Simplicial Sets, Realization & The Bar-Cobar Constructions

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Let Z be a topological space. One way to understand Z is via understanding all the singular chains $S_*(Z)$ of Z and how they relate to each other.

Remark 2.0.1 (Face & degeneracy maps for $S_*(Z)$)**.** There are natural functions one can define on $X = S_*(Z)$ by using the combinatorics of the standard n-simplex Δ^n. Recall that

$$\Delta^n := \{(e_0, \dots, e_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} e_i = 1, e_i \geq 0\}.$$

Consequently, we have maps

$$\begin{aligned} d^i : \Delta^{n-1} &\longrightarrow \Delta^n \\ (e_0, \dots, e_{n-1}) &\longmapsto (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) \end{aligned}$$

and

$$\begin{aligned} \rho^i : \Delta^{n+1} &\longrightarrow \Delta^n \\ (e_0, \dots, e_{n+1}) &\longmapsto (e_0, \dots, e_{i-1}, e_i + e_{i+1}, \dots, e_{n+1}). \end{aligned}$$

Using these two maps, we may define the following maps on singular chains for each $0 \leq i \leq n$:

$$\begin{aligned} \partial_i : X_n &\longrightarrow X_{n-1} \\ \sigma &\longmapsto \sigma \circ d^i \end{aligned}$$

and

$$\begin{aligned} s_i : X_n &\longrightarrow X_{n+1} \\ \sigma &\longmapsto \sigma \circ \rho^i. \end{aligned}$$

The maps ∂_i and s_i are called the face and degeneracy maps of X, respectively.

Lemma 2.0.2. *The face and degeneracy maps of $X = S_*(Z)$ satisfies the following identities (called simplicial identities):*

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \textit{ if } i < j \\ s_i s_j &= s_{j+1} s_i \textit{ if } i \leq j \\ \partial_i s_j &= \begin{cases} s_{j-1} \partial_i & \textit{if } i < j, \\ \mathrm{id} & \textit{if } i = j, j+1, \\ s_j\, \partial_{i-1} & \textit{if } i > j+1. \end{cases} \end{aligned}$$

Proof. It follows at once that we have to show the following co-simplicial identities:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \text{ if } i < j \\ \rho^j \rho^i &= \rho^i \rho^{j+1} \text{ if } i \leq j \\ \rho^j d^i &= \begin{cases} d^i \rho^{j-1} & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j, j+1, \\ d^{i-1} \rho^j & \text{if } i > j+1. \end{cases} \end{aligned}$$

These are immediate from the definitions. $\square$

This motivates the following definition.

Definition 2.0.3 (**Simplicial set**)**.** A simplicial set is a sequence of sets $X = \{X_n\}_{n\geq 0}$ together with maps one for each $0 \leq i \leq n$:

$$\partial_i : X_n \longrightarrow X_{n-1} \;\&\; s_i : X_n \longrightarrow X_{n+1}$$

which satisfies the simplicial identities as stated in Lemma 2.0.2. A simplicial map $f : X \to Y$ is a collection of maps $\{f_n : X_n \to Y_n\}_{n\geq 0}$ which are natural w.r.t. face and degeneracy:

$$f_{n-1}\,\partial_i = \partial_i\, f_n \;\&\; f_{n+1}s_i = s_i f_n.$$

We hence get a category of simplicial sets and simplicial maps, denoted $\mathrm{s}\mathcal{S}\mathrm{et}$.

Remark 2.0.4. By Lemma 2.0.2, $S_*(Z)$ for any space Z is a simplicial set and it thus follows that we have a functor

$$S : \mathcal{T}\mathrm{op} \longrightarrow \mathrm{s}\mathcal{S}\mathrm{et}\,.$$

One of the main goals of this paper is to establish that upto homotopy, this map loses no further information.

Remark 2.0.5 (The Kan filler condition)**.** Let Δ^n be the standard topological n-simplex. Note that

$$\partial_i\Delta^n = \mathrm{Im}\left(d^i\right) = \{(e_0,\ldots,e_{i-1},0,e_i,\ldots,e_{n-1}) \mid (e_0,\ldots,e_{n-1}) \in \Delta^{n-1}\}.$$

We define the k^{th}-horn of Δ^n for $0 \leq k \leq n$ as

$$\Lambda^n_k := \bigcup_{i\neq k}^{n} \partial_i\Delta^n,$$

that is, the union of all the faces of Δ^n except the one opposite to k^{th}-vertex. Note that we have n many inclusion maps one for each $0 \leq i \leq n$ and $i \neq k$

$$\begin{aligned} \iota_i : \Delta^{n-1} &\longrightarrow \Lambda^n_k \subseteq \Delta^n \\ (e_0,\ldots,e_{n-1}) &\longmapsto (e_0,\ldots,e_{i-1},0,e_i,\ldots,e_{n-1}). \end{aligned}$$

An important observation is that there is a retraction

$$r_k : \Delta^n \twoheadrightarrow \Lambda^n_k.$$

Indeed, consider a line passing through the k^{th}-vertex $v_k = (0,\ldots,0,1,0,\ldots,0)$ and pick a point on it outside the simplex Δ^n, say p. Using p, define r_k on $x \in \Delta^n$ as that point on Λ^n_k which is obtained by intersection of the horn with the line joining p and x. This map is clearly identity on the horn.

It follows that for any space and any map $\sigma : \Lambda^n_k \to Z$, composing with r_k gives a singular n-simplex $\sigma \circ r_k : \Delta^n \to Z$. If $\tau_0,\ldots,\tau_{k-1},\tau_{k+1},\ldots,\tau_n \in S_{n-1}(Z)$ are n many $n-1$-singular simplices such that they can glue to form a map from the k^{th}-horn $\Lambda^n_k \to Z$, then by above discussion it would follow that we get an n-simplex $\tau \in S_n(Z)$.

We wish to rigorously state the last condition on gluing n many $n-1$-simplices to a horn Λ^n_k.

Lemma 2.0.6. *Let $\tau_0, \ldots, \tau_{k-1}, \tau_{k+1}, \ldots, \tau_n \in S_{n-1}(Z)$ be n many $n-1$-singular simplices of space Z and $0 \leq k \leq n$. Then the following are equivalent:*

1. *The simplices τ_i glues to a map $\tau : \Lambda^n_k \to Z$ where $\tau\iota_i = \tau_i$ for $0 \leq i \leq n$ and $i \neq k$.*
2. *The simplices τ_i satisfies the following conditions:*

$$\partial_i \, \tau_j = \partial_{j-1} \, \tau_i \text{ for } i < j, i \neq k, j \neq k.$$

Proof. (1. $\Rightarrow$ 2.) We observe that $\partial_i \, \tau_j = \tau\iota_j d^i$ and $\partial_{j-1} \, \tau_i = \tau\iota_i d^{j-1}$. Hence we need only show that

$$\iota_j d^i = \iota_i d^{j-1}.$$

This is a simple check.

(2. $\Rightarrow$ 1.) Define maps on the image of each ι_i by τ_i:

$$\begin{aligned} \tilde{\tau}_i : \mathrm{Im}\,(\iota_i) &\longrightarrow Z \\ (e_0, \ldots, e_{i-1}, 0, e_i, \ldots, e_{n-1}) &\longmapsto \tau_i(e_0, \ldots, e_{i-1}, e_i, \ldots, e_{n-1}). \end{aligned}$$

By pasting lemma, we need only check that for $\tilde{\tau}_i, \tilde{\tau}_j$, $i < j$, we have

$$\tilde{\tau}_i|_{\mathrm{Im}(\iota_j)} = \tilde{\tau}_j|_{\mathrm{Im}(\iota_i)}.$$

Pick $p \in \mathrm{Im}\,(\iota_i) \cap \mathrm{Im}\,(\iota_j)$. Then $p = (p_0, \ldots, p_n)$ where $p_i = p_j = 0$. Hence, we have by definitions that

$$\begin{aligned} \tilde{\tau}_i(p) &= \tau_i(p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_n) \\ &= \tau_i d^{j-1}(p_0, \ldots, p_{i-1}, p_{i+1}, p_{j-1}, p_{j+1}, \ldots, p_n) \\ &= \tau_j d^i(p_0, \ldots, p_{i-1}, p_{i+1}, p_{j-1}, p_{j+1}, \ldots, p_n) \\ &= \tau_j(p_0, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n) \\ &= \tilde{\tau}_j(p_0, \ldots, p_{i-1}, p_{i+1}, p_{j-1}, p_j, p_{j+1}, \ldots, p_n) \\ &= \tilde{\tau}_j(p), \end{aligned}$$

as required. ∎

This motivates the following condition.

Definition 2.0.7 (**Horns, Kan extension condition & Kan complexes**)**.** Let X be a simplicial set. An (n,k)-horn for $0 \leq k \leq n$ is a collection of n many $n-1$-simplices $x_0, \ldots, x_{k-1}, x_k, \ldots, x_n \in X_{n-1}$ such that for all $i < j$, $i \neq k$, $j \neq k$, we have

$$\partial_i \, x_j = \partial_{j-1} \, x_i.$$

The simplicial set X is said to satisfy the Kan extension condition if for all (n,k)-horns $\{x_i\}$ of X, there exists an n-simplex $x \in X_n$ such that for all $i \neq k$,

$$\partial_i x = x_i.$$

A simplicial set satisfying Kan extension condition is called a Kan complex, or sometimes an ∞-groupoid.

Corollary 2.0.8. *For any space Z, the simplicial set $S_*(Z)$ is a Kan complex.* □

Remark 2.0.9. By the above result, one may consequently study Kan complexes in themselves, thinking of them as a generalization of spaces. This is a fruitful endeavour, which ends with one establishing that homotopy theory of Kan complexes is "same" as that of CW-complexes.

We next wish to establish a more functorial way of constructing simplicial sets.

Definition 2.0.10 (Ordinal category). Let $\mathbf{\Delta}$ be the category whose objects are defined as

$$[n] := 0 < 1 < 2 < \cdots < n$$

the toset of first n non-negative integers and maps $f : [n] \to [m]$ are defined to be monotone. There are two distinguished classes of maps for each n and $0 \leq i \leq n$:

$$d^i : [n-1] \longrightarrow [n] \;\&\; \rho^i : [n+1] \longrightarrow [n]$$

where

$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases} \;\&\; \rho^i(k) = \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i. \end{cases}$$

These maps are called coface and codegeneracy maps, respectively.

An important aspect of the category $\mathbf{\Delta}$ is that all monotone maps can be generated by coface and codegeneracy maps.

Remark 2.0.11. Let $f : [n] \to [m]$ be a monotone map. Observe that if $i \in [m]$ is such that $f^{-1}(i)$ is of size l, then by monotonicity, we must have $f(k) = f(k+1) = \cdots = f(k+l-1) = i$. Observe that f partitions n via its fibers. Let $\{i_0, \ldots, i_k\}$ be the ordered image of f and let $n_p = \left|f^{-1}(i_p)\right|$. Consequently, we may consider the monotone map $g : [n] \to [k]$ where $g(f^{-1}(i_p)) = \{p\}$ for each $0 \leq p \leq k$. Clearly, a composition of certain cofaces d^i will give a map $d_f : [k] \to [m]$ such that $d_f g = f$. It hence suffices to show that g can be written as a composite of certain codegeneracies ρ^i. To this end, by induction it suffices to show that the map $a : [n] \to [0]$ is a composite of codegeneracies. Such a composite is given by $a = \rho^0 \ldots \rho^{n-2}\rho^{n-1}$.

Now if one wishes to define a functor $F : \mathbf{\Delta} \longrightarrow \mathcal{C}$, then by Remark 2.0.11, it is sufficient to define F only on the cofaces and codegeneracies. The following is a simple observation from the definitions.

Lemma 2.0.12. *The coface and codegeneracy maps d^i and ρ^j satisfies the cosimplicial identities of Lemma 2.0.2.* □

Lemma 2.0.13. *The following are equivalent:*

1. *X is a simplicial set.*
2. *X is a presheaf*

$$X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$$

Consequently, $\mathrm{s}\mathcal{S}\mathrm{et}$ *is equivalent to the category of presheaves of sets over* $\mathbf{\Delta}$*.*

Proof. (1. $\Rightarrow$ 2.) We define a functor

$$\begin{aligned}[n] &\mapsto X_n \\ d^i &\mapsto \partial_i \\ \rho^i &\mapsto s_i.\end{aligned}$$

The fact that is indeed a functor follows from the decomposition of a map in $\mathbf{\Delta}$ into composition of cofaces followed by codegeneracies as in Remark 2.0.11.

(2. $\Rightarrow$ 1.) Define $X_n = X([n])$ and $\partial_i = X(d^i)$ and $s_i = X(\rho^i)$. Then, $\{X_n, \partial_i, s_i\}$ is a simplicial set by Lemma 2.0.12. □

Definition 2.0.14 (**Simplicial object**)**.** Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a presheaf $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{C}$. Equivalently, its a sequence of objects $\{X_n\}$ of $\mathcal{C}$ together with arrows $\partial_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying the simplicial identities. The category of simplicial objects in $\mathcal{C}$ is denoted by $s\mathcal{C}$.

Our main goal in the rest of this section is to develop basic homotopy theory as for topological spaces, but for Kan complexes. In particular, one of our aim is to define and study homotopy groups of a Kan complex. Also recall that studying homotopy theory in topological spaces amounts to studying fibrations and cofibrations of spaces. We will introduce those notions in the simplicial setting, using which we will establish classical results on homotopy theory in the simplicial setting.

2.1 Homotopy groups of a Kan complex