Example 3: Multiaxial plasticity (rate independent)

$$\begin{array}{ll} \sigma_{ij} & = C_{ijkl}\,(\varepsilon_{kl} - \varepsilon^{\mathrm{pl}}_{kl}) \\ \dot\varepsilon^{\mathrm{pl}}_{kl} & = \Lambda \frac{\partial \Phi}{\partial \sigma_{kl}} \\ \Phi(\sigma_{ij}) \le 0\,, & \Lambda \ge 0,\ \Phi\Lambda = 0 \end{array} \xrightarrow{Euler-backward} \begin{array}{ll} \sigma^{m+1}_{ij} & = C_{ijkl}\,(\varepsilon^{m+1}_{kl} - \varepsilon^{\mathrm{pl}\,m+1}_{kl}) \\ \varepsilon^{\mathrm{pl}\,m+1}_{kl} & = \varepsilon^{\mathrm{pl}\,m}_{kl} + \Delta\Lambda\,\left.\frac{\partial \Phi}{\partial \sigma_{kl}}\right|^{m+1} \\ \Phi(\sigma^{m+1}_{ij}) \le 0\,, & \Delta\Lambda \ge 0,\ \Phi\Delta\Lambda = 0 \end{array} \tag{1}$$

Plastic corrector in general: nonlinear system of equations for $\varepsilon^{\mathrm{pl}\,m+1}_{kl}$, $\Delta\Lambda$ and hardening variables

Mises plasticity with isotropic hardening

$$\Phi = \sigma_{\mathrm{eq}} - \sigma_{\mathrm{y}}(\bar\varepsilon^{\mathrm{pl}}) \qquad \text{with} \qquad \sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\sigma^{\mathrm{d}}_{kl}\sigma^{\mathrm{d}}_{kl}} \qquad \to \dot\varepsilon^{\mathrm{pl}}_{kl} = \Lambda \frac{3}{2}\,\frac{\sigma^{\mathrm{d}}_{kl}}{\sigma_{\mathrm{eq}}} \tag{2}$$

$$\dot{\bar\varepsilon}^{\mathrm{pl}} = \sqrt{\frac{2}{3}\dot\varepsilon^{\mathrm{pl}}_{kl}\dot\varepsilon^{\mathrm{pl}}_{kl}} = \Lambda \tag{3}$$

with isotropic elastic behavior $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = \frac{K}{3}\delta_{ij}\delta_{kl} + 2\mu I^{\mathrm{d}}_{ijkl}$ with $I^{\mathrm{d}}_{ijkl} = 1/2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - 1/3\delta_{ij}\delta_{kl}$

Euler-backward discretization of flow rule:

$$\sigma^{m+1}_{ij} = C_{ijkl}\left(\varepsilon^{m+1}_{kl} - \varepsilon^{\mathrm{pl}\,m}_{kl} - \Delta\Lambda\,\frac{3}{2}\left.\frac{\sigma^{\mathrm{d}}_{kl}}{\sigma_{\mathrm{eq}}}\right|^{m+1}\right) = \underbrace{C_{ijkl}\left(\varepsilon^{m+1}_{kl} - \varepsilon^{\mathrm{pl}\,m}_{kl}\right)}_{\sigma^{\mathrm{tr}}_{ij}} - 3\mu\Delta\Lambda\,\left.\frac{\sigma^{\mathrm{d}}_{ij}}{\sigma_{\mathrm{eq}}}\right|^{m+1} \tag{4}$$

split into hydrostatic and deviatoric part

$$\sigma^{m+1}_{kk} = \sigma^{\mathrm{tr}}_{kk}\,, \qquad \sigma^{\mathrm{d}\,m+1}_{ij} = \sigma^{\mathrm{d\,tr}}_{ij} - 3\mu\Delta\Lambda\frac{\sigma^{\mathrm{d}\,m+1}_{ij}}{\sigma_{\mathrm{eq}}{}^{m+1}} \tag{5}$$

rearranging

$$\sigma_{\mathrm{eq}}{}^{m+1} = \sigma_{\mathrm{eq}}{}^{\mathrm{tr}} - 3\mu\Delta\Lambda \tag{6}$$

$$\sigma^{\mathrm{d}\,m+1}_{ij} = \frac{\sigma_{\mathrm{eq}}{}^{m+1}}{\sigma_{\mathrm{eq}}{}^{\mathrm{tr}}}\sigma^{\mathrm{d\,tr}}_{ij} = \frac{\sigma_{\mathrm{eq}}{}^{\mathrm{tr}} - 3\mu\Delta\Lambda}{\sigma_{\mathrm{eq}}{}^{\mathrm{tr}}}\sigma^{\mathrm{d\,tr}}_{ij} \tag{7}$$

- hydrostatic part is purely elastic
- stress deviator $\sigma^{\mathrm{d}\,m+1}_{ij}$ is colinear to its trial value $\sigma^{\mathrm{d\,tr}}_{ij}$

⇒ only necessary in plastic corrector: (numerical) determination of $\Delta\Lambda$ from yield condition $\Phi^{m+1} = \sigma_{\mathrm{eq}}{}^{\mathrm{tr}} - 3\mu\Delta\Lambda - \sigma_{\mathrm{y}}(\bar\varepsilon^{\mathrm{pl}\,m} + \Delta\Lambda) = 0$

"radial return mapping"

- stress update: $\sigma^{m+1}_{ij} = \frac{1}{3}\delta_{ij}\sigma^{\mathrm{tr}}_{kk} + \frac{\sigma_{\mathrm{eq}}{}^{\mathrm{tr}} - 3\mu\Delta\Lambda}{\sigma_{\mathrm{eq}}{}^{\mathrm{tr}}}\sigma^{\mathrm{d\,tr}}_{ij}$
- algorithmically consistent tangent stiffness:
  - implicit differentiation of $\Phi^{m+1} = 0$: $1 - \frac{\partial\Delta\Lambda}{\partial\sigma_{\mathrm{eq}}{}^{\mathrm{tr}}}(3\mu + h) = 0$

$$C^{\mathrm{t}}_{ijkl} = \frac{\mathrm{d}\sigma^{m+1}_{ij}}{\mathrm{d}\varepsilon^{m+1}} = \frac{1}{3}K\delta_{ij}\delta_{kl} + 2\mu\frac{\sigma_{\mathrm{eq}}{}^{\mathrm{tr}} - 3\mu\Delta\Lambda}{\sigma_{\mathrm{eq}}{}^{\mathrm{tr}}}I^{\mathrm{d}}_{ijkl} - 3\mu\frac{3\mu}{3\mu + h}\frac{\sigma_{\mathrm{eq}}{}^{\mathrm{tr}} - \Delta\Lambda h}{(\sigma_{\mathrm{eq}}{}^{\mathrm{tr}})^3}\sigma^{\mathrm{d\,tr}}_{ij}\sigma^{\mathrm{d\,tr}}_{kl}$$

⇒ to Voigt notation $\underline{\underline{\mathbf{C}}}_{\mathrm{t}}$