SIMPLICIAL SETS, REALIZATION & THE BAR-COBAR CONSTRUCTIONS

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1. Introduction

2. Simplicial sets

Let Z be a topological space. One way to understand Z is via understanding all the singular chains $S_*(Z)$ of Z and how they relate to each other.

Remark 2.0.1 (Face & degeneracy maps for $S_*(Z)$)**.** There are natural functions one can define on $X = S_*(Z)$ by using the combinatorics of the standard n-simplex Δ^n. Recall that

$$\Delta^n := \{(e_0, \dots, e_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} e_i = 1, e_i \geq 0\}.$$

Consequently, we have maps

$$\begin{aligned} d^i : \Delta^{n-1} &\longrightarrow \Delta^n \\ (e_0, \dots, e_{n-1}) &\longmapsto (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) \end{aligned}$$

and

$$\begin{aligned} \rho^i : \Delta^{n+1} &\longrightarrow \Delta^n \\ (e_0, \dots, e_{n+1}) &\longmapsto (e_0, \dots, e_{i-1}, e_i + e_{i+1}, \dots, e_{n+1}). \end{aligned}$$

Using these two maps, we may define the following maps on singular chains for each $0 \leq i \leq n$:

$$\begin{aligned} \partial_i : X_n &\longrightarrow X_{n-1} \\ \sigma &\longmapsto \sigma \circ d^i \end{aligned}$$

and

$$\begin{aligned} s_i : X_n &\longrightarrow X_{n+1} \\ \sigma &\longmapsto \sigma \circ \rho^i. \end{aligned}$$

The maps ∂_i and s_i are called the face and degeneracy maps of X, respectively.

Date: January 29, 2025.

Lemma 2.0.2. *The face and degeneracy maps of $X = S_*(Z)$ satisfies the following identities (called simplicial identities):*

$$\partial_i\partial_j = \partial_{j-1}\partial_i \text{ if } i < j$$
$$s_i s_j = s_{j+1}s_i \text{ if } i \le j$$
$$\partial_i s_j = \begin{cases} s_{j-1}\partial_i & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j, j+1, \\ s_j\,\partial_{i-1} & \text{if } i > j+1. \end{cases}$$

Proof. It follows at once that we have to show the following co-simplicial identities:

$$d^j d^i = d^i d^{j-1} \text{ if } i < j$$
$$\rho^j \rho^i = \rho^i \rho^{j+1} \text{ if } i \le j$$
$$\rho^j d^i = \begin{cases} d^i \rho^{j-1} & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j, j+1, \\ d^{i-1}\rho^j & \text{if } i > j+1. \end{cases}$$

These are immediate from the definitions. □

This motivates the following definition.

Definition 2.0.3 (Simplicial set). A simplicial set is a sequence of sets $X = \{X_n\}_{n\ge 0}$ together with maps one for each $0 \le i \le n$:

$$\partial_i : X_n \longrightarrow X_{n-1} \;\&\; s_i : X_n \longrightarrow X_{n+1}$$

which satisfies the simplicial identities as stated in Lemma 2.0.2. A simplicial map $f : X \to Y$ is a collection of maps $\{f_n : X_n \to Y_n\}_{n\ge 0}$ which are natural w.r.t. face and degeneracy:

$$f_{n-1}\,\partial_i = \partial_i\, f_n \;\&\; f_{n+1}s_i = s_i f_n.$$

We hence get a category of simplicial sets and simplicial maps, denoted $\mathrm{s}\mathcal{S}\mathrm{et}$.

Remark 2.0.4. By Lemma 2.0.2, $S_*(Z)$ for any space Z is a simplicial set and it thus follows that we have a functor

$$S : \mathcal{T}\mathrm{op} \longrightarrow \mathrm{s}\mathcal{S}\mathrm{et}\,.$$

One of the main goals of this paper is to establish that upto homotopy, this map loses no further information.

Remark 2.0.5 (The Kan filler condition)**.** Let Δ^n be the standard topological n-simplex. Note that

$$\partial_i\Delta^n = \mathrm{Im}\left(d^i\right) = \{(e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) \mid (e_0, \dots, e_{n-1}) \in \Delta^{n-1}\}.$$

We define the k^{th}-horn of Δ^n for $0 \le k \le n$ as

$$\Lambda^n_k := \bigcup_{i\ne k}^{n} \partial_i\Delta^n,$$

that is, the union of all the faces of Δ^n except the one opposite to k^{th}-vertex. Note that we have n many inclusion maps one for each $0 \leq i \leq n$ and $i \neq k$

$$\begin{aligned}\iota_i : \Delta^{n-1} &\longrightarrow \Lambda^n_k \subseteq \Delta^n \\ (e_0, \dots, e_{n-1}) &\longmapsto (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}).\end{aligned}$$

An important observation is that there is a retraction

$$r_k : \Delta^n \twoheadrightarrow \Lambda^n_k.$$

Indeed, consider a line passing through the k^{th}-vertex $v_k = (0, \dots, 0, 1, 0, \dots, 0)$ and pick a point on it outside the simplex Δ^n, say p. Using p, define r_k on $x \in \Delta^n$ as that point on Λ^n_k which is obtained by intersection of the horn with the line joining p and x. This map is clearly identity on the horn.

It follows that for any space and any map $\sigma : \Lambda^n_k \to Z$, composing with r_k gives a singular n-simplex $\sigma \circ r_k : \Delta^n \to Z$. If $\tau_0, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n \in S_{n-1}(Z)$ are n many $n-1$-singular simplices such that they can glue to form a map from the k^{th}-horn $\Lambda^n_k \to Z$, then by above discussion it would follow that we get an n-simplex $\tau \in S_n(Z)$.

We wish to rigorously state the last condition on gluing n many $n-1$-simplices to a horn Λ^n_k.

Lemma 2.0.6. *Let $\tau_0, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n \in S_{n-1}(Z)$ be n many $n-1$-singular simplices of space Z and $0 \leq k \leq n$. Then the following are equivalent:*

1. *The simplices τ_i glues to a map $\tau : \Lambda^n_k \to Z$ where $\tau\iota_i = \tau_i$ for $0 \leq i \leq n$ and $i \neq k$.*
2. *The simplices τ_i satisfies the following conditions:*
$$\partial_i \tau_j = \partial_{j-1} \tau_i \text{ for } i < j, i \neq k, j \neq k.$$

Proof. (1. $\Rightarrow$ 2.) We observe that $\partial_i \tau_j = \tau\iota_j d^i$ and $\partial_{j-1} \tau_i = \tau\iota_i d^{j-1}$. Hence we need only show that

$$\iota_j d^i = \iota_i d^{j-1}.$$

This is a simple check.

(2. $\Rightarrow$ 1.) Define maps on the image of each ι_i by τ_i:

$$\begin{aligned}\tilde{\tau}_i : \operatorname{Im}(\iota_i) &\longrightarrow Z \\ (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) &\longmapsto \tau_i(e_0, \dots, e_{i-1}, e_i, \dots, e_{n-1}).\end{aligned}$$

By pasting lemma, we need only check that for $\tilde{\tau}_i, \tilde{\tau}_j$, $i < j$, we have

$$\tilde{\tau}_i|_{\operatorname{Im}(\iota_j)} = \tilde{\tau}_j|_{\operatorname{Im}(\iota_i)}.$$

Pick $p \in \operatorname{Im}(\iota_i) \cap \operatorname{Im}(\iota_j)$. Then $p = (p_0, \dots, p_n)$ where $p_i = p_j = 0$. Hence, we have by definitions that

$$\begin{aligned}\tilde{\tau}_i(p) &= \tau_i(p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_j, p_{j+1}, \dots, p_n) \\ &= \tau_i d^{j-1}(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_{j+1}, \dots, p_n) \\ &= \tau_j d^i(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_{j+1}, \dots, p_n) \\ &= \tau_j(p_0, \dots, p_{i-1}, p_i, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n) \\ &= \tilde{\tau}_j(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_j, p_{j+1}, \dots, p_n) \\ &= \tilde{\tau}_j(p),\end{aligned}$$

as required. □

This motivates the following condition.

Definition 2.0.7 (**Horns, Kan extension condition & Kan complexes**). Let X be a simplicial set. An (n,k)-horn for $0 \le k \le n$ is a collection of n many $n-1$-simplices $x_0, \dots, x_{k-1}, x_k, \dots, x_n \in X_{n-1}$ such that for all $i < j$, $i \neq k$, $j \neq k$, we have

$$\partial_i \, x_j = \partial_{j-1} \, x_i.$$

The simplicial set X is said to satisfy the Kan extension condition if for all (n,k)-horns $\{x_i\}$ of X, there exists an n-simplex $x \in X_n$ such that for all $i \neq k$,

$$\partial_i x = x_i.$$

A simplicial set satisfying Kan extension condition is called a Kan complex, or sometimes an ∞-groupoid.

The following result follows at once from Remark 2.0.5.

Corollary 2.0.8. *For any space Z, the simplicial set $S_*(Z)$ is a Kan complex.* □

Remark 2.0.9. By the above result, one may consequently study Kan complexes in themselves, thinking of them as a generalization of spaces. This is a fruitful endeavour, which ends with one establishing that homotopy theory of Kan complexes is "same" as that of CW-complexes.

We next wish to establish a more functorial way of constructing simplicial sets.

Definition 2.0.10 (**Ordinal category**). Let $\mathbf{\Delta}$ be the category whose objects are defined as

$$[n] := 0 < 1 < 2 < \cdots < n$$

the toset of first n non-negative integers and maps $f : [n] \to [m]$ are defined to be monotone. There are two distinguished classes of maps for each n and $0 \le i \le n$:

$$d^i : [n-1] \longrightarrow [n] \;\&\; \rho^i : [n+1] \longrightarrow [n]$$

where

$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \ge i \end{cases} \;\&\; \rho^i(k) = \begin{cases} k & \text{if } k \le i \\ k-1 & \text{if } k > i. \end{cases}$$

These maps are called coface and codegeneracy maps, respectively.

An important aspect of the category $\mathbf{\Delta}$ is that all monotone maps can be generated by coface and codegeneracy maps.

Remark 2.0.11. Let $f : [n] \to [m]$ be a monotone map. Observe that if $i \in [m]$ is such that $f^{-1}(i)$ is of size l, then by monotonicity, we must have $f(k) = f(k+1) = \cdots = f(k+l-1) = i$. Observe that f partitions n via its fibers. Let $\{i_0, \dots, i_k\}$ be the ordered image of f and let $n_p = \left|f^{-1}(i_p)\right|$. Consequently, we may consider the monotone map $g : [n] \to [k]$ where $g(f^{-1}(i_p)) = \{p\}$ for each $0 \le p \le k$. Clearly, a composition of certain cofaces d^i will give a map $d_f : [k] \to [m]$ such that $d_f g = f$. It hence suffices to show that g can be written as a composite of certain codegeneracies ρ^i. To this end, by induction it suffices to show that the map $a : [n] \to [0]$ is a composite of codegeneracies. Such a composite is given by $a = \rho^0 \dots \rho^{n-2} \rho^{n-1}$.

Now if one wishes to define a functor $F : \mathbf{\Delta} \longrightarrow \mathcal{C}$, then by Remark 2.0.11, it is sufficient to define F only on the cofaces and codegeneracies. The following is a simple observation from the definitions.

Lemma 2.0.12. *The coface and codegeneracy maps d^i and ρ^j satisfies the cosimplicial identities of Lemma 2.0.2.* □

Lemma 2.0.13. *The following are equivalent:*

(1) *X is a simplicial set.*
(2) *X is a presheaf*

$$X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$$

Consequently, $s\mathcal{S}\mathrm{et}$ *is equivalent to the category of presheaves of sets over $\mathbf{\Delta}$.*

Proof. (1. ⇒ 2.) We define a functor

$$[n] \mapsto X_n$$
$$d^i \mapsto \partial_i$$
$$\rho^i \mapsto s_i.$$

The fact that is indeed a functor follows from the decomposition of a map in $\mathbf{\Delta}$ into composition of cofaces followed by codegeneracies as in Remark 2.0.11.

(2. ⇒ 1.) Define $X_n = X([n])$ and $\partial_i = X(d^i)$ and $s_i = X(\rho^i)$. Then, $\{X_n, \partial_i, s_i\}$ is a simplicial set by Lemma 2.0.12. □

Definition 2.0.14 (Simplicial object). Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a presheaf $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{C}$. Equivalently, its a sequence of objects $\{X_n\}$ of $\mathcal{C}$ together with arrows $\partial_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying the simplicial identities. The category of simplicial objects in $\mathcal{C}$ is denoted by $s\mathcal{C}$.

Remark 2.0.15 (Fiber products and quotients)**.** Fiber product of simplicial sets exists. Indeed, recall that in a presheaf category, the limits and colimits exists and are defined pointwise. Consequently, if $f : X \to Z$ and $g : Y \to Z$ are two simplicial maps, then the fiber product $X \times_Z Y$ is a simplicial set whose set of n-simplices is $X_n \times_{Z_n} Y_n$ via the maps $f_n : X_n \to Z_n$ and $g_n : Y_n \to Z_n$. Furthermore, the face and degeneracy maps are given by those of X and Y applied componentwise.

If $Z \subseteq X$ is a sub-simplicial set, then we can define a simplicial set X/Z whose set of n-simplices is X_n/Z_n together with the face and degeneracies which descends from X. Note that there is a simplicial map $q : X \to X/Z$.

Our main goal in the rest of this section is to develop basic homotopy theory as for topological spaces, but for Kan complexes. In particular, one of our aim is to define and study homotopy groups of a Kan complex. Also recall that studying homotopy theory in topological spaces amounts to studying fibrations and cofibrations of spaces. We will introduce those notions in the simplicial setting, using which we will establish classical results on homotopy theory in the simplicial setting.

2.1. **Homotopy of simplices in a Kan complex.** Let Z be a space and $X = S_*(Z)$ be its singular simplicial set, which we now know is a Kan-complex by Corollary 2.0.8. We want a notion of homotopy of two n-simplices purely in terms of simplices of X. The following definition is thus made.

Definition 2.1.1 (Homotopy of simplices). Let X be a simplicial set and $x, x' \in X_n$ be two n-simplices. Further suppose that they satisfy the compatibility condition: $\partial_i\, x = \partial_i\, x'$ for each

$0 \leq i \leq n$. Then, x and x' are said to be homotopic if there exists an $n+1$-simplex $y \in X_n$ such that

$$\begin{aligned} \partial_n y &= x, \\ \partial_{n+1} y &= x', \\ \partial_i y &= s_{n-1} \partial_i x = s_{n-1} \partial_i x' \;\forall\; 0 \leq i \leq n-1. \end{aligned}$$

Here is the lemma which recovers the usual notion of homotopy when $X = S_*(Z)$.

Proposition 2.1.2. *Let Z be a space and $X = S_*(Z)$. If two n-simplices $x, x' \in X_n$ are homotopic, then there exists $H : \Delta^n \times I \to Z$ such that*

$$\begin{aligned} H_0 &= x \\ H_1 &= x' \\ \partial_i H_t &= \partial_i x = \partial_i x' \;\forall\; 0 \leq i \leq n. \end{aligned}$$

Proof. There exists an $n+1$-simplex $y : \Delta^{n+1} \to Z$ such that the conditions of Definition 2.1.1 holds. We define H as follows:

$$\begin{aligned} H : \Delta^n \times I &\longrightarrow Z \\ ((e_0, \ldots, e_n), t) &\longmapsto y(e_0, \ldots, e_{n-1}, te_n, e_n - te_n). \end{aligned}$$

This is clearly a continuous map. Indeed, we have that $H_0 = \partial_n y = x$ and $H_1 = \partial_{n+1} y = x'$. Furthermore, for any $0 \leq i \leq n$, we have

$$\begin{aligned} \partial_i H_t(e_0, \ldots, e_{n-1}) &= H_t d^i(e_0, \ldots, e_{n-1}) \\ &= y(e_0, \ldots, e_{i-1}, 0, e_i, \ldots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) \\ &= yd^i(e_0, \ldots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) = \partial_i y(e_0, \ldots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) \\ &= s_{n-1} \partial_i x(e_0, \ldots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) \\ &= xd^i \rho^{n-1}(e_0, \ldots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) \\ &= xd^i(e_0, \ldots, e_{n-2}, e_{n-1}) = \partial_i x(e_0, \ldots, e_{n-1}) \end{aligned}$$

and similarly for x'. Finally, to see that $\partial_n H_t = \partial_n x = \partial_n x'$, we simply observe the following:

$$\begin{aligned} \partial_n H_t(e_0, \ldots, e_{n-1}) &= H_t(e_0, \ldots, e_{n-1}, 0) = y(e_0, \ldots, e_{n-1}, 0, 0) \\ &= \partial_n y(e_0, \ldots, e_{n-1}, 0) = x(e_0, \ldots, e_{n-1}, 0) \\ &= \partial_{n+1} y(e_0, \ldots, e_{n-1}, 0) = x'(e_0, \ldots, e_{n-1}, 0), \end{aligned}$$

as required. □

Proposition 2.1.3. *Let X be a Kan complex. Then the relation of homotopy of pairs of compatible n-simplices is an equivalence relation for all $n \geq 0$.*

Proof. Reflexivity is clear as for $x \in X_n$, we may take the homotopy to be $y = s_n x$ and verify by the simplicial identities that y is indeed a homotopy $x \sim x$. We will show symmetry and transitivity in one go by showing the following: $x \sim x'$ and $x \sim x''$ implies $x' \sim x''$. It is easy to see that both symmetry and transitivity follows from this. We now prove this implication.

Let $y' : x \sim x'$ and $y'' : x \sim x''$. Thus,

$$\begin{aligned} \partial_n y' = x,\ \partial_{n+1} y' = x',\ \partial_i y' = s_{n-1} \partial_i x = s_{n-1} \partial_i x' \;\forall\; 0 \leq i \leq n-1 \\ \partial_n y'' = x,\ \partial_{n+1} y'' = x'',\ \partial_i y'' = s_{n-1} \partial_i x = s_{n-1} \partial_i x'' \;\forall\; 0 \leq i \leq n-1. \end{aligned}$$

As X is a Kan complex, we will employ the Kan condition to obtain the required homotopy. Consider the $n+2$ many $n+1$-simplices $\{z_i\}_{i=0}^{n+1}$ where

$$z_i = \begin{cases} \partial_i\, s_n s_n x' & \text{if } 0 \leq i \leq n-1, \\ y' & \text{if } i = n, \\ y'' & \text{if } i = n+1. \end{cases}$$

We claim that $\{z_i\}$ forms an $(n+2, n+2)$-horn. Indeed, we need only check $\partial_i\, z_j = \partial_{j-1}\, z_i$ for $i < j < n+2$. We may check this case by case.

(1) If $0 \leq i < j \leq n-1$, then we have

$$\partial_i\, z_j = \partial_i\, \partial_j\, s_n s_n x' = \partial_{j-1}\, \partial_i\, s_n s_n x' = \partial_{j-1}\, z_i.$$

(2) If $0 \leq i \leq n-1$ and $n \leq j \leq n+1$, then we have (we show for $j = n$, for $j = n+1$, same observation will work)

$$\partial_i\, z_j = \partial_i\, y' = s_{n-1}\, \partial_i\, x'$$

and

$$\partial_{j-1}\, z_i = \partial_{n-1}\, \partial_i\, s_n s_n x' = \partial_{n-1}\, s_{n-1}\, \partial_i\, s_n x' = \partial_i\, s_n x' = s_{n-1}\, \partial_i\, x',$$

as required.

(3) If $i = n$ and $j = n+1$, then

$$\partial_n\, z_{n+1} = \partial_n\, y'' = x = \partial_n\, z_n.$$

Hence by Kan condition on X, there exists an $n+2$-simplex z such that $\partial_i\, z = z_i$ for all $i \neq n+2$. Let $h = \partial_{n+2}\, z$. We claim that $h : x' \sim x''$. Indeed, observe that for $0 \leq i \leq n-1$, we have

$$\begin{aligned} \partial_n\, h &= \partial_n\, \partial_{n+2}\, z = \partial_{n+1}\, \partial_n\, z = \partial_{n+1}\, z_n = \partial_{n+1}\, y' = x' \\ \partial_{n+1}\, h &= \partial_{n+1}\, \partial_{n+2}\, z = \partial_{n+1}\, \partial_{n+1}\, z = \partial_{n+1}\, z_{n+1} = \partial_{n+1}\, y'' = x'' \\ \partial_i\, h &= \partial_i\, \partial_{n+2}\, z = \partial_{n+1}\, \partial_i\, z = \partial_{n+1}\, \partial_i\, s_n s_n x' = \partial_{n+1}\, s_{n-1} s_{n-1}\, \partial_i\, x' = s_{n-1}\, \partial_i\, x' = s_{n-1}\, \partial_i\, x''. \end{aligned}$$

This completes the proof. □

Recall that for a space Z, the notion of homotopy relative to a subspace $L \subseteq Z$ is needed to define relative homotopy groups. We hence define homotopy of two n-simplices relative to a sub-simplicial set L of X.

Definition 2.1.4 (**Relative homotopy**)**.** Let X be a simplicial set and $L \subseteq X$ be a sub-simplicial set. Two n-simplices $x, x' \in X_n$ are said to be homotopic rel L if $\partial_i\, x = \partial_i\, x'$ for all $1 \leq i \leq n$, $\partial_0\, x, \partial_0\, x' \in L_{n-1}$ and there is an $n+1$-simplex $w \in K_{n+1}$ such that

$$\partial_n\, w = x,\ \partial_{n+1}\, w = x',\ \partial_i\, w = s_{n-1}\, \partial_i\, x = s_{n-1}\, \partial_i\, x'\ \forall\ 0 \leq i \leq n-1$$

where furthermore $\partial_0\, w \in L_n$ and is a homotopy between $\partial_0\, x$ and $\partial_0\, x'$.

By the same technique of Proposition 2.1.3 (constructing an appropriate $(n+2, n+2)$-horn in X), one can show the following (for more details, see [**?**]).

Proposition 2.1.5. *Let X be a Kan complex and $L \subseteq X$ be a sub-Kan complex. The relation of homotopic rel L is an equivalence relation.*

Any subset of simplices $S_n \subseteq X_n$ of a simplicial set X defines a unique sub-simplicial set of X.

Definition 2.1.6 (**Generated sub-simplicial set**). Let X be a simplicial set and $S_n \subseteq X_n$ be a subset for each $n \geq 0$. The simplicial set generated by $\{S_n\}_{n\geq 0}$ is the smallest sub-simplicial set $\tilde{S}$ of X such that $\tilde{S}_n \supseteq S_n$ for each $n \geq 0$.

Remark 2.1.7. Now let Y be a simplicial set and $\tilde{S}$ be as above. To define a simplicial map $f : \tilde{S} \to Y$, it is sufficient to define maps $f_n : S_n \to Y_n$ and extend f_n to $\tilde{S}_n$ inductively as follows. Suppose f_n is defined on $x \in X_n$, then set $f_{n-1}(\partial_i\, x) = \partial_i\, f_n(x)$ and $f_{n+1}(s_i x) = s_i(f_n(x))$. Indeed, as any other element of $\tilde{S}_n$ is obtained by composites of faces and degeneracies on some element of S_k, hence one can extend $\{f_n\}$ to a map $f : \tilde{S} \to Y$.

We next show that this relation is well-behaved with respect to maps of simplicial sets.

Lemma 2.1.8. *Let $f : (X, K) \to (Y, L)$ be a simplicial map of pairs and $x, x' \in K_n$ for some $n \geq 0$. If $x \sim x'$ rel K, then $f(x) \sim f(x')$ rel L.*

Proof. Let $w\ inX_{n+1}$ be a homotopy rel K between x and x'. We claim that $f(w) \in Y_{n+1}$ is a homotopy rel L for $f(x)$ and $f(x')$. Indeed, $\partial_n\, f_{n+1}(w) = f_n\, \partial_n(w) = f_n(x)$, $\partial_{n+1}\, f_{n+1}(w) = f_n\, \partial_{n+1}(w) = f_n(x')$ and for $0 \leq i \leq n-1$, we have

$$\partial_i\, f_{n+1}(w) = f_n\, \partial_i(w) = f_n(s_{n-1}\, \partial_i\, x) = s_{n-1} f_{n-1}(\partial_i\, x) = s_{n-1}\, \partial_i\, f_n(x).$$

Similarly, one shows that $\partial_i\, f_{n+1}(w) = s_{n-1}\, \partial_i\, f_n(x')$. We need only show that $\partial_0\, f(w) \in L_n$ and is a homotopy between $\partial_0\, f(x)$ and $\partial_0\, f(x')$. This also follows from similar steps as above. $\square$

2.2. **Homotopy classes of simplices.** We now define homotopy groups of a Kan complex. To be able to derive the homotopy long exact sequence, we will define relative homotopy groups as well.

Definition 2.2.1 (**Kan triples & homotopy classes**). Let K be a Kan complex and $\phi \in K_0$ be a vertex. Then (K, ϕ) is called a pointed Kan complex. We identify ϕ with $\tilde{\phi}$, the sub-simplicial set generated by ϕ. Note that $\tilde{\phi}$ will have exactly one simplex in each dimension. A Kan triple is a tuple (K, L, ϕ) where $L \subseteq K$ is a sub-Kan complex and $\phi \in L_0$ is a vertex. Define $\partial(K, \phi)_n := \{x \in K_n \mid \partial_i\, x = \phi,\ 0 \leq i \leq n\}$ and

$$\pi_n(K, \phi) := \partial(K, \phi)_n / \sim$$

where $x \sim x'$ is the homotopy equivalence relation. Similarly, we may define $\partial(K, L, \phi)_n = \{x \in K_n \mid \partial_0\, x \in L_{n-1}, \partial_i\, x = \phi,\ 1 \leq i \leq n\}$ and thus

$$\pi_n(K, L, \phi) := \partial(K, L, \phi)_n / \sim$$

where $x \sim x'$ is the homotopy rel L equivalence relation. Note that $[\phi] \in \pi_n(K, L, \phi)$ is a distinguished element, making $\pi_n(K, L, \phi)$ a pointed set. By Lemma 2.1.8, a simplicial map gives rise to a map on π_n.

Remark 2.2.2. Recall that in the case of usual spaces, a pair (X, A) is well-behaved homologically if it is a cofibration; $i : A \to X$ satisfies homotopy extension property. It turns out that in $\mathbb{S}$et, notion of subcomplex is sufficient for pairs (K, L) to be well-behaved. In-fact, we will see that subcomplexes exactly forms the subcategory of cofibrations for a model category structure on s$\mathbb{S}$et. The hard part will be to study fibrations in s$\mathbb{S}$et.

Our goal now is to derive the analog of homotopy long exact sequence of pairs for a Kan triple.

Theorem 2.2.3. *Let (K, L, ϕ) be a Kan triple with inclusions $i : (L, \phi) \hookrightarrow (K, \phi)$ and $j : (K, \phi, \phi) \hookrightarrow (K, L, \phi)$. Then there is a long exact sequence of sets induced by inclusions:*

$$\cdots \to \pi_{n+1}(K, L, \phi) \to \pi_n(L, \phi) \xrightarrow{i} \pi_n(K, \phi) \xrightarrow{j} \pi_n(K, L, \phi) \to \cdots$$

where ∂ is defined as

$$\begin{aligned}\partial : \pi_{n+1}(K, L, \phi) &\to \pi_n(L, \phi)\\ [x] &\mapsto [\partial_0\, x].\end{aligned}$$

Proof. As all other proofs use similar ideas (finding the right horn to fill using Kan condition), we show the exactness at $\pi_n(L, \phi)$. We first show $i\,\partial = \phi$. Pick $x \in \partial(K, L, \phi)_{n+1}$. To show $\partial_0\, x \in L_n$ is null-homotopic in K_n, i.e. there is a homotopy $w : \partial_0\, x \sim \phi$ rel ϕ in K. We construct a Λ^{n+2}_0-horn, whose 0^{th}-face will be the required homotopy. Indeed, it is easy to see that the $n+2$ many $n+1$-simplices

$$\{\phi, \phi, \ldots, \phi, x\}$$

satisfy the Kan condition for K and thus gives a $z \in K_{n+2}$ such that $\partial_i\, z = \phi$ for $1 \leq i \leq n+1$ and $\partial_{n+2}\, z = x$. Let $w = \partial_0\, z$. We claim that it is the required homotopy. Indeed, we have for $0 \leq i \leq n-1$ the following

$$\begin{aligned}\partial_n\, w &= \partial_n\, \partial_0\, z = \partial_0\, \partial_{n+1}\, z = \phi\\ \partial_{n+1}\, w &= \partial_{n+1}\, \partial_0\, z = \partial_0\, \partial_{n+2}\, z = \partial_0\, x\\ \partial_i\, w &= \partial_i\, \partial_0\, z = \partial_0\, \partial_{i+1}\, z = \phi,\end{aligned}$$

as required.

Next, we show that $\mathrm{Ker}\,(i) \subseteq \mathrm{Im}\,(\partial)$. Pick $x \in \partial(L, \phi)_n$ such that $x \in K_n$ is null homotopic rel ϕ. This gives a homotopy $w \in K_{n+1}$ such that $w : x \sim \phi$ rel ϕ. We wish to construct $y \in \partial(K, L, \phi)_{n+1}$ such that $\partial_0\, y \sim x$ rel ϕ in L. Consider the following $n+2$ many $n+1$-simplices of K

$$\{w, \phi, \phi, \ldots, \phi\}$$

where, it is easily established that they form a Λ^{n+2}_{n+2}, giving rise to $z \in K_{n+2}$. Let $y = \partial_{n+2}\, z$. We claim that $y \in \partial(K, L, \phi)_{n+1}$ and $\partial_0\, y \sim x$ rel ϕ in L. Indeed, we see that for $1 \leq i \leq n+1$

$$\begin{aligned}\partial_0\, y &= \partial_0\, \partial_{n+2}\, z = \partial_{n+1}\, \partial_0\, z = \partial_{n+1}\, w = x\\ \partial_i\, y &= \partial_i\, \partial_{n+2}\, z = \partial_{n+1}\, \partial_i\, z = \phi,\end{aligned}$$

as required. □

In the definition of homotopy of simplices, one may ease out the condition that the top two boundaries of the homotopy yields the simplices between which it is the boundary.

Proposition 2.2.4. *Let (K, ϕ) be a Kan pair and $h \in K_{n+1}$ be an $n+1$-simplex such that $\partial_i\, h = \phi$ for all $i \neq k, k+1$. Then there is a homotopy $\hat{h} : \partial_k\, h \sim \partial_{k+1}\, h$.*

Proof. Consider the following sequence of $n+2$ many $n+1$-simplices $(i \neq k)$

$$z_i = \begin{cases} \phi & \text{if } i \neq k+1, k+2, k+3\\ s_{k+1}\, \partial_{k+1}\, h & \text{if } i = k+1\\ h & \text{if } i = k+2\\ s_k\, \partial_{k+1}\, h & \text{if } i = k+3.\end{cases}$$

We claim that this is a Λ^{n+2}_k-horn for K. This is a simple check. Consequently, there exists an $n+2$-simplex $z \in K_{n+2}$ such that $\partial_i\, z = z_i$ for $i \neq k$. Let $\hat{h}_1 = \partial_k\, z$. Observe that for $i \neq k, k+1$, we have

$$\begin{aligned}
\partial_{k+1}\hat{h}_1 &= \partial_{k+1}\partial_k z = \partial_k\partial_{k+2} z = \partial_k h \\
\partial_{k+2}\hat{h}_1 &= \partial_{k+2}\partial_k z = \partial_k\partial_{k+3} z = \partial_k s_k \partial_{k+1} h = \partial_{k+1} h \\
\partial_i \hat{h}_1 &= \partial_i \partial_k h = \begin{cases} \partial_k \partial_{i-1} h = \phi & \text{if } i \le k-1 \\ \partial_{k+1}\partial_i h = \phi & \text{if } i \ge k+2. \end{cases}
\end{aligned}$$

Thus, $\hat{h}_1 \in K_{n+1}$ is an $n+1$-simpliex satisfying the same hypotheses as h but for k replaced by $k+1$. Inducting over k gives the required homotopy $\partial_k h \sim \partial_{k+1} h$. □

2.3. **The simplicial sphere.** We construct analogs of important spaces in topology but in simplicial sets.

Construction 2.3.1 (Standard simplicial sets)**.** Recall that if Δ^n is the topological n-simplex, then $\Delta^n \cong D^n$ and $\partial\Delta^n \cong S^{n-1}$. Consequently, $(\Delta^n, \partial\Delta^n) \cong (D^n, S^{n-1})$ as pairs. We first generalize the notion of D^n to simplicial sets. Indeed, consider the simplicial set given by

$$\Delta^n := h_{[n]},$$

where $h_{[n]} \in \mathrm{s}\mathcal{S}\mathrm{et}$ is the representable functor on $\mathbf{\Delta}$ determined by $[n]$

$$\begin{aligned}
h_{[n]} : \mathbf{\Delta}^{\mathrm{op}} &\longrightarrow \mathcal{S}\mathrm{et} \\
[m] &\longmapsto \mathrm{Hom}_{\mathbf{\Delta}}([m],[n]).
\end{aligned}$$

The face and degeneracy maps are clear from the definition; they are hom-duals of coface and codegeneracy maps. We call Δ^n the standard n-simplicial set and they play the role of n-disc in simplicial sets.

There is a very important description of simplices of Δ^n, which is often very useful.

Lemma 2.3.2. *Let us denote*

$$\mathrm{Inc}(m,n) := \{(a_0,\ldots,a_m) \mid 0 \le a_0 \le \cdots \le a_m \le n\}.$$

Then there is a bijection

$$\mathrm{Inc}(m,n) \cong \Delta^n(m).$$

Under this identification, we have

$$\begin{aligned}
\partial_i(a_0,\ldots,a_m) &= (a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_m) \\
s_i(a_0,\ldots,a_m) &= (a_0,\ldots,a_{i-1},a_i,a_i,a_{i+1},\ldots,a_m).
\end{aligned}$$

Proof. Take any m-simplex of Δ^n, say x. Then $x \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$, i.e. x is a monotone map $x : [m] \to [n]$. Let $a_i = x(i)$ for each $0 \le i \le m$. Then $(a_0,\ldots,a_m)$ forms the required element of $\mathrm{Inc}(m,n)$. Converse is also easy. As $\partial_i x = x \circ d^i$ and $s_i(x) = x \circ \rho^i$, the other statement follows at once. □

Corollary 2.3.3. *For each $n \ge 0$ and $m > n$, every m-simplex of Δ^n is degenerate.*

Proof. By Lemma 2.3.2, any m-simplex of Δ^n is of form $(a_0,\ldots,a_m)$ where $0 \le a_0 \le \cdots \le a_m \le n$. As $m > n$, thus some $a_i = a_{i+1}$, immediately leading to the simplex being degenerate. □

We would like to identify $\mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(\Delta^n, X)$ for a simplicial set X. Yoneda lemma tells us what we need.

Lemma 2.3.4. *Let X be a simplicial set. Then the following data are equivalent:*

(1) *$x \in X_n$ is an n-simplex of X.*
(2) *$\bar{x} : \Delta^n \to X$ is a simplicial map.*

Proof. It follows from Yoneda lemma that the map

$$\varphi : \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X) \longrightarrow X_n$$
$$\bar{x} \longmapsto \bar{x}_n(\mathrm{id})$$

is a bijection. More explicitly, if $x \in X_n$, we may define a map $f : \Delta^n \to X$ by

$$f_m : \Delta^n(m) \longrightarrow X_m$$
$$\vec{a} = (a_0, \dots, a_m) \longmapsto X(\vec{a})(x)$$

where $X(\vec{a}) = X_n \to X_m$, the map induced by X on $\vec{a} : [m] \to [n]$. □

Remark 2.3.5 (Δ^n is generated by id)**.** In computations with the standard n-simplices, its an important step to reduce to dealing with only one simplex of Δ^n. To this end, one observes that Δ^n is generated by $(0, 1, \dots, n) \in \Delta^n(n)$, i.e. the id : $[n] \to [n]$. This is simple to establish as by Remark 2.0.11, any $f \in \Delta^n(m)$ is composite of d^i and ρ^j. As face and degeneracies of Δ^n are just composites with d^i and ρ^j, it follows at once that any simplex of Δ^n is generated by id. Hence for purposes of many constructions with Δ^n, it is sufficient to work with the simplex id $\in \Delta^n(n)$.

An obvious question is the following: Is Δ^n a Kan complex? A small amount of thought tells no.

Remark 2.3.6 (Δ^n is not a Kan complex)**.** Choose $a_0 < b_0$ in $[n]$ and consider 1-simplices of Δ^n given by

$$z_0 = (a_0, a),\ z_1 = (b_0, a).$$

It is immediate to see that $\{z_0, z_1\}$ satisfies the $(2, 2)$-horn condition. If Δ^n is a Kan complex, then there exists a 2-simplex z of Δ^n such that $\partial_i z = z_i$ for $i = 0, 1$. Denoting $z = (p_0, p_1, p_2)$ for $0 \le p_0 \le p_1 \le p_2 \le n$, we deduce that we must have

$$(p_1, p_2) = (a_0, a),\ (p_0, p_2) = (b_0, a).$$

It follows that $p_1 = a_0$ and $p_0 = b_0$. As $p_0 \le p_1$, we must have $b_0 \le a_0$, a contradiction to our beginning assumption. This shows that Δ^n is not a Kan complex.

There are many important sub simplicial sets of Δ^n, just like there are many important subspaces of the disc D^n. We discuss few of them now.

Construction 2.3.7 (Boundaries and horns)**.** For $0 \le p \le n$, define $\partial_p \Delta^n$ to be the sub-simplicial set of Δ^n such that for $m \ge 0$, the m-simplices of $\partial_p \Delta^n$ is given by

$$(\partial_p \Delta^n)(m) := \{(a_0, \dots, a_m) \mid 0 \le a_0 \le \dots \le a_m \le n,\ a_i \ne p\ \forall i\},$$

Define $\partial \Delta^n$ to be the sub-simplicial set of Δ^n such that for $m \ge 0$, the m-simplices of $\partial \Delta^n$ is given by

$$(\partial \Delta^n)(m) := \{(a_0, \dots, a_m) \mid 0 \le a_0 \le \dots \le a_m \le n, \exists 0 \le p \le n \text{ s.t. } a_i \ne p\ \forall i\},$$
$$= \bigcup_{p=0}^{n} (\partial_p \Delta^n)(m).$$

i.e. the set of non-decreasing maps $[m] \to [n]$ which are not surjective. We hence write $\partial \Delta^n = \cup_{0 \le p \le n} \partial_p \Delta^n$ and call it the boundary simplicial set of Δ^n and $\delta_p \Delta^n$ the p^{th}-boundary simplicial set of Δ^n.

We similarly define horn Λ^n_k for $0 \leq k \leq n$ as a sub-simplicial set of Δ^n as follows

$$(\Lambda^n_k)(m) := \{(a_0, \ldots, a_m) \mid 0 \leq a_0 \leq \cdots \leq a_m \leq n,\ a_i = k \text{ for some } i\},$$

i.e. the set of non-decreasing maps $[m] \to [n]$ which always have k in the image. Note that there are n many $n-1$ non-degenerate simplices of Λ^n_k, each corresponding to the unique coface map $d^i : [n-1] \to [n]$ for $0 \leq i \leq n$, $i \neq k$. The pair $(\Delta^n, \partial\,\Delta^n)$ acts similar to the pair (D^n, S^{n-1}) in topology.

For example, we expect that an (n,k)-horn in a simplicial set X is equivalent to a map $\Lambda^n_k \to X$. Indeed, this is what we show.

Lemma 2.3.8. *Let X be a simplicial set and $0 \leq k \leq n$. Then the following are equivalent:*

(1) *The collection $\{z_0, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n\} \subseteq X_{n-1}$ forms a (n,k)-horn of X.*
(2) *There is a simplicial map $z : \Lambda^n_k \to X$ such that $z(d^i) = z_i$ for each $0 \leq i \leq n$, $i \neq k$.*

Proof. It is immediate that 1. $\Rightarrow$ 2. from the easy observation that Λ^n_k is generated by the $n-1$-simplices $\{d^i\}_{0 \leq i \leq n, i \neq k}$ and the Remark 2.1.7. For 2. $\Rightarrow$ 1., observe that for $z_i = z(d^i)$, we need only check the horn condition. This is immediate from simplicial identities. ☐

We may define simplicial n-sphere now as follows.

Definition 2.3.9 (Simplicial n-sphere). For each $n \geq 0$, the simplicial n-sphere is defined to be the quotient $\mathbb{S}^n := \Delta^n / \partial\,\Delta^n$. Observe that $\mathbb{S}^n(m) = \{\text{pt.}\}$ for $m < n$.

2.4. **Homotopy of maps.** Our goal in the next few sections is to revisit the notion of homotopy of simplices and show a type of coherence result that will yield that homotopy classes of maps from the simplicial n-sphere to a Kan complex X is indeed in bijection with homotopy classes of n-simplices. Let us begin by homotopy between two simplicial maps.

Definition 2.4.1 (Simplicial homotopy). Let K, L be two simplicial sets and $f, g : K \to L$ be two simplicial maps. A homotopy h from f to g is a collection of maps $\{h^q_i : K_q \to L_{q+1}\}_{0 \leq i \leq q, q \geq 0}$ which satisfy the following identities:

$$\partial_0\, h_0 = f,\ \partial_{q+1}\, h_q = g$$

$$\partial_i\, h_j = \begin{cases} h_{j-1}\, \partial_i & \text{if } i < j \\ \partial_i\, h_{i-1} & \text{if } i = j \\ \partial_i\, h_i & \text{if } i = j+1 \\ h_j\, \partial_{i-1} & \text{if } i > j+1 \end{cases}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j \\ h_j s_{i-1} & \text{if } i > j. \end{cases}$$

The usual notions of homotopy of pairs and deformation retracts are immediate.

While this definition is more useful, the following shows how it was arrived at.

Proposition 2.4.2. *Let $f, g : X \to Y$ be two simplicial maps. Then the following are equivalent:*

(1) *Maps f and g are homotopic.*
(2) *There exists a simplicial map $H : X \times \Delta^1 \to Y$ such that $H_0 = g$ and $H_1 = f$, where $H_0, H_1 : X \to Y$ are given on q-simplices by $x \mapsto H_q(x,0), H_q(x,1)$ respectively where $0, 1 \in \Delta^1(q)$ denotes the constant sequences.*

Proof. (1. $\Rightarrow$ 2.) Let $h_i^q : X_q \to Y_{q+1}$ be a homotopy from f to g. We construct $H : X \times \Delta^1 \to Y$ as follows on q-simplices:

$$H_q : X_q \times \Delta^1(q) \longrightarrow Y_{q+1}$$
$$(x, i) \longmapsto \begin{cases} \partial_{i+1}\, h_i^q(x) & \text{if } 0 < i \leq q, \\ \partial_0\, h_0^q(x) & \text{if } i = 0. \end{cases}$$

We denote $i \in \Delta^1(q)$ to be the sequence $(0, \ldots, 0, 1, \ldots, 1)$ where $0 \leq i \leq q$ denotes the no. of 0s in the seqeunce. Observe that H_q maps $(x, q) \mapsto \partial_{q+1}\, h_q^q(x) = g_q(x)$ and $(x, 0) \mapsto \partial_0\, h_0^q(x) = f_q(x)$, as required.

(2. $\Rightarrow$ 1.) Define $h_i^q : X_q \to Y_{q+1}$ as $x \mapsto H_{q+1}(s_i x, i{+}1)$ for $0 \leq i \leq q$. We now establish the relevant identities. First, observe that we have $\partial_0\, h_0^q(x) = \partial_0\, H_{q+1}(s_0 x, 0) = H_q(\partial_0\, s_0 x, \partial_0\, 1) = H_q(x, 0) = g_q(x)$ and $\partial_{q+1}\, h_q^q(x) = \partial_{q+1}\, H_{q+1}(s_q x, q+1) = H_q(\partial_{q+1}\, s_q x, \partial_{q+1}\, q+1) = H_q(x, q) = f_q(x)$. The remaining identities are straightforward to establish. It also is straightforward to establish that both these constructions are invertible. $\square$

Remark 2.4.3 (On homotopy being an equivalence relation)**.** An important aspect of this notion is the question whether it is an equivalence relation on set of simplicial maps $K \to L$. One can understand this best by trying to interpret a homotopy as a 1-simplex in the hypothetic simplicial set of all simplicial maps, L^K. Indeed, we can define this simplicial set quite easily. Denote

$$(L^K)_q := \mathrm{Hom}_{\mathsf{sSet}}\,(K \times \Delta^q, L)$$

together with $\partial_i(f) := f \circ (\mathrm{id} \times d^i)$ and $s_i(f) := f \circ (\mathrm{id} \times \rho^i)$ for some $f : K \times \Delta^q \to L$. Clearly, this makes L^K a simplicial set. What does homotopy being an equivalence relation translates to L^K?

Proposition 2.4.4. *Let K, L be simplicial sets and $f, g : K \to L$ be two simplicial maps.*

(1) *A homotopy $\{h_q^i : K_q \to L_{q+1}\}_{0 \leq i \leq q, q \geq 0}$ from f to g is equivalent to a 1-simplex h in L^K such that $\partial_0\, h = f$ and $\partial_1\, h = g$.*

(2) *If L^K is a Kan complex, then simplicial homotopy is an equivalence relation.*

Proof. (1). We first show how to construct a 1-simplex of L^K from a homotopy h. To this end, first observe that for $n > 1$, an n-simplex of Δ^1 is a non-decreasing seqeunce $(a_0, \ldots, a_n)$ in $\{0, 1\}$. It follows that there is a $0 \leq i \leq n$ such that $a_i = 0$ and $a_{i+1} = 1$. Consequently, we have

$$(a_0, \ldots, a_n) = s_{n-1} \ldots s_{i+1} s_{i-1} \ldots s_0(0, 1)$$

where $(0, 1) \in \Delta^1(1)$ is the unique non-degenerate 1-simplex. Consequently, we may define

$$f : K \times \Delta^1 \longrightarrow L$$

which on $n > 1$-simplices is given by (we denote $i \in \Delta^1(n)$ to be the sequence $(0, \ldots, 0, 1, \ldots, 1)$ where $0 \leq i \leq n$ denotes the no. of 0s in the seqeunce.)

$$f_n : K_n \times \Delta^1(n) \longrightarrow L_n$$
$$(x, i) \longmapsto \partial_{i+1}\, h_i^n(x).$$

We need only check naturality. It is easy to establish.

Conversely, we may begin from a 1-simplex $f : K \times \Delta^1 \to L$ and define the homotopy h as

$$h_i^q : K_q \longrightarrow L_{q+1}$$
$$x \longmapsto f_{q+1}(s_i x, i)$$

One can then check that h is indeed a simplicial homotopy.

(2). By item (1), we know that $f \simeq f$ by the 1-simplex $s_0(f) \in (L^K)_1$. Next, suppose $h : f \simeq g$ and $h' : f \simeq g'$, then we wish to construct $h'' : g \simeq g'$. Consider the collection of 2 many 1-simplices $\{h, h'\}$ of L^K. Observe that $\{h', h\}$ forms a Λ^2_2-horn. As L^K is a Kan complex, therefore there exists a 2-simplex z of L^K extending the horn. Let $h'' = \partial_2 z$. Then, $\partial_0 \partial_2 z = \partial_1 \partial_0 z = \partial_1 h' = g'$ and $\partial_1 \partial_2 z = \partial_1 \partial_1 z = \partial_1 h = g$. Thus h'' is a homotopy from g' to g, as required. $\square$

Our main goal now is to give an equivalent formulation of homotopy classes of maps as classes of simplicial homotopy from simplicial sphere to the given Kan complex. To this end, we first need to establish that the homotopy classes of simplices in both ways is same.

Lemma 2.4.5. *For a Kan pair (K, ϕ) and $x, x' \in \partial(K, \phi)_n$ two compatible n-simplices with corresponding maps being $\bar{x}, \bar{x'} : (\Delta^n, \partial \Delta^n) \to (K, \phi)$, the following are equivalent:*

1. *The simplices x,x' are homotopic.*
2. *The simplicial maps $\bar{x}, \bar{x'}$ are homotopic rel $\partial \Delta^n$.*

Proof. (1. $\Rightarrow$ 2.) Let $h \in K_{n+1}$ be a homotopy $x \sim x'$. Define a homotopy $H^q_i : \Delta^n(q) \to K_{q+1}$ for $0 \le i \le q$ as follows. First, observe that since Δ^n is generated by the identity n-simplex, thus it suffices to define the map $H^n_i : \Delta^n(n) \to K_{n+1}$ on $\mathrm{id} = (0, 1, \dots, n)$ which should satisfy the identities for homotopy for id. We define it as follows

$$H^n_i(\mathrm{id}) := \begin{cases} s_i x & \text{if } 0 \le i \le n-1 \\ h & \text{if } i = n. \end{cases}$$

It is then easy to see that H^n_i satisfies the earlier relations.

(2. $\Rightarrow$ 1.) Let $H^q_i : \Delta^n(q) \to K_{q+1}$, $0 \le i \le q$ be a homotopy from $\bar{x}$ to $\bar{x'}$. We wish to construct a homotopy $h \in K_{n+1}$ from x to x'. Recall that $\bar{x}_n(\mathrm{id}) = x$ and $\bar{x'}_n(\mathrm{id}) = x'$. Denote for each $0 \le i \le n$ the following $n+1$-simplex

$$z_i = H^n_i(\mathrm{id}).$$

Observe that since H is a homotopy rel $\partial \Delta^n$, therefore we have $\partial_i z_j = \phi$ for $i \neq j, j+1$. It follows from Proposition 2.2.4 that $\partial_i z_i \sim \partial_{i+1} z_i$ for $0 \le i \le n$. As $\partial_0 z_0 = \partial_0 H^n_0(\mathrm{id}) = \bar{x}(\mathrm{id}) = x$ and $\partial_{n+1} H^n_n(\mathrm{id}) = \bar{x'}(\mathrm{id}) = x'$, it follows that $x \sim x'$, as needed. $\square$

2.5. **Homotopy groups.**

3. Fibrations of simplicial sets