∞-Categories

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Contents

In this chapter we give an overview of the language of ∞-categories. The ultimate goal of this chapter is to define the ∞-category of ∞-groupoids and state the Yoneda lemma in ∞-categorical setting. Therefore, if you believe that ∞-groupoids are spaces (homotopy hypothesis), then we would construct the ∞-category of spaces by the end of this chapter. In the process, we will learn about the techniques invovled in manipulating ∞-categories. In this chapter, we will only work with the category of compactly generated spaces and by **Top** we mean category of compactly generated spaces.

1 Simplicial sets

The goal of simplicial sets is to obtain a combinatorial approximation of topological spaces. We describe simplicial sets as a presheaf over simplex category. One can motivate themselves why this definition is the correct definition for combinatorially handling topological spaces by reading the review paper cite[Bergner's simplicial set paper]. We give some basic properties of such objects. We also give a general important result about presheaves, "every presheaf is a colimit of representable presheaves". This is of fundamental importance in the development of ∞-categories.

Remark 1.0.1. (*Notations*)

1. As we will be frequently constructing and dealing with presheaf category over a category $\mathbf{C}$, therefore instead of only denoting it by $\mathbf{PSh}(\mathbf{C})$, we will also be denoting it by $\widehat{\mathbf{C}}$, depending on the convenience of the given situation.
2. We will denote objects of an ordinary 1-category $\mathbf{C}$ by lowercase alphabets like $a, b, c, .., x, y, z$, morphisms in $\mathbf{C}$ by lowercase letters like $f, g, h, \ldots$ and functors $\mathbf{C} \to \mathbf{D}$ by uppercase alphabets like $A, B, C, ..., X, Y, Z$ and also by lowercase alphabets like $f, g, h, \ldots$.
3. Let $\mathbf{C}$ be a category and $c \in \mathbf{C}$ be an object. We denote by $h_c : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ the contravariant hom-functor given by $a \mapsto \mathrm{Hom}_{\mathbf{C}}(a, c)$.
4. We will denote ∞-categories by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$.

Let us also as a reminder put the usual Yoneda lemma.

Lemma 1.0.2. *(Yoneda lemma) Let $\mathbf{A}$ be a category, $a \in \mathbf{A}$ be an object and F be a presheaf over $\mathbf{A}$. Then, there is a natural isomorphism*

$$\mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F) \xrightarrow{\cong} F(a).$$

Proof. Consider the following maps

$$\begin{aligned} \varphi : \mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F) &\longrightarrow F(a) \\ \alpha &\longmapsto \alpha_a(\mathrm{id}_a). \end{aligned}$$

and

$$\begin{aligned} \xi : F(a) &\longrightarrow \mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F) \\ x &\longmapsto \beta : h_a \to F \end{aligned}$$

where β is defined as follows. Consider any $a' \in \mathbf{A}$ and any $f \in h_a(a')$. Define $\beta_{a'} : h_a(a') \to F(a')$ by mapping as $f \mapsto F(f)(x)$.

One then sees that $\xi \circ \varphi = \mathrm{id}$ by naturality of α and that $\varphi \circ \xi = \mathrm{id}$ by funtoriality of F. $\square$

Corollary 1.0.3. *Let $\mathbf{A}$ be a category and define* $\mathrm{Yon} : A \to \widehat{\mathbf{A}}$ *to be the Yoneda functor given by $a \mapsto h_a$ and $a \to b \mapsto h_a \to h_b$. Then,* Yon *is fully-faithful.*

Proof. We have by Yoneda lemma (Lemma 1.0.2) that $\mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, h_b) \cong h_b(a) = \mathrm{Hom}_{\mathbf{A}}(a, b)$. $\square$

1.1 Extension of functor by colimits-I

The following is a fundamental result in category theory.

Theorem 1.1.1. *Let $\mathbf{A}$ be a small category and $\mathbf{C}$ be a locally small category with all small colimits. Then, for any functor*

$$f : \mathbf{A} \to \mathbf{C}$$

we get two functors

$$f_! : \widehat{\mathbf{A}} \longrightarrow \mathbf{C}$$

and

$$\begin{aligned} f^* : \mathbf{C} &\longrightarrow \widehat{\mathbf{A}} \\ c &\longmapsto \mathrm{Hom}_{\mathbf{C}}\left(f(-), c\right) \end{aligned}$$

such that

1. *$f_!$ is left adjoint of f^**

$$\begin{tikzcd} \widehat{\mathbf{A}} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, shift left, "f^*"] \end{tikzcd} .$$

The functor $f_!$ is called the extension of f by colimits.

2. *Denoting* $\mathrm{Yon} : \mathbf{A} \hookrightarrow \widehat{\mathbf{A}}$ *to be the Yoneda embedding, we get that the following commutes upto a unique natural isomorphism*

$$\begin{tikzcd} \widehat{\mathbf{A}} \arrow[r, "f_!"] \arrow[dr, phantom, "\Rightarrow"] & \mathbf{C} \\ \mathbf{A} \arrow[u, hook, "\mathrm{Yon}"] \arrow[ur, "f"'] & \end{tikzcd} .$$

That is, $f_!(h_a) \cong f(a)$ where $h_a = \mathrm{Hom}_{\mathbf{A}}\left(-, a\right)$.

Proof. The main heart of the proof is the fact that every presheaf is a colimit of representables (Proposition 1.1.8 of cite[Cis]).

1. We define $f_! : \widehat{\mathbf{A}} \to \mathbf{C}$ as follows. Pick any $X \in \widehat{\mathbf{A}}$. Consider the functor $\varphi_X : \mathbf{A}/X \to \widehat{\mathbf{A}}$ from the category of elements of X given by $(a, s) \mapsto h_a$ and on maps by $f : (a, s) \to (b, t)$ by $h_f : h_a \to h_b$. By Proposition 1.1.8 of cite[Cis], we have $\varinjlim_{(a,s)} h_a = X$. Define $f_!(X) = \varinjlim_{(a,s)} f(a)$ which exists in $\mathbf{C}$ as $\mathbf{C}$ has all small colimits. Consequently, we obtain the following natural isomorphisms by Yoneda lemma (Lemma 1.0.2) and limit preserving properties of contravariant homs

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}\left(f_!(X), c\right) \cong \mathrm{Hom}_{\mathbf{C}}\left(\varinjlim_{(a,s)} f(a), c\right) &\cong \varprojlim_{(a,s)} \mathrm{Hom}_{\mathbf{C}}\left(f(a), c\right) \\ &\cong \varprojlim_{(a,s)} \mathrm{Hom}_{\widehat{\mathbf{A}}}\left(h_a, f^*(c)\right) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}}\left(\varinjlim_{(a,s)} h_a, f^*(c)\right) \\ &\cong \mathrm{Hom}_{\widehat{\mathbf{A}}}\left(X, f^*(c)\right). \end{aligned}$$

2. Since $\mathrm{Hom}_{\mathbf{C}}\left(f_!(h_a), c\right) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}}\left(h_a, f^*c\right) \cong f^*(c)(a) = \mathrm{Hom}_{\mathbf{C}}\left(f(a), c\right)$ for all objects $c \in \mathbf{C}$, therefore the result follows by Corollary 1.0.3. □

The above theorem will take a central place in constructions of this chapter. Indeed, let us point the following applications of this theorem before stating its proof.

Corollary 1.1.2. *Let $\mathbf{A}$ be a small category and $\mathbf{C}$ be a category with small colimits. If $F : \widehat{\mathbf{A}} \to \mathbf{C}$ is a colimit preserving functor. Then*

1. *there exists $f : \mathbf{A} \to \mathbf{C}$ such that $f_! \cong F$ naturally,*
2. *F has a right adjoint.*

Proof. We first define the required f. For any object $a \in \mathbf{A}$, define $f(a) := F(h_a)$. Then by Theorem 1.1.1, 1, it follows that $f_!(h_a) \cong f(a) = F(h_a)$. Since we know that every presheaf is presented as a colimit of representable functors indexed by its category of elements, thus we get that for any $X \in \widehat{\mathbf{A}}$, $F(X) \cong f_!(X)$ naturally. The second conclusion also follows from Theorem 1.1.1, 2. $\square$

The following result will be important in order to define simplicial mapping spaces.

Proposition 1.1.3. *For any small category $\mathbf{A}$, the presheaf category $\widehat{\mathbf{A}}$ is Cartesian closed where the internal hom object is defined by*

$$\underline{\mathrm{Hom}}\,(X,Y)(a) := \mathrm{Hom}_{\widehat{\mathbf{A}}}\,(X \times h_a, Y)$$

and on morphisms as

$$\underline{\mathrm{Hom}}\,(X,Y)(f) := \mathrm{Hom}_{\widehat{\mathbf{A}}}\,(X \times h_f, Y).$$

Proof. We wish to show that $\underline{\mathrm{Hom}}\,(-,-)$ acts as internal hom object in $\widehat{\mathbf{A}}$. This can be seen by establishing following natural bijections

$$\mathrm{Hom}_{\widehat{\mathbf{A}}}\,(T \times X, Y) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}}\,(T, \underline{\mathrm{Hom}}\,(X,Y))$$

This follows from contemplating the functor $f_X : \mathbf{A} \to \widehat{\mathbf{A}}$ for all $X \in \widehat{\mathbf{A}}$ given by $a \mapsto X \times \mathrm{Yon}(-)$, together with Theorem 1.1.1. Indeed, first observe that $\widehat{\mathbf{A}}$ is locally small with all small colimits. Second, observe from the proof of Theorem 1.1.1 that for any $Z \in \widehat{\mathbf{A}}$, we have that $f_{X!} : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}$ takes any $Z \in \widehat{\mathbf{A}}$ and maps it to $f_{X!}(Z) = \varinjlim_{(a,s)} f_X(a)$ where (a,s) varies over $\mathbf{A}/Z$, the category of elements of Z. Since $f_X(a) = X \times h_a$ and filtered colimits commute with finite limits, therefore we get a natural isomorphism $f_{X!}(Z) \cong X \times Z$. Consequently, the adjunction of Theorem 1.1.1 completes the proof. $\square$

1.2 Category of simplicial sets

Consider the category of all finite sets $[n]$ with $n+1$ elements with linear order $0 < 1 < \cdots < n$ and mappings being the non-decreasing maps. Denote this category by $\mathbf{\Delta}$ and call it the *simplex category*. A *simplicial set* is then a presheaf over $\mathbf{\Delta}$. Let the category of all simplicial sets be denoted **sSet**.

There are two important class of maps in $\mathbf{\Delta}$.

Definition 1.2.1 (Face and degeneracy maps). For each $n \in \mathbb{N}$, we have $n+1$ face maps

$$d^i : [n-1] \to [n]$$
$$j \mapsto \begin{cases} j & \text{if } j \leq i \\ j+1 & \text{if } j > i \end{cases}$$

where $0 \leq i \leq n$ and n degeneracy maps

$$s^i : [n] \to [n-1]$$
$$j \mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

where $0 \leq i \leq n-1$.

Remark 1.2.2. By Yoneda embedding, we will allow ourselves to abuse the notation by writing $d^i : [n-1] \to [n]$ as the unique map $d^i : \Delta^{n-1} \to \Delta^n$ and $s^i : [n] \to [n-1]$ as the unique map $s^i : \Delta^n \to \Delta^{n-1}$ in **sSet**. For a simplicial set X we may thus interpret $x \in X_n$ as $x : \Delta^n \to X$. The above give maps which we denote as

$$\begin{aligned} d_i : X_n &\longrightarrow X_{n-1} \\ x &\longmapsto x \circ d_i \end{aligned}$$

for each $0 \le i \le n$ also called the face maps and

$$\begin{aligned} s_i : X_{n-1} &\longrightarrow X_n \\ x &\longmapsto x \circ s_i \end{aligned}$$

for each $0 \le i \le n-1$ also called degeneracy maps.

It is quite easy to observe, but very important for applications, the following relations satisfied by face and degeneracy maps. All these are immediate from definition given above.

Proposition 1.2.3. *The following relations hold in* $\mathbf{\Delta}$ *(and thus in* **sSet** *):*

1. *If $i < j$, then $d^j d^i = d^i d^{j-1}$.*
2. *If $i \le j$, then $s^j s^i = s^i s^{j+1}$.*
3. *We have*

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ d^{i-1} s^j & \text{if } j+1 < i. \end{cases}$$

□

Consequently, for a simplicial set, dual relations hold.

Proposition 1.2.4. *Let X be a simplicial set. Then the face and degeneracy maps of X satisfies the following relations:*

1. *If $i < j$, then $d_i d_j = d_{j-1} d_i$.*
2. *If $i \le j$, then $s_i s_j = s_{j+1} s_i$.*
3. *We have*

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } j+1 < i. \end{cases}$$

□

Remark 1.2.5. One may keep the following picture in mind while working with a simplicial set X:

$$\begin{tikzcd}
{[0]} \arrow[r, tail, shift left=3, "d^0"] \arrow[r, tail, shift left=1, "d^1"'] & {[1]} \arrow[l, two heads, shift left=2, "s^0"] \arrow[r, tail, shift left=4, "d^0"] \arrow[r, tail, shift left=2, "d^1"] \arrow[r, tail, "d^2"'] & {[2]} \arrow[l, two heads, shift left=3, "s^0"] \arrow[l, two heads, shift left=1, "s^1"'] \arrow[r, tail, shift left=2, "d^i"] & \cdots \arrow[l, two heads, shift left=2, "s^i"] \\
X_0 \arrow[r, shift right=2, "s_0"'] & X_1 \arrow[l, shift right=3, "d_0"'] \arrow[l, shift right=1, "d_1"] \arrow[r, shift right=3, "s_0"'] \arrow[r, shift right=1, "s_1"] & X_2 \arrow[l, shift right=4, "d_0"'] \arrow[l, shift right=2, "d_1"] \arrow[l, "d_2"] \arrow[r, shift right=2, "s_i"'] & \cdots \arrow[l, shift right=2, "d_i"']
\end{tikzcd}.$$

Remark 1.2.6. There is a functor

$$\begin{aligned} |-| : \mathbf{\Delta} &\longrightarrow \mathbf{Top} \\ [n] &\longmapsto |\Delta^n| \end{aligned}$$

where $|\Delta^n|$ is the standard topological n-simplex in $\mathbb{R}^{n+1}$ and for $f : [n] \to [m]$ in $\mathbf{\Delta}$, we have

$$\begin{aligned} |f| : |\Delta^n| &\to |\Delta^m| \\ (t_0, \ldots, t_n) &\longmapsto \left(\sum_{i \in f^{-1}(0)} i, \ldots, \sum_{i \in f^{-1}(m)} i \right). \end{aligned}$$

Example 1.2.7 (*Singular chains*)**.** An important example of a simplicial set is that of $\mathrm{Sing}(X)$ defined as

$$\begin{aligned} \mathrm{Sing}(X) : \mathbf{\Delta}^{op} &\to \mathbf{Set} \\ [n] &\mapsto \mathrm{Hom}_{\mathbf{Top}}\left(|\Delta^n|, X\right). \end{aligned}$$

The main point is that $\mathrm{Sing}(X)$ as a simplicial set knows all about the homotopy type of space X. Consequently, we will denote $X([n]) := \mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathbf{Top}}\left(|\Delta^n|, X\right)$.

Example 1.2.8 (*Nerve of a category*)**.** Let $\mathbf{C}$ be a category. Define the nerve of $\mathbf{C}$ as

$$\begin{aligned} N\mathbf{C} : \mathbf{\Delta}^{op} &\to \mathbf{Set} \\ [n] &\mapsto \mathrm{Hom}_{\mathbf{Cat}}\left([n], \mathbf{C}\right) \end{aligned}$$

where $[n]$ is a category as it is a poset. Consequently, $N\mathbf{C}_n$ is the set of all n-composable arrows of $\mathbf{C}$.

Theorem 1.2.9. *Nerve construction is a fully-faithfull embedding of categories into simplicial sets.*

1.3 Operations on simplicial sets

Define product, coproducts, subspaces, unions, quotients, limits, colimits and mapping objects. **TODO!!**

Example 1.3.1 (*Standard Δ^n, boundaries $\partial_i \Delta^n$, $\partial \Delta^n$ and horn Λ^n_i*)**.** Denote $\Delta^n = N[n]$ to be the nerve of the category $[n]$. That is,

$$\Delta^n([m]) = \Delta^n_m = \mathrm{Hom}_{\mathbf{\Delta}}\left([m], [n]\right) = h_{[n]}([m])$$

which are exactly all representable presheaves over $\mathbf{\Delta}$. These are the combinatorial analogues of the topological n-simplex $|\Delta^n|$ and we tend to think about Δ^n using the intuition gained from the topological one. There are some important simplicial subsets of Δ^n.

Let $E \subseteq [n]$ be a totally ordered subset. Define $\Delta^E = NE$ to be a simplicial subset of Δ^n. From this we derive the following simplicial subsets of Δ^n. The first is the i^{th}*-boundary of* Δ^n for $0 \leq i \leq n$ given by

$$\partial_i \Delta^n = \bigcup_{i \notin E \subsetneq [n]} \Delta^E \cong \Delta^{n-1}.$$

The second is the *boundary of* Δ^n given by

$$\partial\Delta^n = \bigcup_{E \subsetneq [n]} \Delta^E = \bigcup_{i=0}^{n} \partial_i \Delta^n$$

The third is the i^{th}-horn of Δ^n denoted Λ^n_i given by

$$\Lambda^n_i = \bigcup_{i \in E \subsetneq [n]} \Delta^E.$$

Remark 1.3.2. Let $n \geq 1$, $0 \leq i \leq n$ and $m \geq 0$. Note that we have

$$\begin{aligned}
(\partial_i\Delta^n)_m &= \left\{\begin{matrix}\text{Order preserving maps } f : [m] \to \\ [n] \text{ which are not surjective and } i \notin \\ \text{Im}\,(f).\end{matrix}\right\} \\
(\partial\Delta^n)_m &= \left\{\begin{matrix}\text{Order preserving maps } f : [m] \to \\ [n] \text{ which are not surjective.}\end{matrix}\right\} \\
(\Lambda^n_i)_m &= \left\{\begin{matrix}\text{Order preserving maps } f : [m] \to \\ [n] \text{ which are not surjective and } i \in \\ \text{Im}\,(f).\end{matrix}\right\}.
\end{aligned}$$

For two simplicial sets, we can define the internal hom using the Proposition 1.1.3.

Definition 1.3.3 (Homotopy & mapping complex). Let S, T be a simplicial set. Then $\underline{\text{Hom}}\,(S,T)$ denotes the following simplicial set

$$[n] \mapsto \text{Hom}_{\mathbf{sSet}}\,(S \times \Delta^n, T).$$

An n-simplex of $\underline{\text{Hom}}\,(S,T)$ is defined to be an n-homotopy from S to T. A 1-homotopy H is also referred to as a homotopy from $H|_{S\times\{0\}} =: f$ to $H|_{S\times\{1\}} =: g$.

1.4 Basic properties

We discuss some properties of simplicial sets which would be useful later on. Some of these might be taken as exercises on combinatorial manipulations with simplicial sets.

We first begin with a simple observation.

Definition 1.4.1 (n-degenerate). A simplicial set X is said to be n-degenerate if for all $m > n$, all m-simplices in X_m are degenerate.

Example 1.4.2. Each standard simplicial sets Δ^n are n-degenerate. Indeed, its m-simplices for $m > n$ are

$$\Delta^n_m = \{\text{Order preserving maps } f : [m] \to [n]\}.$$

But as $m > n$, therefore every such f is necessarily non-injective. It follows that each simplex in Δ^n_m is in the image of $s_i : X_{m-1} \to X_m$ for some $0 \leq i \leq m-1$.

For similar reasons, the i^{th}-boundary $\partial_i\Delta^n$, boundary $\partial\Delta^n$ and horns Λ^n_i for $0 \leq i \leq n$ are all $n-1$-degenerate.

Lemma 1.4.3. *Let X be an n-degenerate simplicial set and Y be a simplicial set. Then any collection of functions $\{\varphi_m : X_m \to Y_m\}_{0\le m\le n}$ such that for any $f : [k] \to [l]$ in $\mathbf{\Delta}$ with $0 \le k, l \le n$, the following square commutes*

$$\begin{tikzcd} X_l \arrow[r, "\varphi_l"] \arrow[d, "f^*"'] & Y_l \arrow[d, "f^*"] \\ X_k \arrow[r, "\varphi_k"'] & Y_k \end{tikzcd},$$

the collection $\{\varphi_m\}_{0\le m\le n}$ lifts to a unique map of simplicial sets $\varphi : X \to Y$.

Remark 1.4.4. As a consequence of Lemma 1.4.3, in order to give a map of simplicial sets from an n-degenerate simplicial set S, it suffices to construct the required map only on m-simplices for $0 \le m \le n$.

Proof of Lemma 1.4.3. Let $\{\varphi_m\}_{0\le m\le n}$ be as given. We wish to define $\varphi_{m'}$ for $m' > n$. We proceed by induction on m'. Suppose $\varphi_{m'-1}$ is given to us. Since we have the following diagram

$$\begin{tikzcd} X_{m'-1} \arrow[r, "\varphi_{m'-1}"] \arrow[d, "s_i"'] & Y_{m'-1} \arrow[d, "s_i"] \\ X_{m'} \arrow[r, "\varphi_{m'}"'] & Y_{m'} \end{tikzcd}$$

and that every element of $X_{m'}$ is in image of s_i (guaranteed by Proposition 1.2.4, 3), it follows that there is a unique choice of $\varphi_{m'}$ to fit in the above diagram, as required. □

Lemma 1.4.5. *Let $n \ge 1$ and $0 \le i \le n$. Then,*
1. *$\partial_i\Delta^n$ has exactly 1 non-degenerate $n-1$-simplex,*
2. *$\partial\Delta^n$ has exactly $n+1$ non-degenerate $n-1$-simplices,*
3. *Λ^n_i has exactly n non-degenerate $n-1$-simplices.*

Proof. These three items are immediate from Remark 1.3.2. □

The following is an important adjunction which will be consistently used in later sections. Moreover, it is generally good to keep in mind all the time while working with simplicial sets so that one transfer intution from topological spaces to that of simplicial sets, as we would need to do time and time again (for example when dealing with homotopy of simplicial sets).

Theorem 1.4.6 (*Geometric realization*)**.** *The singular functor* $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ *has a left adjoint* $|-| : \mathbf{sSet} \to \mathbf{Top}$

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, shift left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift left, "\mathrm{Sing}"] \end{tikzcd}.$$

The functor $|-|$ is called the geometric realization and for a simplicial set X, we have

$$|X| = \coprod_{n\ge 0} X_n \times |\Delta^n| / \sim$$

where $\sim$ is generated by $(f^*x, t) \sim (x, |f|t)$ for all $f : [n] \to [m]$ in $\mathbf{\Delta}$, $x \in X_m$ and $t \in |\Delta^n|$.

Proof. The main idea is that any map $X \to \mathrm{Sing}(Y)$ is a natural transform of presheaves. One observes that the naturality conditions on this morphism is equivalently represented in terms of a map $|X| \to Y$. **TODO**. $\square$

Example 1.4.7. As an example, one can show that

$$\Delta^1 \times \Delta^1 \cong \Delta^2 \cup_{\Delta^1} \Delta^2.$$

Observe that even though Δ^1 has all simplices of dimension ≥ 2 as degenerate, yet $\Delta^1 \times \Delta^1$ has two non-degenerate 2-dimensional simplices.

A consequence of the above isomorphism is that the geometric realization of $\Delta^1 \times \Delta^1$ is exactly I^2, the unit square, which is the product of the geometric realization of Δ^1 with itself. Indeed, this is an instantiation of the general result in Theorem 1.4.9.

Remark 1.4.8. It is immediate to observe from Theorem 1.4.6 that geometric realization of Δ^n is exactly the standard topological n-simplex. Similarly, $\partial\Delta^n$ and $|\Lambda^n_i|$ are homeomorphic to exactly the pictures that we used in our mind to understand them.

Finally, the main result is as follows.

Theorem 1.4.9. *Let X, Y be simplicial sets. Then the natural map*

$$|X \times Y| \to |X| \times |Y|$$

is a homeomorphism.

We show that nerve of a category satisfies lifting property which would prove to be useful later on while discussing ∞-categories.

Proposition 1.4.10. *Let $X = N\mathbf{C}$ be thr nerve of a small category $\mathbf{C}$. Then for any $0 < i < n$, the following lifting problem is uniquely filled:*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "!"'] & \end{tikzcd}.$$

Proposition 1.4.11. *Let S be a simplicial set and X a Kan complex. Then $\underline{\mathrm{Hom}}(S, X)$ is a Kan complex, denoted* $\mathrm{Map}(S, X)$*.*

1.5 Eilenberg-Zilber categories

This is an abstraction of the type of combinatorial proofs that we would like to make in the simplex category $\mathbf{\Delta}$. Indeed, as we would have to work with surjections and injections in $\mathbf{\Delta}$ primarily, which interacts with the the size of $[n]$ (which we would define to be n in a minute) as an injection would only increase the size and a surjection would only decrease it, therefore we need a systematic toolset to work with these things. In particular, if we denote $\mathbf{\Delta}_+$ to be the subcategory of all injective maps, $\mathbf{\Delta}_-$ to be the subcategory of all surjective maps in $\mathbf{\Delta}$ and $d : \mathrm{Ob}(\mathbf{\Delta}) \to \mathbb{N}$ the size map, then we have the following properties about the tuple $(\mathbf{\Delta}, \mathbf{\Delta}_+, \mathbf{\Delta}_-, d)$:

1. All bijections are in both $\mathbf{\Delta}_+$ and $\mathbf{\Delta}_-$.
2. The dimension map d takes bijective cardinals to the same natural.
3. Let $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$ which is not bijective. If σ is injective (i.e. in $\mathbf{\Delta}_+$), then $d(a) < d(b)$ and if σ is surjective (i.e. in $\mathbf{\Delta}_-$), then $d(a) > d(b)$.
4. Any map $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$ factors as a surjection followed by an injection.
5. For any surjective map $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$, there exists a section $\pi : [m] \to [n]$, i.e. such that $\pi\sigma = \mathrm{id}_{[n]}$.
6. If $\sigma : [n] \to [m]$ is surjective, then the set of sections of σ uniquely determines the map σ.

Remark 1.5.1. We need this abstraction of properties of $\mathbf{\Delta}$ so that with the same techniques, we can work with bisimplicial sets, which is important if one wishes to consider simplicial objects in **sSet**.

These considerations about $\mathbf{\Delta}$ motivates the following definition.

Definition 1.5.2. (**Eilenberg-Zilber categories**) A category $\mathbf{A}$ is said to be an Eilenberg-Zilber category (or much simply, *EZ cateogry*), if there exists subcategories $\mathbf{A}_+, \mathbf{A}_-$ and a function $d : \mathrm{Ob}(\mathbf{A}) \to \mathbb{N}$ which satisfies the following axioms:
1. All isomorphisms of $\mathbf{A}$ are in both $\mathbf{A}_+$ and $\mathbf{A}_-$.
2. If a, a' are isomorphic objects in $\mathbf{A}$, then $d(a) = d(a')$.
3. Let $\sigma : a \to a'$ not be an isomorphism. If σ is in $\mathbf{A}_+$, then $d(a) < d(a')$. If σ is in $\mathbf{A}_-$, then $d(a) > d(a')$.
4. For any map $\sigma : a \to a'$ in $\mathbf{A}$, there exists unique factorization of σ into maps $p : a \to c$ in $\mathbf{A}_-$ and $i : c \to a'$ in $\mathbf{A}_+$
$$\begin{tikzcd} & c \arrow[dr, "i"] & \\ a \arrow[ur, "p"] \arrow[rr, "\sigma"'] & & a' \end{tikzcd}.$$
5. If $\sigma : a \to a'$ is a map in $\mathbf{A}_-$, then there exists a section $\pi : a' \to a$, i.e. $\pi\sigma = \mathrm{id}_a$.
6. If σ, σ' are two maps in $\mathbf{A}_-$ such both of them has the same collection of sections, then $\sigma = \sigma'$.

The main thrust of this section is to discuss presheaves over an EZ category $\mathbf{A}$, keeping in mind the prototypical case of $\mathbf{A} = \mathbf{\Delta}$. Indeed, the main result and its corollaries will serve first as a practice for the type of arguments we shall need later and also as a tool to be consistently used later in constructions with simplicial sets (which are, presheaves over $\mathbf{\Delta}$).

Example 1.5.3. By previous discussion, it is clear that the simplex category $\mathbf{\Delta}$ is an EZ category.

Let $\mathbf{A}$ be an EZ category and X a presheaf over $\mathbf{A}$. Then the category of elements of X, $\mathbf{A}/X$, is an EZ category again. Indeed, define $(\mathbf{A}/X)_+$ exactly as those pairs (a, s) where $a \in \mathbf{A}_+$ and $(\mathbf{A}/X)_-$ exactly as those pairs (a, s) where $a \in \mathbf{A}_-$. Further for an object (a, s), define $d(a, s) = d(a)$, where the latter d is coming from the EZ structure on $\mathbf{A}$.

1.6 Kan complexes and homotopy groups

In the beginning section, we showed how simplicial sets can be viewed as combinatorial version of usual spaces. However, in order to do homotopy theory in this combinatorial setting, we need to isolate a class of simplicial sets which are right for this kind. Indeed, these will be Kan complexes.

We wish to define $\pi_n(X,x)$ where $x \in X_0$ is a 0-simplex of a simplicial set. To this end, we immediately run into problems as $\pi_0(X,x)$ should be the equivalence class of all those 0-simplices which are boundaries of a 1-simplex. But it is immediately clear that this is not an equivalence relation! Indeed, consider Δ^1. Then the above relation is not symmetric as we have $0 \to 1$ as a 1-simplex but there is no $1 \to 0$ in Δ^1_1. Similarly, if we try to prove transitivity of the above relation, we land up in the following situation. Let $x \to y, y \to z$ be two 1-simplices in X. Then, we wish to find a 1-simplex $x \to z$ in X. Note that given $x \to y$ and $y \to z$, we have a map $\Lambda^2_1 \to X$ and we wish to wish to fill the following diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}.$$

So we need those simplicial sets, where the above dotted arrow always exists.

Definition 1.6.1 (Kan complex). A simplicial set X is a Kan complex if for any $n \geq 0$ and any $0 \leq i \leq n$, the following lifting diagram is filled

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}.$$

Example 1.6.2. Clearly, Δ^n are not Kan complexes for $n \geq 0$. Moreover, $N\mathbf{C}$ is a Kan complex if and only if $\mathbf{C}$ is a groupoid. This is immediate.

The prototypical (and in some sense, the only example) of Kan complexes are those obtained by Theorem 1.4.6.

Proposition 1.6.3. *Let X be a topological space. Then* $\mathrm{Sing}(X)$ *is a Kan complex.*

Proof. Let $n \geq 0$, $0 \leq i \leq n$. Then a map $\Lambda^n_i \to \mathrm{Sing}(X)$ is equivalent to a map $|\Lambda^n_i| \to X$ by adjunction of Theorem 1.4.6. Consequently, we wish to fill the following diagram

$$\begin{tikzcd} {|\Lambda^n_i|} \arrow[r] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}.$$

But this is immediate as $|\Lambda^n_i| \hookrightarrow |\Delta^n|$ is a retraction. □

Another example of a Kan complex is the mapping complex.

Definition 1.6.4 (Mapping complex). Let X, Y be two spaces. The mapping complex, denoted $\mathrm{Map}(X,Y)$, is the one whose n-simplices are

$$[n] \mapsto \mathrm{Hom}_{\mathbf{Top}}(X \times |\Delta^n|, Y).$$

Note that $\mathrm{Map}(X, Y)$ has 0-simplices as continuous maps, 1-simplices as homotopies and so on. The name is justified by the following result.

Corollary 1.6.5. *Let X, Y be spaces. Then the mapping complex $\mathrm{Map}(X, Y)$ is a Kan complex.*

Proof. As all spaces are compactly generated, therefore we have $\mathrm{Hom}_{\mathbf{Top}}(X \times |\Delta^n|, Y) \cong \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y^X)$. Consequently, $\mathrm{Map}(X, Y) \cong \mathrm{Sing}(Y^X)$, which is a Kan complex by Proposition 1.6.3. □

One can now check that the relation mentioned in the beginning on X_0 is indeed an equivalence relation and would thus yield the definition of $\pi_0(X, x)$, however, we would like to define all homotopy groups in one go. We first define the notion of two simplicies being homotopic.

Definition 1.6.6 (Homotopic rel ∂). Let X be a Kan complex and $\sigma, \tau \in X_n$ be two n-simplicies. Then σ and τ are homotopic rel ∂ if there exists an $n+1$-simplex $\psi \in X_{n+1}$ such that $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$ and $\partial_n\psi = \sigma$, $\partial_{n+1}\psi = \tau$ and for all $0 \leq i \leq n-1$, $\partial_i\psi = \sigma d^i s^{n-1} = \tau d^i s^{n-1}$. We can diagramatically represent these conditions as the commutativity of the the diagrams

$$\begin{tikzcd} X & \Delta^n \arrow[l, "\tau"'] \\ \Delta^n \arrow[u, "\sigma"] & \partial\Delta^n \arrow[l, hook'] \arrow[u, hook] \end{tikzcd},$$

$$\begin{tikzcd} \Delta^{n+1} \arrow[r, "\psi"] & X & \Delta^{n+1} \arrow[l, "\psi"'] \\ \Delta^n \arrow[u, "d^n"] \arrow[ur, "\sigma"'] & & \Delta^n \arrow[u, "d^{n+1}"'] \arrow[ul, "\tau"] \end{tikzcd}$$

and for $0 \leq i \leq n-1$

$$\begin{tikzcd} \Delta^{n+1} \arrow[r, "\psi"] & X & \Delta^n \arrow[l, "\tau"'] \\ \Delta^n \arrow[u, "d^i"] \arrow[rr, "s^{n-1}"'] & & \Delta^{n-1} \arrow[u, "d^i"'] \end{tikzcd}$$

in **sSet**.

Indeed, we see that this generalizes our previous notion of homotopy relative to boundary as follows.

Lemma 1.6.7. *Let $X = \mathrm{Sing}(Y)$ be the Kan complex associated to a space Y. Then two n-simplices $\sigma, \tau \in X_n$ are homotopic rel ∂ if and only if $\sigma, \tau : |\Delta^n| \to Y$ are homotopic relative to $|\partial\Delta^n| \hookrightarrow |\Delta^n|$ in the classical sense.*

Proof. Let $\sigma, \tau : \Delta^n \to X$ be two n-simplices. These are homotopic rel ∂ if the diagrams in Definition 1.6.6 commutes in **sSet**. By the adjunction of Theorem 1.4.6, this is equivalent to commutativity of the following diagrams in **Top**

$$\begin{tikzcd} Y & {|\Delta^n|} \arrow[l, "\tau"'] \\ {|\Delta^n|} \arrow[u, "\sigma"] & {|\partial\Delta^n|} \arrow[l, hook'] \arrow[u, hook] \end{tikzcd},$$

$$\begin{tikzcd}
\left|\Delta^{n+1}\right| \arrow[r, "\psi"] & X & \left|\Delta^{n+1}\right| \arrow[l, "\psi"'] \\
\left|\Delta^{n}\right| \arrow[u, "d^n"] \arrow[ur, "\sigma"'] & & \left|\Delta^{n}\right| \arrow[u, "d^{n+1}"'] \arrow[ul, "\tau"]
\end{tikzcd}$$

and for $0 \le i \le n-1$

$$\begin{tikzcd}
\left|\Delta^{n+1}\right| \arrow[r, "\psi"] & Y & \left|\Delta^{n}\right| \arrow[l, "\tau"'] \\
\left|\Delta^{n}\right| \arrow[u, "d^i"] \arrow[rr, "s^{n-1}"'] & & \left|\Delta^{n-1}\right| \arrow[u, "d^i"']
\end{tikzcd} .$$

But this is equivalent to the data of a homotopy $H : |\Delta^n| \times I \to Y$ rel boundary. Indeed, since we want $H_0 = \sigma$, $H_1 = \tau$ and $H_t|_{|\partial\Delta^n|} = \sigma|_{|\partial\Delta^n|} = \tau|_{\partial\Delta^n}$ for all $t \in I$, therefore we naturally get that H factors through $\left|\Delta^{n+1}\right|$. This can be visualized for a 3-simplex immediately. □

Finally, when X is a Kan complex, then this is an equivalence relation. This is where combinatorial relations between face and degeneracy maps of a simplicial set as provided in Proposition 1.2.3 becomes handy.

Proposition 1.6.8. *Let X be a Kan complex. The homotopy rel ∂ is an equivalence relation on each X_n for $n \ge 1$.*

Proof. We first show reflexivity. Consider $\sigma : \Delta^n \to X$ an n-simplex. We claim that $\psi = \sigma s^n : \Delta^{n+1} \to X$ works as the homotopy from σ to σ. Indeed, we see that $\psi d^n = \sigma s^n d^n = \sigma$ and $\psi d^{n+1} = \sigma s^n d^{n+1} = \sigma$ by Proposition 1.2.3. Similarly, for $0 \le i \le n-1$, we have $\psi d^i = \sigma s^n d^i = \sigma d^i s^{n-1}$ by the same result. This establishes reflexivity.

We now show symmetry. Let $\sigma \sim \tau$ for some $\sigma, \tau : \Delta^n \to X$ with $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$. Then, there exists $\psi : \Delta^{n+1} \to X$ with $\psi d^n = \sigma$, $\psi d^{n+1} = \tau$ and $\psi d^i = \sigma d^i s^{n-1} = \tau d^i s^{n-1}$ for $0 \le i \le n-1$. We wish to show that $\tau \sim \sigma$. We will use the fact that X is a Kan complex (so that all horns can be filled). In particular, we will construct a horn $\kappa : \Lambda^{n+2}_n \to X$, filling which will give us the required homotopy from τ to σ. Indeed, let κ be obtained from ψ by adding degeneracies such that $\kappa d^{n+2} = \psi$ and $\kappa d^{n+1} = \sigma s^n$, that is, the degenerate $n+1$-simplex obtained by repeating the n^{th}-vertex of σ. For $0 \le i \le n-1$, we keep $\kappa d^i = \psi d^i s^{n-1}$. Filling the horn κ yields an $n+2$-simplex $\tilde{\kappa} : \Delta^{n+2} \to X$ whose n^{th}-face is exactly a homotopy from τ to σ. Indeed, denote $\phi = \tilde{\kappa} d^n$. Then, $\phi d^n = \tilde{\kappa} d^n d^n = \tilde{\kappa} d^{n+1} d^n = \sigma$. Similarly, $\phi d^{n+1} = \tau$. These follow from the observation that $d^n d^n$ is the unique n-simplex of $\tilde{\kappa}$ not containing the vertices n and $n+1$ in $\tilde{\kappa}$. For $0 \le i \le n-1$, by Proposition 1.2.3, we have $\phi d^i = \tilde{\kappa} d^n d^i = \tilde{\kappa} d^i d^{n-1}$, which in turn is $\kappa d^i d^{n-1} = \psi d^i s^{n-1} d^{n-1} = \psi d^i = \sigma d^i s^{n-1}$, as needed. This shows symmetry.

Finally, we wish to show transitivity. Let $\psi, \psi' \in X_{n+1}$ be homotopies $\sigma \sim \tau$ and $\tau \sim \eta$ respectively for some $\sigma, \tau, \eta \in X_n$. We wish to construct a homotopy $\psi'' \in X_{n+1}$ between σ and η. Indeed, we obtain a horn $\kappa : \Lambda^{n+2}_{n+2} \to X$ whose n^{th} and $n+1^{\text{th}}$ boundaries are ψ and ψ' respectively and the rest boundaries are required degeneracies. Filling this horn up by the Kan condition gives $\tilde{\kappa}$ and its $n+2^{\text{th}}$-boundary is the required homotopy. □

Consequently, we define homotopy groups of a Kan complex as follows.

Definition 1.6.9 (**Homotopy groups of a Kan complex**)**.** Let X be a Kan complex and $x \in X_0$ be a base point. Then for $n \geq 1$ define

$$\pi_n(X, x_0) = \{\sigma \in X_n \mid \sigma|_{\partial\Delta^n} = c_{x_0}\} / \sim$$

where $\sim$ is the homotopy rel ∂. For $n = 0$, define

$$\pi_0(X) = X_0 / \sim$$

where

$$x \sim y \iff \exists \gamma \in X_1 \text{ s.t. } \gamma d^1 = x \ \& \ \gamma d^0 = y.$$

We now show that $\pi_n(X, x)$ is indeed a group.

Construction 1.6.10 (*Composition and group operation on* $\pi_n(X, x)$)**.** Let X be a Kan complex and $x \in X$ be a point in it (i.e a 0-simplex). Let $\sigma, \tau \in X_n$ be two n-simplices such that $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n} = c_x$. We construct the composition $\sigma \cdot \tau$ of σ and τ as the following n-simplex. Construct the following horn $\kappa : \Lambda^{n+1}_n \to X$ whose $n-1^{\text{th}}$-boundary is σ, $n+1^{\text{th}}$-boundary is τ[1] and for $0 \leq i \leq n-2$, $\kappa d^i = c_x$. It follows from horn-filling condition of X that we get an $n+1$-simplex $\psi : \Delta^{n+1} \to X$ extending the horn κ. Consequently, we define the *concatenation* $\sigma \cdot \tau$ as the n^{th}-boundary of ψ, that is,

$$\sigma \cdot \tau = \psi d^n.$$

We claim that the operation

$$\begin{aligned} \cdot : \pi_n(X, x) \times \pi_n(X, x) &\longrightarrow \pi_n(X, x) \\ ([\alpha], [\beta]) &\longmapsto [\alpha \cdot \beta] \end{aligned}$$

is a well-defined function, that is, $[\alpha \cdot \beta]$ only depends on $[\alpha]$ and $[\beta]$.

Indeed, suppose $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$ in $\pi_n(X, x)$. Then, we have a homotopy rel ∂ denoted $\psi \in X_{n+1}$ from α to α' and $\chi \in X_{n+1}$ from β to β'. We wish to construct $\phi \in X_{n+1}$ which is a homotopy rel ∂ from $\alpha \cdot \beta$ to $\alpha' \cdot \beta'$. As usual, we obtain this homotopy by constructing a higher horn and filling it by Kan condition.

To correctly denote the simplex to be constructed, we first observe that if $\delta \sim \epsilon$ is a homotopy rel ∂ of two n-simplices with $\delta|_{\partial\Delta^n} = \epsilon|_{\partial\Delta^n} = x$, then $\delta \cdot x \sim \epsilon$. Indeed, consider an $n+2$-horn whose $n+2$-boundary is the composition simplex $\delta \cdot x$, $n+1$-boundary is ϵs_0 and all i-boundaries for $0 \leq i \leq n-1$ are degeneracies of δ. Filling this yields the n-boundary as the required homotopy.

We now show that if $\beta \sim \beta'$, then $\alpha \cdot \beta \sim \alpha \cdot \beta'$. This would complete the proof. Indeed, this is immediate by considering an $n+2$-horn given by $\kappa : \Lambda^{n+2}_{n+1} \to X$ such that κd^{n+2} is the homotopy $\beta \sim \beta'$, κd^n is the composition $\alpha \cdot \beta$, κd^{n-1} is the composition $\alpha \cdot \beta'$ and the κd^i for $0 \leq i \leq n-2$ are all x.

Consequently, we get that $\cdot$ is a well defined operation on $\pi_n(X, x)$. We will later show that $\cdot$ makes $\pi_n(X, x)$ into a group. Moreover, we will show that $\pi_n(X, x) \cong \pi_n(|X|, x)$ and $\pi_n(\text{Sing}(Y), y) \cong \pi_n(Y, y_0)$!

[1] If one is thinking of paths, then it is important to note that $\sigma \cdot \tau$ is the one where τ is traversed first and then σ. One has to let go of the past notation because here simplices are not merely going to be paths, homotopies and so on, but rather more general objects like arrows of a category, 2-arrows and so on, so to be consistent with the notion of composition, it is best that we change the order in which we concatenate paths.

1.7 The fundamental group of a Kan complex

Let X be a Kan complex and $x_0 \in X$ be a 0-simplex. The fundamental group $\pi_1(X, x_0)$ is explicitly given by the following

$$\begin{aligned}\pi_1(X, x_0) &= \{\sigma : \Delta^1 \to X \mid \sigma|_{\partial\Delta^1} = x_0\}/\sim \\ &= \{\sigma \in X_1 \mid d_0(\sigma) = d_1(\sigma) = x_0\}/\sim\end{aligned}$$

where $\sigma \sim \tau$ if and only if there exists a homotopy rel ∂ denoted $H : \Delta^2 \to X$, from σ to τ. That is, $d_1(H) = \sigma$, $d_2(H) = \tau$ and $d_0(H) = s_0(d_0(\sigma)) = s_0(x_0) = \mathrm{id}_{x_0}$. This is a group where the operation is

$$\begin{aligned}\cdot : \pi_1(X, x_0) \times \pi_1(X, x_0) &\longrightarrow \pi_1(X, x_0) \\ ([\sigma], [\tau]) &\longmapsto [\sigma \cdot \tau]\end{aligned}$$

where $\sigma \cdot \tau$ is a 1-simplex which is their composition (Construction 1.6.10), obtained by the 1-boundary of the 2-simplex δ obtained by filling the horn

$$\kappa : \Lambda^2_1 \to X$$

whose $\partial_0 \Lambda^2_1 = \sigma$ and $\partial_1 \Lambda^2_1 = \tau$. More precisely, we define κ_1 as follows (this is sufficient by Example 1.4.2 and Lemma 1.4.3):

$$\begin{aligned}\kappa_1 : (\Lambda^2_1)_1 &\longrightarrow X_1 \\ \{0, 1\} &\longmapsto \sigma \\ \{1, 2\} &\longmapsto \tau \\ \{1, 1\} &\longmapsto s_0(d_0(\sigma)).\end{aligned}$$

In this section, following the usual terminology, we will write $\sigma \cdot \tau$ as $\tau * \sigma$ for $\sigma, \tau \in X_1$ with $d_0(\sigma) = d_1(\sigma) = x_0$.

We now prove some basic results about $\pi_1(X, x_0)$. It is a good exercise to show that $\pi_1(X, x_0)$ is a group.

Theorem 1.7.1. *Let X be a Kan complex and $x_0 \in X_0$. Then $\pi_1(X, x_0)$ is a group.*

Proof. We first show that for any three $\alpha, \beta, \gamma \in X_1$ with their boundaries being x_0, we have $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$. Indeed, let H be the witness of composition $\alpha * \beta$, K that of $\beta * \gamma$, L that of $(\alpha * \beta) * \gamma$ and P that of $\alpha * (\beta * \gamma)$. Now consider the following horn $\kappa : \Lambda^3_2 \to X$ given by following on non-degenerate 2-simplices:

$$\begin{aligned}\kappa_2 : (\Lambda^3_2)_2 &\longrightarrow X_2 \\ \{0, 1, 2\} &\longmapsto H \\ \{1, 2, 3\} &\longmapsto K \\ \{0, 2, 3\} &\longmapsto L.\end{aligned}$$

By Kan condition, the above horn is filled and its 2-boundary yields a 2-simplex $\chi \in X_2$ such that $d_0(\chi) = \beta * \gamma$, $d_1(\chi) = (\alpha * \beta) * \gamma$ and $d_2(\chi) = \alpha$. We now construct the required homotopy by filling another 3-horn. Indeed, consider a horn $\lambda : \Lambda^3_1 \to X$ such that $d_0(\lambda) = s_1(\beta * \gamma)$, $d_2(\lambda) = P$ and $d_3(\lambda) = \chi$. This fills to give its 1-boundary as the required homotopy.

The identity element being c_{x_0} and the existence of inverses are also immediate results of horn filling and is thus omitted. $\square$

A Kan complex is path-connected if $\pi_0(X) = 0$.

1.8 ∞-categories

1.9 Theorem of Boardman-Vogt

2 Classical homotopical algebra

2.1 Model categories

We discuss now a general setup in which one can "do" homotopy theory.

Definition 2.1.1 (**Model categories**). Let $\mathbf{C}$ be a category and $W, C, F \subseteq \mathbf{C}$ be subcategories of $\mathbf{C}$ which are called weak equivalences ($\approx$), cofibrations ($\rightarrowtail$) and fibrations ($\twoheadrightarrow$) respectively. We call $W \cap C$ weak/acyclic cofibrations and $W \cap F$ weak/acyclic fibrations. Then, the tuple $(\mathbf{C}, W, C, F)$ is a model category if it satisfies the following axioms:

1. The category $\mathbf{C}$ has all finite limits and colimits.
2. Weak equivalences satisfies 2 out-of 3 property.
3. For any $f : X \to Y$ in $\mathbf{C}$, we have two factorizations

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "\approx"'] & & Y \\ & Z \arrow[ur, two heads] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail] & & Y \\ & Z \arrow[ur, two heads, "\approx"'] & \end{tikzcd}$$

   one a weak cofibration followed by fibration and another one a cofibration followed by a weak fibration.
4. We have

$$rlp(W \cap C) = F$$
$$C = llp(W \cap F)$$

   where for a subcategory $S \subseteq \mathbf{C}$, the collection $rlp(S)$ denotes the collection of all maps $X \to Y$ in $\mathbf{C}$ satisfying right lifting property wrt S, i.e., such that for *any* commutative square

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\in S"'] & X \arrow[d, "f"] \\ W \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

   where left vertical arrow is in S, there exists a lift $h : W \to X$ as shown which makes all diagrams commute. Similarly, one defines $llp(S)$.

Definition 2.1.2 (**Cofibrant/fibrant objects**). An object X in a model category $\mathbf{C}$ is cofibrant (fibrant) if the unique map from initial object $\emptyset \to X$ (to terminal object $X \to \text{pt.}$) is a cofibration (fibration).