SIMPLICIAL SETS, REALIZATION & THE BAR-COBAR CONSTRUCTIONS

Contents

Date: January 22, 2025.

1 Introduction

2 Simplicial sets

Let Z be a topological space. One way to understand Z is via understanding all the singular chains $S_*(Z)$ of Z and how they relate to each other.

Remark 2.0.1 (Face & degeneracy maps for $S_*(Z)$)**.** There are natural functions one can define on $X = S_*(Z)$ by using the combinatorics of the standard n-simplex Δ^n. Recall that

$$\Delta^n := \{(e_0, \ldots, e_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} e_i = 1, e_i \geq 0\}.$$

Consequently, we have maps

$$\begin{aligned} d^i : \Delta^{n-1} &\longrightarrow \Delta^n \\ (e_0, \ldots, e_{n-1}) &\longmapsto (e_0, \ldots, e_{i-1}, 0, e_i, \ldots, e_{n-1}) \end{aligned}$$

and

$$\begin{aligned} \rho^i : \Delta^{n+1} &\longrightarrow \Delta^n \\ (e_0, \ldots, e_{n+1}) &\longmapsto (e_0, \ldots, e_{i-1}, e_i + e_{i+1}, \ldots, e_{n+1}). \end{aligned}$$

Using these two maps, we may define the following maps on singular chains for each $0 \leq i \leq n$:

$$\begin{aligned} \partial_i : X_n &\longrightarrow X_{n-1} \\ \sigma &\longmapsto \sigma \circ d^i \end{aligned}$$

and

$$\begin{aligned} s_i : X_n &\longrightarrow X_{n+1} \\ \sigma &\longmapsto \sigma \circ \rho^i. \end{aligned}$$

The maps ∂_i and s_i are called the face and degeneracy maps of X, respectively.

Lemma 2.0.2. *The face and degeneracy maps of $X = S_*(Z)$ satisfies the following identities (called simplicial identities):*

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \text{ if } i < j \\ s_i s_j &= s_{j+1} s_i \text{ if } i \leq j \\ \partial_i s_j &= \begin{cases} s_{j-1} \partial_i & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j, j+1, \\ s_j \, \partial_{i-1} & \text{if } i > j+1. \end{cases} \end{aligned}$$

Proof. It follows at once that we have to show the following co-simplicial identities:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \text{ if } i < j \\ \rho^j \rho^i &= \rho^i \rho^{j+1} \text{ if } i \leq j \\ \rho^j d^i &= \begin{cases} d^i \rho^{j-1} & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j, j+1, \\ d^{i-1} \rho^j & \text{if } i > j+1. \end{cases} \end{aligned}$$

These are immediate from the definitions. □

This motivates the following definition.

Definition 2.0.3 (**Simplicial set**)**.** A simplicial set is a sequence of sets $X = \{X_n\}_{n\geq 0}$ together with maps one for each $0 \leq i \leq n$:

$$\partial_i : X_n \longrightarrow X_{n-1} \;\&\; s_i : X_n \longrightarrow X_{n+1}$$

which satisfies the simplicial identities as stated in Lemma 2.0.2. A simplicial map $f : X \to Y$ is a collection of maps $\{f_n : X_n \to Y_n\}_{n\geq 0}$ which are natural w.r.t. face and degeneracy:

$$f_{n-1}\,\partial_i = \partial_i\, f_n \;\&\; f_{n+1}s_i = s_i f_n.$$

We hence get a category of simplicial sets and simplicial maps, denoted $\mathrm{s}\mathcal{S}\mathrm{et}$.

Remark 2.0.4. By Lemma 2.0.2, $S_*(Z)$ for any space Z is a simplicial set and it thus follows that we have a functor

$$S : \mathcal{T}\mathrm{op} \longrightarrow \mathrm{s}\mathcal{S}\mathrm{et}\,.$$

One of the main goals of this paper is to establish that upto homotopy, this map loses no further information.

Remark 2.0.5 (The Kan filler condition)**.** Let Δ^n be the standard topological n-simplex. Note that

$$\partial_i\Delta^n = \mathrm{Im}\left(d^i\right) = \{(e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) \mid (e_0, \dots, e_{n-1}) \in \Delta^{n-1}\}.$$

We define the k^{th}-horn of Δ^n for $0 \leq k \leq n$ as

$$\Lambda^n_k := \bigcup_{i\neq k}^{n} \partial_i\Delta^n,$$

that is, the union of all the faces of Δ^n except the one opposite to k^{th}-vertex. Note that we have n many inclusion maps one for each $0 \leq i \leq n$ and $i \neq k$

$$\begin{aligned} \iota_i : \Delta^{n-1} &\longrightarrow \Lambda^n_k \subseteq \Delta^n \\ (e_0, \dots, e_{n-1}) &\longmapsto (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}). \end{aligned}$$

An important observation is that there is a retraction

$$r_k : \Delta^n \twoheadrightarrow \Lambda^n_k.$$

Indeed, consider a line passing through the k^{th}-vertex $v_k = (0, \dots, 0, 1, 0, \dots, 0)$ and pick a point on it outside the simplex Δ^n, say p. Using p, define r_k on $x \in \Delta^n$ as that point on Λ^n_k which is obtained by intersection of the horn with the line joining p and x. This map is clearly identity on the horn.

It follows that for any space and any map $\sigma : \Lambda^n_k \to Z$, composing with r_k gives a singular n-simplex $\sigma \circ r_k : \Delta^n \to Z$. If $\tau_0, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n \in S_{n-1}(Z)$ are n many $n-1$-singular simplices such that they can glue to form a map from the k^{th}-horn $\Lambda^n_k \to Z$, then by above discussion it would follow that we get an n-simplex $\tau \in S_n(Z)$.

We wish to rigorously state the last condition on gluing n many $n-1$-simplices to a horn Λ^n_k.

Lemma 2.0.6. *Let $\tau_0, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n \in S_{n-1}(Z)$ be n many $n-1$-singular simplices of space Z and $0 \leq k \leq n$. Then the following are equivalent:*

1. *The simplices τ_i glues to a map $\tau : \Lambda^n_k \to Z$ where $\tau\iota_i = \tau_i$ for $0 \leq i \leq n$ and $i \neq k$.*
2. *The simplices τ_i satisfies the following conditions:*

$$\partial_i\,\tau_j = \partial_{j-1}\,\tau_i \text{ for } i < j, i \neq k, j \neq k.$$

Proof. (1. $\Rightarrow$ 2.) We observe that $\partial_i \tau_j = \tau\iota_j d^i$ and $\partial_{j-1} \tau_i = \tau\iota_i d^{j-1}$. Hence we need only show that

$$\iota_j d^i = \iota_i d^{j-1}.$$

This is a simple check.

(2. $\Rightarrow$ 1.) Define maps on the image of each ι_i by τ_i:

$$\begin{aligned}\tilde{\tau}_i : \mathrm{Im}\,(\iota_i) &\longrightarrow Z \\ (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) &\longmapsto \tau_i(e_0, \dots, e_{i-1}, e_i, \dots, e_{n-1}).\end{aligned}$$

By pasting lemma, we need only check that for $\tilde{\tau}_i, \tilde{\tau}_j$, $i < j$, we have

$$\tilde{\tau}_i|_{\mathrm{Im}(\iota_j)} = \tilde{\tau}_j|_{\mathrm{Im}(\iota_i)}\,.$$

Pick $p \in \mathrm{Im}\,(\iota_i) \cap \mathrm{Im}\,(\iota_j)$. Then $p = (p_0, \dots, p_n)$ where $p_i = p_j = 0$. Hence, we have by definitions that

$$\begin{aligned}\tilde{\tau}_i(p) &= \tau_i(p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_j, p_{j+1}, \dots, p_n) \\ &= \tau_i d^{j-1}(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_{j+1}, \dots, p_n) \\ &= \tau_j d^i(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_{j+1}, \dots, p_n) \\ &= \tau_j(p_0, \dots, p_{i-1}, p_i, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n) \\ &= \tilde{\tau}_j(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_j, p_{j+1}, \dots, p_n) \\ &= \tilde{\tau}_j(p),\end{aligned}$$

as required. $\square$

This motivates the following condition.

Definition 2.0.7 (**Horns, Kan extension condition & Kan complexes**). Let X be a simplicial set. An (n, k)-horn for $0 \leq k \leq n$ is a collection of n many $n-1$-simplices $x_0, \dots, x_{k-1}, x_k, \dots, x_n \in X_{n-1}$ such that for all $i < j$, $i \neq k$, $j \neq k$, we have

$$\partial_i\, x_j = \partial_{j-1}\, x_i.$$

The simplicial set X is said to satisfy the Kan extension condition if for all (n, k)-horns $\{x_i\}$ of X, there exists an n-simplex $x \in X_n$ such that for all $i \neq k$,

$$\partial_i x = x_i.$$

A simplicial set satisfying Kan extension condition is called a Kan complex, or sometimes an ∞-groupoid.

The following result follows at once from Remark 2.0.5.

Corollary 2.0.8. *For any space Z, the simplicial set $S_*(Z)$ is a Kan complex.* $\square$

Remark 2.0.9. By the above result, one may consequently study Kan complexes in themselves, thinking of them as a generalization of spaces. This is a fruitful endeavour, which ends with one establishing that homotopy theory of Kan complexes is "same" as that of CW-complexes.

We next wish to establish a more functorial way of constructing simplicial sets.

Definition 2.0.10 (**Ordinal category**)**.** Let $\mathbf{\Delta}$ be the category whose objects are defined as

$$[n] := 0 < 1 < 2 < \cdots < n$$

the toset of first n non-negative integers and maps $f : [n] \to [m]$ are defined to be monotone. There are two distinguished classes of maps for each n and $0 \leq i \leq n$:

$$d^i : [n-1] \longrightarrow [n] \;\&\; \rho^i : [n+1] \longrightarrow [n]$$

where

$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases} \;\&\; \rho^i(k) = \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i. \end{cases}$$

These maps are called coface and codegeneracy maps, respectively.

An important aspect of the category $\mathbf{\Delta}$ is that all monotone maps can be generated by coface and codegeneracy maps.

Remark 2.0.11. Let $f : [n] \to [m]$ be a monotone map. Observe that if $i \in [m]$ is such that $f^{-1}(i)$ is of size l, then by monotonicity, we must have $f(k) = f(k+1) = \cdots = f(k+l-1) = i$. Observe that f partitions n via its fibers. Let $\{i_0, \ldots, i_k\}$ be the ordered image of f and let $n_p = \left|f^{-1}(i_p)\right|$. Consequently, we may consider the monotone map $g : [n] \to [k]$ where $g(f^{-1}(i_p)) = \{p\}$ for each $0 \leq p \leq k$. Clearly, a composition of certain cofaces d^i will give a map $d_f : [k] \to [m]$ such that $d_f g = f$. It hence suffices to show that g can be written as a composite of certain codegeneracies ρ^i. To this end, by induction it suffices to show that the map $a : [n] \to [0]$ is a composite of codegeneracies. Such a composite is given by $a = \rho^0 \ldots \rho^{n-2}\rho^{n-1}$.

Now if one wishes to define a functor $F : \mathbf{\Delta} \longrightarrow \mathcal{C}$, then by Remark 2.0.11, it is sufficient to define F only on the cofaces and codegeneracies. The following is a simple observation from the definitions.

Lemma 2.0.12. *The coface and codegeneracy maps d^i and ρ^j satisfies the cosimplicial identities of Lemma 2.0.2.* □

Lemma 2.0.13. *The following are equivalent:*

(1) *X is a simplicial set.*
(2) *X is a presheaf*

$$X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$$

Consequently, $\mathrm{s}\mathcal{S}\mathrm{et}$ *is equivalent to the category of presheaves of sets over $\mathbf{\Delta}$.*

Proof. (1. $\Rightarrow$ 2.) We define a functor

$$[n] \mapsto X_n$$
$$d^i \mapsto \partial_i$$
$$\rho^i \mapsto s_i.$$

The fact that is indeed a functor follows from the decomposition of a map in $\mathbf{\Delta}$ into composition of cofaces followed by codegeneracies as in Remark 2.0.11.

(2. $\Rightarrow$ 1.) Define $X_n = X([n])$ and $\partial_i = X(d^i)$ and $s_i = X(\rho^i)$. Then, $\{X_n, \partial_i, s_i\}$ is a simplicial set by Lemma 2.0.12. □

Definition 2.0.14 (**Simplicial object**)**.** Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a presheaf $X : \mathbf{\Delta}^{\text{op}} \longrightarrow \mathcal{C}$. Equivalently, its a sequence of objects $\{X_n\}$ of $\mathcal{C}$ together with arrows $\partial_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying the simplicial identities. The category of simplicial objects in $\mathcal{C}$ is denoted by $s\mathcal{C}$.

Our main goal in the rest of this section is to develop basic homotopy theory as for topological spaces, but for Kan complexes. In particular, one of our aim is to define and study homotopy groups of a Kan complex. Also recall that studying homotopy theory in topological spaces amounts to studying fibrations and cofibrations of spaces. We will introduce those notions in the simplicial setting, using which we will establish classical results on homotopy theory in the simplicial setting.

2.1 Homotopy of simplices in a Kan complex. Let Z be a space and $X = S_*(Z)$ be its singular simplicial set, which we now know is a Kan-complex by Corollary 2.0.8. We want a notion of homotopy of two n-simplices purely in terms of simplices of X. The following definition is thus made.

Definition 2.1.1 (**Homotopy of simplices**)**.** Let X be a simplicial set and $x, x' \in X_n$ be two n-simplices. Further suppose that they satisfy the compatibility condition: $\partial_i\, x = \partial_i\, x'$ for each $0 \leq i \leq n$. Then, x and x' are said to be homotopic if there exists an $n+1$-simplex $y \in X_n$ such that

$$\begin{aligned}\partial_n\, y &= x,\\ \partial_{n+1}\, y &= x',\\ \partial_i\, y &= s_{n-1}\,\partial_i\, x = s_{n-1}\,\partial_i\, x' \;\forall\; 0 \leq i \leq n-1.\end{aligned}$$

Here is the lemma which recovers the usual notion of homotopy when $X = S_*(Z)$.

Proposition 2.1.2. *Let Z be a space and $X = S_*(Z)$. If two n-simplices $x, x' \in X_n$ are homotopic, then there exists $H : \Delta^n \times I \to Z$ such that*

$$\begin{aligned}H_0 &= x\\ H_1 &= x'\\ \partial_i\, H_t &= \partial_i\, x = \partial_i\, x' \;\forall\; 0 \leq i \leq n.\end{aligned}$$

Proof. There exists an $n+1$-simplex $y : \Delta^{n+1} \to Z$ such that the conditions of Definition 2.1.1 holds. We define H as follows:

$$\begin{aligned}H : \Delta^n \times I &\longrightarrow Z\\ ((e_0, \dots, e_n), t) &\longmapsto y(e_0, \dots, e_{n-1}, te_n, e_n - te_n).\end{aligned}$$

This is clearly a continuous map. Indeed, we have that $H_0 = \partial_n\, y = x$ and $H_1 = \partial_{n+1}\, y = x'$. Furthermore, for any $0 \leq i \leq n-1$, we have

$$\begin{aligned}\partial_i\, H_t(e_0, \dots, e_{n-1}) &= H_t d^i(e_0, \dots, e_{n-1})\\ &= y(e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1})\\ &= yd^i(e_0, \dots, e_{n-1}) = \partial_i\, y(e_0, \dots, e_{n-1})\\ &= x(e_0, \dots, e_{n-1}) = x'(e_0, \dots, e_{n-1}).\end{aligned}$$

Finally, to see that $\partial_n H_t = \partial_n x = \partial_n x'$, we simply observe the following:

$$\begin{aligned}\partial_n H_t(e_0, \dots, e_{n-1}) &= H_t(e_0, \dots, e_{n-1}, 0) = y(e_0, \dots, e_{n-1}, 0, 0)\\ &= \partial_n y(e_0, \dots, e_{n-1}, 0) = x(e_0, \dots, e_{n-1}, 0)\\ &= \partial_{n+1} y(e_0, \dots, e_{n-1}, 0) = x'(e_0, \dots, e_{n-1}, 0),\end{aligned}$$

as required. □

Proposition 2.1.3. *Let X be a Kan complex. Then the relation of homotopy of pairs of compatible n-simplices is an equivalence relation for all $n \geq 0$.*

Proof. Reflexivity is clear as for $x \in X_n$, we may take the homotopy to be $y = s_n x$ and verify by the simplicial identities that y is indeed a homotopy $x \sim x$. We will show symmetry and transitivity in one go by showing the following: $x \sim x'$ and $x \sim x''$ implies $x' \sim x''$. It is easy to see that both symmetry and transitivity follows from this. We now prove this implication.

Let $y' : x \sim x'$ and $y'' : x \sim x''$. Thus,

$$\begin{aligned}\partial_n y' = x,\ \partial_{n+1} y' = x',\ \partial_i y' = s_{n-1} \partial_i x = s_{n-1} \partial_i x' \ \forall\ 0 \leq i \leq n-1\\ \partial_n y'' = x,\ \partial_{n+1} y'' = x'',\ \partial_i y'' = s_{n-1} \partial_i x = s_{n-1} \partial_i x'' \ \forall\ 0 \leq i \leq n-1.\end{aligned}$$

As X is a Kan complex, we will employ the Kan condition to obtain the required homotopy. Consider the $n+2$ many $n+1$-simplices $\{z_i\}_{i=0}^{n+1}$ where

$$z_i = \begin{cases} \partial_i s_n s_n x' & \text{if } 0 \leq i \leq n-1,\\ y' & \text{if } i = n,\\ y'' & \text{if } i = n+1.\end{cases}$$

We claim that $\{z_i\}$ forms an $(n+2, n+2)$-horn. Indeed, we need only check $\partial_i z_j = \partial_{j-1} z_i$ for $i < j < n+2$. We may check this case by case.

(1) If $0 \leq i < j \leq n-1$, then we have

$$\partial_i z_j = \partial_i \partial_j s_n s_n x' = \partial_{j-1} \partial_i s_n s_n x' = \partial_{j-1} z_i.$$

(2) If $0 \leq i \leq n-1$ and $n \leq j \leq n+1$, then we have (we show for $j = n$, for $j = n+1$, same observation will work)

$$\partial_i z_j = \partial_i y' = s_{n-1} \partial_i x'$$

and

$$\partial_{j-1} z_i = \partial_{n-1} \partial_i s_n s_n x' = \partial_{n-1} s_{n-1} \partial_i s_n x' = \partial_i s_n x' = s_{n-1} \partial_i x',$$

as required.

(3) If $i = n$ and $j = n+1$, then

$$\partial_n z_{n+1} = \partial_n y'' = x = \partial_n z_n.$$

Hence by Kan condition on X, there exists an $n+2$-simplex z such that $\partial_i z = z_i$ for all $i \neq n+2$. Let $h = \partial_{n+2} z$. We claim that $h : x' \sim x''$. Indeed, observe that for $0 \leq i \leq n-1$, we have

$$\begin{aligned}\partial_n h &= \partial_n \partial_{n+2} z = \partial_{n+1} \partial_n z = \partial_{n+1} z_n = \partial_{n+1} y' = x'\\ \partial_{n+1} h &= \partial_{n+1} \partial_{n+2} z = \partial_{n+1} \partial_{n+1} z = \partial_{n+1} z_{n+1} = \partial_{n+1} y'' = x''\\ \partial_i h &= \partial_i \partial_{n+2} z = \partial_{n+1} \partial_i z = \partial_{n+1} \partial_i s_n s_n x' = \partial_{n+1} s_{n-1} s_{n-1} \partial_i x' = s_{n-1} \partial_i x' = s_{n-1} \partial_i x''.\end{aligned}$$

This completes the proof. □

Recall that for a space Z, the notion of homotopy relative to a subspace $L \subseteq Z$ is needed to define relative homotopy groups. We hence define homotopy of two n-simplices relative to a sub-simplicial set L of X.

Definition 2.1.4 (**Relative homotopy**). Let X be a simplicial set and $L \subseteq X$ be a sub-simplicial set. Two n-simplices $x, x' \in X_n$ are said to be homotopic rel L if $\partial_i x = \partial_i x'$ for all $1 \le i \le n$, $\partial_0 x, \partial_0 x' \in L_{n-1}$ and there is an $n+1$-simplex $w \in K_{n+1}$ such that

$$\partial_n w = x,\ \partial_{n+1} w = x',\ \partial_i w = s_{n-1} \partial_i x = s_{n-1} \partial_i x' \ \forall\ 0 \le i \le n-1$$

where furthermore $\partial_0 w \in L_n$ and is a homotopy between $\partial_0 x$ and $\partial_0 x'$.

By the same technique of Proposition 2.1.3 (constructing an appropriate $(n+2, n+2)$-horn in X), one can show the following (for more details, see [**?**]).

Proposition 2.1.5. *Let X be a Kan complex and $L \subseteq X$ be a sub-Kan complex. The relation of homotopic rel L is an equivalence relation.*

Any subset of simplices $S_n \subseteq X_n$ of a simplicial set X defines a unique sub-simplicial set of X.

Definition 2.1.6 (**Generated sub-simplicial set**). Let X be a simplicial set and $S_n \subseteq X_n$ be a subset for each $n \ge 0$. The simplicial set generated by $\{S_n\}_{n\ge 0}$ is the smallest sub-simplicial set $\tilde{S}$ of X such that $\tilde{S}_n \supseteq S_n$ for each $n \ge 0$.

We next show that this relation is well-behaved with respect to maps of simplicial sets.

Lemma 2.1.7. *Let $f : (X, K) \to (Y, L)$ be a simplicial map of pairs and $x, x' \in K_n$ for some $n \ge 0$. If $x \sim x'$ rel K, then $f(x) \sim f(x')$ rel L.*

Proof. Let $w\ inX_{n+1}$ be a homotopy rel K between x and x'. We claim that $f(w) \in Y_{n+1}$ is a homotopy rel L for $f(x)$ and $f(x')$. Indeed, $\partial_n f_{n+1}(w) = f_n \partial_n(w) = f_n(x)$, $\partial_{n+1} f_{n+1}(w) = f_n \partial_{n+1}(w) = f_n(x')$ and for $0 \le i \le n-1$, we have

$$\partial_i f_{n+1}(w) = f_n \partial_i(w) = f_n(s_{n-1} \partial_i x) = s_{n-1} f_{n-1}(\partial_i x) = s_{n-1} \partial_i f_n(x).$$

Similarly, one shows that $\partial_i f_{n+1}(w) = s_{n-1} \partial_i f_n(x')$. We need only show that $\partial_0 f(w) \in L_n$ and is a homotopy between $\partial_0 f(x)$ and $\partial_0 f(x')$. This also follows from similar steps as above. ☐

2.2 Homotopy classes of simplices. We now define homotopy groups of a Kan complex. To be able to derive the homotopy long exact sequence, we will define relative homotopy groups as well.

Definition 2.2.1 (**Kan triples & homotopy classes**). Let K be a Kan complex and $\phi \in K_0$ be a vertex. Then (K, ϕ) is called a pointed Kan complex. We identify ϕ with $\tilde{\phi}$, the sub-simplicial set generated by ϕ. Note that $\tilde{\phi}$ will have exactly one simplex in each dimension. A Kan triple is a tuple (K, L, ϕ) where $L \subseteq K$ is a sub-Kan complex and $\phi \in L_0$ is a vertex. Define $\partial(K, \phi)_n := \{x \in K_n \mid \partial_i x = \phi,\ 0 \le i \le n\}$ and

$$\pi_n(K, \phi) := \partial(K, \phi)_n / \sim$$

where $x \sim x'$ is the homotopy equivalence relation. Similarly, we may define $\partial(K, L, \phi)_n = \{x \in K_n \mid \partial_0 x \in L_{n-1}, \partial_i x = \phi,\ 1 \le i \le n\}$ and thus

$$\pi_n(K, L, \phi) := \partial(K, L, \phi)_n / \sim$$

where $x \sim x'$ is the homotopy rel L equivalence relation. Note that $[\phi] \in \pi_n(K, L, \phi)$ is a distinguished element, making $\pi_n(K, L, \phi)$ a pointed set. By Lemma 2.1.7, a simplicial map gives rise to a map on π_n.

Remark 2.2.2. Recall that in the case of usual spaces, a pair (X, A) is well-behaved homologically if it is a cofibration; $i : A \to X$ satisfies homotopy extension property. It turns out that in sSet, notion of subcomplex is sufficient for pairs (K, L) to be well-behaved. In-fact, we will see that subcomplexes exactly forms the subcategory of cofibrations for a model category structure on sSet. The hard part will be to study fibrations in sSet.

Our goal now is to derive the analog of homotopy long exact sequence of pairs for a Kan triple.

Theorem 2.2.3. *Let (K, L, ϕ) be a Kan triple with inclusions $i : (L, \phi) \hookrightarrow (K, \phi)$ and $j : (K, \phi, \phi) \hookrightarrow (K, L, \phi)$. Then there is a long exact sequence of sets induced by inclusions:*

$$\cdots \to \pi_{n+1}(K, L, \phi) \to \pi_n(L, \phi) \xrightarrow{i} \pi_n(K, \phi) \xrightarrow{j} \pi_n(K, L, \phi) \to \cdots$$

where ∂ is defined as

$$\begin{aligned} \partial : \pi_{n+1}(K, L, \phi) &\to \pi_n(L, \phi) \\ [x] &\mapsto [\partial_0 x]. \end{aligned}$$

Proof. As all other proofs use similar ideas (finding the right horn to fill using Kan condition), we show the exactness at $\pi_n(L, \phi)$. We first show $i\,\partial = \phi$. Pick $x \in \partial(K, L, \phi)_{n+1}$. To show $\partial_0 x \in L_n$ is null-homotopic in K_n, i.e. there is a homotopy $w : \partial_0 x \sim \phi$ rel ϕ in K. We construct a Λ^{n+2}_0-horn, whose 0^{th}-face will be the required homotopy. Indeed, it is easy to see that the $n + 2$ many $n + 1$-simplices

$$\{\phi, \phi, \ldots, \phi, x\}$$

satisfy the Kan condition for K and thus gives a $z \in K_{n+2}$ such that $\partial_i z = \phi$ for $1 \leq i \leq n + 1$ and $\partial_{n+2} z = x$. Let $w = \partial_0 z$. We claim that it is the required homotopy. Indeed, we have for $0 \leq i \leq n - 1$ the following

$$\begin{aligned} \partial_n w &= \partial_n \partial_0 z = \partial_0 \partial_{n+1} z = \phi \\ \partial_{n+1} w &= \partial_{n+1} \partial_0 z = \partial_0 \partial_{n+2} z = \partial_0 x \\ \partial_i w &= \partial_i \partial_0 z = \partial_0 \partial_{i+1} z = \phi, \end{aligned}$$

as required.

Next, we show that $\mathrm{Ker}\,(i) \subseteq \mathrm{Im}\,(\partial)$. Pick $x \in \partial(L, \phi)_n$ such that $x \in K_n$ is null homotopic rel ϕ. This gives a homotopy $w \in K_{n+1}$ such that $w : x \sim \phi$ rel ϕ. We wish to construct $y \in \partial(K, L, \phi)_{n+1}$ such that $\partial_0 y \sim x$ rel ϕ in L. Consider the following $n + 2$ many $n + 1$-simplices of K

$$\{w, \phi, \phi, \ldots, \phi\}$$

where, it is easily established that they form a Λ^{n+2}_{n+2}, giving rise to $z \in K_{n+2}$. Let $y = \partial_{n+2} z$. We claim that $y \in \partial(K, L, \phi)_{n+1}$ and $\partial_0 y \sim x$ rel ϕ in L. Indeed, we see that for $1 \leq i \leq n + 1$

$$\begin{aligned} \partial_0 y &= \partial_0 \partial_{n+2} z = \partial_{n+1} \partial_0 z \, \partial_{n+1} w = x \\ \partial_i y &= \partial_i \partial_{n+2} z = \partial_{n+1} \partial_i z = \phi, \end{aligned}$$

as required. $\square$

2.3 The simplicial sphere. We construct analogs of important spaces in topology to simplicial sets.

Construction 2.3.1 (Standard simplicial sets)**.** Recall that if Δ^n is the topological n-simplex, then $\Delta^n \cong D^n$ and $\partial\,\Delta^n \cong S^{n-1}$. Consequently, $(\Delta^n, \partial\,\Delta^n) \cong (D^n, S^{n-1})$ as pairs. We first generalize the notion of D^n to simplicial sets. Indeed, consider the simplicial set given by

$$\Delta^n := h_{[n]},$$

where $h_{[n]} \in \mathrm{s}\mathcal{S}\mathrm{et}$ is the representable functor on $\mathbf{\Delta}$ determined by $[n]$

$$\begin{aligned} h_{[n]} : \mathbf{\Delta}^{\mathrm{op}} &\longrightarrow \mathcal{S}\mathrm{et} \\ [m] &\longmapsto \mathrm{Hom}_{\mathbf{\Delta}}\left([m],[n]\right). \end{aligned}$$

The face and degeneracy maps are clear from the definition; they are hom-duals of coface and codegeneracy maps. We call Δ^n the standard n-simplicial set and they play the role of n-disc in simplicial sets.

There is a very important description of simplices of Δ^n, which is often very useful.

Lemma 2.3.2. *Let us denote*

$$\mathrm{Inc}(m,n) := \{(a_0, \ldots, a_m) \mid 0 \le a_0 \le \cdots \le a_m \le n\}.$$

Then there is a bijection

$$\mathrm{Inc}(m,n) \cong \Delta^n(m).$$

Under this identification, we have

$$\begin{aligned} \partial_i(a_0, \ldots, a_m) &= (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) \\ s_i(a_0, \ldots, a_m) &= (a_0, \ldots, a_i, a_i, a_{i+2}, \ldots, a_m). \end{aligned}$$

Proof. Take any m-simplex of Δ^n, say x. Then $x \in \mathrm{Hom}_{\mathbf{\Delta}}\left([m],[n]\right)$, i.e. x is a monotone map $x : [m] \to [n]$. Let $a_i = x(i)$ for each $0 \le i \le m$. Then $(a_0, \ldots, a_m)$ forms the required element of $\mathrm{Inc}(m,n)$. Converse is also easy. As $\partial_i\, x = x \circ d^i$ and $s_i(x) = x \circ \rho^i$, the other statement follows at once. □

We would like to identify $\mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}\left(\Delta^n, X\right)$ for a simplicial set X. Yoneda lemma tells us what we need.

Lemma 2.3.3. *Let X be a simplicial set. Then the following data are equivalent:*

(1) *$x \in X_n$ is an n-simplex of X.*
(2) *$x : \Delta^n \to X$ is a simplicial map.*

Proof. It follows from Yoneda lemma that the map

$$\begin{aligned} \varphi : \mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}\left(\Delta^n, X\right) &\longrightarrow X_n \\ x &\longmapsto x_n(\mathrm{id}) \end{aligned}$$

is a bijection. More explicitly, if $x \in X_n$, we may define a map $f : \Delta^n \to X$ by

$$\begin{aligned} f_m : \Delta^n(m) &\longrightarrow X_m \\ \vec{a} = (a_0, \ldots, a_m) &\longmapsto X(\vec{a})(x) \end{aligned}$$

where $X(\vec{a}) = X_n \to X_m$, the map induced by X on $\vec{a} : [m] \to [n]$. □

2.4 Homotopy of maps. Our goal in the next few sections is to revisit the notion of homotopy of simplices and show a type of coherence result that will yield that homotopy classes of maps from the simplicial n-sphere to a Kan complex X is indeed in bijection with homotopy classes of n-simplices. Let us begin by homotopy between two simplicial maps.

Definition 2.4.1 (**Simplicial homotopy**)**.** Let K, L be two simplicial sets and $f, g : K \to L$ be two simplicial maps. A homotopy h from f to g is a collection of maps $\{h_q^i : K_q \to L_{q+1}\}_{0 \leq i \leq q, q \geq 0}$ which satisfy the following identities:

$$\partial_0 \, h_0 = f, \; \partial_{q+1} \, h_q = g$$

$$\partial_i \, h_j = \begin{cases} h_{j-1} \, \partial_i & \text{if } i < j \\ \partial_i \, h_{i-1} & \text{if } i = j \\ \partial_i \, h_i & \text{if } i = j+1 \\ h_j \, \partial_{i-1} & \text{if } i > j+1 \end{cases}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j \\ h_j s_{i-1} & \text{if } i > j. \end{cases}$$

The usual notions of homotopy of pairs and deformation retracts are immediate.

Remark 2.4.2 (On homotopy being an equivalence relation)**.** An important aspect of this notion is the question whether it is an equivalence relation on set of simplicial maps $K \to L$. One can understand this best by trying to interpret a homotopy as a 1-simplex in the hypothetic simplicial set of all simplicial maps, L^K. Indeed, we can define this simplicial set quite easily. Denote

$$(K^L)_q := \mathrm{Hom}_{\mathrm{sSet}} \left(K \times \Delta^q, L\right)$$

together with $\partial_i(f) := f \circ (\mathrm{id} \times d^i)$ and $s_i(f) := f \circ (\mathrm{id} \times \rho^i)$ for some $f : K \times \Delta^q \to L$. Clearly, this makes L^K a simplicial set. What does homotopy being an equivalence relation translates to L^K?

Lemma 2.4.3. *If L^K*

2.5 Homotopy groups.