Exercise 1: One-dimensional plasticity

The one-dimensional form of a phenomenological plasticity model is defined by the following set of equations:

Helmholtz free energy:	$\Psi(\varepsilon_\mathrm{e}, \varepsilon_\mathrm{p}, \alpha) = \frac{1}{2}\mathrm{E}\,\varepsilon_\mathrm{e}^2 + \frac{1}{2}\mathrm{H}\,\varepsilon_\mathrm{p}^2 + \frac{1}{2}\mathrm{h}\,\alpha^2$
strain decomposition:	$\varepsilon = \varepsilon_\mathrm{e} + \varepsilon_\mathrm{p}$
yield surface and KKT conditions:	$\Phi(\xi, R) = \lvert\underbrace{\sigma - X}_{\xi}\rvert - \sigma_0 - R \leq 0, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda}\Phi = 0$

Problem 1.1

A proper material model has to fulfill the second law of thermodynamics. Generalized standard materials, for example the one given above, do so by construction.

(a) Determine the laws of state from the Clausius-Planck inequality $\mathcal{D} = \sigma\dot{\varepsilon} - \rho\dot{\Psi} \geq 0$. Employ the abbreviations X and R for the driving forces conjugate to plastic strain ε_p and the isotropic hardening variable α, respectively.

(b) Verify that the associated evolution equations for the internal variables, given as $\dot{\varepsilon}_\mathrm{p} = \dot{\lambda}\,\frac{\partial\Phi}{\partial\xi}$ and $\dot{\alpha} = -\dot{\lambda}\,\frac{\partial\Phi}{\partial R}$, are sufficient to fulfill the reduced Clausius-Planck inequality $\mathcal{D}^\mathrm{red} = \xi\dot{\varepsilon}_\mathrm{p} - R\dot{\alpha} \geq 0$ and therefore guarantee thermodynamical consistency. Use the special form of the yield surface given above.

Remark: Assume that the material parameters $\mathrm{E}, \mathrm{H}, \mathrm{h}$ and σ_0 are positive.

Problem 1.2

In the case of elastic-plastic loading, i.e. $\Phi = 0$, the Lagrange multiplier $\dot{\lambda}$ can be obtained from the consistency condition $\dot{\Phi} = 0$ in closed form due to the linear hardening behavior considered.

(a) Verify that in this case the Lagrange multiplier is defined as $\dot{\lambda} = \frac{\mathrm{sign}(\sigma - X)\,\mathrm{E}\,\dot{\varepsilon}}{\mathrm{E}+\mathrm{H}+\mathrm{h}}$.

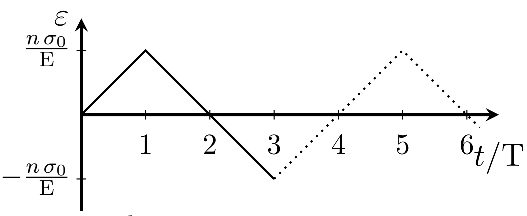
(b) Integrate the evolution equations of the internal variables in closed form for the loading history specified below, employing the explicit definition of the Lagrange multiplier given in (a) and the initial conditions $\lambda(t=0) = \alpha(t=0) = \varepsilon^\mathrm{p}(t=0) = 0$.

(c) Sketch the stress-strain diagram for the loading history specified below and the three different hardening cases: (i) isotropic hardening, (ii) kinematic hardening and (iii) mixed hardening.

	E in MPa	σ_0 in MPa	H in MPa	h in MPa	n
(i)	20000	200	0	1000	3
(ii)	20000	200	1000	0	3
(iii)	20000	200	500	500	3

The loading history is given by the piecewise linear, triangle wave function, defined as

$$\varepsilon(t) = \begin{cases} \frac{n\,\sigma_0}{\mathrm{E}}\,\frac{t}{\mathrm{T}} & 0 \leq t \leq \mathrm{T} \\ \frac{n\,\sigma_0}{\mathrm{E}}\left[2 - \frac{t}{\mathrm{T}}\right] & \mathrm{T} \leq t \leq 3\mathrm{T} \end{cases},$$



where only the time interval $0 \leq t \leq 3\mathrm{T}$ is of interest and $n > 1$.

Homework problems:

Problem 1.3

A one-dimensional, phenomenological plasticity model with linear isotropic and and nonlinear kinematic hardening of Armstrong-Frederick type is defined by the following set of equations:

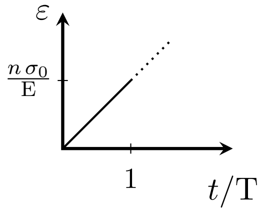
elastic law:	$\sigma = \mathrm{E}\,\varepsilon_\mathrm{e}$	
strain decomposition:	$\varepsilon = \varepsilon_\mathrm{e} + \varepsilon_\mathrm{p}$	
yield surface and KKT conditions:	$\Phi(\xi, R) = \lvert\underbrace{\sigma - X}_{\xi}\rvert - \sigma_0 - R \leq 0, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda}\Phi = 0$	
evolution equations of internal variables:	$\dot{\alpha} = -\dot{\lambda}\,\frac{\partial \Phi}{\partial R}$	$\dot{\varepsilon}_\mathrm{p} = \dot{\lambda}\,\frac{\partial \Phi}{\partial \xi}$
increase in yield stress:	$R = \mathrm{h}\,\alpha$	
evolution of back-stress:	$\dot{X} = \mathrm{H}\,\dot{\varepsilon}_\mathrm{p} - \mathrm{c}X\,\dot{\alpha}$	

(a) Integrate the evolution equation for the back-stress in closed form, denoting the initial conditions by X_0 and λ_0.

(b) Derive the ordinary differential equation for the Lagrange multiplier $\dot{\lambda}$ (as a function of $\dot{\varepsilon}$) from the consistency condition $\dot{\Phi} = 0$.

(c) Use the results from (a) and (b) to obtain a closed form expression of the Lagrange multiplier in terms of the applied strain for a linear increasing strain loading defined as

$\varepsilon(t) = \frac{n\,\sigma_0}{\mathrm{E}}\frac{t}{\mathrm{T}} \quad 0 \leq t \leq \mathrm{T}$



and the initial conditions $\lambda_0 = X_0 = 0$.