 5.3 Suggested Problems from Real Tests**

- Test 1-C-18
- Test 2-C-9
- Test 3-C-36
- Test 5-NC-7
- Test 5-C-13
- Test 9-NC-14
- Test 9-C-13
- Apr 2018-NC-4
- Apr 2018-C-24
- Apr 2019-C-10
- Mar 2021-NC-8
- May 2021 (US)-C-22

**View related real-test problems in SMTEx**

SMTEx





## Section 5.3 Practice Problems

**1** 1600.io 5.3

Hurston has 500 beads. He wants to make at least 4 bracelets and at least 5 necklaces. Each bracelet requires 20 beads, and each necklace requires 40 beads. If $b$ represents the number of bracelets and $n$ represents the number of necklaces, which of the following systems of inequalities represents this situation?

A) $b + n \leq 500$
$b \geq 4$
$n \geq 5$

B) $2b + 4n \leq 500$
$b \geq 4$
$n \geq 5$

C) $2b + 4n \leq 500$
$b \geq 20$
$n \geq 40$

D) $20b + 40n \leq 500$
$b \geq 4$
$n \geq 5$

**2** 1600.io 5.3

Gina is a car dealer who sells two types of cars: a \$35,000 car and a \$65,000 car. Last month, her goal was to sell at least 40 cars. While she did not meet her goal, the total value of the cars she sold was over \$1,800,000. Which of the following systems of inequalities describes $x$, the possible number of \$35,000 cars, and $y$, the possible number of \$65,000 cars, that Gina sold last month?

A) $x + y \geq 40$
$35{,}000x + 65{,}000y > 1{,}800{,}000$

B) $x + y < 40$
$35{,}000x + 65{,}000y < 1{,}800{,}000$

C) $x + y \geq 40$
$35{,}000x + 65{,}000y < 1{,}800{,}000$

D) $x + y < 40$
$35{,}000x + 65{,}000y > 1{,}800{,}000$

**3** 1600.io 5.3

$$y \leq -x + a$$
$$y \leq x + b$$

In the $xy$-plane, if $(0, 0)$ is a solution to the system of inequalities above, which of the following relationships between $a$ and $b$ must be true?

A) $a - b = 0$

B) $a \leq b$

C) $a \geq b$

D) $a + b \geq 0$

## 5.4 Absolute Value

**Absolute value refers to the *distance from zero* of a value**. As such, it is **always a positive value or zero**. The absolute value of 7 is 7 because its distance from zero is 7; the absolute value of $-9$ is 9 because its distance from zero is 9.

The absolute value of an expression is indicated by enclosing the expression in a pair of vertical bars. So, $|5|$ indicates the absolute value of 5, which is 5; $|-5|$ indicates the absolute value of $-5$, which is also 5.

This last example demonstrates a very important characteristic of the absolute value operation: because absolute value represents distance from zero, **there are always *two* values that have the same absolute value**: a positive number and the negative of that number (which happens to be called the additive inverse). The only exception to this is that zero itself is neither negative nor positive, so only the absolute value of zero yields zero.

The fact that there can be two values of an expression that yield the same absolute value means that **when solving an equation for a variable, you need to take into account both of these values**. Using the previous example, if we're told that $|x| = 5$, we must realize that this means that $x$ could be 5 or $-5$:

$$|x| = 5$$
$$\Downarrow$$
$$x = 5$$
$$x = -5$$

**This splitting of the given equation into two equations is the primary mechanism for solving problems that involve an absolute value**. To illustrate this, let's look at a more realistic example of a test problem.

**Example 5.4-1**

**1** 1600.io 5.4

$$|3x - 2| = 7$$

If $a$ and $b$ are the solutions to the equation above, what is the value of $|a - b|$ ?

**Solution 1**

This solution makes clever use of equation combination to solve for the expression $a - b$ directly rather than solving for $a$ and $b$ individually first.

1. Make two equations from the original, one where the expression $3x - 2$ is set equal to 7, and one where it is set equal to $-7$.

$$3x - 2 = 7$$
$$3x - 2 = -7$$

2. Since one solution value of $x$ is $a$ and the other solution value of $x$ is $b$, plug $a$ in for $x$ in the first equation and plug $b$ in for $x$ in the second equation. Note that it does not matter which value goes into which equation (it doesn't matter if $a$ or $b$ is the greater value), because when we take the absolute value of the difference between them, the result would be the same regardless.

$$3a - 2 = 7$$
$$3b - 2 = -7$$





3. Subtract the second equation from the first in order to arrive at an equation containing the expression $3a - 3b$, from which we can factor out a 3 and arrive at the desired expression $a - b$.

$$\begin{array}{rl} & 3a - 2 = 7 \\ - & (3b - 2 = -7) \\ \hline & 3a - 3b = 14 \end{array}$$

4. Factor a 3 out of both terms on the left side of the equation in order to isolate the expression $a - b$.

$$3a - 3b = 14$$
$$3(a - b) = 14$$

5. Divide both sides of the equation by 3 in order to find the value of $a - b$.

$$3(a - b) = 14$$
$$a - b = \frac{14}{3}$$

6. The value of $a - b$ is $\frac{14}{3}$, so $|a - b| = \frac{14}{3}$.

7. The answer is $\frac{14}{3}$.

**Solution 2**

1. Make two equations from the original, one where the expression $3x - 2$ is set equal to 7, and one where it is set equal to $-7$.

$$3x - 2 = 7$$
$$3x - 2 = -7$$

2. Solve each equation to find the two solutions. Start with the first equation, $3x - 2 = 7$.

$$3x - 2 = 7$$
$$3x = 9$$
$$x = 3$$

One of the solutions is 3.

3. Now solve the second equation, $3x - 2 = -7$ to find the second solution.

$$3x - 2 = -7$$
$$3x = -5$$
$$x = \frac{-5}{3}$$

The second solution is $\frac{-5}{3}$.

4. We are asked for the value of $|a - b|$. We know that $a$ and $b$ are the solutions of the equation, but which solution is $a$ and which one is $b$? The good news is that it doesn't matter which solution you choose for $a$ because taking the absolute value of the expression $a - b$ will ensure that we get the same value either way. Let's substitute 3 for $a$ and $\frac{-5}{3}$ for $b$.

$$|a - b|$$
$$\left|3 - \left(\frac{-5}{3}\right)\right|$$
$$\left|\frac{9}{3} + \frac{5}{3}\right|$$
$$\left|\frac{14}{3}\right|$$
$$\frac{14}{3}$$

5. The answer is $\frac{14}{3}$.

**Notes**

If you are skeptical about the respective values of $a$ and $b$ not being important to this problem, then you can try swapping them by plugging in $\frac{-5}{3}$ for $a$ and 3 for $b$, and you will see (if you do it correctly) that in the end we will have $\left|\frac{-14}{3}\right|$, which also evaluates to $\frac{14}{3}$.

Be aware, however, that some problems may have added conditions such as $a > b$. In these cases, you do have to assign values to the variables that agree with any conditions like $a > b$ that are given to you in the problem statement.

**SkillDrill 5.4-1**

**Directions:** Find the solution(s) to each absolute value equation.

**1.** $|x + 2| = 5$ **2.** $|-4x + 1| = 7$ **3.** $|2x - 5| = 17$ **4.** $|3x - 3| = 0$





Absolute value inequalities again work in a similar way to absolute value equations, except that when we produce the second resulting inequality, we have to remember to flip the inequality sign (as if we multiplied by $-1$). If you think about the meaning of absolute value, which is that it represents distance from zero, this makes perfect sense: a negative number's absolute value increases when the number becomes more negative, smaller, or farther from zero on the left side of the number line, while a positive number's absolute value increases when the number becomes more positive, larger, or farther from zero on the right side of the number line.

For example, the inequality $|x - 4| > 2$ tells us that $x - 4$ is either greater than 2 or less than $-2$, because any values greater than 2 or less than $-2$ have an absolute value that is greater than 2.

$$|x - 4| > 2 \quad \Rightarrow \quad \begin{cases} x - 4 & > 2 \\ x - 4 & < -2 \end{cases}$$

Absolute value inequalities that contain the greater-than sign result in two discontiguous simple inequalities; that is, there will be one inequality that's unbounded on the lower end and bounded by a negative value on the upper end, and there will be a second inequality that has a positive lower bound, and it will be unbounded on the upper end. In-between the two ranges will be the values that are excluded. For example, $|x - 4| > 2$ is equivalent to the two inequalities shown just above, and together they cover the range $-\infty$ to $-2$ (exclusive) and 2 (exclusive) to $\infty$, thereby excluding the range $-2$ to 2.

By contrast, absolute value inequalities with less-than signs also produce two inequalities, but they end up collapsing into a compound inequality because they define one contiguous range of values. For example, the inequality $|x - 4| < 2$ tells us that $x - 4$ must be less than 2 and also greater than $-2$.

$$|x - 4| < 2 \quad \Rightarrow \quad \begin{cases} x - 4 & < 2 \\ x - 4 & > -2 \end{cases}$$

These two inequalities will collapse into one compound inequality because the value of $x$ has a lower bound at $-2$ and an upper bound at 2, which produces the compound inequality $-2 < x - 4 < 2$.

**SkillDrill 5.4-2**

**Directions:** Find all values of $x$ that are solutions to each absolute value inequality.
Challenge problems are boxed.

**1.** $|x + 2| < 5$ **2.** $|-4x + 1| \geq 7$ **3.** $|2x - 5| \leq 17$ **4.** $\boxed{|3x - 3| > 0}$

**Section 5.4 Suggested Problems from Real Tests**

- Test 1-C-8
- Test 2-C-21
- Test 4-NC-1
- Test 4-C-26
- Test 6-C-28
- Test 10-NC-17
- Apr 2017-NC-16
- Apr 2018-C-31
- May 2021 (Int)-C-18

View related real-test problems in SMTEx

## Section 5.4 Practice Problems

**1** 1600.io 5.4

For what value of $n$ is $|n-4|+2$ equal to 0 ?

A) 2

B) 3

C) 4

D) There is no such value of $n$

**2** 1600.io 5.4

$$|8-x|=3$$

The value of one solution to the equation above is 5. What is the value of the other solution?

**3** 1600.io 5.4

Two different points on a number line are both 2 units from the point with coordinate $-5$. The solution to which of the following equations gives the coordinates of both points?

A) $|x+5|=2$

B) $|x-5|=2$

C) $|x+2|=5$

D) $|x-2|=5$





## CHAPTER 5 RECAP

- Solve inequalities like you solve equations, but remember to flip the sign if you multiply or divide by a negative number.
- The pointy or small end of the inequality sign is the lesser side of the inequality; the open or big end of the inequality sign is the greater side of the inequality.
- Plugging in values is an easy way to check answers for inequality problems.
- In real world problems, correct answers often have to be integers, particularly if the problem involves items that cannot be divided into fractional amounts (books, shirts, etc.).
- Compound inequalities can be solved in one step as long as you are careful about flipping both inequality signs if you multiply or divide by negative numbers. Just make sure to apply the same operation to all three parts of the compound inequality.
- A point is in the solution set of a system of inequalities only if it satisfies both inequalities.
- When graphing linear inequalities in the $xy$-plane, use a solid line if there is a $\leq$ or $\geq$ sign in the inequality (because points on the line **are** solutions to the inequality), and use a dashed line if there is a $<$ or $>$ sign in the inequality (because points on the line **are NOT** solutions to the inequality). Shade **above** the line to show the solution region when the $y$-variable is **greater than** the linear expression. Shade **below** the line to show the solution region when the $y$-variable is **less than** the linear expression.
- Solutions to a system of linear inequalities must satisfy all the inequalities in the system. If the inequalities are graphed on the same $xy$-plane, the solutions are located within the overlapping shaded regions and on any solid lines bounding those regions.
- You can solve absolute value equations by making two equations: one where the expression in the absolute value bars is set equal to the original value on the other side of the equation, and one where the expression is set equal to the negative of that value. For example, $|x|=5$ yields both $x=5$ and $x=-5$.

  Note that while the expression *within* the absolute value bars can have any value, positive or negative, **the absolute value of the expression can never be negative, by definition**. For example, while $|x|=5$ yields two possible solutions ($x=5$ and $x=-5$), the equation $|x|=-5$ yields **zero solutions** because the absolute value of an expression can never be negative.
- $|x+a|>b \quad \Rightarrow \quad \begin{cases} x+a & >b \\ x+a & <-b \end{cases}$
- $|x+a|<b \quad \Rightarrow \quad \begin{cases} x+a & <b \\ x+a & >-b \end{cases} \Rightarrow \quad -b<x+a<b$

## Additional Problems

**1** 1600.io 5.1

The average annual cable television cost for a certain home is \$1,347. The homeowner plans to spend \$2,400 to install satellite television. The homeowner estimates that the average annual satellite television cost will then be \$1,031. Which of the following inequalities can be solved to find $t$, the number of years after installation at which the total amount of television cost savings will exceed the installation cost?

A) $2{,}400 < 1{,}347 - 1{,}031t$

B) $2{,}400 > (1{,}347 - 1{,}031)t$

C) $2{,}400 < (1{,}347 - 1{,}031)t$

D) $2{,}400 > \frac{1{,}347}{1{,}031}t$

**2** 1600.io 5.1

Carol has \$12.00 to spend on peaches and plums. Peaches cost \$0.80 each, and plums cost \$0.45 each. If there is no tax on this purchase and she buys 12 peaches, what is the maximum number of whole plums she can buy?

**3** 1600.io 5.1

At the beginning of a recipe, Leslie had 8 cups of wine in a measuring cup. The first step of the recipe consisted of Leslie pouring $x$ cups of the wine into a saucepan and $y$ cups of the wine into a different saucepan. There remained at least 2 cups of wine in the measuring cup after the first step. Which of the following inequalities can be used to correctly represent this situation?

A) $2 - x - y \geq 4$

B) $2 - x + y \geq 4$

C) $8 - x - y \geq 2$

D) $8 - x + y \geq 2$

**4** 1600.io 5.3

$$y \geq 2x - 2$$
$$x - y < 2$$

Which of the following ordered pairs $(x, y)$ satisfies the system of inequalities above?

A) $(0, -2)$

B) $(-1, -3)$

C) $(1, 2)$

D) $(3, 3)$





**5** 1600.io 5.1

When 45 is decreased by $4x$, the result is greater than 25. What is the greatest possible integer value for $x$?

**6** 1600.io 5.3

$$y \leq -24x + 4500$$
$$y \leq 6x$$

In the $xy$-plane, if a point with coordinates $(a, b)$ lies in the solution set of the system of inequalities above, what is the maximum possible value of $b$?

**7** 1600.io 5.4

Which of the following expressions is equal to 0 for some value of $x$?

A) $|x - 10| + 2$

B) $|x + 7| - 2$

C) $|x - 2| + 1$

D) $|x - 4| + 0.5$

**8** 1600.io 5.1

A delivery van carries only 50-pound packages and 80-pound packages. For each delivery trip, the van must carry at least 20 packages, and the total weight of the packages can be at most 1,400 pounds. What is the maximum number of 80-pound packages that the van can carry per trip?

A) 13

B) 14

C) 17

D) 18

**9** 1600.io 5.3

$$y < -x + 2$$
$$2x > -7$$

Which of the following consists of the $y$-coordinates of all the points that satisfy the system of inequalities above?

A) $y < \frac{11}{2}$

B) $y < \frac{9}{2}$

C) $y < \frac{7}{2}$

D) $y > \frac{-7}{2}$

**10** 1600.io 5.1

$$4x - 12y < 36$$

Which of the following inequalities is equivalent to the inequality above?

A) $x - y < 9$

B) $3x - y < 3$

C) $3y - x < 9$

D) $x - 3y < 9$

**11** 1600.io 5.3

Students in a baking class are working in groups to make both a small cake and a large cake. A large cake requires 6 eggs and 2 cups of milk, and a small cake requires 4 eggs and 1 cup of milk. There are 66 eggs and 17 cups of milk available, and each group must have enough eggs and cups of milk to bake one large and one small cake. What is the maximum number of groups that could participate in this baking class?

**12** 1600.io 5.3

Charlotte needs to buy at least 12 new pieces of furniture for her office. The furniture will be made up of chairs, which cost \$64 each, and tables, which cost \$210 each. Her budget for the furniture is no more than \$2,200. She must buy at least 6 chairs and at least 3 tables. Which of the following systems of inequalities represents the conditions described if $x$ is the number of chairs and $y$ is the number of tables?

A) $64x + 210y \geq 2{,}200$
$x + y \leq 12$
$x \geq 6$
$y \geq 3$

B) $64x + 210y \leq 2{,}200$
$x + y \leq 12$
$x \leq 6$
$y \leq 3$

C) $64x + 210y \geq 2{,}200$
$x + y \geq 12$
$x \leq 6$
$y \leq 3$

D) $64x + 210y \leq 2{,}200$
$x + y \geq 12$
$x \geq 6$
$y \geq 3$

**13** 1600.io 5.1

Polly can spend up to a total of \$25 on beans and tortillas for a burrito dinner. Beans cost \$0.89 per can, and tortillas cost \$3.59 per pack. Which of the following inequalities represents this situation, where $b$ is the number of cans of beans Polly can buy and $t$ is the number of packs of tortillas Polly can buy? (Assume there is no sales tax.)

A) $0.89b + 3.59t \leq 25$

B) $0.89b - 3.59t \leq 25$

C) $0.89t + 3.59b \geq 25$

D) $0.89t - 3.59b \geq 25$





**14** 1600.io 5.4

A hole digger estimates that a hole will take $x$ minutes to dig, where $x > 240$. The goal is for the estimate to be within 20 minutes of the time it will actually take to dig the hole. If the digger meets the goal and it takes $y$ minutes to dig the hole, which of the following inequalities represents the relationship between the estimated time and the actual completion time?

A) $y \leq x - 20$

B) $y \geq x + 20$

C) $x - y \leq 20$

D) $-20 \leq y - x \leq 20$

**15** 1600.io 5.1

$$T = \frac{S}{B}$$

The formula above is Julian's law for download speed with time $T$, in seconds, download size $S$, in megabits, and bandwidth $B$, in megabits per second (Mbps). A music pirate has a bandwidth of 40 Mbps, and his download size will be generated by $n$ two-megabit songs that produce a total download size of $2n$ megabits. If the download must take no longer than 0.67 seconds, what is the greatest number, $n$, of two-megabit songs that can be downloaded?

**16** 1600.io 5.1

Olivier currently has 100 followers on a social media app. His goal is to have at least 2,200 followers. If he wants to meet this goal in 30 weeks, what is the minimum number of followers per week, on average, he should gain?

A) 69

B) 70

C) 71

D) 73

**17** 1600.io 5.1

$$\frac{a+b}{a} = c$$

In the equation above, if $a$ is positive and $b$ is negative, which of the following must be true?

A) $c < 0$

B) $c < 1$

C) $c = 1$

D) $c > 1$

**18** 1600.io 5.1

Every Sunday, Susie makes seashell sculptures to sell at the flea market. Each sculpture costs her \$0.50 to make, and she sells the sculptures for \$6.50 each. She also pays a vendor's fee of \$80 every Sunday to set up her booth. What is the least number of seashell sculptures Susie needs to sell every Sunday to cover the cost of the vendor's fee?

A) 14

B) 13

C) 12

D) 11

**19** 1600.io 5.2

A brewer is gathering the ingredients to make 17 batches of ale and 1 batch of stout. The stout will require one-third pound of hops. The brewer needs a total of more than 5 but less than 6 pounds of hops. What is one possible value for the fraction of one pound of hops required for each batch of ale?

**20** 1600.io 5.1

The posted weight limit for a ladder in a warehouse is 300 pounds. A worker will climb the ladder while carrying $x$ identical boxes each weighing 4 pounds. If the weight of the worker is 215 pounds, what is the maximum possible value for $x$ that will keep the combined weight of the worker and boxes below the ladder's posted weight limit?

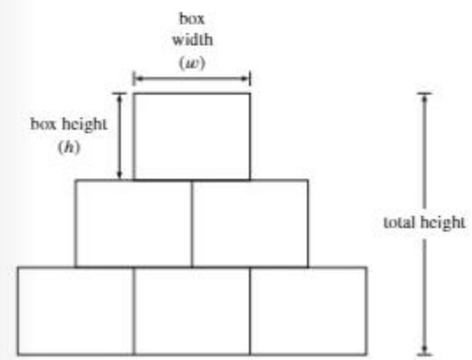
**21** 1600.io 5.4

If $|3x - 2| = 7$ and $|y + 4| = 1$, what is one possible value of $|xy|$?





**Questions 22 and 23 refer to the following information.**




Note: Figure not drawn to scale.

In a box-stacking competition, legal box pyramids must conform to the box pyramid formula $3h + w = 35$, where $h$ is the height of each box, in inches, and $w$ is the width of each box, in inches. The dimensions of every box in the pyramid are the same.

The height of each level in the pyramid is equal to the height of the boxes in the pyramid. For example, there are 3 levels in the figure above, each with a height of $h$. The total height of the box pyramid is the sum of each level's height as shown in the figure.

**22** 1600.io 5.2

Some competitions require that the width of the boxes used to stack must be no more than 14 inches, and the height of the boxes used to stack must be no more than 10 inches. According to the box pyramid formula, which of the following inequalities represents the set of all possible values for the box height that meets this requirement?

A) $h \leq 10$

B) $0 \leq h \leq 10$

C) $7 \leq h \leq 10$

D) $10 \leq h \leq 21$

**23** 1600.io 5.2

A box pyramid stacker wants to use the box pyramid formula to design a pyramid with a total height of 11 feet, a box height between 8 and 9 inches, and an even number of levels. With the stacker's constraints, which of the following must be the box width, in inches, of the boxes in the pyramid?

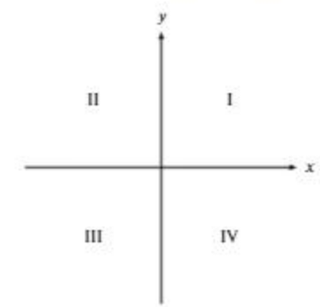
A) 8.25

B) 8.6

C) 10.25

D) 16

**24** 1600.io 5.1




If the system of inequalities $y \leq -3x + 1$ and $y < \frac{1}{4}x + 2$ is graphed in the $xy$-plane above, which quadrant contains no solutions to the system?

A) Quadrant I

B) Quadrant II

C) Quadrant IV

D) There are solutions in all four quadrants.

## Answer Key

**SkillDrill 5.1-1**

1. $2b + 3f < 56$
2. $h + 3c \geq 60$
3. $x \geq 2y + 3$
4. $3n - 5 \leq 35$
5. $8s + 20b < 112$

**SkillDrill 5.1-2**

1. $x \leq 2, x_{max} = 2$
2. $x \geq 8, x_{min} = 8$
3. $x < \frac{20}{7}, x_{max} = 2$
4. $x > 2, x_{min} = 3$
5. $x \geq \frac{2}{5}, x_{min} = 1$
6. $x \leq -10, x_{max} = -10$
7. $x > 0, x_{min} = 1$
8. $x < 6, x_{max} = 5$

**SkillDrill 5.1-3**

1. Point I only
2. Points I, II, and III
3. None
4. Points II and III

**Section 5.1 Practice Problems**

1. B
2. C
3. B
4. D

**SkillDrill 5.2-1**

1. $0 < x < \frac{8}{5}$
2. $\frac{-3}{4} < x < 3$
3. $-1 < x < \frac{-1}{7}$
4. $-2 < x < \frac{7}{2}$

**Section 5.2 Practice Problems**

1. $5 \leq h \leq 6$
2. A

**SkillDrill 5.3-1**

1. Points I and II
2. Point III only
3. None
4. Points I and II

**Section 5.3 Practice Problems**

1. D
2. D
3. D

**SkillDrill 5.4-1**

1. $x = -7, x = 3$
2. $x = \frac{-3}{2}, x = 2$
3. $x = -6, x = 11$
4. $x = 1$

**SkillDrill 5.4-2**

1. $-7 < x < 3$
2. $x \leq \frac{-3}{2}$ or $x \geq 2$
3. $-6 \leq x \leq 11$
4. $x < 1$ or $x > 1$
   which means $x \neq 1$

**Section 5.4 Practice Problems**

1. D
2. 11
3. A

**Additional Problems**

1. C
2. 5
3. C
4. C
5. 4
6. 900
7. B
8. A
9. A
10. D
11. 5
12. D
13. A
14. D
15. 13
16. B
17. B
18. A
19. $\frac{14}{51} < a < \frac{1}{3}$ or $.275 \leq a \leq .333$
20. 21
21. 5, $\frac{25}{3}$, 8.33, 9, or 15
22. C
23. C
24. D



# Exponents and Radicals/Roots 6

## 6.1 What are Exponents?

Up until this point, we have dealt exclusively with linear equations, but in order to round out the skills needed to master this test, we need to go beyond just $x$ and $y$, and learn how to solve equations that have $x^n$ terms, where $n$ is any real number acting as the power of $x$.

For those who are unaware, that little $n$ above the $x$ in the expression $x^n$ is the **exponent** and $x$ is the **base**. The value of such a term is a **power** of the base, and the expression $x^n$ is read as "the $n$-th power of $x$" or "$x$ to the $n$-th power" and refers to the product of multiplying the base, $x$, $n$ times. The bases and exponents can be any number (including fractions, zero, and negative numbers, which we will talk more about later).

For example, the expression $3^4$ is read as "the fourth power of three" or "three to the fourth power" (often shortened to "three to the fourth") and is equal to $3 \times 3 \times 3 \times 3$ (the product of multiplying the base, 3, four times) or 81, and the expression $2^5$ is read as "the fifth power of two" or "two to the fifth power" ("two to the fifth") and is equal to $2 \times 2 \times 2 \times 2 \times 2$ (the product of multiplying the base, 2, five times) or 32.

If the base is raised to the second power, such as in the expression $x^2$, we will usually read the expression as "$x$ squared" instead of "$x$ to the second power" because the area of a square with side length $x$ would be equal to $x^2$. Similarly, if the base is raised to the third power, such as in the expression $x^3$, we will usually read the expression as "$x$ cubed" instead of "$x$ to the third power" because the volume of a cube with side length $x$ is equal to $x^3$.

### Exponents

Exponential terms, or powers, are of the form $a^b$, where $a$ is the base number and $b$ is the exponent. The value of $a^b$ is found by multiplying the base ($a$) by itself $b$ times.

For example, $2^4$ is equal to $2 \times 2 \times 2 \times 2$ or 16.

Take note that **any base (including a base of 0) raised to the zeroth power is equal to 1**, and **any base raised to the first power is equal to itself**.

| | | | | | |
|---|---|---|---|---|---|
| $2^0$ | $= 1$ | $3^0$ | $= 1$ | $40^0$ | $= 1$ |
| $2^1$ | $= 2$ | $3^1$ | $= 3$ | $40^1$ | $= 40$ |
| $2^2 = 2 \times 2$ | $= 4$ | $3^2 = 3 \times 3$ | $= 9$ | $40^2 = 40 \times 40$ | $= 1600$ |
| $2^3 = 2 \times 2 \times 2$ | $= 8$ | $3^3 = 3 \times 3 \times 3$ | $= 27$ | $40^3 = 40 \times 40 \times 40$ | $= 64{,}000$ |

The base **1 raised to any power is always equal to 1**.

The base **0 raised to any *positive* power is always equal to 0**. Remember that we already established that any base (including 0) raised to the zeroth power is 1 ($0^0 = 1$). Negative powers of 0, which are discussed later in the chapter, result in undefined values.

**Positive numbers raised to any power are always positive**.

**Negative numbers raised to even powers are positive** because all negative factors will be be present in pairs (the product of two negative factors is positive). For example, $(-3)^2 = -3 \times -3 = 9$. When you are using your calculator, make sure to put the negative base in parentheses just in case; entering $-3^2$ may get a result of $-9$, which is different from entering $(-3)^2$, which will yield 9.

**Negative numbers raised to odd powers are negative** because there will always be one unpaired negative factor. For example, $(-3)^3 = -3 \times -3 \times -3 = -27$.

For any values of $x$ (where $x$ is not equal to 0, 1, or $-1$), we know that **if $x^a = x^b$, then $a = b$**, because $x$ raised to one power cannot be equal to $x$ raised to a different power.

Of course, we can add terms like $2^3$ and $2^2$, where the base is a number, by evaluating the two exponential terms and then summing the two numbers.

$$2^3 + 2^2$$
$$8 + 4$$
$$12$$

Note that **if the base is an unknown value such as $x$, we CANNOT simply combine terms with different exponents**. For example, $x^3 + x^2$ can be rewritten in various ways, but we cannot collapse these two into one term with $x$ raised to a power, even though the base is the same in both terms.

However, **if the bases and exponents are the same, the terms can be grouped with addition**. Just as $2x + 3x = 5x$, and $2xy + 3xy = 5xy$, so too does $2x^2 + 3x^2 = 5x^2$.

**SkillDrill 6.1-1**

**Directions:** Evaluate the given exponential expression as a quantity with no exponent (an exponent of 1). For example, rewrite $2^2$ as 4.

**1.** $3^3$ **2.** $4^2$ **3.** $10^0$ **4.** $(-3)^4$

**5.** $0^{20}$ **6.** $1^{100}$ **7.** $(-5)^3$ **8.** $17^1$

**9.** $2^2 + 2^3$ **10.** $3^3 - 4^2$ **11.** $0^0 + 0^1$ **12.** $1^3(2^5)$





**Directions:** Rewrite the expressions in the form $a^b$ if possible and evaluate any numerical values. For example, rewrite $x \cdot x$ as $x^2$ and rewrite $2 \cdot 2 \cdot 2 \cdot x \cdot x$ as $8x^2$.

If there is more than one base involved in a multiplication, rewrite the expressions in the form $a^b \cdot c^d \cdot \ldots \cdot y^z$. For example, rewrite $x \cdot x \cdot y \cdot y$ as $x^2y^2$.

Combine added terms if possible. For example, you should go no further if you arrive at an expression like $x^2 + x^3$ or $x^2 + y^2$; however, you should combine terms like $2^4 + 5^2$ as 41 or $x^2 + x^2$ as $2x^2$.

**13.** $x \cdot x$ **14.** $4 \cdot 4 \cdot y \cdot y$ **15.** $x + x$ **16.** $x \cdot x \cdot x \cdot y$

**17.** $(a \cdot a) + (a \cdot a) + (a \cdot a)$ **18.** $(b \cdot b) - (b \cdot b \cdot b)$ **19.** $x \cdot y + y \cdot x$ **20.** $x \cdot x \cdot y + x \cdot y \cdot y$

**21.** $a \cdot a \cdot b \cdot b \cdot c \cdot c$ **22.** $2 \cdot x \cdot y \cdot 2 \cdot x$ **23.** $d \cdot x + d \cdot d \cdot d$ **24.** $5 \cdot x \cdot x - 3 \cdot x \cdot x$

## 6.2 Exponent Rules

### Rule 1: To Multiply Terms with the Same Base, Add Their Exponents

**When two terms with the same base are multiplied by each other, their exponents will be added together**. You probably already use this rule without realizing it when you multiply terms that have an implied exponent of 1; it will not come as any surprise that $x \cdot x = x^2$, but you might not have realized that this could be written as $x^1 \cdot x^1 = x^{1+1} = x^2$. The rule is being used right there, hiding in plain sight.

Let's look at an example where the exponents are something other than 1:

$$2^3 \cdot 2^2 = 2^{3+2} = 2^5$$

To demonstrate why this is true, let's expand out both of the terms and recombine them.

$$2^3 \times 2^2$$
$$(2 \times 2 \times 2) \times (2 \times 2)$$
$$2 \times 2 \times 2 \times 2 \times 2$$
$$2^5$$

**Add Exponents When Multiplying Terms with the Same Base**

When two exponential terms with the same base, $a$, are multiplied by each other, the exponents are added to each other.

$$a^b \cdot a^c = a^{b+c}$$

For example, $3^4 \cdot 3^5 = 3^{4+5} = 3^9$. Similarly, $x^5 \cdot x^2 = x^{5+2} = x^7$.

In order for the exponents to be added, the bases MUST be the same.

As has been previously shown, $x^1 = x$. If we have an expression that requires the multiplication of terms with the same base, and one or more of those terms has a variable without an explicitly indicated exponent, it can be considered to be raised to the first power and written as such to facilitate the adding of the exponents (you probably won't do this if you are comfortable working with exponents, but we'll do it for demonstration purposes).

For example, to evaluate the expression $6x(2x^3)$, we can rewrite $6x$ as $6x^1$ in order to make it clear what the exponents will be when multiplying the two $x$ terms. The coefficients 6 and 2 will be multiplied by each other normally, and we will use the exponent addition rule to find the product of the $x$ terms.

$$6x(2x^3)$$
$$6x^1(2x^3)$$
$$6(2)(x^1 \cdot x^3)$$
$$12(x^{1+3})$$
$$12x^4$$

**SkillDrill 6.2-1**

**Directions:** Rewrite the expressions in the form $a^b$ if possible and evaluate any numerical values. For example, rewrite $x^2 \cdot x^3$ as $x^5$ and rewrite $2^2 \cdot 2^3 \cdot x^2 \cdot x^3$ as $32x^5$.

If there is more than one base involved in a multiplication, rewrite the expression in the form $a^b \cdot c^d \cdot \ldots \cdot y^z$. For example, rewrite $x^2 \cdot x \cdot y^4 \cdot y^6$ as $x^3y^{10}$.

Combine added terms if possible. For example, you should go no further if you arrive at an expression like $x^2 + x^3$ or $x^2 + y^2$; however, you should combine terms like $2^4 + 5^2$ as 41 or $x^2 + x^2$ as $2x^2$.

Challenge problems are boxed.

**1.** $x^2 \cdot x^2$ **2.** $4^2 \cdot 4 \cdot y^3 \cdot y^4$ **3.** $x \cdot x^4 + x^2 \cdot x^3$ **4.** $x^2 \cdot x^5 \cdot x^7 \cdot y^5$

**5.** $(a^2 \cdot a^4) \cdot (a^4 \cdot a)$ **6.** $(b \cdot b^3) - (b^2 \cdot b \cdot b^2)$ **7.** $(x^2 \cdot y^5) \cdot (x^3 \cdot y^2)$ **8.** $2^0 \cdot 2^0$

**9.** $\boxed{x^3 \cdot x^5 \cdot y + x \cdot y \cdot y^4}$ **10.** $\boxed{2^3 \cdot a^3 \cdot a^3 \cdot 2^2 \cdot b^2 \cdot b^4 \cdot 2 \cdot c \cdot c}$

**11.** $\boxed{3 \cdot d^4 \cdot d^3 \cdot x \cdot x^2 + 3^2 \cdot d^2 \cdot d^5 \cdot x^2 \cdot x}$ **12.** $\boxed{5 \cdot x^3 \cdot x^6 - 5 \cdot x^8 \cdot x}$





**Example 6.2-1**

**1** 1600.io 6.2

Which of the following is equivalent to $3x\left(x^2 + 4x\right)$?

A) $15x^2$
B) $7x^3 + x^2$
C) $3x^3 + 4x$
D) $3x^3 + 12x^2$

**Solution**

1. Distribute $3x$ to both terms in the parentheses.

$$3x\left(x^2 + 4x\right)$$
$$3x\left(x^2\right) + 3x(4x)$$

2. The $3x$ terms are really 3 times $x^1$ (the 3 is not part of the base), and by the same token, the $4x$ term can be considered to be 4 times $x^1$. The constants can just be multiplied by each other, but we need to make use of the exponent addition rule to combine and simplify the exponential terms.

$$3x\left(x^2\right) + 3x(4x)$$
$$3x^1\left(x^2\right) + 3x^1\left(4x^1\right)$$
$$3\left(x^1 \cdot x^2\right) + 3(4)\left(x^1 \cdot x^1\right)$$
$$3x^{1+2} + 12x^{1+1}$$
$$3x^3 + 12x^2$$

3. The answer is D.

## Rule 2: To Raise an Exponential Term to Another Power, Multiply the Exponents

**When an exponential term is raised to another power, the exponents are multiplied**. For example, $\left(2^3\right)^2 = 2^{3(2)} = 2^6$.

This can be demonstrated as follows:

$$\left(2^3\right)^2$$
$$(2 \times 2 \times 2) \times (2 \times 2 \times 2)$$
$$2 \times 2 \times 2 \times 2 \times 2 \times 2$$
$$2^6$$

If there are multiple terms to which the exponentiation applies, and the terms are all subject only to multiplication or division, every such term's exponent gets multiplied by the exponent of the whole expression; that is, the exponent is distributed through the exponents of the multiplied terms.

For example,

$$(xy)^2 = x^2y^2$$

This can be demonstrated as follows:

$$(xy)^2$$
$$(xy)\cdot(xy)$$
$$x\cdot y\cdot x\cdot y$$
$$x\cdot x\cdot y\cdot y$$
$$x^2y^2$$

As another example, if we have $\left(x^2y^4\right)^3$, we can distribute the multiplication by the whole expression's exponent of 3 to the exponent of each of the terms in the base expression because they are multiplied by each other.

$$\left(x^2y^4\right)^3$$
$$\left(x^2\right)^3\left(y^4\right)^3$$
$$x^{2(3)}y^{4(3)}$$
$$x^6y^{12}$$

On the other hand, if the expression has addition or subtraction, you CANNOT distribute the multiplication of the exponent.

For example, if we have $(x+y)^2$, we can't distribute the multiplication of the whole expression's exponent of 2 to get $x^2+y^2$ because the terms inside the original expression are added to each other (we'll see how to evaluate $(x+y)^2$ correctly in the next chapter).

This idea is more easily demonstrated using numerical constants instead of variables. For example, we can show that $(2+5)^2 \neq 2^2+5^2$.

Add the two terms in the parentheses to see that $(2+5)^2$ must be equal to $(7)^2$, which is 49. However, $2^2+5^2 = 4+25$, which is 29, so the two expressions are not equal to each other.

## Multiply Exponents When Raising an Exponential Term to a Power

When an exponential term $a^b$ is raised to another power $c$, the exponents are multiplied.

$$\left(a^b\right)^c = a^{bc}$$

One interesting consequence of this fact is that $\left(a^b\right)^c = \left(a^c\right)^b$, because multiplication is commutative. Note that this holds true all the time when the base, $a$ is non-negative, but may have weird consequences when $a < 0$. Most likely, you will not have to deal with any of those consequences on the test, and the rule should be good enough as presented above.

If there are multiple terms to which the exponentiation applies, and the terms are all subject only to multiplication or division, every such term's exponent gets multiplied by the exponent of the whole expression:

$$(ab)^c = a^cb^c$$

$$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$$





**SkillDrill 6.2-2**

**Directions:** Rewrite the expressions in the form $a^b$ if possible and evaluate any numerical values. For example, rewrite $\left(x^2\right)^4$ as $x^8$ and rewrite $\left(2^2\cdot x^3\right)^3$ as $64x^9$.

If there is more than one base involved in a multiplication, rewrite the expression in the form $a^b\cdot c^d\cdot\ldots\cdot y^z$. For example, rewrite $\left(x^2\cdot y^4\right)^6$ as $x^{12}y^{24}$.

Combine added terms if possible, but do not expand or factor. For example, you should go no further if you arrive at an expression like $\left(x^2+x^3\right)^2$ or $\left(x^2+y^2\right)^3$; however, you should combine terms like $\left(2^4+3^2\right)^2$ as 625 or $\left(x^2+x^2\right)^4$ as $16x^8$.

Challenge problems are boxed.

1. $\left(x^3\right)^2$
2. $\left(2^2\cdot 2\right)^2$
3. $\left(a^2\cdot a^4\right)^3$
4. $(x^2\cdot y^5)^7$
5. $\left(2b\cdot b^3\right)^4-(b\cdot b)^8$
6. $\boxed{\left(x^3\cdot y^5\right)^2+(x\cdot y\cdot y)^5}$
7. $\boxed{\left(x\cdot x^4+x^2\cdot x^3\right)^3}$
8. $\boxed{\left((x^2)^4+(y^5)^2\right)^2}$

**Example 6.2-2**

**2** 1600.io 6.2

If $\left(x^{16}\right)^a = \left(x^4\right)^4$, and $x > 1$, what is the value of $a$?

A) 0

B) $\frac{1}{2}$

C) 1

D) 2

**Solution**

1. Distribute the exponents on each side of the equation.

$$x^{16a} = x^{16}$$

2. The two exponents must be equal since $x$ raised to one power cannot be equal to $x$ raised to a different power (the exceptions of $x$ being equal to $-1$, 0, or 1 are taken out of play due to the restriction that $x > 1$). Therefore, $16a = 16$, and dividing both sides of this equation by 16 tells us that $a = 1$.

3. The answer is C.

## Rule 3: Fractional Exponents Can Be Written as Roots and Vice Versa

When a base is raised to an exponent that is a fraction, the expression can be written as a radical (or root) expression (which is denoted by the symbol $\sqrt{\phantom{x}}$) and vice versa.

The square root is probably the most familiar radical. Taking the square root of a number is the same as raising the number to the $\frac{1}{2}$ power:

$$a^{\frac{1}{2}} = \sqrt{a}$$

Taking a square root is the opposite of squaring a number. For example, $\sqrt{4} = 2$ because $2^2 = 4$; notice that when squaring, the exponent is 2, and when taking a square root, the exponent is the reciprocal of 2, or $\frac{1}{2}$. While it is technically true that $(-2)^2 = 4$ as well, by convention, **any even roots of numbers are always positive only** (we will discuss this in more depth in a later chapter).

Note that this rule about even roots being positive by convention applies regardless of how the root is written; just as $\sqrt{4}$ evaluates to 2 only (not also to $-2$), $4^{\frac{1}{2}}$ also evaluates to 2 only, because it means exactly the same thing. This can produce puzzling outcomes; for example, if we apply the exponentiation of a term with exponents rule to simplify $\left(x^2\right)^{\frac{1}{2}}$, we would conclude that $\left(x^2\right)^{\frac{1}{2}} = x^{\frac{2}{2}} = x^1 = x$, and this seems to make sense (squaring and square-rooting appear to be reverse operations of each other), but this is not necessarily true, because the squaring operation $x^2$ produces the same result regardless of whether $x$ is positive or negative. When we then apply the square root operation (raising $x^2$ to the $\frac{1}{2}$ power), and we end up with only the positive root by convention, that wouldn't match the original value of $x$ if $x$ were a negative number. For example, while $\left(2^2\right)^{\frac{1}{2}} = 2$, interestingly enough, $\left((-2)^2\right)^{\frac{1}{2}}$ also evaluates to 2, and, of course, $-2 \neq 2$. This information will not change how any problems are solved on the test, but you might spot an unexpected absolute value operation in an answer choice (which deals with this behavior for you), so students do not need to be concerned with navigating such cases. Nevertheless, it is an interesting phenomenon to be aware of.

Let's return to our discussion of roots/radicals and fractional exponents. Just as the square (second) root can be written in two ways, the cubed root of $a$ is written as $\sqrt[3]{a}$ (notice the small 3 above the hook which differentiates it from the square root), and is the same as raising $a$ to the $\frac{1}{3}$ power:

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

Taking a cubed root of a number is the opposite of cubing a number. For example $\sqrt[3]{8} = 2$ because $2^3 = 8$, and $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$. Again, these complementary operations are reflected in the exponents being the reciprocals of each other: cubing a number means the exponent will be 3, and taking the cube root means the exponent will be $\frac{1}{3}$.

In general, it is true that $x^{\frac{1}{c}} = \sqrt[c]{x}$.

But what happens when the numerator of the fraction is not 1? For example, the expression $a^{\frac{4}{3}}$ could be written as either $\left(a^4\right)^{\frac{1}{3}}$ or as $\left(a^{\frac{1}{3}}\right)^4$ (an infinite number of less useful representations exist also) because in both these cases the exponents would multiply to $\frac{4}{3}$. Therefore, the expression $a^{\frac{4}{3}}$ can be rewritten as either $\sqrt[3]{a^4}$ or $\left(\sqrt[3]{a}\right)^4$. Note once again that there are exceptions to this rule when $a < 0$, but the rule will hold as long as the base, $a$, is non-negative (you should not have to deal with any other cases, or at least not often).





### Fractional Exponents & Roots/Radicals

When a positive base $a$ is raised to an exponent that is a fraction $\frac{b}{c}$, the expression can be written as a radical (or root) expression (which is denoted by the symbol $\sqrt{\phantom{x}}$) and vice versa.

$$a^{\frac{b}{c}} = \sqrt[c]{a^b} = \left(\sqrt[c]{a}\right)^b$$

In order to apply the rules for multiplying and exponentiating terms with exponents, we first have to have terms with exponents! Therefore, if you have any terms that are written in radical format, convert them to exponent form so those rules can be applied. Remember, when a radical is converted to exponent form, you'll get a fractional exponent. **Always rewrite radical expressions as fractional exponents before attempting to apply exponent rules and simplify expressions.**

For example, the expression $x\sqrt{x}$ can be simplified to just $x$ raised to a power if we start by rewriting $\sqrt{x}$ as $x^{\frac{1}{2}}$ and we then apply the rule for adding exponents when two terms with the same base are multiplied.

$$x\sqrt{x}$$
$$x^1 \cdot x^{\frac{1}{2}}$$
$$x^{1+\frac{1}{2}}$$
$$x^{\frac{2}{2}+\frac{1}{2}}$$
$$x^{\frac{3}{2}}$$

Working in the reverse direction, if the exponent is an improper fraction (a fraction that has a larger numerator than denominator), the exponent can be split into a whole number part and the remaining proper fraction, which allows the term to be re-written as the product (multiplication) of the base with an integer exponent and a radical.

For example, if we have $x^{\frac{8}{3}}$, we can rewrite the improper fraction exponent of $\frac{8}{3}$ as the sum of a whole number and a proper fraction, $\frac{6}{3} + \frac{2}{3}$ or $2 + \frac{2}{3}$, giving us $x^{\left(2+\frac{2}{3}\right)}$. Recall that when multiplying terms with exponents, we add the exponents, and the reverse is true, too: we can convert a base with an addition expression in the exponent into a multiplication. Therefore, we can rewrite this expression as $x^2 \cdot x^{\frac{2}{3}}$. We can then convert the second term into a radical, giving us $x^2\sqrt[3]{x^2}$.

**SkillDrill 6.2-3**

**Directions:** Rewrite the radical expressions in the form $a^b$, where $b$ is a fraction, if possible, and evaluate any numerical values. For example, rewrite $\sqrt[3]{x^2}$ as $x^{\frac{2}{3}}$.

Challenge problems are boxed.

1. $\sqrt{a}$
2. $\sqrt[3]{a^2}$
3. $\sqrt[3]{8h^3}$
4. $\sqrt[9]{x^3}$
5. $\sqrt{16x^4}$
6. $\sqrt[4]{a^3}$
7. $\boxed{\sqrt[4]{16x^3}}$
8. $\sqrt[5]{x^2y^{10}}$

**Directions:** Rewrite the following improper fractions as an integer plus a proper fraction in lowest terms. For example, rewrite $\frac{14}{4}$ as $3 + \frac{1}{2}$.

**9.** $\frac{4}{3}$ **10.** $\frac{7}{5}$ **11.** $\frac{5}{2}$ **12.** $\frac{13}{10}$

**13.** $\frac{9}{6}$ **14.** $\frac{11}{5}$ **15.** $\frac{10}{4}$ **16.** $\frac{23}{5}$

Just as exponents are applied across multiplied terms—for example, $(xy)^n = x^n y^n$—so too are radicals (which are just an alternative representation of exponents). For example, the expression $\sqrt{4x}$ can be split into $\sqrt{4}\sqrt{x}$ and then rewritten as $2\sqrt{x}$.

Sometimes, you'll encounter a base in a square root and the base won't be a perfect square (that is, the square root of the base isn't an integer), so you can't just evaluate the term and get a nice round number. In order to match the formats of given answer choices, you may need to split the number into factors (looking for perfect square factors like 4, 9, 25, etc.) which will allow you to rewrite the expression to make it more useful.

For example, 24 is not a perfect square (there is no integer which, when squared, equals 24), so we can't evaluate $\sqrt{24}$ and end up with a nice integer result. However, there might be a factor of 24 that *is* a perfect square, and we could therefore pull that factor out of the radical by taking its convenient, integer square root in order to make the expression more useful. Indeed, by examining the factors of 24 (which are 2, 3, 4, 6, 8, and 12), we notice that 4 *is* a perfect square (4 is equal to $2^2$). If we factor 24 into 4(6), we can exfiltrate (as they say in the CIA) the 4 from within the radical.

The square root is distributed to both factors; remember that radicals and exponents are just two different ways of writing the same thing so the same rules apply regardless of which notation is used. Therefore, the radical can distribute to both the 4 and the 6 because they are multiplied by each other.

$$\sqrt{24} = \sqrt{4(6)} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$$

Note that this is exactly equivalent to distributing a fractional exponent; it's just another way of writing the terms.

$$[4(6)]^{\frac{1}{2}} = 4^{\frac{1}{2}}6^{\frac{1}{2}} = 2 \cdot 6^{\frac{1}{2}}$$

This is true for other roots as well, not just for square roots. For example, if we have $\sqrt[3]{16}$, we can't evaluate it, because 16 isn't a perfect cube (there's no integer which, when cubed, equals 16). However, there might be a factor of 16 that *is* a perfect cube, and we could therefore pull that factor out of the radical in order to make the expression more useful. Indeed, by examining the factors of 16 (which are 2, 4, and 8), we notice that 8 *is* a perfect cube (it's 2 cubed), so if we factor 16 into 8(2), we can exfiltrate 8 from within the radical

$$\sqrt[3]{16} = \sqrt[3]{8(2)} = \sqrt[3]{8}\sqrt[3]{2} = 2\sqrt[3]{2}$$

Again, because radicals and exponents are just two ways of writing the exact same thing, the above is completely equivalent to the following:

$$16^{\frac{1}{3}} = [8(2)]^{\frac{1}{3}} = 8^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 2 \cdot 2^{\frac{1}{3}}$$

Be aware that **it does you no good to rewrite the base as an expression involving addition or subtraction**, because the radical—which, you'll recall, is just another way to write an exponent—**CANNOT** be distributed to added terms. For example, $\sqrt{10}$ should NOT be rewritten as $\sqrt{4+6}$ because the radical cannot be distributed to added terms; that is, $\sqrt{4+6} \neq \sqrt{4} + \sqrt{6}$.





## Splitting Radical Expressions

Radicals are another way to represent fractional exponents, so radicals can be distributed across terms that are multiplied.

When taking a square root of a number that is not a perfect square, look for any factors of the number that are perfect squares and rewrite the number as the perfect square factor times another number so that the square root can be distributed to both factors. For example, $\sqrt{18} = \sqrt{9(2)} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$

When taking a cube root of a number that is not a perfect cube, look for any factors of the number that are perfect cubes and rewrite the number as the perfect cube factor times another number so that the cube root can be distributed to both factors. For example, $\sqrt[3]{54} = \sqrt[3]{27(2)} = \sqrt[3]{27}\sqrt[3]{2} = 3\sqrt[3]{2}$

**SkillDrill 6.2-4**

**Directions:** Factor the following integers into a perfect square and another number. For example, rewrite 27 as 9 · 3, where 9 is the perfect square and 3 is the other number.

Some items have multiple answers; see if you can find them all.

**1.** 24 **2.** 12 **3.** 45 **4.** 50

**5.** 32 **6.** 98 **7.** 72 **8.** 48

**Directions:** Factor the following integers into a perfect cube and another number. For example, rewrite 24 as 8 · 3, where 8 is the perfect cube and 3 is the other number. (The most likely perfect cubes that you will encounter are $8 = 2^3$ and $27 = 3^3$, but possibly $64 = 4^3$ and $125 = 5^3$).

**9.** 32 **10.** 80 **11.** 54 **12.** 250

**Directions:** Simplify the square root expression by splitting the radical into the square root of a perfect square times the square root of another number. For example, rewrite $\sqrt{27}$ as $\sqrt{9}\sqrt{3}$, where 9 is the perfect square and 3 is the other number, and then simplify to $3\sqrt{3}$ as your answer. If the other number can also be split and simplified, then continue the process until the number in the radical sign has no more perfect square factors.

**13.** $\sqrt{24}$ **14.** $\sqrt{12}$ **15.** $\sqrt{45}$ **16.** $\sqrt{50}$

**17.** $\sqrt{32}$ **18.** $\sqrt{98}$ **19.** $\sqrt{72}$ **20.** $\sqrt{48}$

**Directions:** Rewrite the exponential expressions in the form $a^b\sqrt[c]{a^d}$, where $b$, $c$, and $d$ are whole numbers, and evaluate any numerical values. If the numerator of a fractional exponent is larger than its denominator (an "improper" fraction), break the fraction into a whole number and a proper fraction in lowest terms, then split off and convert the proper fraction portion of the exponent into a radical expression. For example, rewrite $\left(x^5\right)^{\frac{1}{2}}$ as $x^2\sqrt{x}$

$$\left(x^5\right)^{\frac{1}{2}} \Rightarrow x^{\frac{5}{2}} \Rightarrow x^{2+\frac{1}{2}} \Rightarrow x^2\left(x^{\frac{1}{2}}\right) \Rightarrow x^2\sqrt{x}$$

and rewrite $\left(32x^4\right)^{\frac{1}{3}}$ as $2x\sqrt[3]{4x}$.

$$\left(32x^4\right)^{\frac{1}{3}} \Rightarrow 32^{\frac{1}{3}}x^{\frac{4}{3}} \Rightarrow (8(4))^{\frac{1}{3}}x^{1+\frac{1}{3}} \Rightarrow 8^{\frac{1}{3}}4^{\frac{1}{3}}x\left(x^{\frac{1}{3}}\right) \Rightarrow 2x\sqrt[3]{4x}$$

**21.** $x^{\frac{1}{2}}$ **22.** $y^{\frac{2}{3}}$ **23.** $a^{\frac{3}{2}}$ **24.** $b^{\frac{11}{3}}$

**25.** $x^{\frac{4}{3}}$ **26.** $a^{\frac{7}{2}}$ **27.** $\boxed{\left(27x^5\right)^{\frac{1}{2}}}$ **28.** $\boxed{\left(16b^8\right)^{\frac{1}{3}}}$

**Example 6.2-3**

**3** 1600.io 6.2

$$\sqrt{16x^4}$$

If $x > 0$, which of the following is equivalent to the given expression?

A) $4x$

B) $4x^2$

C) $16x$

D) $16x^2$





**Solution**

1. The first step when we see a term composed of both constants and variables in a radical (or, equivalently, such a term being exponentiated) is usually to separate the constant part and the variable part so that we can attempt to evaluate or simplify each of those parts individually. When we do this, we take advantage of the principle that when terms are multiplied and then raised to a power, the result is equivalent to first raising each term to that power, and then multiplying those results (this is because exponentiation is a form of multiplication, and multiplications can be done in any order because multiplication is commutative, but you knew that). Here, we'll split the constant, 16, and the variable, $x^4$, into separate radicals that are then multiplied. Conveniently, 16 is a perfect square, so once $\sqrt{16}$ is split off, we can evaluate it and replace it with 4.

   Split the radical expression into two parts, one with the constant, 16, in a radical sign, and one with $x^4$ in a second radical sign. Evaluate the radical containing the constant (the square root of 16 is 4).

$$\sqrt{16x^4}$$
$$\sqrt{16}\sqrt{x^4}$$
$$4\sqrt{x^4}$$

2. Rewrite the radical expression as an exponential expression with a fractional exponent. Remember, the square root is just the second root, so taking a square root of an expression is the same as raising the expression to the $\frac{1}{2}$ power.

$$4\sqrt{x^4}$$
$$4\left(x^4\right)^{\frac{1}{2}}$$
$$4x^{4\left(\frac{1}{2}\right)}$$
$$4x^{\frac{4}{2}}$$
$$4x^2$$

3. The answer is B.

**Notes**

Sometimes, you can simplify the radical with the variable without having to convert it to an exponential term, as when it's simple and obvious. For example, we wouldn't normally rewrite $\sqrt{x^2}$ as $\left(x^2\right)^{\frac{1}{2}}$, because we can recognize that the square root of $x^2$ is equal to $x$ (or $|x|$ if the test makers don't include the caveat that $x > 0$); similarly, though it's not quite as obvious, you might recognize that here, $\sqrt{x^4}$ is equal to $x^2$, but it's still important to be adept at using the more generalized procedure shown above.

### Rule 4: To Divide Terms with the Same Base, Subtract the Denominator's Exponent From the Numerator's

**When a term is divided by another term with the same base, subtract the exponent of the term in the denominator from the exponent of the term in the numerator.**

For example, $\frac{3^4}{3^3} = 3^{4-3} = 3^1 = 3$, which can be demonstrated by expanding the numerator and denominator.

$$\frac{3^4}{3^3} = \frac{3 \times 3 \times 3 \times 3}{3 \times 3 \times 3} = \frac{3 \times \cancel{3} \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times \cancel{3}} = 3$$

Similarly, $\frac{x^5}{x^2} = x^{5-2} = x^3$.

In order for the exponents to be subtracted, the bases MUST be the same, just as when multiplying numbers with exponents by adding their exponents.

#### Subtract Exponents When Dividing Terms with the Same Base

When two exponential terms with the same base, $a$, are divided by each other, the exponent of the term in the denominator is subtracted from the exponent of the term in the numerator.

$$\frac{a^b}{a^c} = a^{b-c}$$

**SkillDrill 6.2-5**

**Directions:** Rewrite the expressions in the form $a^b$, where $b$ can be any fraction or whole number, and evaluate any numerical values. For example, rewrite $\frac{x^3}{x^2}$ as $x$ and rewrite $\frac{32x^6}{4\sqrt{x}}$ as $8x^{\frac{11}{2}}$. Do not convert fractional exponents back into radical expressions.

Challenge problems are boxed.

1. $\frac{x^5}{x^3}$
2. $\frac{y^7}{y^2}$
3. $\frac{32a^3b^4}{8ab}$
4. $\frac{a^3a^7}{a^2}$
5. $\frac{3x^2}{\sqrt{3x}}$
6. $\frac{\sqrt{y}}{\sqrt[4]{y}}$
7. $\boxed{\frac{\sqrt[3]{a^2b^3}}{\sqrt{ab}}}$
8. $\boxed{\frac{x^4\sqrt{y}}{x\sqrt[5]{y^2}}}$





**Example 6.2-4**

**4** 1600.io 6.2

If $\frac{\sqrt[3]{x^2}}{\sqrt[8]{x^5}} = x^{\frac{a}{b}}$ for all positive values of $x$, what is the value of $\frac{a}{b}$?

**Solution**

1. Rewrite both radical terms as exponential terms on the left side of the equation.

$$\frac{\sqrt[3]{x^2}}{\sqrt[8]{x^5}} = x^{\frac{a}{b}}$$

$$\frac{x^{\frac{2}{3}}}{x^{\frac{5}{8}}} = x^{\frac{a}{b}}$$

2. Since two terms with the same base are being divided, the terms can be combined by subtracting the exponents. Rewrite the fractions in terms of a common denominator in order to carry out the subtraction.

$$\frac{x^{\frac{2}{3}}}{x^{\frac{5}{8}}} = x^{\frac{a}{b}}$$

$$x^{\frac{2}{3}-\frac{5}{8}} = x^{\frac{a}{b}}$$

$$x^{\frac{16}{24}-\frac{15}{24}} = x^{\frac{a}{b}}$$

$$x^{\frac{1}{24}} = x^{\frac{a}{b}}$$

3. Excepting the bases $-1$, 0, and 1, a base raised to one power cannot be equal to the same base raised to a different power, and here, the base, $x$, can be any positive number, so therefore the exponents must be equal.

$$\frac{1}{24} = \frac{a}{b}$$

4. The answer is $\frac{1}{24}$.

The subtraction of the exponents leads us to the idea of negative exponents and what it means for a number to be raised to a negative power.

You might wonder what a negative exponent even means; $3^2$ makes sense, because it's just $3 \cdot 3$; similarly, $3^3$ is $3 \cdot 3 \cdot 3$; and so on. We just multiply base by itself the number of times indicated by the exponent. What, then, does $3^{-2}$ mean? How do we multiply something a negative number of times?

Let's look at what's happening when we start with an exponent of 0 and then incrementally increase that exponent:

$$3^0 = 1$$

To increase the exponent by 1 on the left side of the equation, we multiply both sides by 3:

$$3^1 = 1 \cdot 3$$

To increase it again, we again multiply by 3:

$$3^2 = 3 \cdot 3$$

So far, so good. Now, let's reverse course and decrease the exponent by one. To do that, we need to undo the previous operation, so we must perform the reverse operation of multiplication, which, of course, is division:

$$3^1 = \frac{3 \cdot 3}{3} = \frac{3}{1} = 3$$

Let's decrease the exponent again, which means dividing by 3 again:

$$3^0 = \frac{3}{3} = 1$$

Things are about to get interesting, because our exponent is going to become negative. Let's repeat the same division operation as we once again decrement the exponent:

$$3^{-1} = \frac{1}{3}$$

Let's do it one more time just for clarity:

$$3^{-2} = \frac{\frac{1}{3}}{3} = \frac{1}{3 \cdot 3} = \frac{1}{3^2}$$

## Negative Exponents

When a base $a$ is raised to a negative exponent $-b$, the expression is equal to its reciprocal with a positive exponent.

$$a^{-b} = \frac{1}{a^b}$$

For example, if $a = 3$ and $b = 2$, then $a^{-b} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$.

By the same token, we can observe that the same principle applies when the term with the negative exponent is in the denominator (this is just another arrangement of the equation above):

$$\frac{1}{a^{-b}} = a^b$$

As you can see, the general principle is very simple—we can create an equivalent term by taking the reciprocal and negating the sign of the exponent.

Importantly, when performing these transformations, you only take the reciprocal of the exponentiated variable or constant, but not of any coefficient. So, this is correct:

$$3x^{-4} = \frac{3}{x^4}$$

but this is not:

$$3x^{-4} \neq \frac{1}{3x^4}$$





This applies regardless of whether the exponentiated term starts out in the numerator or the denominator, so this is correct:

$$\frac{1}{7r^{-3}} = \frac{r^3}{7}$$

but not this:

$$\frac{1}{7r^{-3}} \neq \frac{7r^3}{1}$$

Generally speaking, this principle that converts a negative exponent into a positive one is useful anytime you have a term with a negative exponent, because converting that term into one with a positive exponent facilitates evaluating it. Here are a few examples:

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25} \qquad 16 \cdot 2^{-3} = \frac{16}{2^3} = \frac{16}{8} = 2 \qquad \frac{1}{4^{-3}} = 4^3 = 64 \qquad \frac{1}{9 \cdot 3^{-4}} = \frac{3^4}{9} = \frac{81}{9} = 9$$

**SkillDrill 6.2-6**

**Directions:** Rewrite any terms with negative exponents as terms with positive exponents. Continue on from there to rewrite the expressions in the form $\frac{a^b}{c^d}$, where $b$ and $d$ can be any fractions or whole numbers, based on any of the established exponent rules. Do not convert fractional exponents back into radical expressions, and express any fractions in lowest terms.

Challenge problems are boxed.

**1.** $x^{-2}$

**2.** $\frac{1}{y^{-3}}$

**3.** $\frac{4x^{-2}}{12y^{-3}}$

**4.** $\frac{r^{-3}}{r^4}$

**5.** $\frac{x^2}{x^{\frac{-1}{2}}}$

**6.** $\boxed{\frac{x^{-7}y^{\frac{-2}{5}}}{x^3y^{\frac{-1}{5}}}}$

**7.** $\boxed{\frac{a^{-4}b^{\frac{3}{4}}}{a^{-1}b^{\frac{-1}{4}}}}$

**8.** $\boxed{3x^{-6}\sqrt[3]{27y^{-4}}}$

"Equivalent to" problems are challenging, because there are many possible equivalent expressions, but only one is present in the answer choices. So, one should examine the answer choices to ascertain what commonality they have in order to know in which direction to set out. Start by reviewing the answer choices to determine what attributes they share, so you'll know at least what *not* to have in any answer candidate. In the following example, we see that there are no negative exponents in the answer choices, so we should set out to eliminate those. Then, note that radical notation is used for fractional exponents, so be sure to match that as well.

**Example 6.2-5**

**5** 1600.io 6.2

The expression $\frac{x^{-1}y^{\frac{1}{3}}}{x^{\frac{1}{3}}y^{-1}}$, where $x > 1$ and $y > 1$, is equivalent to which of the following?

A) $\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$

B) $\frac{y\sqrt[3]{y}}{\sqrt[3]{x^2}}$

C) $\frac{y\sqrt[3]{y}}{x\sqrt[3]{x}}$

D) $\frac{\sqrt[4]{y^3}}{x^3}$

**Solution**

1. Since none of the answer choices contain negative exponents and they all make use of radical notation for fractional exponents, we should look to eliminate negative and fractional exponents by rewriting them.
2. Begin by using the negative exponent rule to move $x^{-1}$ to the denominator as $x^1$ and $y^{-1}$ to the numerator as $y^1$.

$$\frac{x^{-1}y^{\frac{1}{3}}}{x^{\frac{1}{3}}y^{-1}} = \frac{y^1y^{\frac{1}{3}}}{x^1x^{\frac{1}{3}}}$$

3. Rewrite the terms wth fractional exponents as radical expressions.

$$\frac{y^1y^{\frac{1}{3}}}{x^1x^{\frac{1}{3}}} = \frac{y\sqrt[3]{y}}{x\sqrt[3]{x}}$$

4. The answer is C.

**Section 6.2 Suggested Problems from Real Tests**

- Test 3-NC-3
- Test 7-NC-11
- Test 10-NC-5
- Test 10-C-6
- Apr 2017-NC-12
- May 2017-NC-14
- May 2019 (US)-C-8
- May 2019 (Int)-NC-20
- 2019-NC-14
- 2020-NC-15
- 2021-NC-1
- 2021 (Int)-NC-8

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## 6.3 Methods of Solving Exponent Equations

Throughout the chapter we have seen exponent equations where both sides boiled down to the same base being raised to a power, so we knew that the two exponents must be equal. However, there are sometimes equations which do not fit that pattern: the bases of the exponential terms may not always be the same,

### Raising Both Sides to a Power

When solving equations with exponential terms, we can apply exponent rules in order to isolate a variable term that has the correct exponent, usually done by raising both sides of the equation to a power that will transform (through multiplication) the exponent on the variable so that it matches the exponent we want.

Let's take a look at a simple example. If we know that $x = 4$, then we can find the value of $x^2$ by squaring both sides of the equation.

$$\begin{aligned} x &= 4 \\ x^2 &= 4^2 \\ x^2 &= 16 \end{aligned}$$

By the same principle, if we are told $x^3 = 27$, then we can find the value of $x$ (which can also be represented as $x^1$) by raising both sides of the equation to the one-third power, which would cause the exponents on the left side of the equation to multiply to 1.

$$\begin{aligned} x^3 &= 27 \\ \left(x^3\right)^{\frac{1}{3}} &= 27^{\frac{1}{3}} \\ x^{3\left(\frac{1}{3}\right)} &= 27^{\frac{1}{3}} \\ x^1 &= 3 \\ x &= 3 \end{aligned}$$

In the following example, we will work through the problem in two ways, one more direct than the other.

**Example 6.3-1**

**1** 1600.io 6.3

For a positive real number $x$, where $x^5 = 2$, what is the value of $x^{25}$?

A) $\sqrt[5]{25}$

B) 5

C) 16

D) 32

**Solution 1**

1. We need to find a way to transform $x^5$ into $x^{25}$ by applying some exponent rule to both sides of the given equation. We should raise both sides of the equation to the fifth power because, based on exponent rules, raising $x^5$ to the fifth power will cause the exponents to multiply on the left side of the equation, producing the desired term $x^{25}$ on the left side of the equation.

$$\begin{aligned} x^5 &= 2 \\ \left(x^5\right)^5 &= 2^5 \\ x^{5(5)} &= 32 \\ x^{25} &= 32 \end{aligned}$$

2. The answer is D.

**Solution 2**

1. In this approach, we'll first solve for $x$, and then we'll raise both sides of the equation to the 25th power. This is a longer procedure than the one shown above, but some students might be more comfortable with this method.

   We can isolate the term $x$ or $x^1$ on the left side of the equation if we raise both sides of the equation to the one-fifth power.

$$\begin{aligned} x^5 &= 2 \\ \left(x^5\right)^{\frac{1}{5}} &= 2^{\frac{1}{5}} \\ x^{5\left(\frac{1}{5}\right)} &= 2^{\frac{1}{5}} \\ x &= 2^{\frac{1}{5}} \end{aligned}$$

2. Now that we know the value of $x$, we can raise it to the twenty-fifth power to find the value of $x^{25}$.

$$\begin{aligned} x &= 2^{\frac{1}{5}} \\ x^{25} &= \left(2^{\frac{1}{5}}\right)^{25} \\ x^{25} &= 2^{\frac{1}{5}(25)} \\ x^{25} &= 2^5 \\ x^{25} &= 32 \end{aligned}$$

3. The answer is D.





**Notes**

You might have wondered if we could have multiplied both sides of the equation by $x^{20}$ to apply the exponent addition rule, but though this would be a mathematically valid operation, it would not be helpful to do so because even if we were to transform the exponent on one side of the equation, we would fail to isolate the term we are looking for because we would also produce an $x^{20}$ term on the other side of the equation.

$$\begin{aligned} x^5 &= 2 \\ x^5 \cdot x^{20} &= 2x^{20} \\ x^{5+20} &= 2x^{20} \\ x^{25} &= 2x^{20} \end{aligned}$$

This is mathematically valid, but it does not get us any closer to solving the problem.

## Rewriting Exponential Terms In Terms of Another Base

In order to apply exponent rules to combine terms, the terms must all have the same base, so sometimes it is useful to rewrite numbers in terms of their factors, specifically those which may be roots of the number. For example, the number 16 has the factors 2, 4, and 8, and both 2 and 4 are roots of 16 because $2^4 = 16$ and $4^2 = 16$.

We can take advantage of the fact that $\left(a^b\right)^c = a^{bc}$ (and remember that the rules can be applied in both directions in order to write exponential terms in the way that is most useful for the situation at hand) in order to rewrite an exponential term with a different base that allows us to either match answer choices or make equations easier to solve.

It could be advantageous to write relatively large bases in terms of smaller bases in order to rewrite exponential terms with fractional exponents (without using a calculator) in order to match answer choices. For example, if we wanted to simplify the term $9^{\frac{3}{4}}$, we could rewrite the 9 as $3^2$, allowing us to break the exponent up and rewrite $9^{\frac{3}{4}}$ as a radical expression with a base of 3.

| | |
|---|---|
| | $9^{\frac{3}{4}}$ |
| Replace 9 with $3^2$: | $\left(3^2\right)^{\frac{3}{4}}$ |
| Apply exponentiation of exponent rule by multiplying exponents: | $3^{2\left(\frac{3}{4}\right)}$ |
| | $3^{\frac{6}{4}}$ |
| Reduce fraction in exponent: | $3^{\frac{3}{2}}$ |
| Split improper fraction into integer and proper fraction: | $3^{\left(\frac{2}{2}+\frac{1}{2}\right)}$ |
| | $3^{\left(1+\frac{1}{2}\right)}$ |
| Apply (in reverse) multiplication of terms with exponents rule: | $3 \cdot 3^{\frac{1}{2}}$ |
| Convert fractional exponent part to radical notation: | $3\sqrt{3}$ |

It is also advantageous to write terms in the another base whenever you have two different base numbers that share a common factor. For example, we can combine the terms in the expression below if we rewrite 27 as a base of 3 raised to the third power, allowing us to apply exponent rules to combine the terms (particularly if the problem asks us to find the value of $3x + 4y$, which will be the exponent when the terms are combined).

$$27^x \cdot 3^{4y} = 3^5$$
$$\left(3^3\right)^x \cdot 3^{4y} = 3^5$$
$$3^{3x} \cdot 3^{4y} = 3^5$$
$$3^{3x+4y} = 3^5$$

Changing the base was useful in this example because it allowed us to write both sides of the equation as 3 raised to a power, which allows us to very easily see that $3x + 4y$ must be equal to 5 (since 3 to one power can't be equal to 3 to a different power).

**Example 6.3-2**

**2** 1600.io 6.3

If $4x - y = 8$, what is the value of $\frac{16^x}{2^y}$ ?

A) $2^8$

B) $8^2$

C) $4^2$

D) The value cannot be determined from the information given.

**Solution**

1. The first equation might not seem useful at first, but when you are given information like this at the beginning of the problem, it will be useful later. The expression $4x - y$ should appear at some point in the course of solving the problem, so keep an eye out for it.
2. The expression $\frac{16^x}{2^y}$ has to be rewritten to be useful: the two bases are not equal, so we cannot simplify unless we have the numerator and denominator both in the same base.
3. Since $16 = 2^4$, we can replace the 16 with $2^4$. This way, the bases of the numerator and denominator will both be 2.
$$\frac{16^x}{2^y} = \frac{\left(2^4\right)^x}{2^y} = \frac{2^{4x}}{2^y}$$
4. Now that the numerator and denominator are written in terms of the same base, we can subtract the exponents in order to simplify.
$$\frac{2^{4x}}{2^y} = 2^{4x-y}$$
5. The first given equation tells us that $4x - y = 8$, so substitute 8 in for $4x - y$ to find the value of the expression.
$$2^{4x-y} = 2^8$$
6. The answer is A.

Section 6.3 Suggested P

- Test 1-NC-14
- Test 5-NC-12

View related real-test problems in SMTEx

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- Apr 2021-C-36
- May 2021 (US)-NC-20





## 6.4 Table of Exponent Rules

| Rule | Example | Demonstration |
|---|---|---|
| $a^b \cdot a^c = a^{b+c}$ | $3^2 \cdot 3^4 = 3^6$ | $3^2 \cdot 3^4$<br>$(3 \times 3) \times (3 \times 3 \times 3 \times 3)$<br>$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6$ |
| $\left(a^b\right)^c = a^{bc}$ | $\left(4^2\right)^3 = 4^6$ | $\left(4^2\right)^3$<br>$(4 \times 4)^3$<br>$(4 \times 4) \times (4 \times 4) \times (4 \times 4)$<br>$4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$ |
| Powers ARE distributed across multiplied or divided factors. | | ✓ $(ab)^c = a^c b^c$<br>✓ $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$ |
| $(ab)^c = a^c b^c$ | $(2 \cdot 3)^3 = 2^3 \cdot 3^3$ | $(2 \cdot 3)^3$<br>$(2 \cdot 3) \times (2 \cdot 3) \times (2 \cdot 3)$<br>$2 \times 3 \times 2 \times 3 \times 2 \times 3$<br>$2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \cdot 3^3$ |
| Powers are NOT distributed to added or subtracted terms:<br><br>Carrying out the multiplication of expressions like this is discussed in the next chapter. | | ✗ $(a+b)^2 \neq a^2 + b^2$<br>✓ $(a+b)^2 = (a+b)(a+b)$ |
| $(a+b)^c \neq a^c + b^c$ | $(2+3)^3 \neq 2^3 + 3^3$ | $(2+3)^3 \neq 2^3 + 3^3$<br>$(5)^3 \neq 8 + 27$<br>$125 \neq 35$ |
| $\sqrt[c]{a^b} = a^{\frac{b}{c}}$ | $\sqrt[3]{8^4} = 8^{\frac{4}{3}}$ | |
| $\frac{a^b}{a^c} = a^{b-c}$ | $\frac{4^5}{4^3} = 4^2$ | $\frac{4^5}{4^3} = \frac{4 \times 4 \times \cancel{4} \times \cancel{4} \times \cancel{4}}{\cancel{4} \times \cancel{4} \times \cancel{4}} = \frac{4 \times 4}{1} = 4^2$ |
| $a^{-b} = \frac{1}{a^b}$ | $2^{-2} = \frac{1}{2^2}$ | |
| $\frac{1}{a^{-b}} = a^b$ | $\frac{1}{2^{-2}} = 2^2$ | |

## CHAPTER 6 RECAP

- Exponential terms consist of a base $a$ and exponent $b$ and are in the form $a^b$.
- Exponents tell you how many factors of the base should be multiplied by each other. For example, $2^5 = 2\times2\times2\times2\times2$ (there are 5 factors of 2 multiplied by each other).
- $a^b a^c = a^{b+c}$
- $\left(a^b\right)^c = a^{bc}$
- $(ab)^c = a^c b^c$ BUT WATCH OUT FOR THIS MISTAKE: $(a+b)^c \neq a^c + b^c$
- $a^{\frac{b}{c}} = \sqrt[c]{a^b} = \left(\sqrt[c]{a}\right)^b$ Note that the rule will hold as long as the base, $a$, is non-negative.
- Rewrite radical expressions as terms with fractional exponents.
- Look for perfect square or perfect cube factors when taking square roots or cube roots of numbers, respectively.
- You can solve exponential equations by raising both sides to a power such that the exponents of the variable term will multiply to the correct power (the power you are looking for) on one side of the equation.
- $\frac{a^b}{a^c} = a^{b-c}$
- $a^{-b} = \frac{1}{a^b}$ and $\frac{1}{a^{-b}} = a^b$
- Rewrite expressions in terms of the same base if possible so that you can use the preceding rules to simplify. For example, if you are asked for the value of $\frac{25^3}{5^2}$, the simplest way to evaluate the expression (without a calculator) would be to rewrite 25 as $5^2$ and progress from there, applying exponent rules to arrive at an easily calculable value.

$$\frac{25^3}{5^2} \Rightarrow \frac{\left(5^2\right)^3}{5^2} \Rightarrow \frac{5^6}{5^2} \Rightarrow 5^{6-2} \Rightarrow 5^4 \Rightarrow 625$$

Note that "simplifying" is an inexact term because any exponential term can be written in an infinite number of ways. For example, $2^5$ could be written as any of the following:

$$2^2 \cdot 2^3 \qquad 2^{29} \cdot 2^{-24} \qquad \left(2^{30}\right)^{\frac{1}{6}} \qquad \frac{1}{2^{-5}} \qquad \text{etc...}$$

Simplifying exponential terms, as we use it, refers to rewriting the expressions in a way that is useful in solving any particular problem on the test. Take the questions on a case by case basis. As long as you have a good understanding of the rules, anything is possible, but some options will be clearly more advantageous.





## Additional Problems

**1** 6.2

$$\sqrt[3]{8x^9}$$

If $x > 0$, which of the following is equivalent to the given expression?

A) $\frac{8}{3}x^3$

B) $2x^2$

C) $2x^3$

D) $8x^3$

**2** 6.3

Which of the following is equivalent to $8^{\frac{5}{6}}$ ?

A) $\sqrt{2}$

B) $\sqrt[5]{8}$

C) $4\sqrt{2}$

D) $5\sqrt{2}$

**3** 6.2

Which of the following is equal to $a^{\frac{3}{4}}$, for all values of $a$ ?

A) $\sqrt[4]{a^3}$

B) $\sqrt[4]{a^{\frac{1}{3}}}$

C) $\sqrt[3]{a^4}$

D) $\sqrt[3]{a^{\frac{1}{4}}}$

**4** 6.2

Which of the following is equivalent to $r^{\frac{3}{4}} \cdot \sqrt[3]{r}$, where $r > 0$ ?

A) $r^{\frac{1}{4}}$

B) $r^{\frac{1}{3}}$

C) $r^{\frac{4}{7}}$

D) $r^{\frac{13}{12}}$

**5** 6.2

Which of the following is an equivalent form of $\sqrt[4]{f^{8a}k}$, where $f > 0$ and $k > 0$ ?

A) $f^{4a}k^{-3}$

B) $f^{2a}k^{\frac{1}{4}}$

C) $f^{\frac{1}{2a}}k^{-3}$

D) $f^{\frac{1}{2a}}k^{4}$

**6** 6.2

Which of the following expressions is equivalent to $5a^4(a+2)$ ?

A) $7a^5$

B) $15a^5$

C) $5a^5 + 2$

D) $5a^5 + 10a^4$

**7** 1600.io 6.3

If $a^{-\frac{1}{6}} = x$, where $a > 0$, what is $a$ in terms of $x$ ?

A) $-\sqrt[6]{x}$

B) $\sqrt[6]{x}$

C) $-\frac{1}{x^6}$

D) $\frac{1}{x^6}$

**8** 1600.io 6.2

Which of the following expressions is equivalent to $\left(16x^4\right)^{\frac{1}{4}}$ ?

A) $\sqrt[4]{4x}$

B) $2|x|$

C) $4|x|$

D) $16x$

**9** 1600.io 6.2

An exponential function $f$ is defined by $f(t) = b^t$, where $b$ is a constant greater than 1. If $f(9) = 64 \cdot f(6)$, what is the value of $b$ ?

**10** 1600.io 6.2

Which of the following expressions is equivalent to $\left(27x^8y^4\right)^{\frac{1}{3}}$, where $x \geq 0$ and $y \geq 0$ ?

A) $3x^2y^{\frac{4}{3}}$

B) $3x^{\frac{8}{3}}y^{\frac{4}{3}}$

C) $9x^2y^4$

D) $9x^{\frac{8}{3}}y^4$

**11** 1600.io 6.3

If $a^{\frac{b}{3}} = 27$ for positive integers $a$ and $b$, what is one possible value of $b$ ?

**12** 1600.io 6.3

Which of the following expressions is equivalent to $\left(-9x^3\right)^{\frac{2}{3}}$ ?

A) $-3x^2 \cdot \sqrt[3]{3}$

B) $-x^2 \cdot \sqrt[3]{81}$

C) $3x^2 \cdot \sqrt[3]{3}$

D) $3x^2 \cdot \sqrt[3]{81}$





# Answer Key

### SkillDrill 6.1-1

1. 27
2. 16
3. 1
4. 81
5. 0
6. 1
7. $-125$
8. 17
9. 12
10. 11
11. 1
12. 32
13. $x^2$
14. $16y^2$
15. $2x$
16. $x^3y$
17. $3a^2$
18. $b^2 - b^3$
19. $2xy$
20. $x^2y + xy^2$
21. $a^2b^2c^2$
22. $4x^2y$
23. $dx + d^3$
24. $2x^2$

### SkillDrill 6.2-1

1. $x^4$
2. $64y^7$
3. $2x^5$
4. $x^{14}y^5$
5. $a^{11}$
6. $b^4 - b^5$
7. $x^5y^7$
8. 1
9. $x^8y + xy^5$
10. $64a^6b^6c^2$
11. $12d^7x^3$
12. 0

### SkillDrill 6.2-2

1. $x^6$
2. 64
3. $a^{18}$
4. $x^{14}y^{35}$
5. $15b^{16}$
6. $x^6y^{10} + x^5y^{10}$
7. $8x^{15}$
8. $\left(x^8 + y^{10}\right)^2$

### SkillDrill 6.2-3

1. $a^{\frac{1}{2}}$
2. $a^{\frac{2}{3}}$
3. $2h$
4. $x^{\frac{1}{3}}$
5. $4x^2$
6. $a^{\frac{3}{4}}$
7. $2x^{\frac{3}{4}}$
8. $x^{\frac{2}{3}}y^2$
9. $1 + \frac{1}{3}$
10. $1 + \frac{2}{5}$
11. $2 + \frac{1}{2}$
12. $1 + \frac{3}{10}$
13. $1 + \frac{1}{2}$
14. $2 + \frac{1}{5}$
15. $2 + \frac{1}{2}$
16. $4 + \frac{3}{5}$

### SkillDrill 6.2-4

1. $4 \cdot 6$
2. $4 \cdot 3$
3. $9 \cdot 5$
4. $25 \cdot 2$
5. $4 \cdot 8$ or $16 \cdot 2$
6. $49 \cdot 2$
7. $9 \cdot 8$ or $36 \cdot 2$
8. $4 \cdot 12$ or $16 \cdot 3$
9. $8 \cdot 4$
10. $8 \cdot 10$
11. $27 \cdot 2$
12. $125 \cdot 2$
13. $2\sqrt{6}$
14. $2\sqrt{3}$
15. $3\sqrt{5}$
16. $5\sqrt{2}$
17. $4\sqrt{2}$
18. $7\sqrt{2}$
19. $6\sqrt{2}$
20. $4\sqrt{3}$
21. $\sqrt{x}$
22. $\sqrt[3]{y^2}$
23. $a\sqrt{a}$
24. $b^3\sqrt[3]{b^2}$
25. $x\sqrt[3]{x}$
26. $a^3\sqrt{a}$
27. $3x^2\sqrt{3x}$
28. $2b^2\sqrt[3]{2b^2}$

**SkillDrill 6.2-5**

1. $x^2$
2. $y^5$
3. $4a^2b^3$
4. $a^8$
5. $3^{\frac{1}{2}}x^{\frac{3}{2}}$
6. $y^{\frac{1}{4}}$
7. $a^{\frac{1}{6}}b^{\frac{1}{2}}$
8. $x^3y^{\frac{1}{10}}$

**SkillDrill 6.2-6**

1. $\frac{1}{x^2}$
2. $y^3$
3. $\frac{y^3}{3x^2}$
4. $\frac{1}{r^7}$
5. $x^{\frac{5}{2}}$
6. $\frac{1}{x^{10}y^{\frac{1}{5}}}$
7. $\frac{b}{a^3}$
8. $\frac{9}{x^6y^{\frac{4}{3}}}$

**Additional Problems**

1. C
2. C
3. A
4. D
5. B
6. D
7. D
8. B
9. 4
10. B
11. 1, 3, or 9
12. C



# Introduction to Polynomials 7

## 7.1 What are Polynomials?

Let's talk about polynomials. **A polynomial is a mathematical expression that consists of a set of terms added together, where each term contains a variable raised to a power**. Each term can have a numerical coefficient (multiplier) in front of it; if there's no explicit coefficient shown, the coefficient is just 1.

We'll limit our discussion to polynomials that only have one variable, such as $x$, in each term. These happen to be called univariate polynomials, but you don't need to memorize that unless you want to be shunned at social gatherings. The **degree** of a polynomial is the highest power that the variable is raised to in any term in the expression.

We've previously discussed one simple flavor of polynomial: the linear expression, which can be written in the form $mx + b$. For example:

$$2x + 6$$

This polynomial expression has two terms. You might think that it doesn't qualify as a polynomial, because though the first term has a variable ($x$) and a coefficient ($m$, which is 2 in this example), it doesn't show $x$ raised to any power; furthermore, the second term, $b$ (which is 6 in this example), doesn't even have $x$ in it anywhere. But if you recall that a number raised to the first power, such as $x^1$, is the same as just the base (here, $x$), you'll realize that the expression could be written as $mx^1 + b$. For our specific example:

$$2x^1 + 6$$

But what about that constant term (represented here by 6)? Well, another fact we've discussed is that a number raised to the zeroth power is equal to 1, so $x^0 = 1$, which means that we could write the same expression as $2x^1 + 6x^0$.

When we write the polynomial this way, we can see that the second term is in exactly the same general form as the first term, complete with coefficient (6), variable ($x$), and exponent (0). Now our expression is more obviously a single-variable polynomial, with a series of terms that have that variable raised to some power. (Note that we conventionally place the terms in order of decreasing degree, so here, the term of degree 1 comes first, followed by the term of degree 0; this keeps things neat and organized.) Just for completeness, let's consider the ultra-simple constant expression $b$ (for example, $-24$). Believe it or not, this does qualify as a single-variable polynomial, because it could be thought of as $bx^0$ (for example: $-24x^0$).

This fact is not terribly useful, though it might help you understand how constants, variables, and variables raised to powers can all relate to polynomials. Specifically, you should understand that a term that has a variable with no exponent isn't really any different from a term that has an exponent—we just leave off the exponent if it's 1. Similarly, a term that doesn't have a variable at all (a constant by itself) might appear to be of a different type than the other terms in a polynomial, but in a univariate polynomial, it's really just a coefficient for the $x^0$ term.

Let's quickly look at another example, just to be sure this principle is clear:

$$x^2 + 2x - 24$$

Here, we see there are three terms, but they all look different; what's with polynomials and their seemingly endless unique term formats? The first term has no coefficient, the second one has no exponent, and the third one has no variable! Well, we now know that this is simply the following super-consistent series of terms, but with a few math-writing shorthand rules applied:

$$1x^2 + 2x^1 - 24x^0$$

Ah, order has been restored: each term has a coefficient, the variable, and an exponent, with those exponents obediently marching down 2, 1, 0 (writing the terms in order of descending degree is the **Standard Form** for polynomials—weirdly, when it comes to linear polynomials in this form, we refer to it as Slope-Intercept Form and call something entirely different Standard Form). It just so happens that when we apply some conventions, each term in this three-term univariate polynomial—also known as a **quadratic**—takes on a different form:

1. We drop the coefficient of 1 from the first term, because doing so doesn't change the value of the term (anything multiplied by 1 remains unchanged)
2. We drop the exponent of 1 from the second term's variable, because any base raised to the first power is just the base
3. We drop the whole variable component from the third term, because anything raised to the zeroth power is 1, so we don't need to keep that extraneous stuff either; that leaves just the numerical coefficient, which we happen to refer to as a constant

You should be able to see now that polynomials can be built out of terms that consist of a coefficient and a variable raised to some power, though we might not need to explicitly write some of those components with every term.

### Standard Form Polynomials

A polynomial is simply a mathematical expression that consists of a set of terms added together, where each term contains a variable raised to a power. When written in Standard Form, they are expressed in decreasing order of the power of the $x$ terms.

The Standard Form of a quadratic polynomial (the highest power of $x$ is 2) takes the form

$$ax^2 + bx + c$$

where $a$, $b$, and $c$ are constants.

The Standard Form of a cubic polynomial (the highest power of $x$ is 3) takes the form

$$ax^3 + bx^2 + cx + d$$

where $a$, $b$, $c$, and $d$ are constants.

The same pattern continues for higher degree polynomials (where the highest power of $x$ is more than 3).

## Combining Like Terms

As we've already seen, terms of the same variable (same base, same exponent) can be added together. In the simplest case, we know that $x + x = 2x$. In the last chapter, we briefly discussed how in expressions with multiple different variable terms, the terms of the same variables can be added together. For example, we can add the expressions $x^3 + 3x^2 - x - 7$ and $5x^3 + 8x^2 + 9$ by **combining the like terms**: add $x^3$ and $5x^3$; add $3x^2$ and $8x^2$; there is no $x$ term in the second expression, so the $-x$ from the first expression stands alone; add the constants $-7$ and 9.

$$(x^3 + 3x^2 - x - 7) + (5x^3 + 8x^2 + 9)$$
$$\boxed{x^3 + 5x^3} + \boxed{3x^2 + 8x^2} - x\boxed{-7 + 9}$$
$$6x^3 + 11x^2 - x + 2$$





Some problems on the test are just a simple matter of adding (or subtracting) expressions by combining like terms.

**Example 7.1-1**

**1** 1600.io 7.1

Which expression is equivalent to $(5x^2 - 4) - (-6x^2 + 3x - 5)$?

A) $11x^2 + 3x - 1$

B) $11x^2 - 3x + 1$

C) $-x^2 + 3x - 9$

D) $-x^2 - 3x - 9$

**Solution**

1. Drop the parentheses around the first expression and distribute the negative sign through the second expression (be careful about which terms have negative coefficients).

$$(5x^2 - 4) - (-6x^2 + 3x - 5)$$
$$5x^2 - 4 + 6x^2 - 3x + 5$$

2. Group and combine like terms.

$$5x^2 - 4 + 6x^2 - 3x + 5$$
$$\boxed{5x^2 + 6x^2} - 3x\boxed{-4 + 5}$$
$$11x^2 - 3x + 1$$

3. The answer is B.

**Section 7.1 Suggested Problems from Real Tests**

- Test 1-NC-5
- Test 3-C-6
- Test 5-NC-6
- Test 6-C-1
- May 2018-NC-2
- Mar 2020-NC-6
- Oct 2020-NC-2
- Mar 2021-NC-10
- Apr 2021-NC-4

View related real-test problems in SMTEx

SMTEx

## 7.2 Multiplying Polynomials

Polynomials can be multiplied by each other to produce new, higher degree polynomials, and we will need to build on the knowledge of exponent rules (established in the previous chapter) in order to multiply variable terms and combine like terms.

We already know how to multiply a numerical value and an expression with multiple terms by distributing the value across the terms. For example:

$$3(x^2 - 4x + 2)$$
$$3\left(x^2\right) - 3(4x) + 3(2)$$
$$3x^2 - 12x + 6$$

By the same principle, we can also distribute variable terms across multiple terms (we saw this in a few examples in the last chapter), applying exponent rules when terms with the same base variable $x$ are multiplied by each other. For example, let's replace the 3 in the previous example with an $x$ (or $x^1$) term, and find the resulting expression. We'll also rewrite the terms inside the parentheses as $x$-based variable terms:

$$x(x^2 - 4x + 2)$$
$$x^1(x^2 - 4x^1 + 2x^0)$$
$$x^1(x^2) - x^1(4x^1) + x^1(2x^0)$$
$$x^{1+2} - 4x^{1+1} + 2x^{1+0}$$
$$x^3 - 4x^2 + 2x$$

You want to get comfortable multiplying variable terms of polynomials so that you don't always have to write the intermediate steps (writing $x$-terms as $x^1$, for example).

**Example 7.2-1**

**1** 1600.io 7.2

$$x(x^2 - 1) + (3x^2 - x)$$

Which of the following expressions is equivalent to the expression above?

A) $4x^2 - 2x$

B) $4x^2$

C) $x^3 + 3x^2$

D) $x^3 + 3x^2 - 2x$





**Solution**

1. Distribute the $x$ to both terms in the first parenthetical expression, and combine multiplied variable terms using exponent rules. The parentheses can be removed from around the second parenthetical expression because no multipliers need to be distributed.

$$x(x^2 - 1) + (3x^2 - x)$$
$$x(x^2) - x(1) + 3x^2 - x$$
$$x^3 - x + 3x^2 - x$$

2. Combine like terms and write the variable terms in descending order of degree to simplify the expression to match the format of the answer choices.

$$x^3 - x + 3x^2 - x$$
$$x^3 + 3x^2 - x - x$$
$$x^3 + 3x^2 - 2x$$

3. The answer is D.

Polynomial expressions with multiple terms can also be multiplied by multi-term polynomials (instead of by just a single variable term as in the previous example). For example, what if we wanted to write the product of $2x + 3$ and $x + 2$ as a polynomial in Standard Form (in terms of descending degree variable terms)? When we want to carry out the multiplication of polynomial expressions, **each term in the first expression has to be distributed to all the terms in the the second expression**.

For example, if we want to multiply $2x + 3$ and $x + 2$, we need to distribute $2x$ to both terms in the expression $x + 2$ and distribute 3 to both terms in the expression $x + 2$.

$$(2x + 3)(x + 2) = 2x(x + 2) + 3(x + 2)$$
$$(2x + 3)(x + 2) = 2x(x) + 2x(2) + 3(x) + 3(2)$$
$$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6$$
$$(2x + 3)(x + 2) = 2x^2 + 7x + 6$$

**Example 7.2-2**

**2** 1600.io 7.2

$$(x + 4)(4x - 2)$$

Which of the following is equivalent to the expression above?

A) $x^2 + 20x + 2$

B) $5x^2 + 14x + 2$

C) $4x^2 + 14x - 8$

D) $4x^2 + 16x - 8$

**Solution**

1. Distribute the $x$ from the first expression to both terms in the second expression, and then distribute the 4 from the first expression to both terms in the second expression.

$$(x+4)(4x-2) = x(4x) - x(2) + 4(4x) - 4(2)$$
$$(x+4)(4x-2) = 4x^2 - 2x + 16x - 8$$
$$(x+4)(4x-2) = 4x^2 + 14x - 8$$

2. The answer is C.

**SkillDrill 7.2-1**

**Directions:** Multiply the polynomials and write the resulting expression in Standard Polynomial Form.

Challenge problems are boxed.

**1.** $x(x-3)$ **2.** $(x+1)(x+2)$ **3.** $(x-4)(x-5)$ **4.** $(x-1)(x+2)$

**5.** $(x-3)(x-9)$ **6.** $\boxed{(2x+10)(x+7)}$ **7.** $\boxed{(x^2+1)(x^2+2)}$ **8.** $\boxed{(x^2+2x+1)(3x+4)}$

Two-term expressions like $a+b$ can be squared (or raised to any power, though squaring is particularly relevant to this test), and there is a shortcut to finding the resulting expanded Standard Form polynomial that is particularly useful on this test (and which comes in handy when factoring polynomials, covered in the next chapter). Before we get to that, remember that squaring an expression just means that we will multiply the expression by itself, so $(a+b)^2$ can also be written as $(a+b)(a+b)$.

$$(a+b)^2 = (a+b)(a+b)$$
$$(a+b)^2 = a(a) + a(b) + b(a) + b(b)$$
$$(a+b)^2 = a^2 + ab + ab + b^2$$
$$(a+b)^2 = a^2 + 2ab + b^2$$

We can use the result that $(a+b)^2 = a^2 + 2ab + b^2$ to devise a general rule for squaring two-term expressions. For example, in the expression $(x+2)^2$, where $x$ is in the same position as $a$ was in the previous example, and 2 is in the same position as $b$ was in the previous example, we should be able to shortcut the process of distributing all the terms step-by-step and just say that $(x+2)^2$ is equal to $x^2 + 2(x)(2) + 2^2$, which simplifies to $x^2 + 4x + 4$. Let's quickly verify that result by simplifying this expression the long way.

$$(x+2)^2 = (x+2)(x+2)$$
$$(x+2)^2 = x(x) + x(2) + 2(x) + 2(2)$$
$$(x+2)^2 = x^2 + 2x + 2x + 2^2$$
$$(x+2)^2 = x^2 + 4x + 4$$





## Perfect Square Binomials

Any expression (including just a simple value) that can be produced by squaring another expression is called a **perfect square**. For example, 9 is a perfect square because it can be produced by squaring 3, and $x^2$ is a perfect square because it's the result of squaring $x$. More relevantly for the current discussion, a polynomial can be a perfect square when it can be formed by squaring another expression; let's look at the perfect square produced by squaring a binomial (two-term polynomial).

When a binomial expression, $a+b$ is squared, the result is equal to $a^2 + 2ab + b^2$ (as was just shown).

$$(a+b)^2 = a^2 + 2ab + b^2$$

Though the following is consistent with the "formula" above (and does not need its own rule), it is helpful to show that, similarly, when $a-b$ is squared, the result is equal to $a^2 - 2ab + b^2$.

$$(a-b)^2 = (a-b)(a-b)$$
$$(a-b)^2 = a(a) + a(-b) - b(a) - b(-b)$$
$$(a-b)^2 = a^2 - ab - ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

**SkillDrill 7.2-2**

**Directions:** Square the expressions—either by multiplying the expression by itself or by using the fact that $(a+b)^2 = a^2 + 2ab + b^2$—in order to write the resulting expression in Standard Polynomial Form.

Challenge problems are boxed.

**1.** $(x+3)^2$ **2.** $(x+1)^2$ **3.** $(x-4)^2$ **4.** $(x-1)^2$

**5.** $\boxed{(2x+10)^2}$ **6.** $\boxed{\left(x-\frac{3}{2}\right)^2}$ **7.** $\boxed{(x^2+1)^2}$ **8.** $\boxed{\left(\frac{1}{5}x-\frac{1}{4}\right)^2}$

There is another useful shortcut when multiplying a binomial by its **conjugate**. The conjugate of a binomial is formed by negating the second term of the original binomial. For example, let's see what happens when we multiply $a+b$ by its conjugate, $a-b$ (notice that this expression differs from the first in that the second term, $b$, was negated to form the conjugate).

$$(a+b)(a-b) = a(a) + a(-b) + b(a) + b(-b)$$
$$(a+b)(a-b) = a^2 - ab + ab - b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

Notice that the two middle terms in the expansion canceled each other, leaving us with an arrangement called the **Difference of Squares**, which just means that one square is subtracted from another.

Once again, we can generalize this to a pattern we can use as a shortcut when multiplying a binomial by its conjugate. For example, we can predict that $(x+2)(x-2)$—where $x$ is in the same position as $a$ was in the previous example, and 2 is in the same position as $b$ was in the previous example—will result in the polynomial $x^2 - 4$ (because 4 is $2^2$). Indeed, if we multiply the expressions the long way, we will find that our prediction is correct.

$$(x+2)(x-2) = x(x) + x(-2) + 2(x) + 2(-2)$$
$$(x+2)(x-2) = x^2 - 2x + 2x - 2^2$$
$$(x+2)(x-2) = x^2 - 4$$

Note, importantly, that it doesn't matter whether the original binomial used addition or subtraction; the conjugate will always use the opposite operation, so the results of the multiplication will be the same: $(a+b)(a-b) = (a-b)(a+b)$. Also, note that the operation in the final difference of squares binomial will **always** be subtraction regardless of the operation in the original binomial due to the reason given above.

### Difference of Squares

When a binomial expression $a+b$ is multiplied by its conjugate, $(a-b)$, the result is equal to $a^2 - b^2$ (as was just shown).

$$(a+b)(a-b) = a^2 - b^2$$

**SkillDrill 7.2-3**

**Directions:** Find the product of the binomial conjugates, either by multiplying the expressions or by using the fact that $(a+b)(a-b) = a^2 - b^2$, and write the resulting expression in Standard Polynomial Form.
Challenge problems are boxed.

1. $(x+3)(x-3)$
2. $(x+1)(x-1)$
3. $(x-4)(x+4)$
4. $(x-2)(x+2)$
5. $\boxed{\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}$
6. $\boxed{\left(2x-\frac{3}{5}\right)\left(2x+\frac{3}{5}\right)}$
7. $\boxed{(3x^3+\sqrt{6})(3x^3-\sqrt{6})}$
8. $\boxed{\left(\frac{1}{5}x^2-\frac{1}{4}x\right)\left(\frac{1}{5}x^2+\frac{1}{4}x\right)}$

**Example 7.2-3**

**3** 1600.io 7.2

Which of the following is equivalent to $(3x+9)^2 - 9x^2$ ?

A) $9(3x+9)$
B) $27(2x+3)$
C) $54(x+1)$
D) $81(x+1)$





**Solution**

1. Using the "shortcut" for squaring binomials, we can quickly rewrite the given expression in order to match Standard Form for polynomials.

$$(3x+9)^2 - 9x^2 = (3x^2) + 2(3x)(9) + (9)^2 - 9x^2$$
$$(3x+9)^2 - 9x^2 = 9x^2 + 54x + 81 - 9x^2$$
$$(3x+9)^2 - 9x^2 = 54x + 81$$

2. The given expression simplifies to $54x + 81$. If you are comfortable factoring numbers from terms, you can factor 27 from both $54x$ and 81 in order to match the format of the answer choices, or, alternatively, you can multiply out each of the answer choices to see which one yields $54x + 81$.

$$54x + 81 = 27(2x+3)$$

3. The answer is B.

**Section 7.2 Suggested Problems from Real Tests**

- Test 4-NC-5
- Test 5-C-8
- Test 6-NC-15
- Test 7-NC-7
- Test 9-C-16

View related real-test problems in SMTEx

- 2019-C-3
- 2019-NC-2
- 2021 (US)-NC-3
- May 2018-NC-5

## 7.3 Polynomial Equations and Functions

Polynomial expressions can also be found in equations. First, a reminder: an expression is not the same thing as an equation. This is an expression, which is a series of terms that are added to (or subtracted from) each other:

$$x^2 + 2x - 24$$

This is an equation which has the above expression on one side of the equals sign:

$$x^2 + 2x - 24 = 0$$

In some polynomial equation problems, you may have Standard Form polynomials on both sides of the equation, as in the following equation:

$$x^2 + 2x - 24 = ax^2 + bx + c$$

**If you are told that an equation is true for all values of $x$, then the two sides of the polynomial equation must be the same, so we can simply match the coefficients of the terms of the same degree to find the values of $a$, $b$, and $c$.** In the example above, $a$ must be equal to 1 because the coefficient of the $x^2$ term on the other side of the equation is 1; $b$ must be equal to 2 because the coefficient of the $x$ term on the other side is 2; and $c$ must be equal to $-24$ because the constant term on the other side of the equation is $-24$.

Sometimes you may need to take steps to write both sides of a polynomial equation in standard form. Place all terms with numerical coefficients on one side of the equation and all terms with unknown coefficients on the other side of the equation. Once both sides of the equation are in the same form, all you have to do is match the coefficients of each term.

**Example 7.3-1**

**1** 1600.io 7.3

$$-2x(5x+1)+6(5x+1)=ax^2+bx+c$$

In the equation above, $a$, $b$, and $c$ are constants. If the equation is true for all values of $x$, what is the value of $b$ ?

**Solution**

1. Distribute the $-2x$ to both terms in the first set of parentheses, and then distribute the 6 to both terms in the second set of parentheses. Combine any like terms on the left side of the equation.

$$\begin{aligned}-2x(5x+1)+6(5x+1)&=ax^2+bx+c\\-10x^2-2x+30x+6&=ax^2+bx+c\\-10x^2+28x+6&=ax^2+bx+c\end{aligned}$$

2. Since $b$ is the coefficient of the $x$ term on the right side of the equation, it must be the same as the coefficient of the $x$ term on the left side of the equation. Therefore, $b=28$.

3. The answer is 28.

**Notes**

If you are comfortable that this is simply a coefficient-matching problem, you can ignore any of the terms on the left side that will not be $x$ (or $x^1$) terms, only multiplying $-2$ by 1 (which would give us the coefficient of $x$ from the first distribution) and 6 by 5 (which would give us the coefficient of $x$ from the second distribution) giving you $-2(1)+6(5)=-2+30=28$.

Evaluating quadratic functions is the same as evaluating any function. Remember that for any function $f(x)$, you can substitute (replace) every $x$ with whatever is put into the function, and the result is the $y$-value. For example, the value of $f(4)$ would be the $y$-value of the point on the graph of the function when the $x$-value is 4.

**Example 7.3-2**

**2** 1600.io 7.3

In the $xy$-plane, the point $(6,-11)$ lies on the graph of the function $f$. If $f(x)=k-x^2$, where $k$ is a constant, what is the value of $k$ ?

**Solution**

1. Plug in 6 for $x$.

$$\begin{aligned}f(x)&=k-x^2\\f(6)&=k-(6)^2\\f(6)&=k-36\end{aligned}$$





2. We know that $f(6)=-11$ because the point $(6,-11)$ is on the graph of the function. Therefore, the $y$-value (or $f(x)$ value) is $-11$ when the $x$-value is 6. Replace $f(6)$ with $-11$, then solve for $k$.

$$\begin{aligned}f(6)&=k-36\\-11&=k-36\\25&=k\end{aligned}$$

3. The answer is 25.

**Section 7.3 Suggested Problems from Real Tests**

- Test 1-NC-10
- Test 2-NC-17
- Test 2-C-33
- Test 3-C-33
- Test 5-C-34
- Test 7-NC-20
- May 2019 (US)-NC-18
- May 2019 (US)-C-2
- 2019-NC-17
- 2019-C-4
- 2020-NC-3
- 2021-C-5
- May 2021 (US)-C-37
- May 2021 (Int)-C-34

View related real-test problems in SMTEx

SMTEx

## CHAPTER 7 RECAP

- The Standard Form for a univariate (single variable) polynomial puts the terms in order from highest to lowest power, so the Standard Form of a quadratic (second-degree polynomial) equation is $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants.
- When adding or subtracting expressions, group and combine like terms of the same variable (same base and exponent).
- When multiplying expressions, distribute all terms in the first expression to all terms in the second expression.
- When a binomial (two-term polynomial) expression $a + b$ is squared, the result is equal to $a^2 + 2ab + b^2$.

$$(a + b)^2 = a^2 + 2ab + b^2$$

Similarly, when $a - b$ is squared, the result is equal to $a^2 - 2ab + b^2$.

$$(a - b)^2 = a^2 - 2ab + b^2$$

- When a binomial expression $(a + b)$ is multiplied by its conjugate, $(a - b)$, the result is equal to the difference of squares $a^2 - b^2$.

$$(a + b)(a - b) = a^2 - b^2$$

- If the two sides of an equation are polynomials written in Standard Form, and you are told that the equation is true for all values of the variable (that is, the polynomials are **equivalent**), you can match the coefficients of corresponding terms to find the values of any unknown coefficients.
- Evaluating polynomial functions is the same as evaluating any function: substitute (replace) each instance of $x$ (or whatever the parameter variable is) with the value for which you want to evaluate the function.





## Additional Problems

**1** 1600.io 7.3

In the $xy$-plane, the point $(2, 0)$ lies on the graph of the function $f(x) = 2x^2 - bx + 15$. What is the value of $b$ ?

**2** 1600.io 7.2

Which of the following expressions is equivalent to $x^2 + 13x + 42$ ?

A) $(x + 6)(x + 7) + 4$

B) $(x + 6)(x + 7) + 4x$

C) $(x + 5)(x + 8) + 2$

D) $(x + 5)(x + 8) + 2x$

**3** 1600.io 7.2

$$(5x + 5)(ax - 2) - x^2 + 10$$

In the expression above, $a$ is a constant. If the expression is equivalent to $bx$, where $b$ is a constant, what is the value of $b$ ?

A) $-11$

B) $-10$

C) $-9$

D) $-5$

**4** 1600.io 7.3

$$-3(x + a) + 5(x^2 - a) = 5x^2 - 3x - 6$$

In the equation above, $a$ is a constant. If the equation is true for all values of $x$, what is the value of $a$ ?

**5** 1600.io 7.2

$$(5x^4 + 6x)(x^4 - 3x)$$

Which of the following is equivalent to the expression above?

A) $4x^4 + 9x$

B) $6x^4 + 3x$

C) $5x^8 - 9x^5 - 18x^2$

D) $6x^8 - 9x^5 - 18x^2$

**6** 1600.io 7.2

Which of the following is an equivalent form of $(2.5x - 1.8)^2 - (3.8x^2 - 4.3)$ ?

A) $1.2x^2 + 0.7$

B) $1.2x^2 + 7.9$

C) $2.45x^2 - 9x + 1.06$

D) $2.45x^2 - 9x + 7.54$

**7** 7.2

$$7x^2 - 4(2 - x) - 3x(x + 2)$$

Which of the following polynomials is equivalent to the expression above?

A) $4x^2 + 2x - 8$

B) $4x^2 - 2x - 8$

C) $7x^2 + x - 8$

D) $7x^2 - 5x - 8$

**8** 7.3

$$(ax - 5)(2x^2 + bx + 3) = 4x^3 - 4x^2 - 9x - 15$$

The equation above is true for all $x$, where $a$ and $b$ are constants. What is the value of $ab$ ?

A) $-14$

B) $-4$

C) $4$

D) $6$

**9** 7.3

In the $xy$-plane, the point $(3, 30)$ lies on the graph of the function $f$. If $f(x) = k + 2x^2$, where $k$ is a constant, what is the value of $k$ ?

**10** 7.3

The function $f$ is defined by $f(x) = x^2 + 4x + 7$. What is the value of $f(-2)$ ?

A) $-5$

B) $3$

C) $11$

D) $13$

**11** 7.2

$$2(3x + 2)(5x + 2)$$

Which of the following is equivalent to the expression above?

A) $70x$

B) $16x^2 + 8$

C) $30x^2 + 8$

D) $30x^2 + 32x + 8$

**12** 7.3

$$-5x(x - 3) + 3(x - 2) = ax^2 + bx + c$$

In the equation above, $a$, $b$, and $c$ are constants. If the equation is true for all values of $x$, what is the value of $a + b + c$ ?

A) $7$

B) $11$

C) $12$

D) $13$





**13** 7.3

If $y = x^2 + ax - a$, where $a$ is constant, and $y = 7$ when $x = 3$, what is the value of $a$ ?

A) $-8$

B) $-1$

C) $1$

D) $8$

**14** 7.2

If $y = x + \frac{3}{2}$ and $z = 4x - 3$, which of the following is equivalent to $y + yz$ ?

A) $4x^2 + 3x - \frac{9}{2}$

B) $4x^2 + 3x + 3$

C) $4x^2 + 4x - 3$

D) $4x^2 + 4x + 4$

**15** 7.3

$$-2zw^2 + 2w + 15 = -4z$$

In the equation above, what is the value of $z$ when $w = -3$ ?

**16** 7.2

Which of the following is equivalent to $\left(a - \frac{b}{2}\right)^2$ ?

A) $a^2 - \frac{b^2}{2}$

B) $a^2 - \frac{b^2}{4}$

C) $a^2 - \frac{ab}{2} + \frac{b^2}{4}$

D) $a^2 - ab + \frac{b^2}{4}$

**17** 7.1

Which of the following is equivalent to the sum of $5x^5 + 2x^4$ and $3x^5 + 4x^4$ ?

A) $14x^5$

B) $8x^{10} + 6x^8$

C) $15x^5 + 8x^4$

D) $8x^5 + 6x^4$

**18** 7.1

Which of the following expressions is equivalent to $2q^2 + r^2 + 6r - 3q + 4(q^2 - r)$ ?

A) $6r^2 + 3q^2$

B) $r^2 + 6q^2 - qr$

C) $r^2 + 6q^2 - 3q + 2r$

D) $r^2 + 6q^2 - 3q + 5r$

## Answer Key

**SkillDrill 7.2-1**

1. $x^2 - 3x$
2. $x^2 + 3x + 2$
3. $x^2 - 9x + 20$
4. $x^2 + x - 2$
5. $x^2 - 12x + 27$
6. $2x^2 + 24x + 70$
7. $x^4 + 3x^2 + 2$
8. $3x^3 + 10x^2 + 11x + 4$

**SkillDrill 7.2-2**

1. $x^2 + 6x + 9$
2. $x^2 + 2x + 1$
3. $x^2 - 8x + 16$
4. $x^2 - 2x + 1$
5. $4x^2 + 40x + 100$
6. $x^2 - 3x + \frac{9}{4}$
7. $x^4 + 2x^2 + 1$
8. $\frac{1}{25}x^2 - \frac{1}{10}x + \frac{1}{16}$

**SkillDrill 7.2-3**

1. $x^2 - 9$
2. $x^2 - 1$
3. $x^2 - 16$
4. $x^2 - 4$
5. $x^2 - \frac{1}{4}$
6. $4x^2 - \frac{9}{25}$
7. $9x^6 - 6$
8. $\frac{1}{25}x^4 - \frac{1}{16}x^2$

**Additional Problems**

1. $\frac{23}{2}$ or 11.5
2. C
3. C
4. $\frac{3}{4}$ or 0.75
5. C
6. D
7. B
8. D
9. 12
10. B
11. D
12. A
13. B
14. C
15. $\frac{9}{14}$, .642, or .643
16. D
17. D
18. C



# Solving Quadratic Equations 8

On the test, you will often need to find the solutions to a quadratic equation. There are several techniques that you can use for this; each has strengths and weaknesses associated with the characteristics of the problem you are faced with, but it is important to understand that any method can solve any problem—it might just be difficult or exceedingly time-consuming if the problem is a bad fit for the selected method. Here's a table that summarizes the solving methods we'll go on to explain in detail:

| Method | Form(s) of Most Susceptible Expressions | Other Clues | Strengths | Weaknesses |
|---|---|---|---|---|
| **Simple Factoring** | $x^2 + bx + c = 0$ | Answer choices are whole numbers, do not contain $\sqrt{\ }$ | Simple, fast, low error risk | Requires some guesswork, very difficult for some values of $b$ and $c$ |
| **Product-Sum (ac) Factoring** | $ax^2 + bx + c = 0$ | Answer choices are whole or rational numbers, do not contain $\sqrt{\ }$ | Simple, fairly fast, fairly low error risk | Requires some guesswork, very difficult for some values of $a$, $b$ and $c$ |
| **Completing the Square** | $x^2 + bx + c = 0$ or $ax^2 + bx + c = 0$ | $a \neq 1$; Answer choices contain $\sqrt{\ }$ | Directly solves any quadratic | Somewhat time-consuming, moderate error risk |
| **Quadratic Formula** | $x^2 + bx + c = 0$ or $ax^2 + bx + c = 0$ | Answer choices contain $\sqrt{\ }$ | Directly solves any quadratic | Time-consuming, relatively high error risk |

## 8.1 What is Factoring?

A very important technique that can be used in manipulating expressions and solving equations is called **factoring**. When we factor, we convert an expression, or a part of an expression, into a set of terms, called factors, that are multiplied; this set of multiplied terms is equivalent to the expression being replaced.

For example, here's one way we could factor 10:

$$10 = 2 \cdot 5$$

Here, 2 and 5 are factors of 10.

Here's an example with some variables:

$$4x + 4y = 4(x + y)$$

Here, we've factored $4x + 4y$ into two factors: 4 and $(x + y)$. You can see that if we multiply out (distribute) the 4 on the right hand side of this equation, we'll get the left hand side:

$$4(x + y)$$
$$4(x) + 4(y)$$
$$4x + 4y$$

You can see that **factoring is really just the inverse operation of distribution**. We're just dividing each term by some expression (here, just the constant 4), then multiplying the whole thing by that constant so its value is unchanged.

Notice that in the previous example, we "factored out" the number 4 because it was a coefficient of both the $x$-term and the $y$-term. However, the factor doesn't need to be explicitly shown for it to be factored out. For example, if we had

$$4x + 8y$$

we can still factor out 4 by dividing both terms by 4, and then multiplying what remains by 4:

$$4x + 8y$$

$$4\left(\frac{4x}{4} + \frac{8y}{4}\right)$$

$$4(x + 2y)$$

The terms in the expression being factored don't even need to be evenly divisible by the factor being pulled out. We could factor

$$4x + 3y$$

as

$$4\left(x + \frac{3y}{4}\right)$$

if that would be useful—such as when we need to isolate the variable $x$ with no coefficient. Speaking of $x$, we can factor out a variable, too; if we start with

$$4x + 7xy$$

that can be factored as

$$x(4 + 7y)$$

Importantly, note that if the variable has no coefficient, that's really an implied coefficient of 1, so

$$x + 7xy$$

can be factored as

$$x(1 + 7y)$$

This is sometimes confusing for students, because they wonder where that 1 comes from. This is likely because the implicit coefficient of 1 is, well, implicit—it's not shown, so it has to be imagined when factoring. That is, the first expression should be thought of as

$$1x + 7xy$$





which should make it much more obvious how we factor that and obtain

$$x(1 + 7y)$$

A factor doesn't need to be a single term; it can be an expression. This should already be apparent from the above examples, where the second term is an expression, but here's another example. Let's start with

$$(x - 3)(x + y) - (x - 3)(z)$$

Notice that there are two terms here, with the second being subtracted from the first, and each term has $(x - 3)$ as a factor. We can factor out $(x - 3)$ from both terms to get

$$(x - 3)[(x + y) - z]$$

As a reminder, factors are a set of expressions that, when multiplied, yield the original expression.

## Factors Help Us Find Solutions

Factoring has many uses, but one of the most important is that if we are presented with an equation we need to solve, we can move all the terms to one side of the equation (leaving 0 on the other side), and then we can try to factor the resulting expression. Then, **if we can figure out the values of the variable(s) that would make any of the factors equal 0, we know the whole expression will equal 0 (because 0 times anything is 0)**, and we will have solved the equation (which, remember, has a 0 on the other side). We're going to explore the applications of this principle that apply to the math section of the test in what follows, but first, let's dig into a few more essential concepts.

An expression is not the same thing as an equation. This is a **quadratic expression**, which is a univariate (single variable) polynomial expression consisting of a series of terms being added to (or subtracted from) each other, where the degree of the polynomial is 2:

$$x^2 + 2x - 24$$

This is a **quadratic equation** which has the above expression on one side of the equals sign:

$$x^2 + 2x - 24 = 0$$

**You solve equations, not expressions**. Expressions merely have a value, not a truth or falsity; equations are solved by finding the value(s) of any variable(s) that make the equation true.

Solving polynomial equations of degree zero is… meaningless! Polynomials of degree zero only have a term in the following form:

$$ax^0$$

Because $x^0 = 1$, as discussed above, this is equivalent to just $a$, and that's a constant. Therefore, an equation with such a sad little polynomial would be something like $3 = 3$. There's no variable at all here, so there's nothing to solve; the equation is either true or false, all the time.

For something with a little more substance, let's look at a polynomial of degree 1:

$$2x^1 - 24x^0$$

By now, you'll know we can write this more compactly by applying a couple of those exponent rules, so we'll drop the exponent of 1 and the variable raised to the zeroth power:

$$2x - 24$$

Let's set up a simple equation using this first-degree polynomial:

$$2x - 24 = 0$$

Now, let's solve this using simple algebra. There are several ways we could proceed; for example, we could isolate the $x$ term on one side as our first step, and, it's true, that would ordinarily be the most straightforward way to begin. However, because we want to illustrate a specific principle, we'll take a different approach. First, we'll get an $x$ term with no coefficient by factoring out that coefficient of 2:

$$2\left(x - \frac{24}{2}\right) = 0$$

$$2(x - 12) = 0$$

At this point, we have a multiplication of two terms, which are 2 and $(x - 12)$. As we explained earlier, each term involved in a multiplication is a factor of the result of that multiplication. The equations tells us that the product of these two factors equals 0. Here's the most important consequence of the factoring: if the product of the factors equals 0, it means that if either factor is 0, the equation is true, because 0 times anything is still 0. Obviously, 2 doesn't equal zero, but $(x - 12)$ will equal zero when $x = 12$, so $x = 12$ is the solution to the equation.

More generally, we can see that once we get to a setup where we have an equation with a fully factored form on one side and 0 on the other side…

$$a(x - z) = 0$$

…we know that $x = z$, because that will make the second factor $(x - z)$ equal 0, which will make the whole left side equal 0.

The same principle applies if we end up with addition in the expression on the left, effectively changing the sign of the constant term; we just change the sign in the solution:

$$a(x + z) = 0$$

…which means that $x = -z$ is a zero. Simple enough.

**Note that, importantly, this simple technique relies completely on arranging the equation so that 0 is on one side**; whenever we can get just a series of multiplications (factored form) on one side and a 0 on the other side of an equation, we can exploit this, because **if any one of the terms (factors) involved in the multiplication evaluates to 0, the equation will be true**.

## Expressions vs. Equations

Let's introduce and clarify some terms. It should already be clear that **an expression (as contrasted with an equation) doesn't have solutions, because it doesn't make any claims to be equal to anything in particular.** However, we do say that an **expression or a function can have zeros, which are the solutions that would exist if the expression or function were set equal to zero**, thus creating an equation. We also call these solutions **roots**.

Of course, **if we're presented with an equation with zero on one side, the zeros or roots of the expression on the other side are the solutions of the whole equation, and, more casually, we can just refer to the zeros or roots of the whole equation.** If we have an **equation that has something other than zero on one side, it can certainly have solutions (which can also be called roots in this situation, confusingly enough), but the equation as a whole cannot properly be said to have zeros** (though the polynomial expression on the other side of the equation can have zeros).

One final terminological note: **be careful not to mix up the term roots as used in this context with the word as used in terms such as "square root," "cube root," and so on.**





We could solve the generic form of a first-degree polynomial equation, which would give us a formula that would apply to any such equation. Let's try making that formula, starting with the simple equation setup above, where the right side is zero:

$$ax + b = 0$$

We'll factor out that $a$ by simply dividing each term on the left by $a$, and multiplying the entire original expression by $a$ to counter that division:

$$a\left(x + \frac{b}{a}\right) = 0$$

From what we learned above, we know that if the factor $x + \frac{b}{a} = 0$, the equation is true, and therefore this means

$$x = \frac{-b}{a}$$

That's pretty simple. Let's test it with the example we've been developing, which we solved earlier to find $x = 12$:

$$2x - 24 = 0$$

Here, $a = 2$ and $b = -24$. Our formula says $x = \frac{-b}{a}$. Substituting gives us the following result:

$$x = \frac{-(-24)}{2} = \frac{24}{2} = 12$$

That checks out, thank goodness. Now, we're not suggesting that you memorize this formula; it's not necessary to know it when solving simple equations of the form shown here, which you were probably already able to dispatch with little difficulty using a bit of simple algebra. However, we did want to step through its derivation to lay some groundwork for what is to follow.

Now, let's take a look at second-degree univariate polynomial (quadratic) expressions and equations. Remember, second-degree polynomials are just like the first-degree expressions we've been working with, except they will have one more term at the beginning that contains a variable raised to the second power, so second-degree polynomials will look like this when written in Standard Form:

$$ax^2 + bx^1 + cx^0$$

Again, by convention, we'll always be writing this by dropping the extraneous bits:

$$ax^2 + bx + c$$

The simplest equation we can make just has zero on the right, which will help us when it comes to solving (this should give you an idea of where we are going with this!):

$$ax^2 + bx + c = 0$$

Of course, any equation that has just zeroth-, first-, and second-degree terms in any order and on either side can be put into the form shown above simply by combining all the like terms on one side of the equation and placing them in decreasing order of degree.

For example, if you're presented with some mathematical jambalaya such as…

$$-8 + 4x - 2 + 3x^2 = 14 - 3x + x^2 + 5x$$

…just let out a big sigh and start collecting like terms and herding them over to one side in descending degree order, leaving nothing but zero on the other side:

$$3x^2 - x^2 + 4x + 3x - 5x - 8 - 2 - 14 = 0$$
$$2x^2 + 2x - 24 = 0$$

## Standard Form Quadratic Equation

The Standard Form of a quadratic equation, where $a$, $b$, and $c$ are constants, is the following:

$$y = ax^2 + bx + c$$

Most often the value of $a$ is 1 in problems on the test. For example, in the quadratic equation $y = x^2 + 3x + 2$, the value of $a$ is 1 because the $x^2$ term has a coefficient of 1 (which is why there is no number shown in front of the $x^2$).

Usually, in the course of solving quadratic problems on the test, we will set the $y$-value equal to 0 (or make one side of the quadratic equation equal to 0) because it enables us to find the zeros or solutions of the quadratic equation using the most common techniques. If the $y$-value is already set equal to 0, the problem may ask you for solutions to the equation, which, as previously noted, are the same as the zeros in this situation. Though solution is not an interchangeable term for zero, the most frequently utilized setup for solving quadratics involves a quadratic expression on one side of the equation and a value of 0 on the other side of the equation.

Just as we discovered that there's a formula for solving a first-degree polynomial (linear) equation (remember that the solution of $0 = ax + b$ is $x = \frac{-b}{a}$), there's also a formula for second-degree equations: it's commonly referred to as the **Quadratic Formula**, and you've probably heard of it.

## Quadratic Formula

For a quadratic equation in Standard Form, $0 = ax^2 + bx + c$, the solutions can be found with the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which we choose to split into two fractions as below (the expression $\frac{-b}{2a}$ helps reinforce an important feature of quadratic graphs, the $x$-value of the vertex, which will be discussed later):

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Note, importantly, that this formula can produce not just one but two different values due to the presence of the $\pm$ symbol, which indicates that both the positive and negative square roots must be separately determined.

Though the Quadratic Formula can be used to solve any quadratic equation by substituting the values of $a$, $b$, and $c$, there is a fair amount of work involved to evaluate the formula, and that means it can be time-consuming, and the procedure is susceptible to errors due to the number of steps needed, the presence of the square root operation, and the nesting of operations. On the test, time and accuracy are precious, and the use of the Quadratic Formula can use up a lot of time and introduce errors.

All of this brings us to the question: are there simpler methods for finding the solutions to a quadratic equation?

There are. The Quadratic Formula is a valid solving method, and it does have some strengths in certain situations, so we will discuss it in depth later on, but first, we're going to explain easier, more streamlined methods for solving quadratics.

Let's start with the one called **factoring**, which is by far the most frequently applicable technique when it comes to the test. This method relies on the fact that just as a first-degree expression can be expressed as the product of a constant and a simple two-term (binomial) expression of the form $(x + p)$, a quadratic expression can be expressed as the product of a constant and two simple first-degree expressions.





First-degree expression:

$$a(x + p)$$

Second-degree (quadratic) expression:

$$a(x + p)(x + q)$$

Let's multiply out that factored expression just to show that it results in a quadratic (second-degree univariate polynomial):

$$a(x + p)(x + q)$$
$$a(x)(x) + a(p)(x) + a(q)(x) + apq$$
$$ax^2 + a(p + q)x + apq$$

That's definitely a quadratic expression, as it has a term $ax^2$ with an exponent of 2, a term $a(p + q)x$ with an implied exponent of 1, and a final term with an exponent of 0, represented by the constant term $apq$.

In the overwhelming majority of cases, quadratic expressions that appear on the test don't have any $a$ coefficient on the second-degree term, so they correspond to the simpler Factored Form

$$(x + p)(x + q)$$

which, when multiplied out into Standard Form, gives

$$(x)(x) + qx + px + pq$$
$$x^2 + (p + q)x + pq$$

Now we are prepared to solve quadratic equations using factoring. If we have—or can get—a quadratic equation into the factored form

$$(x + p)(x + q) = 0$$

then we can determine a value of $x$ that makes either of the factors equal 0. **If any one of the factors is equal to 0, then the whole expression will evaluate to 0, making the equation true**; put another way, we will have found a solution.

Here, we can see that $x = -p$ will make the first factor, $(x + p)$, equal to 0, so the whole left side will be 0, and thus the equation will be true. Similarly, $x = -q$ will make the second factor, $(x + q)$, equal to 0, and the equation will again be true. Therefore, there are, in the general case, two solutions to quadratic equations in the above factored form (if $p$ and $q$ happen to be the same, there will be just one solution, sometimes called a "double root"):

$$x = -p$$
$$x = -q$$

It should be noted that $p$ and $q$ can be negative numbers themselves, in which case the solutions would be positive values.

Note also that either $p$ or $q$ (or both) could be equal to 0, in which case the factor that corresponds to that solution is $(x - 0)$, which is the same as a factor of $x$, so $x = 0$ is one solution. **When quadratics have a factor of $x$ ($p$ or $q$ is equal to 0), there is no constant term (the value of $c$ is 0) in the Standard Form representation of that expression**. For example, an expression with factors $x$ and $x - 4$ would be the following in Standard Form (as a result of distributing $x$ to both terms in the factor $x - 4$).

$$x(x - 4) = x^2 - 4x$$

Note once again that in the expression $x^2 - 4x$ (which is in Standard Form for quadratics, $ax^2 + bx + c$) the value of $c$ is 0 (there is no constant term written after the $x$-term).

If both $p$ and $q$ are 0, then both factors are simply $x$, and the resulting quadratic expression is $x \cdot x$ or $x^2$, for which there is no $x$-term or constant term (the values of $b$ and $c$ for this Standard Form expression are both 0). In this scenario, the quadratic equation degenerates to $x^2 = 0$, and the one solution is $x = 0$.

When we want to solve quadratic equations using the factoring method, our goal is to have the quadratic expression written as the product of two binomials, which are just expressions with two terms that have the same variable (or a constant).

For example, the quadratic equation $y = x^2 + 3x + 2$ can also be written as $y = (x + 2)(x + 1)$. We will learn how to produce the Factored Form shortly, but for now, you can verify that the two equations are the same by multiplying the two factors and seeing that the result will be the same Standard Form equation.

## Factored Form of Quadratic Equation

In general, the Factored Form of a quadratic equation, where $a$, $p$, and $q$ are constants, is the following:

$$0 = a(x + p)(x + q)$$

Most often on this test, the value of $a$ will be 1 (most of the time when factoring is a good solution method, the value of the $x^2$ coefficient in the Standard Form of the quadratic is 1).

When we are asked to find the **solutions** of a quadratic equation, we want to have all of the individual terms for each degree of the variable including the constant term (if the equation is still in Standard Form) or the factors (if the equation is in Factored Form) on one side of the equation and a 0 on the other side of the equation. Once the equation has a 0 on one side of the equation and a polynomial written in Factored Form on the other side of the equation, we know that the product of the factors is 0 and, therefore, this equation will be true whenever any one of the factors is equal to 0 (any number times 0 equals 0).

Given the equation $0 = a(x + p)(x + q)$, where the factors of the quadratic equation are $(x + p)$ and $(x + q)$, we know that $-p$ and $-q$ are the solutions to the equation because the factor $(x + p)$ will be equal to 0 when $x = -p$, and the factor $(x + q)$ will be equal to 0 when $x = -q$.

Note that if 0 is one of the solutions (because either $p$ or $q$ is 0), then $x$, which is the same as the factor $(x - 0)$, is one of the factors.

All of these ideas extend beyond quadratics to other polynomials with more than two factors. For example, in the polynomial equation, $0 = (x + j)(x + k)(x + \ell)$, the right side of the equation has factors $(x + j)$, $(x + k)$, and $(x + \ell)$. Therefore, $-j$, $-k$, and $-\ell$ are all solutions to the equation because each of those values of $x$ would cause one of the factors to be equal to 0, and thus the entire expression would equal 0.

At this point, you should be able to easily solve any equation that has a quadratic expression in Factored Form set equal to 0. Let's look at a quick example:

$$(x + 6)(x - 4) = 0$$

Here, the solutions should be immediately apparent because of the Factored Form; we can see right away that $x = -6$ or $x = 4$ will make the equation true, because either value of $x$ will produce a value of 0 for one of the factors, and if any factor is 0, the whole expression will be 0, and the equation will be valid.

You might encounter an expression that doesn't exactly match the Factored Form, but which nonetheless consists of a series of factors. The central principle of finding the roots still applies; set each factor equal to 0 and solve for the variable to find one of the solutions. For example, if you are given

$$(4x - 12)(x + 3)^3 = 0$$

set the first factor, $4x - 12$, equal to 0 and solve for $x$:

$$\begin{aligned} 4x - 12 &= 0 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$





Then, set the second factor equal to 0 and solve for $x$:

$$\begin{aligned} (x + 3)^3 &= 0 \\ x + 3 &= 0 \\ x &= -3 \end{aligned}$$

Don't let such a variation throw you off; just keep in mind the principle that if any factor is 0, the equation is valid, so just find those values of the variable that make each factor equal 0.

**Example 8.1-1**

**1** 1600.io 8.1

What is the sum of the solutions to $(x - 4)(x + 0.3) = 0$?

A) $-4.3$

B) $-3.7$

C) $3.7$

D) $4.3$

**Solution**

1. We need to find values of $x$ that make each factor equal to 0 in order to find both solutions.
2. The first factor, $(x - 4)$, will be equal to 0 when $x$ is 4. To determine this, set the factor equal to 0 and solve for $x$.

$$\begin{aligned} x - 4 &= 0 \\ x &= 4 \end{aligned}$$

3. The first solution is 4 since the factor $x - 4$ is equal to 0 when $x = 4$.
4. The second factor, $(x + 0.3)$, will be equal to 0 when $x$ is $-0.3$, so the second solution is $-0.3$.

$$\begin{aligned} x + 0.3 &= 0 \\ x &= -0.3 \end{aligned}$$

5. The sum of the two solutions is $4 + (-0.3)$.

$$4 + (-0.3) = 4 - 0.3 = 3.7$$

6. The answer is C.

**Section 8.1 Suggested Problems from Real Tests**

- Test 6-C-20
- Mar 2018-NC-7
- Mar 2018-NC-10
- Mar 2019-C-30

View related real-test problems in SMTEx

SMTEx

## Section 8.1 Practice Problems

**1** 1600.io 8.1

$$7(x+1)(x-8)=0$$

What positive value of $x$ satisfies the equation above?

**2** 1600.io 8.1

$$(3x-2)(x+4)^2=0$$

What is the solution set to the equation above?

A) $\left\{-\frac{2}{3}, 4\right\}$

B) $\left\{\frac{2}{3}, -4\right\}$

C) $\left\{-\frac{2}{3}, -4, 4\right\}$

D) $\left\{\frac{2}{3}, -4, 4\right\}$

## 8.2 How to Factor

On the test, we are not usually provided with a quadratic polynomial in Factored Form. We need to learn to factor quadratic polynomials that are written in Standard Form, $y = ax^2 + bx + c$. This is easiest when the value of $a$ is 1.

Let's pick back up with our example from the previous section:

$$(x+6)(x-4)=0$$

Clearly, we already know the solutions to this equation (based on Factored Form), but let's put this same equation into Standard Form, and work forward from there. We can multiply out the above example to put it into Standard Form:

$$\begin{aligned}(x+6)(x-4)&=0\\(x)(x)-4x+6x+6(-4)&=0\\x^2+2x-24&=0\end{aligned}$$

Of course, the solutions are the same ($x=-6$ or $x=4$), because we've only transformed the quadratic expression from Factored Form into Standard Form (the two are equivalent even though they look different). But what if we had to start with the equation in Standard Form, where the solutions aren't so obvious? What would we do if we were presented with this at the outset?

$$x^2+2x-24=0$$

We know that there's a Quadratic Formula for finding the solutions, but it's time-consuming and somewhat complicated. If we could simply convert the quadratic expression to Factored Form, the solutions would be at hand, so let's take a look at that procedure.

Remember, when we multiply out a factored quadratic

$$(x+p)(x+q)$$

we get

$$x^2+(p+q)x+pq$$





This means that when we consider the Standard Form of a quadratic expression to be $x^2+bx+c$ (where the value of the $x^2$ coefficient is 1), we see that $p+q=b$ and $pq=c$ (match the terms that occupy the same positions in this example expression and the Standard Form expression).

We know that if we can determine $p$ and $q$, we can get the solutions easily because $x=-p$ and $x=-q$. Though the Standard Form shown above doesn't directly isolate $p$ and $q$ like the Factored Form does, we do have some valuable information: **the sum of $p$ and $q$ equals the $b$ coefficient**, and **the product of $p$ and $q$ equals the constant term $c$**. Therefore, if we can figure out two numbers that add up to $b$ and that multiply to yield $c$, we can solve a quadratic equation where the polynomial is in Standard Form. In our current example, $x^2+2x-24=0$, the $b$ coefficient is 2, and $c$ is $-24$, so that means if we can figure out the two numbers that add up to 2 and that multiply to $-24$, we will have identified the $p$ and $q$ that we need to create the factored form, and thus to solve the equation.

In this case, we won't yet show the generalized, exhaustive procedure for finding all the candidates for $p$ and $q$ and testing them, but we'll observe that $6+(-4)=2$, and $6(-4)=-24$. That means that our $p$ and $q$ are 6 and $-4$ (it doesn't matter which is which), and we can rewrite our quadratic equation in Factored Form as $(x+6)(x-4)=0$.

From our previous discussion, you should be able to see right away that this means $x=-6$ is a solution (because it makes the first factor equal 0), and also that $x=4$ is a solution (same, but for the second factor).

**The key to solving a quadratic equation by factoring the expression is this process of figuring out those two numbers that add to make the $b$ coefficient while also multiplying to equal the $c$ coefficient (again, assuming that the $x^2$ coefficient is 1).**

For example, to find the factors of the quadratic expression in the equation $y=x^2+3x+2$, we need to find two numbers that add to 3 and multiply to 2. The numbers 1 and 2 add to 3 (we know that $1+2=3$) and multiply to 2 (we know that $1(2)=2$), so the factors are $(x+1)$ and $(x+2)$.

$$\begin{aligned}y&=x^2+3x+2\\y&=(x+1)(x+2)\end{aligned}$$

Remember that the main purpose we have for factoring quadratics on this test is to find solutions to a quadratic equation, and our goal is to have 0 on one side of the equation and factors on the other (again, if the product of the factors is equal to 0, then the solutions are the values of $x$ that make any of the individual factors equal to 0).

Generally, the reason we'd like to factor the quadratic expression in an equation is to find the zeros of the expression; recall that the zeros are the values of the variable that result in a value of 0 for the entire expression. We can set up an equation that expresses this by replacing $y$ with 0.

$$\begin{aligned}y&=(x+1)(x+2)\\0&=(x+1)(x+2)\end{aligned}$$

The factor $(x+1)$ tells us that $-1$ is a solution, because when $x=-1$, $(x+1)=0$. We can check by substituting $-1$ for $x$ in the original equation and verifying that $y$ is equal to 0.

$$\begin{aligned}y&=x^2+3x+2\\y&=(-1)^2+3(-1)+2\\y&=1-3+2\\y&=0\end{aligned}$$

The factor $(x + 2)$ tells us that $-2$ is a solution, because when $x = -2$, $(x + 2) = 0$. We can check by substituting $-2$ for $x$ in the original equation and verifying that $y$ is equal to 0.

$$y = x^2 + 3x + 2$$
$$y = (-2)^2 + 3(-2) + 2$$
$$y = 4 - 6 + 2$$
$$y = 0$$

## Factoring Quadratic Equations

The Standard Form of a quadratic equation, where $a$, $b$, and $c$ are constants, is the following:

$$y = ax^2 + bx + c$$

On the test, when the coefficient of the $x^2$ term in a quadratic polynomial is 1, such as in the equation $y = x^2 + bx + c$, it is usually easy to factor (we will examine other solution methods later in the chapter to see what to do when this is not the case). **You need to determine the two numbers, $p$ and $q$, whose sum is $b$ and whose product is $c$.**

When the values of $p$ and $q$ are found, the quadratic expression can be rewritten as $y = (x + p)(x + q)$.

If we are trying to find zeros of the quadratic, we should substitute 0 for $y$ and find the values that would make each factor equal to zero (because if any factor is 0, the entire expression has a value of 0). If the factors are $(x + p)$ and $(x + q)$, then the zeros are $x = -p$ and $x = -q$, respectively.

If you are using the factoring method to find solutions, you don't need to actually write out the Factored Form of the quadratic; once you determine the values of $p$ and $q$, you know that the solutions are $-p$ and $-q$.

One tip for making finding the factors easier is to look at the signs of the $b$ and $c$ terms. If the $c$ term is positive, both signs for $p$ and $q$ will be the same as each other (either both positive or both negative) because a positive times a positive is positive, and a negative times a negative is also positive. To determine whether both signs should be positive, look to the $b$ term: if the $b$ term is positive, both $p$ and $q$ should have positive signs, because adding two negative numbers can't produce a positive result; if the $b$ term is negative, both $p$ and $q$ should have negative signs.

If the $c$ term is negative, then $p$ and $q$ should have opposite signs (one being positive, and the other negative). If the signs for $p$ and $q$ are mixed, the sign of the $b$ term will determine which of $p$ and $q$ should have the negative sign. If $b$ is positive, then the larger value of $p$ and $q$ should be positive (and the smaller should thus be negative) because adding a larger positive number and a smaller negative number yields a positive result; if $b$ is negative, then the larger value of $p$ and $q$ should be negative (and the smaller should therefore be positive) because adding a smaller positive number and a larger negative number yields a negative result.

When a quadratic expression in Standard Form has no constant term written at the end (it consists of only an $x^2$-term and $x$-term), factoring is actually simpler, but this throws off students who are expecting three terms in factorable quadratic expressions. For example, let's look at the quadratic expression $x^2 - 4x$. The most straightforward way to approach factoring this expression is to recognize that both terms, $x^2$ and $-4x$, share a common factor of $x$, so we can factor $x$ out from both terms.

$$x^2 - 4x$$
$$(x)(x) - 4(x)$$
$$x(x - 4)$$

However, even if you fail to recognize this simplified factoring method available with these two-term quadratics, the good news is that the more generalized factoring method explained previously still works as long as you understand that the absence of a constant term just means that you can consider the constant term to be 0.





Let's again look at the quadratic expression $x^2 - 4x$, which is in Standard Form where $a = 1$, $b = -4$, and $c = 0$ (remember, the constant term appears not to be there, but that just means its value is 0). Based on how we factored other quadratic expressions above, in order to find the factors of this equation, we need two numbers that add to $-4$ and multiply to 0. In order to multiply to 0, one of the two number must be 0, so $(x + 0)$ is a factor. Since 0 and $-4$ add to $-4$, then $-4$ must be the other number, meaning that $(x + (-4))$, or $(x - 4)$, is a factor.

$$x^2 - 4x$$
$$(x + 0)(x - 4)$$

Since $(x + 0)$ is the same as $x$, the expression $x^2 - 4x$ in Factored Form is $x(x - 4)$.

Hopefully, you will not freeze up if you encounter one of these on the test because you could apply these generalized principles. However, it's considerably simpler to just factor out the variable as explained.

### SkillDrill 8.2-1

**Directions:** Find two numbers whose sum is $S$ and whose product is $P$.

Challenge questions are boxed.

**1.** $S = 7$, $P = 10$ **2.** $S = -7$, $P = 10$ **3.** $S = -4$, $P = -12$ **4.** $S = 1$, $P = -12$

**5.** $S = 13$, $P = 40$ **6.** $S = 4$, $P = -77$ **7.** $\boxed{S = -9, P = -52}$ **8.** $\boxed{S = -20, P = 96}$

**Directions:** Write each of the following polynomial expressions in Factored Form.

Challenge questions are boxed.

**9.** $x^2 + 3x - 4$ **10.** $x^2 - 10x + 21$ **11.** $a^2 + 18a + 80$ **12.** $b^2 - b - 30$

**13.** $a^2 - 3a$ **14.** $x^2 + x$ **15.** $\boxed{x^4 + 18x^2 + 80}$ **16.** $\boxed{x^5 + 18x^3 + 80x}$

**Directions:** Find the solutions to the polynomial equations below by factoring.

Challenge problems are boxed.

**17.** $x^2 + 3x - 4 = 0$ **18.** $x^2 - 8x + 7 = 0$ **19.** $x^2 + 9x + 20 = 0$ **20.** $x^2 - 10x + 16 = 0$

**21.** $x^2 - x - 12 = 0$ **22.** $x^2 + x - 2 = 0$ **23.** $a^2 + 3a - 28 = 0$ **24.** $b^2 - 14b + 48 = 0$

**25.** $\boxed{a^2 - 2a + 8 = 16}$ **26.** $\boxed{b^2 + 4b - 6 = 5b}$ **27.** $\boxed{x^2 + 22x + 140 = 20}$ **28.** $\boxed{x^2 - x - 12 = 3x}$

**Example 8.2-1**

**1** 1600.io 8.2

$$x^2 + 5x - 24 = 0$$

If $a$ is a solution of the equation above and $a > 0$, what is the value of $a$?

**Solution 1**

1. In order to factor this equation to find the solutions, we need to start by finding two numbers that add to 5 and multiply to $-24$.
2. The numbers 8 and $-3$ add to 5 and multiply to $-24$. Therefore, one of the factors is $(x+8)$ and the other factor is $(x-3)$.

$$x^2 + 5x - 24 = 0$$
$$(x+8)(x-3) = 0$$

3. Since $(x+8)$ is a factor, the equation will be true when $x + 8 = 0$, so $x = -8$ is a solution, but the question tells us that the solution $a$ must be greater than 0 (a positive number). Since $(x-3)$ is also a factor, the equation will be true when $x - 3 = 0$. Therefore, $x = 3$ is also a solution. The value of the solution $a$ must be 3 since 3 is the only solution that is greater than 0.
4. The answer is 3.

**Solution 2**

1. Use the Quadratic Formula to find the solutions. For this equation, $a = 1$, $b = 5$, and $c = -24$.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(5)}{2(1)} \pm \frac{\sqrt{(5)^2 - 4(1)(-24)}}{2(1)}$$
$$x = \frac{-5}{2} \pm \frac{\sqrt{25 + 96}}{2}$$
$$x = \frac{-5}{2} \pm \frac{\sqrt{121}}{2}$$
$$x = \frac{-5}{2} \pm \frac{11}{2}$$

2. The two possible solutions are $x = \frac{-5}{2} + \frac{11}{2}$ and $x = \frac{-5}{2} - \frac{11}{2}$, but only the first one will be a positive value, so we can ignore the second one since we are told that we are looking for a solution that is greater than 0. Therefore, the only solution that answers the questions is $x = \frac{-5}{2} + \frac{11}{2} = \frac{6}{2} = 3$.
3. The answer is 3.





**Notes**

Hopefully this example helps show why it is good to know conventional factoring. Overreliance on the Quadratic Formula is a time killer.

**Section 8.2 Suggested Problems from Real Tests**

- Test 1-NC-15
- Test 1-C-25
- Test 3-NC-16
- Test 5-NC-3
- Apr 2021-C-18

**View related real-test problems in SMTEx**

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## Section 8.2 Practice Problems

**1** 1600.io 8.2

$$x^2 + 2x - 15 = 0$$

If $a$ is a solution of the equation above and $a > 0$, what is the value of $a$?

**2** 1600.io 8.2

A bricklayer is designing a rectangular patio. The length of the patio is to be 4 feet longer than the width. If the area of the patio will be 192 square feet, what will be the length, in feet, of the patio?

## 8.3 Other Methods of Finding Roots

Not all quadratic expressions are easily factorable, so we need other tools to deal with those situations. We have already seen the Quadratic Formula being used to find solutions, and that option is always on the table, but if the only tool you have is a hammer, then every problem will look like a nail. In reality, there are subtle differences between quadratics problems, and you really want to use the appropriate tools in order to minimize the time needed for solving while also minimizing the chances of an error creeping in.

### Divide Through and Factor when $a \neq 1$

Sometimes the coefficient of the $x^2$ term is not 1, but you might find that all of the coefficients in the equation share a common factor that can be divided out, making it easy to factor the quadratic expression (it's important that we are dividing both sides by a constant, not a variable—dividing by a variable can have greater ramifications that are discussed in a later chapter).

**Example 8.3-1**

**1** 1600.io 8.3

What are the solutions of the quadratic equation

$3x^2 - 12x - 15 = 0$?

A) $x = -1$ and $x = 5$

B) $x = -1$ and $x = -5$

C) $x = 1$ and $x = 5$

D) $x = 1$ and $x = -5$

**Solution 1**

1. Divide both sides of the equation by 3 (or multiply by $\frac{1}{3}$ if you prefer to think of it that way) to make the $x^2$ coefficient equal to 1. Remember to divide all three terms on the left side (distribute the $\frac{1}{3}$ to all of the terms).

$$3x^2 - 12x - 15 = 0$$

$$\frac{3x^2 - 12x - 15}{3} = \frac{0}{3}$$

$$x^2 - 4x - 5 = 0$$

2. In order to factor the equation, we need to find two numbers that add to $-4$ and multiply to $-5$.
3. The numbers $-5$ and 1 add to $-4$ and multiply to $-5$, so one of the factors is $(x - 5)$ and the other factor is $(x + 1)$.

$$x^2 - 4x - 5 = 0$$
$$(x - 5)(x + 1) = 0$$

4. Since $(x - 5)$ is a factor, the equation will be true when $x - 5 = 0$. Therefore, we know that $x = 5$ is a solution to the quadratic equation.





5. Since $(x + 1)$ is a factor, the equation will also be true when $x + 1 = 0$. Therefore, we know that $x = -1$ is a solution to the quadratic equation.
6. The two solutions are $x = -1$ and $x = 5$.
7. The answer is A.

**Solution 2**

Step 1 is the same as in Solution 1.

2. In the Quadratic Formula (which you really shouldn't need for this problem because the expression should be easily factorable after we divided by 3), plug in 1 for $a$, $-4$ for $b$, and $-5$ for $c$.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4)}{2(1)} \pm \frac{\sqrt{(-4)^2 - 4(1)(-5)}}{2(1)}$$

$$x = \frac{4}{2} \pm \frac{\sqrt{16 + 20}}{2}$$

$$x = 2 \pm \frac{\sqrt{36}}{2}$$

$$x = 2 \pm \frac{6}{2}$$

$$x = 2 \pm 3$$

3. The two solutions are $x = 2 - 3 = -1$ and $x = 2 + 3 = 5$.
4. The answer is A.

**Notes**

In Solution 1, you could have stopped writing after Step 3: once you find the numbers $p$ and $q$ that sum to $b$ and multiply to $c$, you know that the solutions must be $-p$ and $-q$, which are $-1$ and 5 in this example.

In Solution 2, you do not have to start by factoring out 3, but it makes evaluating the Quadratic Formula easier because the numbers are smaller and more manageable. In practice, after dividing through by 3, you would check if conventional factoring seems like a viable option (which of course it is in this case).

You could do this problem by plugging in each set of answer choices and making sure that both values of $x$ in a choice will make the left side of the equation equal to 0 (weak).

## Product Sum ($ac$) Method when $a \neq 1$

Another method for finding the roots of quadratics is the **Product Sum ($ac$) Method**, which is particularly useful when the $x^2$ coefficient, $a$, is not equal to 1 and dividing all of the terms by $a$ will not produce an easily factorable expression. This method is simply the generalized factoring procedure that can be applied when the $a$ coefficient is not 1; in fact, you'll see that when $a = 1$, this method degenerates to the simpler factoring process that we explained previously.

When factoring a quadratic equation $ax^2 + bx + c = 0$ using the Product Sum ($ac$) Method, we need to find a product number and a sum number, just as with the somewhat simpler task of factoring a quadratic expression where the $a$-coefficient is 1.

As we saw, when the $a$-coefficient is 1, we factor by finding the two numbers whose sum is equal to $b$ and whose product is equal to $c$. **In the situation where the $a$-coefficient isn't 1, the sum must still be $b$, but the product needs to be $a$ times $c$ (that is, $ac$, which is where the method gets its nickname)**. As with the simpler factoring procedure, you need to find the two values that add up to the sum number and multiply to the product number.

For example, in the equation $0 = 8x^2 - 2x - 15$, the sum number is $-2$, and the product number is $8(-15)$, or $-120$. The two numbers that add up to $-2$ and multiply to $-120$ are $-12$ and 10.

**Once we have the two values that satisfy the product and sum constraints, the solutions are just the negatives of those values divided by $a$.**

Here, once you've found $-12$ and 10, the roots are $-\left(\frac{-12}{a}\right) = -\left(\frac{-12}{8}\right)$, which simplifies to $\frac{3}{2}$, and $-\left(\frac{10}{a}\right) = -\left(\frac{10}{8}\right)$, which simplifies to $\frac{-5}{4}$, and you're done (you can plug these back in to the original to check that they are indeed zeros).

### Product Sum ($ac$) Method

Here's a concise description of the procedure for finding roots using this method (we call this the $ac$ method because that serves as a reminder of the most critical step):

1. Find two numbers $r$ and $s$ that add up to $b$ and that multiply to $ac$.
2. The roots are $\frac{-r}{a}$ and $\frac{-s}{a}$.

If you need to construct the complete Factored Form (which is not necessary for solving), it's

$$a\left(x + \frac{r}{a}\right)\left(x + \frac{s}{a}\right)$$

Alternatively, the Factored Form can be written as

$$a(x + p)(x + q)$$

where $p = \frac{r}{a}$ and $q = \frac{s}{a}$.

It should again be noted that this is just factoring, generalized to the case where $a \neq 1$.





**SkillDrill 8.3-1**

**Directions:** Divide out a common factor from all of the coefficients or use the Product Sum ($ac$) Method to find the solutions to the following polynomial equations. Beware the frumious bandersnatch (non-zero right-hand side).

**1.** $4x^2 + 12x + 8 = 0$ **2.** $3x^2 - 15x + 18 = 0$ **3.** $-5x^2 + 15x + 22 = 2$ **4.** $-2x^2 - 10x + 25 = -3$

**5.** $4x^2 - x - 3 = 0$ **6.** $6x^2 + 7x + 2 = 0$ **7.** $8x^2 - 15x - 2 = 0$ **8.** $2x^2 + 17x + 22 = 1$

## Generalized Quadratic Solving Methods

There are equations that have solutions that are not whole numbers or even rational numbers (fractions in which the numerator and denominator are whole numbers)—for example, they may contain square roots—which makes finding factors nearly impossible with simple factoring or the Product Sum ($ac$) Method. The test rarely includes these kinds of equations, but, for such situations, there are other methods available for solving quadratic equations when conventional factoring is too difficult.

## Completing the Square

We've already discussed one such method: the Quadratic Formula. This method is very popular, and you most likely knew about it already. However, if you can't factor a quadratic equation easily, there is yet another option besides the Quadratic Formula called **Completing the Square**. This method is based around creating perfect square expressions like the ones we talked about earlier in this chapter. This method often seems rather mysterious at first, but it is very formulaic and easy to repeat once you get used to it.

In fact, like the more familiar Quadratic Formula (which is, in fact, derived from Completing the Square), Completing the Square can be used to find solutions to any quadratic equation, regardless of the coefficients and whether or not we can find numbers $p$ and $q$ that sum to $b$ and multiply to $ac$.

As mentioned earlier, sometimes it is too difficult or even impossible to come up with two whole numbers whose sum is equal to $b$ and whose product is equal to $ac$ (there could be no solutions, which will be discussed in the coming chapters, or the solutions may not be whole numbers or even fractions). Also, when the coefficient of the $x^2$ term is not 1, factoring could be a tricky option (even if it is viable through the Product Sum ($ac$) Method). The goal of Completing the Square is to use the typical unwrapping solving approach—keep peeling apart an expression until the variable is isolated. There's one special move here when there are two $x$-terms of different degree, so an operation is used that can eliminate that mix of variables of differing degree, leaving just one variable term of degree 1; everything else is boring algebra. It should be noted again that the Quadratic Formula is derived from this technique, so they are equivalent in theory, but in practice, this method can be quicker and can involve fewer simplification steps than using the Quadratic Formula (not to mention it can be a good way to find the Vertex Form of quadratic equations and to find the equations of circles, both of which are discussed in later chapters).

Let's walk through the method using a simple equation (one that can be factored conventionally, as we saw in Example 8.3-1): $3x^2 - 12x - 15 = 0$.

1. Divide both sides of the equation by the $a$ coefficient to eliminate the coefficient of the $x^2$ term.

$$3x^2 - 12x - 15 = 0$$
$$x^2 - 4x - 5 = 0$$

2. Move the constant $c$ to other side to isolate the $x$ terms.

$$x^2 - 4x - 5 = 0$$
$$x^2 - 4x = 5$$

3. Replace the left side of the equation with $\left(x + \frac{b}{2}\right)^2$ (where $b$ is the coefficient of the $x$ term) and balance the extra constant $\left(\frac{b}{2}\right)^2$ produced by this perfect square by adding $\left(\frac{b}{2}\right)^2$ to the right side of the equation (this step seems like magic now, but if you're interested in seeing an explanation, ask George to make a video about it or sneak a peek at the appendix to this chapter).
Because the term $\frac{b}{2}$ is used twice, it pays to simplify and reduce it first before deploying it so you don't have to do that twice; here, $\frac{b}{2} = \frac{-4}{2}$ and it reduces to $-2$. Note that this balancing constant $\left(\frac{b}{2}\right)^2$ will always be positive because squares of real numbers are always positive, so you don't need to think about what sign to use when you put this value on the right side, but we'll show it with the correct sign so you're not confused. From this point forward, it's straightforward algebra to solve for the variable now that it only appears as a first-degree term. Start that process by combining the constant terms on the right side (creating a common denominator as necessary if you've got a fraction or two there so you can complete the addition).

$$x^2 - 4x = 5$$
$$\left(x + \frac{b}{2}\right)^2 = 5 + \left(\frac{b}{2}\right)^2 \quad \text{where} \quad \frac{b}{2} = \frac{-4}{2} = -2$$
$$(x-2)^2 = 5 + (-2)^2$$
$$(x-2)^2 = 5 + 4$$
$$(x-2)^2 = 9$$

4. Take the square root of both sides, being careful to include both positive and negative square roots.

$$(x-2)^2 = 9$$
$$x - 2 = \pm\sqrt{9}$$
$$x - 2 = \pm 3$$

5. Move the constant to right side of the equation to isolate and solve for $x$.

$$x - 2 = \pm 3$$
$$x = 2 \pm 3$$

6. Enumerate the solutions.

$$x = 2 \pm 3$$
$$x = 2 + 3, \quad x = 2 - 3$$
$$x = 5, \quad x = -1$$





As you can see, there are just six steps needed to get to the solutions with no guesswork required. Though our example could also have been solved by using factoring, the value of Completing the Square is that it can be used on *any* quadratic equation regardless of whether it can be easily factored, and it works just as well when a quadratic expression has an $a$ coefficient that does not divide neatly into the $b$ and $c$ coefficients. For example, here's a problem that's a bit messier, but the solving steps are exactly the same; Completing the Square doesn't care about $a$ coefficients or fractions.

For this walkthrough, let's use the same example equation that we used for the Product Sum ($ac$) Method: $8x^2 - 2x - 15 = 0$.

1. Divide both sides of the equation by the $a$ coefficient to eliminate the coefficient of the $x^2$ term; don't be put off by the appearance of a couple of fractions; you're in high school.

$$8x^2 - 2x - 15 = 0$$
$$x^2 - \frac{2}{8}x - \frac{15}{8} = 0$$
$$x^2 - \frac{1}{4}x - \frac{15}{8} = 0$$

2. Move the constant to other side to isolate the $x$ terms.

$$x^2 - \frac{1}{4}x - \frac{15}{8} = 0$$
$$x^2 - \frac{1}{4}x = \frac{15}{8}$$

3. Replace the left side of the equation with $\left(x + \frac{b}{2}\right)^2$ (where $b$ is the coefficient of the $x$ term) and balance the extra constant that expression produces by adding $\left(\frac{b}{2}\right)^2$ to the right side of the equation; you'll be using the term $\frac{b}{2} = \frac{\frac{-1}{4}}{2}$ in two places in the equation, and it can be simplified to $\frac{-1}{8}$, so take care of that before deploying it in both locations—there's no need to do that math twice. Complete this phase by combining the constants on the right side; this time, you'll need to form a common denominator.

$$x^2 - \frac{1}{4}x = \frac{15}{8}$$
$$\left(x + \frac{b}{2}\right)^2 = \frac{15}{8} + \left(\frac{b}{2}\right)^2 \quad \text{where} \quad \frac{b}{2} = \frac{\frac{-1}{4}}{2} = \frac{-1}{8}$$
$$\left(x - \frac{1}{8}\right)^2 = \frac{15}{8} + \left(\frac{-1}{8}\right)^2$$
$$\left(x - \frac{1}{8}\right)^2 = \frac{15}{8} + \frac{1}{64}$$
$$\left(x - \frac{1}{8}\right)^2 = \frac{120}{64} + \frac{1}{64}$$
$$\left(x - \frac{1}{8}\right)^2 = \frac{121}{64}$$

4. Take the square root of both sides, being careful to include both positive and negative square roots.

$$\left(x-\frac{1}{8}\right)^2=\frac{121}{64}$$

$$x-\frac{1}{8}=\pm\sqrt{\frac{121}{64}}$$

$$x-\frac{1}{8}=\pm\frac{11}{8}$$

5. Move the constant to right side of the equation to isolate and solve for $x$.

$$x-\frac{1}{8}=\pm\frac{11}{8}$$

$$x=\frac{1}{8}\pm\frac{11}{8}$$

6. Enumerate the solutions.

$$x=\frac{1}{8}\pm\frac{11}{8}$$

$$x=\frac{1}{8}-\frac{11}{8},\ x=\frac{1}{8}+\frac{11}{8}$$

$$x=\frac{-10}{8},\ x=\frac{12}{8}$$

$$x=\frac{-5}{4},\ x=\frac{3}{2}$$

This method works just as well when the solutions will involve radicals; you just need to follow the steps and the solutions will emerge.

### Completing the Square

1. Divide both sides of the equation by the $a$ coefficient to eliminate the coefficient of the $x^2$ term.
2. Move the constant to other side to isolate the $x$ terms.
3. Replace the left side of the equation with $\left(x+\frac{b}{2}\right)^2$ (where $b$ is the coefficient of the $x$ term) and balance the extra constant by adding $\left(\frac{b}{2}\right)^2$ to the right side of the equation; combine constants.
4. Take the square root of both sides, being careful to include both positive and negative square roots.
5. Move the constant to right side of the equation to isolate and solve for $x$.
6. Enumerate the solutions.





## Quadratic Formula

If you are having trouble factoring a quadratic equation by any of the previously outlined methods (or if you are weak), you may want to make use of the **Quadratic Formula** presented in Section 8.1 of this chapter (some people prefer to use it all the time, and while it will work, it is often inefficient and error-prone, so we recommend not relying on it as the only tool in your quadratic-solving toolkit). It has an advantage over conventional factoring and the Product Sum ($ac$) Method in that, like Completing the Square, the Quadratic Formula will always help you find the solutions to a quadratic equation (as long as you don't mess up any plugging in, square-rooting, multiplying, dividing, or simplifying steps).

**SkillDrill 8.3-2**

**Directions:** Use Completing the Square or the Quadratic Formula (weak) to find the solutions to the following polynomial equations.

Challenge problems are boxed.

**1.** $x^2+4x-4=0$ **2.** $x^2+x-1=0$ **3.** $x^2-3x+5=8$ **4.** $x^2-6x+9=8$

**5.** $2x^2-8x-6=6$ **6.** $\boxed{3x^2+3x-2=0}$ **7.** $2a^2+10a+5=0$ **8.** $\boxed{4b^2-9b-2=0}$

**Example 8.3-2**

**2** 1600.io 8.3

Which of the following is a solution to the equation $3x^2-4x-4=0$?

A) $-2$

B) $-\frac{2}{3}$

C) $\frac{1}{3}$

D) $4$

**Solution 1**

This solution makes use of Product Sum ($ac$) Method.

1. The product number, which is equal to $ac$ for a quadratic expression in Standard Form, is equal to $3(-4)$ or $-12$. The sum number, which is equal to $b$ for a quadratic expression in Standard Form, is $-4$. We need to find values, $r$ and $s$, that add to $-4$ and multiply to $-12$.

2. The values $-6$ and $2$ add to $-4$ and multiply to $-12$, so $r = -6$ and $s = 2$ (it doesn't matter which is which). We can speed our way to this determination by observing that because the product target of $-12$ is a negative number, we know that one of the two numbers will be positive and the other will be negative; because the sum number is negative, we know that the number with the larger absolute value must be the negative number because we want to subtract a larger number from a smaller one to get a negative result.
3. At this point we are done because the roots are equal to $\frac{-r}{a}$ and $\frac{-s}{a}$ (we don't need to write the equation in Factored Form).
4. Since $r = -6$ and $a = 3$, then one of the solutions is $x = \frac{-(-6)}{3}$, which reduces to $x = 2$. This does not match any answer choices.
5. Since $s = 2$ and $a = 3$, then the other solution is $x = \frac{-2}{3}$. This matches answer choice B.
6. The answer is B.

**Solution 2**

This solution makes use of Completing the Square.

1. Divide both sides of the equation by 3 in order to eliminate the coefficient of the $x^2$ term.

$$3x^2 - 4x - 4 = 0$$

$$x^2 - \frac{4}{3}x - \frac{4}{3} = 0$$

2. Move the constant term to the other side of the equation in order to isolate the $x$ terms: add $\frac{4}{3}$ to both sides of the equation.

$$x^2 - \frac{4}{3}x - \frac{4}{3} = 0$$

$$x^2 - \frac{4}{3}x = \frac{4}{3}$$

3. Replace the left side of the equation with $\left(x - \frac{4}{3(2)}\right)^2$ and balance the extra constant on the other side by adding $\left(\frac{4}{2(3)}\right)^2$ to the right side of the equation.

$$x^2 - \frac{4}{3}x = \frac{4}{3}$$

$$\left(x - \frac{4}{3(2)}\right)^2 = \left(\frac{4}{2(3)}\right)^2 + \frac{4}{3}$$

$$\left(x - \frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^2 + \frac{4}{3}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{4}{9} + \frac{4}{3}$$





4. Combine the constants on the right side of the equation (make a common denominator of 9).

$$\left(x - \frac{2}{3}\right)^2 = \frac{4}{9} + \frac{4}{3}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{4}{9} + \frac{12}{9}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{16}{9}$$

5. Take the square root of both sides, being careful to include both positive and negative square roots.

$$\left(x - \frac{2}{3}\right)^2 = \frac{16}{9}$$

$$x - \frac{2}{3} = \pm\sqrt{\frac{16}{9}}$$

$$x - \frac{2}{3} = \pm\frac{4}{3}$$

6. Move the constant to right side of the equation (add $\frac{2}{3}$ to both sides of the equation) to isolate and solve for $x$.

$$x - \frac{2}{3} = \pm\frac{4}{3}$$

$$x = \frac{2}{3} \pm \frac{4}{3}$$

$$x = \frac{2}{3} + \frac{4}{3},\ \ x = \frac{2}{3} - \frac{4}{3}$$

$$x = \frac{6}{3},\ \ x = \frac{-2}{3}$$

$$x = 2,\ \ x = \frac{-2}{3}$$

7. The two solutions are $x = 2$ and $x = \frac{-2}{3}$, but only $\frac{-2}{3}$ is available as an answer choice.
8. The answer is B.

**Solution 3**

This solution makes use of Quadratic Formula.

1. There is no real value in dividing by 3 to start this problem, because it does not help simplify the terms that will be plugged into the Quadratic Formula. Substitute 3 for $a$, $-4$ for $b$, and $-4$ for $c$, then chug away.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4)}{2(3)} \pm \frac{\sqrt{(-4)^2 - 4(3)(-4)}}{2(3)}$$

$$x = \frac{4}{6} \pm \frac{\sqrt{16 + 48}}{6}$$

$$x = \frac{2}{3} \pm \frac{\sqrt{64}}{6}$$

$$x = \frac{2}{3} \pm \frac{8}{6}$$

$$x = \frac{2}{3} \pm \frac{4}{3}$$

$$x = \frac{2}{3} + \frac{4}{3},\ x = \frac{2}{3} - \frac{4}{3}$$

$$x = \frac{6}{3},\ x = \frac{-2}{3}$$

$$x = 2,\ x = \frac{-2}{3}$$

2. The two solutions are $x = 2$ and $x = \frac{-2}{3}$, but only $\frac{-2}{3}$ is available as an answer choice.
3. The answer is B.

**Notes**

Alternatively, you could just plug in each answer choice to see which one makes the equation true (still weak).

All roads lead to Rome, but the roads are not always equal. If it's unfamiliar to you, Completing the Square may seem like a worse option for this problem than Quadratic Formula, but if you get good at Completing the Square, all of the steps can be performed in your head, and, again, ultimately Completing the Square not only forms the basis of the Quadratic Formula, but also has uses outside of these factoring problems.

**Example 8.3-3**

**3** 1600.io 8.3

$$x^2 - 6x + 4 = 0$$

Which of the following is a solution to the equation above?

A) $x = -3 + \sqrt{5}$

B) $x = -3 + \sqrt{11}$

C) $x = 3 + \sqrt{5}$

D) $x = 3 + \sqrt{11}$





**Solution 1**

Since we cannot find whole numbers that add to $-6$ and multiply to 4, we should use the Completing the Square or the Quadratic Formula to factor this equation and find solutions. In this problem, you don't even need to try to factor, because if you examine the answer choices, you'll see that all the options are expressions, not single whole number values, and they all have a square root term, so the correct answer will have a square root term. That would mean that at least one of the factors would be of the form $x - (r + \sqrt{s})$, so $r + \sqrt{s}$ would need to be one of the values you'd need to find if you use factoring, but it's impractical to search for anything other than whole numbers. More to the point, whenever you see a problem where the answer choices are of the form $d + \sqrt{e}$ (or $d - \sqrt{e}$), you should probably use Completing the Square or the Quadratic Formula to find the solution(s), not factoring. This is a great example of why it can be extremely informative to examine the answer choices before commencing work.

First, we'll use Completing the Square.

1. Move the constant term to the other side of the equation by subtracting 4 from both sides of the equation.

$$x^2 - 6x + 4 = 0$$
$$x^2 - 6x = -4$$

2. Replace $x^2 - 6x$ with $(x - 3)^2$ and add $3^2$ to the other side of the equation to neutralize the unwanted constant generated by the squaring operation.

$$x^2 - 6x = -4$$
$$(x - 3)^2 = -4 + 3^2$$
$$(x - 3)^2 = -4 + 9$$
$$(x - 3)^2 = 5$$

3. Take the square root of both sides of the equation, being careful to retain both the positive and negative square roots on the right side.

$$(x - 3)^2 = 5$$
$$x - 3 = \pm\sqrt{5}$$

4. Add 3 to both sides of the equation in order to isolate $x$.

$$x - 3 = \pm\sqrt{5}$$
$$x = 3 \pm \sqrt{5}$$
$$x = 3 + \sqrt{5},\ x = 3 - \sqrt{5}$$

5. Out of the two solutions, only $x = 3 + \sqrt{5}$ appears as an answer choice.
6. The answer is C.

**Solution 2**

As stated in Solution 1, based on the answer choices, there is no point in searching for factors, so for this solution, we will use the Quadratic Formula.

1. The value of $a$ is 1; the value of $b$ is $-6$; the value of $c$ is 4.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6)}{2(1)} \pm \frac{\sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{6}{2} \pm \frac{\sqrt{36 - 16}}{2}$$

$$x = 3 \pm \frac{\sqrt{20}}{2}$$

2. The radical can be simplified since $\sqrt{20} = \sqrt{4(5)} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$.

$$x = 3 \pm \frac{\sqrt{20}}{2}$$

$$x = 3 \pm \frac{\sqrt{4}\sqrt{5}}{2}$$

$$x = 3 \pm \frac{2\sqrt{5}}{2}$$

$$x = 3 \pm \sqrt{5}$$

3. The two solutions are $x = 3 + \sqrt{5}$, which is choice C, and $x = 3 - \sqrt{5}$.

4. The answer is C.

**Notes**

While many people like to have a formula to fall back on, it is undeniable that the numbers are much easier to work with in Solution 1, using Completing the Square. You WILL accept Completing the Square as your primary method of finding solutions to Quadratic Equations that aren't easily factorable.

**Section 8.3 Suggested Problems from Real Tests**

- Test 3-NC-14
- Test 4-NC-15
- Test 7-C-6
- May 2018-NC-9
- May 2019 (US)-NC-8
- 2019 (US)-NC-9
- 2021-NC-18
- 2021-C-32
- May 2021 (Int)-NC-14

View related real-test problems in SMTEx

SMTEx





## Section 8.3 Practice Problems

**1** 1600.io 8.3

What are the solutions to $3x^2 + 18x + 9 = 0$ ?

A) $x = -3 \pm \sqrt{6}$

B) $x = -3 \pm 2\sqrt{3}$

C) $x = -6 \pm \sqrt{6}$

D) $x = -6 \pm 2\sqrt{3}$

**2** 1600.io 8.3

$$y = -16t^2 + 48t + 160$$

The equation above gives the height of an object above the ground, $y$ in feet, $t$ seconds after it is launched from the roof of a building. How many seconds after it is launched does the object reach the ground?

**3** 1600.io 8.3

$$6w^2 + 10w + 1 = 0$$

Which of the following values is a solution to the equation above?

A) $\frac{-5 + \sqrt{19}}{6}$

B) $\frac{-5 + \sqrt{31}}{6}$

C) $\frac{5 + \sqrt{19}}{6}$

D) $\frac{5 + \sqrt{31}}{6}$

**4** 1600.io 8.3

$$2x^3 - 13x^2 - 7x$$

Which of the following is NOT a factor of the polynomial above?

A) $x$

B) $x - 7$

C) $2x - 7$

D) $2x + 1$

## 8.4 Simplified Factoring Situations

There are some patterns that appear often on the test in quadratic equations. If you learn to spot these patterns, you can save a lot of time.

### Difference of Squares

A common form of quadratic equation on this test is **the Difference of Squares**. Learning to spot quadratic equations in this form will save time and effort because you will immediately know how to factor them.

Generally, when multiplying a pair of binomials of the form $(x + a)$ and $(x + b)$, we end up with a three-term quadratic expression:

$$(x + a)(x + b)$$
$$x^2 + ax + bx + ab$$
$$x^2 + (a + b)x + ab$$

Notice that the middle $(x)$ term's coefficient is $(a + b)$. If that term evaluates to 0, there will be no middle $(x)$ term at all. That will happen when $b = -a$, because $(a + (-a)) = 0$.

Let's substitute $-a$ for $b$ to show this:

$$(x + a)(x - a)$$
$$x^2 + ax - ax - a^2$$

You should be able to see where this is leading already. Here's the final step after those two middle terms cancel out:

$$x^2 - a^2$$

What this means is that any expressions in the form $x^2 - a^2$ can be factored as $(x + a)(x - a)$

**Factoring the Difference of Squares**

Any expression of the form $a^2 - b^2$, which is the difference of two squares, can be factored as $(a + b)(a - b)$.

In the following example, we will see that both terms are perfect squares:

$$x^2 - 9 = (x)^2 - (3)^2$$

Since this equation fits the difference of squares pattern, we know that the factors must be $(x + 3)$ and $(x - 3)$.

$$x^2 - 9 = (x + 3)(x - 3)$$

In this example, we will see that the we can use any number as the second term, and the situation works the same even when the constant is not a perfect square:

$$x^2 - 6$$
$$(x)^2 - (\sqrt{6})^2$$
$$(x + \sqrt{6})(x - \sqrt{6})$$





In this final example, we will see that this pattern is true for the difference of any two squares (even cases where the exponents of $x$ terms are are greater than 2):

$$x^4 - 81 = (x^2)^2 - 9^2 = (x^2 + 9)(x^2 - 9)$$

Did you notice that one of these factors, $(x^2 - 9)$, is also a difference of two squares expression? We can factor this expression further. The other factor, $(x^2 + 9)$, is not a difference of squares (the terms within it are being added, not subtracted), and we cannot factor it any further (it does not have any real solutions).

$$x^4 - 81$$
$$(x^2 + 9)(x^2 - 9)$$
$$(x^2 + 9)(x + 3)(x - 3)$$

**SkillDrill 8.4-1**

**Directions:** Write the following difference of squares polynomials as the product of factors. If a factor is itself a difference of squares, decompose it further.

Challenge questions are boxed.

1. $x^2 - 4$
2. $x^2 - 81$
3. $a^2 - b^2$
4. $a^2 - 25$
5. $\boxed{9a^2 - 4b^2}$
6. $\boxed{x^4 - 81}$

**Example 8.4-1**

**1** 1600.io 8.4

If $x + y = 12$ and $x - y = 4$, what is the value of $x^2 - y^2$ ?

A) 4
B) 8
C) 48
D) 196

**Solution 1**

1. Since $x^2 - y^2$ is in a difference of squares pattern, it can be factored as $(x + y)(x - y)$. You can set the expression $x^2 - y^2$ equal to $(x + y)(x - y)$.

$$x^2 - y^2 = (x + y)(x - y)$$

2. Because we are told that $x + y = 12$ and $x - y = 4$, we can substitute 12 in for $(x + y)$ and 4 in for $(x - y)$.

$$\begin{aligned} x^2 - y^2 &= (x + y)(x - y) \\ x^2 - y^2 &= (12)(4) \\ x^2 - y^2 &= 48 \end{aligned}$$

3. The answer is C.

**Solution 2**

This is a slower solution that does not make use of the difference of two squares. The problem can simply be looked at as a system of linear equations.

1. Add the two given equations, $x + y = 12$ and $x - y = 4$, in order to eliminate $y$ and solve for $x$.

$$\begin{array}{rl} x + y &= 12 \\ + \quad x - y &= 4 \\ \hline 2x \quad &= 16 \\ & \Downarrow \\ x \quad &= 8 \end{array}$$

2. Substitute 8 in for $x$ in the first equation, $x + y = 12$, to solve for $y$.

$$\begin{aligned} x + y &= 12 \\ (8) + y &= 12 \\ y &= 4 \end{aligned}$$

3. Since $x = 8$ and $y = 4$, we can solve for the value of $x^2 - y^2$ by replacing $x$ with 8 and $y$ with 4.

$$\begin{aligned} x^2 - y^2 &= 8^2 - 4^2 \\ x^2 - y^2 &= 64 - 16 \\ x^2 - y^2 &= 48 \end{aligned}$$

4. The answer is C.





## Perfect Square Expressions

For some factorable quadratic expressions, the two factors are actually the same. For example, if we factor the the expression $x^2 + 8x + 16$, we need to find two numbers that add to 8 and multiply to 16. Those numbers are 4 and 4 (the same number), so the two factors are $(x + 4)$ and $(x + 4)$, which means that $x^2 + 8x + 16 = (x + 4)(x + 4)$. Written another way, the expression $x^2 + 8x + 16$ is a perfect square that is equivalent to $(x + 4)^2$.

Since $(x + a)^2 = x^2 + 2ax + a^2$ (as we saw in the previous chapter), **any expressions in the form $x^2 + 2ax + a^2$ can be factored as $(x + a)^2$.**

Note that the particular letters are not important, and any of the letters can represent either a variable or a constant; don't conclude that $x$ must be matched by a variable or that $c$ must represent a constant. In fact, we could define the pattern with any letters we choose; for example:

$$(g + h)^2 = g^2 + 2gh + h^2$$

Note that the order of the letters $g$ and $h$ in the middle term $2gh$ is reversed from what we saw in the middle term of the previous expansion of $(x + a)^2$, where the middle term was written as $2ax$. Due to the commutative property of multiplication, $2ax$ is equal to $2xa$, just as $2gh$ is equal to $2hg$. In the first example, since we assumed $x$ was a variable and $a$ was a constant (as will be the case in many factoring problems), by convention, we wrote the constant first. In the second example, since we assumed both $g$ and $h$ are constants, we wrote the constants in alphabetical order.

Don't let the simple reordering of constants and variables get in the way of spotting this pattern. You can check for a match by seeing if the middle term of the expansion is equal to the product of 2, the square root of the first term, and the square root of the last term.

Another expression that fits the pattern is as follows:

$$\begin{aligned} &x + 2\sqrt{x}\sqrt{a} + a \\ &(\sqrt{x} + \sqrt{a})^2 \end{aligned}$$

Or, equivalently (because it doesn't matter what letters we pick),

$$\begin{aligned} &g + 2\sqrt{g}\sqrt{h} + h \\ &(\sqrt{g} + \sqrt{h})^2 \end{aligned}$$

This form might seem scarier with those radicals in the middle term, but it's actually rather useful when trying to determine if certain expressions qualify as perfect squares, such as those that don't have first or last terms with explicit exponents of 2. The key here is to see if the middle term is the product of 2 and the square roots of each of the other two terms.

Sometimes, when either $x$ or $a$ has a known value or includes a coefficient, it is harder to see that the middle term fits the pattern. For example, the expression $x^2 + 6x + 9$ fits the pattern, where $x = x$ and $a = 3$, which means that the middle term is equal to $2ax$ (in this case $2ax$ should equal $2(3)(x)$ or $6x$ based on the values of $x$ and $a$). Using the first version of the pattern,

$$\begin{aligned} &x^2 + 6x + 9 \\ &(x)^2 + 2(3)(x) + (3)^2 \\ &(x + 3)^2 \end{aligned}$$

It might be easier to see how we could determine that the expression matches the pattern by using the second version.

$$x^2 + 6x + 9$$
$$x^2 + 2(\sqrt{x^2})(\sqrt{9}) + 9$$
$$x^2 + 2(x)(3) + 9$$
$$\left(\sqrt{x^2} + \sqrt{9}\right)^2$$
$$(x+3)^2$$

The pattern can be disguised fairly well if you are aren't looking for it. In the pattern $x^2 + 2ax + a^2$, the values of $x$ and $a$ might be entire expressions with coefficients and unknown constants.

For example the expression $4p^4 + 12p^2q^2 + 9q^4$ also fits the pattern $x^2 + 2ax + a^2$ even though it looks vastly more complicated. If we take the value of $x$ to be $2p^2$, which, when squared, equals $4p^4$, and we take the value of $a$ to be $3q^2$, which, when squared, equals $9q^4$, then we can see that it does fit the pattern, where the middle term is equal to $2ax$ (which in this case is $2\left(2p^2\right)\left(3q^2\right)$ or $12p^2q^2$ based on the values of $a$ and $x$), and can be factored as $\left(2p^2 + 3q^2\right)^2$.

$$4p^4 + 12p^2q^2 + 9q^4$$
$$\left(2p^2\right)^2 + 2\left(2p^2\right)\left(3q^2\right) + \left(3q^2\right)^2$$
$$\left(2p^2 + 3q^2\right)^2$$

This is another case where you might find it helpful to use the second form of the pattern to determine if the expression is a perfect square and exactly what the components would be.

Using the pattern

$$g + 2\sqrt{g}\sqrt{h} + h$$

we identify $4p^4$ as $g$ and $9q^4$ as $h$, giving us

$$4p^4 + 2\left(\sqrt{4p^4}\right)\left(\sqrt{9q^4}\right) + 9q^4$$
$$4p^4 + 2\left(2p^2\right)\left(3q^2\right) + 9q^4$$
$$4p^4 + 12p^2q^2 + 9q^4$$
$$\left(\sqrt{4p^4} + \sqrt{9q^4}\right)^2$$
$$\left(2p^2 + 3q^2\right)^2$$

The following is actually a result of the same pattern as above, where the value of $a$ happens to be negative. Nonetheless, for completeness, we will demonstrate the following recurring pattern.

Since $(x-a)^2 = x^2 - 2ax + a^2$, **any expressions in the form $x^2 - 2ax + a^2$ can be factored as $(x-a)^2$.**





For example, the expression $x^2 - 2xy + y^2$ fits the pattern $x^2 - 2ax + a^2$, so it can be factored as $(x-y)^2$.

$$x^2 - 2xy + y^2$$
$$(x)^2 - 2(x)(y) + (y)^2$$
$$(x-y)^2$$

Similar to the above pattern, these expressions can be well disguised if you aren't aware of them. For example the expression $4p^4 - 12p^2q^2 + 9q^4$ also fits the pattern $x^2 - 2ax + a^2$, where $x = 2p^2$ and $a = 3q^2$ (or, alternatively, the pattern $x - 2\sqrt{x}\sqrt{a} + a$, where $x = 4p^4$ and $a = 9q^4$).

$$4p^4 - 12p^2q^2 + 9q^4$$
$$\left(2p^2\right)^2 - 2\left(2p^2\right)\left(3q^2\right) + \left(3q^2\right)^2$$
$$\left(2p^2 - 3q^2\right)^2$$

## Factoring Perfect Square Expressions

Any expression of the form $a^2 + 2ab + b^2$ can be factored as $(a+b)^2$.

Any expression of the form $a^2 - 2ab + b^2$ can be factored as $(a-b)^2$.

**SkillDrill 8.4-2**

**Directions:** Write the following perfect square polynomial expressions in Factored Form.

1. $x^2 + 2x + 1$
2. $p^2 - 6p + 9$
3. $y^2 + 8y + 16$
4. $9x^2 - 18x + 9$
5. $\boxed{4a^2 + 16ab + 16b^2}$
6. $\boxed{b^4 + 2b^2 + 1}$

**Example 8.4-2**

**2** 1600.io 8.4

If $a^2 + b^2 = z$ and $ab = y$, what of the following is equivalent to $9z + 18y$ ?

A) $(a+3b)^2$

B) $(3a+3b)^2$

C) $(9a+9b)^2$

D) $(9a+18b)^2$

**Solution 1**

1. To find the value of $9z + 18y$, substitute $a^2 + b^2$ for $z$ and $ab$ for $y$.

$$9z + 18y = 9(a^2 + b^2) + 18(ab)$$
$$9z + 18y = 9a^2 + 9b^2 + 18ab$$

2. It might be tempting to factor out 9 at some point in this process, as the alert student is always on the lookout for ways to factor and simplify expressions in the course of solving a problem, but an examination of the form of all the answer choices shows that there is no factored-out constant in any of them, so this temptation must be resisted in this situation. We should reorder the expression on the right side of the equation, in order to "polarize" the expression such that the squared terms, $9a^2$ and $9b^2$, are separated by the term $18ab$, in order to match the general structure of squaring binomials (after all, all of the answer choices tell us that the value of $9z + 18y$ must be the result of squaring a binomial).

$$9z + 18y = 9a^2 + 9b^2 + 18ab$$
$$9z + 18y = 9a^2 + 18ab + 9b^2$$

3. The expression on the right side is a perfect square expression and can be factored easily. Even though it looks intimidating, it matches the form of the perfect square expression $g^2 + 2gh + h^2$, where $g$ in the template corresponds to $3a$ in the expression we have, and $h$ corresponds to $3b$, so we know how to factor it.

$$9z + 18y = 9a^2 + 18ab + 9b^2$$
$$9z + 18y = (3a)^2 + 2(3a)(3b) + (3b)^2$$
$$9z + 18y = (3a + 3b)^2$$

4. The answer is B.

**Solution 2**

This is a slower solution that does not make use of factoring perfect square expressions; instead, it expands each answer choice and tests it.

1. To find the value of $9z + 18y$, substitute $a^2 + b^2$ in for $z$ and $ab$ in for $y$.

$$9z + 18y = 9(a^2 + b^2) + 18(ab)$$
$$9z + 18y = 9a^2 + 18ab + 9b^2$$

2. Expand (multiply out) each answer choice to see which one evaluates to $9a^2 + 18ab + 9b^2$.

3. Start with choice A, $(a + 3b)^2$.

$$(a + 3b)^2 = (a + 3b)(a + 3b)$$
$$(a + 3b)^2 = a^2 + 3ab + 3ab + 9b^2$$
$$(a + 3b)^2 = a^2 + 6ab + 9b^2$$





4. Since choice A does not evaluate to $9a^2 + 18ab + 9b^2$, we can eliminate answer choice A, and move on to choice B, $(3a + 3b)^2$.

$$(3a + 3b)^2 = (3a + 3b)(3a + 3b)$$
$$(3a + 3b)^2 = 9a^2 + 9ab + 9ab + 9b^2$$
$$(3a + 3b)^2 = 9a^2 + 18ab + 9b^2$$

5. Choice B does evaluate to $9a^2 + 18ab + 9b^2$, so this is the correct choice.

6. The answer is B.

**Notes**

After Step 1 in either solution, we could have taken a shortcut by noticing that the first term in the expanded expression we need to find is $9a^2$. Looking at just the $a$ terms in the answer choices, it is clear that only choice B would expand such that $9a^2$ would be the first term.

**Section 8.4 Suggested Problems from Real Tests**

- Test 1-NC-16
- Test 2-NC-4
- Test 2-NC-7
- Test 5-NC-10
- Test 5-C-30
- May 2017-NC-7
- Mar 2021-NC-2
- Mar 2021-C-26
- May 2021 (Int)-NC-7

View related real-test problems in SMTEx

## Section 8.4 Practice Problems

**1** 1600.io 8.4

$$x^4 - 18x^2 + 81$$

Which of the following is equivalent to the expression above?

A) $(x - 9)^4$
B) $(x - 3)^4$
C) $(x^2 + 9)(x + 3)(x - 3)$
D) $(x - 3)^2(x + 3)^2$

**2** 1600.io 8.4

If $t > 0$ and $t^2 - 36 = 0$, what is the value of $t$ ?

**3** 1600.io 8.4

$$36x^2 - 16 = (px + t)(px - t)$$

In the equation above, $p$ and $t$ are constants. Which of the following could be the value of $p$ ?

A) 4
B) 6
C) 16
D) 36

**4** 1600.io 8.4

In the equations $a = x + 3$ and $b = x - 3$, $a$ and $b$ are constants. When the product $ab$ is written in the form $x^2 - c$, where $c$ is a constant, what is the value of $c$ ?

## 8.5 Systems of Quadratic Equations and Linear Equations

Just like linear equations, quadratic equations can be represented in graphs, and when the graph of a quadratic equation intersects the graph of another equation, the intersection points are called solutions to the system; these points represent values of the $x$ and $y$ variables that make all the equations true. For now we will not need to see the graphs of quadratic equations because, just as with systems of linear equations, systems of quadratic equations can be solved without the graphs by using some of the same techniques we used to solve systems of linear equations. However, these quadratic systems will be more frequently solved using substitution (or setting the polynomials equal to each other if the equations are already solved in terms of the same variable) rather than by using the elimination method often preferred for solving systems of linear equations.

If you are given a system of equations consisting of one linear equation and one quadratic equation, or less commonly, two quadratic equations, you will use substitution to collapse the system of two equations into one quadratic equation which you can then solve. Often, the two equations are already solved for $y$ in terms of $x$, so you can just set the expressions containing the $x$-variables equal to each other. Once you have one equation, you can write it in Standard Form so that one side of the equation is only 0 and the other side of the equation is a quadratic expression. Once the quadratic equation is in Standard Form, you can proceed to find the solutions using your technique of choice.

It is possible that there are no values of $x$ and $y$ that satisfy both the linear equation and the quadratic equation, or, put into graphical terms, the graph of a line does not intersect the graph of a quadratic equation (a case that will be discussed in the coming chapters) and it is also possible to to solve systems of two quadratic equations by similar methods (again, this is discussed in later chapters as there are more possibilities for the number of solutions). In this chapter, we are only presenting systems with one linear and one polynomial equation and which are solvable by this method.

**Example 8.5-1**

**1** 1600.io 8.5

$$y = x^2$$
$$y = 8x + 20$$

The system of equations above is graphed in the $xy$-plane. The graphs of the equations intersect at a point $(x, y)$ where $x > 0$ and $y > 0$. What is the $y$-coordinate of this point of intersection?

A) 2

B) 10

C) 36

D) 100

**Solution**

1. Set the two polynomials equal to each other because they are both already solved for $x$ in terms of $y$.

$$x^2 = 8x + 20$$

2. Bring all the terms to the left side of the equation so that we have a standard form of a quadratic on the left side that we can factor.

$$x^2 - 8x - 20 = 0$$





3. We'll attempt simple factoring, so we need to find two numbers that add to $-8$ and multiply to $-20$.

4. The values $-10$ and 2 add to $-8$ and multiply to $-20$, so the factors are $(x - 10)$ and $(x + 2)$.

$$x^2 - 8x - 20 = 0$$
$$(x - 10)(x + 2) = 0$$

5. Since $(x - 10)$ is a factor, then $x = 10$ is a solution. Because $(x + 2)$ is also a factor, $x = -2$ is also a solution, but since we only care about the solution where $x > 0$, it is irrelevant that $x = -2$ is also a solution. Note that we didn't need to actually write out the factors to arrive at the solutions; once we determine that $-10$ and 2 satisfied the factoring constraints, we can conclude that the negatives of those numbers, 10 and $-2$, are the solutions.

6. It is important to be absolutely sure of what the question is asking for; wrongly assuming that it is the positive solution that is being sought will cause the student to select choice C, which the test-maker has thoughtfully supplied to avoid tipping off students that the are not yet done solving the problem, but the problem asks for the $y$-coordinate of the solution point. Since the solution occurs when $x = 10$, plug in 10 for $x$ in either one of the two given equation to find the value of $y$. We will use the first equation, $y = x^2$.

$$y = 10^2$$
$$y = 100$$

7. The answer is D.

**Section 8.5 Suggested Problems from Real Tests**

- Test 5-NC-9
- Test 6-C-34
- Test 7-C-35
- Mar 2018-NC-14
- Apr 2018-C-25
- May 2018-NC-11
- Mar 2019-C-21
- May 2019 (Int)-C-32
- Oct 2020-C-17

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SMTEx

### Section 8.5 Practice Problems

**1** 1600.io 8.5

$$y = -6$$
$$y + 22 = x^2$$

If $(x_1, y_1)$ and $(x_2, y_2)$ are solutions to the system of equations above, what are the values of $x_1$ and $x_2$ ?

A) $-2\sqrt{7}$ and $2\sqrt{7}$

B) $-\sqrt{22}$ and $\sqrt{22}$

C) $-4$ and 4

D) $-6$ and 6

**2** 1600.io 8.5

$$y = 2x + 4$$
$$y = x^2 + 6x + 8$$

What is the $y$-coordinate of the point of intersection, in the $xy$-plane, of the graphs of the equations above?

A) $-2$

B) 0

C) 2

D) 8

**3** 1600.io 8.5

$$y = x^2 - 12x + 12$$
$$y = 12 - x$$

If the ordered pair $(x, y)$ satisfies the system of equations above, what is one possible value of $x$ ?

**4** 1600.io 8.5

$$x^2 + y = 16$$
$$x - y = 4$$

Which value is a $y$-coordinate of a solution to the system of equations above?

A) $-9$

B) $-8$

C) $-5$

D) $-4$

## 8.6 Extra Tools for Factoring and Quadratics Problems

If a question asks you to find the sum or product of the solutions (or zeros, $x$-coordinates of points of intersection, "values that satisfy," or roots) of a quadratic expression, equation, function, or a system of equations that includes a quadratic equation, you don't necessarily need to find the individual solutions to answer the question. If the quadratic expression or equation is given to you in (or is easily converted into) Standard Form, then there are useful shortcuts that can help you answer sum and product of solutions questions very easily.

Remember, a quadratic *equation* of the form $ax^2 + bx + c = 0$ has solutions or roots; a quadratic *equation* of the form $y = ax^2 + bx + c$ has roots or zeros; a quadratic *function* of the form $f(x) = ax^2 + bx + c$ has roots or zeros; and a quadratic *expression* of the form $ax^2 + bx + c$ has roots or zeros. These formulas apply to any of those characteristics, so generalize the summaries below accordingly. We empathize with the confusion this terminology can elicit, but we are not responsible for the proliferation of these overlapping terms.

### Sum of Solutions

For quadratic equations in the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are constants, the **sum of the solutions** is equal to $\frac{-b}{a}$.

$$\text{Sum of Solutions} = \frac{-b}{a}$$

### Product of Solutions

For quadratic equations in the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are constants, the **product of the solutions** is equal to $\frac{c}{a}$.

$$\text{Product of Solutions} = \frac{c}{a}$$





**SkillDrill 8.6-1**

**Directions:** Find the sum $s$ and product $p$ of the solutions or roots for each of the following quadratics. Use the formulas above instead of finding the individual solutions.

**1.** $x^2 + 4x + 3 = y$ **2.** $x^2 + 12x + 32 = 0$ **3.** $x^2 - 8x + 6 = 1$ **4.** $2x^2 - 18x + 9 = 0$

**5.** $3x^2 + 2x - 5 = y$ **6.** $x^2 - x + 17 = -3$ **7.** $5x^2 + 8x - 30 = 0$ **8.** $9x^2 - 81x + 9 = 0$

### Quadratics Not in $x$ and U-Substitution

In most cases, quadratic expressions will be based in a simple variable such as $x$. It's important (and interesting) to understand that as long as the *structure* of a quadratic expression is maintained, you can have a quadratic in a more complicated term, such as a variable with a coefficient, an exponentiated variable, or even an expression. For example, just as $y = x^2 + 5x + 6$ is a quadratic equation in $x$, so too is $y = (7x)^2 + 5(7x) + 6$ a quadratic equation in $7x$. The equation $y = \left(x^4\right)^2 + 5\left(x^4\right) + 6$ is a quadratic in $x^4$, and the equation $y = (x-3)^2 + 5(x-3) + 6$ is a quadratic equation in $(x-3)$.

Because these are all quadratic equations, they are all susceptible to the solving methods that apply to quadratic equations as long as the term that serves as the basis for the quadratic expression is used wherever you'd use the simple variable. Be on the lookout for complicated-seeming equations that are really just quadratics in something other than a simple variable; they are easily solved when recognized.

We have looked at factoring with respect to single variable terms, but when certain expressions occur multiple times in an equation on this test, you can probably shortcut the problem by factoring with respect to that expression rather than a single variable. For example, if we encounter the equation $(x-4)^2 + 2(x-4) - 15 = 0$, one method of finding solutions is to expand all the terms and rewrite the equation in Standard Form in terms of $x$.

$$\begin{aligned}(x-4)^2 + 2(x-4) - 15 &= 0\\ x^2 - 8x + 16 + 2x - 8 - 15 &= 0\\ x^2 - 6x - 7 &= 0\\ (x-7)(x+1) &= 0\\ x = 7,\ \ x &= -1\end{aligned}$$

However, we could reduce the number of steps if we realize that the equation $(x-4)^2 + 2(x-4) - 15 = 0$ is already in Standard Form in terms of the expression $x - 4$ instead of just $x$. The process is the same as factoring when a single variable like $x$ is the repeated term, but we are using $(x-4)$ as the term we are factoring in terms of. We still need to find two numbers that add to 2 (the $b$-coefficient) and multiply to -15 (the constant term $c$). These numbers are $-3$ and 5, so our factors will be $[(x-4)-3]$ and $[(x-4)+5]$.

$$\begin{aligned}(x-4)^2 + 2(x-4) - 15 &= 0\\ [(x-4)-3][(x-4)+5] &= 0\\ (x-7)(x+1) &= 0\\ x = 7,\ \ x &= -1\end{aligned}$$

Notice that we ended up with the same factors (as we must), but we skipped the steps of expanding out the multiplied expressions and combining like terms, which would be an avoidable source of error and a waste of time.

For some people, it can be easier to temporarily replace the repeated expression with a variable. By custom, $u$ is generally used for this purpose, and that gives the name **U-Substitution** to this technique. This makes the equation more approachable because the equation is transformed to look like a familiar quadratic equation in terms of a single variable rather than an expression. Once $u$ is substituted for the repeated expression, factor the equation normally and then replace the value of $u$ (reverse the substitution) at the end.

For example, in the equation above, we could say that $u = x - 4$, and we'd then substitute $u$ for the repeated expression $x - 4$, allowing us to factor with respect to a single variable without having to expand and recombine terms. To reiterate, you do have to substitute $x - 4$ back in for $u$ at the end in order to find solution values of $x$ (and not for our temporary placeholder value $u$, which is equal to $x - 4$).

$$\begin{aligned}(x-4)^2 + 2(x-4) - 15 &= 0\\ u^2 + 2u - 15 &= 0\\ (u-3)(u+5) &= 0\\ u = 3,\ \ u &= -5\end{aligned}$$

We've now solved for $u$, but ultimately, we need to solve for $x$. Remember that $u = x - 4$, so if we want to find the solution values of $x$, we need to replace $u$ with $x - 4$.

$$\begin{aligned}u = 3,\ \ u &= -5\\ x - 4 = 3,\ \ x - 4 &= -5\\ x = 7,\ \ x &= -1\end{aligned}$$

The key to identifying equations that are susceptible to this streamlined technique is to see if there's an expression that appears with an exponent of 2 in one term and that also appears with no exponent (that is, with an implied exponent of 1) in a second term. The following example should clarify how to factor in terms of an expression rather than a single variable and how to use U-Substitution to find the solutions of a quadratic equation.

There is an additional discussion of these quadratics in non-simple terms in the "Wormholes" chapter, if you're interested in a deeper dive.

**Example 8.6-1**

**1** 1600.io 8.6

If $(x+5)^2 - 4(x+5) + 4 = 0$, what is the value of $x$ ?

A) $-5$

B) $-3$

C) $2$

D) $5$





**Solution 1**

1. Factor the equation with respect to the expression $x + 5$ instead of just a single variable term like $x$. This is a perfect square expression (matches the form $p^2 - 2pq + q^2 = (p-q)^2$, where $p = x + 5$ and $q = 2$), so we can factor it easily.

$$\begin{aligned}(x+5)^2 - 4(x+5) + 4 &= 0\\ (x+5)^2 - 2(x+5)(2) + (2)^2 &= 0\\ [(x+5) - 2]^2 &= 0\\ (x+3)^2 &= 0\end{aligned}$$

2. Since $(x + 3)$ is a factor, $x = -3$ is a solution.

3. The answer is B.

**Solution 2**

In this solution, we substitute $u$ for the expression $x + 5$ so that the factoring step is more reminiscent of factoring in terms of a single variable.

1. Since the expression $x + 5$ shows up as both a squared term and (with a coefficient) as a first-degree term, we can declare a new variable $u$, and say that $u = x + 5$. Now we can replace $(x + 5)$ with $u$ in the original equation in order to avoid distributing and combining many terms.

$$\begin{aligned}(x+5)^2 - 4(x+5) + 4 &= 0\\ u^2 - 4u + 4 &= 0\end{aligned}$$

2. This is a perfect square expression (matches the form $p^2 - 2pq + q^2$ where $p = u$ and $q = 2$) that can be factored easily.

$$\begin{aligned}u^2 - 4u + 4 &= 0\\ (u)^2 - 2(u)(2) + (2)^2 &= 0\\ (u-2)^2 &= 0\end{aligned}$$

3. Since $(u - 2)$ is a factor, $u = 2$ is a solution.

4. This is the crucial step with U-Substitution. After you solve for $u$, **you must never forget to rewrite $u$ in terms of the original variable used**, thus reversing the substitution. Since $u = x + 5$, replace $u$ with $x + 5$ in order to solve for $x$.

$$\begin{aligned}u &= 2\\ x + 5 &= 2\\ x &= -3\end{aligned}$$

5. The answer is B.

**Solution 3**

This is a solution that does not make use of U-Substitution. It is not necessarily slower, but is possibly more error prone since you have to expand and combine many more terms in the early steps.

1. Expand and simplify the original equation.

$$(x+5)^2 - 4(x+5) + 4 = 0$$
$$x^2 + 10x + 25 - 4x - 20 + 4 = 0$$
$$x^2 + 6x + 9 = 0$$

2. This is a perfect square expression that can be factored easily.

$$x^2 + 6x + 9 = 0$$
$$(x+3)^2 = 0$$

3. Since $(x+3)$ is a factor, $x = -3$ is a solution.
4. The answer is B.

**Notes**

If you do not recognize the perfect square pattern in Solution 3, you can factor the quadratic expression (or any other perfect square quadratic expression) by conventional factoring. You just need to look for numbers that multiply to 9 and add to 6, which happen to be 3 and 3. Notice that this will still get you the factors $x+3$ and $x+3$, but with a tiny bit of extra thought. Ultimately, factoring perfect square expressions is just a minor extension to conventional factoring, and offers a small time savings on may problems.

Alternatively, you could plug in each answer choice and see what value of $x$ makes the equation true. Generally, plugging in answer choices can be used to solve many problems on the test, but we DO NOT recommend it because it does not help you gain any understanding of the topics at hand, and it usually wastes time and leads to more errors.

## Factoring by Grouping

**Factoring by Grouping** is a method of factoring that is used to help factor certain types of equations in which the degree is greater than 2. You can break a Standard Form polynomial into sets of expressions that all share a common factor, forming subsets of terms that make the polynomial easier to factor and solve. For example, in the equation $x^5 - 3x^4 + 3x - 9 = 0$, we can break the polynomial expression into two subsets for which the terms in each of the subsets share a common factor. We can factor $x^4$ from both of the first two terms.

$$x^5 - 3x^4 + 3x - 9 = 0$$
$$x^4(x-3) + 3x - 9 = 0$$

From the third and fourth original terms, we can factor out a 3.

$$x^4(x-3) + 3x - 9 = 0$$
$$x^4(x-3) + 3(x-3) = 0$$





Now, we discover that we have had some good fortune bestowed on us by the omnipotent powers that craft these problems so that mortals can solve them: because both expression share a common factor of $(x-3)$, we can factor $(x-3)$ from both subsets, showing us that $x = 3$ must be one of the solutions.

$$x^4(x-3) + 3(x-3) = 0$$
$$(x-3)(x^4+3) = 0$$

Note that the factor $x^4 + 3$ provides no more real number solutions because it will never equal 0 (because $x^4$ cannot be equal to $-3$, as no real number raised to an even power will ever be negative). Non-real (complex) solutions are discussed elsewhere in this text, if you're curious about that.

The fact that $(x-3)$ was a factor of both of the subsets may seem like a coincidence, but as alluded to earlier, the problems on the test are very well designed, and the way that this problem was solvable with factoring by grouping was not an accident. If you are asked to find solutions to a higher-degree polynomial on the test, you should be able to solve the problem by factoring by grouping, but you can, in a moment of weakness, also find the solutions graphically on your calculator (*only* if it is on the calculator section: NO CHEATING!!—we'll discuss the graphs of polynomials in another chapter), or simply plug in answer choices and see which one makes the equation true.

Take it under advisement that the only time factoring by grouping has been required on the test is for polynomials where the leading $x$-term has an exponent greater than 2 (that is, for polynomials of degree 3 or higher), and there are four terms in the polynomial expression. If you encounter this setup, factor anything you can from the first set of two terms (the two highest-degree terms) and factor anything you can from the second set of two terms (the two lowest-degree terms). What should happen is that you will discover that both pairs of terms have a common factor, allowing you to factor the entire expression and thus to find the roots. These problems will be specifically constructed so they have that characteristic.

**Example 8.6-2**

**2** 1600.io 8.6

$$x^3 - 4x^2 + 2x - 8 = 0$$

For what real value of $x$ is the equation above true?

**Solution**

1. Group the first two terms as one expression (the two highest-degree terms) and group the third and fourth terms as another expression (the two lowest-degree terms).

$$x^3 - 4x^2 + 2x - 8 = 0$$
$$(x^3 - 4x^2) + (2x - 8) = 0$$

2. We can factor $x^2$ out of the first set of two terms.

$$(x^3 - 4x^2) + (2x - 8) = 0$$
$$x^2(x-4) + (2x-8) = 0$$

3. We can factor 2 out of the second set of two terms.

$$x^2(x-4) + (2x-8) = 0$$
$$x^2(x-4) + 2(x-4) = 0$$

4. We have found that $(x-4)$ is a factor of both sets of terms. Factor $(x-4)$ from both sets of terms.

$$x^2(x-4)+2(x-4)=0$$
$$(x-4)(x^2+2)=0$$

5. Since $(x-4)$ is a factor, then $x=4$ is a solution.

6. The factor $(x^2+2)$ produces no real solutions. Set the expression equal to 0 and try to solve for $x$.

$$x^2+2=0$$
$$x^2=-2$$

Since no real number squared can be equal to a negative number, there are no more real solutions. We will talk more about real roots and other types of roots in the coming chapters. The only real solution is $x=4$.

7. The answer is 4.

Section 8.6 Suggested P

- Test 2-NC-13
- Test 4-NC-18
- May 2017-NC-5

View related real-test problems in SMTEx

SMTEx

- Mar 2021-C-24
- May 2021 (US)-NC-19
- May 2021 (US)-C-29

## Section 8.6 Practice Problems

**1** 1600.io 8.6

What is the sum of all values of $m$ that satisfy $3m^2-18m+9=0$?

A) $-6$

B) $-2\sqrt{6}$

C) $2\sqrt{6}$

D) 6

**2** 1600.io 8.6

$$x(x+4)=21$$

What is the product of the solutions to the given equation?

A) 21

B) 10

C) $-4$

D) $-21$

**3** 1600.io 8.6

If $t>0$ and $(2t)^2-2(2t)-48=0$, what is the value of $t$?

**4** 1600.io 8.6

Which of the following is an equivalent form of the expression $(4x-3)^2-4(4x-3)$?

A) $(4x-3)(4x-3)$

B) $(4x-3)(4x-7)$

C) $4x^2-28x+21$

D) $16x^2-40x-3$





## CHAPTER 8 RECAP

- Factoring is the inverse operation of distribution.
- When factoring a quadratic expression in Standard Form where the coefficient of the $x^2$ term is 1, $x^2+bx+c=0$, we need to find two numbers, $p$ and $q$, such that $p+q=b$ (the sum of the numbers is $b$) and $pq=c$ (the product of the numbers is $c$). The solutions to the equation are $-p$ and $-q$, and we can rewrite the expression in Factored Form. One of the factors is $(x+p)$ and the other factor is $(x+q)$.

$$(x+p)(x+q)=0$$

- When the coefficient of the $x^2$ term is NOT 1 (when $a\neq 1$), first try dividing all of the terms by $a$ to see if the resulting expression is easily factorable.
- Sometimes, when the coefficient of the $x^2$ term is NOT 1 (when $a\neq 1$), you can use the Product Sum ($ac$) Method, which is a generalized form of factoring.

  When factoring a quadratic expression in Standard Form $ax^2+bx+c=0$ using the Product Sum ($ac$) Method, we need to find two numbers, $r$ and $s$, such that $r+s=b$ (the sum of the numbers is $b$) and $rs=ac$ (the product of the numbers is $ac$). The solutions to the equation are $\frac{-r}{a}$ and $\frac{-s}{a}$, and we can rewrite the expression in Factored Form. One of the factors is $\left(x+\frac{r}{a}\right)$ and the other factor is $\left(x+\frac{s}{a}\right)$.

$$a\left(x+\frac{r}{a}\right)\left(x+\frac{s}{a}\right)=0$$

- If you cannot find numbers that satisfy the conditions for factoring, you can always use Completing the Square:
  1. Divide both sides of the equation by the $a$ coefficient to eliminate the coefficient of the $x^2$ term.
  2. Move the constant to other side to isolate the $x$ terms.
  3. Replace the left side of the equation with $\left(x+\frac{b}{2}\right)^2$ (where $b$ is the coefficient of the $x$ term) and balance the extra constant on the other side by adding $\left(\frac{b}{2}\right)^2$ to the right side of the equation, pre-calculating $\frac{b}{2}$ so you don't have to do that twice.
  4. Take the square root of both sides, being careful to include both positive and negative square roots.
  5. Move the constant to right side of the equation to isolate and solve for $x$.
  6. Enumerate the solutions.
- The Quadratic Formula is constructed by using Completing the Square, and both methods can be used to find the solutions of any quadratic equation.
- The Quadratic Formula can be used to find the solutions of quadratic equations in Standard Form, $0=ax^2+bx+c$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

  or, as we prefer,

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

- The Difference of Squares $x^2-a^2$ can be quickly factored as $(x-a)(x+a)$.

- Perfect square quadratics of the form $x^2 + 2ax + a^2$ can be quickly factored as $(x + a)^2$.
- Perfect square quadratics of the form $x^2 - 2ax + a^2$ can be quickly factored as $(x - a)^2$.
- If you are given a system of equations consisting of one linear equation and one quadratic equation, or less commonly, two quadratic equations, you will use substitution to collapse the system of two equations into one quadratic equation which you can then solve.
- The sum of the solutions of a quadratic equation in Standard Form, $ax^2 + bx + c = 0$, is equal to $\frac{-b}{a}$.
- The product of the solutions of a quadratic equation in Standard Form, $ax^2 + bx + c = 0$, is equal to $\frac{c}{a}$.
- When an expression appears multiple times in a quadratic equation, you can probably shortcut the problem by factoring with respect to that expression rather than a single variable via U-Substitution, replacing the expression with the variable $u$, and then factoring normally with respect to $u$. Don't forget to finish up by solving for the original variable.
- Factoring by grouping is extremely rarely necessary on the test. The dead giveaway will be that the polynomial you are asked to solve has more than three terms (it probably would be four terms, just as a practical matter). If this is the case, find a factor that is common to the first set of two terms and the second set of two terms.





## Additional Problems

**1** 8.5

$$(x - 2)^2 = 3x - 2$$

What is one possible solution to the equation above?

**2** 8.2

$$x(x - 3) = 4$$

Which of the following lists all solutions to the quadratic equation above?

A) $\sqrt{7}$

B) $-1$ and $4$

C) $1$ and $-4$

D) $4$ and $7$

**3** 8.2

If $(ax + 4)(bx + 5) = 14x^2 + cx + 20$ for all values of $x$, and $a + b = 9$, what are the two possible values for $c$ ?

A) 2 and 7

B) 8 and 35

C) 10 and 28

D) 38 and 43

**4** 8.4

If $4x - 12y = 8z$, which of the following expressions is equivalent to $x^2 - 6xy + 9y^2$ ?

A) $8z$

B) $2z^2$

C) $4z^2$

D) $64z^2$

**5** 8.4

The expression $\frac{1}{5}x^2 - 3$ can be rewritten as $\frac{1}{5}(x - k)(x + k)$, where $k$ is a positive constant. What is the value of $k$ ?

A) 3

B) 15

C) $\sqrt{3}$

D) $\sqrt{15}$

**6** 8.6

$$x^2 - 12x + 27 = 3x + 1$$

What is the sum of the solutions to the given equation?

A) $-15$

B) $-12$

C) $12$

D) $15$

**7** 1600.io 8.1

$$(x-7)^6 = 0$$

What value of $x$ makes the equation above true?

**8** 1600.io 8.4

If $u + t = -3$ and $u - t = 2$, what is the value of $(u+t)(u^2 - t^2)$ ?

**9** 1600.io 8.5

In the $xy$-plane, the graph of the equation $y = -4x + 5$ intersects the graph of the equation $y = x^2$ at two points. What is the sum of the $x$-coordinates of the two points?

A) $-6$

B) $-4$

C) $4$

D) $6$

**10** 1600.io 8.5

In the $xy$-plane, the graph of $y = 5x^2 - 6x$ intersects the graph of $y = x$ at the points $(0, 0)$ and $(a, a)$. What is the value of $a$ ?

**11** 1600.io 8.5

$$x^2 - 9x + 26 = y$$
$$x = y - 2$$

The system of equations above is graphed in the $xy$-plane. Which of the following is the $y$-coordinate of an intersection point $(x, y)$ of the graphs of the two equations?

A) $-6$

B) $-4$

C) $4$

D) $6$

**12** 1600.io 8.3

$$x^2 - \frac{k}{2}x = 4p$$

In the quadratic equation above, $k$ and $p$ are constants. What are the solutions for $x$ ?

A) $x = \frac{k}{2} \pm \frac{\sqrt{k^2 + 64p}}{2}$

B) $x = \frac{k}{2} \pm \frac{\sqrt{k^2 + 4p}}{2}$

C) $x = \frac{k}{4} \pm \frac{\sqrt{k^2 + 64p}}{4}$

D) $x = \frac{k}{4} \pm \frac{\sqrt{k^2 + 4p}}{4}$





**13** 1600.io 8.5

$$y = x^2$$
$$5y + 20 = 5(x + 4)$$

If $(x, y)$ is a solution of the system of equations above and $x > 0$, what is the value of $xy$ ?

A) $1$

B) $4$

C) $5$

D) $16$

**14** 1600.io 8.2

$$x^3(x^2 - 13) = -36x$$

If $x > 0$, what is one possible solution to the equation above?

**15** 1600.io 8.5

$$y = 3x + 6$$
$$y = (x + 4)(x + 2)$$

The system of equations above is graphed in the $xy$-plane, At which of the following points do the graphs of the equations intersect?

A) $(-4, -6)$

B) $(-4, 0)$

C) $(-1, -2)$

D) $(-1, 3)$

**16** 1600.io 8.4

If $x^4 - y^4 = -26$ and $x^2 - y^2 = -2$, what is the value of $x^2 + y^2$ ?

A) $1$

B) $2$

C) $13$

D) $16$

**17** 1600.io 8.5

$$x - y = 1$$
$$x + y = x^2 - 5$$

Which ordered pair is a solution to the system of equations above?

A) $\left(1 + \sqrt{7}, \sqrt{7}\right)$

B) $\left(\sqrt{7}, -1 + \sqrt{7}\right)$

C) $\left(\sqrt{5}, -\sqrt{5}\right)$

D) $\left(1 + \sqrt{5}, \sqrt{5}\right)$

**18** 1600.io 8.5

$$(x + 4)^2 + (y - 3)^2 = 51$$
$$y = -4x + 4$$

Which of the following could be the $x$-coordinate of a solution to the system of equations above?

A) $\sqrt{2}$

B) $\frac{\sqrt{34}}{4}$

C) $\frac{8 + \sqrt{642}}{17}$

D) $\frac{20 + \sqrt{723}}{17}$

**19** 8.4

Which of the following is equivalent to $\sqrt[4]{x^2 - 6x + 9}$, where $x > 0$ ?

A) $(x-3)^2$

B) $(x-3)$

C) $(x-3)^{\frac{1}{2}}$

D) $(x-3)^{\frac{1}{4}}$

**20** 8.3

In the equation $(ax-2)^2 = 16$, $a$ is a constant. If $x = -2$ is one solution to the equation, what is a possible value of $a$ ?

A) $-9$

B) $-3$

C) $-1$

D) $6$

**21** 8.4

If $\frac{x^{a^2}}{x^{b^2}} = x^8$, $x > 1$, and $a + b = 2$, what is the value of $a - b$ ?

A) 4

B) 6

C) 8

D) 10

**22** 8.2

$$h = -4.9t^2 + 30t$$

The equation above expresses the approximate height $h$, in meters, of a ball $t$ seconds after it is launched vertically upward from the ground with an initial velocity of 30 meters per second. After approximately how many seconds will the ball hit the ground?

A) 5.5

B) 6.0

C) 6.5

D) 7.0

**23** 8.1

$$P(x) = x^2 - 7x + k$$

In the function above, $k$ is a constant. If 3 is a zero of the function, what is the value of $k$ ?

A) $-12$

B) $-3$

C) 4

D) 12





# Answer Key

### Section 8.1 Practice Problems

1. 8
2. B

### SkillDrill 8.2-1

1. 5 and 2
2. $-5$ and $-2$
3. $-6$ and 2
4. $-3$ and 4
5. 5 and 8
6. $-7$ and 11
7. $-13$ and 4
8. $-12$ and $-8$
9. $(x-1)(x+4)$
10. $(x-7)(x-3)$
11. $(a+8)(a+10)$
12. $(b-6)(b+5)$
13. $a(a-3)$
14. $x(x+1)$
15. $(x^2+8)(x^2+10)$
16. $x(x^2+8)(x^2+10)$
17. $x = -4$ and $x = 1$
18. $x = 1$ and $x = 7$
19. $x = -5$ and $x = -4$
20. $x = 2$ and $x = 8$
21. $x = -3$ and $x = 4$
22. $x = -2$ and $x = 1$
23. $a = -7$ and $a = 4$
24. $b = 6$ and $b = 8$
25. $a = -2$ and $a = 4$
26. $b = -2$ and $b = 3$
27. $x = -12$ and $x = -10$
28. $x = -2$ and $x = 6$

### Section 8.2 Practice Problems

1. 3
2. 16

### SkillDrill 8.3-1

1. $x = -2, x = -1$
2. $x = 2, x = 3$
3. $x = -1, x = 4$
4. $x = -7, x = 2$
5. $x = \frac{-3}{4}, x = 1$
6. $x = \frac{-1}{2}, x = \frac{-2}{3}$
7. $x = \frac{-1}{8}, x = 2$
8. $x = -7, x = \frac{-3}{2}$

### SkillDrill 8.3-2

1. $x = -2 \pm 2\sqrt{2}$
   or $x = -4.828, x = 0.828$
2. $x = \frac{-1}{2} \pm \frac{\sqrt{5}}{2}$
   or $x = -1.618, x = 0.618$
3. $x = \frac{3}{2} \pm \frac{\sqrt{21}}{2}$
   or $x = -0.791, x = 3.791$
4. $x = 3 \pm 2\sqrt{2}$
   or $x = 0.172, x = 5.828$
5. $x = 2 \pm \sqrt{10}$
   or $x = -1.162, x = 5.162$
6. $x = \frac{-1}{2} \pm \frac{\sqrt{33}}{6}$
   or $x = -1.457, x = 0.457$
7. $a = \frac{-5}{2} \pm \frac{\sqrt{15}}{2}$
   or $a = -4.436, a = -0.564$
8. $b = \frac{9}{8} \pm \frac{\sqrt{113}}{8}$
   or $b = -0.203, b = 2.454$

### Section 8.3 Practice Problems

1. A
2. 5
3. A
4. C

### SkillDrill 8.4-1

1. $(x-2)(x+2)$
2. $(x-9)(x+9)$
3. $(a-b)(a+b)$
4. $(a-5)(a+5)$
5. $(3a-2b)(3a+2b)$
   or $9\left(a - \frac{2}{3}b\right)\left(a + \frac{2}{3}b\right)$
6. $(x-3)(x+3)(x^2+9)$

### SkillDrill 8.4-2

1. $(x+1)^2$
2. $(p-3)^2$
3. $(y+4)^2$
4. $(3x-3)^2$ or $9(x-1)^2$
5. $(2a+4b)^2$ or $4(a+2b)^2$
6. $(b^2+1)^2$

### Section 8.4 Practice Problems

1. D
2. 6
3. B
4. 9

### Section 8.5 Practice Problems

1. C
2. B
3. 0 or 11
4. A

### SkillDrill 8.6-1

1. $s = -4, p = 3$
2. $s = -12, p = 32$
3. $s = 8, p = 5$
4. $s = 9, p = \frac{9}{2}$
5. $s = \frac{-2}{3}, p = \frac{-5}{3}$
6. $s = 1, p = 20$
7. $s = \frac{-8}{5}, p = -6$
8. $s = 9, p = 1$

**Section 8.6 Practice Problems**

1. D
2. D
3. 4
4. B

**Additional Problems**

1. 1 or 6
2. B
3. D
4. C
5. D
6. D
7. 7
8. 18
9. B
10. $\frac{7}{5}$ or 1.4
11. D
12. C
13. A
14. 2 or 3
15. D
16. C
17. D
18. A
19. C
20. B
21. A
22. B
23. D





# Appendix: Proofs and Alternative Reasoning

## Explanation of Product Sum ($ac$) Method

Note: In the discussion that follows, we assume that the coefficients $a$, $b$, and $c$ are integers (or can be converted into integers by multiplication or reducing).

We've seen that when the $a$ coefficient is 1 in a quadratic equation in Standard Form $0 = ax^2 + bx + c$, we can express the quadratic expression in Factored Form as follows:

$$(x + p)(x + q)$$

This leads to the zeros being $x = -p$ and $x = -q$.

When expanded, this Factored Form produces the following Standard Form expression:

$$x^2 + qx + px + pq$$
$$x^2 + (p + q)x + pq$$

We see that $p + q = b$, and $pq = c$. Therefore, to determine the values of $p$ and $q$, we need to find two numbers that sum to $b$ and whose product is $c$.

Here's a compact example:

$$0 = x^2 - 2x - 15$$
$$a = 1,\ b = -2,\ c = -15$$

$$x^2 - 2x - 15 = (x + p)(x + q)$$
$$x^2 - 2x - 15 = x^2 + qx + px + pq$$
$$x^2 - 2x - 15 = x^2 + (p + q)x + pq$$
$$\Downarrow$$
$$p + q = b = -2$$
$$pq = c = -15$$
$$\Downarrow$$
$$p = 3,\ q = -5 \qquad \text{(or } p = -5, q = 3\text{; the assignment is arbitrary)}$$
$$\Downarrow$$
$$x^2 - 2x - 15 = (x + 3)(x - 5)$$
$$\text{Zeros: } x = -3,\ x = 5$$

When the $a$ coefficient is *not* one, but the $b$ and $c$ coefficients are evenly divisible by $a$, we can simply divide through by $a$ to end up with a quadratic expression whose $a$ coefficient is 1, and we can still use the above procedure to find the zeros. Example:

$$0 = 3x^2 - 6x - 45$$
$$\frac{0}{3} = \frac{x^2 - 2x - 15}{3}$$
$$0 = x^2 - 2x - 15$$

At this point, we have the same equation as we did above, and the solving proceeds the same way.

The situation in which this procedure usually becomes too difficult to employ occurs when dividing through by the $a$ coefficient does *not* produce integer $b$ and $c$ coefficients, but instead produces irreducible fractions, making the determination of $p$ and $q$ much harder; directly finding two values (one or both of which must also be a fraction for their sum and/or product to be a fraction) that sum to $b$ and whose product is $c$ is usually far too time-consuming. Here's an example, shown symbolically and with specific values:

$$0 = ax^2 + bx + c \qquad 0 = 8x^2 - 2x - 15$$

We can certainly eliminate the $a$ coefficient (that is, have it become 1) by dividing through by $a$, which is 8:

$$\frac{0}{a} = \frac{ax^2 + bx + c}{a} \qquad \frac{0}{3} = \frac{8x^2 - 2x - 15}{8}$$

$$0 = x^2 + \frac{b}{a}x + \frac{c}{a} \qquad 0 = x^2 - \frac{2}{8}x - \frac{15}{8}$$

Now, the coefficient of the $x^2$ term is 1 (which is good), but the $x$ coefficient is $\frac{b}{a} = \frac{-2}{8}$ and the constant is $\frac{c}{a} = \frac{15}{8}$, which is not at all good, because they're fractions; how are we to go about finding the two values $p$ and $q$ that sum to $\frac{-2}{8}$ and that multiply to $\frac{15}{8}$ ?

Though it is of course *possible* to determine those two values given enough time, it would be far easier if we only had to deal with integers for the sum and product goals, as we can use the same methodology we'd use when we have a simpler quadratic factoring task. We can accomplish exactly that by using two fractions, but they will both have the same denominator, $a$ (which in our example is 8), as that is the denominator of the fractional coefficients, and so we only need to concern ourselves with the integer numerators, so we'll only be working with integers. We'll use $r$ and $s$ to denote those numerators, so the two fractions we'll use instead of $p$ and $q$ are $\frac{r}{a}$ and $\frac{s}{a}$.

To elaborate, rather than use $p$ and $q$, which yield the following Factored Form and equivalent Standard form,

$$(x + p)(x + q) = x^2 + (p + q)x + q$$

we'll use $\frac{r}{a}$ and $\frac{s}{a}$, so we'll have this Factored Form and Standard Form:

$$\left(x + \frac{r}{a}\right)\left(x + \frac{s}{a}\right) = x^2 + \left(\frac{r+s}{a}\right)x + \frac{rs}{a^2}$$

Remember, when we divided through by $a$, here's what we got:

$$0 = x^2 + \frac{b}{a}x + \frac{c}{a} \qquad 0 = x^2 - \frac{2}{8}x - \frac{15}{8}$$

Matching coefficients with our new Standard Form, $x^2 + \left(\frac{r+s}{a}\right)x + \frac{rs}{a^2}$, we see

$x$ coefficient:

$$\frac{r+s}{a} = \frac{b}{a}$$
$$r + s = b \quad \Rightarrow \quad r + s = -2$$

Constant:

$$\frac{rs}{a^2} = \frac{c}{a}$$
$$rs = \frac{a^2c}{a}$$
$$rs = ac \quad \Rightarrow \quad rs = 8(-15) = -120$$





We now have our product and sum goal values: **the sum of $r$ and $s$ must be $b$** ($-2$ in our example), and **the product of $r$ and $s$ must be $ac$** (which is $-120$ in our example) (this is where this method gets its name, by the way).

For our example, we can satisfy these constraints with the values 10 and $-12$, which we can assign arbitrarily to the two variables; we'll set $r = 10$ and $s = -12$. We can now substitute those values in our Factored Form equation:

$$0 = \left(x + \frac{r}{a}\right)\left(x + \frac{s}{a}\right)$$
$$0 = \left(x + \frac{10}{8}\right)\left(x - \frac{12}{8}\right)$$
$$0 = \left(x + \frac{5}{4}\right)\left(x - \frac{3}{2}\right)$$

Solving,

$$x + \frac{5}{4} = 0, x - \frac{3}{2} = 0$$
$$x = \frac{-5}{4}, x = \frac{3}{2}$$

As always when using factoring, if you don't actually need the Factored Form (or any of the constituent factors), you can skip writing that out and just go right to the solutions; when using $p$ and $q$, the solutions are $-p$ and $-q$, but here, we have to remember that $r$ and $s$ are the numerators of fractions where the denominators are $a$, so the solutions are $-\left(\frac{r}{a}\right)$ and $-\left(\frac{s}{a}\right)$.

## Detailed Walkthrough of Completing the Square

We will find the solutions of the equation $3x^2 - 12x - 15 = 0$, using Completing the Square, but with a little more explanation of each step. You don't need to look at this if you are willing to accept the "magic" step where we replace $x^2 + bx$ with $\left(x + \frac{b}{2}\right)^2$ and add $\left(\frac{b}{2}\right)^2$ to the other side of the equation.

We are starting with the following equation:

$$3x^2 - 12x - 15 = 0$$

Divide by the $a$ coefficient (which is 3 in this example) to eliminate coefficient of $x^2$ term:

$$x^2 - 4x - 5 = 0$$

Now we have a normal quadratic equation of the form

$$x^2 + bx + c = 0$$

where $b = -4$ and $c = -5$.

Move the constant $c$ (which is $-5$ in this example) to other side to isolate the $x$ terms:

$$x^2 - 4x = 5$$

At this point, all the $x$ terms are on one side, and there are no other terms on that side, so we need to replace the remaining expression with one that has only one $x$ term of degree 1 (that is, it has an exponent of 1, which does not need to be written because it's implied when no exponent is shown) so we can eventually isolate and thus solve for $x$.

How can we craft a term that only has a first-degree $x$ term but that will be equivalent to a second-degree binomial? Here's one way: we know from squaring binomials of the form $x + d$ that these operations will produce quadratic expressions, which have both first- and second-degree variable terms:

$$(x + d)^2 = x^2 + 2dx + d^2$$

Notice that the $b$ coefficient of the resulting quadratic expression is equal to $2d$; put another way, $d = \frac{b}{2}$. Also, note that the $c$ coefficient is equal to $d^2$, and because $d = \frac{b}{2}$, the $c$ coefficient is equal to $\left(\frac{b}{2}\right)^2$. Therefore, here's another way we could write this equation:

$$(x + d)^2 = x^2 + 2dx + d^2$$
$$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2$$

So back to our example. We have

$$x^2 - 4x = 5$$

We're just looking at the simple quadratic expression $x^2 - 4x$ on the left side for now; it's in the form

$$x^2 + bx$$

where $b = -4$. Looking at our earlier work, we know we can get a quadratic from an expression of the form

$$\left(x + \frac{b}{2}\right)^2$$

Here, that would be

$$\left(x + \frac{-4}{2}\right)^2$$
$$\left(x - \frac{4}{2}\right)^2$$
$$(x - 2)^2$$

Let's perform that squaring operation to see if we get what we're looking for—the two-term quadratic $x^2 - 4x$:

$$(x - 2)^2$$
$$x^2 - 4x + 4$$





So close…we have produced the desired expression $x^2 - 4x$, but we also ended up with a third term. You might have had a feeling that was coming! We didn't account for the fact that there would be an extra, constant term, representing $\left(\frac{b}{2}\right)^2$ (here evaluating to $\left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$), but we don't want that constant, and it messes up our brilliant plan. However, there is a simple solution: instead of replacing

$$x^2 - 4x$$

with just

$$(x - 2)^2$$

which produces that extra constant 4, we'll counteract that unwanted effect by simply subtracting that same unwanted value. Therefore, instead of replacing $x^2 - 4x$ with $(x - 2)^2$, we'll replace it with

$$(x - 2)^2 - 4$$

That is, we know we're going to get an extra 4, so we neutralize that by also subtracting 4.

As a check, let's multiply things out and combine like terms to see if our fix has had the desired effect:

$$(x - 2)^2 - 4$$
$$x^2 - 4x + 4 - 4$$
$$x^2 - 4x$$

Success! We now have an expression we can use as a replacement for the two-term quadratic on the left side of the equation, and it will leave us with only a single $x$ term with no exponent, which sets us up to continue our quest for the value of $x$.

Let's go ahead and perform that substitution on our original equation:

$$x^2 - 4x = 5$$
$$(x - 2)^2 - 4 = 5$$

Because we're trying to have only $x$ terms on the left side, we'll move that correcting constant $-4$ to the other side by adding 4 to both sides:

$$(x - 2)^2 - 4 = 5$$
$$(x - 2)^2 = 4 + 5$$

Because we're always going to perform the previous step, we can streamline the procedure by combine the last two steps when performing this substitution by simply adding the unwanted constant $\left(\frac{b}{2}\right)^2 = 4$ to the other side of the equation directly, so we can go right from

$$x^2 - 4x = 5$$

to

$$(x - 2)^2 = 5 + 4$$

Let's look at what we did in the last few steps with the symbolic representation alongside. We started with a quadratic equation

$$x^2 - 4x - 5 = 0 \qquad x^2 + bx + c = 0$$

We moved $c$ to the other side:

$$x^2 - 4x = 5 \qquad x^2 + bx = -c$$

We replaced the second-degree binomial on the left side with a term having only a first-degree variable, and we balanced the unwanted constant that's generated by adding it to the other side of the equation:

$$(x-2)^2 = 4+5 \qquad \left(x+\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c$$

Proceeding with our example, let's combine those constants on the right side

$$(x-2)^2 = 4+5$$
$$(x-2)^2 = 9$$

To further unwrap the left side where the jewel we seek, $x$, is nestled, we'll take the square root of both sides, being careful to include both positive and negative square roots:

$$(x-2)^2 = 9$$
$$x-2 = \pm\sqrt{9}$$
$$x-2 = \pm 3$$

Move that remaining constant over to the right side to isolate and thus solve for $x$:

$$x-2 = \pm 3$$
$$x = 2 \pm 3$$
$$x = 2+3,\ x = 2-3$$
$$x = 5,\ x = -1$$

## Proof of the Quadratic Formula based on Completing the Square

The derivation of this formula comes by following the steps of Completing the Square for a Standard Form quadratic equation: $ax^2 + bx + c = 0$.

1. Divide both sides of the equation by $a$ in order to eliminate the coefficient of the $x^2$ term.

$$ax^2 + bx + c = 0$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

2. Move the constant term to the other side of the equation in order to isolate the $x$ terms.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$





3. Replace the left side of the equation with $\left(x+\frac{b}{2a}\right)^2$ and balance the extra constant on the other side by adding $\left(\frac{b}{2a}\right)^2$ to the right side of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$\left(x+\frac{b}{2a}x\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$
$$\left(x+\frac{b}{2a}x\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Combine the constants on the right side of the equation by making a common denominator of $4a^2$.

$$\left(x+\frac{b}{2a}x\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$\left(x+\frac{b}{2a}x\right)^2 = \frac{b^2}{4a^2} - \left(\frac{c}{a}\right)\left(\frac{4a}{4a}\right)$$
$$\left(x+\frac{b}{2a}x\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$
$$\left(x+\frac{b}{2a}x\right)^2 = \frac{b^2-4ac}{4a^2}$$

5. Take the square root of both sides, being careful to include both positive and negative square roots.

$$\left(x+\frac{b}{2a}x\right)^2 = \frac{b^2-4ac}{4a^2}$$
$$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$
$$x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

6. Move the constant to right side of the equation to isolate and solve for $x$.

$$x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$
$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

It is done!

## Proof of the Sum and Product of Roots

We can verify that the sum of the roots is equal to $\frac{-b}{a}$ by considering the Factored Form of a polynomial expression.

A polynomial expression in Standard Form, $ax^2 + bx + c$ can be factored as

$$ax^2 + bx + c = a(x - p)(x - q)$$

where $p$ and $q$ are the zeros or roots. When we distribute the Factored Form, we can see how the Standard Form of the equation takes shape based on the values of the roots (which we will use to prove the Sum of the Roots Formula).

$$ax^2 + bx + c = ax^2 - apx - aqx + apq$$

We can factor out $a$ and $x$ from both of the $x$ terms on the right side of the equation in order to rewrite the expression in a way that matches Standard Form (and then we can match the coefficients to find the sum of the roots in terms of the Standard Form coefficients).

$$ax^2 + bx + c = ax^2 - a(p + q)x + apq$$

The quadratic expression on the right side of the equation is now in Standard Form, and the coefficient of the $x$ term, denoted as $b$ on the other side of the equation, must be equal to $-a(p + q)$.

$$b = -a(p + q)$$

Remember that $p$ and $q$ are the roots of the expression, so $p + q$ is the sum of the roots, and we can find the value of the sum of the roots by dividing both sides of the equation by $-a$.

$$\frac{-b}{a} = p + q$$

Thus, we have shown that the sum of the roots is equal to $\frac{-b}{a}$.

Another proof of this formula comes if we accept the Quadratic Formula as our starting point. The two root values, $x$, can be found with the equation below:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

One of the roots is equal to $\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and the other root is equal to $\frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$. If we add these two roots, we will find that the sum of the roots is $\frac{-b}{a}$.

$$\left(\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) + \left(\frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

$$\frac{-b}{2a} + \cancel{\frac{\sqrt{b^2 - 4ac}}{2a}} - \frac{b}{2a} - \cancel{\frac{\sqrt{b^2 - 4ac}}{2a}}$$

$$\frac{-b}{2a} - \frac{b}{2a}$$

$$\frac{-2b}{2a}$$

$$\frac{-b}{a}$$





Similar to the proof above, based on expanding the Factored Form into Standard Form, we know that

$$ax^2 + bx + c = ax^2 - a(p + q)x + apq$$

By matching coefficients, we can see that the constant terms must be equal:

$$c = apq$$

Remember that since $p$ and $q$ are the roots of the expression, the product of the roots is $pq$, and we can find the value of the product of the roots by dividing both sides of the equation by $a$.

$$\frac{c}{a} = pq$$

Thus, we have shown that the product of the roots is equal to $\frac{c}{a}$.

Similar to the above proof of the sum of roots, we can start with the two root values provided by the Quadratic Formula. The two roots happen to be conjugates of one another, so their product can be calculated more quickly using the Difference of Squares.

$$\left(\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

$$\left(\frac{-b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2$$

$$\frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2}$$

$$\frac{b^2 - b^2 + 4ac}{4a^2}$$

$$\frac{4ac}{4a^2}$$

$$\frac{c}{a}$$

# Extraneous Solutions and Dividing Polynomials

# 9

## 9.1 Phantom Equations and Extraneous Solutions

In the exponents chapter, we saw that we could solve some equations by raising both sides of the equation to a power. For example, if we wanted to find the value of $x$ given the equation $\sqrt[5]{x} = 2$, we could raise both sides to the fifth power to solve for $x$.

$$\sqrt[5]{x} = 2$$
$$\left(\sqrt[5]{x}\right)^5 = (2)^5$$
$$x = 32$$

**Example 9.1-1**

**1** 1600.io 9.1

If $a = 3\sqrt{6}$ and $3a = \sqrt{6x}$, what is the value of $x$ ?

**Solution**

1. Since $a = 3\sqrt{6}$, substitute $3\sqrt{6}$ for $a$ in the second equation.

$$3a = \sqrt{6x}$$
$$3(3\sqrt{6}) = \sqrt{6x}$$
$$9\sqrt{6} = \sqrt{6x}$$

2. Square both sides of the equation to get rid of the radical. Make sure to distribute the exponent on the left side of the equation.

$$(9\sqrt{6})^2 = (\sqrt{6x})^2$$
$$9^2(\sqrt{6})^2 = 6x$$
$$81(6) = 6x$$

3. Instead of multiplying 81 by 6, divide both sides of the equation by 6 to solve for $x$.

$$81(6) = 6x$$
$$81 = x$$

4. The answer is 81.

**Notes**

In Step 2 (or at the end of Step 1), we could have split the right side of the equation into $\sqrt{6}\sqrt{x}$ and canceled the $\sqrt{6}$ on both sides of the equation before squaring both sides.

When there are $x$ terms on both sides of an equation with a radical expression, however, the situation can be a bit more tricky because invalid solutions arise in the process of solving these equations. Specifically, let's limit ourselves to square root equations like $\sqrt{x} = -x$. We can solve this algebraically by squaring both sides of the equation.

$$\sqrt{x} = -x$$
$$\left(\sqrt{x}\right)^2 = (-x)^2$$
$$x = x^2$$

By squaring both sides of the original equation, we created a quadratic equation that can be solved as shown in the last chapter; however, as we will see, one of the solutions to this new equation does not actually satisfy the original equation. Let's solve the equation by getting all the $x$ terms on one side and factoring.

$$x = x^2$$
$$0 = x^2 - x$$
$$0 = x(x-1)$$
$$x = 0,\ x = 1$$

It would seem that both $x = 0$ and $x = 1$ are solutions, but if we substitute 1 for $x$ in the original equation, $\sqrt{x} = -x$, we will arrive at the equation $\sqrt{1} = -1$, which is not true. Remember that by definition, a symbolic reference to a square root (e.g. $\sqrt{x}$ or $x^{\frac{1}{2}}$) indicates only the non-negative (principal) root. This also applies to any other even-numbered root, such as $\sqrt[4]{x}$ or $x^{\frac{1}{4}}$. Therefore, the solution $x = 1$ is invalid.

These invalid solutions are called **extraneous solutions**. As a result of squaring, we sometimes come up with values that violate the starting conditions; **specifically, we might find values of $x$ that cause the square root to be set equal to a negative number or result in an invalid equation**, such as $5 = 1$ (as seen in the example that follows). The square root of a positive number (or of 0) can't be negative, and 5 clearly isn't equal to 1, so if solutions to these equations violate these restrictions, then they are extraneous.

Why does this happen? Other algebraic manipulations we use don't have this weird side-effect. It's because the squaring operation (or, in fact, raising to any even power) can produce **the same result for two different values**: the number being squared, and also the *negative* of that number, because both produce the same positive result when squared. This means that from that point forward in the solving process, you're actually solving not *just* for the original equation, *but also for a second equation where the value you squared has the opposite sign*. Therefore, you might get two solutions, but one of them is for this phantom equation, NOT for the real equation you're trying to solve.

In the example above, we started with this sequence:

$$\sqrt{x} = -x$$

Squaring both sides,

$$\left(\sqrt{x}\right)^2 = (-x)^2$$
$$x = x^2$$





The complication that has now appeared is that the $x^2$ on the right side could have been produced by squaring either $-x$ **or** $x$:

$$(-x)^2 = x^2$$
and
$$(x)^2 = x^2$$

That means that when we proceed with solving $x = x^2$, we're actually solving for not only the original equation

$$\sqrt{x} = -x$$

but *also* for the phantom equation

$$\sqrt{x} = x$$

Indeed, if you plug the extraneous solution we found ($x = 1$) into the phantom equation ($\sqrt{x} = x$) we see that it is a solution:

$$\sqrt{1} = 1$$

In general, if you see there is a radical sign in the original equation and that the question is asking you about the **solution set** of the equation, then you should **just plug in the possible solutions from the answer choices and see which ones are extraneous**. You can solve these equations by squaring both sides and finding solutions algebraically, but for this test, it is rarely worth the time and effort. If you solve for the possible solutions algebraically, you will have to plug back in any of the possible solutions to check if they are extraneous, so you might as well just plug in the possibilities offered to you in the answer choices.

You may see new notation for solution sets that you don't recognize. When you see $\{3, 4\}$, that means that the solution set includes $x = 3$ and $x = 4$ (that is, both 3 and 4 are solutions).

**Example 9.1-2**

**2** 1600.io 9.1

$$\sqrt{2x-2} + 3 = x - 2$$

What is the solution set of the equation above?

A) $\{3\}$

B) $\{9\}$

C) $\{3, 9\}$

D) $\{0, 3, 9\}$

**Solution 1**

Instead of solving the equation, just plug in the values 3, 9, and 0 to see if any violate the initial conditions of the problem.

1. Plug in 3 for $x$.

$$\begin{aligned} \sqrt{2x-2}+3 &= x-2 \\ \sqrt{2(3)-2}+3 &= (3)-2 \\ \sqrt{6-2}+3 &= 1 \\ \sqrt{4}+3 &= 1 \\ \text{✗} \quad 2+3 &= 1 \\ 5 &\neq 1 \end{aligned}$$

2. It is clear that 5 does not equal 1, so 3 is not a valid solution (though if you solved this algebraically you would find that it appears to be). Eliminate choices A, C, and D because all of those solution sets include 3.
3. By process of elimination, choice B is the only one left, but you probably should check the other numbers included in the solution sets (9 and 0) to be sure (9 will work out but 0 will not because the number in the radical will be negative).
4. The answer is B.

**Solution 2**

1. Subtract 3 from both sides of the equation.

$$\begin{aligned} \sqrt{2x-2}+3 &= x-2 \\ \sqrt{2x-2} &= x-5 \end{aligned}$$

2. Square both sides of the equation in order to eliminate the radical. Be aware that this step is where the phantom equation $\sqrt{2x-2} = -(x-5)$ is injected into the problem and this is why we have to check for extraneous solutions at the end.

$$\begin{aligned} \left(\sqrt{2x-2}\right)^2 &= (x-5)^2 \\ 2x-2 &= x^2-10x+25 \end{aligned}$$

3. Group all of the terms on the right side of the equation.

$$\begin{aligned} 2x-2 &= x^2-10x+25 \\ 0 &= x^2-12x+27 \end{aligned}$$

4. Factor the expression on the right side of the equation in order to find the possible solutions.

$$\begin{aligned} 0 &= x^2-12x+27 \\ 0 &= (x-9)(x-3) \end{aligned}$$

5. Since $(x-9)$ is a factor, $x = 9$ is a possible solution. Since $(x-3)$ is a factor, $x = 3$ is a possible solution.
6. Because this problem started with a radical, we need to check both solutions to see if either is extraneous. In the meantime, we can eliminate choice D because 0 is not even a possible solution.





7. Plug in 9 for $x$ to see if it is an extraneous solution. When doing this step, remember that $\sqrt{16}$ is equal to 4 but is not equal to $-4$.

$$\begin{aligned} \sqrt{2x-2}+3 &= x-2 \\ \sqrt{2(9)-2}+3 &= (9)-2 \\ \sqrt{18-2}+3 &= 7 \\ \sqrt{16}+3 &= 7 \\ 4+3 &= 7 \\ \text{✓} \quad 7 &= 7 \end{aligned}$$

8. We have found that 9 is a valid solution, so we can eliminate choice A, which does not include 9 in the solution set.
9. Plug in 3 for $x$ to see if it is an extraneous solution. When doing this step, remember that $\sqrt{4}$ is equal to 2 but is not equal to $-2$ (this is the most common mistake people make when using this method to solve this problem).

$$\begin{aligned} \sqrt{2x-2}+3 &= x-2 \\ \sqrt{2(3)-2}+3 &= (3)-2 \\ \sqrt{6-2}+3 &= 1 \\ \sqrt{4}+3 &= 1 \\ \text{✗} \quad 2+3 &= 1 \\ 5 &\neq 1 \end{aligned}$$

10. We have found that 3 is an extraneous solution. Therefore, the only valid solution is 9.
11. The answer is B.

**Notes**

Notice that Solution 2 takes longer, and in the end, we still had to plug in the possible solutions to check for any extraneous solutions.

Because $x = 3$ was found to be an extraneous solution to the original equation, it follows that 3 must be a solution to the phantom equation $\sqrt{2x-2} = -(x-5)$ injected into the problem when we squared both sides of an equation during the solving process (to reiterate, 3 is still not a valid solution to the original equation). To prove that, we can plug in 3 for $x$ in that phantom equation:

$$\begin{aligned} \sqrt{2x-2} &= -(x-5) \\ \sqrt{2(3)-2} &= -(3-5) \\ \sqrt{6-2} &= -3+5 \\ \sqrt{4} &= 2 \\ 2 &= 2 \end{aligned}$$

If you are not given choices to plug in because the question is a free response question, make sure to **check any solutions you do find because those solutions may be extraneous**.

If you're wondering if there is a method for dealing with solving equations with radicals without having to test all the possible solutions to see if any are extraneous, the answer is yes. You can preserve the radical throughout the solving process and terminate any branch of the solving path if the radical would be negative (radicals cannot be negative). This framework prohibits the injection of the phantom equation.

Let's look at the previous example problem and solve it using this method. Though it is not strictly necessary, it's most convenient to bottle up the radical using U-Substitution so the terms remain compact throughout the solving procedure. Here's the equation we're asked to solve:

$$\sqrt{2x-2}+3=x-2$$

Substitute the radical term, $\sqrt{2x-2}$, with $u$. Because $u$ represents a radical, it can never be negative, and that will be used to filter out any invalid solution path.

$$u=\sqrt{2x-2}$$

Solve for $x$ in preparation for substituting for $x$ on the right side of the equation so the only variable in the equation is $u$.

$$\begin{aligned} u &= \sqrt{2x-2} \\ u^2 &= 2x-2 \\ u^2+2 &= 2x \\ \frac{u^2+2}{2} &= x \end{aligned}$$

Now that we have expressions in terms of $u$ for both the radical and for $x$, we can substitute for $\sqrt{2x-2}$ and $x$ in the original equation.

$$\begin{aligned} \sqrt{2x-2}+3 &= x-2 \\ u+3 &= \frac{u^2+2}{2}-2 \end{aligned}$$

Rearrange to get a quadratic equation in $u$ in Standard Form in preparation for factoring.

$$\begin{aligned} u+3 &= \frac{u^2+2}{2}-2 \\ 0 &= \frac{u^2+2}{2}-2-u-3 \\ 0 &= \frac{u^2+2}{2}-u-5 \\ 0 &= (u^2+2)-2(u)-2(5) \\ 0 &= u^2+2-2u-10 \\ 0 &= u^2-2u-8 \end{aligned}$$

Factor.

$$0=(u+2)(u-4)$$

Solve for $u$. First let's find the solution that we get from the factor $(u+2)$.

$$\begin{aligned} u+2 &= 0 \\ u &= -2 \end{aligned}$$





Remember, $u$ **cannot** be negative, so this solution path is terminated. This is where a potential extraneous solution is squashed before it can weasel its way into our solution set. Proceed with the second solution.

$$\begin{aligned} u-4 &= 0 \\ u &= 4 \end{aligned}$$

Reverse the U-Substitution to find the value of $x$, using the equation for $x$ in terms of $u$ that we made at the start.

$$\begin{aligned} \frac{u^2+2}{2} &= x \\ \frac{4^2+2}{2} &= x \\ \frac{16+2}{2} &= x \\ \frac{18}{2} &= x \\ 9 &= x \end{aligned}$$

We do not need to test this solution to see if it is extraneous, because this method only produces correct solutions. Therefore, the solution set is $\{9\}$.

**Section 9.1 Suggested Problems from Real Tests**

- Test 1-NC-20
- Test 2-NC-5
- Test 4-NC-9
- Test 5-NC-5
- Test 6-NC-9
- Test 8-NC-7
- Apr 2017-C-20
- May 2017-NC-19
- Apr 2019-NC-5
- Apr 2019-C-20
- May 2019 (Int)-NC-16
- Oct 2019-NC-18
- Oct 2019-C-13
- Apr 2021-NC-19

View related real-test problems in SMTEx

SMTEx

## Section 9.1 Practice Problems

**1** 1600.io ? 9.1

$$\sqrt{x-3}=12$$

What value of $x$ satisfies the equation above?

**2** 1600.io ? 9.1

$$\sqrt{k+4}-x=0$$

In the equation above, $k$ is a constant. If $x=5$, what is the value of $k$ ?

A) 1

B) 3

C) 9

D) 21

**3** 1600.io 9.1

$$\left(12-2\sqrt{x}\right)^2=\left(3+\sqrt{x}\right)^2$$

What is the solution to the equation above?

A) $x=81$

B) $x=25$

C) $x=9$

D) $x=3$

**4** 1600.io 9.1

$$\sqrt{x+a}=x-5$$

If $a=1$, what is the solution set of the equation above?

A) $\{3,8\}$

B) $\{-1\}$

C) $\{3\}$

D) $\{8\}$

## 9.2 Rational Functions

Operations that are performed on constants or variables can also be performed on entire polynomial expressions. We can add or subtract two polynomials, $f(x)$ and $g(x)$, combining like terms to produce a new polynomial $h(x)$. For example, if $f(x)=2x^2+3x$ and $g(x)=x^2-12$, then the sum of $f(x)$ and $g(x)$ is found as follows:

$$h(x)=f(x)+g(x)$$
$$h(x)=(2x^2+3x)+(x^2-12)$$
$$h(x)=2x^2+3x+x^2-12$$
$$h(x)=3x^2+3x-12$$

As we saw in the last chapter, we can also multiply two polynomials. For example, we know that $x+2$ and $x-4$ are both first-degree polynomials, and we showed how two first-degree polynomials could be multiplied to produce a second-degree polynomial.

$$(x+2)(x-4)$$
$$x^2-2x-8$$

In this chapter, we are going to look at the division of polynomials, sometimes called **Rational Functions**. Terms in math are said to be **rational** when they can be expressed as a *ratio* (they're "ratio-nal") or, equivalently, as a fraction. Note that this has nothing to do with whether the number or expression is capable of logical thinking. A **rational number** can be expressed as a ratio or fraction of two integers, such as 3/5 or 23/1. A **rational function** can be expressed as a fraction where both the numerator and the denominator are polynomials, such as

$$\frac{x^2+2x+3}{x+1} \qquad \frac{x+1}{x-5} \qquad \frac{1}{3x^3}$$

A number that can't be expressed as a fraction with integer numerator and denominator is said to be **irrational**. Some examples of irrational numbers are $\sqrt{2}$ and $\pi$.

In a way, factoring has prepared us for the task of polynomial division (factoring is akin to dividing out a factor), though the two processes are not exactly the same, as true division carries some implications that were not present when we factored. We will discuss these implications in more detail later in the chapter, but for now, we just need to get comfortable rewriting (simplifying, if you like, though again, this is an inexact term) rational expressions to match answer choices.





One necessary skill is the ability to make common denominators in order to combine terms. We know how to do this for numerical fractions. For example, we can write the sum $2+\frac{3}{5}$ as an improper fraction by multiplying 2 by the fraction $\frac{5}{5}$, which is the same as multiplying by 1 (because $\frac{5}{5}=1$), allowing us to represent 2 as a fraction with 5 in the denominator without changing its value.

$$2+\frac{3}{5}$$
$$2\left(\frac{5}{5}\right)+\frac{3}{5}$$
$$\frac{10}{5}+\frac{3}{5}$$
$$\frac{13}{5}$$

The process of finding a common denominator is no different when the fractions involve polynomials. For example, we can turn the sum $2+\frac{3}{x+5}$ into a single fraction if we multiply 2 by $\frac{x+5}{x+5}$ (by exactly the same reasoning as that used in the previous example).

$$2+\frac{3}{x+5}$$
$$2\left(\frac{x+5}{x+5}\right)+\frac{3}{x+5}$$
$$\frac{2x+10}{x+5}+\frac{3}{x+5}$$
$$\frac{2x+13}{x+5}$$

**Example 9.2-3**

**3** 1600.io 9.2

$$\frac{1}{3x+2}+6$$

Which of the following expressions is equivalent to the expression above, for $x>0$ ?

A) $\frac{3x+12}{3x+2}$

B) $\frac{3x+13}{3x+2}$

C) $\frac{18x+12}{3x+2}$

D) $\frac{18x+13}{3x+2}$

**Solution**

1. Since all of the answer choices are a single fraction, write the terms with a common denominator by multiplying 6 by $\frac{3x+2}{3x+2}$, and then combine the two fractions.

$$\frac{1}{3x+2}+6$$

$$\frac{1}{3x+2}+6\left(\frac{3x+2}{3x+2}\right)$$

$$\frac{1}{3x+2}+\frac{18x+12}{3x+2}$$

$$\frac{18x+13}{3x+2}$$

2. The answer is D.

**Notes**

The information that $x > 0$ was not necessary to know in order to answer this problem. It is a necessary restriction to eliminate the one value of $x$ for which this equation would not be true, which is when $x = \frac{-2}{3}$. This value of $x$ would cause the denominator to be equal to 0, which is a situation we will discuss a little more later on in this chapter.

In general, if limits are put on the values of variables, the limits exist to ensure that the conclusions drawn are mathematically accurate for all considered cases, but they do not affect the solution method and should simply be checked once solutions are found to verify consistency with the limits placed on the problem.

If you find that you do have to make a common denominator, multiply the numerator and denominator of each fraction by the denominator of the other fraction, just as you would if the denominators were numbers.

For example, if you had to simplify $\frac{1}{x+2}+\frac{1}{x+3}$, multiply the first fraction by $\frac{x+3}{x+3}$ and multiply the second fraction by $\frac{x+2}{x+2}$.

$$\frac{1}{x+2}+\frac{1}{x+3}$$

$$\frac{1}{x+2}\left(\frac{x+3}{x+3}\right)+\frac{1}{x+3}\left(\frac{x+2}{x+2}\right)$$

$$\frac{x+3}{(x+2)(x+3)}+\frac{x+2}{(x+3)(x+2)}$$

$$\frac{2x+5}{(x+2)(x+3)}$$

Most of the time when we are working with equations rather than standalone expressions, we are given the freedom to completely avoid dealing with common denominators by **getting rid of all fractions. Multiply both sides of the equation by all of the denominators** (make sure to cancel any expressions before multiplying when possible).

In the following example, instead of making common denominators on the right side of the equation, our first step will be to combine terms with the same denominator, and then multiply both sides of the equation by that denominator in order to eliminate the fractions.





**Example 9.2-4**

4 1600.io 9.2

$$\frac{4(x-3)}{x+2}=3-\frac{1}{x+2}$$

What is the solution to the equation above?

A) 13

B) 15

C) 17

D) 19

**Solution 1**

1. Instead of trying to make a common denominator on the right side of the equation, combine the terms with the same denominator by adding $\frac{1}{x+2}$ to both sides of the equation. Since we are combining the fraction terms, we should also distribute the 4 to the expression $x-3$ in order to combine the terms in the numerator expression.

$$\frac{4(x-3)}{x+2}=3-\frac{1}{x+2}$$

$$\frac{4x-12}{x+2}+\frac{1}{x+2}=3$$

$$\frac{4x-12+1}{x+2}=3$$

$$\frac{4x-11}{x+2}=3$$

2. Multiply both sides of the equation by $x+2$ in order to eliminate the fraction.

$$\frac{4x-11}{x+2}=3$$

$$\frac{4x-11}{\cancel{x+2}}(\cancel{x+2})=3(x+2)$$

$$4x-11=3x+6$$

3. Collect and combine the $x$ terms on the left side of the equation by subtracting $3x$ from both sides of the equation, and collect and combine the constant terms on the right side of the equation by adding 11 to both sides of the equation.

$$4x-11=3x+6$$
$$4x-3x=6+11$$
$$x=17$$

4. The answer is C.

**Solution 2**

In this solution, we opt to make a common denominator in the first step. This method is not necessarily longer, but is perhaps more elaborate, leaving room for errors.

1. Make the two fractions on the right side of the equation have common denominators (which will also match the denominator of the fraction on the left side of the equation). Multiply 3 by $\frac{x+2}{x+2}$ in order to write it in terms of a common denominator, $x+2$, without changing its value.

$$\frac{4(x-3)}{x+2} = 3 - \frac{1}{x+2}$$

$$\frac{4x-12}{x+2} = 3\left(\frac{x+2}{x+2}\right) - \frac{1}{x+2}$$

$$\frac{4x-12}{x+2} = \frac{3x+6}{x+2} - \frac{1}{x+2}$$

$$\frac{4x-12}{x+2} = \frac{3x+6-1}{x+2}$$

$$\frac{4x-12}{x+2} = \frac{3x+5}{x+2}$$

2. Both sides of the equation are written in the same form, so we can simply match equivalent terms (or you can now choose to eliminate the fractions by multiplying both sides by $x+2$, which will cancel both denominators). The two numerators must be the same.

$$4x - 12 = 3x + 5$$

3. Combine the $x$ terms on the left side of the equation by subtracting $3x$ from both sides of the equation, and combine the constant terms on the right side of the equation by adding 12 to both sides of the equation.

$$4x - 12 = 3x + 5$$
$$4x - 3x = 5 + 12$$
$$x = 17$$

4. The answer is C.

The following example is very similar to the previous example. We could solve this problem by making common denominators, but since there are more terms (none of which have the same denominator), it is more evident how getting rid of the fractions instead of working with them simplifies the work (the more terms you have to copy down line by line, the more potential there is for errors).





**Example 9.2-5**

**5** 1600.io 9.2

$$\frac{2}{x-4} + \frac{1}{x+4} = \frac{rx+t}{(x-4)(x+4)}$$

The equation above is true for all $x > 4$, where $r$ and $t$ are positive constants. What is the value of $rt$ ?

A) $-12$

B) $0$

C) $8$

D) $12$

**Solution 1**

1. Instead of trying to make a common denominator, simply get rid of all fractions by multiplying both sides of the equation by $(x-4)(x+4)$. Even though all three fractions have different denominators, multiplying by $(x-4)(x+4)$ will cancel all three of the denominators (the denominator of the fraction on the right side is the product of the other two denominators). Remember to distribute.

$$\frac{2}{x-4} + \frac{1}{x+4} = \frac{rx+t}{(x-4)(x+4)}$$

$$(x-4)(x+4)\left(\frac{2}{x-4} + \frac{1}{x+4}\right) = \cancel{(x-4)}\cancel{(x+4)}\left(\frac{rx+t}{\cancel{(x-4)}\cancel{(x+4)}}\right)$$

$$\cancel{(x-4)}(x+4)\left(\frac{2}{\cancel{x-4}}\right) + (x-4)\cancel{(x+4)}\left(\frac{1}{\cancel{x+4}}\right) = rx + t$$

$$(x+4)(2) + (x-4)(1) = rx + t$$
$$2x + 8 + x - 4 = rx + t$$
$$3x + 4 = rx + t$$

2. Based on matching coefficients, the value of $r$ is 3 and the value of $t$ is 4. Therefore, the value of $rt$ is equal to 3(4) or 12.

3. The answer is D.

**Solution 2**

1. Make the two fractions on the left side of the equation have common denominators. Multiply the first fraction by $\frac{x+4}{x+4}$, and multiply the second fraction by $\frac{x-4}{x-4}$. When simplifying, don't expand the expressions in the denominators.

$$\frac{x+4}{x+4}\left(\frac{2}{x-4}\right)+\frac{x-4}{x-4}\left(\frac{1}{x+4}\right)=\frac{rx+t}{(x-4)(x+4)}$$

$$\frac{2(x+4)}{(x+4)(x-4)}+\frac{x-4}{(x+4)(x-4)}=\frac{rx+t}{(x-4)(x+4)}$$

$$\frac{2x+8}{(x+4)(x-4)}+\frac{x-4}{(x+4)(x-4)}=\frac{rx+t}{(x-4)(x+4)}$$

2. Since the fractions have the same denominator, we can combine the numerators.

$$\frac{2x+8}{(x+4)(x-4)}+\frac{x-4}{(x+4)(x-4)}=\frac{rx+t}{(x-4)(x+4)}$$

$$\frac{2x+8+x-4}{(x+4)(x-4)}=\frac{rx+t}{(x-4)(x+4)}$$

$$\frac{3x+4}{(x+4)(x-4)}=\frac{rx+t}{(x-4)(x+4)}$$

3. Both sides of the equation are written in the same form, so we can simply match equivalent terms. The two numerators must be the same.

$$3x+4=rx+t$$

4. The $x$-coefficients must match, so $r=3$. Similarly, the constant terms must match, so $t=4$.
5. Since $r=3$ and $t=4$, then $rt$ is equal to $3(4)$, or 12.
6. The answer is D.

**Section 9.2 Suggested Problems from Real Tests**

- Test 1-NC-13
- Test 5-NC-19
- Test 6-C-31
- Test 8-C-8
- Mar 2018-NC-11
- 2019-C-29
- 2019 (US)-NC-7
- 2019 (US)-C-20
- Mar 2021-NC-6

**View related real-test problems in SMTEx**

SMTEx

## Section 9.2 Practice Problems

**1** 1600.io 9.2

$$x-3=\frac{6}{x-3}$$

In the equation above, which of the following is a possible value of $x-3$ ?

A) $\sqrt{6}$

B) 3

C) $2\sqrt{3}$

D) $3+\sqrt{6}$

**2** 1600.io 9.2

$$\frac{5x+23}{(x+4)^2}-\frac{5}{x+4}$$

The expression above is equivalent to $\frac{a}{(x+4)^2}$, where $a$ is a positive constant and $x\neq -4$. What is the value of $a$ ?





## 9.3 Undefined Values and Extraneous Solutions for Rational Functions

As we stated earlier, dividing polynomials is not quite the same as factoring because there are added restrictions (this was also alluded to in Example 9.2-3). We already know that fractions with 0 as the denominator are undefined (you can't divide by 0). So, if you are given a rational function and asked for what values of $x$ the function is undefined, you need to find the zeros of the polynomial expression in the denominator; remember, the zeros of a polynomial are simply the values of the variable that make the whole polynomial equal to zero.

**Example 9.3-6**

**6** 1600.io 9.3

Which of the following is a value of $x$ for which the expression $\frac{-7}{x^2+12x-13}$ is undefined?

A) $-7$

B) $-1$

C) 0

D) 1

**Solution**

1. Set the quadratic expression in the denominator equal to 0 so that we can factor it and find the zeros.

$$x^2+12x-13=0$$

2. Two numbers that add to 12 and multiply to $-13$ are 13 and $-1$, so the factors are $(x+13)$ and $(x-1)$.

$$x^2+12x-13=0$$
$$(x+13)(x-1)=0$$

3. Since $(x+13)$ is a factor, $x=-13$ is a zero of the denominator (which would cause the denominator to be 0, producing an undefined value for the fraction), but it is not one of the choices.
4. Since $(x-1)$ is a factor, $x=1$ is a zero of the polynomial in the denominator. When the value of $x$ is 1, the denominator will be 0, making the expression undefined.
5. The answer is D.

When there are terms with common denominators on both sides of the equation, you can solve for $x$ algebraically by grouping those terms with common denominators together on one side of the equation and then cross multiplying in order to eliminate denominators. We used this procedure with no complications in earlier examples in this chapter in order to solve certain problems.

However, when we do multiply both sides of the equation by expressions that were in the denominators, we disguise and lose sight of the added restrictions on possible values of $x$; since these expressions are no longer in the denominator, it is not clear that the zeros of those expressions could cause undefined values. When we multiply the expressions, we create new polynomials that do not obviously have the same restrictions.

Occasionally, in the process of solving rational function equations algebraically, we arrive at extraneous solutions for $x$ which satisfy the new polynomials we created as a result of multiplying, but, when they are plugged back into the original equation, will cause denominators in the original equation to be equal to 0, thus producing undefined values.

As with problems involving radical equations, if you are presented with possible solution sets as answer choices, there is a good chance that some of the values in those sets are values that you would find if you were to solve the problem algebraically but which don't truly satisfy the original equation. This is because they would cause a denominator to be equal to 0, leading to undefined values. **Rational function questions with solution sets for answer choices are often easiest to solve by plugging in the answer choices**.

**We have to stress that the questions in this chapter represent some of the *ONLY* cases on the test where plugging in answer choices is the best solving method**. This strategy makes sense for these questions because even if you take the time to solve the problem algebraically, you still have to check for extraneous solutions by plugging those solutions back into the original equation.

**Example 9.3-7**

**7** 1600.io 9.3

$$\frac{x-2}{x-4} = \frac{1}{x} + \frac{2}{x-4}$$

What is the solution set of the equation above?

A) $\{1\}$

B) $\{0, 4\}$

C) $\{1, 4\}$

D) $\{2, 4\}$

**Solution 1**

1. Since this is a rational function problem where we are asked about the solution set of the equation, we should start by plugging in possible solutions from the answer choices. In general, you want to start with any possible solution values that appear frequently in the answer choices because if you can eliminate those values, you'll substantially narrow the set of possible answer choices. In this case, 4 is a possible solution in three of the answer choices, so we should start by plugging in 4 for $x$. It should be immediately apparent that plugging in 4 will produce undefined values because the denominator of two of the expressions is $x - 4$. (It should be noted that 0 also should not be in the answer set because $x$ is a denominator that would be equal to 0 when $x = 0$.)

$$\frac{x-2}{x-4} = \frac{1}{x} + \frac{2}{x-4}$$

$$\frac{4-2}{4-4} = \frac{1}{4} + \frac{2}{4-4}$$

$$\frac{2}{0} = \frac{1}{4} + \frac{2}{0}$$

2. Since choices B, C, and D include 4 as a possible solution, all of these choices can be eliminated (though you can check what happens when $x$ equals 0, 1, or 2 to see which ones are valid solutions—0 will also produce an undefined value, and 2 is not even an extraneous solution—1 will work of course).

3. The answer is A.





**Solution 2**

1. Combine the two terms with a denominator of $x - 4$ by subtracting $\frac{2}{x-4}$ from both sides of the equation.

$$\frac{x-2}{x-4} = \frac{1}{x} + \frac{2}{x-4}$$

$$\frac{x-2}{x-4} - \frac{2}{x-4} = \frac{1}{x}$$

$$\frac{x-4}{x-4} = \frac{1}{x}$$

2. On the left side of the equation, the numerator and the denominator are the same, so the fraction reduces to 1. Noticing this now actually helps eliminate an extraneous solution that would arise from cross multiplying and distributing terms (which we will see in Solution 3).

$$\frac{\cancel{x-4}}{\cancel{x-4}} = \frac{1}{x}$$

$$1 = \frac{1}{x}$$

3. Multiply both sides of the equation by $x$ to get $x$ out from the denominator and solve for $x$.

$$1 = \frac{1}{x}$$

$$x = 1$$

4. Check that 1 is a valid solution by plugging 1 in for $x$.

$$\frac{x-2}{x-4} = \frac{1}{x} + \frac{2}{x-4}$$

$$\frac{1-2}{1-4} = \frac{1}{1} + \frac{2}{1-4}$$

$$\frac{-1}{-3} = 1 + \frac{2}{-3}$$

$$\frac{1}{3} = 1 - \frac{2}{3}$$

$$\frac{1}{3} + \frac{2}{3} = 1$$

$$\frac{3}{3} = 1$$

$$\checkmark \quad 1 = 1$$

5. When $x = 1$, no initial constraints of the problem are violated, so 1 is a valid solution. Only choices A and C contain 1 as part of the solution set; however, we did not find that 4 was a valid solution by this method (as we said earlier, we luckily avoid solving for this extraneous solution by reducing the fraction in the previous step). Choice C is questionable, and we should double check that $x = 4$ is not also a solution (these extra checks are why we advocate plugging in right away on this particular type of problem).

6. If we tried to test $x = 4$, we would see that plugging in 4 would cause the denominators of the fractions in the given equation to be equal to 0, so 4 should not be included in the solution set. Eliminate choice C.

7. The answer is A.

**Solution 3**

In Step 2 of Solution 2, suppose we had not reduced the fraction to 1 when we saw $\frac{x-4}{x-4}$. This is the step where we will introduce an extraneous solution to the problem because by multiplying by $x - 4$, we disguise the fact that there will be undefined values if $x = 4$.

1. Combine the two terms with a denominator of $x - 4$ by subtracting $\frac{2}{x-4}$ from both sides of the equation.

$$\frac{x-2}{x-4} = \frac{1}{x} + \frac{2}{x-4}$$
$$\frac{x-2}{x-4} - \frac{2}{x-4} = \frac{1}{x}$$
$$\frac{x-4}{x-4} = \frac{1}{x}$$

2. Multiply both sides of the equation by each of the denominators (cross multiply) in order to eliminate fractions.

$$\frac{x-4}{x-4} = \frac{1}{x}$$
$$(x-4)(x) = (1)(x-4)$$
$$x^2 - 4x = x - 4$$

3. Bring all the terms to the left side of the equation so that we have a quadratic expression set equal to 0 (which we need to do in order to find solutions).

$$x^2 - 4x = x - 4$$
$$x^2 - 4x - x + 4 = 0$$
$$x^2 - 5x + 4 = 0$$

4. Factor the quadratic expression in order to find the solutions (zeros). In case you still aren't comfortable factoring, we'll remind you that we need to find two numbers that add to $-5$ and multiply to 4 (alternatively, you could use your solving technique of choice, of course). Those numbers are $-1$ and $-4$, so the factors are $(x - 1)$ and $(x - 4)$.

$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$

5. Since $x - 4$ and $x - 1$ are factors, then $x = 4$ and $x = 1$ are possible solutions, but we need to verify that they are not extraneous.





6. If we try to plug 4 into the original equation, we will see that it would cause the denominators of two of the fractions to be equal to 0.

$$\frac{x-2}{x-4} = \frac{1}{x} + \frac{2}{x-4}$$
$$\frac{4-2}{4-4} = \frac{1}{4} + \frac{2}{4-4}$$
$$\frac{2}{0} = \frac{1}{4} + \frac{2}{0}$$

7. Since $x = 4$ would result in undefined values in the original equation, 4 is an extraneous solution, and should not be included in the solution set. You can eliminate choices B, C, and D (which all include 4 in the solution set), leaving only choice A as a possibility. To be safe, you should verify that 1 (the only value in the solution set in choice A) is actually a valid solution.

8. Check that 1 is a valid solution by plugging 1 in for $x$ in the original equation.

$$\frac{x-2}{x-4} = \frac{1}{x} + \frac{2}{x-4}$$
$$\frac{1-2}{1-4} = \frac{1}{1} + \frac{2}{1-4}$$
$$\frac{-1}{-3} = 1 + \frac{2}{-3}$$
$$\frac{1}{3} = 1 - \frac{2}{3}$$
$$\frac{1}{3} + \frac{2}{3} = 1$$
$$\frac{3}{3} = 1$$
$$\checkmark \quad 1 = 1$$

9. When $x = 1$, no initial constraints of the problem are violated, so 1 is a valid solution.

10. The answer is A.

**Section 9.3 Suggested Problems from Real Tests**

- Test 1-C-36
- Test 8-C-14
- Test 10-NC-6
- Mar 2018-NC-12

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## Section 9.3 Practice Problems

**1** 1600.io 9.3

$$\frac{x^2-4}{x-2}=-2$$

What are all the values of $x$ that satisfy the equation above?

A) $-4$

B) $-2$

C) $2$

D) $-4$ and $2$

**2** 1600.io 9.3

$$\frac{4x^2}{x^2-16}-\frac{2x}{x+4}=\frac{3}{x-4}$$

What value of $x$ satisfies the equation above?

A) $-4$

B) $-\frac{3}{2}$

C) $\frac{3}{2}$

D) $4$

## 9.4 Polynomial Long Division, Polynomial Remainder Theorem, and Matching Coefficients

When faced with dividing one polynomial by another, the first thing you should try is factoring. If you find a common factor to both the numerator and the denominator, you can cancel those factors to simplify the fraction.

**Example 9.4-8**

**8** 1600.io 9.4

$$f(x)=x^3-4x$$

$$g(x)=x^2+5x+6$$

Which of the following expressions is equivalent to $\frac{f(x)}{g(x)}$, for $x>-2$?

A) $\frac{1}{x+3}$

B) $\frac{x-2}{x+3}$

C) $\frac{x(x-2)}{x+3}$

D) $\frac{x(x+2)}{x+3}$





**Solution**

1. Since $f(x)=x^3-4x$ and $g(x)=x^2+5x+6$, then we can write the following equation:

$$\frac{f(x)}{g(x)}=\frac{x^3-4x}{x^2+5x+6}$$

2. Factor the numerator and denominator to see if there are any common factors that can be canceled in order to simplify; notice that after we factor $x$ from the numerator in the first step below, the difference of squares term $(x^2-4)$ appears in the numerator, so we factor that further.

$$\frac{f(x)}{g(x)}=\frac{x^3-4x}{x^2+5x+6}$$

$$\frac{f(x)}{g(x)}=\frac{x(x^2-4)}{(x+2)(x+3)}$$

$$\frac{f(x)}{g(x)}=\frac{x(x+2)(x-2)}{(x+2)(x+3)}$$

3. The numerator and denominator both have a factor of $(x+2)$, so we can cancel $(x+2)$.

$$\frac{f(x)}{g(x)}=\frac{x\cancel{(x+2)}(x-2)}{\cancel{(x+2)}(x+3)}$$

$$\frac{f(x)}{g(x)}=\frac{x(x-2)}{x+3}$$

4. The answer is C.

### Polynomial Long Division

In the previous example, we did not have to divide any further in order to match an answer choice, so we didn't need to know any other ways to perform division. While polynomial long division is not a necessary skill to succeed on the test, it can be useful, and it is helpful in explaining why certain other viable solution methods are valid.

Polynomial division can be accomplished by using long division (for cases when factoring isn't easy to do). The process works much like long division of numbers. Though as noted, polynomial long division has never been a necessary technique for the test, it is useful to know, and it makes it easier to understand the Remainder Theorem (we'll get to it) and the associated techniques for dealing with factoring and dividing polynomials.

Here's how the long division operation looks symbolically, with the names of each of the components:

$$\text{divisor}\,\overline{)\,\text{dividend}}\;\;\frac{\text{quotient}}{}\;+\;\frac{\text{remainder}}{\text{divisor}}$$

Written using fraction notation, this would be

$$\frac{\text{dividend}}{\text{divisor}}=\text{quotient}+\frac{\text{remainder}}{\text{divisor}}$$

For example, a simple numerical long division might look like this:

$$4\,\overline{)\,11}\;\;\frac{2}{}\;+\;\frac{3}{4}$$

Written using fraction notation, this would be

$$\frac{11}{4} = 2 + \frac{3}{4}$$

which indicates that 4 goes into 11 twice (the quotient is 2) with a remainder of 3, or, put another way, 4 goes into 11 two and three-quarter times.

Polynomial long division uses the same framework as numerical long division. For now, let's walk through the process of long division by dividing $x^3 - 2x + 4$ by $x + 2$ (most of the time on the test, long division and the Remainder Theorem will be applicable when dividing by binomial first degree polynomials like $x + 2$, which play an important role in factoring and finding zeros).

In order to start the process, let's place the **divisor**, $x + 2$, outside of the division bracket, and the **dividend**, $x^3 - 2x + 4$, inside of the division bracket. When you are performing long division with a dividend that has any degree of $x$ missing, you should hold its place inside the division bracket by adding the term with a coefficient of 0 or by leaving extra space to indicate the position of the zero term so you can line up the quotient (answer) terms with those of the same degree in the dividend. For example, since the dividend is $x^3 - 2x + 4$, you should also include the term $0x^2$ and write the dividend polynomial as $x^3 + 0x^2 - 2x + 4$.

$$x + 2 \overline{\big)\ x^3 + 0x^2 - 2x + 4}$$

The first step is seeing how many times the leading term, $x$, of the divisor, $x + 2$, goes into the first term of the dividend, $x^3 + 0x^2 - 2x + 4$. Since $x$ goes into the leading term, $x^3$, $x^2$ times (since $x$ times $x^2$ equals $x^3$), write $x^2$ above the $x^2$ term of the dividend; the placeholder term $0x^2$ that we inserted into the dividend keeps things aligned and minimizes the chances of an error.

$$\begin{array}{r} \mathbf{x^2} \\ x + 2 \overline{\big)\ x^3 + 0x^2 - 2x + 4} \end{array}$$

Multiply the divisor, $x + 2$, by $x^2$, then subtract the product from the dividend, $x^3 + 0x^2 - 2x + 4$, carrying down only the result of the subtraction and the subsequent term from the dividend. Note that because you chose the quotient term $x^2$ so that when multiplied by $x$, it would equal the first term, $x^3$, of the dividend, it's therefore not strictly necessary to perform that part of the multiplication and subsequent subtraction, because you already know that the first term will drop out ($x^3 - x^3 = 0$); nonetheless, those redundant steps do serve as a useful double-check on your work.

$$\begin{array}{rl} & \quad\ \ x^2 \\ x + 2 & \overline{\big)\ x^3 + 0x^2 - 2x + 4} \\ & \underline{-(\mathbf{x^3 + 2x^2})} \quad \Downarrow \\ & \qquad\quad \mathbf{-2x^2 - 2x} \end{array}$$

Next, see how many times $x$ (again, the leading term of the divisor) goes into $-2x^2$ (the result of the subtraction). Since $x$ goes into $-2x^2$ exactly $-2x$ times, write $-2x$ above the $x$ term of the dividend.

$$\begin{array}{rl} & \quad\ \ x^2 - \mathbf{2x} \\ x + 2 & \overline{\big)\ x^3 + 0x^2 - 2x + 4} \\ & \underline{-(x^3 + 2x^2)} \\ & \qquad\quad -2x^2 - 2x \end{array}$$

Multiply the divisor, $x + 2$, by $-2x$ then subtract the result from $-2x^2 - 2x$ (be very careful about distributing the negative sign) and carry down the next (and final) term from the dividend.

$$\begin{array}{rl} & \quad\ \ x^2 - 2x \\ x + 2 & \overline{\big)\ x^3 + 0x^2 - 2x + 4} \\ & \underline{-(x^3 + 2x^2)} \quad \Downarrow \\ & \qquad\quad -2x^2 - 2x \quad \Downarrow \\ & \qquad\quad \underline{-(\mathbf{-2x^2 - 4x})} \quad \Downarrow \\ & \qquad\qquad\qquad \mathbf{2x + 4} \end{array}$$





Since $x$ goes into $2x$ twice, write $+2$ above the constant term of the dividend.

$$\begin{array}{rl} & \quad\ \ x^2 - 2x + \mathbf{2} \\ x + 2 & \overline{\big)\ x^3 + 0x^2 - 2x + 4} \\ & \underline{-(x^3 + 2x^2)} \\ & \qquad\quad -2x^2 - 2x \\ & \qquad\quad \underline{-(-2x^2 - 4x)} \\ & \qquad\qquad\qquad 2x + 4 \end{array}$$

Multiply $x + 2$ by 2 and subtract the product from $2x + 4$.

$$\begin{array}{rl} & \quad\ \ x^2 - 2x + 2 \\ x + 2 & \overline{\big)\ x^3 + 0x^2 - 2x + 4} \\ & \underline{-(x^3 + 2x^2)} \\ & \qquad\quad -2x^2 - 2x \\ & \qquad\quad \underline{-(-2x^2 - 4x)} \\ & \qquad\qquad\qquad 2x + 4 \\ & \qquad\qquad\qquad \underline{\mathbf{-(2x + 4)}} \\ & \qquad\qquad\qquad\qquad \mathbf{0} \end{array}$$

Notice that this final subtraction results in a 0. This 0 indicates that there is no remainder, which means that $x + 2$ divides evenly into $x^3 - 2x + 4$. Here's how this division operation looks when written using fraction notation:

$$\frac{x^3 - 2x + 4}{x + 2} = x^2 - 2x + 2$$

This means that both the divisor $x + 2$ and the quotient $x^2 - 2x + 2$ are factors of $x^3 - 2x + 4$. This can be made more obvious by multiplying both sides of the above equation by the divisor $x + 2$, yielding

$$x^3 - 2x + 4 = (x + 2)(x^2 - 2x + 2)$$

To verify this, we can multiply these two factors to show that their product is $x^3 - 2x + 4$.

$$\begin{aligned} &(x + 2)(x^2 - 2x + 2) \\ &x\left(x^2\right) + x(-2x) + x(2) + 2\left(x^2\right) + 2(-2x) + 2(2) \\ &x^3 - 2x^2 + 2x + 2x^2 - 4x + 4 \\ &x^3 \cancel{-2x^2} + 2x + \cancel{2x^2} - 4x + 4 \\ &x^3 + 2x - 4x + 4 \\ &x^3 - 2x + 4 \end{aligned}$$

To reiterate: when the division was complete, the remainder was 0 because $x + 2$ was a factor of $x^3 - 2x + 4$; that is, when a polynomial is divided by one of its factors, the result is just another factor—no remainder is produced.

## The Steps of Polynomial Long Division

1. See how many times the leading term of the divisor goes into the $x$-term of the greatest remaining power in the dividend.
2. Write that term on top of the division symbol directly above the dividend term of matching degree as part of the resulting quotient.
3. Multiply the divisor by that term and subtract it from the dividend (remember to distribute the negative sign).
4. Carry down the next term in the dividend.
5. Repeat the process seeing how many times the leading term of the divisor goes into the first term of what remains of the dividend as a result of the previous subtraction step.
6. Stop when only a constant is left. That constant is the remainder.

Before you practice this skill, walk yourself through the same steps on the following long divisions (all of which also have no remainder (the remaining constant is 0). The repeated process is always the same:

$$\frac{x^2+8x+15}{x+3}$$

$$\begin{array}{r|l} & \quad x \quad +5 \\ \hline x+3 & x^2+8x+15 \\ & \underline{-(x^2+3x)} \quad \Downarrow \\ & \qquad 5x+15 \\ & \qquad \underline{-(5x+15)} \\ & \qquad\qquad 0 \end{array}$$

So

$$x^2+8x+15=(x+3)(x+5)$$

$$\frac{2x^2-7x-4}{2x+1}$$

$$\begin{array}{r|l} & \quad x \quad -4 \\ \hline 2x+1 & 2x^2-7x-4 \\ & \underline{-(2x^2+x)} \quad \Downarrow \\ & \qquad -8x-4 \\ & \qquad \underline{-(-8x-4)} \\ & \qquad\qquad 0 \end{array}$$

So

$$2x^2-7x-4=(2x+1)(x-4)$$

$$\frac{9x^3+23x-18}{3x-2}$$

$$\begin{array}{r|l} & \quad 3x^2 \quad +2x \quad +9 \\ \hline 3x-2 & 9x^3+0x^2+23x-18 \\ & \underline{-(9x^3-6x^2)} \quad \Downarrow \quad \Downarrow \\ & \qquad 6x^2+23x \quad \Downarrow \\ & \qquad \underline{-(6x^2-4x)} \quad \Downarrow \\ & \qquad\qquad 27x-18 \\ & \qquad\qquad \underline{-(27x-18)} \\ & \qquad\qquad\qquad 0 \end{array}$$

So

$$9x^3+23x-18=(3x-2)(3x^2+2x+9)$$





**SkillDrill 9.4-1**

**Directions:** Perform Polynomial Long Division in order to find the quotient of the following 0-remainder divisions. Remember to insert placeholder terms into the dividend as needed.

**1.** $\dfrac{x^2-7x+12}{x-3}$

**2.** $\dfrac{x^2+3x-10}{x-2}$

**3.** $\dfrac{x^3+6x^2-2x-7}{x+1}$

**4.** $\dfrac{x^3-5x-12}{x-3}$

**5.** $\dfrac{x^4-4x^3-2x^2+12x-16}{x-4}$

**6.** $\dfrac{4x^3+6x^2+8x+12}{2x+3}$

### Polynomial Division with a Remainder

Remember that factoring polynomials allowed us to find zeros of those polynomials. Let's look back to our previous example: we divided $x^3-2x+4$ by $x+2$ and found that the remainder was 0, which also means that $x+2$ is a factor of $x^3-2x+4$. The fact that $x+2$ is a factor of $x^3-2x+4$ indicates that $x=-2$ is a zero of the polynomial because when $x=-2$, $x+2=0$, and therefore the entire polynomial will equal 0.

If we plug $-2$ in for $x$ in the equation $y=x^3-2x+4$, we can verify that the $y$-value is 0 (this is obvious if we write $x^3-2x+4$ in Factored Form, $y=(x+2)(x^2-2x+2)$, because the factor $x+2$ would be equal to 0).

$$\begin{aligned} y &= x^3-2x+4 \\ y &= (-2)^3-2(-2)+4 \\ y &= -8+4+4 \\ y &= 0 \end{aligned}$$

Let's try dividing $x^3 - 2x + 4$ by another binomial, $x - 1$, to see what happens and what to do with nonzero remainders. The process is exactly the same as the one above, but there will be a nonzero constant left over once we run out of $x$-terms in the dividend.

$$
\begin{array}{r|rrrr}
 & & x^2 & +x & -1 \\
\hline
x-1 & x^3 & +0x^2 & -2x & +4 \\
 & -(x^3 & -x^2) & \Downarrow & \Downarrow \\
\hline
 & & x^2 & -2x & \Downarrow \\
 & & -(x^2 & -x) & \Downarrow \\
\hline
 & & & -x & +4 \\
 & & & -(-x & +1) \\
\hline
 & & & & 3
\end{array}
$$

When we arrive at this step, we cannot proceed any further as the $x$-term from the divisor does not divide evenly into 3 (technically, $x$ goes into 3 a total of $\frac{3}{x}$ times, but that is not helpful to us), so 3 is the remainder, and we can represent the division in fraction form.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$\frac{x^3 - 2x + 4}{x - 1} = x^2 + x - 1 + \frac{3}{x - 1}$$

**SkillDrill 9.4-2**

**Directions:** Perform Polynomial Long Division in order to write the result of the divisions in the form

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{x^2 + 3x + 5}{x + 2}$

2. $\dfrac{x^2 - x - 7}{x + 2}$

3. $\dfrac{x^2 - 7x + 13}{x - 3}$

4. $\dfrac{3x^2 + 11x - 9}{3x - 1}$

5. $\dfrac{x^3 - 5x^2 + 10x - 17}{x - 3}$

6. $\dfrac{2x^3 - x^2 - 7x - 1}{2x + 1}$





## Polynomial Remainder Theorem

Let's look at polynomial division symbolically alongside the example we used above.

Remember, this is the symbolic representation of long division:

$$\text{divisor}\,\overline{)\,\text{dividend}}^{\,\text{quotient}} + \frac{\text{remainder}}{\text{divisor}}$$

In our example, here's how that looks:

$$x - 1\,\overline{)\,x^3 + 0x^2 - 2x + 4}^{\,x^2 + x - 1} + \frac{3}{x - 1}$$

There's nothing fancy going on here; it's just division with a remainder.

Let's rewrite those two representations as equations using fraction notation for the division operation so that we can do a little algebra:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$\frac{x^3 - 2x + 4}{x - 1} = x^2 + x - 1 + \frac{3}{x - 1}$$

Now, let's rearrange things so the dividend is isolated; we'll simply multiply both sides by the divisor, which is $x - 1$:

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$
$$x^3 - 2x + 4 = (x - 1)(x^2 + x - 1) + 3$$

At this point, we can observe that **if the divisor equals 0, the dividend will equal the remainder**, because the first term on the right side will drop out:

$$\text{dividend} = (0)(\text{quotient}) + \text{remainder} = \text{remainder}$$
$$x^3 - 2x + 4 = (0)(x^2 + x - 1) + 3 \qquad = 3$$

This means that **we can easily find the remainder of a division operation by simply finding the value of the variable that makes the divisor equal 0**. In our example, the divisor $x - 1$ will equal 0 when $x = 1$ because $1 - 1 = 0$. That means that if we plug in 1 for $x$ in the dividend (represented here as a function), it should evaluate to 3; let's try it:

$$f(x) = x^3 - 2x + 4$$
$$f(1) = (1)^3 - 2(1) + 4$$
$$f(1) = 1 - 2 + 4$$
$$f(1) = 3$$

Generalizing, we can say that if we have a polynomial function $f(x)$ divided by a binomial $x - r$, $f(r)$ will evaluate to the remainder of the division. Symbolically, that means if we have a division of the form

$$\frac{f(x)}{x - r}$$

then

$$f(r) = \text{remainder}$$

In our example, we have our dividend as

$$f(x) = x^3 - 2x + 4$$

and our divisor is the binomial

$$x - 1$$

so

$$r = 1$$

Therefore, the remainder of

$$\frac{f(x)}{x-1}$$

is equal to $f(1)$, meaning

$$f(1) = \text{remainder} = 3$$

This represents the **Polynomial Remainder Theorem**, which is an exotic-sounding term for the principle explained above: **if we have a polynomial function $f(x)$ divided by a binomial $x - r$, $f(r)$ will evaluate to the remainder of the division**.

## Polynomial Remainder Theorem

**When a polynomial $f(x)$ is divided by a binomial, $x - r$, the remainder is equal to $f(r)$.**

This shows that **when $x - r$ is a factor of $f(x)$, then $x = r$ is a zero of $f(x)$.**

For example, if you are told that $x - 4$ is a factor of a polynomial, then you know that $x = 4$ is a zero of that polynomial.

Similarly, **if $f(r) = 0$** ($x = r$ is a zero of the function), **then $x - r$ is a factor of $f(x)$.**

For example, if you are told that $f(4) = 0$, then you know that $x - 4$ is a factor of the polynomial.

In a more general sense, **if you are told that the remainder of dividing $f(x)$ by $x - r$ is $R$, then you know that $f(r) = R$.**

For example, if dividing $f(x)$ by $x + 5$ results in a remainder of $-7$, then you know that $f(-5) = -7$.

Similarly, **if you are told that $f(r) = R$, then you know that the remainder of dividing $f(x)$ by $x - r$ is $R$.**

For example, if you are told that $f(-5) = -7$, then you know that the remainder of dividing $f(x)$ by $x + 5$ is $-7$.

---

On the test, the above definition is all you need to equip yourself to quickly answer the problems that are susceptible to rapid solving via the Remainder Theorem. For completeness, we will also explain the more general case where this theorem can be applied, though knowledge of this is not especially important for the test as presently constructed, as alternative solving methods, such as polynomial long division or factoring, are typically as good or better in situations where this more general version of the theorem applies.

**When a polynomial $f(x)$ is divided by a binomial, $ax - r$, the remainder is equal to $f\left(\frac{r}{a}\right)$.**

This shows that **when $ax - r$ is a factor of $f(x)$, then $x = \frac{r}{a}$ is a zero of $f(x)$.** Note that this means that $f(x)$ will equal





the remainder for the value of $x$ that makes the divisor $ax - r$ evaluate to 0; when $a = 1$, as in all the previous examples and discussions, we can simply use $r$ as the value of $x$ that makes the binomial evaluate to 0, but more generally, we need to use $\frac{r}{a}$.

For example, if you are told that $2x - 1$ is a factor of a polynomial, then you know that $x = \frac{1}{2}$ is a zero of that polynomial.

Similarly, **if $f\left(\frac{r}{a}\right) = 0$** ($x = \frac{r}{a}$ is a zero of the function), **then $ax - r$ is a factor of $f(x)$.**

For example, if you are told that $f\left(\frac{-2}{3}\right) = 0$, then you know that $3x + 2$ is a factor of the polynomial.

In a more general sense, **if you are told that the remainder of dividing $f(x)$ by $ax - r$ is $R$, then you know that $f\left(\frac{r}{a}\right) = R$.**

For example, if dividing $f(x)$ by $3x + 5$ results in a remainder of $-7$, then you know that $f\left(\frac{-5}{3}\right) = -7$.

Similarly, **if you are told that $f\left(\frac{r}{a}\right) = R$, then you know that the remainder of dividing $f(x)$ by $ax - r$ is $R$.**

For example, if you are told that $f\left(\frac{-5}{3}\right) = -7$, then you know that the remainder of dividing $f(x)$ by $3x + 5$ is $-7$.

Awareness of the Polynomial Remainder Theorem allows you to handle questions that deal with polynomial long division and remainders by plugging values into functions rather than actually performing long division.

### SkillDrill 9.4-3

**Directions:** Use the Polynomial Remainder Theorem to find the remainder when $f(x)$ is divided by $x - r$. The theorem states that the remainder will be $f(r)$.

Challenge problems are boxed.

1. $\frac{x^2 + 3x + 6}{x + 1}$
2. $\frac{x^2 - 2x + 1}{x - 4}$
3. $\frac{x^2 - 3x - 5}{x - 5}$
4. $\frac{x^2 + x - 6}{x - 2}$
5. $\frac{x^3 - 2x^2 + 3}{x - 2}$
6. $\frac{x^3 - 5x - 3}{x + 2}$
7. $\frac{x^2 + 3x - 20}{x - 1}$
8. $\boxed{\frac{x^2 + 4x + 2}{2x + 1}}$

For the following example, we will use two different solutions. The first solution makes use of Polynomial Remainder Theorem; the second solution employs long division.

**Example 9.4-9**

**9** 1600.io 9.4

Which of the following expressions is equivalent to

$$\frac{x^2-3x-10}{x-4}\ ?$$

A) $x-7-\frac{38}{x-4}$

B) $x-7-\frac{18}{x-4}$

C) $x+1-\frac{14}{x-4}$

D) $x+1-\frac{6}{x-4}$

**Solution 1**

This solution requires some thought up front, but by far has the fewest steps involving copying down terms correctly, making it the most efficient method for this problem.

1. For this solution, we will use the truths related to the Polynomial Remainder Theorem to choose the answer choice that shows the correct remainder. Notice that this problem involves dividing a polynomial $f(x)$, where $f(x)=x^2-3x-10$, by a polynomial of the form $x-r$, where $r=4$, and all of the answer choices are written in the form

$$\text{quotient}+\frac{\text{remainder}}{\text{divisor}}$$

where **the value of the remainder is different in each answer choice, so if we can simply determine the remainder, we can select the correct answer choice**—we don't even need to figure out what the quotient is.

We know from Polynomial Remainder Theorem that $f(r)$ is equal to the remainder, so if we plug in 4 for $x$ in the expression $x^2-3x-10$, we will get the correct value of the remainder.

2. Plug in 4 for $x$ in the expression $x^2-3x-10$.

$$f(x)=x^2-3x-10$$
$$f(4)=(4)^2-3(4)-10$$
$$f(4)=16-12-10$$
$$f(4)=-6$$

3. Since for $f(x)=x^2-3x-10$, the value of $f(4)$ is $-6$, we know that the remainder when we divide $x^2-3x-10$ by $x-4$ will be $-6$, which is only found in one of the answer choices.

4. The answer is D.





**Solution 2**

1. Use polynomial long division to find the result of dividing $x^2-3x-10$ by $x-4$.

$$\begin{array}{r|l} & x\ \ +1 \\ \hline x-4 & x^2-3x\ -10 \\ & -(x^2-4x)\ \ \Downarrow \\ \hline & x\ -10 \\ & -(x\ \ -4) \\ \hline & -6 \end{array}$$

2. Since the result is $x+1$ (shown above the division bracket) with a remainder of $-6$, we know that the following equation is true.

$$\frac{x^2-3x-10}{x-4}=x+1-\frac{6}{x-4}$$

3. The answer is D.

Long division and Polynomial Remainder Theorem are not always the best ways to do polynomial division problems. The following is an example of yet another variation on the idea of matching coefficients. Simply group terms based on common denominators, and then cross multiply. Match the coefficients once both sides of the equation are in the same form.

**Example 9.4-10**

**10** 1600.io 9.4

The equation $\frac{-12x^2+28x-44}{ax-4}=-6x+2-\frac{36}{ax-4}$ is true for all values of $x\neq\frac{4}{a}$, where $a$ is a constant. What is the value of $a$ ?

A) $-6$

B) $-2$

C) $2$

D) $6$

**Solution 1**

1. In order for the first term of the quotient to be $-6x$, which is the result of dividing $-12x^2$ (the first term of the dividend) by $ax$ (the first term of the divisor), the value of $a$ must be 2.

$$\frac{-12x^2}{ax} = -6x$$
$$-12x^2 = (-6x)(ax)$$
$$-12\cancel{x^2} = -6a\cancel{x^2}$$
$$\frac{-12}{-6} = a$$
$$2 = a$$

2. The answer is C.

**Solution 2**

1. Group the terms with the denominator $ax - 4$ on the left side of the equation.

$$\frac{-12x^2 + 28x - 44}{ax - 4} = -6x + 2 - \frac{36}{ax - 4}$$

$$\frac{-12x^2 + 28x - 44}{ax - 4} + \frac{36}{ax - 4} = -6x + 2$$

$$\frac{-12x^2 + 28x - 8}{ax - 4} = -6x + 2$$

2. Multiply both sides of the equation by $ax - 4$.

$$\frac{-12x^2 + 28x - 8}{ax - 4} = -6x + 2$$

$$-12x^2 + 28x - 8 = (-6x - 2)(ax - 4)$$

3. You can distribute if you like, but really we are only concerned about the coefficients of the $x^2$ terms. Since $-6x$ times $ax$ has to be equal to $-12x^2$, the value of $a$ must be 2.

$$-12x^2 = (-6x)(ax)$$
$$-12x^2 = -6ax^2$$
$$-12 = -6a$$
$$2 = a$$

4. The answer is C.

**Notes**

The more comfortable you get with the topics covered in this chapter, the easier it is to think of faster and faster methods for handling these problems, most of which can be done in quite a number of ways.





**Section 9.4 Suggested Problems from Real Tests**

- Test 1-C-29
- Test 2-NC-15
- Test 3-NC-13
- Test 4-C-25
- Apr 2019-C-34
- Mar 2020-C-14
- Oct 2020-C-23

View related real-test problems in SMTEx

## Section 9.4 Practice Problems

**1** 1600.io 9.4

$$\frac{x^2 + 20x + 99}{x + 11}$$

If the expression above is equivalent to an expression of the form $x + a$, where $x \neq -11$, what will be the value of $a$ ?

**2** 1600.io 9.4

The expression $\frac{4x - 1}{x + 6}$ is equivalent to which of the following?

A) $\frac{4 - 1}{6}$

B) $4 - \frac{1}{6}$

C) $4 - \frac{1}{x + 6}$

D) $4 - \frac{25}{x + 6}$

## 9.5 Solving in Terms of Variables

This section covers problems that involve solving equations with multiple variables or unknown constants for one variable or unknown constant in terms of the others. The equations in these problems can be of any form (not just linear or quadratic) and may involve squaring or taking square roots of both sides in scientific equations. Essentially, this boils down to performing algebraic steps in the correct order to rearrange a provided equation as required, but you will not be bothered with the properties of numbers or polynomials at all, and you don't have to worry about plugging anything in or evaluating numbers. These problems are much easier than anything else in this chapter as long as you simply unwrap the equations one step at a time.

The equation $p = \frac{2}{3}t$ contains two unknowns and is currently solved for $p$ in terms of $t$. If we want to solve for $t$ in terms of $p$ we would just multiply both sides of the equation by $\frac{3}{2}$ in order to isolate $t$: $\frac{3}{2}p = t$. Just to be sure you understand what just happened, we could have multiplied both sides by the reciprocal of $\frac{2}{3}$ (as we did), we could have divided both sides by $\frac{3}{2}$, or we could have performed the two separate steps of multiplying both sides by 3 and dividing both sides by 2 (in either order); these are all equivalent in their effect.

Let's look at a slightly more complicated equation with multiple unknown values.

$$A = \frac{B + 3C + D}{5}$$

If we wanted to solve for $C$ in terms of the other unknowns ($A$, $B$, and $D$), we should start by multiplying both sides of the equation by 5 because the $C$ is currently grouped into the numerator and this step would allow us to isolate the numerator group first.

$$5A = B + 3C + D$$

Now that the numerator group is isolated, the term $3C$ is the next grouping that $C$ belongs to, so subtract $B$ and $D$ from both sides of the equation.

$$5A - B - D = 3C$$

Finally, we can solve for $C$ in terms of the other unknowns by dividing both sides of the equation by 3.

$$\frac{5A - B - D}{3} = C$$

Similarly, we can solve an equation for an *expression* rather than a single term as in the example below.

**Example 9.5-11**

**11** 9.5

$$\text{James's formula: } A = \frac{\sqrt{\ell w}}{10}$$

$$\text{McNeil's formula: } A = \frac{8 + w}{5}$$

The formulas above are used by cobblers to estimate the body surface area $A$, in square feet, of snakes whose weight $w$ ranges between 10 and 40 pounds and whose length $\ell$ is measured in inches.

If James's and McNeil's formulas give the same estimate for $A$, which of the following expressions is equivalent to $\sqrt{\ell w}$ ?

A) $\frac{8 + w}{2}$

B) $\frac{8 + w}{50}$

C) $\frac{(8 + w)^2}{2}$

D) $2(8 + w)$

**Solution**

Take note that we are looking to isolate the entire expression $\sqrt{\ell w}$, so be sure to stop when that expression is isolated.

1. The problem states that the two formulas give the same value of $A$ for a certain length and width, so set the two expressions from the right sides of the formulas equal to each other.

$$\frac{\sqrt{\ell w}}{10} = \frac{8 + w}{5}$$





2. Multiply both sides of the equation by 10 in order to isolate the expression $\sqrt{\ell w}$ on the left side of the equation and reduce the fraction.

$$\frac{\sqrt{\ell w}}{10} = \frac{8 + w}{5}$$

$$\sqrt{\ell w} = 10\left(\frac{8 + w}{5}\right)$$

$$\sqrt{\ell w} = 2(8 + w)$$

3. The answer is D.

**Notes**

Notice that we did not distribute the 2 in Step 2 because we already had a match to an answer choice.

If the term you need to solve for is "trapped" in a radical, you will need to raise both sides to a power (usually squaring both sides).

**Example 9.5-12**

**12** 9.5

$$T = \sqrt{\frac{4\pi^2 L}{g}}$$

The formula above is used to calculate the period, $T$, of a pendulum given the length, $L$, of the pendulum and the acceleration due to gravity, $g$. Which of the following correctly expresses the acceleration due to gravity in terms of the other variables?

A) $g = \sqrt{\frac{4\pi^2 L}{T}}$

B) $g = \frac{\sqrt{4\pi^2 L}}{T}$

C) $g = \frac{4\pi^2 L}{T^2}$

D) $g = \frac{T^2}{4\pi^2 L}$

**Solution**

1. We need to isolate the term $g$, the acceleration due to gravity, which is in the radical sign. Our first step should be to square both sides of the equation in order to "free the $g$ from the radical prison."

$$T = \sqrt{\frac{4\pi^2 L}{g}}$$

$$T^2 = \frac{4\pi^2 L}{g}$$

2. Multiply both sides of the equation by $g$ in order to get $g$ out from the denominator.

$$T^2 = \frac{4\pi^2 L}{g}$$

$$gT^2 = 4\pi^2 L$$

3. Divide both sides of the equation by $T^2$ in order to solve for $g$ in terms of the other variables.

$$gT^2 = 4\pi^2 L$$

$$g = \frac{4\pi^2 L}{T^2}$$

4. The answer is C.

In other cases, the term you want to solve for is raised to a power in the initial equation, so you will end up taking a root of both sides of the equation in order to solve for that term.

**Example 9.5-13**

**13** 1600.io 9.5

The volume of a cone is given by the formula $V = \frac{1}{3}hr^2$, where $r$ is the radius of the base of the cone and $h$ is the height of the cone. Which of the following gives the radius of the base of the cone in terms of the volume of the cone and the height of the cone?

A) $\frac{3V}{h}$

B) $\frac{h}{3V}$

C) $\sqrt{\frac{3V}{h}}$

D) $\sqrt{\frac{h}{3V}}$

**Solution**

1. We need to isolate the term $r$, the radius of the base of the cone, which is a squared term in the original equation. First we need to isolate the term $r^2$ on one side of the equation by itself. Start by multiplying both sides of the equation by 3 in order to get rid of the fraction.

$$V = \frac{1}{3}hr^2$$

$$3V = hr^2$$





2. Divide both sides of the equation by $h$ in order to isolate $r^2$.

$$3V = hr^2$$

$$\frac{3V}{h} = r^2$$

3. Finally, in order to solve for $r$ (which is currently raised to the second power), take the square root of both sides of the equation.

$$\frac{3V}{h} = r^2$$

$$\sqrt{\frac{3V}{h}} = r$$

4. The answer is C.

If the variable or unknown constant is found in multiple terms, you will need to **group those terms and then factor the variable or constant out of those terms**. For example, we can solve the equation $wx + wy = zy$ for $w$ in terms of $x$, $y$, and $z$. First factor the $w$ from both terms on the left side of the equation: $w(x + y) = zy$. All we have to do now is divide both sides of the equation by the expression $x + y$ in order to isolate $w$: $w = \frac{zy}{x + y}$.

If the variable or unknown constant you need to isolate is "trapped" in the denominator of a fraction, you should **multiply both sides of the equation by the denominator** in order to get rid of the fractions.

In the example below, there are multiple terms that contain the constant we want to solve for, and we have to multiply by a denominator in order to put all the terms on the same level.

**Example 9.5-14**

**14** 1600.io 9.5

$$R = \frac{U}{D + U}$$

James McNeil uses the formula above to calculate a snake photo's rating, $R$, based on the number of upvotes, $U$, and downvotes, $D$, the photo gets on social media. Which of the following expresses the number of upvotes in terms of the other variables?

A) $U = \frac{RD}{1 - R}$

B) $U = \frac{RD}{R - 1}$

C) $U = \frac{D}{1 - R}$

D) $U = \frac{D}{R - 1}$

**Solution**

1. Multiply both sides of the equation by the expression $D + U$ from the denominator of the fraction on the right side of the equation in order to free the variable $U$ from the denominator and get rid of fractions.

$$R = \frac{U}{D+U}$$
$$R(D+U) = U$$

2. Distribute $R$ on the left side of the equation.

$$R(D+U) = U$$
$$RD + RU = U$$

3. Subtract $RU$ from both sides of the equation in order to group the terms with $U$.

$$RD + RU = U$$
$$RD = U - RU$$

4. Factor $U$ from both terms on the right side of the equation in order to isolate $U$.

$$RD = U - RU$$
$$RD = U(1 - R)$$

5. Divide both sides of the equation by $1 - R$ in order to isolate and solve for $U$ in terms of the other variables.

$$RD = U(1 - R)$$
$$\frac{RD}{1-R} = U$$

6. The answer is A.

## Section 9.5 Suggested Problems from Real Tests

- Test 1-NC-7
- Test 1-C-9
- Test 2-NC-12
- Test 2-C-22
- Test 3-C-13
- Test 4-NC-6
- Test 5-C-10
- Test 6-NC-7
- Test 6-C-17
- Test 7-C-19
- Test 7-C-20
- Apr 2018-NC-9
- Mar 2019-NC-11
- Apr 2019-NC-8
- Apr 2019-NC-10
- Apr 2019-C-16
- May 2019 (US)-NC-6
- 2019 (US)-C-36
- 2019 (Int)-NC-4
- 2020-NC-14
- 2020-NC-6
- Oct 2020-C-10
- Mar 2021-NC-16
- May 2021 (US)-NC-8
- May 2021 (Int)-NC-4

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SMTEx





## Section 9.5 Practice Problems

**1** 1600.io 9.5

$$\frac{1}{r}mv^2 = ma$$

Uniform circular motion is represented by the equation above, where $m$ represents mass, $r$ represents the radius, $v$ represents the velocity, and $a$ represents the acceleration. According to the equation for uniform circular motion, which of the following is equivalent to the velocity, $v$?

A) $ra$

B) $\frac{1}{r}am^2$

C) $\sqrt{ra}$

D) $\sqrt{\frac{1}{r}ma}$

**2** 1600.io 9.5

James McNeil uses the formula $n = 10\ell w$ to estimate the number of snakes, $n$, needed to fill a snakepit that is $\ell$ feet long and $w$ feet wide. Which of the following correctly expresses $w$ in terms of $n$ and $\ell$?

A) $w = \frac{10}{n\ell}$

B) $w = \frac{n}{10\ell}$

C) $w = \frac{\ell}{10n}$

D) $w = \frac{n}{10+\ell}$

**3** 1600.io 9.5

$$A = 6s^2$$

The formula above gives the surface area, $A$, of a cube in terms of the length of its sides, $s$. Which of the following gives the length of the sides of the cube in terms of its surface area?

A) $s = \sqrt{\frac{A}{6}}$

B) $s = \sqrt{\frac{6}{A}}$

C) $s = \frac{\sqrt{A}}{6}$

D) $s = \frac{\sqrt{6}}{A}$

**4** 1600.io 9.5

Sludge volume, $V_S$, in a pond can be determined using the equation $V_S = \frac{V_P}{1-W}$, where $V_P$ is the poison volume and $W$ is the hydrocrit (the fraction of sludge volume that is water). Which of the following correctly expresses the hydrocrit in terms of the sludge volume and the poison volume?

A) $W = V_S - V_P$

B) $W = \frac{V_P}{V_S}$

C) $W = 1 + \frac{V_P}{V_S}$

D) $W = 1 - \frac{V_P}{V_S}$

## CHAPTER 9 RECAP

- Most of the problems that are covered in this chapter have several completely valid solution methods, and any one method will not be the fastest in all cases. It is important to note the benefits and drawbacks of each, and it is better to use a method in which you're confident over one that may be faster but that requires a greater "leap of faith" for you.
- When a given equation has an expression in a radical sign, move everything to the other side of the equation before squaring both sides of the equation to eliminate the radical.
- When squaring both sides of an equation, extraneous solutions are often created. Remember that even though both $x = 2$ and $x = -2$ are valid solutions to the equation $x^2 = 4$, when an equation is given to you with an existing square root, the value of that square root, by definition, is non-negative (this is also true for all other even roots).
- For questions that ask you to pick the correct solution set for an equation containing a radical, you will need to double check any solutions you find algebraically by plugging those solutions back into the equation (unless you are using the sophisticated U-Substitution solving approach). Because of this need to check answers on these problems, you are usually better off not working algebraically on these problems. You will save time by checking the answer choices right away (it is rare that plugging in is the preferred solution method, so revel in it).
- Rational functions consist of one polynomial being divided by another. For some questions, you may need to combine terms through addition or subtraction. To do so, you may need to write terms so that they have common denominators.
- In equations with rational functions, you may be able to make quick assessments based on the Polynomial Remainder Theorem to determine what the remainder would be. If the remainders in the answer choices are not unique or if the leading coefficient of the divisor is not known, you may be able to discern based on the first term of the quotient what the coefficient of the divisor must be.

  If you are not comfortable with these methods, you should combine any terms that share a common denominator before doing anything else. After that, instead of making common denominators on terms that don't already share them, simply multiply both sides of the equations by the different denominators of individual expressions in order to eliminate fractions. In this process, you may end up finding solutions that would have caused the denominator of one or more of the terms in the given equation to be equal to 0 (causing an undefined value), so make sure to double check that any solutions you find will not cause this problem.
- The value of a rational function is undefined when the denominator is equal to 0. Use factoring skills to determine what values of $x$ will cause the denominator to be 0.
- Polynomial Long Division is rarely necessary (there are always multiple routes through a problem), but you should have a handle on the process just in case. In part, we presented Polynomial Long Division to set up the explanation of the Polynomial Remainder Theorem, a tool that can often be used to solve problems more easily than can long division. Another technique for performing polynomial division is called Synthetic Division, which we have not presented herein, but if you're interested, you can easily find explanations of it through the Google machine.
- The Polynomial Remainder Theorem states that when a polynomial $f(x)$ is divided by a binomial $x - r$, the remainder of the division, $R$, is equal to $f(r)$. Therefore, when $f(r) = 0$, $f(x)$ is divisible by $x - r$. For example, if we wanted to check if $f(x)$ is divisible by $x - 4$, we could plug 4 into $f(x)$. If $f(4) = 0$, then $f(x)$ is divisible by $x - 4$.

  In cases where the divisor is of the form $ax - r$, the Polynomial Remainder Theorem states that when a polynomial $f(x)$ is divided by a binomial $ax - r$, the remainder of the division, $R$, is equal to $f\left(\frac{r}{a}\right)$. Therefore, when $f\left(\frac{r}{a}\right) = 0$, $f(x)$ is divisible by $ax - r$. For example, if we wanted to check if $f(x)$ is divisible by $2x - 3$, we could plug $\frac{3}{2}$ into $f(x)$. If $f\left(\frac{3}{2}\right) = 0$, then $f(x)$ is divisible by $2x - 3$.





- In science or geometry equations, you often need to solve for one variable in terms of the others.

  If the variable you need to solve for is "trapped" in a radical sign, start by moving everything outside of the radical to the other side of the equation before you raise both sides of the equation to any power to eliminate the radical. Luckily, you do not need to worry about extraneous solutions to these when you square both sides of an equation.

  Similarly, if the variable you are solving for is raised to a power in the given equation, move everything else to the other side of the equation before taking a root of both sides.

  If there are multiple terms containing the variable you need to solve for, group those terms and factor the variable out from those terms.

  If the variable you are solving for is in the denominator, multiply both sides of the equation by the denominator to "free" the variable from the denominator.

## Additional Problems

**1** 1600.io 9.1

$$\sqrt{9x} = x - 4$$

What are all values of $x$ that satisfy the equation above?

I. 1

II. 16

A) I only
B) II only
C) I and II
D) Neither I nor II

**2** 1600.io 9.1

$$\sqrt{x+62} - 4\sqrt{x+2} = 0$$

What value of $x$ satisfies the equation above?

A) 2
B) 4
C) 18
D) 20

**3** 1600.io 9.1

What is the set of all solutions to the equation $\sqrt{x+42} = -x$ ?

A) $\{-6, 7\}$
B) $\{-6\}$
C) $\{7\}$
D) There are no solutions to the given equation.

**4** 1600.io 9.4

In the $xy$-plane, what is the $y$-coordinate of the $y$-intercept of the graph of the equation $y = \frac{5x+20}{x-4}$ ?

A) $-5$
B) $-4$
C) $-1$
D) 4





**5** 1600.io 9.2

$$\frac{x}{x-10} = \frac{4x}{4}$$

Which of the following represents all the possible values of $x$ that satisfy the equation above?

A) 0 and 9
B) 0 and 11
C) 9 and 11
D) 11

**6** 1600.io 9.1

$$x - 3 = \sqrt{x-1}$$

Which of the following values of $x$ is a solution to the equation above?

A) $-2$
B) 2
C) 5
D) 9

**7** 1600.io 9.2

If $x > 7$, which of the following is equivalent to $\dfrac{1}{\frac{1}{x+5} + \frac{1}{x+7}}$ ?

A) $2x + 12$

B) $x^2 + 12x + 35$

C) $\dfrac{2x+12}{x^2+12x+35}$

D) $\dfrac{x^2+12x+35}{2x+12}$

**8** 1600.io 9.3

$$h(x) = \frac{1}{(x-3)^2 + 6(x-3) + 9}$$

For what value of $x$ is the function $h$ above undefined?

**9** 1600.io 9.1

$$\sqrt{3k^2+4} - x = 0$$

If $k > 0$ and $x = 4$ in the equation above, what is the value of $k$ ?

A) 0

B) 2

C) 4

D) 6

**10** 1600.io 9.4

Which of the following is equivalent to $\frac{5x^2+9x}{5x+4}$ ?

A) $x$

B) $x+5$

C) $x - \frac{4}{5x+4}$

D) $x + 1 - \frac{4}{5x+4}$

**11** 1600.io 9.1

$$\sqrt{x} + 3 = 10$$

Which of the following is the solution to the equation above?

A) 7

B) 13

C) 49

D) 97

**12** 1600.io 9.4

$$f(x) = 4x^3 + 10x^2 + 6x$$
$$g(x) = 2x^2 + 5x + 3$$

The polynomials $f(x)$ and $g(x)$ are defined above. Which of the following polynomials is divisible by $2x+5$ ?

A) $h(x) = f(x) + g(x)$

B) $k(x) = f(x) + 5g(x)$

C) $m(x) = 2f(x) + 5g(x)$

D) $p(x) = 5f(x) + 2g(x)$





**13** 1600.io 9.4

For a polynomial $p(x)$, the value of $p(6)$ is 4. Which of the following must be true about $p(x)$ ?

A) $x - 4$ is a factor of $p(x)$.

B) $x - 2$ is a factor of $p(x)$.

C) $x + 4$ is a factor of $p(x)$.

D) The remainder when $p(x)$ is divided by $x - 6$ is 4.

**14** 1600.io 9.1

$$\sqrt[4]{x^4} = x$$

Which of the following values of $x$ is NOT a solution to the equation above?

A) $-2$

B) 0

C) 1

D) 2

**15** 1600.io 9.1

If $r > 0$ and $\sqrt[3]{\frac{25r}{2}} = \frac{1}{2}r$, what is the value of $r$ ?

**16** 1600.io 9.4

$$\frac{x^2 - 2c}{x - b}$$

In the expression above, $b$ and $c$ are positive integers. If the expression is equivalent to $x + b$ and $x \neq b$, which of the following could be the value of $c$ ?

A) 1

B) 2

C) 4

D) 16

**17** 9.1

If $\sqrt{3x} = 6$, what is the value of $x$ ?

A) 2
B) 6
C) 12
D) 36

**18** 9.1

Which of the following is a solution to the equation $\sqrt{22 + 2x} + 1 = x$ ?

I. $-3$

II. 5

III. 7

A) I only
B) II only
C) III only
D) I and III

**19** 9.1

If $\sqrt{x - 3} = 9$, what is the value of $x$ ?

**20** 9.5

$$W = \frac{288x^2}{2}$$

In the equation above, the potential energy, $W$, of a spring with a spring constant of 288 units is given in terms of the distance the spring is stretched, $x$. If the equation is rewritten in the form $x = a\sqrt{W}$, where $a$ is a positive constant, what is the value of $a$ ?





**21** 9.1

$$\sqrt{x - 7} = 7 - \sqrt{x}$$

If $x$ is the solution to the equation above, what is the value of $\sqrt{x - 7}$ ?

A) $\sqrt{7}$

B) 3

C) 7

D) 16

**22** 9.2

$$\frac{1}{x^2 - 18x + 81} = 9$$

If $x$ is a solution to the given equation, which of the following is a possible value of $x - 9$ ?

A) $\frac{1}{3}$

B) $\frac{10}{3}$

C) $\frac{26}{3}$

D) $\frac{28}{3}$

**23** 9.1

$$x - 3\sqrt{x} - 4 = 0$$

What value of $x$ satisfies the equation above?

**24** 9.4

$$\frac{x - 5}{(x - 5)^2}$$

Which of the following expressions is equivalent to the given expression, where $x \neq 5$ ?

A) $x - 5$

B) $x^2 - 10x + 25$

C) $\frac{1}{x - 5}$

D) $\frac{1}{x^2 - 10x + 25}$

**25** 1600.io 9.1

If $\sqrt{x} + \sqrt{16} = \sqrt{49}$, what is the value of $x$ ?

A) $\sqrt{3}$

B) 3

C) 9

D) 33

**26** 1600.io 9.2

A group of coworkers decided to divide the $300 cost of a team-building retreat equally among themselves. When one of the coworkers decided not to go on the retreat, those remaining still divided the $300 cost equally, but each coworker's share of the cost increased by $10. How many coworkers were in the group originally?

**27** 1600.io 9.5

$$x = a\sqrt{1 - \frac{y^2}{b^2}}$$

For an ellipse centered at the origin with half-width $a$ and half-height $b$, the $x$-coordinate, $x$, of a point on the ellipse can be determined with the formula above, where $y$ is the $y$-coordinate of that point. The formula above expresses $x$ in terms of $a$, $b$, and $y$. Which of the following gives the $y$-coordinate of the point in terms of the other quantities?

A) $y = b\sqrt{1 + \frac{x^2}{a^2}}$

B) $y = b\sqrt{1 - \frac{x^2}{a^2}}$

C) $y = b^2\left(1 + \frac{x^2}{a^2}\right)^2$

D) $y = b^2\left(1 - \frac{x^2}{a^2}\right)^2$

**28** 1600.io 9.5

$$abd + bcd = bc - cd$$

In the equation above, $a$, $b$, and $c$ are each greater than 1. Which of the following is equivalent to $d$ ?

A) $ab + c$

B) $\frac{1}{a + 2c}$

C) $\frac{1}{abc}$

D) $\frac{bc}{ab + bc + c}$





# Answer Key

**Section 9.1 Practice Problems**

1. 147
2. D
3. C
4. D

**Section 9.2 Practice Problems**

1. A
2. 3

**Section 9.3 Practice Problems**

1. A
2. C

**SkillDrill 9.4-1**

1. $x - 4$
2. $x + 5$
3. $x^2 + 5x - 7$
4. $x^2 + 3x + 4$
5. $x^3 - 2x + 4$
6. $2x^2 + 4$

**SkillDrill 9.4-2**

1. $x + 1 + \frac{3}{x+2}$
2. $x - 3 - \frac{1}{x+2}$
3. $x - 4 + \frac{1}{x-3}$
4. $x + 4 - \frac{5}{3x-1}$
5. $x^2 - 2x + 4 - \frac{5}{x-3}$
6. $x^2 - x - 3 + \frac{2}{2x+1}$

**SkillDrill 9.4-3**

1. 4
2. 9
3. 5
4. 0
5. 3
6. −1
7. −16
8. $\frac{1}{4}$ or 0.25

**Section 9.4 Practice Problems**

1. 9
2. D

**Section 9.5 Practice Problems**

1. C
2. B
3. A
4. D

**Additional Problems**

1. B
2. A
3. B
4. A
5. B
6. C
7. D
8. 0
9. B
10. D
11. C
12. B
13. D
14. A
15. 10
16. B
17. C
18. C
19. 84
20. $\frac{1}{12}$ or .083
21. B
22. A
23. 16
24. C
25. C
26. 6
27. B
28. D

**25** 1600.io ? 9.1

If $\sqrt{x} + \sqrt{16} = \sqrt{49}$, what is the value of $x$ ?

A) $\sqrt{3}$

B) 3

C) 9

D) 33

**26** 1600.io ? 9.2

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**27** 1600.io ? 9.5

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C) $y = b^2\left(1 + \frac{x^2}{a^2}\right)^2$

D) $y = b^2\left(1 - \frac{x^2}{a^2}\right)^2$

**28** 1600.io ? 9.5

$$abd + bcd = bc - cd$$

In the equation above, $a$, $b$, and $c$ are each greater than 1. Which of the following is equivalent to $d$ ?

A) $ab + c$

B) $\frac{1}{a + 2c}$

C) $\frac{1}{abc}$

D) $\frac{bc}{ab + bc + c}$





# Answer Key

**Section 9.1 Practice Problems**

1. 147
2. D
3. C
4. D

**Section 9.2 Practice Problems**

1. A
2. 3

**Section 9.3 Practice Problems**

1. A
2. C

**SkillDrill 9.4-1**

1. $x - 4$
2. $x + 5$
3. $x^2 + 5x - 7$
4. $x^2 + 3x + 4$
5. $x^3 - 2x + 4$
6. $2x^2 + 4$

**SkillDrill 9.4-2**

1. $x + 1 + \frac{3}{x+2}$
2. $x - 3 - \frac{1}{x+2}$
3. $x - 4 + \frac{1}{x-3}$
4. $x + 4 - \frac{5}{3x-1}$
5. $x^2 - 2x + 4 - \frac{5}{x-3}$
6. $x^2 - x - 3 + \frac{2}{2x+1}$

**SkillDrill 9.4-3**

1. 4
2. 9
3. 5
4. 0
5. 3
6. $-1$
7. $-16$
8. $\frac{1}{4}$ or 0.25

**Section 9.4 Practice Problems**

1. 9
2. D

**Section 9.5 Practice Problems**

1. C
2. B
3. A
4. D

**Additional Problems**

1. B
2. A
3. B
4. A
5. B
6. C
7. D
8. 0
9. B
10. D
11. C
12. B
13. D
14. A
15. 10
16. B
17. C
18. C
19. 84
20. $\frac{1}{12}$ or .083
21. B
22. A
23. 16
24. C
25. C
26. 6
27. B
28. D

# The Graphs of Quadratic Equations and Polynomials

# 10

## 10.1 Standard Form: General Shape of Graphs

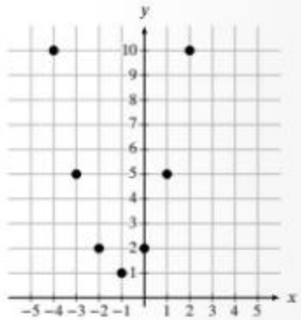
The graphs of quadratic equations (or functions) are called **parabolas**, and they are U-shaped graphs that have a maximum or minimum point. The **range** of a function is the set of all possible $y$-values that can be produced by a function (we'll see a different definition for range when we discuss statistics in a later chapter). For quadratics, the range is bounded on one end by the maximum or minimum $y$-value of the function at its vertex, and is unbounded (goes to infinity) on the other end.

Let's consider the quadratic functions $f(x) = x^2 + 2x + 2$ and $g(x) = -2x^2 + 8x - 8$. The tables below show the function values for several values of $x$. We will use the points given in the table to construct the graphs of these functions.

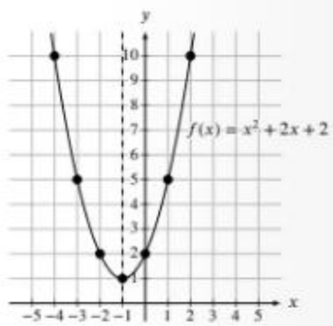
| $x$ | $y = f(x)$ | $y = g(x)$ |
|---|---|---|
| −4 | 10 | −72 |
| −3 | 5 | −50 |
| −2 | 2 | −32 |
| −1 | 1 | −18 |
| 0 | 2 | −8 |
| 1 | 5 | −2 |
| 2 | 10 | 0 |
| 3 | 17 | −2 |
| 4 | 26 | −8 |
| 5 | 37 | −18 |

You may notice that the series of values for each function shows a pattern in which the $y$-values reach a maximum or minimum and then begin repeating in reverse order. This is due to the fact that **parabolas are symmetric about their maximum or minimum points** (the maximum and minimum points for each function do appear in the table, as these two functions were chosen because their maximum and minimum points both have integer $x$-values), and we will learn more about this symmetry in the coming parts of this chapter when we discuss Factored Form and especially Vertex Form. This maximum or minimum point is called the **vertex** of the parabola because it's where the "axis of symmetry" line meets the parabola, just as the point in a triangle or other figure at which two lines meet is called a vertex. All parabolas are symmetrical—the graph on one side of the axis of symmetry is the mirror image of the graph on the other side.

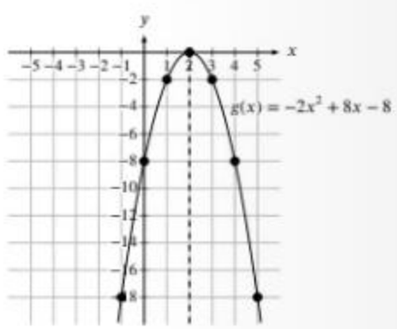
To see what the graphs of these equations look like, let's graph some of the points on each function. For $f(x)$ let's use the following points: $(-4, 10)$, $(-3, 5)$, $(-2, 2)$, $(-1, 1)$, $(0, 2)$, $(1, 5)$, and $(2, 10)$.




When we connect the points, we should use a smooth curve instead of a bunch of straight lines between the points, thus forming a U-shaped graph for $f(x)$. You should notice that the $y$-value reaches its lowest value at the point $(-1, 1)$, which is the vertex of the parabola, the point where the graph of $f(x)$ intersects the line of the axis of symmetry, $x = -1$ (notice that the graph is mirrored over this line). We will discuss later in this chapter methods of finding the lowest (or highest) points of the graphs of parabolas, but we would say that the **range** of this function is $y \geq 1$ because the vertex, at which the $y$-value is 1, is the lowest point on the graph, and there is no highest point because the $y$-values increase infinitely.




Let's do the same for $g(x)$ by graphing, and then connecting with a smooth curve, the following points: $(-1, -18)$, $(0, -8)$, $(1, -2)$, $(2, 0)$, $(3, -2)$, $(4, -8)$, and $(5, -18)$. (Note that the $y$-axis is on a different scale than the $x$-axis to keep the diagram compact.)








Once again, we see a U-shaped graph called a parabola, but in this case, the graph is pointed in the opposite direction, so that the vertex point, $(2, 0)$, represents the maximum $y$-value of the equation (the range is $y \leq 0$); again, the vertex point is where the graph of $g(x)$ intersects the line of the axis of symmetry, $x = 2$.

There are three main forms of quadratic equations that will be discussed in this chapter, and each one can be used to easily find key features of the graphs (and, conversely, the graphs can be used to easily form these equations based on which features and points are evident in the graph), but for now we are only interested in the Standard Form.

When quadratics are written in Standard Form, $y = ax^2 + bx + c$, each of the terms in the quadratic expression can be used to tell us some information about what the graph looks like. There are problems on the test that probe exactly this knowledge, and a solid grasp on these principles helps you make simple, rapid assessments about what a graph for a given quadratic equation will look like in general terms and what attributes the equation that models a particular graph will have.

Let's start with the leading $x^2$ terms of quadratic equations. Whether the graph opens upward or downward depends on the coefficient of the $x^2$ term. **Positive coefficients of the $x^2$ term make the graph open upwards, and negative coefficients of the $x^2$ term make the graph open downwards** (we saw this with our previous examples of $f(x) = x^2 + 2x + 2$ and $g(x) = -2x^2 + 8x - 8$).

This is because $x^2$ alone is always positive (or zero), so as $x$ becomes larger in absolute value (that is, more positive or more negative), $x^2$ becomes larger, too. Because $x^2$, or $x$ to the second power, literally has more "power" than the other terms of lower degree (smaller exponents), it is the $x^2$ term's value that dominates the value of the whole polynomial as $x$ gets sufficiently large in absolute value (very positive or very negative).

When the coefficient of the $x^2$ term is positive, this means that the quadratic expression's value, which represents $y$ when graphing, will become greater and greater as the value of $x$ increases on the right side of the vertex, and it *also* means that the value of the expression (the $y$-value) will become larger and larger as the value of $x$ decreases on the left side of that same vertex point. The result is that the graph of the parabola opens upwards and the vertex represents the minimum point.

The converse is true when the coefficient of the $x^2$ term is negative; that ensures that the value of the $x^2$ term is always negative (or zero), and the $y$-value will decrease on both sides of the vertex point, which produces a parabola that opens downward with the vertex representing the maximum point.

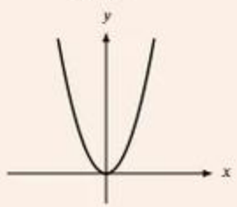
Similarly, the same $x^2$ coefficient determines how wide or narrow the graph looks. Coefficients with higher absolute values (bigger numbers, whether positive or negative) will make the graph stretch vertically (look narrower) because the $y$-values are changing more rapidly as the $x$-values change. Conversely, coefficients with lower absolute values will make the graph flatten vertically (look wider) because the $y$-values are changing more slowly as the $x$-values change.

## Direction of Parabola

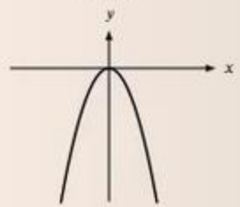
For a quadratic equation in Standard Form, $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, the coefficient of the $x^2$ term, $a$, dictates the direction and elongation of the parabola.

Positive values of $a$ make the graph open upwards; negative values of $a$ make the graph open downwards. Note that when, as is frequently the case, there is no $a$ coefficient (or negative sign) explicitly shown, that simply means that $a = 1$; in that case, the parabola opens upwards, because 1 is positive. In the less-common case that the quadratic has a negative sign in front of the $x^2$ term, the implied a coefficient is $-1$, and therefore, the parabola opens downwards.

$y = x^2$

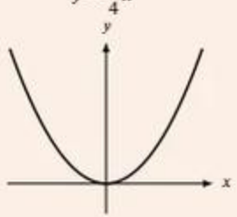



$y = -x^2$

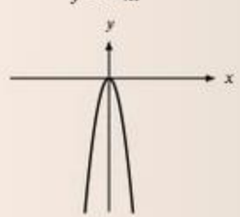



As the absolute values of $a$ decreases, the parabola gets flatter because the $x^2$ term grows more slowly, which in turn slows the absolute value growth of $y$-values. As the absolute value of $a$ increases, the parabola gets narrower (more pointy looking) because the $x^2$ term grows more rapidly, which in turn accelerates the absolute value growth of $y$-values.

$y = \frac{1}{4}x^2$




$y = -4x^2$




Notice that the graph of the equation $y = -4x^2$ is narrower than the graph of $y = \frac{1}{4}x^2$ because $|-4| > \left|\frac{1}{4}\right|$; that is, the $y$-value changes more rapidly due to the $x^2$ term's coefficient having a higher absolute value.

The attribute of a quadratic equation in Standard Form that is easiest to determine is the value of the $y$-intercept. The $y$-intercept is the value of the equation when the $x$-value is 0. **The constant term, $c$, is the $y$-intercept of a quadratic equation in Standard Form** (just as the constant term showed us the $y$-intercept of lines in Slope-Intercept Form) because the other terms, which have $x$ in them, become 0 when $x$ is 0.

For example, the $y$-intercept of $y = x^2 + 2x + 3$ is found by setting $x = 0$: $y = 0^2 + 2(0) + 3 = 3$. As you can see, when the $x$-coordinate is 0, as it is at the $y$-intercept, the value of the $y$-coordinate will always equal the constant $c$.

The value of $b$, the coefficient of the $x$ term, can also be useful when sketching the graph of a quadratic equation. From Standard Form, we can already determine which way a parabola opens and what its $y$-intercept is, but we do not necessarily know which side of the $y$-axis the maximum or minimum point of the graph (the vertex) should be on.

**The $b$ value tells you the slope of the graph as it passes through the $y$-intercept** (the slope of a parabola is constantly changing, unlike that of a line, which has a constant slope). You can draw a **tangent line** (a line that just touches the graph at one point)





through the $y$-intercept to help visualize this. For a parabola $y = ax^2 + bx + c$, the equation of the tangent line at the $y$-intercept is simply $y = bx + c$ (as if the $x^2$ term has gone to zero).

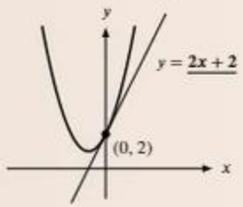
For now, you will have to accept this fact without explanation because it is most easily demonstrated with Calculus, though the upcoming discussion on the Vertex Form of a parabola could shed some light on how the value of the $x$ coefficient determines the position of the vertex.

Once you know the slope of the graph at the $y$-intercept, you can tell what the graph looks like as it crosses the $y$-axis and which side the maximum or minimum point (the vertex) is on, though later we will discuss other methods of locating the vertex (again, once you understand the Vertex Form, it is clear how the value of $b$ is linked to the position of the vertex).

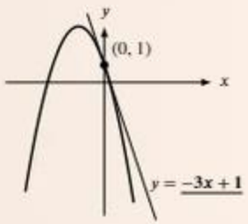
## $y$-intercept from Standard Form

For a quadratic equation in Standard Form, $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, the constant term, $c$, gives the value of the $y$-intercept, and the $x$ coefficient, $b$, gives the slope of the graph (the slope of a line tangent to the graph) as it goes through the $y$-intercept.

$y = x^2 + \underline{2x + 2}$




$y = -x^2 \underline{-3x + 1}$




As you can see, knowing the sign of the $b$ coefficient (which gives the sign of the slope of the tangent line at the $y$-intercept) in conjunction with the sign of the $a$ coefficient (which determines whether the parabola opens up or down) lets you determine the sign of the $x$ coefficient of the vertex; put another way, the $b$ coefficient tells you which side of the $y$-axis the vertex lies on.

If the parabola opens upwards (the $a$ coefficient is positive) and the slope of the tangent line is positive (the $b$ coefficient is positive), the vertex's $x$ coefficient is negative (the figure above on the left shows this situation). More generally, the sign of the $x$-coordinate of the vertex is opposite the sign of the product $ab$; as another example, in the figure above on the right, the product $ab$ is positive (it's $-1(-3) = 3$), so the $x$-coordinate of the vertex is negative. Note that if there is no middle term, the effective value of $b$ is 0, and the product $ab$ is therefore 0; this means that the vertex lies directly on the $y$-axis.

**SkillDrill 10.1-1**

**Directions:** Based on the Standard Form quadratic equations given below, identify the direction the parabola opens (up or down), on which side of the $y$-axis the vertex lies (right, left, or 0), and the $y$-intercept.

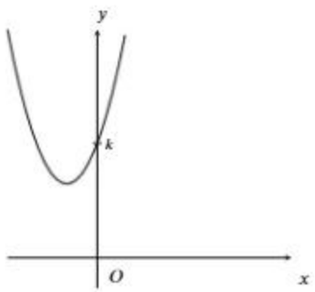
For example, given the equation $y = x^2 - x + 3$, your answer should be "up, right, 3" (you can use shorthand such as ↑ and ↓ for the direction the parabola opens and +, 0, or − for which side of the $y$-axis the vertex is on).

**1.** $y = x^2 + 3x - 6$ **2.** $y = -2x^2 - 2x + 1$ **3.** $y = 3x^2 - 3x + 5$ **4.** $y = -x^2 - 8$

**5.** $y = 4x^2 - 2x - 4$ **6.** $y = -10x^2 + 5x - 3$ **7.** $y = x^2 + 20$ **8.** $y = -x^2 - 4x + 2$

**Example 10.1-1**

**1** 1600.io 10.1




The graph of $y = 3x^2 + 8x + 15$ is shown. If the graph crosses the $y$-axis at the point $(0, k)$, what is the value of $k$?

A) 3

B) 5

C) 8

D) 15

**Solution 1**

1. Since $k$ is the $y$-value of the equation when $x = 0$, $k$ is the $y$-intercept of the equation.
2. The constant in the equation is also the $y$-intercept, so the value of $k$ is 15.
3. The answer is 15.





**Solution 2**

1. Plug in 0 for $x$ to find the value of the $y$-intercept, which is the point on the graph where the $x$-value is 0.

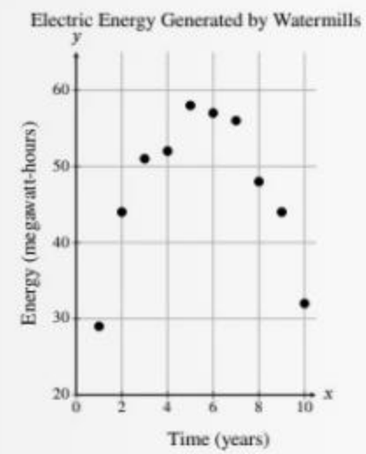
$$y = 3x^2 + 8x + 15$$
$$y = 3(0)^2 + 8(0) + 15$$
$$y = 15$$

2. The answer is 15.

**Example 10.1-2**

**2** 1600.io 10.1

The scatterplot below shows the amount of electric energy generated, in megawatt-hours, by watermills over a 10-year period.




Of the following, which best models the data in the scatterplot?

A) $y = 1.234x^2 + 13.57x - 20.67$

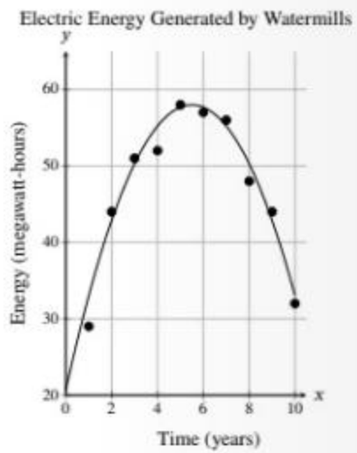
B) $y = -1.234x^2 - 13.57x - 20.67$

C) $y = 1.234x^2 + 13.57x + 20.67$

D) $y = -1.234x^2 + 13.57x + 20.67$

**Solution**

1. The scatterplot points can clearly be modeled by a parabola—not to mention that all of the choices are quadratic equations. If you want to, you can sketch in a parabola to approximate the curve of best fit. Scatterplots have a whole chapter dedicated to them later, but the main idea is that the points shown roughly fit the shape of a graph, in this case a parabola, and the curve will go through the cluster of points, with some points above the curve of best fit and some points below the curve of best fit as shown in the picture below.




2. The parabola opens downward, so the $x^2$ coefficient must be negative. Eliminate choices A and C.
3. We can tell for sure that the graph will have a positive slope as it passes through the $y$-intercept, meaning that the $x$ coefficient must be positive. Choice B can be eliminated.
4. The answer is D.

**Notes**

**The $y$-axis in this picture starts at 20 and not at 0.** Students who do not notice the actual starting point of the $y$-axis often think that a $y$-intercept of 20.67 would be too high because at first glance, it appears that the parabola of best fit intersects the "origin" of the graph. For this reason, they will then immediately jump to choices A and B based on the $y$-intercept alone (even though $-20.67$ is just as far away from the origin as is 20.67).

For this problem, it might be easier to make a decision based on the $x^2$ and $x$ coefficients alone, but if you correctly read the axis, you will realize that 20.67 is a much better estimate than $-20.67$ for the value of the $y$-intercept.

Section 10.1 Suggested

- Test 6-C-30
- Test 7-C-24
- Test 9-C-28

View related real-test problems in SMTEx

SMTEx

2020-NC-4
2020-C-19
2021 (US)-NC-12





## Section 10.1 Practice Problems

**1** 1600.io 10.1

$$h(t) = -9t^2 + 90t + 31$$

The function above models the height $h$, in centimeters, of an object above ground $t$ seconds after being thrown straight up in the air. What does the number 31 represent in the function?

A) The initial speed, in centimeters per second, of the object

B) The maximum speed, in centimeters per second, of the object

C) The initial height, in centimeters, of the object

D) The maximum height, in centimeters, of the object

**2** 1600.io 10.1

If $y = 4x^2 + 8x + 3$ is graphed in the $xy$-plane, which of the following characteristics of the graph is displayed as a constant or coefficient in the equation?

A) $y$-coordinate of the vertex

B) $y$-intercept

C) $x$-intercept of the line of symmetry

D) $x$-intercept(s)

## 10.2 Factored Form: Zeros of the Quadratic

We have seen quadratic equations (or functions) in Standard Form and Factored Form, and both can tell us certain things about what the graphs of these equations will look like.

Not all quadratic equations are easily factorable, and not all parabolas intersect the $x$-axis (have real zeros or $x$-intercepts). But when quadratics can be factored, they will definitely have $x$-intercepts, and the Factored Form makes it easy to visualize the graph of a quadratic equation because when a quadratic equation is written in Factored Form, $y = a(x + p)(x + q)$, we know that its zeros are $-p$ and $-q$ (it has $x$-intercepts at $(-p, 0)$ and $(-q, 0)$).

Remember, the main utility of factoring is in finding the values of $x$ that make the $y$-value equal to 0. Once we know the factors, we know the zeros because the values of $x$ that make the factors equal to 0 will cause the $y$-value to be equal to 0.

The zeros correspond to points on the graph where the $y$-value is 0 (these are $x$-intercepts). **If we know that $(x + p)$ is a factor of a polynomial, then we know that $-p$ is a zero of the polynomial**. For example, if we know that $x - 4$ is a factor of a quadratic, then we know that $x = 4$ is a zero of the function because plugging in 4 for $x$ would make that factor equal to 0. Therefore, the graph intersects the $x$-axis at $x = 4$.

The reverse is also true: **if we see that the graph of a polynomial touches the $x$-axis when $x = z$ (or are told in words that $z$ is a zero of the function), then we know that $x - z$ is a factor of the quadratic**.

### Zeros from Factored Form

For a quadratic equation in Factored Form, $y = a(x + p)(x + q)$, where $a$, $p$, and $q$ are constants, the zeros of the equation are $-p$ and $-q$. That means that when $x = -p$ or $x = -q$, the quadratic expression will evaluate to zero, and therefore $y$ will equal zero.

If $x = -p$,

$$y = a(x + p)(x + q)$$
$$y = a(-p + p)(-p + q)$$
$$y = a(0)(-p + q)$$
$$y = 0$$

On the $xy$-plane, we can represent this zero of this equation with the $xy$-coordinate pair (also known as a point) $(-p, 0)$ because $x = -p$ and $y = 0$.

If $x = -q$,

$$y = a(x+p)(x+q)$$
$$y = a(-q+p)(-q+q)$$
$$y = a(-q+p)(0)$$
$$y = 0$$

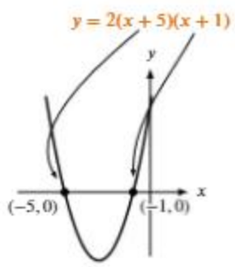
On the $xy$-plane, we can represent this zero of this equation with the $xy$-coordinate pair (also known as a point) $(-q, 0)$ because $x = -q$ and $y = 0$.

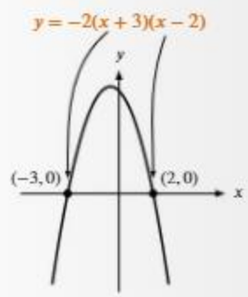
In summary, when $x = -p$ or $x = -q$, $y$ is zero, so we say that $-p$ and $-q$ are zeros of the quadratic expression. Graphically, when a point satisfying an equation has $y = 0$, the point lies on the $x$-axis, so the graph of the equation will intersect the $x$-axis at that point.

Note that we can still tell the direction the parabola opens from Factored Form. **If $a$ is positive, the graph will open upwards, and if $a$ is negative, the graph will open downwards**. Recall that when a quadratic expression is in Standard Form, the coefficient $a$ of the $x^2$ term tells us which way the parabola opens. You can see that when the Factored Form is expanded to Standard Form, $a$ is again the coefficient of the $x^2$ term and thus indicates which way the parabola opens:

$$y = a(x+p)(x+q)$$
$$y = \boldsymbol{ax^2} + aqx + apx + apq$$

Note that when, as is frequently the case, there is no $a$ coefficient (or negative sign) explicitly shown, that simply means that $a = 1$; in that case, the parabola opens upwards, because 1 is positive. In the case that the quadratic has a negative sign in front of the $x^2$ term, the implied a coefficient is $-1$, and therefore, the parabola opens downwards.




It should also be noted that **values of $a$ with larger absolute values will stretch the parabola more, meaning that the vertex is farther from the $x$-axis for Factored Form quadratics with larger absolute values of the $a$ coefficient** (unless the vertex is actually on the $x$-axis, meaning the $y$-coordinate of the vertex is 0, in which case the parabola will stretch with larger $a$-values, but the vertex will remain on the $x$-axis because the $y$-coordinate value of 0 will not change regardless of what value of $a$ it's multiplied by).





**SkillDrill 10.2-1**

**Directions:** Based on the Factored Form quadratic equations given below, identify the zeros. For example, given the equation $y = (x-1)(2x+3)$, your answers should be 1 and $\frac{-3}{2}$.

**1.** $y = (x-4)(x+2)$ **2.** $y = (x+3)(x+5)$ **3.** $y = (x-1)(x-3)$ **4.** $y = (3x-1)(2x-5)$

**5.** $y = (x+24)^2$ **6.** $y = 2x(x+10)$ **7.** $y = 3(x-4)^2$ **8.** $y = -4(x+2)(4x+7)$

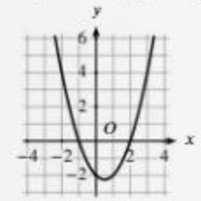
**Example 10.2-1**

**1** 1600.io 10.2

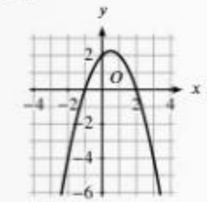
$$f(x) = (x+2)(x-k)$$

The function $f$ is defined above. If $k$ is a positive integer, which of the following could represent the graph of $y = f(x)$ in the $xy$-plane?

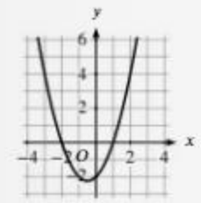
A)



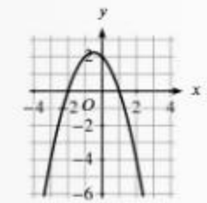
B)



C)



D)



**Solution**

1. Since there is an implied leading constant of 1, which is positive, the graph must open upwards, so we can eliminate choices B and D.
2. Since $(x+2)$ is a factor, $-2$ is a zero of the function, the graph must pass through the point $(-2, 0)$. Choice A does not pass through the point $(-2, 0)$, so choice A can be eliminated.
3. The answer is C.

The same relationship between factors and zeros holds true for all other polynomials (not just quadratics) as well, like linear $x^1$ functions, cubic $x^3$ functions, and quartic $x^4$ functions. The factors of *any* polynomial tell you the zeros of the function.

## The Factor Theorem

For any polynomial, if you are told that $(x + p)$ is a factor, then $-p$ is a zero of the function (the $y$-value is 0 when $x = -p$), and you can plug in $-p$ for $x$ and set the $y$-value equal to 0.

Similarly, if you are told that $z$ is a zero of the function (the $y$-value is 0 when $x = z$), then $(x - z)$ is a factor, and you can rewrite the function with $(x - z)$ as one of the factors.

The proof of this is closely linked to the Polynomial Remainder Theorem which was discussed in the previous chapter (not to mention all of our previous work with factoring polynomials).

If $x - r$ is a factor ($r$ is a zero of the function) and $x - s$ is another factor ($s$ is also a zero of the same function), then the product of those factors, $(x - r)(x - s)$, is also a factor.

**Example 10.2-2**

**2** 1600.io 10.2

$$f(x) = (x + 2)(x - 3)(5x + 4)$$

The function $f$ is defined above. Which of the following is NOT an $x$-intercept of the graph of the function in the $xy$-plane?

A) $(-2, 0)$

B) $\left(\frac{-5}{4}, 0\right)$

C) $\left(\frac{-4}{5}, 0\right)$

D) $(3, 0)$

**Solution**

1. An $x$-intercept means a point where the $y$-value is 0, which means that $x$-intercepts represent zeros of the function by definition. Since $(x + 2)$ is a factor, then $x = -2$ is a zero of the function, so the graph must pass through the point $(-2, 0)$, which is choice A, so choice A can be eliminated.
2. Since $(x - 3)$ is another factor, $x = 3$ is also a zero of the function, so the graph must pass through the point $(3, 0)$, which is choice D, so we can eliminate choice D.
3. Since $(5x + 4)$ is the last factor, $x = \frac{-4}{5}$ is the third and final zero of the function (if you're unsure why this is, simply set $5x + 4$ equal to 0 and solve for $x$), so the graph must pass through the point $\left(\frac{-4}{5}, 0\right)$; that's choice C, so we can eliminate that as well.
4. Because we've eliminated the choices corresponding to all three of the zeros, only choice B remains, so it must be the answer.
5. The answer is B.

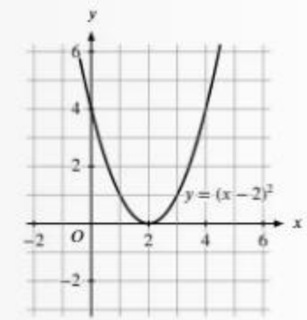
We've factored perfect square quadratics in previous chapters, finding that they had one squared factor, indicating that there was





only one distinct zero for the quadratic, not two as we see in the general case. What are the implications of this for the graph of such a function?

If we started with an upward-opening parabola with two zeros and then gradually shifted the graph upwards, the two zeros of a quadratic would get closer and closer together until the two points actually are the same (when the vertex is on the $x$-axis). Since the two zeros, and thus the two factors, would be the same, the equation is a perfect square. For example, the quadratic equation $y = x^2 - 4x + 4$ can be written as $y = (x - 2)^2$ or as $y = (x - 2)(x - 2)$, so the equation has one "double zero" at $x = 2$, and the graph looks like it "bounces" off of the point $(2, 0)$.



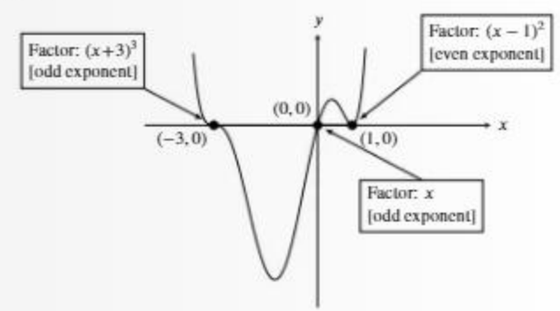

## Odd and Even Degree Factors

For any polynomial, when a factor, $x - z$, is raised to an **odd** power, the graph goes through the point $(z, 0)$. That is, the curve goes from above the $x$-axis to below it, or it goes from below the $x$-axis to above it, through the point $(z, 0)$.

When a factor, $x - z$, is raised to an **even** power, the graph "bounces" off of the $x$-axis at the point $(z, 0)$ instead of passing through. That is, the curve remains above or below the $x$-axis on both sides of that point.

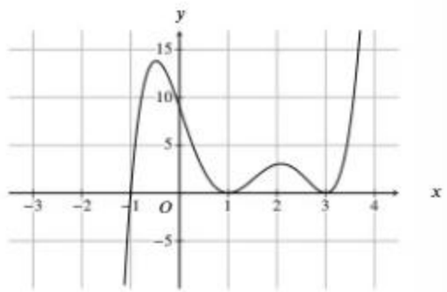
This behavior is explained by the fact that raising a value to an even power always produces a non-negative number regardless of the sign of the original value (e.g. $5^2 = 25$, and $(-5)^2 = 25$). Conversely, raising a value to an odd power preserves the sign of the original value (e.g. $5^3 = 125$, but $(-5)^3 = -125$).

For example, the polynomial $f(x) = x(x - 1)^2(x + 3)^3$ has three factors: $x$, $x - 1$, and $x + 3$. Since the factor $x$ (which corresponds to a zero at $x = 0$) is raised to the first power (an **odd** degree), the graph will **cross** the $x$-axis point $(0, 0)$. Since the factor $x - 1$ (which means there's a zero at $x = 1$) is raised to the second power (an **even** degree) the graph will **bounce off** the $x$-axis at the point $(1, 0)$. Finally, since the factor $x + 3$ (giving a zero at $x = -3$) is raised to the third power (an **odd** degree), the graph will **cross** the $x$-axis at the point $(-3, 0)$.



**Example 10.2-3**

**3** 1600.io 10.2




The graph of the function $f$ is shown in the $xy$-plane above, where $y = f(x)$. Which of the following functions could define $f$ ?

A) $f(x) = (x+3)(x+1)(x-1)^2$

B) $f(x) = (x+3)^2(x+1)^2(x-1)$

C) $f(x) = (x-3)(x-1)(x+1)^2$

D) $f(x) = (x-3)^2(x-1)^2(x+1)$

**Solution**

1. Since the zeros of the function are $x = -1$, $x = 1$, and $x = 3$, the factors of the function must include $(x+1)$, $(x-1)$, and $(x-3)$. Eliminate choices A and B.
2. The graph bounces off the $x$-axis when $x = 1$ and $x = 3$, so the factors $(x-1)$ and $(x-3)$ must have even exponents. This condition is true in choice D, but not in choice C.
3. The answer is D.

## Section 10.2 Suggested Problems from Real Tests

- Test 3-NC-7
- Test 3-C-12
- Test 4-C-12
- Test 7-NC-10
- Test 8-NC-11
- Test 8-C-19
- Test 9-C-26
- Test 9-C-27
- May 2017-C-22
- Apr 2018-NC-12
- May 2018-C-36
- Apr 2019-C-36
- May 2019 (US)-NC-11
- May 2019 (US)-C-22
- May 2019 (Int)-C-23
- Mar 2020-C-28
- Mar 2021-NC-11
- Mar 2021-NC-14
- Apr 2021-C-4
- May 2021 (US)-C-27

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## Section 10.2 Practice Problems

**1** 1600.io 10.2

In the $xy$-plane, the graph of the function $f(x) = x^2 + 6x + 5$ has two $x$-intercepts. What is the distance between the $x$-intercepts?

A) 2

B) 4

C) 5

D) 6

**2** 1600.io 10.2

$$ax^3 + bx^2 + cx + d = 0$$

In the equation above, $a$, $b$, $c$, and $d$ are constants. If the equation has roots $-3$, $-2$, and 4, which of the following is a factor of $ax^3 + bx^2 + cx + d$ ?

A) $x - 3$

B) $x - 2$

C) $x + 2$

D) $x + 4$

**3** 1600.io 10.2

$$h(x) = -9x^2 + 90x + 31$$

The quadratic function above models the height above the ground $h$, in centimeters, of a ball $x$ seconds after it had been thrown vertically. If $y = h(x)$ is graphed in the $xy$-plane, which of the following represents the real-life meaning of the positive $x$-intercept of the graph?

A) The time at which the ball hits the ground

B) The time at which the ball reaches its maximum height

C) The maximum height of the ball

D) The initial height of the ball

**4** 1600.io 10.2

$$h(x) = x^3 + ax^2 + bx + c$$

The function $h$ is defined above, where $a$, $b$, and $c$ are integer constants. If the zeros of the function are $-3$, 1, and 4, what is the value of $c$ ?

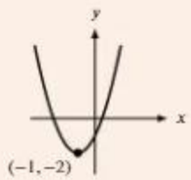
## 10.3 Vertex Form: Maximums and Minimums

Another form of quadratic equation is the **Vertex Form**. The **vertex of a parabola is the maximum or minimum point on the parabola, and the Vertex Form displays the $x$- and $y$-coordinates of the vertex as constants in the equation**. We will discuss how to find the Vertex Form shortly; for now, let's just see what it looks like and what it tells us.

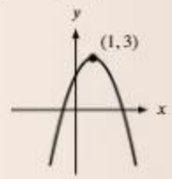
### Vertex Form

For any quadratic of the form $y = a(x - h)^2 + k$, where $a$, $h$, and $k$ are constants, the vertex of the parabola is located at the point $(h, k)$, and the value of $a$ dictates the direction and elongation of the parabola. Positive values of $a$ make the graph open upward, and negative values make the graph open downward.

$y = (x + 1)^2 - 2$




$y = -(x - 1)^2 + 3$




Notice that the $x$-coordinate, $h$, is the opposite of the sign shown in the parentheses. In the graphs above, $(x + 1)^2$ tells us that the $x$-value of the vertex is $-1$; $(x - 1)^2$ tells us that the $x$-value of the vertex is 1.

Also notice that the constant term in this form is NOT the $y$-intercept (as it is in Standard Form), but instead is the $y$-value of the vertex. This is easy to understand if you simply substitute $h$ (the $x$-coordinate of the vertex) for $x$; then, the equation becomes $y = a(h - h)^2 + k$, or simply $y = k$, so $k$ is the $y$-coordinate of the vertex. Though the $y$-intercept does not appear directly in the Vertex Form, you can always determine it by simply substituting $x = 0$ and evaluating the expression to find the $y$-value, which represents the $y$-intercept.

It should be noted that the range of a quadratic in Vertex Form is easy to determine. If the parabola opens upward, it has a minimum point, and the range of the function is $y \geq k$. If the parabola opens downward, it has a maximum point, and the range of the function is $y \leq k$.

In a way, you can think of the Vertex Form as a quadratic equation written in terms of a double zero, $(x - h)(x - h)$, which is shifted up or down depending on the value of $k$ and then elongated or flattened based on the value of $a$.

When Standard Form quadratics have no $x$-term (because the $b$ coefficient is 0), they are simultaneously written in Vertex Form already. The $x$-value of the vertex is 0, and the $y$-value of the vertex is equal to the constant term (which is also the $y$-intercept for these particular cases).





**SkillDrill 10.3-1**

**Directions:** Based on the Vertex Form quadratic equations given below, identify the vertex point, state whether the vertex is a maximum or minimum (max or min), and find the $y$-intercept by plugging in 0 for $x$.

For example, given the equation $y = (x + 2)^2 + 4$, your answer should be "$(-2, 4)$, min, 8."

1. $y = (x - 1)^2 - 3$
2. $y = -(x + 5)^2 + 10$
3. $y = (x + 2)^2 + 3$
4. $y = (x - 3)^2$
5. $y = -2(x - 1)^2 + 1$
6. $y = x^2 + 5$
7. $y = -3x^2$
8. $y = -2(x - 2)^2 - 5$

**Example 10.3-1**

**1** 1600.io 10.3

$$y = 3(x + 4)^2 + a$$

In the equation above, $a$ is a constant. The graph of the equation in the $xy$-plane is a parabola. Which of the following is true about the parabola?

A) Its minimum occurs at $(-4, a)$.

B) Its minimum occurs at $(4, a)$.

C) Its maximum occurs at $(-4, a)$.

D) Its maximum occurs at $(4, a)$.

**Solution**

1. The quadratic equation is given in Vertex Form. The positive constant 3 tells us that the graph opens upward, so the parabola will have a minimum value, but no maximum value. Therefore, we can eliminate choices C and D.
2. The expression $(x+4)^2$ tells us that the $x$-coordinate of the vertex is $-4$, and the constant term $a$ tells us that $y$-coordinate of the vertex is $a$, so the minimum occurs at the point $(-4, a)$.
3. The answer is A.

**Notes**

By convention, we write the Vertex Form symbolically as $y = a(x - h)^2 + k$, so we refer to the coefficient of the first term as the $a$ coefficient; here, that's 3. Don't be confused by the use of the letter $a$ as the constant term in this particular problem, which is represented symbolically in the Vertex Form as the letter $k$.

## Vertex Form to Standard Form and Factored Form

We can find the Standard Form from the Vertex Form by simply distributing and expanding. For example, given the following quadratic equation in Vertex Form

$$y = (x-3)^2 - 4$$

we can square the expression $(x-3)$ in order to expand it into more terms and then combine like terms to end up with the Standard Form of the same equation.

$$y = (x-3)^2 - 4$$
$$y = x^2 - 6x + 9 - 4$$
$$y = x^2 - 6x + 5$$

We can use the Vertex Form to find zeros of the function (though the zeros aren't immediately apparent as they are in Factored Form), and we can convert from Vertex Form to Factored Form (when real number factors exist) as well.

One way to find zeros or to find the Factored Form would be to convert from Vertex Form to Standard Form and then factor the expression.

Let's expand the Vertex Form equation $y = 2(x-2)^2 - 72$ into Standard Form so that we can then factor the expression and use it to find the zeros.

$$y = 2(x-2)^2 - 72$$
$$y = 2(x^2 - 4x + 4) - 72$$
$$y = 2x^2 - 8x + 8 - 72$$
$$y = 2x^2 - 8x - 64$$

Since all of the constants and coefficients are even, we can factor out a 2 to make factoring easier.

$$y = 2x^2 - 8x - 64$$
$$y = 2(x^2 - 4x - 32)$$

In order to factor the quadratic expression in parentheses, we need numbers that sum to $-4$ and multiply to $-32$. Those numbers are $-8$ and 4, so the factors are $(x-8)$ and $(x+4)$.

$$y = 2(x^2 - 4x - 32)$$
$$y = 2(x-8)(x+4)$$

Setting $y = 0$, we will see that the zeros are $x = -4$ and $x = 8$ based on the factors. This is a bit of a roundabout way to find zeros from Vertex Form.

However, we can take a more direct route if we use the Vertex Form to find the zeros of the quadratic equation (we did this a few chapters ago when we solved quadratic equations by Completing the Square), and, if desired, we can then write the Factored Form based on the factors that correspond to those zeros. Solving a Vertex Form quadratic equation just involves some straightforward algebra.

Again using the equation $y = 2(x-2)^2 - 72$, which is in Vertex Form ($y = a(x-h)^2 + k$), we can find the zeros of the equation, $p$ and $q$, and use those to write the equation in Factored Form, $y = a(x+p)(x+q)$.

Set the $y$-value to zero.

$$0 = 2(x-2)^2 - 72$$





Add 72 to both sides of the equation.

$$72 = 2(x-2)^2$$

Divide both sides of the equation by 2.

$$36 = (x-2)^2$$

Take the square root of both sides, remembering to account for both the positive and negative square roots.

$$\pm 6 = x - 2$$

Add 2 to both sides of the equation.

$$2 \pm 6 = x$$

Enumerate the solutions.

$$x = 2 + 6, \quad x = 2 - 6$$
$$x = 8, \quad x = -4$$

Since 8 is one of the zeros, $x - 8$ is a factor, and since $-4$ is one of the zeros, $x + 4$ is factor. Remembering to account for the value of $a$, which is 2, we can rewrite the equation in Factored Form: $y = 2(x-8)(x+4)$.

Once again, the primary point of discussion in this chapter is about the Vertex Form's utility as a representation of a quadratic equation that displays the coordinates of the vertex point as constants in the expression. That said, when you need to find zeros of a quadratic equation in Vertex Form (or, less commonly, rewrite the equation in Factored Form), there is no need to take a roundabout approach.

**SkillDrill 10.3-2**

**Directions:** Convert the following Vertex Form quadratic equations to Standard Form by expanding the squared term and combining like terms.

**1.** $y = (x-1)^2 - 3$ **2.** $y = -(x+5)^2 + 10$ **3.** $y = -2(x-1)^2 + 1$ **4.** $y = x^2 + 5$

**Directions:** Use the Vertex Form directly (without converting to Standard Form first) to find the zeros of the quadratic.

**5.** $y = (x-1)^2 - 4$ **6.** $y = -(x+5)^2 + 9$ **7.** $y = -2(x-1)^2 + 32$ **8.** $y = x^2 - 5$

**9.** $y = (x-1)^2 - 3$ **10.** $y = -(x+5)^2 + 10$ **11.** $y = -4(x+1)^2 + 1$ **12.** $y = 5(x-6)^2 - 3$

**Example 10.3-2**

**2** 1600.io 10.3

$$h(x) = 3(x-3)^2 - 27$$

The quadratic function $h$ is defined as shown. In the $xy$-plane, the graph of $y = h(x)$ intersects the $x$-axis at the points $(0, 0)$ and $(t, 0)$, where $t$ is a constant. What is the value of $t$ ?

A) 1

B) 3

C) 6

D) 9

**Solution 1**

1. We know that the vertex, which has an $x$-value of 3 based on the Vertex Form equation given, lies on the axis of symmetry (the line $x = 3$). The two zeros must be equidistant from the axis of symmetry.

   One zero exists when $x = 0$ (shown by the point $(0, 0)$), and that point is a distance of 3 units to the left of the axis of symmetry. Therefore, the other zero, the point $(t, 0)$, must be a distance of 3 units to the right of the axis of symmetry. That means that $t = 6$.

2. The answer is C.

**Solution 2**

1. Based on the fact that the graph of $h(x)$ intersects the $x$-axis at the point $(t, 0)$, we know that $x = t$ is one of the zeros of the function. Set $h(x)$ equal to 0 and substitute $t$ for $x$, then solve for $t$.

$$\begin{aligned} h(x) &= 3(x-3)^2 - 27 \\ 0 &= 3(t-3)^2 - 27 \\ 27 &= 3(t-3)^2 \\ 9 &= (t-3)^2 \\ \pm 3 &= t - 3 \\ 3 \pm 3 &= t \\ 3 + 3 = t,\ \ 3 - 3 &= t \\ 6 = t,\ \ 0 &= t \end{aligned}$$

2. The solution $t = 0$ is already shown when we are told that $(0, 0)$ is one of the $x$-intercepts (and 0 is not an answer choice). Therefore $t = 6$ is the zero of the equation that will give us the other $x$-intercept.

3. The answer is C.





**Section 10.3 Suggested Problems from Real Tests**

- Test 1-C-30
- Test 6-NC-11
- Test 10-C-19
- May 2019 (US)-C-25
- May 2019 (Int)-NC-12
- May 2019 (Int)-C-30
- Apr 2021-NC-3
- 2021-NC-11
- 2021 (Int)-NC-11
- 2021 (Int)-NC-13
- 2021 (Int)-NC-20
- May 2021 (Int)-C-24

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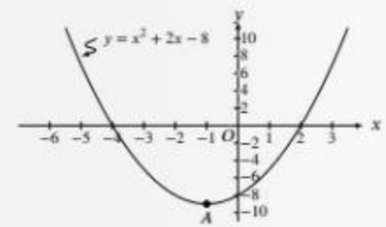
## Section 10.3 Practice Problems

**1** 1600.io 10.3




Which of the following is an equivalent form of the equation of the graph shown in the $xy$-plane above, from which the $y$-coordinate of vertex $A$ can be identified as a constant in the equation?

A) $y = (x+1)^2 - 9$

B) $y = x(x+2) - 8$

C) $y = (x-4)(x+2)$

D) $y = (x+4)(x-2)$

**2** 1600.io 10.3

The expressions $x^2 + bx + 14$ and $(x-4)^2 + c$, where $b$ and $c$ are constants, are equivalent. What is the value of $b + c$ ?

A) 10

B) 4

C) −6

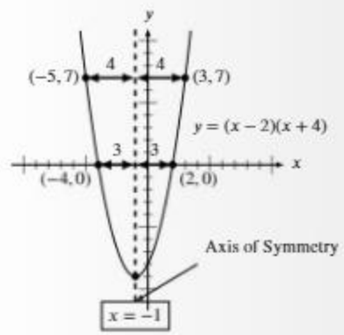
D) −10

## 10.4 Vertex Form from Factored Form

If you are given a quadratic equation in Factored Form, it is easy to find the vertex because **the $x$-coordinate of the vertex is exactly halfway between the two roots**. In fact, all parabolas are symmetrical—the graph on one side of the vertex is the mirror image of the graph on the other side. The vertex is the one and only point on the parabola that lies on the axis of symmetry; both $x$-intercepts (which correspond to the roots) are equidistant from the axis of symmetry. In fact, any two points on the parabola that have the same $y$-coordinate are equidistant from the axis of symmetry (thus the name!).

For example, the $x$-coordinate of the vertex of the parabola $y = (x-2)(x+4)$ is $-1$ because $-1$ is exactly halfway between the two roots, $x = -4$ and $x = 2$, so the axis of symmetry is at $x = -1$, and, as explained, the vertex lies on that axis of symmetry. You can average the two roots' values (add the roots and divide by 2) to find the value that is midway between them:

$$x_{vertex} = \frac{-4+2}{2} = \frac{-2}{2} = -1$$




If you need to know the $y$-coordinate of the vertex, you can just plug the $x$-value of the vertex into the Factored Form. There is no need to expand the Factored Form into the Standard Form. In this example, we would plug $-1$ in for $x$ in order to find the $y$-value of the vertex.

$$y_{vertex} = (x-2)(x+4)$$
$$y_{vertex} = (-1-2)(-1+4)$$
$$y_{vertex} = (-3)(3)$$
$$y_{vertex} = -9$$

Once you have the coordinates of the vertex, you can write the equation in Vertex Form. Since the $x$-value of the vertex is $-1$ and the $y$-value of the vertex is $-9$, the Vertex Form of this parabola is $y = (x+1)^2 - 9$.

**Remember to account for the value of $a$.** In our example, the value of $a$ was 1, so we didn't have to worry about it. However, if the Standard Form has any other value of $a$, you need to include it as the coefficient of the $(x-h)^2$ term when writing the Vertex Form.

## $x$-value of the Vertex Based on Roots

For any quadratic function $y = f(x)$, the $x$-value of the vertex is exactly halfway between the two zeros and can be found by averaging the two zeros of the function (add the roots and divide by 2). Recall that when we're talking about functions, the terms "zero" and "root" mean the same thing and thus are interchangeable; we're using both terms here to make sure you're paying attention. Also, eat your vegetables.

For example, for the equation $y = (x+3)(x+7)$, the roots are $-3$ and $-7$, the $x$-value of the vertex (denoted below as $h$) is $-5$ because the average of $-3$ and $-7$ is $-5$.

$$h = \frac{-3+(-7)}{2}$$

$$h = \frac{-3-7}{2}$$

$$h = \frac{-10}{2}$$

$$h = -5$$





## $y$-value of the Vertex Based on Roots

Once you have the $x$-value of the vertex, you can simply plug it into the quadratic equation to find the $y$-value of the vertex.

The $x$-value of the vertex (denoted as $h$) of the equation $y = (x+3)(x+7)$ is $-5$, as we found above. Plug $-5$ in for $x$ in order to find the $y$-value of the vertex (denoted as $k$ below).

$$y = (x+3)(x+7)$$
$$k = (h+3)(h+7)$$
$$k = (-5+3)(-5+7)$$
$$k = (-2)(2)$$
$$k = -4$$

**SkillDrill 10.4-1**

**Directions:** Use the Factored Form to find the coordinates of the vertex of the parabola, and convert the quadratic equation to Vertex Form.

Challenge problems are boxed.

**1.** $y = (x-4)(x+2)$

**2.** $y = (x+3)(x+5)$

**3.** $y = -(x-1)(x-3)$

**4.** $y = (x+6)^2$

**5.** $y = 2x(x+10)$

**6.** $y = 3(x-4)^2$

**7.** $\boxed{y = 4\left(x - \frac{1}{2}\right)\left(x - \frac{5}{2}\right)}$

**8.** $\boxed{y = -8(x+1)\left(x + \frac{1}{2}\right)}$

### Which Form to Use For Particular Information

There is a particular style of question on the test that asks us to rewrite quadratic equations in different forms. Usually, several of the choices will be different forms of the same quadratic equation; however, the questions require that you pick one particular form based on the type of information that each form shows.

## Choosing the Correct Form of Quadratic Equation or Function

When asked to pick an equivalent form of a quadratic equation or function, you must choose an answer that is not only equivalent, but that also shows the features as stated in the question.

- **Roots, Zeros, or $x$-Intercepts**
  If you are asked for an equivalent form of the equation or function that displays the roots, zeros, or $x$-intercepts of the function as constants or coefficients, you want to pick a choice that is in Factored Form. For a quadratic equation in Factored Form $y = a(x+p)(x+q)$, $-p$ and $-q$ are the zeros/roots/$x$-intercepts (note that "root," "zero," and "$x$-intercept" all mean the same thing in this context).
- **Maximum, Minimum, or Vertex**

If you are asked for an equivalent form of the equation or function that displays the maximum or minimum of the graph (or the coordinates of the vertex) as constants or coefficients, you want to pick an answer choice that is in Vertex Form. For a quadratic equation in Vertex Form $y = a(x-h)^2+k$, $(h, k)$ is the vertex; it will represent the maximum or minimum point of the parabola.

- **y-intercept**
  If you are asked for an equivalent form of the equation or function that displays the $y$-intercept of the parabola as a constant or coefficient, you want to pick a choice that is in Standard Form. For a quadratic equation in Standard Form $y = ax^2 + bx + c$, the constant $c$ is the $y$-intercept, because when $x = 0$, the other terms drop out.

**Example 10.4-1**

**1** 1600.io 10.4

$$f(x) = (x + 2)(x - 8)$$

Which of the following is an equivalent form of the function $f$ above in which the minimum value of $f$ appears as a constant or coefficient?

A) $f(x) = x^2 - 16$

B) $f(x) = x^2 - 6x - 16$

C) $f(x) = (x + 3)^2 + 11$

D) $f(x) = (x - 3)^2 - 25$

**Solution**

1. Since we want an equivalent form that shows the minimum value, we need to select a choice that is in Vertex Form. Eliminate choices A and B.

   Note that we don't have to worry about whether the function has a minimum value or a maximum value, because it must have only one of those, and if the problem asks for a minimum, the function has a minimum at the vertex. The same principle applies if the problem asks for a maximum—the function will have one at the vertex. Here, you can verify that the function has a minimum because the parabola opens up as indicated by the (implied) $a$ coefficient of 1, which is positive.

2. The two roots of the function are $-2$ and 8. The value halfway between the two (the average of the two values) is 3, so the $x$-value of the vertex is 3.

$$h = \frac{-2+8}{2}$$

$$h = \frac{6}{2}$$

$$h = 3$$





3. Substitute 3 in for $x$ to find the $y$-value of the vertex, which will be the minimum value of the function $f$ (remember that $f(x)$ is equal to the $y$-value for a given value of $x$).

$$f(x) = (x + 2)(x - 8)$$
$$k = f(h) = (h + 2)(h - 8)$$
$$k = f(3) = (3 + 2)(3 - 8)$$
$$k = (5)(-5)$$
$$k = -25$$

4. We must choose the answer in which the $y$-value of the vertex (the minimum value of the function), $-25$, appears as a constant or coefficient. Choice D is the only choice that shows $-25$ as a constant.

5. The answer is D.

**Notes**

We could have eliminated choice C (leaving only choice D) as soon as we found that the $x$-value of the vertex is 3 because choice C, which is in Vertex Form, indicates that the $x$-value of the vertex would be $-3$.

**Section 10.4 Suggested Problems from Real Tests**

- Test 2-C-7
- Test 3-NC-12
- Test 4-C-28
- Test 7-NC-12
- Mar 2021-C-14

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## Section 10.4 Practice Problems

**1** 1600.io ? 10.4

The function $f$ is defined by $f(x) = (x + 6)(x + 2)$. The graph of $f$ in the $xy$-plane is a parabola. Which of the following intervals contains the $x$-coordinate of the vertex of the graph of $f$ ?

A) $-7 < x < -6$

B) $-7 < x < -5$

C) $-6 < x < 2$

D) $2 < x < 6$

**2** 1600.io ? 10.4

$$y = a(x - 3)(x + 5)$$

In the quadratic equation above, $a$ is a nonzero constant. The graph of the equation in the $xy$-plane is a parabola with vertex $(c, d)$. Which of the following is equal to $d$ ?

A) $-16a$

B) $-15a$

C) $-12a$

D) $-2a$

## 10.5 Finding the Vertex from Standard Form

If you are given a quadratic equation in Standard Form, it is easy to find the vertex using a simple method.

### $x$-value of the Vertex Based on Standard Form

For any quadratic in Standard Form, $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, the $x$-value of the vertex (denoted as $h$ below) is equal to $\frac{-b}{2a}$.

$$h = \frac{-b}{2a}$$

For example, for the quadratic $y = -2x^2 + 4x - 8$, the $x$-value of the vertex is equal to 1.

$$h = \frac{-4}{2(-2)}$$

$$h = \frac{-4}{-4}$$

$$h = 1$$

You might recall that $\frac{-b}{a}$ represents the sum of the zeros, so it makes sense that the $x$-coordinate of the vertex, which represents the *average* of the zeros, would be $\frac{-b}{2a}$.

### $y$-value of the Vertex Based on Standard Form

Once you have the $x$-value of the vertex, you can plug that value into the equation to find the $y$-value of the vertex.

For example, for the quadratic $y = -2x^2 + 4x - 8$, plug in 1, which is the $x$-value of the vertex (denoted as $h$), in order to find the $y$-value of the vertex (denoted as $k$).

$$y = -2x^2 + 4x - 8$$
$$k = -2h^2 + 4h - 8$$
$$k = -2(1)^2 + 4(1) - 8$$
$$k = -2 + 4 - 8$$
$$k = -6$$

### Vertex Form Based on Standard Form

Once you have found the values of $h$ and $k$ (and you already know the value of $a$ from the Standard Form equation), you can plug those values into the general Vertex Form equation, which is $y = a(x - h)^2 + k$.





In our example, $y = -2x^2 + 4x - 8$, we know that $a = -2$, $h = 1$, and $k = -6$, so the Vertex Form is as follows:

$$y = a(x - h)^2 + k$$
$$y = -2(x - 1)^2 - 6$$

### Vertex Form by Completing the Square

There is a streamlined method for converting directly from Standard Form to Vertex form without having to first find the $x$-coordinate of the vertex, then plug that $x$-value into the quadratic equation and evaluate it to determine the $y$-coordinate, then construct the Vertex Form once the $x$- and $y$-values are in hand. The direct method relies on the same principles used in the Completing the Square quadratic solving method discussed elsewhere, and it's both rapid and has the benefit of producing both the $x$- and $y$-coordinates of the vertex at the same time.

Here's the procedure that applies when there is no $a$ coefficient (that is, when there is an implied $a$ coefficient of 1):

1. Replace the first two terms (the $x^2$ term and the $bx$ term) with

$$\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

   The purpose of subtracting the second term is to eliminate the unwanted constant $\left(\frac{b}{2}\right)^2$ that's created when squaring the first term. (You might observe that this correcting term could be simplified as $\frac{b^2}{4}$, but you'll already have the term $\frac{b}{2}$ evaluated in the first term, so it's simpler to just square that value, and it's also easier to remember that the second term is just the negative of the constant term generated by expanding the first term.)

2. Combine like terms.

To demonstrate the simplicity of this new method, we will first find the vertex of an example function by separately determining the $x$- and $y$-coordinates so you can see how much more efficient the streamlined method is when later applied to the same problem.

$$y = x^2 - 6x - 16$$

Here, $a = 1$, $b = -6$, and $c = 16$.

The $x$-coordinate, $h$, of the vertex is $\frac{-b}{2a}$.

$$h = \frac{-b}{2a}$$

$$h = \frac{-(-6)}{2(1)}$$

$$h = \frac{6}{2}$$
$$h = 3$$

Plug in $x = 3$ to find the $y$-coordinate, $k$, of the vertex.

$$k = (3)^2 - 6(3) - 16$$
$$k = 9 - 18 - 16$$
$$k = -25$$

The vertex $(h, k)$ is $(3, -25)$. Now we can construct the Vertex Form.

$$y = a(x-h)^2 + k$$
$$y = (x-3)^2 - 25$$

Now, let's use the streamlined Completing the Square procedure for comparison.

1. Replace the first two terms with $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$. Recall that for the equation $y = x^2 - 6x - 16$, the value of $b$ is $-6$, so $\frac{b}{2} = \frac{-6}{2} = -3$.

$$\mathbf{y = x^2 - 6x} - 16$$
$$\mathbf{y = (x-3)^2 - (-3)^2} - 16$$
$$\mathbf{y = (x-3)^2 - 9} - 16$$

2. Combine like terms.

$$y = (x-3)^2 - 25$$

Voilà! The equation is now in Vertex Form $y = a(x-h)^2 + k$; here, $h = 3$ and $k = -25$, so the vertex is at $(3, -25)$. Notice how rapidly we were able to reach our goal.

The same principle can also be used when $a$ is not 1. Here's the more general procedure:

### Vertex Form by Completing the Square when $a \neq 1$

1. Replace the first two terms (the $ax^2$ term and the $bx$ term) with

$$a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2$$

Again, the purpose of subtracting the second term is to eliminate the unwanted constant $a\left(\frac{b}{2a}\right)^2$ that's created when expanding the first term. (You could simplify the second term symbolically to $\frac{b^2}{4a}$, though you'll already have the value of $\frac{b}{2a}$ in hand and it might be simpler to just square that and multiply it by $a$; it's also easier to remember that the second term is just the negative of the constant term generated by expanding the first term.)

2. Combine like terms.





Let's demonstrate this using the example shown above when we explained how to find the vertex's $x$- and $y$-coordinates separately.

$$y = -2x^2 + 4x - 8$$

Here, $a = -2$, $b = 4$, and $c = 8$.

1. Replace the first two terms with $a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2$. Recall that for the equation $y = -2x^2 + 4x - 8$, $a = -2$ and $b = 4$, so $\frac{b}{2a} = \frac{4}{2(-2)} = \frac{4}{-4} = -1$.

$$\mathbf{y = -2x^2 + 4x} - 8$$
$$\mathbf{y = -2(x-1)^2 - [-2(-1)^2]} - 8$$
$$\mathbf{y = -2(x-1)^2 + 2} - 8$$

2. Combine like terms.

$$y = -2(x-1)^2 - 6$$

The equation is now in Vertex Form $y = a(x-h)^2 + k$; here, $h = 1$ and $k = -6$, so the vertex is at $(1, -6)$.

**SkillDrill 10.5-1**

**Directions:** Convert the Standard Form quadratic to Vertex Form. You can first find the $x$-coordinate then plug in to find the $y$-coordinate and assemble the equation, or you can try to use the streamlined direct Completing the Square procedure.

Challenge problems are boxed.

**1.** $y = x^2 + 2x - 4$ **2.** $y = x^2 - 6x + 20$ **3.** $y = x^2 + 10x - 15$ **4.** $y = x^2 - 4x - 7$

**5.** $y = x^2 - 8x + 3$ **6.** $y = x^2 + 20x + 100$ **7.** $y = x^2 + 14x + 32$ **8.** $y = x^2 - 10$

**9.** $\boxed{y = x^2 + 5x - 1}$ **10.** $\boxed{y = x^2 - 3x + 2}$ **11.** $\boxed{y = 2x^2 + 8x + 10}$ **12.** $\boxed{y = -3x^2 + 12x + 12}$

**Example 10.5-1**

**1** 1600.io 10.5

The fuel economy $M(s)$, in kilometers per liter, of a moped traveling $s$ kilometers per hour is modeled by the function below, where $30 \leq s \leq 100$.

$$M(s) = -\frac{1}{20}s^2 + 6s + 230$$

According to the model, at what speed, in kilometers per hour, does the moped obtain its greatest fuel economy?

A) 30

B) 60

C) 75

D) 90

**Solution**

1. The moped will achieve its greatest fuel economy at the vertex (maximum) of the graph of the function. We need to find the speed (the $s$-value) of the moped at the vertex.

2. The $s$-value of the vertex (remember that $s$ is acting like the $x$-variable here), denoted as $h$, can be found using the formula $h = \frac{-b}{2a}$, where $b$ is the coefficient of the $s$-term, 6, and $a$ is the coefficient of the $s^2$-term, $-\frac{1}{20}$.

$$h = \frac{-b}{2a}$$
$$h = \frac{-6}{2\left(-\frac{1}{20}\right)}$$
$$h = \frac{-6}{-\frac{2}{20}}$$
$$h = \frac{-6}{-\frac{1}{10}}$$
$$h = -6(-10)$$
$$h = 60$$

3. The answer is B.

**Notes**

If we needed to know the maximum fuel economy, achieved at that optimal speed of 60 $\frac{\text{km}}{\text{h}}$, we would plug in 60 for $s$ and solve for $M(60)$.

$$M(s) = -\frac{1}{20}s^2 + 6s + 230$$
$$M(60) = -\frac{1}{20}(60)^2 + 6(60) + 230$$
$$M(60) = -\frac{3600}{20} + 360 + 230$$
$$M(60) = -180 + 590$$
$$M(60) = 410$$

Consequently, knowing that the vertex point is (60, 410), we could, if desired, rewrite the equation in Vertex Form:

$$M(s) = -\frac{1}{20}(s - 60)^2 + 410$$



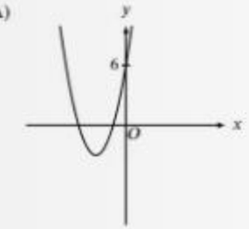


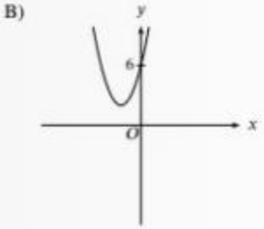
In the next example, it is not obvious that we need the Vertex Form to get the correct answer. Earlier in this chapter, we talked about what we can tell about the graph of a quadratic equation in Standard Form, which is good for quick visualizations of the general direction and location of the parabola. However, in the following question, that knowledge gets us down to two choices, but we will need to know about the position of the vertex in order to make a final decision.

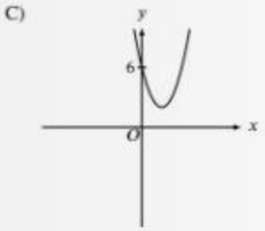
**Example 10.5-2**

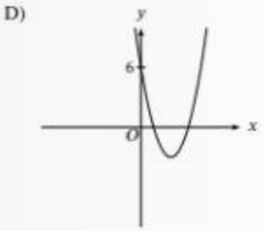
2 1600.io 10.5

Which of the following could be the graph of $y = x^2 + 4x + 6$ ?

A)



B)



C)



D)



**Solution 1**

The Standard Form of the provided equation tells us that the parabola opens upwards due to the positive (implied) $a$-coefficient, and that the $y$-intercept (the $c$ constant) is 6, but all the answer choices satisfy those constraints, so we need to use another approach to identifying the correct graph. Notice that the vertex of each parabola is in a different quadrant of the $xy$-coordinate plane; that means if we can determine the vertex, we can select the proper graph. Because finding the vertex would lead directly to the correct answer choice, we can use the streamlined procedure for converting from Standard Form directly to Vertex Form to rapidly solve this problem without even needing to eliminate any answer choices at the outset.

1. Replace the first two terms of the Standard Form quadratic with $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$. In the equation $y = x^2 + 4x + 6$, the value of $b$ is 4, so $\frac{b}{2} = 2$.

$$y = \mathbf{x^2 + 4x} + 6$$
$$y = (\mathbf{x + 2})^2 - \mathbf{2^2} + 6$$
$$y = (\mathbf{x + 2})^2 - \mathbf{4} + 6$$

2. Combine like terms.

$$y = (x+2)^2 - 4 + 6$$
$$y = (x+2)^2 + 2$$

3. This Vertex Form of the quadratic tells us that the vertex is at $(-2, 2)$, and only choice B matches that constraint.
4. The answer is B.

**Solution 2**

1. Based on Standard Form, we know that the $y$-intercept must be 6 (the constant term is 6), which is shown in every choice, and thus does not help us eliminate any choices.
2. Based on Standard Form, we know that the vertex of the parabola must be on the left side of the $y$-axis because the parabola opens up due to the (implied) a coefficient of 1, and the coefficient of the $x$ term is 4, which is positive (the line that is tangent to the parabola at the $y$-intercept has a slope of 4). We can eliminate choices C and D based on this information, but we cannot adequately decide between choices A and B. For one thing, we don't know the scale of the axes and we don't know if this quadratic has zeros (because we cannot easily factor the equation).
3. We can find the coordinates of the vertex to help decide between choices A and B. The $x$-value of the vertex, denoted $h$, is equal to $\frac{-b}{2a}$, where $b = 4$ and $a = 1$.

$$h = \frac{-b}{2a}$$
$$h = \frac{-4}{2(1)}$$
$$h = -2$$

4. We know that the $x$-value of the vertex is $-2$, but this is still insufficient to decide between choices A and B. Since we have no indication of the scale of the $x$-axis, we cannot tell which one of the vertices of these graphs has an $x$-coordinate of $-2$. Therefore, we should substitute $-2$ for $x$ in order to find the $y$-value of the vertex.

$$y = x^2 + 4x + 6$$
$$y = (-2)^2 + 4(-2) + 6$$
$$y = 4 - 8 + 6$$
$$y = 2$$

5. The vertex is the point $(-2, 2)$, which is above the $x$-axis, eliminating choice A (in which the vertex is clearly below the $x$-axis) and leaving us with only choice B (in which the vertex is clearly above the $x$-axis, and also to the left of the $y$-axis).
6. The answer is B.

Section 10.5 Suggested

- Test 3-NC-10
- Test 9-NC-13
- Mar 2018-C-27

View related real-test problems in SMTEx

SMTEx





## Section 10.5 Practice Problems

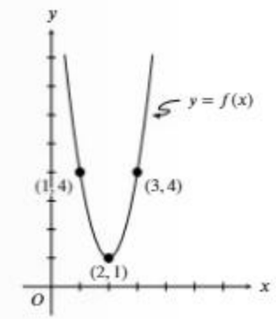
**1** 1600.io 10.5

$$y = x^2 + 4x + 16$$

The graph of the equation above in the $xy$-plane is a parabola. Which of the following equivalent forms of the equation includes the $x$- and $y$-coordinates of the vertex as constants?

A) $y = x(x+4) + 16$
B) $y - 16 = x(x+4)$
C) $y = (x+2)^2 + 12$
D) $y = x^2 + 4(x+3)$

**2** 1600.io 10.5




The graph of the function $f$ in the $xy$-plane above is a parabola. Which of the following defines $f$?

A) $f(x) = 2(x+2)^2 + 1$
B) $f(x) = (x-2)^2 + 1$
C) $f(x) = 3(x+2)^2 + 1$
D) $f(x) = 3(x-2)^2 + 1$

## 10.6 Number of Intersections = Number of Solutions

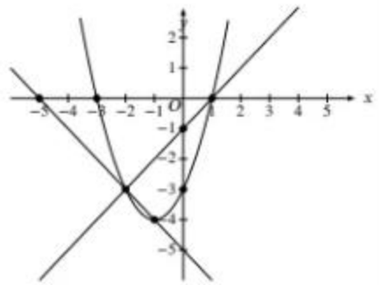
We already have talked about Systems of Linear Equations and Systems of Quadratic and Linear Equations. We know that the solutions to systems consist of the intersection points of all of the equations in the system because those points represent $x$- and $y$-values that satisfy all the equations in the system. We have even seen that the graphs of quadratic equations (parabolas) can have intersections with the $x$-axis that indicate the roots or zeros of those equations (which, in turn, correspond to the factors of those equations).

In the next chapter we will discuss the number of solutions to quadratic equations in a slightly different way, but in the final section of this chapter, we want to look at how the number of real solutions to any system of equations can be found by looking at the graphs.

In short, **the number of real solutions to a system of equations is equal to the number of intersections of the graphs of the equations in the system**. If there are more than two equations in the system (more than two lines, parabolas, or any other curves), **all of the graphs must intersect at a common point in order for that point to represent a solution to the system because the $x$- and $y$-values of that point satisfy *all* the equations in the system**.

**Example 10.6-1**

1 1600.io 10.6




A system of three equations is graphed in the $xy$-plane above. How many solutions does the system have?

A) None

B) One

C) Two

D) Three

**Solution**

1. Note that this problem does not ask how many $x$-intercepts are shown in the picture or how many intersections there are between any of the graphs. In order to be a solution to the system, all three of the graphs have to intersect at the same point. While two graphs intersect at points $(1, 0)$ and $(-1, -4)$, the only point that all three graphs go through is $(-2, -3)$. Therefore, there is only one solution to the system of equations shown in the picture.

2. The answer is B.

Of course, if graphs can be drawn accurately (the $xy$-plane is provided to you with a scaled grid and the graphs intersect on points where gridlines meet), you can use the graph to determine solutions to the system.

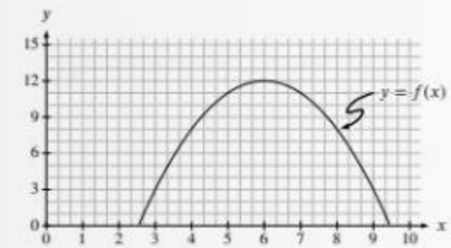
In the following example, it is possible to add a line to the picture in order to determine the solutions, but you must be careful about the scales of the axes if you are going to take this route.





**Example 10.6-2**

2 1600.io 10.6




The graph of the function $f$, defined by $f(x) = -(x-6)^2 + 12$, is shown in the $xy$-plane above. If the function $g$ (not shown) is defined by $g(x) = -x + 12$, what is one possible value of $a$ such that $f(a) = g(a)$ ?

**Solution 1**

The recommended approach here is to be safe and solve the problem with substitution rather than use the graph. Using the graph is technically faster, but there is great risk of doing it incorrectly when taking that route. The axes are scaled differently in the picture you are given, and honestly, you should double check your answer even if you do use the graph and draw the line in.

1. We have a system of equations consisting of a quadratic equation and a linear equation. If we want to find the solution point, we should use substitution.

   We want to know the value of $a$ (which is an $x$-value) where $f(a) = g(a)$. By substituting $a$ for $x$, we know that

   $$f(a) = -(a-6)^2 + 12$$

   and that

   $$g(a) = -a + 12$$

   Set $f(a)$ equal to $g(a)$, and substitute their respective expressions.

   $$f(a) = g(a)$$
   $$-(a-6)^2 + 12 = -a + 12$$

2. Expand the expression on the left side of the equation and combine like terms to make a Standard Form quadratic equation.

$$-(a-6)^2 + 12 = -a + 12$$

Subtract 12 from both sides of the equation: $-(a-6)^2 = -a$

Multiply both sides of the equation by $-1$: $(a-6)^2 = a$

Expand the expression on the left side of the equation: $a^2 - 12a + 36 = a$

Subtract $a$ from both sides of the equation: $a^2 - 13a + 36 = 0$

3. Factor to find the two values of $a$ for which $f(a) = g(a)$. We need to find two numbers that multiply to 36 and add to $-13$, and those numbers are $-4$ and $-9$, so the factors are $(a-4)$ and $(a-9)$.

$$a^2 - 13a + 36 = 0$$
$$(a-4)(a-9) = 0$$
$$a = 4, \quad a = 9$$

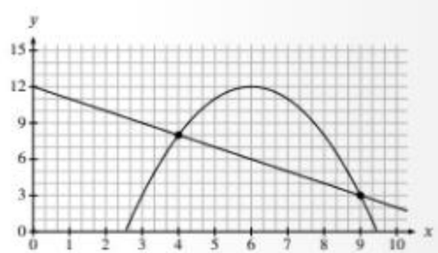
4. The answer is 4 or 9.

**Solution 2**

Below is an incredibly risky solution method. For one, there is no specific reason to know that the answers will be integers and thus precisely discernible even from a perfectly-drawn line. On what basis could one assume that the solutions lie at intersection points of the grid? Nevertheless, many do try to do this problem using this method, and most fail for the reason outlined below.

In this solution, we will draw in the line $g(x) = -x + 12$ and pick out intersection points. This method is faster but many people draw the line incorrectly. If you try to draw the slope of $-1$ by going diagonally down each square of the grid, you will get a wrong answer because the axes are scaled differently. In the $y$-direction, each box does represent a change of 1, but in the $x$-direction, a change of 1 is achieved by moving three boxes over (each tick is 1 apart, but is divided into three equal parts).

1. Since we want to know the values of $a$ for which $f(a) = g(a)$, we can draw the line $g(x) = -x + 12$, and find the $x$-values of the intersection points.
   When drawing the line, be careful to slope the line correctly. Based on the grid, a slope of $-1$ is achieved by moving 1 down as we move 3 boxes to the right.




2. The intersections are at the points $(4, 8)$ and $(9, 3)$, so the values of $a$ (remember $a$ is an $x$-value) for which $f(a) = g(a)$ are 4 and 9.

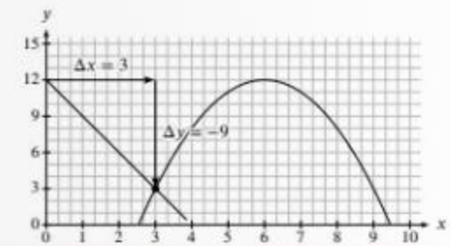
3. The answer is 4 or 9.





**Notes**

Just so you know exactly what most people do incorrectly when trying to solve this problem by the method in Solution 2, we have shown below how people mistakenly believe they are drawing the line $g(x) = -x + 12$ while they are in fact drawing the line $y = -3x + 12$, which intersects the parabola at the point $(3, 3)$. Notice that although the line shows a change of one square at a time, the squares do not represent a 1 to 1 change. This highlights why it is absolutely essential to examine the axes' scaling (and starting values, for that matter) when analyzing a graph.




**Section 10.6 Suggested**

- Test 4-NC-13
- Test 4-C-30
- Test 8-C-35

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## Section 10.6 Practice Problems

**1** 10.6

In the $xy$-plane, the parabola with the equation $y = (x - 13)^2$ intersects the line with equation $y = 36$ at two points, $A$ and $B$. What is the length of $\overline{AB}$ ?

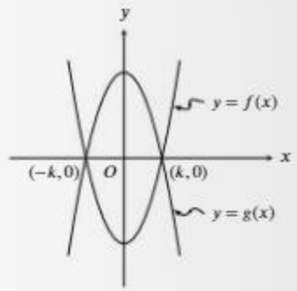
A) $\sqrt{13}$

B) 12

C) 13

D) 23

**2** 10.6




The functions $f$ and $g$, defined by $f(x) = 9x^2 - 1$ and $g(x) = -9x^2 + 1$, are graphed in the $xy$-plane above. The graphs of $f$ and $g$ intersect at the points $(k, 0)$ and $(-k, 0)$. What is the value of $k$ ?

A) 3

B) 1

C) $\frac{1}{3}$

D) $\frac{1}{9}$





## CHAPTER 10 RECAP

### Standard Form

- The Standard Form of a quadratic equation is $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants.
- For quadratics in Standard Form $y = ax^2 + bx + c$, the value of $a$ (the $x^2$ coefficient) dictates the direction and elongation of the parabola. High absolute values result in graphs that are stretched vertically. Low absolute values result in flatter graphs. Positive values cause the graph to open upwards. Negative values cause the graph to open downwards.
- Every parabola that represents a function $y = f(x)$ is symmetric about a vertical line through its vertex, which is called the axis of symmetry.
- For a quadratic in Standard Form $y = ax^2 + bx + c$, the $x$-value of the vertex is equal to $\frac{-b}{2a}$.
- Plug the $x$-value of the vertex into the Standard Form in order to easily find the $y$-value of the vertex. This also allows you to construct the Vertex Form of the parabola from the Standard Form.
- To rapidly convert a quadratic equation with an (implied) $a$-coefficient of 1 from Standard Form to Vertex Form, thus finding the vertex, use the Completing the Square procedure by replacing $x^2 + bx$ with $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$. If there is a non-1 $a$-coefficient, replace $ax^2 + bx$ with $a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2$.
- The range of a function is the set of all $y$-values that can be produced by the function. For parabolas that open upwards (the $a$ coefficient is positive), the range is all $y$-values greater than or equal to the $y$-value of the vertex. For parabolas that open downwards (the $a$ coefficient is negative), the range is all $y$-values less than or equal to the $y$-value of the vertex.
- For quadratics in Standard Form $y = ax^2 + bx + c$, the value of $c$ (the constant term) is the $y$-intercept.
- For quadratics in Standard Form $y = ax^2 + bx + c$, the value of $b$ (the $x$ coefficient) is the slope of the tangent line through the $y$-intercept. The sign of the slope of this tangent line, in conjunction with the sign of the $a$ coefficient, will indicate on which side of the $y$-axis the vertex lies.

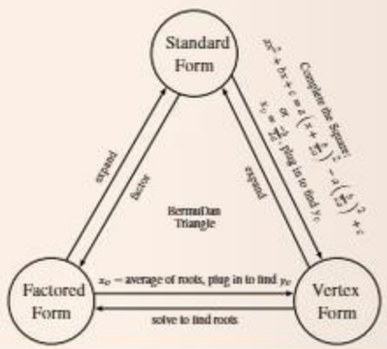
### Factored Form

- The Factored Form of a quadratic is $y = a(x + p)(x + q)$, where $a$, $p$, and $q$ are constants.
- For quadratics in Factored Form $y = a(x + p)(x + q)$, $-p$ and $-q$ are roots. If $(x - z)$ is a factor of a quadratic, then $z$ is a root of the function and vice versa.
- If $(x - r)$ is a factor and $(x - s)$ is also a factor, then the product of those factors, $(x - r)(x - s)$, is also a factor.
- The $x$-value of the vertex is exactly halfway between the two root values and can be found by averaging the two roots (add the roots and divide by 2).
- Plug the $x$-value of the vertex into the Factored Form in order to easily find the $y$-value of the vertex. This allows you to construct the Vertex Form of the parabola from the Factored Form.
- To convert from Factored Form to Standard Form, expand the terms and recombine them in order of decreasing degree.
- Polynomials will "bounce" off of the $x$-axis when the exponent of a factor is even; they will go through the $x$-axis if the exponent of a factor is odd.

**Vertex Form**

- The Vertex Form of a quadratic is $y = a(x-h)^2 + k$, where $a$, $h$, and $k$ are constants.
- For quadratics in Vertex Form $y = a(x-h)^2 + k$, the vertex of the graph is $(h, k)$ and the value of $a$ dictates the direction and elongation of the parabola.
- To convert from Vertex Form to Standard Form, expand the terms and recombine them in order of decreasing degree.

As a final note, the different forms of quadratic equations or functions can tell us different things about the graphs of those equations or functions. The Standard Form of a quadratic shows its $y$-intercept as a constant. The Factored Form of a quadratic shows its roots as constants. The Vertex Form of a quadratic shows its maximum or minimum value as a constant and shows the coordinates of its vertex as a pair of constants.








## Additional Problems

**1** 1600.io 10.4

$$h(x) = -(x-3)(x+3)$$

The function $h$ is defined as shown. For what value of $x$ does the function $h$ reach its maximum value?

A) $-3$

B) $0$

C) $3$

D) $9$

**2** 1600.io 10.4

The graphs in the $xy$-plane of the following quadratic equations each have $x$-intercepts of $-1$ and $3$. The graph of which equation has its vertex farthest from the $x$-axis?

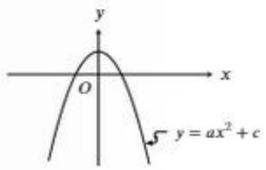
A) $y = -5(x+1)(x-3)$

B) $y = \frac{-1}{4}(x+1)(x-3)$

C) $y = \frac{1}{2}(x+1)(x-3)$

D) $y = 3(x+1)(x-3)$

**3** 1600.io 10.3




The vertex of the parabola in the $xy$-plane above is $(0, c)$. Which of the following is true about the parabola with the equation $y = -a(x+b)^2 + c$ ?

A) The vertex is $(b, c)$ and the graph opens upward.

B) The vertex is $(b, c)$ and the graph opens downward.

C) The vertex is $(-b, c)$ and the graph opens upward.

D) The vertex is $(-b, c)$ and the graph opens downward.

**4** 1600.io 10.2

| $x$ | $f(x)$ |
|---|---|
| 0 | 2 |
| 3 | 4 |
| 6 | $-2$ |
| 8 | 0 |

The function $f$ is defined by a polynomial. Some values of $x$ and $f(x)$ are shown in the table above. Which of the following must be a factor of $f(x)$ ?

A) $x - 2$

B) $x - 3$

C) $x - 5$

D) $x - 8$

**5** 10.2

$$y = (x - k)(x - h)^2$$

The equation above is graphed in the $xy$-plane. If $h$ and $k$ are positive constants and $h \neq k$, how many distinct $x$-intercepts does the graph have?

A) 1
B) 2
C) 3
D) 4

**6** 10.2

Which of the following equivalent forms of the function $f(x) = 9x^2 + 9x - 18$ is the most suitable to indicate the $x$-coordinates of the $x$-intercepts of the graph of $y = f(x)$ in the $xy$-plane?

A) $(3x - 3)(3x + 6)$
B) $3(x - 1)(3x + 6)$
C) $9(x - 1)(x + 2)$
D) $9(x^2 + x - 2)$

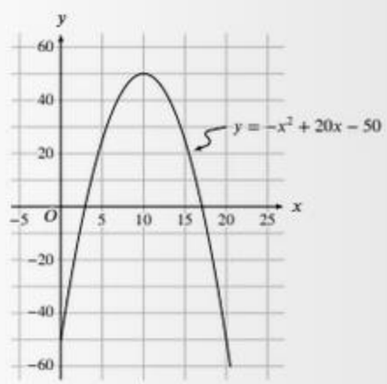
**7** 10.3

$$c(t) = -0.27(t - 26.5)^2 + 304.2$$

The function $c$ above models the calorie-burning rate of a certain runner, in calories per day (cal/day), in terms of the running time $t$, in minutes per day (min/day). What is the meaning of $(26.5, c(26.5))$ in this context?

A) The calorie-burning rate increased by $c(26.5)$ cal/day for every 26.5 min/day increase in running time.
B) The running time increases by $c(26.5)$ min/day for every 26.5 cal/day increase in calorie-burning rate.
C) The calorie-burning rate of 26.5 cal/day results in a running time of $c(26.5)$ min/day.
D) The running time of 26.5 min/day results in a calorie-burning rate of $c(26.5)$ cal/day.

**8** 10.3




The graph above models the daily profit $y$, in dollars, that a website expects to make from selling socks for a price of $x$ dollars. Based on the model, what is the maximum daily profit, in dollars? (Disregard the $ sign when writing your answer.)



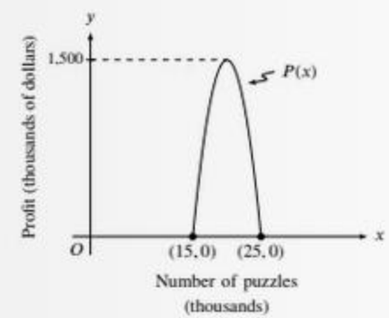


**9** 10.2

The polynomial $p^4 - 6p^3 + 5p^2 + 24p - 36$ can be written as $(p^2 - 4)(p - 3)^2$. What are all the roots of the polynomial?

A) 2 and 3
B) 3 and 4
C) −3, −2, and 2
D) −2, 2, and 3

**10** 10.4




A company produces puzzles and sells them online and in stores. The quadratic function $P$ models the company's monthly profits $P(x)$, in thousands of dollars, when $x$ puzzles, in thousands, are produced and sold. The graph of $P(x)$, where $15 \leq x \leq 25$, is shown in the $xy$-plane above. How many puzzles must the company produce and sell in order to earn the maximum profit estimated by the model?

A) 15,000
B) 20,000
C) 25,000
D) 1,500,000

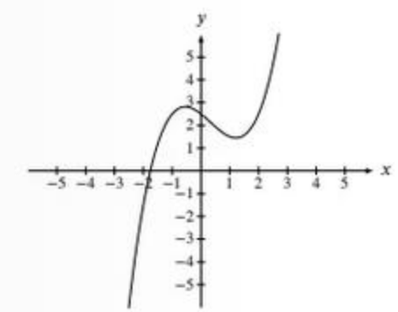
**11** 10.4

$$y = x^2 - 8x + 15$$

The equation above represents a parabola in the $xy$-plane. Which of the following equivalent forms of the equation displays the $x$-intercepts of the parabola as constants or coefficients?

A) $y - 15 = x^2 - 8x$
B) $y + 1 = (x - 4)^2$
C) $y = x(x - 8) + 15$
D) $y = (x - 3)(x - 5)$

**12** 10.6




The function $f(x) = \frac{1}{2}x^3 - \frac{1}{2}x^2 - x + \frac{5}{2}$ is graphed in the $xy$-plane above. If $k$ is a constant such that the equation $f(x) = k$ has three real solutions, which of the following could be the value of $k$ ?

A) −2
B) 0
C) 2
D) 3

**13** 1600.io 10.3

In the $xy$-plane, a parabola has vertex $(4, 2)$ and intersects the $x$-axis at two points. If the equation of the parabola is written in the form $y = -ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, which of the following could be a value of $c$ ?

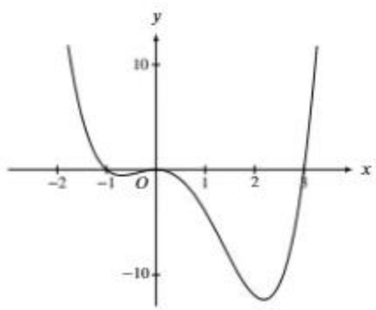
A) $-30$

B) $2$

C) $16$

D) $34$

**14** 1600.io 10.2




Which of the following could be the equation of the graph above?

A) $y = x(x - 1)(x + 3)$

B) $y = x^2(x - 1)(x + 3)$

C) $y = x(x - 3)(x + 1)$

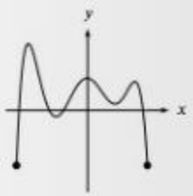
D) $y = x^2(x - 3)(x + 1)$

**15** 1600.io 10.2

If the function $f$ has five distinct zeros, which of the following could represent the complete graph of $f$ in the $xy$-plane?

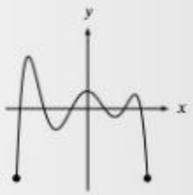
A)

B)




C)

D)




**16** 1600.io 10.2

| $x$ | $p(x)$ |
|---|---|
| $-2$ | $0$ |
| $-1$ | $2$ |
| $0$ | $4$ |
| $1$ | $0$ |
| $2$ | $-3$ |

The table above gives selected values of a polynomial function $p$. Based on the values in the table, which of the following must be a factor of $p$ ?

A) $(x - 4)$

B) $(x + 4)$

C) $(x - 2)(x + 1)$

D) $(x + 2)(x - 1)$





**17** 1600.io 10.2

In the $xy$-plane, the graph of function $f$ has $x$-intercepts at $-4$, $-3$, and $3$. Which of the following could define $f$ ?

A) $f(x) = (x - 4)(x - 3)(x + 3)$

B) $f(x) = (x - 4)(x - 3)^2$

C) $f(x) = (x - 3)(x + 3)(x + 4)$

D) $f(x) = (x + 3)^2(x + 4)$

**18** 1600.io 10.2

$$f(x) = -200x^2 + 10{,}000x$$

The number of views $f(x)$ that a content creator receives on a video is given by the function $f$ above, where $x$ is the length, in minutes, of the video. The graph of $y = f(x)$ in the $xy$-plane intersects the $x$-axis at 0 and $a$. What does $a$ represent?

A) The number of views when the length of the video is 0 minutes

B) The length, in minutes, of the video that will result in maximum views

C) The length, in minutes, of the video that will result in 0 views

D) The maximum views that the content creator can receive on the video

**19** 1600.io 10.2

In the $xy$-plane, the graph of $y = x^2 + bx + c$, where $b$ and $c$ are constants, has $x$-intercepts at $x = -5$ and $x = -2$. What is the value of $b$ ?

**20** 1600.io 10.2

Parabola $D$ in the $xy$-plane has equation $x - 2y^2 - 12y - 14 = 0$. Which equation shows the $x$-intercept(s) of the parabola as constants or coefficients?

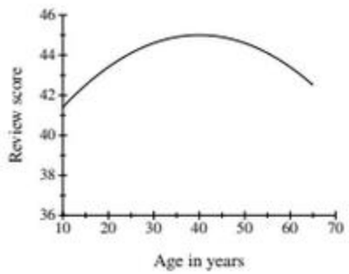
A) $x = 2y^2 + 12y + 14$

B) $x = 2(y + 3)^2 - 4$

C) $x + 4 = 2(y + 3)^2$

D) $y = -\sqrt{\frac{x + 4}{2}} - 3$

**21** 1600.io ? 10.3




A website received movie reviews from people aged 10 to 65. The graph above shows the quadratic function $R$, which models their review scores as a function of their age $x$, in years, Which of the following could define $R$ ?

A) $R(x) = \frac{1}{250}(x-45)^2 + 40$

B) $R(x) = \frac{1}{250}(x-40)^2 + 45$

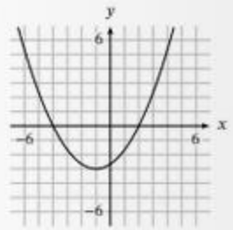
C) $R(x) = -\frac{1}{250}(x-45)^2 + 40$

D) $R(x) = -\frac{1}{250}(x-40)^2 + 45$

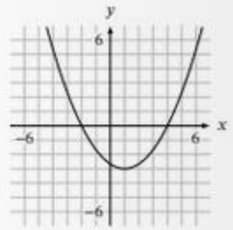
**22** 1600.io ? 10.2

The range of the polynomial function $f$ is the set of real numbers greater than or equal to $-3$. If the zeros of $f$ are $-2$ and 4, which of the following could be the graph of $y = f(x)$ in the $xy$-plane?

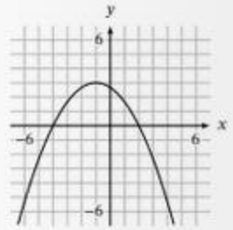
A)



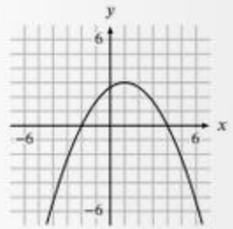

B)




C)




D)








# Answer Key

### SkillDrill 10.1-1

1. up, left, −6
2. down, left, 1
3. up, right, 5
4. down, 0, −8
5. up, right, -4
6. down, right, −3
7. up, 0, 20
8. down, left, 2

### Section 10.1 Practice Problems

1. C
2. B

### SkillDrill 10.2-1

1. −2 and 4
2. −5 and −3
3. 1 and 3
4. $\frac{1}{3}$ and $\frac{5}{2}$
5. −24 only
6. −10 and 0
7. 4 only
8. −2 and $\frac{-7}{4}$

### Section 10.2 Practice Problems

1. B
2. C
3. A
4. 12

### SkillDrill 10.3-1

1. $(1, -3)$, min, −2
2. $(-5, 10)$, max, −15
3. $(-2, 3)$, min, 7
4. $(3, 0)$, min, 9
5. $(1, 1)$, max, −1
6. $(0, 5)$, min, 5
7. $(0, 0)$, max, 0
8. $(2, -5)$, max, −13

### SkillDrill 10.3-2

1. $y = x^2 - 2x - 2$
2. $y = -x^2 - 10x - 15$
3. $y = -2x^2 + 4x - 1$
4. $y = x^2 + 5$
5. $x = -1$ and $x = 3$
6. $x = -8$ and $x = -2$
7. $x = -3$ and $x = 5$
8. $x = \pm\sqrt{5}$
9. $x = 1 \pm \sqrt{3}$
10. $x = -5 \pm \sqrt{10}$
11. $x = \frac{-3}{2}$ and $x = \frac{-1}{2}$
12. $x = 6 \pm \sqrt{\frac{3}{5}}$

### Section 10.3 Practice Problems

1. A
2. D

### SkillDrill 10.4-1

1. $(1, -9)$, $y = (x-1)^2 - 9$
2. $(-4, -1)$, $y = (x+4)^2 - 1$
3. $(2, 1)$, $y = -(x-2)^2 + 1$
4. $(-6, 0)$, $y = (x+6)^2$
5. $(-5, -50)$, $y = 2(x+5)^2 - 50$
6. $(4, 0)$, $y = 3(x-4)^2$
7. $\left(\frac{3}{2}, -4\right)$, $y = 4\left(x - \frac{3}{2}\right)^2 - 4$
8. $\left(\frac{-3}{4}, \frac{1}{2}\right)$, $y = -8\left(x + \frac{3}{4}\right)^2 + \frac{1}{2}$

### Section 10.4 Practice Problems

1. C
2. A

### SkillDrill 10.5-1

1. $y = (x+1)^2 - 5$
2. $y = (x-3)^2 + 11$
3. $y = (x+5)^2 - 40$
4. $y = (x-2)^2 - 11$
5. $y = (x-4)^2 - 13$
6. $y = (x+10)^2$
7. $y = (x+7)^2 - 17$
8. $y = x^2 - 10$
9. $y = \left(x + \frac{5}{2}\right)^2 - \frac{29}{4}$
10. $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$
11. $y = 2(x+2)^2 + 2$
12. $y = -3(x-2)^2 + 24$

### Section 10.5 Practice Problems

1. C
2. D

### Section 10.6 Practice Problems

1. B
2. C

**Additional Problems**

1. B
2. A
3. C
4. D
5. B
6. C
7. D
8. 50
9. D
10. B
11. D
12. C
13. A
14. D
15. C
16. D
17. C
18. C
19. 7
20. A
21. D
22. B

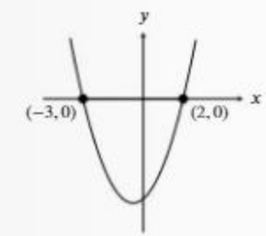


# Number of Zeros/Imaginary and Complex Numbers

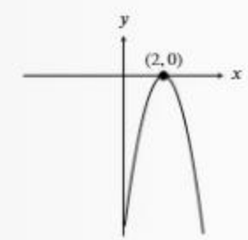
# 11

## 11.1 Number of Zeros of Quadratics

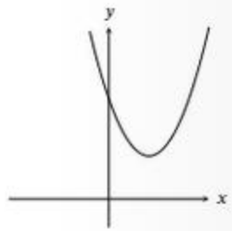
We've seen in the last few chapters that the zeros ($x$-intercepts) of a polynomial function correspond to the factors of that polynomial. Focusing on quadratics, we know that factorable quadratic equations can have **two** real factors. For example, the quadratic (written here in Factored Form) $y = (x - 2)(x + 3)$ has two zeros ($x$-intercepts) at $x = 2$ and $x = -3$, and its graph crosses the $x$-axis at the points $(2, 0)$ and $(-3, 0)$. We could also say that the equation $(x - 2)(x + 3) = 0$ has two solutions: $x = 2$ and $x = -3$.




We also have seen quadratics with only **one** factor (of degree 2) that only have a single zero or $x$-intercept. For example, the quadratic (shown in Vertex Form) $y = -2(x - 2)^2$ has one zero at $x = 2$, and its graph "bounces off" the $x$-axis at the point $(2, 0)$. Put another way, $-2(x - 2)^2 = 0$ has one solution: $x = 2$.




What happens when a quadratic has **no** factors and, when graphed, doesn't even intersect the $x$-axis? For example, look at the Standard Form quadratic $y = x^2 - 4x + 7$, which never touches the $x$-axis. That is, there's no value on the $x$-axis that makes $x^2 - 4x + 7 = 0$ true.




Examining the graph above, it appears that the quadratic function it represents has no zeros in the sense that there are no values on the $x$-axis for which the $y$-value will be 0 when the function is evaluated. This means, for example, that try as you might, you won't be able to factor it, because you won't be able to find two numbers whose sum is the $b$ coefficient and whose product is the $c$ coefficient.

But if there are no roots to this function, and we have methods such as Completing the Square and the Quadratic Formula that can find the roots of *any* quadratic function, what happens when we deploy those all-powerful solving tools on an equation that apparently has no solutions? Does the universe collapse to a singularity? Buckle up, because we're about to find out.

Let's attempt to find the roots of the function $f(x) = x^2 - 4x + 7$ using both Completing the Square and the Quadratic Formula.

| **Completing the Square** | **Quadratic Formula** |
|---|---|
| $x^2 - 4x + 7 = 0$ | $a = 1, b = -4, c = 7$ |
| $x^2 - 4x = -7$ | $x = \frac{-(-4)}{2(1)} \pm \frac{\sqrt{(-4)^2 - 4(1)(7)}}{2(1)}$ |
| $(x-2)^2 = 4 - 7$ | $x = \frac{4}{2} \pm \frac{\sqrt{16-28}}{2}$ |
| $x - 2 = \pm\sqrt{-3}$ | $x = 2 \pm \frac{\sqrt{-12}}{2}$ |
| $x = 2 \pm \sqrt{-3}$ | $x = 2 \pm \frac{\sqrt{4(-3)}}{2}$ |
| $x = 2 + \sqrt{-3}, x = 2 - \sqrt{-3}$ | $x = 2 \pm \frac{2\sqrt{-3}}{2}$ |
| | $x = 2 \pm \sqrt{-3}$ |
| | $x = 2 + \sqrt{-3}, x = 2 - \sqrt{-3}$ |

Assuming we're all still here, you can see that we've found the zeros, and they have a distinctive form, as they consist of two terms that can't be further combined: a number and **a square root of a negative number**.

We previously explained that the square of a number can't be negative, regardless of the sign of the original number, so the square root of a negative number can't be a real number. And, indeed, that is exactly the label we apply to the types of numbers we've been dealing with up until now: **real numbers**.

A real number is an integer, such as 7 or −13; a fraction made from integers (a rational number), such as $\frac{3}{5}$ or 0.7 (which can be expressed as $\frac{7}{10}$); or an irrational number, such as $\sqrt{2}$ or $\pi$. As noted, any real number when squared produces a non-negative





result, so that $\sqrt{-3}$ term we got when finding the roots of the quadratic above must not represent a real number. Instead, it represents what is called an **imaginary number. Square roots of negative numbers are imaginary numbers**, and to facilitate referring to and manipulating them, we use a special symbol that represents the square root of −1: we say that

$$\sqrt{-1} = i$$

where $i$ stands for "imaginary." This is handy because we can then form any negative square root by multiplying $i$ by a real number; for example, here we can write

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

Note that by convention, we write the real number first as a coefficient, then the imaginary number next. Sometimes, depending on the value we're working with, we can completely eliminate the square root; for example, if we had $\sqrt{-4}$, we could convert that to

$$\sqrt{4(-1)} = \sqrt{4}\sqrt{-1} = 2i$$

As you can see, we can multiply an imaginary number by a real number to get a term like $\sqrt{3}i$ or $2i$ (and, it follows, the same is true of division). We already know that we can add real numbers together, of course; $5 + 7 = 12$. Similarly, we can add imaginary numbers: $5i + 7i = 12i$. Here, $i$ behaves like a variable or a letter that stands in for a constant. However, we can't add a real number and an imaginary number to get a single term; something like $2 + \sqrt{-3}$ can't be further simplified, as **a real term and an imaginary term can't be combined into a single term through addition or subtraction**; once you've combined all the real terms and all the imaginary terms in an expression, you will end up with one real term and one imaginary term in the general form $a + bi$, where $a$ and $b$ are real constants. This is what resulted from our attempts to solve the example quadratic we're working with, where we got $x = 2 + \sqrt{-3}$ (or $x = 2 + \sqrt{3}i$) and $x = 2 - \sqrt{-3}$ ($x = 2 - \sqrt{3}i$). An expression like this, with one real term added to an imaginary term in the form $a + bi$, is called a **complex number**, so **a quadratic function whose graph never intersects the $x$-axis has two roots that are complex numbers**. (A minor technicality: real numbers are complex numbers where the coefficient of the imaginary part is 0.)

Let's summarize: A quadratic function can have **two real roots**, and the graph of such a function will intersect the $x$-axis (which represents real numbers) at two distinct points. A quadratic function can have **one real root** (a double root), and so the graph will touch the $x$-axis at just one point. Finally, a quadratic function can have **two complex roots**, and its graph won't intersect the real number $x$-axis at all. (Note that you might hear the phraseology "two imaginary roots" thrown around elsewhere in reference to the last case, but that's not really right; the roots are complex numbers, which can include both a real part and an imaginary part, not just imaginary numbers alone.)

We will discuss more about imaginary and complex numbers later in the chapter. For now, however, let's make use of the Quadratic Formula to explain the logic behind a simple test for finding the number and types of zeros for any quadratic, which is often the fastest route through quadratic problems that ask you about the number of zeros. (We're using the terms "zero" and "root" depending on our mood, but remember that these mean the same thing in this context; we just want you to get used to seeing either term.)

For a quadratic equation in Standard Form, $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, the zeros of the equation can be found using the Quadratic Formula.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula produces two results, which reflects the fact that there is an expression, $b^2 - 4ac$, under a square root symbol, and both the positive and negative roots must be used. Because only a positive number has different positive and negative real square roots (e.g. $\sqrt{4} = 2$, so $\pm\sqrt{4}$ equals both 2 and $-2$), and zero has just zero as its one square root ($\pm\sqrt{0}$ is just 0), we can use that $b^2 - 4ac$ expression to figure out whether there will be two real zeros, just one, or none (in which case the two zeros will result from taking the square root of a negative number, and thus they will include an imaginary number term). Because the expression $b^2 - 4ac$ helps us discriminate among these possibilities, we call it **the discriminant**.

## The Discriminant

For a quadratic equation in Standard Form, $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, the discriminant is the expression $b^2 - 4ac$, which is found in the radical in the Quadratic Formula.

It is simple enough to find the number of solutions to a quadratic equation if you can factor the equation because the number of unique factors is also the number of zeros. However, when a quadratic is not easily factorable or you suspect it doesn't have any real solutions ($x$-intercepts), the easiest way to tell how many solutions the quadratic has is to make use of the **discriminant**. **The value of the discriminant, $b^2 - 4ac$, will indicate the number and type of zeros of a quadratic equations**.

- **$b^2 - 4ac > 0$: When the value of the discriminant is positive, there are TWO REAL number zeros**.

  Note that the fact that there are two real zeros does *not* mean that those zeros are both integers or that the quadratic would be easily factorable.

- **$b^2 - 4ac = 0$: When the value of the discriminant is equal to 0, there is ONE REAL number zero (a double zero)**.

- **$b^2 - 4ac < 0$: When the value of the discriminant is negative, there are NO REAL zeros, but there are TWO COMPLEX NUMBER zeros**.

  The two complex roots are **complex conjugates** of one another (this term will be discussed more in the next section), which means that if $a + bi$ is a root, then $a - bi$ is the other root.

We will look more closely at complex numbers and imaginary solutions momentarily, but for now, we just need to know how to determine the number and type of zeros for quadratic equations (or systems of equations).

**Example 11.1-1**

**1** 1600.io 11.1

$$3x^2 - 6x = t$$

In the equation above, $t$ is a constant. If the equation has no real solutions, which of the following could be the value of $t$ ?

A) $-4$

B) $-3$

C) $3$

D) $4$





**Solution**

1. Subtract $t$ from both sides of the equation to make a quadratic equation in Standard Form.
$$3x^2 - 6x = t$$
$$3x^2 - 6x - t = 0$$

2. We need to find a value of $t$ that causes there to be no real solutions, so we should set the discriminant less than 0. The value of $b$ is $-6$; the value of $a$ is 3; the value of $c$ is $-t$.
$$b^2 - 4ac < 0$$
$$(-6)^2 - 4(3)(-t) < 0$$
$$36 + 12t < 0$$

3. Subtract 36 from both sides of the inequality.
$$36 + 12t < 0$$
$$12t < -36$$

4. Divide both sides of the inequality by 12 to solve for $t$.
$$12t < -36$$
$$t < -3$$

5. The value of $t$ must be less than $-3$, and choice A, $-4$, is the only choice that is less than $-3$.

6. The answer is A.

It should be noted that the discriminant is not necessary for every problem in which you are asked about the number of roots or zeros for a quadratic. **The use of the discriminant should be a last resort**.

In general, you should try to find the number of zeros in the following ways:

- If the quadratic is in Standard Form and can be easily factored, then there are two real zeros if the two factors are different; if the two factors are the same (the quadratic expression is a perfect square), there is one real zero.

- If the quadratic is in Standard Form, you can convert it to Vertex Form by Completing the Square. Once it is in Vertex Form, you know the sign of the $y$-coordinate of the vertex point and whether the quadratic opens up or down and therefore whether (and how many times) it will intersect the $x$-axis, so you can easily tell the number and type of zeros without the discriminant.

- If the quadratic is in Standard Form, cannot be easily factored, and you're in denial about Completing the Square being awesome, then you can use the discriminant to determine the number of roots (just because you can't factor it doesn't mean it does not have real zeros).

- If the quadratic is in Factored Form with two *different* factors, then it has two real zeros.

- If the quadratic is a perfect square expression (there is one factor squared), then it has one real zero.

- If the quadratic is in Vertex Form, then you can visualize or draw a quick picture that will tell you how many real roots there are. You will know the sign of the $y$-coordinate of the vertex point and whether the quadratic opens up or down, so the discriminant is unnecessary.

  For example, the quadratic $y = -2(x - 1)^2 + 4$ has its vertex at $(1, 4)$, which is **above** the $x$-axis, and the parabola **opens downward** because the $a$-coefficient is $-2$, so the parabola will intersect the $x$-axis twice, meaning that there are two real zeros or roots.

All of the same principles apply when working with a system of equations. If you can graph both of the equations easily, then the number of intersections will tell you the number of solutions.

If you use substitution to collapse the system into one quadratic equation, then the number of real zeros of that new equation will tell you the number of solutions to the system. You can use the same procedures described previously for determining the characteristics of the zeros of the collapsed equation.

All of the methods explained above have test problems for which they are well-suited and ones for which they are ill-suited. Luckily, problems involving the number of solutions to a quadratic are not very common, so if you do encounter them on the test, hopefully the rest of your skills are sharp enough that you can afford to spend a little time trying more than one method to verify your findings on these problems.

**SkillDrill 11.1-1**

**Directions:** Using any method, determine the number and type of distinct roots for the following quadratics, which are given in Standard, Factored, or Vertex Form (you do not have to find the values of those zeros).

**1.** $y = (x-1)^2 - 3$ **2.** $y = x^2 - 5x + 4$ **3.** $y = x^2 - 3x + 4$ **4.** $y = x^2 - 8x + 16$

**5.** $y = -2(x+4)(x+2)$ **6.** $y = -(x+5)^2 + 10$ **7.** $y = 2(x-1)^2 + 1$ **8.** $y = x^2 + 5$

**9.** $y = (x+6)^2$ **10.** $y = -x^2 - 6x - 9$ **11.** $y = -x^2 - 5x - 8$ **12.** $y = -x^2 + 7x - 12$

**Section 11.1 Suggested Problems from Real Tests**

- Test 2-C-29
- Test 4-NC-11
- Test 5-NC-4
- Test 6-NC-13

**View related real-test problems in SMTEx**

SMTEx

- 2019 (Jan)-C-34
- 2020-C-26
- 2020-NC-14
- 2020-C-27
- Apr 2019-C-17
- May 2021 (US)-C-15





## Section 11.1 Practice Problems

**1** 11.1

$$x^2 + bx + 9 = 0$$

In the quadratic equation shown, $b$ is a constant. For what values of $b$ does the equation have only one solution?

A) $-3$ only

B) $-6$ only

C) $-3$ and $3$

D) $-6$ and $6$

**2** 11.1

$$y = 5$$
$$y = ax^2 + b$$

In the system of equations above, $a$ and $b$ are constants. For which of the following values of $a$ and $b$ does the system of equations have exactly two real solutions?

A) $a = -2, b = 4$

B) $a = -2, b = 5$

C) $a = -2, b = 6$

D) $a = 2, b = 5$

## 11.2 Imaginary and Complex Numbers

We previously explored the idea that a quadratic function might have two complex roots, and we explained that complex numbers consist of a real part and an imaginary part that includes the special imaginary number $i$, which represents $\sqrt{-1}$. Let's dig into that topic more deeply, as there are problems on the test for which a good working knowledge of imaginary and complex numbers is required.

### Imaginary Numbers

There exists an imaginary number called $i$ that is equal to $\sqrt{-1}$.

Because we designate $i$ as the square root of $-1$, we know that $i^2 = -1$. Let's look at some higher powers of $i$:

| $i^3$ | $i^4$ | $i^5$ |
|---|---|---|
| $i^3 = i \cdot i \cdot i$ | $i^4 = i \cdot i \cdot i \cdot i$ | $i^5 = i \cdot i \cdot i \cdot i \cdot i$ |
| $i^3 = i^2 \cdot i$ | $i^4 = i^2 \cdot i^2$ | $i^5 = i^4 \cdot i$ |
| $i^3 = -1 \cdot i$ | $i^4 = -1(-1)$ | $i^5 = 1 \cdot i$ |
| $i^3 = -i$ | $i^4 = 1$ | $i^5 = i$ |

You'll notice that $i^5 = i$, or, put more clearly, $i^5 = i^1$; we're back where we started. This indicates that a cycle is beginning again, and therefore there will be a repetition of the four-value sequence shown above as the exponent increases in increments of 1. In fact, $i^6 = i^2$, $i^7 = i^3$, $i^8 = i^4$, and so on…
To summarize:

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$
- $i^5 = \sqrt{-1}$
- $i^6 = -1$
- $i^7 = -i$
- $i^8 = 1$
- $i^9 = \sqrt{-1}$
- $i^{10} = -1$
- $i^{11} = -i$
- $i^{12} = 1$

If asked to evaluate the value of $i$ terms with large exponents, find a multiple of 4 that is close to that exponent (any time $i$ is raised to a power that is a multiple of 4, it will evaluate to 1) and count up or down from there to see which of the four primary powers of $i$ it corresponds to.

For example $i^{26}$ is equal to $i^2$ (or $-1$) because 26 is 2 more than 24 (a multiple of 4). Similarly, $i^{104}$ is equal to $i^4$ (or 1) because 104 is a multiple of 4.

**Example 11.2-1**

**1** 1600.io 11.2

In the complex number system, what is the value of the expression $13i^4 - 7i^2 + 2$ ? (Note: $i = \sqrt{-1}$)

**Solution**

1. Since $i^4 = 1$, replace $i^4$ with 1. Since $i^2 = -1$, replace $i^2$ with $-1$.

$$\begin{aligned} &13i^4 - 7i^2 + 2 \\ &13(1) - 7(-1) + 2 \\ &13 + 7 + 2 \\ &22 \end{aligned}$$

2. The answer is 22.

To reiterate what we explained earlier, complex numbers consist of a real part and an imaginary part and are written in the form $a + bi$, where $a$ and $b$ are real constants; $a$ is the real part of the complex number, and $b$ is the coefficient of the imaginary part $i$ of the complex number.

Complex numbers can be added to (or subtracted from) each other by simply combining like terms—add the real parts to each other and add the imaginary parts together (treat $i$ like a variable). As previously explained, you can't add a real number and an imaginary number to get a single value.





**Example 11.2-2**

**2** 1600.io 11.2

$$(2 + 6i) - (1 - 3i)$$

In the complex number system, which of the following is equivalent to the expression above? (Note: $i = \sqrt{-1}$)

A) 4
B) $3 + 3i$
C) $1 + 9i$
D) $1 - 9i$

**Solution**

1. Combine the real terms and combine the imaginary terms. Remember to distribute the negative sign.

$$\begin{aligned} &(2 + 6i) - (1 - 3i) \\ &2 + 6i - 1 + 3i \\ &(2 - 1) + (6i + 3i) \\ &1 + 9i \end{aligned}$$

2. The answer is C.

**SkillDrill 11.2-1**

**Directions:** Write the complex number expressions in the form $a + bi$.

**1.** $(2 + 3i) + (3 + 4i)$
**2.** $(4 + i) - (6 + 3i)$
**3.** $i^2 - i$
**4.** $i^3 + 2i$
**5.** $(5 - 3i) + (8 - 6i)$
**6.** $3i^{12} + 2i^7$
**7.** $4i^6 - 2i^{13}$
**8.** $-3i^4 + 2i^3 - 5i^2 + i$

Going one step further, complex numbers can be multiplied by each other much the same way polynomials are multiplied by each other: multiply each value in one expression by each value in the other expression.

**Example 11.2-3**

**3** 1600.io 11.2

$$(2 + 6i)(1 - 4i) = a + bi$$

In the equation above, $a$ and $b$ are real numbers and $i = \sqrt{-1}$. What is the value of $a$ ?

**Solution**

1. Multiply the two complex numbers in parentheses to expand the terms. Make sure you write the $i^2$ term before you attempt to substitute $-1$ for $i^2$—many students make a mistake here trying to do too many things at once in their heads. It is too easy to make a mistake with the negative sign.

$$\begin{aligned}(2+6i)(1-4i) &= a+bi\\ 2(1)+2(-4i)+6i(1)+6i(-4i) &= a+bi\\ 2-8i+6i-24i^2 &= a+bi\\ 2-2i-24i^2 &= a+bi\end{aligned}$$

2. Since $i^2=-1$, replace $i^2$ with $-1$.

$$\begin{aligned}2-2i-24(-1) &= a+bi\\ 2-2i+24 &= a+bi\\ 26-2i &= a+bi\end{aligned}$$

3. By matching corresponding components of the complex numbers on each side of the equation, we can see that $a=26$.
4. The answer is 26.

**Notes**

The astute student will notice that because the problem asks only for the value of the real number $a$ in the expression $a+bi$, one could ignore the multiplications that will produce imaginary numbers (that is, where exactly one of the operands contains $i$) and only perform the multiplications that will produce real values; that's $2(1)$ and $6i(-4i)$, whose sum is $2-24i^2=2-(24(-1))=26$. You must be very attentive when taking optimized approaches of this sort, but they are valid for the proficient student.

Finally, the last thing you will be tested on involving complex numbers occurs when there is a complex number in the denominator of a fraction. To solve any known problem from previous tests where there is a complex number in the denominator, you will need to find a way to rewrite the fraction so that there are no imaginary numbers in the denominators of any fraction.

We would like to have some operation that will eliminate the imaginary number in the denominator. What can we multiply a complex number by that will result in the imaginary number becoming a real number? Merely multiplying a fraction by $\frac{i}{i}$ won't do it, because then the $a$ term (the real part of the complex number) in the denominator's complex number will become $ai$ (the product of the real part and $i$ will be imaginary), but if we multiply the denominator by the same expression but with the addition (or subtraction) operation negated, we'll get two $i$ terms that will have the same absolute value, but they will be opposite in sign, so they'll cancel out (recall the difference of squares and its factorization).

So, if we have an expression, $a+bi$, we'd multiply it by $a-bi$:

$$\begin{aligned}&(a+bi)(a-bi)\\ &a^2-a(bi)+bi(a)-bi(bi)\\ &a^2+abi-abi-b^2i^2\end{aligned}$$

You'll notice that the two middle terms cancel, leaving:

$$a^2-b^2i^2$$





We do still have a term with $i$ in it, but because it's $i^2$, we can replace that with $-1$, so we no longer have any imaginary numbers, which was our goal:

$$\begin{aligned}&a^2-\left(b^2\right)(-1)\\ &a^2+b^2\end{aligned}$$

Note that because both $a$ and $b$ are real constants, we'll be able to calculate a single real value for the denominator when we apply this technique (the numerator is on its own!).

This expression $(a-bi)$ that we can use to eliminate the imaginary number is called the **complex conjugate** of the original complex number.

For example, if we had $\frac{3-8i}{2-4i}$, we could multiply the fraction by the denominator's complex conjugate, which is $2+4i$, over itself (so we're just multiplying the fraction by 1) in order to write the value with a real number denominator. Here's the calculation:

$$\begin{aligned}&\frac{3-8i}{2-4i}\cdot\frac{2+4i}{2+4i}\\ &\frac{(3-8i)(2+4i)}{(2-4i)(2+4i)}\\ &\frac{3(2)+3(4i)-8i(2)-8i(4i)}{2(2)+2(4i)-4i(2)-4i(4i)}\\ &\frac{6+12i-16i-32i^2}{4\cancel{+8i}\cancel{-8i}-16i^2}\\ &\frac{6-4i-32i^2}{4-16i^2}\end{aligned}$$

Notice how the two middle terms in the denominator cancel out (exactly like they do with the difference of squares). Replacing $i^2$ with $-1$, we get

$$\begin{aligned}&\frac{6-4i-32(-1)}{4-16(-1)}\\ &\frac{6-4i+32}{4+16}\\ &\frac{38-4i}{20}\end{aligned}$$

Note that the fraction above may be split into two terms (and reduced) in order to match the general complex number form $a+bi$:

$$\frac{38-4i}{20}=\frac{38}{20}-\frac{4i}{20}=\frac{19}{10}-\frac{i}{5}$$

To summarize, **in order to rewrite the fraction so that there are no imaginary numbers in the denominator, you need to multiply both the numerator and denominator by the complex conjugate of the denominator**.

## Complex Conjugate

For a complex number $a + bi$, there exists a complex conjugate $a - bi$.

For example, the complex conjugate of $2 - 4i$ is $2 + 4i$, and vice versa.

To reiterate, multiplying by the conjugate of the denominator results in a difference of squares expression where the $i$ terms will cancel out, leaving only a real number in the denominator.

**SkillDrill 11.2-2**

**Directions:** Write the complex number expressions in the form $a + bi$. Multiply both the numerator and denominator by the complex conjugate of the denominator. Express any fractions in lowest terms.

1. $\frac{2+3i}{3+4i}$
2. $\frac{4+i}{6-3i}$
3. $\frac{-2-3i}{-1-4i}$
4. $\frac{1+i}{2-2i}$
5. $\frac{8-4i}{2-i}$
6. $\frac{5+6i}{6-5i}$
7. $\frac{3+2i}{3-2i}$
8. $\frac{10+10i}{6-8i}$

**Example 11.2-4**

**4** 1600.io 11.2

Which of the following complex numbers is equivalent to $\frac{7+i}{4-3i}$ ? (Note: $i = \sqrt{-1}$)

A) $\frac{7}{4} - \frac{1}{3}i$

B) $\frac{7}{4} + \frac{1}{3}i$

C) $5 + 5i$

D) $1 + i$

**Solution**

1. Multiply the numerator and denominator by the complex conjugate of the denominator. The denominator is $4 - 3i$, so its conjugate is $4 + 3i$.

$$\frac{7+i}{4-3i}$$

$$\frac{(7+i)(4+3i)}{(4-3i)(4+3i)}$$





2. Multiply the complex numbers in the numerator and denominator. Combine like terms to simplify.

$$\frac{(7+i)(4+3i)}{(4-3i)(4+3i)}$$

$$\frac{7(4)+7(3i)+i(4)+i(3i)}{4(4)+4(3i)-3i(4)-3i(3i)}$$

$$\frac{28+21i+4i+3i^2}{16+12i-12i-9i^2}$$

$$\frac{28+25i+3i^2}{16-9i^2}$$

3. Since $i^2 = -1$, replace $i^2$ with $-1$.

$$\frac{28+25i+3(-1)}{16-9(-1)}$$

$$\frac{28+25i-3}{16+9}$$

$$\frac{25+25i}{25}$$

4. Factor 25 out of the numerator since both terms are divisible by 25. Since the factor of 25 in the numerator will cancel with the denominator, we will be left with a complex number that has a separate real part and imaginary part.

$$\frac{25+25i}{25}$$

$$\frac{25(1+i)}{25}$$

$$\frac{\cancel{25}(1+i)}{\cancel{25}}$$

$$1+i$$

5. The answer is D.

**Section 11.2 Suggested Problems from Real Tests**

- Test 1-NC-2
- Test 2-NC-11
- Test 4-NC-14
- Test 6-NC-3
- 2019-NC-14
- 2019 (US)-NC-14
- May 2018-NC-20

View related real-test problems in SMTEx

SMTEx

## Section 11.2 Practice Problems

**1** 1600.io 11.2

Which of the following complex numbers is equal to $(1 + 11i) - (8i^2 - 2i)$, for $i = \sqrt{-1}$ ?

A) $9 + 13i$

B) $7 + 9i$

C) $-7 - 9i$

D) $-9 - 13i$

**2** 1600.io 11.2

$$\frac{2 + 4i}{9 - i}$$

If the expression above is rewritten in the form $a + bi$, where $a$ and $b$ are real numbers, what is the value of $a$ ?

A) $\frac{7}{41}$

B) $\frac{2}{9}$

C) $\frac{19}{41}$

D) $\frac{1}{2}$





## CHAPTER 11 RECAP

- For quadratics in Standard Form, $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants, the discriminant, $b^2 - 4ac$, indicates the number of real zeros.
- When the discriminant is positive, there are 2 real zeros or roots.
- When the discriminant is equal to 0, there is one real zero or root.
- When the discriminant is negative, there are no real zeros or roots, but there are 2 complex zeros or roots, and they are the complex conjugates of one another.
- If a Standard Form quadratic is factorable (or is written in Factored Form to begin with), then it has either one (if the quadratic is a perfect square expression) or two real zeros or roots. There is no need to use the discriminant to check for the number of zeros if you can factor the quadratic.
- If a Vertex Form quadratic is given (or you can use Completing the Square to rewrite a Standard Form quadratic in Vertex Form), you can simply visualize or sketch the parabola to determine the number of intersections with the $x$-axis (these are $x$-intercepts, which will represent the zeros or roots) because you will know the position of the vertex and whether the parabola opens up or down. Once again, you can forego the use of the discriminant.
- There is an imaginary number $i$, and $i = \sqrt{-1}$. It follows that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$; as with all bases, $i^0 = 1$. For powers higher than 4, the pattern repeats for every set of 4. For example, the next four powers of $i$ are as follows:

$$i^5 = i$$
$$i^6 = -1$$
$$i^7 = -i$$
$$i^8 = 1$$

  If asked to evaluate the value of $i$ terms with large exponents, find a multiple of 4 that is close to that exponent and count up or down from there to see which of the four primary powers of $i$ it corresponds to. For example $i^{35}$ is equal to $i^3$ because 35 is 3 more than 32 (a multiple of 4). Similarly, $i^{73}$ is equal to $i$ because 73 is 1 more than 72 (a multiple of 4).
- The complex conjugate of a complex number $a + bi$ is $a - bi$.
- If there is a complex number in the denominator of a fraction, multiply the numerator and denominator by the complex conjugate of the denominator.

## Additional Problems

**1** 11.1

In the $xy$-plane, a line that has the equation $y = c$ for some constant $c$ intersects a parabola at exactly one point. If the parabola has the equation $y = -4x^2 + 3x$, what is the value of $c$ ?

**2** 11.1

$$y = x^2 - 2x + 7$$
$$y - 2x - 3 = 0$$

How many solutions are there to the system of equations above?

A) There are exactly 4 solutions.

B) There are exactly 2 solutions.

C) There is exactly 1 solution.

D) There are no solutions.

**3** 11.1

Which of the following equations has a graph in the $xy$-plane with no $x$-intercepts?

A) $y = 3x + 2$

B) $y = -4x^2$

C) $y = x^2 + 6x + 10$

D) $y = x^2 - 6x - 7$

**4** 11.1

$$x^2 - ax + 20 = 0$$

In the equation above, $a$ is a constant and $a > 0$. If the equation has two integer solutions, what is a possible value of $a$ ?





**5** 11.2

$$i^4 + (-i)^4$$

In the complex number system, what is the value of the given expression? (Note: $i = \sqrt{-1}$)

A) $-2$

B) $0$

C) $2$

D) $2i$

**6** 11.1

If $4x^2 + bx + 4 = 0$, where $b$ is a constant, has exactly one solution, what is a possible value of $b$ ?

A) 4

B) 8

C) 16

D) 64

**7** 11.2

$$(10 - 2i)(4 + 5i) = a + bi$$

In the equation above, $a$ and $b$ are real numbers and $i = \sqrt{-1}$. What is the value of $b$ ?

**8** 11.2

In the complex number system, what is the value of the expression $3i^4 - 5i^2 + 1$ ? (Note: $i = \sqrt{-1}$)

**9** 1600.io 11.1

$$x = 2y + 2$$
$$y = (3x - 2)(x + 4)$$

How many ordered pairs $(x, y)$ satisfy the system of equations shown above?

A) 0

B) 1

C) 2

D) Infinitely many

**10** 1600.io 11.2

What is the sum of the complex numbers $3 + 7i$ and $1 + 5i$, where $i = \sqrt{-1}$ ?

A) 16

B) $16i$

C) $3 + 35i$

D) $4 + 12i$

**11** 1600.io 11.1

$$kx - y = 3$$
$$y = x^2 - k$$

In the system of equations above, $k$ is a constant. When the equations are graphed in the $xy$-plane, the graphs intersect at exactly two points. Which of the following CANNOT be the value of $k$ ?

A) 2

B) 3

C) 4

D) 5

**12** 1600.io 11.2

What is the sum of the complex numbers $5 + 3i$ and $7 + 2i^2$ ? (Note: $i = \sqrt{-1}$)

A) $10 + 3i$

B) $12 - i$

C) $12 + 5i^3$

D) $14 + 3i$





# Answer Key

### SkillDrill 11.1-1

1. 2 real roots
2. 2 real roots
3. 2 complex roots
4. 1 real root
5. 2 real roots
6. 2 real roots
7. 2 complex roots
8. 2 complex roots
9. 1 real root
10. 1 real root
11. 2 complex roots
12. 2 real roots

### Section 11.1 Practice Problems

1. D
2. C

### SkillDrill 11.2-1

1. $5 + 7i$
2. $-2 - 2i$
3. $-1 - i$
4. $i$
5. $13 - 9i$
6. $3 - 2i$
7. $-4 - 2i$
8. $2 - i$

### SkillDrill 11.2-2

1. $\frac{18}{25} + \frac{i}{25}$
2. $\frac{7}{15} + \frac{2i}{5}$
3. $\frac{14}{17} - \frac{5i}{17}$
4. $\frac{i}{2}$
5. 4
6. $i$
7. $\frac{5}{13} + \frac{12i}{13}$
8. $\frac{-1}{5} + \frac{7i}{5}$

### Section 11.2 Practice Problems

1. A
2. A

### Additional Problems

1. $\frac{9}{16}$, .562, or .563
2. C
3. C
4. 9, 12, or 21
5. C
6. B
7. 42
8. 9
9. C
10. D
11. A
12. A

# Ratios, Probability, and Proportions 12

## 12.1 Ratios and Probability

**Ratios** are used to show the relative size of two or more values. Fractions are one common way of expressing a ratio between exactly two values. For example, the fraction $\frac{4}{3}$ is a ratio, but if we were to use **ratio notation**, which juxtaposes the values in a list delimited by a character that looks like a colon, it would be expressed as $4:3$, which would most commonly be read as "4 to 3."

### Representations of Ratios

The ratio $\frac{a}{b}$ can be represented in several other ways.

- $a:b$
  For example, $\frac{2}{5}$ can be written as $2:5$, as in "the ratio of the length to the height of the box is $2:5$."
- $a$ to $b$
  For example, $\frac{2}{5}$ can be written as 2 to 5.
- $a$ in $b$ (used with probabilities and subsets)
  For example, $\frac{2}{5}$ can be written as 2 in 5, as in "2 in 5 days are rainy" or "the probability is 2 in 5 that the next card dealt will be an ace."
- Decimal representation
  For example, $\frac{2}{5}$ can be written as 0.4, as in "the probability of hitting a red light on the way home is 0.4."

Fractions are limited to two parts, the numerator and denominator, but ratio notation can be used to show the relative values of any number of terms. For example, the side lengths of a triangle might be in a ratio of $3:4:5$ ("3 to 4 to 5").

### Reducing Ratios

Reducing ratios works exactly like reducing fractions. Just as the fraction $\frac{4}{10}$ reduces to $\frac{2}{5}$, so too does the ratio $4:10$ reduce to $2:5$.

The process works the same way for ratios with more than two terms. For example, a ratio of $6:8:10$ can be reduced by dividing all of the terms by 2, resulting in a reduced ratio of $3:4:5$.

## Parts to Parts and Parts to Whole

If you are told that a snack mixture contains 2 parts of peanuts and 5 parts of pretzels, then there are a total of 7 parts in the mixture.

If you are asked about the ratio of peanuts to pretzels, then you would say "the amount of peanuts is $\frac{2}{5}$ the amount of pretzels," or "the ratio of peanuts to pretzels is $\frac{2}{5}$."

On the other hand, if you are asked about the portion of the *whole* mixture that is made up of peanuts, then you would say "the amount of peanuts is $\frac{2}{7}$ of the entire mixture (peanuts make up two-sevenths of the mixture)."

Which fraction or ratio you use is dependent on the context of the question: are you asked to compare parts to parts, or parts to the whole?

## Probabilities from Tables and Graphs

One of the major question types on the test asks us to find the **probability** of an event occurring. Probabilities are just ratios or fractions (sometimes in the form of percentages) in which the numerator is the number of chances for one event to occur and the denominator is the number of chances for any possible event to occur in a given scenario. For instance, if there are 10 balls in a bucket, 4 of which are red, the probability of choosing a red ball is $\frac{4}{10}$, or $\frac{2}{5}$ (probabilities can be reduced just like normal fractions), or 0.4 (more on decimal representations when we get to percentages). The denominator is 10 because there are 10 balls that could be pulled from the bucket. The numerator is 4 because there are 4 red balls that could be pulled from the bucket.

These problems are not mathematically difficult, but can be troublesome if you have difficulty interpreting the (usually verbose) questions. You need to quickly discern which values to pull from a table without feeling like you are just pulling numbers randomly. A simple method for finding the right values is by making statements in your own words about which numbers you need.

Usually we like to start by determining the reference total number of events or entities (people, things, etc.), which will act as the denominator. This reference number is often found in an "if" statement in the question prompt. For example, if a question contains the phrase, "If a plumber from California is chosen at random..." then the denominator in the fraction is the number of plumbers from California. Another common identifying phrase is of the form, "what fraction of the plumbers from California...," which tells us that the plumbers from California are the subset of people from which we are choosing randomly. Based on either phrase, we need to find the number of plumbers in Calfornia—not the total number of plumbers from any state, and not the number of electricians from California. It may seem like a dumb tip, but turning parts of the question into statements that tell you what numbers you need to find makes it less like a guessing game and more of a hunt for something specific.

When it comes to finding the numerator, it is important to realize that the numerator must be smaller than the denominator because we are always looking for a subset of the reference group that was used for the denominator. For example, let's complete the example phrase we just started: "If a plumber from California is chosen at random, what is the probability that he will have more than 4 years of experience?" For the numerator, we first need to make sure that we are pulling only from the pool of people that fit into the reference category—they must be plumbers from California. Then, within that particular group—plumbers from California—we need just those *who have more than 4 years of experience*.

Once you know the numbers you are looking for, you need to be able to find them in a **two-way table**, which is a table whose rows and columns are labeled separately to provide two different pieces of information about each cell in the table. The labels of the rows and columns will guide you to the specific value that you want to use in your answer. Let's start with a simple example that requires you to carefully choose a number for the numerator by finding the boxes in the table that meet all the given conditions.





**Example 12.1-1**

**1** 1600.io 12.1

| | Color | | | |
|---|---|---|---|---|
| | White | Black | Red | Total |
| Sedan | 10 | 4 | 1 | 15 |
| SUV | 3 | 5 | 2 | 10 |
| Total | 13 | 9 | 3 | 25 |

The table above shows the distribution of sedans and SUVs in a neighborhood classified according to color. If one of the vehicles is selected at random, what is the probability that it will be a black sedan?

A) $\frac{4}{25}$

B) $\frac{4}{15}$

C) $\frac{9}{25}$

D) $\frac{4}{9}$

**Solution**

1. Let's figure out which number to use as the denominator first. Since we are randomly selecting out of all the vehicles, the total number of vehicles (regardless of type or color) is the reference number to use as the denominator. This number is in the intersection of the Total column and Total row, so the denominator is 25.

| | Color | | | |
|---|---|---|---|---|
| | White | Black | Red | Total |
| Sedan | 10 | 4 | 1 | 15 |
| SUV | 3 | 5 | 2 | 10 |
| Total | 13 | 9 | 3 | 25 |

2. Now we need to find the number that will be the numerator. We want the numerator to be the number of black sedans, so we should find the cell that is at the intersection of the Sedan row and the Black column. The numerator should be 4.

| | Color | | | |
|---|---|---|---|---|
| | White | Black | Red | Total |
| Sedan | 10 | 4 | 1 | 15 |
| SUV | 3 | 5 | 2 | 10 |
| Total | 13 | 9 | 3 | 25 |

3. The numerator (the number of black sedans) is 4 and the denominator (the total number of vehicles) is 25, so the probability that a car from the neighborhood chosen at random will be a black sedan is $\frac{4}{25}$.

4. The answer is A.

The most common mistake students make with probability problems is choosing the wrong number to be the denominator: they wrongly assume that the overall total number of items or events is *always* the denominator. You must pay attention to any limitations that are snuck into the question that restrict you to only a subset of the total items or events.

**Example 12.1-2**

**2** 1600.io 12.1

Number of Train Arrivals at
Matawan Station in a Month

| | On Time | Delayed | Total |
|---|---|---|---|
| Northbound | 833 | 432 | 1265 |
| Southbound | 646 | 299 | 945 |
| Total | 1,479 | 731 | 2,210 |

Based on the table above, what fraction of the southbound trains were delayed?

A) $\frac{432}{731}$

B) $\frac{299}{731}$

C) $\frac{299}{945}$

D) $\frac{646}{945}$

**Solution**

1. First, let's figure out which number to use as the denominator. We are asked for some fraction of the southbound trains only. Therefore, we do not want the total number of all trains as the denominator; instead we should use the total number of southbound trains as the denominator (look for the intersection of the Total column and the Southbound row). The denominator will be 945.

Number of Train Arrivals at
Matawan Station in a Month

| | On Time | Delayed | Total |
|---|---|---|---|
| Northbound | 833 | 432 | 1265 |
| Southbound | 646 | 299 | 945 |
| Total | 1,479 | 731 | 2210 |





2. Now we need to find the number that will be the numerator. We know that the reference group is made up only of southbound trains, so we are limited to the cells in the Southbound row. We want the numerator to be the number of delayed southbound trains (look for the intersection of Delayed column and Southbound row), which is 299.

Number of Train Arrivals at
Matawan Station in a Month

| | On Time | Delayed | Total |
|---|---|---|---|
| Northbound | 833 | 432 | 1265 |
| Southbound | 646 | 299 | 945 |
| Total | 1,479 | 731 | 2210 |

3. The numerator (the number of delayed southbound trains) is 299 and the denominator (the total number of southbound trains) is 945, so the fraction of southbound trains that were delayed is $\frac{299}{945}$.
4. The answer is C.

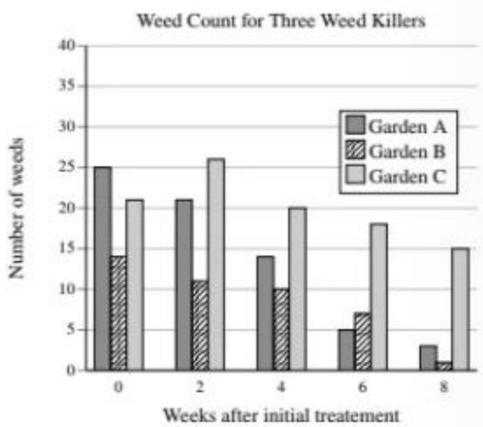
Occasionally, the answer choices are presented as decimals instead of fractions, but the process works exactly the same as in previous problems. You just need to perform the actual division of the numbers in the fraction—**the only time you should really need a calculator with the probability problems is when you are forced to work with decimals**.

## Reading Bar Graphs for Information

Sometimes, you are required to read values from bar graphs and make ratios from those values. Pay attention to the axes and the keys, which will tell you what is being measured and what each bar represents. Often, you are forced to make estimates of the values because there are not grid lines for every possible value. Make a judgment based on the interval length. If a bar looks exactly halfway between 40 and 50, then 45 is a good estimate for the value. If the bar is slightly above halfway, then something like 46, 47, or 48 is probably a better estimate. Usually on this test, you don't have to be too precise in order to get a reasonable estimate and find the correct answer.

**Example 12.1-3**

3 1600.io 12.1




Three gardens were each treated with different weed killers over an 8-week period to test the effectiveness of three weed killers. Gardens A, B, and C were treated with weed killers A, B, and C, respectively. Each weed killer was applied every 2 weeks to one of the three gardens over the 8-week period. The bar graph above shows the weed counts for each of the gardens 0, 2, 4, 6, and 8 weeks after the initial treatment.

Of the following, which is the closest to the ratio of the total number of weeds in all three gardens in week 8 to the total number of weeds in all three gardens at the time of the initial treatment?

A) 1 to 2

B) 2 to 5

C) 1 to 3

D) 1 to 5





**Solution**

1. Start by quickly analyzing the axes. The horizontal axis shows the number of weeks after treatment and indicates that the weeds were counted every 2 weeks. The vertical axis shows the number of weeds, and the interval between gridlines is 5. There is a cluster of 3 bars for each number of weeks represented on the axis. The height of each bar tells the number of weeds for a specific garden in a specific week.

2. To make the ratio that we are asked to find, we first need to know the total number of weeds in all three gardens after 8 weeks, so we have to add up the values shown in the last grouping of bars. After 8 weeks, Garden A (the dark gray bar) appears to have about 3 weeds, Garden B (the striped bar) appears to have about 1 weed, and Garden C (the light gray bar) has exactly 15 weeds. In total there are about $3 + 1 + 15$, or 19, weeds in the three gardens after 8 weeks.

3. Next, we have to figure out the total number of weeds when the treatment was started, which would be 0 weeks after treatment (the first grouping of bars). At 0 weeks, Garden A (the dark gray bar) has exactly 25 weeds, Garden B (the striped bar) appears to have about 14 weeds, and Garden C (the light gray bar) appears to have about 21 weeds. In total there are about $25 + 14 + 21$, or 60, weeds in the three gardens at the time the treatment starts.

4. The ratio of the total number of weeds in all three gardens 8 weeks after treatment to the total number of weeds 0 weeks after treatment is 19 to 60. You may notice that all of the answer choices are reduced ratios, so you can think about tweaking your numerator or denominator slightly to make a ratio that can be reduced. Notice that the question doesn't demand the *exact* ratio; it instead asks for which answer is *closest* to the actual ratio, so changing the 19 to 20 is justifiable because if the ratio is 20 to 60, it can be reduced to 1 to 3.

5. The answer is C.

**Notes**

If you do not feel comfortable making the adjustments to your estimates at the end of the problem, you can find the decimal representation of your answer $\left(\frac{19}{60} = 0.31\overline{6}\right)$, write each answer choice as a decimal, and then compare. Since $\frac{1}{3} = 0.\overline{3}$, choice C is the closest match to our estimate.

**Section 12.1 Suggested Problems from Real Tests**

- Test 1-C-21
- Test 1-C-23
- Test 2-C-16
- Test 3-C-2
- Test 4-C-7
- Test 4-C-9
- Test 5-C-14
- Test 6-C-8
- Test 7-C-1
- Test 7-C-38
- Test 8-C-16
- Test 9-C-9
- Test 9-C-33
- Test 10-C-13
- Apr 2018-C-13
- May 2018-C-29
- May 2018-C-38
- Mar 2019-C-16
- Mar 2019-C-19
- Apr 2019-C-18
- Mar 2020-C-24
- Oct 2020-C-5
- Oct 2020-C-34
- Mar 2021-C-6
- Apr 2021-C-19
- Apr 2021-C-21
- Apr 2021-C-38
- May 2021 (US)-C-5
- May 2021 (US)-C-6
- May 2021 (US)-C-25
- May 2021 (Int)-C-14
- May 2021 (Int)-C-32
- May 2021 (Int)-C-36

View related real-test problems in SMTEx

SMTEx

## Section 12.1 Practice Problems

**1** 1600.io 12.1

Cupcake and Topping Selections

| | | Flavor | |
|---|---|---|---|
| | | Vanilla | Chocolate |
| Topping | Sprinkles | 5 | 7 |
| | Cookie Crumbs | 8 | 12 |

The table above shows the flavors of cupcake and the toppings chosen by the people at a party. Each person chose one flavor of cupcake and one topping. Of the people who chose a vanilla cupcake, what fraction chose sprinkles as a topping?

A) $\frac{5}{32}$

B) $\frac{5}{13}$

C) $\frac{13}{32}$

D) $\frac{8}{13}$

**2** 1600.io 12.1

A store received a shipment of 1,000 sets of earbuds, 12 of which were defective. If a set of earbuds is randomly selected from this shipment, what is the probability that it is defective?

A) 0.012

B) 0.12

C) 1.2

D) 12

**3** 1600.io 12.1

| | Age | | Total |
|---|---|---|---|
| Gender | Under 25 | 25 and older | |
| Male | 20 | 4 | 24 |
| Female | 16 | 8 | 24 |
| Total | 36 | 12 | 48 |

The table above shows the distribution of age and gender for 48 people who entered a raffle. If the raffle winner will be selected at random, what is the probability that the winner will be either a female under age 25 or a male age 25 or older?

A) $\frac{16}{48}$

B) $\frac{20}{48}$

C) $\frac{28}{48}$

D) $\frac{36}{48}$

**4** 1600.io 12.1

| Ointment | Number of rashes | | |
|---|---|---|---|
| | Worsened | Improved | Total |
| A | 40 | 60 | 100 |
| B | 80 | 20 | 100 |

The table above shows the results of an experiment involving the effect of two ointments, A and B, on rashes. Based on the results, what fraction of the rashes that worsened received treatment A ?

A) $\frac{1}{5}$

B) $\frac{1}{3}$

C) $\frac{2}{5}$

D) $\frac{1}{2}$





## 12.2 Proportions

Sometimes, we will have situations where two pairs of values form ratios that must be the same. For example, if a table is two and a half times as long as it is wide, then its length-to-width ratio is 2.5:1; multiplying this ratio by 2 to eliminate the decimal, it becomes 5:2 or $\frac{5}{2}$. If the table is 10 feet long, we know that its width must be 4 feet, because $\frac{5}{2} = \frac{10}{4}$. If we make other tables of other lengths that obey the same length-to-width ratio, no matter how long a table is, it will have that length-to-width ratio of 5:2.

Because the length-to-width ratio of any such table must be 5:2, we can create an equation to find either dimension of any particular table if the other dimension is known. The general equation here would be

$$\frac{\text{length}}{\text{width}} = \frac{5}{2}$$

We're simply setting two ratios equal to each other. Now, if we're told that a table is 3 feet wide, and we'd like to know how long it is, we can write, using $x$ as the length,

$$\frac{x}{3} = \frac{5}{2}$$

and we can solve for $x$ by multiplying both sides by 3 and calculating a decimal value.

$$x = \frac{3(5)}{2}$$

$$x = \frac{15}{2}$$

$$x = 7.5$$

When we set two ratios equal to each other, the resulting equation is called a **proportion**. Proportions are an extremely useful tool and they have very broad applicability both to math problems generally and to problems on this test specifically, so it's vital that you become comfortable with spotting when they can be used and with setting them up and solving them.

Mathematically, it doesn't matter which ratio goes on which side of the equation, nor does it matter what you choose for the numerator and denominator, as long as those terms correspond properly in the two ratios. For example, we could have set up the previous proportion in any of these ways:

$$\frac{x}{3} = \frac{5}{2}$$

$$\frac{5}{2} = \frac{x}{3}$$

$$\frac{3}{x} = \frac{2}{5}$$

$$\frac{2}{5} = \frac{3}{x}$$

However, when it comes to solving, some forms have advantages over others. For example, let's use

$$\frac{3}{x} = \frac{2}{5}$$

First, we need to get $x$ in the numerator, so we'll multiply both sides by $x$.

$$\frac{3}{1} = \frac{x(2)}{5}$$

Next, we need to eliminate the denominator on the right side to isolate the $x$ term, so we multiply both sides by 5.

$$\frac{5(3)}{1} = \frac{2x}{1}$$
$$15 = 2x$$

Finally, we want to eliminate the coefficient of $x$, so we divide both sides by 2.

$$\frac{15}{2} = x$$
$$7.5 = x$$

Notice how many steps there are, how the unknown changes sides of the equation and moves from a denominator to a numerator, how intermediate values might need to be computed, and how the final "sentence" isn't quite as satisfying as one where the variable is on the left side. By contrast, if we arrange our proportion with the unknown in the numerator of the ratio on the left side, we only need to make one move (the denominator on the left moves to the numerator on the right), and then we have our complete calculation ready for evaluation on the right side (on problems with larger numbers, it can be punched into a calculator with no parentheses or storage of intermediate results needed); the unknown never changes position, and the final "sentence" reads nicely as "$x$ is equal to 7.5."

$$\frac{x}{3} = \frac{5}{2}$$
$$x = \frac{3(5)}{2}$$
$$x = \frac{15}{2}$$
$$x = 7.5$$

For these reasons, we suggest always setting up a proportion using this template whenever you're afforded the opportunity.

One scenario in which proportions are the right solving tool is when you have a physical object or geographical area, and the object is being reproduced as a (smaller) scale model, or the area is represented as a map (which should *definitely* be smaller than the area it shows!). In these situations, the model or map will have a scale, which is nothing more than a ratio that describes the size of features on the model or map relative to the size of the corresponding features on the actual object or area. For example, if a model car is built at a $1:20$ scale, any measurement on the model will be $\frac{1}{20}^{\text{th}}$ of the same measurement on the real car; when a scale like this is given, the meaning of the constants in the ratio are **model or map : real object or geographical area**. We can set up the following proportion, where $m$ is a measurement for the model car and $r$ is a measurement for the real car, and we can use it to find the measurements of either the model car or the real car:

$$\frac{m}{r} = \frac{1}{20}$$

If the windshield of the model car is 2 inches tall, then we can find the measure of the real car's windshield by plugging in 2 for $m$, then cross multiplying and solving for $r$.

$$\frac{2}{r} = \frac{1}{20}$$
$$2(20) = 1(r)$$
$$40 = r$$





This shows us that if the windshield on the $\frac{1}{20}^{\text{th}}$ model is 2 inches, then the windshield on the real car is 40 inches tall (it is 20 times the height of the model car's windshield).

We could have set up the proportion with $r$, the value we want to solve for, as the numerator on the left side of the equation, for the most "satisfying" solve.

$$\frac{r}{m} = \frac{20}{1}$$
$$\frac{r}{2} = 20$$
$$r = 40$$

Once a proportion is set up correctly, it can be used to quickly calculate corresponding values.

Note that in some instances, the scale will be expressed in a mix of units rather than as a unitless ratio, so a map might have a scale of 2 inches per mile, for example. This poses no problem if you're using proportions, because any values provided to you for a map distance will be in inches, and values supplied for actual distances will be in miles. In these situations, you do not need to convert to a common unit measure.

To illustrate, if two towns on that map with a scale of 2 inches per mile are 11 inches apart, we can use a straightforward proportion to figure out the distance $d$ between the real towns:

$$\frac{\text{Distance in miles}}{\text{Measurement on map in inches}} = \frac{1\text{ mile}}{2\text{ inches}}$$
$$\frac{d}{11} = \frac{1}{2}$$
$$d = \frac{11(1)}{2}$$
$$d = 5.5$$

The towns are 5.5 miles apart.

Proportions also come into play with situations that involve simple linear relationships that have no starting value or offset as would be represented by the $b$ value in the Slope-Intercept Form equation $y = mx + b$, so that equation degenerates to just $y = mx$, and $m$ will be a constant, so we can say that the ratio of $\frac{y}{x} = m$. This, in turn, means that if we have the values of $y$ and $x$ for one scenario, we can use that ratio in a proportion that allows us to solve for one of those values in a second scenario. Situations of this type typically involve rates, such as the number of miles that a car travels each hour, or the amount a worker gets paid each week; knowing the rate, or any example of a specific pair of the values, gives us the ratio that controls the situation.

For example, if we're told that a helicopter ascends at a constant rate of 50 feet per second, and we're asked to determine how may seconds, $s$, it takes to ascend from the ground to a height of 450 feet, we can set up the following proportion, placing the unknown in the numerator of the left side ratio as is our preference:

$$\frac{s}{450} = \frac{1}{50}$$
$$s = \frac{450(1)}{50}$$
$$s = 9$$

If we weren't given the rate of ascent directly, but we were instead provided with a specific example of an ascent, we could perform the same procedure but using the values from the example, because they must conform to the same ratio; let's say we're told that the helicopter takes 23 seconds to ascend 1,150 feet, and we're again asked how many seconds it would take to ascend 450 feet:

$$\frac{s}{450} = \frac{23}{1{,}150}$$
$$s = \frac{450(23)}{1{,}150}$$
$$s = 9$$

Notice that there's just one move, and then the chain calculation $250 \times 23 \div 1150$ could be executed on a calculator if this were on the calculator section of the test.

One interesting characteristic of proportions is that they can be "rotated," meaning they still hold when looked at, um, sideways. Here, we made ratios with time and distance and equated them to other ratios of time and distance, but we could also have made ratios of time relative to time and distance relative to distance. Let's look at that symbolically, using $s$ and $d$ for time and distance, respectively; we'll employ subscripts so it's clear what's going on. We made proportions of the following form:

$$\frac{s_1}{d_1} = \frac{s_2}{d_2}$$

Let's multiply both sides by $d_1$.

$$\frac{s_1}{1} = \frac{d_1(s_2)}{d_2}$$

Now, let's divide both sides by $s_2$:

$$\frac{s_1}{s_2} = \frac{d_1}{d_2}$$

Now we have ratios that each have the same units: seconds to seconds and feet to feet, instead of having rates in feet per second. A proportion of this form is just as valid as the first arrangement, and sometimes, it's more natural to think about a specific situation one way or the other; you have your choice.

**Example 12.2-1**

**1** 1600.io 12.2

At his summer job, Elan earns the same amount of money for each hour he works. If he earns \$330 for working 30 hours, how much does he earn for 6 hours?

A) \$11

B) \$55

C) \$60

D) \$66

**Solution**

1. The ratio of money earned, $m$, to hours worked, $h$, will always be equal to $\frac{330}{30}$ (which reduces to 11) because Elan earns the same amount for each hour worked. If you can reduce the fractions first, you should do so before multiplying by another number in order to keep values low and manageable without a calculator.





$$\frac{m}{h} = \frac{330}{30}$$
$$\frac{m}{h} = \frac{33\cancel{0}}{3\cancel{0}}$$
$$\frac{m}{h} = 11$$

2. Substitute 6 for $h$ and then solve for $m$ to see how much money Elan makes for 6 hours of work.

$$\frac{m}{h} = 11$$
$$\frac{m}{6} = 11$$
$$m = 6(11)$$
$$m = 66$$

3. The answer is D.

**Notes**

We made use of a proportion, but you'll notice that the right side of the equation isn't written as a ratio in fraction form; that's because the denominator would be 1, so we omitted it. However, it might help to think of it (or even to write it) as $\frac{m}{h} = \frac{11}{1}$, and then to substitute 6 for $h$ to get $\frac{m}{6} = \frac{11}{1}$ and then solve for $m$, if you want to maintain the general form of a proportion.

This problem could also be done by writing a linear equation to represent the amount of money he earned and then solving for the slope of the line and using the equation to find the money he makes for working 6 hours.

Occasionally, proportion equations are given to you in the form $y = kx$ or $y = \frac{k}{x}$, where $k$ is a constant. Usually, all you have to do is plug in given values to find the value of the constant and then, if necessary, make use of the equation again to find a different $x$ or $y$ value.

**Example 12.2-2**

**2** 1600.io 12.2

If $y = kx$, where $k$ is a constant, and $y = 28$ when $x = 7$, what is the value of $y$ when $x = 5$ ?

A) 7

B) 20

C) 21

D) 26

**Solution 1**

1. The ratio of $\frac{y}{x}$ is $\frac{28}{7}$, so we can set up a proportion and substitute 5 for $x$.

$$\frac{y}{x} = \frac{28}{7}$$
$$\frac{y}{5} = 4$$
$$y = 5(4)$$
$$y = 20$$

2. The answer is B.

**Solution 2**

1. Plug in 28 for $y$ and 7 for $x$ in the given equation in order to find the value of the constant $k$.

$$y = kx$$
$$28 = k(7)$$
$$4 = k$$

2. Substitute 4 for $k$ to make the equation $y = 4x$. Now, we can plug in 5 for $x$ to find the value of $y$ when $x$ is 5.

$$y = 4x$$
$$y = 4(5)$$
$$y = 20$$

3. The answer is B.

## Ratios in Squared Relationships

The proportion equations we've seen so far are closely tied to linear relationships in that one variable changes in direct proportion to the changes in the other variable. However, sometimes one variable changes in relation to the *square* of another variable. For example, the area of a circle is calculated using the equation $A = \pi r^2$. The value of the variable $A$, the area, does not change linearly with changes in $r$, the radius; instead, the area changes according to the **square** of the radius. When the radius is 1, the area is $\pi(1^2) = \pi$. When the radius is doubled ($r = 2$), the area *DOES NOT double*: it *quadruples* ($A = \pi(2^2) = 4\pi$).

In a lot of these problems, you are not given specific values for the variables, but you are told how the variable in one situation is related to the variable in another situation. For example, you may be told that a small circle has a radius $r$ and the radius of a bigger circle is twice the radius of the smaller circle, and you are then asked to find the ratio of the area of the smaller circle to the area of the bigger circle. At this point, you may be tempted to choose dummy values (like 2 and 4) to plug in for the radii. This is not incorrect, but it does lead to possible confusion when reconciling which values correspond to which situation.

The smarter method is to make use of subscripts to differentiate the variables. For example, use $A_s$ to represent the area of the smaller circle and $r_s$ to represent the radius of the smaller circle. The area of the smaller circle is found with the equation $A_s = \pi r_s^2$. Use $A_b$ to represent the area of the bigger circle and $r_b$ to represent the radius of the bigger circle. The area of the bigger circle is found with the equation $A_b = \pi r_b^2$.





When it comes time to find the value of the ratio $\frac{A_s}{A_b}$ (the ratio of the area of the smaller circle to the area of the bigger circle), we can replace $r_b$ with $2r_s$ since we know that $r_b$ must be twice as big as $r_s$ (note that the entire expression $2r_s$ must be squared when we replace $r_b$). When we do this, we have a fraction that will reduce because the variable terms will cancel out—there was no need to plug in values for either radius.

$$\frac{A_s}{A_b} = \frac{\pi r_s^2}{\pi r_b^2}$$
$$\frac{A_s}{A_b} = \frac{\pi r_s^2}{\pi (2r_s)^2}$$
$$\frac{A_s}{A_b} = \frac{\pi r_s^2}{4\pi r_s^2}$$
$$\frac{A_s}{A_b} = \frac{\cancel{\pi r_s^2}}{4\cancel{\pi r_s^2}}$$
$$\frac{A_s}{A_b} = \frac{1}{4}$$

To reiterate, plugging in dummy values is not necessarily wrong, and you may be more comfortable doing that, but it leaves the door open to mistakes that can be avoided if you just work with the variables.

**Example 12.2-3**

**3** 1600.io 12.2

$$K = \frac{1}{2}mv^2$$

The kinetic energy $K$ generated by an object moving with velocity $v$ can be found using the formula above, where $m$ is the mass of the object. A baseball coach uses the formula to find the kinetic energy of a ball moving with velocity $v$ and the same ball moving with velocity $2.5v$. What is the ratio of the kinetic energy of the faster ball to the kinetic energy of the slower ball?

**Solution**

1. Let's use subscripts to differentiate the variables. Use $K_s$ to represent the kinetic energy of the slower ball and $v_s$ to represent the velocity of the slower ball. The kinetic energy of the slower ball is found with the equation $K_s = \frac{1}{2}mv_s^2$. Use $K_f$ to represent the kinetic energy of the faster ball and $v_f$ to represent the velocity of the faster ball. The kinetic energy of the faster ball is found with the equation $K_f = \frac{1}{2}mv_f^2$.

2. The ratio of the kinetic energy of the faster ball to the kinetic energy of the slower ball can be found with the following equation.

$$\frac{K_f}{K_s} = \frac{\frac{1}{2}mv_f^2}{\frac{1}{2}mv_s^2}$$

3. We are told that the velocity of the slower ball, $v_s$, is equal to $v$, so we can replace $v_s$ with $v$. We are told that the velocity of the faster ball, $v_f$, is equal to $2.5v$, which we should represent as the fraction $\frac{5}{2}v$ because working with fractions instead of decimals helps wean us off of calculator dependence. We can replace $v_f$ with $\frac{5}{2}v$ (remember that the entire expression $\frac{5}{2}v$ will have to be squared). Simplify by canceling out any terms that are common to the numerator and denominator.

$$\frac{K_f}{K_s} = \frac{\frac{1}{2}mv_f^2}{\frac{1}{2}mv_s^2}$$

$$\frac{K_f}{K_s} = \frac{\cancel{\frac{1}{2}}\cancel{m}\left(\frac{5}{2}v\right)^2}{\cancel{\frac{1}{2}}\cancel{m}(v)^2}$$

$$\frac{K_f}{K_s} = \frac{\frac{25}{4}v^2}{v^2}$$

$$\frac{K_f}{K_s} = \frac{\frac{25}{4}\cancel{v^2}}{\cancel{v^2}}$$

$$\frac{K_f}{K_s} = \frac{\frac{25}{4}}{1} = \frac{25}{4}$$

4. The answer is $\frac{25}{4}$.

**Notes**

You can take a very quick shortcut on problems like this, but you lose the advantage of precision that using subscripts provides. In this case, since the kinetic energy varies **directly** with the square of the velocity, because the squared velocity term $v^2$ is in the **numerator** of the expression, we could simply say that the ratio of the kinetic energy of the faster ball to that of the slower ball will be $2.5^2$ to 1, or 6.25 to 1 (which, as a fraction, is $\frac{25}{4}$).

The only caveat is that you have to be sure about which way the ratio works. For instance, according to the relevant formula, $F = G\left(\frac{m_1m_2}{d^2}\right)$, the force of gravity varies **inversely** with the square of the distance between two objects because the squared distance term $d^2$ is in the **denominator**, and therefore the force of gravity *decreases* as two objects move farther apart (unlike how kinetic energy *increases* as velocity increases), so you have to be aware that the force of gravity between two farther objects will be *smaller* than the force of gravity between two closer object of the same mass. In that case, if two objects are 2.5 times farther apart than two objects of the same mass, then the ratio of the force of gravity between the two farther objects to that between the two closer objects would be 1 to $2.5^2$, or $\frac{4}{25}$, instead (again, the force of gravity decreases as the distance between the objects increases because the $d^2$ term is in the denominator of the formula expression).

It is easy to mix up the numerator and denominator if you are not careful, so we advocate using subscripts and being very deliberate with your work. Some who understand squared (and other power) relationships will not be swayed. If you choose to use the shortest method, be prepared to double check your work and make sure that you set up the relationship the right way.





**Section 12.2 Suggested Problems from Real Tests**

- Test 1-C-2
- Test 1-C-27
- Test 2-C-2
- Test 2-C-4
- Test 2-C-23
- Test 2-C-31
- Test 4-C-35
- Test 5-C-32
- Test 7-C-15
- May 2017-C-2
- May 2017-C-4
- May 2017-C-16
- May 2018-C-31
- Mar 2019-C-26
- Apr 2019-C-29
- Apr 2019-C-38
- May 2019 (US)-C-7
- Oct 2019-C-2
- Oct 2020-C-2
- Mar 2021-NC-12
- Mar 2021-C-4

View related real-test problems in SMTEx

SMTEx

## Section 12.2 Practice Problems

**1** 1600.io 12.2

Child A is currently 24 inches tall, and child B is currently 32 inches tall. The ratio of the heights of Child A to Child B is equal to the ratio of the heights of Child C to Child D. If Child C is 30 inches tall, what is the height of Child D, in inches?

A) 20.0
B) 22.5
C) 38.0
D) 40.0

**2** 1600.io 12.2

A farmer selects 6 melons at random for inspection out of every 500 melons grown. At this rate, how many melons will be inspected if the farm grows 10,000 melons?

A) 100
B) 120
C) 140
D) 160

**3** 1600.io 12.2

In the $xy$-plane, the point $(4, 6)$ lies on the graph of $y = \frac{k}{x}$, where $k$ is a constant. Which of the following points must also lie on the graph?

A) $(2, 3)$
B) $(2, 4)$
C) $(8, 3)$
D) $(8, 12)$

**4** 1600.io 12.2

Lydia sold boxes of matches and bundles of firewood. The ratio of the number of boxes of matches she sold to the number of bundles of firewood she sold was 4 to 1. If Lydia sold 12 boxes of matches, how many bundles of firewood did she sell?

A) 3
B) 12
C) 16
D) 48

## 12.3 Ratio Word Problems

When it comes to "word" problems on this test, many students freeze immediately because the solution method is not immediately apparent as it is when you are given equations or are told that you are dealing with a particular type of mathematical entity (a line, a quadratic function, etc.). You have to be willing to take the information in a word problem one piece at a time and try to write equations, even if you are not sure how they are going to help you solve the problem. Once you have one or two equations that are definitely correct, you can see about making substitutions to reduce the number of unknown terms and simplify the equation to make it useful.

**Example 12.3-1**

**1** 1600.io 12.3

Polk High School randomly selected freshman, sophomore, junior, and senior students for a survey about potential changes to the vaping policy. Of students selected for the survey, $\frac{1}{6}$ were freshmen and $\frac{1}{4}$ were sophomores. Half of the remaining selected students were juniors. If 288 students were selected for the survey, how many were seniors?

A) 70
B) 84
C) 140
D) 168

**Solution 1**

1. The fraction of students that are freshmen and sophomores is hard to make use of without an equation in which to ground the question. We do know that 288 students were selected for the survey and that those students were made up of freshmen, sophomores, juniors and seniors. We can write an equation that represents the total number of students. If $f$ is the number of freshmen, $p$ is the number of sophomores (we want to use $s$ for seniors because we want to solve for the number of seniors later), $j$ is the number of juniors, and $s$ is the number of seniors, then the total of all the students can be represented by the equation $288 = f + p + j + s$.

2. If $\frac{1}{6}$ of all the students are freshmen, then the number of freshmen is $\frac{1}{6}(288)$, or 48. Replace $f$ with 48, then simplify the equation by combining numerical values on the left side of the equation.

$$288 = f + p + j + s$$
$$288 = 48 + p + j + s$$
$$240 = p + j + s$$





3. If $\frac{1}{4}$ of all the students are sophomores, then the number of sophomores is $\frac{1}{4}(288)$, or 72. Replace $p$ with 72, then simplify the equation by combining numerical values on the left side of the equation.

$$240 = p + j + s$$
$$240 = 72 + j + s$$
$$168 = j + s$$

4. The remaining students are either juniors or seniors. Since half of the remaining students are juniors, that also means that half of the remaining students are seniors. Therefore, $j = s$, and we can replace $j$ with $s$ and solve the equation for $s$, the number of seniors who participated in the survey.

$$168 = j + s$$
$$168 = s + s$$
$$168 = 2s$$
$$84 = s$$

5. The answer is B.

**Solution 2**

This solution is a slightly less formal version of Solution 1. It is probably safer to work within equations whenever possible, but this version of the solution is no less valid.

1. If freshmen make up $\frac{1}{6}$ of the survey, where 288 is the total number of students who participated, then the number of freshmen who participated is equal to $\frac{1}{6}(288)$, or 48.

2. If sophomores make up $\frac{1}{4}$ of the survey, where 288 is the total number of students who participated, then the number of sophomores who participated is equal to $\frac{1}{4}(288)$, or 72.

3. The number of juniors and seniors in the survey is equal to 288 minus the number of freshmen and sophomores, so there are $288 - 48 - 72$, or 168, juniors and seniors in the survey.

4. We are told that juniors are half of this 168, so seniors must be the other half of this 168 of the respondents. Let's use $s$ to represent the number of seniors. The number of seniors is equal to $\frac{1}{2}(168)$, or 84.

5. The answer is B.

**Solution 3**

In this solution, we'll figure out what fraction of the survey respondents were seniors and then multiply that fraction by the total number of students.

1. First, we need to find what fraction of the respondents was made up of freshmen and sophomores combined. Freshmen made up $\frac{1}{6}$ of the respondents and sophomores made up $\frac{1}{4}$ of the respondents, so to find the total fraction of the survey made up of freshmen and sophomores, we need to add the two fractions (we will need to make a common denominator of 12 to do so).

$$\frac{1}{6} + \frac{1}{4} = \frac{2}{12} + \frac{3}{12} = \frac{5}{12}$$

2. Since freshmen and sophomores made up $\frac{5}{12}$ of the respondents, juniors and seniors must make up the remaining $\frac{7}{12}$ of the respondents ($1 - \frac{5}{12} = \frac{12}{12} - \frac{5}{12} = \frac{7}{12}$).

3. We are told that juniors are half of this remaining $\frac{7}{12}$ of the respondents, so seniors must be the other half of this $\frac{7}{12}$ of the respondents. Multiply $\frac{7}{12}$ by $\frac{1}{2}$ to find that seniors must make up $\frac{7}{24}$ of the total number of respondents.

4. Let's use $s$ to represent the number of seniors. We know that $s$ is equal to $\frac{7}{24}$ of the total number of students in the survey, which was 288 students.

$$s = \frac{7}{24}(288)$$
$$s = 84$$

5. The answer is B.

**Example 12.3-2**

**2** 1600.io 12.3

| | Eye Color | |
|---|---|---|
| Gender | Brown | Green |
| Female | | |
| Male | | |
| Total | 28 | 11 |

The incomplete table above summarizes the number of brown-eyed and green-eyed students by gender for the seventh-grade students of Parkway Central Middle School. There are 4 times as many brown-eyed female students as there are green-eyed female students, and there are 2 times as many brown-eyed male students as there are green-eyed male students. If there is a total of 28 brown-eyed students and 11 green-eyed students in the class, which of the following is closest to the probability that a green-eyed student selected at random is female? (Note: Assume that none of the seventh-grade students are both brown-eyed and green-eyed because Max Scherzer graduated already.)

A) 0.273

B) 0.375

C) 0.429

D) 0.727





**Solution**

1. First, we need to write the given information as short equations with subscripts to help us figure out what to do. If $f_g$ is the number of green-eyed female students, then we know that that the number of brown-eyed female students is equal to $4f_g$ since we are told that the number of brown-eyed female students is 4 times the number of green-eyed female students.

2. If $m_g$ is the number of green-eyed male students, then we know that the number of brown–eyed male students is equal to $2m_g$ since the number of brown-eyed male students is 2 times the number of green-eyed male students.

3. Fill in the table with the new variable designations in order to help write a few more equations.

| | Eye Color | |
|---|---|---|
| Gender | Brown | Green |
| Female | $4f_g$ | $f_g$ |
| Male | $2m_g$ | $m_g$ |
| Total | 28 | 11 |

4. From the table, we can write a system of equations, forming each equation by summing the values in the columns.

   From the "Brown" column, we know that $4f_g + 2m_g = 28$.

   From the "Green" column, we know that $f_g + m_g = 11$.

   With those equations written in terms of the numbers of green-eyed male and female students, the system of equations is the following:

$$4f_g + 2m_g = 28$$
$$f_g + m_g = 11$$

5. We need to know the number of green-eyed female students, $f_g$, so multiply the bottom equation by $-2$ in order to combine the equations and eliminate $m_g$.

$$\begin{aligned} 4f_g + 2m_g &= 28 \\ -2(f_g + m_g) &= -2(11) \\ &\Downarrow \\ 4f_g + 2m_g &= 28 \\ -2f_g - 2m_g &= -22 \\ \hline 2f_g \quad &= 6 \\ &\Downarrow \\ f_g \quad &= 3 \end{aligned}$$

6. We need to choose numbers for the numerator and denominator to make the probability that is asked for: the probability that a green-eyed student selected at random is female.

   First off, the denominator should be 11, because we are choosing someone at random out of only the green-eyed population of the school.

   The numerator should be 3 because that is the number of green-eyed female students.

   Therefore, the probability that a green-eyed student selected at random is female is equal to $\frac{3}{11}$, which is equal to 0.273 when written in decimal form.

7. The answer is A.

**Notes**

You could find the number of brown-eyed male students and brown-eyed female students based on the numbers of green-eyed students if you needed to. The number of brown-eyed female students is 4 times the number of green-eyed female students, or 4(3), which is 12. Since the total number of green-eyed students is 11 and the table tells us that 3 of those are female, then we know that there are 8 green-eyed male students. The number of brown-eyed male students is 2 times the number of green-eyed male students, or 2(8), which is 16.

Again, you do not have to use carefully-chosen, subscripted variables as placeholders: you can make the same system of equations slightly faster if you just use $x$ and $y$ to represent the number of green-eyed female students and male students, respectively (then the number of brown-eyed female students and male students would be $4x$ and $2y$, respectively). However, in the interest of remembering what everything stands for, we promote carefully and deliberately working through the problem to avoid having to double check what our variables represent at the end.

**Section 12.3 Suggested Problems from Real Tests**

- Test 3-C-29
- Test 4-C-34
- Test 5-C-5
- Mar 2019-C-6
- 2019 (US)-C-30
- 2020-C-34
- 2021-C-31

**View related real-test problems in SMTEx**

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## Section 12.3 Practice Problems

**1** 1600.io 12.3

The cost of a product in Ontario is approximately $\frac{7}{5}$ of its price in Newfoundland. The cost of a product in Alberta is approximately $\frac{4}{5}$ of its price in Newfoundland. If an object costs \$200 in Newfoundland, approximately how many more dollars does it cost in Ontario than it costs in Alberta?

A) 280

B) 160

C) 120

D) 80

**2** 1600.io 12.3

Aditya has 5 green shirts and $b$ blue shirts in his dresser, and these are the only shirts in his dresser. If Aditya selects a shirt at random from his dresser, which of the following gives the probability that Aditya will select a blue shirt?

A) $\frac{5}{b}$

B) $\frac{b}{5}$

C) $\frac{b}{5+b}$

D) $\frac{5}{5+b}$





# CHAPTER 12 RECAP

- A fraction $\frac{a}{b}$ is a form of ratio that can also be expressed with the notation $a:b$, which can be read as "$a$ to $b$" or "$a$ in $b$." Sometimes ratios will be expressed as decimal numbers; for example, $\frac{1}{2}$ is equivalent to the decimal 0.5.
- Fractions are limited to two parts, the numerator and denominator, but ratio notation can be used to show the relative values of any number of terms. For example, the side lengths of a triangle might be in a ratio of $3:4:5$.
- Given a ratio of $a:b$, you may need to make fractions for use in solving problems.

  Depending on the problem, it may be beneficial to use a **part-to-part** fraction of $\frac{a}{b}$ or $\frac{b}{a}$, which helps when comparing the amount of one part to the amount of another part.

  Other times, you might need to use a **part-to-whole** fraction like $\frac{a}{a+b}$ or $\frac{b}{a+b}$, where the denominator is the sum of the parts (the whole), which helps when comparing the amount of one part to the total, or whole, amount.
- Reducing ratios works exactly like reducing fractions. Just as the fraction $\frac{4}{10}$ reduces to $\frac{2}{5}$, so too does the ratio $4:10$ reduce to $2:5$.

  The process works the same for ratios with more than two terms. For example, a ratio of $6:8:10$ can be reduced by dividing all of the terms by 2, resuting in a reduced ratio of $3:4:5$.
- Probability is the number of desired outcomes divided by the total number of possible outcomes in a given scenario.
- Choose values from tables very carefully, making sure to restrict yourself appropriately based on the selection criteria.

  Usually we like to start by determining the **reference** total number of events or entities (people, things, etc.), which will act as the **denominator**. **This reference number is often found in an "if" statement in the question prompt**. For example, if a question contains the phrase, "If a plumber from California is chosen at random..." then the denominator in the fraction is the number of plumbers from California.

  **The numerator must be smaller than the denominator** because we are always looking for a subset of the reference group that was used for the denominator. For example, let's complete the example phrase we just started: "If a plumber from California is chosen at random, what is the probability that he will have more than 4 years of experience?" For the numerator, we first need to make sure that we are pulling only from the pool of people that fit into the reference category—they must be plumbers from California. Then, within that particular group—plumbers from California—we need just those *who have more than 4 years of experience*.
- The intersection between columns and rows in two-way tables will guide you to the specific values you need to solve problems.
- Estimate carefully from bar graphs, especially if the number of gridlines does not make for precise guessing. Pay attention to the axes and legends.
- Proportions are formed by setting two ratios equal to each other. Proportions are easy ways to solve for values that have constant rates of change (instead of using linear equations).

  For example, if a table manufacturer makes tables of varying sizes but the length to width ratio of their tables is always 5 to 2, then you can write the following proportion:

  $$\frac{\text{length}}{\text{width}} = \frac{5}{2}$$

  Once you have the proportion set up, you can find how long any table would have to be given a known width, or vice versa.

- When using proportions to solve for unknown values, place the unknown value in the numerator of the fraction on the left side of the equation in order to give yourself the smoothest path. For example, if we knew how long we wanted a table to be but had to find out how wide it would have to be, we can write the same proportion as the following

$$\frac{\text{width}}{\text{length}} = \frac{2}{5}$$

  allowing us to more easily find the corresponding width.
- Sometimes relationship problems involve squared or cubed relationships rather than linear relationships, so ratios and proportions will not apply. For example, the area of circle is given by the formula $A = \pi r^2$, where $A$ is the area and $r$ is the radius. Doubling the radius does not double the area. Due to the squared relationship, doubling the radius actually quadruples the area of the circle.
- When finding the ratio of two variable quantities that are unknown, you can use subscripts and cancel terms instead of plugging in dummy values.
- Some problems require many steps before you can calculate a probability. You may be required to use lines, tables, lists, or other tools to contextualize the problem and eventually figure out how to find the probability that you are looking for.





## Additional Problems

**1** 1600.io ? 12.1

Customer Purchases at a Coffee Shop

| | Donut purchased | Donut not purchased | Total |
|---|---|---|---|
| Coffee purchased | 40 | 180 | 220 |
| Coffee not purchased | 20 | 10 | 30 |
| Total | 60 | 190 | 250 |

On Monday, a local coffee shop had 250 customers. The table above summarizes whether or not the customers on Monday purchased coffee, a donut, both, or neither. Based on the data in the table, what is the probability that a coffee shop customer selected at random on that day did not purchase coffee?

A) $\frac{10}{30}$

B) $\frac{10}{190}$

C) $\frac{20}{30}$

D) $\frac{30}{250}$

**2** 1600.io ? 12.2

A scale drawing of a tower uses the scale 3 inches = 1 foot. In the drawing, one wall has a length of 24 inches. What is the actual length, in feet, of this wall?

**3** 1600.io ? 12.3

In a study of finch migration habits, 30 male finches and 20 female finches have been tagged. If 40 more female finches are tagged, how many more male finches must be tagged so that $\frac{2}{3}$ of the total number of finches in the study are male?

**4** 1600.io ? 12.1

The painting *The Fluffy Bunny* by Mathieu Timeau is rectangular in shape with height 41 inches and width 65.2 inches. If a reproduction was made where each dimension is $\frac{1}{5}$ the corresponding original dimension, what is the height of the reproduction, in inches?

**5** 12.2

A geologist estimates that a certain river is widening at a rate of 3.5 inches per year. According to the geologist's estimate, how long will it take, in years, for the river to widen by 42 inches?

**6** 12.2

$$h = 8p$$

A wolf tracker uses the formula above to estimate the height, $h$, in centimeters, of a wolf from its paw to its shoulder, based on the length $p$, in centimeters, of the wolf's pawprint.

The length $p$ of a mother wolf's pawprint is 3 times the length of a baby wolf's pawprint. What is the ratio of the height of the mother to the height of the baby?

A) 8 to 1

B) 3 to 1

C) 8 to 3

D) 1 to 3

**7** 12.2

If the ratio of $0.75 : x$ is equivalent to $4.5 : 7.5$, what is the value of $x$ ?

A) 0.45

B) 1

C) 1.05

D) 1.25

**8** 12.2

A jelly bean tank measures 10 feet by 10 feet. Ten students each mark off a randomly selected region of the tank; each region is a square and has side lengths of 1 foot, and no two regions overlap. The students count the red jelly beans contained in the tank to a depth of 1 foot below the surface in each region. The results are shown in the table below.

| Region | Number of red jelly beans | Region | Number of red jelly beans |
|---|---|---|---|
| A | 1,304 | F | 1,281 |
| B | 1,492 | G | 1,152 |
| C | 1,347 | H | 1,291 |
| D | 1,518 | I | 1,323 |
| E | 1,107 | J | 1,276 |

Which of the following is a reasonable approximation of the number of red jelly beans to a depth of 1 foot beneath the surface in the entire tank?

A) 130

B) 1,300

C) 13,000

D) 130,000





**9** 12.1

Mike's Bikes Sales in May, 2018

| Bike Style | Bike Color | | | | |
|---|---|---|---|---|---|
| | Red | Black | Blue | White | Total |
| Mountain | 2 | 5 | 2 | 0 | 9 |
| BMX | 1 | 4 | 1 | 1 | 7 |
| Racing | 6 | 0 | 8 | 10 | 24 |
| Total | 9 | 8 | 11 | 11 | 40 |

The table above represents the 40 bikes that Mike sold at his bike shop in May, 2018, categorized by color and bike style. What proportion of the bikes are blue racing bikes?

A) $\frac{1}{5}$

B) $\frac{11}{40}$

C) $\frac{1}{3}$

D) $\frac{8}{11}$

**10** 12.1

Change in Subscribers of 50 Magazines

| | Increased in May | Decreased in May | Total |
|---|---|---|---|
| Increased in June | 14 | 12 | 26 |
| Decreased in June | 6 | 18 | 24 |
| Total | 20 | 30 | 50 |

The two-way table categorizes the change in subscribers in May and June for 50 magazines. If one of the magazines that decreased in subscribers in May is chosen at random, what is the probability that the magazine also decreased in subscribers in June?

A) 0.40

B) 0.48

C) 0.60

D) 0.75

**11** 12.2

The distance between two locations on a map is 8 inches (in). If 1 in on the map corresponds to an actual distance of 25 kilometers, what is the actual distance, in kilometers, between the two locations?

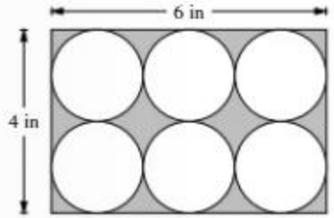
A) 0.32

B) 3.125

C) 200

D) 250

**12** 12.2




Shaelynn, an ice cube tray designer, is designing a tray to hold 6 ice cubes shaped as cylinders. Her initial sketch of the top view of the base of the tray is shown above.

If the length and width of the tray in the sketch were doubled, at most how many more ice cubes could the new container hold?

**13** 12.2

If 20 one-euro coins were stacked on top of each other in a column, the column would be approximately $4\frac{5}{6}$ cm tall. At this rate, which of the following is closest to the number of one-euro coins it would take to make a 10-cm-tall column?

A) 40
B) 50
C) 100
D) 120

**14** 12.2

Arnav created a batch of purple juice by mixing 4 liters of red juice with 3 liters of blue juice. He must mix a second batch using the same ratio of red and blue juice as the first batch. If he uses 6 liters of red juice for the second batch, how much blue juice should Arnav use?

A) Exactly 6 liters
B) 2 liters more than the amount of blue juice used in the first batch
C) 0.75 times the amount of blue juice used in the first batch
D) 0.75 times the amount of red juice used in the second batch

**15** 12.2

Franklin created a scale model of a bus where 1 inch on the model equals 4 feet on the bus. The length of the model is 9.3 inches. Franklin wants to make a new model where the scale of 1 inch on the model equals 2 feet on the bus. Which of the following best describes how the length of the new model will compare to the length of the first model?

A) The length of the new model will be 2 inches longer than the first model.
B) The length of the new model will be 2 inches shorter than the first model.
C) The length of the new model will be 2 times as long as the length of the first model.
D) The length of the new model will be $\frac{1}{2}$ as long as the length of the first model.

**16** 12.2

The relationship between $x$ and $y$ can be written as $y = mx$, where $m$ is a constant. If $y = 23$ when $x = a$, what is the value of $y$ when $x = 3a$ ?





**17** 12.2

| $x$ | $y$ |
|---|---|
| 1 | 6 |
| 3 | 18 |
| 5 | 30 |
| 60 | $k$ |

In the table above, the ratio of $y$ to $x$ for each ordered pair is constant. What is the value of $k$ ?

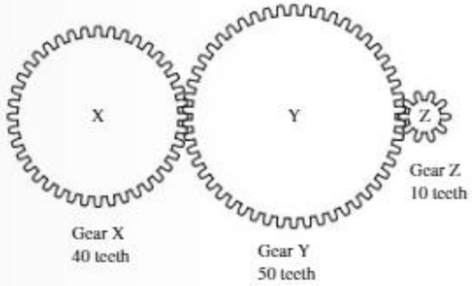
A) 42
B) 54
C) 180
D) 360

**18** 12.3

One serving of a certain brand of peanut butter provides 200 calories, 160 of which are from fat. One serving of a certain brand of high-riboflavin cookies provides 140 calories, 14 of which are from fat. How many more calories from fat are provided by a 50-calorie serving of the peanut butter than are provided by a 50-calorie serving of the cookies?

**19** 12.3

A gear ratio $r:s$ is the ratio of the number of teeth of two connected gears. The ratio of the number of revolutions per minute (rpm) of two gear wheels is $s:r$. In the diagram below, Gear X is turned by a hand crank. The turning of Gear X causes Gears Y and Z to turn as well.




If gear X is rotated by the hand crank at a rate of 40 rpm, what is the number of revolutions per minute for Gear Z?

A) 10
B) 80
C) 160
D) 400

**20** 12.2

There are four atoms of hydrogen and one atom of carbon in one molecule of methane gas. How many atoms of hydrogen are there in 33 molecules of methane gas?

**21** 1600.io 12.1

English Monarchs
from 1000 to 2000

| Ages | Number |
|---|---|
| 0–9 | 4 |
| 10–19 | 11 |
| 20–29 | 13 |
| 30–39 | 15 |
| 40–49 | 5 |
| 50–59 | 4 |
| 60–69 | 1 |

The table above gives the number of English monarchs from 1000 to 2000 whose age at the time they first took the throne is within the interval listed. Of those monarchs who were younger than 40 years old when they first took the throne, what fraction were less than 20 years old?

A) $\frac{13}{43}$

B) $\frac{15}{43}$

C) $\frac{28}{43}$

D) $\frac{28}{53}$

**22** 1600.io 12.3

A researcher surveyed 300 adults selected at random from the town of Freehold and 400 adults selected at random from the town of Howell. Each person surveyed was asked whether he or she owns a goat. Some of the results are shown in the partially completed table below.

| | Owns goat | Does not own goat | Total |
|---|---|---|---|
| Freehold | 60 | 240 | 300 |
| Howell | $x$ | $y$ | 400 |

The researcher found that an adult surveyed in Howell is four times as likely to own a goat as an adult surveyed in Freehold. Of the following, which could be the value of $x$ ?

A) 80

B) 100

C) 240

D) 320





# Answer Key

**Section 12.1 Practice Problems**

**1.** B
**2.** A
**3.** B
**4.** B

**Section 12.2 Practice Problems**

**1.** D
**2.** B
**3.** C
**4.** A

**Section 12.3 Practice Problems**

**1.** C
**2.** C

**Additional Problems**

**1.** D
**2.** 8
**3.** 90
**4.** $\frac{41}{5}$ or 8.2
**5.** 12
**6.** B
**7.** D
**8.** D
**9.** A
**10.** C
**11.** C
**12.** 18
**13.** A
**14.** D
**15.** C
**16.** 69
**17.** D
**18.** 35
**19.** C
**20.** 132
**21.** B
**22.** D

# Percentages 13

## 13.1 Introduction to Percentages

### Basics of Percentage Calculations and the Three Scenarios

Percentages are a particularly convenient way to express an amount relative to a reference, base, total, starting, or original amount. There are several ways to express such relative amounts; we can use **fractions**, as when we cut a pizza into 8 slices, and John eats 2 of them. We can then say John ate $\frac{2}{8}$ (or $\frac{1}{4}$) of the pizza. We can use the **decimal value** of the corresponding fraction: John ate 0.25 times the whole pizza, because $\frac{2}{8} = 0.25$. When we're discussing situations like this, the fraction is a part of the whole, and thus its value is less than 1 (it's a "proper fraction"), so the decimal value of the fraction will start with a decimal point on the left (the 0 in 0.25 doesn't contribute to the number's value—it just looks a bit neater when written that way). Therefore, purely as a convenience, we often use the decimal value of such a fraction multiplied by 100, so instead of having something rather awkward such as 0.25 parts out of 1, we have 25 parts out of 100. When a decimal is scaled in this way so that the resulting number represents a number of parts out of 100 (100 is "cent," in Latin), we say that this value is a **percent**" (literally, "per one hundred"), and we use the symbol "%" to indicate this, so we'd write $\frac{1}{4} = 0.25 = 25\%$. Just as $\frac{1}{4}$ means 1 part out of 4, and 0.25 means 0.25 parts out of 1, 25% means 25 parts out of 100, and they all indicate exactly the same portion of some reference, base, total, starting, or original amount.

Note that in any situation involving percentages, there are three fundamental components:

1. The reference, base, total, starting, or original amount of something
   In our example, that was 8 slices of pizza.
2. The *relative* portion or share of the reference amount, which will be given or requested as a percentage
   That was 25%.
3. The *actual value* of the relative portion or share, which will be in the same units as the reference value
   That was 2 slices.

When we have a linear relationship like this among three values and we know any two of them, we can calculate the third one. In our example, we were given the **starting amount** and the **actual value** of the portion of interest, and we calculated the **relative portion** and expressed it as a percentage. This was simple; because the percentage is just a simple conversion (multiplication by 100) of the decimal equivalent of a fraction consisting of the portion (2 slices) over the total (8 slices), we could just say

$$\frac{\text{portion of the total}}{\text{total}} = \text{relative amount}$$

$$\frac{2}{8} = \frac{1}{4},\ 0.25,\ \text{or } 25\%$$

It's important to remember that when you need to use a percentage in a calculation that will produce an amount that's not also a percentage, the percentage you're using must first be converted to its fractional or decimal equivalent. Remember, the percentage symbol means "per 100" so, if you like, you can think of the % sign as representing "/100" or "×0.01." When you see, for example, 25%, you can consider it to represent the term $\frac{25}{100}$ or $25 \times 0.01$, both of which are equivalent to 0.25. That understanding allows you to readily perform arithmetic computations involving percentages.

Notice that the difference between the decimal and percentage representations is that the decimal point has been moved two places to the **left**—that is the effect of **dividing by 100**.

Similarly, if we are presented with a decimal, we can convert it into a percentage by multiplying by 100% (treat the % sign as a variable or unit). For example, 0.25 is equivalent to $(0.25)(100\%) = 25\%$. Notice that the difference between the decimal and percentage representations is that the decimal point has been moved two places to the **right**—that is the effect of **multiplying by 100**.

## Percent to Decimal

Convert a percentage into a decimal number by **dividing the percentage by 100**; an easy way to do this is to move the decimal point in a percentage **two places to the left** (inserting zeros as needed).

For example, 32% is 0.32, and 1.24% is 0.0124.

## Decimal to Percent

Convert a decimal number into a percentage by **multiplying the decimal by 100**; an easy way to do this is to move the decimal point of a decimal number **two places to the right** (inserting zeros as needed).

For example, 0.25 is 25%, and 0.01 is 1%.

**SkillDrill 13.1-1**

**Directions:** Convert the percentage into a decimal by dividing by 100% (or moving the decimal point two places to the left).

**1.** 47% **2.** 31% **3.** 30% **4.** 2%

**5.** 40.2% **6.** 2.35% **7.** 0.6% **8.** 210%

**Directions:** Convert the decimal into a percentage by multiplying by 100% (or moving the decimal point two places to the right).

**9.** 0.42 **10.** 0.15 **11.** 0.8 **12.** 0.7

**13.** 0.125 **14.** 0.236 **15.** 0.004 **16.** 1.06

To finish off our discussion of the three percentage scenarios, let's return to our pizza scenario but with different given information.





If we're told that a pizza has 8 slices, and John ate 25% of them, we could be asked how many slices $s$ of the pizza John ate. Again,

$$\frac{\text{portion of the total}}{\text{total}} = \text{relative amount}$$

This time, though, we don't want to find the relative amount—it's given to us as 25%. We want to know the actual value of the portion of the total, so we would rearrange the equation to isolate that quantity.

$$\text{portion of the total} = (\text{relative amount})(\text{total})$$

Putting in the information we're given,

$$s = (25\%)(8)$$

(When we have a percentage involved in multiplication like this, we wouldn't say, "25% *times* 8;" we'd say, "25% *of* 8.")

It's a simple matter now to solve for $s$, but we have to remember that 25% means $\frac{25}{100}$ or $25 \times 0.01$, so our calculation will look like this:

$$s = \left(\frac{25}{100}\right)(8) \qquad \text{or} \qquad s = (25 \times 0.01)(8)$$

which works out to

$$s = 0.25(8)$$
$$s = 2$$

In this situation, where we want to know the value of a percentage of some total as framed by a question such as "what's 25 percent of 8," you might find it more comfortable to write the solving equation so that it models the way you'd express the answer to the question in words: "25 percent of 8 is $s$:"

$$(25\%)(8) = s$$

This is purely a matter of personal preference; you can put either expression on either side, because unlike $b^2 - 4ac$, equals signs don't discriminate (sorry).

There is only one other possible combination of two knowns and one unknown: we're given the percentage and the actual value the percentage represents, and we need to find the original or total amount. This is the most confusing scenario for students, especially when posed as part of a word problem, because of the awkwardness of the phrasing and the seemingly backwards way we get to our solution—it's as though we're starting with the answer and trying to figure out the question (the original amount). Let's list the three possible percentage scenarios, then address this toughest one; we'll use $t$ for the total amount, $s$ for the share of the total, and $P$ for the percentage involved:

1. $s$ is what percentage of $t$? (Solve for $P$, the relative amount)
   This seems simple, as the answer is merely the value of $s$ relative to $t$, or $\frac{s}{t}$, converted to a percentage.
2. What is $P$ percent of $t$? (Solve for $s$, the portion of the total)
   Again, this is straightforward; $P$ percent of $t$ is $\left(\frac{P}{100}\right)(t)$.
3. $s$ is $P$ percent of what total amount? (Solve for $t$, the total)
   Uh…

Don't be flustered by problems of the third form. You just need to know at least one form of the equation that relates the three quantities (we've shown two of them above), so all you need to do is to plug in what you're given and solve for the unknown. For example, we could be asked, 2 is 25% of what total amount? Let's use the first form of the equation we explained above:

$$\frac{\text{portion of the total}}{\text{total}} = \text{relative amount}$$

Substituting what we're given,

$$\frac{2}{\text{total}} = 25\%$$

Now, all we need to do is a little rearranging to isolate the total, which is what we're asked to find:

$$\frac{2}{25\%} = \text{total}$$

Remember, when we need to perform a calculation with a percentage (and the result is not also a percentage), we must convert the percentage to a decimal, so we can write,

$$\frac{2}{0.25} = 8 = \text{total}$$

We could just as easily use the second form of the relative amount equation:

$$\text{portion of the total} = (\text{relative amount})(\text{total})$$

Substituting (conversion from percentage to decimal has already been done),

$$2 = (0.25)(\text{total})$$

Isolating the total,

$$\frac{2}{0.25} = 8 = \text{total}$$

The important point here is that **you only need to know one version of the equation that expresses the relationship among the three quantities involved in a percentage calculation** in order to solve for any of the three quantities, so don't be knocked off your game when you encounter a problem involving that tricky third type of scenario; the math will always have your back.

### Use Your Head

When working with percentages, don't blindly accept the results of the calculations you perform without considering whether they make sense. If you're finding what percentage 18 is of 20, and your calculation yields 111%, don't just plow ahead without considering whether your answer should be greater or less than 100% (hint: 18 is less than 100% of 20). In most cases, you can easily get a sense of what a reasonable, if inexact, answer will be before you perform the computation, so take advantage of that to check yourself.

Another way to model percentage situations is through the use of proportions. This is not the most expeditious method, but some students find it the easiest to remember and to use in practice, and there's nothing wrong with adopting that preference if it allows you to reliably and confidently tackle percentage problems.

The principle is simply to set up a proportion with the percentage and 100% on one side of the proportion, and the portion and the whole on the other side; you don't need to convert the percentages to decimals, because you're comparing percentages to percentages in the fraction. For example, we can model the situation where John eats 25% of his 8-slice pizza as follows:

$$\frac{25\%}{100\%} = \frac{2}{8}$$





You can use a setup like this with any of the three variations of a percentage situation. You don't need to actually write the % signs, of course, and as always, we recommend arranging the proportion so that the unknown is in the numerator to simplify the arithmetic involved in solving, and we like putting the unknown on the left side just so it reads more like a sentence when solved. Let's look at the 3rd-form problem shown above: 2 is 25% of what total $t$? We'd set this up like this:

$$\frac{t}{2} = \frac{100}{25}$$

Solving for $t$,

$$t = \frac{2(100)}{25} = \frac{200}{25} = 8$$

## Probabilities and Percentages

As discussed above, percentages are an alternative way of expressing the values of fractions. Some types of quantities might commonly be expressed as a fraction, a decimal value, or a percentage. For example, in the previous chapter we talked about probabilities, which are fractions describing what portion of an entire group (where the denominator is the reference or total number) is made up by a certain sub-group within that larger group (where the numerator is the size of this sub-group).

When we model a probability as a fraction, that fraction might be more usefully expressed as a decimal or a percentage, as this allows easier comparison to other probabilities than does the fractional representation; we instantly know that 0.75 or 75% is more than 0.73 or 73%, but is $\frac{3}{4}$ more or less than $\frac{68}{92}$? For example, if 150 out of 200 students have brown eyes, then the probability that a randomly chosen student has brown eyes is $\frac{150}{200}$, which can be reduced to $\frac{3}{4}$ (when written as a fraction, a probability such as this is sometimes expressed in words as "three in four" or "three out of four"). Expressed as a decimal, the probability is 0.75, and, as always, a decimal can be converted into a percentage by multiplying by 100 or by just moving the decimal point two places to the right as explained previously. To complete the example, 0.75 expressed as a percentage is equivalent to $0.75 \times 100\% = 75\%$.

### Using Fractions instead of Decimals

Typically, we advocate using your calculator as little as possible, so it should come as no surprise that we recommend using fractions instead of decimals whenever possible when performing calculations.

If a percentage ends in a 0 or a 5, the odds are the problem is easier to perform by hand with a fraction. For example, write 25% as $\frac{1}{4}$; write 30% as $\frac{3}{10}$; write 80% as $\frac{4}{5}$; and you can even write 65% as $\frac{13}{20}$.

A tip: To rapidly divide a larger number by 5, first divide it by 10, then multiply the result by 2, as this is equivalent to dividing by 5; referring to the last example just above, to divide 65 by 5 to figure out how many 20ths it represents, divide it by 10 to get 6.5, then multiply by 2 to get 13, so 65% is equivalent to $\frac{13}{20}$.

**SkillDrill 13.1-2**

**Directions:** Write the percentage as a fraction in lowest terms.

**1.** 50% **2.** 75% **3.** 70% **4.** 5%

**5.** 60% **6.** 120% **7.** 110% **8.** 205%

**Example 13.1-1**

**1** 1600.io 13.1

Weekly Hours

| | Office A | Office B |
|---|---|---|
| Project J | 84 | 32 |
| Project K | 50 | 110 |

A company has only two projects, J and K, and only works on those projects in offices A and B. The table above shows the number of hours worked on each project during a week.

Which of the following is closest to the percent of the total number of hours worked on projects J and K combined that are worked at Office A weekly?

A) 30%

B) 49%

C) 63%

D) 72%

**Solution**

1. In the end, the percentage we need to find for the answer is based on the following fraction:

$$\frac{\text{portion of the total}}{\text{total}} = \text{relative amount}$$

In this problem, the "portion of the total" is the total number of hours worked on both projects *at Office A*, the "total" is the total number of hours worked on both projects *at both locations*, and the "relative amount" is the percent of the total hours worked that were worked *at Office A*.

$$\frac{\text{The number of hours worked on Projects J and K at Office A}}{\text{The total number of hours worked on Projects J and K}} = \text{Percent of hours worked at Office A}$$

2. We have to start by finding the numerator and denominator for the fraction, which will eventually be turned into a percentage. The numerator will be the total number of hours worked at Office A on both Project J and Project K (add the cells in the first column):

$$84 + 50 = 134$$

3. The denominator will be the total number of hours worked at both offices on both projects (add all 4 cells in the table):

$$84 + 50 + 32 + 110 = 276$$

4. Divide the numerator by the denominator to find the decimal representation of the fraction.

$$\frac{\text{The number of hours worked on Projects J and K at Office A}}{\text{The total number of hours worked on Projects J and K}} = \text{Percent of hours worked at Office A}$$

$$\frac{134}{276} = 0.4855072464$$





5. The result of the division is a repeating decimal (it repeats after 23 digits), but we should round it to two places to match the number of digits in the answer choices (remember that the question asks us not for the exact percentage but for which choice is closest). Round the decimal to 0.49.

6. Move the decimal point two places to the right to convert the decimal into a percentage.

$$0.49 = 49\%$$

7. The answer is B.

There are a lot of ways to handle percentage problems depending on how comfortable you are working with them.

For example, if you want to know what 40% of 10 is, you can set up a proportion, $\frac{40}{100} = \frac{x}{10}$, and solve for $x$, or you can just multiply 0.4 by 10 right out of the gate because $x\%$ of $y$ means $\frac{x}{100} \cdot y$.

## Multiplying by Percentage

To find $P\%$ of a value $x$, multiply the decimal or fraction representation of the percentage by $x$.

For example, to find 30% of 50, you would multiply 0.3 (or $\frac{3}{10}$) by 50 to get 15.

**SkillDrill 13.1-3**

**Directions:** Find the value that is $x\%$ of $y$ by multiplying $y$ by the decimal representation of $x\%$. You may use a calculator for this drill.

**1.** 50% of 100 **2.** 80% of 200 **3.** 21% of 82 **4.** 3% of 280

**5.** 0.4% of 1000 **6.** 18.4% of 330 **7.** 120% of 40 **8.** 100% of 20

Writing equations with percentages can seem perplexing because often it is unclear what gets multiplied or divided by what. The key is to rephrase the questions in a straightforward way, trying to consistently match the format of the following phrase:

***A* is *P* percent of *B***

The corresponding equation, which you can use to solve for the unknown value is

$$A = \frac{P}{100}B$$

Equivalently, we can use lowercase $p$ for the decimal equivalent of $P$ percent directly, writing

$$A = pB$$

For example, if we are asked what percentage of 32 is 8, we could make the statement,

$$8 \text{ is } P \text{ percent of } 32$$

and then write the equation

$$8 = p \cdot (32)$$

Solving for $p$, we see that $p = \frac{8}{32} = 0.25$, so the equivalent percentage, $P$, would be $0.25(100) = 25\%$.

**Example 13.1-2**

**2** 1600.io 13.1

A sample of sugar water is 4.2% sugar by mass and contains 1,000 grams of sugar. Which of the following is closest to the mass, in grams, of the sample of sugar water?

A) 23,800

B) 22,800

C) 958

D) 42

**Solution 1**

1. Let's begin by rephrasing so we know exactly what we want to write down: the mass of sugar, $s$, in the sample is 4.2% (written in decimal form as 0.042) of the mass of the whole sample, $w$.

$$s = 0.042w$$

2. Since the mass of the sugar is 1,000, plug in 1,000 for $s$.

$$1{,}000 = 0.042w$$

3. Divide both sides of the equation by 0.042 to solve for $w$, the mass of the whole sample.

$$\frac{1{,}000}{.042} = w$$
$$23{,}810 \approx w$$

4. The answer that is closest to 23,810 is A.

**Solution 2**

1. We can use a proportion to solve this problem. The ratio of the mass of the sample of sugar water to the mass of the sugar in that sample is constant; therefore, on one side of the equals sign we can put a variable, $w$, which is the total mass of the sugar water, over 1,000, which is the mass of just the sugar, and on the other side of the equals sign we can put 100 (the percentage of the sugar water that is sugar water—i.e. all of it) over 4.2 (the percentage of the sugar water that is just sugar). Notice that we placed the unknown in the numerator to make solving the proportion as easy as possible, and also note that the units of % and grams (g) will cancel on their respective sides of the equation.

$$\frac{w \text{ g}}{1{,}000 \text{ g}} = \frac{100\%}{4.2\%} \Bigg\} \frac{\text{total}}{\text{sugar}}$$





2. Multiply both sides of the equation by 1,000 in order to solve for $w$. Use your calculator to do the entire calculation in one step (don't round twice).

$$\frac{w}{1{,}000} = \frac{100}{4.2}$$
$$w = \frac{100{,}000}{4.2}$$
$$w \approx 23{,}810$$

3. The answer that is closest to 23,810 is A.

Note that percentages can be applied to entire expressions. For example, Doug makes \$30 per hour at work. In one week, Doug works $h$ hours and gets paid a bonus of \$200, so his pay for the week is $30h + 200$. Doug's car breaks down and the repairs cost him 50% of his pay from this week. We can find out how much it cost (in terms of $h$) by multiplying the entire expression representing his pay by 0.5: the repairs cost $0.5(30h + 200)$, and we can distribute the 0.5 to rewrite the expression as $15h + 100$.

Some percentage word problems on the test will require you to write simple expressions (usually linear) and then multiply the entire expression by a percentage, and they should not give you any trouble if your fundamentals are solid.

**Section 13.1 Suggested Problems from Real Tests**

- Test 1-C-13
- Test 2-C-5
- Test 3-C-5
- Test 4-NC-12
- Test 5-C-25
- Test 6-C-6
- Test 6-C-38
- Test 7-NC-8
- May 2017-C-23
- May 2017-C-27
- May 2018-C-20
- Oct 2019-C-18
- Mar 2020-C-9
- Mar 2020-C-12
- Apr 2021-C-5
- May 2021 (US)-C-35
- May 2021 (Int)-C-7
- May 2021 (Int)-C-35

View related real-test problems in SMTEx

SMTEx

## Section 13.1 Practice Problems

**1** 1600.io 13.1

At Holmdel High School, approximately 8 percent of enrolled juniors and 12 percent of enrolled seniors had 1600.io memberships. If there were 424 juniors and 505 seniors enrolled at Holmdel High School last year, which of the following is closest to the total number of juniors and seniors at Holmdel High School last year who had 1600.io memberships?

A) 34

B) 61

C) 95

D) 186

**2** 1600.io 13.1

Pearl is working this winter as part of a retail staff. She earned \$9 per hour for the first 10 hours she worked this week. Because of her performance, her manager raised her salary to \$10 per hour for the rest of the week. Pearl saves 70% of her earnings from each week. What is the least number of hours she must work for the rest of the week to save at least \$210 for the week?

A) 39

B) 30

C) 21

D) 6

**3** 1600.io 13.1

Annette performed an experiment where she tossed two coins, one penny and one quarter, at the same time and recorded whether each one landed on heads or tails. Her results are shown in the table below.

| | | Penny | |
|---|---|---|---|
| | | Heads | Tails |
| Quarter | Heads | 50 | 46 |
| | Tails | 47 | 57 |

For what percent of the tosses did the quarter Annette tossed land on tails?

A) 49%

B) 51%

C) 52%

D) 54%

**4** 1600.io 13.1

| Number of sports | Percent of students |
|---|---|
| None | 25% |
| One | 45% |
| Two | 20% |
| More than two | 10% |

The table above shows the distribution of the number of sports that students at a high school participate in. If the number of students who participate in one sport is 80 more than the number of students who participate in two sports, what is the total number of students who attend the high school?

A) 160

B) 240

C) 320

D) 400

## 13.2 Percent Increase/Percent Decrease

If an item is discounted by 20%, you are only paying 80% of the original price because 100% − 20% = 80% (we will use the term **complementary** to describe two percentages that add to 100, so 80% is the complement of 20%). Therefore, if a problem tells you that something **decreases by, declines by, is reduced by, is discounted by, shrinks by, or loses 20%, you will need to multiply the original number by 0.8, not 0.2**, to compute the remaining amount.

For example, if a jacket originally priced at $j$ dollars was on sale for 10% off, and you paid $p$ dollars for the jacket, you would write $p = 0.9j$ because the $p$ dollars you paid was 100% − 10% = 90% of the original price $j$.

### Percent Decrease

When a value is decreased by a percentage, you need to multiply the value by 1 MINUS the percent decrease in decimal form. For example, when a value $x$ is decreased by 20%, we need to multiply $x$ by $1 - 0.2 = 0.8$ because the new value is 100% − 20% = 80% of the original value. When a value $x$ is decreased by 13.6%, we need to multiply $x$ by $1 - 0.136 = 0.864$ because the new value is 100% − 13.6% = 86.4% of the original price.

For the same jacket, if you are told that you *save* $s$ dollars for buying it at the discounted price of 10% off, then you would probably be more interested in writing $s = 0.1j$ (because the $s$ dollars saved is 10% of the original price $j$). Be careful that on such problems, you **check whether you need to find the amount of the decrease or the amount of the remaining quantity** so you know which calculation to apply.





Most often on the test, you will encounter the first situation, and you should be aware that a percentage reduction problem usually means that you want the complementary percentage: 40% reduction means you still have to pay 60% of the price; 13% reduction means you still have to pay 87% of the price; etc.

Similarly, percent growth should be done relative to 1 or 100%. If something grows by 4%, there is now 104% of the original amount (all 100% of the original plus 4% more). For example, if you invest $d$ dollars, and it grows by 4%, then you now have $1.04d$.

### Percent Increase

When a value is increased by a percentage, you need to multiply the value by 1 PLUS the percent increase in decimal form to get the total amount after the increase. For example, when a value $x$ is increased by 20%, we need to multiply $x$ by $1 + 0.20 = 1.2$ because the new value is 100% + 20% = 120% of the original value. When a value $x$ is increased by 13.6%, we need to multiply $x$ by $1 + 0.136 = 1.136$ because the new value is 100% + 13.6% = 113.6% of the original price. When a value $x$ is increased by 200%, we need to multiply $x$ by $1 + 2.00 = 3$ because the new value 300% of the original price (this seems crazy at first, but 100% of the original price plus 200% of that price is indeed 300% of the original price).

### Percent More ≠ Percent Less

It is important to note that 150 is 50% *more* than 100 because $1.5(100) = 150$, but 100 is obviously NOT 50% *less* than 150—50% (or half) of 150 is 75. This is simply because 50% of 100 doesn't equal 50% of 150; the percentage is, of course, *relative* to the reference value.

Write your equations based on what the question tells you to write. Look for the words "is" or "equals" to tell you on which side of the equation to write certain terms. If $x$ *is* 80% of $y$, write the equation $x = 0.8y$. **It is incorrect to conclude that since $x$ is 20% less than $y$ that $y$ is therefore 20% more than $x$.** Given the statement "$x$ is 80% of $y$," you should NOT write $y = 1.2x$.

**SkillDrill 13.2-1**

**Directions:** Given a percentage change from the old, or reference, value $r$, to a new value, $n$, write an equation of the form $n = pr$, where $p$ is the decimal representation of the percent change.

For example, given the prompt "20% increase," your answer should be the equation $n = 1.2r$ to show that the value of $n$, the new value, is 120% of $r$, the reference value.

**1.** 50% increase **2.** 80% decrease **3.** 7% increase **4.** 5% decrease

**5.** 12.2% increase **6.** 10.5% decrease **7.** 3.2% increase **8.** 5.4% decrease

**9.** 0.4% increase **10.** 18.4% decrease **11.** 120% increase **12.** 100% decrease

**Example 13.2-1**

**1** 1600.io 13.2

Last year, Gus's bees produced 88 gallons of honey. This year, Gus increased the number of bees by 50%. If his bees produce honey this year at the same rate per bee as last year, how many gallons of honey can Gus expect to produce this year?

**Solution**

We'll choose $g$ to represent the number of gallons of honey, and to distinguish between the amounts for each of the two years involved, we will use subscripts to clearly mark the new amount, $g_n$ (the $n$ stands for new) and the old amount, $g_o$ (guess what the $o$ stands for… no it stands for original, not old).

1. Since the bees produce honey at the same rate per bee as last year, the only difference is that there are more bees. If the number of bees grows by 50%, the gallons of honey will also grow by 50%. The new number of gallons of honey, $g_n$, will be equal to 150% of the original number of gallons, $g_o$, which can be written as $g_n = 1.5g_o$.

2. The original number of gallons of honey, 88, should be plugged in for $g_o$ so we can solve for $g_n$.

$$g_n = 1.5(88)$$
$$g_n = 132$$

3. The answer is 132.

**Notes**

Note that a 50% increase means that half of the original amount is added on to the original to reach the new amount, so we could have just added 44 (which is half of 88) to 88 to see that the new amount is $88 + 44 = 132$.

This way of doing the problem in your head is perfectly acceptable and shows a good command of percentages and their meaning; however, we are also interested in promoting safe practices for the test, and writing full equations and showing calculations on paper is an important component of those practices, which help minimize careless errors.

The decimal 1.5 is equivalent to the fraction $\frac{3}{2}$ (which could be another way to make the arithmetic in this problem easier to do without a calculator), so the equation in Step 1 could be written as $g_n = \frac{3}{2}g_o$.

Percent change can be confusing because many people do not know which value to use as a denominator when asked to find the percent increase or decrease of a quantity. There is a formula for percent changes, and **the denominator is always the original value**.





## Relative Change Formula

The percent change between values can be calculated with the following formula. Recognize that the result will be a decimal, which you can convert to a percentage as outlined earlier in the chapter (multiplying by 100 and adding the % sign).

$$\text{Relative Change} = \frac{\text{New Value} - \text{Original Value}}{\text{Original Value}}$$

**If the decimal comes out *positive* (because the New Value is *larger* than the Original Value), the percent change represents an *increase*.**

**If the decimal comes out *negative* (because the New Value is *smaller* than the Original Value), the percent change represents a *decrease*.**

For example, let's say that in the year 2000, there were 400 members in a club, and in 2001 there were 440 members in the club. We can find the percent change using the Relative Change Formula.

$$\text{Relative Change} = \frac{\text{New Value} - \text{Original Value}}{\text{Original Value}}$$

$$\text{Relative Change} = \frac{440 - 400}{400}$$

$$\text{Relative Change} = \frac{40}{400}$$

$$\text{Relative Change} = 0.1$$

Since the result of the formula is 0.1, that means that the number of club members *increased* by 10%.

While using the formula is not wrong, we like to suggest tackling these problems in a way that is more consistent with what we've seen so far in this chapter because it eliminates mistakes and cuts down on the number of different things we need to remember.

Given the same scenario, it is useful to rephrase the question as a statement: "440 is $p$ percent of 400", because that translates into a simple equation: $440 = p(400)$. If we divide both sides of the equation by 400, we find that $p = 1.1$, which tells us that 440 is 110% of 400. Therefore, there was a $110\% - 100\% = 10\%$ increase in the number of club members.

Both approaches yield the same result in slightly different ways, but the second method (not using the Relative Change Formula) is preferable because it reinforces the way of thinking that is essential to percentage problems more generally.

**Example 13.2-2**

**2** 1600.io 13.2

A news article on recreation habits in Canada reported that in 2006, children spent an average of 7.2 hours playing sports. The article also reported the average for 2007 to be 6.3 hours per week. Based on the article, how did the average number of hours that children spent playing sports per week change from 2006 to 2007?

A) It decreased by 14.3%.

B) It decreased by 12.5%.

C) It decreased by 9.0%.

D) It increased by 12.5%.

**Solution 1**

1. Start by reframing the question to make it easy to write an equation. The new average, 6.3 hours, is $p$ percent of the old average, 7.2 hours.

$$6.3 = p \cdot (7.2)$$

2. Solve for $p$ by dividing both sides of the equation by 7.2.

$$6.3 = p \cdot (7.2)$$

$$\frac{6.3}{7.2} = p$$

$$0.875 = p$$

3. Since 6.3 is 87.5% of 7.2, the average number of hours decreased by 12.5%($1 - 0.875 = 0.125$ or $100 - 87.5 = 12.5$).

4. The answer is B.

**Solution 2**

1. Use the Relative Change Formula to find the percent change. Make sure that the denominator is the old value, 7.2.

$$\text{Relative Change} = \frac{\text{New Value} - \text{Original Value}}{\text{Original Value}}$$

$$\text{Relative Change} = \frac{6.3 - 7.2}{7.2}$$

$$\text{Relative Change} = \frac{-0.9}{7.2}$$

$$\text{Relative Change} = -0.125$$

2. Since the decimal is negative, there is a decrease, and converting the decimal to a percentage shows that the value decreased by 12.5%.

3. The answer is B.





## Changing Reference Values

In some problems, a quantity changes by one percentage and then by another percentage (both can be increases or decreases, or the changes can be mixed). **These successive percentage changes (regardless of sign) cannot be combined into one change by summing the individual changes.**

**IF A PROBLEM STATES THAT THE PRICE OF SOMETHING IS RAISED BY 5% AND THEN DECREASED BY 10%, THAT DOES NOT MEAN THAT THE PRICE WAS DROPPED BY 5% OVERALL.**

Percentages represent a relative share or portion of some reference amount. **Two percentages that apply to the *same* reference amount can be added and subtracted**. For example, if John eats 25% of the 8 slices in a pizza, and George eats 50% of the 8 slices in a pizza, together they will have eaten 25% + 50% = 75% of the slices in the pizza, which would be 0.75(8) = 6 slices. **This is valid because both percentages refer to the same reference amount of 8 slices.**

However, **if the reference amount is *different* for two percentages**, as happens when we're actually increasing or decreasing the reference amount based on percentage changes to it, **we cannot add the percentages**, because that would combine two relative amounts *that apply to two different references*. For example, if John ate 25% of the pizza, then later George ate 50% of the *remaining* pizza, John will have eaten 2 slices (25% of the **starting** value of 8), and George will have eaten 3 slices (50% of the **remaining** 6 slices). Together, they will have eaten 2 + 3 = 5 slices, not the 6 slices we would calculate if we simply added their percentages together. Again, this is due to the percentages applying to different reference amounts; John's 25% applied to 8, while George's 50% applied to 6.

Scenarios where this comes up usually involve a sequence of percentage calculations, where the first event causes a change in the reference amount, as seen in the pizza example; John's eating of 25% of the pizza caused a change in the reference amount from 8 to 6, so George's consumption of 50% applied to a *different* reference amount.

This principle seems straightforward, but it lurks in certain types of problems in a somewhat less obvious way. For example, let's say that after John decreases the amount of pizza by 25% by eating two slices, he's sad, so he "borrows" some of Dan's pizza to increase the amount of his own pizza by 25%. You might think that decreasing something by 25% and then increasing it by 25% would get you right back where you started, and John would have 8 slices again, but be careful: this is one of those stealthy examples of invalidly trying to add two percentages (−25% and +25%) that have **different reference values**.

When John eats 25% of his 8-slice pizza, he then has 6 slices. When he subsequently increases the amount of pizza by 25% through Dan's unknowing generosity, that will be 25% of 6 slices, and that's a different, smaller amount than the 25% of the 8 slices that he'd previously eaten: 25% of 6 is 0.25(6) = 1.5, so when he's done with his stealthy operation, John won't have 8 slices again—he'll only have 6 + 1.5 = 7.5 slices.

With these increase/decrease situations, it doesn't matter which happens first; the important feature of such scenarios is that **there is a sequence of percentage operations, each one modifying the results of the previous one, so the reference value is changing after each operation**. As another example, let's say George starts with a whole pizza with 8 slices, and he decides he's really hungry, so he increases the amount of pizza he buys by 50%. Since 50% of 8 is 4, George now has 8 + 4 = 12 slices. What happens if George now eats 50% of his pizza? We can't simply say that because George added 50% and subtracted 50%, he'll be right where he started with 8 slices, because **the reference value will change from 8 slices initially to 12 slices after the increase**. Instead, we have to apply each operation in sequence: 8 + (50% of 8) = 12, and then 12 − (50% of 12) = 6. Again, note that George does not end up where he started, because **the increase of 50% of 8 isn't the same as the subsequent decrease of 50% of 12**.

The tl;dr is that **whenever you have a situation where percentage changes (regardless of sign) happen in a sequence**, thus changing the underlying reference value at each step, **you must apply the percentage operations (typically multiplications) in sequence—you cannot simply add the percentage changes together and then apply the resulting percentage change to the original reference value.**

One common situation in which this comes up involves markups and discounts. When you have two price changes expressed as percentages, you need to multiply two percentage changes by the original amount in order to find the final price. Let's say that a book is in high demand, so the price is raised 5% from its original cover price, $e$. The new price of the book is $1.05e$.

When you go to buy the book, you bring a 10% off coupon. You are only going to pay 90% *of the new marked up price*. The amount you pay is $(0.9)(1.05)e$, or $0.945e$, NOT $0.95e$, because the reference amount changes after the 5% increase is applied, so you cannot add the two percentages together to get a net change of −5%; **you must multiply the original price by the markup and then multiply that product by the discount**.

## Successive Percent Changes

If a value changes by a certain percentage and then changes again by a certain percentage, you need to *multiply* the starting value by the percentages, NOT add the percentages together before applying the result to the original value.

For example, if a value $x$ decreases by 20% and is then increased by 20%, the end value is NOT the same as the original value.

You must first multiply $x$ by 0.8 to reflect the 20% decrease, and then multiply the new value $(0.8x)$ by 1.2 to reflect the 20% increase. Therefore, the new value is $1.2(0.8x)$, which evaluates to $0.96x$ (NOT just $x$).

**Example 13.2-3**

**3** 1600.io 13.2

The budget for a school hockey team was $5,400 in the year 2000. The budget increased by 11% from 2000 to 2001 and then decreased by 13% from 2001 to 2002. Which of the following expressions represents the budget, in dollars, for the school hockey team in 2002?

A) $(1.13)(0.89)(5,400)$

B) $(1.13)(1.11)(5,400)$

C) $(0.87)(1.11)(5,400)$

D) $(0.87)(0.89)(5,400)$

**Solution**

1. Between 2000 and 2001, the budget grew by 11%, so we should multiply the year 2000 budget, which was $5,400, by 1.11 to find the year 2001 budget.

$$(1.11)(5,400)$$

2. Between 2001 and 2002, the budget decreased by 13%, so we should multiply the year 2001 budget, which was $(1.11)(5,400)$, by $1 - 0.13 = 0.87$ to find the year 2002 budget.

$$(0.87)(1.11)(5,400)$$

3. The answer is C.

**Notes**

Remember that multiplication is commutative, so the order in which the multiplications are performed or written is irrelevant. You may well have written the final expression as $(5,400)(1.11)(0.87)$, in the order in which the events occurred. This expression of course is equivalent to the one in the solution above, and both lead us to answer C.





### Section 13.2 Suggested Problems from Real Tests

- Test 1-C-20
- Test 1-C-26
- Test 2-C-17
- Test 3-C-22
- Test 3-C-27
- Test 4-C-22
- Test 5-C-24
- Test 6-C-24
- Test 7-C-12
- Test 7-C-30
- Test 8-C-22
- Test 9-C-35
- Test 9-C-38
- Test 10-C-12
- Test 10-C-16
- Mar 2018-C-25
- Apr 2018-C-22
- Apr 2018-C-38
- May 2018-C-37
- Mar 2019-C-20
- Mar 2019-C-38
- Apr 2019-C-9
- Oct 2019-C-34
- Mar 2020-C-5
- Oct 2020-C-24
- Oct 2020-C-36
- Oct 2020-C-37
- Mar 2021-C-11
- Mar 2021-C-30
- Mar 2021-C-37
- Apr 2021-C-33
- Apr 2021-C-35
- May 2021 (US)-C-24
- May 2021 (US)-C-34
- May 2021 (US)-C-36
- May 2021 (Int)-C-6

View related real-test problems in SMTEx

SMTEx

## Section 13.2 Practice Problems

**1** 1600.io 13.2

Federico would owe $10,300 in taxes each year if he were not eligible for any tax deductions. This year, Federico is eligible for tax deductions that reduce the amount of taxes he owes by $3,811.00. If these tax deduction reduce the taxes Federico owes this year by $d$%, what is the value of $d$ ?

**2** 1600.io 13.2

The number $y$ is 25% greater than the number $x$. The number $z$ is 30% less than $y$. The number $z$ is how many times $x$ ?

**3** 1600.io 13.2

A customer's monthly cable bill was $82.68. After switching cable providers, her monthly bill is now $78.10. To the nearest tenth of a percent, by what percent did the amount of the customer's cable bill decrease?

A) 4.6%

B) 5.2%

C) 5.5%

D) 5.9%

**4** 1600.io 13.2

| Year | Subscriptions sold |
|---|---|
| 2018 | 3,400 |
| 2019 | 4,420 |

The manager of an online video library service received the report above on the number of subscriptions sold by the service. The manager estimated that the percent increase from 2018 to 2019 would be half the percent increase from 2019 to 2020. How many subscriptions did the manager expect would be sold in 2020?

A) 4,930

B) 5,083

C) 5,746

D) 7,072

## 13.3 Mixtures and Concentrations

There is a small subset of interesting percentage problems that deal with components of mixtures or solutions.

For example, if we know that we are mixing two glucose solutions together, we can write an equation that represents the total amount of glucose in the mixture based on the percentage of glucose in each of the two solutions. If Solution A is 20% glucose by mass and we have $A$ kilograms of the solution, then the amount of glucose in Solution A is $0.2A$.

Similarly, if Solution B is 30% glucose by mass and we have $B$ kilograms of the solution, then the amount of glucose in Solution B is $0.3B$.

If we form a new solution, Solution C, by mixing all of Solution A and all of Solution B, and we are told that solution C contains $k$ kilograms of glucose, then $k$ (the amount of glucose in Solution C) is equal to the sum of the glucose in Solution A and the glucose in Solution B.

$$k = 0.2A + 0.3B$$

**Example 13.3-1**

**1** 1600.io 13.3

A candy lover buys two kinds of gummy bears. Brand A contains 30% green bears by weight and Brand B contains 60% green bears by weight. Together, the bears bought by the candy lover contain a total of 150 pounds of green bears. Which equation models this relationship, where $x$ is the number of pounds of Brand A and $y$ is the number of pounds of Brand B?

A) $0.3x + 0.6y = 150$

B) $0.6x + 0.3y = 150$

C) $30x + 60y = 150$

D) $60x + 30y = 150$

**Solution**

1. The candy lover buys $x$ pounds of Brand A, and 30% of that weight is made up of green bears. Expressed as a decimal, the weight of green bears in Brand A is $0.3x$.
2. The candy lover buys $y$ pounds of Brand B, and 60% of that weight is made up of green bears. Expressed as a decimal, the weight of green bears in Brand B is $0.6y$.
3. The total weight of green bears, which we are told is 150 pounds, can be found by adding the weight of green bears from Brand A ($0.3x$) to the weight of green bears from Brand B ($0.6y$).

$$0.3x + 0.6y = 150$$

4. The answer is A.





These questions can also require you to solve for some of the unknown values, but in order to do that, you will need to write a system of equations: **one equation that represents the total amount of *one particular ingredient*** (as in the previous example) and **one equation that represents the *total amount of the mixture*** (in the previous example, this would be just adding the total weights of the mixtures).

Let's say the candy lover from the previous example mixes two bags (Bag A and Bag B) of gummy bears into an empty bag (Bag C). We'll be given the percentages of red gummy bears by weight for each of Bag A, Bag B, and Bag C (the final mixture), and the total weight of Bag C must be equal to the total weight of the two bags that make it up (Bag A and Bag B); for now, we will disregard the total weights of any of the bags (usually one of the weights will be given, but let's focus on writing the equations—the substitution of values is the easy part).

Bag A weighs $A$ pounds and is 15% red gummy bears by weight; Bag B weighs $B$ pounds and contains 20% red gummy bears by weight; the final mixture, Bag C, weighs $C$ pounds and contains 18% red gummy bears by weight. This might, at first, seem like an overwhelming amount of information, but if you approach the problem methodically, you'll see that it's straightforward, and each piece of information has its place.

We can write a system of equations (which will condense into one equation) to represent this situation. One equation will represent the total weights of the bags, and the other equation will represent the total weight of red gummy bears based on their percentage of the weight of each of the bags.

The first equation, representing the total weights of the bags, is $A + B = C$ because the weight of Bag C is equal to the sum of the weights of the bags that were mixed together.

Next, we need to write an equation that shows the total weight of red gummy bears only. The weight of red bears in Bag A is equal to $0.15A$ because 15% of the weight of Bag A comes from red gummy bears. The weight of red bears in Bag B is equal to $0.2B$ because 20% of the weight of bag B comes from red gummy bears. The sum of these two amounts can be set equal to $0.18C$ because Bag C contains all of the red gummy bears from both of those bags and 18% of the weight of Bag C comes from red gummy bears.

$$0.15A + 0.2B = 0.18C$$

Now we have a system of equations representing the total weight of the bags and the total weight of red gummy bears.

$$A + B = C$$
$$0.15A + 0.2B = 0.18C$$

Since we know that $A + B = C$ (from the first equation), we can substitute $A + B$ in for $C$ in the second equation in order to condense the system into one equation.

$$0.15A + 0.2B = 0.18C$$
$$0.15A + 0.2B = 0.18(A + B)$$

Mixture problems appear rarely on the test, but have proven to be very difficult for those unfamiliar with them. Those that have appeared so far on the test may give you the equation you will be working with or require you to write your own equation. Above, we laid out the equation writing portion of these problems, and it is very much in line with writing linear equations in general, with only the minor addition of percentages. After the equations are given or written to you, the problems have only gone slightly further by giving you a value to plug in for either one of the components or the total size of the combined mixture. Once you have the equation and a value to plug in, you are dealing with relatively simple problems.

**Example 13.3-2**

**2** 1600.io 13.3

How many ounces of a 15% saline solution must be added to 10 ounces of a 45% saline solution to obtain a 35% saline solution?

**Solution**

The percent salinity refers to the amount of salt in a saline solution (a solution of salt in water), so we will be writing an equation that models the total amount of salt that is present in the total amount of saline solution.

1. Since 15% of the first solution, we'll call it Solution A, is salt, and we have $A$ ounces of this solution, then the amount of salt in Solution A is $0.15A$ (multiply the decimal representation of the percentage by the total amount of solution).
2. Since 45% of the second solution, Solution B, is salt, and we have 10 ounces of this solution, then the amount of salt in Solution B is $0.45(10)$, or 4.5 ounces.
3. Since Solution A and Solution B are going to be mixed, the total amount of salt in the resulting mixture will be the sum of the amounts of salt in each solution, so the total amount of salt is $0.15A + 4.5$.
4. We need to finish the equation representing the total amount, so now we need to find the expression for the other side of the equation.

   The new mixture solution is 35% salt, and the total amount of the new solution is $A + 10$ ounces (the new mixture consists of all of the volume of both Solution A and Solution B), so the total amount of salt in the new mixture is $0.35(A + 10)$.
5. Set the two expressions for the total amount of salt equal to each other.

$$0.15A + 4.5 = 0.35(A + 10)$$

6. Now that we have a single equation with one variable, the rest is simple algebra. Distribute 0.35 on the right side of the equation.

$$0.15A + 4.5 = 0.35(A + 10)$$
$$0.15A + 4.5 = 0.35A + 3.5$$

7. Group the $A$ terms on the right side of the equation (subtract $0.15A$ from both sides of the equation), and group the constant terms on the left side of the equation (subtract 3.5 from both sides of the equation).

$$0.15A + 4.5 = 0.35A + 3.5$$
$$1 = 0.2A$$

8. Divide both sides of the equation by 0.2 (or multiply both sides of the equation by 5 because $0.2 = \frac{1}{5}$) in order to solve for $A$, the number of ounces of Solution A to add to the mix.

$$1 = 0.2A$$
$$5 = A$$

9. The answer is 5.





**Section 13.3 Suggested**

- Test 6-NC-2
- Test 6-NC-19

**View related real-test problems in SMTEx**

SMTEx

## Section 13.3 Practice Problems

**1** 1600.io 13.3

$$0.10x + 0.40y = 0.30(x + y)$$

Emmett will mix $x$ liters of a 10% by mass acid solution with $y$ liters of a 40% by mass acid solution in order to create a 30% by mass acid solution. The equation above represents this situation. If Emmett uses 100 liters of the 40% by mass acid solution, how many liters of the 10% by mass acid solution must he use?

A) 50

B) 67

C) 100

D) 150

**2** 1600.io 13.3

How many ounces of a 35% acid solution must be added to 5 ounces of a 10% acid solution to obtain a 25% acid solution?

# CHAPTER 13 RECAP

- Percentages are a particularly convenient way to express an amount relative to a reference, base, total, starting, or original amount.
- In any situation involving percentages, there are three fundamental components:
  1. The reference, base, total, starting, or original amount of something
  2. The relative portion or share of the reference amount, which will be given or requested as a percentage
  3. The actual value of the relative portion or share, which will be in the same units as the reference value
- You can represent any of the situations with either of these easily-understood equation forms, filling in any two known quantities in order to solve for the remaining unknown value:

$$\frac{\text{portion of the total}}{\text{total}} = \text{relative amount} \qquad \text{OR} \qquad \text{portion of total} = (\text{relative amount})(\text{total})$$

- You can always orient yourself in percentage problems by putting all the provided information into the form, "$A$ is $P$% of $B$," which translates to the second equation form above written as $A = pB$, where the lowercase $p$ is the decimal representation of $P$.
- Convert from decimal to percentage by multiplying the decimal by 100; this is easily accomplished by moving the decimal point two places to the right, adding zeros as needed.
- Convert from percentage to decimal by dividing the percentage by 100, easily done by moving the decimal point two places to the left, adding zeros as needed.
- Two percentages that add to 100 are **complementary**—80% (0.8 in decimal) is the complement of 20% (or 0.2).
- When a value is **decreased** by a percentage, you need to multiply the value by the complementary decimal value of the percentage, which is **1 MINUS the decrease in decimal form**, to find the remaining amount. For example, if a problem tells you that something **loses, decreases by, declines by, is reduced by, is discounted by, or shrinks by 20%, you will need to multiply the number by 0.8, not 0.2, because the new value is 80% of the original value**.
- When a value is **increased** by a percentage, you need to multiply the value by the decimal value of the percentage, which is **1 PLUS the increase in decimal form**, to get the total amount after the increase. For example, if a problem tells you that something **grows or increases by 20%, you will need to multiply the number by 1.2 because the new value is 120% of the original value**.
- Percent More $\neq$ Percent Less
  It is important to note that 150 is 50% *more* than 100 because $1.5(100) = 150$, but 100 is obviously NOT 50% *less* than 150—50% (or half) of 150 is 75.

  Write your equations based on what the question tells you to write. Look for the words "is" or "equals" to tell you on which side of the equation to write certain terms. If $x$ *is* 80% of $y$, write the equation $x = 0.8y$. **It is INCORRECT to infer that if $x$ is 20% less than $y$, then $y$ is 20% more than $x$.** Given the statement "$x$ is 80% of $y$," you should NOT write $y = 1.2x$, but rather $x = 0.8y$, which means $y = \frac{x}{0.8}$ or $y = 1.25x$.
- You can calculate percent increases and decreases using the same equation $A = pB$, where $A$ is the new amount and $B$ is the original amount (read as "the new amount, $A$, is $P$ percent of the old amount, $B$"). The decimal $p$ that is solved for is relative to 1 (indicating what percent $A$ is of $B$).

  If $p = 1.2$, there was a 20% growth from $B$ to $A$ (that is, $A$ is 120% of $B$).

  If $p = 0.85$, there was a 15% decrease from $B$ to $A$ (that is, $A$ is only 85% of $B$, so $B$ decreased by 15%).
- If you don't want to use the uniform percentage equation because you want to avoid interpreting the meaning of the decimal, you can use the Relative Change Formula.

$$\text{Relative Change} = \frac{\text{New Value} - \text{Original Value}}{\text{Original Value}}$$





The result of this formula is very straightforward to interpret. The decimal is the percent change from the original value, and the sign (positive or negative) tells you whether the change is an increase of decrease.

- If a value changes by a certain percentage and then changes again by a certain percentage, you need to *multiply* the original value by the percentages, NOT add the percentages and then apply the result to the original value, because the reference value changes during the sequence.

  For example, if an item cost \$100 initially, but the cost is reduced by 10% (which changes the price to \$90), and then you use a 10% off coupon, you are taking 10% off of the new listed price of \$90, not the original price. Therefore, the price you pay is 90% of 90% of \$100:

$$\text{price} = 0.9(0.9)(100) = .9(90) = 81$$

- The percentage equations in solution and mixture problems are just linear equations with decimals.

  The amount of a substance in a solution/mixture is equal to the percentage of the solution that is made up of that substance times the total amount of the solution/mixture. For example, if there are $A$ kilograms of a saline solution (a solution of salt in water) that is 10% salt by mass, then $0.1A$ is the amount of salt in that solution.

  When mixing solutions to form a new solution, add the amounts of a substance in the ingredients (the percentage of the solution that is made up of that substance times the amount of the solution) for the two solutions and set them equal to the amount of that substance in the resulting mixture (which, again, is the percentage of the solution that is made up of that substance times the total amount of the solution). For example, if we mix $A$ kilograms of a 10% saline solution and $B$ kilograms of a 20% saline solution to form a new 15% saline solution, we can write the following equation

$$0.1A + 0.2B = 0.15(A + B)$$

  because the amount of salt in the first solution is 10% of the total amount of that solution (which is $A$ kilograms), the amount of salt in the second solution is 20% of the total amount of that solution (which is $B$ kilograms), and the amount of salt in the new solution is 15% of the total amount of that solution (which is $A + B$ kilograms).

## Additional Problems

**1** 1600.io 13.2

A customer paid $27.00 for a tie after an 8 percent sales tax was added. What was the price of the tie before the sales tax was added?

A) $24.84

B) $25.00

C) $26.92

D) $29.16

**2** 1600.io 13.2

Recommended Daily Intake of Iodine

| Age | Iodine ($\mu g$) |
|---|---|
| 1-4 years | 60 |
| 4-8 years | 80 |
| 9-13 years | 120 |
| 14-18 years | 150 |
| 19 years and up | 180 |

The table above show the recommended amount of iodine, in micrograms ($\mu g$) per day, for people of all ages according to Good Health Corp. International.

Neil's recommended daily intake of iodine is 50% greater than that of his eleven-year-old brother. What is the least possible age, in years, of Neil?

**3** 1600.io 13.1

Frank bought a trolley pass that had an initial value of $70. For every trolley ride Frank takes, $3.50, the cost of one trolley ride, is subtracted from the value of the pass. What percent of the initial value of Frank's trolley pass is the cost of one trolley ride?

A) 3.5%

B) 5%

C) 10%

D) 95%

**4** 1600.io 13.1

The amount of money a singer earns is directly proportional to the number of people who walk past him on the street. The singer earns $20 when 100 people walk past.

The singer uses 51% of the money earned to pay the costs involved in traveling to the street corner. The rest of the money earned is the singer's profit. What is the profit the singer makes on a day when 100 people walk past?

A) $8.00

B) $9.80

C) $10.00

D) $10.20





**5** 1600.io 13.1

Survey Results

| Answer | Percent |
|---|---|
| Never | 27.4% |
| Rarely | 34.2% |
| Often | 16.3% |
| Always | 22.1% |

The table above shows the results of a survey in which smartphone gamers were asked how often they would watch video advertisements in order to gain in-game currency for free. Based on the table, which of the following is closest to the probability that a smartphone gamer answered "Rarely," given that the smartphone gamer did not answer "Always"?

A) 0.22

B) 0.34

C) 0.35

D) 0.44

**6** 1600.io 13.2

In 2010, the populations of Island X and Island Y were equal. From 2005 to 2010, the population of Island X increased by 20% and the population of Island Y decreased by 20%. If the population of Island X was 10,000 in 2005, what was the population of Island Y in 2005?

A) 15,000

B) 12,500

C) 10,000

D) 9,600

**7** 1600.io 13.2

What number is 25% greater than 80 ?

A) 60

B) 64

C) 100

D) 105

**8** 1600.io 13.2

A number $n$ is increased 9%. If the result is 436, what is the value of $n$ ?

A) 299

B) 399

C) 400

D) 475

**9** 1600.io 13.2

What is 120% of 3,000?

A) 6,000

B) 3,600

C) 600

D) 360

**10** 1600.io 13.2

Interstate Tourist Arrivals, in thousands

| State | March | April |
|---|---|---|
| Alabama | 4.2 | 3.1 |
| New Jersey | 56.2 | 58.1 |
| New York | 250.2 | 400.5 |
| Utah | 1.7 | 5.9 |
| California | 300.1 | 300.2 |
| Maine | 12.3 | 14.6 |

The table above shows the number of interstate tourist arrivals, rounded to the nearest tenth of a thousand, to six states in both March and April of a year.

The number of interstate tourist arrivals in California in March was 12.5% greater than in February. The number of interstate tourist arrivals in California was $k$ thousand more in March than in February. What is the value of $k$ to the nearest integer?

**11** 1600.io 13.2

A rectangle was altered by decreasing its length by 20% and increasing its width by $p$ percent. If these alterations increased the area of the rectangle by 20%, what is the value of $p$?

A) 0

B) 20

C) 40

D) 50

**12** 1600.io 13.2

An online article on television habits in Germany reported that in 2005 viewers spent an average of 520.0 hours per year watching television. The article also reported the average for 2006 to be 496.6 hours per year. Based on the article, how did the average number of hours that viewers spent watching television per year change from 2005 to 2006?

A) It decreased by 23.4%.

B) It decreased by 5.5%.

C) It decreased by 4.5%.

D) It increased by 4.7%.

**13** 1600.io 13.2

In each revolution of a tire, a bike travels a distance equal to the circumference of one of its tires. The circumference of each tire is equal to $2\pi r$, where $r$ is the radius of the tire.

Alonzo gets new tires for his bike. The radius of each of his old tires is 8 inches, and the radius of each of his new tires is 12.5% larger than the radius of one of his old tires. What is the circumference of each new tire, to the nearest tenth of an inch?





**14** 1600.io 13.2

Jafari sold $x$ tennis rackets in 2007. The number of tennis rackets he sold in 2008 was 137% greater than in 2007, and the number of tennis rackets he sold in 2009 was 38% greater than in 2008. Which of the following expressions represents the number of tennis rackets Jafari sold in 2009?

A) $(0.38)(1.37x)$

B) $(0.38)(2.37x)$

C) $(1.38)(1.37x)$

D) $(1.38)(2.37x)$

**15** 1600.io 13.2

Victor works as a sales associate at a used car dealership. He is normally paid 30% of the total retail value of the cars he sells, but he can also earn a bonus. When he earns a bonus, he is paid an additional 5% of his normal pay. During one pay period, Victor sold $8,400 in used cars and earned a bonus. How much was he paid, in dollars, for this pay period? (Disregard the $ sign when writing your answer.)

**16** 1600.io 13.2

Paola bought a dress at a store that gave her a 30 percent discount off its original price. The total amount she paid to the cashier was $p$ dollars, including a 9 percent sales tax on the discounted price. Which of the following represents the original price of the dress in terms of $p$?

A) $0.79p$

B) $\frac{p}{0.79}$

C) $(0.7)(1.09)p$

D) $\frac{p}{(0.7)(1.09)}$

**17** 1600.io 13.2

On February 1st, there were 3,500 rats in a lab. On March 1st, there were 25% fewer rats in the lab than there were on February 1st. On April 1st, there were 40% more rats in the lab than there were on March 1st. How many rats were in the lab on April 1st?

A) 4,025

B) 3,675

C) 2,625

D) 1,575

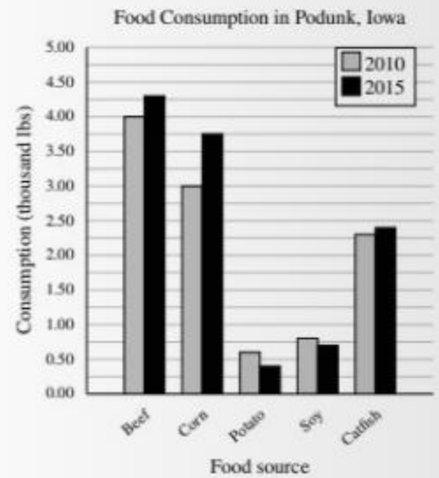
**18** 1600.io 13.1

A company is forming a committee to discuss plans for the launch of a new web service. Of those invited to join the committee, 15% are executives, 40% are programmers, 15% are graphic designers, and the remaining 12 people are marketing representatives. How many more programmers were invited to join the committee than executives?

**19** 1600.io 13.2

Akeelah is ordering laptop computers for her classroom. The laptop computers cost $180 each, and tax is an additional 10% of the total cost of the computers. If she can spend no more than $4,000 on the laptop computers, what is the maximum number of computers that Akeelah can purchase?

**20** 1600.io 13.2




The bar graph above shows food consumption in thousands of pounds (lbs) in Podunk, Iowa, by food, for several foods in the years 2010 and 2015.

Of the following, which best approximates the percent increase in consumption of corn in Podunk from 2010 to 2015?

A) 17%

B) 20%

C) 25%

D) 75%





**Questions 21 and 22 refer to the following information.**

Sports Participation in the 2018–2019 School Year

| Class | Football | Track | Baseball | Soccer | Total |
|---|---|---|---|---|---|
| Juniors | 22 | 15 | 8 | 7 | 52 |
| Seniors | 28 | 13 | 10 | 6 | 57 |
| Total | 50 | 28 | 18 | 13 | 109 |

Of all juniors and seniors who attended a particular high school during the 2018–2019 school year, 109 participated in the sports listed in the table above. Each of the 109 students participated in only one of the four school sports listed. The table shows the distribution of the 109 students by class and sport participation.

**21** 1600.io 13.1

The baseball team was composed of freshmen, sophomores, juniors, and seniors. If 40% of the students on the baseball team were seniors, how many students were on the baseball team?

A) 18

B) 20

C) 25

D) 32

**22** 1600.io 13.1

Of the number of juniors and seniors on the track team, the 25% who have a pet dog represents $\frac{1}{6}$ of the total number of juniors and seniors who have a pet dog. How many juniors and seniors have a pet dog?

A) 42

B) 35

C) 24

D) 7

**Questions 23 and 24 refer to the following information.**

Christopher bought a can of Greasy Stank slop. The nutrition facts on the can state that a serving size of the slop is $\frac{7}{8}$ cup and provides 280 calories, 100 of which are calories from fat. In addition, each serving of the slop provides 160 milligrams of calcium, which is 8% of the daily allowance for adults.

**23** 1600.io 13.1

If $p$ percent of an adult's daily allowance of calcium is provided by $x$ servings of Greasy Stank slop per day, which of the following expresses $p$ in terms of $x$ ?

A) $p = 0.08x$

B) $p = 8x$

C) $p = \frac{0.08}{x}$

D) $p = (0.08)^x$

**24** 1600.io 13.3

On Friday, Christopher will mix Greasy Stank slop with Unctuous Stank slop for his breakfast. Unctuous Stank slop provides 300 calories per cup. If the total number of calories in one cup of Christopher's mixture is 315, how much Unctuous Stank slop is in one cup of the mixture?

A) $\frac{1}{8}$ cup

B) $\frac{1}{4}$ cup

C) $\frac{1}{2}$ cup

D) $\frac{7}{8}$ cup

**25** 1600.io 13.1

Oscar and Kevin each ordered a hamburger at a diner. The price of Oscar's hamburger was $x$ dollars, and the price of Kevin's hamburger was $2 more than the price of Oscar's hamburger. If Oscar and Kevin split the cost of the hamburgers evenly, and each paid a 10% tip, which of the following expressions represents the amount, in dollars, each of them paid? (Assume there is no sales tax.)

A) $0.05x + 0.2$

B) $0.10x + 0.4$

C) $1.1x + 1.1$

D) $1.1x + 2.2$

**26** 1600.io 13.1

Where Do People Get Most of Their Political Information

| Source | Percent of those surveyed |
|---|---|
| Internet | 57% |
| Television | 21% |
| Megaphone-wielder | 8% |
| Politician | 5% |
| Newspapers | 3% |
| Other/none of the above | 6% |

The table above shows a summary of 1,500 responses to a survey question. Based on the table, how many of those surveyed got most of their political information from either the Internet or television?

A) 855

B) 990

C) 1,034

D) 1,170

**27** 1600.io 13.1

A biologist set up an experiment to study the tendency of a gorilla to eat the last banana when presented with a series of bananas. In the experiment, 400 gorillas were presented with a set of five bananas arranged in random order. Each gorilla was allowed to eat one of the bananas. Of the first 200 participants, 84 ate the last banana in the set. Among the remaining 200 participants, $g$ gorillas chose the last banana in the set. If more than 40% of all participants chose the last banana in the set, which of the following inequalities best describes the possible values of $g$ ?

A) $g + 84 > 0.40(400)$, where $g \leq 200$

B) $g - 84 > 0.40(400)$, where $g \leq 200$

C) $g > 0.40(400 + 84)$, where $g \leq 200$

D) $g > 0.40(400 - 84)$, where $g \leq 200$

**28** 1600.io 13.2

There were approximately 178,000 moose wrangling jobs in Canada in 2014. The Bureau of Obscure Jobs has projected that this number will increase by 19% from 2014 to 2024. Of the following, which is closest to the number of moose wrangling jobs the bureau has projected for Canada in 2024?

A) 195,000

B) 197,000

C) 211,800

D) 213,600





**29** 1600.io 13.1

Bikes Owned in Town X

| Bike Color | Percent of owned bikes |
|---|---|
| Black | 11% |
| Blue | 9% |
| Gray | 9% |
| Chrome | 24% |
| Red | 36% |
| Other | 11% |

The table above shows the distribution of color for the 5,000 bikes owned in Town X. Based on the table, how many more red bikes than gray bikes are owned in Town X?

**30** 1600.io 13.2

The sum of three numbers is 630. One of the numbers, $x$, is 25% more than the sum of the other two numbers. What is the value of $x$ ?

A) 280

B) 350

C) 394

D) 473

**31** 1600.io 13.1

According to the 2010 Census, the adult population aged 25 years or greater of the United States was 203,891,983. In 2010, a survey was conducted among a randomly chosen sample of adults aged 25 years or greater in the United States about their preference to watch news or sports. The table below displays a summary of the survey results.

Viewing Preferences

| | News | Sports | No preference | Total |
|---|---|---|---|---|
| 25–40 years old | 158 | 215 | 39 | 412 |
| 41–55 years old | 194 | 222 | 71 | 487 |
| 56–70 years old | 297 | 246 | 17 | 560 |
| Greater than 70 years old | 188 | 110 | 43 | 341 |
| Total | 837 | 793 | 170 | 1,800 |

Which of the following is closest to the difference between the percentage of adults aged 25–55 years who responded "sports" and the percentage of adults aged 56 years or greater who responded "sports"?

A) 5%

B) 9%

C) 10%

D) 19%

**32** 1600.io 13.2

Kinsleigh enlarged the area of a photograph to 150% of its original size. The original dimensions of the photograph were 3 inches by 5 inches. What is the area of the enlarged photograph, in square inches?

A) 22.5

B) 33.75

C) 37.5

D) 93.75

## Answer Key

**SkillDrill 13.1-1**

1. 0.47
2. 0.31
3. 0.3
4. 0.02
5. 0.402
6. 0.0235
7. 0.006
8. 2.1
9. 42%
10. 15%
11. 80%
12. 70%
13. 12.5%
14. 23.6%
15. 0.4%
16. 106%

**SkillDrill 13.1-2**

1. $\frac{1}{2}$
2. $\frac{3}{4}$
3. $\frac{7}{10}$
4. $\frac{1}{20}$
5. $\frac{3}{5}$
6. $\frac{6}{5}$
7. $\frac{11}{10}$
8. $\frac{41}{20}$

**SkillDrill 13.1-3**

1. 50
2. 160
3. 17.22
4. 8.4
5. 4
6. 60.72
7. 48
8. 20

**Section 13.1 Practice Problems**

1. C
2. C
3. C
4. C

**SkillDrill 13.2-1**

1. $n = 1.5r$
2. $n = 0.2r$
3. $n = 1.07r$
4. $n = 0.95r$
5. $n = 1.122r$
6. $n = 0.895r$
7. $n = 1.032r$
8. $n = 0.946r$
9. $n = 1.004r$
10. $n = 0.816r$
11. $n = 2.2r$
12. $n = 0$

**Section 13.2 Practice Problems**

1. 37
2. $\frac{7}{8}$ or .875
3. C
4. D

**Section 13.3 Practice Problems**

1. A
2. $\frac{15}{2}$ or 7.5

**Additional Problems**

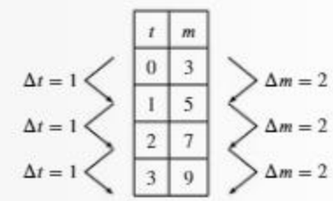
1. B
2. 19
3. B
4. B
5. D
6. A
7. C
8. C
9. B
10. 33
11. D
12. C
13. 56.5
14. D
15. 2646
16. D
17. B
18. 10
19. 20
20. C
21. C
22. A
23. B
24. B
25. C
26. D
27. A
28. C
29. 1350
30. B
31. B
32. A



# Exponential Relationships 14

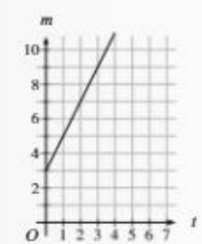
## 14.1 Basic Exponential Equations

Linear Equations are used to represent situations where two values change proportionally. For example, a club that starts with 3 members and adds 2 members every year could be represented by the linear equation $m = 2t + 3$, where $m$ is the number of members, and $t$ is the number of years since the club started. We could visualize the number of members with a table like the following.




| $t$ | $m$ |
|---|---|
| 0 | 3 |
| 1 | 5 |
| 2 | 7 |
| 3 | 9 |

We can also represent the same relationship in a graph of the linear equation.




Lines are used when there is a consistent change based on adding or subtracting the same value. For this example, the number of members always grows by 2 every year.

Exponential equations are used to represent a different type of consistent change: a multiplicative change. **Exponential equations model relationships where the value of a function is repeatedly multiplied or divided by the same number as the input value increases.**

For example, let's say a club starts with 3 members $m$ and doubles the number of members every year $t$ (the number of members is multiplied by 2 every year). This club will have 6 members after the first year, 12 members after the second year, 24 members after the third year, etc.

| $t$ | $m$ |
|---|---|
| 0 | 3 |
| 1 | 6 |
| 2 | 12 |
| 3 | 24 |

$\Delta t = 1$, $\Delta t = 1$, $\Delta t = 1$

$\Delta m = 3$, $\Delta m = 6$, $\Delta m = 12$

The amount by which the number of members increases each year is not constant: the club added 3 members in the first year, then 6 members in the second year, and 12 members in the third year.

You can see that the number of members doubles every year, so, in order to see why these are called **exponential relationships**, let's think a little more about this pattern, and then write the function $m = f(t)$ (the number of members, $m$, is a function of the number of years since the club's inception, $t$) that demonstrates the relationship between $m$ and $t$.

In the beginning (0 years after the club started), there are 3 members, so $f(0) = 3$. After 1 year, there are 6 members, or $3 \times 2$, members, so $f(1) = 3 \cdot 2$. After 2 years, there are 12, or $3 \times 2 \times 2$, members, and so on.

$$\begin{aligned} m &= f(0) &&= 3 = 3 \\ m &= f(1) &&= 3 \cdot 2 = 6 \\ m &= f(2) &&= 3 \cdot 2 \cdot 2 = 12 \\ m &= f(3) &&= 3 \cdot 2 \cdot 2 \cdot 2 = 24 \end{aligned}$$

The number of members is equal to 3 (the starting amount) multiplied by 2 each year for every year after the formation of the club. In effect, **the number of years, $t$, acts like an exponent** because it tells us how many times we have to multiply 3 by 2. Let's rewrite that list using the exponentiation of the factor of 2.

$$\begin{aligned} m &= f(0) &&= 3 = 3 \\ m &= f(1) &&= 3 \cdot 2 = 6 \\ m &= f(2) &&= 3 \cdot 2^2 = 12 \\ m &= f(3) &&= 3 \cdot 2^3 = 24 \end{aligned}$$

For illustrative purposes, let's temporarily suspend the conventions for dropping exponents when they're 1 and for omitting terms that evaluate to 1 so that each equation will have the same form (remember that $2^0 = 1$, so the value of $m = f(0)$ can be written as $3 \cdot 2^0$). You should now see a distinct pattern emerge.

$$\begin{aligned} m &= f(\mathbf{0}) &&= 3 \cdot 2^{\mathbf{0}} = 3 \\ m &= f(\mathbf{1}) &&= 3 \cdot 2^{\mathbf{1}} = 6 \\ m &= f(\mathbf{2}) &&= 3 \cdot 2^{\mathbf{2}} = 12 \\ m &= f(\mathbf{3}) &&= 3 \cdot 2^{\mathbf{3}} = 24 \end{aligned}$$

In each of the rows, the value of $t$ is bolded to show how the value of $t$ that is plugged into the function is the same as the exponent of the 2. Therefore, the exponential function that represents this relationship is the following:

$$m = f(t) = 3 \cdot 2^t$$

Now, we can see that **the input variable $t$ is used as an exponent in the function**, so we call this type of relationship an **exponential relationship**.





## General Form of Exponential Equations

Exponential Equations can be represented simply with the general form shown below, where $a$ is the initial amount, $b$ is the value by which the initial amount is multiplied over every interval, and $x$ is the number of intervals (usually time periods).

$$y = a \cdot b^x$$

In the club member example, the number of members is multiplied each year by a constant value that's **greater than 1** (here, 2), and therefore the number of members is **increasing** every year, so this relationship is termed an **increasing exponential relationship** (also known as **exponential growth**).

When an amount is multiplied repeatedly by a value that's **less than 1 and greater than 0** (or, equivalently, when an amount is divided repeatedly by a value that's greater than 1), and thus the amount is **decreasing**, we have a **decreasing exponential relationship** (also known as **exponential decay**).

For example, a certain deer population is halved due to Chronic Wasting Disease each year. If the group starts with 320 deer, then after the first year there will be only 160 deer left, then 80 the next year, and so on. In this situation, the amount is multiplied by 0.5 each year, or, equivalently, the amount is divided by 2 each year. We can represent this decreasing exponential relationship with the following function, where $d$, the number of deer, is a function of $t$, the number of years since the population size was first measured.

$$d(t) = 320 \cdot \left(\frac{1}{2}\right)^t$$

## Exponential Change vs. Linear Change

Linear Change involves a consistent change in $y$-values for the same change in $x$-values because the same amount is **added** to or **subtracted** from the $y$-value for each constant change in $x$.

Exponential Change involves different changes in $y$-values for the same change in $x$-values. The $y$-value is consistently **multiplied** or **divided** by the same amount.

## Increasing Exponential Change

When the $y$-values get **larger** as $x$ increases in an exponential relationship, the relationship shows **increasing exponential change**. For example, if the $y$-value is tripled every time $x$ increases by 1, then the $y$-values will keep growing, and there is an increasing exponential relationship.

## Decreasing Exponential Change

When the $y$-values get **smaller** as $x$ increases in an exponential relationship, the relationship shows **decreasing exponential change**. For example, if the $y$-value is halved every time $x$ increases by 1, then the $y$-values will keep shrinking, and there is a decreasing exponential relationship.

**Example 14.1-1**

**1** 1600.io 14.1

The first year Alyssa organized an almond milk festival, she invited 25 people. For each of the next 10 years, she invited three times the number of people she had invited the previous year. If $f(n)$ is the number of people invited to the festival $n$ years after Alyssa began organizing the event, which of the following statements best describes the function $f$?

A) The function $f$ is a decreasing linear function.

B) The function $f$ is an increasing linear function.

C) The function $f$ is a decreasing exponential function.

D) The function $f$ is an increasing exponential function.

**Solution**

1. The number of invited people keeps tripling, so it is getting larger, which means that it is an increasing function. Eliminate choices A and C.
2. The number of people invited to the festival is multiplied by 3 every year. The first year there are 25 people invited; in the second year, there are 75 people invited (an extra 50 people); in the third year, there are 225 people invited (an extra 150 people). Since the change is multiplicative and not additive, it is an exponential function.
3. The answer is D.

**Notes**

The value starts at 25 and repeatedly is multiplied by 3 (for every year, there is one more factor of 3), so the function will be $f(n) = 25 \cdot 3^n$. This is clearly an exponential function because the input value $n$ appears as an exponent.

## Percentage Growth and Decay in Exponential Equations

Much of the time, the rate of change (the repeatedly multiplied factor) in exponential equations is given in the form of a percentage. If a town's population grows by 2% every year after the year 2000, then the function representing this relationship, where $P$ is the initial population of the town and $t$ is the number of years after 2000, is $f(t) = P(1.02)^t$ (remember that a 2% increase corresponds to multiplication by 1.02). Note that this percentage increase, represented by multiplication by 1.02, is applied to the new population every year, so it is a multiplicative change and not an additive change.

If the population *decreased* by 2% every year, the function would be $f(t) = P(0.98)^t$ because every year, there is only 98% of the population of the previous year.





**SkillDrill 14.1-1**

**Directions:** Write a function $f$ that represents growth or decay for the given starting amount, percentage change, and interval. Use $d$ for days, $w$ for weeks, $m$ for months, and $t$ for years.

For example, given the conditions "\$100 that grows by 6% each year," you should write the function $f(t) = 100(1.06)^t$.

**1.** \$50 that grows by 4% per week

**2.** \$200 that grows by 10% per day

**3.** \$742 that decreases by 4% per month

**4.** \$100 that decreases by 13% per year

**5.** \$50 that grows by 120% per week

**6.** \$200 that grows by 0.25% per day

**7.** \$742 that decreases by 4.2% per month

**8.** \$100 that decreases by 51% per year

**Example 14.1-2**

**2** 1600.io 14.1

Hightown's population has grown at an average rate of 2.4 percent per year since 1960. There were approximately 10 thousand people in Hightown in 1960. Which of the following functions represents Hightown's population, $P$, in thousands of people, $t$ years since 1960?

A) $P(t) = 1.24t + 10$

B) $P(t) = 1.024t + 10$

C) $P(t) = 10(1.24)^t$

D) $P(t) = 10(1.024)^t$

**Solution**

1. The population growth is modeled by an exponential function because the town's population is always 2.4% more than it was the previous year (the initial population is repeatedly multiplied by a percentage greater than 100%).
   Eliminate choices A and B, which show linear changes based on a constant growth that is unrelated to the size of the population (the percentages show 24% and 2.4%, respectively, but are multiplied by the number of years since 1960, so the linear growth is based on a percentage of a linearly increasing number of years).
2. A growth of 2.4% is equivalent to multiplying by 1.024.
   Answer choice D is the one that has the rate of growth represented correctly as 1.024. (Note that 1.24, found in choice C, would represents a 24% growth.)
3. The answer is D.

## Converting Exponents to the Proper Units

### A Note about Interest Compounding

The interest problems on the test involve what's called **compound interest**, where periodically, an amount of interest is calculated based on the current principal amount, and the interest is added to that principal amount, increasing it. Though in the real world there are situations where the interest rate might have one time period basis (such as an interest rate of 5% per year), but the compounding might happen over a different, shorter period (such as monthly), these scenarios do not appear on the test in a way that requires you to create or evaluate the equations that model them. Rather, on test questions requiring creating or evaluating an interest equation, the period to which the interest rate applies (such as an annual rate) also reflects how often the interest amount is calculated (so the interest would both be calculated and added to the principal, or compounded, annually). The discussion that follows applies to this simpler situation and it will cover the compound interest problems you'll encounter on the test.

When the period or interval to which the growth or decay rate applies is provided directly, we can use the basic exponential equation $y = ab^x$, where $x$ represents a value **in units of the rate period**. For example, if an investor makes an initial investment of \$100 in an account with an interest rate of 7% **per year**, we can model the growth of the account value as a function $A(t) = 100(1.07)^t$, where $t$ is the number of **years** since the initial investment was made. Put another way, the exponent's units are **years** because that's the period of the rate, which is 7% per **year**. This is necessary for the function to produce the correct values.

Sometimes, however, we would like to model a growth or decay situation as a function of a unit with a different scale than the one that defines the rate. For example, we might want to know the value of that same investment account as a function $A(m)$ of a number of months $m$ instead of a number of years $t$. In this type of situation, the essential principle that must be understood is that *the units of the exponent do not change*. Here, that means that **the exponent must still be in years**, even if we aren't directly given a quantity in units of years. This means that **we must convert the quantity we're given into the units of the exponent**; here, that means we must convert $m$ **months** into an expression that represents a number of **years**; because the rate period is a **year**, the exponent of the rate must also be in **years**.

To convert from units with one scale to units with a different scale, such as from months to years (both measure the same thing—time—but they're scaled differently) we can set up a simple proportion that takes the following form:

$$\frac{\text{rate period variable}}{\text{supplied period variable}} = \text{ratio of rate units to supplied period units}$$

In our example, this would be

$$\frac{t}{m} = \frac{1 \text{ year}}{12 \text{ months}}$$

Remember, we have our original function in terms of the rate period units, which is years in our example (the interest is 7% **per year**), but we need it in terms of months. Therefore, we simply need to replace the exponent $t$ with an equivalent term that uses $m$. That's easily accomplished by solving the proportion for the exponent in terms of $m$, which will produce the exponent term for our function:

$$t = \frac{m}{12}$$

The original function which is in terms of years is

$$A(t) = 100(1.07)^t$$

To obtain the equivalent function $A(m)$, we simply replace the exponent $t$ by substituting the equivalent term $\frac{m}{12}$ for $t$:

$$A(m) = 100(1.07)^{\frac{m}{12}}$$

The key principle, again, is that the original exponent $t$ represented a number of **years**, and the new exponent $\frac{m}{12}$ also represents a number of **years**; that's essential, because the rate to which the exponent applies is expressed as a certain percentage change **per**





**year**, so **regardless of the units of the input time period, the exponent's units must be the same as the rate period**, which is why we must perform this units conversion when we are working with a different interval than that of the rate.

After you create the converted exponent term, you can informally check it by substituting the number of intervals in the new units that corresponds to the original exponent to see if the new exponent evaluates to the correct value. Here, the original interval is one year, so the exponent, $t$, is 1 when $t = 1$. There are 12 months in a year, so we can substitute 12 for $m$ in our new exponent term $\frac{m}{12}$ and evaluate it: $\frac{m}{12} = \frac{12}{12} = 1$, so it passes that test. Note, importantly, that this test is not foolproof; for example, if you had erroneously come up with $\frac{12}{m}$ as the new exponent, substituting 12 for $m$ would still yield $\frac{12}{12} = 1$, so this test would not expose the error.

Note that if you see the exponential equation $A(m) = 100(1.07)^{\frac{m}{12}}$, it should be interpreted as "the initial \$100 grows by 7% every 12 months." This is because for the principal amount to be multiplied by 1.07 (representing an increase of 7%), the exponent of the rate must equal 1, and $\frac{m}{12}$ equals 1 when $m = 12$; therefore, the principal will grow by 7% every 12 months. This makes perfect sense, of course; the interest rate is 7% per year, and there are 12 months per year. Being able to interpret exponential equations with coefficients in the exponent is important to some questions on the test.

Now let's flip the script and suppose the investment grew by 7% each month instead of each year, so the equation would be

$$A(m) = 100(1.07)^m$$

If we wanted to write this equation in terms of $t$, the number of years since the initial investment, we'd have to convert from $t$ years into months to maintain the units of the exponent, which must be in months. Again, a simple proportion will do the job: $\frac{m}{t} = \frac{12}{1}$. We want to replace $m$ with an exponent in terms of $t$, so we solve for $m$ to get $m = 12t$, and our exponent will be $12t$ months.

$$A(t) = 100(1.07)^{12t}$$

This equation should be interpreted as "the initial \$100 grows by 7% 12 times per year." Notice how the exponent differed depending on whether the function was initially written in terms of the larger or smaller unit of measurement.

## Working with Non-Standard Rate Periods

Let's apply the same process to an example which doesn't use the same conventional units of time.

If a deer population is initially 200 and it grows by 10% every 4 years (four-year periods are not a typical measurement of time that can strictly be thought of as a normal unit), then the population size, $P$, can be represented with the following equation, where the number of four-year periods is represented by $f$.

$$P(f) = 200(1.1)^f$$

Note that the 1.1 represents the 10% growth rate, and the exponent is just $f$ to show that when $f = 1$ (one four-year period has passed) the population has grown by 10% in that time. What if we wanted to write this in terms of the total number of years, $t$, instead of the number of four-year periods, $f$ ? Well, we know that there are 4 years to every four-year period, so we can write the proportion $\frac{f}{t} = \frac{1}{4}$, and solving for $f$, we see that $f = \frac{t}{4}$, so we can substitute $\frac{t}{4}$ for $f$ to rewrite the equation as a function of the number of years (rather than the number of four-year periods).

$$P(t) = 200(1.1)^{\frac{t}{4}}$$

Once again, this makes sense because the exponent will be 1 when $t = 4$, showing that there will be 10% growth of the initial population only after 4 full years have passed.

Again, your ability to interpret this equation as a statement is important. If you see the exponential equation $P(t) = 200(1.1)^{\frac{t}{4}}$, it should be interpreted as "the initial population of 200 grows by 10% every 4 years."

You might be presented with a non-standard rate period and then asked to create (or identify) an exponential equation in terms of a standard unit. For example, you could be told that a bacteria colony in a petri dish starts with 500 bacteria and increases in number by 45% every 18 minutes, a period that we'll call $p$; the function $B$ that models this situation would be

$$B(p) = 500(1.45)^p$$

Next, you might be asked for a growth function $B(h)$ that gives the number of bacteria as a function of hours $h$ instead of as a function of 18-minute periods $p$. The number of 18-minute periods in an hour is $\frac{60}{18}$, so we can write the following proportion:

$$\frac{p}{h} = \frac{\frac{60}{18}}{1}$$

Solving for $p$ so we can substitute an equivalent term that instead uses $h$, we get

$$p = \frac{60h}{18} = \frac{10h}{3}$$

We can now craft the function by substituting $\frac{10h}{3}$ for $p$ in the exponent:

$$B(h) = 500(1.45)^{\frac{10h}{3}}$$

## Exponential Equations with Modified Exponents

In exponential equations where the initial equation's exponent is written in terms of one unit of measurement, but you are supplied with the period or interval variable or value in different units, you should use a proportion showing the relative values of the units to determine the value of the exponent when expressed in the original units.

For example, there are four quarters in one year, so we can use the proportion $\frac{q}{t} = \frac{4}{1}$, where $q$ is the number of quarters and $t$ is the number of years, to convert measurements in one unit to the other.

In the worst case on the test, you may have to know something like how many 18-minute periods are in one hour in order to make this proportion. You will have to make a quick calculation by division in order to set up the proportion you need. Since there are 60 minutes in an hour and $\frac{60}{18} = \frac{10}{3}$ (indicating that there are $\frac{10}{3}$, which is about 3.33, 18-minute periods in one hour), then the proportion for the number of 18-minute periods, $p$, to the number of hours, $h$, is $\frac{p}{h} = \frac{\frac{10}{3}}{1}$, which simplifies to $\frac{p}{h} = \frac{10}{3}$. Solving for $p$, because we would want the exponent in units of $p$-length periods, we get $p = \frac{10h}{3}$, so our exponent would be $\frac{10h}{3}$ if it needs to be expressed in terms of $h$ hours.





### SkillDrill 14.1-2

**Directions:** Rewrite each exponential function in terms of the supplied input variable, where $d$ is in days, $w$ is in weeks, $m$ is in months, $q$ is in quarters, and $t$ is in years (assume there are 365 days in a year and exactly 52 weeks in a year).

For example, if you are given $f(t) = 100(1.05)^t$ and asked for $f(m)$, write $f(m) = 100(1.05)^{\frac{m}{12}}$.

1. $f(q) = 50(1.04)^q$, find $f(t)$
2. $f(d) = 200(1.1)^d$, find $f(w)$
3. $f(m) = 742(0.96)^m$, find $f(t)$
4. $f(t) = 100(0.87)^t$, find $f(q)$
5. $f(q) = 50(1.04)^q$, find $f(m)$
6. $f(t) = 200(1.0025)^t$, find $f(d)$
7. $f(w) = 742(0.958)^w$, find $f(t)$
8. $f(w) = 100(0.49)^w$, find $f(d)$

**Directions:** Interpret each equation as a statement, where $d$ is in days, $w$ is in weeks, $m$ is in months, $q$ is in quarters, and $t$ is in years. For example, if you are given $f(t) = 100(1.05)^{\frac{t}{4}}$, your answer should be "100 grows by 5% every 4 years." If you are given $f(t) = 100(.95)^{4t}$, your answer should be "100 decays by 5% four times every year."

9. $f(t) = 50(1.04)^{\frac{t}{6}}$
10. $f(w) = 200(1.1)^{7w}$
11. $f(m) = 742(0.96)^{\frac{m}{3}}$
12. $f(d) = 100(0.87)^{\frac{d}{12}}$
13. $f(q) = 80(2.1)^{2q}$
14. $f(t) = 200(1.0025)^{\frac{t}{10}}$
15. $f(d) = 742(0.958)^{4d}$
16. $f(w) = 100(0.49)^{\frac{w}{8}}$

### Example 14.1-3

3 1600.io 14.1

$$S = 12(1.12)^t$$

The equation above models the number of students, $S$, of a learning center $t$ years after the center opens. Of the following, which equation models the number of students of the learning center $q$ quarters after the center opens?

A) $S = 12(1.12)^{4q}$

B) $S = 12(1.03)^q$

C) $S = 12(1.03)^{\frac{q}{4}}$

D) $S = 12(1.12)^{\frac{q}{4}}$

**Solution**

1. The initial equation is written with the exponent in terms of the number of years, $t$, but we want to write the equation in terms of the number of quarters, $q$. Since there are four quarters in one year, we can write the proportion $\frac{t}{q} = \frac{1}{4}$.
2. Multiply both sides of the equation by $q$ to get $t = \frac{q}{4}$.
3. Rewrite the equation by substituting $\frac{q}{4}$ for $t$.

$$S = 12(1.12)^t$$
$$S = 12(1.12)^{\frac{q}{4}}$$

4. The answer is D

**Section 14.1 Suggested Problems from Real Tests**

- Test 1-C-37
- Test 2-NC-14
- Test 3-C-21
- Test 3-C-28
- Test 4-C-13
- Test 4-C-37
- Test 7-C-27
- Test 8-C-23
- Test 8-C-29
- Test 10-C-22
- Test 10-C-26
- May 2018-NC-10
- Mar 2019-NC-7
- Mar 2019-C-28
- May 2019 (US)-C-24
- Mar 2019 (Int)-C-19
- Oct 2020-NC-12
- Oct 2020-C-16
- Mar 2021-NC-15
- Mar 2021-C-19
- Mar 2021-C-27
- Apr 2021-C-16
- Apr 2021-C-30
- May 2021 (US)-NC-15
- May 2021 (US)-C-9
- May 2021 (US)-C-26
- May 2021 (Int)-C-28

View related real-test problems in SMTEx

SMTEx

## Section 14.1 Practice Problems

**1** 1600.io 14.1

Which of the following describes an exponential relationship between the pair of variables listed?

A) Each second $s$, a bike's speed $B$ decreases at a constant rate of 2 meters per second.

B) The depth $d$ of water remaining in a pool decreases by 1 inch each hour $h$ as the water is being splashed out at a constant rate.

C) For every 4-foot increase in the distance $d$ from a trashcan, the intensity of the odor $I$ decreases by 18%.

D) With every 10-inch increase in depth $d$ below the surface of maple syrup, the pressure $p$ on an object increases by 1.2 pounds per square inch.

**2** 1600.io 14.1

A radioactive substance decays at an annual rate of 22 percent. If the initial amount of the substance is 450 grams, which of the following functions $f$ models the remaining amount of the substance, in grams, $t$ years later?

A) $f(t) = 450(0.22)^t$

B) $f(t) = 450(0.78)^t$

C) $f(t) = 0.22(450)^t$

D) $f(t) = 0.78(450)^t$





**3** 1600.io 14.1

In planning budgets for a town's council, an accountant estimates that, starting from the present, the population of the town will decrease by 5 percent every 10 years. If the present population of the town is 11,310, which of the following expressions represents the accountant's estimate of the population of the city $t$ years from now?

A) $11{,}310(0.05)^{10t}$

B) $11{,}310(0.05)^{\frac{t}{10}}$

C) $11{,}310(0.95)^{10t}$

D) $11{,}310(0.95)^{\frac{t}{10}}$

**4** 1600.io 14.1

$$P(t) = 24(3)^{\frac{t}{5}}$$

The number of microscopic organisms in a petri dish grows exponentially with time. The function $P$ above models the number of organisms after growing for $t$ days in the petri dish. Based on the function, which of the following statements is true?

A) The predicted number of organisms in the dish quintuples every three days.

B) The predicted number of organisms in the dish quintuples every day.

C) The predicted number of organisms in the dish triples every five days.

D) The predicted number of organisms in the dish triples every day.

## 14.2 Solving Exponential Equations

Sometimes you need to solve for values using exponential equations, and it is just a matter of writing the equation correctly and substituting values for the correct terms. **It has never been required to use logarithms to solve any problem on this test**, so if you do not know what they are, that is OK. If you do know what logarithms are, there has only been one known case where using logarithms provided a simpler (forward) solution (and this problem could be readily solved by plugging in values, so discussing logarithms in depth is not necessary). In the worst case, you will end up with an equation where a variable raised to a power is equal to a number. Remember that we know how to handle these cases based on what we know about exponents; for example, if we have $x^5 = 200$, we just need to raise both sides to the one-fifth power to solve for $x$ (no logarithms needed). You won't even have to reach for a calculator on most of these problems.

**Example 14.2-1**

**1** 1600.io 14.2

Matthew deposited $x$ dollars in his investment account on January 1, 1901. The amount of money in the account quadrupled each year until Matthew had 16,000 dollars in his investment account on January 1, 1904. What is the value of $x$?

**Solution 1**

Because we have a situation with a starting amount that is being multiplied by the same value over and over again, we can identify this situation as being modeled by an exponential relationship. Therefore, we can solve the problem by creating an exponential equation, substituting the given values, and then solving for the unknown, which is the starting amount in the account $x$.

1. Write an equation to represent the amount of money in the account based on the general exponential equation. Remember, exponential equations can be represented by $y = a \cdot b^x$, where $a$ is the initial amount, $b$ is the value by which the initial amount is multiplied over every interval, and $x$ is the number of intervals. Here, if $A(t)$ is the amount of money in the account $t$ years (where $t$ is the number of intervals) after 1901, $r$ is the value that the initial investment is multiplied by every year (over each interval), and $x$ is the initial investment, then the equation representing the situation is $A(t) = xr^t$.

2. We do not know $x$, the initial investment; that's what we are trying to find. But we do know that $r$ is 4 since the amount quadruples (is multiplied by 4) every year, and we know that $t$ is 3 because 1904 is 3 years after the account was opened. We also know that $A(3)$ is 16,000. Plug these values in and the solve for $x$.

$$A(t) = xr^t$$
$$A(3) = x \cdot (4)^3$$
$$16{,}000 = x \cdot (4)^3$$
$$16{,}000 = x \cdot (64)$$
$$\frac{16{,}000}{64} = x$$
$$250 = x$$

3. The answer is 250.

**Solution 2**

For this solution, we will forego the actual equation and simply list the values of the function by working backwards until we find the value of the initial investment.

1. Make a table of values that shows the year and amount in the account. The value in 1904 is 16,000, and the initial investment, represented by $x$ (which we want to find), is the value of the account in 1901.

| Year | Amount |
|---|---|
| 1901 | $x$ |
| 1902 | |
| 1903 | |
| 1904 | 16,000 |

2. Start from the last row and work your way up the table, dividing the amount by 4 each time you move up a row (because each amount should be 4 times the value in the row above it). We're working backwards here, so we need to use the inverse operation of the multiplication by 4 that happens when going forwards, and the inverse of multiplying by 4 is dividing by 4.

| Year | Amount |
|---|---|
| 1901 | $x = 250$ |
| 1902 | 1,000 |
| 1903 | 4,000 |
| 1904 | 16,000 |

3. The answer is 250.





**Section 14.2 Suggested Problems from Real Tests**

- Test 1-C-38
- Test 4-C-14
- Test 4-C-38
- Test 6-C-37
- Test 8-C-38
- Test 10-NC-18
- 2020-NC-20
- 2020-C-9
- 2021-C-32
- 2021 (Int)-C-20
- May 2019 (US)-NC-12
- Oct 2020-NC-3

View related real-test problems in SMTEx

## Section 14.2 Practice Problems

**1** 1600.io 14.2

$$h(x) = 3^x$$

The function $h$ is defined above. What is $h(4) - h(3)$?

A) 1
B) 3
C) 27
D) 54

**2** 1600.io 14.2

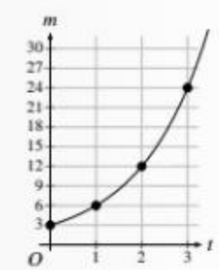
A certain colony of bacteria began with two cells, and the population doubled every 30 minutes. What was the population of the colony after 3 hours?

A) 256
B) 128
C) 64
D) 16

## 14.3 Graphs of Exponential Equations

Let's once again go back to the example of a club from earlier in the chapter. The club starts with 3 members and doubles every year. We can form the graph of this relationship by plotting the points shown in the table below and connecting those points with a smooth, continuous curve.

| $t$ | $m$ |
|---|---|
| 0 | 3 |
| 1 | 6 |
| 2 | 12 |
| 3 | 24 |

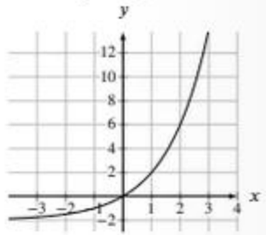



It may look like this is half of a parabola graph, but exponential graphs are not symmetrical; the one end goes up towards infinity (becomes seemingly vertical), but without plotting some more points to the left of the $y$-axis, we can't really tell yet what the general shape of such a curve will be.

positive and negative, and are not present on this test). That is, an exponential expression with a negative sign in front is correctly interpreted as $(-a)(b^{cx})$.

Exponential graphs can be shifted up or down if there is a constant added after the exponential expression. For example, the graph of $y = 2(2)^x - 2$ is shifted down 2 units compared to the first graph shown in the examples above.

$y = 2(2)^x - 2$




The graphs of exponential equations always have the same basic shape (one side is almost horizontal and the other side is almost vertical), but they can come in a few different orientations based on whether the initial values and exponents are positive or negative or if the rate is greater than or less than 1.

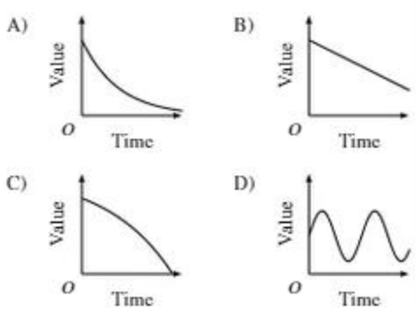
Most of the time, you will not have to deal with all of the different orientations, but if you do, plugging in a few test values should give you the general shape (you can also check on your graphing calculator if you have one available to you—these questions will probably appear in the calculator section).

You only need to understand the basic shape that comes with exponential changes and to know what decreasing relationships (**Exponential Decay**) and increasing relationships (**Exponential Growth**) look like.

**Example 14.3-1**

**1** 1600.io 14.3

In 1890, Philadelphia, Pennsylvania began withdrawing money from a trust fund set up by Benjamin Franklin 100 years earlier. If the value of the trust fund decreased by half every $n$ years, which of the following graphs best models the value of the trust fund over time?








**Solution**

1. We know that we need to pick a graph that looks exponential (has both a horizontal and vertical asymptote) because the value of the fund keeps decaying by half, which indicates an exponential decay (the value is repeatedly multiplied by $\frac{1}{2}$ or 0.5). Eliminate choices B and D.
2. Early on, a decrease of half the value will be a greater change (half of a greater value is a bigger change), but over time, removing half of the fund will be a smaller and smaller amount taken away. For example, if the account started with \$100, then \$50 would be removed the first year, leaving \$50 in the account. The next year, only \$25 would be withdrawn, leaving only \$25 in the account, and so on. The largest changes to the amount of money in the account would occur in the earliest years.
3. Choice A shows a rapid decrease early on, which slows down over time. On the other hand, Choice C decreases slowly at first, and then decreases more rapidly over time, which is not the effect that the withdrawals would have (half of the remaining amount would get smaller and smaller as the amount left in the fund decreases).
4. The answer is A.

**Section 14.3 Suggested Problems from Real Tests**

- Test 2-C-27
- Test 4-C-15
- Test 4-C-20
- Test 9-C-5
- Test 9-C-6
- Test 9-C-21
- Test 10-NC-7
- Apr 2017-C-2
- Oct 2019-NC-3
- Mar 2020-NC-12
- Mar 2021-NC-13
- 2021-NC-14
- 2021-C-15
- 2021 (US)-NC-11
- 2021 (Int)-NC-9
- May 2021 (Int)-C-13
- May 2021 (Int)-C-29

View related real-test problems in SMTEx

SMTEx

## Section 14.3 Practice Problems

**1** 1600.io 14.3

What is the $y$-intercept of the graph of $y = 2^x$ in the $xy$-plane?

A) $(0, 1)$

B) $(0, 2)$

C) $(1, 2)$

D) $(-100, 0)$

**2** 1600.io 14.3

The graph of the exponential function $g$ in the $xy$-plane passes through the points $(0, 3)$, $(1, 9)$, and $(2, 27)$. Which of the following is NOT true?

A) A line can be drawn that does not intersect the graph of $g$.

B) A line can be drawn that intersects the graph of $g$ at exactly one point.

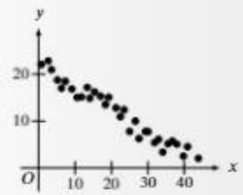
C) A line can be drawn that intersects the graph of $g$ at exactly two points.

D) A line can be drawn that intersects the graph of $g$ at exactly three points.

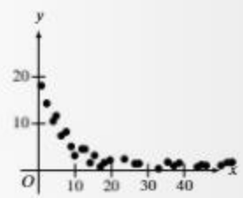
**3** 1600.io 14.3

Which of the following scatterplots shows a relationship that is appropriately modeled with the equation $y = ab^x$, where $a > 1$ and $b > 1$ ?

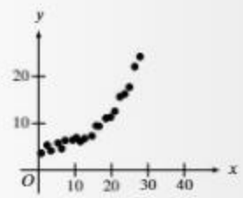
A)



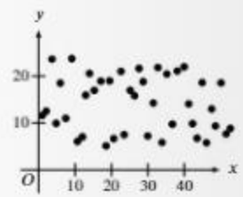

B)




C)




D)








## CHAPTER 14 RECAP

- Exponential change is characterized by an initial value that is repeatedly multiplied or divided by the same amount.
- Most exponential equations you will have to write or interpret will deal with percent increase (growth) or decrease (decay) and will be of the form $y = a \cdot b^x$, where $a$ is the initial value, $b$ is the rate of change (the amount that the value is multiplied by over each interval), and $x$ is the number of intervals (usually a time interval).
- In exponential equations where the initial equation's exponent is written in terms of one unit of measurement, but you are supplied with the period or interval variable or value in different units, you should use a proportion showing the relative values of the units to determine the value of the exponent when expressed in the original units.

  For example, there are four quarters in one year, so we can use the proportion $\frac{q}{t} = \frac{4}{1}$, where $q$ is the number of quarters and $t$ is the number of years, to convert measurements in one unit to the other.

  As a check, verify that the expected value of the interval in the new units will make the exponent equal to 1 to confirm that you have the correct conversion between the two measures of time. For example, if a value doubles every year based on the equation $y = a \cdot 2^t$, where $t$ is the number of years, then it should also take 4 quarters for the amount to double based on the equation $y = a \cdot 2^{\frac{q}{4}}$, where $q$ is the number of quarters (and there are 4 quarters in a year). Substituting 4 for $q$, the equation becomes $y = a \cdot 2^{\frac{4}{4}} = a \cdot 2^1$ which matches $a \cdot 2^t$ when $t = 1$. Note, importantly, that this test is not foolproof; for example, if you had erroneously come up with $\frac{4}{q}$ as the new exponent, substituting 4 for $q$ would still yield $\frac{4}{4} = 1$, so this test would not expose the error.
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  For example, if the exponential term is $x^5$, raise both sides of the equation to the one-fifth power, which will leave the base of the exponential term raised to the first power because the exponents will multiply to 1: $\left(x^5\right)^{\frac{1}{5}} = x$.

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**1** 14.3

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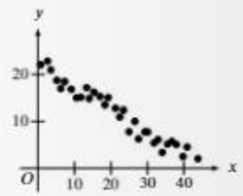
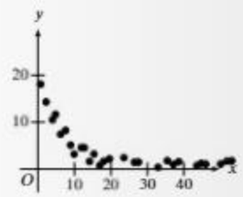
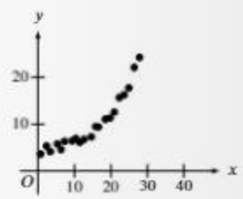


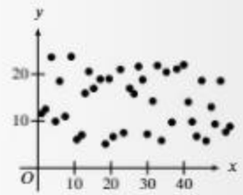

B)




C)




D)





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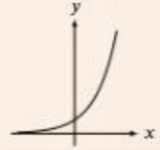
It is useful to know the general shapes of exponential graphs, where one side of the graph is almost horizontal and the other side is almost vertical. Plugging in test values and graphing a few points should give you a reasonable understanding of the graph for any particular exponential equation.

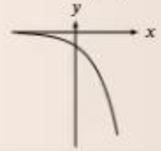
In general for equations of the form $y = ab^{cx}$:

- Positive values of $a$ indicate that the graph has a positive $y$-intercept and will be entirely above the $x$-axis.
- Negative values of $a$ indicate that the graph has a negative $y$-intercept and will be entirely below the $x$-axis.
- When $b \geq 1$ and $c$ is positive AND when $0 < b < 1$ and $c$ is negative, the graph levels off to the left side and goes to infinity (or negative infinity when $a$ is negative) on the right side (the $y$-value changes slowly, then changes rapidly).

$y = 2(2)^x$ or $y = 2\left(\frac{1}{2}\right)^{-x}$




$y = -2(2)^x$ or $y = -2\left(\frac{1}{2}\right)^{-x}$




- When $b \geq 1$ and $c$ is negative AND when $0 < b < 1$ and $c$ is positive, the graph levels off to the right side and goes to infinity (or negative infinity when $a$ is negative) on the left side (the $y$-value changes rapidly, then changes slowly).

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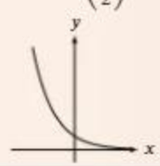


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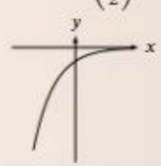







## Additional Problems

**1** 14.1

Algernon modeled the growth over several hundred years of a gator population by estimating the number of teeth per square mile that were deposited each year within layers of a swamp's bottom. He estimated that there were 430 teeth per square mile the first year, with a 2.4% annual increase in the number of teeth per square mile thereafter. Which of the following functions models $N(t)$, the number of teeth per square mile $t$ years after the first year the teeth were deposited?

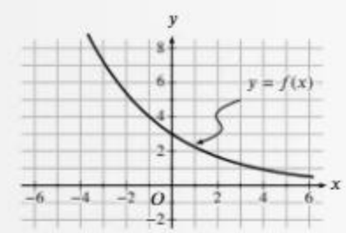
A) $N(t) = 430^t$

B) $N(t) = 430^{1.024t}$

C) $N(t) = 1.24(430)^t$

D) $N(t) = 430(1.024)^t$

**2** 14.3




The graph of $y = f(x)$ is shown in the $xy$-plane. What is the value of $f(0)$?

A) 0

B) 2

C) 3

D) 4

**Questions 3 and 4 refer to the following information.**

The stock price of one share in a certain company is worth \$280 today. A stock analyst believes that the stock will lose 11 percent of its value each week for the next four weeks. The analyst uses the equation $V = 280(r)^w$ to model the value, $V$, of the stock after $w$ weeks.

**3** 14.1

What value should the analyst use for $r$?

**4** 14.2

To the nearest dollar, what does the analyst believe the value of the stock will be at the end of four weeks? (Note: Disregard the \$ sign when writing your answer.)

**5** 14.2

Jon purchased a painting that had a value of \$400 at the time of purchase. Each year, the value of the painting is estimated to increase 20% over its value the previous year. The estimated value of the painting, in dollars, 2 years after purchase can be represented by the expression $400a$, where $a$ is a constant. What is the value of $a$ ?

**6** 14.1

A printer is initially loaded with 4,000 sheets of paper. The printer starts printing a very long document and prints at a constant rate. After 10 minutes, it has used 20% of the paper. Which of the following equations models the number of sheets of paper, $p$, remaining in the printer $m$ minutes after the printer started printing?

A) $p = 4{,}000 - 80m$

B) $p = 4{,}000 - 20m$

C) $p = 4{,}000(0.8)^{\frac{m}{10}}$

D) $p = 4{,}000(0.2)^{\frac{m}{10}}$

**7** 14.2

$$2{,}000\left(1+\frac{r}{1{,}200}\right)^{12}$$

The expression above gives the amount of money, in dollars, generated in a year by a \$2,000 deposit in a bank account that pays an annual rate of $r$%, compounded monthly. Which of the following expressions shows how much additional money is generated at an interest rate of 4.2% than at an interest rate of 2% ?

A) $2{,}000\left(1+\frac{4.2}{1{,}200}\right)^{12} - 2{,}000\left(1+\frac{2}{1{,}200}\right)^{12}$

B) $2{,}000\left(1+\frac{4.2-2}{1{,}200}\right)^{12}$

C) $2{,}000\left(1+\frac{\frac{4.2}{2}}{1{,}200}\right)^{12}$

D) $\dfrac{2{,}000\left(1+\frac{4.2}{1{,}200}\right)^{12}}{2{,}000\left(1+\frac{2}{1{,}200}\right)^{12}}$

**8** 14.1

$$P = 312(1.04)^{\frac{t}{4}}$$

The equation above can be used to model the population, in thousands, of a certain city $t$ years after 1980. According to the model, the population is predicted to increase by 4% every $n$ months. What is the value of $n$ ?

A) 3

B) 4

C) 16

D) 48

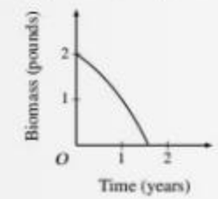




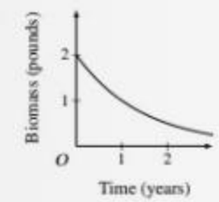
**9** 14.3

The mass of living organisms in a pond is defined to be the biomass of the pond. If the biomass in a pond halves each year, which of the following graphs could model the biomass in the pond as a function of time? (Note: In each graph below, $O$ represents (0, 0).)

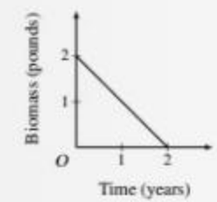
A)



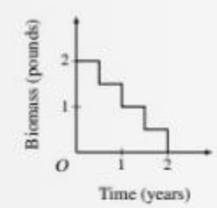

B)




C)




D)




**10** 14.3

The graph of the exponential function $h$ in the $xy$-plane, where $y = h(x)$, has a $y$-intercept of $d$, where $d$ is a negative constant. Which of the following could define the function $h$ ?

A) $h(x) = 2(d)^x$

B) $h(x) = 2(x)^d$

C) $h(x) = d(2)^x$

D) $h(x) = d(-x)^2$

**11** 14.2

Retirement funds decrease over time as money is withdrawn. The amount $M$, in thousands of dollars, in a particular retirement account $t$ years after the beginning of retirement is shown in the table below.

| Number of years, $t$ | Amount, $M$ (thousand \$) |
|---|---|
| 0 | 800.00 |
| 5 | 688.64 |
| 10 | 592.78 |
| 15 | 510.27 |

If this relationship is modeled by the function $M(t) = a \cdot 10^{bt}$, which of the following could be the values of $a$ and $b$ ?

A) $a = 80$ and $b = -0.187$

B) $a = 80$ and $b = 0.187$

C) $a = 800$ and $b = -0.0130$

D) $a = 800$ and $b = 0.0130$

**12** 1600.io 14.2

| Days after virus introduction | Number of bees |
|---|---|
| 0 | 60,000 |
| 7 | 38,487 |
| 14 | 24,634 |
| 21 | 15,687 |
| 28 | 10,111 |

When a virus is introduced to a colony of bees, the bees begin dying. The table above shows the number of bees in the hive at 7-day intervals for the 28 days immediately following a one-time virus introduction.

The bee population in the hive $d$ days after the virus is introduced is modeled by the function $B$ defined by $B(d) = 60{,}000r^{\frac{d}{7}}$. If $B$ approximates the values in the table to within 100 bees, what is the value of $r$, rounded to the nearest hundredth?

**13** 1600.io 14.1

$$W(L) = 0.9(1.31)^L$$

The function $W$ gives the estimated weight, $W(L)$, in kilograms, of a wideboi cobra based on its length, $L$, in decimeters. Which of the following is the best interpretation of the number 1.31 in this context?

A) For each increase of 1 decimeter in length, the estimated weight of the cobra, in kilograms, increases by 31%.

B) For each increase of 1 kilogram in weight, the estimated length of the cobra, in decimeters, increases by 31%.

C) For each increase of 1 decimeter in length, the estimated weight of the cobra increases by 1.31 kilograms.

D) For each increase of 1 kilogram in weight, the estimated length of the cobra increases by 1.31 decimeters.

**14** 1600.io 14.2

$$P = 1{,}500x$$
$$P = 100(4^x)$$

A cookie salesman is considering one of two models above to predict the total number of purchases, $P$, made $x$ weeks after the cookie's new advertising campaign begins. How many more purchases are predicted by the exponential model than by the linear model 3 weeks after the advertising campaign begins?

A) 1,900

B) 3,400

C) 4,500

D) 6,400

**15** 1600.io 14.1

Tia measured the temperature of a cup of iced tea placed in a room with a constant temperature of 70 degrees Fahrenheit (°F). The temperature of the iced tea was 21°F at 3:00 p.m. when it started warming. The temperature of the iced tea was 36°F at 3:05 p.m. and 47°F at 3:10 p.m. The iced tea's temperature continued to increase. Of the following functions, which best models the temperature $T(m)$, in degrees Fahrenheit, of Tia's iced tea $m$ minutes after it started warming?

A) $T(m) = 21(1.71)^m$

B) $T(m) = 21(0.58)^m$

C) $T(m) = (70 - 21)(0.73)^{\frac{m}{5}}$

D) $T(m) = 70 - 49(0.69)^{\frac{m}{5}}$





**16** 1600.io 14.1

A chemist forms crystals in a solution as part of an experiment. At the start the experiment, there are 4 crystals in the solution. The chemist observes that the number of crystal doubles every 24 minutes. Which of the following equations models the number, $n$, of crystals $t$ hours after the start of the experiment?

A) $n = 4(2)^{\frac{t}{24}}$

B) $n = 4(2)^{\frac{5t}{2}}$

C) $n = 4\left(1 + \frac{t}{24}\right)$

D) $n = 4\left(1 + \frac{5}{2}t\right)$

**17** 1600.io 14.1

Of the following four types of account withdrawal plans, which option would yield exponential decay of the money in the account?

A) Each successive year, 5% of the initial account balance is withdrawn from the account.

B) Each successive year, 2% of the initial account balance and $500 is withdrawn from the account.

C) Each successive year, 3% of the current value is withdrawn from the account.

D) Each successive year, $500 is withdrawn from the account.

**Questions 18 and 19 refer to the following information.**
Matilda opened a bank account that earns 3 percent interest compounded annually. Her initial deposit was $180, and she uses the expression $\$180(x)^t$ to find the value of the account after $t$ years.

**18** 1600.io 14.1

What is the value of $x$ in the expression?

**19** 1600.io 14.2

Matilda's friend Hayden found an account that earns 3.5% interest compounded annually. Hayden made an initial deposit of $180 into his account at the same time Matilda made a deposit of $180 into her account. After 8 years, how much more money will Hayden's initial deposit have earned than will Matilda's initial deposit? (Round your answer to the nearest cent and ignore the dollar sign when writing your response.)

**20** 14.3

In the $xy$-plane, the graph of a linear equation of the form $y = mx + b$ and the graph of an exponential equation of the form $y = ab^x$ both contain points $(5, -6)$ and $(6, -8)$. If the point $(r, s)$ is on the graph of the linear equation and the point $(r, t)$ is on the graph of the exponential equation, where $3 < r < 7$, and $s < t$, which of the following must be true?

A) $3 < r < 4$

B) $4 < r < 5$

C) $5 < r < 6$

D) $6 < r < 7$

**21** 14.1

There were no people named Helen in Middletown before 1892, when 32 Helens moved into Middletown. By 1960, the population of people named Helen had reached 30 thousand. If the population had grown exponentially, this would correspond to a 10.6% increase, on average, in the population each year. Which of the following functions best models the population $p(t)$ of Helens $t$ years after 1892?

A) $p(t) = 1.106(32)^t$

B) $p(t) = 32(1.106)^t$

C) $p(t) = (32 \cdot 1.106)^t$

D) $p(t) = 32(2)^{1.106t}$

**22** 14.1

From 1960 through 1969, the number of Helens born in Middletown declined each year by approximately 24% of the number born the preceding year. In 1960, approximately 3 thousand Helens were born in Middletown. Of the following, which best models $J$, the number of thousands of Helens born in Middletown, $t$ years after 1960?

A) $J = 3(0.24)^t$

B) $J = 3(0.76)^t$

C) $J = 3(1.24)^t$

D) $J = 3(1.76)^t$

**23** 14.1

$$S(n) = 60{,}000\,a^n$$

The function $S$ above models the annual salary, in dollars, of an employee $n$ years after starting a job, where $a$ is a constant. If the employee's salary increases by 7% each year, what is the value of $a$ ?

A) 0.07

B) 0.7

C) 1.07

D) 1.7





**24** 14.1

The population of Helens in a state is estimated over the course of twenty decades, as shown in the table.

| Time (decades) | Population |
|---|---|
| 0 | 2,000,000 |
| 5 | 200,000 |
| 10 | 20,000 |
| 15 | 2,000 |
| 20 | 200 |

Which of the following best describes the relationship between time and the estimated population of Helens during the twenty decades?

A) Exponential decay

B) Exponential growth

C) Decreasing linear

D) Increasing linear

**25** 14.1

In which of the following tables is the relationship between the values of $x$ and and their corresponding $y$-values nonlinear?

A)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 5 | 6 | 7 | 8 |

B)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 0 | 2 | 4 | 6 |

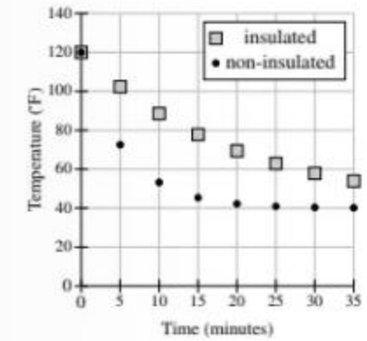
C)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 1 | 2 | 4 | 8 |

D)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 0 | 3 | 6 | 9 |

**26** 14.3




Two samples of slime of equal mass are heated to 120 degrees Fahrenheit (°F). One sample is poured into an insulated container, and the other sample is poured into a non-insulated container. The samples are then left for 35 minutes to cool in a refrigerator having a temperature of 40°F. The graph above shows the temperature of each sample at 5-minute intervals. Which of the following statements correctly compares the average rates at which the temperatures of the two samples change?

A) In every 5-minute interval, the magnitude of the rate of change of temperature of the non-insulated sample is greater than that of the insulated sample.

B) In every 5-minute interval, the magnitude of the rate of change of temperature of the insulated sample is greater than that of the non-insulated sample.

C) In the intervals from 0 to 5 minutes and 5 to 10 minutes, the rates of change of temperature of the non-insulated sample are of greater magnitude, whereas in the intervals from 20 to 25 minutes and 25 to 30 minutes, the rates of change of temperature of the insulated samples are of greater magnitude.

D) In the intervals from 0 to 5 minutes and 5 to 10 minutes, the rates of change of temperature of the insulated sample are of greater magnitude, whereas in the intervals from 20 to 25 minutes and 25 to 30 minutes, the rates of change of temperature of the non-insulated samples are of greater magnitude.

**27** 14.1

$$s(n) = 3.61(1.04^n)$$

The function above can be used to estimate the number of shellphones, $s(n)$, in millions, in Atlantis for $0 \le n \le 56$, where $n$ is the number of years after 1980. Which of the following is the best interpretation of the number 3.61 in this context?

A) The estimated number of shellphones, in millions, in 1980

B) The estimated number of shellphones, in millions, $n$ years after 1980

C) The estimated increase in the number of shellphones, in millions, each year after 1980

D) The estimated percent by which the number of shellphones increased from each year to the next after 1980

**28** 14.1

The half-life of a certain substance in a vacuum is about 30 years. Which of the following exponential functions best models the amount $A(t)$, in grams, of the substance that remains $t$ years after 40 grams of the substance is placed in the vacuum? (The half-life is the length of time needed for an amount of the substance to decrease to one-half of that amount.)

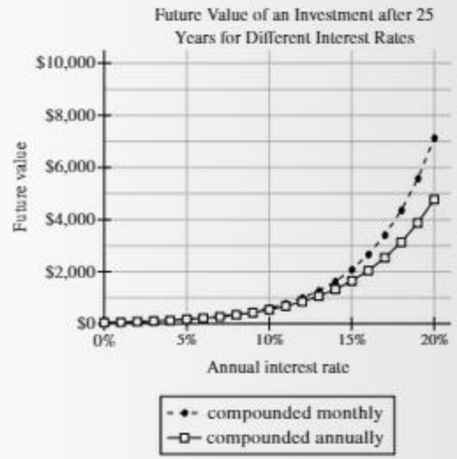
A) $A(t) = 40\left(\frac{1}{2}\right)^{30t}$

B) $A(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{30}}$

C) $A(t) = 30\left(\frac{1}{2}\right)^{40t}$

D) $A(t) = 30\left(\frac{1}{2}\right)^{\frac{t}{40}}$

**29** 14.3




An initial investment of \$50 is made at a constant annual interest rate. The graphs above show the corresponding future value $v$, in dollars, of the investment for different annual interest rates, $r$, after 25 years. One graph shows the value when the interest is compounded monthly, and the other graph shows the value when the interest is compounded annually. Which of the following statements is true?

A) As $r$ increases at a constant rate, $v$ increases more rapidly if interest is compounded monthly rather than annually.

B) As $r$ increases at a constant rate, $v$ increases more rapidly if interest is compounded annually rather than monthly.

C) As $r$ increases at a constant rate, the difference in interest compounded monthly and interest compounded annually increases at a constant rate.

D) If $r = 15\%$ and interest is compounded annually, a \$50 investment will be worth \$2,000 after 25 years.





**30** 14.2

$$f(x) = 5^x - 5$$

The function $f$ is defined by the equation above. Which of the following points in the $xy$-plane is an $x$-intercept of the graph of the equation $y = f(x)$?

A) $(-1, f(-1))$

B) $(0, f(0))$

C) $(1, f(1))$

D) $(5, f(5))$

**31** 14.1

The wholesale price of a pound of beans decreased by 5% from the previous month for 9 consecutive months. If $x$ is the number of months since the price began to drop and $y$ is the cost of a pound of beans, which of the following equations could model the cost of beans over this time period?

A) $y = 0.95x + 2.31$

B) $y = 1.05x + 2.31$

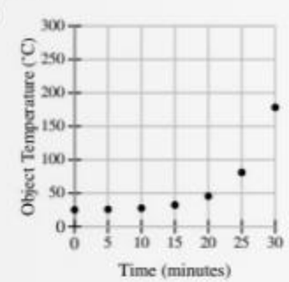
C) $y = 2.31(0.95)^x$

D) $y = 2.31(1.05)^x$

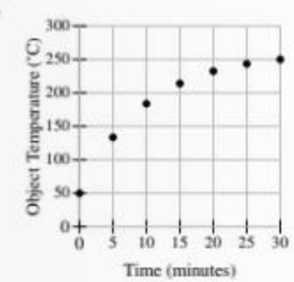
**32** 14.3

The temperature, in degrees Celsius (°C), of a cool object in an oven is recorded every five minutes. The temperature of the object increases rapidly at first, then increases more slowly as the object's temperature approaches the temperature of the oven. Which of the following graphs could represent the temperature of this object over time?

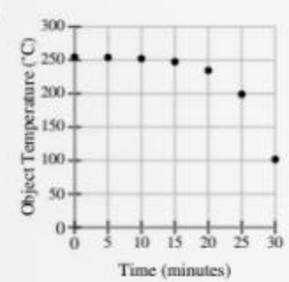
A)



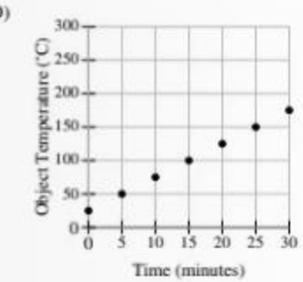

B)




C)




D)



## Answer Key

**SkillDrill 14.1-1**

1. $f(w) = 50(1.04)^{w}$
2. $f(d) = 200(1.1)^{d}$
3. $f(m) = 742(0.96)^{m}$
4. $f(t) = 100(0.87)^{t}$
5. $f(w) = 50(2.2)^{w}$
6. $f(d) = 200(1.0025)^{d}$
7. $f(m) = 742(0.958)^{m}$
8. $f(t) = 100(0.49)^{t}$

**SkillDrill 14.1-2**

1. $f(t) = 50(1.04)^{4t}$
2. $f(w) = 200(1.1)^{7w}$
3. $f(t) = 742(0.96)^{12t}$
4. $f(q) = 100(0.87)^{\frac{q}{4}}$
5. $f(m) = 50(1.04)^{\frac{m}{3}}$
6. $f(d) = 200(1.0025)^{\frac{d}{365}}$
7. $f(t) = 742(0.958)^{52t}$
8. $f(d) = 100(0.49)^{\frac{d}{7}}$
9. 50 grows by 4% every 6 years
10. 200 grows by 10% seven times every week
11. 742 decays by 4% every 3 months
12. 100 decays by 13% every 12 days
13. 80 grows by 110% twice every quarter
14. 200 grows by .25% every 10 years
15. 742 decays by 4.2% four times every day
16. 100 decays by 51% every 8 weeks

**Section 14.1 Practice Problems**

1. C
2. B
3. D
4. C

**Section 14.2 Practice Problems**

1. D
2. B

**Section 14.3 Practice Problems**

1. A
2. D
3. C

**Additional Problems**

1. D
2. C
3. .89
4. 176
5. 1.44
6. A
7. A
8. D
9. B
10. C
11. C
12. .64
13. A
14. A
15. D
16. B
17. C
18. 1.03
19. 9.01
20. C
21. B
22. B
23. C
24. A
25. C
26. C
27. A
28. B
29. A
30. C
31. C
32. B



# Scatterplots and Line Graphs 15

## 15.1 Scatterplots and Lines of Best Fit

We've already seen a few scatterplots scattered throughout the chapters, but it's time to take a deeper dive into the topic. In general, scatterplots show a collection of points from real data that can be used to interpret the potential relationships between variables. Each point represents a single person or object or entity for which we know the two pieces of data measured on the axes. For example, if the $x$-axis of a scatterplot shows age and the $y$-axis shows height, then each point on the scatterplot could represent one person and tell us that person's age and height.

On the test, the data set represented on a scatterplot will often be approximated by a modeling function, which is usually linear and thus appears as a line, but sometimes it's a parabola or exponential graph. The modeling function, or **line of best fit**, roughly fits the path described by the points, but does not have to go through any specific points. Usually, there are about an equal number of points above and below the line of best fit, and some points may lie right on the line. You may be given an equation for the line of best fit or be asked to find the equation of the line of best fit, which can be done by methods that have been outlined earlier in the book (use points on the best fit graph to find a linear, quadratic, or exponential equation to use for approximation). Once you have a best fit equation, you can use it to *predict* $y$-values for given $x$-values; it is important to note that these are simply approximations based on a model function that does not necessarily provide exact values from the original data set.

Before we do any calculations, we just need to look at some of the terminology used to describe the possible relationships of points on a scatterplot.

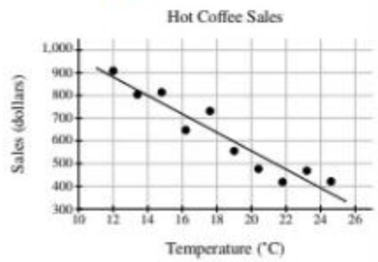
### Correlation Strength / Positive and Negative Correlations

A **strong correlation** means the dots (data points) are tightly clustered around a line; a **weak correlation** means that the dots are more spread out. A **positive correlation** means that as the $x$-values of the data points increase, the $y$-values tend to increase, too, so the **line of best fit will slope upwards from left to right (it will have a positive slope)**. Unsurprisingly, a **negative correlation** simply reverses this relationship, so as the $x$-values of the data points increase, the $y$-values tend to decrease, so the **line of best fit will slope downwards from left to right (it will have a negative slope)**.

A lot of scatterplot questions are solved by picking points on the line of best fit or writing the equation of the line of best fit. In the rare circumstance where you need to create your own, you don't have to perform the relatively complicated calculations that would be needed to find the exact right line of best fit; a rough approximation based on drawing a reasonable line (roughly the same number of dots above and below) and finding the slope and $y$-intercept of that line will do just fine for the types of questions that appear on the test, which are just assessing basic knowledge of this topic.

**Example 15.1-1**

**1** 1600.io 15.1




The scatterplot above shows a coffee stand's hot coffee sales $d$, in dollars, and the low temperature $t$, in degrees Celsius (°C), on 10 different days. A line of best fit for the data is also shown. Which of the following could be an equation of the line of best fit?

A) $d = -0.02t + 933$

B) $d = -8t + 933$

C) $d = -41t + 994$

D) $d = -41t + 1{,}391$

**Solution**

1. Recall that Slope-Intercept Form of a line is $y = mx + b$, where $m$ is the slope and $b$ is the $y$-intercept. First, calculate the slope. Pick two points on the line of best fit (*not* scatterplot points) that are near gridline intersections and relatively far apart (the farther apart the two points are, the more precise your slope calculation will be). Use these points to find the slope of the line. It looks like (14, 800) and (24, 400) are points on or very close to the line of best fit.

2. Between these points, the $y$-values decrease by about 400, so $\Delta y = -400$, and the $x$-values increase by 10, so $\Delta x = 10$. Therefore the slope, which is $\frac{\Delta y}{\Delta x}$, is about $\frac{-400}{10}$, which is $-40$. This slope is close to $-41$, so eliminate choices A and B. This leaves us with choices C and D, which differ only in the $y$-intercept.

3. You now have the slope; to produce a complete linear equation, you must also find the $y$-intercept. Be careful when approximating the $y$-intercept because **the $x$-axis does not start at 0—it starts at 10**—so you can't just eyeball the line and estimate where it would hit the $y$-axis. If you want, you can plug in the slope (use $-41$ based on the answer choices) and the point (24, 400) into the general Slope-Intercept Form of the line to find an approximate $y$-intercept.

$$y = mx + b$$
$$400 = -41(24) + b$$
$$400 = -984 + b$$
$$1{,}384 = b$$

4. The $y$-intercept should be close to 1,384, so choice D is better than choice C.

5. The answer is D.





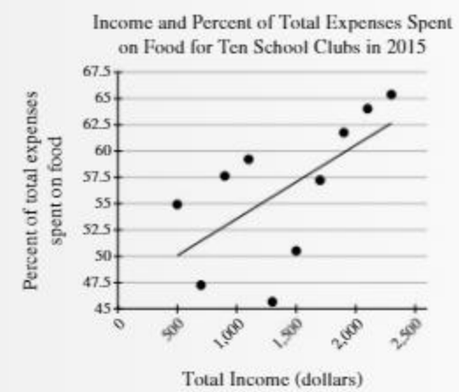
**Notes**

Alternatively, in place of finding the $y$-intercept in Step 3, you could look at the choices for the $y$-intercept, which are around 950 and close to 1,400 , and say with a fair bit of certainty that the $y$-intercept will not be close to 950 because the line looks like it would hit a $y$-value of 950 when the temperature is around 10 degrees (where the grid starts) and we still have 10 more units to go to the left along the $x$-axis before the line would hit the true $y$-axis which is at $x = 0$ (the $y$-values of the line change by about 400 from where $x = 12$ to where $x = 22$). That would leave us with choice D.

Some problems want you to look at specific points of data and judge how far they are from the line of best fit. You need to see how far above or below the line a scatterplot point is by reading the difference in the $y$-value of the point and the $y$-value predicted by the line of best fit at the same $x$-coordinate. Data points *on* the line are the most accurately predicted, and points that are far above or below the line are not accurately predicted by the line of best fit.

**Example 15.1-2**

**2** 1600.io 15.1




The scatterplot above shows data for ten school clubs along with the line of best fit. For the club with the lowest percentage of total expenses spent on food, which of the following is closest to the difference of the actual percent and the percent predicted by the line of best fit?

A) 40%

B) 20%

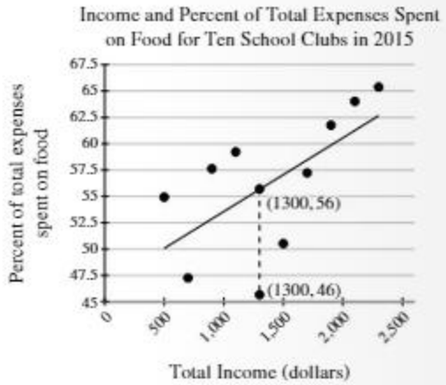
C) 10%

D) 5%

**Solution**

1. The point with the lowest percentage of expenses spent on food is the dot that is lowest down on the graph. We can't find its exact value, but let's say it's about (1300, 46). Note that the actual $x$-value (representing the income value) doesn't matter; you're just going to look vertically straight up above that point to find the approximate $y$-value of the line of best fit.

2. At an income of 1,300 dollars for a club, the line of best fit predicts that the percentage of the expenses spent on food should be about 56%.




3. The difference between the actual percentage (about 46%) and the predicted percentage (about 56%) is 10%.

4. The answer is C.

**Notes**

If you take a quick look at the answer choices prior to setting out on this problem, you'll notice that the values are very far apart, so very little precision is going to be needed when calculating the difference between the actual value and the predicted value for the relevant point. Therefore, it would be perfectly valid to refer to the closest horizontal gridlines and see that the difference is going to be very close to $55\% - 45\% = 10\%$.

There are a couple of useful things to notice about the graph you are given. For one, the horizontal axis is not the true $x$-axis where $y = 0$; the line $y = 45$ acts as the horizontal axis in the graph. For this problem, this fact would come into play if you mistakenly assumed that the lowest point's percentage value is almost 0.

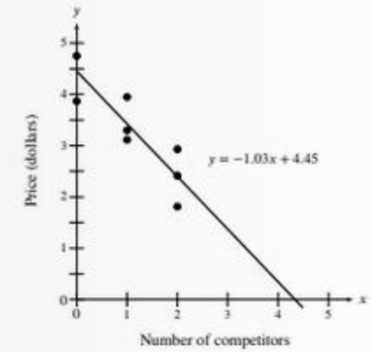
The next important feature is that the horizontal gridlines are spaced every 2.5 percentage points, which could lead to some confusing decisions if you try to do this problem by simply counting gridlines.

When it comes to real world situations represented by linear functions, we are able to make definitive statements about the meaning of $x$- and $y$-intercepts. However, we cannot be so precise when we are asked to interpret the graphs of lines of best fit. It is important to remember that lines of best fit are not exact models, and so we can only make very general claims about the meanings of their slopes and intercepts.

For example, if the price of paper towels in a store is predicted based on the number of competitors in the area, the $y$-intercept would be the predicted price of paper towels in a store with no competitors. The scatterplot below shows the prices of paper towels and number of competitors for 8 stores around the state (each dot corresponds to one store) as well as the line of best fit and its equation.








If the line of best fit represented a perfect model, a store with no competitors *should* charge the price predicted by the $y$-intercept of the line of best fit, which would be \$4.45; however, the scatterplot shows that neither store with zero competitors charges exactly that price (one charges less than \$4.00 and the other charges about \$4.75).

Similarly, if the line were a perfect predictor, the $x$-intercept would represent the number of competitors a store would need to have in order to charge nothing for paper towels because the price, or $y$-value, would be 0 (not exactly a winning business model).

If we continue along the line of best fit, we can see that it crosses the $x$-axis somewhere between 4 and 5 competitors, which would mean that, if the model were absolutely perfect, stores with 5 competitors would have to *pay* people to take paper towels (even worse than giving them away for free!).

Regardless of how badly the store wants to beat its competition, it is not likely that the store would give products away for free or even pay people to take them (even though no one would stop them). Therefore, based solely on the line of best fit, we **cannot make any definite claims** about what a store *would* do. It is more likely that the model simply doesn't work well once a store has a lot of competitors. In other words, the line of best fit can do a decent job of making predictions of the price of paper towels for stores whose number of competitors is in a certain limited domain of $x$-values (we see that the stores with between zero and two competitors follow the pattern), but eventually the model makes no sense and at best is simply a way of estimating what will happen (note that only one store appears to be charging exactly what the line of best fit predicts).

**In short, modest claims about what the line of best fit tells you are better than definitive claims because the lines of best fit are merely approximations.**

**Section 15.1 Suggested Problems from Real Tests**

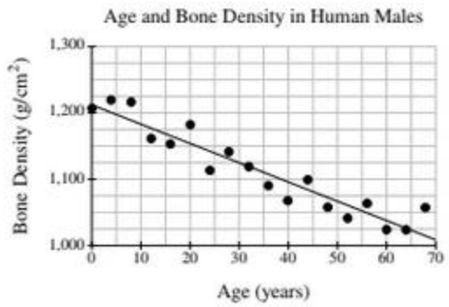
- Test 1-C-5
- Test 2-C-14
- Test 3-C-20
- Test 4-C-10
- Test 4-C-11
- Test 4-C-21
- Test 4-C-27
- Test 5-C-17
- Test 7-C-7
- Test 7-C-8
- Test 7-C-18
- Test 7-C-21
- Test 8-C-4
- Test 8-C-18
- Test 8-C-21
- Test 9-C-29
- Test 9-C-30
- Test 10-C-24
- May 2017-C-21
- May 2017-C-37
- May 2017-C-38
- Mar 2018-C-11
- Apr 2018-C-9
- Apr 2018-C-32
- May 2018-C-16
- May 2018-C-25
- Mar 2019-C-9
- Apr 2019-C-28
- May 2019 (Int)-C-6
- May 2019 (Int)-C-7
- May 2019 (Int)-C-22
- Oct 2019-C-12
- Oct 2019-C-20
- Mar 2020-C-4
- Mar 2020-C-20
- Mar 2020-C-37
- Mar 2020-C-38
- Oct 2020-C-6
- Oct 2020-C-31
- Mar 2021-C-15
- Mar 2021-C-20
- Apr 2021-C-8
- Apr 2021-C-9
- Apr 2021-C-12
- May 2021 (US)-C-8
- May 2021 (US)-C-17
- May 2021 (US)-C-32
- May 2021 (Int)-C-2

View related real-test problems in SMTEx

SMTEx

## Section 15.1 Practice Problems

**Questions 1 and 2 refer to the following information.**




The scatterplot above shows the bone densities of 18 human males, in grams per squared centimeter, with respect to their age in years. The line of best fit is also shown.

**1** 1600.io 15.1

According to the scatterplot, which of the following statements is true about the relationship between a human male's age and his bone density?

A) The bone density of a human male that is twice as old as another human male is half the bone density of that other human male.

B) The age of a human male is unrelated to bone density.

C) Human males that are older tend to have greater bone densities.

D) Human males that are older tend to have lesser bone densities.

**2** 1600.io 15.1

A doctor has a human male patient that is 45 years old. According to the line of best fit, which of the following best approximates the bone density of the human male, in grams per square centimeter?

A) 1,000

B) 1,060

C) 1,080

D) 1,100

**3** 1600.io 15.1

The scatterplot above represents the salary $y$, in thousands of euros, and the number of years of experience, $x$, for each of eight employees at a company. A line of best fit for the data is also shown. Which of the following could be an equation of the line of best fit?

A) $y = \frac{3}{4}x$

B) $y = \frac{3}{4}x + \frac{61}{2}$

C) $y = \frac{4}{3}x + \frac{61}{2}$

D) $y = \frac{4}{3}x + 33$





## 15.2 Line Graphs Without Modeling Equations

There are line graphs on the test that are not based on continuous linear relationships, quadratic relationships, or exponential relationships. A woman who is jogging may run very quickly at the start, then slow down, then speed up again, stop to tie her shoe and drink water, run slowly, and then sprint to close out her run. Therefore, her speed is not constant, and so she runs different distances in different amounts of time.

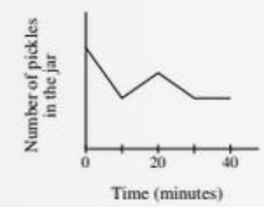
On the test, one of the more common questions is about what periods on a graph show the fastest increases or decreases (steepest slope) or what periods show that no change is occurring (flat or horizontal portions of the graph).

**Example 15.2-1**

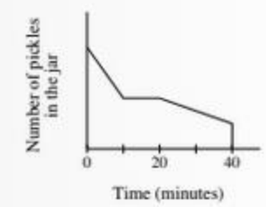
**1** 1600.io 15.2

Hannah eats pickles while she studies. She eats half of the pickles during the first 10 minutes of studying. After eating half of the pickles, she stops eating for the next 10 minutes. Then she gradually eats the pickles until she purposely spills all of the remaining pickles. Which of the following graphs could represent the situation?

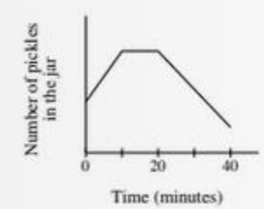
A)



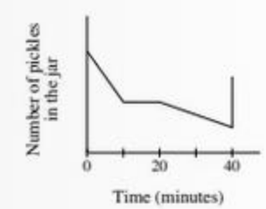

B)




C)




D)




**Solution**

1. Between 0 and 10 minutes, the number of pickles must decline by half. Eliminate Choice C, which shows an increase (instead of a decrease) in the number of pickles during the first 10 minutes.

2. Between 10 and 20 minutes, Hannah stops eating pickles, so the graph should be horizontal (showing no change in the number of pickles in the jar). Eliminate Choice A, which shows an increase in the number of pickles during this time (let's not think about what an increase in pickles might mean).

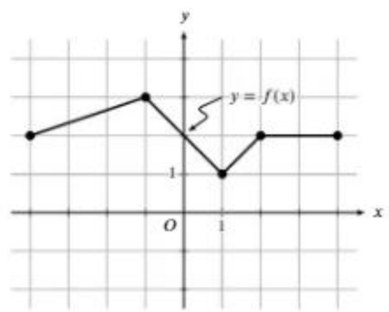
3. Both choices B and D show a gradual decline between 20 and 40 minutes, but choice D indicates that there is a sudden infusion of pickles into the jar at 40 minutes. In the question, it states the opposite: Hannah suddenly spilled the rest of the pickles, so the graph should jump straight down to the $x$-axis at 40 minutes, as it does in choice B.
4. The answer is B.

## Line Graphs/Reading and Fitting Points

Some questions just want you to read $y$, or $f(x)$, values from line graphs. There is no big secret here—if you are asked to find the value of $f(4)$, find the $y$-value when $x$ is 4.

**Example 15.2-2**

**2** 1600.io 15.2




The complete graph of the function $f$ is shown in the $xy$-plane above. Which of the following are equal to 2 ?

I. $f(-3)$

II. $f(0)$

III. $f(3)$

A) II only
B) I and II
C) II and III
D) I, II, and III

**Solution**

1. Check each Roman numeral item to see if the value is 2.
2. The value of $f(-3)$ is the $y$-value when $x$ is $-3$. The line is clearly above 2 when $x = -3$, so item I does not have a value of 2 (the point is $\left(-3, \frac{7}{3}\right)$ if you're curious).



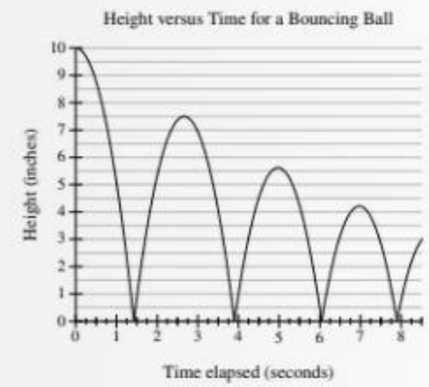


3. The value of $f(0)$ is the $y$-value when $x$ is 0 (also known as the $y$-intercept). The $y$-value is 2 when $x = 0$ (the graph goes through the point $(0, 2)$), so item II satisfies the condition.
4. The value of $f(3)$ is the $y$-value when $x$ is 3. The point $(3, 2)$ is on the graph, so $f(3) = 2$. Therefore, item III is also equal to 2.
5. The answer is C because, out of the given Roman numeral items, only $f(0)$ (item II) and $f(3)$ (item III) are equal to 2.

More than once, there has been a question about a bouncing ball and how many times the ball is at a certain height. In order to solve these problems, all you have to do is see how many times the graph reaches a certain $y$-value that is asked for in the problem. These are simple questions, but people get confused by them all the time. The ball is dropped from an initial height (which is the $y$-intercept), and each time the graph touches the $x$-axis, the ball is hitting the ground because its height above the ground is 0.

**Example 15.2-3**

**3** 1600.io 15.2

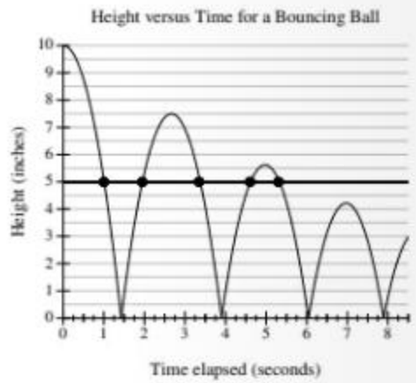



As part of an experiment, a ball was dropped and allowed to bounce repeatedly off the ground until it was caught. The graph above represents the relationship between the time elapsed after the ball was dropped and the height of the ball above the ground. After it was dropped, how many times was the ball at a height of 5 inches?

A) One
B) Two
C) Four
D) Five

**Solution**

1. Draw a horizontal line at $y = 5$ and see how many times the graph intersects the horizontal line.




2. There are 5 crossing points. Some people forget to include the first one because they mistakenly think that "after it was dropped" means "after it bounces the first time."
3. The answer is D.

**Notes**

If you dig into the physics of this completely fake scenario, you would see that this takes place on one of the moons of planet 1600—1600.io—where there is less gravity.

**Section 15.2 Suggested Problems from Real Tests**

- Test 1-C-1
- Test 1-C-17
- Test 1-C-33
- Test 2-C-26
- Test 3-C-1
- Test 3-C-3
- Test 5-C-1
- Test 5-C-20
- Test 6-C-3
- Test 6-C-12
- Test 6-C-23
- Apr 2017-C-15
- May 2017-C-10
- Mar 2018-C-13
- Apr 2018-C-10
- Apr 2019-C-21
- May 2019 (US)-C-9
- May 2019 (US)-C-37
- May 2019 (US)-C-38
- Oct 2019-C-1
- Mar 2020-C-2
- Oct 2020-C-8
- Oct 2020-C-35
- Mar 2021-C-10
- May 2021 (US)-C-4
- May 2021 (Int)-C-37

**View related real-test problems in SMTEx**

SMTEx

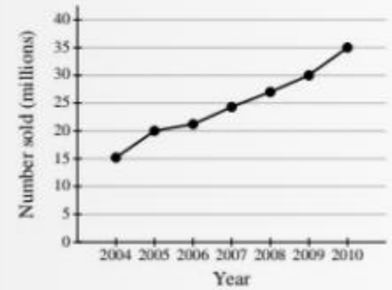




## Section 15.2 Practice Problems

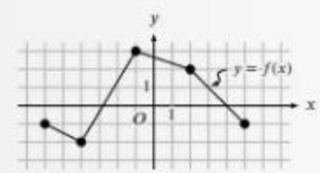
**1** 1600.io 15.2

Number of Game Consoles Sold Worldwide Each Year from 2004 to 2010




According to the line graph above, the number of game consoles sold in 2005 is what fraction of the number sold in 2010?

**2** 1600.io 15.2

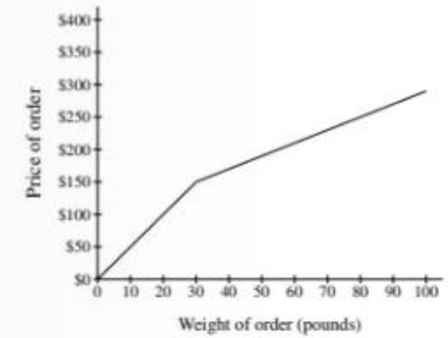



The complete graph of the function $f$ is shown in the $xy$-plane above. For what value of $x$ is the value of $f(x)$ at its minimum?

A) $-6$
B) $-4$
C) $-1$
D) $5$

**3** 1600.io 15.2

Cereal Price




The graph above shows the price that a cereal company charges for an order of cereal, depending on the weight of the order. Based on the graph, which of the following statements must be true?

A) The company charges the same price per pound, regardless of the order size.
B) The company charges less per pound for orders greater than 30 pounds than for orders less than 30 pounds.
C) The company charges less per pound for orders greater than 150 pounds than for orders less than 150 pounds.
D) The company charges more per pound for orders greater than 30 pounds than for orders less than 30 pounds.

## CHAPTER 15 RECAP

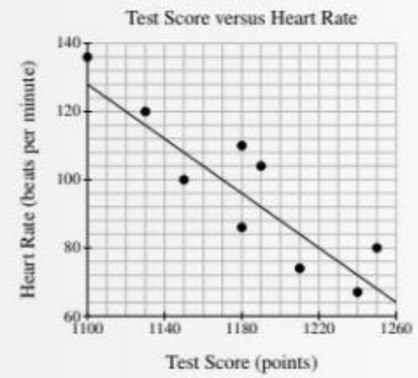
- Find the equation of a line of best fit for a scatterplot in the same ways that you did for any regular line.
- Find the difference between actual and predicted values by seeing how far above or below the line of best fit a data point is.
- Do not make overly definite statements about what the slopes and intercepts of scatterplots tell us. They are merely models and sometimes only fit very roughly over particular intervals.
- Most line graph questions are simply related to the slope of the line on certain intervals. Steep lines mean something changed rapidly during an interval. Horizontal lines means something stayed the same for an interval.





## Additional Problems

**1** 1600.io ? 15.1




Reed takes a practice test once a week for nine straight weeks. The scatterplot above shows his score on and corresponding heart rate after each test. The line of best fit for the data is also shown. For the test on which he scored an 1190, Reed's actual heart rate was about how many beats per minute more than the rate predicted by the line of best fit?

A) 4

B) 5

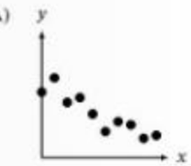
C) 12

D) 15

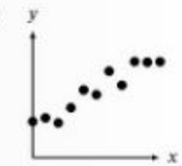
**2** 1600.io ? 15.1

Which of the following shows a strong negative association between $x$ and $y$?

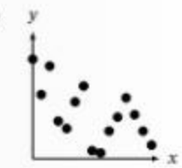
A)



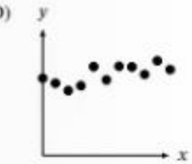

B)




C)

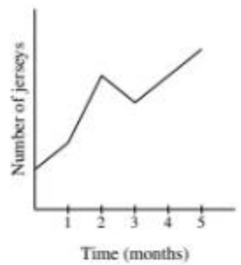



D)



**3**

Peter collects, sells, and trades basketball jerseys, and he tracks the number of jerseys in his collection on the graph below.




On what interval did the number of jerseys increase the fastest?

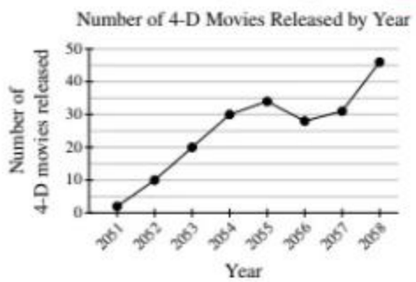
A) Between 0 and 1 months

B) Between 1 and 2 months

C) Between 2 and 3 months

D) Between 3 and 5 months

**4**




According to the line graph above, between which two consecutive years was there the greatest change in the number of 4-D movies released?

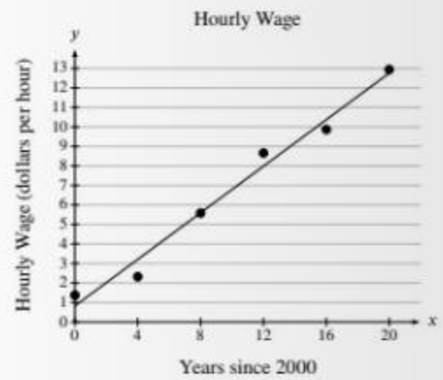
A) 2052–2053

B) 2053–2054

C) 2054–2055

D) 2057–2058

**5**




The scatterplot above shows the hourly wage of waiters at a restaurant every 4 years between 2000 and 2020. A line of best fit is shown and its equation is $y = 0.58x + 0.86$. What does the line of best fit predict about the increase in the hourly wage over the 20-year period?

A) Every 4 years between 2000 and 2020, the average increase in hourly wage was 0.58 dollars.

B) Every 4 years between 2000 and 2020, the average increase in hourly wage was 0.86 dollars.

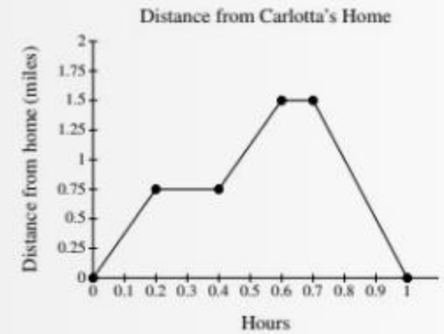
C) Each year between 2000 and 2020, the average increase in hourly wage was 0.58 dollars.

D) Each year between 2000 and 2020, the average increase in hourly wage was 0.86 dollars.





**6**




Carlotta rode her unicycle along a straight road from her home to a coffee shop and drank a coffee. She then continued along the same road to a book store to buy a book. Finally, she returned home on the same road after buying the book. Carlotta's distance from home during the hour she was out is shown in the graph above. How many total miles did she ride her unicycle?

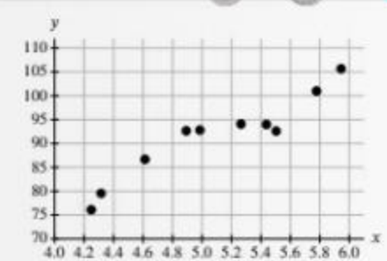
A) 1.5

B) 2.5

C) 3

D) 4.5

**7**




A set of data is represented by the scatterplot in the portion of the $xy$-plane shown. Which of the following linear equations best fits the data?

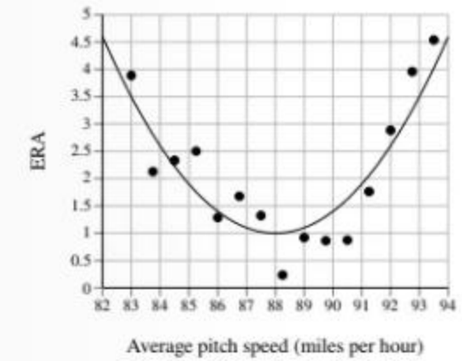
A) $y = -18.5 + 1.4x$

B) $y = 18.5 + 1.4x$

C) $y = -18.5 + 14x$

D) $y = 18.5 + 14x$

**Questions 8 and 9 refer to the following information.**




The scatterplot above shows the ERA (a measure of performance) for 15 college baseball pitchers with different average pitch speeds. The graph of a quadratic model for the data is also shown.

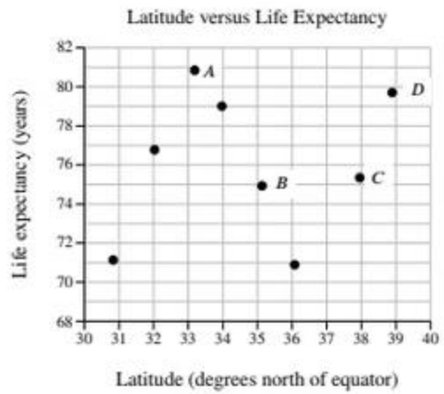
**8**

For what fraction of the speeds does the model underestimate the numerical value of the ERA?

**9**

The quadratic model predicts the ERA to be 2 for how many different average pitch speeds?

**Questions 10 and 11 refer to the following information.**




A researcher created the scatterplot above to examine the relationship between the latitude and life expectancy of people in 8 cities.

**10** 1600.io ? 15.1

What is the approximate life expectancy, in years, for a person in the city that is the most degrees north of the equator?

A) 80.9
B) 79.8
C) 78.9
D) 75.3

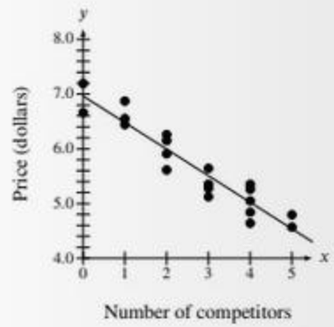
**11** 1600.io ? 15.1

Of the labeled points, which represents the city for which the ratio of life expectancy to latitude is the greatest?

A) *A*
B) *B*
C) *C*
D) *D*

**Questions 12 and 13 refer to the following information.**

A researcher is investigating the relationship between the price of a screwdriver at a hardware store and the number of competitors the store has. He defines a store's competitor as another similar store within a 5-mile radius of the store he selects. He selects 20 hardware stores across a county at random, and for each store, he records the number of its competitors and its price for a screwdriver. The results are shown, along with the line of best fit, in the scatterplot below.




**12** 1600.io ? 15.1

Another hardware store in the same county sells a screwdriver for $5.21. If the store's price is less than that predicted by the line of best fit, what is the greatest number of competitors the store could have?

A) 1
B) 2
C) 3
D) 4

**13** 1600.io ? 15.1

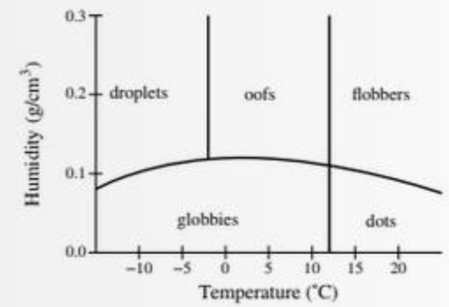
The line of best fit passes through the point (15, −0.31). Which of the following can be concluded from this?

A) A hardware store cannot have more than 14 competitors.
B) A hardware store with 14 competitors cannot decrease its price any further.
C) A hardware store with 14 competitors can no longer sell screwdrivers.
D) The line of best fit will not model the price well for a store with a large number of competitors.





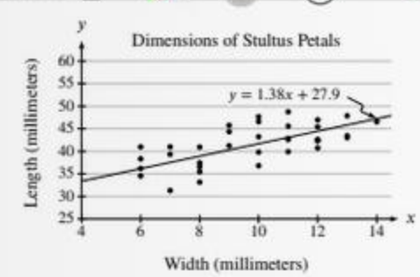
**14** 1600.io ? 15.2




The figure above shows a graph with five regions that correspond to temperature, in degrees Celsius (°C), and humidity conditions, in grams of water vapor per cubic centimeter of air (g/cm$^3$), that will result in different gelatin blob shapes when the gelatin is set in a refrigerator. Based on the graph, which of the following is a combination of temperature and humidity at which droplets will be formed?

A) 5 °C and 0.15 g/cm$^3$
B) 0 °C and 0.20 g/cm$^3$
C) −5 °C and 0.16 g/cm$^3$
D) −10 °C and 0.08 g/cm$^3$

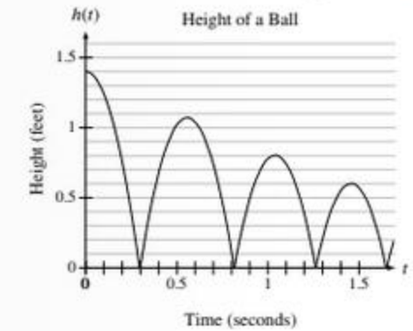
**15** 1600.io ? 15.1




The scatterplot above shows data collected on the lengths and widths of *Stultus folium* petals. A line of best fit for the data is also shown. Based on the line of best fit, if the width of a *Stultus folium* petal is 15 millimeters, what is the predicted length, in millimeters, of the petal?

A) 48.60
B) 47.50
C) 33.42
D) 27.90

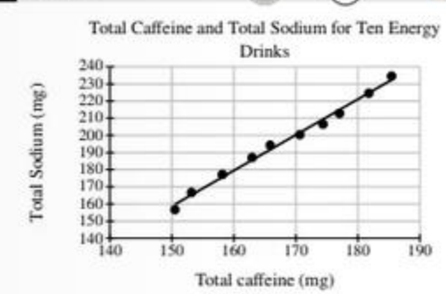
**16** 1600.io ? 15.2




A ball was dropped from a height of 1.4 feet and hit the floor several times. The graph above represents the height $h$, in feet, of the ball $t$ seconds after it was dropped. Of the following, which best approximates the maximum height, in feet, of the ball between the second and third time it hit the floor?

A) 0.6
B) 0.8
C) 1.1
D) 1.4

**17** 1600.io ? 15.1




The scatterplot above shows the number of milligrams of both total caffeine and total sodium for ten energy drinks. The line of best fit for the data is also shown. According to the line of best fit, which of the following is closest to the predicted increase in total sodium, in milligrams, for every increase of 1 milligram in total caffeine?

A) 2.0
B) 2.5
C) 3.0
D) 3.5

**18** 1600.io ? 15.2

The graph above shows Shreya's distance from her home during a 3-hour walk. She stopped for 30 minutes during her walk to pet a dog. Based on the graph, which of the following is closest to the time she finished petting the dog and continued her hike?

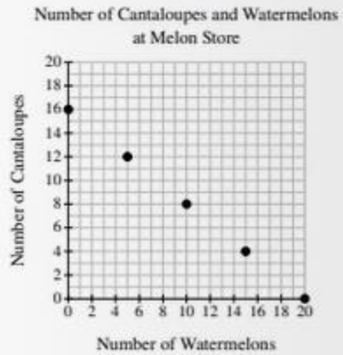
A) 12:40 P.M.

B) 1:10 P.M.

C) 1:45 P.M.

D) 2:00 P.M.

**19** 1600.io ? 15.1

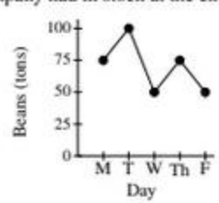



The points plotted in the coordinate plane above represent the possible numbers of cantaloupes and watermelons that someone can buy at the Melon Store in order to spend exactly $40.00 total on the two types of melons. The price of each cantaloupe is the same and the price of each watermelon is the same. What is the price, in dollars, of 1 cantaloupe? (Disregard the $ sign when writing your answer.)

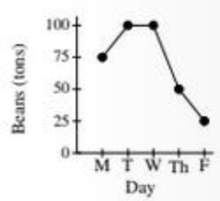
**20** 1600.io ? 15.2

A bean company had 75 tons of beans in stock at the end of the day on Monday. On Tuesday the company made no shipments and received a delivery of 25 tons of beans. On Wednesday the company shipped 50 tons of beans and received no deliveries. On Thursday the company made no shipments and received no deliveries. On Friday the company shipped 25 tons of beans and received no deliveries. Which of the following represents the number of tons of beans the company had in stock at the end of each day?

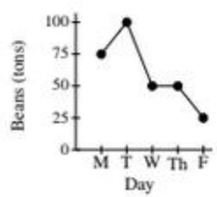
A)



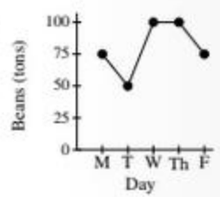

B)




C)



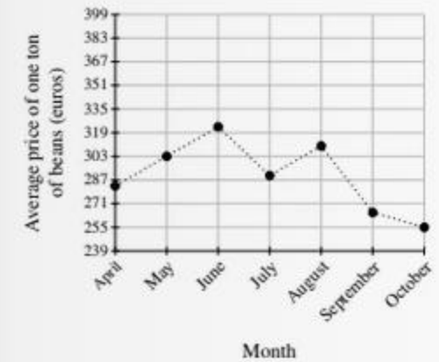

D)








**Questions 21 and 22 refer to the following information.**




The line graph above shows the average price of one ton of beans, in euros, for each of seven months in 1986.

**21** 1600.io ? 15.2

Between which two consecutive months shown did the average price of one ton of beans decrease the most?

A) May to June

B) June to July

C) July to August

D) August to September

**22** 1600.io ? 15.2

In 1986, the average price of one ton of beans increased by 5.46% from March (not shown) to April. Which of the following is closest to the price of one ton of beans in March 1986?

A) 250

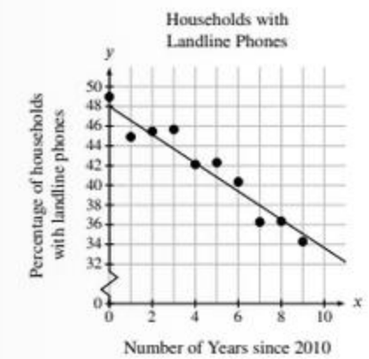
B) 265

C) 295

D) 310

**Questions 23 and 24 refer to the following information.**

A phone company recorded the percentage of households in New Jersey that had landline phones from 2010 to 2020. In the scatterplot below, $x$ represents the number of years since 2010 and $y$ represents the percentage of households with landline phones. The line of best fit for the data is shown.




**23** 1600.io ? 15.1

Which of the following is closest to the equation of the line of best fit shown?

A) $y = -\frac{7}{5}x + 48$

B) $y = -\frac{7}{5}x - 48$

C) $y = -\frac{7}{5}x$

D) $y = 48x - \frac{7}{5}$

**24** 1600.io ? 15.1

Which of the following is the best interpretation of the slope of the line of best fit for these data?

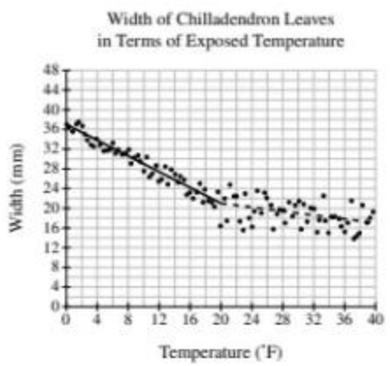
A) The actual decrease in the number of households with landline phones each year

B) The predicted decrease in the number of households with landline phones each year

C) The actual decrease in the percentage of households with landline phones each year

D) The predicted decrease in the percentage of households with landline phones each year

**Questions 25–27 refer to the following information.**




Chilladendron leaves curl up at temperatures above freezing, and they uncurl and widen as the temperature decreases. The scatterplot above shows the results of a study in which the widths of some chilladendron leaves from the same plant were measured at various times over a certain period. The widths $w$, in millimeters (mm), were plotted against the temperatures $T$, in degrees Fahrenheit (°F), at the times the leaves were measured. The solid line segment is a line of best fit that models the relationship between the temperatures and widths of the leaves when $0 \le T < 20$. The dashed line segment is a line of best fit that models the relationship when $20 \le T \le 40$.

**25** 1600.io 15.1

Based on the solid line segment, which of the following is closest to the temperature at which the width of a chilladendron leaf is predicted to be 26 millimeters?

A) 12°F

B) 14°F

C) 16°F

D) 18°F

**26** 1600.io 15.1

At a time when the temperature was 36°F, the width of one of the chilladendron leaves was measured. By approximately how much does the actual measurement differ from the predicted value?

A) It is approximately 6 millimeters greater than the predicted value.

B) It is approximately 3 millimeters greater than the predicted value.

C) It is approximately 3 millimeters less than the predicted value.

D) It is approximately 6 millimeters less than the predicted value.

**27** 1600.io 15.1

Based on the dashed line segment, which of the following could be the predicted width, in millimeters, of a chilladendron leaf when the temperature is $T$°F, where $28 \le T \le 40$ ?

A) 22

B) 20

C) 18

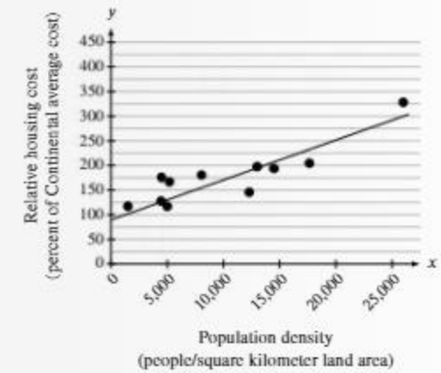
D) 16





**28** 1600.io 15.1

The relative housing cost for a European city is defined to be the ratio $\frac{\text{average housing cost for the city}}{\text{Continental average housing cost}}$, expressed as a percent.




The scatterplot above shows the relative housing cost and the population density for several large European cities in the year 2010. The line of best fit is also shown and has the equation $y = 0.0107x + 89$. Which of the following best explains how the number 89 in the equation relates to the scatterplot?

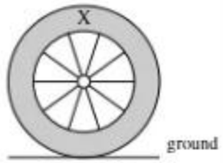
A) In 2010, the lowest housing cost in Europe was about € 89 per month.

B) In 2010, the lowest housing cost in Europe was about 89% of the highest housing cost.

C) In 2010, even in cities with low population densities, housing costs were never below 89% of the Continental average.

D) In 2010, even in cities with low population densities, housing costs were likely at least 89% of the Continental average.

**29** 1600.io 15.2



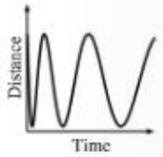

In the figure above, X is a mark on the side of a tire of a bike moving at a speed of 10 feet per second. The bike, starting at speed, will experience a deceleration for some period of time. Which of the following could represent the distance between the mark X and the ground after the bike starts to decelerate and the tire makes its first few revolutions?

A)

B)

C)

D)








# Answer Key

**Section 15.1 Practice Problems**

1. D
2. C
3. B

**Section 15.2 Practice Problems**

1. $\frac{4}{7}$, $\frac{8}{14}$, or .571
2. B
3. B

**Additional Problems**

1. C
2. A
3. B
4. D
5. C
6. C
7. D
8. $\frac{8}{15}$ or .533
9. 2
10. B
11. A
12. C
13. D
14. C
15. A
16. B
17. A
18. B
19. 2.5, or $\frac{5}{2}$
20. C
21. D
22. B
23. A
24. D
25. B
26. C
27. C
28. D
29. D

# Functions 16

## 16.1 Evaluating Functions ("Plugging In")

We've already seen function notation used for lines, parabolas, higher-degree polynomials, and exponents, and there are some questions that simply require you to plug a value into those functions.

**Example 16.1-1**

**1** 1600.lo 16.1

$$f(x) = x^3 + 2x^2 - 2x + 1$$

For the function $f$ defined above, what is the value of $f(-2)$?

A) $-11$

B) $-5$

C) $-3$

D) $5$

**Solution**

1. Substitute $-2$ for $x$ to find $f(-2)$ by replacing all instances of $x$ with $-2$. Make sure to handle the exponents before you start multiplying by coefficients.

$$f(-2) = (-2)^3 + 2(-2)^2 - 2(-2) + 1$$
$$f(-2) = -8 + 2(4) + 4 + 1$$
$$f(-2) = -8 + 8 + 5$$
$$f(-2) = 5$$

2. The answer is D.

Occasionally, you will have to evaluate functions that are not polynomial or exponential. The idea is the same: replace all the instances of $x$ with whatever is being plugged into the function.

**Example 16.1-2**

**2** 1600.io 16.1

If $f(x) = \dfrac{x^2 - 8x + 3}{x - 3}$, what is $f(-3)$?

A) $-6$

B) $-3$

C) $3$

D) $6$

**Solution**

1. Substitute $-3$ for $x$ to find $f(-3)$ by replacing all instances of $x$ with $-3$.

$$f(-3) = \frac{(-3)^2 - 8(-3) + 3}{-3 - 3}$$

$$f(-3) = \frac{9 + 24 + 3}{-6}$$

$$f(-3) = \frac{36}{-6}$$

$$f(-3) = -6$$

2. The answer is A.

## Composite Functions

You may be given two functions and be tasked with plugging the value from one into the other. Sometimes, these are found in the form of **composite functions**, where one function is plugged into another.

Usually, the easiest thing to do is to substitute a value into one function and then replace its value in the other function.

**Example 16.1-3**

**3** 1600.io 16.1

$$g(x) = 4x + 2$$
$$h(x) = 3 - g(x)$$

The functions $g$ and $h$ are defined above. What is the value of $h(0)$?

A) $-3$

B) $1$

C) $3$

D) $5$





**Solution 1**

1. Notice that the second equation, $h(x)$, is defined in terms of $g(x)$. When we plug in 0 for $x$ to find $h(0)$, make sure to replace each instance of $x$ with 0, including the $x$ that is plugged into the function $g$.

$$h(x) = 3 - g(x)$$
$$h(0) = 3 - g(0)$$

2. We should first find the value of $g(0)$ since the value of $h(0)$ depends on it. Plug in 0 for $x$ in the function $g$ to find $g(0)$ by replacing all instances of $x$ with 0.

$$g(x) = 4x + 2$$
$$g(0) = 4(0) + 2$$
$$g(0) = 2$$

3. We know the value of $g(0)$ is 2, so replace $g(0)$ with 2 to find the value of $h(0)$.

$$h(0) = 3 - g(0)$$
$$h(0) = 3 - 2$$
$$h(0) = 1$$

4. The answer is B.

**Solution 2**

1. Notice that the second equation, $h(x)$, is defined in terms of $g(x)$. We can replace $g(x)$ in the second equation with the expression that $g(x)$ is equal to, $4x + 2$, and then simplify the resulting expression for $h(x)$.

$$h(x) = 3 - g(x)$$
$$h(x) = 3 - (4x + 2)$$
$$h(x) = 3 - 4x - 2$$
$$h(x) = 1 - 4x$$

2. Plug in 0 for $x$ to find $h(0)$ by replacing all instances of $x$ with 0.

$$h(x) = 1 - 4x$$
$$h(0) = 1 - 4(0)$$
$$h(0) = 1$$

3. The answer is B.

You may encounter composite functions where the value of one function acts as the input value for another function. The expression $f(g(x))$, which reads as "$f$ of $g$ of $x$," where $g$ is the inside function and $f$ is the outside function, means that the value of $g(x)$ should be used as the input to the function $f$. When you encounter this, first find the value of the inside function, then plug that into the outside function.

**Example 16.1-4**

**4** 1600.io 16.1

A function $f$ satisfies $f(1) = 4$ and $f(4) = 8$. A function $g$ satisfies $g(1) = 3$ and $g(8) = 4$. What is the value of $f(g(8))$ ?

A) 1

B) 3

C) 4

D) 8

**Solution**

1. Since $g$ is the inside function of $f(g(8))$, first find the value of $g(8)$. We are told that $g(8)$ is 4.
2. In the expression $f(g(8))$, replace $g(8)$ with 4.

$$f(g(8)) = f(4)$$

3. We are told that the value of $f(4)$ is 8.
4. The answer is D.

**Notes**

As occasionally happens, the problem supplies some extraneous information (that $f(1) = 4$ and $g(1) = 3$), which is not of any use when solving. Be prepared for this; you do not always have to use all the provided information.

There has been one problem that took the idea of plugging one function's value into another function one step further. In what could be called an **iterative** or **recursive** function, the test gave you an example of a function whose value is defined in terms of previous values of that same function. In other words, you needed to find the value of the function for one input, and then use the resulting value to plug back into the same function to find the next value.

It might seem confusing, and it probably snagged a lot of students because the concept appeared on one practice test and then was never used again (to our knowledge). However, in the interest of covering all the bases, here is a similar problem.





**Example 16.1-5**

**5** 1600.io 16.1

A biologist is breeding a rare species of mouse in a controlled environment and currently has 200 of those mice. The population of this species that the biologist expects to have next year, $N_{\text{next year}}$, can be estimated from the number of mice this year, $N_{\text{this year}}$, by the equation below.

$$N_{\text{next year}} = N_{\text{this year}} + 0.4\left(N_{\text{this year}}\right)\left(1 - \frac{N_{\text{this year}}}{K}\right)$$

The constant $K$ in this formula is the number of mice the environment is able to support.

According to the formula, what will be the number of mice two years from now if $K = 400$ ? (Round your answer to the nearest whole number.)

**Solution**

1. We cannot start by simply substituting 2 (for the number of years) anywhere in this function. We need to find the number of mice after the first year, and then plug that value back in to find the number of mice after two years.
2. The number of mice 0 years after the breeding begins (the starting number of mice), which we will call $N_0$, is 200. To find $N_1$, the number of mice after one year, we need to use $N_1$ in the place of $N_{\text{next year}}$ and $N_0$ in the place of $N_{\text{this year}}$.

$$N_{\text{next year}} = N_{\text{this year}} + 0.4\left(N_{\text{this year}}\right)\left(1 - \frac{N_{\text{this year}}}{K}\right)$$

$$N_1 = N_0 + 0.4\left(N_0\right)\left(1 - \frac{N_0}{K}\right)$$

3. Replace $N_0$ with 200 and replace $K$ with 400, which is given in the problem statement. Solve for the value of $N_1$ (this step can be done without a calculator if you are comfortable with decimals and fractions).

$$N_1 = N_0 + 0.4\left(N_0\right)\left(1 - \frac{N_0}{K}\right)$$

$$N_1 = 200 + 0.4(200)\left(1 - \frac{200}{400}\right)$$

$$N_1 = 200 + 80\left(1 - \frac{1}{2}\right)$$

$$N_1 = 200 + 80\left(\frac{1}{2}\right)$$

$$N_1 = 200 + 40$$

$$N_1 = 240$$

4. The number of mice after two years, $N_2$, depends on the number of mice after 1 year, $N_1$. Pretend we have travelled forward a year and the number of mice after one year, $N_1$, is now $N_{\text{this year}}$ and $N_2$ will be $N_{\text{next year}}$. We can use $N_1$ to find $N_2$, by replacing $N_{\text{next year}}$ with $N_2$ and replacing $N_{\text{this year}}$ with $N_1$.

$$N_{\text{next year}} = N_{\text{this year}} + 0.4\left(N_{\text{this year}}\right)\left(1 - \frac{N_{\text{this year}}}{K}\right)$$

$$N_2 = N_1 + 0.4\left(N_1\right)\left(1 - \frac{N_1}{K}\right)$$

5. Replace $N_1$ with 240 (which we found in Step 3) and replace $K$ with 400 (again, this is given). Solve for the value of $N_2$. For this one, you definitely want to use your calculator, but make sure not to round any intermediate decimal values—make use of your calculator's "Ans" button or be rigorous about plugging in all the values at once.

$$N_2 = N_1 + 0.4\left(N_1\right)\left(1 - \frac{N_1}{K}\right)$$

$$N_2 = 240 + 0.4\,(240)\left(1 - \frac{240}{400}\right)$$

$$N_2 = 278.4$$

6. We are told to round the value to the nearest whole number, which is 278.

7. The answer is 278.

**Section 16.1 Suggested Problems from Real Tests**

- Test 2-C-10
- Test 2-C-37
- Test 2-C-38
- Test 3-C-16
- Test 6-NC-8

**View related real-test problems in SMTEx**

SMTEx

2017-C-28
2018-C-17
2018-C-7
2019-NC-5

## Section 16.1 Practice Problems

**1** 1600.io 16.1

$$f(x) = x(x+2)$$

The function $f$ is defined above. If the function $g$ is defined by $g(x) = f(x) + 4$, what is the value of $g(7)$ ?

A) 11
B) 28
C) 63
D) 67

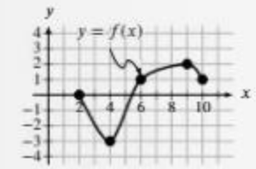
**2** 1600.io 16.1

If $f(x) = x^2 - 6x + 10$ and $c$ is a positive integer less than 5, what is one possible value of $f(c)$ ?





**3** 1600.io 16.1

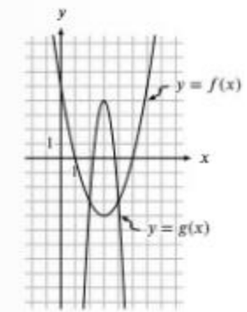



| $x$ | $g(x)$ |
|---|---|
| −3 | 4 |
| −1 | 2 |
| 0 | −1 |
| 1 | 6 |
| 2 | 9 |
| 4 | −3 |

The complete graph of the function $f$ and a table of values for the function $g$ are shown above. The minimum value of $f$ is $k$. What is the value of $g(k)$ ?

A) −3
B) 2
C) 4
D) 9

**4** 1600.io 16.1




Graphs of the functions $f$ and $g$ are shown in the $xy$-plane above. For which of the following values of $x$ does $f(x) + g(x) = 0$ ?

A) 1
B) 2
C) 3
D) 4

## 16.2 Plugging Expressions into Functions

A lot of students feel good about plugging numbers into functions but freeze up as soon as they have to plug an expression into a function even though the two processes are exactly the same. This is surely due to the fact that it makes sense to replace a variable with a value, or with an expression containing a different variable, but it's confusing to replace a variable with an expression that contains that very same variable.

For example, there's nothing odd about thinking of plugging in 4 for $x$ (saying $x = 4$), then evaluating accordingly. However, it seems weird to say $x = 2x$, or $x = x + 2$! It makes us want to solve for $x$, which might well be impossible; how can $x = x + 2$ ? What is this madness?

It's important to realize that it doesn't matter whether the same variable letter is used; there is no connection or relationship between the specific variable letter used in the function's definition (the **formal parameter,** or just the **parameter**) and the variable provided in a term or expression that is to be used as the input to the function (the **actual parameter**, or the **argument**). **The term or expression simply replaces the variable used in the function definition everywhere that it appears**.

For example, if we're given the function definition

$$f(x) = 3x + 2$$

and we're asked to find $f(2x)$, this could seem perplexing; does this mean that $x = 2x$ ?

No. The function definition could use *any* letter for the placeholder variable (the parameter); by convention, we usually use $x$, and that helps us do things like graph the function using the conventional $xy$-plane, but that's all—there's no mathematical reason to use any particular letter. For example, we could have used $w$ for the variable in the definition (the parameter), writing

$$f(w) = 3w + 2$$

Note, importantly, that *this is exactly the same definition as the version that used x*—it just uses a different variable letter. But now when we want to find $f(2x)$, it's not at all confusing to say we need to plug in $2x$ for $w$, and even to conceptualize this as $w = 2x$, and then to replace every occurrence of $w$ with $2x$, giving

$$f(2x) = 3(2x) + 2$$

(Strictly speaking, though you can think about it as $w = 2x$, we're not really saying that variable $w$ has the same *value* as term $2x$, but rather that the term $2x$, which is the argument, should replace the parameter represented by $w$, but the effect is the same.) **Whatever is in the parentheses of the function replaces every instance of $x$ in the function**. For example if $f(x) = 3x + 2$, then to find $f(2x)$, we simply replace each $x$ with $2x$. If you need to, add an intermediate step, where you first replace every instance of $x$ with a blank set of parentheses:

$$\begin{aligned} f(x) &= 3x + 2 \\ f(\quad) &= 3(\quad) + 2 \\ f(2x) &= 3(2x) + 2 \\ f(2x) &= 6x + 2 \end{aligned}$$

Here is an example that has you plugging a binomial expression into a quadratic equation.

**Example 16.2-1**

**1** 1600.Io 16.2

If $f(x) = 3x^2 + 5$ and $f(x + a) = 3x^2 - 24x + 53$, what is the value of $a$?

A) $-24$
B) $-4$
C) $4$
D) $24$

**Solution**

1. There are not two different functions in this problem. Both equations are based off of the function $f$, but they have different expressions acting as the input. If we plug $x + a$ into the $f(x)$ equation, then both equations will have the same function with the same input value. When both equations are written in terms of the same input value then we can simply match coefficients between the two versions of the same equation.

2. In order to set up a situation where we can match coefficients, we should plug $x + a$ into the function $f(x)$ that is defined in the first equation. Once again, we can use blank parentheses in place of $x$ as an intermediate step to make the process less confusing. Replace each instance of $x$ with $x + a$.

$$\begin{aligned} f(x) &= 3x^2 + 5 \\ f(\qquad) &= 3(\qquad)^2 + 5 \\ f(x + a) &= 3(x + a)^2 + 5 \end{aligned}$$





3. Expand the squared binomial term and simplify so that the expression is written in Standard Form to match the form of the other given equation.

$$\begin{aligned} f(x + a) &= 3(x + a)^2 + 5 \\ f(x + a) &= 3(x^2 + 2ax + a^2) + 5 \\ f(x + a) &= 3x^2 + 6ax + 3a^2 + 5 \end{aligned}$$

4. We now have $f(x + a)$ written in Standard Form in two equations (the one provided to us and the one we just formed), and we simply have to match the coefficients because they must be the same equation.

$$\begin{aligned} f(x + a) &= 3x^2 - 24x + 53 \\ f(x + a) &= 3x^2 + 6ax + 3a^2 + 5 \end{aligned}$$

5. The $x^2$ terms already match.
6. By matching the $x$ terms, $-24x$ and $6ax$, we can see that $-24 = 6a$.
7. Divide both sides by 6 to see that $-4 = a$
8. The answer is B.

**Notes**

If you want to double check yourself, you can match the two constant terms, 53 and $3a^2 + 5$, as well. Plug in $-4$ for $a$ to make sure that $3a^2 + 5$ is equal to 53.

$$\begin{aligned} 3a^2 + 5 &= 3(-4)^2 + 5 \\ 3a^2 + 5 &= 3(16) + 5 \\ 3a^2 + 5 &= 48 + 5 \\ 3a^2 + 5 &= 53 \end{aligned}$$

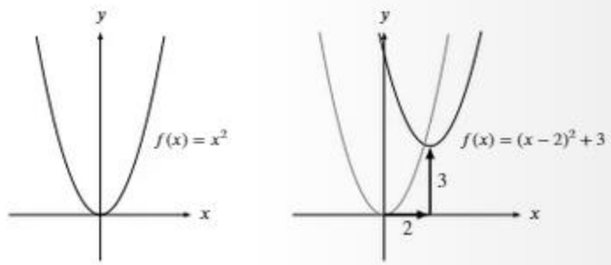
This check verifies that all the coefficients and the constant are equal when $a = -4$.

Note also that when $a = 4$ (choice C), the constant expression, $3a^2 + 5$, would still have a value of 53; however, the coefficient of the $x$ terms would no longer match, so $a \neq 4$ (matching the $x$-coefficients was both easier and less error prone in this instance, and in general, you should **exercise caution while matching exponents when there are squared or other even-exponent components involved** because of the disappearing negative signs). Once again, plugging in answer choices is usually not the recommended solution because there are often unforeseen consequences based on answer choices which satisfy some conditions, but not all.

## Shifting Functions and Working with Uncommon Graphs

The graphs of functions can be shifted in the $x$-direction (horizontally) and in the $y$-direction (vertically), and, though you may not have realized it at the time, we have already seen both shifts in the Vertex Form of quadratic equations. The simplest parabola, called the **standard parabola**, is $f(x) = x^2$, and its vertex is the point $(0, 0)$; for a parabola in vertex form, $f(x) = (x - h)^2 + k$, the vertex has been shifted horizontally $h$ units and vertically $k$ units to the point $(h, k)$.

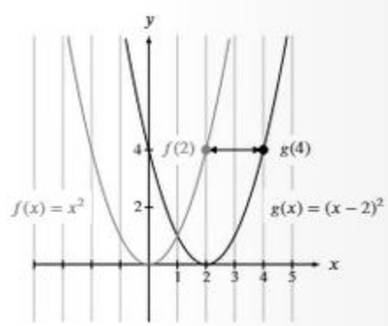
For example, the quadratic function $f(x) = (x - 2)^2 + 3$ has a vertex $(2, 3)$, and you can see that compared to $f(x) = x^2$, the graph has been shifted to the right 2 units and up 3 units.




Because, by convention, $x$ is used as (or as part of) the input to a function, and the $x$-axis lies horizontally in the $xy$-plane, shifting the value of $x$ by adding to or subtracting from it will result in a shift in the graph along that horizontal $x$-axis. This is true for any function in the $xy$-plane, not just for parabolas.

Because the Vertex Form of quadratics is a familiar reference point, let's think about quadratics again for a moment. If $f(x) = x^2$, then $f(x - 2) = (x - 2)^2$ (simply replace $x$ with $x - 2$), and we know based on Vertex Form that the graph of $f(x - 2)$ is shaped the same as the graph of $f(x)$, but shifted 2 units to the right (the vertex is at $(2, 0)$ instead of $(0, 0)$). Going forward, let's define $f(x - 2)$ as a new function called $g$, where $g(x) = (x - 2)^2$.

The sign in front of the horizontal shift is the opposite of what you might expect at first, but it makes sense when you think about it. If a graph is shifted 2 units to the right, then each point on the shifted graph corresponds to a point on the original graph when the $x$-value was 2 less. Consider again that when $f(x) = x^2$ and $g(x) = (x - 2)^2$, it is true that $g(4) = f(2)$: plugging 4 into $g(x)$ results in $(4 - 2)^2$, or $2^2$, and plugging 2 into $f(x)$ also results in $2^2$.




By similar reasoning, if $g(x) = f(x + 2)$, the graph of $g(x)$ is shifted two units to the left from the graph of $f(x)$.

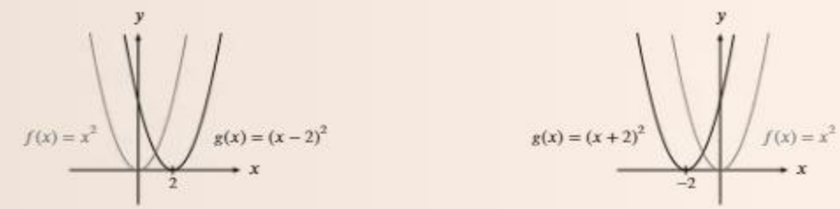




Another way to conceptualize this principle that might be intuitively appealing is to think about the **graph being "delayed" when the value of $x$ is reduced by subtracting a constant from it**. As the value of $x$ advances (increases) from left to right, **the value of the term containing $x$ in the function lags behind due to the subtraction, so the graph appears "later," meaning farther to the right**. The converse also applies: **when a constant value is added to $x$ in the function expression, the value of the term containing $x$ is "ahead" of the value of $x$ alone, so the graph appears "sooner," meaning it's shifted to the left**.

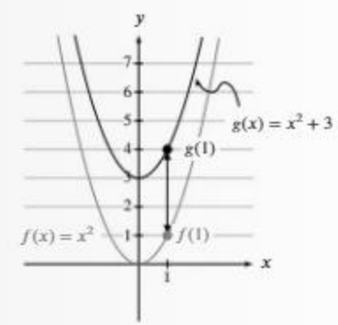
### Horizontal Shifts

For functions $f$ and $g$, where $g(x) = f(x - h)$, the graph of $g$ is shifted $h$ units horizontally from the graph of $f$.




Note that right shifts have a minus sign and left shifts have a plus sign.

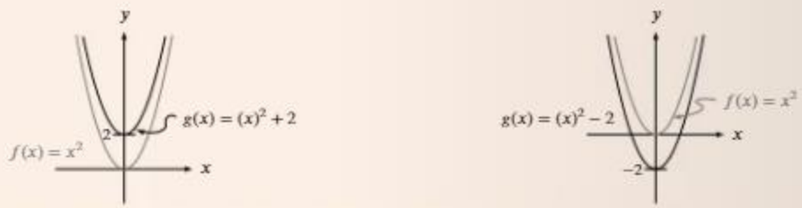
Vertical shifts reflect changes in the value of the function, because that's plotted on the $y$-axis, which is vertical, and therefore these vertical shifts are attributable to values being added to or subtracted from the expression that defines the function. For example, if $g(x) = f(x) + 3$, then each $y$-value on the graph of $g$ will be 3 above the corresponding point on the graph of $f$.




These shifts are more commonly tested than horizontal shifts and are a lot simpler because the signs correspond to what the shift looks like.

## Vertical Shifts

For functions $f$ and $g$, where $g(x) = f(x) + k$, the graph of $g$ is shifted $k$ units vertically from the graph of $f$.



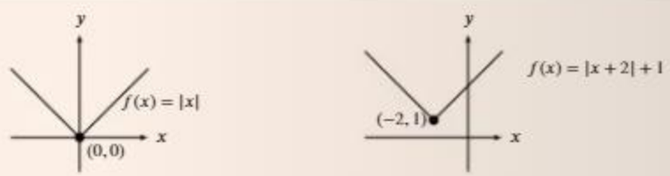

There are not too many questions that deal with the topic of function shifting outside of the Vertex Form of quadratics. Generally, questions so far have been focused on how vertical shifts affect $y$-values. It is not highly likely that you will encounter the topic, but you should be aware of a few basic graphs and how they are affected by shifting.

## Shifting Common Functions

Pay attention to how the functions on the right are changed and how that affects the position of the point compared to the standard graphs on the left. Given any function graph, similar changes to the function will result in similar shifts of the graph.

### Quadratics

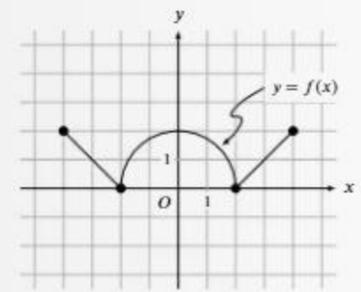
### Absolute Value




### Cubics

**Example 16.2-2**

**2** 16.2



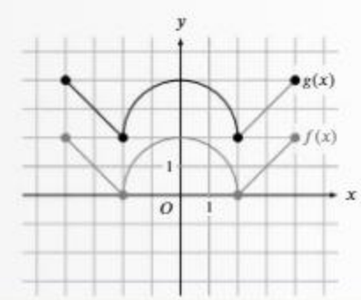

The figure above shows the complete graph of the function $f$ in the $xy$-plane. The function $g$ (not shown) is defined by $g(x) = f(x) + 2$. What is the maximum value of the function $g$?

**Solution**

1. The question wants us to find the maximum $y$-value of the function $g(x)$. Since every point on $g$ is shifted up 2 units from the corresponding point on $f$, then the maximum value of $g$ must also be 2 units above the maximum value of $f$.
2. From the graph of $f(x)$, we can see that the greatest $y$-value reached by the graph is 2 (true of three points on the graph). Since the maximum value of $f$ is 2, the maximum value of $g$ must be 4.
3. The answer is 4.

**Notes**

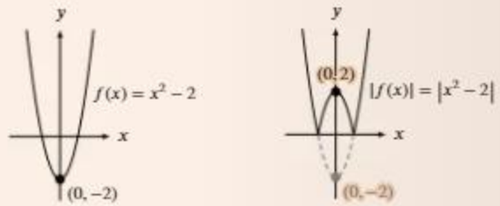
We can draw the graph of $g$ by shifting all of the key points on the graph of $f$ up by 2 units if we want to see the maximum value in a graph.



When we take the absolute value of a function, any $y$-values that would be negative become positive, so the graph of $|f(x)|$ is formed by reflecting over the $x$-axis **only** those portions of the graph of $f(x)$ that would be below the axis.

## Absolute Value of a Function

The graph of $|f(x)|$ will be the same as the graph of $f(x)$ *except* that any negative portions of the graph of $f(x)$ will be reflected over the $x$-axis.




If you have to evaluate the graph of an unknown type of function based on given points, eliminate any answer choices that don't contain every one of the given points (if you choose carefully, you may be able to eliminate many of the choices quickly).

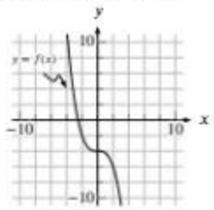
### Example 16.2-3

**3** 1600.io 16.2

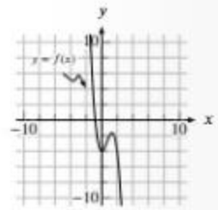
| $x$ | $f(x)$ |
|---|---|
| −2 | 28 |
| −1 | 2 |
| 0 | −4 |
| 1 | −2 |
| 2 | −4 |

The table gives some values of $x$ and the corresponding values of $f(x)$ for polynomial function $f$. Which of the following could be the graph of $f$ in the $xy$-plane, where $y = f(x)$ ?

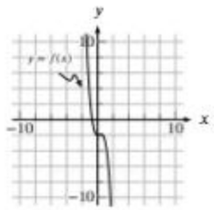
A)



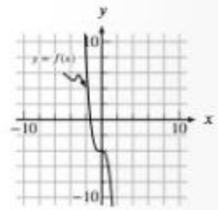

B)




C)




D)








**Solution**

1. All of the answer choices are shaped similarly, so we will have to look carefully to find a graph that matches point by point. Each row in the table represents the $x$- and $y$-value of a point, so we can eliminate any graphs which do not go through each of the points shown in the table. **Be aware that the gridlines on the graph are spaced 2 units apart**.

   The easiest point to check first would be the $y$-intercept, $(0, -4)$ (which is the intersection of the $y$-axis and the second gridline below the $x$-axis—remember the gridlines are spaced 2 units apart). Only choice C does not have the $y$-intercept at $(0, -4)$, so we can eliminate choice C.

2. The next easiest point to check is $(2, -4)$, because this is another point that will be placed on the intersection of gridlines (the first gridline to the right of the $y$-axis and the second gridline below the $x$-axis). Only choice B contains this point, so this must be the graph we are looking for.

3. The answer is B.

**Notes**

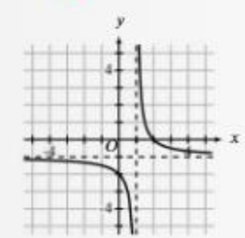
There are many ways to find the right answer. For example, you could take the point $(-2, 28)$ in the first row in the table and use that to eliminate choices A and D. You could then use the $y$-intercept, $(0, -4)$, to identify choice B as the correct answer.

If you are given an unfamiliar graph type and asked to find the function, pick a few easily identifiable points from the graph (points at gridline intersections, labeled points, and $y$-intercepts are usually particularly helpful points to choose) and see which equation fits all of those points (plug them into the functions).

Though this might seem like a violation of our commandment not to solve math problems by plugging in test values, it's really not. Here, we are using *provided* values and checking them against possible solutions. This is a completely different procedure from the unreliable cop-out students sometimes try whereby they pick random numbers and evaluate expressions as a way around actually doing math.

### Example 16.2-4

**4** 1600.io 16.2




The graph of $y = f(x)$ is shown in the $xy$-plane above. Which of the following could define $f$ ?

A) $f(x) = \dfrac{1}{x-1} - 1$

B) $f(x) = \dfrac{1}{x-1} + 1$

C) $f(x) = \dfrac{1}{x+1} - 1$

D) $f(x) = \dfrac{1}{x+1} + 1$

**Solution**

The graph shown in the questions is a hyperbola, which has been seen only once so far on the test. This type of graph does not warrant its own chapter in this book. When we want to find the equation for this graph, we simply need to check points (though knowledge of function shifting and asymptotes could help you do this question faster).

1. In order to see which equation is correct, we can plug points from the graphs into each answer choice in order to see which function corresponds to the graph.

   A useful first point to check is the $y$-intercept, $(0, -2)$.

2. To check the point $(0, -2)$, plug 0 in for $x$ and see if the resulting $y$-value, $f(0)$, is equal to $-2$. If not, eliminate the choice.

   - Choice A:

$$f(x) = \frac{1}{x-1} - 1$$
$$f(0) = \frac{1}{0-1} - 1$$
$$\checkmark \quad f(0) = -2$$

   - Choice B:

$$f(x) = \frac{1}{x-1} + 1$$
$$f(0) = \frac{1}{0-1} + 1$$
$$\times \quad f(0) = 0$$

   Eliminate Choice B.

   - Choice C:

$$f(x) = \frac{1}{x+1} - 1$$
$$f(0) = \frac{1}{0+1} - 1$$
$$\times \quad f(0) = 0$$

   Eliminate Choice C.

   - Choice D:

$$f(x) = \frac{1}{x+1} + 1$$
$$f(0) = \frac{1}{0+1} + 1$$
$$\times \quad f(0) = 2$$

   Eliminate Choice D.

3. The answer is A because no other choice even has the correct $y$-intercept.





**Notes**

Smart selection of test points allowed us to eliminate all of the wrong choices using only one sample point; we could also have checked the $x$-intercept, $(2, 0)$, which would also have identified choice A as the only function for which that point is a solution.

Note that the use of the asymptotes could have also allowed us to pick the correct choice expeditiously if we are familiar with what asymptotes represent.

The vertical asymptote (marked by the dashed line $x = 1$) tells us that the function is undefined when $x = 1$, meaning that the function must have a factor of $x - 1$ in the denominator of a fraction (this denominator would equal 0 when $x = 1$, which would cause the function to be undefined at that value), leaving us with only choices A and B.

The horizontal asymptote (marked by the dashed line $y = -1$) tells us that the function approaches a $y$-value of $-1$ as the $x$-values go to infinity and negative infinity because the fraction part of the equation will approach (though never equal) 0 when the $x$-values are very large or very small (similar to how exponential graphs level off near, but never exactly reach, a certain $y$-value). When the fraction part gets close to 0, we are left with just the constant part of the equation. Looking at the correct answer, $f(x) = \frac{1}{x-1} - 1$, we see that as $x$ goes to infinity or negative infinity, the value of the fraction $\frac{1}{x-1}$ approaches 0 (because the denominator has a very high absolute value compared to the numerator), so the entire function approaches $-1$ as $x$ gets very positive or very negative, which is why the graph levels off at the horizontal asymptote $y = -1$.

**Section 16.2 Suggested Problems from Real Tests**

- Test 2-NC-10
- Test 4-NC-4
- Test 5-NC-14
- Test 9-NC-18
- Test 9-C-24
- May 2018-C-32
- May 2019 (Int)-NC-15
- Mar 2020-NC-7
- May 2021 (US)-NC-9
- May 2021 (Int)-C-22

View related real-test problems in SMTEx

## Section 16.2 Practice Problems

**1** 1600.io 16.2

The function $f$ is defined by $f(x) = x^3$, and the function $g$ is defined by $g(x) = x^3 - 2$. Which of the following translations of the graph of $f$ in the $xy$-plane results in the graph of $g$ ?

A) A translation 2 units upward

B) A translation 2 units downward

C) A translation 2 units to the right

D) A translation 2 units to the left

**2** 1600.io 16.2

Which of the following equations has a graph in the $xy$-plane for which $y$ is always less than or equal to 3 ?

A) $y = x^3 - 4$

B) $y = -x^2 + 4$

C) $y = -(x+4)^2$

D) $y = |x - 4| - 4$

**3** 16.2

If $f(x) = -4x + 3$, what is $f(-2x)$ equal to?

A) $8x^2 + 3$

B) $8x + 3$

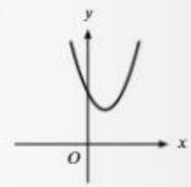
C) $-8x - 3$

D) $-8x + 3$

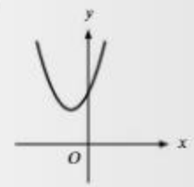
**4** 16.2

The quadratic function $f$ is defined by $f(x) = (x - 1)^2 + 2$. In the $xy$-plane, which of the following could be the graph of $y = f(x)$ shifted 2 units to the left?

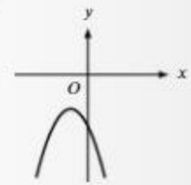
A)




B)




C)




D)

## CHAPTER 16 RECAP

- When plugging into a function $f(x)$, replace each instance of $x$ with the value or expression that is being plugged into the function. Replace each $x$ with a set of blank parentheses as an intermediate step if you have trouble plugging into the function.
- When working with composite functions, plug the input value into the inside function first and then plug the resulting value into the outside function.
- The graph of a function $g(x) = f(x - h) + k$ is shifted $h$ units horizontally and $k$ units vertically from the graph of $f(x)$.

  The graph is "delayed" when the value of $x$ is reduced by subtracting a constant from it. As the value of $x$ advances (increases) from left to right, the value of the term containing $x$ in the function lags behind due to the subtraction, so the graph appears "later," meaning farther to the right.

  The converse also applies: when a constant value is *added* to $x$ in the function expression, the value of the term containing $x$ is "ahead" of the value of $x$ alone, so the graph appears "sooner," meaning it's shifted to the left.
- If you have to evaluate the graph of an unknown type of function based on supplied points, eliminate any answer choices that don't contain every one of those points (if you choose your first point carefully, you may be able to eliminate many of the choices right away).
- If you are given an unfamiliar graph type and are asked to find the corresponding function, plug identifiable points, such as the $y$-intercept or $x$-intercept, into each of the answer choices, eliminating any that don't fit all of the tested points.

**3** 16.2

If $f(x) = -4x + 3$, what is $f(-2x)$ equal to?

A) $8x^2 + 3$

B) $8x + 3$

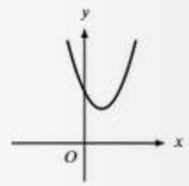
C) $-8x - 3$

D) $-8x + 3$

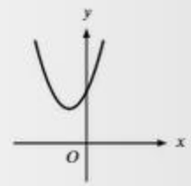
**4** 16.2

The quadratic function $f$ is defined by $f(x) = (x - 1)^2 + 2$. In the $xy$-plane, which of the following could be the graph of $y = f(x)$ shifted 2 units to the left?

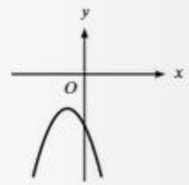
A)




B)




C)




D)

## CHAPTER 16 RECAP

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- The graph of a function $g(x) = f(x - h) + k$ is shifted $h$ units horizontally and $k$ units vertically from the graph of $f(x)$.

  The graph is "delayed" when the value of $x$ is reduced by subtracting a constant from it. As the value of $x$ advances (increases) from left to right, the value of the term containing $x$ in the function lags behind due to the subtraction, so the graph appears "later," meaning farther to the right.

  The converse also applies: when a constant value is *added* to $x$ in the function expression, the value of the term containing $x$ is "ahead" of the value of $x$ alone, so the graph appears "sooner," meaning it's shifted to the left.
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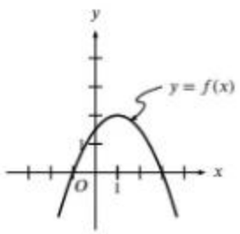
- If you are given an unfamiliar graph type and are asked to find the corresponding function, plug identifiable points, such as the $y$-intercept or $x$-intercept, into each of the answer choices, eliminating any that don't fit all of the tested points.

## Additional Problems

**1** 1600.io 16.1




| $x$ | $g(x)$ |
|---|---|
| −1 | −3 |
| 0 | 0 |
| 1 | 1 |
| 2 | 2 |
| 3 | 0 |

The graph of the function $f$ is shown in the $xy$-plane above, and selected values for the function $g$ are shown in the table. For which of the following values of $x$ is $g(x) > f(x)$ ?

A) −1

B) 0

C) 1

D) 2

**2** 1600.io 16.2

In the $xy$-plane, which of the following changes to the graph of the equation $y = x^2 + 5$ will result in the graph of the equation $y = \left(x^2 + 5\right) - 9$ ?

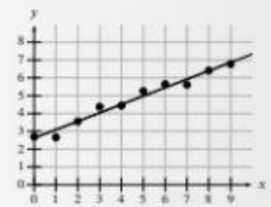
A) A shift 9 units upward

B) A shift 9 units downward

C) A shift 9 units to the left

D) A shift 9 units to the right

**3** 1600.io 16.2




The scatterplot above shows ten data points in the $xy$-plane. A line of best fit is also shown for the data. If each data point is shifted 4 units downward and a new line of best fit for the shifted points is drawn, how will the value of the $y$-intercept of the new line compare with that of the line shown?

A) It will decrease.

B) It will increase.

C) It will remain the same.

D) There will no longer be a $y$-intercept.

**4** 1600.io 16.1

| $x$ | $w(x)$ | $t(x)$ |
|---|---|---|
| 1 | 2 | −4 |
| 2 | 3 | −1 |
| 3 | −2 | 8 |
| 4 | 2 | 2 |
| 5 | −3 | 5 |

The table above shows some values of the functions $w$ and $t$. For which value of $x$ is $w(x) + t(x) = 2x$ ?

A) 1

B) 2

C) 3

D) 4



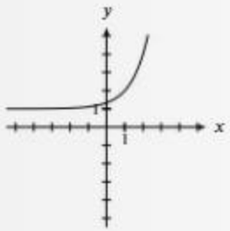


**5** 1600.io 16.2

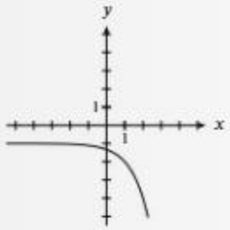
$$f(x) = 3^{(x-1)} + 1$$

The function $f$ is defined by the equation above. Which of the following is the graph of $y = f(-x)$ in the $xy$-plane?

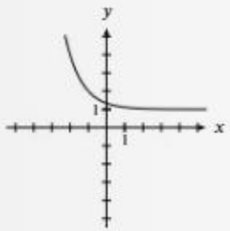
A)



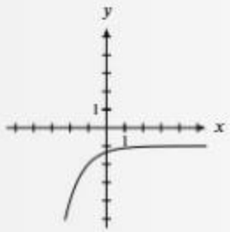

B)




C)




D)




**6** 1600.io 16.1

The functions $f$ and $g$ are defined by $f(x) = 16x$ and $g(x) = x^2$. For what value of $x$ does $f(x) - g(x) = 64$ ?

A) −8

B) −4

C) 4

D) 8

**7** 1600.io 16.2

In the $xy$-plane, the graph of $y = (x + 3)^2 - 2$ is the image of the graph $y = (x + 7)^2 - 2$ after a translation of how many units to the right?

**8** 16.2

The function $f$ is defined by $f(r) = (r-3)(r+6)^2$. If $f(h-8) = 0$, what is one possible value of $h$ ?

**9** 16.1

A biologist is breeding a rare species of mouse in a controlled environment and currently has 200 of those mice. The population of this species that the biologist expects to have next year, $N_{\text{next year}}$, can be estimated from the number of mice this year, $N_{\text{this year}}$, by the equation below.

$$N_{\text{next year}} = N_{\text{this year}} + 0.4\left(N_{\text{this year}}\right)\left(1 - \frac{N_{\text{this year}}}{K}\right)$$

The constant $K$ in this formula is the number of mice the environment is able to support.

The biologist would like to increase the number of mice that the environment can support so that the population of the species will increase more rapidly. If the biologist's goal is that the number of mice will increase from 200 this year to 260 next year, how many mice must the modified environment support?

**10** 16.2

The function $f$ has the property that, for all $x$, $5f(x) = f(5x)$. If $f(20) = 40$, what is the value of $f(4)$ ?

**11** 16.2

For the function $f$, if $f(5x) = x - 10$ for all values of $x$, what is the value of $f(10)$ ?

A) $-10$

B) $-8$

C) $0$

D) $5$



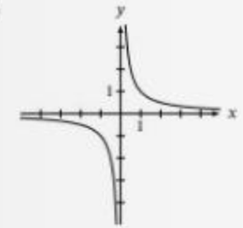


**12** 16.2

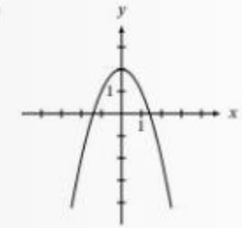
| $x$ | $f(x)$ |
|---|---|
| 1 | $a$ |
| 2 | $b$ |
| 3 | $c$ |

For the function $f$, the table above shows some values of $x$ and their corresponding values of $f(x)$ in terms of the constants $a$, $b$, and $c$. If $a > b > c$, which of the following could NOT be the graph of $y = f(x)$ in the $xy$-plane?

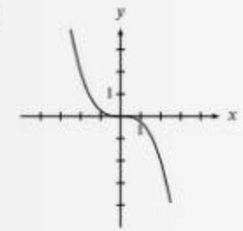
A)



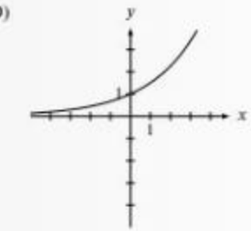

B)




C)




D)



## Answer Key

**Section 16.1 Practice Problems**

1. D
2. 1, 2, or 5
3. C
4. C

**Section 16.2 Practice Problems**

1. B
2. C
3. B
4. B

**Additional Problems**

1. D
2. B
3. A
4. C
5. C
6. D
7. 4
8. 2 or 11
9. 800
10. 8
11. B
12. D



# Statistics 17

## 17.1 Data Sets: Mean, Median, Mode, and Range

The mathematical concepts presented in this chapter are among the easiest to grasp, but the questions are often considered among the hardest on the test because they usually contain long sentences with many pieces of information. It is important to tackle these problems **one phrase at a time**. Do not let yourself be overwhelmed by trying to read and absorb all of the information in one quick pass; you need to break the problem down and process each phrase before moving on. If you do that, you will find that these problems are actually very straightforward and simple to solve.

Many of the problems ask you to find a simple property of a data set. Data sets are most often presented in tables or lists, but can also be found in the form of bar graphs or word problems. Whatever the presentation of the data, you need to be able to find the properties outlined below.

### Properties of Data Sets

Let's consider two data sets and find the four key properties (mean, median, mode, and range) for both data sets. The following data sets, $A$ and $B$, are written in set notation (between curly brackets).

$$A = \{4, 2, 8, 10, 1, 9, 8\}$$
$$B = \{2, 0, 0, 4, 5, 5\}$$

The first step in data analysis problems is to rewrite any data sets in order from least to greatest (unless they are prohibitively long). This will help in solving for the range and median of the data.

$$A = \{1, 2, 4, 8, 8, 9, 10\}$$
$$B = \{0, 0, 2, 4, 5, 5\}$$

### Mean or Average

The **mean** (sometimes called the arithmetic mean) of a data set is the average of all the values in the data set. The mean is calculated by **adding all of the values in the data set and then dividing by the number of values in the data set**.

$$\text{Mean} = \text{Average} = \frac{\text{Sum of Values in Data Set}}{\text{Size of Data Set}}$$

For example, the mean of set $A$ is found with the following calculation. Notice that we write $2(8)$ instead of $8 + 8$ when we are calculating the sum of the values in the data set. You are less prone to errors when calculating the averages of very large data sets by this method instead of repeating the same additions over and over again. The size of the data set is 7 because there are seven members or elements in the data set.

$$\text{Mean}_A = \frac{1 + 2 + 4 + 2(8) + 9 + 10}{7}$$

$$\text{Mean}_A = \frac{42}{7} = 6$$

The mean of set $B$ is found with the following calculation. The size of the data set is 6 because there are six elements in the set, and notice that we write 2(0) and 2(5) instead of 0 + 0 and 5 + 5, respectively (we could, of course, skip the 0 elements when summing, but the Math Book Police might frown on that).

$$\text{Mean}_B = \frac{2(0) + 2 + 4 + 2(5)}{6}$$

$$\text{Mean}_B = \frac{16}{6} = \frac{8}{3}$$

Any new value added to a data set can change the mean if the value is not equal to the mean of the original data set. If the new value is *lower* than the original mean, the new mean will *decrease*. Conversely, if the new value is *greater* than the original mean, the new mean will *increase*. The more the new value differs from the original mean, the more the mean will change when the new value is added to the data set (more extreme new values have a greater "pull" on the average). If you need to calculate the new mean, add the new value to the original sum of the values in the data set, and make sure to add 1 to the size of the data set before dividing.

## Median

The **median** of a data set is the **value of the element at the middle position in the data set (when the data is ordered by value)**.

**When the size of a data set is odd, the median is equal to the middle value**. To determine which value in the list is the middle of a data set with an odd number of values, **add 1 to the size of the data set and then divide by 2 to find the position of the median value in the set**.

For example, the median of a data set consisting of 99 numbers will be the 50th number in the data set because $\frac{99+1}{2} = \frac{100}{2} = 50$.

For set $A$, the size of the data set is an odd number, 7, so the median is equal to the 4th value in the list ($\frac{7+1}{2} = \frac{8}{2} = 4$). The fourth value in set $A$ is 8, so the median of set $A$ is 8.

$$A = \{1, 2, 4, 8, 8, 9, 10\}$$

However, **when the size of a data set is even, we need to average the two values occupying the middle positions in the data set** by adding the two middle values and dividing the result by 2. In order to locate the two middle positions of a data set with an even number of values, **divide the size of the data set by 2 to find the position of the first of the two middle values. The second of the middle values occupies the next position in the set**. Once you know the two middle positions, you need to average the values in those positions in order to find the median.

For example if a data set contains 100 numbers, then the median will be the average of the 50th number and the 51st number in the data set because $\frac{100}{2} = 50$, which tells us that the first of the middle values is in the 50th position and the second of the middle values is in the 51st position in the set.

For set $B$, the size of the data set is 6 (an even number), so the median will be found from the average of the 3rd and 4th values in the set ($\frac{6}{2} = 3$, which means we need to average the 3rd and 4th numbers in the set to find the median). The 3rd number in set $B$ is 2 and the 4th number in the set is 4, so the median of set $B$ is found with the following calculation.

$$B = \{0, 0, 2, 4, 5, 5\}$$

$$\text{Median}_B = \frac{2+4}{2} = \frac{6}{2} = 3$$





## Mode

The **mode of a data set is the value that appears the most often in the data set**.

For set $A$, the most common number in the list is 8; this value appears twice, and no other value is repeated at all, so the mode is equal to 8.

For set $B$, both 0 and 5 appear in the list twice. In this case, we say data set $B$ is **bimodal** because it has two values that appear more than any other value in the set (but the same amount of times as each other). Though a data set *can* have more than one mode, you will never be faced with a situation on the test where there is more than one mode, so you don't need to concern yourself with that scenario.

If no value appears more often than any other, there is **no mode**. Sets with no mode are more likely to appear on the test than sets with multiple modes, but most of the time, there will be a single mode for any sets on the test.

Importantly, the concept of mode is never mentioned in the specifications for material that will appear on the test, and has only made a single appearance in a known test question.

## Range

The **range** of a data set is the difference between the greatest (highest) and least (lowest) values in the set (and it is therefore always non-negative). To find the range, simply **subtract the smallest value in the set from the largest value in the set**.

For set $A$, the largest value is 10, and the smallest value is 1, so the range is equal to $10 - 1$, or 9.

For set $B$, the largest value is 5, and the smallest value is 0, so the range is equal to $5 - 0$, or 5.

**SkillDrill 17.1-1**

**Directions:** Find the mean, median, mode, and range of the given data sets.

**1.** $\{1, 2, 3, 2, 8, 9, 10, 1, 2\}$

**2.** $\{-4, -5, 2, 6, 6, -2, 14, 0\}$

**Example 17.1-1**

**1** 1600.io 17.1

Median Ages of Populations of Selected Counties, 2015

| County | Median age of population (years) |
|---|---|
| Essex | 34.2 |
| Sussex | 42.1 |
| Wessex | 40.6 |
| Burlington | 28.3 |
| Jackson | 50.8 |

What is the range, in years, of the median ages of the populations for the counties in the table above?

**Solution**

1. First of all, do not be confused by the table showing median values. We do not have to calculate any medians, we simply need to find the range of the data presented in the table because we are looking for the range of the median ages in the counties.
2. The range is equal to the difference between the greatest value and least value in the set. The greatest value is 50.8, and the least value is 28.3. Therefore, the range is equal to 50.8 – 28.3, or 22.5.
3. The answer is 22.5.

**Example 17.1-2**

**2** 1600.io 17.1

Interstate Tourist Arrivals, in thousands

| State | March | April |
|---|---|---|
| Alabama | 4.2 | 3.1 |
| New Jersey | 56.2 | 58.1 |
| New York | 250.2 | 400.5 |
| Utah | 1.7 | 5.9 |
| California | 300.1 | 300.2 |
| Maine | 12.3 | 14.6 |
| Colorado | 150.2 | 161.2 |

The table above shows the number of interstate tourist arrivals, rounded to the nearest tenth of a thousand, to seven tourist destinations in both March and April of a year.

Based on the information given in the table, how much greater, in thousands, was the median number of interstate tourist arrivals to these seven tourist destinations in April than the median number in March, to the nearest tenth of a thousand?





**Solution**

1. For this question, we need to make two sets: one for the March data and one for the April data. Make sure to write the sets in order from least to greatest.

$$\text{March} = \{1.7, 4.2, 12.3, 56.2, 150.2, 250.2, 300.1\}$$
$$\text{April} = \{3.1, 5.9, 14.6, 58.1, 161.2, 300.2, 400.5\}$$

2. Since the size of each data set is 7, the median of each data set is the 4th number in each set (to find the position of the median for an odd-sized data set, add 1 to the size and divide by 2: $\frac{7+1}{2} = 4$).

$$\text{March} = \{1.7, 4.2, 12.3, 56.2, 150.2, 250.2, 300.1\}$$
$$\text{April} = \{3.1, 5.9, 14.6, 58.1, 161.2, 300.2, 400.5\}$$

3. The median for April was 58.1 thousand, and the median for March was 56.2 thousand, so the difference to the nearest tenth of a thousand is 58.1 – 56.2, or 1.9.
4. The answer is 1.9.

**Notes**

It is not strictly necessary to rewrite the set in order from least to greatest, but it is a safe way to make sure that you don't overlook anything if you have time. Alternatively, you can find the medians, which are in this case the fourth lowest values in each set (or column), by simply crossing out the three smallest values in each column. The next smallest value in each column will be the fourth smallest in the set and will represent the median of that set. This process is quicker, but we've seen enough students perform this process carelessly that we recommend listing the values in order to make sure each value is accounted for.

Interstate Tourist Arrivals, in thousands

| State | March | April |
|---|---|---|
| Alabama | ~~4.2~~ | ~~3.1~~ |
| New Jersey | **56.2** | **58.1** |
| New York | 250.2 | 400.5 |
| Utah | ~~1.7~~ | ~~5.9~~ |
| California | 300.1 | 300.2 |
| Maine | ~~12.3~~ | ~~14.6~~ |
| Colorado | 150.2 | 161.2 |

**Example 17.1-3**

**3** 1600.io 17.1

Race Summary

| Split Number | Race Segment (meters) | Split time (seconds) | Total race time at end of split (seconds) |
|---|---|---|---|
| 1 | 0-800 | 148 | 148 |
| 2 | 800-1600 | 156 | 304 |
| 3 | 1600-2400 | 154 | 458 |
| 4 | 2400-3200 | 142 | 600 |

A track team entered a 3200-meter relay race. The team's coach is analyzing the race based on the team's split times, as shown in the table above. A split time is the time it takes to complete an 800-meter segment of the race.

By the end of the month, the coach wants the team to reduce its mean split time by 10% as compared to this race. At the end of the month, what should the team's mean split time be, in seconds?

**Solution**

1. The mean split time for this race can be found by adding the four split times and dividing by 4. Note that the bottom right cell of the table contains the sum of the split times because the total race time at the end of split 4 is the sum of the four individual split times.

$$\text{Mean}_{\text{race}} = \frac{600}{4} = 150$$

2. Since the coach wants the team to reduce their average split time by 10%, the new split time will be equal to 0.9 times the average split time for this race (90% of the average split time for this race).

$$\text{Mean}_{\text{new}} = 0.9(\text{Mean}_{\text{race}})$$
$$\text{Mean}_{\text{new}} = 0.9(150)$$
$$\text{Mean}_{\text{new}} = 135$$

3. The answer is 135.

**Notes**

In Step 2 of the solution, it may be easier to find 10% of the mean split time and then subtract instead of finding 90% of the mean split time.

$$0.1(150) = 15$$
$$150 - 15 = 135$$

For consistency, we wrote the solution above by finding 90% directly, but when performing the math in your head, most likely you would make some use of the fact 10% of 150 is 15: either subtract 15, which is 10% of 150, from 150, leaving 90% of its original value; or multiply 15, which is 10% of 150, by 9 to find the value of 90% of 150.





Sometimes you do not have all of the data in a set, but you need to write average equations that will allow you to solve for unknown values. Make sure to set the equation up correctly: the sum of values divided by the size of the data set is equal to the average.

**Note that the sum of scores can always be found by multiplying the size of a data set by the average**. We know that

$$\text{Average} = \frac{\text{Sum of Values}}{\text{Size of Data Set}}$$

If we multiply both sides of the equation by the size of the data set, we can prove that the sum of values is also equal to the average times the size of the data set:

$$(\text{Size of Data Set}) \times (\text{Average}) = \text{Sum of Values}$$

This is useful for word problems that tell you about the averages of data sets without listing out all of the individual values in the set.

For example, if you know that the average of 10 scores is 40 (even though you do not know any of the individual scores), the sum of those 10 values is 10(40), or 400.

**SkillDrill 17.1-2**

**Directions:** Find the value of $x$ given a data set and the mean of that data set.

**1.** $\{1, 2, 3, 2, x\}$, Mean = 4

**2.** $\{-4, 30, 56, 100, -40, x\}$, Mean = 24

Average problems are sometimes about maximizing or minimizing values. Typically, **in order to maximize one value, you usually need to minimize some other values, and vice versa**. For example, when you are trying to calculate the minimum score you can get on a test and still pass a class, you may need to assume you're going to get the best possible score of 100 for the remaining tests (wishful thinking) in order to find the minimum possible score on the upcoming test.

**Example 17.1-4**

**4** 1600.io 17.1

An escape room receives customer satisfaction ratings between 0 and 100, inclusive. In the first 8 ratings the escape room received, the average (arithmetic mean) of the ratings was 82. What is the least value the escape room can receive for the 9th rating and still be able to have an average of at least 90 for the first 15 ratings?

**Solution**

1. Notice that we have three kinds of ratings: the first eight ratings, which average 82, the ninth rating, which is unknown, but subject to a constraint, and the tenth through fifteenth ratings (6 samples) which are completely unspecified.

   The constraint on the ninth value is that it must be the smallest value that allows for it to be *possible* (but not certain) that the average of all 15 ratings is 90. Put another way, if the ninth value is any smaller, then no matter how high the unspecified values are for the tenth through fifteenth ratings, the average can't reach 90. What this tells us is that, first, we should jack up those 6 unspecified ratings to their maximum allowable value of 100, which will allow the ninth value to be as small as possible while still producing a 90 average across all 15 ratings.

   The eight ratings that average 82 must sum to $8(82) = 656$ (since the average of the first 8 ratings $= \frac{\text{sum of first 8 ratings}}{8} = 82$). The maximum possible sum of the unspecified ratings 10 through 15 is $6(100) = 600$. We can write the sum of all the ratings as an equation:

   $$\text{Sum of All Ratings} = (\text{Sum of Ratings 1 through 8}) + (\text{Ninth Rating}) + (\text{Sum of Max Ratings 10 through 15})$$

   Using $n$ to represent the ninth rating, and substituting the values we know and have chosen:

   $$\text{Sum}_{\text{ratings}} = 8(82) + n + 6(100) = 656 + n + 600 = n + 1256$$

2. In this question, we want to set up an inequality rather than an equation. The average score must be greater than or equal to 90. The average is equal to the sum of the ratings, which is $n + 1256$, divided by the number of ratings, which is 15.

   $$\text{Average Rating} \geq 90$$
   $$\frac{\text{Sum}_{\text{ratings}}}{\text{Number}_{\text{ratings}}} \geq 90$$
   $$\frac{n + 1256}{15} \geq 90$$

3. Solve for $n$ in order to find the minimum possible value for the 9th rating.

   $$\frac{n + 1256}{15} \geq 90$$
   $$n + 1256 \geq 15(90)$$
   $$n + 1256 \geq 1350$$
   $$n \geq 94$$

4. The answer is 94.

**Section 17.1 Suggested Problems from Real Tests**

- Test 1-C-14
- Test 2-C-18
- Test 3-C-32
- Test 3-C-35
- Test 4-C-29
- Test 5-C-27
- Test 6-NC-10
- Test 6-C-36
- Test 7-C-22
- Test 10-C-33
- Apr 2017-C-35
- May 2017-C-33
- Mar 2018-C-15
- Mar 2019-C-13
- Mar 2019-C-33
- Apr 2019-C-34
- May 2019 (US)-C-32
- Mar 2020-C-32
- Oct 2020-C-3
- Oct 2020-C-14
- Mar 2021-C-34
- Apr 2021-C-11
- Apr 2021-C-37
- May 2021 (US)-C-7

View related real-test problems in SMTEx

SMTEx





## Section 17.1 Practice Problems

**1** 17.1

$x, 100, 250, 400, 450$

If the mean of the five numbers above is 250, what is the value of $x$ ?

**2** 17.1

The maximum value of a data set containing 35 positive integers is 61. A new data set consisting of 36 positive integers is created by including 65 in the original data set. Which of the following measures must be 4 greater for the new data set than for the original data set?

A) The median

B) The range

C) The mean

D) The standard deviation

**3** 17.1

The mean lap time of 10 runners on a track team was 55 seconds. If the highest individual lap time is removed, the mean lap time of the remaining 9 runners becomes 50 seconds. What was the highest lap time, in seconds?

A) 45

B) 50

C) 60

D) 100

**4** 17.1

The weights, in kilograms, for 40 rats in a restaurant were reported, and the mean, median, range, and standard deviation for the data were found. The rat with the highest reported weight was found to actually weigh 4 kilograms more than its reported weight. What value remains unchanged if the four values are reported using the corrected weight?

A) Standard deviation

B) Range

C) Mean

D) Median

## 17.2 Frequency Tables, Histograms, and Standard Deviation

Rather than showing each of the values of a data set as an individual row, tables can be used to show how often certain values appear in a data set, with each row consisting of a value in one column and the number of times that value appears in the set (the frequency) in another column.

From these **frequency tables**, we *could* find all the key properties of the data set by writing the set in order from least to greatest as we did when reordering the small data sets in the previous section. However, frequency tables are often used as a way to condense very large data sets, so when solving problems involving a frequency table, it is *not practical* to actually list out all the values in a data set. It is straightforward to find most of the data set properties from frequency tables without any need for listing: for the **mean** (average), we multiply each value by how many times it appears (its frequency) and then sum those products and divide by the total number of elements; to determine the **mode**, we simply look for the value that has the highest frequency; and for the **range**, we subtract the highest value from the lowest. However, when faced with finding the **median** from a frequency table, many students are uncertain how to proceed. When finding the median of a large data set, you need to start by finding which value in the set will be in the middle position.

For example, if a frequency table is used to display data for 100 items in a data set, we already know that the median will be the average of the 50th and 51st values in the set. To determine what those values are, we step through the rows in the frequency table from top to bottom (lowest to highest element values), adding the number of occurrences of each value to a running sum, until we hit or exceed the position (or **index**) of the value (or the lower of the two values) we need to determine the median; here, that would mean the 50th value. Importantly, the entries in the table must be sorted from least to greatest before initiating this stepping-through procedure; in the problems seen on the test to date, frequency tables are already provided to you sorted in this way.

**SkillDrill 17.2-1**

**Directions:** Find the mean, median, mode, and range of the given data sets based on the frequency tables provided. Round to two decimal places if necessary.

**1.**

| Value | Frequency |
|---|---|
| 1 | 2 |
| 2 | 5 |
| 3 | 7 |
| 4 | 4 |
| 5 | 3 |

**2.**

| Value | Frequency |
|---|---|
| 11 | 20 |
| 24 | 50 |
| 37 | 51 |
| 50 | 11 |
| 63 | 8 |





**Example 17.2-1**

1 1600.io 17.2

Ages of 200 People Enrolled in a Hot Yoga Studio

| Age | Frequency |
|---|---|
| 18 | 34 |
| 19 | 21 |
| 23 | 37 |
| 25 | 38 |
| 30 | 46 |
| 45 | 24 |

The table above shows the distribution of ages of the 200 people enrolled in a hot yoga studio. Which of the following gives the correct order of the mean, median and mode of the ages?

A) mode < mean < median

B) median < mode < mean

C) median < mean < mode

D) mean < median < mode

**Solution**

1. The mode is the easiest value to find because the mode will be the age with the highest frequency (the frequency tells us how many times a value appears in the data set). The highest value in the frequency column is 46, which corresponds to the age 30. Therefore, the mode is 30.

2. To find the mean, we need to add together every person's age and then divide by the number of people. Note, importantly, that you do not have to add up the Frequency column to determine the total number of people (though that would work); that total is given right to you in the title of the table! Don't overlook these small but very helpful bits of information on the test that might be in parts of a problem that you think are irrelevant.

   Luckily, we do not have to add 200 individual values to find the sum of the ages. For example, the first 34 values are 18, so we can just multiply 18 by 34, repeat this procedure for each row in the table, and then sum the values from each row. To find the mean we will divide this sum by 200, which is the number of people in the studio.

$$\text{Mean} = \frac{34(18) + 21(19) + 37(23) + 38(25) + 46(30) + 24(45)}{200} = \frac{5272}{200} = 26.36$$

   Since the mean is less than the mode, we can eliminate choices A and B, which indicate that the opposite is true.

3. To find the median, we need to find the two middle values and average them (there are an even number of values in the set). Since there are 200 people in the set, the median will be the average of the ages of the 100th and 101st people in the set. Starting at the top row, step through each row and add the frequencies until we get to the 100th value in the set.

   There are 34 people in the set after the first row is accounted for. There are 55 people in the set after the first two rows are accounted for ($34 + 21 = 55$). There are 92 people in the set after the first three rows are accounted for ($55 + 37 = 92$). Adding the fourth row, which represents 38 more values, we will be adding the 93rd through 130th people ($92+38 = 130$), but you shouldn't need to actually perform this addition to realize that if you're already at sample number 92, you'll go way over 100 if you add 38, so you can quickly decide that the 100th and 101st values must be 25 (both the 100th and 101st people in the set would be represented in this fourth row of the table, so they both must be 25 years old).

Since both the 100th and 101st people are 25 years old, the average of their ages is also 25. Therefore, the median age is 25. Now we know that the median is 25, the mean is 26.36, and the mode is 30, so median < mean < mode.

4. The answer is C.

There is another piece of information, called the **Standard Deviation**, that is sometimes used to describe data sets on the test, though you do not need to know how to actually calculate the value; you just need to understand it conceptually. The **standard deviation is a way to measure how closely clustered a data set is about its mean** (how spread out the data is). When all or most of the values in a data set are **very close to the mean value, that data set has a low standard deviation** (narrow spread). When the values are **further away from the mean or more spread out from the mean, the data set has a high standard deviation** (wide spread). For example, let's compare the following two data sets, which are presented in frequency tables, below.

| Set *A* | |
|---|---|
| Value | Frequency |
| 1 | 1 |
| 2 | 5 |
| 3 | 10 |
| 4 | 3 |
| 5 | 2 |

Lower Standard Deviation

| Set *B* | |
|---|---|
| Value | Frequency |
| 1 | 3 |
| 2 | 6 |
| 3 | 4 |
| 4 | 4 |
| 5 | 4 |

Higher Standard Deviation

Both data sets consist of 21 values, and for both data sets the mean value is 3. However, Set *A* has a lower standard deviation than does Set *B* because the values in Set *A* are more concentrated at and around the mean value. In Set *B*, the values are more spread out, and most of the values are further away from the mean rather than equal to or close to the mean. We'll return to these data sets shortly with a graphic representation that makes these characteristics visually obvious.

**Example 17.2-2**

**2** 1600.io 17.2

The tables below give the distribution of low temperatures in degrees Celsius (°C) for City A and City B over the same 31 days in August.

City A

| Temperature (°C) | Frequency |
|---|---|
| 18 | 2 |
| 19 | 4 |
| 20 | 20 |
| 21 | 4 |
| 22 | 1 |

City B

| Temperature (°C) | Frequency |
|---|---|
| 18 | 4 |
| 19 | 7 |
| 20 | 7 |
| 21 | 6 |
| 22 | 7 |

Which of the following is true about the data shown for these 31 days?

A) The standard deviation of temperatures in these cities cannot be calculated with the data provided.

B) The standard deviation of temperatures in City A is the same as that of City B.

C) The standard deviation of temperatures in city A is larger.

D) The standard deviation of temperatures in City B is larger.

**Solution**

1. Do not be fooled by choice A. Just because you don't know *how* to calculate the standard deviation exactly does not mean that it cannot be found (it's pretty easy to do so on your calculator if you need to, which you won't).





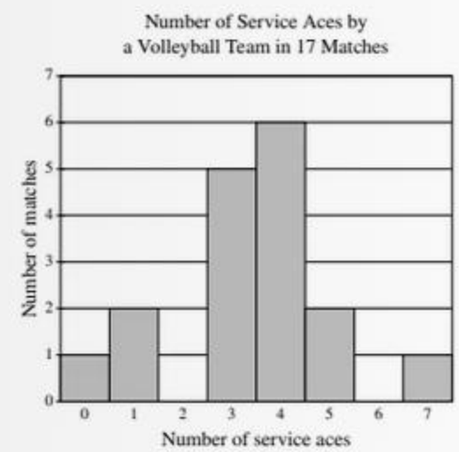
2. It is clear that the temperatures in City A are much more tightly concentrated than those in City B. Since most of the values in City A are very closely clustered around 20°C, the temperature in City A has a lower standard deviation than that of City B. Therefore, City B has a larger standard deviation than City A (notice how the temperatures in the table all occur with roughly equal frequency).

3. The answer is D.

Often, instead of frequency tables, bar graphs are used to convey the frequency information for data sets. Each bar will correspond to one of the values in the data set, and the height of the bar will show how many times that value appears in the set (the frequency). All of the key properties of a data set can be found based on these bar graphs as well.

**Example 17.2-3**

**3** 1600.io 17.2




Based on the graph above, in how many of the matches played did the volleyball team get service aces equal to the median number of service aces for the 17 matches?

**Solution**

1. Since there are 17 matches represented in the data set (an odd-sized set), the median value will be the 9th value in the set (add 1 to 17 and divide by 2). Add the frequencies shown by the heights of the bars, starting from the smallest (leftmost) value, until the number of values equals or exceeds 9. The first bar shows us 1 value (the height of the bar is 1, which means that the team had 0 service aces in only 1 match). The second bar gives us 2 more values (the height of the bar is 2), for a total of 3 values. The third bar gives us 5 more values (the height of the bar is 5), for a total of 8. Finally, the fourth bar, corresponding to 4 service aces (found along the $x$-axis), gives us 6 more values (the height of the bar is 6, meaning the team had 4 service aces in 6 different matches). The 9th value (the median) is represented in this fourth bar, corresponding to 4 service aces. Therefore, the median of the data set (the 9th value if the set were listed in increasing order) is 4 service aces.

2. Note that it is essential to read the question with care so you are aware that you are asked to find the **number of matches** in which the team had the median number of service aces. You are *not* merely being asked what the median number of service aces is; you don't want to do all the work correctly just to get the answer wrong due to failure to pay attention to the specifics of the question. The bar for 4 service aces tells us that the team achieved this feat in 6 matches (the height of the bar is 6).

3. The answer is 6.

Once again, bar graphs share some similarity with frequency charts in that they can be used to estimate relative standard deviations of two data sets. Let's return to our sample data sets from before, where Set A has a lower standard deviation than does Set B (even though the two sets have the same mean, median, range, and number of values).

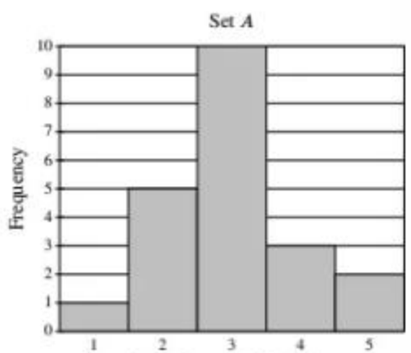
| Set *A* | |
|---|---|
| Value | Frequency |
| 1 | 1 |
| 2 | 5 |
| 3 | 10 |
| 4 | 3 |
| 5 | 2 |

Lower Standard Deviation

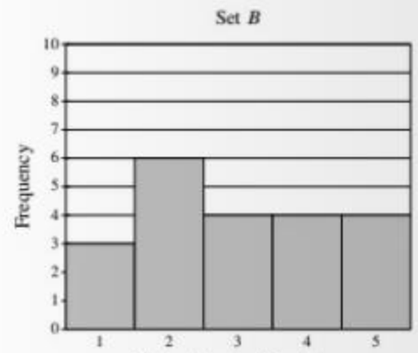
| Set *B* | |
|---|---|
| Value | Frequency |
| 1 | 3 |
| 2 | 6 |
| 3 | 4 |
| 4 | 4 |
| 5 | 4 |

Higher Standard Deviation

When bar graphs are used to show frequency distributions of data sets, they are called **histograms**. If all of the tallest bars in a histogram are closely clustered around the average value, the data set will have a low standard deviation. If the heights of the bars are more evenly distributed (or taller at the ends of the data set), the data set will have a high standard deviation.




Lower Standard Deviation




Higher Standard Deviation





**Section 17.2 Suggested Problems from Real Tests**

- Test 1-C-7
- Test 1-C-12
- Test 2-C-19
- Test 4-C-23
- Test 6-C-22
- Test 8-C-17
- Test 8-C-28
- Test 9-C-34
- Test 10-C-11
- Apr 2018-C-17
- May 2018-C-19
- May 2019 (Int)-C-28
- Oct 2019-C-14
- Oct 2019-C-25
- Oct 2019-C-27
- Mar 2021-C-1
- Mar 2021-C-7
- Mar 2021-C-12
- Apr 2021-C-23
- Apr 2021-C-31
- May 2021 (US)-C-23
- May 2021 (US)-C-30
- May 2021 (Int)-C-23
- May 2021 (Int)-C-25
- May 2021 (Int)-C-26

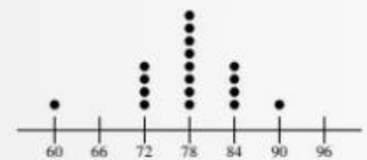
View related real-test problems in SMTEx

SMTEx

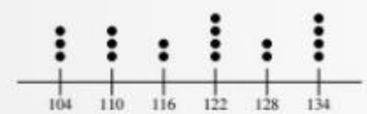
## Section 17.2 Practice Problems

**1** 1600.io 17.2

The 18 students in a baking class conducted an experiment in which they each recorded their blood sugar levels, in mg/dL, before and after eating a slice of cake. The dot plots below display the results.




Blood sugar level, in mg/dL, before eating




Blood sugar level, in mg/dL, after eating

Let $s_1$ and $r_1$ be the standard deviation and range, respectively, of the data before eating, and let $s_2$ and $r_2$ be the standard deviation and range, respectively, of the data after eating. Which of the following is true?

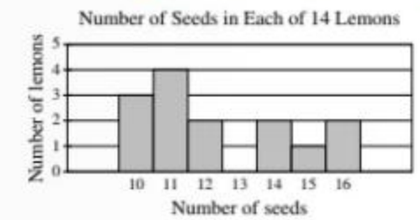
A) $s_1 \neq s_2$ and $r_1 = r_2$

B) $s_1 > s_2$ and $r_1 > r_2$

C) $s_1 = s_2$ and $r_1 < r_2$

D) $s_1 < s_2$ and $r_1 > r_2$

**2** 1600.io 17.2




Based on the histogram above, of the following, which is closest to the average (arithmetic mean) number of seeds per lemon?

A) 12

B) 13

C) 14

D) 15

**3** 1600.io 17.2

A data set of 37 different numbers has a mean of 45 and a median of 45. A new data set is created by subtracting 3 from each number in the original data set that is less than the median and adding 3 to each number in the original data set that is greater than the median. Which of the following measures does NOT have the same value in both the original and new data sets?

A) Standard Deviation

B) Sum of the numbers

C) Mean

D) Median

## 17.3 Box Plots

When **box plots** (also called box and whisker plots) began showing up on the test, many students were baffled by the strange graphs. However, once you understand what each part of the graph means, you'll see that box and whisker plots are a clear and effective way of representing several useful properties of a data set.

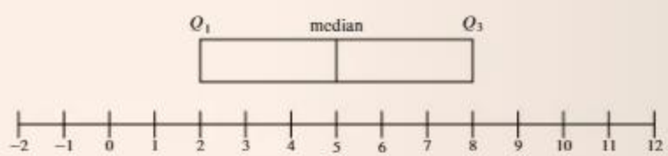
Generally, it's good to understand that plots and charts are intended to make numerical data easy to understand, not to obfuscate it. Once you learn the meaning of the features in a plot or chart type, answering the questions is straightforward, because the charts are chosen specifically to convey the requested information more naturally than would raw lists of numbers. Don't view unfamiliar charts as your enemy, intended to scramble information, but as the opposite: sensible methods of visually conveying data. Once you accept this, it will ease your path to the understanding and appreciation of these figures.

### Box Plots

Box plots are graphical representations of data sets that are drawn on a number line. They are used to show the **median, range, and quartiles** of a data set, as well as to give some idea about how tightly grouped the values of a data set are and whether or not the data is symmetrical or skewed to one side of the median.

We need to discuss the meaning of the quartiles in order to understand the construction of a box plot. When a data set (listed in increasing order) is split into four roughly equal-sized groupings (as close to equal as possible), each grouping represents a quarter, or 25%, of the data. The median of a data set is the value directly in the middle of a data set (it's the second quartile or $Q_2$), so 50% of the data is less than the median (left of the median if the box plot is in horizontal orientation or below it if the box plot is in vertical orientation), and 50% of the data is greater than the median (right of the median if the box plot is in horizontal orientation or above it if the box plot is in vertical orientation). The lower 50% of the data appears to the left of (or below) the median, and the **first quartile, often denoted $Q_1$, is the median of the lower half of the data**. Similarly, the upper 50% of the data appears to the right of (or above) the median, and the **third quartile, often denoted $Q_3$, is the median of the upper half of the data**. Note that there is no single "correct" method of calculating the quartiles (there are many different formulas which can give slightly differing results), but defining the quartiles as the medians of the upper and lower halves of the data set (the method used in the upcoming example) should be suitable for anything you encounter on this test.

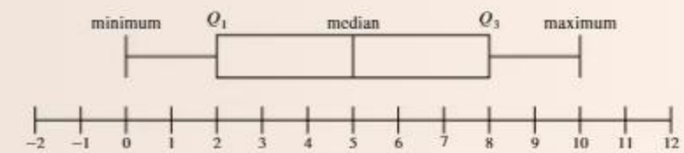
A box plot is formed by drawing short vertical lines for the median and quartiles $Q_1$ and $Q_3$ and then connecting the tops and bottoms of those lines to form a box. For example, if the median of a data set is 5, the first quartile is 2, and the third quartile is 8, the box would look like the one in the picture below.



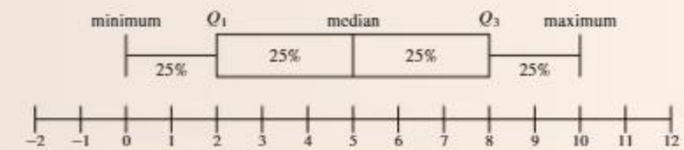





In order to add the whiskers, we just need to draw vertical lines for the maximum and minimum values in the set, and then draw a line from the maximum and minimum to the nearest quartile. Let's say the minimum value in our data set is 0 and the maximum value is 10.




About 25% of the data set will be found between each vertical line on the box and whisker plot.




Let's use the following two sets of 11 numbers (already in increasing order) to make box plots by finding the median, quartiles, and range of the data sets.

$$A = \{14, 15, 15, 16, 16, 17, 18, 18, 19, 19, 20\}$$
$$B = \{10, 12, 12, 15, 17, 19, 20, 20, 20, 21, 22\}$$

Because both data sets contain 11 values, the median of both sets is the 6th value in the set ($11 + 1 = 12$, and $\frac{12}{2} = 6$). The median of Set $A$ (the 6th value) is 17, and the median of Set $B$ is 19.

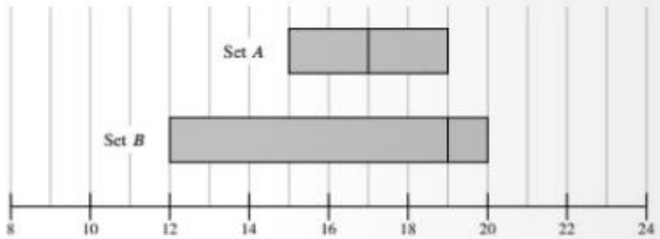
$$A = \{14, 15, 15, 16, 16, 17, 18, 18, 19, 19, 20\}$$
$$B = \{10, 12, 12, 15, 17, 19, 20, 20, 20, 21, 22\}$$

In both data sets, there are 5 values to the left of the median, so $Q_1$, which is the median of the lower half of the data, will be the 3rd value in the lower half of the data ($\frac{5+1}{2} = 3$). The first quartile for Set $A$ is 15, and the first quartile for Set $B$ is 12.

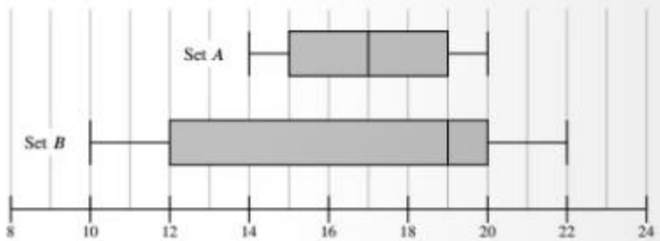
Similarly, there are 5 values to the right of the median, so $Q_3$, which is the median of the upper half of the data, will be the 3rd value in the upper half of the data. The third quartile for Set $A$ is 19, and the third quartile for Set $B$ is 20.

$$A = \{14, 15, 15, 16, 16, 17, 18, 18, 19, 19, 20\}$$
$$B = \{10, 12, 12, 15, 17, 19, 20, 20, 20, 21, 22\}$$

Now we can draw the boxes for the two data sets, marking the median, $Q_1$, and $Q_3$ for both.



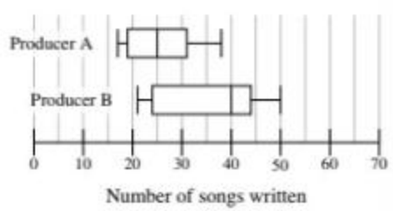

Finally, we need the minimums and maximums of each set in order to add the whiskers. The minimum of Set $A$ is 14, and the maximum of Set $A$ is 20. The minimum of Set $B$ is 10, and the maximum of Set $B$ is 22.




It is clear that Set $A$ is more tightly clustered than Set $B$ due to the smaller range and smaller difference between $Q_1$ and $Q_3$. Also, Set $A$ is symmetrical about its median, while Set $B$ is tightly clustered on its right side (the upper 50% of the data ranges only between 19 and 22) but spread wide on its left side (the lower 50% of the data ranges from 10 to 19).

**Example 17.3-1**

1 1600.io 17.3




The box plots above summarize the distribution of the number of songs written each day by two country music producers for a year. By how many songs does the median number of songs written each day by Producer B exceed the median number by Producer A?





**Solution**

1. The median of each data set is represented by the line located inside the box. For Producer A, this line is at 25, so the median number of songs written each day by Producer A is 25.
2. For Producer B, this line is at 40, so the median number of songs written each day by Producer B is 40.
3. To find how much the median number of songs per day for Producer B exceeds that for Producer A, subtract Producer A's median from Producer B's median:

$$\text{median}_B - \text{median}_A = 40 - 25 = 15$$

4. The answer is 15.

**Section 17.3 Suggested Problems from Real Tests**

- Mar 2018-C-28
- Mar 2018-C-29
- Mar 2019-C-32
- May 2019 (US)-C-5
- May 2019 (Int)-C-1
- May 2019 (Int)-C-27
- May 2021 (Int)-C-19

View related real-test problems in SMTEx

## Section 17.3 Practice Problems

1 1600.io 17.3

Bowling Ball Weight (lbs) of 16 Bowlers

The box plot above summarizes the weights of the bowling balls used by 16 bowlers. Based on the box plot, of the following, which is the best estimate of the median bowling ball weight, in lbs, of the 16 bowlers?

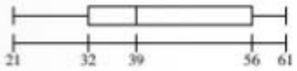
A) 10

B) 13

C) 15

D) 16

2 1600.io 17.3




Which of the following statements about the data represented in the box plot above must be true?

A) There are more data between 39 and 61 than between 21 and 39.

B) There are no data between 56 and 61.

C) The mean of the data is 39.

D) The range of the data is 40.

## 17.4 Survey Design and Interpreting Results

When people want to make predictions about a **population**, they can take the results of a survey of a **subset** of that population, which is called a **sample**, and generalize those results to the whole population. The predictions will not necessarily be perfect, so we need to keep in mind that we should avoid making definitive statements. Nevertheless, some questions carefully ask you about what to *expect* from a population based on the results of a survey while avoiding saying that the prediction is perfectly accurate. Survey problems that ask for a numerical value generally represent percentage or proportion questions, where the percentage or proportion found in a survey is then applied to the whole population.

**Example 17.4-1**

**1** 1600.io 17.4

In Town X, Mrs. Pumblechook's knitting club consisting of 32 knitters was surveyed and 21.9% of the knitters reported they had at least three cats. The average knitting club size in the town is 32. If the knitters in Mrs. Pumblechook's knitting club are representative of knitters in the town's knitting clubs and there are 1,200 knitting clubs in the town, which of the following best estimates the number of knitters in the town who have fewer than three cats?

A) 8,400

B) 17,500

C) 30,000

D) 38,400

**Solution 1**

1. Since Mrs. Pumblechook's knitting club is representative of all the knitting clubs in town, we can use the results of the survey to make estimates about all of the knitting clubs.

   Notice that the survey results directly tell us the percentage of people with at least 3 cats rather than what the problems asks for, which is the number of people with fewer than 3 cats, so we need to keep that in mind when we do our calculations.

2. The average knitting club is the same size as Mrs. Pumblechook's club, so we can determine the number of knitters in her club with fewer than 3 cats and use that as the average number of knitters with fewer than 3 cats in every club in town.

   Multiply the number of knitters in her club, 32, by the decimal representation of the percentage of the knitters that reported owning at least 3 cats, which is 0.219:

   $$0.219(32) = 7.008$$

   In this case, since the percentage came from a survey made up of an integer number of people, it is safe to round this to 7 people in the club who have at least 3 cats.

   Since 7 people in the club have at least 3 cats, that means the remaining 25 people ($32 - 7 = 25$) in the club have fewer than 3 cats.





   Here, we could instead have first taken the complementary percentage of 21.9%, which is 78.1%, and then used that figure to directly determine the number of knitters with fewer than 3 cats in the club: ($.781(32) = 24.992 \approx 25$). Once again, we can assume that her club is representative of all the clubs. Since the average club also has 32 people, then the average club should have 25 people with fewer than 3 cats.

3. Since we know the average number of people in a knitting club with fewer than 3 cats is 25, we can multiply 25 by the number of clubs to find an estimate of the total number of knitters in the town's knitting clubs who have fewer than 3 cats:

   $$25(1{,}200) = 30{,}000$$

4. The answer is C.

**Solution 2**

1. Since Mrs. Pumblechook's knitting club is representative of all the knitting clubs in town, we can use the results of the survey to make estimates about all of the knitting clubs.

2. First, we need to see how many total knitters there are in knitting clubs in the town so that we can then apply the percentage to that whole group. We are told that there are 1,200 knitting clubs, which on average have 32 knitters. Based on what we know about averages, the total number of knitters can be determined by multiplying the average number of knitters in a club, 32, by the number of clubs, 1,200.

   $$\text{average number of members per club} = \frac{\text{sum of members in all clubs}}{1{,}200} = 32$$

   The total number of knitters in all the clubs in town is $32(1{,}200) = 38{,}400$.

3. Now that we have the total number of knitters, we can multiply this amount by the decimal representation of the percentage of knitters that have fewer than 3 cats. As reported in the representative survey, the percentage of people with at least 3 cats is 21.9%, so the percentage of knitters with fewer than 3 cats is 78.1%, which is 0.781 when written as a decimal.

   Multiply 0.781 by 38,400 to find an estimate of how many knitters in all the clubs in town have fewer than 3 cats.

   $$0.781(38{,}400) = 29{,}990.4$$

   No answers match this amount exactly, but answer C is by far the closest.

4. The answer is C.

The key to the survey design questions is to remember that **the sample set used in a well-structured survey is random enough to make accurate predictions about an entire population**. It is important that you can identify the proper population to which survey results can be applied based on randomness (or lack thereof) in the sample set.

**When you are asked about potential problems with how a survey was conducted, the answer will most likely be that the survey participants were not chosen randomly enough**.

If we want to know how residents of a town feel about a proposed law, we need to ask as diverse a group of people in the town as possible—a group that, while it is necessarily a subset of the population, is nonetheless representative of the makeup of the population overall. As soon as too many people in a sample have anything else in common aside from living in that town, we are really performing a study of people in the town *who also share another trait*. For instance, if we went to the park and asked residents there how they felt about a law that will be voted on by the entire town, our sample population is skewed towards people who go to the park in that town, and so we have conducted a survey of parkgoers in the town rather than a survey of the residents of the town as a whole.

**Example 17.4-2**

**2** 1600.io 17.4

A researcher conducted a survey to determine whether people in a certain large town prefer watching robot battles on television or attending robot battles in person. The researcher asked 249 people who visited a local coffee shop on a Monday, and 8 people refused to respond. Which of the following factors makes it least likely that a reliable conclusion can be drawn about the robot-battle-watching preferences of all people in the town?

A) The number of people who refused to respond

B) Where the survey was conducted

C) Population size

D) Sample size

**Solution**

1. A sample size of 249 is sufficient to have an accurate survey, regardless of the size of the town's population. Eliminate choices C and D.
2. If a vast majority of the people surveyed refused to answer, then it might be a problem because we would have very few responses, but only 8 people out of the 249 surveyed felt that their robot-battle-watching preferences were too sensitive to share with researchers. Eliminate choice A.
3. One thing we know for sure is that all of the people surveyed visited the same coffee shop on a Monday, which means that they have another trait in common, aside from the fact that they were all in that large town. Therefore, the location of the survey is the factor that makes it least likely that we will draw a reliable conclusion about all of the people in the town: essentially the survey tells us about the robot-battle-watching preferences of people in this town *who go to this coffee shop on a Monday*.
4. The answer is B.

When asked what can be inferred from survey results, make sure that you keep in mind who was surveyed and **apply the results ONLY to the population that shares the traits that are common to everyone in the sample set**. Just as important is to **avoid overly strong or definitive claims**.

For example, a study was conducted on a certain hair growth supplement. The study shows that 95% of the people who used the supplement grew their hair back, while only 5% of the people who didn't use the supplement had hair regrowth. While the results are promising, we cannot say that the supplement *was the cause* of the hair growth. All we can say is that the people who used the supplement were more likely to regrow their hair than those who didn't. There is a subtle but important difference between correlation (two things both being present or occurring) and causation (one thing actually *causing* the other thing), and it takes a lot more to prove causation. **It is safer to choose more general or modest conclusions rather than absolute or unqualified ones when asked to make inferences about a survey**.





**Example 17.4-3**

**3** 1600.io 17.4

To determine if smelling lavender oil reduces stress for men, researchers interviewed a random sample of 4,800 men who had no history of stress. Study participants were identified as either regular or occasional lavender oil sniffers. Ten years later, researchers interviewed the men again. They found that the proportion of men who experienced frequent stress was significantly lower for men identified as regular lavender oil sniffers. Which of the following is the most appropriate conclusion of the study?

A) Lavender oil causes a reduction in stress for men but not necessarily for women.

B) Lavender oil causes a reduction in stress for men and women.

C) There is an association between lavender oil use and stress for men, but it is not necessarily a cause-and-effect relationship, and the association may not exist for women.

D) There is an association between lavender oil use and stress for men and women, but it is not necessarily a cause-and effect relationship.

**Solution**

1. Only men are surveyed, so we cannot make any claims one way or the other about the association between smelling lavender oil and its effect on women's stress. Eliminate answers B and D.
2. We do not know what other factors could be in play in regards to the habits of the men surveyed. Perhaps those with less stress who regularly sniffed lavender oil are the type of people who also tend to choose low-stress jobs. Therefore, we cannot claim a cause-and-effect relationship because the study, as far as we know, did not account for other possible factors that could reduce stress. Eliminate choice A as well.
3. Notice that choice C, the only remaining choice, simply states that there is a correlation between lavender oil sniffing and lower stress levels in men (not women), but stops short of saying that lavender oil sniffing is the *cause* of the lower stress level.
4. The answer is C.

**Sample size** is asked about less often on the test because we usually aren't given the precise size of the group we are trying to study, but sample size does play a role in the accuracy of a survey.

The **margin of error** accounts for the potential difference between **actual values for an entire population** and the **predicted values based on a sample** of the whole population. The margin of error helps describe a range (relative to a predicted value based on a sample) of values in which the true value **for the whole population** is *likely* to be found.

It is unlikely that the average value of a sample will be *exactly* the same as the average value of the entire population, but as the sample size gets closer to the population size, the margin of error gets lower (the true value for the whole population is limited to a smaller range of values around the value predicted based on the sample). **A small margin of error means that the sample size was sufficiently large to predict the results for the whole population fairly accurately**. If a random survey finds that 70% of the residents are in favor of a new law, with a margin of error of 3.6%, then we know that the true percentage of residents in favor of the law is *most likely* between 66.4% (which is 3.6 percentage points less than 70%) and 73.6% (which is 3.6 percentage points above 70%).

It is essential to understand that the **margin of error applies to the population, not to the sample** from which the measurement was derived. It does *not* represent any uncertainty about the measurement taken of the sample group (the values are known exactly for the sample), nor about variations from sample to sample, but rather about **how accurately the measurement from the sample group can be applied to the entire population from which the sample group was drawn**.

**Example 17.4-4**

**4** 1600.io 17.4

An ecologist selected a random sample of 50 beavers from a river and found that the mean weight of the beavers in the sample was 42 pounds (lbs), with an associated margin of error of 3.1 lbs. Which of the following is the best interpretation of the ecologist's findings?

A) All beavers in the sample have a weight between 38.9 lbs and 45.1 lbs.

B) Most beavers in the river have a weight between 38.9 lbs and 45.1 lbs.

C) Any weight between 38.9 lbs and 45.1 lbs is a plausible value for the mean weight of the beavers in the sample.

D) Any weight between 38.9 lbs and 45.1 lbs is a plausible value for the mean weight of beavers in the river.

**Solution**

1. The margin of error, unlike the range, doesn't provide any information at all about the individual weights of the beavers in the sample. The margin of error tells us about the likely window for the mean weight of *all* the beavers in the river. Eliminate choices A and B.
2. Choice C seems good at first, but the mean weight of the beavers in the sample is known exactly, not estimated. The ecologist weighed and averaged the weights of the 50 beavers in the sample, so the mean weight of those beavers is an exact figure. Remember, the margin of error applies to the accuracy of the measurement from the sample group *when applied to the entire population*. Eliminate choice C.
3. Choice D correctly identifies that the margin of error provides a window of values in which the true mean of the beaver weights across the entire population is likely to fall. Unlike choice C, choice D correctly generalizes the results of the random sample to make a statement about all of the beavers in that river.
4. The answer is D.





**Section 17.4 Suggested Problems from Real Tests**

- Test 2-C-13
- Test 2-C-20
- Test 3-C-15
- Test 4-C-4
- Test 5-C-15
- Test 5-C-22
- Test 6-C-7
- Test 6-C-21
- Test 7-C-4
- Test 8-C-24
- May 2017-C-14
- May 2017-C-25
- Mar 2018-C-4
- Mar 2019-C-15
- Mar 2019-C-18
- Apr 2019-C-5
- Apr 2019-C-8
- May 2019 (US)-C-18
- May 2019 (US)-C-26
- May 2019 (Int)-C-10
- May 2019 (Int)-C-15
- Oct 2019-C-10
- Oct 2019-C-27

View related real-test problems in SMTEx

SMTEx

## Section 17.4 Practice Problems

**1** 1600.io 17.4

A store manager asked a random sample of shoppers how long they shopped. Based on the responses, the estimated mean was found to be 12.5 minutes, with an associated margin of error of 2.5 minutes. Which of the following is the best conclusion from these data?

A) It is likely that all shoppers shopped for between 10 and 15 minutes.

B) It is likely that most shoppers shopped for 12.5 minutes.

C) It is plausible that the mean time spent shopping by all shoppers is between 10 and 15 minutes.

D) It is not possible that any shopper shopped for less than 10 minutes.

**2** 1600.io 17.4

Students in a school were surveyed to determine whether they are satisfied with the mattresses in the dorms. A random sample of 100 students was selected. All 100 responded, and 61% said they are satisfied. Based on this information, which of the following statements must be true?

I. Of all the school's students, 61% would say they are satisfied with the mattresses in the dorms.

II. If another random sample of 100 students were surveyed, 61% would say they are satisfied.

A) Neither

B) I only

C) II only

D) I and II

**3** 1600.io 17.4

A market researcher selected 400 people at random from a group of people who indicated that they liked a certain social media influencer. The 400 were given a book written by the influencer and then asked whether they liked or disliked the book. Of those surveyed, 75% said they liked the book. Which of the following inferences can appropriately be drawn from this survey result?

A) At least 75% of readers will like this book.

B) At least 75% of people who use social media will like this book.

C) Most people who like this influencer will like this book.

D) Most people who use social media will like this book.

**4** 1600.io 17.4

To determine whether members of 1600.io would vote in favor of Greg lowering the price of the most popular shirt in his online store, Dan surveyed 50 members who had bought the shirt for full price. He found that 40 of those surveyed reported that they would not vote in favor of the price reduction. Which of the following statements must be true?

A) When the actual vote is taken, 80 percent of the votes will not be in favor of the price reduction.

B) The sampling method is flawed and may produce biased results.

C) The sampling method is not flawed and is likely to produce unbiased results.

D) No prediction should be made about the vote on the proposal because the sample size is too small.

## CHAPTER 17 RECAP

- When trying to find the key properties of a data set, you may find it useful to rewrite the data set in order from least to greatest to help prevent mistakes.
- The **mean** is the average of the values in a data set. The mean is found by dividing the sum of the values in the data set by the size of the data set.

$$\text{Mean} = \frac{\text{Sum of Data Set}}{\text{Size of Data Set}}$$

- The **median** is located in the middle of the data set.

  For an odd-sized data set, the position of the median is found by adding 1 to the size of the data set and then dividing by 2. For example, the median of a data set with 49 values is found in position 25 (the 25th value in the data set) because $\frac{49+1}{2} = \frac{50}{2} = 25$.

  For an even-sized data set, the median is found by averaging the two middle values. The positions of these two values are found by dividing the size of the data set by 2; the result gives the position of the first middle value, and the other middle value is found at the next position in the data set. For example, the median of a data set with 100 values is found by averaging the values of the 50th and 51st values in the data set because $\frac{100}{2} = 50$.
- The **mode** of a data set is the value that appears most frequently in a data set.

  Though a data set *can* have more than one mode, you will never be faced with such a situation on the test where this will matter, so you don't need to concern yourself with that scenario.
- The **range** of a data set is the difference between the largest and smallest value in a data set. Subtract the smallest value from the largest value to find the range.
- Frequency tables and histograms are used to show how many times particular values occur in a data set. From these representations, we are able to calculate all of the key properties of data sets. To find the mean, we need to find the sum of the data, which can be determined by multiplying each value in the data set by its frequency (or the height of its bar) and summing the results, then dividing by the number of samples. The median is found by counting the number of elements until you reach the position of the median (in the middle of the set).
- **Standard deviation** is a measure of how closely clustered the values in a data set are (how close to the mean of the data most of the values are). Tightly clustered data sets will have a lower standard deviation than will data sets that are more spread out.
- In order to draw valid conclusions from surveys, the samples must be sufficiently large (so far there has never been a sample that was too small on any released tests because, regardless of the population size, even seemingly small sample sizes are fairly accurate), and most importantly, truly random. If the subjects in a survey share some trait (other than just being members of the larger population), then the survey's results are restricted to just the population that shares that trait and cannot be applied to the larger population to which those subjects belong.
- The **margin of error** accounts for the potential difference between the true value for an entire population and the value found based on a survey sample. The true value for the whole population is most likely (but not definitely) found inside the margin of error around the value found for the survey.
- Do not make overly definitive claims based on a survey. The results only apply to the population that shares the traits common to those sampled. A cause-and effect relationship is highly unlikely unless you are specifically told that all other variables have been controlled for. You are a lot safer merely making a claim just that there is a relationship between two things.





## Additional Problems

**1** 1600.io 17.1

Percent of Residents Who Watched *Citizen Kane*

| State | Percent of residents |
|---|---|
| State A | 5.4% |
| State B | 12.1% |
| State C | 7.2% |
| State D | 1.6% |
| State E | 2.1% |
| State F | 3.4% |
| State G | 3.3% |

A survey was given to residents of all 50 states asking if they had ever watched the movie *Citizen Kane*. The results from 7 of the states are given in the table above. The median percent of residents who have watched *Citizen Kane* for all 50 states was 3.45%. What is the difference between the median percent of residents who watched *Citizen Kane* for these 7 states and the median for all 50 states?

A) 0.05%

B) 0.15%

C) 1.56%

D) 1.95%

**2** 1600.io 17.1

1, 15, 7, 6, $G$, 3, 8

In the data set shown, $G$ is an integer. If the median of the data set is 6 and $G > 3$, what is a possible value of $G$?

**3** 1600.io 17.1

Ticket Prices by Row Number

| Row number | Ticket price |
|---|---|
| 1–5 | \$40 |
| 6–10 | \$20 |
| 11–20 | \$10 |

The price of a ticket to a wrestling event is based on the row the seat is in, as shown in the table above. A group wants to purchase 20 tickets for the event.

They will purchase 8 tickets for seats in row 1.
They will purchase 2 tickets for seats in row 5.
They will purchase 4 tickets for seats in row 8.
They will purchase 6 tickets for seats in row 15.

What is the average (arithmetic mean) ticket price, in dollars, for the 20 tickets? (Disregard the \$ sign when writing your answer.)

**4** 1600.io 17.1

If $x$ is the average (arithmetic mean) of $m$ and 4, $y$ is the average of $3m$ and 12, and $z$ is the average of $8m$ and 8, what is the average of $x$, $y$, and $z$ in terms of $m$?

A) $m + 6$

B) $2m + 4$

C) $4m + 8$

D) $9m + 12$

**5** 1600.io 17.4

In order to determine if treatment A is successful in improving hearing, a research study was conducted. From a large population of people with poor hearing, 500 participants were selected at random. Half of the participants were randomly assigned to receive treatment A, and the other half did not receive treatment A. The resulting data showed that participants who received treatment A had significantly improved hearing as compared to those who did not receive treatment A. Based on the design and results of the study, which of the following is an appropriate conclusion?

A) Treatment A is likely to improve the hearing of people who have poor hearing.

B) Treatment A improves hearing better than all other available treatments.

C) Treatment A will improve the hearing of anyone who takes it.

D) Treatment A will cause a substantial improvement in hearing.

**6** 1600.io 17.1

| Heights of Cacti (in meters) | | | | |
|---|---|---|---|---|
| 4 | 4 | 5 | 5 | 5 |
| 6 | 6 | 7 | 7 | 7 |
| 7 | 8 | 8 | 9 | 15 |

The table above lists the heights, to the nearest meter, of a random sample of saguaro cacti. The outlier measurement of 15 meters is an error. Of the mean, median, and range of the values listed, which will change the most if the 15-meter measurement is removed from the data?

A) Median

B) Mean

C) Range

D) They will all change by the same amount.

**7** 1600.io 17.4

A study was done on the lengths of different types of snakes in a forest. A random sample of snakes were caught and marked in order to ensure that none were measured more than once. The sample contained 110 northern copperheads, of which 45% were longer than 30 inches. Which of the following conclusions is best supported by the sample data?

A) The average length of all snakes in the forest is approximately 30 inches.

B) The majority of all snakes in the forest are shorter than 30 inches.

C) Approximately 45% of all snakes in the forest are longer than 30 inches.

D) Approximately 45% of all northern copperheads in the forest are longer than 30 inches.

**8** 1600.io 17.4

The members of a state senate wanted to assess the opinions of all state residents about renovating the truck stops on the state highways. The senate surveyed a sample of 2,800 state residents who drive trucks. The survey showed that the majority of those sampled were in favor of the renovations. Which of the following is true about the state senate's survey?

A) The survey sample should have included more residents who drive trucks.

B) The survey sample should have consisted entirely of residents who do not drive trucks.

C) The survey sample is biased because it is not representative of all state residents.

D) It shows the majority of state residents are in favor of the truck stop renovations.





**9** 1600.io 17.2

| | | | | | | |
|---|---|---|---|---|---|---|
| Data set A | 12,550 | 35,790 | 124,360 | 250,280 | 420,510 | 520,300 |
| Data set B | 12,360 | 24,540 | 25,220 | 51,000 | 78,650 | 92,330 |

Which of the following is true about the standard deviations of the two data sets in the table above?

A) The standard deviation of data set A is equal to the standard deviation of data set B.

B) The standard deviation of data set A is larger than the standard deviation of data set B.

C) The standard deviation of data set B is larger than the standard deviation of data set A.

D) There is not enough information available to compare the standard deviations of the two data sets.

**10** 1600.io 17.1

A group of 20 students took a certain quiz. Every student received a score equal to an integer from 1 to 20, inclusive. For the 20 students, the mean score was 11. If more than half of the students received a score greater than 12, which of the following is true about the mean score of the remaining students?

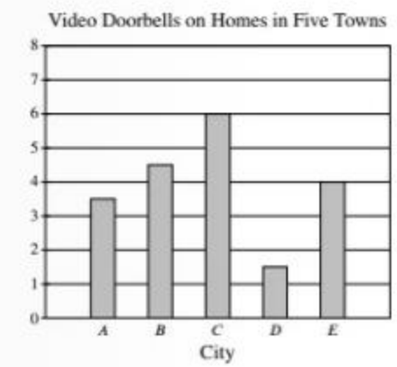
A) It must be greater than 12.

B) It must be between 11 and 12.

C) It must be equal to 11.

D) It must be less than 11.

**11** 1600.io 17.2




The number of homes with video doorbells in 5 towns is shown in the graph above. If the total number of video doorbells is 1,950, what is an appropriate label for the vertical axis of the graph?

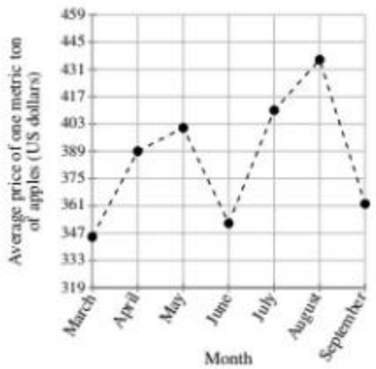
A) Number of doorbells (in tens)

B) Number of doorbells (in hundreds)

C) Number of doorbells (in thousands)

D) Number of doorbells (in tens of thousands)

**12** 17.1




The line graph above shows the average price of one metric ton of apples, in dollars, for each of seven months in 1989.

Which of the following is closest to the median price, in dollars, of the seven recorded prices of one metric ton of apples?

A) 352

B) 362

C) 389

D) 401

**13** 17.1

If $a$ is the mean and $b$ is the median of 15 consecutive odd integers, what is the value of $|b - a|$ ?

**14** 17.2

Of 100 people who bet on a certain basketball game, 74 won more than 0 but less than 1,000 dollars, 25 won between 1,000 and 10,000 dollars, and the remaining gambler won 243,000 dollars. Which of the following statements about the mean and median of the 100 gamblers' winnings is true?

A) The mean and median are equal.

B) The median is greater than the mean.

C) The mean is greater than the median.

D) There is not enough information to determine whether the mean or the median is greater.

**15** 17.4

A sample of 50 accountants was selected at random from a certain company. The 50 accountants completed a survey about the break room, and 13 thought the break room had good snacks. Which of the following is the largest population to which the results of the survey can be applied?

A) The 50 employees who were surveyed

B) All accountants at the company

C) All employees at the company

D) All accountants in the country in which the company is located





**16** 17.1

Bradan is preparing for a juggling tournament. His goal is to juggle an average of at least 180 minutes a day for 4 days. He juggled 150 minutes the first day, 200 minutes the second day, and 220 minutes the third day. Which inequality can be used to represent the number of minutes, $x$, Bradan could juggle on the 4th day to meet his goal?

A) $150 + 200 + 220 \geq x(180)$

B) $150 + 200 + 220 + x \geq 4(180)$

C) $\frac{150 + 200 + 220}{3} + x \geq 180$

D) $\frac{150}{4} + \frac{200}{4} + \frac{220}{4} + x \geq 180$

**17** 17.4

Near the end of a livestreamed makeup tutorial, the influencer invited viewers to respond to a poll on social media that asked,"Do you think I should review more lipstick on my channel?" At the end of the livestream, the influencer reported that 34% responded "Yes," and 60% responded "No." Which of the following best explains why the results are unlikely to represent the sentiments of the population of the United States?

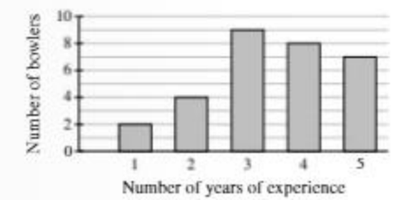
A) The influencer did not allow viewers enough time to respond to the poll.

B) The percentages do not add up to 100%, so any possible conclusions from the poll are invalid.

C) There were not 50% "Yes" responses and 50% "No" responses.

D) Those who responded to the poll were not a random sample of the population of the United States.

**18** 17.2




The graph above shows the distribution of the number of years of experience for 30 bowlers in a bowling league at a particular bowling alley. If a 31st bowler with 4 years of experience is added to the league and to the data set, what will be the effect on the mean and median of the data set?

A) The mean and median will both remain the same.

B) The mean will increase and the median will stay the same.

C) The mean will remain the same and the median will increase.

D) The mean and median will both increase.

**19** 17.4

A survey was conducted using a sample of bowlers selected at random from the Greater Utah Bowling League. The bowlers surveyed were asked to name the manufacturer of their current bowling balls. What is the largest population to which the results of the survey can be generalized?

A) All athletes in the United States.

B) All bowlers in the United States.

C) All bowlers in the Greater Utah Bowling League.

D) All bowlers in Utah.

**Questions 20 and 21 refer to the following information.**

In fall 2016, three separate studies on the chess ability of middle-aged men were conducted in the town of Howell. In each study, every man in a group of middle-aged men played the same chess computer and received a score on the game. The possible scores on the game are the whole numbers from 0 to 50, inclusive. The distribution of the scores for each of the studies is shown in the table below.

| Score range | Study I | Study II | Study III |
|---|---|---|---|
| 0–9 | 20 | 45 | 81 |
| 10–19 | 46 | 87 | 117 |
| 20–29 | 15 | 32 | 49 |
| 30–39 | 10 | 22 | 29 |
| 40–50 | 9 | 14 | 24 |
| Mean score | 18.7 | 19.2 | 17.1 |
| Total number of participants | 100 | 200 | 300 |

The participants for the studies were selected as follows.

- For Study I, 100 middle-aged men were selected at random from all middle-aged men in Howell.
- For Study II, 200 middle-aged men from Howell volunteered to participate.
- For Study III, 300 middle-aged men were selected at random from all middle-aged men in Howell.

No middle-aged man participated in more than one of the studies.

**20** 1600.io 17.2

Which of the following could be the median score in Study III?

A) 8

B) 13

C) 20

D) 31

**21** 1600.io 17.4

The results of which studies can appropriately be generalized to all middle-aged men in Howell in fall 2016?

A) Study II only

B) Studies I and II only

C) Studies I and III only

D) Studies I, II, and III

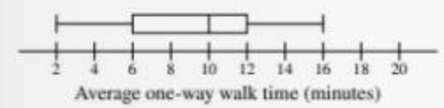




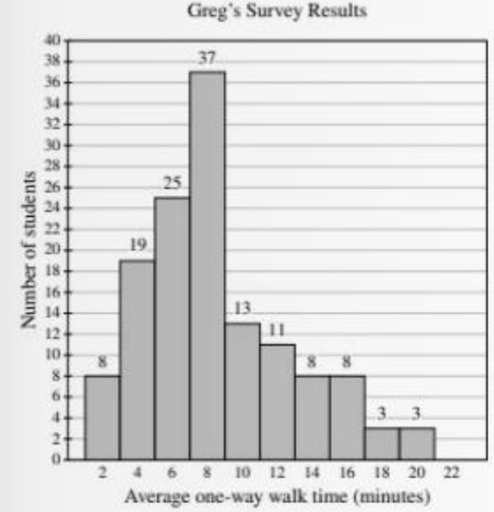
**Questions 22 and 23 refer to the following information.**

For a particular college campus with 1,640 students, Zendaya and Greg each conducted a survey about the average one-way walk times, in minutes, between the students' dorms and math classes. Both Zendaya and Greg selected students at random, emailed surveys, and collected data from the returned surveys. For both surveys, respondents were asked to report their average walk times to the nearest 2 minutes. Zendaya collected data from 300 students, and Greg collected data from 135 students. The results from Zendaya's and Greg's returned surveys are summarized below.

Zendaya's Survey Results




Greg's Survey Results




**22** 1600.io 17.3

If $Z$ is the median walk time of the students who responded to Zendaya's survey and $G$ is the median walk time of the students who responded to Greg's survey, what is the value of $Z - G$ ?

A) 0

B) 2

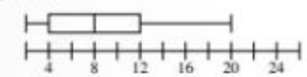
C) 4

D) 8

**23** 1600.io 17.3

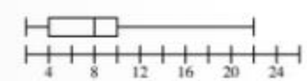
Which of the following box plots could represent Greg's survey data?

A)

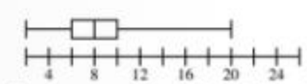



B)

C)




D)



**24** 17.4

For a contest to choose a new soda flavor, consumers could use either social media or a text message to vote for their favorite of two flavors. The flavor receiving more than 50% of the vote won. As estimated 5% of consumers voted, and 80% of the votes were cast on social media. Flavor 2 earned 60% of the votes cast on social media and 5% of the votes cast using a text message. Based on this information, which of the following is an accurate conclusion?

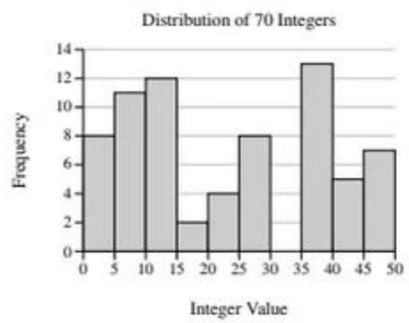
A) If all consumers had voted, Flavor 2 would have won.

B) Consumers voting by social media were likely younger than consumers voting by text message.

C) Consumers voting by social media were more likely to prefer Flavor 2 than were consumers voting by text message.

D) If all consumers who voted had voted by social media instead of by text message, Flavor 2 would have won.

**25** 17.2




The histogram summarizes the distribution of a data set composed of 70 integers. The first bar represents the number of integers that are at least 0 but less than 5. The second bar represents the number of integers that are at least 5 but less than 10, and so on. What is a possible value of the median of the data set?

**26** 17.2

Two independent surveys asked random samples of 200 bowlers about the time they spend bowling each day. The results of the surveys are detailed in the table below.

Daily Bowling Time

| Survey | Mean (hours) | Standard deviation (hours) |
|---|---|---|
| A | 3.6 | 0.5 |
| B | 3.2 | 0.5 |

Which statement is true based on the results of these surveys?

A) There is greater variation in the distribution of the times people bowl in Survey A.

B) There is greater variation in the distribution of the times people bowl in Survey B.

C) The variation in the distribution of the times people bowl is the same in both surveys.

D) It is impossible to determine the variations in the distribution of the times people bowl because the means are different.

**27** 17.2

| Value | Frequency |
|---|---|
| 1 | $a$ |
| 2 | $3a$ |
| 3 | $5a$ |
| 4 | $3a$ |
| 5 | $a$ |

The frequency distribution above summarizes a set of data, where $a$ is a positive integer. How much greater is the mean of the set of data than the median?

A) 0

B) 1

C) 3

D) 5





**28** 17.2

A bird hatchery has three cages for holding birds before they are introduced to the wild. Eight birds less than 3 months old are placed in cage A. Nine birds that are at least 3 months old but no more than 8 months old are placed in cage B. Ten birds older than 8 months are placed in cage C. Which of the following could be the median of the ages, in months, of these 27 birds?

A) 1.5

B) 8

C) 8.5

D) 11

**29** 17.4

A box containing 1,000 buttons of assorted colors is purchased from a sewing store. To estimate the percent of blue buttons in the box, a sample of buttons is selected at random. The percent of blue buttons in the box was estimated to be 26% with an associated margin of error of 4%. If $b$ is the actual number of blue buttons in the box, which of the following is most plausible?

A) $b < 220$

B) $40 < b < 260$

C) $220 < b < 300$

D) $b > 300$

**30** 17.4

A media outlet wants to predict how residents of Keansburg, New Jersey will react to a new soda tax proposed in the city council. Which of the following study designs is most likely to provide reliable results for the media outlet?

A) Interviewing a group of retirees randomly selected from a large retirement home in Keansburg

B) Surveying a group of 2,000 randomly selected New Jersey residents

C) Mailing a questionnaire to each of 120 randomly selected residents of Keansburg

D) Surveying a group of 150 randomly selected Keansburg residents

**31** 17.4

Two hundred 1600.io members will be selected to participate in a survey about selecting a new item for the online store. Which of the following methods of choosing the 200 members would result in a random sample of members of the site?

A) Obtain a numbered list of all 1600.io members. Use a random number generator to select 200 members from the list. Give the survey to those 200 members.

B) Obtain a list of all 1600.io members sorted alphabetically. Give the survey to the last 200 members on the list.

C) Tell all 1600.io members that volunteers are needed to take the survey. Give the survey to the first 200 members who volunteer.

D) Obtain a list of all 1600.io members who are attending a live event online. Give the survey to the first 200 members on the list.

**32** 17.1

A survey was taken of the weight of pigs in a state, and it was found that the mean pig weight was 850 pounds and the median pig weight was 650 pounds. Which of the following situations could explain the difference between the mean and median pig weights in the state?

A) The pigs have weights that are close to each other.

B) There are a few pigs that weigh much less than the rest.

C) There are a few pigs that weigh much more than the rest.

D) Many of the pigs have weights between 650 pounds and 850 pounds.

**33** 17.1

| | Weights (pounds) | | | | | |
|---|---|---|---|---|---|---|
| Arun | 1.6 | 1.5 | 4.3 | 3.5 | 1.5 | 1.4 |
| Carolina | $x$ | 3.5 | 1.4 | 1.9 | 4.0 | 1.6 |

Arun and Carolina each collected six bags of recyclables, and the weights of the bags are shown in the table above. The mean weight of the bags Carolina collected is 0.3 pounds more than the mean of the weights of the bags Arun collected. What is the value of $x$ ?

**34** 17.4

A sample of 800 bowlers was selected at random and asked how much time they spend polishing their bowling balls each day. Of the bowlers selected, 340 spend less than 1 hour polishing their bowling balls each day. If the conclusion was drawn that "approximately 11.2 thousand bowlers spend less than 1 hour polishing their bowling balls each day," which of the following is closest to the population, in thousands, of bowlers?

A) 4.76

B) 11.2

C) 26.35

D) 42.50

**35** 17.4

A researcher surveyed a random sample of retirees from a large retirement home about how often they see their grandchildren. Using the sample data, the researcher estimated that 14% of the retirees in the population saw their grandchildren at least once per year. The margin of error for this estimation is 3%. Which of the following is the most appropriate conclusion about all retirees at the retirement home, based on the given estimate and margin of error?

A) At least 14% but no more than 17% of the retirees see their grandchildren at least once per year.

B) The researcher is between 11% and 17% sure that most retirees see their grandchildren at least once per year.

C) It is unlikely that less than 14% of the retirees see their grandchildren at least once per year.

D) It is plausible that the percentage of retirees who see their grandchildren at least once per year is between 11% and 17%.





**Questions 36 and 37 refer to the following information.**

An animal shelter owner chose 400 people at random from each of two towns and asked each person how many pets he or she has. The results are shown in the table below.

People's Pet Survey

| Number of pets | Podunk | Swampa |
|---|---|---|
| 0 | 20 | 20 |
| 1 | 40 | 30 |
| 2 | 80 | 60 |
| 3 | 100 | 110 |
| 4 | 160 | 180 |

There are a total of 1,600 residents in Podunk and 2,000 residents in Swampa.

**36** 17.2

What is the median number of pets for all the people surveyed?

A) 1

B) 2

C) 3

D) 4

**37** 17.4

Based on the survey data, which of the following most accurately compares the expected total number of people with 0 pets in the two towns?

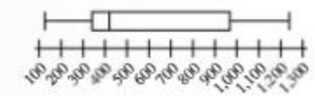
A) The total number of people with 0 pets in Swampa is expected to be 400 more than in Podunk.

B) The total number of people in Swampa with 0 pets is expected to be 20 more than in Podunk.

C) The total number of people with 0 pets in Podunk is expected to be 20 more than in Swampa.

D) The total number of people with 0 pets is expected to be equal in the two towns.

**38** 17.3




Bowling Alleys

The number of bowling alleys in 20 states in America is summarized in the box plot above. Which of the following is closest to the median number of bowling alleys?

A) 341

B) 420

C) 968

D) 1,239

**39** 17.4

| Sample | Percent in favor | Margin of Error |
|---|---|---|
| A | 68% | 3.5% |
| B | 61% | 1.2% |

The results of two random samples of votes for a new shirt design are shown above. The samples were selected from the same population, and the margins of error were calculated using the same method. Which of the following is the most appropriate reason that the margin of error for sample A is greater than the margin of error for sample B?

A) Sample A had a larger sample size.

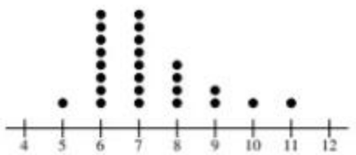
B) Sample A had a smaller sample size.

C) Sample A had a higher percentage of favorable responses.

D) Sample A has a smaller number of votes that could not be recorded.

40 1600.io 17.2

Age of 25 Bowlers When They Began Bowling




The dot plot above gives the ages, in years, at which 25 bowlers began bowling. Which of the following is true about the mean and median of the data?

A) The mean is less than the median.

B) The mean is greater than the median.

C) The mean is equal to the median.

D) The relationship between the mean and the median cannot be determined from the dot plot.

41 1600.io 17.3

| List A | List B |
|---|---|
| 20 | 30 |
| 30 | 40 |
| 40 | 40 |
| 50 | 50 |
| 60 | 50 |
| 70 | 60 |

The table above shows two lists of numbers. Which of the following is a true statement comparing list A and list B?

A) The means are the same, and the standard deviations are different.

B) The means are the same, and the standard deviations are the same.

C) The means are different, and the standard deviations are different.

D) The means are different, and the standard deviations are the same.

42 1600.io 17.4

A pollster recently surveyed 3,000 people who were selected at random from a large country and asked each of the adults, "Are you satisfied with the quality of health care in the country?" Of those surveyed, 56 percent responded that they were satisfied with the quality of health care in the country. Based on the results of the survey, which of the following statements must be true?

I. Of all adults in the country, 56 percent are satisfied with the quality of health care in the country.

II. If 3,000 adults selected at random from a different country were surveyed, 56 percent of them would report they are satisfied with the quality of health care in the country.

III. If another 3,000 adults selected at random from the country were surveyed, 56 percent of them would report they are satisfied with the quality of health care in the country.

A) None

B) II only

C) I and II only

D) I and III only





# Answer Key

**SkillDrill 17.1-1**

1. Mean = $4.\overline{2}$
   Median = 2
   Mode = 2
   Range = 9
2. Mean = 2.125
   Median = 1
   Mode = 6
   Range = 19

**SkillDrill 17.1-2**

1. $x = 12$
2. $x = 2$

**Section 17.1 Practice Problems**

1. 50
2. B
3. D
4. D

**SkillDrill 17.2-1**

1. Mean = 3.05
   Median = 3
   Mode = 3
   Range = 4
2. Mean = 31.15
   Median = 30.5
   Mode = 37
   Range = 52

**Section 17.2 Practice Problems**

1. A
2. A
3. A

**Section 17.3 Practice Problems**

1. C
2. D

**Section 17.4 Practice Problems**

1. C
2. A
3. C
4. B

**Additional Problems**

1. A
2. 4, 5, or 6
3. 27
4. B
5. A
6. C
7. D
8. C
9. B
10. D
11. B
12. C
13. 0
14. C
15. B
16. B
17. D
18. D
19. C
20. B
21. C
22. B
23. D
24. C
25. 20, 20.5, 21, 21.5, 22, 22.5, 23, 23.5, or 24
26. C
27. A
28. B
29. C
30. D
31. A
32. C
33. 3.2 or $\frac{16}{5}$
34. C
35. D
36. C
37. B
38. B
39. B
40. B
41. A
42. A

# Unit Conversions 18

## 18.1 Simple Conversions with Proportions

Many word problems that are linked to real world situations are focused solely on **units of measurement** and your ability to convert a measurement given in one set of units into another set of units by using proportions. For example, if you are told that George is 76 inches (in) tall and asked to find his height in centimeters (cm), then, based on the fact that 1 in = 2.54 cm (the same object that measures 1 in on an English/US/Imperial system ruler will measure 2.54 cm on a metric ruler), we can set up a proportion in order to find the value of $h_{cm}$, George's height in cm. The ratio of George's height in centimeters, $h_{cm}$, to his height in inches, $h_{in.}$ must be equal to the ratio of 2.54 centimeters to 1 inch. (As usual, we advise arranging the proportion so that the unknown quantity—here, $h_{cm}$—is in the numerator of the first [left-side] fraction; that eliminates the need to involve that unknown in any arithmetic, it ensures that the unknown never switches sides in the proportion equation, it allows the unknown to be evaluated with a simple chain calculation, and it produces a final equation with the unknown on the left, providing a natural solution "sentence.")

$$\frac{h_{cm}}{h_{in.}} = \frac{2.54\text{ cm}}{1\text{ in}}$$

$$\frac{h_{cm}}{76\text{ in}} = \frac{2.54\text{ cm}}{1\text{ in}}$$

Now that the proportion is set up, we can multiply both sides of the equation by 76 in in order to solve for $h_{cm}$ (the units of inches will cancel).

$$h_{cm} = 76\,\cancel{\text{in}}\left(\frac{2.54\text{ cm}}{1\,\cancel{\text{in}}}\right)$$

$$h_{cm} = 193.04\text{ cm}$$

**Units will cancel each other and exponentiate just like any constants or variables** (units are not actually variables or constants in the truest sense of the words—they are reference measurements—but for our sake, we can just treat them as if they are constants that exponentiate). As we saw above, dividing centimeters by centimeters cancels out the units. Areas are measured in squared units—if an index card measures 3 in × 5 in, then its area (found by multiplying the length by the width) is equal to 15 $\text{in}^2$.

There are two broad classes of unit conversions. The first, an example of which is discussed above, involves two different units of measure that pertain to **the same underlying type of quantity**: inches and centimeters are both measures of length. Other examples include different currencies which represent money, units of weight such as kilograms and pounds, and so on.

The second type of conversion involves units of **different types of quantities that nonetheless have a specified linear relationship**. For example, if we know that a car can travel 25 miles for each gallon of fuel consumed, we can convert between gallons and miles, or from miles to gallons, even though miles and gallons measure different things. If we also know the price in dollars of a gallon of fuel, we now can convert among cost in dollars, amount of fuel in gallons, and distance in miles.

**The procedures for performing both of these types of conversions is the same**; don't be confused by seeing situations where you're asked to convert among dissimilar types of measurements. **As long as you know the conversion ratios and the units that apply to the numerators and denominators of each, you can perform any needed conversion**.

**SkillDrill 18.1-1**

**Directions:** Use a proportion to convert from one unit to another unit using the following unit conversions. Round or truncate answer to two decimal places when applicable:

- Length: 1 ft = 30.48 cm
- Length: 1 mi = 5,280 ft
- Weight: 1 kg = 2.20 lbs
- Money: \$1 = €0.84
- Volume: 1 L = 0.264 gal
- Fuel Cost: \$3/gal
- Fuel Efficiency: 25 mi/gal

Challenge problems are boxed.

**1.** 20 ft to cm **2.** 6 kg to lbs **3.** \$4.60 to € **4.** 7.54 L to gal

**5.** 100 cm to ft **6.** 1 lb to kg **7.** €35.36 to \$ **8.** 11 gal to L

**9.** If Calvin drives 125 mi in his car, which has the fuel efficiency shown above, how much fuel did Calvin use for this trip?

**10.** For the same trip Calvin made in the previous problem, if the cost of fuel is as shown above, how much did the fuel cost Calvin, in dollars?

**11.** If Calvin took his car to Europe and bought €30 of fuel (at the same cost as shown above), how many liters of fuel did he purchase?

**12.** If Calvin took his car (which has the fuel efficiency shown above) to Europe and bought €30 of fuel (at the same cost as shown above), how many miles could he drive with the fuel he purchased?

**Example 18.1-1**

**1** 1600.io 18.1

BTU and joules are units of measure of energy. They are directly proportional such that 2 BTU is equal to 2,110 joules. How much energy, in joules, is equal to 3 BTU?

**Solution**

1. Set up a proportion showing that the ratio of the energy in joules, $E_J$, to the energy in BTU, $E_{BTU}$, is constant. Use the energy in joules (which is what we're asked to find) as the numerator on both sides of the proportion to streamline solving.





$$\frac{E_J}{E_{BTU}} = \frac{2{,}110\,J}{2\,BTU}$$

$$\frac{E_J}{3\,BTU} = \frac{2{,}110\,J}{2\,BTU}$$

2. Multiply both sides of the equation by 3 BTU in order to solve for $E_J$ (the units of BTU will cancel).

$$\frac{E_J}{3\,BTU} = \frac{2{,}110\,J}{2\,BTU}$$

$$E_J = 3\,\cancel{BTU}\left(\frac{2{,}110\,J}{2\,\cancel{BTU}}\right)$$

$$E_J = \frac{6{,}330\,J}{2}$$

$$E_J = 3{,}165\,J$$

3. The answer is 3,165.

**Section 18.1 Suggested Problems from Real Tests**

- Test 2-NC-3
- Test 2-C-32
- Test 3-NC-1
- Test 3-C-37
- Test 4-C-3
- Test 5-C-9
- Test 5-C-31
- Test 6-C-9
- Test 7-C-10
- Test 8-C-37
- Test 9-C-2
- Mar 2018-C-20
- Apr 2018-C-3
- Apr 2018-C-15
- May 2018-C-9
- May 2018-C-14
- May 2019 (Int)-C-8
- Oct 2019-C-23
- Mar 2020-C-6
- Mar 2021-C-2
- Mar 2021-C-36
- Apr 2021-C-6
- Apr 2021-C-7
- Apr 2021-C-25
- May 2021 (US)-C-1
- May 2021 (US)-C-3
- May 2021 (Int)-C-12
- May 2021 (Int)-C-21

**View related real-test problems in SMTEx**

SMTEx

## Section 18.1 Practice Problems

**1** 1600.io 18.1

The load capacity of a certain clothes dryer is 15 kilograms. What is the approximate load capacity of the same clothes dryer in pounds?
(1 kilogram = 2.2046 pounds)

A) 2.2
B) 6.8
C) 12.8
D) 33.1

**2** 1600.io 18.1

In normal weather conditions, a particular type of electric car motor uses an average of 4.4 kilowatt-hours of energy per mile driven. The distance from Chicago to New Orleans is about 925 miles. Approximately how many kilowatt-hours of energy will the car use for a trip from Chicago to New Orleans in normal weather conditions?

A) 200
B) 400
C) 1,800
D) 4,000

**3** 1600.io ? 18.1

If $h$ hours and 25 minutes is equal to 205 minutes, what is the value of $h$ ?

**4** 1600.io ? 18.1

Maui spent 25% of his 10-hour workday browsing social media. How many minutes of his workday did he spend browsing social media?

A) 25

B) 90

C) 120

D) 150

## 18.2 Factor-Label Method

For some problems, you have to perform multiple conversions in order to complete a problem. For these problems, you can complete a series of single proportions in order to arrive at the final, desired units, or you can set up a string of ratio factors in order to systematically cancel the units until the desired units are achieved. This process is called the **Factor-Label Method** or **Dimensional Analysis**, and it can be used to convert units or simply to check that the units used in a problem will eventually result in a quantity that is measured in the correct units (you may have used this method in Chemistry or Physics classes and hated it).

Suppose we wanted to convert from inches to meters, but we do not have a direct conversion. All we know is that 1 in = 2.54 cm and that 100 cm = 1 m. We can string ratio factors together to cancel inches first and then cancel centimeters, leaving us with units of meters.

$$76\,\text{in}\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)\left(\frac{1\,\text{m}}{100\,\text{cm}}\right) = 1.93\,\text{m}$$

**When you need to use a calculator, make sure to chain all of the operations together (either in one long string of multiplication and division, making careful use of parentheses, or by using the "ANS" key to continue from one stage of the process to the next) due to the rounding that occurs when the calculator is asked to display intermediate results**. The cumulative error introduced due to rounding at each step in the process can affect the results just enough to produce an incorrect answer.

Most of the information given to you in these problems is necessary, but occasionally there are pieces of information that will not be used. In order to determine what information is useful and what is not, you need to assess whether or not the units of each piece of information will help you cancel units and arrive at the final, desired units.

**To find a starting point, we must, oddly enough, pick out information that tells us about what the end goal of a situation is**. Once we know what the desired outcome is in one set of units, we can begin converting those units into the units that we are asked for. For example, if we're told that someone wants to know how many fewer miles they need to drive to save a certain amount of money on gas (our end goal is saving an amount of money, in dollars), then we can use that dollar amount as our starting point, and string together factors that will cancel the units appropriately. If someone wants to see how long it will take to read a book (the end goal is reading one book, which could be measured in pages or chapters or words), we can start by looking at the length of the book and use measurements of reading speed to convert the measure of book length into a measure of reading time.

Once you have found a starting point, all you need to do is string together factors that will cancel one type of unit and introduce a new type of unit that can in turn be cancelled by another factor until the final units are found. You might find it helpful when planning your approach to first write out the sequence of units that will connect the starting value to the desired final value; for example, if you are asked for the cost of driving 175 miles in a car that goes 25 miles per gallon of fuel, and fuel costs \$3 per gallon, your conversion plan would likely be

miles ⇒ gallons ⇒ dollars





and you can then construct the conversion factor chain as

$$175\,\text{miles}\left(\frac{1\,\text{gallon}}{25\,\text{miles}}\right)\left(\frac{\$3}{1\,\text{gallon}}\right)$$

**SkillDrill 18.2-1**

**Directions:** J. Ernest is writing a mathematics textbook, and he has to write both the instructional material and a set of questions for each chapter in the book. Using the (wildly optimistic) information about J. Ernest's writing habits in the table below, find the desired information based on the additional information you are told. Round or truncate to two decimal places when applicable.

| | |
|---|---|
| Average words per chapter (excluding questions) | 12,320 |
| Average typing speed (words per minute) | 45.3 |
| Questions per chapter | 40 |
| Average time to write a question (minutes) | 8.6 |
| Daily time spent writing (hours) | 10 |

1. How many days, working his average number of hours, will J. Ernest take to write 4 chapters (without questions)?
2. Working his average number of hours, how many complete questions can J. Ernest write in 3 days?
3. When he's super hyped up, J. Ernest can think and write significantly faster than usual. In one 1.5–day span (working the usual 10 hours per day), he wrote 6 average-length chapters (without questions). During this productive period, what was his average typing speed in words per second?
4. J. Ernest has a backlog of 10 chapters for which to write questions. Working at his average rate, how many work days will it take J. Ernest to write the questions?

**Example 18.2-1**

**1** 1600.io 18.2

| | |
|---|---|
| Number of hours Kerry wrote textbook per day | 4 |
| Number of chapters in the textbook | 24 |
| Number of pages in the textbook | 320 |
| Number of words per chapter (average) | 4,562 |
| Number of images per chapter | 11 |
| Number of words Kerry writes per minute | 20 |

Kerry wrote a textbook. The table above shows information about the textbook, Kerry's writing speed, and the amount of time she spent writing the textbook each day. If Kerry wrote at the rates given in the table, which of the following is closest to the number of days it took Kerry to write the entire textbook?

A) 1

B) 23

C) 92

D) 300

**Solution**

1. First, we need to identify the key detail that tells us about the length of the book. This is a good starting point because it's essential to have a measure of the length of the book when determining how long it took to write. We know that the book is 320 pages long, but none of the other rows in the table have units of pages at all. Therefore, we cannot convert the length of the book in pages into any other useful measures. However, we can also see that the book contains 24 chapters, which is another way to measure the length of a book, and there are other rows in the table that contain data measured in units based on chapters.

2. Looking through the table, we need to find data that is measured in some units per chapter so that we can convert chapters into some other unit. In the fourth row, we can see that there are an average of 4,562 words per chapter, so multiply the number of chapters by the ratio of words to chapters in order to cancel units of chapters and get units of words. This will tell us how many words are in the entire book.

$$24\,\text{chapters}\left(\frac{4{,}562\text{ words}}{1\text{ chapter}}\right)$$

3. Now we need to convert units of words into some other units by finding data that is measured in some units (preferably time) per word. In the final row of the table, we see that Kerry writes 20 words per minute. Instead of using the factor $\frac{20\text{ words}}{1\text{ minute}}$, we should flip the fraction so that the units of words will cancel. This will tell us how many minutes it took Kerry to write the book.

$$24\,\text{chapters}\left(\frac{4{,}562\text{ words}}{1\text{ chapter}}\right)\left(\frac{1\text{ minute}}{20\text{ words}}\right)$$

4. Now we have to convert the number of minutes into hours to find out how many hours it took Kerry to write the book using the factor $\frac{1\text{ hour}}{60\text{ minutes}}$. The units of minutes will cancel.

$$24\,\text{chapters}\left(\frac{4{,}562\text{ words}}{1\text{ chapter}}\right)\left(\frac{1\text{ minute}}{20\text{ words}}\right)\left(\frac{1\text{ hour}}{60\text{ minutes}}\right)$$

5. Be careful at this final step of converting the number of hours it took to write the book into the number of days it took to write the book. We are not going to use the fact that 24 hours = 1 day because Kerry did not spend whole days writing. We need to use the factor $\frac{1\text{ day}}{4\text{ hours}}$ because Kerry only spent 4 hours each day writing. Carry out the multiplication and division of the factors to find the number of days it took Kerry to write the textbook.

$$24\,\text{chapters}\left(\frac{4{,}562\text{ words}}{1\text{ chapter}}\right)\left(\frac{1\text{ minute}}{20\text{ words}}\right)\left(\frac{1\text{ hour}}{60\text{ minutes}}\right)\left(\frac{1\text{ day}}{4\text{ hours}}\right)$$

$$\frac{24(4{,}562)(1)(1)(1)}{20(60)(4)}\text{days}$$

$$\frac{109{,}488}{4{,}800}\text{days}$$

$$22.81\text{ days}$$

6. Round the number of days to 23 to find that Kerry took about 23 days to write the textbook.

7. The answer is B.





**Notes**

If you decided to plan out the conversion chain before constructing the actual computation, you would have written

chapters ⇒ words ⇒ minutes ⇒ hours ⇒ days

Notice that table rows 3 and 5 were never used in the calculation because they provided no useful unit conversions.

This methodology can be overwhelming to students who have horrifying flashbacks to stoichiometry problems in Chemistry class, so we will present another example. This represents a problem type that requires students to write an equation and strategically disregard some of the information given to them.

The first solution to this problem is actually based on a single proportion, but setting up the proportion takes a bit more thought than do the problems in the first section of this chapter.

**Example 18.2-2**

**2** 1600.io 18.2

Chris drives an average of 150 kilometers each week. His car can travel an average of 40 kilometers per liter of petrol. Chris would like to reduce his weekly expenditure on petrol by €10. Assuming petrol costs €3 per liter, which equation can Chris use to determine how many fewer average kilometers, $k$, he should drive each week?

A) $\frac{40}{3}k = 140$

B) $\frac{40}{3}k = 10$

C) $\frac{3}{40}k = 140$

D) $\frac{3}{40}k = 10$

**Solution 1**

1. We can set up a simple proportion to solve this problem, based on the cost per km of driving, but we first have to relate two of the measurements that are not presented as if they are directly connected.

2. Since petrol costs 3 euros per liter and Chris's car can go 40 kilometers on one liter of petrol, Chris can drive 40 km for a cost of €3. We can set up a proportion to see how many kilometers, $k$, would cost him €10, the amount of money he wants to save.

$$\frac{€3}{40\text{ km}} = \frac{€10}{k\text{ km}}$$

3. Multiply both sides of the equation by $k$ in order to match the format of the given answer choices (we can drop the units as well because they do not affect the answer choices).

$$\frac{€3}{40\text{ km}} = \frac{€10}{k\text{ km}}$$

$$\frac{3}{40}k = 10$$

4. The answer is D.

**Solution 2**

1. First we need to identify the key detail that will allow us to start converting units. We know that Chris wants to end up saving €10 per week, so that will be our starting data.
2. The only other piece of information that we can use to cancel units of € is that petrol costs €3 per liter (L), so multiply €10 by the factor $\frac{1\text{ L}}{€3}$ in order to cancel units of € and see how many fewer liters of petrol Chris must use in order to save €10.

$$\cancel{€}10\left(\frac{1\text{ L}}{\cancel{€}3}\right)$$

3. Now that we know how many liters of petrol fewer Chris is using per week, we can convert this into the number of kilometers fewer he should drive each week. The only other information that we can use to cancel units of liters and get units of kilometers is his car's fuel efficiency, which is 40 kilometers per liter of petrol. Multiply by the factor $\frac{40\text{ km}}{1\text{ L}}$ in order to cancel units of liters and get units of kilometers.

$$\cancel{€}10\left(\frac{1\cancel{\text{L}}}{\cancel{€}3}\right)\left(\frac{40\text{ km}}{1\cancel{\text{L}}}\right)$$

$$10\left(\frac{40}{3}\right)\text{ km}$$

4. The expression we have set up will convert from a savings of €10 per week into how many fewer kilometers Chris should drive each week, which is what the variable $k$ stands for, so set this expression equal to $k$. Ignore the units at this point because we have already verified that they will tell us what we need to know. Do not multiply the numbers just yet because the format of the answer choices shows us that this step would not be helpful.

$$k = 10\left(\frac{40}{3}\right)$$

5. In order to match the format of the answer choices, we must isolate the 10 on the right side of the equation, so multiply both sides of the equation by $\frac{3}{40}$ in order to eliminate the fraction on the right side of the equation.

$$\left(\frac{3}{40}\right)k = 10\left(\frac{\cancel{40}}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{40}}\right)$$

$$\frac{3}{40}k = 10$$

6. The answer is D.

**Notes**

Notice that the number of kilometers Chris drives per week is irrelevant to the problem because it has nothing to do with how many fewer kilometers he should drive in order to save a specific amount of money (as opposed to, say a percentage of his current costs).





**Section 18.2 Suggested Problems from Real Tests**

- Test 1-C-6
- Test 1-C-34
- Test 2-C-11
- Test 3-C-14
- Test 3-C-19
- Test 3-C-38
- Test 4-C-33
- Test 5-NC-15
- Test 10-C-37
- Apr 2017-C-14
- May 2017-C-7
- 2018-C-32
- 2019 (US)-C-31
- 2019-C-19
- 2020-C-18
- Oct 2020-C-26
- May 2021 (Int)-C-8

View related real-test problems in SMTEx

SMTEx

## Section 18.2 Practice Problems

**1** 1600.io 18.2

In 1894, after the repeal of the Sherman Silver Purchase Act, each ounce of silver was worth \$0.60, and the largest known mass of silver found in Colorado was worth \$17,664. What was the weight, in pounds, of this mass of silver? (16 ounces = 1 pound)

**2** 1600.io 18.2

A certain box requires 8 centimeters of string to be tied shut. What is the maximum number of boxes of this type that can be tied shut with 24 meters of string? (1 meter = 100 cm)

A) 300

B) 33

C) 30

D) 3

**3** 1600.io 18.2

How many cups, each with a capacity of 10 fluid ounces, can be filled with water from a cooler that contains 20 pints of water? (1 pint = 16 fluid ounces)

**4** 1600.io 18.2

The cost of using the internet in a cafe is \$0.10 per minute. Which of the following equations represents the total cost $c$, in dollars, for $h$ hours of internet use?

A) $c = \frac{60h}{0.10}$

B) $c = \frac{0.10h}{60}$

C) $c = 0.10(60h)$

D) $c = 0.10h + 60$

## 18.3 The Distance/Speed/Time Equation

The distance covered by objects moving at a constant speed can be represented by a linear equation because the change in distance is constant for a given change in time. Since speed is measured in units of distance divided by units of time, the distance covered by an object in a given time can be found by multiplying the speed of the object by the amount of time it traveled at that speed (the units will cancel).

There's nothing complicated or mysterious about this. If a car travels 50 miles per hour, it will go 50 miles in one hour, 100 miles in two hours, 150 miles in three hours, and so on.

### The Distance/Speed/Time Equation

The distance $d$ covered by an object moving at a constant speed $s$ for a time $t$ can be found with the following linear equation.

$$d = st$$

If you are asked about the average speed of an object, you don't have to do anything tricky. Just divide the total distance traveled by the total amount of time it took to travel that far.

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$

Many questions about speed can be done using the methods outlined in the previous two sections of this chapter, or by simply using linear equations. However, sometimes it is useful to refer to the speed equation in order to work through problems involving speed or other rates. Often unit conversions will have to be incorporated as well.

**SkillDrill 18.3-1**

**Directions:** Given any two of the distance $d$, the speed $s$, and the time $t$ that an object traveled, find the remaining quantity in the desired units. Round to two decimal places when applicable.

Note that the units of time used are hours (hr), minutes (min), and seconds (s, not to be confused with the variable $s$ that represents speed), and that there are 60 seconds in a minute and 60 minutes in an hour. For speeds measured in "miles per hour," we use the units mph.

For some problems, you may need to use the following additional conversions:

- 1 mile = 1.61 km
- 1 mile = 5,280 ft

**1.** $d = 40\,\text{km}$, $t = 8\,\text{hr}$, Find $s$ in $\frac{\text{km}}{\text{hr}}$

**2.** $d = 26\,\text{km}$, $s = 30\,\frac{\text{km}}{\text{hr}}$, Find $t$ in hours

**3.** $t = 3.4\,\text{hr}$, $s = 60\,\frac{\text{km}}{\text{hr}}$, Find $d$ in km

**4.** $d = 21$ miles, $t = 8\,\text{hr}$, Find $s$ in $\frac{\text{km}}{\text{hr}}$

**5.** $d = 865\,\text{km}$, $s = 30\,\frac{\text{ft}}{\text{s}}$, Find $t$ in minutes

**6.** $t = 48$ min, $s = 75$ mph, Find $d$ in km

**7.** $d = 1{,}500$ ft, $t = 3$ min, Find $s$ in mph

**8.** $t = 52.1$ s, $s = 300\,\frac{\text{km}}{\text{hr}}$, Find $d$ in feet





**Example 18.3-1**

**1** 1600.io 18.3

Abigail took a nonstop train from Boston to New York, a total ride distance of 218 miles. The train drove at a speed of 72 miles per hour for the first 100 minutes of the ride and at a speed of 60 miles per hour for the remainder of the ride. To the nearest minute, for how many minutes did the train travel at a speed of 60 miles per hour?

**Solution**

1. To begin, we need to figure out how much of the total distance was covered during the first 100 minutes of the train ride (when the train was traveling at a speed of 72 mph). Using the distance equation, $d = st$, we can see how many miles were covered during the first part of the train ride. Let $d_1$ be the distance covered during the first part of the trip, $s_1$ be the speed during the first part of the trip, and $t_1$ be the time of the first part of the trip.

$$d_1 = s_1 t_1$$
$$d_1 = \left(72\frac{\text{mi}}{\text{hr}}\right)(100\,\text{min})$$

2. We have a problem here because none of the units can cancel: the time is given in units of minutes while the speed is given in units of miles per hour. We must therefore include the factor $\frac{1\,\text{hr}}{60\,\text{minutes}}$ in order to cancel units leaving only miles.

$$d_1 = \left(72\frac{\text{mi}}{\cancel{\text{hr}}}\right)(100\,\cancel{\text{min}})\left(\frac{1\,\cancel{\text{hr}}}{60\,\cancel{\text{min}}}\right)$$
$$d_1 = 72\,\text{mi}\left(\frac{100}{60}\right)$$
$$d_1 = 120\,\text{mi}$$

3. Since the first part of the trip covered a distance of 120 miles, then $d_2$, the distance covered during the second part of the trip (when the train was moving 60 mph) must be equal to the distance of the full trip (218 mi) minus $d_1$.

$$d_2 = 218 - d_1$$
$$d_2 = 218 - 120$$
$$d_2 = 98\,\text{mi}$$

4. The time taken for the second part of the trip, $t_2$, can be found by using the distance formula where $d_2$, the distance covered by the train in the second part of the trip, is equal to 98 mi and $s_2$, the speed of the train during the second part of the trip, is equal to $60\,\frac{\text{mi}}{\text{hr}}$. This calculation will result in units of hours, so we will convert from hours to minutes after the calculation is completed.

$$d_2 = s_2 t_2$$

$$\frac{d_2}{s_2} = t_2$$

$$\frac{98\,\cancel{\text{mi}}}{60\,\frac{\cancel{\text{mi}}}{\text{hr}}} = t_2$$

$$\frac{98}{60}\,\text{hr} = t_2$$

5. Multiply the time for the second part of the trip, $t_2$, which is in units of hours, by the factor $\frac{60\,\text{min}}{1\,\text{hr}}$ in order to find the time in units of minutes.

$$t_2 = \frac{98}{\cancel{60}}\,\cancel{\text{hr}}\left(\frac{\cancel{60}\,\text{min}}{1\,\cancel{\text{hr}}}\right)$$

$$t_2 = 98\,\text{min}$$

6. The answer is 98.

**Notes**

At the beginning of the problem, we could have converted the speeds from miles per hour to miles per minute by multiplying the two speeds by the factor $\frac{1\,\text{hr}}{60\,\text{min}}$, which would save us from tacking the unit conversions on at the end of the calculations.

Also note that the last two steps could have been skipped if you realized that objects traveling at 60 mph are covering a distance of 1 mile every minute because there are 60 minutes in each hour.

**Section 18.3 Suggested Problems from Real Tests**

- Test 2-C-15
- Test 3-C-9
- Test 5-C-37
- Test 5-C-38
- Test 6-C-29
- Apr 2018-C-35
- Oct 2019-NC-16
- Mar 2020-C-10
- Mar 2020-C-25
- Mar 2021-C-8
- Mar 2021-C-9
- May 2021 (US)-NC-17

**View related real-test problems in SMTEx**

SMTEx

## Section 18.3 Practice Problems

**1** 1600.io 18.3

On December 1, 1456, James McNeil set off on his midnight ride from Jamestown to Jamesburg. If he had ridden straight to Jamesburg without stopping, he would have traveled 40 miles in 78 minutes. In such a ride, what would the average speed of his moose have been, to the nearest tenth of a mile per hour?

**2** 1600.io 18.3

In 1910, James McNeil crawled from Podunk to Maple Springs, stopping to rest for only 13 hours of the 196 hours it took him to reach Maple Springs. If his average speed while traveling without resting was 4 miles per hour, how many miles did James McNeil travel?

A) 732

B) 759

C) 772

D) 785





## CHAPTER 18 RECAP

- When converting from one unit to another unit based on a direct conversion rate, you can simply use a proportion to solve the problem.
- Use the Factor-Label method when a direct conversion from one unit to another is not available. String together factors in order to cancel one unit at a time until the desired units are achieved.
- Identify a **starting point** for word problems with unit conversions by picking out information that tells us about the **end goals** of the people involved. In order to determine the steps needed after the starting point is identified, try to cancel or convert unwanted units, one-by-one, based on additional information in the problem.
- It's possible that not all of the measurements given are necessary in any given unit conversion problem. Analyze which measurements are given in units that will not be cancellable based on the other information given in the question.
- The speed equation is $d = st$, where $d$ is distance, $s$ is speed, and $t$ is time.

  This is no different than a normal linear equation (though students tend to have a lot more difficulty with these problems than with any other type of line problems), and these problems also have a lot in common with the rest of the unit-based problems in this chapter.

## Additional Problems

**1** 1600.io 18.1

Two units of weight used in ancient McNellia were spronks and jits, where 1 spronk is equivalent to 6 jits. The Great James statue in the capital city of McNellia is approximately 150 spronks tall. Which of the following best approximates the height, in jits, of the Great James statue?

A) 0.04

B) 25

C) 650

D) 900

**2** 1600.io 18.2

Phyllo dough, which is used to make baklava, is so thin that a sheet weighing one gram can cover up to $2\frac{1}{2}$ volleyball courts. If a volleyball court has an area of approximately $\frac{1}{25}$ acres, about how many acres could 50 grams of phyllo dough cover?

A) 5

B) 125

C) 2,500

D) 3,125

**Questions 3 and 4 refer to the following information.**

Nutritional Information for 1-Cup Servings of Beans

| Bean | Calories | Total fat (grams) | Protein (grams) |
|---|---|---|---|
| Black | 228 | 1.0 | 16.2 |
| Pinto | 252 | 1.2 | 16.0 |
| Kidney | 248 | 0.6 | 19.4 |
| Navy | 254 | 0.8 | 15.0 |
| Small Red | 220 | 1.0 | 12.4 |

The table above shows the calories, grams of fat, and grams of protein in 1-cup servings of selected beans.

**3** 1600.io 18.1

How many more grams of protein are in one quart of pinto beans than are in one quart of navy beans? (1 quart = 4 cups)

A) 0.5

B) 2

C) 4

D) 8

**4** 1600.io 18.1

Jolene purchases 1-quart boxes of each of the five beans shown in the table. Of the following, which best approximates the average (arithmetic mean) number of calories per box? (1 quart = 4 cups)

A) 240

B) 960

C) 1,200

D) 2,400





**Questions 5 and 6 refer to the following information.**

The sailfish can reach speeds of up to 70 miles per hour while swimming, making it the fastest fish in the ocean.

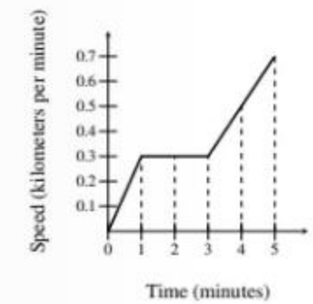
**5** 1600.io 18.2

What is a sailfish's maximum speed while swimming, in feet per second? (Round your answer to the nearest whole number. 1 mile = 5280 feet)

**6** 1600.io 18.3

If a sailfish swam at its maximum speed for a quarter of a mile to evade a shark, how many seconds would the swim take? (Round your answer to the nearest second.)

**7** 1600.io 18.3




The graph above models the speed, $s$, of a cyclist during the first 5 minutes of travel time, $t$. What was the total distance traveled from $t = 1$ to $t = 3$ ?

A) 0.3 kilometer

B) 0.6 kilometer

C) 0.9 kilometer

D) 1.2 kilometers

**8** 1600.io 18.3

A motor powers a model boat so that after starting from rest the boat travels $s$ feet in $t$ minutes, where $s = 31t\sqrt{t}$. Which of the following gives the average speed of the boat, in feet per minute, over the first $t$ minutes after it starts?

A) $31\sqrt{t}$

B) $31t$

C) $\frac{31}{\sqrt{t}}$

D) $\frac{31}{t}$

**9** 1600.io 18.1

If a 5-pound cake is sliced in half and each half is sliced into fourths, what is the weight, in ounces, of each of the slices? (1 pound = 16 ounces)

A) 5

B) 8

C) 10

D) 16

**10** 1600.io 18.1

A homeopath prescribes a treatment in which a patient takes 10 milligrams of an herbal extract every 8 hours for 10 days. According to the prescription, how many milligrams of the extract should the patient take in a 24-hour period?

A) 30

B) 40

C) 80

D) 400

**11** 1600.io 18.1

A carpet installer will carpet $f$ floors with the same size and shape in a building using a specific carpet. The installer's fee can be calculated by the expression $fKlw$, where $f$ is the number of floors, $K$ is a constant with units of dollars per square foot, $l$ is the length of each floor in feet, and $w$ is the width of each floor in feet. If a customer asks the installer to use a more expensive carpet, which of the factors in the expression would change?

A) $f$

B) $K$

C) $l$

D) $w$

**12** 1600.io 18.2

A local radio station sells time slots for programs in 2-hour intervals. If the station operates 24 hours per day, every day of the week, what is the total number of 2-hour time slots the station can sell for Friday, Saturday, and Sunday?





**13** 1600.io 18.3

On a 380-mile trip, Chantelle drove at an average speed of 40 miles per hour for the first $x$ hours. She then completed the trip, driving at an average speed of 60 miles per hour for the remaining $y$ hours. If $x = 2$, what is the value of $y$?

**14** 1600.io 18.3

A beam of light from the Sun travels to Jupiter at a speed of $1 \times 10^9$ feet per second. If the distance between the Sun and Jupiter is $4.80 \times 10^8$ miles, which of the following is closest to the number of minutes it will take for the light from the Sun to reach Jupiter?
(1 mile = 5,280 feet)

A) 1

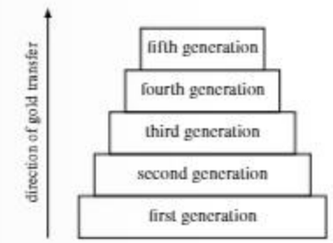
B) 25

C) 42

D) 48

**15** 1600.io 18.1

The gold pyramid below shows five generations of a certain family and the direction of gold transfer between those generations.




On average, 50% of the gold of one generation is transferred to the next generation in the family. Based on the gold pyramid, if the first generation had 6,000 pounds (lbs) of gold, approximately how much of this gold, in kilograms, is transferred to the fourth generation in this family? (1 kilogram = 2.20 lbs)

A) 3,300

B) 1,650

C) 825

D) 340

**16** 1600.io 18.2

A 25-foot ribbon and a 10-foot ribbon were each cut completely into 5-inch pieces. How many fewer 5-inch pieces resulted from the 10-foot ribbon than from the 25-foot ribbon? (12 inches = 1 foot)

A) 3

B) 30

C) 36

D) 60

**17** 18.2

Pure maple syrup has a density of 1.37 kilograms per liter. A store sells pure maple syrup at a price of $6.00 per kilogram. What is the selling price, in dollars per liter, for pure maple syrup purchased from this store? (Disregard the $ sign when writing your answer. For example, if your answer is $2.34, write 2.34.)

**18** 18.1

Amélie is planning a rave in a 1,950-square-meter room. If there should be no more than 3 square meters per person, what is the minimum number of people that could attend this event?

A) 300
B) 650
C) 2,700
D) 5,850

**19** 18.2

A gelatin artist purchases gelatin powder in 20-liter drums. The gelatin powder is mixed with water to make several batches of solution. Each batch of solution is made by mixing 4 deciliters of gelatin powder with water. What is the maximum number of batches of solution that can be made from two 20-liter drums of gelatin powder? (1 liter = 10 deciliters)

A) 40
B) 50
C) 80
D) 100

**20** 18.2

Which of the following speeds is equivalent to 180 kilometers per hour? (1 kilometer = 1,000 meters)

A) 5 meters per second
B) 50 meters per second
C) 300 meters per second
D) 648 meters per second

**21** 18.1

Geologists discovered a new type of igneous rock at the peak of the world's tallest volcano, Ojos del Salado. The peak is 4.3 miles above sea level. Approximately how many kilometers above sea level is the peak? (1 kilometer $\approx$ 0.6214 miles)

A) 1
B) 3
C) 5
D) 7





**22** 18.1

Gertrude's motorcycle contains 2 liters of fuel. She plans to ride the motorcycle $k$ kilometers using the fuel currently in the motorcycle. If the motorcycle can go 30 kilometers per liter of fuel, which inequality gives the the possible values of $k$ ?

A) $k \leq (2)(30)$
B) $k \geq (2)(30)$
C) $2 \leq 30k$
D) $2 \geq 30k$

**23** 18.3

$$d = 85t$$

The equation above can be used to calculate the distance, $d$, in kilometers, traveled by a motorcycle moving at a speed of 85 kilometers per hour over a period of $t$ hours. For any positive constant $k$, the distance the motorcycle would have traveled after $16k$ hours is how many times the distance the motorcycle would have traveled after $4k$ hours?

A) $4k$
B) $12k$
C) 4
D) 12

**24** 18.3

Chemtrail, a superhero, flew 960 miles. Part of the flight was in dense air, where his average speed was 40 miles per minute, and the rest was in thin air, where his average speed was 60 miles per minute. The flight took 20 minutes. What distance, in miles, did Chemtrail fly through dense air?

A) 120
B) 240
C) 480
D) 640

**Questions 25 and 26 refer to the following information.**

Mr. David's Weekend Evening Bike Ride

| Segment of ride | Distance (miles) | Average cycling speed with no cramp delay (mph) |
|---|---|---|
| From home to top of hill | 1.2 | 10 |
| From top of hill to valley | 4.6 | 25 |
| From valley to river | 2.4 | 15 |

Mr. David rides his bike from his home to the river every weekend evening. The table above shows the distance, in miles, and his average cycling speed, in miles per hour (mph), when he doesn't have any cramps, for each segment of his ride.

**25** 18.3

One evening, Mr. David rode directly from his home to the river in 36 minutes. What was his average speed, in miles per hour, during his ride that evening?

**26** 18.3

If Mr. David starts his ride at 7:30 p.m., he can ride at his average cycling speed with no cramps for each segment of the ride. If he starts his ride at 8:00 p.m., the ride time from his home to the top of the hill increases by 40% due to cramps, but the ride time for each of the other segments of his ride does not change. Based on the table, how many more minutes does Mr. David take to arrive at the river if he starts his ride at 8:00 p.m. than if he starts his ride at 7:30 p.m.? (Round your answer to the nearest minute.)

**Questions 27 and 28 refer to the following information.**

Polly completed a 6-kilometer swimming and crawling exercise. She swam for $5t$ kilometers and she crawled for $2\left(\frac{11}{8} - t\right)$ kilometers, where $t$ is the total amount of time, in hours, Polly spent swimming. The equation $5t + 2\left(\frac{11}{8} - t\right) = 6$ models the situation.

**27** 18.3

Which of the following is the best interpretation of the value 5 in the equation that models this situation?

A) Polly swam at a speed of 5 kilometers per hour.

B) Polly crawled at a speed of 5 kilometers per hour.

C) Polly swam for 5 minutes.

D) Polly crawled for 5 minutes.

**28** 18.1

What was the total distance that Polly spent swimming and crawling, in miles? (Use 1 kilometer = 0.62 miles)

A) 3.10

B) 3.72

C) 8.06

D) 9.68

**29** 18.1

An object with a length of 1.0 meter measures approximately 3.3 feet. An object having which of the following lengths, in meters, measures closest to 3.0 feet?

A) 0.30

B) 0.91

C) 1.1

D) 9.9

**30** 18.1

The planet Jupiter completes one orbit around the Sun every 12 years. Which of the following functions $p$ models the number of orbits of Jupiter in $t$ years?

A) $p(t) = 12 + t$

B) $p(t) = 12t$

C) $p(t) = \frac{12}{t}$

D) $p(t) = \frac{t}{12}$

**31** 18.2

A Centurion army battle tank uses approximately 6 liters of fuel per kilometer driven. If the tank has 1,200 liters of fuel at the beginning of a trip and drives at an average speed of 30 kilometers per hour, which of the following functions estimates the amount of remaining fuel $F(t)$, in liters, $t$ hours after the trip began?

A) $F(t) = 1{,}200 - 5t$

B) $F(t) = 1{,}200 - 30t$

C) $F(t) = 1{,}200 - 180t$

D) $F(t) = \frac{1{,}200 - 30t}{6}$





# Answer Key

### SkillDrill 18.1-1

1. 609.6 cm
2. 13.2 lbs
3. €3.86
4. 1.99 gal
5. 3.28 ft
6. 0.45 kg
7. \$42.09 or \$42.10
8. 41.67 L
9. 5 gal
10. \$15
11. 45.09 L

    Hint:
    € to \$,
    \$ to gal,
    gal to L
12. 297.61 mi or 297.62 mi

    Hint:
    € to \$,
    \$ to gal,
    gal to mi

### Section 18.1 Practice Problems

1. D
2. D
3. 3
4. D

### SkillDrill 18.2-1

1. 1.81 days

   Hint: $\text{chapters}\left(\frac{\text{words}}{\text{chapter}}\right)\left(\frac{\text{minute}}{\text{words}}\right)\left(\frac{\text{hour}}{\text{minutes}}\right)\left(\frac{\text{day}}{\text{hours}}\right)$
2. 209 questions

   Hint: $\text{days}\left(\frac{\text{hours}}{\text{day}}\right)\left(\frac{\text{minutes}}{\text{hour}}\right)\left(\frac{\text{question}}{\text{minutes}}\right)$
3. 1.36 or 1.37 words per second

   Hint: $\frac{\text{chapters}}{\text{days}}\left(\frac{\text{words}}{\text{chapter}}\right)\left(\frac{\text{day}}{\text{hours}}\right)\left(\frac{\text{hour}}{\text{minutes}}\right)\left(\frac{\text{minute}}{\text{seconds}}\right)$
4. 5.73 days

   Hint: $\text{chapters}\left(\frac{\text{questions}}{\text{chapter}}\right)\left(\frac{\text{minutes}}{\text{question}}\right)\left(\frac{\text{hour}}{\text{minutes}}\right)\left(\frac{\text{day}}{\text{hours}}\right)$

### Section 18.2 Practice Problems

1. 1840
2. A
3. 32
4. C

### SkillDrill 18.3-1

1. $5\ \frac{\text{km}}{\text{hr}}$
2. 0.87 h
3. 204 km
4. $4.23\ \frac{\text{km}}{\text{hr}}$
5. 1,575.98 min
6. 96.6 km
7. 5.68 mph
8. 14,238.51 ft

### Section 18.3 Practice Problems

1. 30.8
2. A

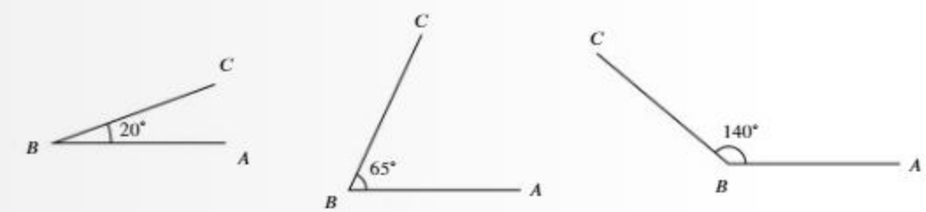
### Additional Problems

1. D
2. A
3. C
4. B
5. 103
6. 13
7. B
8. A
9. C
10. A
11. B
12. 36
13. 5
14. C
15. D
16. C
17. 8.22
18. B
19. D
20. B
21. D
22. A
23. C
24. C
25. 13.6 or 13.7
26. 3
27. A
28. B
29. B
30. D
31. C

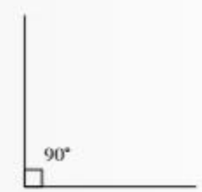
# Angles, Triangles, and Trigonometry 19

## 19.1 Angle Basics

The most familiar and most commonly used unit of measurement for angles is the degree, which is often denoted by the degree symbol (°). Below are some examples of angles shown with their corresponding measurements. Angles are formed by the intersection of two lines; the point at which the two lines meet is called the **vertex** (plural, **vertices**). Note that the angles in question can be denoted as either $\angle B$ (when there are not multiple angles shown at vertex $B$) or as $\angle ABC$ (where $\overline{AB}$ and $\overline{BC}$ are the two lines that intersect to form the angle, and the middle letter in the designation tells you the vertex at which the angle in question is located).

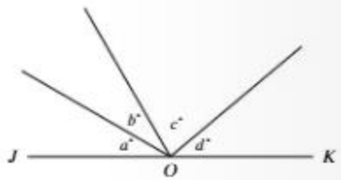



A 90° angle is a **right angle** (the lines forming the angle are perpendicular). Right angles are often marked with a small square instead of an arc and angle measurement.

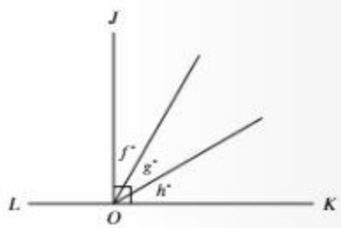



A 180° angle results in a straight line.

**Any angles that combine to form a straight line add up to 180°**, and, if you're interested, it's called a **straight angle**, though you won't see this term used very often. For example, in the picture below, $a + b + c + d = 180$ because the four angles measuring $a°$, $b°$, $c°$, and $d°$ form the straight line $\overline{JK}$. When there are two angles that combine to form a straight line or that otherwise sum to 180°, the two angles are said to be **supplementary** angles. For example, 120° and 60° are supplementary angles. (Though you won't be required to know all the terminology we define in this topic for the test, it's a good idea to know it nonetheless, so indulge us.)



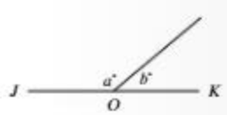

Similarly, and unsurprisingly, any angles that form a right angle sum to 90°. For example, in the picture below, $\angle JOK$ is a right angle. Therefore, $f + g + h = 90$ because the three angles measuring $f°$, $g°$, and $h°$ form right angle $\angle JOK$. When there are two angles that combine to form a right angle or that otherwise sum to 90°, the two angles are said to be **complementary** angles. For example, 25° and 65° are complementary angles. By the way, we also know that $\angle LOJ$ is a right angle because the angles must add up to 180° to form $\overline{LK}$.




Angles that measure more than 0° and less than 90° are called **acute angles**. Angles that measure more than 90° and less than 180° are called **obtuse angles**. For completeness, we'll mention that angles greater than 180° are called **reflex** angles.

**SkillDrill 19.1-1**

**Directions:** Based on the figure below, which is not drawn to scale, find the value of $b$ given the value of $a$ (the angles are measured in degrees, and they are supplementary, forming $\overline{JK}$).

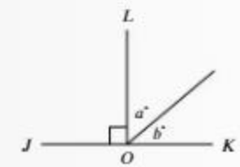



**1.** $a = 120$ **2.** $a = 45$ **3.** $a = 96$ **4.** $a = 143$





**Directions:** Based on the figure below, which is not drawn to scale, find the value of $b$, in degrees, given the value of $a$ (the angles are measured in degrees, and they are complementary, forming right angle $\angle KOL$).



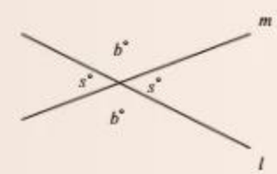

**5.** $a = 20$ **6.** $a = 45$ **7.** $a = 17$ **8.** $a = 38$

## 19.2 Finding Angle Measures

The simplest angle questions require you to find the measures of angles given the measures of other angles in a diagram.

### Vertical Angles

Whenever two lines intersect, four angles are formed, and the angles opposite each other, called **Vertical Angles** (even when they're horizontal!), are equal. In the picture below, four angles are formed by the intersection of line $l$ and line $m$. The angles labeled $s°$ (for smaller) are vertical angles and thus are equal to each other. Similarly, the angles labeled $b°$ (for bigger) are vertical angles and are therefore equal to each other.

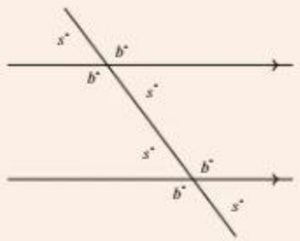


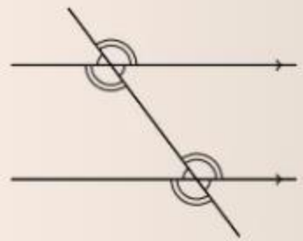

As previously explained, angles that form a line sum to 180°. Therefore, $s + b = 180$.

When a diagonal (non-perpendicular) line runs across two parallel lines, eight angles are formed. The four larger (obtuse) angles are all equal to each other. Similarly, the four smaller (acute) angles are also equal to each other.

## Angles Formed by Crossing Parallel Lines

All of the obtuse (greater than 90°) angles in the first diagram below are equal to each other and are labeled $b°$ (for bigger) and all of the acute (less than 90°) angles are equal to each other and are labeled $s°$ (for smaller). The use of arcs in the corners can be used to show which angles are equal to each other. For example, in the second figure below, the angles marked with one arc are equal to each other, and the angles marked with 2 arcs are equal to each other. The arrow head markings along lines show that the lines are parallel. Lines marked with the same number of arrow heads are parallel.



Again, any angles that form a line sum to 180°. Therefore, $b + s = 180$.

### Example 19.2-1

**1** 1600.io 19.2

In the figure above, lines $l$ and $m$ are parallel and lines $s$ and $t$ are parallel. If the measure of $\angle 1$ is 20°, what is the measure of $\angle 2$?

A) 20°

B) 70°

C) 140°

D) 160°

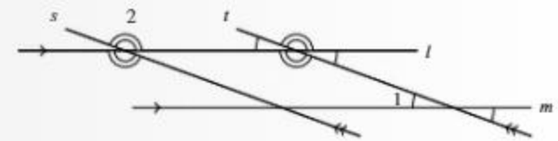
### Solution

1. Since lines $m$ and $l$ are parallel, all the small angles formed by the intersection with line $t$ will also be equal to $\angle 1$.

2. Since lines $s$ and $t$ are parallel, all the big angles formed by the intersection with line $l$ will also be equal to $\angle 2$.








3. At the intersection of lines $t$ and $l$, it is clear that the small angle (marked with a single arc and equal to $\angle 1$) and the big angle (marked with a double arc and equal to $\angle 2$) form a line, so the sum of the measures of $\angle 1$ and $\angle 2$ must be 180°.

$$\angle 1 + \angle 2 = 180°$$

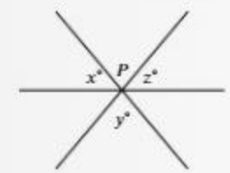
4. Since $\angle 1 = 20°$, substitute 20° in for $\angle 1$ then solve for $\angle 2$.

$$\angle 1 + \angle 2 = 180°$$
$$20° + \angle 2 = 180°$$
$$\angle 2 = 160°$$

5. The answer is D.

### Example 19.2-2

**2** 1600.io 19.2




Note: Figure not drawn to scale.

In the figure above, three lines intersect at point $P$. If $x = 35$ and $y = 45$, what is the value of $z$?

A) 100

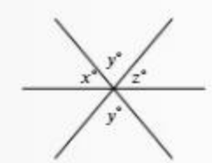
B) 80

C) 70

D) 10

### Solution

1. The angle that is opposite the angle marked $y°$ is its vertical angle, so the two angles are equal.




2. Above the horizontal line, the three angles marked $x°$, $y°$, and $z°$ form a straight line across when combined, so the sum

of these angles is 180°.

$$x + y + z = 180$$

3. Since $x = 35$ and $y = 45$, substitute 35 for $x$ and 45 for $y$, then solve for $z$.

$$\begin{aligned} x + y + z &= 180 \\ 35 + 45 + z &= 180 \\ 80 + z &= 180 \\ z &= 100 \end{aligned}$$

4. The answer is A.

**Section 19.2 Suggested Problems from Real Tests**

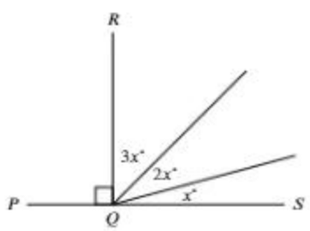
- Test 1-C-3
- Test 3-NC-11
- Test 10-NC-8
- Mar 2019-C-2
- May 2021 (int)-C-1

**View related real-test problems in SMTEx**

SMTEx

## Section 19.2 Practice Problems

**1** 1600.io 19.2




In the figure above, point $Q$ lies on $\overline{PS}$. What is the value of $5x$ ?

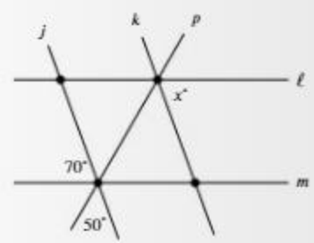
A) 15

B) 45

C) 75

D) 90

**2** 1600.io 19.2




In the figure shown, line $j$ is parallel to line $k$ and line $\ell$ is parallel to line $m$. What is the value of $x$ ?

A) 50

B) 60

C) 70

D) 120





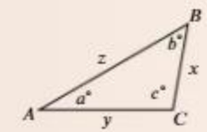
## 19.3 Triangles and Polygons

Understanding triangles is key to the geometry questions on the test (including many circle problems). The questions about triangles are often simply an extension of the angle rules covered already in the chapter, but there are a few important additional facts and rules.

### Angles Inside Triangles

**The interior angles of any triangle sum to 180°**. The length of each side is positively correlated with the size of the angle opposite that side, so the **largest side of a triangle is always across from the largest angle of the triangle, and the smallest side is across from the smallest angle**.

In the triangle below (drawn to scale), $a + b + c = 180$ because the sum of any triangle's interior angles is 180°. Since $\angle C$ is the largest angle, $z$ is the longest side. Similarly, since $\angle A$ is the smallest angle, $x$ is the shortest side.




When none of the sides (and therefore none of the angles) are equal to each other, as in the example above, the triangle is called a **scalene triangle**. In other cases, some or all of the sides/angles are equal; these special cases are described below.

Note that the triangle depicted above is an **oblique** triangle because none of the angles are right angles.

### Isosceles Triangles

**In isosceles triangles, two of the angles are equal to each other, and the sides across from those angles are therefore also equal to each other.**

If you know that two angles in a triangle are equal, then set the two sides across from those angles equal to each other. Similarly, if you know that two sides of a triangle are equal, or congruent, to each other, set the two angles across from those sides equal to each other.

You might have noticed the short lines drawn through the two equal legs of the triangle depicted above. Sometimes, **congruent sides of shapes are marked with small dashes, called hatch (or hash or tick) marks**.

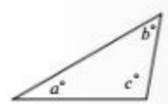
For example, in the parallelogram below, the two sides marked with a single hatch mark are equal to each other, and the two sides marked with a double hatch mark are equal to each other. A minor detail: if two lines are marked with a different number of hatch marks, it does not mean that they're of different lengths, though as a practical matter, they usually will be. The **hatch marks only indicate congruence (equality), but not incongruence (inequality)**.

## Equilateral Triangles

**In equilateral triangles, all of the sides are equal to each other, and therefore all of the angles are also equal**. Because all three of the angles are equal and they must sum to 180°, each angle is $\frac{180^\circ}{3} = 60^\circ$.

**SkillDrill 19.3-1**

**Directions:** Based on the figure below, which is not drawn to scale, find the value of $c$ given the values of $a$ and $b$ (all angles are measured in degrees).




**1.** $a = 30, b = 60$ **2.** $a = 40, b = 74$ **3.** $a = 96, b = 53$ **4.** $a = 143, b = 15$

**Directions:** Based on the isosceles triangle below, which is not drawn to scale, find the value of $b$ and $c$ given the value of $a$ (all angles are measured in degrees).

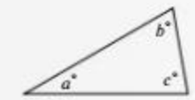
**5.** $a = 20$ **6.** $a = 45$ **7.** $a = 37$ **8.** $a = 59$





**Example 19.3-1**

**1** 1600.io 19.3




Note: Figure not drawn to scale.

In the triangle above, $a = 26$. What is the value of $b + c$?

**Solution**

1. The interior angles of a triangle must add up to 180°, so $a + b + c = 180$.
2. Since $a = 26$, substitute 26 for $a$, then solve for $b + c$.

$$\begin{aligned} a + b + c &= 180 \\ 26 + b + c &= 180 \\ b + c &= 154 \end{aligned}$$

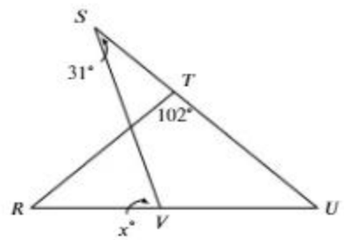
3. The answer is 154.

The previous example was straightforward; there was just a single equation, and the solution involved just one logical step. Often, however, you won't be able to solve problems of this general type in a single step; rather, you'll have to march through a series of steps by successively adding more and more information to what is initially provided using the various rules that apply. Each time you add a piece of information, you can then use that information to find another piece, and so on, until the desired value is finally found.

Even if you're unsure about the complete sequence of deductions that will take place on the path to the solution, you should still begin to solve for whatever you can and see where that takes you. The odds are that at some point, you'll either see how you can get where you need to go in one more well-chosen step, or you'll simply be left with only one more unknown, and you can solve for it. Don't freeze up just because the solution path is not immediately obvious; just get to work, and each time you find some new information, see if it helps you find what you're looking for.

**Example 19.3-2**

**2** 1600.io 19.3




In the figure above, $RT = TU$. What is the value of $x$?

A) 78

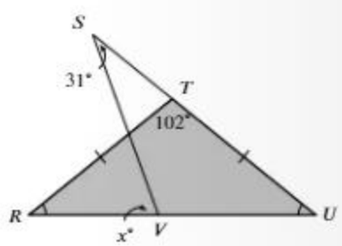
B) 70

C) 63

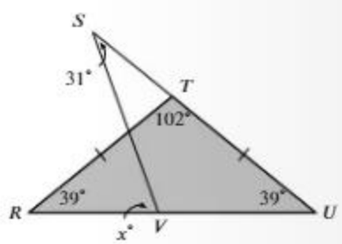
D) 39

**Solution 1**

1. Since we are told that $RT = TU$, we know that $\triangle RTU$ is isosceles and $\angle R = \angle U$.




2. Since the angles in a triangle (in this case $\triangle RTU$) must add up to 180, we know that $102^\circ + \angle R + \angle U = 180^\circ$. Therefore, $\angle R + \angle U = 78^\circ$. Since the two angles are equal, they each make up half of the $78^\circ$, so $\angle R = 39^\circ$ and $\angle U = 39^\circ$.

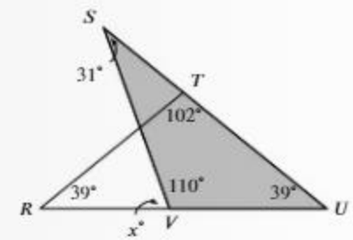







3. Let's turn our attention to $\triangle SVU$ because we now know two of the angles that form that triangle. Since the angles in a triangle must add up to $180^\circ$, we know that $31^\circ + 39^\circ + \angle SVU = 180^\circ$. We can use this equation to solve for $\angle SVU$, which forms a line with (is supplementary to) the angle that measures $x^\circ$ (so we know that the two angles also add up to $180^\circ$).

$$31^\circ + 39^\circ + \angle SVU = 180^\circ$$
$$70^\circ + \angle SVU = 180^\circ$$
$$\angle SVU = 110^\circ$$



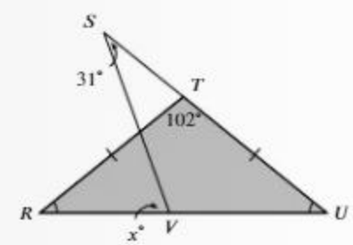

4. Since $\angle SVU$ and the angle measuring $x^\circ$ must sum to $180^\circ$ because they form a line (they're supplementary), the value of $x$ must be 70.

$$x^\circ + 110^\circ = 180^\circ$$
$$x^\circ = 70^\circ$$

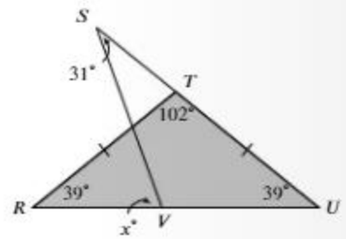
5. The answer is B.

**Solution 2**

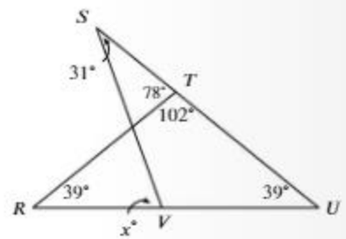
1. Since we are told that $RT = TU$, we know that $\triangle RTU$ is isosceles and $\angle R = \angle U$.



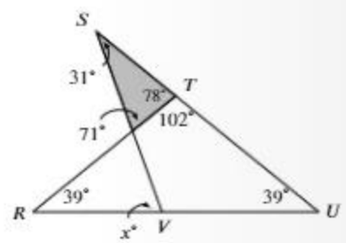

2. Since the angles in a triangle (in this case $\triangle RTU$) must add up to 180, we know that $102^\circ + \angle R + \angle U = 180^\circ$. Therefore, $\angle R + \angle U = 78^\circ$. Since the two angles are equal, they each make up half of the $78^\circ$, so $\angle R = 39^\circ$ and $\angle U = 39^\circ$.



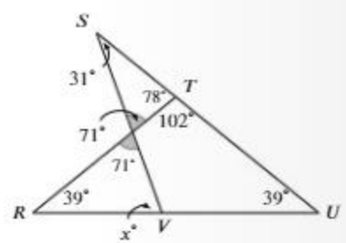

3. If we look over to point $T$, we can use the fact that angles that form a line sum to 180° in order to find the measure of $\angle RTS$. Since $\angle RTU$ measures 102°, then its supplementary angle $\angle RTS$ must equal 78° (since $180 - 102 = 78$).




4. We now know two of the angles in the triangle formed by vertices $S$ and $RTS$ and the intersection of lines $\overline{SV}$ and $\overline{RT}$. The three angles in this triangle have to sum to 180°. The remaining angle must be equal to 71° because $180-78-31 = 71$.



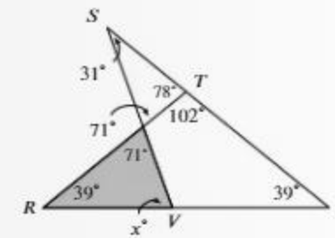

5. The angle we just found has a vertical angle that must also be equal to 71°.








6. Finally, since angles in a triangle add up to 180°, we can solve for $x$ by adding up the angles in the small triangle in the bottom left (with vertices $R$ and $RVS$).




$$39^\circ + 71^\circ + x^\circ = 180^\circ$$
$$110^\circ + x^\circ = 180^\circ$$
$$x^\circ = 70^\circ$$

7. The answer is B.

For any other **polygons** (shapes that consist only of straight lines and angles—no curves) with more than three sides, you can find the sum of the angles in the shape by using the **Sum of Interior Angles Formula** or simply by seeing how many triangles could be drawn inside the shape by connecting one vertex (corner) to all other nonadjacent vertices. Knowing the sum of the angles inside of a polygon is sometimes necessary to find the measures of individual angles inside of polygons.

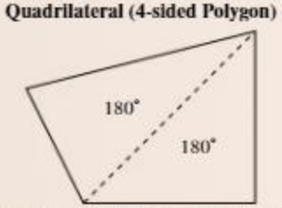
## Sum of Interior Angles

For a polygon with $n$ sides and angles, the sum of the interior angles can be found using the following formula:

$$\text{Sum of Interior Angles} = 180(n-2)$$

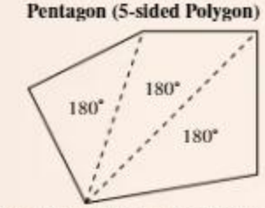
Note that 180° is the sum of the angles inside a triangle, and $n-2$ is the number of triangles that can be drawn from one vertex to all nonadjacent vertices in a polygon.

**Quadrilateral (4-sided Polygon)**




Sum of Interior Angles = $180(n-2)$
Sum of Interior Angles = $180(4-2)$
Sum of Interior Angles = 360

**Pentagon (5-sided Polygon)**

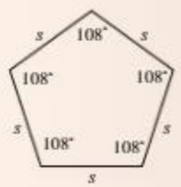



Sum of Interior Angles = $180(n-2)$
Sum of Interior Angles = $180(5-2)$
Sum of Interior Angles = 540

## Regular Polygons

In a **Regular Polygon**, all sides are equal, and all angles are equal. The measure of any angle is equal to the sum of the interior angles divided by the number of angles.

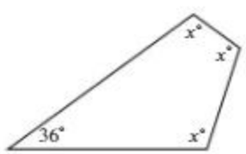
For example, a regular pentagon has five sides and five angles, where each side has the same length, $s$, and each interior angle measures $\frac{540^\circ}{5}$, or $108^\circ$ (divide the sum of the interior angles by the number of angles).




It is worth noting that **regular hexagons** (six-sided polygons) can be split into 6 equilateral triangles if we draw a line from the center of the figure to each of the six vertices (note that the sum of the interior angles is still $720^\circ$ as expected, because we are not drawing a line from one vertex to all other vertices). This can be extremely helpful for determining the areas or side lengths of regular hexagons (this will become clearer when we discuss special right triangles such as "30-60-90" triangles whose interior angles measure $30^\circ$, $60^\circ$, and $90^\circ$, as equilateral triangles can be split down the middle into two identical 30-60-90 triangles).

**Example 19.3-3**

**3** 1600.io 19.3




In the figure above, what is the value of $x$ ?

A) 36

B) 48

C) 90

D) 108





**Solution**

1. The shape shown in the figure is a quadrilateral with 4 sides, so we can use the Sum of the Interior Angles Formula, with $n = 4$, to find that the interior angles sum to $360^\circ$.

$$\text{Sum of Interior Angles} = 180(n-2)$$
$$\text{Sum of Interior Angles} = 180(4-2)$$
$$\text{Sum of Interior Angles} = 180(2)$$
$$\text{Sum of Interior Angles} = 360$$

2. Add the four interior angles and set them equal to 360 in order to solve for $x$.

$$36 + x + x + x = 360$$
$$3x = 324$$
$$x = 108$$

3. The answer is D.

**Section 19.3 Suggested Problems from Real Tests**

- Test 3-NC-18
- Test 5-NC-20
- Test 7-NC-17
- Test 7-C-14
- Apr 2017-NC-5
- May 2019 (Int)-NC-1
- May 2019 (Int)-C-9
- Oct 2019-C-15
- Apr 2021-NC-17
- May 2021 (US)-NC-5

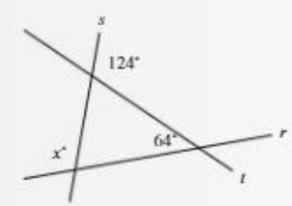
View related real-test problems in SMTEx

SMTEx

## Section 19.3 Practice Problems

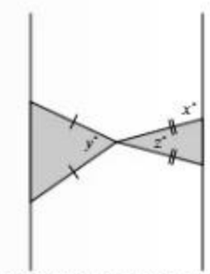
**1** 1600.io 19.3

Intersecting lines $r$, $s$ and $t$ are shown below.




What is the value of $x$ ?

**2** 1600.io 19.3

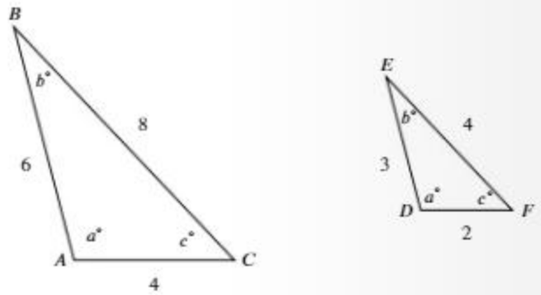



Note: Figure not drawn to scale.

Two isosceles triangles are shown above. If $180 - z = 3y$ and $y = 50$, what is the value of $x$ ?

## 19.4 Similar Triangles

Two triangles are called **Similar Triangles** when they have the same interior angles. Put another way, each angle in one triangle has a corresponding match in the other triangle.




In the figure above, $\triangle ABC$ is similar to $\triangle DEF$ because all of the angles in $\triangle ABC$ have a matching angle in $\triangle DEF$. Notice that the triangles are different sizes, however. **In similar triangles, all of the angles have a match, but the sides are not *necessarily* equal. However, sides across from matching angles (corresponding sides) are always in the same proportion to each other.**

In similar triangles, the sides across from matching angles correspond to each other, and the ratio of corresponding sides is always equal. In the figure shown above, $\overline{AB}$ corresponds to $\overline{DE}$; $\overline{BC}$ corresponds to $\overline{EF}$; and $\overline{AC}$ corresponds to $\overline{DF}$. In these particular triangles, the sides in $\triangle ABC$ are all twice the length of the corresponding sides in $\triangle DEF$.

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{2}{1}$$

It is also true that the ratio of any two sides of one triangle is equal to the ratio of the corresponding two sides in a similar triangle. For example, the ratio of the length of $\overline{AB}$ to the length of $\overline{AC}$ must be equal to the ratio of the corresponding sides in $\triangle DEF$, which are $\overline{DE}$ and $\overline{DF}$, respectively. We know that $\frac{AB}{AC} = \frac{6}{4}$, which reduces to $\frac{3}{2}$, and we know that $\frac{DE}{DF} = \frac{3}{2}$.

$$\frac{AB}{AC} = \frac{DE}{DF} = \frac{3}{2}$$

If your eyes glazed over from the above explanation, the central principle is simply that **similar triangles are exactly the same *shape*, though they might not be the same *size*.**

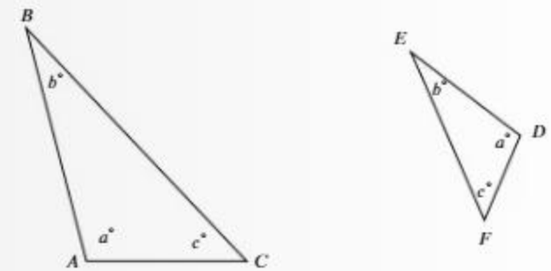
Due to these relationships among the angles and side lengths in a pair of similar triangles, we can make determinations about unknown angles and side lengths if we can recognize that two triangles in a test problem are similar. Therefore, it is important to be able to spot similar triangles. Note, importantly, that you cannot determine that two triangles are similar merely by looking at their shape as drawn—only an examination of given values for angles and/or side lengths, the presence of shared sides or angles, and geometric principles that produce congruent angles can provide definitive information that proves that two triangles are similar.

Conversely, two triangles might, at first glance, appear not to be similar, but that initial impression can be deceiving due to the rotation or flipping-over of one triangle relative to the other.





Similar triangles are not always shown in the same orientation—one might be rotated and/or flipped over, making the identification of the corresponding angles and sides a bit of a challenge—but the sides across from matching angles are still proportional. In the figures below, $\triangle ABC$ and $\triangle DEF$ are similar because all the angles in one triangle have a matching angle in the other triangle even though they might not appear to be similar due to the differing orientations.




One more useful principle: similarity of shapes is transitive, so if $\Delta A$ is similar to $\Delta B$, and $\Delta B$ is similar to $\Delta C$, $\Delta A$ is also similar to $\Delta C$. This comes in handy as one way of seeing all the similar triangles when a figure is divided, as we'll see a bit later on.

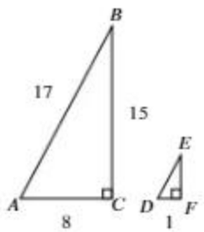
**SkillDrill 19.4-1**

**Directions:** Triangles $\Delta ABC$ and $\Delta DEF$ are similar. Given the measurements of two sides of $\Delta ABC$ and one side of $\Delta DEF$, determine the length of the side of $\Delta DEF$ that you are asked to find (the figures are not drawn to scale).

1. $AB = 3$, $BC = 6$, $DE = 1$, Find $EF$
2. $AB = 42$, $BC = 70$, $DE = 6$, Find $EF$
3. $AB = 25$, $AC = 10$, $DE = 15$, Find $DF$
4. $BC = 3$, $AC = \frac{3}{4}$, $EF = \frac{3}{4}$, Find $DF$

**Example 19.4-1**

**1** 1600.io 19.4




Note: Figure not drawn to scale.

Triangles $ABC$ and $DEF$ above are similar. How much longer than segment $EF$ is segment $DE$?

A) $\frac{1}{8}$

B) $\frac{1}{4}$

C) 1

D) 2

**Solution**

1. Since the two triangles are similar, all the corresponding sides are in the same proportion to each other.
2. Since side $\overline{DF}$ corresponds to side $\overline{AC}$ and $\frac{DF}{AC} = \frac{1}{8}$, which can be rewritten as $DF = \frac{1}{8}AC$, then the other sides in $\triangle DEF$ must also be $\frac{1}{8}$ of their corresponding sides in $\triangle ABC$.

$$DE = \frac{1}{8}AB \qquad EF = \frac{1}{8}BC$$

$$DE = \frac{1}{8}(17) = \frac{17}{8} \qquad EF = \frac{1}{8}(15) = \frac{15}{8}$$

3. Subtract $EF$ from $DE$ to see how much longer segment $\overline{DE}$ is than segment $\overline{EF}$.

$$DE - EF = \frac{17}{8} - \frac{15}{8}$$

$$DE - EF = \frac{2}{8} = \frac{1}{4}$$

4. The answer is B.

**Notes**

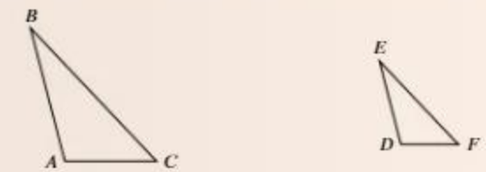
Rather than separately finding the lengths of $\overline{DE}$ and $\overline{EF}$, you could have observed that the difference between the corresponding sides in the larger triangle, $\overline{AB}$ and $\overline{BC}$, is 2, and therefore the difference between the lengths of $\overline{EF}$ and $\overline{DE}$ in the smaller triangle, which has side lengths $\frac{1}{8}$ of those in the larger triangle, is $\frac{1}{8}(2) = \frac{1}{4}$.





## Using a Constant for Unknown Sides

If you are given two similar triangles and are told the ratio of their sides but are not given the measure of either side outright, use a constant in writing the side lengths to avoid making mistakes about the actual lengths of the sides.




For example, if you are told that the two triangles above are similar and that $AB = 2DE$ (or that $\frac{AB}{DE} = 2$), do not assume that $DE = 1$ and $AB = 2$. Label side $\overline{DE}$ with a constant—let's say $d$—and label side $\overline{AB}$ with $2d$.

Even though it is sometimes uncomfortable to not be sure about the actual lengths of the sides, by using constants instead of randomly picking values that fit the proportion, you will avoid making incorrect assumptions about the lengths that might be inconsistent with information you are given later in the problem.

Problems involving similar triangles occur with some regularity on the test. Though on rare occasion, as in the previous example, you will be explicitly told that two triangles are similar, it is far more common that you'll have to figure out that there are similar triangles within the problem so that you can then use the properties of such triangles to find the needed unknown value(s).

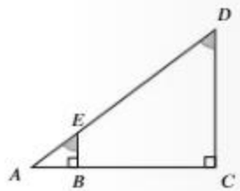
We've already discussed that all three angles will be the same in both similar triangles, and that all three sides will be the in the same ratios between the two triangles, but it turns out that you don't need all of that angle or side information to determine that a pair of triangles are similar. Perhaps the most important principle here involves our knowledge that the sum of the interior angles in a triangle is always the same: 180°. That means that **if we know (or can determine) that there are just two angles that are the same in both triangles, the third angle must also be the same** because the three angles in each triangle must sum to the same value (180°). This, in turn, means that **if we are told, or we find, that two angles are the same, we immediately know we're dealing with similar triangles**. Though the congruence of the third angle can generally be found independently, it's a big time-saver if you can quickly spot the features of a figure that convey the fact that two angles are equal.

Often, this principle comes into play when a larger triangle—typically, a right triangle—is subdivided with a line (explicitly, or as a step in the solution) that creates another, smaller right triangle (or two), and there are several other common setups as well. Let's look at some of these ways similar triangles can be produced within a single contiguous figure as we go beyond the previous situations where we simply had two separate, unconnected similar triangles.

## Divided Triangles

Similar triangles are commonly depicted as one triangle inside of another, where you can show that the triangles must be similar because all the angles in both triangles will have a match.

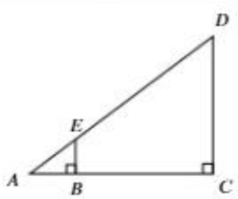
In the figure below, you can instantly determine that $\triangle ACD$ is similar to $\triangle ABE$ because they both contain $\angle A$ and a right angle, which, as explained above, means the third angles in each triangle ($\angle AEB$ and $\angle ADC$) must also be equal. (Note that when this setup appears in a test problem you might not be explicitly told that $\angle ABE$ is a right angle; instead, you might be given the fact that $\overline{BE}$ is parallel to $\overline{CD}$, which results in $\angle ABE$ being a right angle.)




Be sure to label these pictures very carefully because often you will be given the lengths of segments that are **not a complete side** of any triangle. For example, if you are told the length of segment $\overline{DE}$, remember that it is *not* the length of any complete side in $\triangle ACD$ (or in any smaller triangle here, for that matter). In order to make any similar triangle calculations, you would need to compare the length of side $\overline{AE}$ in $\triangle ABE$ to the *full* length of side $\overline{AD}$ (which is equal to $AE + DE$) in $\triangle ACD$.

**Example 19.4-2**

**2** 1600.io 19.4




In the figure above, $CD = 3$, and the length of line segment $\overline{AB}$ is a third the length of line segment $\overline{BC}$. What is the length of line segment $\overline{BE}$?

A) $\frac{1}{3}$

B) $\frac{3}{4}$

C) 1

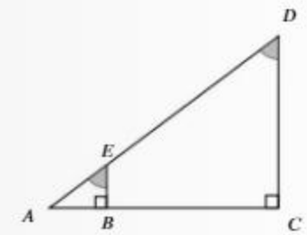
D) $\frac{4}{3}$





**Solution**

1. The triangles $\triangle ABE$ and $\triangle ACD$ must be similar. They both contain $\angle A$ and a right angle, which means that the third angles ($\angle AEB$ and $\angle ADC$) are also equal to each other. Since all three angles have a match, the triangles are similar.



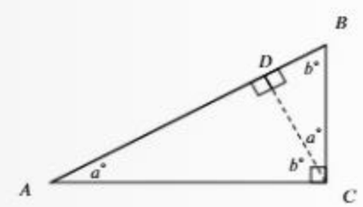

2. We are told that the length of $\overline{AB}$ is a third the length of $\overline{BC}$. Because there is a part-to part ratio of 1:3, then the ratio consists of 4 total parts, meaning that the ratio of $AB$ to $AC$ is a part to whole ratio of 1:4 (the length of $\overline{AB}$ is a fourth of the corresponding side of the large triangle, $\overline{AC}$).

   We can prove the part-to-whole ratio is true. We can see from the picture that $AC = AB + BC$ and we are told $\frac{AB}{BC} = \frac{1}{3}$, so $BC = 3(AB)$. Therefore, replacing $BC$ with $3(AB)$ in the equation $AC = AB + BC$, we see that $AC = AB + 3(AB) = 4(AB)$. Since $AC = 4AB$, then $\frac{AB}{AC} = \frac{AB}{4(AB)} = \frac{1}{4}$.

3. Each side in the small triangle $\Delta ABE$ is one fourth of its corresponding side in the large triangle $\Delta ACD$. Because the length of $\overline{CD}$ is 3, then the length of $\overline{BE}$ is $\frac{1}{4}(3) = \frac{3}{4}$.

4. The answer is B.

Another way that right triangles can be divided with a single line actually creates three similar triangles. In the figure below, a line is drawn from the right angle vertex $C$ that intersects the hypotenuse at a right angle at point $D$. The larger right triangle is split into two smaller right triangles, $\Delta ADC$ and $\Delta CDB$, both of which are similar to the original triangle $\Delta ACB$.




Notice that the two acute angles in $\Delta ACB$ measure $a°$ and $b°$, so $a + b + 90 = 180$. When the two right triangles $\Delta ADC$ and $\Delta CDB$ are formed by adding line segment $\overline{CD}$, $\Delta ADC$ has a 90° angle and an $a°$ angle, and $\Delta CDB$ has a 90° angle and a $b°$ angle. Therefore, $\angle ACD$ must measure $b°$ to complete the 180° in $\Delta ADC$, and $\angle BCD$ must measure $a°$ to complete the 180° in $\Delta CDB$. Remember that because the angles in any triangle must sum to the same value (180°), **if we determine that two triangles have two congruent angles, the third angle must also be congruent and the triangles are similar**. Here, each smaller triangle has a right angle, as does the larger outer triangle, and each small triangle shares one angle with the outer triangle, so both smaller triangles are therefore similar to the outer triangle. Furthermore, because similarity is transitive, we can also ascertain that the two smaller triangles are similar to each other even without determining that they have a second congruent angle in addition to the right angle.

The corresponding sides from the three triangles can be used to make proportions that help us find unknown lengths. The three

sides of all three triangles must all be in the same ratio to each other, and so we can choose any two side lengths and the lengths of their corresponding sides in another triangle to make those proportions.

In these similar triangles, segment $\overline{AB}$ is the hypotenuse of the large triangle $\Delta ACB$, segment $\overline{AC}$ is the hypotenuse of $\Delta ADC$ and segment $\overline{BC}$ is the hypotenuse of $\Delta CDB$, so these all correspond to each other.

The sides across from the angles measuring $a^\circ$, segments $\overline{BC}$ (in $\Delta ACB$), $\overline{CD}$ (in $\Delta ADC$), and $\overline{BD}$ (in $\Delta CDB$), all correspond to each other.

Similarly, the sides across from the angles measuring $b^\circ$, segments $\overline{AC}$ (in $\Delta ACB$), $\overline{AD}$ (in $\Delta ADC$), and $\overline{CD}$ (in $\Delta CDB$), all correspond to each other.

Therefore, the ratios of the corresponding sides in each triangle is the same (the hypotenuses of the three triangles are in the same ratio to each other as are the short legs and long legs of each triangle)

$$\text{All of these are equal ratios} \Rightarrow \begin{cases} \overline{AB} : \overline{AC} : \overline{BC} & \text{hypotenuse } \Delta ACB : \text{hypotenuse } \Delta ADC : \text{hypotenuse } \Delta CDB \\ \overline{AC} : \overline{AD} : \overline{CD} & \text{long leg } \Delta ACB : \text{long leg } \Delta ADC : \text{long leg } \Delta CDB \\ \overline{BC} : \overline{CD} : \overline{BD} & \text{short leg } \Delta ACB : \text{short leg } \Delta ADC : \text{short leg } \Delta CDB \end{cases}$$

and the ratios of the sides of each triangle is the same (the three sides of $\Delta ACB$ are in the same ratio to each other as the sides of $\Delta ADC$ and $\Delta CDB$)

$$\text{All of these are equal ratios} \Rightarrow \begin{cases} \overline{AB} : \overline{AC} : \overline{BC} & \text{hypotenuse } \Delta ACB : \text{long leg } \Delta ACB : \text{short leg } \Delta ACB \\ \overline{AC} : \overline{AD} : \overline{CD} & \text{hypotenuse } \Delta ADC : \text{long leg } \Delta ADC : \text{short leg } \Delta ADC \\ \overline{BC} : \overline{CD} : \overline{BD} & \text{hypotenuse } \Delta CDB : \text{long leg } \Delta CDB : \text{short leg } \Delta CDB \end{cases}$$

You may have noticed that the side ratios wind up being exactly the same regardless of which perspective you take (the ratio of the sides within each triangle is the same as the ratio of the corresponding sides from each triangle); this is a peculiar feature of this setup, but it does not apply to just any group of three similar triangles.

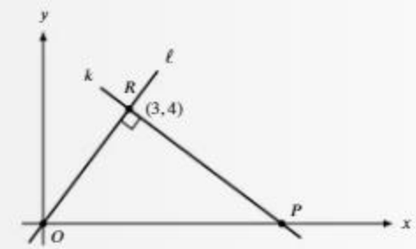




In the following problem, a version of which has previously appeared in the extra practice for Chapter 3 (because slope-based solutions make use of the same proportions involved in similar triangles), the dividing line that creates a set of similar triangles is not shown, but since the two lines in the problem are perpendicular (intersect at a right angle), we can use the information above to solve the problem in a way that makes use of similar triangles.

**Example 19.4-3**

**3** 1600.io 19.4




In the $xy$-plane above, lines $k$ and $\ell$ are perpendicular. What is the $x$-coordinate of point $P$?

A) 5

B) $\frac{16}{3}$

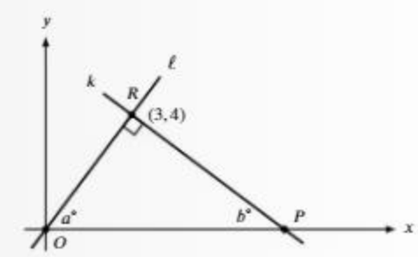
C) 7

D) $\frac{25}{3}$

**Solution**

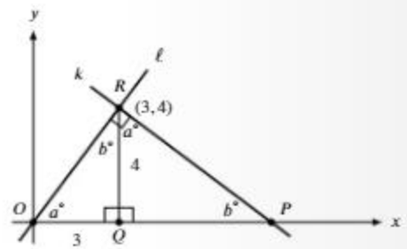
1. Though this problem can be solved using linear equations, in this case we want to use similar triangles, so let's start by labeling $\angle ROP$ with an $a^\circ$ and $\angle RPO$ with a $b^\circ$ (the remaining $\angle ORP$ measures $90^\circ$ because lines $k$ and $\ell$ are perpendicular).

   We know that the angles in a triangle must sum to $180^\circ$, so $a + b + 90 = 180$.



2. If we draw a vertical line from point $R$ to the $x$-axis, it will intersect the $x$-axis at a right angle, and we'll call this intersection point $Q$. The line $\overline{RQ}$ splits the larger right triangle into two smaller right triangles that must both be similar to the larger triangle because $\Delta ROQ$ has an angle measuring $a^\circ$ and a right angle, so the remaining angle must measure $b^\circ$, and $\Delta RPQ$ has an angle measuring $b^\circ$ and a right angle, so the remaining angle must measure $a^\circ$. Furthermore, since all three angles are the same in triangles $\Delta ORQ$ and $\Delta RPQ$, those two triangles are similar. Alternatively, we can simply observe that each of the smaller triangles has two angles that are the same as angles in the larger outer triangle (one right angle and one shared angle), and therefore both smaller triangles are similar to the larger triangle and hence are also similar to each other.

   We should also label $\overline{RQ}$ with a length of 4 because the line goes up from a $y$-value of 0 (at point $Q$) to a $y$-value of 4 (at point $R$), and we should label $\overline{OQ}$ with a length of 3 because the line goes from an $x$-value of 0 (at point $O$) to an $x$-value of 3 (at point $Q$).




3. In order to find the length of line $\overline{OP}$ (and thus the $x$-coordinate of point $P$), we can first figure out the length of $\overline{QP}$ by using a proportion.

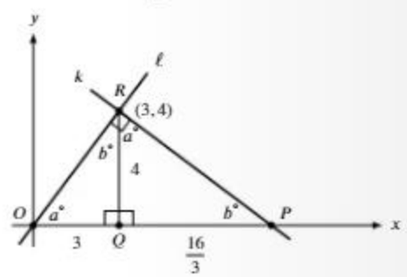
   Sides $\overline{QR}$ (in $\Delta ORQ$) and $\overline{QP}$ (in $\Delta RPQ$) are both across from angles marked $a^\circ$, and sides $\overline{OQ}$ (in $\Delta ORQ$) and $\overline{QR}$ (in $\Delta RPQ$) are both across from angles marked $b^\circ$. We can write the following proportion in order to find the length of $\overline{QP}$, substituting the lengths of $\overline{OQ}$ and $\overline{QR}$, which are 3 and 4, respectively.

$$\frac{QP}{QR} = \frac{QR}{OQ}$$

$$\frac{QP}{4} = \frac{4}{3}$$

$$QP = \frac{16}{3}$$

4. Since the length of $\overline{OQ}$ is 3, and the length of $\overline{QP}$ is $\frac{16}{3}$, then the length of $\overline{OP}$ must be equal to $3 + \frac{16}{3} = \frac{9}{3} + \frac{16}{3} = \frac{25}{3}$.








5. Because the horizontal line $\overline{OP}$ starts at point $(0,0)$ and has a length of $\frac{25}{3}$, point $P$ is located at $\left(\frac{25}{3}, 0\right)$, so its $x$-coordinate is $\frac{25}{3}$.

6. The answer is D.

**Solution 2**

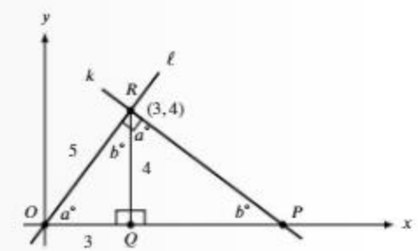
In this solution, we again make use of similar triangles, but it also requires knowledge of the Pythagorean Theorem or Special Right triangles, which are covered later in this chapter, in order to solve for the length of $\overline{OP}$ directly.

The first two steps of the solution are the same as in Solution 1 where we establish that after drawing line $\overline{QR}$, all three resulting triangles are similar.

3. $\Delta OQR$ is a 3:4:5 special right triangle because its legs, $\overline{OQ}$ and $\overline{QR}$, measure 3 and 4, respectively. Therefore, the length of $\overline{OR}$ is 5.

   You can also find this using the Pythagorean Theorem:

$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = c^2$$
$$9 + 16 = c^2$$
$$25 = c^2$$
$$5 = c$$




4. As a reminder, the large triangle $\Delta OPR$ is similar to triangle $\Delta ORQ$ because they have three matching angles; they share an angle marked $a^\circ$, and both have a right angle, so the third angle must be congruent as well. Sides $\overline{OP}$ (in $\Delta OPR$) and $\overline{OR}$ (in $\Delta ORQ$) are the hypotenuses of their respective triangles, and sides $\overline{OR}$ (in $\Delta OPR$) and $\overline{OQ}$ (in $\Delta ORQ$) are both across from angles marked $b^\circ$. We can write the following proportion in order to find the length of $\overline{OP}$, substituting the lengths of $\overline{OQ}$ and $\overline{OR}$, which are 3 and 5, respectively.

$$\frac{OP}{OR} = \frac{OR}{OQ}$$

$$\frac{OP}{5} = \frac{5}{3}$$

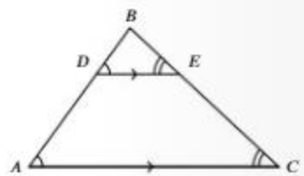
$$OP = \frac{25}{3}$$

5. Because the horizontal line $\overline{OP}$ starts at point $(0,0)$ and has a length of $\frac{25}{3}$, point $P$ is located at $\left(\frac{25}{3}, 0\right)$, so its $x$-coordinate is $\frac{25}{3}$.

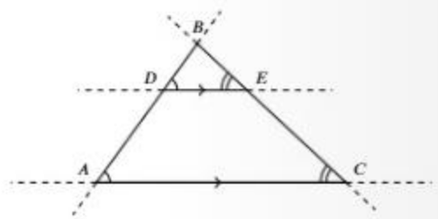
6. The answer is D.

Non-right triangles often appear on the test divided by a line that is parallel to one of their sides (usually a line that is horizontal). Because the parallel lines both intersect the other two sides of the triangles, we know that the angles in the larger triangle and the smaller triangle will all be congruent, and, therefore, the two triangles are similar.

For example, in the figure below, line $\overline{DE}$ is parallel to line $\overline{AC}$. The angle $\angle ABC$ is in both $\Delta ABC$ and $\Delta DBE$. By the nature of parallel lines which are intersected by sides $\overline{AB}$ and $\overline{BC}$, we also know that $\angle BAC = \angle BDE$ and that $\angle BCA = \angle BED$.



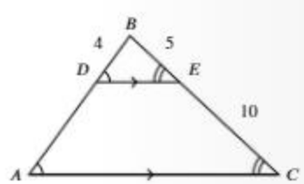

It is often easier for students to visualize these angle equivalences if they extend the parallel lines $\overline{AC}$ and $\overline{DE}$ and the diagonal lines $\overline{AB}$ and $\overline{BC}$ so that it looks more like what we saw earlier in the chapter (parallel lines that both intersect another line).




If you see this kind of drawing, you should recognize that $\Delta ABC$ and $\Delta DBE$ are similar because all three angles are already shown to be congruent (two would have been sufficient because the third angles would have to match once two angles are known to be congruent). Since the triangles are similar, you can use proportions to solve for unknown side lengths.

$$\frac{AB}{DB} = \frac{AC}{DE} = \frac{BC}{BE}$$

For example, in the figure below, $DB = 4$, $BE = 5$, and $EC = 10$, and we can use the proportions above to solve for the length of $\overline{AB}$; we'll make use of the proportion $\frac{AB}{DB} = \frac{BC}{BE}$.








Since $\Delta ABC$ is similar to $\Delta DBE$, we know that $\frac{AB}{DB} = \frac{BC}{BE}$. Note that we must first recognize that $CB = CE + BE = 10 + 5 = 15$, and then we can substitute the lengths of the line segments $\overline{DB}$, $\overline{BC}$, and $\overline{BE}$.

$$\frac{AB}{DB} = \frac{BC}{BE}$$

$$\frac{AB}{4} = \frac{15}{5}$$

$$\frac{AB}{4} = 3$$

$$AB = 12$$

## Triangle Proportionality Theorem

More generally, we can state that **when a triangle is divided by a line parallel to one side, the other two sides are divided proportionally**; this is called the **Triangle Proportionality Theorem**. This is particularly useful when the problems are arranged in such a way that you need to solve for a segment such as $\overline{AD}$, which is not the side of either of the triangles shown: you do not need to make whole triangle ratios if this is the case.

For instance, given the previous figure shown, where we know $DB = 4$, $BE = 5$, and $EC = 10$, we can solve for $AD$ directly, without first finding the length of $\overline{AB}$ and then subtracting $DB$ (based on what we just did, we already know $AD = AB - BD = 12 - 4 = 8$, but let's ignore that for a moment), by using the Triangle Proportionality Theorem and setting up the proportion:

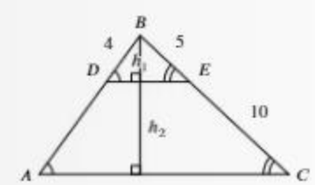
$$\frac{AD}{DB} = \frac{CE}{EB}$$

$$\frac{AD}{4} = \frac{10}{5}$$

$$\frac{AD}{4} = 2$$

$$AD = 8$$

**A trapezoid is a quadrilateral in which only two sides are parallel to each other** (unlike a rectangle or square). You should take note that the heights of $\Delta DBE$ and trapezoid $ADEC$ are in the same 2:1 ratio; if you'd like to see why that's the case, just drop a vertical line down from the peak of the triangle at $B$ to its base. Now, $\overline{DE}$ will be proportionally dividing $\overline{AB}$ and this new line, the segments of which will represent the heights of the small triangle and the trapezoid below.

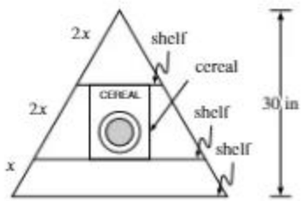



$$\frac{h_2}{h_1} = \frac{AD}{DB} = \frac{CE}{EB} = \frac{10}{5} = \frac{2}{1}$$

This realization can be most helpful when problems feature trapezoids (sometimes as part of divided triangles and sometimes without any triangles in sight). One interesting use of this fact is shown in the following problem, which has us solving for the height of a trapezoidal shelf in a triangular cabinet.

**Example 19.4-4**

**4** 1600.io 19.4

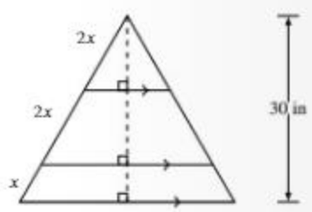



Gabbie bought an old, weird house with a stupid triangular cabinet. The total height of the cabinet is 30 inches, and there are three parallel shelves as shown above. What is the maximum height, in inches, of a cereal box that can stand upright on the middle shelf?

**Solution**

Since all of the shelves are parallel to each other, we can invoke the Triangle Proportionality Theorem to find the heights of the shelves. The relative heights of the shelves will be proportional to the relative lengths of the slanted segments.

1. If we draw a vertical line from top vertex of the shelf to the bottom shelf, the entire left side of the structure is one large right triangle and the horizontal shelves divide the vertical line into segments whose lengths are the heights of the individual shelves; the sum of these heights is 30 inches.

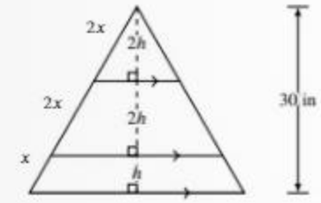



2. By the Triangle Proportionality Theorem, the heights of each section of the vertical line are in the same proportions as the lengths of the slanted side segments that make up the large triangle's entire left slanted side.

   The length of the whole slanted side of the shelf is $5x$, and the portion of the slant that belongs to the middle shelf has a length of $2x$. The length of the middle portion of the slant makes up $\frac{2x}{5x} = \frac{2}{5}$ of the slant for the whole shelf, so the height of the middle shelf must also be $\frac{2}{5}$ of the height of the whole shelf.







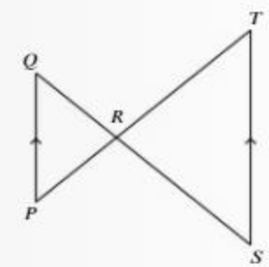

The whole shelf has a height of 30 inches, so the middle shelf has a height of $\frac{2}{5}(30) = 12$ inches.

3. The answer is 12.

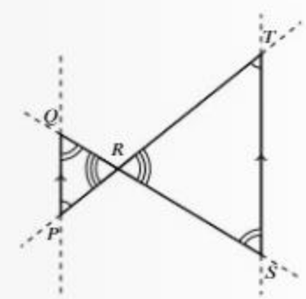
## Similar Triangles with Vertical Angles

Similar triangles sometimes arise in arrangements such as that shown in the figure below, with the two similar triangles touching at one vertex (where there is a vertical angle) and each having a side that is parallel to a side in the other triangle.

For example, in the figure below, the angles meeting at point $R$ are vertical angles (and thus are equal to each other), and side $\overline{PQ}$ is parallel to side $\overline{ST}$.




We can tell that these triangles are similar if we look at lines $\overline{QS}$ and $\overline{PT}$ as diagonals crossing parallel lines $\overline{PQ}$ and $\overline{ST}$ and mark the angles that must match each other.




In the figure above, we can tell that the vertical angles (at point $R$) are equal, $\angle P = \angle T$, and $\angle Q = \angle S$. Therefore, side $\overline{PR}$ corresponds to side $\overline{RT}$, side $\overline{QR}$ corresponds to side $\overline{RS}$, and side $\overline{PQ}$ corresponds to side $\overline{ST}$. The ratio of corresponding sides is constant.

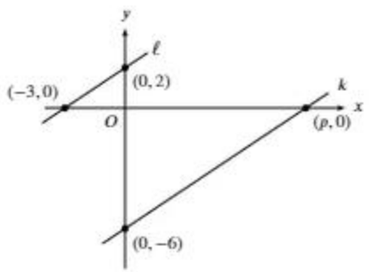
$$\frac{PR}{RT} = \frac{QR}{RS} = \frac{PQ}{ST}$$

Note, importantly, that the corresponding angles (and, therefore, the corresponding sides) are on opposite sides, top-to-bottom; for example, $\angle Q$ (which is on top) corresponds with $\angle S$ (which is on the bottom) not with $\angle T$ (which is on top). Students often get confused about this, so it can be helpful to draw those extensions of the parallel lines (here, $\overline{PQ}$ and $\overline{ST}$) to help remind you of which angles are equal and therefore which sides correspond.

The following question could just as easily have been shown in our chapter on lines. If we did this problem in that chapter, we would have thought about it in the context of parallel lines by setting the slopes of parallel lines equal to each other then building an equation for the lower line and plugging in the rightmost point to solve for $p$. Here, we'll apply what we know about similar triangles to use a different approach.

**Example 19.4-5**

**5** 1600.io 19.4




In the $xy$-plane above, line $\ell$ is parallel to line $k$. What is the value of $p$?

A) 3

B) 4

C) 6

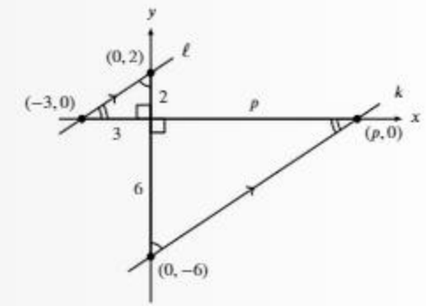
D) 9

**Solution**

1. The two triangles formed by the lines and the portions of the axes between the lines and the origin are similar triangles because they have vertical angles (both are right angles) at the origin and the diagonal lines are parallel to each other, so the angles formed between the parallel lines and the axes are congruent as marked. Label the lengths of each segment in order to make a proportion that can help us solve for $p$.








2. The ratio of the vertical legs of each triangle, $\frac{6}{2}$, is the same as the ratio of the horizontal legs of each triangle, $\frac{p}{3}$, so set the two equal to each other and solve for $p$; as usual with proportions, we'll put the unknown $p$ in the numerator of the left side for easiest solving and to produce the most natural mathematical "sentence."

$$\frac{p}{3} = \frac{6}{2}$$

$$\frac{p}{3} = 3$$

$$p = 9$$

3. The answer is D.

**Section 19.4 Suggested Problems from Real Tests**

- Test 1-NC-17
- Test 2-NC-6
- Test 2-NC-18
- Test 4-NC-16
- Test 6-C-16

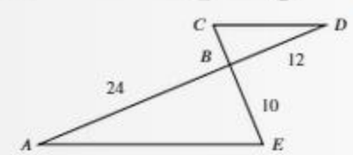
View related real-test problems in SMTEx

SMTEx

- Oct 2020-C-32
- Mar 2021-C-22
- Apr 2021-C-13
- May 2021 (Int)-NC-6

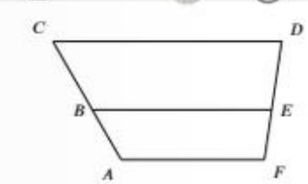
## Section 19.4 Practice Problems

**1** 1600.io 19.4




In the figure above, $\overline{AE} \parallel \overline{CD}$ and segment $\overline{AD}$ intersects segment $\overline{CE}$ at $B$. What is the length of segment $\overline{CE}$?

**2** 1600.io 19.4




In the figure above, $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ are parallel. Points $B$ and $E$ lie on $\overline{AC}$ and $\overline{FD}$, respectively. If $AB = 4$, $BC = 5.5$, and $FE = 3.5$, what is the length of $\overline{ED}$ to the nearest tenth?

A) 2.6

B) 4.8

C) 5.2

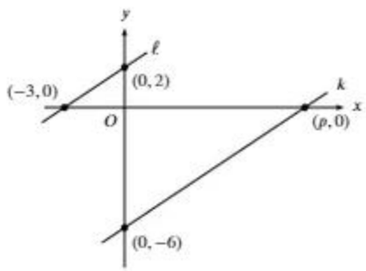
D) 6.3

Note, importantly, that the corresponding angles (and, therefore, the corresponding sides) are on opposite sides, top-to-bottom; for example, $\angle Q$ (which is on top) corresponds with $\angle S$ (which is on the bottom) not with $\angle T$ (which is on top). Students often get confused about this, so it can be helpful to draw those extensions of the parallel lines (here, $\overline{PQ}$ and $\overline{ST}$) to help remind you of which angles are equal and therefore which sides correspond.

The following question could just as easily have been shown in our chapter on lines. If we did this problem in that chapter, we would have thought about it in the context of parallel lines by setting the slopes of parallel lines equal to each other then building an equation for the lower line and plugging in the rightmost point to solve for $p$. Here, we'll apply what we know about similar triangles to use a different approach.

**Example 19.4-5**

**5** 1600.io 19.4




In the $xy$-plane above, line $\ell$ is parallel to line $k$. What is the value of $p$ ?

A) 3

B) 4

C) 6

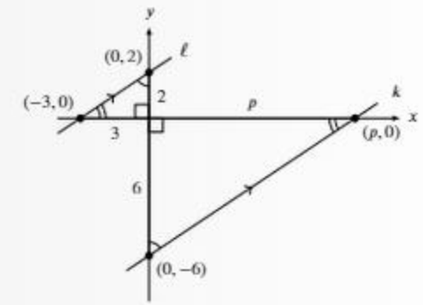
D) 9

**Solution**

1. The two triangles formed by the lines and the portions of the axes between the lines and the origin are similar triangles because they have vertical angles (both are right angles) at the origin and the diagonal lines are parallel to each other, so the angles formed between the parallel lines and the axes are congruent as marked. Label the lengths of each segment in order to make a proportion that can help us solve for $p$.








2. The ratio of the vertical legs of each triangle, $\frac{6}{2}$, is the same as the ratio of the horizontal legs of each triangle, $\frac{p}{3}$, so set the two equal to each other and solve for $p$; as usual with proportions, we'll put the unknown $p$ in the numerator of the left side for easiest solving and to produce the most natural mathematical "sentence."

$$\frac{p}{3} = \frac{6}{2}$$

$$\frac{p}{3} = 3$$

$$p = 9$$

3. The answer is D.

**Section 19.4 Suggested Problems from Real Tests**

- Test 1-NC-17
- Test 2-NC-6
- Test 2-NC-18
- Test 4-NC-16
- Test 6-C-16

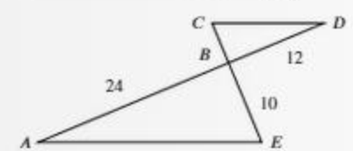
View related real-test problems in SMTEx

SMTEx

- Oct 2020-C-32
- Mar 2021-C-22
- Apr 2021-C-13
- May 2021 (Int)-NC-6

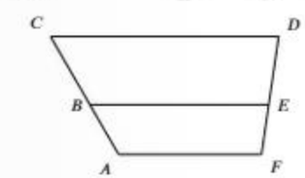
## Section 19.4 Practice Problems

**1** 1600.io 19.4




In the figure above, $\overline{AE} \parallel \overline{CD}$ and segment $\overline{AD}$ intersects segment $\overline{CE}$ at $B$. What is the length of segment $\overline{CE}$ ?

**2** 1600.io 19.4




In the figure above, $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ are parallel. Points $B$ and $E$ lie on $\overline{AC}$ and $\overline{FD}$, respectively. If $AB = 4$, $BC = 5.5$, and $FE = 3.5$, what is the length of $\overline{ED}$ to the nearest tenth?

A) 2.6

B) 4.8

C) 5.2

D) 6.3

## 19.5 Trigonometric Ratios

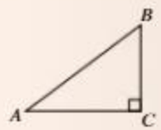
There are three important side length ratios in right triangles that form the basis of the trigonometry questions on the test.

### The Three Sides of Right Triangles

In a right triangle, the longest side, which is across from the right angle, is called the **Hypotenuse**. In the right triangle below, side $\overline{AB}$ is the hypotenuse.

The side referred to as the **Opposite** side is the side across from a specified angle. In the triangle below, side $\overline{BC}$ is the opposite side for $\angle A$, and side $\overline{AC}$ the opposite side for $\angle B$.

The side referred to as the **Adjacent** side is the side that, along with the hypotenuse, forms a specified angle. In the triangle below, side $\overline{AC}$ is the adjacent side for $\angle A$, and side $\overline{BC}$ is the adjacent side for $\angle B$.




The figure above will be referenced in the following discussions of each of the three basic trigonometric ratios.

### Sine

**The sine of an angle $\theta$, denoted as $\sin\theta$ or $\sin(\theta)$, is the ratio of the opposite side to the hypotenuse, and the sines of any congruent angles are always equal (equal angles have the same sine)**. To be clear, only the measure of an angle determines the ratio of the two involved side lengths, and thus the sine of that angle is dependent only on the measure of the angle. The *specific* lengths of the two sides are irrelevant, as it is only the *ratio* of their lengths that is determined by the angle.

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

For example, in the triangle above, the sine of $\angle A$ is equal to $\frac{BC}{AB}$, and the sine of $\angle B$ is equal to $\frac{AC}{AB}$.

### Cosine

**The cosine of an angle $\theta$, denoted as $\cos\theta$ or $\cos(\theta)$, is the ratio of the adjacent side to the hypotenuse, and the cosines of any congruent angles are always equal (equal angles have the same cosine)**.

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

For example, in the triangle above, the cosine of $\angle A$ is equal to $\frac{AC}{AB}$, and the cosine of $\angle B$ is equal to $\frac{BC}{AB}$.





### Tangent

**The tangent of an angle $\theta$, denoted as $\tan\theta$ or $\tan(\theta)$, is the ratio of the opposite side to the adjacent side, and the tangents of any congruent angles are always equal (equal angles have the same tangent)**.

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

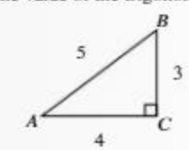
For example, in the triangle above, the tangent of $\angle A$ is equal to $\frac{BC}{AC}$, and the tangent of $\angle B$ is equal to $\frac{AC}{BC}$.

Because **S**ine is **O**pposite over **H**ypotneuse, **C**osine is **A**djacent over **H**ypotenuse, and **T**angent is **O**pposite over **A**djacent, the mnemonic device **SOH-CAH-TOA** is commonly used by students to remember those ratios.

It is essential to understand that these trigonometric functions' values are derived from the side ratios **in a right triangle**, and their **values for a given angle don't change** if the angle appears in a figure that is not a right triangle; the **sine of an angle is the sine of that angle regardless of what type of figure includes that angle**. However, you can **only calculate these functions from a figure when the angle is part of a right triangle**; DON'T make the mistake of calculating a trigonometric function from the side length ratios in an oblique (non-right) triangle!

**SkillDrill 19.5-1**

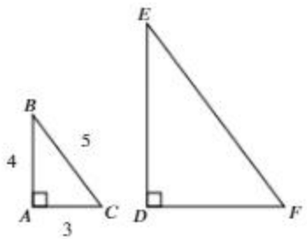
**Directions:** Given the triangle below, determine the value of the trigonometric ratio that you are asked to find.




1. Find $\cos A$
2. Find $\sin A$
3. Find $\tan A$
4. Find $\cos B$
5. Find $\sin B$
6. Find $\tan B$

**Example 19.5-1**

**1** 1600.io 19.5




In the figure above, triangle $ABC$ is similar to triangle $DEF$. What is the value of $\cos(E)$?

A) $\frac{4}{3}$

B) $\frac{4}{5}$

C) $\frac{3}{4}$

D) $\frac{3}{5}$

**Solution**

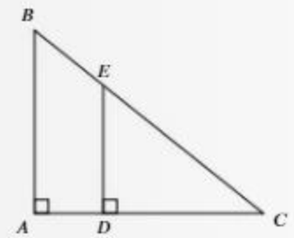
1. Since $\angle E$ corresponds to $\angle B$ in the similar triangles, we know that $\cos(E) = \cos(B)$.
2. Since the cosine is the ratio of the adjacent side (the side $\overline{AB}$, which measures 4 units, is adjacent to $\angle B$) to the hypotenuse (the hypotenuse, across from the 90° angle, measures 5 units), the cosine of $\angle B$ (and therefore, the cosine of $\angle E$) is equal to $\frac{4}{5}$.
3. The answer is B.





**Example 19.5-2**

**2** 1600.io 19.5

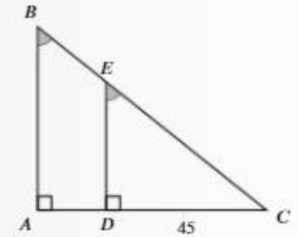



Note: Figure not drawn to scale.

In the figure above, $ABC$ and $DEC$ are right triangles. If $CD = 45$ and the tangent of angle $ABC$ is 1.25, what is the length of segment $\overline{ED}$?

**Solution**

1. The triangles $\triangle ABC$ and $\triangle DEC$ must be similar. They both contain $\angle C$ and a right angle, which means that the third angles ($\angle ABC$ and $\angle DEC$) are also equal to each other. Since all three angles have a match, the triangles are similar.




2. Since $\angle ABC$ and $\angle DEC$ have the same measure, then the tangent of $\angle ABC$, which is equal to 1.25 (if you are comfortable with fractions, you may choose to represent this as $\frac{5}{4}$ so you don't need your calculator for this problem), is equal to the tangent of $DEC$.

$$\tan \angle ABC = \tan \angle DEC$$
$$1.25 = \tan \angle DEC$$

3. The tangent of an angle is equal to the ratio of the opposite side to the adjacent side. The side opposite of $\angle DEC$ is $\overline{CD}$ (which we are told has a length of 45), and the side adjacent to $\angle DEC$ is $\overline{ED}$ (whose length we need to find). We can substitute $\frac{45}{ED}$ for $\tan \angle DEC$ and solve for $ED$.

$$1.25 = \tan \angle DEC$$

$$1.25 = \frac{45}{ED}$$

$$1.25ED = 45$$

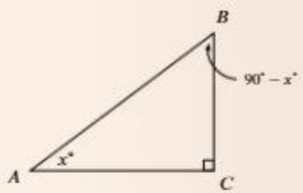
$$ED = \frac{45}{1.25}$$

$$ED = 36$$

4. The answer is 36.

## $\sin(x) = \cos(90 - x)$ and $\cos(x) = \sin(90 - x)$

**The vast majority of problems that involve sines and cosines on this test make use of this fact**. Because the two angles that are not the right angle in a right triangle must sum to 90°(all three must add up to 180°, and the right angle measures 90°, so the other two angles must add up to 90° as well), then if one of the angles measures $x°$, the other angle measures $90° - x°$. The side **opposite** of the angle measuring $x°$ is also **adjacent** to the angle measuring $90° - x°$; similarly, the side **adjacent** to the angle measuring $x°$ is also **opposite** of the angle measuring $90° - x°$.




In the right triangle shown above, $\sin(x) = \frac{BC}{AB}$ and $\cos(90 - x) = \frac{BC}{AB}$ as well, so you can see that $\mathbf{\sin(x) = \cos(90 - x)}$.

Similarly, $\cos(x) = \frac{AC}{AB}$ and $\sin(90 - x) = \frac{AC}{AB}$, thus demonstrating that $\mathbf{\cos(x) = \sin(90 - x)}$.

By the same token, if it is true for any two angles $x$ and $y$ that $\sin x = \cos y$ or $\cos x = \sin y$, then $y = 90 - x$ and $x = 90 - y$ (the two angles must be complementary, which means they sum to 90°).

Keep these principles in mind because there is a high likelihood that any problems that mention sine and cosine will integrate these facts in some manner.

### Example 19.5-3

**3** 1600.io 19.5

In a right triangle, one angle measures $x°$, where $\sin x° = \frac{3}{5}$. What is $\cos(90 - x)$ ?





### Solution

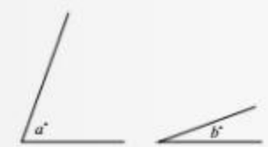
1. Since the $\sin(x) = \cos(90 - x)$, the answer is $\frac{3}{5}$.

### Notes

This problem could be done in a longer way based on some other right triangle properties (which will be discussed shortly), as could many of the trigonometry problems; however, knowing this one fact saves a lot of time and avoids potential mistakes on many problems.

### Example 19.5-4

**4** 1600.io 19.5




Note: Figure not drawn to scale.

The angles shown above are acute and $\cos(a°) = \sin(b°)$.
If $a = 5k + 15$ and $b = 4k - 24$, what is the value of $k$ ?

A) 5

B) 5.5

C) 9

D) 11

### Solution

1. Since the cosine of $a$ is equal to the sine of $b$, then it must be true that $a = 90 - b$ and $b = 90 - a$. We can use either equation (which are the same equation in terms of different constants) to eliminate one of the constants, but let's use $a = 90 - b$, substituting in the expressions for $a$ and $b$ that are in terms of $k$ so that we can solve for $k$.

$$a = 90 - b$$
$$5k + 15 = 90 - (4k - 24)$$
$$5k + 15 = 90 - 4k + 24$$
$$9k + 15 = 114$$
$$9k = 99$$
$$k = 11$$

2. The answer is D.

**Notes**

Even though the two angle are not shown in right triangles, the fact remains that when the sine of one angle equals the cosine of another angle, those angles must add up to 90° because of how these functions are defined.

**Section 19.5 Suggested Problems from Real Tests**

View related real-test problems in SMTEx

- Test 1-NC-19
- Test 3-C-23
- Test 4-NC-17
- Test 8-C-36
- May 2018-C-36
- Mar 2021-C-33
- Apr 2021-NC-8
- May 2021 (US)-NC-6
- May 2021 (US)-C-12
- May 2021 (US)-C-21

## Section 19.5 Practice Problems

**1** 1600.io ? 19.5

The two acute angles of a right triangle have degree measures of $x$ and $y$. If $\sin x = \frac{3}{5}$, what is the value of $\cos y$ ?

**2** 1600.io ? 19.5

In a right triangle, the tangent of one of the two acute angle is $\sqrt{3}$. What is the tangent of the other acute angle?

A) $-\sqrt{3}$

B) $-\frac{1}{\sqrt{3}}$

C) $\frac{1}{\sqrt{3}}$

D) $\sqrt{3}$

**3** 1600.io ? 19.5

In the triangle above, the sine of $x^\circ$ is 0.8. What is the cosine of $y^\circ$ ?

**4** 1600.io ? 19.5

If $\cos y^\circ = b$, which of the following must be true for all values of $y$ ?

A) $\sin y^\circ = b$

B) $\cos(90^\circ - y^\circ) = b$

C) $\sin(90^\circ - y^\circ) = b$

D) $\cos(y^2)^\circ = b^2$





## 19.6 Perimeter, Area, Pythagorean Theorem, and Special Right Triangles

The **perimeter of a polygon is the sum of the lengths of the outside edges**. This is a not a very deep concept, but the term perimeter has been used several times on the test, so it needs to be mentioned.

For example, in the figures below, the perimeter of the triangle is 9 because $2 + 3 + 4 = 9$, and the perimeter of the rectangle is 10 because $2 + 3 + 2 + 3 = 2(2) + 2(3) = 4 + 6 = 10$.

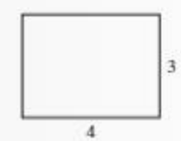
Be sure that when a problem requires you to calculate the *perimeter* of a figure, you don't mistakenly calculate the *area* instead; we're specifically cautioning the reader about this because we have seen this specific error many times.

With that out of the way, let's move on to the much more interesting measurement of **area**.

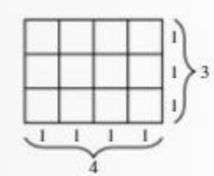
The formula that gives the area of a triangle is rarely needed, but it has appeared on the test, and it is also useful in proving an important theorem in this section.

Before we get back into triangles, let's look for a moment at rectangles (of which squares are a subset) and the way the area of rectangles is calculated.

Let's look at a rectangle whose width is 4 units and whose height is 3 units (any units of length will do); we'd refer to this as a "4 by 3" or "4 × 3" rectangle.




Area is measured in square units, so let's look at how we can determine the number of square units in our figure. The rectangle can be divided into 1 unit squares, each of which has an area of 1 unit squared, which is our unit of measure for area. The number of 1-unit-squared squares in the rectangle is the area of the rectangle.




In a $4 \times 3$ rectangle, there are $4 \times 3 = 12$ squares (each with an area of 1 unit squared), so the area of the rectangle is 12 units squared.

## Area of a Rectangle

**For a rectangle with length $\ell$ and width $w$, the area $A$ is found with the following formula:**

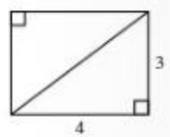
$$A = \ell w$$

Area is measured in units of length squared such as inches squared ($in^2$) or feet squared ($ft^2$).

A square is simply a special type of rectangle for which all four sides are equal, and so we normally call the side length $s$, and write the area as the following:

$$A = s^2$$

If we draw a diagonal across the rectangle, we can divide it into two equal halves, each of which is a right triangle.

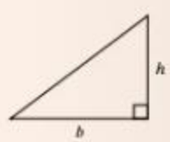



Since the area of the rectangle has been split in half, each right triangle has an area equal to half of that of the rectangle, thus leading us to the formula for the area of a triangle.

## Area of a Triangle

**For a triangle with a base length $b$ and height $h$, the area $A$ is found with the following formula:**

$$A = \frac{1}{2}bh$$




The base and height are simple enough to identify when you are given a right triangle that is already rotated to a convenient orientation, with the base side horizontal. However, triangles can be rotated in any orientation, and not all triangles are right triangles (and yet the formula still holds true). In general, if a triangle is in a confusing orientation, you should redraw it so that one side of the triangle is horizontal and can clearly be called the base of the triangle.

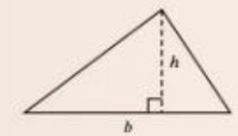
For triangles that are not right triangles, you often have to do some work to determine the height of the triangle, which is where some of the other concepts in this chapter will come into play. Before we get into actually finding the heights, however, you should know that the **height is always perpendicular to the base**.



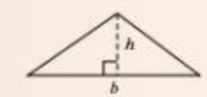


## Area of Acute and Obtuse Triangles

For acute triangles (all angles smaller than 90°) you usually have to draw in another line perpendicular to the base that divides the triangle into two right triangles.



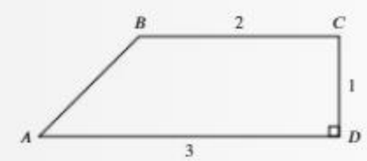

For obtuse triangles (one angle is larger than 90°), depending on the orientation, you may need to draw the perpendicular line for the height outside of the triangle as in the first picture below (note that the base line is extended for clarity so the right angle between the extended base line and the height line can be shown, but the length of the base is just the length of bottom side of the triangle not including the dashed extension). As a good alternative, you can always rotate the triangle (as in the second picture below), so that the perpendicular height line drops from the largest angle.

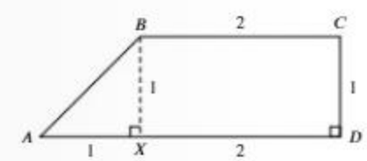



Occasionally, you need to work with the area of other figures such as trapezoids and parallelograms (and regular hexagons, which we already saw could be divided into six congruent equilateral triangles whose height can be calculated quickly if you're familiar with 30-60-90 triangles, which will be discussed shortly).

You do not need to remember a formula for the area of trapezoids and parallelograms because you can divide trapezoids and parallelograms into right triangles and rectangles that make calculating the area easy.

For example, the trapezoid shown below can be divided into a rectangle and a right triangle by drawing a vertical line from point $B$ to $\overline{AD}$ (we'll call the intersection of this vertical line and $\overline{AD}$ the point $X$). The new line splits $\overline{AD}$ into two segments: the rightmost segment, $\overline{XD}$, measures the same as $\overline{BC}$, and the remaining segment, $\overline{AX}$, which is the base of the right triangle, is equal to 1 because $3 - 2 = 1$.



The area of the trapezoid is equal to the sum of the areas of the rectangle and the right triangle.

$$A_{trap} = A_{rect} + A_{tri}$$

$$A_{trap} = \ell_{rect} w_{rect} + \frac{1}{2} b_{tri} h_{tri}$$

$$A_{trap} = (BC)(CD) + \frac{1}{2}(AX)(BX)$$

$$A_{trap} = (2)(1) + \frac{1}{2}(1)(1)$$

$$A_{trap} = 2 + \frac{1}{2}$$

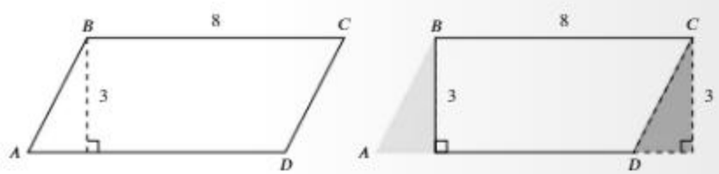
$$A_{trap} = \frac{4}{2} + \frac{1}{2}$$

$$A_{trap} = \frac{5}{2}$$

You might also find it intuitively appealing to learn that another way to compute the area of a trapezoid is to take the average of the lengths of the two unequal sides and multiply that by the height. Here, the average is $\frac{3+2}{2} = \frac{5}{2}$, and the height is 1, so the area is $1\left(\frac{5}{2}\right) = \frac{5}{2}$.

Parallelograms are interesting because you can "cut off" a right triangle from one side and "move" the right triangle to the other side of the parallelogram, filling in the previously blank corner and forming a rectangle.

For example, in the figure below, we can form a right triangle by drawing a vertical line from point $B$ to $\overline{AD}$, and then moving this triangle to the other side to form a rectangle. Even though we removed part of $\overline{AD}$ temporarily when we "cut off" the right triangle, we immediately replaced the missing piece and so the length of the bottom horizontal line is still 8 (by the nature of parallelograms, $\overline{BC}$ must also have a length of 8).




The parallelogram has been transformed into a rectangle whose area is $(3)(8) = 24$.





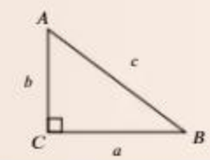
## Pythagorean Theorem and Special Right Triangles

Before we do any area-based problems, let's talk about how distances are determined using right triangles and the Pythagorean Theorem.

### Pythagorean Theorem

For a right triangle with legs measuring $a$ units and $b$ units and hypotenuse measuring $c$ units, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides:

$$a^2 + b^2 = c^2$$



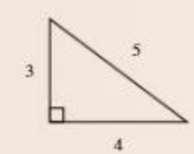

So far on the test, the use of the Pythagorean Theorem can be completely avoided (though it is still useful to know in case you do not remember the following information) because almost every right triangle used in known problems is a member of one of two special subsets of right triangles: **Side-Based Special Right Triangles** (also known as **Pythagorean Triples**) and **Angle-Based Special Right Triangles**. Using these special right triangles allows you to find unknown side lengths (when two are known) without ever having to use the Pythagorean Theorem or any weird trigonometry, and the angle-based special right triangles even allow you to find two unknown side lengths based on one known side without resorting to the use of a calculator to do any complicated trigonometry.

### Side-Based Special Right Triangles: Pythagorean Triples

There are some special right triangles in which all three sides of the triangle are whole numbers (or are in whole number ratios with each other). Let's explain by looking at the most common Pythagorean Triple first.

### 3:4:5 Triangles

In the simplest example, in a right triangle with legs of length 3 and 4, the hypotenuse has length 5.

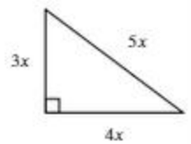



$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$
$$25 = 25$$

If you are shown a right triangle where one leg is 3 and the other is 4, you should realize *immediately* that the hypotenuse is 5. If you are given a right triangle where the hypotenuse is 5 (it is important that the hypotenuse must be the longest side) and one of the legs is 3, then you know the other leg is 4 (and vice versa).

Once again, it is important not to assume a 3:4:5 triangle exists where it doesn't. **If the hypotenuse is 4 and one of the legs is 3, the other leg is NOT 5 because the hypotenuse must be the longest side**. In this case, you'd have to use the Pythagorean Theorem to find the length of the other leg (spoiler alert: it's $\sqrt{7}$).

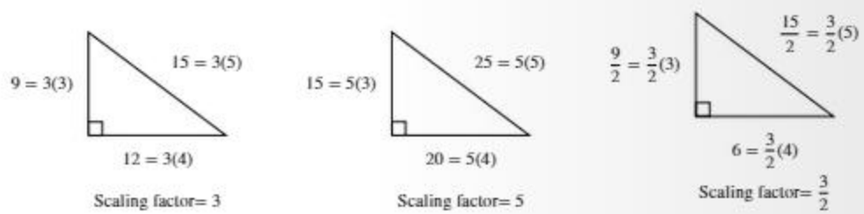
**Note that the sides do not have to actually be 3, 4, and 5**; those values just represent the *relative* lengths of the sides in lowest terms. **The sides can be any set of values in the same ratio: $3x$, $4x$, and $5x$**, where each side length can be represented as one of the terms in the underlying 3:4:5 ratio multiplied or scaled by the same factor, here expressed as $x$. Note that this scaling factor does not have to be an integer; fractional or irrational side lengths can be involved in a ratio expressible as a set of integers, though the test tends to stick to whole number side lengths.




$$a^2 + b^2 = c^2$$
$$(3x)^2 + (4x)^2 = (5x)^2$$
$$9x^2 + 16x^2 = 25x^2$$
$$25x^2 = 25x^2$$

Because the sides just have to be in a 3 to 4 to 5 ratio, you may see 6, 8, 10 triangles (each side corresponds to the values in the 3:4:5 ratio multiplied by the factor 2), 9, 12, 15 triangles (the scaling factor is 3), and so on.

All of the triangles below qualify as 3:4:5 right triangles.




### Other Common Pythagorean Triples

Everything that is true for 3:4:5 triangles is also true for 5:12:13 triangles, 7:24:25 triangles, 8:15:17 triangles, and 20:21:29 triangles.

**3:4:5 5:12:13 7:24:25 8:15:17 20:21:29**

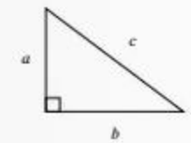
There are others, but not even all of these have been spotted on the test so far, so a greater list is unnecessary.





**SkillDrill 19.6-1**

**Directions:** Given the side-based special right triangle below (which is not drawn to scale) and the lengths of two of the sides, find the length of the remaining side.




**1.** $a = 3$, $b = 4$; find $c$ **2.** $a = 12$, $c = 20$; find $b$ **3.** $b = 48$, $c = 50$; find $a$ **4.** $a = 36$, $b = 15$; find $c$

**5.** $a = 80$, $b = 60$; find $c$ **6.** $a = \frac{3}{2}$, $b = 2$; find $c$ **7.** $b = 45$, $c = 51$; find $a$ **8.** $a = 33$, $c = 55$; find $b$

**Example 19.6-1**

**1** 1600.io 19.6

In the right triangle above, the tangent of $\angle A$ is $\frac{12}{5}$.
What is the cosine of $\angle B$ ?

A) $\frac{5}{13}$

B) $\frac{12}{13}$

C) $\frac{13}{12}$

D) $\frac{13}{5}$

### Solution 1

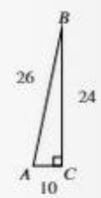
Note that in this problem, the **actual lengths of the sides are irrelevant** if we make smart use of the tangent information that is given to us. We only need to deal with the **ratio of relative side lengths** in lowest terms when we are looking for the value of trigonometric ratios.

1. Since we are told that $\tan A = \frac{12}{5}$, we know that this must be a 5:12:13 triangle (the two legs are in a ratio of 12:5, so the hypotenuse must be 13 times whatever scale factor is common to all the sides).
2. The side adjacent to $\angle B$ corresponds to the "12 side" of the triangle, so $\cos B = \frac{12}{13}$.
3. The answer is B.

### Solution 2

1. In order to find the value of $\cos B$, we need to know the length of the side adjacent to $\angle B$, which is $\overline{BC}$. The tangent of $\angle A$ is the ratio of side $\overline{BC}$ (opposite) to side $\overline{AC}$ (adjacent), which must be equal to $\frac{12}{5}$. To find $BC$, set up a proportion by setting the two fractions equal to each other, and substitute 10, the length of side $\overline{AC}$, in for $AC$, then solve for $BC$.

$$\tan(\angle A) = \frac{12}{5}$$
$$\frac{BC}{AC} = \frac{12}{5}$$
$$\frac{BC}{10} = \frac{12}{5}$$
$$BC = 10\left(\frac{12}{5}\right)$$
$$BC = 2(12)$$
$$BC = 24$$




2. The cosine of $\angle B$ is equal to the ratio of side $\overline{BC}$ (the adjacent side, which has a length of 24) to side $\overline{AB}$ (the hypotenuse, which has a length of 26).

$$\cos(\angle B) = \frac{BC}{AB}$$
$$\cos(\angle B) = \frac{24}{26}$$
$$\cos(\angle B) = \frac{12}{13}$$

3. The answer is B.





### Solution 3

If you choose to use the Pythagorean Theorem, you may need a calculator for this problem (or you could find perfect square factors of 576 to simplify the process). The other two solutions make better use of the given information which speeds up the process and limits your need to divert your eyes from the work you can do directly on paper.

1. We can find the length of leg $\overline{BC}$ using the Pythagorean Theorem.

$$AB^2 + BC^2 = AC^2$$
$$(10)^2 + BC^2 = (26)^2$$
$$100 + BC^2 = 676$$
$$BC^2 = 576$$
$$BC = \sqrt{576}$$
$$BC = 24$$

2. The cosine of $\angle B$ is equal to the ratio of side $\overline{BC}$ (the adjacent side, which has a length of 24) to side $\overline{AB}$ (the hypotenuse, which has a length of 26).

$$\cos(\angle B) = \frac{BC}{AB}$$
$$\cos(\angle B) = \frac{24}{26}$$
$$\cos(\angle B) = \frac{12}{13}$$

3. The answer is B.

### Notes

It should be noted that Solution 2 and especially Solution 3 can be made simpler if you realize that we are only interested the *relative* lengths of the sides for the final answer. For example, in Solution 3, you could use the Pythagorean Theorem with the relative lengths of $AC$ and $AB$ (in lowest terms), which are 5 and 13, respectively.

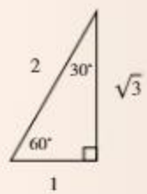
$$AB^2 + BC^2 = AC^2$$
$$(5)^2 + BC^2 = (13)^2$$
$$25 + BC^2 = 169$$
$$BC^2 = 144$$
$$BC = \sqrt{144}$$
$$BC = 12$$

## Angle-Based Special Right Triangles

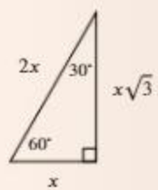
Some right triangles appear frequently on the test, not only because they arise from dividing regular polygons (which we will see shortly) but also because their side lengths are in easy-to-write-and-remember ratios.

### 30-60-90 Triangles

The first important angle-based special right triangle is a 30-60-90 right triangle, which has a 30°, 60°, and 90° angle. Its sides, from shortest to longest, are in a ratio of $1:\sqrt{3}:2$ (you might find this easier to memorize by reordering these values and taking advantage of the "1-2-3" pattern in $1:2:\sqrt{3}$—just be sure you know that this maps to short side : hypotenuse : long side and does not go in increasing order).




The shorter leg, which is across from the smallest angle (the 30° angle), is usually treated as the basis for the other lengths, so it is often represented with a single constant such as $x$. Across from the 60° angle is the longer leg, and its length is always the length of the shorter leg times $\sqrt{3}$, so it is marked $x\sqrt{3}$. Finally, the hypotenuse, across from the right angle, is twice the shorter leg, so it has a length of $2x$.

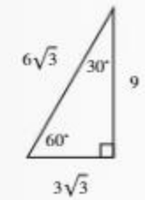



If we know that the shortest side measures 2 units, then we know that the longer leg (across from the 60° angle) measures $2\sqrt{3}$ units, and the hypotenuse measures 4 units.

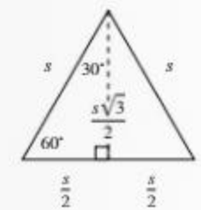




One variation that tends to confuse students is when the hypotenuse or the shortest side have a length that includes a factor $\sqrt{3}$ (rather than being an integer) because students assume that any side with a $\sqrt{3}$ showing must be the longer leg. For example, in the triangle below, the shortest side measures $3\sqrt{3}$ units, so the hypotenuse measures $6\sqrt{3}$ units (twice the length of the shorter leg), and the longer leg (across from the 60° angle, whose length must be $\sqrt{3}$ times the length of the shorter leg) actually measures $3\sqrt{3}(\sqrt{3}) = 3(3) = 9$ units.



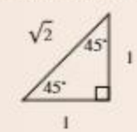

This type of special right triangle is especially common on the test, presented in both direct and indirect ways. **One of the most common ways that 30-60-90 triangles are hidden is in equilateral triangles**. If you divide an equilateral triangle with side lengths $s$ by drawing a line from any vertex to its opposite side and intersecting that opposite side at a right angle (such a line is called an **altitude**), the triangle is divided into two congruent (equal in shape and size) 30-60-90 triangles where the short legs of these triangles have length $\frac{s}{2}$ and the height of the triangles is $\frac{s\sqrt{3}}{2}$, in keeping with the ratios established above.



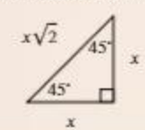

Note that this could be useful when dealing with regular hexagons, which can be divided into 6 equilateral triangles, which can then be divided into 30-60-90 triangles.

### 45-45-90 Triangles

The second important angle-based special right triangle is a 45-45-90 right triangle. Since **two angles are the same, these triangles are isosceles, meaning that both legs are the same length**. The sides are in a ratio of $1:1:\sqrt{2}$.

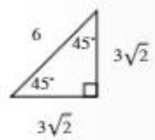



The legs are usually treated as the basis for the hypotenuse, so their length is usually just represented with a single constant, such as $x$. The hypotenuse, across from the right angle, has a length of $x\sqrt{2}$ (this can be proven with the Pythagorean Theorem).

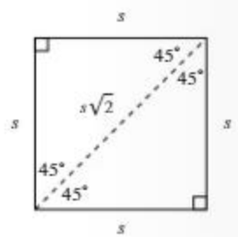



If we know that the legs measure 2 units, then we know that the hypotenuse measures $2\sqrt{2}$ units.

One variation that tends to confuse students is when the length of the legs include a factor $\sqrt{2}$ (rather than being an integer) because students assume that any side with a $\sqrt{2}$ showing must be the hypotenuse. For example, in the triangle below, the legs measure $3\sqrt{2}$ units, so the hypotenuse actually measures $3\sqrt{2}(\sqrt{2}) = 3(2) = 6$ units.

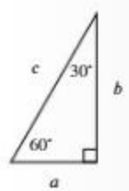



This type of special right triangle is especially common on the test, presented in both direct and indirect ways. **One of the most common ways that 45-45-90 triangles are hidden is in squares**. If you divide a square with side lengths $s$ in half diagonally from one vertex to its opposite, the square is divided into two congruent 45-45-90 triangles where the legs have length $s$ and the hypotenuse has length $s\sqrt{2}$.




**SkillDrill 19.6-2**

**Directions:** Given the 30-60-90 special right triangle below and the length of one of the sides, find the length of the two remaining sides.




**1.** $a = 3$; find $b$ and $c$ **2.** $a = 10$; find $b$ and $c$ **3.** $b = 5\sqrt{3}$; find $a$ and $c$ **4.** $b = 3$; find $a$ and $c$

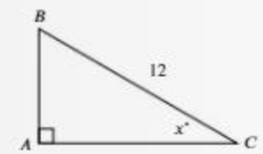
**5.** $c = 4$; find $a$ and $b$ **6.** $c = 7$; find $a$ and $b$ **7.** $a = \frac{3\sqrt{3}}{4}$; find $b$ and $c$ **8.** $b = 11\sqrt{3}$; find $a$ and $c$





**Example 19.6-2**

**2** 1600.io 19.6



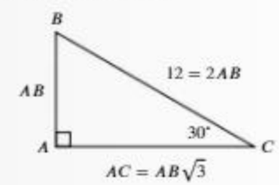

Note: Figure not drawn to scale.

In right triangle $ABC$ above, $BC = 12$. If the cosine of $x°$ is $\frac{\sqrt{3}}{2}$, what is the length of $\overline{AB}$?

**Solution 1**

1. The cosine of $x°$ is equal to the adjacent side, $\overline{AC}$, divided by the hypotenuse, $\overline{BC}$, whose length is 12.

   You should realize that this is a 30-60-90 triangle because the ratio of one side to the hypotenuse is $\sqrt{3}$ to 2, so $x = 30$. The length of the hypotenuse, therefore, is twice the length of the other side, $\overline{AB}$, because the side-length ratio for a 30-60-90 triangle is $1 : \sqrt{3} : 2$ (short leg : long leg : hypotenuse).




2. Since the hypotenuse has a length of 12, the length of $\overline{AB}$ is equal to 6.
3. The answer is 6.

**Solution 2**

1. The cosine of $x°$ is equal to the adjacent side, $\overline{AC}$, divided by the hypotenuse, $\overline{BC}$, whose length is 12.

   Set $\frac{AC}{BC}$ equal to $\frac{\sqrt{3}}{2}$ and solve for $AC$.

$$\cos x° = \frac{\sqrt{3}}{2}$$
$$\frac{AC}{BC} = \frac{\sqrt{3}}{2}$$
$$\frac{AC}{12} = \frac{\sqrt{3}}{2}$$
$$AC = 6\sqrt{3}$$

2. We can use the Pythagorean Theorem to find the length of $\overline{AB}$.

$$AB^2 + AC^2 = BC^2$$
$$AB^2 + (6\sqrt{3})^2 = (12)^2$$
$$AB^2 + 36(3) = 144$$
$$AB^2 + 108 = 144$$
$$AB^2 = 36$$
$$AB = 6$$

3. The answer is 6.

### Section 19.6 Suggested Problems from Real Tests

- Test 2-C-30
- Test 3-NC-20
- Test 5-C-19
- Test 6-NC-18
- Test 7-C-36
- Test 8-NC-5
- Test 9-NC-19
- Test 10-C-30
- Apr 2017-C-28
- Apr 2017-C-29
- Apr 2018-C-30
- Apr 2018-C-33
- Mar 2019-NC-6
- Apr 2019-NC-11
- Mar 2020-NC-11
- Mar 2020-C-16
- Mar 2020-C-35
- Oct 2020-C-7
- Mar 2021-NC-17
- Mar 2021-C-25
- May 2021 (US)-C-31
- May 2021 (US)-C-38
- May 2021 (Int)-NC-19
- May 2021 (Int)-C-17
- May 2021 (Int)-C-31

View related real-test problems in SMTEx

SMTEx

## Section 19.6 Practice Problems

**1** 1600.io 19.6

Triangle $ABC$ and $DEF$ are similar triangles, where $\overline{AB}$ and $\overline{DE}$ are corresponding sides. If $DE = \frac{3}{2}AB$ and the perimeter of triangle $ABC$ is 30, what is the perimeter of triangle $DEF$?

A) 20
B) 30
C) 35
D) 45

**2** 1600.io 19.6

Note: Figure not drawn to scale.

In the right triangle above, $x = 60$. What is the length of side $\overline{AB}$?

A) 4
B) 5
C) 6
D) It cannot be determined from the information given.



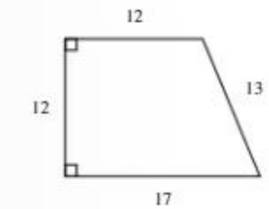


**3** 1600.io 19.6

The length of a rectangular card is 9 times the width of the card. If the area of the card is 36 square inches, what is the width of the card, in inches?

A) 2
B) 4
C) 6
D) 18

**4** 1600.io 19.6




What is the area, in square units, of the figure shown?

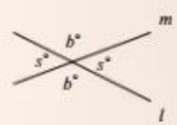
A) 30
B) 144
C) 169
D) 174

# CHAPTER 19 RECAP

- Angles are formed by the intersection of two lines; the point at which the two lines meet is called the **vertex** (plural, **vertices**).
- Angles are most commonly measured in **degrees**, which is denoted by the degree symbol (°).
- A 90° angle is a **right angle** (the lines forming the angle are perpendicular). Right angles are often marked with a small square instead of an arc and angle measurement.

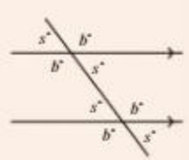
  Any angles that together form a right angle sum to 90°. When two angles form a 90° angle they are called **complementary angles**.
- A 180° angle results in a straight line.

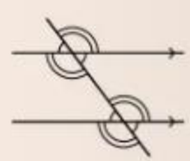
  Any angles that together form a straight line sum to 180°. When two angles form a 180° angle they are called **supplementary angles**.
- Angles that measure more than 0° and less than 90° are called **acute angles**. Angles that measure more than 90° and less than 180° are called **obtuse angles**. Angles greater than 180° are called **reflex** angles.
- Whenever two lines intersect, four angles are formed, and the angles opposite each other, called **Vertical Angles**, are equal.




  As previously explained, angles that form a line sum to 180°. Therefore, $s + b = 180$.
- To indicate congruence (equality of measure) for sides and angles, those features can be marked with dashes or arcs, called hatch (or hash or tick) marks. Sides marked with the same number of hatch marks are congruent, and angles marked with the same number of arcs are congruent.

  Parallel lines can be marked by arrow heads. Lines marked with the same number of arrow heads are parallel.
- When a diagonal (non-perpendicular) line runs across two parallel lines, eight angles are formed. The four larger (obtuse) angles are all equal to each other. Similarly, the four smaller (acute) angles are also equal to each other.



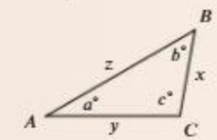







- **The interior angles of any triangle sum to 180°.** The length of each side is positively correlated with the size of the angle opposite that side, so the **largest side of a triangle is always across from the largest angle of the triangle, and the smallest side is across from the smallest angle**.

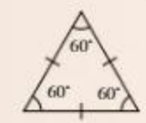
  In the triangle below (drawn to scale), $a + b + c = 180$ because the sum of any triangle's interior angles is 180°. Since $\angle C$ is the largest angle, $z$ is the longest side. Similarly, since $\angle A$ is the smallest angle, $x$ is the shortest side.




- **In isosceles triangles, two of the angles are equal to each other, and the sides across from those angles are therefore also equal to each other.**

- **In equilateral triangles, all of the sides are equal to each other, and therefore all of the angles are also equal.**

  Because all three of the angles are equal and they must sum to 180°, each angle is $\frac{180°}{3} = 60°$.

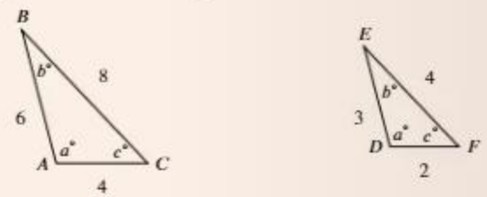



- For a polygon with $n$ sides and angles, the sum of the interior angles can be found using the following formula:

$$\text{Sum of Interior Angles} = 180(n - 2)$$

- In a **regular polygon**, all sides are equal, and all angles are equal. The measure of any angle is equal to the sum of the interior angles divided by the number of angles.
- **Regular hexagons** (six-sided polygons) can be split into 6 equilateral triangles if we draw a line from the center of the figure to each of the six vertices.

- Two triangles are called **Similar Triangles** when they have the same interior angles. Put another way, each angle in one triangle has a corresponding match in the other triangle.

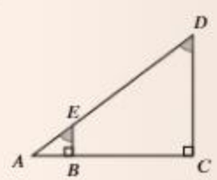



In the figure above, $\triangle ABC$ is similar to $\triangle DEF$ because all of the angles in $\triangle ABC$ have a matching angle in $\triangle DEF$. Notice that the triangles are different sizes, however. **In similar triangles, all of the angles have a match, but the sides are not *necessarily* equal. However, sides across from matching angles are always in the same proportion to each other.**

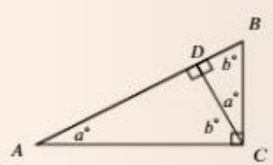
In the figure shown above, $\overline{AB}$ corresponds to $\overline{DE}$; $\overline{BC}$ corresponds to $\overline{EF}$; and $\overline{AC}$ corresponds to $\overline{DF}$. In these particular triangles, the sides in $\triangle ABC$ are all twice the length of the corresponding sides in $\triangle DEF$.

- **If we know (or can determine) that there are two angles that are the same in two triangles, the third angle must also be the same** because the three angles in each triangle must sum to the same value (180°). This, in turn, means that **the two triangles are similar triangles**.
- Similar triangles are commonly depicted as one triangle inside of another, where you can show that the triangles must be similar because all the angles in both triangles will have a match.

In the figure below, $\triangle ACD$ is similar to $\triangle ABE$ because they both contain $\angle A$ and a right angle, which means the third angles in each triangle must also be equal.




- You can use part-to-whole or part-to-part ratios to solve for lengths in divided triangles. When a **triangle is divided by a line parallel to one side, the other two sides are divided proportionally**; this is the **Triangle Proportionality Theorem**.
- Right triangles can be divided with a single line that creates three similar triangles. In the figure below, a line is drawn from the right angle vertex $C$ that intersects the hypotenuse at a right angle at point $D$. The larger right triangle is split into two smaller right triangles, $\Delta ADC$ and $\Delta CDB$, both of which are similar to the original triangle $\Delta ACB$ and thus to each other as well.

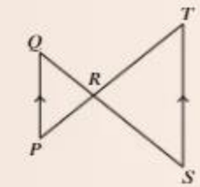







- Similar triangles arise in arrangements such as that shown in the figure below, with the two similar triangles touching at one vertex (where there is a vertical angle) and each having a side that is parallel to a side in the other triangle.

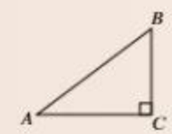
In the figure below, the angles meeting at point $R$ are vertical angles (and thus are equal to each other), and side $\overline{PQ}$ is parallel to side $\overline{ST}$.




- In a right triangle, the side that is across from the right angle is called the **hypotenuse** ($\overline{AB}$ in the triangle below).

The side across from an acute angle is called the **opposite** side. In the triangle below, side $\overline{BC}$ is the opposite side for $\angle A$, and side $\overline{AC}$ is the opposite side for $\angle B$.

The side that forms an angle with the hypotenuse is called the **adjacent** side of that angle. In the triangle below, side $\overline{AC}$ is the adjacent side for $\angle A$, and side $\overline{BC}$ is the adjacent side for $\angle B$.



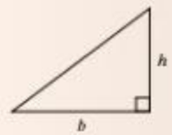

| | | |
|---|---|---|
| sine | $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ | $\sin A = \frac{BC}{AB}$ |
| | | $\sin B = \frac{AC}{AB}$ |
| cosine | $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ | $\cos A = \frac{AC}{AB}$ |
| | | $\cos B = \frac{BC}{AB}$ |
| tangent | $\cos\theta = \frac{\text{opposite}}{\text{adjacent}}$ | $\tan A = \frac{BC}{AC}$ |
| | | $\tan B = \frac{AC}{BC}$ |

- The trig ratios can be remembered using the mnemonic **SOH-CAH-TOA**, which distills the information that **S**ine is **O**pposite over **H**ypotneuse, **C**osine is **A**djacent over **H**ypotenuse, and **T**angent is **O**pposite over **A**djacent.
- Trigonometric functions' values are derived from the side ratios **in a right triangle**, and their **values for a given angle don't change** if the angle appears in a figure that is not a right triangle; the **sine of an angle is the sine of that angle regardless of what type of figure** includes that angle. However, you can **only calculate these functions from a figure when the angle is part of a right triangle**.
- $\sin(x) = \cos(90 - x)$ and $\cos(x) = \sin(90 - x)$
- The **perimeter** of a polygon is the sum of the lengths of the outside edges.
- **Area** is a measure of the space that a shape covers on a plane. Area is measured in units of length squared.
- The area of a rectangle of length $\ell$ and width $w$ is given by $A = \ell w$.

The area of a square of side length $s$ is given by $A = s^2$.

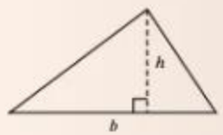
- For a triangle with a base length $b$ and height $h$, the area $A$ is found with the following formula:

$$A = \frac{1}{2}bh$$

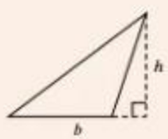


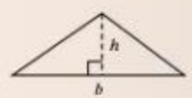

- The height of a triangle is always perpendicular to the base (and is drawn as an **altitude**).

  For acute triangles (all angles smaller than 90°) you usually have to draw in another line perpendicular to the base that divides the triangle into two right triangles.




  For obtuse triangles (one angle is larger than 90°), depending on the orientation, you may need to draw the perpendicular line for the height outside of the triangle as in the first picture below. Alternatively, you can always rotate the triangle (as in the second picture below) so that the perpendicular height line goes into the largest angle.

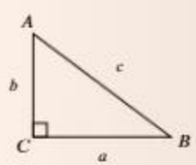



- If you need to find the area of other polygons, you should divide them into triangles and rectangles so that you can sum the areas of the individual sections.
- If you know two of the side lengths of a right triangle, you can use the **Pythagorean Theorem** to find the length of the third side.

  The Pythagorean Theorem states the following for a right triangle with legs measuring $a$ units and $b$ units and hypotenuse measuring $c$ units:

$$a^2 + b^2 = c^2$$

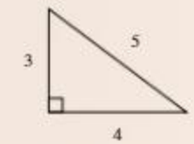






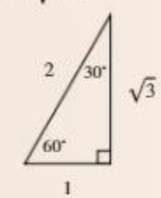

- There are some special right triangles in which all three sides of the triangle are whole numbers (or are in whole number ratios with each other). These are called **Pythagorean Triples**.
- The most common Pythagorean Triple is the **3:4:5 triangle**.

  In a right triangle with legs of length 3 and 4, the hypotenuse has length 5.

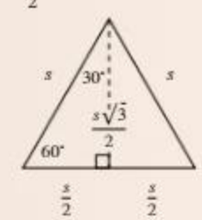



$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$
$$25 = 25$$

  The actual side lengths can be any set of values that conform to the 3:4:5 ratio.
- Other Pythagorean Triples include 5:12:13 triangles (occasionally seen), and the far rarer 7:24:25 triangles, 8:15:17 triangles, and 20:21:29 triangles.
- The test makes extensive use of **angle-based special right triangles** because their sides are in easy-to-write-and-remember ratios to each other.
- The first important angle-based special right triangle is a **30-60-90 right triangle**, which has a 30°, 60°, and 90° angle. Its sides, from shortest to longest, are in a ratio of $1:\sqrt{3}:2$.



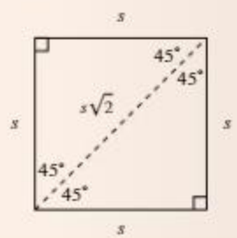

- **One of the most common ways that 30-60-90 triangles are hidden is in equilateral triangles**. If you divide an equilateral triangle with side lengths $s$ in half by drawing a line from any vertex to its opposite side and intersecting that opposite side at a right angle, the triangle is divided into two 30-60-90 triangles where the short legs of these triangles have length $\frac{s}{2}$ and the height of the triangles is $\frac{s\sqrt{3}}{2}$.




  Note that this can be useful when dealing with regular hexagons, which can be divided into 6 equilateral triangles, which can then be divided into 30-60-90 triangles.

The second important angle-based special right triangle is a **45-45-90 right triangle**. Since **two angles are the same, these triangles are isosceles, meaning that both legs are the same length**. The sides are in a ratio of $1:1:\sqrt{2}$.

**One of the most common ways that 45-45-90 triangles are hidden is in squares**. If you divide a square with side lengths $s$ in half diagonally from one vertex to its opposite, the square is divided into two 45-45-90 triangles where the legs have length $s$ and the hypotenuse has length $s\sqrt{2}$.





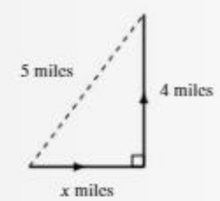



## Additional Problems

**1** 1600.io 19.4

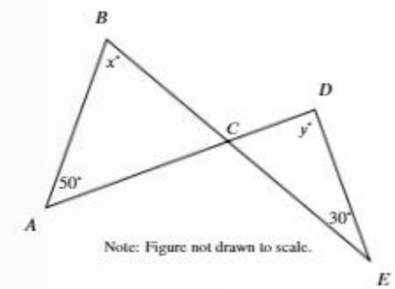
In triangle $ABC$, angle $A$ measures 23°, angle $B$ measures 83°, and angle $C$ measures 74°. Triangle $ABC$ is similar to triangle $XYZ$, such that $\frac{XY}{AB} = \frac{YZ}{BC} = \frac{XZ}{AC} = 5$. What is the measure, in degrees, of angle $X$ ?

**2** 1600.io 19.6




The solid lines in the figure above represent the route of a driver, and the dashed line represents the distance from her starting point to the point at which the driver stopped. What is the value of $x$ ?

**3** 1600.io 19.3




Note: Figure not drawn to scale.

In the figure above, $\overline{AD}$ intersects $\overline{BE}$ at $C$. If $x = 70$, what is the value of $y$ ?

A) 90
B) 80
C) 70
D) 50

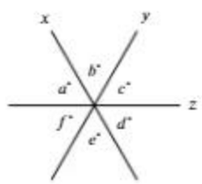
**4** 1600.io 19.6

In triangle $LMN$, the measure of $\angle M$ is 90°, $MN = 24$, and $LN = 26$. Triangle $QRS$ is similar to triangle $LMN$, where vertices $Q$, $R$, and $S$ correspond to vertices $L$, $M$, and $N$, respectively, and each side of triangle $QRS$ is $\frac{1}{5}$ the length of the corresponding side of triangle $LMN$. What is the value of $\sin S$ ?

**5** 19.6

Triangle $ABC$ has right angle $B$. If $\sin C = \frac{3}{5}$, what is the value of $\tan A$ ?

**6** 19.2




Note: Figure not drawn to scale.

In the figure above, lines $x$, $y$, and $z$ intersect at a point. If $a - c = b - f$, which of the following must be true?

I. $a = e$

II. $d = e$

III. $c = d$

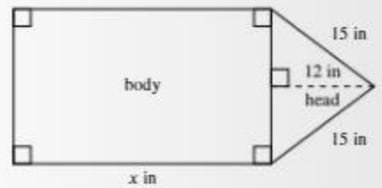
A) I and II only

B) I and III only

C) II and III only

D) I, II, and III

**7** 19.6




The figure above is the blueprint drawn by a carpenter for an arrow sign he is making. The arrow's head has a length of 12 inches (in) and two edges each of length 15 in. If the arrow's body has an area of 540 square inches, what is the value of $x$ ?

**8** 19.3

In triangle $XYZ$, the measure of angle $X$ is 40°. If triangle $XYZ$ is isosceles, which of the following is NOT a possible measure of angle $Y$ ?

A) 40

B) 70

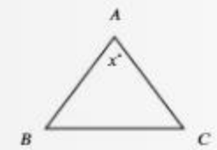
C) 80

D) 100





**9** 19.3




In the given triangle, $AB = AC$ and $\angle ABC$ has a measure of 53°. What is the value of $x$ ?

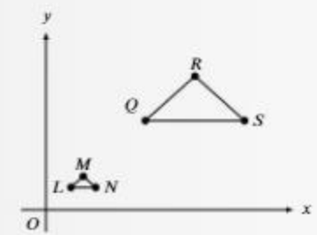
A) 84

B) 74

C) 53

D) 37

**10** 19.4




In the $xy$-plane above, a dilation with center $O$ and scale factor 4 transforms triangle $LMN$ to triangle $QRS$. Which of the following is NOT true?

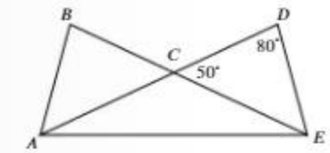
A) The measure of angle $R$ is 4 times the measure of angle $M$.

B) The perimeter of triangle $QRS$ is 4 times the perimeter of triangle $LMN$.

C) Angle $N$ is congruent to angle $S$.

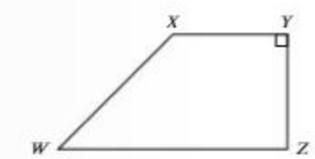
D) The length of $\overline{LN}$ is $\frac{1}{4}$ the length of $\overline{QS}$.

**11** 19.3




In the figure above, $\overline{AD}$ and $\overline{BE}$ intersect at point $C$, $BC = DC$, and $AC = CE$. What is the measure, in degrees, of $\angle DAE$ ? (Disregard the degree symbol when writing your answer.)

**12** 19.6




In quadrilateral $WXYZ$ above, $\overline{WZ} \parallel \overline{XY}$ and $YZ = \frac{1}{\sqrt{2}} XW$. What is the measure of angle $X$ ?

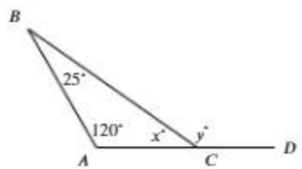
A) 135

B) 120

C) 90

D) 45

**13** 1600.io 19.3




In triangle $ABC$ above, side $\overline{AC}$ is extended to point $D$. What is the value of $y - x$ ?

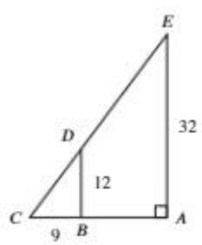
A) 120

B) 110

C) 45

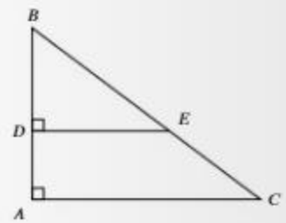
D) 35

**14** 1600.io 19.6



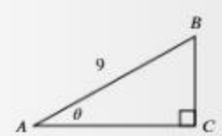

In the figure above, $\overline{BD}$ is parallel to $\overline{AE}$. What is the length of $\overline{CE}$ ?

**15** 1600.io 19.6




In the figure above, $\tan B = \frac{4}{3}$. If $BC = 25$ and $DA = 6$, what is the length of $\overline{DE}$ ?

**16** 1600.io 19.6

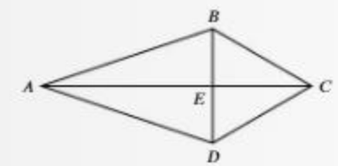



In the right triangle above, $\sin\theta = \frac{2}{3}$. If $AC = \sqrt{n}$, what is the value of $n$ ?





**17** 1600.io 19.6




In the figure above, $AB = AD$, $BC = CD$, $BE = 4$, $BC = 8$, and $AC = 12$. What is the area of triangle $ABD$ ?

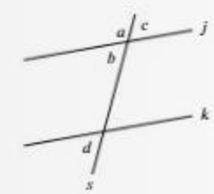
A) $96 - 32\sqrt{3}$

B) $48 - 16\sqrt{3}$

C) $24 - 8\sqrt{3}$

D) $12 - 4\sqrt{3}$

**18** 1600.io 19.2




In the figure above, line $s$ intersects lines $j$ and $k$. Which of the following statements, if true, would imply that lines $j$ and $k$ are parallel?

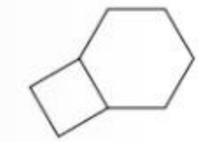
A) $a + b = 180$

B) $a = d$

C) $b = c$

D) $b = d$

**19** 1600.io 19.6



The figure above shows a regular hexagon with sides of length $a$ and a square with sides of length $a$. If the area of the hexagon is $216\sqrt{3}$ square centimeters, what is the area, in square centimeters, of the square?

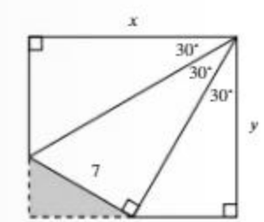
A) $144\sqrt{3}$

B) 144

C) $12\sqrt{3}$

D) 12

**20** 1600.io 19.6




The figure above shows that the shaded triangular region with a hypotenuse of 7 inches (in) has been removed from a rectangular board with dimension $x$ in by $y$ in. Of the following, which best approximates the area, in square inches, of the board before the piece was removed?

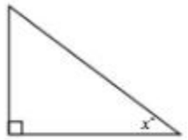
A) 127

B) 117

C) 74

D) 11

**21** 1600.io 19.6




Note: Figure not drawn to scale.

In the figure above, $\sin(90^\circ - x^\circ) = \frac{4}{5}$. What is the value of $\sin x^\circ$ ?

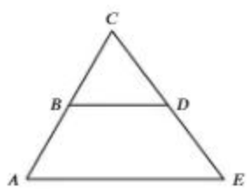
A) $\frac{5}{3}$

B) $\frac{5}{4}$

C) $\frac{4}{5}$

D) $\frac{3}{5}$

**22** 1600.io 19.3




Note: Figure not drawn to scale.

In the figure above, segments $\overline{AE}$ and $\overline{BD}$ are parallel. If angle $BDC$ measures $53^\circ$ and angle $ACE$ measures $67^\circ$, what is the measure of angle $CAE$ ?

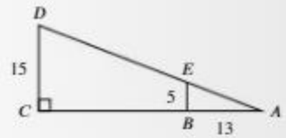
A) $53^\circ$

B) $60^\circ$

C) $67^\circ$

D) $120^\circ$

**23** 1600.io 19.4




In the figure above, $\Delta ACD$ is a right triangle and $\overline{BE}$ is parallel to $\overline{CD}$. What is the perimeter of $\Delta ACD$ to the nearest tenth of a unit?

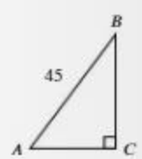
A) 31.9

B) 67.9

C) 90.0

D) 95.8

**24** 1600.io 19.6

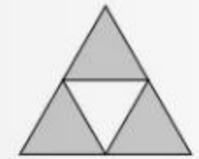



Triangle $ABC$ above is a right triangle, and $\sin(B) = \frac{3}{5}$. What is the length of side $\overline{BC}$ ?





**25** 1600.io 19.6



An artist is making a drawing for a fantasy game enthusiast. The drawing is shown in the figure above. The drawing is in the shape of a triangle and consists of four congruent equilateral triangles. If the perimeter of the drawing is 48 centimeters, what is the combined area of the shaded regions, in square centimeters, of the drawing?

A) $64\sqrt{3}$

B) $48\sqrt{3}$

C) $32\sqrt{3}$

D) $24\sqrt{3}$

**26** 1600.io 19.6

The length, in inches, of the sides and height of a parallelogram are shown in the figure. What is the area, in square inches, of the parallelogram?

## Answer Key

**SkillDrill 19.1-1**

1. 60
2. 135
3. 84
4. 37
5. 70
6. 45
7. 73
8. 52

**Section 19.2 Practice Problems**

1. C
2. C

**SkillDrill 19.3-1**

1. 90
2. 66
3. 31
4. 22
5. $b = 140, c = 20$
6. $b = 90, c = 45$
7. $b = 106, c = 37$
8. $b = 62, c = 59$

**Section 19.3 Practice Problems**

1. 120
2. 105

**SkillDrill 19.4-1**

1. 2
2. 10
3. 6
4. $\frac{3}{16}$

**Section 19.4 Practice Problems**

1. 15
2. B

**SkillDrill 19.5-1**

1. $\frac{4}{5}$
2. $\frac{3}{5}$
3. $\frac{3}{4}$
4. $\frac{3}{5}$
5. $\frac{4}{5}$
6. $\frac{4}{3}$

**Section 19.5 Practice Problems**

1. $\frac{3}{5}$ or 0.6
2. C
3. 0.8 or $\frac{4}{5}$
4. C

**SkillDrill 19.6-1**

1. 5
2. 16
3. 14
4. 39
5. 100
6. $\frac{5}{2}$
7. 24
8. 44

**SkillDrill 19.6-2**

1. $b = 3\sqrt{3}, c = 6$
2. $b = 10\sqrt{3}, c = 20$
3. $a = 5, c = 10$
4. $a = \sqrt{3}, c = 2\sqrt{3}$
5. $a = 2, b = 2\sqrt{3}$
6. $a = \frac{7}{2}, b = \frac{7\sqrt{3}}{2}$
7. $b = \frac{9}{4}, c = \frac{3\sqrt{3}}{2}$
8. $a = 11, c = 22$

**Section 19.6 Practice Problems**

1. D
2. C
3. A
4. D

**Additional Problems**

1. 23
2. 3
3. A
4. $\frac{5}{13}$, .384 or .385
5. $\frac{4}{3}$ or 1.33
6. A
7. 30
8. C
9. B
10. A
11. 25
12. A
13. B
14. 40
15. 12
16. 45
17. B
18. D
19. B
20. A
21. D
22. B
23. D
24. 36
25. B
26. 36

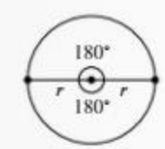


# Circles and Volume 20

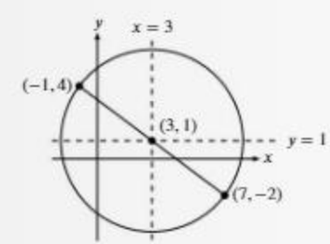
## 20.1 Circles and Angles

A circle is a shape formed by **all of the points that are the same distance away from one central point** (the center of the circle). The **radius** (plural, *radii*), marked $r$ in the picture below, is the distance from any point on the circle to the center point.

An angle that has as its vertex the center point of a circle is called a **central angle**. **The complete central angle of a circle (and, therefore, the sum of all central angles that divide up a circle) measures 360°**. One way to make this evident is as follows. If we split the circle in half by drawing in a **diameter** (a line from one point on the arc to its opposite point on the arc that goes directly through the center of the circle and whose length, $d$, is equal to twice the radius: $d = 2r$), we can see that the circle is split into two congruent **semicircles**, and since 180° angles form a line, we can see that each half of the arc measures 180°, and thus the whole circle contains 360°.




It's essential to understand that you can draw any number of radii from the center point to the circle, and the sum of all the resulting central angles will always be 360°. As a quick aside, note that **if you are presented with a circle drawn in the $xy$-plane, and you are given the coordinates of two endpoints of a diameter, the center point of the circle is the midpoint between those endpoints, and this center point's $x$- and $y$-coordinates can be found by averaging the $x$- and $y$-coordinates of the endpoints, respectively.**




$$x_{center} = \frac{(-1)+7}{2} \qquad y_{center} = \frac{4+(-2)}{2}$$

$$x_{center} = \frac{6}{2} \qquad y_{center} = \frac{2}{2}$$

$$x_{center} = 3 \qquad y_{center} = 1$$

$$\text{Center} = (x_{center}, y_{center})$$

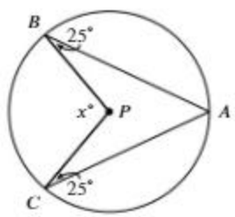
$$\text{Center} = (3, 1)$$

In many circle problems, we encounter triangles that connect points on the circle to the center point. In general, **if you see a point on a circle without a radius drawn to that point, you should draw in a radius to connect any marked points on the circle to the center point and label any lines from the center to a point on the circle with an $r$ to show that they are all radii**.

In some problems, you have to make triangles inside of a circle in order to find the measure of some angles. If a **triangle is formed with one vertex at the center of the circle and the other two vertices at points on the circle, then the two sides of the triangle that extend from the center point are each equal to the radius, and thus the triangle is isosceles**. Therefore, if you know any one of the angles inside the triangle, you can find the measure of the other two angles.

**Example 20.1-1**

**1** 1600.io 20.1

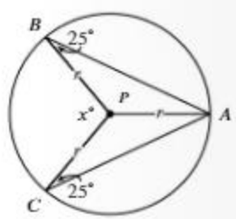



Point $P$ is the center of the circle in the figure above.
What is the value of $x$ ?

**Solution**

1. The first line you may consider drawing into the given figure is a line from $B$ to $C$, but that won't get us anywhere in this problem, since all we'd be doing is adding a triangle for which we know none of the angles.

   Instead, we should heed the advice given earlier to connect any marked points on the circle to the center of the circle, so draw a line from $P$ to $A$, and label all of the radii with an $r$. It doesn't matter that we don't know the length of the radius; labeling the sides helps us realize that we are going to form two isosceles triangles (two equal sides, and thus two equal angles), $\triangle ABP$ and $\triangle ACP$.



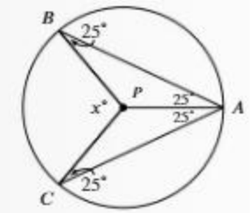

2. Since $\triangle ABP$ has two sides with length $r$, the triangle is isosceles, and the angles opposite of those equal sides must be



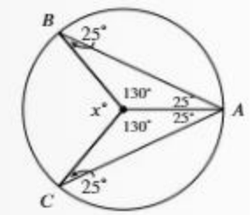


equal to each other. Therefore, $\angle BAP$ must also be 25°.

   Similarly, since $\triangle ACP$ has two sides with length $r$, the triangle is isosceles, and the angles opposite of those equal sides must be equal to each other. Therefore, $\angle CAP$ must also be 25°.




3. Since there are 180° in a triangle, $\angle APB$ and $\angle CPA$ must both be equal to $180° - 25° - 25° = 130°$. (Note that in the diagram below, we've omitted the label for center point $P$ for spacing reasons.)

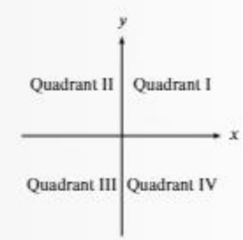



4. Finally, since the central angles of the circle must sum to 360°, the angle marked $x°$ must measure $360° - 130° - 130° = 100°$.

5. The answer is 100.

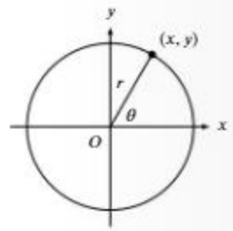
## Circles and Trigonometry

So far we have only talked about trigonometry as it relates to isolated triangles; however, when we graph angles on the coordinate plane, we see something new. To set the stage, it's useful to divide the $xy$-plane into four **quadrants** as shown below.

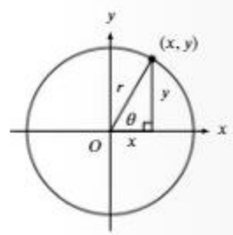


Throughout the following discussion, we'll be using a series of line segments of length $r$ that emanate from the origin, so they will all lie on a circle of radius $r$ centered at the origin.

When we graph a line segment of length $r$ having one end at the origin, extending to a point $(x, y)$, and lying at an angle $\theta$ (theta) measured relative to the positive $x$-axis, we don't necessarily see a triangle initially.




But if we draw a vertical line from the end of the angled line at $(x, y)$ to the $x$-axis and add a segment along the $x$-axis from the end of the vertical line to the origin, we will form a right triangle to work with. The lengths of the legs of this right triangle are given to us by the coordinates of the point $(x, y)$ at the end of the segment that we started with (which will form the hypotenuse of the right triangle): since the end point is $(x, y)$, the horizontal leg has length $x$, and the vertical leg has length $y$. Note, importantly, that **because $x$ and $y$ represent coordinate values, they can be positive or negative, but $r$, the length of the line segment, is always positive**.



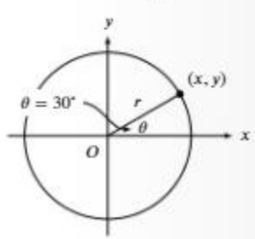

The angle ratios of a right triangle drawn on the coordinate plane are still calculated the same way (remember SOH-CAH-TOA).

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

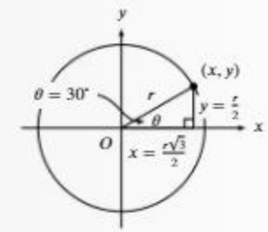
This is simple enough for angles in the first quadrant like the 30° angle shown below.



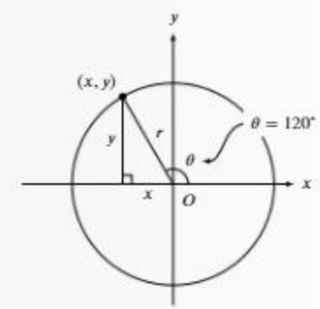




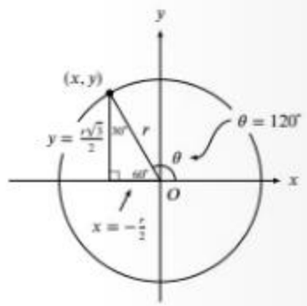

Simply draw a vertical leg down from the endpoint of the radius to the positive $x$-axis. The resulting triangle is a 30-60-90 triangle, so we know that the leg across from the 30° angle is half the length of the radius and that the leg across from the 60° angle has a length that is $\sqrt{3}$ times the length of the shortest leg. Therefore, the hypotenuse, marked by the radius, measures $r$, the value of $y$ is $\frac{r}{2}$, and the value of $x$ is $\frac{r\sqrt{3}}{2}$.




For larger angles, the process is very similar but we draw the triangles so that they make a right triangle with either the positive or negative $x$-axis. For example, if the radius that marks an angle $\theta$ falls in the second quadrant, such as the 120° angle shown in the figure below, draw the vertical leg from the endpoint of the radius down to the negative $x$-axis.

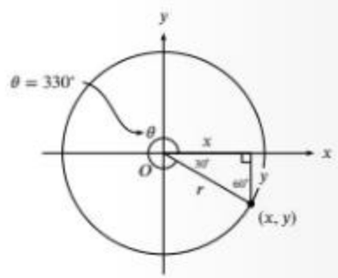



The angle at the origin and *inside* the right triangle is the angle we'll use to figure out the relationships among the sides, and in this case that angle measures 60° because it has to be supplementary to the 120° angle ($180 - 120 = 60$). Note that the endpoint of the radius, $(x, y)$, is located in the second quadrant, so while the $y$-value is still positive, the $x$-value is negative. Due to the presence of that 60° angle, the resulting triangle is again a 30-60-90 triangle, so we know that the short leg, which is across from the 30° angle, is half the length of the radius (as noted, this leg is along the negative $x$-axis) and the long leg, which is across from the 60° angle, has a length that is $\sqrt{3}$ times the length of the short leg. Therefore, the hypotenuse, marked by the radius, measures $r$, the value of $x$ is $-\frac{r}{2}$ (remember that **this value must be negative** due to its location on the $xy$-plane), and the value of $y$ is $\frac{r\sqrt{3}}{2}$.

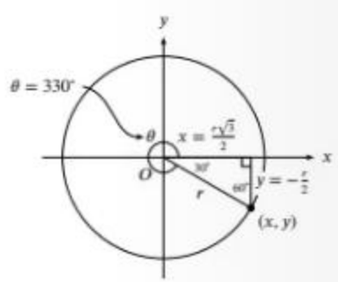



Since $\cos\theta = \frac{x}{r}$, then $\cos(120) = \frac{-\frac{r}{2}}{r} = -\frac{1}{2}$.

Let's look at one more possibility just to really drive home the point. For the 330° angle shown below, we would draw the vertical leg from the endpoint of the radius up to the positive $x$-axis, and the angle at the origin and inside the triangle must measure 30° (because 360 − 330 = 30, representing the full circle minus the 330° angle).




The endpoint of the radius will have a positive $x$-value and a negative $y$-value. Since this is another 30-60-90 triangle, the side across from the 30° angle will be half of the hypotenuse, and because $y$ must be negative, we know $y = -\frac{r}{2}$. The leg across from the 60° angle is $\sqrt{3}$ times the other leg, and it must also be positive, so we know $x = \frac{r\sqrt{3}}{2}$.



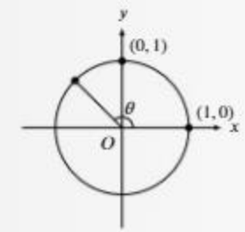

Since $\sin\theta = \frac{y}{r}$, then $\sin(330) = \frac{-\frac{r}{2}}{r} = -\frac{1}{2}$. Since $\cos\theta = \frac{x}{r}$, then $\cos(330) = \frac{\frac{r\sqrt{3}}{2}}{r} = \frac{\sqrt{3}}{2}$.





**Example 20.1-2**

**2** 1600.io 20.1




In the given figure, $\theta$ is an angle. If $\sin\theta = \frac{\sqrt{2}}{2}$, what is $\cos\theta$ ?

A) $\frac{\sqrt{2}}{2}$

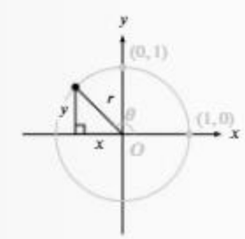
B) $\frac{1}{2}$

C) $-\frac{1}{2}$

D) $-\frac{\sqrt{2}}{2}$

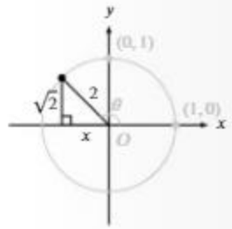
**Solution 1**

1. Start by making a right triangle by drawing a vertical line from the endpoint of the slanted line (the line showing the radius that is oriented at an angle of $\theta$ from the positive $x$-axis). Label the radius as $r$, and label the horizontal and vertical legs as $x$ and $y$, respectively.




2. While we can use the given trigonometric ratio to solve for the *actual* value of $y$, there is no need to because we can use the Pythagorean Theorem to find the *relative* length of the horizontal leg, which is good enough to calculate the cosine of angle $\theta$; because trig ratios represent relative lengths, we don't need to compute actual lengths.

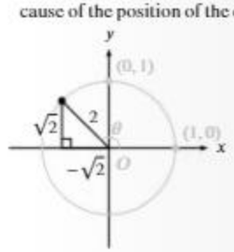
   We are given that $\sin\theta = \frac{\sqrt{2}}{2}$, and remembering that $\sin\theta = \frac{y}{r}$, we know that the *relative* lengths of $y$ and $r$ are $\sqrt{2}$ and 2, respectively (if you look at the picture, you will see that the *value* of $r$ is 1, not 2, but we are using relative lengths here).




In the Pythagorean Theorem, plug in $\sqrt{2}$ for $y$ and 2 for $r$ so that we can solve for the relative value of $x$ (based on the picture, remember that **$x$ must be a negative value because the horizontal leg of our right triangle is on the negative $x$-axis**).

$$x^2 + y^2 = r^2$$
$$x^2 + (\sqrt{2})^2 = (2)^2$$
$$x^2 + 2 = 4$$
$$x^2 = 2$$
$$x = -\sqrt{2}$$

(Use the negative square root because of the position of the endpoint)




3. Using the relative lengths of $x$ and $r$, we can find the cosine of $\theta$ by plugging in $-\sqrt{2}$ for $x$ and 2 for $r$.

$$\cos\theta = \frac{x}{r}$$

$$\cos\theta = -\frac{\sqrt{2}}{2}$$

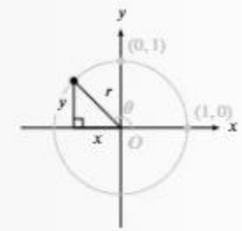
4. The answer is D.





**Solution 2**

1. Start by making a right triangle by drawing a vertical line from the endpoint of the line representing a radius that is oriented at an angle of $\theta$ from the positive $x$-axis. Label this radius as $r$, and label the horizontal and vertical legs as $x$ and $y$, respectively.




2. We know that $r$ must be equal to 1 because the center point is (0, 0) and points (0, 1) and (1, 0) are on the circle (both of these are 1 unit away from the origin).

3. We can solve for the vertical leg, $y$, using a proportion. We know that $\sin\theta = \frac{y}{r}$, and the problem tells us the value of $\sin\theta$ is equal to $\frac{\sqrt{2}}{2}$, so we can substitute $\frac{\sqrt{2}}{2}$ for $\sin\theta$ on the left side of the equation and substitute 1 for $r$ on the right side of the equation in order to solve for $y$.

$$\sin\theta = \frac{y}{r}$$

$$\frac{\sqrt{2}}{2} = \frac{y}{1}$$

$$\frac{\sqrt{2}}{2} = y$$

4. If you are very keen, you may recognize that this is a 45-45-90 right triangle because the hypotenuse's length is equal to $\sqrt{2}$ times the length of the leg. The legs of 45-45-90 triangles are congruent, so the horizontal leg also has a length of $\frac{\sqrt{2}}{2}$.

   Be careful not to jump to a wrong answer! The given angle is drawn in the coordinate plane, and the endpoint of the angle has a negative $x$-coordinate. The correct way to label the horizontal leg is $\frac{-\sqrt{2}}{2}$.

   If your 45-45-90 recognition skills aren't particularly sharp, you can use the Pythagorean Theorem to solve for the length of the horizontal leg (substitute $\frac{\sqrt{2}}{2}$ for $y$ and 1 for $r$, but remember again to use the negative value for the length of that leg because of the position of the endpoint to the left of the $y$-axis).

$$x^2 + y^2 = r^2$$

$$x^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1^2$$

$$x^2 + \frac{2}{4} = 1$$

$$x^2 + \frac{1}{2} = 1$$

$$x^2 = 1 - \frac{1}{2}$$

$$x^2 = \frac{1}{2}$$

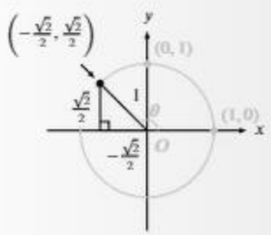
$$x = -\sqrt{\frac{1}{2}} = -\frac{1}{\sqrt{2}}$$ (Use the negative square root because of the position of the endpoint)

5. None of the answer choices have a radical in the denominator, so we need to remove the square root from the denominator by multiplying the right side of the equation by $\frac{\sqrt{2}}{\sqrt{2}}$, which will not change the value because $\frac{\sqrt{2}}{\sqrt{2}}$ is equal to 1.

$$x = -\frac{1}{\sqrt{2}}$$

$$x = -\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$x = -\frac{\sqrt{2}}{2}$$



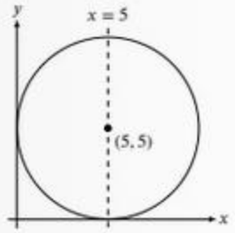

6. For an angle $\theta$ on the $xy$-plane with endpoint $(x, y)$ and radius $r$, the cosine of the angle is equal to $\frac{x}{r}$. Since $x = -\frac{\sqrt{2}}{2}$ and $r = 1$, $\cos\theta = -\frac{\frac{\sqrt{2}}{2}}{1}$, which we can simplify to find $\cos\theta = -\frac{\sqrt{2}}{2}$.

7. The answer is D.

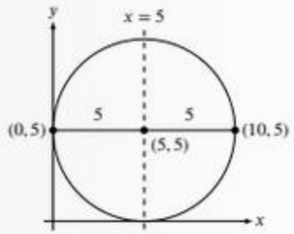




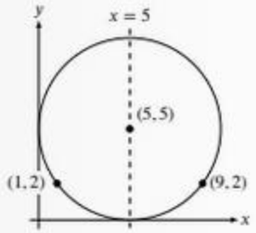
Circles, by definition, are symmetrical around their center point; that is, all points on the circle are equidistant from the center. As a consequence of this, if the circle is divided in half with a vertical line through the center point, that line will represent an axis of symmetry (as will any line through the center point). Because it goes through the center point, this vertical axis of symmetry line has the same $x$-coordinate as that center point.



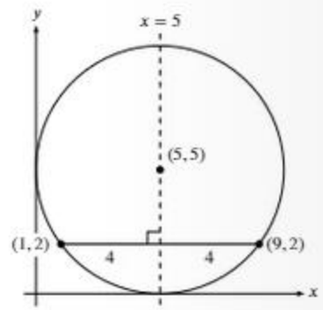

Pairs of points on the circle that have the same $y$-coordinate will be equidistant from this vertical axis of symmetry, and the distance between such points and the axis of symmetry is simply the difference in the points' $x$-coordinates and the axis of symmetry's (and thus the center point's) $x$-coordinate. This is true for any pair of points on the circle with the same $y$-coordinates, but the most obvious case of this symmetry is seen when we have a horizontal line that goes right through the center point. That will simply be a diameter, which, of course, is composed of two radii.



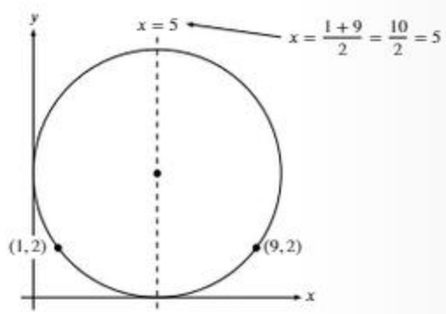

Just a little less obvious, but still apparent, is the fact that, as mentioned earlier, any pair of points on the circle with the same $y$-coordinates will also be equidistant from the axis of symmetry.



To find the distance from each point in a pair such as this to the axis of symmetry, we simply find the difference between the $x$-coordinate of one of the points and the $x$-coordinate of the center point (which, again, is also the $x$-coordinate of the axis of symmetry). The distance to the line of symmetry will be the same for both points due to symmetry.




If we're not given the coordinates of the center point, but we have the coordinates of both points on the circle, we can take advantage of our knowledge about the symmetry of the arrangement to find the $x$-coordinate of the axis of symmetry, which is also the $x$-coordinate of the center point of the circle: we simply **average the $x$-coordinates of the two points on the circle to find the value midway between them**, because we know they are equidistant from the axis of symmetry by definition.

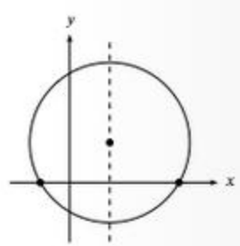



$$x_{center} = \frac{1+9}{2}$$

$$x_{center} = \frac{10}{2}$$

$$x_{center} = 5$$

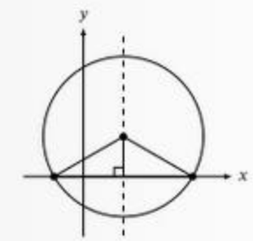
A special case of this principle arises when we're given a circle that intersects the $x$-axis and we are told the coordinates of the points of intersection. Again, these two points will be equidistant from the vertical axis of symmetry, meaning the midpoint between them will lie on the axis of symmetry and thus will have the same $x$-coordinate as does the center point of the circle through which the axis of symmetry passes.



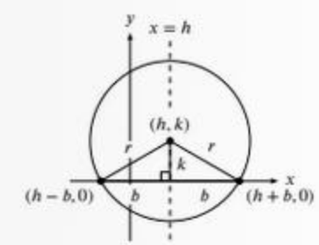




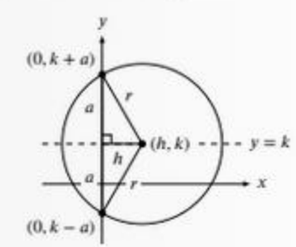

In this situation, we can draw in a right triangle connecting the center point, one of the intersection points, and the point where the axis of symmetry crosses the $x$-axis; if we have information that tells use the length of two of the sides of this triangle, we can find the length of the third side by using the Pythagorean Theorem (or, equivalently, by recognizing a Pythagorean triple such as a 3:4:5 or 5:12:13 triangle). (We've shown both possible triangles in the following figures, but you only need one of them to proceed.)




The hypotenuse of this triangle is a radius, so if we're given the radius of the circle, we know the length of the hypotenuse (this might be obvious to you, but students often overlook this essential fact and then can't make any progress on a problem). If we're given the coordinates of the intersection points of the circle and the $x$-axis, we can use our knowledge that the distance between each such point and the axis of symmetry is half the distance between the intersection points to get the length of the base of the triangle, which lies along the $x$-axis; this can, in turn, provide us with the $x$-coordinate of the axis of symmetry, which is also the $x$-coordinate of the center point of the circle through which it passes. If we're provided with the coordinates of the center point of the circle (or at least the $y$-coordinate), we know that the $y$-coordinate is equal to the length of the vertical leg of the triangle, which lies along the vertical axis of symmetry.



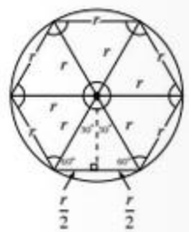

Note that because a circle is symmetrical with respect to the center point, *any* line through the center represents an axis of symmetry, so all of the above principles would also apply if we had a circle that intersected the $y$-axis, except that everything would be rotated clockwise by 90°, and $x$ and $y$ would be interchanged in all the relationships.




The following example makes use of the above properties of a circle to find the value of an unknown.

Noting that two of the sides of each triangle are equal to $r$, we can see that the other two angles in each triangle (which must add to $180^\circ - 60^\circ = 120^\circ$) must be equal to each other, so these two angles both also measure $60^\circ$ (because $\frac{120^\circ}{2} = 60^\circ$). All of the triangles are equilateral!

Once we know the length of any line in the figure, we can find any other measurement we might need (here, it's very handy to remember that equilateral triangles can be split into two 30-60-90 triangles).



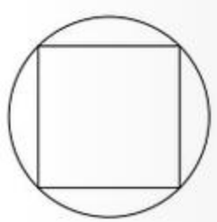
**Example 20.1-4**

4 1600.io 20.1

A square is inscribed in a circle with radius $3\sqrt{2}$ centimeters. What is the perimeter of the square in centimeters?

**Solution**

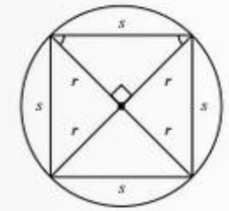
1. Start by drawing a square inscribed inside of a circle.







2. Draw radii from the center of the circle to each of the vertices of the square. Each central angle will measure $\frac{360^\circ}{4} = 90^\circ$ because squares are regular polygons. Don't forget to label the radii with an $r$ (you could use the actual length of $r$, which is given to us, but usually, seeing that all of these lines are radii is the best reminder that these new triangles are all isosceles) and label each side of the square with an $s$.



3. Since each resulting triangle is an isosceles right triangle (two sides are equal to $r$, and the central angle is $90^\circ$), the remaining angles must add to $90^\circ$ and be equal to each other; therefore, each of the other angles is equal to $45^\circ$. Each triangle is a 45-45-90 triangle for which the hypotenuse is a side of the inscribed square.

   Recall that for a 45-45-90 triangle in which the legs measure $x$, the hypotenuse has a length of $x\sqrt{2}$. For these triangles, since the legs have a measurement of $r$ (which is equal to $3\sqrt{2}$), the hypotenuse $s$ (which is also a side of the inscribed square) has a length of $(3\sqrt{2})(\sqrt{2}) = 3(2) = 6$.

   Alternatively, you can use the side-length ratio for a 45-45-90 triangle of $1:1:\sqrt{2}$ (leg:leg:hypotenuse), then set up a proportion using the known side length to find the length of the hypotenuse $s$:

$$\frac{\text{hypotenuse}}{\text{leg}} = \frac{\sqrt{2}}{1}$$

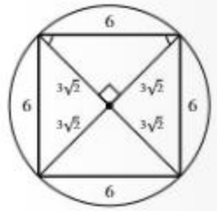
$$\frac{s}{3\sqrt{2}} = \frac{\sqrt{2}}{1}$$

$$s = 3\sqrt{2}(\sqrt{2})$$
$$s = 3(2)$$
$$s = 6$$

   As another alternative, you can always rely on the Pythagorean Theorem:

$$(3\sqrt{2})^2 + (3\sqrt{2})^2 = s^2$$
$$9(2) + 9(2) = s^2$$
$$36 = s^2$$
$$6 = s$$

4. Since all four sides of the square must have a length of 6 units, the perimeter of the square (the sum of all the sides of the square) is $4(6) = 24$.




5. The answer is 24.

**Section 20.1 Suggested**

- Test 5-C-36
- Test 9-C-31
- May 2017-NC-20

**View related real-test problems in SMTEx**

SMTEx

## Section 20.1 Practice Problems

**1** 1600.io 20.1

A circle in the $xy$-plane has a diameter with endpoints $(-7, -5)$ and $(17, 5)$. If the point $(0, b)$ lies on the circle and $b > 0$, what is the value of $b$ ?

**2** 1600.io 20.1

Note: Figure not drawn to scale.

In the figure above, the circle has center $A$, and line segment $CB$ is tangent to the circle at point $C$. If $AB = 2.0$ and $CB = 1.2$, what is the length of the diameter of the circle?





## 20.2 Circumference, Arc Length, and Radians

There is a very important relationship that exists for circles: **the length of the circumference of a circle is approximately 3.14 times the diameter of the circle** (the exact value is irrational [it cannot be expressed as a fraction with integer numerator and denominator] and it has an infinite number of decimal places). **This ratio between circumference and diameter is represented by the Greek letter $\pi$ (pi)**, and thus $\pi \approx 3.14$. Written mathematically,

$$\frac{\text{circumference}}{\text{diameter}} = \pi$$

More usefully, as this allows us to calculate the circumference from the diameter, we rearrange things and write,

$$\text{circumference} = \pi(\text{diameter})$$

The radius of a circle is half the diameter, and we're generally given the radius rather than the diameter when working with a circle, so this relationship is more commonly expressed as

$$\text{circumference} = 2\pi(\text{radius})$$

We can write this more compactly, with $C$ being circumference and $r$ being radius, as

$$C = 2\pi r$$

You can think of this as meaning that it would require $2\pi$ radiuses to wrap perfectly around the circumference of a circle. This idea will be very helpful later in the section, where we discuss radians.

### Circumference

The circumference $C$ of a circle with a radius $r$ is the distance along the outside of the circle, and it is found with the following equation:

$$C = 2\pi r$$

Since the diameter $d$ of the circle is twice the radius, or $2r$, the formula is sometimes written as the following:

$$C = \pi d$$

Make sure you are careful to distinguish between the radius and diameter because both are given and asked for in problems.

Note that you are given the formula for circumference in the box at the start of each math section.

**SkillDrill 20.2-1**

**Directions:** Find the circumference of a circle in terms of $\pi$ given the radius of the circle (there are no specific units of length needed).

**1.** $r = 3$ **2.** $r = 2$ **3.** $r = \frac{1}{2}$ **4.** $r = \sqrt{6}$

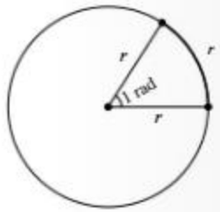
**Directions:** Given the circumference $C$ of a circle, find the radius of the circle.

**5.** $C = 2\pi$ **6.** $C = 5\pi$ **7.** $C = 12\pi$ **8.** $C = \frac{\pi}{2}$

Degrees are usually a more familiar way for most people to visualize angles, but technically, the decision that there are 360 degrees

in a circle is an arbitrary designation (the number 360 has 24 divisors, which makes it easy to divide in many different ways).

However, there exists a much more natural measurement of angles: **radians**. When the geniuses who figured out geometry thought about how to calculate the length of the circumference of a circle, they realized that there must be some angle that exists for which **the length along the intercepted arc is equal to the radius of the circle, and that angle was defined as measuring 1 radian (often abbreviated to "rad")**.




So, **for an angle measuring 1 radian, the length of the intercepted arc is equal to the radius, by definition**. Importantly, this is a simple linear relationship; for example, a central angle of 2 radians intercepts an arc that measures 2 radiuses, and so on. This means that **if we know how many radians a central angle measures, we instantly know how many radiuses the intercepted arc measures** and, by the same token, **if we know how many radiuses long an arc measures, we immediately know the measure of the central angle that intercepts that arc: the number of radians of angle measure is the same as the number of radiuses in arc length**. This is the key power of the radian as a unit of central angle measure for circles.

Recall that the circumference of a circle measures $2\pi r$, or $2\pi$ times the radius of the circle. Put another way, the length of the circumference is $2\pi$ radiuses. Because 1 radian intercepts an arc of 1 radius in length (by definition), it follows that **because the circumference measures $2\pi$ radiuses, the central angle of a complete circle is $2\pi$ radians**.

This information leads us directly to the numerical relationship between degrees and radians: we already learned that the central angle of a circle is 360°, and we now know that the central angle of a circle is $2\pi$ radians. Therefore, we can use the following proportion when we want to convert between the two units:

$$\frac{\text{degrees}}{\text{radians}} = \frac{360^\circ}{2\pi \text{ rad}}$$

As always, we recommend setting up any proportion with the unknown as the numerator on the left side for easiest solving and the most natural resulting mathematical "sentence." For example, if we want to convert 3 radians to degrees, we can write,

$$\frac{x^\circ}{3 \text{ rad}} = \frac{360^\circ}{2\pi \text{ rad}}$$

To convert 90° to radians, we can set up,

$$\frac{x \text{ rad}}{90^\circ} = \frac{2\pi \text{ rad}}{360^\circ}$$

If you like, you can take advantage of some simplifications and optimizations available here. First, you might have noticed that there's 360 in the numerator and 2 in the denominator of the ratio between degrees and radians; we can simplify that to end up with

$$\frac{\text{degrees}}{\text{radians}} = \frac{180^\circ}{\pi \text{ rad}}$$

This just makes the arithmetic easier. (Note that this also corresponds with the fact that a semicircle [half of a circle] has a central angle of 180° or $\pi$ radians.)





As a further optimization, we can pre-isolate the unknown in the proportions by multiplying both sides by the numerator beneath the unknown; that gives us the following two conversion formulas:

$$\text{degrees} = \text{radians}\left(\frac{180}{\pi}\right)$$

$$\text{radians} = \text{degrees}\left(\frac{\pi}{180}\right)$$

If you're comfortable with these streamlined formulas, you can use them, but it's important to know that **you can always construct the proportion** if you are uncertain or confused. All you have to remember is the number of degrees and radians in a full circle (or semicircle) and you can then construct a proportion for solving for the unknown measure.

## Converting Angle Measurements

Note that radians can be written as a unit, rad, but often the units are not written at all.

Since $\pi$ radians is equal to 180°, we can convert between the two units using this proportion.

To convert from degrees to radians, multiply the angle measurement in degrees by the fraction $\frac{\pi}{180^\circ}$.

To convert from radians to degrees, multiply the angle measurement in radians by the fraction $\frac{180^\circ}{\pi}$.

**SkillDrill 20.2-2**

**Directions:** Convert the angles from degrees to radians. All of the following conversions can be performed without a calculator and written in terms of $\pi$ instead of as decimals.

**1.** 180° **2.** 60° **3.** 45° **4.** 120°

**5.** 210° **6.** 36° **7.** 300° **8.** 360°

**Directions:** Convert the angles from radians to degrees (where the angle measure in degrees is between 0 and 360, inclusive).

**9.** $\pi$ rad **10.** $2\pi$ rad **11.** $\frac{4\pi}{5}$ rad **12.** $\frac{4\pi}{3}$ rad

**13.** $\frac{\pi}{3}$ rad **14.** $\frac{3\pi}{2}$ rad **15.** $\frac{11\pi}{6}$ rad **16.** $\frac{3\pi}{4}$ rad

**Example 20.2-1**

**1** 1600.io 20.2

An angle with a measure of $\frac{4\pi}{3}$ radians has a measure of $d$ degrees, where $0 \le d \le 360$. What is the value of $d$ ?

**Solution 1**

1. All we have to do here is convert from radians to degrees. Multiply the angle measured in radians by $\frac{180^\circ}{\pi}$.

$$d = \left(\frac{4\cancel{\pi}}{3}\right)\left(\frac{180^\circ}{\cancel{\pi}}\right)$$

$$d = 4(60^\circ) = 240^\circ$$

2. The answer is 240.

**Solution 2**

1. Use the proportion of the angle measure in degrees to radians to solve for $d$.

$$\frac{d}{\frac{4\pi}{3}} = \frac{180}{\pi}$$

$$d = \left(\frac{4\cancel{\pi}}{3}\right)\left(\frac{180^\circ}{\cancel{\pi}}\right)$$

$$d = 4(60^\circ) = 240^\circ$$

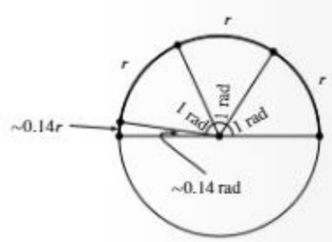
2. The answer is 240.

**Notes**

We could have used the circumference ratio $\frac{360}{2\pi}$ on the right side of the proportion; you should use whichever you're most comfortable with.

## Arc Length

Let's look back to the picture we made when discussing what a radian was so that we can look at how radians help us determine the distance around the outside of the circle.








Since a radian is an angle for which the arc length is equal to the radius, and there are about 3.14 or $\pi$ radians in half a circle, then there are $2\pi$ radians in a whole circle, which means that the distance around the entire circle is equal to $2\pi$ (the number of radians in a circle) times the radius of the circle, which gives us the formula for the **circumference** of a circle.

### Arc Length in General

Remember that **if we know how many radians are in a central angle, then we also know that the intercepted arc length is simply equal to the radius times the number of radians**: *radians* of angle produce *radiuses* of arc length (yes, it's really that simple!). Therefore, the formula for the arc length $L$ for a portion of a circle with an angle $\theta$ (measured in radians, NOT degrees) and radius $r$ is given by the following simple equation:

$$L = r\theta$$

If you are given a central angle measure in radians, **do not convert this angle to degrees to calculate the intercepted arc length**. Such a conversion is completely unnecessary and only serves as a needless detour on the road to the solution; it only wastes time and increases the probability of error. It is essential to become comfortable with working in radians alone; again, remember that **radians of angle produce radiuses of intercepted arc**.

### Fraction of a Whole Circle

If you are given the measurement of the central angle in degrees, you have two options for calculating the arc length. You can convert the angle from degrees to radians, or you can calculate the arc length as a fraction of the whole circumference by using a proportion.

Since there are $360^\circ$ in a circle, then a central angle of $x^\circ$ will signify a fraction of the circle. For example, $90^\circ$ is one-fourth of the circle because $\frac{90}{360} = \frac{1}{4}$. For a $90^\circ$ angle, the arc length will be one-fourth of the full circumference.

In general, when the central angle is $x^\circ$, then the fraction of the circle that is marked by the central angle is

$$\frac{x}{360}$$

One key correlation is that the length of the arc swept out by the central angle can be calculated (without converting to radians) by using a proportion. The ratio of the arc length $L$ to the whole circumference $C$ is equal to the ratio of the central angle to $360^\circ$.

$$\frac{L}{C} = \frac{x}{360}$$

Multiplying both sides by $C$, we get:

$$L = \frac{x}{360}C$$

If we are not given the circumference, but we do know the radius, we can use our circumference formula and substitute $2\pi r$ for $C$ to end up with the formula

$$L = \frac{x}{360}(2\pi r)$$

Though helpful, **it's not necessary to memorize these formulas; it is sufficient to remember the principles so that you can construct an appropriate proportion and calculate the circumference from the radius if needed**. Note that while arc length can be found directly when you have the angle in radians, the *ratio* of an arc length to the circumference of the circle is still equal to the ratio of the central angle $x$ in radians to the number of radians in the whole circle, which is $2\pi$.

**SkillDrill 20.2-3**

**Directions:** Find the arc length swept by an angle of $\theta$ radians in a circle of radius $r$. There is no need to convert any measurements or use any proportions: simply multiply the angle (in radians) by the radius as per the generalized arc length formula.

**1.** $r = 3, \theta = \pi$ **2.** $r = 5, \theta = \frac{\pi}{2}$ **3.** $r = \frac{1}{2}, \theta = \frac{2\pi}{3}$ **4.** $r = 6, \theta = \frac{7\pi}{4}$

**Directions:** Find what fraction of a circle is formed by a central angle of $\theta$, measured in degrees or radians (if no units are given, the angle is in radians).

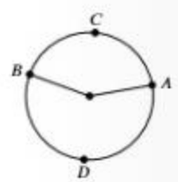
**5.** $\theta = \pi$ **6.** $\theta = \frac{4\pi}{3}$ **7.** $\theta = \frac{2\pi}{5}$ **8.** $\theta = \frac{3\pi}{2}$

**9.** $\theta = 180^\circ$ **10.** $\theta = 45^\circ$ **11.** $\theta = 30^\circ$ **12.** $\theta = 240^\circ$

**Directions:** Find the arc length swept by an angle of $\theta$ degrees in a circle of radius $r$. Either convert the angle measurement to radians, or multiply the circumference by the fraction of the circle that is formed by the given central angle.

**13.** $r = 3, \theta = 180^\circ$ **14.** $r = 5, \theta = 45^\circ$ **15.** $r = \frac{1}{2}, \theta = 210^\circ$ **16.** $r = 6, \theta = 300^\circ$

Note that the measure of an arc between points $A$ and $B$ on a circle can be denoted as $\overset{\frown}{AB}$. If there is a third point $C$ between the points $A$ and $B$, the arc may be represented as $\overset{\frown}{ACB}$. In the figure below, we may choose to refer to $\overset{\frown}{ACB}$ in order to distinguish it from $\overset{\frown}{ADB}$, which is the **complementary arc**.




If an intermediate point such as $C$ or $D$ is not provided, and you see a reference to **minor arc** $\overset{\frown}{AB}$, the reference is to the **smaller** arc between the two named points, which here would be arc $\overset{\frown}{ACB}$, and it is called a **minor arc because it makes up less than half of the circle**. Likewise, **major arc** $\overset{\frown}{AB}$ refers to the **larger** arc between the two points, which here corresponds to arc $\overset{\frown}{ADB}$, and it is called a **major arc because it makes up more than half of the circle**. If there are only two points, and neither major nor minor is specified, you should assume the reference is to the minor arc.





**Example 20.2-2**

**2** 1600.io 20.2

Points $A$ and $B$ lie on a circle with radius 1, and $\overset{\frown}{AB}$ has length $\frac{\pi}{8}$. What fraction of the circumference of the circle is the length of arc $\overset{\frown}{AB}$?

**Solution**

1. Since the radius of the circle is 1, the circumference, $C$, can be found using the equation for the circumference of the circle.

$$C = 2\pi r$$
$$C = 2\pi(1)$$
$$C = 2\pi$$

2. Divide the length of arc $\overset{\frown}{AB}$, which is $\frac{\pi}{8}$, by the circumference $C$, which is $2\pi$, in order to find what fraction of the circumference the length $\overset{\frown}{AB}$ is.

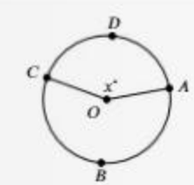
$$\frac{\overset{\frown}{AB}}{C} = \frac{\frac{\pi}{8}}{2\pi}$$

$$\frac{\overset{\frown}{AB}}{C} = \frac{1}{16}$$

3. The answer is $\frac{1}{16}$.

**Example 20.2-3**

**3** 1600.io 20.2




The circle above has center $O$, the length of $\overset{\frown}{ADC}$ is $5\pi$, and $x = 150$. What is the length of arc $\overset{\frown}{ABC}$?

A) $\frac{12}{5}\pi$

B) $\frac{7}{2}\pi$

C) $7\pi$

D) $12\pi$

**Solution 1**

1. Since we know the measure of central angle marked $x^\circ$ is $150^\circ$, we know that the remainder of the central angle (which intercepts major arc $\widehat{ABC}$) measures $360^\circ - 150^\circ = 210^\circ$ (the two angles have to add up to $360^\circ$).
   The ratio of $\widehat{ABC}$ to $\widehat{ADC}$ is equal to the ratio of the central angles that form those intercepted arcs, $210^\circ$ and $150^\circ$, respectively. Write a proportion in order to solve for the length of $\widehat{ABC}$. Recall that the length of $\widehat{ADC}$ is $5\pi$.

$$\frac{\widehat{ABC}}{\widehat{ADC}} = \frac{210}{150}$$

$$\frac{\widehat{ABC}}{\widehat{ADC}} = \frac{7}{5}$$

$$\frac{\widehat{ABC}}{5\pi} = \frac{7}{5}$$

$$\widehat{ABC} = \frac{7}{5}(5\pi)$$

$$\widehat{ABC} = 7\pi$$

2. The answer is C.

**Solution 2**

1. Since we know the measure of central angle marked $x^\circ$ is $150^\circ$, we know that the fraction of the circumference that is formed by minor arc $\widehat{ADC}$ is $\frac{150}{360}$, which reduces to $\frac{5}{12}$.
2. Since the ratio of the full circumference to the length of $\widehat{ADC}$ is 12:5, then the full circumference of the circle must be $12\pi$.

$$\frac{\text{circumference}}{\widehat{ADC}} = \frac{12}{5}$$

$$\frac{\text{circumference}}{5\pi} = \frac{12}{5}$$

$$\text{circumference} = \frac{12}{5}(5\pi)$$

$$\text{circumference} = 12\pi$$

3. Since the circumference is $12\pi$ and the length of $\widehat{ADC}$ is $5\pi$, then the length of $\widehat{ABC}$ must be equal to $12\pi - 5\pi = 7\pi$.
4. The answer is C.

**Notes**

Notice that we did not have to find the radius of the circle in order to solve this problem.



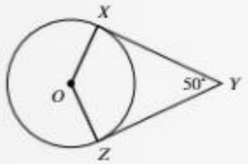


Note that you are not always given the measure of a central angle and must use other knowledge of polygons and terminology in order to find the central angle.

For instance, if you are told that lines are **tangent** to a circle, then you should note immediately that those lines **intersect the circle at exactly one point**, and a **radius drawn from the center of the circle to the point of tangency will intersect the tangent line at a right angle**. In the figure below, line $\ell$ is tangent to the circle at the point $A$.

**Example 20.2-4**

**4** 1600.io 20.2




In the figure above, point $O$ is the center of the circle, line segments $XY$ and $YZ$ are tangent to the circle at points $X$ and $Z$, respectively, and the segments intersect at point $Y$ as shown. If the circumference of the circle is 72, what is the length of the minor arc $\widehat{XZ}$?

**Solution**

1. Notice that there is a quadrilateral $OXYZ$ drawn in the picture, which must have a sum of interior angles of $360^\circ$ as we saw in the previous chapter using the Sum of Interior Angles formula, where $n$ is the number of sides of a polygon (which is 4 in this case):

$$\text{Sum of Interior Angles} = 180(n-2)$$

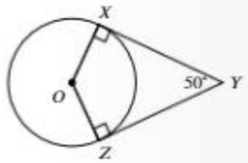
$$\text{Sum of Interior Angles} = 180(4-2)$$

$$\text{Sum of Interior Angles} = 180(2)$$

$$\text{Sum of Interior Angles} = 360$$

The four angles must add to $360^\circ$. We need to know $\angle XOZ$ so we can determine what fraction of a complete circle is represented by arc $\widehat{XZ}$, and we can calculate the measure of the angle if we know the measures of the other three angles.

2. Note that line segments $XY$ and $YZ$ are tangent to the circle, which means that the radii drawn to the points $X$ and $Z$ are perpendicular to the tangent lines (they meet at right angles).




3. We know that angles $\angle OXY$ and $\angle OZY$ measure $90^\circ$ and angle $\angle XYZ$ measures $50^\circ$, so $\angle XOZ$ must be equal to $360^\circ - 2(90^\circ) - 50^\circ = 130^\circ$.

4. Now that we know the measure of angle $\angle XOZ$, we can calculate the arc length of $\widehat{XZ}$ by multiplying the circumference of the whole circle, which is 72, by the fraction of the whole circle that is formed by $\angle XOZ$, which is $\frac{130}{360}$ (use the arc length formula for degrees, where $L$ is the arc length, $x$ is the measure of the central angle, and $C$ is the circumference of the circle).

$$L = \frac{x}{360}C$$

$$\widehat{XZ} = \frac{130}{360}(72)$$

$$\widehat{XZ} = \frac{13}{36}(72)$$

$$\widehat{XZ} = (13)(2)$$

$$\widehat{XZ} = 26$$

5. The answer is 26.

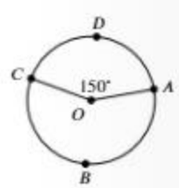
**Notes**

If you don't know or can't recall the arc length formula, you can always set up a simple proportion between the ratio of arc length and circumference and the ratio of the central angle and the central angle of a complete circle:

$$\frac{\widehat{XZ}}{72} = \frac{130}{360}$$

## Inscribed Angles and Arc Length

On occasion, the test will refer to an arc being measured in degrees as opposed to a unit of length. For example, in the figure below, instead of identifying the length of minor arc $\widehat{ADC}$, we can just say that the arc measures $150^\circ$ because the central angle forming that intercepted arc measures $150^\circ$.



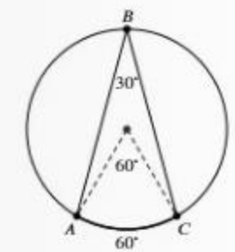





An **inscribed angle** is an angle with its vertex on the circumference of the circle. Admittedly, these appear rarely, but they have been used on the test before, and if you do encounter them, there are two important things to note.

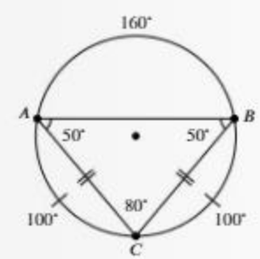
1. **The measure of an inscribed angle is half of the measure of the central angle of its intercepted arc.**

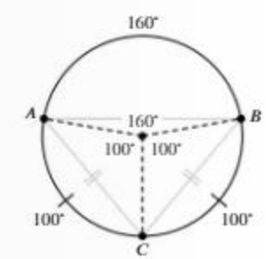
   For example, in the figure below, the measure of $\angle ABC$ is $30^\circ$, and the measure of the central angle that intercepts minor arc $\widehat{AC}$ is $60^\circ$ (its relative length it would be $\frac{60}{360}$ or $\frac{1}{6}$ of the circumference).




2. Just like central angles with the same measure, **inscribed angles with the same measure correspond to equal intercepted arcs.**

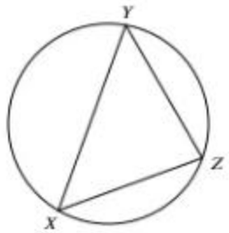
   For example, in the figure below, both $\angle ABC$ and $\angle BAC$ measure $50^\circ$, and therefore, minor arc $\widehat{AC}$ and minor arc $\widehat{BC}$ each measure $2(50^\circ) = 100^\circ$ (if we formed central angles between the endpoints of these arcs, they would measure $100^\circ$, and each arc's length would be $\frac{100}{360}$ or $\frac{5}{18}$ of the circle's circumference). You should also notice that $\angle ACB$ measures $80^\circ$ (there are $180^\circ$ in $\Delta ABC$, and $180 - 2(50) = 80$) and the remaining minor arc $\widehat{AB}$ measures $160^\circ$ (there are $360^\circ$ of arc in a circle, and $360 - 2(100) = 160$).



**Example 20.2-5**

**5** 1600.io 20.2




Note: Figure not drawn to scale.

Triangle $XYZ$ is inscribed in the circle above. If arc $\widehat{XY}$ is congruent to arc $\widehat{YZ}$, and the measure of $\angle Y$ is $20^\circ$, what is the measure of $\angle Z$ ?

A) $20^\circ$

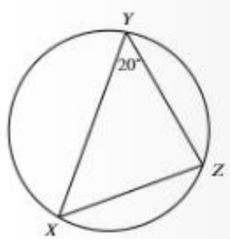
B) $60^\circ$

C) $80^\circ$

D) $140^\circ$

**Solution**

1. The first thing to notice is that this figure is not drawn even close to scale. As shown in the figure, $\angle Y$ (also known as $\angle XYZ$) is clearly larger than $20^\circ$ and arc $\widehat{XY}$ doesn't look anywhere close to congruent to arc $\widehat{YZ}$. Don't let that throw you off! Simply label $\angle Y$ as $20^\circ$.

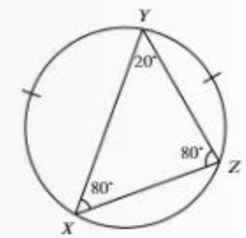



Note: Figure not drawn to scale.





2. Based on what we know about inscribed angles and arcs, if $\widehat{XY}$ is congruent (or equal) to $\widehat{YZ}$ (even though they don't look like it), then the inscribed angles that intercept those arcs, namely $\angle Z$ and $\angle X$, respectively, must be equal to each other, and they make up the remaining $160^\circ$ in triangle $XYZ$ (after subtracting the $20^\circ$ from $180^\circ$ to account for $\angle Y$). Therefore, $\angle X$ and $\angle Z$ are both equal to $\frac{160^\circ}{2} = 80^\circ$.




Note: Figure not drawn to scale.

3. The answer is C.

**Section 20.2 Suggested Problems from Real Tests**

- Test 2-NC-19
- Test 2-C-36
- Test 4-C-24
- Test 4-C-36
- Test 5-NC-2
- Apr 2017-NC-19
- Apr 2018-C-36
- Mar 2019-NC-13
- May 2019 (Int)-NC-11
- May 2019 (Int)-C-21
- Oct 2019-NC-11

View related real-test problems in SMTEx

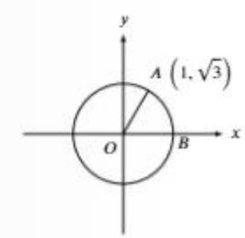
SMTEx

## Section 20.2 Practice Problems

**1** 1600.io 20.2

An arc of a circle measures 1.8 radians. To the nearest degree, what is the measure, in degrees, of this arc? (Disregard the degree sign when writing your answer.)

**2** 1600.io 20.2




In the $xy$-plane above, $O$ is the center of the circle, and the measure of $\angle AOB$ is $\frac{\pi}{a}$ radians. What is the value of $a$ ?

## 20.3 Area of a Circle

The formula for the area of a circle is one of the formulas that is given in the box at the start of both math sections on the test. Its derivation is outside the scope of the math needed for this test, so just learn and remember it or look it up in that information box on the test if you forget.

The fraction of the area of a circle marked off by a central angle (called a **sector**) is the ratio of that central angle to the angle measure of a complete circle (which is 360° or $2\pi$ radians). When part of a circle is marked off by a central angle, the resulting fraction of the whole area is also proportional to the ratio of the central angle to the entire circle.

### Area of a Circle

The formula for the area $A$ of a circle of radius $r$ is the following:

$$A = \pi r^2$$

The area of a sector of a circle marked by a central angle $\theta$ is proportional to the ratio of the central angle to the angle measure of the full circle. For example, a central angle of $\frac{\pi}{2}$ radians or 90° marks off one-fourth of the circle (because $\frac{\frac{\pi}{2}}{2\pi} = \frac{1}{4}$ and $\frac{90}{360} = \frac{1}{4}$), so the area of that portion of the circle is one-fourth of the area of the whole circle.

In general, the area of a sector marked by a central angle measuring $x°$ or $y$ radians is found with the following equations (it is not necessary to memorize these as long as you understand that the area of a sector is proportional to the central angle or fraction of the circumference that defines it):

$$A_{sector} = \frac{x}{360}(\pi r^2)$$

$$A_{sector} = \frac{y}{2\pi}(\pi r^2)$$

**SkillDrill 20.3-1**

**Directions:** Find the area of a circle with radius $r$ in terms of $\pi$.

**1.** $r = 3$ **2.** $r = 5$ **3.** $r = \frac{1}{2}$ **4.** $r = 1$

**Directions:** Find the area of a portion of a circle with radius $r$ formed by a central angle of $\theta$, measured in degrees or radians. Multiply the area of the whole circle by the fraction of the circle that is formed by central angle $\theta$.

**5.** $r = 3, \theta = \pi$ **6.** $r = 5, \theta = \frac{\pi}{2}$ **7.** $r = \frac{1}{2}, \theta = \frac{2\pi}{3}$ **8.** $r = 4, \theta = \frac{7\pi}{4}$

**9.** $r = 3, \theta = 90°$ **10.** $r = 5, \theta = 45°$ **11.** $r = \frac{1}{2}, \theta = 210°$ **12.** $r = 6, \theta = 300°$





**Example 20.3-1**

**1** 1600.io 20.3

In a circle with center $O$, central angle $FOG$ has a measure of $\frac{7\pi}{6}$ radians. The area of the sector formed by central angle $FOG$ is what fraction of the area of the circle?

**Solution**

1. The fraction of the area of a sector formed by a central angle is equal to the ratio of the angle's measure to the full measure of a circle. Since we are given the measure of $\angle FOG$, simply divide its measure by $2\pi$ (which is the full measure, in radians, of a circle).

$$\frac{\text{Area of sector formed by } \angle FOG}{\text{Area of circle}} = \frac{\text{Measure of } \angle FOG}{\text{Measure of circle}}$$

$$\frac{\text{Area of sector formed by } \angle FOG}{\text{Area of circle}} = \frac{\frac{7\cancel{\pi}}{6}}{2\cancel{\pi}}$$

$$\frac{\text{Area of sector formed by } \angle FOG}{\text{Area of circle}} = \frac{7}{12}$$

2. The answer is $\frac{7}{12}$.

Section 20.3 Suggested

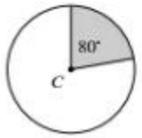
- Test 3-C-34
- Test 7-C-34
- May 2018-C-6

View related real-test problems in SMTEx

SMTEx

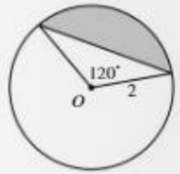
## Section 20.3 Practice Problems

**1** 1600.io 20.3




Point $C$ is the center of the circle above. What fraction of the area of the circle is the area of the shaded region?

**2** 1600.io 20.3




In the figure above, the circle has center $O$. What is the area of the shaded region?

A) $\frac{4\pi}{3}$

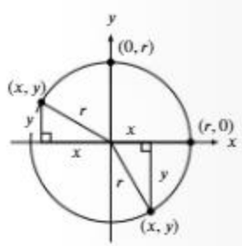
B) $\frac{1}{3}(4\pi - \sqrt{3})$

C) $\frac{1}{3}(4\pi - 2\sqrt{3})$

D) $\frac{1}{3}(4\pi - 3\sqrt{3})$

## 20.4 The Circle Equation

Recall that circles and trigonometry are closely related when it comes to working with angles in the $xy$-plane. For any point $(x, y)$ on a circle, we can draw a right triangle whose legs have length $x$ and $y$ and whose hypotenuse has a length equal to the radius of the circle. For now, let's just consider circles with their center at the origin.




By the Pythagorean Theorem, we can see that $x^2 + y^2 = r^2$ for any of these cases. Even the points which lie directly above, below, to the right, and to the left of the center point fit this formula (despite the fact that we can't technically draw a right triangle to the points that are on a vertical or horizontal diameter line).

For the points on the same vertical line as the center point, the value of $x$ is 0, so the $x^2$ term drops out and we see that $y^2 = r^2$, which also means $y = r$ (or $y = -r$, in the case of points below the origin).

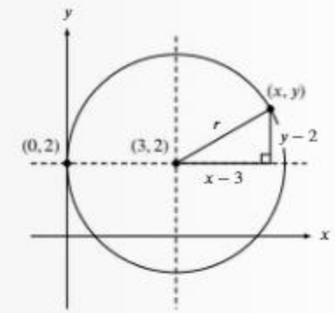
Similarly, for the points on the same horizontal line as the center point, the value of $y$ is 0, so the $y^2$ term drops out and we see that $x^2 = r^2$, which also means $x = r$ (or $x = -r$, for points to the left of the origin).

Therefore, for any point on a circle centered at the origin with radius $r$, all of the points on the outside of the circle fit the equation $x^2 + y^2 = r^2$.





But what happens when we shift the center point? Let's look at a circle centered at (3, 2), with a radius of 3. When we draw a right triangle, we draw it so that the vertical leg intersects the horizontal diameter line of the circle. We can see that the vertical line now has a length of $y - 2$ and the horizontal leg has a length of $x - 3$ (this is like the $\Delta y$ and $\Delta x$ measurements we do when calculating the slopes of lines).




Based on this drawing and by using the Pythagorean Theorem, we can see that for any points $(x, y)$ on the outside of the circle, the equation for that circle would be the following:

$$(x - 3)^2 + (y - 2)^2 = r^2$$

Generalizing, we recognize that the 3 in $x - 3$ represents the $x$-coordinate of the center point, and the 2 in $y - 2$ represents the $y$-coordinate of the center point, so the general form of the equation of a circle with center point $(h, k)$ is

$$(x - h)^2 + (y - k)^2 = r^2$$

The length of the horizontal leg, $(x - h)$, will be negative for points with an $x$-coordinate less than the $x$-coordinate of the center point (and likewise with the vertical leg $(y - k)$, but with respect to the $y$-coordinates), but the squares of those negative lengths will be positive, so the value of each side of the circle equation will always be positive.

Here, because the point (0, 2) is marked on the circle and the center point is at (3, 2), we can tell that the radius must be $3 - 0 = 3$, so, plugging in 3 for $r$, we get the equation $(x - 3)^2 + (y - 2)^2 = 3^2$ which can be rewritten as $(x - 3)^2 + (y - 2)^2 = 9$.

All points strictly **inside** of a circle conform to the inequality $(x - h)^2 + (y - k)^2 < r^2$ (the distance from a point $(x, y)$ inside of a circle to the center of that circle, $(h, k)$, is less than the length of a radius). All points strictly **outside** of a circle conform to the inequality $(x - h)^2 + (y - k)^2 > r^2$ (the distance from a point $(x, y)$ outside of a circle to the center of that circle, $(h, k)$, is greater than the length of a radius).

### Equation of a Circle in the $xy$-plane

The equation for all points $(x, y)$ on a circle **centered at the origin** with radius $r$ is very similar to the Pythagorean Theorem:

$$x^2 + y^2 = r^2$$

For any circle with radius $r$, **centered not at the origin but at a point** $(h, k)$, the equation for all points $(x, y)$ on the circle is slightly modified to reflect the shifted center point:

$$(x - h)^2 + (y - k)^2 = r^2$$

**SkillDrill 20.4-1**

**Directions:** Identify the center point, $(h, k)$, and radius, $r$, of the circles represented by the given equations.

**1.** $x^2 + y^2 = 25$ **2.** $x^2 + y^2 = 9$ **3.** $x^2 + y^2 = 6$ **4.** $(x-1)^2 + (y-2)^2 = 25$

**5.** $(x+4)^2 + (y-4)^2 = 16$ **6.** $(x+3)^2 + y^2 = 9$ **7.** $(x-2)^2 + (y-2)^2 = 11$ **8.** $x^2 + (y+5)^2 = 5^2$

**Directions:** Write the equation of a circle with center point $C$ and radius $r$.

**9.** $C = (0, 0), r = 2$ **10.** $C = (1, 4), r = 6$ **11.** $C = (-5, 0), r = 4$ **12.** $C = (-3, -6), r = 9$

**Example 20.4-1**

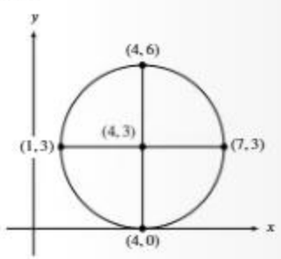
**1** 1600.io 20.4

A circle in the $xy$-plane has the equation $(x-4)^2 + (y-3)^2 = 9$. Which of the following points does NOT lie in the interior of the circle?

A) $(0, 2)$
B) $(2, 2)$
C) $(4, 3)$
D) $(6, 5)$

**Solution 1**

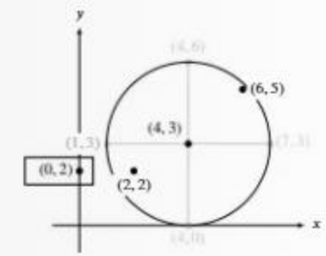
1. Based on the circle equation, we can tell that the center point of the circle is $(4, 3)$ and the radius is $\sqrt{9} = 3$. In order to eliminate some of the choices and narrow down our search, we should sketch the circle by marking the center point, which is $(4, 3)$, and the endpoints of the horizontal and vertical diameters. Since the radius is 3, the endpoints of the horizontal diameter are $(1, 3)$ and $(7, 3)$, so the $x$-coordinate of any point inside the circle must be between 1 and 7 exclusive. Similarly, the endpoints of the vertical diameter are $(4, 0)$ and $(4, 6)$, so the $y$-coordinate of any point inside the circle must be between 0 and 6 exclusive.








2. The point $(0, 2)$ given by answer choice A clearly would not fall inside of the circle because $x$ must be greater than 1 for any point inside the circle as given by the endpoint $(1, 3)$ of the horizontal diameter. Additionally, note that the point $(4, 3)$ given by answer choice C is the center point and is therefore inside of the circle. The other two points given in the answer choices, $(2, 2)$ and $(6, 5)$, are not obviously inside the circle when you draw a rough sketch (without the assistance of beautiful computer assisted graphics), but since we've already determined that $(0, 2)$ cannot possibly be inside the circle and thus must be the correct answer, we can conclude that these two points must be inside the circle based on the process of elimination (you will see in Solution 2 how you could check these points mathematically if you wanted to be sure).




3. The answer is A.

**Solution 2**

1. We can go about this problem by simply plugging in the points. However, it should be noted that the circle equation gives all of the points *on* the circle (all the points that are $\sqrt{9} = 3$ units away from the center point $(4, 3)$), and since we want all of the points *inside* the circle to pass our test, we should look for all the points $(x, y)$ where $(x-4)^2 + (y-3)^2 < 9$; this will tell us all of the points that are *less* than 3 units away from the center point $(4, 3)$, which means they are inside the circle.

2. Choice A, the point $(0, 2)$, fails this test when we plug in 0 for $x$ and 2 for $y$.

$$(x-4)^2 + (y-3)^2 < 9$$
$$(0-4)^2 + (2-3)^2 < 9$$
$$(-4)^2 + (-1)^2 < 9$$
$$16 + 1 < 9$$
$$\times \quad 17 \not< 9$$

3. Since $(0, 2)$ does not lie inside the circle, the answer is A.

**Notes**

For completeness in either solution above, you may want to test the points $(2, 2)$ and $(6, 5)$ in the inequality used in Solution 2 (remember that Choice C, point $(4, 3)$, is the center point of the circle, so we really don't need to run this check on it).

Choice B: $(2, 2)$

$$(x-4)^2+(y-3)^2<9$$
$$(2-4)^2+(2-3)^2<9$$
$$(-2)^2+(-1)^2<9$$
$$4+1<9$$
$$\checkmark \quad 5<9$$

Choice D: $(6, 5)$

$$(x-4)^2+(y-3)^2<9$$
$$(6-4)^2+(5-3)^2<9$$
$$(2)^2+(2)^2<9$$
$$4+4<9$$
$$\checkmark \quad 8<9$$

## Completing the Square to Standardize Circle Equations

Sometimes you are given the equation of a circle in a form that disguises the center point and radius (it kind of looks like a standard form polynomial, but with both $x$- and $y$-terms on one side of the equation). In these cases, you must complete the square for both sets of terms.

For example, you may be given an equation like

$$2x^2+12x+2y^2+16y-98=52$$

and told that it is the equation of a circle in the $xy$-plane. Unfortunately, when the equation is written in this form, we cannot immediately tell the center point or radius of the circle.

Much like we did when completing the square for quadratics, **we should first collect any constant terms on the opposite side of the equation from the $x$- and $y$-terms**. In this case, we'll add 98 to both sides of the equation.

$$2x^2+12x+2y^2+16y-98=52$$
$$2x^2+12x+2y^2+16y=150$$

Next, we should **divide both sides of the equation by the coefficient of the $x^2$ and $y^2$ terms** (which is 2 in this case) so that the squared variable terms have coefficients of 1; if the coefficients are already 1, you obviously don't need to perform this step, but it should be noted that for circles, **the coefficients of the squared terms will always have to match**. Make sure to distribute this division across all terms on **both** sides of the equation.

$$2x^2+12x+2y^2+16y=150$$
$$x^2+6x+y^2+8y=75$$

**Finally, we have to complete the square *twice***, once for the $x$-terms and once for the $y$-terms, so that both are written in terms of a perfect square (because we must produce the form $(x-h)^2+(y-k)^2=r^2$). The expression $x^2+6x$ is a polynomial in Standard Form, $ax^2+bx+c$, where $a=1$, $b=6$ and $c=0$. Remember that when we complete the square, we replace the $x^2+bx$ expressions with $\left(x+\frac{b}{2}\right)^2$ and we add the value of the resulting extraneous constant $\left(\frac{b}{2}\right)^2$ to the opposite side of the equation to keep it balanced.





For the $x$-terms, the value of $b$ is 6, so we should replace $x^2+6x$ with $(x+3)^2$ and add $\left(\frac{6}{2}\right)^2=3^2=9$ to the right side of the equation.

$$x^2+6x+y^2+8y=75$$
$$(x+3)^2+y^2+8y=75+9$$

By the same process, we can complete the square for the $y$-terms. The value of the $y$-coefficient is 8, so we should replace $y^2+8y$ with $(y+4)^2$ and add $\left(\frac{8}{2}\right)^2=4^2=16$ to the right side of the equation.

$$(x+3)^2+y^2+8y=75+9$$
$$(x+3)^2+(y+4)^2=75+9+16$$
$$(x+3)^2+(y+4)^2=100$$

The equation is now in Standard Form for a circle, and the center point is $(-3, -4)$; be mindful that the terms within the parentheses are involved in subtraction in the Standard Form, so if it helps you to get the signs of the center coordinates correct, you can think about this equation as $(x-(-3))^2+(y-(-4))^2=100$. Be careful not to incorrectly conclude that the radius is 100—remember that the equation is $(x-h)^2+(y-k)^2=r^2$, and therefore, $r^2=100$, so $r=10$.

If you only need to determine the coordinates of the center point $(h, k)$, but you don't need the radius, you don't need to perform the entire completing the square procedure; you can observe that $h$ will be the negative of $\frac{1}{2}$ of the $x$-coefficient, and, similarly, $k$ will be the negative of $\frac{1}{2}$ of the $y$-coefficient (after dividing through by the squared terms' coefficient, if it's not 1). Here, the original equation had the terms $6x$ and $8y$ after eliminating the $x^2$ and $y^2$ coefficient, so the $x$-coordinate of the center point is $-\left(\frac{6}{2}\right)=-3$, and the $y$-coordinate is $-\left(\frac{8}{2}\right)=-4$. Neat, eh?

Similarly, if you only need the radius, move any constant to the right side and add to it the squares of half the coefficients (again, after dividing through by the squared terms' coefficient, if it's not 1); the result will be the square of the radius. In the example above, there's a constant of 75, and adding $\left(\frac{6}{2}\right)^2=9$ and $\left(\frac{8}{2}\right)^2=16$ gives $75+9+16=100$, so the radius is 10.

### Completing the Square Twice

In order to write a circle equation in the form $ax^2+bx+ay^2+cy+d=e$ in Standard Form, $(x-h)^2+(y-k)^2=r^2$, follow these steps:

1. Isolate all constant terms on the opposite side of the equation from the $x$- and $y$-terms.
2. Divide both sides of the equation by $a$, the coefficient of the $x^2$ and $y^2$ terms, making sure to distribute the division across all terms. The $x^2$ and $y^2$ coefficients will always match for circle equations, and most often on the test will be 1, so this step will often be unnecessary.
3. Complete the square for both $x$- and $y$-variable polynomial expressions.

   If the $x$-coefficient is $p$ and the $y$-coefficient is $q$, you should rewrite the expression of $x$ terms as $\left(x+\frac{p}{2}\right)^2$, rewrite the expression of $y$ terms as $\left(y+\frac{q}{2}\right)^2$, and add $\left(\frac{p}{2}\right)^2$ and $\left(\frac{q}{2}\right)^2$ to the other side of the equation.
4. The equation is now in Standard Form for circles, and you can therefore identify the center point and radius. Note that the radius is the square root of the constant that is found on the opposite side of the equation from the squared binomials.

**SkillDrill 20.4-2**

**Directions:** Identify the center point, $(h, k)$, and radius, $r$, of the circles represented by the given equations by completing the square to write the equations in the Standard Form for circles in the $xy$-plane: $(x-h)^2 + (y-k)^2 = r^2$.

1. $x^2 - 4x + y^2 - 10y + 4 = 0$
2. $x^2 + 6x + y^2 - 6y - 7 = 11$
3. $x^2 - 2x + y^2 - 4y = 6$
4. $3x^2 + 12x + 3y^2 - 18y - 18 = 24$
5. $2x^2 - 24x + 2y^2 - 24y + 72 = 0$
6. $x^2 - 3x + y^2 - 7y = 0$

**Example 20.4-2**

**2** 1600.io 20.4

In the $xy$-plane, the graph of $2x^2 + 10x + 2y^2 - 2y = 59$ is a circle. What is the radius of the circle?

A) 3.5
B) 6
C) $\sqrt{59}$
D) $\sqrt{75}$

**Solution**

1. We need to write the equation of the circle in its standard representation. There are no constant terms on the variable side of the equation that need moving, so the first step is to divide both sides of the equation by 2 so that the $x^2$ and $y^2$ terms have coefficients of 1 (remember to distribute the division across all of the terms).

$$2x^2 + 10x + 2y^2 - 2y = 59$$

$$x^2 + 5x + y^2 - y = \frac{59}{2}$$

2. Now we need to complete the square twice—once for each pair of variable terms. In order to complete the square for the $x$ terms, we need to replace $x^2 + 5x$ with $\left(x + \frac{5}{2}\right)^2$ and add $\left(\frac{5}{2}\right)^2$, which is equal to $\frac{25}{4}$, to the right side of the equation.

$$x^2 + 5x + y^2 - y = \frac{59}{2}$$

$$\left(x + \frac{5}{2}\right)^2 + y^2 - y = \frac{59}{2} + \frac{25}{4}$$





3. In order to complete the square for the $y$ terms, we need to replace $y^2 - y$ with $\left(y - \frac{1}{2}\right)^2$ and add $\left(-\frac{1}{2}\right)^2$, which is equal to $\frac{1}{4}$, to the right side of the equation. Simplify the right side of the equation by combining the constant terms.

$$\left(x + \frac{5}{2}\right)^2 + y^2 - y = \frac{59}{2} + \frac{25}{4}$$

$$\left(x + \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{59}{2} + \frac{25}{4} + \frac{1}{4}$$

$$\left(x + \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{59}{2} + \frac{26}{4}$$

$$\left(x + \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{59}{2} + \frac{13}{2}$$

$$\left(x + \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{72}{2}$$

$$\left(x + \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 36$$

4. Since the equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$, then in this case $r^2 = 36$, and $r = 6$.
5. The answer is B.

**Notes**

Because we only needed to find the radius of the circle, we could have skipped the steps that produced the coordinates of the center point and just added the constants $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ and $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ to the right side, combined all the constant terms, and taken the square root of the result.

**Section 20.4 Suggested Problems from Real Tests**

- Test 1-C-24
- Test 2-C-24
- Test 5-C-29
- Test 6-C-27
- Test 7-C-29
- May 2017-C-11
- Mar 2021-NC-7
- Apr 2021-NC-15
- Apr 2021-C-17
- May 2021 (US)-NC-18
- May 2021 (Int)-NC-15
- May 2021 (Int)-C-30

**View related real-test problems in SMTEx**

SMTEx

## Section 20.4 Practice Problems

**1** 1600.io 20.4

The equation $(x+3)^2 + (y-5)^2 = 144$ defines a circle in the $xy$-plane. What is the radius of the circle?

**2** 1600.io 20.4

The graph of $x^2 - 2x + y^2 + 4y - 12 = 0$ in the $xy$-plane is a circle. What is the radius of the circle?

A) $\sqrt{7}$
B) $2\sqrt{3}$
C) $\sqrt{17}$
D) $4\sqrt{2}$

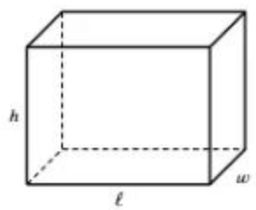
## 20.5 Solids (3D Figures)

The **volume of a solid or three-dimensional figure is the amount of space that the object occupies** and is usually measured in units$^3$ (though the volume of fluids like water and oil might be measured in units like ounces or liters).

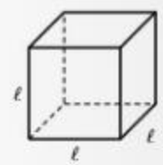
Any time you need to work with volume, the formula for the given shape is provided to you, either included with the problem or present in the information box that starts the test section. Most of the time, volume questions are about either rectangular prisms (boxes) or cylinders.

### Right Rectangular Prisms

A **right rectangular prism** is a box (all edges meet at right angles) with 6 rectangular faces and dimensions of length $\ell$, width $w$, and height $h$. A rectangular prism's dimensions may be told to you in the form $\ell \times w \times h$, which is read as "$\ell$ by $w$ by $h$." A **cube** is a special case of rectangular prism where all three dimensions are equal (there is one common side length $\ell$) and therefore, every face is a square.



Rectangular Prism



Cube

For right rectangular prisms, the volume formula is given to you in the info box at the beginning of test sections. A box has three dimensions—length $\ell$, width $w$, and height $h$—and its volume is the product of the three (similarly to how the area of a rectangle is the product of the length and width):

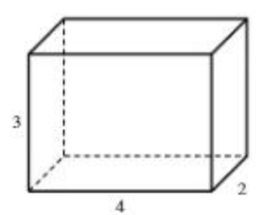
$$V = \ell wh$$

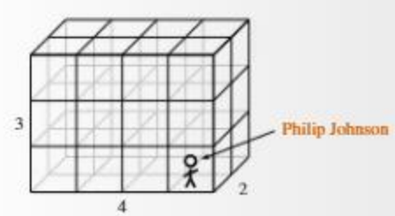
The volume of a cube with side length $\ell$ is simplified because all three dimensions are equal:

$$V = \ell\ell\ell \qquad \text{or} \qquad V = \ell^3$$

It is exactly due to this relationship that when we raise a value to the third power, we say that it is "cubed."

Just as we can divide the area of a rectangle into 1 unit$^2$ squares, we can also divide rectangular prisms into 1 unit$^3$ cubes. For example, the $4 \times 2 \times 3$ rectangular prism shown below has a volume of $4(2)(3) = 24$, and it could be divided into 24 cubes of size 1 unit$^3$.




Note, importantly, that it makes no difference which dimension of the solid is assigned each designation (length, width, and height) in a diagram or sketch; only the orientation of the solid will vary, but the solid will still have the same shape and therefore the volume calculation will produce the same result regardless.





**Example 20.5-1**

**1** 1600.io 20.5

What is the volume, in cubic feet, of a right rectangular prism that has a length of 3 feet, a width of 10 feet, and a height of 8 feet?

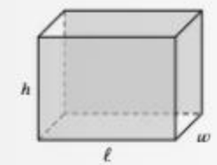
**Solution**

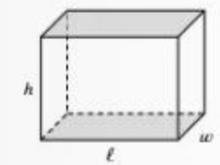
1. The volume of a right rectangular prism is found with the equation $V = \ell wh$, so substitute 3 for $\ell$, 10 for $w$ and 8 for $h$ (all of which are in units of feet) to find the volume in cubic feet.

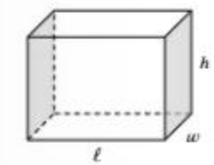
$$V = \ell wh$$
$$V = (3)(10)(8)$$
$$V = 240$$

2. The answer is 240.

Occasionally you will be asked for the **surface area** of right rectangular prisms or cubes, which represents the **sum of the areas of all the faces of the solid**. To find the surface area of a right rectangular prism, you simply add up the areas of all 6 faces of the object. As shown in the figures below, there are two faces that are rectangles measuring $\ell$ by $h$, two measuring $\ell$ by $w$, and two measuring $h$ by $w$. Here, again, it makes no difference how you draw and label your figure; the solid, and thus its surface area, will be the same.




It is not worth remembering the formula for the surface area of a right rectangular prism because it is used infrequently and it is easy to figure it out when needed, but for those interested, the surface area $S$ of a right rectangular prism with length $\ell$, width $w$, and height $h$ is the following:

$$S = 2(\ell h + \ell w + wh)$$

For cubes with sides of length $\ell$, this is once again simplified because all 6 faces are squares with the same area of $\ell^2$. Therefore, the surface area $S$ of a cube with sides of length $\ell$ is the following:

$$S = 6\ell^2$$

**Example 20.5-2**

**2** 1600.io 20.5

The dimensions of a right rectangular prism are 2 inches by 3 inches by 4 inches. What is the surface area, in square inches, of the prism?

A) 12

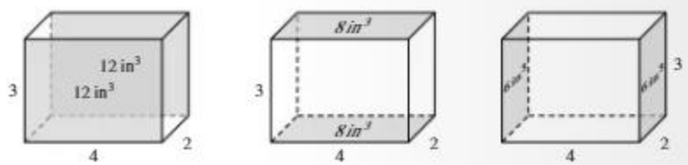
B) 24

C) 26

D) 52

**Solution**

1. The surface area of a right rectangular prism is equal to the sum of the areas of the six faces. Make a quick sketch of a box and label its dimensions with the values given in the problem.




Two of the faces measure 3 by 4 inches and thus each has an area of (3)(4) = 12 square inches, combining for a total of 24 square inches.

Two of the faces measure 2 by 4 inches and thus each has an area of (2)(4) = 8 square inches, combining for a total of 16 square inches.

Finally, two of the faces measure 2 by 3 inches and thus each has an area of (2)(3) = 6 square inches, combining for a total of 12 square inches.

2. Add the combined areas from the previous step to find the surface area of the right rectangular prism.

$$S = 24 + 16 + 12$$
$$S = 52$$

3. The answer is D.

**Notes**

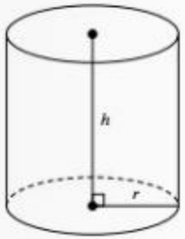
You can modestly reduce the number of calculations needed (and thus the likelihood of an error creeping in) by first summing the areas of one of each of the faces and then doubling the sum: $2(12 + 8 + 6) = 2(26) = 52$.





## Right Circular Cylinders

A **right circular cylinder** is a three-dimensional figure with a circular base of radius $r$ and height of $h$.




The volume of a cylinder is given to you in the info box at the beginning of math sections on the test.

$$V_{cyl} = \pi r^2 h$$

We like to think about the volume of a right circular cylinder as the area of the circular base times the height of the cylinder (conceptually, the volume is formed by stacking a bunch of circles with area $\pi r^2$ on top of each other).

**Example 20.5-3**

**3** 1600.io 20.5

The volume of right circular cylinder A is 32 cubic inches. What is the volume, in cubic centimeters, of a right circular cylinder with half the radius and twice the height of cylinder A?

A) 8

B) 16

C) 32

D) 64

**Solution**

1. Since the volume of a right circular cylinder is given by the equation $V = \pi r^2 h$, we know that for cylinder A, $V_A = \pi r_A^2 h_A$, where $r_A$ is the radius of cylinder A and $h_A$ is the height of cylinder A (we will make use of the fact that $V_A = 32$ later in the problem).

2. We are told about a new cylinder, which we will call cylinder B. Cylinder B has a radius of length $r_B = \frac{r_A}{2}$ (it is half the radius of cylinder A) and a height of $h_B = 2h_A$.

   Since the volume of cylinder B will be given by the equation $V_B = \pi r_B^2 h_B$, we can substitute $\frac{r_A}{2}$ for $r_B$ and $2h_A$ for $r_B$. This step will enable us to find the volume of cylinder B in terms of $V_A$.

$$V_B = \pi r_B^2 h_B$$

$$V_B = \pi \left(\frac{r_A}{2}\right)^2 (2h_A)$$

$$V_B = \pi \left(\frac{r_A^2}{4}\right)(2h_A)$$

$$V_B = \pi \left(\frac{2r_A^2 h_A}{4}\right)$$

$$V_B = \pi \left(\frac{r_A^2 h_A}{2}\right)$$

$$V_B = \frac{1}{2}(\pi r_A^2 h_A)$$

3. We can use a proportion to solve for $V_B$. On one side of our proportion, we should write $\frac{V_B}{V_A}$, and on the other side we should write $\frac{\frac{1}{2}(\pi r_A^2 h_A)}{\pi r_A^2 h_A}$ (these are the volume expressions for both cylinders in terms of the radius and height of cylinder $A$). Substitute 32 for $V_A$ and then solve for $V_B$.

$$\frac{V_B}{V_A} = \frac{\frac{1}{2}(\pi r_A^2 h_A)}{\pi r_A^2 h_A}$$

$$\frac{V_B}{32} = \frac{\frac{1}{2}(\cancel{\pi r_A^2 h_A})}{\cancel{\pi r_A^2 h_A}}$$

$$\frac{V_B}{32} = \frac{1}{2}$$

$$V_B = \frac{1}{2}(32)$$

$$V_B = 16$$

4. The answer is B.

**Notes**

A more intuitive way to approach this problem is to understand that the influence of the radius varies as the square of its value due to the presence of $r^2$ in the volume formula, so if the radius is halved, the overall value of the expression representing the volume will be quartered because $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Then, you can observe that the volume varies directly with the height due to the presence of $h$ in the formula, so if the height is doubled, then volume will double as well; the net result is that the volume will be halved overall because $\left(\frac{1}{4}\right)(2) = \frac{1}{2}$.

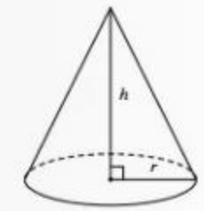
This approach is valid, but taking less strict approaches often leads to careless mistakes, so be careful.





## Right Circular Cones

Right circular cones have a circular base with radius $r$, but unlike a cylinder, cones taper to a point as they reach their height $h$.




The volume of a cone of height $h$ whose base has radius $r$ is given by the following equation:

$$V = \frac{1}{3}\pi r^2 h$$

You may notice that this volume equation is very similar to that of a cylinder and is just the volume of a cylinder times $\frac{1}{3}$.

**Example 20.5-4**

**4** 1600.io 20.5

A right circular cone has a volume of $\frac{1}{3}\pi$ cubic centimeters and a height of 25 centimeters. What is the radius, in centimeters, of the base of the cone?

A) $\frac{1}{5}$

B) $\frac{1}{\sqrt{5}}$

C) $\sqrt{5}$

D) 5

**Solution**

1. The equation for the volume $V$ of a right circular cone with a base radius of $r$ and a height of $h$ is $V = \frac{1}{3}\pi r^2 h$. Substitute $\frac{1}{3}\pi$ for $V$, and 25 for $h$, then solve for $r$.

$$V = \frac{1}{3}\pi r^2 h$$

$$\cancel{\frac{1}{3}}\cancel{\pi} = \cancel{\frac{1}{3}}\cancel{\pi} r^2(25)$$

$$1 = 25r^2$$

$$\frac{1}{25} = r^2$$

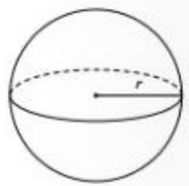
$$\pm\frac{1}{5} = r$$

2. The radius must be positive, so $r = \frac{1}{5}$.
3. The answer is A.

## Spheres and Right Rectangular Pyramids

The volume equations for spheres and rectangular pyramids are given in the info box at the beginning of each math section on the test, but neither shape has appeared on any known tests so far.

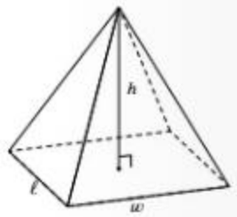
A **sphere** is the shape formed by all points in three dimensions that are the same distance, $r$ (a radius), away from the center point.




The formula for the volume a sphere of radius $r$ is given by the following equation:

$$V = \frac{4}{3}\pi r^3$$

A **right rectangular pyramid** has a rectangular base of dimensions $\ell$ by $w$ and four triangular faces that meet at a point directly above the center of the base at a height of $h$.




The formula for the volume of a right rectangular pyramid of height $h$ whose base measures $\ell$ by $w$ is the following:

$$V = \frac{1}{3}\ell wh$$

You may notice that this volume equation is very similar to that of a right rectangular prism and is just the volume of a right rectangular prism times $\frac{1}{3}$ (analogous to the relationship between right circular cylinders and cones).





## Density

The concept of density is sometimes referenced on the test and is more a matter of understanding how to make use of the units than of understanding the physical concept. The **density $D$ of an object is equal to the mass of the object $m$ divided by its volume $V$:**

$$D = \frac{m}{V}$$

When density has appeared on the test so far, a reference to this above equation has usually been given, but you could also construct the equation by yourself if you are told that the density of an object is measured in units like kilograms per cubic meter because, as the units indicate, the density must be found by dividing a mass (measured in kilograms) by a volume (measured in cubic centimeters).

**Example 20.5-5**

**5** 1600.io 20.5

A gelatin block in the shape of a right rectangular prism has dimensions 50 centimeters by 60 centimeters by 70 centimeters. The block has a density of 1.1 grams per cubic centimeter. What is the mass of the block, in grams? (Density is mass per unit volume.)

A) 330
B) 3,300
C) 23,100
D) 231,000

**Solution**

1. The equation for density is $D = \frac{m}{V}$, where $D$ is the density of the block, $m$ is the mass of the block, and $V$ is the volume of the block. In order to solve for $m$, we need to multiply both sides of the equation by $V$ to get $DV = m$.
2. Since the block is a right rectangular prism, its volume $V$ is equal to the product of its three given dimensions. Substitute 1.1 for $D$ and substitute $(50 \times 60 \times 70)$ for $V$ (the units will be $\frac{\text{g}}{\text{cm}^3}$ as desired, so there is no need for conversions).

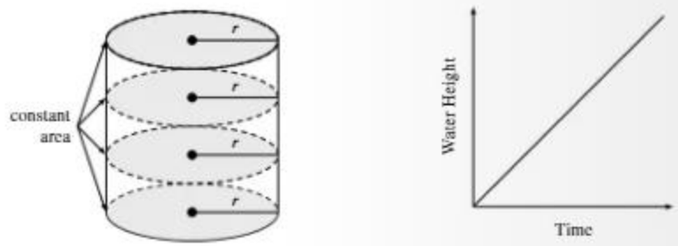
$$DV = m$$
$$(1.1)(50 \times 60 \times 70) = m$$
$$231{,}000 = m$$

3. The answer is D.

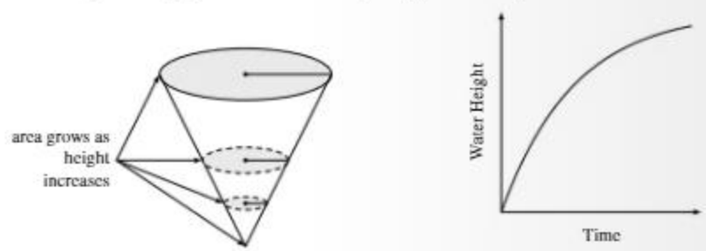
## Filling an Object with Water

There has only been a single appearance on the test of a problem that asked about the way the height of a liquid changes when filling a container, but many students were befuddled by it, so we'll do a quick rundown of the idea. As long as the **cross-sectional area** (a horizontal slice through an object) of an object is constant, the height of liquid in the object will increase at the same rate.

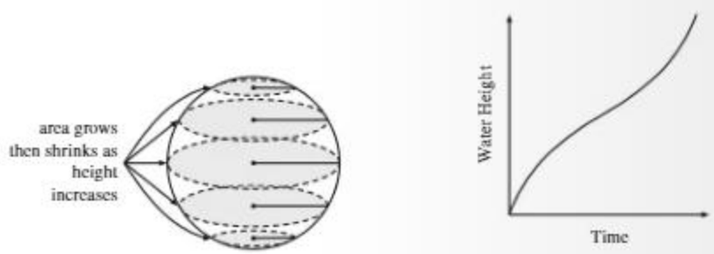
For example, if we filled a right cylinder with water poured at a steady rate, the height of water in the glass will increase linearly with time because the cross-sectional area is always the same (each layer or "slice" of water will take the same time to fill).




However, for a shape like a cone with its point on the bottom, the cross-sectional area gets bigger as we go up the cone (we go from a point on the bottom to a circle of radius $r$ at the top), so the height will increase more and more slowly as the liquid is poured in (in simple terms, the water has to spread out more to the sides before it can go up). Therefore, the graph of the height of water over time will look more like an exponential graph that starts out steep but gets less steep as the cross-sectional area keeps increasing.




If we were to fill a sphere with water, the cross-sectional area of the object first grows, then reaches its maximum in the middle of the sphere, and then shrinks back. The graph of the height of the water over time would look more like the following, steep at first but with a decreasing (but still positive) slope, then midway along, the slope begins increasing again (kind of like a cubic graph).








### Section 20.5 Suggested Problems from Real Tests

- Test 1-C-35
- Test 3-C-25
- Test 4-C-18
- Test 5-NC-11
- Test 5-C-26
- Test 6-C-33
- Test 9-NC-16
- Test 10-C-15
- May 2019 (Int)-NC-9
- Oct 2019-NC-13
- Mar 2021-NC-12
- Apr 2021-C-10
- Apr 2021-C-22

View related real-test problems in SMTEx

SMTEx

## Section 20.5 Practice Problems

**1** 1600.io 20.5

A box in the shape of a right rectangular prism has a volume of 40 cubic inches. If the dimensions of the box are 5 inches by 8 inches by $h$ inches, what is the value of $h$ ?

A) 0

B) 1

C) 2

D) 3

**2** 1600.io 20.5

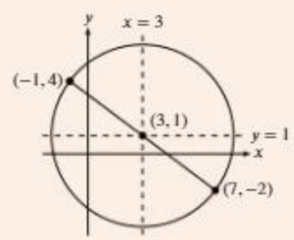
A drinkware company produces cylindrical glasses, each with an internal radius of 1.5 inches and an internal height between 4.5 inches and 5 inches. What is one possible volume, rounded to the nearest cubic inch, of a cylindrical glass produced by this company?

# CHAPTER 20 RECAP

A circle is a shape formed by **all of the points that are the same distance away from one central point** (the center of the circle). The **radius**, marked $r$ in the picture below, is the distance from any point on the circle to the center point.

A **diameter** is a line from one point on the arc to its opposite point on the arc that goes directly through the center of the circle and whose length, $d$, is equal to twice the radius: $d = 2r$.

The center point of the circle is the midpoint between the endpoints of any diameter. This center point's $x$- and $y$-coordinates can be found by averaging the $x$- and $y$-coordinates, respectively, of the endpoints of any diameter.




$$x_{center} = \frac{(-1)+7}{2} \qquad y_{center} = \frac{4+(-2)}{2}$$

$$x_{center} = \frac{6}{2} \qquad y_{center} = \frac{2}{2}$$

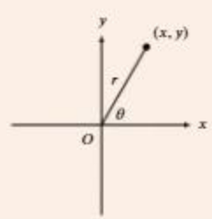
$$x_{center} = 3 \qquad y_{center} = 1$$

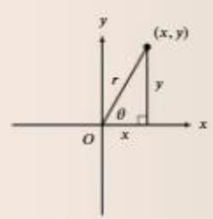
$$\text{Center} = (x_{center}, y_{center})$$

$$\text{Center} = (3, 1)$$

If a triangle is formed with one vertex at the center of the circle and the other two vertices at points on the circle, then two sides of the triangle are equal to the radius, and thus the triangle is isosceles.

When we graph a line segment of length $r$ having one end at the origin, extending to a point $(x, y)$, and lying at an angle $\theta$ measured relative to the positive $x$-axis, we can form a triangle as shown below. The horizontal leg has length $x$, and the vertical leg has length $y$. The values of $x$ and $y$ can be positive or negative, while $r$ is always positive.

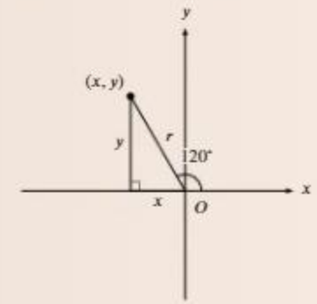


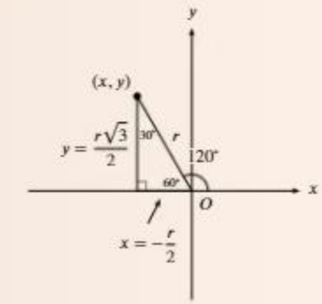








When determining absolute or relative side lengths for such a triangle, be sure to assign the correct sign. In the example below, the short leg lies along the negative $x$-axis, so its length will be negative. It is essential to have the proper signs for the side lengths when constructing trigonometric ratios for angles represented on the coordinate plane.

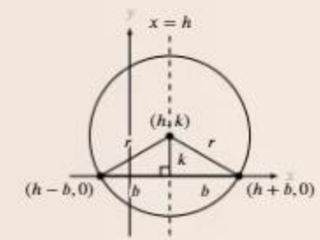


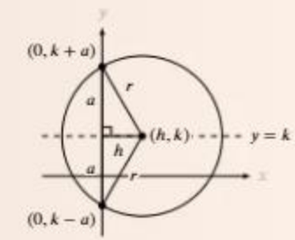




The angle ratios of a right triangle drawn on the coordinate plane are still calculated the same way (remember SOH-CAH-TOA).

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

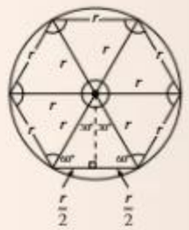
Pairs of points on the circle that have the same $y$-coordinate will be equidistant from the vertical axis of symmetry, and the distance between such points and the axis of symmetry is simply the difference in the points' $x$-coordinates and the axis of symmetry's (and thus the center point's) $x$-coordinate, as seen below, left. With respect to the horizontal axis of symmetry, the same principle applies for pairs of points that have the same $x$-coordinate, as seen on the right below.




If we're not given the coordinates of the center point, but we have the coordinates of two points on the circle with either the same $x$- or $y$-value, **average the $x$- or $y$-coordinates of the two points on the circle to find the corresponding coordinate of the center point**.

A polygon can be **inscribed** in a circle, which means that **all of the vertices of the polygon are points on the circle**. If the **inscribed polygon is a convex regular polygon, you should draw radii to all of the polygon's vertices, thus creating a set of triangles**; all of the **central angles created will be equal to each other and will divide the circle into equal segments**. All of the triangles will be isosceles (and possibly equilateral in the case of the hexagon shown below) because (at least) two of the sides of each triangle will be equal to the radius of the circle.



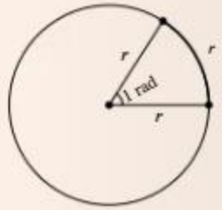

The circumference $C$ of a circle with a radius $r$ is found with the following equation:

$$C = 2\pi r$$

Since the diameter $d$ of the circle is twice the radius, or $2r$, the formula is sometimes written as the following:

$$C = \pi d$$

For an angle measuring 1 **radian**, the length of the intercepted arc is equal to the **radius**, by definition.




**The number of radians of angle measure is defined as the number of radiuses in intercepted arc length**, so if we know how many radians a central angle measures, we know how many radiuses the intercepted arc measures, and if we know how many radiuses long an arc measures, we know the measure in radians of the central angle that intercepts that arc.

There are 360° in a circle, and there are $2\pi$ radians in a circle, so we can use the following proportion when we want to convert between the degrees and radians:

$$\frac{\text{degrees}}{\text{radians}} = \frac{360^\circ}{2\pi \text{ rad}} \quad \text{or, equivalently} \quad \frac{\text{degrees}}{\text{radians}} = \frac{180^\circ}{\pi \text{ rad}}$$

Rearranging gives us the following two conversion formulas:

$$\text{degrees} = \text{radians}\left(\frac{180}{\pi}\right) \qquad \text{radians} = \text{degrees}\left(\frac{\pi}{180}\right)$$





***Radians* of angle produce *radiuses* of arc length**. Therefore, the formula for the length of an arc intercepted by an angle $\theta$ (measured in radians) is given by

$$L = r\theta$$

If you are given a central angle measure in radians, **do not convert this angle to degrees to calculate the intercepted arc length**.

If you are given the measurement of the central angle in degrees, you can calculate the intercepted arc length by converting the angle from degrees to radians or you can find the arc's fraction of the whole circumference by using a proportion.

Since there are 360° in a circle, the fraction of the circle that is marked by a central angle measuring $x°$ is

$$\frac{x}{360}$$

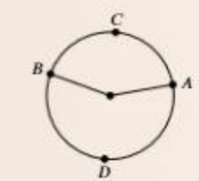
The ratio of the arc length $L$ to the whole circumference $C$ is equal to the ratio of the central angle to 360°.

$$\frac{L}{C} = \frac{x}{360} \qquad \text{solving for } L \qquad L = \frac{x}{360}C$$

If you are not given the circumference, but you do know the radius, you can use the circumference formula and substitute $2\pi r$ for $C$ to end up with the formula

$$L = \frac{x}{360}(2\pi r)$$

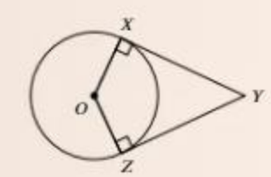
The measure of an arc between points $A$ and $B$ on a circle can be denoted as $\widehat{AB}$. If there is a third point $C$ between the points $A$ and $B$, the arc may be represented as $\widehat{ACB}$. In the figure below, you may choose to refer to **minor** (smaller) **arc** $\widehat{ACB}$ in order to distinguish it from **major** (larger) **arc** $\widehat{ADB}$, which is the **complementary** arc.



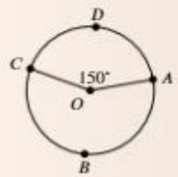

If there are only two points, and neither major nor minor is specified, assume the reference is to the minor arc.

If you are told that lines are **tangent** to a circle, then you should note immediately that those lines **intersect the circle at exactly one point**, and a **radius drawn from the center of the circle to the point of tangency will intersect the tangent line at a right angle**.

For example, in the figure below, lines $\overline{XY}$ and $\overline{ZY}$ are tangent to the circle.

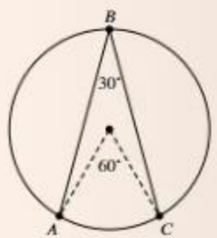


An arc can be specified by the corresponding central angle (in degrees or radians). For example, in the figure below, we can say that the arc $\widehat{ADC}$ measures 150° because the central angle forming that intercepted arc measures 150°.




An **inscribed angle** is an angle with its vertex on the circumference of the circle. **The measure of an inscribed angle is half of the measure of the central angle of its intercepted arc**.

For example, in the figure below, the measure of $\angle ABC$ is 30°, and the measure of the central angle that intercepts minor arc $\widehat{AC}$ is 60° (its relative length it would be $\frac{60}{360}$ or $\frac{1}{6}$ of the circumference).




The formula for the area $A$ of a circle of radius $r$ is the following:

$$A = \pi r^2$$

The area of a sector of a circle marked by a central angle $\theta$ is proportional to the ratio of the central angle to the angle measure of a full circle (360° or $2\pi$ radians). Therefore, the area of a sector marked by a central angle measuring $x°$ or $y$ radians is found with the following equations:

$$A_{sector} = \frac{x}{360}(\pi r^2) \qquad \text{or} \qquad A_{sector} = \frac{y}{2\pi}(\pi r^2)$$

The equation for all points $(x, y)$ on a circle **centered at the origin** with radius $r$ is derived from the Pythagorean Theorem:

$$x^2 + y^2 = r^2$$

For any circle with radius $r$, **centered at a point** $(h, k)$, the equation for all points $(x, y)$ on the circle is slightly modified to reflect the shifted center point:

$$(x - h)^2 + (y - k)^2 = r^2$$

All points strictly inside of a circle conform to the inequality $(x - h)^2 + (y - k)^2 < r^2$. All points strictly outside of a circle conform to the inequality $(x - h)^2 + (y - k)^2 > r^2$.





In order to write a circle equation in the form $ax^2 + bx + ay^2 + cy + d = e$ in Standard Form, $(x - h)^2 + (y - k)^2 = r^2$, follow these steps:

1. Isolate all constant terms on the opposite side of the equation from the $x$- and $y$-terms.
2. Divide both sides of the equation by $a$, the coefficient of the $x^2$ and $y^2$ terms, making sure to distribute the division across all terms. The $x^2$ and $y^2$ coefficients will always match for circle equations, and most often on the test will be 1, so this step will often be unnecessary.
3. Complete the square for both $x$- and $y$-variable polynomial expressions.

   If the $x$-coefficient is $p$ and the $y$-coefficient is $q$, you should rewrite the $x$ terms as $\left(x + \frac{p}{2}\right)^2$, rewrite the $y$ terms as $\left(y + \frac{q}{2}\right)^2$, and add $\left(\frac{p}{2}\right)^2$ and $\left(\frac{q}{2}\right)^2$ to the other side of the equation.
4. The equation is now in Standard Form for circles, and you can therefore identify the center point and radius.

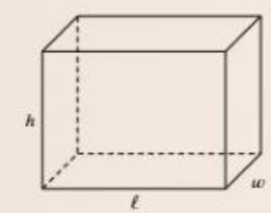
If you only need to determine the coordinates of the center point $(h, k)$, but you don't need the radius, you don't need to perform the entire completing the square procedure; you can observe that $h$ will be the negative of $\frac{1}{2}$ of the $x$-coefficient, and, similarly, $k$ will be the negative of $\frac{1}{2}$ of the $y$-coefficient (after dividing through by the squared terms' coefficient, if it's not 1).

Similarly, if you only need the radius, move any constant to the right side and add to it the squares of half the coefficients (again, after dividing through by the squared terms' coefficient, if it's not 1); the result will be the square of the radius.

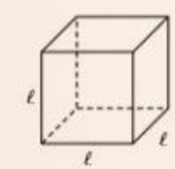
The **volume of a solid or three-dimensional figure is the amount of space that the object occupies** and is usually measured in units$^3$ (though the volume of fluids like water and oil might be measured in units like ounces or liters).

A **right rectangular prism** is a box (all faces meet at right angles) with 6 rectangular faces and dimensions of length $\ell$, width $w$, and height $h$.

A **cube** is a special case of rectangular prism where all three dimensions are equal (there is one common side length $\ell$) and therefore, every face is a square.



Rectangular Prism



Cube

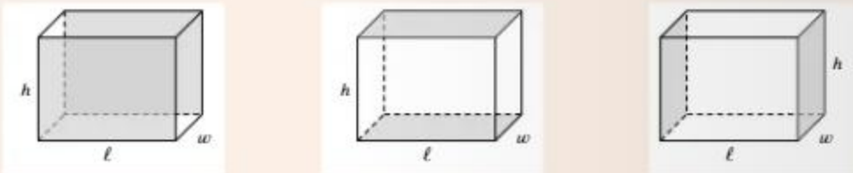
For right rectangular prisms, the volume formula is given to you in the info box at the beginning of test sections.

$$V = \ell wh$$

The volume of a cube with side length $\ell$ is simplified because all three dimensions are equal:

$$V = \ell\ell\ell \qquad \text{or} \qquad V = \ell^3$$

To find the surface area of a right rectangular prism, you simply add up the areas of all 6 faces of the object. As shown in the figures below, there are two faces that are rectangles measuring $\ell$ by $h$, two faces that are rectangles measuring $\ell$ by $w$, and two faces measuring $h$ by $w$.



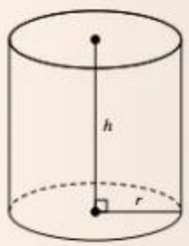

The surface area $S$ of a right rectangular prism with length $\ell$, width $w$, and height $h$ is the following:

$$S = 2(\ell h + \ell w + wh)$$

For cubes with sides of length $\ell$, all six faces are squares with the same area of $\ell^2$. Therefore, the surface area $S$ of a cube with sides of length $\ell$ is the following:

$$S = 6\ell^2$$

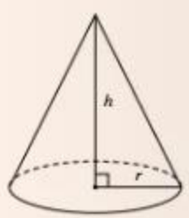
A **right circular cylinder** is a three-dimensional figure with a circular base of radius $r$ and height of $h$.




The volume of a cylinder is given to you in the info box at the beginning of math sections on the test, and it's the area of the circular base times the height of the object.

$$V_{cyl} = \pi r^2 h$$

Right circular cones have a circular base with radius $r$, but unlike cylinders, cones taper to a point as they reach their height $h$.



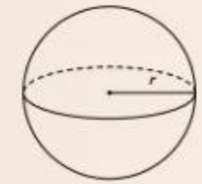

The volume of a cone of height $h$ whose base has radius $r$ is given by the following equation:

$$V = \frac{1}{3}\pi r^2 h$$





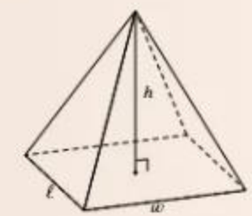
A **sphere** is the shape formed by all points in three dimensions that are the same distance, $r$ (a radius), away from the center point.




The formula for the volume a sphere of radius $r$ is given by the following equation:

$$V = \frac{4}{3}\pi r^3$$

A **right rectangular pyramid** has a rectangular base of dimensions $\ell$ by $w$ and four triangular faces that meet at a point directly above the center of the base at a height of $h$.




The formula for the volume of a right rectangular pyramid of height $h$ whose base measures $\ell$ by $w$ is the following:

$$V = \frac{1}{3}\ell w h$$

The **density $D$ of an object is equal to the mass of the object $m$ divided by its volume $V$:**

$$D = \frac{m}{V}$$

When filling a container with a liquid, the height of the liquid in the container changes at a rate that coincides with the cross-sectional area of the container. If the container is a cylinder, whose cross-sectional area is the same at all heights, the height of the liquid changes linearly. For a spherical container, the height of the liquid will increase at a declining rate until the container is half full, after which the height will rise at an increasing rate.

## Additional Problems

**1** 20.4

In the $xy$-plane, the points $(3, 5)$ and $(-3, -5)$ are the endpoints of a diameter of a circle. Which of the following is an equation of the circle?

A) $x^2 + y^2 = 34$

B) $x^2 + y^2 = 16$

C) $(x-3)^2 + (y-5)^2 = 64$

D) $(x-3)^2 + (y-5)^2 = 16$

**2** 20.2

An angle measure of 720 degrees was written in radians as $x\pi$. What is the value of $x$ ?

**3** 20.4

A circle in the $xy$-plane has center $(4, 2)$ and radius 3. Which of the following is an equation of the circle?

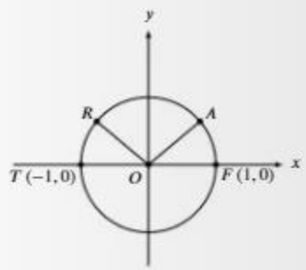
A) $(x+4)^2 + (y+2)^2 = 9$

B) $(x-4)^2 + (y-2)^2 = 9$

C) $(x+4)^2 + (y+2)^2 = 3$

D) $(x-4)^2 + (y-2)^2 = 3$

**4** 20.2




In the $xy$-plane above, points $F$, $A$, $R$, and $T$ lie on the circle with center $O$. The degree measures of angles $FOA$ and $ROT$ are each 40°. What is the radian measure of angle $AOR$ ?

A) $\frac{1}{9}\pi$

B) $\frac{2}{9}\pi$

C) $\frac{1}{3}\pi$

D) $\frac{5}{9}\pi$

**5** 20.4

Which of the following is an equation of the circle in the $xy$-plane that has center $(0, 0)$ and radius 9 ?

A) $x^2 + y^2 = 9$

B) $x^2 + y^2 = 18$

C) $x^2 + y^2 = 81$

D) $x^2 + y^2 = 324$





**6** 20.4

Which of the following is an equation of a circle in the $xy$-plane with center $(0, 5)$ and a radius with endpoint $\left(\frac{8}{3}, 3\right)$ ?

A) $x^2 + (y+5)^2 = \frac{100}{9}$

B) $x^2 + (y-5)^2 = \frac{100}{9}$

C) $x^2 + (y+5)^2 = \frac{10}{3}$

D) $x^2 + (y-5)^2 = \frac{10}{3}$

**7** 20.3

The area enclosed by a circle is $36\pi$ square inches. What is the length, in inches, of the radius of the circle?

A) 3

B) 3.6

C) 6

D) 18

**8** 20.5

Faye has identical tubes in the shape of a right circular cylinder with internal diameter of 2 inches. She pours water from a gallon bottle into each tube until it is full. If the height of water in each tube is 8 inches, what is the largest number of full water tubes that she can pour from one gallon of water? (Note: There are 231 cubic inches in 1 gallon.)

A) 2

B) 3

C) 9

D) 10

**9** 20.2

An instrument shows the number of revolutions per minute made by each tire of a bicycle. In each revolution, the bicycle travels a distance equal to the circumference of one of its tires. The circumference of each tire is equal to $2\pi r$, where $r$ is the radius of the tire.

If the radius of each tire on Imelda's bicycle is 0.24 meter, what is the approximate speed of Imelda's bicycle, to the nearest kilometer per hour, when the instrument is showing 100 revolutions per minute? (1 kilometer = 1000 meters)

**10** 20.5

An ice cream shop owner determined that right circular cones with a height that is 6 inches longer than the radius offer the optimal number of cones to be displayed in a case. Which of the following expresses the volume, $V$, in cubic inches, of such cones, where $r$ is the radius, in inches?

A) $V = 2\pi r^2$

B) $V = \frac{1}{3}\pi(r+6)^3$

C) $V = \frac{1}{3}\pi r^2 + 2\pi r$

D) $V = \frac{1}{3}\pi r^3 + 2\pi r^2$

**11** 1600.io 20.2

The number of radians in a 540-degree angle can be written as $a\pi$, where $a$ is a constant. What is the value of $a$ ?

**12** 1600.io 20.4

Which of the following equations describes a circle with radius 6 that passes through the origin when graphed in the $xy$-plane?

A) $(x-3)^2+(y+3)^2=6$

B) $(x-3)^2+(y+3)^2=36$

C) $(x-6)^2+(y-6)^2=36$

D) $\left(x-3\sqrt{2}\right)^2+\left(y-3\sqrt{2}\right)^2=36$

**13** 1600.io 20.5

A right circular cone has a volume of $54\pi$ cubic inches. If the height of the cones is 2 inches, what is the radius, in inches, of the base of the cone?

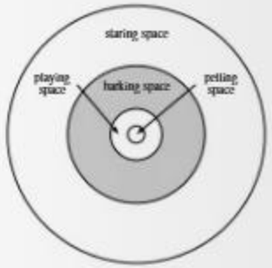
A) $3\sqrt{3}$

B) 9

C) 27

D) 81

**14** 1600.io 20.3




The diagram above represents George O. Williams's concept of space surrounding a dog defined by four nonoverlapping regions. Petting space is the region inside a a circle of radius 1 foot. Playing space is the region within a circle of radius of 3 feet but outside petting space. Barking space is the region within a circle of radius 8 feet but outside playing space. Staring space is the region within a circle of radius of 15 feet but outside barking space. What is the area, in square feet, of the shaded region representing a dog's barking space?

A) $9\pi$

B) $55\pi$

C) $64\pi$

D) $225\pi$

**15** 1600.io 20.4

$$x^2+40x+y^2+18y=-40$$

The equation above defines a circle in the $xy$-plane. What are the coordinates of the center of the circle?

A) $(-40,-18)$

B) $(-20,-9)$

C) $(20,9)$

D) $(40,18)$





**16** 1600.io 20.4

$$x^2+y^2+2x-6y=-1$$

The equation of a circle in the $xy$-plane is shown above. What is the radius of the circle?

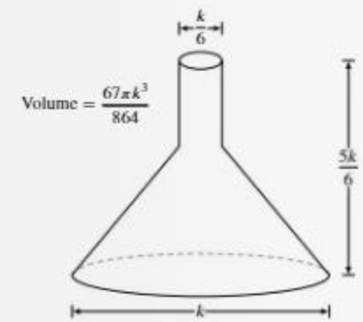
A) 1

B) 3

C) 8

D) 9

**Questions 17 and 18 refer to the following information.**




The flask pictured above can hold a maximum volume of 100 cubic centimeters.

**17** 1600.io 20.5

What is the value of $k$, in centimeters?

A) 2.43

B) 2.89

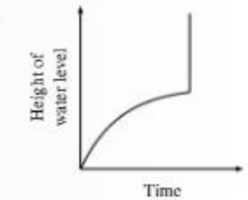
C) 7.43

D) 12.21

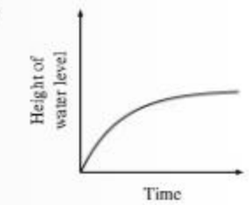
**18** 1600.io 20.5

Water pours into the flask slowly and a constant rate. Which of the following graphs best illustrates the height of the water level in the flask as it fills?

A)

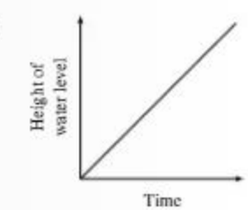



B)

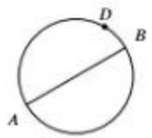



C)

D)



**19** 1600.io 20.2




In the circle above, segment $AB$ is a diameter. If the length of arc $\widehat{ADB}$ is $6\pi$, what is the length of the radius of the circle?

A) 2
B) 3
C) 6
D) 12

**20** 1600.io 20.4

$$(x-7)^2+(y+6)^2=9$$

In the $xy$-plane, the graph of the equation above is a circle. Point $P$ is on the circle and has coordinates $(10, -6)$. If $\overline{PQ}$ is a diameter of the circle, what are the coordinates of point $Q$ ?

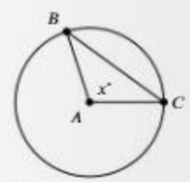
A) $(7, -9)$
B) $(7, -6)$
C) $(7, -3)$
D) $(4, -6)$

**21** 1600.io 20.5

A cube has a surface area of 24 square inches. What is the volume, in cubic inches, of the cube?

A) 4
B) 8
C) 12
D) 64

**22** 1600.io 20.2




Note: Figure not drawn to scale.

In the circle above, point $A$ is the center and the length of arc $\widehat{BC}$ is $\frac{3}{10}$ of the circumference of the circle. What is the value of $x$ ?





**23** 1600.io 20.2

The circumference of Mars is estimated to be 21,345 km at the equator. Which of the following best approximates the diameter, in miles, of Mars' equator? (1 kilometer ≈ 0.62137 miles)

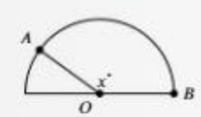
A) 2,034 miles
B) 4,222 miles
C) 5,132 miles
D) 6,794 miles

**24** 1600.io 20.4

$$(x-4)^2+(y-5)^2=36$$

The graph in the $xy$-plane of the equation above is a circle. If the circle is translated downward $a$ units such that the circle is tangent to the $x$-axis, the equation becomes $(x-4)^2+(y-5+a)^2=36$. What is the value of $a$ ?

**25** 1600.io 20.2




Segments $\overline{OA}$ and $\overline{OB}$ are radii of the semicircle above. Arc $\widehat{AB}$ has length $4\pi$ and $OA = 5$. What is the value of $x$ ?

**26** 1600.io 20.5

A hobbyist purchased two slabs of pewter, both in the shape of a right rectangular prism. The table below shows some information about the two slabs.

| | Length | Width | Thickness | Mass |
|---|---|---|---|---|
| Slab 1 | 100 cm | 15 cm | 6 cm | 67,230 g |
| Slab 2 | 150 cm | 10 cm | 4 cm | 44,820 g |

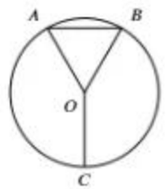
What is the density, in grams per cubic centimeter, of Slab 1 ?

**27** 1600.io 20.5

A cylindrical cup containing beans is filled to the top with hot dog water before being chugged. The base of the cup has an area of 25 $cm^2$, and the height of the cup is 10 cm. If 135 $cm^3$ of hot dog water is needed to fill the cup to the top, which of the following is closest to the total volume of the beans in the cup?

A) 2.5 $cm^3$
B) 115 $cm^3$
C) 160 $cm^3$
D) 250 $cm^3$

**28** 1600.io 20.2




Point $O$ is the center of the circle above, and the measure of $\angle OAB$ is $60^\circ$. If the length of $\overline{OC}$ is 12, what is the length of arc $\widehat{AB}$?

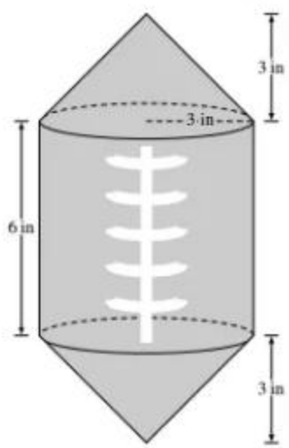
A) $2\pi$

B) $4\pi$

C) $12\pi$

D) $24\pi$

**29** 1600.io 20.5




A futuristic football is built from two right circular cones and a right circular cylinder with internal measurements represented by the figure above. Of the following, which is closest to the volume of the football, in cubic inches?

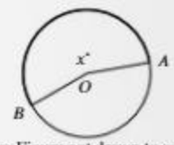
A) 56.1

B) 169.6

C) 197.9

D) 226.2

**30** 1600.io 20.2




Note: Figure not drawn to scale.

In the figure above, the circle has center $O$ and has radius 5. If the length of arc $\widehat{AB}$ (shown in bold) is between 20 and 21, what is one possible integer value of $x$?





# Answer Key

### Section 20.1 Practice Problems

1. 12
2. 3.2 or $\frac{16}{5}$

### SkillDrill 20.2-1

1. $6\pi$
2. $4\pi$
3. $\pi$
4. $2\pi\sqrt{6}$
5. 1
6. $\frac{5}{2}$
7. 6
8. $\frac{1}{4}$

### SkillDrill 20.2-2

1. $\pi$
2. $\frac{\pi}{3}$
3. $\frac{\pi}{4}$
4. $\frac{2\pi}{3}$
5. $\frac{7\pi}{6}$
6. $\frac{\pi}{5}$
7. $\frac{5\pi}{3}$
8. $2\pi$
9. $180^\circ$
10. $360^\circ$
11. $144^\circ$
12. $240^\circ$
13. $60^\circ$
14. $270^\circ$
15. $330^\circ$
16. $135^\circ$

### SkillDrill 20.2-3

1. $3\pi$
2. $\frac{5\pi}{2}$
3. $\frac{\pi}{3}$
4. $\frac{21\pi}{2}$
5. $\frac{1}{2}$
6. $\frac{2}{3}$
7. $\frac{1}{5}$
8. $\frac{3}{4}$
9. $\frac{1}{2}$
10. $\frac{1}{8}$
11. $\frac{1}{12}$
12. $\frac{2}{3}$
13. $3\pi$
14. $\frac{5\pi}{4}$
15. $\frac{7\pi}{12}$
16. $10\pi$

### Section 20.2 Practice Problems

1. 103
2. 3

### SkillDrill 20.3-1

1. $9\pi$
2. $25\pi$
3. $\frac{\pi}{4}$
4. $\pi$
5. $\frac{9\pi}{2}$
6. $\frac{25\pi}{4}$
7. $\frac{\pi}{12}$
8. $14\pi$
9. $\frac{9\pi}{4}$
10. $\frac{25\pi}{8}$
11. $\frac{7\pi}{48}$
12. $30\pi$

### Section 20.3 Practice Problems

1. $\frac{2}{9}, \frac{4}{18}, \frac{8}{36}$, or .222
2. D

### SkillDrill 20.4-1

1. $(0, 0), r = 5$
2. $(0, 0), r = 3$
3. $(0, 0), r = \sqrt{6}$
4. $(1, 2), r = 5$
5. $(-4, 4), r = 4$
6. $(-3, 0), r = 3$
7. $(2, 2), r = \sqrt{11}$
8. $(0, -5), r = 5$
9. $x^2 + y^2 = 4$
10. $(x - 1)^2 + (y - 4)^2 = 36$
11. $(x + 5)^2 + y^2 = 16$
12. $(x + 3)^2 + (y + 6)^2 = 81$

**SkillDrill 20.4-2**

1. $(2, 5)$, $r = 5$
2. $(-3, 3)$, $r = 6$
3. $(1, 2)$, $r = \sqrt{11}$
4. $(-2, 3)$, $r = 3\sqrt{3}$ or $\sqrt{27}$
5. $(6, 6)$, $r = 6$
6. $\left(\frac{3}{2}, \frac{7}{2}\right)$, $r = \frac{\sqrt{58}}{2}$ or $\sqrt{\frac{58}{4}}$ or $\sqrt{\frac{29}{2}}$

**Section 20.4 Practice Problems**

1. 12
2. C

**Section 20.5 Practice Problems**

1. B
2. 32, 33, 34, or 35

**Additional Problems**

1. A
2. 4
3. B
4. D
5. C
6. B
7. C
8. C
9. 9
10. D
11. 3
12. D
13. B
14. B
15. B
16. B
17. C
18. C
19. C
20. D
21. B
22. 108
23. B
24. 11
25. 144
26. 7.47
27. B
28. B
29. D
30. 230, 231, ..., 239, 240



# Wormholes 21

## 21.1 Introduction

The SAT, of course, rewards students for getting math problems right; if you get a problem right, you get points. The test is assessing whether you have the knowledge and skill necessary to get to the correct answer; this is not complicated. However, there is another type of assessment that's also happening, and it's far more subtle: the SAT also offers a different kind of reward for students who go beyond merely knowing the mathematical principles covered by the test, because it gives students who have deeper insight into certain problems a bonus in the form of the time saved in solving those problems.

When a problem's construction has a special feature that allows a direct (or nearly direct) jump from the problem to the solution, skipping all the steps that would be necessary for a general solving approach, we at 1600.io say that the problem has a **wormhole**, named for a hypothesized feature of spacetime that allows near-instant travel between two distant locations in space without going through all the places in-between using conventional space travel. These wormholes are intentionally placed in a few problems on each test as a way to reward the perceptive student who has that extra insight, but in nearly all cases, these problems can also be solved, albeit more slowly, using more general techniques. In this way, these problems almost act like two problems in one—they reward everyone who can get them right, but there's an extra benefit for the students who spot the wormhole and save some time that can be invested elsewhere in the test section. Though the number of wormhole problems varies dramatically from test to test, there have been an average of about three such problems per released exam.

### The Classic Wormhole: Solving for an Expression

The most common type of wormhole problem asks the student to solve for the value of an *expression* that contains one or more variables, **not just for the value of a single variable**. Though these problems can almost always be answered by solving for the individual variables, then constructing the solve-for expression and evaluating it, solving directly for the value of the expression—traveling through the wormhole—is much faster.

Consider this equation:

$$3x = 12$$

If we're asked to solve for $x$, we'd divide both sides by 3:

$$\frac{3x}{3} = \frac{12}{3}$$
$$x = 4$$

Simple enough. Now, let's look at another equation:

$$3(x + 6) = 12$$

What if we're asked to solve for $x + 6$? Of course, you could first solve for $x$, and then compute the value of $x + 6$:

$$
\begin{aligned}
3(x+6) &= 12 \\
3x+18 &= 12 \\
3x &= 12-18 \\
3x &= -6 \\
x &= \frac{-6}{3} \\
x &= -2 \\
x+6 &= 4
\end{aligned}
$$

This works, but there's a wormhole here that allows a much quicker route to a solution. The solve-for, $x+6$, is a component of the original equation, so if we isolate it, we're done. Let's travel through the wormhole and solve for $x+6$ directly.

$$
\begin{aligned}
3(x+6) &= 12 \\
x+6 &= \frac{12}{3} \\
x+6 &= 4
\end{aligned}
$$

Here, **we're asked for the value of an expression, and that expression is available in the original equation**; there's no point in taking apart the expression to get the value of the variable just to have to put it back together again. We like to think of these expressions as math "molecules" that include "atoms," like variables and coefficients; if the original material has the molecule you want, there's no sense in decomposing the molecule into atoms just to have to rebuild the same molecule later. This is precisely the core principle of this class of wormhole: **you don't need to solve for a variable (atom) when it's more direct to get the value of the solve-for expression (molecule)**.

## When the Solve-For Is (or Has) a Factor or Multiple of an Available Term

Sometimes, the molecule isn't explicit in the problem equation, but it has been transformed in such a way that it's easy to reconstitute. Let's look at one of these examples.

**Example 21.1-1**

**1** 1600.io 21.1

$$3x+18=12$$

If $x$ satisfies the equation above, what is the value of $x+6$ ?

**Conventional Solution**

1. Here, the solve-for is $x+6$, but we don't see $x+6$ in the equation. Of course, we could first solve for atom $x$:

$$
\begin{aligned}
3x+18 &= 12 \\
3x\cancel{+18}\cancel{-18} &= 12-18 \\
3x &= -6 \\
\frac{\cancel{3}x}{\cancel{3}} &= \frac{-6}{3} \\
x &= -2
\end{aligned}
$$





2. Then we can build our expression molecule:

$$
\begin{aligned}
x &= -2 \\
x+6 &= -2+6 \\
x+6 &= 4
\end{aligned}
$$

3. The answer is 4.

**Wormhole Solution**

1. If we recognize that **the solve-for expression, $x+6$, though it doesn't appear in that form, is nonetheless a component of the original equation**—here, it's a *factor* of the expression on the left side—we can skip the phase where we solve for $x$, and we can solve for $x+6$ directly:

$$
\begin{aligned}
3x+18 &= 12 \\
3(x+6) &= 12 \\
x+6 &= \frac{12}{3} \\
x+6 &= 4
\end{aligned}
$$

2. The answer is 4.

## The Partial Wormhole

Another variation of this class of wormhole problem has a solve-for that's an expression, but the *whole* expression doesn't appear in the given equation, even in disguised form as above. However, there is a molecule term that's *part* of the solve-for expression that *does* appear in the given equation, so that mini-molecule can be solved for rather than solving for any individual atom; this results in a modest wormhole, but it's advantageous to be able to spot nonetheless.

**Example 21.1-2**

**2** 1600.io 21.1

If $\frac{6s}{2}=9$, what is the value of $6s+9$ ?

A) 12

B) 18

C) 27

D) 117

**Conventional Solution**

1. The strategy is to isolate and solve for $s$, then construct the solve-for expression and evaluate it.

$$\frac{6s}{2} = 9$$
$$6s = 9(2)$$
$$6s = 18$$
$$s = \frac{18}{6}$$
$$s = 3$$

2. Substitute $s = 3$ into the solve-for expression and evaluate.

$$6s + 9 = 6(3) + 9$$
$$6s + 9 = 18 + 9$$
$$6s + 9 = 27$$

3. The answer is C.

**Wormhole Solution**

1. We should notice that the solve-for is an expression, but we don't find the expression $6s + 9$ in the given equation, and $6s + 9$ isn't even a factor of any part of the equation. However, the term with the variable, $6s$, *is* in the equation, so we'll save a step if we solve for the molecule $6s$ rather than solving for atom $s$ and then having to rebuild the term in the solve-for expression:

$$\frac{6s}{2} = 9$$
$$6s = 2(9)$$
$$6s = 18$$

2. Construct and evaluate the solve-for expression.

$$6s = 18$$
$$6s + 9 = 18 + 9$$
$$6s + 9 = 27$$

3. The answer is C.

Sometimes the situation is slightly more complex, and the solve-for has a term that is a multiple of a term in the given equation.





**Example 21.1-3**

**3** 1600.io 21.1

If $4x + 9 = 59$, what is the value of $8x + 9$ ?

**Conventional Solution**

1. Let's isolate the term with $x$, solve for $x$ and then construct the solve-for expression and evaluate it.

$$4x + 9 = 59$$
$$4x = 59 - 9$$
$$4x = 50$$
$$x = \frac{50}{4}$$

2. Build the solve-for expression and substitute the value of $x$.

$$8x + 9 = 8\left(\frac{50}{4}\right) + 9$$
$$8x + 9 = 2(50) + 9$$
$$8x + 9 = 100 + 9$$
$$8x + 9 = 109$$

3. The answer is 109.

**Wormhole Solution**

1. Notice that one term in the solve-for is $8x$, and $4x$, which is a factor of $8x$ (because $8x = 2(4x)$), appears in the given equation; put another way, $8x$ is a multiple of $4x$. That means we can solve for $4x$ rather than for $x$, and it will speed us to the answer.

$$4x + 9 = 59$$
$$4x = 59 - 9$$
$$4x = 50$$

2. Construct and evaluate the solve-for expression.

$$4x = 50$$
$$2(4x) = 2(50)$$
$$8x = 100$$
$$8x + 9 = 100 + 9$$
$$8x + 9 = 109$$

3. The answer is 109.

**Notes**

This is not a major reduction in the number of operations needed, but it does eliminate the need to perform a division, which is one of the more error-susceptible of the arithmetic operations, and there is no need to work with decimals or fractions, so if you spot this modest wormhole, you'll save a little time and reduce your chances of error.

There are other variations of this general class of wormhole problem, and there will surely be more configurations appearing on future tests, so it's important to understand that the categorizations above (and below) represent a broad overview only. You should maintain a flexible mindset when analyzing problems for the possible presence of a wormhole solution.

### The "Tell"

As you've probably figured out, one tip-off (or "tell," as poker players say) for the presence of a wormhole is that **the solve-for is (or involves) an expression**. This same tell applies to one of the types of systems of linear equations wormholes detailed later on.

**Section 21.1 Suggested Problems from Real Tests**

- Test 1-C-4
- Test 2-NC-1
- Test 3-NC-2
- Test 7-NC-16
- Test 9-NC-17

View related real-test problems in SMTEx

SMTEx

- 2021-C-17
- 2021-C-14
- 2021 (US)-C-2
- 2021 (Int)-C-9
- Oct 2019-C-22

## 21.2 Sum/Product of Solutions Wormholes

When a problem asks for the sum or product of the solutions to a quadratic equation (or, equivalently, the sum/product of the $x$-coordinates of the points of intersection of a system of equations consisting of a quadratic equation and a linear equation), this is a tell for a wormhole. The general way to solve such systems involves finding the solutions (and, if the problem mentions solution points, finding both the $x$- and $y$-coordinates of the solution points), then constructing the sum or product expression and evaluating it. However, if you remember that the sum of the solutions to a quadratic equation is given by $\frac{-b}{a}$ and that the product of the solutions is $\frac{c}{a}$ (and $a$ is often 1 in test problems), you can drop into that wormhole and arrive at the answer right away. In a sense, this is just another situation where the problem asks for the value of a molecule, not an atom, and there's a method that bypasses solving for the atoms individually.





### Sum of Solutions

Let's start with an example that shows how exploiting the streamlined procedure for finding the sum of the solutions to a quadratic equation can speed the route to a solution.

**Example 21.2-1**

**1** 1600.io 21.2

In the $xy$-plane, the graph of the equation $y = -6x - 27$ intersects the graph of the equation $y = -x^2$ at two points. What is the sum of the $x$-coordinates of the two points?

A) $-12$

B) $-6$

C) $6$

D) $12$

**Conventional Solution**

Note, first, that the $x$-coordinates of the points of intersection of a quadratic equation and a linear equation (as we have here) represent the solutions to a quadratic equation formed by combining the two given equations, so this problem is, at its core, really asking for the sum of the solutions to a quadratic equation.

1. The first step is to create a single quadratic equation by setting the expressions from the right side of each given equation equal to each other, which is valid because both expressions equal $y$.

$$-6x - 27 = -x^2$$
$$x^2 - 6x - 27 = 0$$

2. Factor.

$$x^2 - 6x - 27 = 0$$
$$(x + 3)(x - 9) = 0$$

3. Find solutions.

$$x + 3 = 0 \qquad x - 9 = 0$$
$$x = -3 \qquad x = 9$$

4. Find the sum of the solutions.

$$-3 + 9 = 6$$

5. The answer is C.

**Wormhole Solution**

1. The first step is to create a single quadratic equation by setting the expressions from the right side of each given equation equal to each other, which is valid because both expressions equal $y$.

$$-6x - 27 = -x^2$$
$$x^2 - 6x - 27 = 0$$

2. The sum of the solutions is provided by the expression $\frac{-b}{a}$.

$$\frac{-b}{a} = \frac{-(-6)}{1} = 6$$

3. The answer is C.

**Notes**

This wormhole is modest, but if you're confident applying the principle, you'll save a bit of time and reduce your chance of errors when factoring and finding solutions.

## Product of Solutions

The following example demonstrates the value of knowing how to rapidly find the product of the solutions to a quadratic equation.

**Example 21.2-2**

**2** 1600.io 21.2

$$t(t - 9) = 22$$

What is the product of the solutions to the given equation?

A) 22
B) 13
C) −9
D) −22





**Conventional Solution**

The solving strategy is to get the equation into Standard Form for a quadratic by distributing on the left side and moving all terms to one side, then to apply a solving technique to find the solutions, and finally to multiply those solutions to get the answer.

1. Convert the equation to a quadratic equation in Standard Form.

$$t(t - 9) = 22$$
$$t^2 - 9t = 22$$
$$t^2 - 9t - 22 = 0$$

2. Factor

$$t^2 - 9t - 22 = 0$$
$$(t + 2)(t - 11) = 0$$

3. Solve for $t$.

$$t + 2 = 0 \qquad t - 11 = 0$$
$$t = -2 \qquad t = 11$$

4. Multiply the solutions to find their product.

$$-2(11) = -22$$

5. The answer is D.

**Wormhole Solution**

1. We should realize that the product of the solutions for a quadratic equation is just $\frac{c}{a}$, and here $a = 1$, so we just need to determine the value of the constant $c$. That just involves placing all terms on one side of the equation, maintaining the implied $a$ coefficient of positive 1 (so we don't want to move the $t$ term to the right side or its sign will switch and become negative).

$$t(t - 9) = 22$$
$$t(t - 9) - 22 = 0$$

2. The value of the constant $c$ is −22 (and, again, the value of $a$ is 1), so the product of the solutions is −22.

3. The answer is D.

**Notes**

In the Wormhole Solution, you could have expanded the terms by distributing $t$ if you needed to see the equation written in true Standard Form in order to confidently apply the wormhole.

**Section 21.2 Suggested**

- Test 2-NC-13
- Mar 2018-NC-14
- May 2019 (Int)-NC-13

**View related real-test problems in SMTEx**

SMTEx

## 21.3 Systems of Linear Equations Wormholes

We might be presented with a system of linear equations and be asked to solve for an expression; in some of these situations, we can solve for the requested expression's value through the right choice of manipulations of the given equations. Again, the fact that the problem asks you to solve for the value of an *expression* is the tell that there might be a wormhole through which you can travel.

In the simplest scenario, a system of equations problem asks for the value of an expression, and an equation that gives the value of that expression can be directly found merely by adding or subtracting the equations.

**Example 21.3-1**

**1** 1600.io 21.3

$$6x - 2y = 500$$
$$3x - 7y = 850$$

Based on the system of equations above, what is the value of $9x - 9y$?

**Conventional Solution**

The strategy will be to manipulate the equations and then combine them in order to eliminate one of the variables, which we can then solve for. Then, we'll substitute that value and solve for the other variable. Finally, we'll construct and evaluate the requested expression.

1. Multiply the second equation by 2 so that the $x$ terms drop out when one equation is subtracted from the other.

$$\begin{aligned} 6x - 2y &= 500 \\ 3x - 7y &= 850 \\ &\Downarrow \\ 6x - 2y &= 500 \\ 2(3x - 7y) &= 2(850) \\ &\Downarrow \\ 6x - 2y &= 500 \\ 6x - 14y &= 1{,}700 \end{aligned}$$





2. Subtract the second equation from the first to eliminate $x$.

$$\begin{aligned} 6x - 2y &= 500 \\ -(6x - 14y) &= -(1{,}700) \\ \hline 12y &= -1{,}200 \\ &\Downarrow \\ y &= -100 \end{aligned}$$

3. Substitute for $y$ in the first equation.

$$\begin{aligned} 6x - 2y &= 500 \\ 6x - 2(-100) &= 500 \\ 6x + 200 &= 500 \\ 6x &= 500 - 200 \\ 6x &= 300 \\ x &= \frac{300}{6} \\ x &= 50 \end{aligned}$$

4. Construct and evaluate the solve-for expression.

$$\begin{aligned} 9x - 9y &= 9(50) - 9(-100) \\ 9x - 9y &= 450 + 900 \\ 9x - 9y &= 1{,}350 \end{aligned}$$

5. The answer is 1,350.

**Wormhole Solution**

1. Noting that the solve-for is an expression, our first thought should be that there's a wormhole available, and, sure enough, simply adding the two equations yields the needed expression and its value:

$$\begin{aligned} 6x - 2y &= 500 \\ +\quad 3x - 7y &= 850 \\ \hline 9x - 9y &= 1{,}350 \end{aligned}$$

2. The answer is 1,350.

A variation of this situation crops up where the *exact* solve-for expression cannot be directly evaluated, but **a multiple or factor of the solve-for can be found directly**, leaving a simple multiplication or division to get to the answer.

**Example 21.3-2**

**2** 1600.io 21.3

If $3x - 5y = -11$ and $6x - 7y = -10$, what is the value of $3x - 4y$?

A) $-7$

B) $-1$

C) $0$

D) $9$

**Wormhole Solution**

We will skip the conventional solution this time; by now, you should have a good idea of what would be involved in solving for each variable and proceeding from there, so let's get right to the…

1. Our first thought would be to check whether a simple manipulation of the system will yield the solve-for expression directly, but it won't. Of course, we could solve for $x$ or $y$, substitute to find the value of the other variable, and then construct and evaluate the solve-for expression. However, if we inspect the system with larceny in our hearts (thinking along with the test-makers a bit), we'll discover that if we add the two equations, we get a useful result:

$$\begin{array}{r} 3x - 5y = -11 \\ +\quad 6x - 7y = -10 \\ \hline 9x - 12y = -21 \end{array}$$

2. Notice that the expression on the left isn't our solve-for expression, but it is three times that expression, so we can get to the final answer by dividing the equation through by 3.

$$\begin{aligned} 9x - 12y &= -21 \\ \frac{9x - 12y}{3} &= \frac{-21}{3} \\ 3x - 4y &= -7 \end{aligned}$$

3. The answer is A.

In some cases, the system might require a little bit of manipulation before it yields its bounty, but the procedure might still be simpler than using the generalized procedure because it does not require ever solving for the individual variables in the system. This does require some insight to rapidly determine what must be done to unlock the treasure, so if you don't find the key to the wormhole's door quickly, it is likely best to proceed with conventional solving.

**Example 21.3-3**





**3** 1600.io 21.3

$$\begin{aligned} 2s + 5t &= 6 \\ 4s - 2t &= 12 \end{aligned}$$

For the system of equations above, what is the value of $s + t$?

A) $-1$

B) $3$

C) $4$

D) $12$

**Wormhole Solution**

1. We see that we're asked for an expression that has the two variables involved in addition, so there is a chance we can get right to the answer without solving for each variable individually, but if we take the first obvious path of adding or subtracting these two equations, we won't get $s + t$ (or any multiple), so that's a dead end. However, you might have the insight to realize that doubling the first equation, then adding them together, will result in a useful expression: a multiple of the solve-for.

$$\begin{aligned} 2(2s + 5t) &= 2(6) \\ 4s - 2t &= 12 \\ &\Downarrow \\ 4s + 10t &= 12 \\ +\quad 4s - 2t &= 12 \\ \hline 8s + 8t &= 24 \\ &\Downarrow \\ \frac{8s + 8t}{8} &= \frac{24}{8} \\ s + t &= 3 \end{aligned}$$

2. The answer is B.

**Notes**

This manipulation of the provided information in such a way that the wormhole appears is an advanced technique for which there is no simple step-by-step procedure, but it's worth understanding; the more of these you see, the better-developed your sense will be for whether this route is available.

In a more complex situation than the ones above, there might be more than one molecule involved in the solve for. Such problems are uncommon, but they do come up.

To lay the groundwork for understanding this more complex situation, let's first consider a simple system of linear equations:

$$\begin{aligned} x + y &= 5 \\ x - y &= -1 \end{aligned}$$

What if we're asked to solve for $xy$? Your first thought should be, "hmm, $xy$? That's an expression, and that might be a tell for a wormhole, so let's see if I can solve for $xy$ directly!" However, there's no obvious route here to do that; $xy$ doesn't appear as a term, and there's no simple transformation of it, either. It's reasonable to conclude that there is no wormhole available, so we should just solve for each variable and then construct and evaluate the solve-for. We'll use elimination, because the system is well-suited to it.

First, let's add the two equations, which will eliminate y:

$$\begin{array}{r} x + y = 5 \\ + \quad x - y = -1 \\ \hline 2x = 4 \end{array}$$

$$\Downarrow$$

$$x = 2$$

Substitute $x = 2$ into either equation to find $y$; we'll use the first equation:

$$\begin{aligned} x + y &= 5 \\ 2 + y &= 5 \\ y &= 5 - 2 \\ y &= 3 \end{aligned}$$

Now that have values for $x$ and $y$, we can evaluate $xy$:

$$xy = 2(3) = 6$$

Just to be clear, there's nothing fancy going on here; it's just a straightforward solving approach to a system of linear equations, and there is no wormhole. However, we are setting the stage for the example that follows.

**Example 21.3-4**

**4** 1600.io 21.3

$$\left(\frac{1}{7}t + 3\right) + \left(\frac{1}{4}u - 2\right) = 5$$
$$\left(\frac{1}{7}t + 3\right) - \left(\frac{1}{4}u - 2\right) = -1$$

For the system of equations above, what is the value of the product $\left(\frac{1}{7}t + 3\right)\left(\frac{1}{4}u - 2\right)$?





**Wormhole Solution**

We *could* take all this nonsense apart by combining like terms, using elimination or substitution to solve for $t$, substituting back in to find $u$, and then constructing the solve-for expression and evaluating it. But if you look closely, you'll notice that **this is actually the *exact* same problem as the one we just worked through**—except instead of the atom $x$, we have the molecule $\left(\frac{1}{7}t + 3\right)$, and instead of $y$, we have $\left(\frac{1}{4}u - 2\right)$. Therefore, we can solve for each of those *molecules*, then construct and evaluate the solve-for expression (which is just the product of those two molecules), without ever having to determine the individual values of $t$ and $u$.

1. First, let's add the two equations, which will eliminate the second molecule, $\left(\frac{1}{4}u - 2\right)$:

$$\begin{array}{r} \left(\frac{1}{7}t + 3\right) + \left(\frac{1}{4}u - 2\right) = 5 \\ + \quad \left(\frac{1}{7}t + 3\right) - \left(\frac{1}{4}u - 2\right) = -1 \\ \hline 2\left(\frac{1}{7}t + 3\right) = 4 \end{array}$$

$$\Downarrow$$

$$\left(\frac{1}{7}t + 3\right) = 2$$

2. Substitute $\left(\frac{1}{7}t + 3\right) = 2$ into either equation to find $\left(\frac{1}{4}u - 2\right)$; we'll use the first equation:

$$\begin{aligned} \left(\frac{1}{7}t + 3\right) + \left(\frac{1}{4}u - 2\right) &= 5 \\ 2 + \left(\frac{1}{4}u - 2\right) &= 5 \\ \left(\frac{1}{4}u - 2\right) &= 3 \end{aligned}$$

3. Now that we have values for $\left(\frac{1}{7}t + 3\right)$ and $\left(\frac{1}{4}u - 2\right)$, we can evaluate the solve for, $\left(\frac{1}{7}t + 3\right)\left(\frac{1}{4}u - 2\right)$:

$$\left(\frac{1}{7}t + 3\right)\left(\frac{1}{4}u - 2\right) = 2(3) = 6$$

4. The answer is 6.

Note that the solving method used in the above example is *exactly the same procedure* we used for the previous problem, all the way down to the values. The given equations have all the molecules we need for the solve-for (which is just the product of the two expressions that appear in each equation), so there's no need to decompose them into atoms when we'd just have to put them back together again.

### The "Tell"

The tell here that this approach is potentially viable is that **the solve-for is composed of expressions that are in the original equations**, so your first thought should be to ascertain if you can solve for those expressions directly. Though a problem in this mold will rarely appear, think about how much time you'd save if you instantly recognized how fast you'll get to the answer by traveling through the wormhole rather than by solving for the atoms.

More generally, the tell for a wormhole in a problem that presents a system of linear equations is the familiar request to **solve for the value of an expression**.

**Section 21.3 Suggested**

- Test 3-NC-6
- Test 9-NC-1
- Test 9-C-17
- May 2021 (Int)-C-33

**View related real-test problems in SMTEx**

SMTEx

## 21.4 Quadratic in Non-Atomic Terms Wormholes

Another classic wormhole problem type presents a daunting-looking polynomial which is actually just a quadratic expression, except that rather than being a familiar quadratic in an atom, like $x$, it's a quadratic in a molecule (expression), such as $\sqrt{x}$, $x^2$, $3k$, or $(x-2)$.

**Example 21.4-1**

**1** 1600.io 21.4

If $(x+7)^2 - 4(x+7) + 3 = 0$, which of the following could be the value of $x$ ?

A) $-7$

B) $-4$

C) $6$

D) $7$

**Conventional Solution**

1. Expand all terms, combine like terms, and then solve the resulting quadratic equation by factoring.

$$\begin{aligned}(x+7)^2 - 4(x+7) + 3 &= 0\\ x^2 + 14x + 49 - 4x - 28 + 3 &= 0\\ x^2 + 10x + 24 &= 0\\ (x+4)(x+6) &= 0\\ x = -4,\ x &= -6\end{aligned}$$

Of the two possible values of $x$, only $-4$ is found as an answer choice.

2. The answer is B.





**Wormhole Solution**

1. Recognize that this is just a simple quadratic in $x+7$:

$$(x+7)^2 - 4(x+7) + 3 = 0$$

2. Factor with respect to the expression $(x+7)$:

$$\begin{aligned}(x+7)^2 - 4(x+7) + 3 &= 0\\ \big((x+7)-1\big)\big((x+7)-3\big) &= 0\\ (x+6)(x+4) &= 0\\ x = -4,\ x &= -6\end{aligned}$$

Of the two possible values of $x$, only $-4$ is found as an answer choice.

3. The answer is B.

In the above example, the basis of the quadratic is readily apparent, as the expression $(x+7)$ is explicitly present in the second- and first-degree terms. Sometimes, however, the quadratic has been cloaked like a Romulan ship to make it even more difficult to identify, but by probing it with a few algebraic photon torpedoes (blame Dan), its true nature can be discerned, and it can be neutralized.

**Example 21.4-2**

**2** 1600.io 21.4

Which of the following is a solution to the equation $16x^2 - 12x - 70 = 0$ ?

A) $-2.5$

B) $-1.5$

C) $1.75$

D) $2.5$

**Conventional Solution**

1. The solutions could be found by using *ac* factoring, completing the square, or the quadratic formula, but simple factoring, which is the fastest method when it is applicable, seems impossible with those coefficients. Let's use Completing the Square.

$$16x^2 - 12x - 70 = 0$$
$$x^2 - \frac{12}{16}x - \frac{70}{16} = 0$$
$$x^2 - \frac{3}{4}x = \frac{70}{16}$$
$$\left(x - \frac{3}{8}\right)^2 = \frac{70}{16} + \frac{9}{64}$$
$$\left(x - \frac{3}{8}\right)^2 = \frac{280}{64} + \frac{9}{64}$$
$$\left(x - \frac{3}{8}\right)^2 = \frac{289}{64}$$
$$x - \frac{3}{8} = \pm\frac{17}{8}$$
$$x = \frac{3}{8} \pm \frac{17}{8}$$
$$x = \frac{3}{8} + \frac{17}{8}, x = \frac{3}{8} - \frac{17}{8}$$
$$x = \frac{20}{8}, x = \frac{-14}{8}$$
$$x = \frac{5}{2}, x = \frac{-7}{4}$$
$$x = 2.5, x = -1.75$$

Of the two possible values of $x$, only 2.5 is found as an answer choice.

2. The answer is D.

**Wormhole Solution**

1. This is a quadratic in a non-atom; the tell here is that the second-degree term, $16x^2$, is a perfect square of $4x$, and $4x$ is also a factor of the first-degree term, so we could re-write the equation as a quadratic in $4x$.

$$16x^2 - 12x - 70 = 0$$
$$(4x)^2 - 3(4x) - 70 = 0$$

2. In this form, the quadratic is easily factored in terms of the expression $4x$.

$$(4x)^2 - 3(4x) - 70 = 0$$
$$(4x + 7)(4x - 10) = 0$$
$$4x + 7 = 0, 4x - 10 = 0$$
$$4x = -7, 4x = 10$$
$$x = \frac{-7}{4}, x = \frac{10}{4}$$
$$x = -1.75, x = 2.5$$

Of the two possible values of $x$, only 2.5 is found as an answer choice.

3. The answer is D.

Remember, a quadratic expression has two terms with variables; one has an exponent that's two times the exponent of the other





one. Typically, of course, you'd have one term with an exponent of 2, like $x^2$, and another term with an exponent of 1, like $-2x$ (the exponent is implicit; we don't need to write it when it's 1). This complies with our observation about the exponents in a quadratic, because 2 is $2 \cdot 1$. However, we could also have a quadratic where the exponent of one variable term is, say, 4, and the exponent of the other variable term is 2, because $4 = 2 \cdot 2$. For example, $x^4 - 13x^2 + 36$ is a quadratic in $x^2$, and it can be rewritten as $(x^2)^2 - 13(x^2) + 36$.

Let's look at an example that illustrates this.

**Example 21.4-3**

**3** 1600.io 21.4

$$x^4 - 13x^2 + 36 = 0$$

What is a solution to the equation above where $x > 0$?

**Wormhole Solution**

1. Notice that the exponent of the variable in the first term, $x^4$, is 4, and that's two times the exponent of the variable in the second term, $-13x^2$, which is 2. It's the variable portion of that second term that represents the base of the quadratic; here, that means we have a quadratic in an expression with an exponent of 2, instead of the more typical quadratic in a term with an exponent of 1, and, specifically, the polynomial here is a quadratic in $x^2$, and we can re-write it to make that more apparent:

$$(x^2)^2 - 13(x^2) + 36 = 0$$

2. Now that we've re-written the polynomial as a quadratic, it is easily factored to find the solutions:

$$(x^2 - 4)(x^2 - 9) = 0$$
$$x^2 = 4, x^2 = 9$$
$$x = \pm 2, x = \pm 3$$

3. The answer is 2 or 3 (remember, the problem specified that $x > 0$—don't overlook those constraints!).

Interestingly, this principle also applies with fractional exponents, not just integer exponents, as the following example will illustrate.

**Example 21.4-4**

**4** 1600.io 21.4

$$x - 5\sqrt{x} - 6 = 0$$

What value of $x$ satisfies the equation above?

**Conventional Solution**

1. The most obvious approach here would be to isolate the term with the radical, then eliminate it by squaring things and then solving.

$$\begin{aligned} x - 5\sqrt{x} - 6 &= 0 \\ x - 6 &= 5\sqrt{x} \\ (x-6)^2 &= (5\sqrt{x})^2 \\ x^2 - 12x + 36 &= 25x \\ x^2 - 12x - 25x + 36 &= 0 \\ x^2 - 37x + 36 &= 0 \\ (x-1)(x-36) &= 0 \\ x = 1,\ x &= 36 \end{aligned}$$

We have two candidates for a solution, but because we squared a term with a square root in it, we might have introduced an extraneous solution, so we must test each possibility in the original equation to eliminate any such extraneous solution(s) (see Chapter 9 for details about when and why this procedure is necessary).

2. First, we'll test $x = 1$ by substituting that value into the original equation.

$$\begin{aligned} x - 5\sqrt{x} - 6 &= 0 \\ 1 - 5\sqrt{1} - 6 &= 0 \\ 1 - 5 - 6 &= 0 \\ \text{✗} \quad -10 &= 0 \end{aligned}$$

3. We've discovered that $x = 1$ is an extraneous solution. Let's test $x = 36$ to verify that it is valid.

$$\begin{aligned} x - 5\sqrt{x} - 6 &= 0 \\ 36 - 5\sqrt{36} - 6 &= 0 \\ 36 - 5(6) - 6 &= 0 \\ 36 - 30 - 6 &= 0 \\ \text{✓} \quad 0 &= 0 \end{aligned}$$

4. The solution $x = 36$ checks out, so the only answer is 36.





**Wormhole Solution**

A better approach is to use the wormhole that the polynomial is actually a quadratic. Wait, you say—this doesn't look anything like a quadratic; there are no exponents at all! However, there really *are* exponents—they're just not explicitly written as such. Let's rewrite the expression with explicit exponents instead of an implicit exponent and a radical symbol:

$$x^1 - 5x^{\frac{1}{2}} - 6 = 0$$

Now it's apparent that the exponent of the first term, 1, is twice the exponent of the middle term, which is $\frac{1}{2}$. That means this is a quadratic, believe it or not. But what's it a quadratic in? The middle term tells us: it's a quadratic in $x^{\frac{1}{2}}$ or $\sqrt{x}$.

Now that we've uncovered the opening to the wormhole, we can proceed to travel through it.

1. Let's rewrite the expression in Standard Form with the typical exponents to make it more quadratic-ey; remember, it's a quadratic in $\sqrt{x}$, so that will serve as the core of the first two terms:

$$(\sqrt{x})^2 - 5\sqrt{x} - 6 = 0$$

2. Now we can factor it as we would any quadratic and find the zeros:

$$\begin{aligned} (\sqrt{x})^2 - 5\sqrt{x} - 6 &= 0 \\ (\sqrt{x}+1)(\sqrt{x}-6) &= 0 \\ &\Downarrow \\ \text{✗} \quad \sqrt{x} &= -1 \\ \text{✓} \quad \sqrt{x} &= 6 \\ x &= 36 \end{aligned}$$

Reject this solution; the square root cannot be negative.

3. The answer is 36.

**Notes**

Notice how much more rapidly we were able to get to the solution, and also how we avoided having to perform the extraneous solution testing because we never squared a square root during the course of solving; as soon as a square root with a negative value appeared, we simply terminated that solution branch.

You might think the above is a contrived example, and that nothing like this would ever come up on the test, but you'd be wrong—problems that incorporate precisely this principle have appeared on recent tests, so it's worth digesting these concepts if you want to maximize the power of your math toolkit.

## The "Tell"

The tell for the presence of a quadratic in a non-atom is an expression that is (or can be transformed into) a three-term univariate (single-variable) expression having two variable terms and a constant, with the appropriate relationship between the exponents of the two variable terms.

**Section 21.4 Suggested**

- Test 1-C-36
- May 2017-NC-5
- May 2018-NC-12

**View related real-test problems in SMTEx**

SMTEx

## 21.5 Polynomial Remainder Theorem Wormholes

One rarely-seen wormhole is based on assessing the student's knowledge of the Polynomial Remainder Theorem (see Chapter 9) and their ability to discern when it is applicable. **The tell for such a situation is a rational expression (a fraction that has a polynomial in the numerator or denominator) where the denominator is of the form $x - a$** (it could be $x + a$, which is just $x - (-a)$, of course). These problems can always be solved by actually performing the division indicated (using polynomial long division or synthetic division, for example), but they're much more rapidly solved by applying the Remainder Theorem, and that time savings is the bonus the test awards those students who know and apply that theorem. Here's an example.

**Example 21.5-1**

**1** 1600.io 21.5

Which of the following expressions is equivalent to $\dfrac{2x^2 + 7x - 17}{x - 2}$?

A) $x - 2 - \dfrac{5}{x - 2}$

B) $x + 22 + \dfrac{2}{x - 2}$

C) $2x + 11 + \dfrac{5}{x - 2}$

D) $2x + 11 + \dfrac{10}{x - 2}$

**Conventional Solution**

1. The division could be worked through via conventional means to find the quotient and remainder expression; let's do that to see what would be involved.

$$
\begin{array}{r|rrrl}
 & & 2x & +11 & \text{R } 5 \\
\hline
x - 2 & 2x^2 & +7x & -17 & \\
 & -(2x^2 & -4x) & & \\
\hline
 & & 11x & -17 & \\
 & & -(11x & -22) & \\
\hline
 & & & 5 &
\end{array}
$$

2. The result is $2x + 11 + \dfrac{5}{x - 2}$.
3. The answer is C.





**Wormhole Solution**

The above method requires eight arithmetic operations, including two division operations, along with some careful housekeeping of the terms along the way. A much faster approach is to spot the tells that the Polynomial Remainder Theorem can be applied here to get right to the answer: **the given expression involves division of a polynomial by an expression of the form $x - a$, and the remainders in the answer choices are all different**. That means that if we can rapidly determine the **remainder** of the division operation—we don't need to waste time finding the quotient—we can identify the correct answer choice.

1. As a reminder, the Remainder Theorem states that when a polynomial $P(x)$ is divided by an expression of the form $x - a$, the value of $P(a)$ is the remainder. Here, that means that the value of the expression $2x^2 + 7x - 17$ when $x = 2$ will represent the remainder. Let's perform that substitution.

$$
\begin{aligned}
P(x) &= 2x^2 + 7x - 17 \\
P(2) &= 2(2^2) + 7(2) - 17 \\
P(2) &= 8 + 14 - 17 \\
P(2) &= 5
\end{aligned}
$$

2. Because $P(2) = 5$, the remainder of the division operation will be 5. Examining the answer choices, we see that only choice C has a remainder of 5, so that's the correct choice (note that the remainder in choice A is $-5$, not 5).

**Notes**

This approach is much more streamlined than is the long division method; we only needed to perform five simple arithmetic operations, and none of them was a division. Of course, the division approach could be speeded somewhat by using Synthetic Division.

### The "Tell"

The tell for a wormhole traversable with the aid of the Polynomial Remainder Theorem is the presence of a fraction composed of polynomials, with the denominator in the form $x - a$ (or, equivalently, $x + b$). Another indicator can be a mention of a remainder.

**Section 21.5 Suggested**

- Test 2-NC-15

**View related real-test problems in SMTEx**

SMTEx

## 21.6 Radical Wormholes

There's a wormhole available in some problems that include a radical expression; such problems typically present possible solutions or solution sets as answer choices. The wormhole involves recognizing that **the radical term cannot be negative**; this sufficiently restricts the solutions that only one of the solutions or sets could possibly qualify (and that one must be the answer because there is no "none of the above" choice), or this principle eliminates one of the answer choices. The latter situation is illustrated by the following example.

**Example 21.6-1**

**1** 1600.io 21.6

$$\sqrt{(3x)^2} = 3x$$

Which of the following values of $x$ is NOT a solution to the equation above?

A) $-2$
B) $0$
C) $1$
D) $2$

**Wormhole Solution**

1. We can instantly make the determination that $x = -2$ cannot be a solution by observing that **the principal square root must not be negative**, and therefore that the term on the right-hand side of the equation, $3x$, likewise must not be negative. This, in turn, means that $x$ must not be negative, and therefore choice A is not a solution, so it's the correct answer.

An example follows of a more complex form of problem where this knowledge about the non-negative nature of a radical can be deployed to open up a useful wormhole.





**Example 21.6-2**

**2** 1600.io 21.6

$$\sqrt{3x-2} + 7 = x + 5$$

What is the solution set of the equation above?

A) $\{1\}$
B) $\{6\}$
C) $\{1, 6\}$
D) $\{-1, -6\}$

**Conventional Solution**

1. Isolate and then eliminate the radical (by squaring both sides of the equation).

$$\sqrt{3x-2} + 7 = x + 5$$
$$\sqrt{3x-2} = x + 5 - 7$$
$$\sqrt{3x-2} = x - 2$$
$$(\sqrt{3x-2})^2 = (x-2)^2$$
$$3x - 2 = x^2 - 4x + 4$$

2. Combine like terms and collect all terms on one side to produce a Standard Form quadratic equation.

$$3x - 2 = x^2 - 4x + 4$$
$$0 = x^2 - 4x - 3x + 4 + 2$$
$$0 = x^2 - 7x + 6$$

3. Factor to find solutions:

$$0 = (x-1)(x-6)$$
$$x - 1 = 0,\ x - 6 = 0$$
$$x = 1,\ x = 6$$

4. Because we squared both sides of an equation in the course of solving, we need to test each candidate solution to eliminate any extraneous solution (see Chapter 9 for details on when and why this procedure is necessary).

   First, let's substitute $x = 1$ into the original equation to see if it is valid.

$$\sqrt{3x-2} + 7 = x + 5$$
$$\sqrt{3(1)-2} + 7 = 1 + 5$$
$$\sqrt{3-2} + 7 = 6$$
$$\sqrt{1} + 7 = 6$$
$$1 + 7 = 6$$
$$\text{✗} \quad 8 = 6$$

   The solution $x = 1$ is extraneous.

5. Let's check $x = 6$.

$$\begin{aligned}\sqrt{3x-2}+7 &= x+5\\ \sqrt{3(6)-2}+7 &= 6+5\\ \sqrt{18-2}+7 &= 11\\ \sqrt{16}+7 &= 11\\ \sqrt{16}+7 &= 11\\ 4+7 &= 11\\ \checkmark \quad 11 &= 11\end{aligned}$$

The solution $x = 6$ is valid.

6. The solution set is $\{6\}$, which is answer choice B.

**Wormhole Solution**

Note that the answer choices are solution sets, so there is a high likelihood that we can rapidly determine which choice complies with the constraint that the radical term must be non-negative.

1. We can rearrange the equation to isolate the radical term, then construct a simple inequality that will uniquely identify which answer choice complies.

$$\begin{aligned}\sqrt{3x-2}+7 &= x+5\\ \sqrt{3x-2} &= x+5-7\\ \sqrt{3x-2} &= x-2\end{aligned}$$

2. Because the radical must be non-negative, we can write,

$$\begin{aligned}x-2 &\geq 0\\ x &\geq 2\end{aligned}$$

Inspecting the answer choices, we see that only choice B has a solution set consisting of values that are greater than or equal to 2, so we know that must be the correct answer even though we never solved the original equation.

3. The answer is B.

The example above demonstrates the amount of time you could save if you recognize the presence of a wormhole. Again, this is not a trick; these problems are intentionally constructed with these wormholes in order to reward the student with the insight to identify them.

### The "Tell"

The tell for this type of wormhole is the presence of a radical in the given equation, though be aware that further analysis is needed to ascertain if the problem is exploitable due to the non-negative nature of the principal square root.





**Section 21.6 Suggested**
- Test 4-NC-9
- Test 8-NC-7

**View related real-test problems in SMTEx**

## 21.7 Perfect Square and Difference of Squares Wormholes

It's a good idea to be alert for the patterns that apply to perfect squares and difference of squares expressions, which are explained in Chapter 8. If you fail to recognize the opportunities for a wormhole-style solution, you could end up doing a lot more work than necessary.

### Perfect Square Wormholes

Recognition of a perfect square expression can be a huge time-saver, so it's worth learning the relevant pattern; it's covered in Chapter 8. Problems that would be arduous and time-consuming to solve if you fail to spot the perfect square have a speedy wormhole that you can take advantage of if you're prepared.

**Example 21.7-1**

**1** 1600.io 21.7

If $4s - 12t = 8u$, which of the following expressions is equivalent to $s^2 - 6st + 9t^2$ ?

A) $4u$
B) $2u^2$
C) $4u^2$
D) $16u^2$

**Conventional Solution**

We can use substitution to get each answer choice in terms of $s$ and $t$ so we can compare them to the given expression.

1. Choice A is $4u$, and we are given $8u$ in terms of $s$ and $t$; $4u$ is $\frac{8u}{2}$, so we can write

$$\begin{aligned}4s-12t &= 8u\\ \frac{4s-12t}{2} &= \frac{8u}{2}\\ 2s-6t &= 4u\end{aligned}$$

This does not match the subject expression $s^2 - 6st + 9t^2$.

2. Choice B is $2u^2$. We have an equation for $8u$ in terms of $s$ and $t$, so we can use that to create an equation for just $u$ so we can substitute for $u$.

$$4s - 12t = 8u$$
$$\frac{4s - 12t}{8} = u$$
$$\frac{s - 3t}{2} = u$$

Now, we can construct an expression for choice B, $2u^2$, in terms of $s$ and $t$.

$$2u^2 = 2\left(\frac{s - 3t}{2}\right)^2$$
$$2u^2 = 2\left(\frac{s^2 - 6st + 9t^2}{4}\right)$$
$$2u^2 = \frac{s^2 - 6st + 9t^2}{2}$$

This does not match the subject expression $s^2 - 6st + 9t^2$.

3. Choice C is $4u^2$, which can be rewritten as $(2u)^2$. That's handy because inspecting the given equation $4s - 12t = 8u$, we see that we can divide through by 4 and rewrite it as $s - 3t = 2u$. That allows us to write $(2u)^2$ in terms of $s$ and $t$. (We could also double the expression for $2u^2$, which was found when checking choice B, because $4u^2 = 2(2u^2)$.)

$$(2u)^2 = (s - 3t)^2$$
$$(2u)^2 = s^2 - 6st + 9t^2$$

That matches the subject expression.

4. The answer is C.

5. Let's check answer choice D, which is $16u^2$. That's equivalent to $(4u)^2$, and we are given that $4s - 12t = 8u$, which can be divided through by 2 and rewritten as $2s - 6t = 4u$, so we can substitute for $4u$.

$$(4u)^2 = (2s - 6t)^2$$
$$(4u)^2 = 4s^2 - 12st + 36t^2$$

As suspected, this does not match the given expression, so D is not the correct answer choice, and the answer is indeed C.





**Wormhole Solution**

1. Recognize that the expression $s^2 - 6st + 9t^2$ is a perfect square because it matches the pattern $x^2 + 2xy + y^2$; here, $x = s$ and $y = -3t$. A perfect square can be factored to $(x + y)^2$; here, that will be $(s - 3t)^2$.

2. We're given $4s - 12t = 8u$, but we need $s - 3t$ in terms of $u$, because all the answer choices are in terms of $u$. We can get $s - 3t$ from the supplied equation by dividing through by 4.

$$4s - 12t = 8u$$
$$\frac{4s - 12t}{4} = \frac{8u}{4}$$
$$s - 3t = 2u$$

3. Now, we can substitute for $s - 3t$ in our perfect square expression.

$$s - 3t = 2u$$
$$(s - 3t)^2 = (2u)^2$$

We must remember to distribute the exponent on the right side so our answer matches the format of the answer choices.

$$(s - 3t)^2 = 2^2u^2$$
$$(s - 3t)^2 = 4u^2$$

4. The answer is C.

## Difference of Squares Wormholes

It's important to know the pattern for a difference of squares, which is explained in Chapter 8. Some problems require knowing the pattern, but there are others where you might not realize that this principle can be used as a wormhole because a conventional solution might seem like the appropriate procedure.

**Example 21.7-2**

**2** 1600.io 21.7

If $x + y = 12$ and $x - y = 3$, what is the value of $x^2 - y^2$?

A) 9

B) 36

C) 81

D) 137

**Conventional Solution**

1. Write the two given linear equations as a system of linear equations and use elimination to solve for one variable.

$$\begin{array}{r} x + y = 12 \\ + \quad x - y = 3 \\ \hline 2x = 15 \end{array}$$

$$\Downarrow$$

$$x = \frac{15}{2}$$

2. Substitute $x = \frac{15}{2}$ into the first equation to find the value of $y$.

$$x + y = 12$$
$$\frac{15}{2} + y = 12$$
$$y = 12 - \frac{15}{2}$$
$$y = \frac{24}{2} - \frac{15}{2}$$
$$y = \frac{9}{2}$$

3. Construct the solve-for expression and evaluate.

$$x^2 - y^2 = \left(\frac{15}{2}\right)^2 - \left(\frac{9}{2}\right)^2$$
$$x^2 - y^2 = \frac{225}{4} - \frac{81}{4}$$
$$x^2 - y^2 = \frac{144}{4}$$
$$x^2 - y^2 = 36$$

4. The answer is B.

**Wormhole Solution**

1. Recognize that $x^2 - y^2$ is a difference of squares and thus is equivalent to $(x + y)(x - y)$. Substitute the given values of $x + y$ and $x - y$ to find the value of $x^2 - y^2$.

$$x^2 - y^2 = (x + y)(x - y)$$
$$x^2 - y^2 = 12(3)$$
$$x^2 - y^2 = 36$$

2. The answer is B.





## A Combination Wormhole Example

The following example is particularly interesting because it manages to combine both a perfect square and a difference of squares. Exploitation of that recognition drastically reduces the amount of work needed to reach the solution when compared with trudging through an alternative procedure as someone had to do to write the following explanation.

**Example 21.7-3**

**3** 1600.io 21.7

$$z^4 - 18z^2 + 81$$

Which of the following is equivalent to the expression above?

A) $(z - 9)^4$

B) $(z - 3)^4$

C) $(z^2 + 9)(z + 3)(z - 3)$

D) $(z - 3)^2(z + 3)^2$

**Conventional Solution**

When asked to identify an equivalent expression, it's tempting to put all the expressions in fully-expanded form so that they can be compared. Here, the expression in the question is already fully expanded, but none of the answer choices are, so let's expand them all and compare them to the given expression.

1. Expand choice A. (Below we use the Binomial Expansion Theorem to generate all the terms for the expansions in choices A and B in one step, but you do not need to know this method because it is not generally useful for this test, and if you don't use the wormhole on this problem, you are already wasting time so you might as well expand the terms very carefully.)

$$(z - 9)^4$$
$$z^4 + 4(-9)z^3 + 6(-9)^2z^2 + 4(-9)^3z + (-9)^4$$
$$z^4 - 36z^3 + 486z^2 - 2916z + 6561$$

This does not match the original expression.

2. Expand choice B.

$$(z - 3)^4$$
$$z^4 + 4(-3)z^3 + 6(-3)^2z^2 + 4(-3)^3z + (-3)^4$$
$$z^4 - 12z^3 + 54z^2 - 108z + 81$$

This does not match the original expression.

3. Expand choice C.

$$(z^2+9)(z+3)(z-3)$$
$$(z^2+9)(z^2-3z+3z-9)$$
$$(z^2+9)(z^2-9)$$
$$z^4-9z^2+9z^2-81$$
$$z^4-81$$

This does not match the original expression.

4. Expand choice D.

$$(z-3)^2(z+3)^2$$
$$(z^2-6z+9)(z^2+6z+9)$$
$$z^4\cancel{+6z^3}\cancel{-6z^3}+9z^2-36z^2+9z^2\cancel{+54z}\cancel{-54z}+81$$
$$z^4-18z^2+81$$

This matches the original expression.

5. The answer is D.

**Wormhole Solution**

1. Recognize that the given expression matches the pattern of a **perfect square**: The middle term, $-18z^2$, is the product of 2, the square root ($z^2$) of the highest-degree term $z^4$, and the negative square root ($-9$) of the constant term 81. Therefore, the expression is equivalent to $(z^2-9)^2$.

$$z^4-18z^2+81=(z^2-9)^2$$

2. Note that this does not match any of the answer choices. However, it can be further rewritten based on the recognition that the base, $z^2-9$, of this new form fits the pattern of a **difference of squares**, and thus it can be rewritten further:

$$(z^2-9)^2$$
$$\big((z+3)(z-3)\big)^2$$
$$(z+3)^2(z-3)^2$$

3. At first glance, you might think that this expression does not match any of the answer choices, but note that it's equivalent to choice D, $(z-3)^2(z+3)^2$, just with the operands in reverse order; that doesn't matter because multiplication is commutative.

4. The answer is D.

**Notes**

As you can see, the conventional approach shown above is extremely tedious and error prone, and it probably would take too long to complete under timed conditions, so spotting the wormhole is nearly essential in this case.





It's important to observe that there are perfect square and difference of square problems on the test that require knowledge of those patterns because there is no practical alternative solving method available. We don't consider those to be wormhole problems, and they're not included in the lists of real-test problems here (though they are listed in the topic chapter). But regardless, it is important to be facile with these concepts so you're prepared when any such problems appear on the test.

**Section 21.7 Suggested Problems from Real Tests**

- Test 2-NC-4
- Test 5-NC-10
- Test 5-C-30
- Test 6-NC-4
- Apr 2019-C-17
- Mar 2021-C-26
- May 2021 (Int)-NC-7

**View related real-test problems in SMTEx**

SMTEx

## 21.8 Coefficient Matching Wormholes

Some test problems present an equation and rely on matching the coefficients of terms on both sides of the equals sign. In some of these problems, the solution can be reached rapidly by only working with the terms that include the variable(s) that form coefficients rather than manipulating every term in the equation or solving for each atom in the solve-for and then building and evaluating the solve-for expression. This type of problem and the associated solving techniques are discussed in Chapter 7, but it's worth including this brief discussion in the context of wormholes to remind you that there are usually fast-lane methods. The tell here is that the value of an expression is the solve-for, and that same expression (or a factor or multiple of it) will appear as a coefficient.

**Example 21.8-1**

**1** 1600.io 21.8

$$(3x+a)(2bx^2-2x-1)=-24x^3-22x^2-7x-2$$

The equation above is true for all $x$, where $a$ and $b$ are constants. What is the value of $2ab$ ?

A) $-44$

B) $-24$

C) $-22$

D) $-16$

**Conventional Solution**

1. Expand the expression on the left side.

$$(3x+a)(2bx^2-2x-1)=-24x^3-22x^2-7x-2$$
$$6bx^3-6x^2-3x+2abx^2-2ax-a=-24x^3-22x^2-7x-2$$

2. Combine like terms on the left side of the equation.

$$6bx^3-6x^2-3x+2abx^2-2ax-a=-24x^3-22x^2-7x-2$$
$$6bx^3+(-6+2ab)x^2+(-3-2a)x-a=-24x^3-22x^2-7x-2$$

3. Solve for $a$ by matching the constant terms from both sides of the equation.

$$-a = -2$$
$$a = 2$$

4. Solve for $b$ by matching the coefficients of the $x^3$ terms.

$$6bx^3 = -24x^3$$
$$6b = -24$$
$$b = \frac{-24}{6}$$
$$b = -4$$

5. Construct and evaluate solve-for expression.

$$2ab = 2(2)(-4)$$
$$2ab = -16$$

6. The answer is D.

**Wormhole Solution**

1. Only distribute the terms that produce products of the same degree as those that produce terms with $2ab$ as a coefficient, and only retain terms of that same degree on the right side of the equation in preparation for coefficient matching. The distribution that produces a term with $2ab$ as a coefficient is $a(2bx^2) = 2abx^2$, so we only need to distribute any other terms that also produce a term with $x$ to the second degree ($x^2$), and we only retain any $x^2$ terms on the right side. This will allow us to match coefficients and solve for $2ab$.

$$(3x + a)(2bx^2 - 2x - 1) = -24x^3 - 22x^2 - 7x - 2$$

Ignore any distributions that produce terms with bases other than $x^2$.

$$\Downarrow$$

$$3x(-2x) + a(2bx^2) = -22x^2$$
$$-6x^2 + 2abx^2 = -22x^2$$

We're only interested in the coefficients, so we can drop (divide through by) the variable term, $x^2$.

$$\Downarrow$$

$$-6 + 2ab = -22$$
$$2ab = -16$$

2. The answer is D.

**Section 21.8 Suggested**

- Test 2-NC-17
- Test 3-C-33
- Test 9-NC-10

**View related real-test problems in SMTEx**

SMTEx





## 21.9 Factoring Wormholes

On occasion, you'll encounter a problem that asks for an equivalent form of a provided expression. Though you *could* expand the given polynomial and every answer choice and compare them, it is far more efficient to spot a factoring opportunity that will lead directly to the correct answer choice in the form that it's written.

**Example 21.9-1**

**1** 1600.io 21.9

Which of the following is an equivalent form of the expression $(4x - 3)^2 - (4x - 3)$ ?

A) $4x^2 - 12x + 12$

B) $16x^2 - 28x + 6$

C) $(4x - 3)(4x - 3)$

D) $(4x - 4)(4x - 3)$

**Conventional Solution**

We'll expand the given expression and compare it to the fully-expanded forms of each answer choice.

1. First, expand the provided expression.

$$(4x - 3)^2 - (4x - 3) = 16x^2 - 12x - 12x + 9 - 4x + 3$$
$$(4x - 3)^2 - (4x - 3) = 16x^2 - 28x + 12$$

Let's compare this expanded form of the supplied expression to each answer choice, expanding as needed.

2. The expanded expression does not match choices A or B, so we can eliminate them.

3. Expand choice C.

$$(4x - 3)(4x - 3) = 16x^2 - 12x - 12x + 9$$
$$(4x - 3)(4x - 3) = 16x^2 - 24x + 9$$

This does not match the expanded form of the supplied expression.

4. Expand choice D.

$$(4x - 4)(4x - 3) = 16x^2 - 12x - 16x + 12$$
$$(4x - 4)(4x - 3) = 16x^2 - 28x + 12$$

This matches the expanded form of the given expression, so the answer is D.

**Wormhole Solution**

This problem setup has a tell for a factoring wormhole: the presence of a repeated sub-expression, $4x-3$, in the given expression, indicating that factoring, rather than expanding, might be the key to the wormhole's door lock.

1. Factor out $4x - 3$.

$$(4x-3)^2 - (4x-3) = (4x-3)\Big((4x-3)-1\Big)$$
$$(4x-3)^2 - (4x-3) = (4x-3)(4x-4)$$

This matches choice D, which has been thinly disguised by re-ordering the factors (be alert for this!), but the wormhole has dropped us next door, and we know D is correct.

**Notes**

Note that if we had failed to match one of the factored choices, we could have proceeded to expand the given expression and to compare it to choices A and B. This would not have taken longer than using the conventional solution would have, and it might even have been a bit quicker because it would only involve one expansion, not two or three.

## The Factor-Preservation Wormhole

A variation on the don't-break-up-the-molecule wormhole involves a problem that does *not* ask for the value of an expression, so the wormhole might be overlooked. However, this type of problem has a term that, if *not* expanded, can be easily eliminated, leaving a much simpler problem. That is, there is a factored expression that should be left as-is rather than being subject to distribution.

**Example 21.9-2**

**2** 1600.io 21.9

$$7(r-5) + 6r = 7(r-5) - 24$$

What value of $r$ satisfies the equation above?

A) $-30$
B) $-4$
C) $-1.2$
D) $4$





**Conventional Solution**

1. Here, it's tempting to start by expanding everything, then combining like terms, isolating the r term, and solving.

$$7(r-5) + 6r = 7(r-5) - 24$$
$$7r - 7(5) + 6r = 7r - 7(5) - 24$$
$$7r - 35 + 6r = 7r - 35 - 24$$
$$13r - 35 = 7r - 59$$
$$13r - 7r = -59 + 35$$
$$6r = -24$$
$$r = -4$$

2. The answer is B.

**Wormhole Solution**

1. Notice that there is a term in factored form, $7(r-5)$, which appears on both sides of the equation:

$$\mathbf{7(r-5)} + 6r = \mathbf{7(r-5)} - 24$$

Rather than perform the distribution, we can keep that term intact and simply subtract it from both sides.

$$\cancel{7(r-5)} + 6r = \cancel{7(r-5)} - 24$$
$$6r = -24$$
$$r = -4$$

2. The answer is B.

### The "Tell"

The tell for this class of wormhole is the **presence of terms in a factored form (or expressions) that appear more than once**; often, it's best not to expand them so you don't collapse the wormhole, forcing you to take the long way to the solution.

Section 21.9 Suggested

- Apr 2018-NC-15
- May 2018-C-24

View related real-test problems in SMTEx

## 21.10 Symmetry Wormholes

Some problems of various types are set up with a symmetrical aspect that opens up a wormhole. Here's an example where the symmetry is present in a data set.

**Example 21.10-1**

**1** 1600.io 21.10

Fountain Pen Production in 1929 by Company

| Company | Production (in thousands) |
|---|---|
| Wahl-Eversharp | 300 |
| Parker | 700 |
| Sheaffer | 900 |
| Waterman | 1,100 |
| Frank Holland | |

If the mean fountain pen production in 1929 of the five companies shown in the table is 900,000, what was the number of fountain pens, in thousands, produced by Frank Holland in 1929?

**Conventional Solution**

1. Let $p$ be the number of fountain pens produced by Frank Holland in 1929 and let $m$ be the mean fountain pen production, in thousands, of the five companies in 1929.

   To compute the mean of a data set, sum the members of the set and divide by the number of members in the set.

$$m = \frac{300 + 700 + 900 + 1{,}100 + p}{5}$$
$$900 = \frac{3{,}000 + p}{5}$$
$$5(900) = 3{,}000 + p$$
$$4{,}500 = 3{,}000 + p$$
$$1{,}500 = p$$

2. The answer is 1,500.





**Wormhole Solution**

1. Inspection of the data set reveals that the third value listed, 900, representing 900,000 pens, is also the (given) mean of the data set, and the previous and next values (700 and 1,100, respectively) are symmetrical around 900 (they're 200 less and 200 more than 900, respectively). Therefore, the remaining two values must also be symmetrical with respect to 900 in order to maintain 900 as the mean of the whole data set. The first value is 300, which is $900 - 300 = 600$ less than 900; therefore, the unknown value must be $900 + 600 = 1{,}500$ to maintain the symmetry that produces 900 as the mean.
2. The answer is 1,500.

Another situation in which symmetry has been planted as the doorway to a wormhole is when the factors of a quadratic expression display a symmetrical character.

**Example 21.10-2**

**2** 1600.io 21.10

$$f(t) = (t + 11)(t - 11)$$

The function $f$ is defined as shown. For what value of $x$ does the function $f$ reach its maximum value?

A) $-11$
B) $0$
C) $11$
D) $121$

**Conventional Solution**

A quadratic function reaches its minimum or maximum value at its vertex, so we need to determine the $t$-value of the vertex.

1. Convert the quadratic expression to Standard Form.

$$f(t) = (t + 11)(t - 11)$$
$$f(t) = t^2 - 121$$

2. Use the formula $\frac{-b}{2a}$ to find the $t$-value of the vertex. For this quadratic function $a = 1$, $b = 0$, and $c = -121$.

   The $t$-value of the vertex, $t_v$ is given by the formula $\frac{-b}{2a}$.

$$t_v = \frac{-b}{2a}$$

$$t_v = \frac{-0}{2(1)}$$

$$t_v = 0$$

3. The answer is B.

**Wormhole Solution**

1. Inspection of the function shows that the $t$-intercepts are negatives of each other, and therefore the axis of symmetry of the parabola, on which the vertex lies, is at $t = 0$.
2. The answer is B.

It's important to be open-minded about the possible settings in which a symmetry-based wormhole can be installed by the test makers, as the two very different examples above demonstrate. There are surely many more variations that will be seen in the future, so be alert.

**Section 21.10 Suggested**

- Test 10-C-33

**View related real-test problems in SMTEx**

## 21.11 The Wormhole-Only Problem Type

On rare occasion, there will be a problem where a wormhole-style approach is the *only* route to a solution, as in the following example.

**Example 21.11-1**

**1** 1600.io 21.11

$$6ax - 6y + 7 = 61$$

Based on the equation above, what is the value of $ax - y$ ?

A) 3
B) 6
C) 9
D) 18





**Wormhole Solution**

1. If you've been paying attention, you should spot the fact that the solve-for is an expression, and the expression can be factored out of the left side of the equation after we isolate the terms with $x$ and $y$.

$$6ax - 6y + 7 = 61$$
$$6ax - 6y = 61 - 7$$
$$6(ax - y) = 54$$
$$ax - y = \frac{54}{6}$$
$$ax - y = 9$$

2. The answer is C.

**Notes**

This problem can *only* be solved through this wormhole-style approach because there is insufficient information to solve for the individual atoms $a$, $x$, and $y$; it's like a shortcut where the regular roads are closed. Of course, if you're attuned to the wormhole tells, such a problem will pose no issue at all for you, as you'll won't need to consider any other approach.

**Section 21.11 Suggested**

- Test 7-C-9

**View related real-test problems in SMTEx**

## 21.12 The Wormhole Imposter

We should also acknowledge that on rare occasion, you will come across a problem where, even though there's a tell for a wormhole (such as the solve-for being an expression), **there is no wormhole**, and you really do have to (or really should) solve for a variable or variables before constructing the value of the solve-for expression. These situations are analogous to that fake tunnel entrance that Wile E. Coyote paints on the side of a mountain to trick the Roadrunner: if you try to go through it, you'll find there's no opening there, so don't spend *too* much time looking for a wormhole—there might not be one, and you'll just be banging your head against a wall.

**Example 21.12-1**

**1** 1600.io 21.12

$$5u + 4v = 8$$
$$2u - 4v = -1$$

For the solution $(u, v)$ to the system of equation above, what is the value of $u - v$ ?

A) $-\frac{7}{4}$

B) $\frac{1}{4}$

C) $\frac{7}{4}$

D) $\frac{23}{4}$

**Notes**

Seeing that the solve-for here is an expression, our Spidey-sense should start to tingle. However, a few moments' contemplation should produce the realization that there's no *simple* procedure by which any manipulation and combination of these equations will yield $u - v$, and we must therefore proceed to solve this problem the old-fashioned way: by finding the values of $u$ and $v$, then assembling the solve-for expression and evaluating it. We won't show the solving procedure here, because the point we're illustrating is that sometimes there's a tell, but no wormhole, so you should be aware of this possibility; don't get bogged down trying to run through a trompe l'oeil tunnel (look it up).

There have been occasions where, with sufficient tinkering, a system of linear equations can be transformed to produce an equation that can be further manipulated to yield an equation that evaluates the solve-for expression (in the example above, multiplying the top equation by 2 and the bottom equation by 9 will get you there, but good luck discovering that faster than someone who solves this conventionally), but this procedure is far more time-consuming than would be just solving for each variable individually. Be mindful of this fact and maintain a flexible mindset.

**Section 21.12 Suggested**

- Test 6-C-11
- May 2017-C-3
- Mar 2018-C-14

**View related real-test problems in SMTEx**

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## 21.13 Related Problem Types

The problem classes in this group aren't really wormholes; rather, they represent situations where there is an optimization available that students might not be alert to, or there's a confusing aspect of the problem that we can shed some light on.

### Given a Key, but Not the Lock

There is a style of problem where some information is provided, but nothing can be done with it right away. In effect, it's the key to a lock that you have to construct, and when you do, you can unlock the solution. Problems of this type can be unsettling, because you feel that you should be using the given information right away, but it doesn't seem to have anything to do with the problem when you start.

**Example 21.13-1**

**1** 1600.io 21.13

If $2p - r = 12$, what is the value of $\frac{9^p}{3^r}$ ?

A) $3^{12}$

B) $9^6$

C) $18^3$

D) The value cannot be determined from the information given.

**Solution**

1. The equation provided doesn't seem to be of any use in evaluating the subject fraction, and that could throw you. An attempt to solve the equation in terms of one of the two variables and then substituting will lead you down a rabbit hole you might not be able to climb out of.

   The important realization here is that the supplied equation represents a key, but you need to transform the given expression to produce the matching lock.

   Recall that in order to apply the exponent rules such as division of exponentiated terms, the terms must have a common base. Here, we can produce a common base of 3 by converting the 9 in the numerator into $3^2$.

$$\frac{9^p}{3^r}$$
$$\frac{\left(3^2\right)^p}{3^r}$$
$$\frac{3^{2p}}{3^r}$$
$$3^{2p-r}$$

2. Look what we have here! The lock has been created (check out the exponent), and the key (the value of $2p-r$) is at hand in that equation we were given in the problem statement. Let's insert the key in the lock to unlock the answer.

$$2p - r = 12$$
$$3^{2p-r} = 3^{12}$$

3. The answer is A.

## Effect of Value of Term on Value of Expression

Sometimes, all you need to identify the correct answer choice is an understanding of the effect of the value of a variable on the value of an entire expression, and you can avoid some annoying arithmetic.

**Example 21.13-2**

**2** 1600.io 21.13

If $\frac{11}{z} = 440$, what is the value of $z$ ?

A) 4840

B) 220

C) 40

D) 0.025

**Solution**

1. You can certainly solve this by rearranging to isolate $z$ and then solving, but inspection of the equation should bring the realization that if $z$ is *greater* than 11, the value of the fraction will be *less than* 1, and, conversely, if $z$ is *less than* 11, the value of the fraction will be *greater* than 1. Because the value of the fraction is 440, $z$ must be less than 11, and only choice D satisfies that constraint.

If you can develop this sort of number sense, it can empower you considerably, and not just on the SAT; it's a great tool to have when dealing with all sorts of mathematical situations that arise in everyday life, too.

## Take Advantage of Particularly Helpful Information

There are occasions where some of the provided information has the superpower to speed you to a solution. This seems to happen most frequently with problems that involve the $xy$-coordinate system where point coordinates are supplied.

**Example 21.13-3**

**3** 1600.io 21.13

Line $m$ in the $xy$-plane contains the points $(3, 5)$ and $(0, 2)$. Which of the following is an equation of line $m$ ?

A) $y = 3x + 3$

B) $y = 3x + 5$

C) $y = x + 3$

D) $y = x + 2$





**Solution**

1. We could, of course, use the two supplied points to compute the slope, then plug in the values in one of the points to ascertain the $y$-intercept to complete the expression representing the function. However, notice that one of the provided points has an $x$-coordinate value of 0; that allows an extremely rapid test to see if only one of the functions is satisfied by that point, as the $x$-term will simply drop out.

   Here, we're given point $(0, 2)$, so we can simply ignore the $x$ term in each answer choice (because it will be 0) and see if the value of $y$ is 2. Only choice D produces the equation $y = 2$, so the answer is D.

   Another way to think about it is that all of the answer choices are in Slope-Intercept Form, and the point $(0, 2)$ tells us that the $y$-intercept of the line must be 2, which is only true for choice D.

**Section 21.13 Suggested**

- Test 1-NC-14
- Test 2-NC-9

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# About the Authors

## J. Ernest Gotta

J. Ernest Gotta graduated from Rutgers University with a bachelor's degree in Mathematics in 2012, and he has been teaching SAT classes since his senior year of college. He was trying to be modest here, but George told him "you're like a legit pro at this," so he'll admit that much is true. His goal with this book was to deliver the same level of insight and appreciation for the mathematics covered by the test as he does in his classes. He thinks he did that, but one of the unexpected benefits was that he grew from LaTeX noob to LaTeX novice over the last three years. The soundtrack of this book is the soothing sounds of Beach House punctuated by frequent requests from J. Ernest's daughter to pour her some water. He is currently working on his DIY skills to get his house in shape.

## Daniel Kirchheimer

Daniel Kirchheimer is the Higher Education Readiness Specialist at 1600.io, which created the low-cost, education-based guided self-study model for college readiness in the guise of standardized test preparation. In that role, he has become a recognized authority on the teaching and learning of the academic subjects assessed by the SAT, and he supports thousands of students by clearly and patiently answering their math, reading, and writing questions in the 1600.io Discord server, in 1600.io livestreams, on Reddit, through online chats and DMs, and via email. After graduating from the Bronx High School of Science, Dan studied computer science and engineering at the University of Pennsylvania with further graduate coursework in multimedia and 3D computer graphics at New York University. He was a member of Penn's computer programing team that placed 4th in the world in the 1984 ACM competition, and he is the creator of IBIIS, the first multimedia database for infrastructure inspection. Daniel was minding his own business developing touchscreen control systems for commercial audiovisual installations when one day, George roped him in by "innocently" asking, "hey Dan—how can I get a T-shirt store on my website?" Daniel grew up in New York City and now lives with his family in New Jersey where he plays blues guitar, coaches youth baseball, and collects, repairs, researches, and writes about vintage fountain pens.

## George Rimakis

After tutoring thousands of students in person for the SAT, and tens of thousands more virtually through 1600.io, George Rimakis discovered that the only secret to doing well on the SAT is to be well-educated in the academic skills that the SAT assesses. Astonishing, right? In keeping with that groundbreaking revelation, George set out to create an entire ecosystem of educational resources at 1600.io that had that principle at their core, and this book is no exception. This project represents the distillation of everything that George and his co-authors have learned in their collective decades of teaching young people high school-level math without pain. George holds a bachelor's degree in Mathematics from Yale University, so hopefully you can trust what he says.

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