Exercise 3: Three-dimensional von Mises plasticity at small strains (linear hardening)

Problem 4.1

Three-dimensional material models often require the application of basic results from tensor algebra and tensor analysis, e.g. tensor products, tensor decomposition and determination of the derivative of tensor functions (scalar- or tensor-valued). In the following, some statements, relevant for the formulation of elastic-plastic material models are given.

(a) The fourth order identity tensor $\mathbb{I} = \delta_{ik}\delta_{jl}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ can be decomposed into the symmetriser $\mathbb{I}^S = \frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ and the antisymmetriser $\mathbb{I}^A = \frac{1}{2}\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right]\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ in an additive form, i.e. $\mathbb{I} = \mathbb{I}^S + \mathbb{I}^A$.
Express the split of the second order tensor $\mathbf{A} = \operatorname{sym}(\mathbf{A}) + \operatorname{skw}(\mathbf{A})$ into its symmetric and skew-symmetric part by means of $\mathbb{I}^S$ and $\mathbb{I}^A$ and determine $\frac{\partial \operatorname{sym}(\mathbf{A})}{\partial \mathbf{A}}$ and $\frac{\partial \operatorname{skw}(\mathbf{A})}{\partial \mathbf{A}}$.

(b) Furthermore, the fourth order identity tensor $\mathbb{I} = \delta_{ik}\delta_{jl}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ can be decomposed into the tensor $\mathbb{J} = \frac{1}{3}\mathbf{I} \otimes \mathbf{I}$ and the deviatoriser $\mathbb{P} = \mathbb{I} - \mathbb{J}$, where $\mathbf{I}$ is the second order identity tensor.
Use these tensors to express the decomposition of the second order tensor $\mathbf{A} = \operatorname{dev}(\mathbf{A}) + \mathbf{A}^\circ$ into its deviatoric and spherical part and determine $\frac{\partial \operatorname{dev}(\mathbf{A})}{\partial \mathbf{A}}$ and $\frac{\partial \mathbf{A}^\circ}{\partial \mathbf{A}}$.

(c) Scalar isotropic tensor functions $f(\mathbf{A})$ can conveniently be expressed in terms of the three invariants

$$
\begin{aligned}
I_1(\mathbf{A}) &= \operatorname{tr}(\mathbf{A}) \\
I_2(\mathbf{A}) &= \frac{1}{2}\left[[\operatorname{tr}(\mathbf{A})]^2 - \operatorname{tr}\left(\mathbf{A}^2\right)\right] \\
I_3(\mathbf{A}) &= \frac{1}{6}[\operatorname{tr}(\mathbf{A})]^3 - \frac{1}{2}\operatorname{tr}(\mathbf{A})\operatorname{tr}\left(\mathbf{A}^2\right) + \frac{1}{3}\operatorname{tr}\left(\mathbf{A}^3\right)\ .
\end{aligned}
$$

Yield functions can more easily expressed in terms of the modified set of invariants, $I_1(\mathbf{A}), J_2(\mathbf{A}) = -I_2(\operatorname{dev}(\mathbf{A}))$ and $J_3(\mathbf{A}) = I_3(\operatorname{dev}(\mathbf{A}))$, because it facilitates the inclusion of pressure dependence into the yield surface.
Evaluate $J_2(\mathbf{A})$ and $J_3(\mathbf{A})$ and determine $\frac{\partial I_1(\mathbf{A})}{\partial \mathbf{A}}$ and $\frac{\partial J_2(\mathbf{A})}{\partial \mathbf{A}}$. Use the result that the derivative of the scalar, isotropic tensor function $f(\mathbf{A}) = \operatorname{tr}\left(\mathbf{A}^k\right)$ is $\frac{\partial f}{\partial \mathbf{A}} = k\left[\mathbf{A}^{k-1}\right]^{\mathrm{T}}$ with $\mathbf{A}^0 = \mathbf{I}$.

(d) Use the results from (c) and the chain rule to show that for the von-Mises yield function, $\Phi(\boldsymbol{\xi}, \beta) = \|\boldsymbol{\xi}\| - \sqrt{\frac{2}{3}}\left[\sigma_{y0} + \beta\right]$, with $\boldsymbol{\xi} = \operatorname{dev}(\boldsymbol{\sigma}) - \boldsymbol{\beta}$, $\|\boldsymbol{\xi}\| = \sqrt{\boldsymbol{\xi}^{\mathrm{T}} : \boldsymbol{\xi}}$ and $\mathbf{A} : \mathbf{B} = \operatorname{tr}\left(\mathbf{A}^{\mathrm{T}} \cdot \mathbf{B}\right)$, the following statements hold

- $\frac{\partial \Phi}{\partial \boldsymbol{\xi}} = \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}$
- $\frac{\partial \Phi}{\partial \boldsymbol{\beta}} = -\frac{\partial \Phi}{\partial \boldsymbol{\xi}}$
- $\frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \frac{\partial \Phi}{\partial \boldsymbol{\xi}}$

(e) Show that $\frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{\partial \Phi}{\partial \operatorname{dev}(\boldsymbol{\sigma})} : \operatorname{dev}(\dot{\boldsymbol{\sigma}})$ holds.

Problem 4.2

The constitutive equations of a three-dimensional, phenomenological plasticity model with linear isotropic and kinematic hardening are given in the following:

elastic law:	$\boldsymbol{\sigma} = \kappa\,\mathrm{tr}\,(\boldsymbol{\varepsilon}_\mathrm{e})\,\mathbf{I} + 2\mu\,\mathrm{dev}\,(\boldsymbol{\varepsilon}_\mathrm{e})$
strain decomposition:	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\mathrm{e} + \boldsymbol{\varepsilon}_\mathrm{p}$
yield function:	$\Phi(\boldsymbol{\xi}, \beta) = \Vert \underbrace{\mathrm{dev}\,(\boldsymbol{\sigma}) - \boldsymbol{\beta}}_{\boldsymbol{\xi}} \Vert - \sqrt{\tfrac{2}{3}}\,[\sigma_{\mathrm{y}0} + \beta]$
increase in yield stress:	$\beta = \mathrm{h}\,\alpha$
back-stress:	$\boldsymbol{\beta} = \mathrm{H}\,\boldsymbol{\alpha} = \mathrm{H}\,\boldsymbol{\varepsilon}_\mathrm{p}$
evolution of internal variables:	$\dot{\boldsymbol{\varepsilon}}_\mathrm{p} = \dot{\lambda}\frac{\partial\Phi}{\partial\boldsymbol{\xi}} = \dot{\lambda}\frac{\boldsymbol{\xi}}{\Vert\boldsymbol{\xi}\Vert} = \dot{\lambda}\mathbf{n}$ with $\boldsymbol{\varepsilon}_\mathrm{p}\,(t=0) = \mathbf{0}$
	$\dot{\alpha} = \sqrt{\tfrac{2}{3}}\dot{\lambda}$ with $\alpha\,(t=0) = 0$
KKT conditions:	$\Phi \leq 0, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda}\Phi = 0$

Due to the linear format of the constitutive relations, analytical solutions of the material model can be obtained for a prescribed strain history.

(a) Determine the Lagrange multiplier $\dot{\lambda}$ from the consistency condition $\dot{\Phi} = 0$ as a function of the applied strain rate tensor $\dot{\boldsymbol{\varepsilon}}$, normal $\mathbf{n}$ and material parameters. Take into account the isochoric nature of the plastic flow.

(b) Considering a uniaxial stress state, i.e. $\boldsymbol{\sigma} = \sigma_{11}\mathbf{e}_1 \otimes \mathbf{e}_1$, the strain rate is prescribed as $\dot{\boldsymbol{\varepsilon}} = \dot{\varepsilon}_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + q\dot{\varepsilon}_{11}\mathbf{e}_2 \otimes \mathbf{e}_2 + q\dot{\varepsilon}_{11}\mathbf{e}_3 \otimes \mathbf{e}_3$, where the parameter q has to be determined from the uniaxial stress condition. Evaluate $\mathrm{dev}\,(\boldsymbol{\sigma})\,, \Vert\mathrm{dev}\,(\boldsymbol{\sigma})\Vert, \mathrm{dev}\,(\dot{\boldsymbol{\varepsilon}})$ and $\mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}})$ for the given stress state and explain why the back-stress takes the form $\boldsymbol{\beta} = \frac{2}{3}\beta_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{1}{3}\beta_{11}\mathbf{e}_2 \otimes \mathbf{e}_2 - \frac{1}{3}\beta_{11}\mathbf{e}_3 \otimes \mathbf{e}_3$. Give an expression for the normal $\mathbf{n}$ using the previous results.

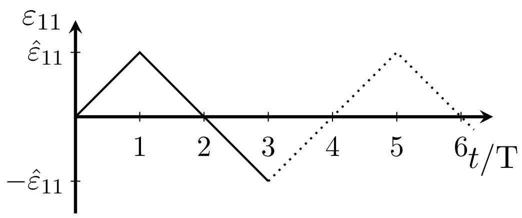
(c) Specialize your results from (a) to the uniaxial stress state and find an expression of the Lagrange multiplier in terms of the parameter q, the prescribed strain rate $\dot{\varepsilon}_{11}$ and the material parameters.

(d) Determine the parameter q from the uniaxial stress condition using the rate form of the elastic law for the case of elastic (q^{el}) and elastic-plastic (q^{ep}) loading. The corresponding values of q are employed to obtain the strain from $\boldsymbol{\varepsilon} = \int \dot{\varepsilon}_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + q\dot{\varepsilon}_{11}\mathbf{e}_2 \otimes \mathbf{e}_2 + q\dot{\varepsilon}_{11}\mathbf{e}_3 \otimes \mathbf{e}_3 \mathrm{d}\tau$.

(e) Compute the stress and the internal variables for the strain history specified below.

The strain history is given by the piecewise linear, triangle wave function, defined as

$$\varepsilon_{11}(t) = \begin{cases} n\hat{\varepsilon}_{11}\frac{t}{\mathrm{T}} & 0 \leq t \leq \mathrm{T} \\ n\hat{\varepsilon}_{11}\left[2 - \frac{t}{\mathrm{T}}\right] & \mathrm{T} < t \leq 3\mathrm{T} \end{cases},$$



where $\hat{\varepsilon}_{11} = \frac{\sigma_{\mathrm{y}0}}{2\mu[1-q^{\mathrm{el}}]}$ and only the time interval $0 \leq t \leq 3\mathrm{T}$ is of interest and $n > 1$.