Exercise 2: Algorithmic treatment of one-dimensional plasticity

Problem 2.2

Applying the Euler-implicit time integration scheme, to the constitutive equations of a one-dimensional plasticity model with linear hardening leads to the following discrete formulation:

elastic law:	$\sigma_{n+1} = \mathrm{E}\,\varepsilon^{\mathrm{e}}_{n+1}$
strain decomposition:	$\varepsilon_{n+1} = \varepsilon^{\mathrm{e}}_{n+1} + \varepsilon^{\mathrm{p}}_{n+1}$
yield function:	$\Phi_{n+1}(\xi_{n+1}, R_{n+1}) = \vert\underbrace{\sigma_{n+1} - X_{n+1}}_{\xi_{n+1}}\vert - \sigma_0 - R_{n+1}$
increase in yield stress:	$R_{n+1} = \mathrm{h}\,\alpha_{n+1}$
back-stress:	$X_{n+1} = \mathrm{H}\,\varepsilon^{\mathrm{p}}_{n+1}$
evolution of internal variables:	$\Delta\varepsilon^{\mathrm{p}} = \varepsilon^{\mathrm{p}}_{n+1} - \varepsilon^{\mathrm{p}}_{n} = \underbrace{\Delta t\dot{\lambda}_{n+1}}_{\Delta\lambda}\,\mathrm{sign}\,(\sigma_{n+1} - X_{n+1})$
	$\Delta\alpha = \alpha_{n+1} - \alpha_n = \Delta\lambda$
discrete KKT conditions:	$\Phi_{n+1} \leq 0, \quad \Delta\lambda \geq 0, \quad \Delta\lambda\Phi_{n+1} = 0$
initial conditions:	$\alpha_0 = 0 \quad \varepsilon^{\mathrm{p}}_0 = 0$

A well established method to solve the discretized evolution equations together with the discrete KKT conditions is the (elastic) predictor - (plastic) corrector scheme. This two-step algorithm emerges from the operator-split of the elastic-plastic constitutive equations. It consists of:

- the *elastic predictor*, which assumes the plastic process to be frozen in the time interval Δt, i.e. $\Delta\lambda = 0$, and leads to the *trial elastic state*, that possibly violates the yield condition
- the *plastic corrector*, which ensures that the final stress state is located on the yield surface, if the *trial elastic state* violates the yield condition

(a) Derive an expression for the stress $\sigma^{\mathrm{trial}}_{n+1}$ at the *trial elastic state*.

(b) Evaluate the yield condition at the *trial elastic state*.

(c) Show that, $\mathrm{sign}\,(\xi_{n+1}) = \mathrm{sign}\,(\xi^{\mathrm{trial}}_{n+1})$ holds. Hint: Use the fact that $a = |a|\,\mathrm{sign}\,(a)$.

(d) Similar to the continuous formulation, the linear hardening in the discrete formulation allows for an explicit evaluation of the incremental Lagrange multiplier $\Delta\lambda$. Derive the corresponding expression based on $\Phi^{\mathrm{trial}}_{n+1}$ and the material parameters.

(e) Code the discrete formulation of the one-dimensional plasticity model by completing the function `stress_computation_one_dim_plasticity_linear_hardening.m` in the Matlab-code provided. Test your implementation and compare your results with the exact solution.

(f) Run the model with mixed hardening (H = 500 MPa and h = 500 MPa) and a strain amplitude of $\hat{\varepsilon} = 3\frac{\sigma_0}{\mathrm{E}}$ employing 15 increments. Explain your results.

(g) Discuss the influence of the isotropic and kinematic hardening on the evolution of the elastic domain.

Problem 3.1

The constitutive equations of a one-dimensional, phenomenological plasticity model with nonlinear isotropic and linear kinematic hardening are given in the following discrete form, resulting from the application of an Euler-implicit time integration scheme:

elastic law:	$\sigma_{n+1} = \mathrm{E}\,\varepsilon^{\mathrm{e}}_{n+1}$
strain decomposition:	$\varepsilon_{n+1} = \varepsilon^{\mathrm{e}}_{n+1} + \varepsilon^{\mathrm{p}}_{n+1}$
yield function:	$\Phi_{n+1}(\xi_{n+1}, R_{n+1}) = \lvert\,\underbrace{\sigma_{n+1} - X_{n+1}}_{\xi_{n+1}}\,\rvert - \sigma_0 - R_{n+1}$
increase in yield stress:	$R_{n+1} = \mathrm{h}\,\alpha_{n+1} + \Delta\mathrm{Y}\left[1 - \exp\left(-\eta\,\alpha_{n+1}\right)\right]$
back-stress:	$X_{n+1} = \mathrm{H}\,\varepsilon^{\mathrm{p}}_{n+1}$
evolution of internal variables:	$\Delta\varepsilon^{\mathrm{p}} = \varepsilon^{\mathrm{p}}_{n+1} - \varepsilon^{\mathrm{p}}_{n} = \underbrace{\Delta t\dot{\lambda}_{n+1}}_{\Delta\lambda}\,\mathrm{sign}\left(\sigma_{n+1} - X_{n+1}\right)$
	$\Delta\alpha = \alpha_{n+1} - \alpha_n = \Delta\lambda$
discrete KKT conditions:	$\Phi_{n+1} \leq 0, \quad \Delta\lambda \geq 0, \quad \Delta\lambda\Phi_{n+1} = 0$
initial conditions:	$\alpha_0 = 0 \quad \varepsilon^{\mathrm{p}}_0 = 0$

A well established method to solve the discretized evolution equations together with the discrete KKT conditions is the (elastic) predictor - (plastic) corrector scheme. This two-step algorithm emerges from the operator-split of the elastic-plastic constitutive equations. It consists of:

- the *elastic predictor*, which assumes the plastic process to be frozen in the time interval Δt, i.e. $\Delta\lambda = 0$, and leads to the *trial elastic state*, that possibly violates the yield condition
- the *plastic corrector*, which ensures that the final stress state is located on the yield surface, if the *trial elastic state* violates the yield condition

(a) Derive an expression for the stress $\sigma^{\mathrm{trial}}_{n+1}$ at the *trial elastic state*.

(b) Evaluate the yield condition at the *trial elastic state*.

(c) Show that, $\mathrm{sign}\left(\xi_{n+1}\right) = \mathrm{sign}\left(\xi^{\mathrm{trial}}_{n+1}\right)$ holds. Hint: Use the fact that $a = |a|\,\mathrm{sign}\,(a)$.

(d) Enforcing the discrete consistency condition, $\Phi_{n+1} = 0$, in the *plastic corrector* step allows for the determination of the incremental Lagrange multiplier $\Delta\lambda$. Express Φ_{n+1} in terms of $\left(\xi^{\mathrm{trial}}_{n+1}, \alpha_n, \Delta\lambda\right)$ and derive the corresponding linearization to enable the solution of this nonlinear equation by means of a Newton-Raphson scheme.

(e) Code the discrete formulation of the one-dimensional plasticity model by completing the function `stress_computation_one_dim_plasticity_nonlinear_hardening.m` in the Matlab-code provided. Test your implementation and compare your results with the exact solution.