The Facets of Geometry

Topological

(Under heavy construction!!)

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Part III

The Topological Viewpoint

Chapter 10

Foundational Homotopy Theory

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We introduce basic players of homotopy theory.

Let us first engage in a discussion of the type of spaces we would like to work with, that is, compactly generated space.

Definition 10.0.0.1 (**Compactly generated spaces**)**.** A space X is said to be compactly generated if it satisfies

1. (*weak Hausdorff*) for any compact Hausdorff space K and a map $g : K \to X$, the image $g(K)$ is closed,
2. (k-*space*) for any $A \subseteq X$, if $g^{-1}(A)$ is closed in K for any $g : K \to X$ where K is a compact Hausdorff space[1], then A is closed in X.

The following are some immediate observations.

Proposition 10.0.0.2. *Let X be a compactly generated space. Then,*

1. *Every compact subspace of X is closed.*
2. *If K is compact Hausdorff and $g : K \to X$ is a map, then $g(K) \subseteq X$ is compact Hausdorff.*
3. *If X is compactly generated and $f : X \to Y$ is a function, then f is continuous if and only if $f|_K$ is continuous for all compact subspaces $K \subseteq X$.*
4. *Any closed subspace of a compactly generated space is compactly generated.*

Proof. **TODO.** □

Example 10.0.0.3. Following are some examples of compactly generated spaces.
1. Any compact Hausdorff space is compactly generated. Indeed, for any compact Hausdorff K and a map $g : K \to X$, we have $g(K)$ is compact in X which is Hausdorff, so closed. Furthermore, if $A \subseteq X$ and $g^{-1}(A)$ is closed in K for any such g, then letting $K = X$ and $g = \text{id}$, we immediately deduce that A is closed, as required.

---

[1] we then call A to be *compactly closed*

2. Any Hausdorff space X which is locally compact is compactly generated. Indeed, for any compact Hausdorff K and a map $g : K \to X$, we have $g(K)$ is compact in X which is Hausdorff, so closed. Furthermore, if $A \subseteq X$ and $g^{-1}(A)$ is closed in K for any such g, then letting $\tilde{X}$ denote the 1-pt. compactification of X, we see that $\tilde{X}$ is compact Hausdorff. Consequently we may consider the map $\mathrm{id} : \tilde{X} \to \tilde{X}$. As any compact Hausdorff space is compactly generated as shown above, therefore $\mathrm{id}^{-1}(A) = A$ is closed by hypothesis, as needed.

3. Hence, every CW-complex is a compactly generated space.

Remark 10.0.0.4. The above example in particular shows that any real or complex manifold is a compactly generated space.

Construction 10.0.0.5. (*k-ification*) Let X be a weak-Hausdorff space. Then, X can be made into a compactly generated space. Define kX to have the same set as X but a finer topology obtained by deeming any compactly closed subspace to be closed in kX. It then follows that

1. kX is compactly generated,
2. the function $\mathrm{id} : kX \to X$ is continuous,
3. X and kX have same compact subsets,
4. for weak Hausdorff spaces X and Y, we have $k(X \times Y) = kX \times kY$.

We now show why we restrict our gaze to only these spaces. In part because the category of compactly generated spaces is well-behaved.

TODO *Category* $\mathbf{Top}^{cg}$ *has limits, colimits and exponential objects (all after k-ification) and that the dual notion of homotopy as a path in function space is same as that of the usual notion.*

Remark 10.0.0.6. From now on in this chapter, we only work with the category of compactly generated spaces, $\mathbf{Top}^{cg}$. Moreover, any construction on spaces that we do is assumed to be k-ified, i.e. functor k is applied to it to always end up with the category of compactly generated spaces.

Next, we introduce constructions that one can do on based spaces. We denote $\mathbf{Top}_*^{cg}$ to be the category of based compactly generated spaces and based maps between them.

Construction 10.0.0.7 (*Based constructions*)**.** Let X and Y be two based spaces. Then, we denote by

1. $[X, Y]$ the based homotopy classes of based maps from X to Y. This is a based set itself, the basepoint being the homotopy class of $c_* : X \to Y$ mapping $x \mapsto *$. If $X \simeq X'$ and $Y \simeq Y'$, then there is a base point preserving bijection $[X, Y] \cong [X', Y']$.
2. $X \wedge Y$ the smash product given by $X \times Y / X \vee Y$ where $X \vee Y = \{*\} \times Y \cup X \times \{*\}$. This is a based space, the base point being the point corresponding to the subspace $X \vee Y$.
3. $\mathrm{Map}_*(X, Y)$ the collection of based maps from X to Y. This is again a based space in compact-open topology where the basepoint is c_*.
4. X_+ the based space obtained by adjoining a distinct point $*$ to X.
5. $X \wedge I_+$ the reduced cylinder of X where X is based. For any based X and unbased Y the based space $X \wedge Y_+$ is naturally homeomorphic to $X \times Y / \{*\} \times Y$.

There is a natural "$\otimes$-Hom" adjunction in $\mathbf{Top}_*^{cg}$.

Theorem 10.0.0.8. *Let X, Y, Z be based spaces in $\mathbf{Top}_*^{cg}$. Then we have a natural isomorphism*

$$\mathrm{Map}_*(X \wedge Y, Z) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)).$$

Proof. (*Sketch*) Let $f : X \wedge Y \to Z$. Then by universal property of quotients, we get a map $\bar{f} : X \times Y \to Z$ which is constant on $X \vee Y$. Now construct

$$\begin{aligned} \tilde{f} : X &\longrightarrow \mathrm{Map}_*(Y, Z) \\ x &\longmapsto y \mapsto \bar{f}(x, y). \end{aligned}$$

The fact that this is based follows from $\bar{f}$ being constant on $X \vee Y$.

Let $g : X \to \mathrm{Map}_*(Y, Z)$ a based map. Then we get

$$\begin{aligned} \bar{g} : X \times Y &\longrightarrow Z \\ (x, y) &\longmapsto g(x)(y). \end{aligned}$$

This is based immediately. Further, on $X \vee Y$, we see that $\bar{g}$ is constant. By universal property of quotients, we get the required $\tilde{g} : X \wedge Y \to Z$. □

This theorem shows the duality between smash products and mapping space constructions.

Construction 10.0.0.9 (*More based constructions*)**.** We now give two constructions each for smash product and mapping space which complement each other.

1. CX the cone of X obtained by $X \wedge I$ where 1 is the basepoint of I.
2. ΣX the suspension of X obtained by $X \wedge S^1$.
3. PX the path space of X obtained by $\mathrm{Map}_*(I, X)$.
4. ΩX the loop space of X obtained by $\mathrm{Map}_*(S^1, X)$.

It follows from Theorem 10.0.0.8 that we have following natural isomorphisms

$$\mathrm{Map}_*(CX, Y) \cong \mathrm{Map}_*(X, PY)$$

and

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y),$$

the latter being the famous *suspension-loop space* adjunction.

In the next few items, we give results which are simple to see but important as technical tools.

Proposition 10.0.0.10. *Let X, Y be based spaces in $\mathbf{Top}_*^{cg}$. Then*

$$\pi_0(\mathrm{Map}_*(X, Y)) \cong [X, Y].$$

In particular, we have

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Proof. (*Sketch*) In $\mathbf{Top}^{cg}$, both left and right notions of homotopy are equivalent. Consequently, a path-component in $\mathrm{Map}_*(X, Y)$ is equivalently the set of based maps $X \to Y$ which are homotopic, as required. □

Every space can be *pointified*.

Definition 10.0.0.11 (**Pointification**)**.** The functor $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$ given by $X \mapsto X_+$ and $f : X \to Y$ mapping to $f_+ : X_+ \to Y_+$ is called the pointification functor.

There are important relationships between based and unbased constructions. We first have the following simple observation.

Lemma 10.0.0.12. *Let X be a based space. We have the following bijection*

$$\left\{\begin{matrix}\text{Based homotopies } h : \\ X \times I \to Y\end{matrix}\right\} \cong \mathrm{Map}_*(X \wedge I_+, Y).$$

□

Remark 10.0.0.13. Let X be an unbased space. All the construction of Construction 10.0.0.9 have an unbased counterpart where smash products are replaced by Cartesian product and Map_* are replaced by Map. In particular,

1. CX the unreduced cone of X obtained by $X \times I/X \times \{1\}$.
2. ΣX the unreduced suspension of X obtained by $X \times S^1/X \times \{1\}$.
3. PX the unbased path space of X obtained by $\mathrm{Map}(I, X)$.
4. ΩX the unbased loop space of X obtained by $\mathrm{Map}(S^1, X)$.

We also call them by same name, if it is clearly understood that the space in question is unbased.

The following is an important observation about pointification and cones.

Lemma 10.0.0.14. *Let X be an unbased space. Then, the unreduced cone of X is isomorphic to the reduced cone on X_+. That is,*

$$CX \cong CX_+.$$

□

10.1 Fundamental group and covering maps

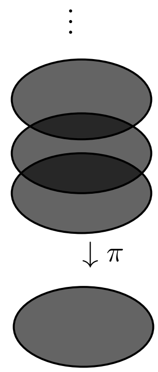
10.1.1 Covering spaces

We will now study a very important concept which is used everywhere in algebraic topology, the concept of covering spaces. This concept captures the notion of when does another space *covers* another space. Even though at this time it may seem completely unrelated to what we've been doing, but we will soon see that using this simple idea we would be able to calculate first homotopy group of S^1. So let us first give the definition of a covering space:

Definition 10.1.1.1. (**Covering space**) Let X be a topological space and suppose $\pi : \tilde{X} \to X$ is a continuous map such that for all $x \in X$, there exists open neighborhood $U_x \ni x$ such that:

1. $\pi^{-1}(U_x) = \coprod_{\alpha \in J_x} V_\alpha$ where V_α's are disjoint open sets in $\tilde{X}$,
2. $\pi|_{V_\alpha} : V_\alpha \to U_x$ is a homeomorphism.

Then, $\pi : \tilde{X} \to X$ is said to be a **covering map** and $\tilde{X}$ is said to be a covering space over X. In this case, the open neighborhood $U_x \subseteq X$ containing x is said to be the **evenly-covered neighborhood** of $x \in X$.



Let us begin with an important example.

Example 10.1.1.2. Well, clearly, the easiest way to get a covering space out of any space is to simply consider that map $X \amalg X \to X$. But that's not interesting.
The most important example of covering spaces that we will consider in this course is the exponential map:

$$\begin{aligned} \exp : \mathbb{R} &\longrightarrow S^1 \\ \theta &\longmapsto e^{2\pi i\theta}. \end{aligned}$$

Let us make sure that this is indeed a covering map. Take any point $e^{2\pi i\theta} \in S^1$ where $0 < \theta \leq 1$. Now consider an open set U of S^1, formed by $B_\epsilon(e^{2\pi i\theta}) \cap S^1$ where $0 < \epsilon < 2$. Denote $U =: e^{2\pi i(\theta-\delta,\theta+\delta)}$ where clearly $0 < \delta < 1/2$. Consider now $\pi^{-1}(U) \subseteq \mathbb{R}$. We will have

$$\pi^{-1}(U) = \coprod_{n\in\mathbb{Z}} (\theta + 2\pi n - \delta, \theta + 2\pi n + \delta).$$

Denote $V_n := (\theta + 2\pi n - \delta, \theta + 2\pi n + \delta)$. Moreover, it is clear that

$$\pi|_{V_n} : V_n \longrightarrow U$$

is a homeomorphism. So indeed π is a covering map of S^1. This is a very famous covering map as well. You should think of it as an infinite spiral (homeomorphic to $\mathbb{R}$) which covers the S^1 in the sense that when you view the spiral from the top, you will see only S^1.

We will use this covering map $\exp : \mathbb{R} \to S^1$ to find the first homotopy group of S^1. The main idea there will be *resolve* complicated loops in S^1 to $\mathbb{R}$, where each loop is homotopic to constant loop at the starting/ending point of the loop(!)

Remark 10.1.1.3. It is clear that every covering map is surjective.

The following is an important example of a covering map.

Lemma 10.1.1.4. *The map $\varphi : S^1 \to S^1$ given by $z \mapsto z^n$ is a covering map.*

Proof. Pick any $z_0 = e^{i\theta_0} \in S^1$. We wish to show that there exists an open set $U_0 \ni z_0$ in S^1 such that

$$\varphi^{-1}(U_0) = \coprod_{k=0}^{n-1} V_k$$

where V_k are open in S^1 and $\varphi|_{V_k} : V_k \to U_0$ is a homeomorphism.

Denote by $\gamma : \mathbb{R} \to S^1$ the continuous surjective map given by $t \mapsto e^{it}$. Thus, $z_0 = \gamma(\theta_0)$. Consider the interval $I_0 = \left(\theta_0 - \frac{\pi}{n}, \theta_0 + \frac{\pi}{n}\right)$. As the map $\gamma : \mathbb{R} \to S^1$ is an open map, therefore we have $U_0 = \gamma(I_0)$ which is an open set of S^1 containing z_0. We claim that U_0 is an evenly covered neighborhood for z_0. Indeed, we see that

$$\begin{aligned}
\varphi^{-1}(U_0) &= \{z \in S^1 \mid z^n \in U_0\} \\
&= \left\{e^{i\theta} \in S^1 \mid e^{ni\theta} \in \gamma(I_0)\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } \gamma(\kappa) = e^{i\kappa} = e^{ni\theta}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } n\theta = \kappa + 2k\pi, \text{ for some } k \in \mathbb{Z}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } \theta = \frac{\kappa}{n} + \frac{2\pi k}{n}, \text{ for some } k \in \mathbb{Z}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \theta \in \coprod_{k \in \mathbb{Z}} \left(\frac{\theta_0}{n} - \frac{\pi}{n^2} + \frac{2\pi k}{n}, \frac{\theta_0}{n} + \frac{\pi}{n^2} + \frac{2\pi k}{n}\right)\right\} \\
&= \coprod_{k=0}^{n-1} \gamma\left(\left(\frac{\theta_0}{n} - \frac{\pi}{n^2} + \frac{2\pi k}{n}, \frac{\theta_0}{n} + \frac{\pi}{n^2} + \frac{2\pi k}{n}\right)\right).
\end{aligned}$$

This completes the proof. □

We next discuss the notion of mapping torus of a map and how van Kampen can be used to compute its fundamental group.

Definition 10.1.1.5 (**Mapping torus**)**.** For any map $f : X \to X$ the mapping torus of f is $T_f := X \times I/\sim$ where $(x, 0) \sim (f(x), 1)$.

Example 10.1.1.6. For $\mathrm{id} : X \to X$, one can check that $T_{\mathrm{id}} = X \times S^1$.

We have the following basic, but useful lemma.

Lemma 10.1.1.7. *Let $\pi : \tilde{X} \to X$ be a covering map. Then, for all $x \in X$ the fiber $\pi^{-1}(x) \subseteq \tilde{X}$ is a discrete subspace of $\tilde{X}$, that is, each $\tilde{x} \in \pi^{-1}(x)$ is both open and closed.*

Proof. To see this, take any $\tilde{x} \in \pi^{-1}(x)$ and an evenly covered neighborhood $U_x \subseteq X$ of x. Since $\pi^{-1}(U_x) = \coprod_{\alpha \in J_x} V_\alpha$, where each V_α is homeomorphic to U_x under $\pi|_{V_\alpha}$. Thus, the unique $\tilde{x}_\alpha \in V_\alpha$ such that $\pi(\tilde{x}_\alpha) = x$ is an element of $\pi^{-1}(x)$, one for each $\alpha \in J_x$. Now an open set of $\pi^{-1}(x)$ is of the form $V \cap \pi^{-1}(x)$ where $V \subseteq \tilde{V}$ is open, therefore $V_\alpha \cap \pi^{-1}(x)$ is open in $\pi^{-1}(x)$. But $V_\alpha \cap \pi^{-1}(x) = \{\tilde{x}_\alpha\}$ because each V_α are disjoint. Therefore $\{\tilde{x}_\alpha\}$ is open in $\pi^{-1}(x)$. Similarly, it is closed in $\pi^{-1}(x)$ by considering the complement of $\cup_{\beta \neq \alpha} V_\beta$ in $\pi^{-1}(x)$. Hence $\pi^{-1}(x)$ is a discrete subspace of $\tilde{X}$. □

10.1.2 Path lifting

Covering maps are important in algebraic topology because they come equipped with a lot of unique lifting properties. We will first spell out the unique path lifting property of covering spaces, which is a baby version of unique homotopy lifting property. Before that, we need some specific property of a path in space X which is covered by a covering space $\tilde{X}$.

Lemma 10.1.2.1. *Let $\gamma : I \to X$ be a path in X and $\pi : \tilde{X} \to X$ be a covering map. Then there exists a partition $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1$ of unit interval I such that for all $i = 0, \dots, k-1$, the image $\gamma([t_i, t_{i+1}]) \subseteq X$ is contained in an evenly-covered neighborhood of X.*

Proof. So first, for all $t \in I$, there exists an evenly-covered neighborhood $U_t \subseteq X$ of $\gamma(t) \in X$. Thus, by continuity of γ, we get that there exists $(a_t, b_t) \subseteq I$ containing $t \in I$ such that $\gamma((a_t, b_t)) \subseteq U_t$. Since each open interval contains a compact interval, therefore we can assume (a_t, b_t) to be $[a_t, b_t]$. So we have a family of closed subintervals $\{[a_t, b_t]\}_{t \in I}$ of I. By compactness of I, we get that there exists a finite subcover $[a_{t_1}, b_{t_1}], \dots, [a_{t_n}, b_{t_n}]$ of I. Now suppose $[a_{t_i}, b_{t_i}]$ and $[a_{t_j}, b_{t_j}]$ intersect, then we can break down $[a_{t_i}, b_{t_i}] \cup [a_{t_j}, b_{t_j}]$ into three disjoint closed intervals $[a_{t_i}, a_{t_j}] \cup [a_{t_j}, b_{t_i}] \cup [b_{t_i}, b_{t_j}]$. Furthermore note that each of the above three have their images contained inside an evenly-covered neighborhood. Since there are only finitely many such intersections, therefore we have a finite disjoint cover of I by closed intervals, each of which has image under γ contained in an evenly covered neighborhood. □

Theorem 10.1.2.2. *(Unique path lifting of covering maps) Let $\pi : \tilde{X} \to X$ be a covering map. Suppose there is a path $\gamma : I \to X$ and a prescribed point $\tilde{\gamma}_0 : \{0\} \to \tilde{X}$ such that $\pi(\tilde{\gamma}_0) = \gamma(0)$, then there exists a unique path $\tilde{\gamma} : I \to \tilde{X}$ such that $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{\gamma}_0$. That is, the following lifting problem is uniquely filled:*

$$\begin{tikzcd} \{0\} \arrow[r, hook, "\tilde{\gamma}_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "\pi"] \\ I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & X \end{tikzcd} .$$

Proof. Let us first construct such a path lift. By Lemma 10.1.2.1, we have a partition of I into $I = \cup_{i=0}^{k-1} [t_i, t_{i+1}]$ of disjoint closed intervals where $\gamma([t_i, t_{i+1}]) \subset U_i \subset X$ and U_i is evenly-covered in X. Now to construct the said $\tilde{\gamma}$, we will have to do it for each $[t_i, t_{t+1}]$, starting from $i = 0$, making use of $\tilde{\gamma}_0 \in \tilde{X}$ that has been already given to us. Now, let us first denote $\pi^{-1}(U_i) = \coprod_{\alpha \in J_i} V^i_\alpha$ for all $i = 0, \dots, k-1$ where $V^i_\alpha \cong U_i$, which is given by the fact that π is a covering map. Also keep in note that $\forall t \in [t_i, t_{i+1}]$, $\gamma(t) \in U_i \subseteq X$ which is evenly-covered.

So let us first define $\tilde{\gamma}$ for $[t_0, t_1] = [0, t_1]$. Since $\pi(\tilde{\gamma}_0) = \gamma(0) \in U_0$, therefore $\tilde{\gamma}_0 \in \pi^{-1}(U_0)$ and hence there is unique $\alpha_0 \in J_0$ such that $\tilde{\gamma}_0 \in V^0_{\alpha_0}$.

$$\begin{aligned} \tilde{\gamma}|_{[t_0,t_1]} : [t_0, t_1] &\longrightarrow \tilde{X} \\ t &\longmapsto \left(\pi|_{V^0_{\alpha_0}}\right)^{-1} (\gamma(t)), \end{aligned}$$

where $\pi|_{V^0_{\alpha_0}} : V^0_{\alpha_0} \to U_0$ is a homeomorphism and we are using it's inverse map in the above definition. Ok, so we first observe that $\tilde{\gamma}|_{[t_0,t_1]}(0) = \left(\pi|_{V^0_{\alpha_0}}\right)^{-1}(\gamma(0)) = \left(\pi|_{V^0_{\alpha_0}}\right)^{-1}(\pi(\tilde{\gamma}_0)) = \tilde{\gamma}_0$. That is, the starting point of path $\tilde{\gamma}$ is indeed $\tilde{\gamma}_0$. So we have constructed a path in $\tilde{X}$ from $\tilde{\gamma}_0$ to $\tilde{\gamma}|_{[t_0,t_1]}(t_1)$. Moreover, this path satsfies that $\pi \circ \tilde{\gamma}|_{[t_0,t_1]} = \gamma|_{[t_0,t_1]}$, which is exactly what we wanted.

Next, let us continue defining $\tilde{\gamma}$ for $[t_1, t_2]$ by using where we left off at $[t_0, t_1]$. This in turn will suggest us how to completely define the whole path $\tilde{\gamma}$. So we first note that $\gamma(t_1) \in U_0 \cap U_1$, therefore the end point of path $\tilde{\gamma}|_{[t_0,t_1]}$ at t_1, takes value in $\pi^{-1}(U_1)$ as well, so let $\tilde{\gamma}|_{[t_0,t_1]}(t_1) \in V^1_{\alpha_1}$. It should be clear by now what we are about to do; now define:

$$\begin{aligned} \tilde{\gamma}|_{[t_1,t_2]} : [t_1, t_2] &\longrightarrow \tilde{X} \\ t &\longmapsto \left(\pi|_{V^1_{\alpha_1}}\right)^{-1}(\gamma(t)). \end{aligned}$$

As usual, we again observe that $\tilde{\gamma}|_{[t_1,t_2]}(t_1) = \tilde{\gamma}|_{[t_0,t_1]}(t_1)$ because we have

$$\begin{aligned} \left(\pi|_{V^1_{\alpha_1}}\right)^{-1}(\gamma(t_1)) &= \left(\pi|_{V^1_{\alpha_1}}\right)^{-1}(\pi(\tilde{\gamma}|_{[t_0,t_1]}(t_1))) \\ &= \tilde{\gamma}|_{[t_0,t_1]}(t_1) \end{aligned}$$

where we conclude second line from first as $\gamma(t_1) \in U_0 \cap U_1$, where $\left(\pi|_{V^1_{\alpha_1}}\right)^{-1}$ is indeed defined. So we have indeed define a path $\tilde{\gamma}|_{[t_1,t_2]}$ whose starting point is same as the ending point of $\tilde{\gamma}|_{[t_0,t_1]}$, so we have defined the $\tilde{\gamma}$ upto $[t_0, t_2]$.

Having done the above, we now give general procedure of continuing the definition of path $\tilde{\gamma}$ till $[t_{k-1}, t_k]$. Suppose $2 \le j \le k-1$ and suppose we have constructed $\tilde{\gamma}|_{[t_{j-1},t_j]} : [t_{j-1}, t_j] \to \tilde{X}$ as of yet. So we know the point $\tilde{\gamma}|_{[t_{j-1},t_j]}(t_j) \in V^{j-1}_{\alpha_{j-1}}$ where $\gamma(t_j) \in U_{j-1} \cap U_j$. We now construct with this information the next piece of path $\tilde{\gamma}|_{[t_j,t_{j+1}]} : [t_j, t_{j+1}] \to \tilde{X}$. Well, the following definition shouldn't be a surprise:

$$\begin{aligned} \tilde{\gamma}|_{[t_j,t_{j+1}]} : [t_j, t_{j+1}] &\longrightarrow \tilde{X} \\ t &\longmapsto \left(\pi|_{V^j_{\alpha_j}}\right)^{-1}(\gamma(t)) \end{aligned}$$

where we again observe that the starting point of the above path is same as $\tilde{\gamma}|_{[t_{j-1},t_j]}(t_j)$. Moreover, it is easy to observe that $\pi \circ \tilde{\gamma}|_{[t_j,t_{j+1}]} = \gamma|_{[t_j,t_{j+1}]}$.

Finally, since there are only finitely many $[t_j, t_{j+1}]$s, therefore we have constructed a path $\tilde{\gamma}$ in $\tilde{X}$ such that it starts from $\tilde{\gamma}_0$ ($\tilde{\gamma}_0 = \tilde{\gamma}(0)$) and when projected back to X under π, we obtain the path γ back ($\pi \circ \tilde{\gamma} = \gamma$). In particular, the end point $\tilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$. The uniqueness of $\tilde{\gamma}$ follows by construction. $\square$

A simple yet useful observation about higher homotopy groups of universal covers is the following.

Lemma 10.1.2.3. *Let (X, x_0) be a path-connected, locally path-connected and semi-locally simply connected space and denote $p : \tilde{X} \to X$ be its universal cover. Then,*

$$p_* : \pi_k(\tilde{X}) \to \pi_k(X)$$

is an isomorphism for all $k \geq 2$.

Proof. We have a homomorphism $p_* : \pi_k(\tilde{X}) \to \pi_k(X)$ for all $k \geq 2$. We shall show that this homomorphism has an inverse. Indeed, we have a map

$$\begin{aligned} \psi : \pi_k(X) &\longrightarrow \pi_k(\tilde{X}) \\ [\gamma] &\longmapsto [\tilde{\gamma}] \end{aligned}$$

where $\tilde{\gamma}$ is the unique lift of γ which exists as S^k and $\tilde{X}$ are simply connected for $k \geq 2$. It follows immediately that $p_* \circ \psi = \text{id}$ and by uniqueness of lifts that $\psi \circ p_* = \text{id}$. Hence p_* is a bijection, as required. □

10.1.3 Homotopy lifting

The Theorem 10.1.2.2 will be the building block for it's generalization, which is the homotopy lifting of covering maps. Let us first define what does it mean for a map to have homotopy lifting property.

Definition 10.1.3.1. (**Homotopy lifting property**) Let $p : E \to B$ be a continuous map. The map p is said to have homotopy lifting property if for any homotopy $H : Y \times I \to B$ and any map $\tilde{H}_0 : Y \times \{0\} \to E$ such that $p \circ \tilde{H}_0 = H(-, 0)$, there exists a homotopy $\tilde{H} : Y \times I \to E$ such that $\tilde{H}(-, 0) = \tilde{H}_0$ and $p \circ \tilde{H} = H$. That is, the following lifting problem is filled:

$$\begin{tikzcd} Y \times 0 \arrow[r, "\tilde{H}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd} .$$

Remark 10.1.3.2. It is clear that path lifting property is obtained from homotopy lifting property by setting $Y = \{0\}$ in the diagram of homotopy lifting problem above.

We then have the following theorem.

Theorem 10.1.3.3. *(Unique homotopy lifting of covering maps) Let $\pi : \tilde{X} \to X$ be a covering map. Then π satisfies unique homotopy lifting property. That is, given any homotopy $H : Y \times I \to X$ and a map $\tilde{H}_0 : Y \to \tilde{X}$ such that $\pi \circ \tilde{H}_0 = H(-, 0)$, there exists a unique homotopy $\tilde{H} : Y \times I \to \tilde{X}$ such that $\tilde{H}(-, 0) = \tilde{H}_0$ and $\pi \circ \tilde{H} = H$. In other words, the following lifting problem is uniquely filled:*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{H}_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "\pi"] \\ Y \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & X \end{tikzcd} .$$

Proof. **[TODO] Proof is quite long and detailed so I will do it when I will get time..** □

10.1.4 $\pi_1(S^1) \cong \mathbb{Z}$

We now prove using the covering map $\exp : \mathbb{R} \to S^1$ that the first homotopy group of S^1 is $\mathbb{Z}$.

Theorem 10.1.4.1. *$\pi_1(S^1) \cong \mathbb{Z}$.*

Proof. Consider the following map which is quite intuitive to define:

$$\begin{aligned} \varphi : \mathbb{Z} &\longrightarrow \pi_1(S^1) \\ n &\longmapsto [\gamma_n] \end{aligned}$$

where $\gamma_n : I \to S^1$ is the loop $\theta \longmapsto e^{2\pi i n\theta}$, that is, γ_n is the loop corresponding to travelling around n-times on the circle S^1. Let us first show that it is indeed a group homomorphism. We see that

$$\begin{aligned} \varphi(n+m) &= [\gamma_{n+m}] \\ &= [\gamma_n * \gamma_m] \\ &= [\gamma_n] * [\gamma_m] \\ &= \varphi(n) * \varphi(m), \end{aligned}$$

so no qualms there.

The major hurdle starts when we try to prove the injectivity and surjectivity. This is where we will need to use the path and homotopy lifting properties of the covering map $\exp : \mathbb{R} \to S^1$ where we indeed verified that exp is a covering map in the example below the definition of covering spaces.

Let us first show surjectivity. So take any $[\gamma] \in \pi_1(S^1)$. We need to show that $\exists n \in \mathbb{Z}$ such that $[\gamma_n] = [\gamma]$. So we have that $\exp(\tilde{x}) = \gamma_n(0)$, which in diagrammatic form is

$$\begin{tikzcd} \{0\} \arrow[r, hook, "\tilde{x}"] \arrow[d, hook] & \mathbb{R} \arrow[d, "\exp"] \\ I \arrow[r, "\gamma"'] & S^1 \end{tikzcd}.$$

Since $\exp : \mathbb{R} \to S^1$ is a covering map, therefore using the unique path lifting property of covering maps (Theorem 10.1.2.2), we get that there is a unique $\tilde{\gamma} : I \to \mathbb{R}$ such that the above lifting problem is filled and then we get $\exp \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{x} \in (\exp)^{-1}(1)$. Now, we also have that $\tilde{\gamma}(1) \in (\exp)^{-1}(1)$. Therefore $\tilde{\gamma}(1) - \tilde{\gamma}(0) =$ total number of times the loop γ crosses $1 = n$, say. So $\tilde{\gamma}$ is homotopic to the straight line joining $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$, that is $\kappa(t) = (1-t)\tilde{\gamma}(0) + t\tilde{\gamma}(1)$. Let this homotopy between κ and $\tilde{\gamma}$ be denoted by $H : I \times I \to \mathbb{R}$. Then $\exp \circ H$ is a homotopy between $\exp \circ \kappa$ and $\exp \circ \tilde{\gamma}$ where the former is the γ_n and the latter is γ. We thus have a homotopy between them and therefore $[\gamma] = [\gamma_n]$.

Let us next show injectivity. So suppose $\varphi(n) = [\gamma_n] = [c_1] = [\gamma_0]$ where $c_1 = \gamma_0 : I \to S^1$ is the constant loop at $1 \in S^1$. We need to show that this implies $n = 0$. We will use homotopy lifting to prove this, that is, we will lift the homotopy which makes γ_n homotopic to c_1 to a homotopy in $\mathbb{R}$ between the lift of γ_n to a constant path. More precisely, consider the homotopy

$$H : I \times I \longrightarrow S^1$$

establishing a homotopy between $H(-,0) = \gamma_n$ and $H(-,1) = \gamma_0$ and moreover $H(0,-) = H(1,-) = 1$. Also consider the map $\tilde{\gamma}_n : I \longrightarrow \mathbb{R}$ given by $t \longmapsto nt$. This is the other map which the lifted homotopy will give a homotopy from to some other map (which we have to figure out). We then observe that $\tilde{\gamma}_n$ is the right map to define here because $\exp \circ \tilde{\gamma}_n(s) = e^{2\pi ins} = \gamma_n(s) = H(s,0)$. Ok so now we lift. Using Theorem 10.1.3.3, the following lifting problem is uniquely solved:

$$\begin{tikzcd} I \times \{0\} \arrow[r, "\tilde{\gamma}_n"] \arrow[d, hook] & \mathbb{R} \arrow[d, "\exp"] \\ I \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & S^1 \end{tikzcd} .$$

So we have a homotopy $\tilde{H} : I \times I \longrightarrow \mathbb{R}$ such that $\tilde{H}(s,0) = \tilde{\gamma}_n(s)$ and, more importantly, $\exp \circ \tilde{H} = H$. Thus, $\exp(\tilde{H}(s,1)) = H(s,1) = 1$, that is, $\mathrm{Im}\left(\tilde{H}(-,1)\right) \subseteq (\exp)^{-1}(1)$. Since fibres of a covering map are necessarily discrete (Lemma 10.1.1.7) and $\tilde{H}(-,1)$ is a continuous map from a connected set I, so it's image has to be connected as well and hence $\mathrm{Im}\left(\tilde{H}(-,1)\right)$ has to be a point inside $(\exp)^{-1}(1)$. What this means is that $\tilde{H}(-,1)$ is a constant map, to a point in $\mathbb{R}$, which we denote as $a \in \mathbb{R}$ such that $\exp(a) = 1$. So $\tilde{H}$ is a homotopy between $\tilde{\gamma}_n$ and c_a (the constant path at a). Moreover, we also have that $\tilde{H}(0,t) = \tilde{H}(1,t)$ for all $t \in I$ because $\tilde{H}$ is a based homotopy. So we get that the map $\tilde{H}(1,t) = \tilde{H}(0,t) = a \in (\exp)^{-1}(1)$ for all $t \in I$ as it is a for $t = 1$. So this **forces** $H(\tilde{s},0) = \tilde{\gamma}_n(s)$ to have starting point and ending point same, equal to a. But this can only happen when $n = 0$ (see definition of $\tilde{\gamma}_n$). We are done. □

10.1.5 Couple of properties of covering spaces

Covering maps are quite nice maps as is shown by Theorem 10.1.3.3. We will consider a couple of important properties that covering spaces hold in this section. The first one being that all fibers of a covering map of a path-connected space (which is discrete, Lemma 10.1.1.7) are bijective (so have same *size*).

Lemma 10.1.5.1. *Let $\pi : \tilde{X} \to X$ be a covering map and let X be a path-connected space*[2]*. Let $x_0, x_1 \in X$ be two points, then there is a set bijection*

$$\pi^{-1}(x_0) \cong \pi^{-1}(x_1).$$

Proof. [**TODO**]. □

Another use of covering spaces is that if $\pi : \tilde{X} \to X$ is a covering map where both the spaces are path-connected, then the fundamental group of $\tilde{X}$ is naturally embedded inside the fundamental group of X.

[2] or work over path-components.

Proposition 10.1.5.2. *Let $\pi : \tilde{X} \to X$ be a covering map where both X and $\tilde{X}$ are path-connected. Then the map*

$$\pi_1(\pi) : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, \pi(\tilde{x}_0))$$

is injective.

10.1.6 Fun applications of $\pi_1(S^1) \cong \mathbb{Z}$

We first have the famous Brouwer's fixed point theorem.

Proposition 10.1.6.1. *(Brouwer's fixed point theorem) For any continuous $f : D^2 \to D^2$, there exists a point $x \in D^2$ such that $f(x) = x$.*

Next is something we know very well but didn't knew that it can be done from the methods we have developed till now:

Proposition 10.1.6.2. *(Fundamental theorem of algebra) Let $p(x) \in \mathbb{C}[x]$. Then there exists a $c \in \mathbb{C}$ such that $x - c$ divides $p(x)$. That is, every complex polynomial has a root in $\mathbb{C}$ (and thus have all roots in $\mathbb{C}$).*

The last one is something we saw in the departmental seminar a week ago, using which we saw that one can prove very non-trivial combinatorial results.

Proposition 10.1.6.3. *(Borsuk-Ulam theorem) If $f : S^2 \to \mathbb{R}^2$ is a continuous map, then there exists a pair of anti-podal points which are mapped to same point under f.*

10.1.7 Covering spaces, group actions and Galois theory of covers

So in this second phase of the course, we will be seeing some more fancy theorems, but the main goal will be to go to some calculative things, like computing homology groups and all that. In any case, we covered covering spaces, but it would be rather incomplete if we don't say something about universal covering and more theorems in that direction. The first theorem we therefore discuss, tells us how a certain type of G-space naturally enriches the quotient map with the structure of a covering space. We first define the type of G-space we wish to look out for.

Definition 10.1.7.1. (**Properly discontinuous action**) Let G be a group and X be a space with a continuous action[3] of G. The action of G is said to be properly discontinuous if for all $x \in X$, there exists an open set $U_x \subseteq X$ containing x such that $gU_x \cap U_x = \emptyset$ for all $g \in G$.

There is another type of action:

Definition 10.1.7.2. (**Free action**) Let G be a group acting continuously on space X. The the action is said to be free if for all $x \in X$, the stabilizer subgroup is trivial, that is, $S_G(x) = \{e\}$.

There are some consequences of the above definition which we collectively state in the the following lemma:

Lemma 10.1.7.3. *Let G be a group and X be a space with continuous G-action.*

[3] this means that the action map $G \times X \to X$ is a continuous map where G is given the discrete topology.

1. *If the action is properly discontinuous, then it is free.*
2. *If G is finite and X is locally finite*[4]*, then the action is free if and only if it is properly discontinuous.*

Proof. 1. Take any $x \in X$. Let U_x be the open set containing x obtained from properly discontinuous action of G. If $g \in S_G(x)$, then $gU_x \cap U_x \neq \emptyset$. Thus $g = e$.

2. R $\Rightarrow$ L is simple. For L $\Rightarrow$ R, we go by contradiction. So suppose the action is free but not properly discontinuous. Take any point $x \in X$. So for any open $U \ni x$ and for any $g \in G$, $gU \cap U \neq \emptyset$. Now, we have a sequence of open sets each containing x, U_n, such that $\cap_n U_n = \{x\}$. Since $gU_n \cap U_n \neq \emptyset$ for each n, therefore we get a sequence $\{x_n\}$ where $x_n \in U_n$ such that $\varprojlim x_n = x$ and $\varprojlim gx_n = x$. Since $g \in G$ can be treated as $g : X \to X$ a homeomorphism, therefore $g(\varprojlim x_n) = x$ that is $gx = x$, a contradiction to the fact that G acts freely[5]. □

Let us now state the theorem of interest.

Theorem 10.1.7.4. *Let G be a group and X be a space with continuous G-action. If the action is properly discontinuous, then the quotient map*

$$q : X \twoheadrightarrow X/G$$

is a covering map.

Before stating the proof, we would like to give some example uses of this theorem.

Example 10.1.7.5. Consider $G = \mathbb{Z}^n$ and $X = \mathbb{R}^n$. There is a canonical action we can define on $\mathbb{R}^n$ using $\mathbb{Z}^n$ given by

$$\begin{aligned} G \times X &\longrightarrow X \\ ((m_1, \dots, m_n), (x_1, \dots, x_n)) &\longmapsto (m_1 + x_1, \dots, m_n + x_n). \end{aligned}$$

The fact that this is a continuous action is trivial to check. We first claim that this action is properly discontinuous. It is simple to see why that's the case; for an $x \in X$ simply take any $0 < a < 1/2$ and define $U = \prod (x_i - a, x_i + a)$. This U is open and for any $m := (m_1, \dots, m_n) \in \mathbb{Z}^n$, $(m + U) \cap U = \emptyset$ for any $m \neq 0$. So indeed the action is properly discontinuous.

Next, we observe that $X/G = \mathbb{R}^n/\mathbb{Z}^n$ is simply homeomorphic to $[0,1]^n/G$ and which is in turn homeomorphic to $([0,1]/0 \sim 1)^n$ and which is just $(S^1)^n$. So that is why the questions regarding $\mathbb{R}/\mathbb{Z}$ are so innumerable in literature, as they quickly form spaces which are quite weird to imagine.

Example 10.1.7.6. (*Configuration space of k-points in space X*) Let X be a space. The configuration space of k points in X, denoted $F_k(X)$, is intuitively the set of all possible positions that k particles moving in X can inhabit. More precisely, we define:

$$F_k(X) = \{(x_1, \dots, x_k) \in \prod_{i=1}^{k} X \mid \forall\, i \neq j = 1, \dots, k,\ x_i \neq x_j\}.$$

[4] this means that for all $x \in X$, there exists a sequence of open sets U_n containing x such that $\bigcap_n U_n = \{x\}$.
[5] this is in-line with what the wonderful man *I.P. Freely* had to say.

This space has an action of S_k, the symmetry group of k letters, given by:

$$\begin{aligned} S_k \times F_k(X) &\longrightarrow F_k(X) \\ (\sigma, x_1, \dots, x_k) &\longmapsto (x_{\sigma(1)}, \dots, x_{\sigma(k)}). \end{aligned}$$

In other words, we just permute the k points which we find in some position in X. For $k = 2$, we get that since $S_2 = \mathbb{Z}_2$, so the only action possible is

$$\begin{aligned} \mathbb{Z}_2 \times F_2(X) &\longrightarrow F_2(X) \\ (0, x_1, x_2) &\longmapsto (x_1, x_2) \\ (1, x_1, x_2) &\longmapsto (x_2, x_1). \end{aligned}$$

In other words, we swap the two points. Then, orbits of the action of $\mathbb{Z}_2$ over $F_2(X)$ will consist of just the point itself and it's swapped counterpart. Hence,

$$F_2(X)/\mathbb{Z}_2 \cong (X \times X)/\sim$$

where $(x_1, x_2) \sim (y_1, y_2)$ iff $x_1 = y_2$ and $x_2 = y_1$. To better understand the situation, suppose $X = S^1$. Then, $F_2(S^1) = S^1 \times S^1/\sim$. Since $S^1 \times S^2 = T^2$, therefore we get $F^2(S^1) = T^2 \setminus \Delta(S^1)$, where $\Delta(S^1)$ is the diagonal subspace of $S^1 \times S^2$. But $T^2 \setminus \Delta(S^1)$ will look like quotient of $I \times I \setminus \Delta(I)$ which looks like two disjoint right triangles together. Now, we can obtain $F_2(S^1)/\sim$ by identifying the two triangles and doing the ensuing identifications of $I \times I$ to reach some weird object.

Example 10.1.7.7. The next example that we do is known for it's weirdness. It is the construction of *lens space*. Consider the odd sphere $S^{2k+1} \subset \mathbb{C}^{k+1}$ for $k \in \mathbb{N}$. Consider the cyclic group $\mathbb{Z}_d$ where we take the following presentation of it: $\mathbb{Z}_d = \langle \xi \rangle$ where ξ is the d^{th} root of unity. We then have the following action of $\mathbb{Z}_d$ on S^{2k+1}:

$$\begin{aligned} \mathbb{Z}_d \times S^{2k+1} &\longrightarrow S^{2k+1} \\ (\xi, z_1, \dots, z_k) &\longmapsto (\xi z_1, \dots, \xi z_k). \end{aligned}$$

This is indeed a valid action. In particular, we claim that this is a free action so that by Lemma 10.1.7.3, 2, this action becomes properly discontinuous and we can then use Theorem 10.1.7.4 to get that S^{2k+1} is a cover of this so-called lens space. To see that it is free, take any $(z_1, \dots, z_k) \in S^{2k+1}$. We see that if $(\xi^n z_1, \dots, \xi^n z_k) = (z_1, \dots, z_k)$, then $\xi^n = 1$. So each stabilizer subgroup is trivial. Hence the action is free. Then, the lens space is defined to be the quotient $S^{2k+1}/\mathbb{Z}_d$. Whatever that may look like, it has a structure of a $2k+1$ dimension manifold, as we have a cover by Theorem 10.1.7.4.

With all these examples out of the way, let us now get to the proof of the theorem at hand.

Proof of Theorem 10.1.7.4. Since the action of G is properly discontinuous, therefore for each $x \in X$, there exists open $U_x \subseteq X$ such that $gU_x \cap U_x = \emptyset$ for all $g \in G$. We claim that for any $[x] \in X/G$, the set $V_x := q(U_x)$ is evenly covered open neighborhood of $[x]$. In order to show this, we first claim the following

$$q^{-1}(V_x) = \coprod_{g \in G} gU_x.$$

Now, since $g : X \to X$ is a homeomorphism, thus $gU_x = g(U_x) \subseteq X$ is open in X. Hence, $q^{-1}(V_x)$ is open in X, if the above claim is true. So in order to see the claim, we see that

$$\begin{aligned} q^{-1}(V_x) &= \{y \in X \mid q(y) \in V_x = q(U_x)\} \\ &= \{y \in X \mid \exists z \in U_x \text{ s.t. } q(y) = q(z)\} \\ &= \{y \in X \mid \exists z \in U_x \text{ s.t. } y = gz \text{ for some } g \in G\} \\ &= \bigcup_{g \in G} gU_x. \end{aligned}$$

So we need only show that $gU_x \cap hU_x =$. This is simple because if it is not the case, then for some $y, z \in U_x$, we get $gy = hz$, so $y = g^{-1}hz$, a contradiction to $U_x \cap g^{-1}hU_x = \emptyset$ by properly discontinuous action of G on X. So indeed the claim is true.

We need only show now that for any $g \in G$, the restriction

$$q|_{gU_x} : gU_x \longrightarrow V_x$$

is a homeomorphism. Firstly, it is rather easy to see that $q(gU_x) = q(U_x)$, after all, q kills all orbits so that $q(gy) = q(y)$. Next, since $q(U_x) =: V_x$, so the above map is well defined. We now only need to show that it is a homeomorphism. For that, we can consider the following inverse:

$$\begin{aligned} w : V_x := q(U_x) &\longrightarrow gU_x \\ q(y) &\longmapsto gy. \end{aligned}$$

This is indeed well-defined. To see this, take any $z \in U_x$ such that $q(y) = q(z)$. Thus there is an $h \in G$ such that $y = hz$. Since $y, z \in U_x$ and U_x is such that $kU_x \cap U_x = \emptyset \ \forall k \in G$, thus, if $q(y) = q(z)$, then $y = z$, hence $gy = gz$. It is now easy to see that w is a continuous inverse of $q|_{gU_x}$, as $gy \mapsto q(gy) \mapsto w(q(gy)) = gy$ and conversely $q(y) \mapsto gy \mapsto q(gy) = y$. This completes the proof. □

10.1.8 Category of covering maps

Let (X, x_0) be a based space. It is easy to see that knowing information about all covers of (X, x_0), would be pretty handy. But how can one do that? Well, we will try to do exactly that in this section. Since we want to handle all covers of X, so it is better we start giving this collection of all covers of (X, x_0) some structure. One structure that it has is that it forms a category.

Definition 10.1.8.1. (**The category Cov** (X, x_0)) Let (X, x_0) be a based map. The category of covering maps of (X, x_0) and homomorphisms between them is defined by:

1. **Objects**: An object of **Cov** (X, x_0) is a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$.
2. **Arrows**: An arrow in **Cov** (X, x_0) is a continuous based map $f : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ such that the following commutes:

$$\begin{tikzcd} (\tilde{X}_1, \tilde{x}_1) \arrow[rr, "f"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd} .$$

It is clear that $\mathbf{Cov}\,(X, x_0)$ is a sub-category of the category $\mathbf{Top}_*$ over (X, x_0), that is, $\mathbf{Cov}\,(X, x_0) \subseteq \mathbf{Top}_*/(X, x_0)$.

We will see in this and the following sections that the main ingredient of our goal to understand a covering space will be, just like in Galois theory, the automorphism group of $(\tilde{X}, \tilde{x}_0)$ in the category $\mathbf{Cov}\,(X, x_0)$. We denote the set of all **automorphisms** of $(\tilde{X}, \tilde{x}_0)$ by $\mathsf{Deck}(\tilde{X}, x_0)$. Note that in the unbased setting, we will denote the automorphism group of $\tilde{X} \in \mathbf{Cov}\,(X, x_0)$ as just $\mathsf{Deck}(X)$.

From now, we will abbreviate a based space (X, x_0) by just X. Similarly for the covering spaces.

For our purposes, we see the following result.

Proposition 10.1.8.2. *Let X be a path connected and locally path connected based space and consider $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ to be two path-connected covers in $\mathbf{Cov}\,(X)$. Let $\varphi : (\tilde{X}_1, p_1) \to (\tilde{X}_2, p_2)$ be a map of covering spaces. Then, φ is a covering map over $(\tilde{X}_2, p_2)$.*

Proof. We break the proof into following steps.

$$\textbf{Act 1} : \textit{The map } \varphi \textit{ is surjective.}$$

Take any point $y \in \tilde{X}_2$. Since $\tilde{X}_2$ is path connected, so there is a path $\eta : I \to \tilde{X}_2$ with $\eta(0) = \tilde{x}_2$ and $\eta(1) = y$. Then we have $z := p_2(y) \in X$. Since X is path-connected, we thus have a path $\gamma : I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = z$. By Theorem 10.1.2.2 on $\tilde{X}_2$, it can be easily seen that η is the unique lift of γ. Now, by Theorem 10.1.2.2 for covering space $\tilde{X}_1$ with starting point $\tilde{x}_1$, we get a path $\tilde{\gamma}_1 : I \to \tilde{X}_1$ such that $\tilde{\gamma}_1(0) = \tilde{x}_1$ and $p_1 \circ \tilde{\gamma}_1 = \gamma$. Moreover, it is unique w.r.t. these properties. Now denote $x := \tilde{\gamma}_1(1) \in \tilde{X}_1$. Now, we have another path $\tilde{\gamma}_2 := \varphi \circ \tilde{\gamma}_1 : I \to \tilde{X}_2$ such that $\tilde{\gamma}_2(0) = \tilde{x}_2$. Moreover, by the fact that $p_2 \circ \varphi = p_1$, we get that $p_2 \circ \tilde{\gamma}_2 = \gamma$. So if we apply Theorem 10.1.2.2 on $\tilde{X}_2$, then the path that we must get should exactly be $\tilde{\gamma}_2$ because it satisfies the conditions that makes the path coming from the theorem unique. But then, $\eta = \tilde{\gamma}_2$. Hence $\tilde{\gamma}_2(1) = \eta(1) = y$. Hence $\varphi(x) = y$. This completes Act 1.

$$\textbf{Act 2} : \textit{Each point of } \tilde{X}_2 \textit{ has an evenly covered neighborhood.}$$

Take any point $y \in \tilde{X}_2$. To get an evenly covered neighborhood of y, we begin with $z := p_2(y) \in X$. Since both $\tilde{X}_1, \tilde{X}_2$ are covering X, therefore there are evenly covered neighborhoods $U_1, U_2 \subseteq X$ containing z. Then $V := U_1 \cap U_2$ is an open set which is an evenly covered neighborhood for both the covers. Now, $(p_2)^{-1}(V) \ni y$. Since $(p_2)^{-1}(V) = \coprod_{i \in J_z} V_i$. Let $y \in V_{i_y}$. We claim that this V_{i_y} will be an evenly covered neighborhood of $y \in \tilde{X}_2$ for φ. Clearly, $(\varphi)^{-1}(V_{i_y}) \cong (p)^{-1}(V) \cong \coprod_{i \in I_z} W_i$ where $p_1|_{W_i} : W_i \to V$ which is a homeomorphism. This concludes Act 2.

This concludes the proof. □

We now define universal covering space of a based space.

Definition 10.1.8.3. (**Universal covering**) Let (X, x_0) be a path-connected and locally path-connected space. A simply connected covering space $(\tilde{X}, \tilde{x}_0)$ is called a universal covering space of (X, x_0).

The justification of the name will come soon, but for the time being, let us develop some more theory of covering spaces, which we would need in order to prove Theorem **??**, which classifies coverings of a space upto isomorphism!

More properties of covering spaces & classification

Let us discuss few more properties of morphisms of covering spaces. It is good to remind ourselves that a space is path-connected and locally path-connected if and only if it is connected and locally path-connected.

Remark 10.1.8.4. It is clear by the definition of covering maps that if X is a locally path-connected space, then any covering space $\tilde{X}$ is also a locally path-connected space. But it is in general not true that if X is connected then $\tilde{X}$ is connected, a simple example is the trivial covering $X \amalg X \to X$. In conclusion, if X is connected and locally path-connected, then $\tilde{X}$ may not be connected but is locally path-connected.

The following lemma shows that to check equality of two maps in $\mathbf{Cov}(X)$ of connected covering spaces, we may check only at one point(!)

Lemma 10.1.8.5. *Let X be a path-connected and locally path-connected space. If $\varphi_0, \varphi_1 : (\tilde{X}_1, p_1) \rightrightarrows (\tilde{X}_2, p_2)$ are two maps of covering spaces in $\mathbf{Cov}(X)$ between connected covers $\tilde{X}_1$ and $\tilde{X}_2$, such that there exists a point $x_1 \in \tilde{X}_1$ for which $\varphi_0(x_1) = \varphi_1(x_1)$, then $\varphi_0 = \varphi_1$.*

Proof. Let $x \in \tilde{X}_1$. We wish to show that $\varphi_0(x) = \varphi_1(x)$. For this, we first denote $z := p_1(x) = p_2 \circ \varphi_0(x) = p_2 \circ \varphi_1(x)$. Hence it is clear that $y_0 := \varphi_0(x), y_1 := \varphi_1(x) \in (p_2)^{-1}(z)$, i.e. $y_0, y_1 \in \tilde{X}_2$ are in the same fiber. We now need to show that the points $y_0, y_1 \in p^{-1}(z)$ are literally the same. Suppose to the contrary that $y_0 \neq y_1$. Let $z \in U \subseteq X$ be an evenly covered neighborhood of z. Now, $(p_2)^{-1}(U) = \coprod_{i \in J} V_i$ where $p_2|_{V_i} : V_i \to U$ is an homeomorphism. Since $y_0 \neq y_1$, therefore, say $y_0 \in V_0$ and $y_1 \in V_1$ where V_0 and V_1 are disjoint in $\tilde{X}_2$. Since φ_0 and φ_1 are continuous, therefore there are open sets $W_0, W_1 \subseteq \tilde{X}_1$ containing x such that $\varphi_0(W_0) \subseteq V_0$ and $\varphi_1(W_1) \subseteq V_1$. Now, denote $W = W_0 \cap W_1$, so we have $\varphi_0(W) \subseteq V_0$ and $\varphi_1(W) \subseteq V_1$. So for each $x \in \tilde{X}_1$, we have an open set $x \in W_x \subseteq \tilde{X}_1$ such that $\varphi_0(W_x) \cap \varphi_1(W_x) = \emptyset$. This contradicts the fact that $x_1 \in \tilde{X}_1$ is not such a point. $\square$

Remark 10.1.8.6. Hence, for any $\varphi \in \mathsf{Deck}(\tilde{X})$ where $\tilde{X}$ is connected, φ doesn't have any fixed points.

The next result is an important one for our purposes, for it generalizes the unique path lifting property of covering maps to that of any path-connected and locally path-connected space, by comparing it's fundamental group.

Theorem 10.1.8.7 (*Unique lifting property*)**.** *Let (X, x_0) be a path-connected and locally path-connected space and let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. Let (Y, y_0) be a path-connected and locally path-connected space. If $\varphi : (Y, y_0) \to (X, x_0)$ is a based map, then there exists a unique lift $\tilde{\varphi} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if $\varphi_*(\pi_1(Y, y_0)) \leq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

More diagrammatically, the following lifting problem is uniquely solved if and only if $\varphi_*(\pi_1(Y, y_0)) \leq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$:

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "\varphi"'] \arrow[ur, dashed, "\exists!\, \tilde{\varphi}"] & (X, x_0) \end{tikzcd}.$$

Proof. (L $\Rightarrow$ R) Since $p \circ \tilde{\varphi} = \varphi$, therefore $\varphi_*(\pi_1(Y, y_0)) = (p \circ \tilde{\varphi})_*(\pi_1(Y, y_0)) = p_*(\tilde{\varphi}_*(\pi_1(Y, y_0))) \leq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

(R $\Rightarrow$ L) We define the following candidate for the lift: for each point $y \in Y$, we join it to y_0 using $\gamma_y : I \to Y$ where $\gamma_y(0) = y_0$ and $\gamma_y(1) = y$, and then lift (Theorem 10.1.2.2) $\varphi \circ \gamma_y$ to a path $\tilde{\gamma}_y$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{\gamma}_y(1) \in p^{-1}(\varphi(y))$. This process gives the following map

$$\begin{aligned} \tilde{\varphi} : Y &\longrightarrow \tilde{X} \\ y &\longmapsto \tilde{\gamma}_y(1). \end{aligned}$$

We complete the rest of the proof in the following acts.

Act 1 : *The map $\tilde{\varphi}$ is well-defined.*

The plan is to use both homotopy an path liftings for this. So what we need to show is that for any other choice $\eta : I \to Y$ with $\eta(0) = y_0$ and $\eta(1) = y$, we get that $\tilde{\eta}_y(1) = \tilde{\gamma}_y(1)$. In order to do this, we first note that we get a loop $\gamma_y * \bar{\eta}_y$ at y_0 in Y, so that we have an element $[\gamma_y * \bar{\eta}_y] \in \pi_1(Y, y_0)$. Now, $\varphi_*([\gamma_y * \bar{\eta}_y]) = [\varphi \circ \gamma_y * \varphi \circ \bar{\eta}_y]$. Now since $\varphi_*(\pi_1(Y, y_0)) \leq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, therefore there exists a loop $[\xi] \in \pi_1(\tilde{X}, \tilde{x}_0)$ such that $[p \circ \xi] = [\varphi \circ \gamma_y * \varphi \circ \bar{\eta}_y]$. Let us denote $[\varphi \circ \gamma_y * \varphi \circ \bar{\eta}_y] =: [\chi]$. So we have $p \circ \xi \simeq \chi$. Now, by Theorem 10.1.3.3, we get that ξ is homotopic to a loop at $\tilde{x}_0$, denoted τ such that $p \circ \tau = \chi$. Now note that $\tilde{\gamma}_y$ joins $\tilde{x}_0$ to a point, say $\omega \in \tilde{X}$ such that $p(\omega) = \varphi(y)$. Since we have a path $\varphi \circ \bar{\eta}_y$ which connects $\varphi(y)$ to x_0 in X, therefore if we lift (Theorem 10.1.2.2) $\varphi \circ \bar{\eta}_y$ to a path $\tilde{\bar{\eta}}_y$ beginning from ω and ending to a point in $p^{-1}(x_0)$ in $\tilde{X}$, we get that we get a unique path $\tilde{\gamma}_y * \tilde{\bar{\eta}}_y$ from $\tilde{x}_0$ to a point in $p^{-1}(x_0)$ in $\tilde{X}$ which is unique w.r.t the property that $p \circ (\tilde{\gamma}_y * \tilde{\bar{\eta}}_y) = \chi$. But, τ is also a path beginning from $\tilde{x}_0$ such that $p \circ \tau = \chi$, hence $\tilde{\gamma}_y * \tilde{\bar{\eta}}_y = \tau$, and thus the lift of $\bar{\eta}_y$ in $\tilde{X}$ starts at ω and ends at $\tilde{x}_0$. So now if we lift η_y in $\tilde{X}$, we get the path $\bar{\tilde{\bar{\eta}}}_y$ because of uniqueness of path lifts. Hence $\tilde{\eta}_y$ is a path from $\tilde{x}_0$ to $\omega =: \tilde{\gamma}_y(1)$. Hence well-definedness of $\tilde{\varphi}$ follows.

Act 2 : *The map $\tilde{\varphi}$ is continuous.*

It is at this point that we will use the hypotheses imposed on Y. We will show that $\tilde{\varphi}$ is locally a continuous map. Take any point $y \in Y$ and let $\varphi(y) \in X$. There is an evenly covered neighborhood of $\varphi(y)$, which we denote by $U \ni \varphi(y)$ so that $p^{-1}(U) = \coprod_{i \in I} V_i$. Denote $\tilde{\varphi}(y) \in V_0$. We also have an open set $\varphi^{-1}(U)$ of Y. Since Y is locally path-connected, let $W \subseteq \varphi^{-1}(U)$ be a path-connected subset of Y containing y. We now claim that $\tilde{\varphi}|_W = \left(p|_{V_0}\right)^{-1} \circ \varphi|_W$. For this, take any point $z \in W$, and since W is path-connected, therefore there exists ξ joining $y \to z$. Since γ_y already joins $y_0 \to y$, therefore we have that $\gamma_y * \xi$ joins $y_0 \to z$. By Act 1, we get

$$\begin{aligned} \tilde{\varphi}(z) &= (\varphi \circ (\widetilde{\gamma_y * \xi}))(1) \\ &= (\varphi \,\tilde{\circ}\, \gamma_y) * (\varphi \,\tilde{\circ}\, \xi)(1). \end{aligned}$$

Now, since $p|_{V_0}$ is a homeomorphism of V_0 to U and since $\varphi(y), \varphi(z) \in U$ are connected by a path $\varphi \circ \xi$, so V_0 also has a path connecting $\tilde{\varphi}(y)$ and $\tilde{\varphi}(z)$. Hence, by uniqueness of path lifts (Theorem 10.1.2.2), we get $(\varphi \,\tilde{\circ}\, \gamma_y) * (\varphi \,\tilde{\circ}\, \xi)(1) = \left(p|_{V_0}\right)^{-1}(\varphi(z))$. We are now gladly done.

Act 3 : *The map $\tilde{\varphi}$ is unique.*

Essentially by construction. If the reader is not convinced, just start doing the brute force verification and you will see why that's the case.

This proof is now complete. □

This theorem is an extremely important result as it will allow us to classify all connected covers of a connected and path-connected space upto isomorphism, as we will soon see. We will in the following few results see the beginnings of the Galois theory of covering spaces.

Lemma 10.1.8.8. *Let (X, x_0) be a path-connected and locally path-connected space and consider* $\mathbf{Cov}(X, x_0)$. *If $(\tilde{X^H}_1, \tilde{x}_1, p_1)$ and $(\tilde{X^H}_2, \tilde{x}_2, p_2)$ are two connected covering spaces over (X, x_0) such that*

$$p_{1*}(\pi_1(\tilde{X^H}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X^H}_2, \tilde{x}_2)) = H \leq \pi_1(X, x_0),$$

then there exists a unique homeomorphism $\varphi : (\tilde{X^H}_1, \tilde{x}_1, p_1) \to (\tilde{X^H}_2, \tilde{x}_2, p_2)$, that is, $(\tilde{X^H}_1, \tilde{x}_1, p_1)$ and $(\tilde{X^H}_2, \tilde{x}_2, p_2)$ are equivalent. In diagrammatic terms,

$$\begin{tikzcd}
(\tilde{X^H}_1, \tilde{x}_1) \arrow[rr, dashed, "\exists!\,\varphi", "\cong"'] \arrow[dr, "p_1"'] & & (\tilde{X^H}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}.$$

Proof. We will use Theorem 10.1.8.7 for this purpose. By the said theorem, where, in the notation of the theorem, we let $Y = \tilde{X^H}_1$ and $\varphi = p_1$, we get that there is a unique map $\varphi : \tilde{X^H}_1 \to \tilde{X^H}_1$ such that $p_2 \circ \varphi = p_1$. This follows because the condition of the theorem is trivially satisfied. We now need only show that it has an inverse. This is also easy because of the equality of the image subgroups; since $H = p_{2*}(\pi_1(\tilde{X^H}_2, \tilde{x}_2)) \subseteq p_{1*}(\pi_1(\tilde{X^H}_1, \tilde{x}_1)) = H$, therefore another application of Theorem 10.1.8.7 yields a unique map $\varpi : (\tilde{X^H}_2, \tilde{x}_2) \to (\tilde{X^H}_1, \tilde{x}_1)$ such that $p_1 \circ \varpi = p_2$. To show that φ and ϖ are inverses of each other, consider the composite $\varphi \circ \varpi : (\tilde{X^H}_2, \tilde{x}_2) \to (\tilde{X^H}_2, \tilde{x}_2)$. Since $\varphi \circ \varpi$ is a unique map w.r.t. the property that $p_2 \circ (\varphi \circ \varpi) = (p_2 \circ \varphi) \circ \varpi = p_1 \circ \varpi = p_2$, but since so is $\mathrm{id}(\tilde{X^H}_2, \tilde{x}_2)$, therefore $\varphi \circ \varpi = \mathrm{id}(\tilde{X^H}_2, \tilde{x}_2)$. Similarly, $\varpi \circ \varphi = \mathrm{id}(\tilde{X^H}_1, \tilde{x}_1)$. This completes the proof. □

Remark 10.1.8.9. Let $\tilde{X}$ be a connected cover of a p.c., l.p.c. space $(X, \tilde{x}_0)$. Then, we would like to know whether for any two choice of $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$, we get an element $\varphi \in \mathsf{Deck}(\tilde{X})$ such that $\varphi(\tilde{x}_1) = \tilde{x}_2$ and $\varphi(\tilde{x}_2) = \tilde{x}_1$. In such a case, we can say that the cover $\tilde{X}$ will be the one with *maximal symmetry*. Now with the result above, we can partly answer that, for if $p_{1*}(\pi_1(\tilde{X}, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}, \tilde{x}_2))$ in $\pi_1(X, x_0)$, then there is a *unique* deck transformation $\varphi \in \mathsf{Deck}(\tilde{X})$ such that $\varphi(\tilde{x}_1) = \tilde{x}_2$ as $p_2 \circ \varphi = p_1$, where $p_i : (\tilde{X}, \tilde{x}_i) \to (X, x_0)$. But the question for the converse remains open and we see how to resolve it in the next big theorem.

We now state one of the major theorems of this course.

Theorem 10.1.8.10. *(Classification of coverings) Let (X, x_0) be a path-connected and locally path-connected space. Then,*

1. *(Based version) Two connected covers $(\tilde{X}_1, \tilde{x}_1, p_1)$ and $(\tilde{X}_2, \tilde{x}_2, p_2)$ are equivalent if and only if*

$$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2)) \text{ in } \pi_1(X, x_0).$$

2. *(Unbased version) Two connected covers $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are equivalent if and only if for any $\tilde{x}_1 \in p_1^{-1}(x_0)$ and $\tilde{x}_2 \in p_2^{-1}(x_0)$, we have that*

$$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) \;\&\; p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2)) \text{ are conjugate subgroups of } \pi_1(X, x_0).$$

Proof. 1. (R $\Rightarrow$ L) This is exactly the Lemma 10.1.8.8 above.
(L $\Rightarrow$ R) Suppose the two covers are equivalent. Then there is a homeomorphism $\varphi : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ such that $p_2 \circ \varphi = p_1$. Let its inverse be $\varpi : (\tilde{X}_2, \tilde{x}_2) \to (\tilde{X}_1, \tilde{x}_1)$, which satisfies $p_1 \circ \varpi = p_2$. The former gives us $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*} \circ \varphi_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) \le p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$. Similarly, the latter gives us $p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2)) = p_{1*} \circ \varpi_*(\pi_1(\tilde{X}_2, \tilde{x}_2)) \le p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1))$. Hence we get the equality.

2. (L $\Rightarrow$ R) Choose $\tilde{x}_i \in p_i^{-1}(x_0)$. We know that there is a homeomorphism $\varphi : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ \varphi = p_1$. Hence $\varphi(\tilde{x}_1) \in p_2^{-1}(x_0)$ and may not be equal to $\tilde{x}_2$. So we have two based covers $(\tilde{X}_2, \tilde{x}_2)$ and $(\tilde{X}_2, \varphi(\tilde{x}_1))$ with the same projection map p_2. Now since $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \varphi(\tilde{x}_1))$ are equivalent, then by 1. above, they induce the same subgroups of $\pi_1(X, x_0)$. So if we can show that the subgroups induced by $(\tilde{X}_2, \varphi(\tilde{x}_1))$ and $(\tilde{X}_2, \tilde{x}_2)$ are conjugates, then we would be done. So we reduce to showing that $p_{2*}(\pi_1(\tilde{X}_2, \varphi(\tilde{x}_1)))$ and $p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$ are conjugates. Since $\tilde{X}_2$ is path-connected, therefore we have a path $\gamma : I \to \tilde{X}_2$ such that $\gamma(0) = \varphi(\tilde{x}_1)$ and $\gamma(1) = \tilde{x}_2$. Now recall from proof of Lemma **??** that the following establishes an isomorphism of groups:

$$\begin{aligned} \Phi : \pi_1(\tilde{X}_2, \tilde{x}_2) &\longrightarrow \pi_1(\tilde{X}_2, \varphi(\tilde{x}_1)) \\ [\xi] &\longmapsto [\gamma * \xi * \bar{\gamma}]. \end{aligned}$$

So, applying p_{2*} on the above map Φ yields

$$\begin{aligned} p_{2*}(\Phi) : p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2)) &\longrightarrow p_{2*}(\pi_1(\tilde{X}_2, \varphi(\tilde{x}_1))) \\ [p_2 \circ \xi] &\longmapsto [(p_2 \circ \gamma) * (p_2 \circ \xi) * (p_2 \circ \bar{\gamma})], \end{aligned}$$

which is also an isomorphism. But this tells us more, that each element of $p_{2*}(\pi_1(\tilde{X}_2, \varphi(\tilde{x}_1)))$ can be written as a conjugate of an element of $p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$ by a fixed element $[p_2 \circ \gamma]$, conditioned on the fact that we somehow show that $[\overline{p_2 \circ \gamma}] = [p_2 \circ \bar{\gamma}]$, but that's a tautology. Hence we are done.

(R $\Rightarrow$ L) We are given that there exists $[\gamma] \in \pi_1(X, x_0)$ for any choice of $\tilde{x}_1$ and $\tilde{x}_2$ such that

$$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = [\bar{\gamma}] p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2)) [\gamma].$$

In order to get a homeomorphism $\varphi : (\tilde{X}_1, \tilde{x}_1, p_1) \to (\tilde{X}_2, \tilde{x}_2, p_2)$, we will use statement 1. above. Since we need a homeomorphism φ such that $p_2 \circ \varphi = p_1$, therefore we may show that $p_{1*}(\pi_1(\tilde{X}_1, \tilde{y}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{y}_2))$ for any $\tilde{y}_i \in p_i^{-1}(x_0)$ and then use 1. to conclude the existence of such φ. To show this, we first lift the loop γ in X to a unique path $\tilde{\gamma}$ in $\tilde{X}_2$ where we start the lift at $\tilde{x}_2$ (Theorem 10.1.2.2). Hence we have a path $\tilde{\gamma} : I \to \tilde{X}_2$ where $\tilde{\gamma}(0) = \tilde{x}_2$ and denote $z := \tilde{\gamma}(1) \in p_2^{-1}(x_0)$. Now, if $[p_2 \circ \xi] \in p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$, then $[\bar{\gamma} * (p_2 \circ \xi) * \gamma]$ is equal to $[(p_2 \circ \bar{\tilde{\gamma}}) * (p_2 \circ \xi) * (p_2 \circ \tilde{\gamma})]$ because

$p_2 \circ \tilde{\gamma} = \gamma$, and then we further get that it is equal to $[p_{2*} \circ (\bar{\tilde{\gamma}} * \xi * \tilde{\gamma})]$ where $[\bar{\tilde{\gamma}} * \xi * \tilde{\gamma}] \in \pi_1(\tilde{X}_2, z)$. Conversely, for any $[p_2 \circ \eta] \in p_{2*}(\pi_1(\tilde{X}_2, z))$, we get the loop $[\alpha] := [\tilde{\gamma} * \eta * \bar{\tilde{\gamma}}] \in \pi_1(\tilde{X}_2, \tilde{x}_2)$ which is such that $[\bar{\gamma} * (p_2 \circ \alpha) * \gamma] = [p_2 \circ \eta]$. Hence indeed, we get that $[\bar{\gamma}]p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))[\gamma] = p_{2*}(\pi_1(\tilde{X}_2, z))$. Since $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = [\bar{\gamma}]p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))[\gamma]$, therefore we get $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, z))$, so we are done as now we can take $\tilde{y}_1 := \tilde{x}_1$ and $\tilde{y}_2 := z$. □

Construction of universal cover

We will show some striking results about the group of deck transformations of the universal cover and the fundamental group of the base space. Before that, let us define a class of connected covers which have in some sense maximal symmetry.

Definition 10.1.8.11. (**Normal covers**) Let (X, x_0) be a path-connected and locally path-connected space. A connected cover $p : \tilde{X} \to X$ is said to be normal if for any two $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ there exists a $\varphi \in \mathsf{Deck}(\tilde{X})$ such that $\varphi(\tilde{x}_1) = \tilde{x}_2$.

Clearly, this induces the following map when $\tilde{X}$ is normal:

$$\begin{aligned} \mathsf{Deck}(\tilde{X}) &\longrightarrow S_{p^{-1}(x_0)} \\ \varphi &\longmapsto \varphi\big|_{p^{-1}(x_0)} . \end{aligned}$$

We will use this map later. The following gives a characterization of normal covers.

Lemma 10.1.8.12. *Let (X, x_0) be a path-connected and locally path-connected space. Then, a connected cover $p : \tilde{X} \to X$ is normal if and only if for all $\tilde{x}_0 \in p^{-1}(x_0)$, we have that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$.*

Proof. (L $\Rightarrow$ R) Take any $[\gamma] \in \pi_1(X, x_0)$ and let $\tilde{\gamma}$ be the unique lift of γ in $\tilde{X}$ starting from $\tilde{x}_0 \in \tilde{X}$ (Theorem 10.1.2.2). Denote $\tilde{x}_1 := \tilde{\gamma}(1) \in p^{-1}(x_0)$ as γ is a lift of a loop so both endpoints are in $p^{-1}(x_0)$. Now, since $\tilde{X}$ is normal, therefore there exists $\varphi \in \mathsf{Deck}(\tilde{X})$ such that $\varphi(\tilde{x}_0) = \tilde{x}_1$. Hence $(\tilde{X}, \tilde{x}_0)$ and $(\tilde{X}, \tilde{x}_1)$ are equivalent connected based covers. Therefore by Theorem 10.1.8.10, 1, we get that $H_i := p_*(\pi_1(\tilde{X}, \tilde{x}_i))$[6], $i = 0, 1$, are exactly equal. Now, $[\bar{\gamma}]H_0[\gamma] = [\bar{\gamma}]p_*(\pi_1(\tilde{X}, \tilde{x}_0))[\gamma] = p_*([\bar{\tilde{\gamma}}]\pi_1(\tilde{X}, \tilde{x}_0)[\tilde{\gamma}]) = p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = H_1 = H_0$ where the third to last equality follows from proof of Lemma **??**. Hence H_0 is a normal subgroup.

(R $\Rightarrow$ L) Take any two points $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$. To find the required deck transformation φ, we see that since $(\tilde{X}, \tilde{x}_1)$ and $(\tilde{X}, \tilde{x}_2)$ are two covers such that $H_1 := p_*(\pi_1(\tilde{X}, \tilde{x}_1))$ and $H_2 := p_*(\pi_1(\tilde{X}, \tilde{x}_2))$ are normal subgroups of $\pi_1(X, x_0)$. Now since $\tilde{X}$ is path-connected, therefore there is a path joining $\tilde{x}_1$ to $\tilde{x}_2$ and let us denote it by $\gamma : I \to \tilde{X}$. Now, we get a loop $\xi := p \circ \gamma : I \to X$, based at x_0, and hence $[\xi] \in \pi_1(X, x_0)$. By uniqueness of path lifts (Theorem 10.1.2.2), we see that the lift of ξ (started at $\tilde{x}_1$) indeed has to be γ. We thus get $[\bar{\xi}]H_1[\xi] = [\bar{\xi}]p_*(\pi_1(\tilde{X}, \tilde{x}_1))[\xi] = p_*([\bar{\gamma}]\pi_1(\tilde{X}, \tilde{x}_1)[\gamma]) = p_*(\pi_1(\tilde{X}, \tilde{x}_2)) = H_2$, where second to last equality follows from proof of Lemma **??**. Thus, H_1 and H_2 are conjugate, but both are normal, therefore $H_1 = H_2$ and by Theorem 10.1.8.10, 1, we are done. □

[6] Should have made this notation earlier?

Let us now briefly outline the construction of universal covering space. Let (X, x_0) be a path-connected, locally path-connected and semi-locally simply connected space[7]. For such a space, the universal cover exists and is unique upto isomorphism (in $\mathbf{Cov}(X, x_0)$). We construct the universal cover by quotienting out $\mathrm{Path}_*(X, x_0)$, the space of all paths starting at x_0, by an equivalence relation given by the following:

$$\gamma \sim \eta \iff [\gamma\bar{\eta}] = [c_{x_0}] \in \pi_1(X, x_0).$$

This is a loaded relation, so let us explain. First, γ and η are two elements of $\mathrm{Path}_*(X, x_0)$, so they are paths both starting from x_0. The fact that we are demanding $[\gamma\bar{\eta}] = [c_{x_0}]$ tells us that we are demanding two things: 1) that γ and $\bar{\eta}$ be joinable, that is both γ and η have same end points, and 2) $\gamma\bar{\eta}$ is homotopy equivalent to constant loop x_0. This is indeed an equivalence relation on $\mathrm{Path}_*(X, x_0)$. Hence, by quotienting $\mathrm{Path}_*(X, x_0)$ by this relation we obtain a quotient, denoted:

$$\tilde{X} := \mathrm{Path}_*(X, x_0)/\sim .$$

This inherits a topology from compact-open topology of $\mathrm{Path}_*(X, x_0)$. Let us only state what is a basis of that topology, because verifying that indeed is so will unnecessarily deviate us from our goal. A basis of $\tilde{X}$ is given by subsets of the following form: for each path-connected, locally path-connected and semi-locally simply connected open subset $U \subseteq X$ and any $[\gamma] \in \tilde{X}$ whose endpoint lies in U, define

$$U_{[\gamma]} := \{[\gamma\alpha] \in \mathrm{Path}_*(X, x_0) \mid \alpha \text{ is contained in } U\}.$$

Such sets $U_{[\gamma]}$ forms a basis of $\tilde{X}$. A basic fact that can be checked about this basis is the following:

$$U_{[\gamma]} \cap U_{[\eta]} \neq \emptyset \implies U_{[\gamma]} = U_{[\eta]}.$$

This is because if $[\gamma\alpha] = [\eta\beta]$, then for any $[\gamma\delta] \in U_{[\gamma]}$, we have $[\gamma\delta] = [\eta\beta\bar{\alpha}\delta] \in U_{[\eta]}$, similarly the converse. We then have the following natural map:

$$\begin{aligned} p : \tilde{X} &\longrightarrow X \\ [\gamma] &\longmapsto \gamma(1). \end{aligned}$$

This is indeed well-defined. Moreover, it's a covering map as for any $x = \gamma(1) \in X$ for some path γ and any p.c., l.p.c., s.l.s.c. open set $U \ni x$, we get $p^{-1}(U) = \coprod_{[\alpha]\in\pi_1(X,x_0)} U_{[\alpha\gamma]}$. Finally, note that $\tilde{X}$ is simply-connected.

Construction of a connected cover from a subgroup

Construction 10.1.8.13. Let (X, x_0) be a connected, path-connected and semi-locally simply connected space. Let $H \leq \pi_1(X, x_0)$ be a subgroup. We will construct a connected cover $(X_H, \tilde{x}_0)$ of X such that $p_*(\pi_1(X_H, \tilde{x}_0)) = H$. This is obtained as follows.

[7]This means that for all $x \in X$, there exists an open set $U \ni x$ which also contains x_0 such that $\iota_*(\pi_1(U, x_0)) = \{0\} \leq \pi_1(X, x_0)$. Note that this doesn't necessarily means that $\pi_1(U, x_0) = \{0\}$(!)

Consider the following map:

$$H \times \mathrm{Path}_* (X, x_0)/ \sim \longrightarrow \mathrm{Path}_* (X, x_0)/ \sim$$
$$([\alpha], [\gamma]) \longmapsto [\alpha * \gamma].$$

This is well-defined because if $([\alpha], [\gamma]) = ([\beta], [\eta])$, then $[\alpha*\gamma] = [\beta*\eta]$ in $\mathrm{Path}_* (X, x_0)/ \sim$ obtained by concatenating the two homotopies. Moreover, we have the following

$$([c_{x_0}], [\gamma]) \mapsto [\gamma]$$
$$([\alpha], [\beta\gamma]) \mapsto [\alpha\beta\gamma].$$

So we have that the group H acts on the universal covering space $\mathrm{Path}_* (X, x_0)/ \sim = \tilde{X}$. Now, consider the quotient $\tilde{X}/H$. Explicitly, this is the quotient of $\tilde{X}$ obtained by the relation

$$[\gamma] \sim_H [\eta] \iff \exists[\alpha] \in H \text{ s.t. } [\gamma] = [\alpha\eta].$$

The above holds if and only if $\gamma(1) = \eta(1)$, hence $\gamma\bar{\eta}$ is a loop of X based at x_0. The relation above can thus be read as:

$$[\gamma] \sim_H [\eta] \iff [\gamma\bar{\eta}] \in H.$$

Now, note that the quotient space $X_H := \tilde{X}/H$ will identify certain decks of the cover. Let us explain. Let $\gamma(1) = x \in X$ for some path γ in X and $U \subseteq X$ be an evenly covered neighborhood of x. Therefore

$$p^{-1}(U) = \coprod_{[\alpha]\in\pi_1(X,x_0)} U_{[\alpha\gamma]}.$$

That is, the cardinality of decks is exactly the order of $\pi_1(X, x_0)$. Now, when we apply the quotient map $q : \tilde{X} \twoheadrightarrow \tilde{X}/H$, we get that

$q(U_{[\xi]})$ and $q(U_{[\eta]})$ are identified if and only if $[\xi] = [\alpha\eta]$ for some $[\alpha] \in \pi_1(X, x_0)$

Hence, applying q on $p^{-1}(U)$ will give us

$$\begin{aligned} q(p^{-1}(U)) &= q\left(\coprod_{[\alpha]\in\pi_1(X,x_0)} U_{[\alpha\gamma]}\right) \\ &= \bigcup_{[\alpha]\in\pi_1(X,x_0)} q(U_{[\alpha\gamma]}) \\ &= \coprod_{[\alpha]\in H} q(U_{[\alpha\gamma]}). \end{aligned}$$

Now since q is a quotient map and $p : \tilde{X} \to X$ is map such that p identifies all elements of an equivalence class of $\tilde{X}/H$, therefore we have a unique map $p_H : X_H \to X$, which is the required covering map corresponding to subgroup H. Moreover, one can show that $p_{H*}(\pi_1(X_H, \tilde{x}_0)) = H$.

10.1.9 Covers of $\mathbb{R}P^2 \times \mathbb{R}P^2$

We will classify all covers of this space, and in the process will portray the power of tools developed so far. We first begin with a section on background calculations. The reader interested only in the classification result may safely jump on to Theorem 10.1.9.4 and may refer back to results in the following section whenever it is used in the proof.

Background calculations

Let us begin by trying to understand the structure of $\pi_1(\mathbb{R}P^2)$.

Lemma 10.1.9.1. *The antipodal action of $\mathbb{Z}_2$ on S^n is a free action. This induces a covering map $p : S^n \to \mathbb{R}P^n$.*

Proof. The action is defined by

$$\begin{aligned} \mathbb{Z}_2 \times S^n &\longrightarrow S^n \\ (0, x) &\longmapsto x \\ (1, x) &\longmapsto -x. \end{aligned}$$

So if $x \in S^n$ is any point, then for any $g \in S_{\mathbb{Z}_2}(x)$, we get $g \cdot x = x$. This implies that either $g = 0$ or $x = -x$. Since there is no point in S^n such that $x = -x$, therefore $g = 0$. So the action is free. Now, since $\mathbb{Z}_2$ is finite and S^n is locally finite, therefore by Lemma 10.1.7.3, 2, we get that this action is properly discontinuous. Now, using Theorem 10.1.7.4, we get that the quotient map $p : S^n \to S^n/\mathbb{Z}_2$ is a covering map. But since $S^n/\mathbb{Z}_2$ is exactly how $\mathbb{R}P^n$ constructed, therefore we have S^n as a cover of $\mathbb{R}P^n$.

Aliter : One can show that we get a covering map $p : S^n \to \mathbb{R}P^n$ by the $\mathbb{Z}_2$ action without using Theorem 10.1.7.4. For this, take any point $[x] \in \mathbb{R}P^n$ where we identify $\mathbb{R}P^n$ as the quotient of S^n by $\mathbb{Z}_2$, so each element of $\mathbb{R}P^n$ represents an equivalence class of two points which are antipodal. To find the required evenly covered neighborhood of $[x]$, we first notice that we get an open subset of S^n, denoted U and it's antipodal version $-U$ such that $x \in U$ and $-x \in -U$ and, most importantly, $U \cap -U = \emptyset$. *This last fact follows most importantly from the fact that the action of $\mathbb{Z}_2$ on S^n is **properly discontinuous**.* Defining p to be the quotient map $p : S^n \to S^n/\mathbb{Z}_2$, we get that $p^{-1}(U) = U$. So we have that p is a 2-sheeted covering of $\mathbb{R}P^n$. This explicit proof shows the importance of the action of the finite group $\mathbb{Z}_2$ being free on S^n. □

Next we calculate the fundamental group of $\mathbb{R}P^2$ and as a result, gets pleasantly surprised in the process.[8]

Lemma 10.1.9.2. $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ *for* $n > 1$.

Proof. Take any $n > 1$. The Lemma 10.1.9.1 tells us that $p : S^n \to \mathbb{R}P^n$ is a covering map for $\mathbb{R}P^n$. We take it as a fact that $\pi_1(S^n) = 0$. Thus, S^n is a simply, path and locally path-connected space where $\mathbb{R}P^n$ is also semi-locally simply connected. Hence by the corollary of **main theorem**

[8] You see, the fact that $\mathbb{R}P^n$ are such weird manifolds to imagine and also the fact that they are not embeddable in $\mathbb{R}^n$ (for $n > 1$) entices and invites one to think that their fundamental group is quite bad and complicated. But it is not so!

of universal covering of a space, we get that $\pi_1(\mathbb{R}P^n) \cong \mathsf{Deck}(S^n)$. It is clear that $\mathsf{Deck}(S^n)$ is just $\mathbb{Z}_2$, as S^n is a 2-sheeted cover of $\mathbb{R}P^n$ (**by Galois equivalence for connected covers**). ☐

Lemma 10.1.9.3. *$\mathbb{R}P^2$ is connected, locally path-connected and semi-locally simply connected.*

Proof. Since $\mathbb{R}^3$ is satisfies all of the three properties and the quotient map $q : \mathbb{R}^3 \twoheadrightarrow \mathbb{R}P^2$ is continuous, so $\mathbb{R}P^2$ is connected. To show that $\mathbb{R}P^2$ is also locally path-connected, take any point $[x] \in \mathbb{R}P^2$, then $l_x := q^{-1}([x]) \subseteq \mathbb{R}^3$ is a line passing through origin in $\mathbb{R}^3$. For any open set $V \ni [x]$ in $\mathbb{R}P^2$, we have $U := q^{-1}(V)$ is open in $\mathbb{R}^3$, containing the line l_x. Choose an $\epsilon > 0$ small enough so that $l_x \times B_\epsilon \subseteq U$. Clearly, $l_x \times B_\epsilon$ is path-connected (it's a solid infinite cylinder with open boundary). Now, since q is a quotient map so $q(l_x \times B_\epsilon)$ is an open set inside V which is path-connected (as it is a continuous image of a path-connected set). Hence $\mathbb{R}P^2$ is both connected and locally path-connected.
Since $\mathbb{R}P^n$ is an n-dimensional manifold, so for each point there is an open neighborhood U which is homeomorphic to an open ball of $\mathbb{R}^n$, which is contractible. Hence $\mathbb{R}P^n$ is semi-locally simply connected. ☐

The classification theorem

Theorem 10.1.9.4. *(Classification of covers of $\mathbb{R}P^2 \times \mathbb{R}P^2$) Each connected cover of $\mathbb{R}P^2 \times \mathbb{R}P^2$ belongs to equivalence class of one of the following:*
1. $\mathbb{R}P^2 \times \mathbb{R}P^2$,
2. $\mathbb{R}P^2 \times S^2$,
3. $S^2 \times \mathbb{R}P^2$,
4. $S^2 \times S^2$,
5. $S^2 \times S^2/\sim$ *where* $\sim$ *is generated by* $(x, y) \sim (-x, -y)$.

Proof. In Lemma 10.1.9.2, we obtained $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$. By Lemma **??**, we get that $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Now, there are the following five subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$:
1. $H_1 = \{(0, 0)\} = \{e\}$,
2. $H_2 = \{(0, 0), (0, 1)\}$,
3. $H_3 = \{(0, 0), (1, 0)\}$,
4. $H_4 = \{(0, 0), (0, 1), (1, 0), (1, 1)\} = \mathbb{Z}_2 \times \mathbb{Z}_2$.
5. $H_5 = \{(0, 0), (1, 1)\}$.

Now, note that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is an abelian group, therefore, each subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is normal. We know the following equivalence:

$$\begin{tikzcd} \{\text{Connected covers of } (X, x_0)\}/\text{equivalence} \arrow[r, bend left, "(\tilde{X}, p) \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0))"] & \{\text{Subgroups of } \pi_1(X, x_0)\}/\text{conjugacy} \arrow[l, bend left, "X_H \longleftarrow\!\shortmid H"] \end{tikzcd}$$

for a path-connected, locally-path connected and semi-locally simply connected space X. Now, remember that X_H for some $H \leq \pi_1(X, x_0)$ is made via quotienting the universal cover of X by the action of H that is obtained by restricting the global action of $\pi_1(X, x_0)$ on $\tilde{X}$, via the deck transformations (we have $\pi_1(X, x_0) \cong \mathsf{Deck}(\tilde{X})$). Hence, X_H will be obtained by identifying the sheets of the universal cover $\tilde{X}$. In our case, $\mathsf{Deck}(\mathbb{R}P^2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $(1, 0)$ acts on $(x, y) \in S^2$

via $(1,0)\cdot(x,y)\mapsto(-x,y)$, similarly for $(0,1),(1,1)$. This gives the required identifications on the sheets and hence the five classes of connected covers as stated above. $\square$

10.2 Cofibrations and cofiber sequences

Most of the long exact sequences appearing in algebraic topology are derived from the topics that we will cover in this chapter. These should rather be seen as an important conceptual tool in order to do computations. We will begin with cofibrations, closed subspaces from whose homotopies can be extended to the whole space, and then fibrations, which can be thought of as generalizations of covering spaces (more generally, fiber bundles) which one studies in a first course in algebraic topology.

Cofibrations can be treated as an intermediary tool for developing more sophisticated concepts in algebraic topology. In particular, we will be using this to derive an exact sequence of groups out of a map of based spaces.

Note that there is little to no difference in based or unbased cofibrations, so we will prove something for unbased context and will use it as it has been proved for based context as well. We will give some remarks towards the end.

10.2.1 Definition and first properties

Definition 10.2.1.1. (**Cofibrations**) A map $i : A \to X$ is a cofibration if it satisfies the homotopy extension property; if $f : X \to Y$ is a continuous map such that there is a homotopy $h : A \times I \to Y$ where $h(-, 0) = f \circ i$, then that homotopy can be lifted to $\tilde{h} : X \times I \to Y$ where $\tilde{h}(-, 0) = f$. More abstractly, if $h \circ i_0 = f \circ i$ in the following diagram, then there exists $\tilde{h}$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"'] \arrow[dr, "i\times\mathrm{id}"] & \\
X \arrow[r, "f"] \arrow[rr, bend right, "i_0"'] & Y & X \times I \arrow[l, dashed, "\tilde{h}"]
\end{tikzcd} .$$

One sees that pushout of a cofibration along any map is a cofibration.

Lemma 10.2.1.2. *Let $i : A \to X$ be a cofibration and $f : A \to B$ be any other map. Then, the pushout $j : B \to B \cup_f X$ is a cofibration.*

Proof. Take any map $g : B \cup_f X \to Y$ and a homotopy $h : B \times I \to Y$ where $h \circ i_0 = g \circ j$. We have the following diagram:

$$\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dddd, "i \text{ (cofibration)}"'] & & & \\
& B \arrow[rr, "i_0"] \arrow[dd, "j"'] & & B \times I \arrow[dl, "j\times\mathrm{id}"] \arrow[dd, "h"] \\
\text{p.o.} & & (B \cup_f X) \times I & \\
& B \cup_f X \arrow[ur, "i_0"] \arrow[rr, "g"'] & & Y \\
X \arrow[ur] & & &
\end{tikzcd} .$$

We wish to show that there is a map $\tilde{h} : (B \cup_f X) \times I \to Y$ which commutes with the diagram shown above. Since we have the following pushout square:

$$\begin{tikzcd}
B \cup_f X & X \arrow[l] \\
B \arrow[u, "j"] \arrow[ur, phantom, "\text{p.o.}"] & A \arrow[l, "f"] \arrow[u, "i"']
\end{tikzcd},$$

therefore after applying functor $- \times I$, which has a right adjoint, so is colimit preserving (we are working in the category of compactly generated spaces which is cartesian closed), we get the following pushout square which is closer to what we have in the first diagram:

$$\begin{tikzcd}
(B \cup_f X) \times I & X \times I \arrow[l] \\
B \times I \arrow[u, "j\times\mathrm{id}"] \arrow[ur, phantom, "\text{p.o.}"] & A \times I \arrow[l, "f\times\mathrm{id}"] \arrow[u, "i\times\mathrm{id}"']
\end{tikzcd}.$$

Now, we get a map h' as below by the virtue of i being a cofibration:

$$\begin{tikzcd}
A \arrow[rrrr, "i_0"] \arrow[dr, "f"] \arrow[dddd, "i\text{ (cofibration)}"'] & & & & A \times I \arrow[dl, "f\times\mathrm{id}"'] \arrow[dddd, "i\times\mathrm{id}"] \\
& B \arrow[rr, "i_0"] \arrow[dd, "j"'] & & B \times I \arrow[dl, "i\times\mathrm{id}"'] \arrow[dd, "h"] & \\
\arrow[r, phantom, "\text{p.o.}"] & {} & (B \cup_f X) \times I & & \\
& B \cup_f X \arrow[ur, "i_0"] \arrow[rr, "g"'] & & Y & \\
X \arrow[ur] \arrow[rrrr, "i_0"'] & & & & X \times I \arrow[ul, dashed, "h'"']
\end{tikzcd}.$$

Next, by the universal property of pushout $(B \cup_f X) \times I$, we get a map $\tilde{h}$

$$\begin{tikzcd}
A \arrow[rrrr, "i_0"] \arrow[dr, "f"] \arrow[dddd, "i\text{ (cofibration)}"'] & & & & A \times I \arrow[dl, "f\times\mathrm{id}"'] \arrow[dddd, "i\times\mathrm{id}"] \\
& B \arrow[rr, "i_0"] \arrow[dd, "j"'] & & B \times I \arrow[dl, "i\times\mathrm{id}"'] \arrow[dd, "h"] & \\
\arrow[r, phantom, "\text{p.o.}"] & {} & (B \cup_f X) \times I \arrow[dr, dashed, "\tilde{h}"] & & \\
& B \cup_f X \arrow[ur, "i_0"] \arrow[rr, "g"'] & & Y & \\
X \arrow[ur] \arrow[rrrr, "i_0"'] & & & & X \times I \arrow[ul, dashed, "h'"']
\end{tikzcd},$$

which satisfies the required commutativity. □

To check that a map $i : A \to X$ is a cofibration, we can reduce to checking the homotopy extension property to the map $X \to Mi$ where Mi is the *mapping cylinder*.

Definition 10.2.1.3 (**Mapping cylinder**)**.** Let $f : X \to Y$ be a map. Then the mapping cylinder of f is the following pushout space

$$\begin{tikzcd} Mf \arrow[dr, phantom, "\ulcorner", very near start] & X \times I \arrow[l] \\ Y \arrow[u] & X \arrow[l, "f"] \arrow[u, "i_0"'] \end{tikzcd} .$$

More explicitly, it is $((X \times I) \amalg Y)/\sim$ where $(x, 0) \sim f(x)$ for all $x \in X$.

Let $f : X \to Y$ be a map. More pictorially, Mf is formed by gluing cylinder $X \times I$ to Y along f. In mind, one pictures a cylinder "popping out" of Y from where $f(X)$ lived in Y, as shown in the following diagram: A based version of mapping cylinder is as follows.

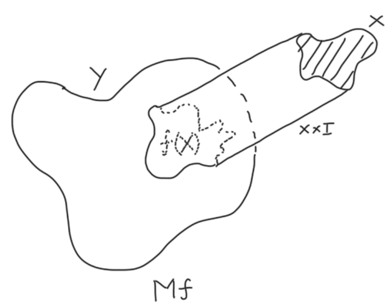


Figure 10.1: Schematic representation of mapping cylinder for $f : X \to Y$.

Definition 10.2.1.4 (**Based mapping cylinder**)**.** Let $f : X \to Y$ be a based map. The based mapping cylinder M_*f is the pushout of reduced cylinder about f:

$$\begin{tikzcd} M_*f \arrow[dr, phantom, "\lrcorner", very near start] & X \wedge I_+ \arrow[l] \\ Y \arrow[u] & X \arrow[l, "f"] \arrow[u, "i_0"'] \end{tikzcd} .$$

Indeed, we have the following result:

Proposition 10.2.1.5. *Let $i : A \to X$ be a map. Then the following are equivalent:*

1. *i is a cofibration.*
2. *i satisfies homotopy extension property for any $f : X \to Y$ and for any Y.*
3. *i satisfies homotopy extension property for the natural map $X \to Mi$ and the homotopy $h : A \times I \to Mi$ obtained from pushout.*

Proof. The only non-trivial part is to show $3 \Rightarrow 2$. Take any map $f : X \to Y$ and any homotopy $h : A \times I \to Y$. Consider

$$\begin{tikzcd}
A \arrow[rrr, "i_0"] \arrow[dd, "i"'] & & & A \times I \arrow[dll] \arrow[dl, "i\times\mathrm{id}"'] \arrow[dd, "h"] \\
& Mi \arrow[drr, dashed, "g"] & X \times I \arrow[l, dashed, "h_1"'] & \\
X \arrow[ur] \arrow[urr, "i_0"] \arrow[rrr, "f"'] & & & Y
\end{tikzcd}.$$

The map h_1 is formed by homotopy extension property of i for $X \to Mi$ and g is formed by universal property of pushout which is Mi. The map $gh_1 : X \times I \to Y$ follows the required commutativity relations. $\square$

Consequently, we have the following result.

Proposition 10.2.1.6. *Any cofibration $i : A \to X$ is an inclusion with closed image.*

Proof. Consider the natural maps $j : X \to Mi$ and $h : A \times I \to Mi$ obtained by the pushout square. Since $hi_0 = ji$, therefore by Proposition 10.2.1.5, 3, we obtain a map $\tilde{h} : X \times I \to Mi$ fitting in the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"] \arrow[dr] & \\
X \arrow[r, "j"'] \arrow[rr, bend right, "i_0"'] & Mi & X \times I \arrow[l, dashed, "\tilde{h}"']
\end{tikzcd}.$$

Let $k : Mi \to X \times I$ be obtained by the following diagram

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"'] \arrow[ddr, "i\times\mathrm{id}"] & \\
X \arrow[r, "j"] \arrow[drr, "i_0"'] & Mi \arrow[dr, dashed, "k"] & \\
& & X \times I
\end{tikzcd}.$$

It follows that $\tilde{h} \circ k : Mi \to Mi$ is id, that is, Mi is a retract of $X \times I$. Consequently, restricting onto $i(A)$, we see that $i(A)$ is a retract of $X \times I$, hence closed as $X \times I$ is compactly generated. It also follows from $\tilde{h} \circ k = \mathrm{id}$ that i is injective. $\square$

We see the following from the proof of the above result.

Corollary 10.2.1.7. *Let $i : A \to X$ be a map. Then the following are equivalent:*
1. *Map $i : A \to X$ is a cofibration.*
2. *Mapping cylinder Mi is a retract of $X \times I$.*

Proof. 1. $\Rightarrow$ 2. is immediate from the proof. For 2. $\Rightarrow$ 1. we seee that if $Mi \hookrightarrow X \times I \twoheadrightarrow Mi$ is a retract, then letting $\tilde{h} : X \times I \twoheadrightarrow Mi$, we have $\tilde{h} \circ i_0 = \mathrm{id}_X$ and $\tilde{h}\Big|_{A\times I} = h$, as needed. $\square$

Let $f : X \to Y$ be an arbitrary map of spaces. We can replace f by a cofibration followed by a homotopy equivalence.

Construction 10.2.1.8 (Replacement by a cofibration and a homotopy equivalence)**.** Let $f : X \to Y$ be a map of spaces. Consider the following commutative triangle:

$$\begin{tikzcd} & Mf \arrow[dr, "r"] & \\ X \arrow[ur, "j"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

where $Mf = Y \cup_f (X \times I)$ is the mapping cylinder and the other two maps are given as follows:

1. Map $j : X \to Mf$ is given by $x \mapsto (x, 1)$. We claim that j is a cofibration. Indeed, if $g : Mf \to Z$ is any map and we have a diagram as in Definition 10.2.1.1, then we can form the required homotopy $\tilde{h} : Mf \times I \to Z$ by defining
$$\tilde{h}([(x,s)],t) := \begin{cases} g(x) & \text{if } x \in Y \\ h(x,st) & \text{if } [(x,s)] \in X \times I. \end{cases}$$
We then see that $\tilde{h}(j \times \mathrm{id})(x,t) = \tilde{h}([(x,1)],t) = h(x,t)$ and that $\tilde{h}i_0([(x,s)]) = \tilde{h}([(x,s)],0) = h(x,0) = g(x)$. So we have the required extension and hence $j : X \to Mf$ is a cofibration.
2. Map $r : Mf \to Y$ is given by $r|_Y = \mathrm{id}_Y$ and $r|_{X\times I}(x,t) = f(x)$ for $t > 0$. We claim that r is a homotopy equivalence. For this, we have a map $i : Y \to Mf$ taking $y \mapsto [y]$. We then see that $ri = \mathrm{id}_Y$ and $ir \simeq \mathrm{id}_{Mf}$. The former is simple and the latter is established by the following homotopy $h : Mf \times I \to Mf$ mapping as $([(x,s)],t) \mapsto [(x,(1-t)s)]$ on $X \times I$ and $(y,t) \mapsto y$ on Y. This is indeed a homotopy from ir to id_{Mf}. Thus, $r : Mf \to Y$ establishes that Y is a deformation retract of the mapping cylinder Mf.

Hence, one can replace a map of spaces $f : X \to Y$ by a cofibration $j : X \to Mf$ followed by a homotopy equivalence $r : Mf \to Y$.

We now discuss an important characterization of cofibrations. For this we define first the following notion.

Definition 10.2.1.9 (**Neighborhood deformation retract**)**.** A pair (X, A) where $A \subseteq X$ is a neighborhood deformation retract (NDR) if there exists a map $u : X \to I$ such that $u^{-1}(0) = A$ and a homotopy $h : X \times I \to X$ such that $h(x,0) = \mathrm{id}_X(x) = x$, $h(a,t) = a$ for all $a \in A$ and all $t \in I$ and $h(x,1) \in A$ if $u(x) < 1$.

Remark 10.2.1.10. Let (X, A) be an NDR-pair. If $u(X) \subseteq [0,1)$, then $A \hookrightarrow X$ is a closed subspace which is a deformation retract of X.

Theorem 10.2.1.11. *Let A be a closed subsapce of X. Then the following are equivalent:*

1. *(X, A) is an NDR-pair.*
2. *$i : A \to X$ is a cofibration.*

We now define the notion of homotopy equivalence under a space. This will come in handy later. Recall that if $\mathbf{C}$ is a category $c \in \mathbf{C}$ is an object, then $\mathbf{C}_{c/}$ denotes the under category at c, i.e., where objects are $i : c \to a$ and maps are commutative triangles

$$\begin{tikzcd} & c \arrow[dl, "i"'] \arrow[dr, "j"] & \\ a \arrow[rr, "f"'] & & b \end{tikzcd} .$$

Definition 10.2.1.12 (Relative homotopy). Let $i : A \to X$ and $j : A \to Y$ be in $\mathbf{Top}^{cg}_{A/}$. Let $f, g : X \rightrightarrows Y$ be maps in $\mathbf{Top}^{cg}_{A/}$. Then $h : X \times I \to Y$ is a homotopy rel A between f and g if $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ and $h(i(a), t) = j(a)$ for all $a \in A$ and $t \in I$.

The notion of homotopy equivalence rel A is special as the Theorem 10.2.1.14 shows, hence we give it the following name.

Definition 10.2.1.13 (Cofiber homotopy equivalence). Let $i : A \to X$ and $j : A \to Y$ be two spaces under A in $\mathbf{Top}^{cg}_{A/}$. If i and j homotopy equivalent under A, then X and Y are said to be cofiber homotopy equivalent.

Theorem 10.2.1.14. *Let $i : A \to X$ and $j : A \to Y$ be two cofibrations under A and $f : X \to Y$ be a map under A. If f is a homotopy equivalence, then f is a cofiber homotopy equivalence.*

Example 10.2.1.15. Let $i : A \to X$ be a cofibration. Then by Construction 10.2.1.8, we have

$$\begin{tikzcd} & Mi \arrow[dr, "r"] & \\ A \arrow[ur, "j"] \arrow[rr, "i"'] & & X \end{tikzcd}$$

where j is a cofibration and r is a homotopy equivalence. Since r is a homotopy equivalence under A, therefore by Theorem 10.2.1.14, r is a cofiber homotopy equivalence. Consequently, there is a homotopy inverse $\kappa : X \to Mi$ of r under A.

The following is a mild generalization of Theorem 10.2.1.14 in the sense that we allow mapping between two cofibration pairs now.

Proposition 10.2.1.16. *Let (X, A) and (Y, B) be two cofibration pairs and let $f : X \to Y$ and $d : A \to B$ be maps such that $f|_A = d$. If f and d are homotopy equivalences, then the map of pairs $(f, d) : (X, A) \to (Y, B)$ is a homotopy equivalence of pairs*[9].

We next portray how a cofibration pair (X, A) in some cases behaves homotopically same as the quotient X/A.

Proposition 10.2.1.17. *Let $i : A \to X$ be a cofibration and A be contractible. Then the quotient map $p : X \twoheadrightarrow X/A$ is a homotopy equivalence.*

[9] as defined in Definition 10.4.1.1.

Proof. As A is contractible, therefore for some $x_0 \in A$, we have a homotopy $h : A \times I \to A$ such that $h_0 = \mathrm{id}_A$ and $h_1 = c_{x_0}$. Consequently, we obtain $\tilde{h}$ as in the commutative square

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"] \arrow[dr, "i\times\mathrm{id}"] & \\ X \arrow[r, "\mathrm{id}"] \arrow[rr, bend right, "i_0"'] & X & X \times I \arrow[l, dashed, "\tilde{h}"'] \end{tikzcd}$$

where we have $\tilde{h}_0 = \mathrm{id}_X$, $\tilde{h}_t(A) \subseteq A$ for all $t \in I$ and $\tilde{h}_1(A) = \{x_0\} \in A$. Consequently, $\tilde{h}_1$ fits in the following diagram

$$\begin{tikzcd} X \arrow[d, two heads, "p"'] \arrow[dr, "\tilde{h}_1"] & \\ X/A \arrow[r, "g"'] & X \end{tikzcd}$$

where $g : X/A \to X$ comes from the universal property of quotients. We claim that g is the required homotopy inverse of p. Indeed, by definition $\tilde{h} : \mathrm{id}_X \simeq g \circ p$. Consequently, we need only show that $\mathrm{id}_{X/A} \simeq p \circ g$. We derive this homotopy from $\tilde{h}$ as well. Indeed, for any $t \in I$, we obtain $\tilde{q}_t$ by universal property of quotients as in

$$\begin{tikzcd} X \arrow[r, "\tilde{h}_t"] \arrow[d, two heads, "p"'] & X \arrow[d, two heads, "p"] \\ X/A \arrow[r, "\tilde{q}_t"'] & X/A \end{tikzcd} .$$

It follows that the homotopy $\tilde{q} : X/A \times I \to X/A$ is such that $\tilde{q}_0 = \mathrm{id}_{X/A}$ and $\tilde{q}_1 = p \circ g$, as needed. □

Let us end this section by discussing how we will tell the same story in the based setting.

Remark 10.2.1.18 (*Based cofibration*)**.** A based map $i : A \to X$ is a based cofibration if it satisfies the based version of homotopy extension property. The following are few remarks which are easily verifiable of the situation in the based case.

1. If a based map $i : A \to X$ is an unbased cofibration, then it is a based cofibration.
2. If $A \subseteq X$ is a closed subspace such that $* \to A$ and $* \to X$ are cofibrations and $i : A \to X$ is a based cofibration, then $i : A \to X$ is an unbased cofibration.
3. A based map $i : A \to X$ is a based cofibration if and only if M_*i is a retract of $X \wedge I_+$.

We see the following example of above remark.

Lemma 10.2.1.19. *Let X be a based space. Then the inclusion $X \hookrightarrow CX$ to the base of the cone*

1. *is a deformation retract,*
2. *is a cofibration.*

Proof. The inclusion map is $x \mapsto [x, 0]$. The fact that X is deformation retract is immediate by the based homotopy $h : CX \times I \to CX$ given by $([x, t], s) \mapsto [x, t(1-s)]$. We will use Remark 10.2.1.18, 3 for showing $i : X \hookrightarrow CX$ is a cofibration. Indeed, consider the map $CX \wedge I_+ \to M_*i$ given by $[[x, t], s] \mapsto [x, s + t]$. The inclusion $M_*i \to Y \wedge I_+$ is the map which on CX is $[x, t] \mapsto [[x, t], 0]$ and on $X \wedge I_+$ is $[x, t] \mapsto [[x, 0], t]$. One checks that this makes M_*i a retract of $CX \wedge I_+$. □

10.2.2 Based cofiber sequences

The main point of cofiber sequences is to obtain an exact sequence of groups, which will prove to be helpful later. All cofibrations in this section are based cofibrations. We first observe that $[\Sigma X, Y]$ is a group.

Proposition 10.2.2.1. *Let X, Y be based spaces. Then*

1. *$[\Sigma X, Y]$ is a group under concatenation,*
2. *$[\Sigma^2 X, Y]$ is an abelian group under the same operation.*

Proof. The concatenation operation here is as follows : for $f, g \in \mathrm{Map}_*(\Sigma X, Y)$, define $f + g$ as

$$(f+g)([(x,t)]) := \begin{cases} f([(x,2t)]) & \text{if } 0 \leq t \leq 1/2 \\ g([x,2t-1]) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

This tells us that $[\Sigma X, Y] \cong [X, \Omega Y]$ is a group. The second statement uses Theorem 10.0.0.8 to observe that a map $\Sigma^2 X \to Y$ is a map $S^1 \wedge S^1 = S^2 \to \mathrm{Map}_*(X, Y)$. Hence we reduce to showing that $[S^2, X]$ is an abelian group, this is well-known. □

Definition 10.2.2.2. (**Homotopy cofiber/Mapping cone**) Let $f : X \to Y$ be a based map and let $j : X \to M_* f$, $x \mapsto (x, 1)$ be it's cofibrant replacement. The homotopy cofiber Cf of f is defined to be the quotient of the based mapping cylinder $M_* f$ of f by the image of the map j taking $x \mapsto (x, 1)$. That is,

$$Cf := M_* f / j(X).$$

Alternatively, it is the pushout $Cf = Y \cup_f CX$.

There is a relationship between unbased cofiber and based cofiber.

Lemma 10.2.2.3. *Let X be an unbased space. Then the unreduced cone of X is isomorphic to the reduced cone of pointification of X. That is,*

$$CX \cong CX_+.$$

Proof. We have

$$\begin{aligned} CX_+ = X_+ \wedge I = \frac{X_+ \times I}{\{\mathrm{pt.}\} \times I \amalg X \times \{1\}} &= \frac{X \times I \amalg \{\mathrm{pt.}\} \times I}{\{\mathrm{pt.}\} \times I \amalg X \times \{1\}} \\ \cong \frac{X \times I}{X \times \{1\}} = CX, \end{aligned}$$

as needed. □

This is an important observation, as it says that unreduced homotopy cofiber is isomorphic to the homotopy cofiber of the poinitification.

Proposition 10.2.2.4. *Let X, Y be unbased spaces and $f : X \to Y$ be an unbased map. Then the unreduced homotopy cofiber of f is isomorphic to the homotopy cofiber of $f_+ : X_+ \to Y_+$. That is,*

$$Cf \cong Cf_+.$$

Proof. By Lemma 10.2.2.3, we can write

$$Cf_+ = Y_+ \cup_{f_+} CX_+ \cong Y_+ \cup_{f_+} CX$$

where $X_+ \to CX$ is the map which takes pt. $\mapsto [x,1]$ as the basepoint of CX is $[x,1]$. Consequently, $Y_+ \cup_{f_+} CX$ is isomorphic to $Y \cup_f CX$. $\square$

Remark 10.2.2.5. It follows from Proposition 10.2.2.4 that there is really no difference between reduced and unreduced cofiber as unreduced cofiber is really a special case of reduced cofiber by pointification.

The following result shows that the homotopy cofiber of a based cofibration is is of the same homotopy type as X/A. This is an important property of cofibrations.

Proposition 10.2.2.6. *Let $i : A \to X$ be a based cofibration between based spaces. Then,*

1. *$Ci/CA \cong X/A$,*
2. *$\pi : Ci \to Ci/CA$ is a based homotopy equivalence.*

Proof. **TODO.** $\square$

Pictorially, one sees that the mapping cone Cf of $f : X \to Y$ is obtained by gluing Y to the cone of X at it's base. We are now ready to construct cofiber sequence of a based map $f : X \to Y$.

Construction 10.2.2.7 (Cofiber sequence)**.** Let $f : X \to Y$ be a based map and denote Cf to be the mapping cone of f. We have a natural map $i : Y \to Cf$ which is the inclusion of Y into the mapping cone. This is a cofibration because it is the pushout (Lemma 10.2.1.2) of the inclusion $X \to CX$ of X into the 0-th level of the cone CX and this inclusion is a cofibration (Lemma 10.2.1.19). The sequence $X \to Y \to Cf$ is called the *short cofiber sequence of f*.

Consider also the map $-\Sigma f : \Sigma X \to \Sigma Y$ which maps $[(x,t)] \mapsto [(f(x), 1-t)]$. We have another natural map from the mapping cone to its quotient by Y given by $\pi : Cf \to Cf/Y \cong \Sigma X$. We then get the following sequence of based maps, called the *long cofiber sequence of map f*:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, tail, "i"] & Cf \arrow[dll, two heads, "\pi"'] \\
\Sigma X \arrow[r, "-\Sigma f"'] & \Sigma Y \arrow[r, "-\Sigma i"] & \Sigma Cf \arrow[dll, "-\Sigma\pi"'] \\
\Sigma^2 X \arrow[r, "\Sigma^2 f"'] & \Sigma^2 Y \arrow[r, "\Sigma^2 i"] & \Sigma^2 Cf \arrow[dll, dash]
\end{tikzcd}$$

The main theorem that will be used continuously elsewhere is that cofiber sequence of a map gives a long exact sequence in homotopy sets. First, recall that for any based space Z, we have the homotopy classes of maps $[X, Z]$. Moreover, $[-, Z]$ is contravariantly functorial as for any based map $f : X \to Y$, we get

$$\begin{aligned}[f, Z] : [Y, Z] &\longrightarrow [X, Z] \\ g &\longmapsto g \circ f.\end{aligned}$$

We are now ready to state the main theorem.

Theorem 10.2.2.8 (Main theorem of cofiber sequences)**.** *Let $f : X \to Y$ be a based map and Z be a based space in $\mathbf{Top}_*^{cg}$. Then the functor $[-, Z]$ applied on the long cofiber sequence of f yields a long exact sequence of based sets:*

$$\begin{tikzcd}
 & & {} \\
{[\Sigma^2 Cf, Z]} \arrow[urr, dash] & {[\Sigma^2 Y, Z]} \arrow[l] & {[\Sigma^2 X, Z]} \arrow[l] \\
{[\Sigma Cf, Z]} \arrow[urr] & {[\Sigma Y, Z]} \arrow[l] & {[\Sigma X, Z]} \arrow[l] \\
{[Cf, Z]} \arrow[urr, "\pi_*"] & {[Y, Z]} \arrow[l, "i^*"] & {[X, Z]} \arrow[l, "f^*"]
\end{tikzcd}$$

The proof of this theorem relies on the following fundamental observation.

Proposition 10.2.2.9. *Let $f : X \to Y$ be a based map and Z be a based space. Consider the short cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} Cf.$$

Then the sequence of based sets

$$[Cf, Z] \longrightarrow [Y, Z] \longrightarrow [X, Z]$$

is exact.

Proof. Let $g \in [Y, Z]$ such that $gf \simeq c_*$ in $[X, Z]$. We wish to show that there is a map $k \in [Cf, Z]$ such that $ki \simeq g$ in $[Y, Z]$. We first have a based homotopy $h : X \times I \to Z$ between gf and c_*. As h is constant on $X \vee I$, therefore we obtain a map $\bar{h} : CX \to Z$. Note that the following pushout diagram commutes so to give a unique map $k : Cf \to Z$

$$\begin{tikzcd}
Z & & \\
 & Cf \arrow[ul, dashed, "k"] & CX \arrow[l] \arrow[ull, bend right, "\bar{h}"'] \\
 & Y \arrow[u, "i"] \arrow[uul, bend left, "g"] \arrow[ur, phantom, "\ulcorner"] & X \arrow[l, "f"] \arrow[u, "i_0"']
\end{tikzcd}$$

Hence we have that $ki = g$, hence we don't even need to construct a homotopy between ki and g. $\square$

We will now show that each term in the cofiber sequence is obtained by taking cofiber of the previous map. For that, we would need the following small result.

Lemma 10.2.2.10. *Let $f : X \to Y$ be a based map. Then,*

1. *We have a natural based homeomorphism $\Sigma Cf \cong C\Sigma f$.*

2. *The suspension functor takes the short cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} Cf$$

to a short cofiber sequence

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma i} \Sigma Cf.$$

Proof. The first one follows from Σ being a left adjoint. The second statement follows from first statement as we have the following isomorphism

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma i"] \arrow[dr, hook] & \Sigma Cf \arrow[d, "\cong"] \\ & & C\Sigma f \end{tikzcd}.$$

This completes the proof. □

Proposition 10.2.2.11. *Let $f : X \to Y$ be a based map. Then each consecutive pair of maps in the long cofiber sequence of f is a short cofiber sequence.*

Proof. Note that the following square commutes

$$\begin{tikzcd} \Sigma Cf \arrow[r, "\Sigma\pi"] \arrow[d, "\cong"'] & \Sigma^2 X \arrow[r, "-\Sigma^2 f"] \arrow[d, "\tau"] & \Sigma^2 Y \arrow[d, equal] \\ C\Sigma f \arrow[r, "\pi'"'] & \Sigma^2 X \arrow[r, "\Sigma^2 f"'] & \Sigma^2 Y \end{tikzcd}$$

where $\tau([x,t,s]) = [x,s,t]$ is a homeomorphism and $\pi' : C\Sigma f \to C\Sigma f/\Sigma Y$ is the quotient map. We claim that τ is homotopic to $-\mathrm{id}$, where $(-\mathrm{id})([x,t,s]) = [x,t,1-s]$. With this claim and Lemma 10.2.2.10, we would reduce to showing that $Y \to Cf \to \Sigma X$ and $Cf \to \Sigma X \to \Sigma Y$ in the cofiber sequence of f are short cofiber sequences.

To see a based homotopy between τ and $-\mathrm{id}$ as based maps $\Sigma^2 X \to \Sigma^2 X$, we see that the following map will work

$$\begin{aligned} h : \Sigma^2 X \times I &\longrightarrow \Sigma^2 X \\ ([x,t,s],r) &\longmapsto [x,(1-r)s+rt,(1-r)t+r(1-s)]. \end{aligned}$$

We now wish to show that the two pairs are short cofiber sequenes. The fact that $Y \to Cf \to \Sigma X$ is a short cofiber sequence is immediate from Proposition 10.2.2.6 as it will yield the following diagram

$$\begin{tikzcd} & & Ci \arrow[d, dash, "\simeq"] \\ Y \arrow[r, "i"'] & Cf \arrow[ur, "\pi'"] \arrow[r, "\pi"'] & \Sigma X \end{tikzcd}.$$

The fact that $Cf \to \Sigma X \to \Sigma Y$ is also a short cofiber sequence follows from the following diagram which can be seen to be commutative, albeit requires a lot of work:

$$\begin{tikzcd} Cf \arrow[r, "\pi"] \arrow[d, equal] & \Sigma X \arrow[r, "-\Sigma f"] \arrow[d, "\simeq"] & \Sigma Y \arrow[d, "\cong"] \\ Cf \arrow[r, "\pi'"'] & Ci \arrow[r, "\pi''"'] & Ci/Cf \end{tikzcd}.$$

This completes the proof. □

10.3 Fibrations and fiber sequences

We now study fibrations, which is a generalization of covering spaces. Indeed, recall that covering spaces satisfies homotopy lifting property. That *becomes* the definition of a fibration. Indeed, one can have a fruitful time reading about fibrations by keeping the basic results about covering spaces in mind. We'll see that familiar objects from geometry are fibrations (fiber bundles, for example).

10.3.1 Definition and first properties

Definition 10.3.1.1 (**Fibrations**). A surjective map $p : E \to B$ is a fibration if it satisfies homotopy lifting property. That if, for any map $f : Y \to E$ and any homotopy $h : Y \times I \to B$ such that $p \circ f = h \circ i_0$, there exists $\tilde{h} : Y \times I \to E$ such that the following commutes

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}.$$

Just as pushouts of cofibrations along any map is a cofibration, we have pullback of a fibration along any map is a fibration.

Lemma 10.3.1.2. *Let $p : E \to B$ be a fibration and $g : A \to B$ be any map. Then the pullback of p along g given by $p' : E \times_B A \to A$ is a fibration.*

Proof. Consider the following diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \times_B A \arrow[r, "\pi"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] & A \arrow[r, "g"'] & B \end{tikzcd}.$$

As p is a fibration, we yield a homotopy $\tilde{h}_1 : Y \times I \to E$ as in

$$\begin{tikzcd} Y \arrow[r, "\pi f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "gh"'] \arrow[ur, dashed, "\tilde{h}_1"] & B \end{tikzcd}.$$

Consequently, we get a pullback diagram

$$\begin{tikzcd} Y \times I \arrow[r, dashed, "!\tilde{h}"] \arrow[rr, bend left, "\tilde{h}_1"] \arrow[dr, "h"'] & E \times_B A \arrow[r, "\pi"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ & A \arrow[r, "g"'] & B \end{tikzcd}$$

which yields $\tilde{h} : Y \times I \to E \times_B A$. We claim that this is the required homotopy extension. We immediately have $p'\tilde{h} = h$ from the above diagram. We need only show that $\tilde{h}i_0 = f$. To this end, consider the following pullback square

$$\begin{tikzcd} Y \arrow[r, dashed, "!\kappa"] \arrow[rr, bend left, "\pi f"] \arrow[dr, "hi_0"'] & E \times_B A \arrow[r, "\pi"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ & A \arrow[r, "g"'] & B \end{tikzcd}$$

which yields a unique $\kappa : Y \to E \times_B A$. It follows that both f and $\tilde{h}i_0$ satisfies the same commutation properties as κ. It follows from uniqueness of κ w.r.t. these properties that $\tilde{h}i_0 = f$, as required. $\square$

We now introduce a sort of intermediary space for further studying fibrations.

Definition 10.3.1.3 (Mapping path space). Let $f : X \to Y$ be a map. The mapping path space Nf is defined to be the following pullback

$$\begin{tikzcd} Nf := X \times_Y Y^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y^I \arrow[r, "p_0"'] & Y \end{tikzcd}$$

where $p_0 : Y^I \to Y$ takes $\gamma \mapsto \gamma(0)$.

Remark 10.3.1.4. Consequently, the mapping path space $Nf = \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(0)\}$. Hence a point in Nf is the data of a point $x \in X$ *upstairs* and a path $\gamma \in Y^I$ starting *downstairs* at the image of x under f.

With regards to mapping path spaces, one important type of function Nf is that of a *path lifters.*

Definition 10.3.1.5 (Path lifters). Let $f : X \to Y$ be a map. Let $k : X^I \to Nf$ be the unique map obtained by the following pullback diagram

$$\begin{tikzcd} X^I \arrow[r, dashed, "k"] \arrow[rr, bend left, "p_0"] \arrow[dr, "f^I"'] & Nf \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ & Y^I \arrow[r, "p_0"'] & Y \end{tikzcd} .$$

A path lifter $s : Nf \to X^I$ is a global section of k, i.e. $k \circ s = \mathrm{id}_{Nf}$.

Remark 10.3.1.6. The main content of a path lifter $s : Nf \to X^I$ is the fact that its a global section of k. That is, if we let $\tilde{\gamma} = s(x,\gamma) \in X^I$, then $k(\tilde{\gamma}) = (p_0(\tilde{\gamma}), f \circ \tilde{\gamma}) = (x,\gamma)$. It follows that $s(x,\gamma) = \tilde{\gamma}$ is a lift of the path $\gamma \in Y^I$ starting at $f(x)$ to a path $\tilde{\gamma} \in X^I$ starting at x. We may keep the following picture in mind (Figure 10.2).

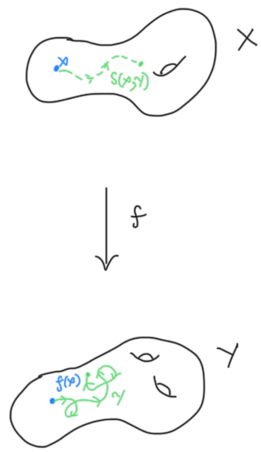


Figure 10.2: Path lifter s taking (x,γ) downstairs to a lift $s(x,\gamma)$ in X upstairs.

Remark 10.3.1.7. (*Covering maps have a unique path lifter*). Recall that a covering space $p : E \to B$ has *unique* homotopy lifting property, hence in particular it is a cofibration. Furthermore recall that a covering space also has *unique* path lifting property, hence in particular it has a unique path-lifter.

We have the following reduction of fibration criterion to mapping path space.

Proposition 10.3.1.8. *Let $p : E \to B$ be a surjective map. Then the following are equivalent:*

1. *p is a fibration.*
2. *p satisfies homotopy lifting property for the natural projection map $Np \to E$.*

Proof. 1. $\Rightarrow$ 2. is definition. For 2. $\Rightarrow$ 1. we proceed as follows. Consider the following diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \arrow[dr, phantom, "\llcorner"] & Np \arrow[l, "\pi"'] \arrow[d, "\eta"] \\
Y \times I \arrow[r, "h"'] & B & B^I \arrow[l, "p_0"]
\end{tikzcd} .$$

We may write $h : Y \times I \to B$ as $h^T : Y \to B^I$. Observe that $p_0 h^T = pf$, leading to the following unique map $\kappa : Y \to Np$ as below

$$\begin{tikzcd}
Y \arrow[r, dashed, "\kappa"] \arrow[rr, bend left, "f"] \arrow[dr, "h^T"'] & Np \arrow[r, "\pi"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\
& B^I \arrow[r, "p_0"'] & B
\end{tikzcd} .$$

Similar to h^T, we also have $\eta^T : Np \times I \to B$. It is immediate from $\eta\kappa = h^T$ that $\eta^T(\kappa \times \mathrm{id}) = h : Y \times I \to B$. Consequently, we have the following commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "\kappa"] \arrow[d, "i_0"'] \arrow[rr, bend left, "f"] & Np \arrow[r, "\pi"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "\kappa\times\mathrm{id}"'] \arrow[rr, bend right, "h"'] & Np \times I \arrow[r, "\eta^T"'] \arrow[ur, dashed, "\tilde{\eta}^T"'] & B
\end{tikzcd}$$

and composing $\tilde{\eta}^T$ with $\kappa \times \mathrm{id}$ yields the required lift of h. □

Proposition 10.3.1.9. *Let $p : E \to B$ be a map. Then the following are equivalent:*

1. *$p : E \to B$ is a fibration.*
2. *There exists a path lifter $s : Np \to E^I$.*

Proof. The forward direction is immediate from dualizing the homotopy lifting property into mappings into path space. For the converse, use Proposition 10.3.1.8. □

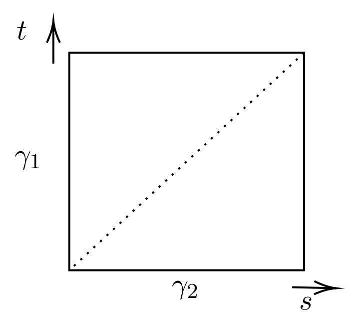
We see that map that the canonical maps $p_0, p_1 : Y^I \to Y$ is a fibration.

Lemma 10.3.1.10. *Let Y be a space. The map*

$$\begin{aligned} p_0 : Y^I &\longrightarrow Y \\ \gamma &\longmapsto \gamma(0) \end{aligned}$$

is a fibration.

Proof. By Proposition 10.3.1.9, it suffices to show that there is a path lifter $s : Np_0 \to Y^{I\times I}$, i.e. a global section of $k : Y^{I\times I} \to Np_0$ mapping $h(s,t) \mapsto (h(s,0), h(0,t))$. Indeed, we define $s((\gamma_1, \gamma_2))$ for $\gamma_i \in Y^I$ such that $\gamma_1(0) = \gamma_2(0)$ by the following homotopy square:



This gives us a map $h \in Y^{I\times I}$ such that $h(0,t) = \gamma_1$ and $h(s,0) = \gamma_2$. This completes the proof. □

Let $f : X \to Y$ be an arbitrary map of spaces. We can replace f by a homotopy equivalence followed by a fibration.

Construction 10.3.1.11 (Replacement by a homotopy equivalence and a fibration)**.** Let $f : X \to Y$ be a map. Consider the following commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\nu"'] & & Y \\ & Nf \arrow[ur, two heads, "\rho"'] & \end{tikzcd}$$

where

$$\begin{aligned} \nu : X &\longrightarrow Nf \\ x &\longmapsto (x, c_{f(x)}) \end{aligned}$$

and

$$\begin{aligned} \rho : Nf &\longrightarrow Y \\ (x,\gamma) &\longmapsto \gamma(1). \end{aligned}$$

We now make the following claims:

1. Map ν is a homotopy equivalence. Indeed, consider the natural projection map $\pi : Nf \to X$ given by $(x,\gamma) \mapsto x$. We claim that π is a homotopy inverse of ν. Indeed, $\pi\nu = \mathrm{id}_X$ is immediate. We claim $\nu\pi \simeq \mathrm{id}_{Nf}$. Indeed, we may consider the homotopy

$$\begin{aligned} h : Nf \times I &\longrightarrow Nf \\ ((x,\gamma),t) &\longmapsto (x,\gamma_t) \end{aligned}$$

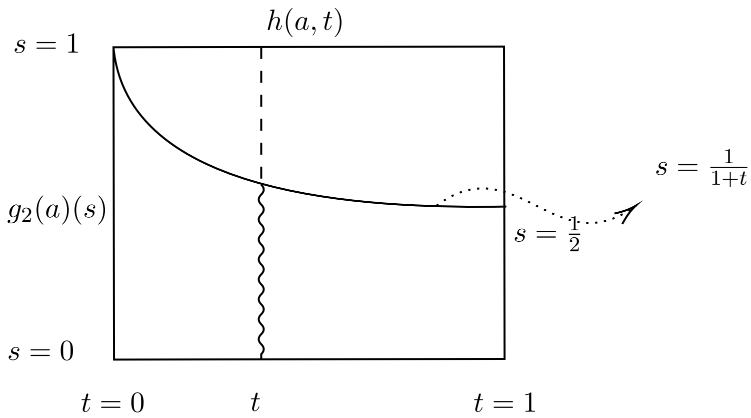
where $\gamma_t(s) = \gamma((1-t)s)$.

2. Map ρ is a fibration. Let $g : A \to Nf$ be a map such that the following square commutes

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"'] & Nf \arrow[d, "\rho"] \\ A \times I \arrow[r, "h"'] & Y \end{tikzcd} .$$

We wish to construct $\tilde{h} : A \times I \to Nf$ which would lift h. Indeed, let $g(a) = (g_1(a), g_2(a))$ where $g_1 : A \to X$ and $g_2 : A \to Y^I$ are the component functions. In order to construct $\tilde{h}$, we need only construct $\alpha : A \times I \to Y^I$ and $\beta : A \times I \to X$ such that the following holds (these are obtained by unravelling $\rho\tilde{h} = h$, $\tilde{h}i_0 = g$ and the respective pullback square):
   (a) $f\beta = p_0\alpha$,
   (b) $\beta(a,0) = g_1(a)$,
   (c) $\alpha(a,0) = g_2(a)$,
   (d) $\alpha(a,t)(1) = h(a,t)$.

   We may immediately set $\beta(a,t) = g_1(a)$. For $\alpha : A \times I \to Y^I$, we may dually write α as $\alpha : A \times I \times I \to Y$ (recall we are in compactly generated spaces, where the dual notion of homotopy is equivalent to the usual one). We construct this α as follows. Fix $a \in A$. We then define the following homotopy



which more explicitly is given by

$$\alpha(a,t,s) = \begin{cases} g_2(a)(s\cdot(1+t)) & \text{if } 0 \le s \le \frac{1}{1+t} \\ h(a, s\cdot(1+t)-1) & \text{if } \frac{1}{1+t} \le s \le 1. \end{cases}$$

One can then observe that this α satisfies conditions (a), (c) and (d) mentioned above.

10.3.2 Bundles and change of fibers

We now see that, under some mild hypothesis, fibration is a local property on base. As a consequence, we will show that under some mild hypothesis any bundle (Definition 8.7.1.2) is a fibration.

An open cover $\{U_\alpha\}$ of B will be called *numerable* if for each α, there is a map $f_\alpha : B \to I$ such that $f_\alpha^{-1}((0,1]) = U_\alpha$ and $\{U_\alpha\}$ is a locally finite cover.

Theorem 10.3.2.1. *Let $p : E \to B$ be a map and $\{U_\alpha\}$ be a numerable open cover of B. Then the following are equivalent:*

1. *$p : E \to B$ is a fibration.*
2. *$p : p^{-1}(U_\alpha) \to U_\alpha$ is a fibration for each α.*

Proof. 1. $\Rightarrow$ 2. is immediate from Lemma 10.3.1.2.

(2. $\Rightarrow$ 1.) The main idea is to patch up the lifts of a homotopy that we obtain by virtue of each $p|_{p^{-1}(U_\alpha)}$ being a fibration. **TODO**. □

We claimed in the beginning that fibrations are upto homotopy generalizations of covering spaces/certain bundles. We know that such objects have homeomorphic fibres (say, when base is path-connected). This fact can be generalized to fibrations which would yield that fibres of a fibration may not be homeomorphic, but will be of same homotopy type!

Construction 10.3.2.2. (*Homotopy invariance of path-lifting for fibrations*). We now show that a path γ in the base gives a map of fibers which is invariant under homotopy class of γ.

In particular, let $p : E \to B$ be a fibration and $\gamma : I \to B$ be a path from b to b' in B. Let E_b and $E_{b'}$ be fibers at b and b' respectively under p. We claim that we get a map $\tilde{\gamma} : E_b \to E_{b'}$ whose homotopy class is independent of the path γ upto homotopy.

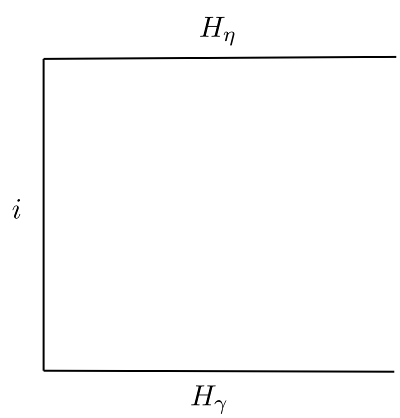
We first construct $\tilde{\gamma} : E_b \to E_{b'}$. Indeed, we have the following diagram

$$\begin{tikzcd} E_b \arrow[rr, "i"] \arrow[d, "i_0"'] & & E \arrow[d, "p"] \\ E_b \times I \arrow[r, "\pi_2"'] \arrow[urr, dashed, "H_\gamma"] & I \arrow[r, "\gamma"'] & B \end{tikzcd}$$

by virtue of fibration p. Observe that $H_{\gamma,1}(e) = H_\gamma(e,1)$ is such that $pH_\gamma(e,1) = \gamma(1) = b'$ for all $e \in E_b$. Consequently, $\tilde{\gamma} = H_{\gamma,1} : E_b \to E_{b'}$ is the required map. This shows the construction of $\tilde{\gamma}$. We now show that its homotopy class is invariant of homotopy class of γ.

Let $\gamma, \eta \in B^I$ be two paths joining b and b' together with a homotopy $h : I \times I \to B$ rel $\{0,1\}$ such that $h_0 = \gamma$ and $h_1 = \eta$, that is h is a homotopy between γ and η through paths joining b and b'. We wish to show that $\tilde{\gamma}$ and $\tilde{\eta}$ are homotopy equivalent as well. To this end, we need to construct a homotopy $\tilde{h} : E_b \times I \to E_{b'}$ satisfying $\tilde{h}_0 = \tilde{\gamma} = H_{\gamma,1}$ and $\tilde{h}_1 = \tilde{\eta} = H_{\eta,1}$.

Fix an $e \in E_b$. Our goal is to fill the right side of this square continuously with $e \in E_b$



where $i : E_b \hookrightarrow E$ the inclusion. To this end, we first observe that there is a homeomorphism of pairs

$$(I \times I, S) \xrightarrow{\alpha} (I \times I, I \times 0)$$

where S is the union of three sides of the square as shown above; $S = I \times \{0,1\} \cup \{0\} \times I$. Using this homeomorphism, we obtain the following square

$$\begin{tikzcd} E_b \times S \arrow[rr, "f"] \arrow[d, "k"'] & & E \arrow[d, "p"] \\ E_b \times I \times I \arrow[r, "\kappa"'] \arrow[urr, dashed, "l"] & I \times I \arrow[r, "h"'] & B \end{tikzcd}$$

where $k = \iota(\mathrm{id} \times \alpha)$ where $\iota : E_b \times (I \times 0) \hookrightarrow E_b \times (I \times I)$ and $\kappa(e,t,s) = \alpha^{-1}(t,s)$. Moreover, $f : E_b \times S \to E$ is defined as in the incomplete square above; on $I \times \{0\}$, f is given by H_γ, on $I \times \{1\}$, f is given by H_η and on $0 \times I$, f is given by i. .Observe that $\kappa k(e,t,s) = (t,s)$. The fact that this is a commutative square is immediate. It follows from p being a fibration that there is a lift $l : E_b \times I \times I \to E$ which fits in the above commutative square. Consequently, we have $pl = h\pi_2$ and $lk = f$. By appropriately composing l with α and replacing l with this composition, we get that $l : E_b \times I \times I \to E_{b'}$ which is given by following schematic homotopy cube, which we leave the reader to draw. Consequently, we get the following map $\tilde{h} : E_b \times I \to E_{b'}$ where

$$\tilde{h}(-, s) := l(-, 1, s) : E_b \times I \to E_{b'}$$

where $l(e, 1, s) \in E_{b'}$ because $h(1, s) \in b'$ (h is a homotopy through paths joining b and b'). Moreover, $\tilde{h}(e, 0) = l(e, 1, 0) = H_{\gamma,1}(e) = \tilde{\gamma}(e)$ and $\tilde{h}(e, 1) = H_{\eta,1}(e) = \tilde{\eta}(e)$. Thus, $\tilde{h}$ is the required homotopy between $\tilde{\gamma}$ and $\tilde{\eta}$.

10.3.3 Based fiber sequences

Just as for cofibrations, we had a long cofiber sequence, similarly we have a long fiber sequence for a map of based spaces. As is customary, for based case, we change the definition of mapping path space of $f : X \to Y$, to $Nf = \{(x, \gamma) \mid f(x) = \gamma(1)\}$. We thus define homotopy fiber of a map and construct the short and long fiber sequences of a map.

Definition 10.3.3.1 (Homotopy fiber/Mapping path space). Let $f : X \to Y$ be a based map of based spaces. The homotopy fiber of f, denoted Ff, is the following pullback space:

$$\begin{tikzcd} Ff \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ PY \arrow[r, "p_1"'] & Y \end{tikzcd}.$$

Remark 10.3.3.2 (Homotopy fiber is the fiber of mapping path space)**.** Let $f : X \to Y$ be a based map. Denote $Nf = X \times_Y Y^I = \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(1)\}$ to be the mapping path space of Y. Then, we have a map

$$\begin{aligned} q : Nf &\longrightarrow Y \\ (x, \gamma) &\longmapsto \gamma(0). \end{aligned}$$

As the base point of Nf is $(*, c_*)$ which is mapped to $*$ under q, thus, q is based as well. Moreover, the fiber of q is

$$q^{-1}(*) = \{(x, \gamma) \mid f(x) = \gamma(1) \ \& \ \gamma(0) = *\}.$$

Hence, $q^{-1}(*) = Ff$, as required.

We first see why this is called homotopy fiber.

Lemma 10.3.3.3. *Let $f : X \to Y$ be a based map of based spaces.*
1. *The map $\pi : Ff \to X$ is a fibration.*
2. *If $\rho : Nf \to Y$ is the fibration replacement of f (Construction 10.3.1.11) where Nf is the mapping path space of f, then*

$$\rho^{-1}(*) = Ff.$$

Proof. For item 1, consider the fibration $p_1 : PY \to Y$ (Lemma 10.3.1.10). By Lemma 10.3.1.2, we see that $\pi : Ff \to X$ as above is a fibration. For item 2, recall that $\rho(x,\gamma) = \gamma(0)$. Thus, we have $\rho^{-1}(*) = \{(x,\gamma) \in Nf \mid \gamma(0) = *,\ \gamma(1) = f(x)\}$. But this is exactly the fiber Ff as PY is the based path space. □

We expect the fiber of a fibration to be homotopy equivalent to the homotopy fiber. Indeed it is true.

Proposition 10.3.3.4. *Let $p : E \to B$ be a based fibration. Then the fiber $F := p^{-1}(*)$ is based homotopy equivalent to homotopy fiber Fp.*

Proof. Let $F = p^{-1}(*)$. Consider the map

$$\begin{aligned} \phi : F &\longrightarrow Fp \\ e &\longmapsto (e, c_*). \end{aligned}$$

Indeed as $p_1(c_*) = * = p(e)$, so $(e, c_*) \in Fp$. To construct a homotopy inverse, we will begin from the mapping path space of p. Recall from Remark 10.3.3.2 that Fp is the fiber of mapping path space $q : Np \to B$, $(e,\gamma) \mapsto \gamma(0)$. Consider the following homotopy

$$\begin{aligned} H : Np \times I &\longrightarrow B \\ ((e,\gamma), t) &\longmapsto \gamma(1-t). \end{aligned}$$

Observe that the following map commutes where the top horizontal map is $(e,\gamma) \mapsto e$, so that we get $\tilde{H}$ as shown:

$$\begin{tikzcd} Np \arrow[r] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Np \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}.$$

Define the following homotopy using $\tilde{H}$:

$$\begin{aligned} G : Fp \times I &\longrightarrow Fp \\ ((e,\gamma), t) &\longmapsto \left(\tilde{H}((e,\gamma),t),\ \gamma|_{[0,1-t]}\right). \end{aligned}$$

Indeed, as $p(\tilde{H}((e,\gamma),t) = H((e,\gamma),t) = \gamma(1-t) = p_1(\gamma|_{[0,1-t]})$, thus G is well-defined. Let $g : Fp \times I \to E$ given by $((e,\gamma),t) \mapsto \tilde{H}((e,\gamma),t)$, that is the first coordinate of homotopy G. Then consider the map

$$\begin{aligned} \psi : Fp &\longrightarrow F \\ (e,\gamma) &\longmapsto g((e,\gamma),1). \end{aligned}$$

Indeed, as $p(\tilde{H}((e,\gamma),1)) = H((e,\gamma),1) = \gamma(1-1) = \gamma(0) = *$ as $(e,\gamma) \in Fp$, thus ψ is well-defined. We claim that ψ is the homotopy inverse of ϕ. Indeed, we have

$$\begin{aligned} \phi \circ \psi : Fp &\longrightarrow Fp \\ (e,\gamma) &\longmapsto (g((e,\gamma),1), c_*). \end{aligned}$$

Observe that $G_1(e,\gamma) = (g((e,\gamma),1), c_*)$ and $G_0 = \mathrm{id}_{Fp}$, so that G forms a homotopy between $\phi\circ\psi$ and id. Conversely, we have

$$\begin{aligned}\psi\circ\phi : F &\longrightarrow F\\ e &\longmapsto g((e,c_*),1) = \tilde{H}((e,c_*),1).\end{aligned}$$

Consider the restriction of G onto the subspace T of elements $((e,c_*),t) \in Fp\times I$. Note that G maps onto T as well. Thus we have $G : T\times I \to T$ and $G_1(e,c_*) = \tilde{H}((e,c_*),1)$ and $G_0 = \mathrm{id}_T$. Moreover, observe that $F \to T$, $e \mapsto (e,c_*)$ is a homeomorphism. Hence the above restriction of G is a homotopy from $\psi\circ\phi$ to id_F. This completes the proof. □

Construction 10.3.3.5 (Fiber sequence)**.** Let $f : X \to Y$ be a based map of based spaces. Consider the following three maps

$$\begin{aligned}\pi : Ff &\longrightarrow X\\ (x,\gamma) &\longmapsto x\\ \iota : \Omega Y &\longrightarrow Ff\\ \gamma &\longmapsto (*,\gamma).\end{aligned}$$

The sequence

$$Ff \xrightarrow{\pi} X \xrightarrow{f} Y$$

is called the *short fiber sequence*.

We can continue the above short fiber sequence into a *long fiber sequence* as follows. Consider the functor $-\Omega : \mathbf{Top}_*^{cg} \to \mathbf{Top}_*^{cg}$ taking X to ΩX and $f : X\to Y$ to $-\Omega f : -\Omega X \to -\Omega Y$ given by $\gamma(t) \mapsto f\circ\gamma(1-t)$. Thus, we get the following sequence of maps

$$\begin{tikzcd}
 & & {} \arrow[dll] \\
\Omega^2 Ff \arrow[r, "\Omega^2\pi"'] & \Omega^2 X \arrow[r, "\Omega^2 f"] & \Omega^2 Y \arrow[dll, "-\Omega\iota"] \\
\Omega Ff \arrow[r, "-\Omega\pi"'] & \Omega X \arrow[r, "-\Omega f"] & \Omega Y \arrow[dll, "\iota"] \\
Ff \arrow[r, two heads, "\pi"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

which we call the long fiber sequence of $f : X \to Y$.

The main theorem is the following, which associates an exact sequence of based sets to the long fiber sequence.

Theorem 10.3.3.6 (Main theorem of fiber sequences)**.** *Let $f : X \to Y$ be a based continuous map of based spaces and Z be a based space. Then, the long cofiber sequence of f induces a long exact* *sequence of based homotopy sets:*

$$\begin{tikzcd}
 & & {} \arrow[dll, out=-170, in=10] \\
{[Z,\Omega^2 Ff]} \arrow[r] & {[Z,\Omega^2 X]} \arrow[r] & {[Z,\Omega^2 Y]} \arrow[dll, out=-170, in=10] \\
{[Z,\Omega Ff]} \arrow[r] & {[Z,\Omega X]} \arrow[r] & {[Z,\Omega Y]} \arrow[dll, out=-170, in=10] \\
{[Z,Ff]} \arrow[r, "\pi_*"'] & {[Z,X]} \arrow[r, "f_*"'] & {[Z,Y]}
\end{tikzcd}$$

Taking $Z = S^0$ and recalling the suspension-loop space adjunction (Proposition 10.0.0.10), we immediately get the following long exact sequence of homotopy groups.

Corollary 10.3.3.7 (Homotopy L.E.S.-1)**.** *Let $f : X \to Y$ be a based map of based space. Then the fiber sequence of f induces the following long exact sequence of homotopy groups (basepoint suppressed):*

$$\begin{tikzcd}
 & & {} \arrow[dll, out=-170, in=10, "\partial"'] \\
\pi_2(Ff) \arrow[r, "\pi_*"'] & \pi_2(X) \arrow[r, "f_*"] & \pi_2(Y) \arrow[dll, out=-170, in=10, "\partial"'] \\
\pi_1(Ff) \arrow[r, "\pi_*"'] & \pi_1(X) \arrow[r, "f_*"] & \pi_1(Y) \arrow[dll, out=-170, in=10, "\partial"'] \\
\pi_0(Ff) \arrow[r, "\pi_*"'] & \pi_0(X) \arrow[r, "f_*"] & \pi_0(Y)
\end{tikzcd}.$$

□

10.3.4 Serre spectral sequence

For any fibration (more generally, for Serre fibration) $p : E \to B$, there is a spectral sequence converging to homology of the total space E.

Theorem 10.3.4.1. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a Serre fibration with fiber F. If B is simply connected, then there is a first quadrant homology spectral seqeunce converging to homology of E:*

$$E^2_{pq} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E).$$

See cite[HopSSeq] for a proof. We will see some applications of the above spectral sequence below.

Theorem 10.3.4.2 (Loop fibration)**.** *Let $\Omega B \to PB \xrightarrow{\pi} B$ be the loop space fibration where $\pi(\gamma) = \gamma(1)$ (see Lemma 10.3.1.10). Then,*

1. $H_1(\Omega B; \mathbb{Z}) \cong H_2(B; \mathbb{Z})$,

2. there is an exact sequence

$$H_4(B) \to H_2(B) \otimes H_2(B) \to H_2(\Omega B) \to H_3(B) \to 0.$$

Theorem 10.3.4.3 (Fibrations over S^n/Wang sequence)**.** *Let $F \xrightarrow{i} E \xrightarrow{\pi} S^n$ be a fibration for $n \geq 2$. Then there is a long exact sequence*

$$\begin{tikzcd}
 & & {} \arrow[dll] \\
H_{q-n+1}(F) \arrow[r, "d^{n-1}"'] & H_q(F) \arrow[r, "i_*"] & H_q(E) \arrow[dll] \\
H_{q-n}(F) \arrow[r, "d^n"'] & H_{q-1}(F) \arrow[r, "i_*"] & H_{q-1}(E) \arrow[dll, dash] \\
{} & &
\end{tikzcd}.$$

Theorem 10.3.4.4 (Sphere fibrations/Gysin sequence)**.** *Let $S^n \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration for $n \geq 1$ and B be simply connected. Then there is a long exact sequence*

$$\begin{tikzcd}
 & & {} \arrow[dll] \\
H_{p-n}(B) \arrow[r] & H_p(E) \arrow[r, "\pi_*"] & H_p(B) \arrow[dll, "d^{n+1}" description] \\
H_{p-n-1}(B) \arrow[r] & H_{p-1}(E) \arrow[r, "\pi_*"] & H_{p-1}(B) \arrow[dll, dash] \\
{} & &
\end{tikzcd}.$$

We discuss some more general properties now.

Useful properties of Serre spectral sequence

Acyclic fiber theorem

Theorem 10.3.4.5 (Acyclic fiber). *Let $f : X \to Y$ be a based map between connected CW-complexes. Then the following are equivalent:*

1. *For all $k \geq 0$, we have*

$$f_* : H_k(X; M) \xrightarrow{\cong} H_k(Y; M)$$

   for every $\pi_1(Y)$-module M[10].
2. *The homotopy fiber Ff of f is acyclic*[11].

Proof. □

[10] That is, M is a left $\mathbb{Z}[\pi_1(Y)]$-module.
[11] that is, Ff has homology of a point.

10.4 Homology theories

Give a remark of viewing singular cohomology as ES-axioms and also for manifolds as sheaf cohomology.

We will begin by introducing (co)homology from an axiomatic point of view and will derive few properties off of it. This will come in handy for discussing the main properties of differential manifolds in (co)homological language, especially characteristic classes and orientations and what not. The main thing that we wish to do is the Hurewicz theorem, which will allow us to connect homotopy groups and homology groups on the one hand, and will allow us to prove the uniqueness of homology theories for CW complexes on the other hand.

All spaces X are assumed to be compactly generated (Definition 10.0.0.1).

We will use the theory of cofibrations and fibrations as developed above quite freely.

10.4.1 Homology theories

We begin with the category of pairs on which homology theories are defined.

Definition 10.4.1.1 (**$\mathbf{Top}_2$**)**.** The $\mathbf{Top}_2$ denotes the category of pairs (X, A) of spaces where $A \hookrightarrow X$ and maps $(X, A) \to (Y, B)$ which consists of the pair $f : X \to Y$ and $g : A \to B$ such that $g = f|_A$. A map of pairs $(f, d) : (X, A) \to (Y, B)$ is said to be a homotopy equivalence if there is a map of pairs $(g, e) : (Y, B) \to (X, A)$ and there are homotopies $H : g \circ f \simeq \mathrm{id}_X$ and $K : f \circ g \simeq \mathrm{id}_Y$ which extends the homotopies $h : e \circ d \simeq \mathrm{id}_A$ and $k : d \circ e \simeq \mathrm{id}_B$ respectively.

Definition 10.4.1.2. (**Homology theory**) A homology theory for an abelian group π is a sequence of functors

$$H_q(-, -; \pi) : \mathbf{Top}_2 \longrightarrow \mathbf{AbGrp}$$

for $q \in \mathbb{Z}$ equipped with natural transformations

$$\partial : H_q(-, -; \pi) \longrightarrow H_{q-1}(-, -; \pi)$$

whose component at (X, A) is given by $\partial : H_q(X, A; \pi) \to H_{q-1}(A, \emptyset; \pi)$. Denote $H_q(X; \pi) := H_q(X, \emptyset; \pi)$. This data must satisfy the following axioms:

1. *(Homology of a point)* : If $X = \{\text{pt.}\}$, then homology must be concentrated at degree 0:

$$H_q(\{\text{pt.}\}; \pi) = \begin{cases} \pi & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

2. (*Homology long exact sequence*) : The trivial inclusions $A \hookrightarrow X$ and $(X, \emptyset) \hookrightarrow (X, A)$ induces

the following long exact sequence:

$$\begin{tikzcd}
& & \cdots \arrow[dll, dashed, "\partial"'] \\
H_q(A;\pi) \arrow[r] & H_q(X;\pi) \arrow[r] & H_q(X,A;\pi) \arrow[dll, dashed, "\partial"'] \\
H_{q-1}(A;\pi) \arrow[r] & H_{q-1}(X;\pi) \arrow[r] & H_{q-1}(X,A;\pi) \arrow[dll, dashed, "\partial"'] \\
\cdots
\end{tikzcd}.$$

3. *(Excision invariance)* : For an excisive triple (X, A, B), that is $A, B \hookrightarrow X$ and $X = A^\circ \cup B^\circ$, the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism at all degree $q \in \mathbb{Z}$:

$$H_q(A, A \cap B; \pi) \xrightarrow{\cong} H_q(X, B; \pi) \ .$$

4. *(Coproduct preserving)* : If (X_i, A_i) is an arbitrary collection of objects in $\mathbf{Top}_2$, then the homology in any degree of their disjoint union is the sum of the corresponding homology groups:

$$\bigoplus_i H_q(X_i, A_i; \pi) \xrightarrow{\cong} H_q\left(\coprod_i (X_i, A_i); \pi\right)$$

where the maps are induced by the inclusions $(X_{i_0}, A_{i_0}) \hookrightarrow \coprod_i (X_i, A_i)$.

5. *(π_*-insensitivity)* : If $f : (X, A) \longrightarrow (Y, B)$ is a weak equivalence, then in all degrees the corresponding homology groups are isomorphic:

$$f_p : H_q(X, A; \pi) \xrightarrow{\cong} H_q(Y, B; \pi).$$

Remark 10.4.1.3. In nature, there are some homology theories which satisfy all of the above axioms except the dimension axiom, that is, the group that they assign to a point is not concentrated in degree 0 (axiom 1. above). A famous example of this is K-theory via the Bott-periodicity theorem. One calls such a homology theory to be a *generalized homology theory.* All results that we will derive here will hold true for a generalized homology theory E_q.

General properties

We now discuss some general properties of homology theories that one can deduce from the axioms.

Proposition 10.4.1.4. *Let π be a group and E_q be a generalized homology theory. Let X be a space.*

1. *If $A \hookrightarrow X \xrightarrow{r} A$ is a retract of X, then the following natural maps form a short-exact sequence of E-homology groups:*

$$0 \to E_q(A) \to E_q(X) \to E_q(X, A) \to 0.$$

2. *$E_q(X, X) \cong 0$.*

Proof. 1. The fact that $E_q(A) \to E_q(X)$ is injective follows from a set theoretic observation; any factorization of identity is a monic followed by an epic. By homology long-exact sequence, we then have that all boundary maps ∂ are trivial. It follows that maps $E_q(X) \to E_q(X, A)$ is surjective. The exactness at middle is given by the homology long-exact sequence.
2. Since X is always a retract of itself, therefore from item 1, it follows that $E_q(X, X) \cong E_q(X)/E_q(X) \cong 0$. $\square$

The following is a long exact sequence in homology that one obtains from a *triplet* (X, A, B) where $X \supseteq A \supseteq B$.

Proposition 10.4.1.5 (Triplet long-exact sequence)**.** *Let (X, A, B) be a triplet and denote $i : (A, B) \hookrightarrow (X, B)$ and $j : (X, B) \hookrightarrow (X, A)$ to be inclusions. Also denote $\partial' : E_q(X, A) \to E_{q-1}(A, B)$ to be the composite $E_q(X, A) \xrightarrow{\partial} E_{q-1}(A) \to E_{q-1}(A, B)$. Then there is a long exact sequence*

$$\begin{tikzcd}
E_q(A,B) \arrow[r, "i_*"] & E_q(X,B) \arrow[r, "j_*"] & E_q(X,A) \arrow[dll, "\partial'"] \\
E_{q-1}(A,B) \arrow[r, "i_*"] & E_{q-1}(X,B) \arrow[r, "j_*"] & E_{q-1}(X,A)
\end{tikzcd}.$$

Proof. This follows from a fairly long diagram chase involving the homology long-exact sequence corresponding to each of the pairs (A, B), (X, B) and (X, A) which one has to expand for degrees q and $q - 1$. From that big diagram, the chase is straightforwad after some reductions and hence is omitted. $\square$

There is an equivalent form of excision which is also quite useful.

Lemma 10.4.1.6 (Excision-II)**.** *Let $(X, A) \in \mathbf{Top}_2$ be a pair and E_q be a homology theory. If $B \subseteq A$ is a subspace such that $\bar{B} \subseteq A^\circ$, then B can be excised, that is, the inclusion*

$$(X - B, A - B) \hookrightarrow (X, A)$$

induces an isomorphism in homology:

$$E_q(X - B, A - B; \pi) \cong E_q(X, A; \pi).$$

Proof. Consider the triple $(X, A, X - B)$. This is an excisive triple since $A^\circ \cup (X - B)^\circ = X$ since $(X - B)^\circ = X - \bar{B}$. Thus by excision axiom, the inclusion

$$j : (X - B, A \cap X - B) \hookrightarrow (X, A)$$

induces isomorphism in E_q. As $A \cap (X - B) = A - B$, we get the desired result. $\square$

10.4.2 Reduced homology

For each homology theory $E_q(-,-)$, we can construct a based version of the theory denoted $\tilde{E}_q(-,\mathrm{pt.})$. For a based space $(X,\mathrm{pt.})$, define the following

$$\tilde{E}_q(X) := E_q(X,\mathrm{pt.}).$$

This tends to remove the effect of the defining group of the homology theory, so to normalize the theory in the sense of Lemma 10.4.2.1, 1. In particular, if E_q satisfies dimension axiom, it follows that $E_0(\mathrm{pt.}) = \pi$. Thus this lemma will tell that $\tilde{E}_0(X) = \tilde{E}_0(X) \oplus \pi$.

Let us spell out some basic relations of this reduced homology $\tilde{E}_q$ to that of original homology E_q.

Proposition 10.4.2.1. *Let π be a group and E_q be a generalized homology theory. Let $(X,\mathrm{pt.})$ be a based space and $(A,\mathrm{pt.}) \hookrightarrow (X,\mathrm{pt.})$ be a based subspace.*

1. *$E_q(X) = \tilde{E}_q(X) \oplus E_q(\mathrm{pt.})$ and the map $\iota_* : E_q(A) \to E_q(X)$ restricted on $E_q(\mathrm{pt.})$ is the identity map $\iota_* : E_q(\mathrm{pt.}) \to E_q(\mathrm{pt.})$.*
2. *There is a long exact sequence*

$$\begin{tikzcd}
 & & \dots \arrow[dll, "\partial"'] \\
\tilde{E}_q(A) \arrow[r] & \tilde{E}_q(X) \arrow[r] & E_q(X,A) \arrow[dll, "\partial"'] \\
\tilde{E}_{q-1}(A) \arrow[r] & \tilde{E}_{q-1}(X) \arrow[r] & E_{q-1}(X,A) \arrow[dll, "\partial"'] \\
\dots & &
\end{tikzcd}.$$

3. *If E_q is an ordinary homology theory, then for any $q \geq 2$, we have*

$$\tilde{E}_q(X) \cong E_q(X).$$

Proof. 1. The following is split exact on the left as the map $\mathrm{pt.} \hookrightarrow X$ is a retract (Proposition 10.4.1.4):

$$0 \to E_q(\mathrm{pt.}) \to E_q(X) \to E_q(X,\mathrm{pt.}) \to 0.$$

Note that the left map here is split by the retraction $r_* : E_q(X) \to E_q(\mathrm{pt.})$. The latter statement follows from the fact that $E_q(-,\emptyset)$ is a functor and thus takes $\mathrm{id}_{\mathrm{pt.}}$ to $\mathrm{id} : E_q(\mathrm{pt.}) \to E_q(\mathrm{pt.})$.
2. Consider $i : A \hookrightarrow X$. Then $E_q(A) \to E_q(X)$ takes $E_q(\mathrm{pt.})$ to $E_q(\mathrm{pt.})$ isomorphically as in item 1. Hence we may quotient it out under the exactness to get the desired sequence.
3. This is immediate from long exact sequence of the pair $(X,\mathrm{pt.})$. $\square$

In-fact, one can obtain the unreduced homology back by reduced homology via a simple use of coproduct preservation axiom.

Lemma 10.4.2.2. *Let X be a space and denote X_+ to be the based space obtained by disjoint union of X with a point pt.. For any generalized homology theory E_q, we have*

$$E_q(X) \cong \tilde{E}_q(X_+).$$

Proof. As $X_+ = X \amalg \{\text{pt.}\}$, therefore by additivity of homology theories, we obtain

$$\begin{aligned}\tilde{E}_q(X_+) &= E_q(X \amalg \{\text{pt.}\}, \text{pt.}) = E_q((X, \text{pt.}) \amalg (\text{pt.}, \text{pt.})) \cong E_q(X, \text{pt.}) \oplus E_q(\text{pt.}, \text{pt.}) \\ &\cong \tilde{E}_q(X) \oplus E_q(\text{pt.}) \cong E_q(X)\end{aligned}$$

where the second-to-last isomorphism follows from Proposition 10.4.2.1, 1 and the last from 4. $\square$

10.4.3 Mayer-Vietoris sequence in homology

We now cover an important calculational tool for generaized homology theories, which relates the homology groups of X with those of A, B and $A \cap B$ where (X, A, B) forms an excisive triad.

Theorem 10.4.3.1 (Mayer-Vietoris for homology)**.** *Let (X, A, B) be an excisive triple and denote $i : A \cap B \hookrightarrow A$, $j : A \cap B \hookrightarrow B$, $k : A \hookrightarrow X$ and $l : B \hookrightarrow X$. Then there is a long exact sequence*

$$\begin{tikzcd} E_q(A \cap B) \arrow[r, "\begin{bmatrix} i_* \\ j_* \end{bmatrix}"] & E_q(A) \oplus E_q(B) \arrow[r, "\begin{bmatrix} k_* - l_* \end{bmatrix}"] & E_q(X) \arrow[dll, "\overline{\partial}"] \\ E_{q-1}(A \cap B) \arrow[r] & E_{q-1}(A) \oplus E_{q-1}(B) \arrow[r] & E_{q-1}(X) \arrow[dll, dashed, no head] \\ {} & & \end{tikzcd}$$

where $\overline{\partial}$ is obtained as the following composite

$$\begin{tikzcd} E_q(X) \arrow[r] \arrow[d, dashed, "\overline{\partial}"'] & E_q(X, B) \arrow[d, "\cong"] \\ E_{q-1}(A \cap B) & E_q(A, A \cap B) \arrow[l, "\partial"] \end{tikzcd}$$

where top horizontal arrow is corresponds to $(X, \emptyset) \hookrightarrow (X, B)$, the right vertical is exicision isomorphism and the bottom horizontal is the boundary map of homology long exact sequence of the pair $(A, A \cap B)$.

Proof. The proof will follow from excision and long exact sequence for homology quite easily. **TODO.** $\square$

10.4.4 Relative homology of cofibrations and suspension isomorphism

There are two important results for homology. The first affirms our intuition that the homology of pair (X, A) ought to behave as homology of X/A, but it works out only when $A \hookrightarrow X$ is a cofibration. The second gives a suspension isomorphism type result akin to that of homotopy groups.

Relative homology of cofibrations

Theorem 10.4.4.1. *Let $i : A \hookrightarrow X$ be a cofibration and E_q a generalized homology theory. Then the quotient map $p : (X, A) \twoheadrightarrow (X/A, \text{pt.}$ induces an isomorphism*

$$p_* : E_q(X, A) \xrightarrow{\cong} E_q(X/A).$$

Suspension isomorphism

Theorem 10.4.4.2. *Let (X, x_0) be a non-degenerately based space, that is, the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration. Let E_q be a generalized homology theory. Then, there is a natural isomorphism*

$$\tilde{E}_q(\Sigma X) \cong \tilde{E}_{q-1}(X).$$

10.4.5 Fundamental theorem of homology theories

We will now see that reduced homology and unreduced homology theories are equivalent. To this end, we first axiomatize reduced homology theory. The category $\mathbf{Top}_*$ denotes the category of well-pointed spaces.

Definition 10.4.5.1 (**Reduced homology theory**)**.** A reduced homology theory for an abelian group π is a sequence of functors

$$\tilde{H}_q(-;\pi) : \mathbf{Top}_* \longrightarrow \mathbf{AbGrp}$$

for $q \in \mathbb{Z}$ which satisfies the following axioms (we suppress π):

1. *(Cofibration exactness)* If $i : A \hookrightarrow X$ is a cofibration, then

   $$\tilde{H}_q(A) \to \tilde{H}_q(X) \to \tilde{H}_q(X/A)$$

   is exact.
2. *(Suspension isomorphism)* For all $q \geq 0$, we have a natural isomorphism

   $$\Sigma : \tilde{H}_q(X) \xrightarrow{\cong} \tilde{H}_{q+1}(\Sigma X).$$

3. *(Additivity)* If $X = \bigvee_{i \in I} X_i$ where each X_i is well-pointed, then the natural inclusions $\iota_i : X_i \hookrightarrow X$ induces an isomorphism

   $$\bigoplus_{i \in I} \tilde{H}_q(X_i) \cong \tilde{H}_q(X)$$

4. *(Weak equivalence)* If $f : X \to Y$ is a weak equivalence, then

   $$f_* : \tilde{H}_q(X) \to \tilde{H}_q(Y)$$

   is an isomorphism.

10.4.6 Singular homology & applications

We define the usual singular homology groups and will mention that it is a homology theory. Once that's set-up, then with the explicit description of chain complexes in singular homology and the ES-axioms and all the surrounding results, we will have a good toolbox to compute homology groups of very many spaces. In-fact, these applications are important to really showcase that if in any situation we have an invariant of any class of objects which is a homology theory, then we can immediately make this invariant very palpatable to calculations, which is very important in aspects where the objects are abstract entities like rings or schemes.

For this section, we may assume that our spaces are not compactly generated.

Definition 10.4.6.1 (**Singular homology**)**.** Let X be a space and fix a field F. Let $S_i(X)$ be the free F-vector space generated by the set of all i-simplices $\{f : \Delta^i \to X \mid f \text{ is continuous}\}$. An element of $S_i(X)$ is called *singular i-chain.* Consider the map $\partial : S_i(X) \to S_{i-1}(X)$ which on an i-simplex σ is given by $\sigma \mapsto \sum_{j=0}^{i}(-1)^j \partial_j \sigma$ where $\partial_j \sigma$ is the σ restricted to the face opposite to j^{th}-vertex. It follows that $\partial^2 = 0$. Thus, we have a chain complex $(S_i(X), \partial)$, called the singular chain complex. The homology of this chain complex is defined to be the singular homology of X, denoted $H_i(X;\mathbb{Z})$ or simply $H_i(X)$. A map $f : X \to Y$ on spaces yields a map on singular complex $f_\sharp : S_\bullet(X) \to S_\bullet(Y)$. As map of complexes induces map on homology, we get $f_* : H_\bullet(X) \to H_\bullet(Y)$.

Let (X, A) be a pair. We define the relative singular i-chains to be

$$S_\bullet(X, A) := S_\bullet(X)/S_\bullet(A).$$

The boundary map of $S_\bullet(X)$ descends to a boundary map on $S_\bullet(X, A)$ by properties of quotients and thus we define the singular homology of a pair (X, A) to be homology of the complex $S_\bullet(X, A)$ denoted $H_i(X, A; \mathbb{Z})$.

In the following result, we state some important first properties of singular homology.

Theorem 10.4.6.2 (Singular homology is a homology theory)**.** *Let X be a space.*

1. *If $\{X_k\}$ is the collection of path-components of X, then*
$$H_i(X;\mathbb{Z}) \cong \bigoplus_k H_i(X_k\ \mathbb{Z}).$$

2. *Singular homology satisfies dimension axiom:*
$$H_i(\{\text{pt.}\};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

3. *X is path-connected if and only if*
$$H_0(X;\mathbb{Z}) \cong \mathbb{Z}.$$

4. *Singular homology has long exact sequence of pairs, that is, if (X, A) is a pair, then there is a long exact sequence obtained by inclusions $A \hookrightarrow X$ and $(X, \emptyset) \hookrightarrow (X, A)$ as follows:*
$$\begin{tikzcd} & & \cdots \arrow[dll, dashed, "\partial"'] \\ H_q(A;\pi) \arrow[r] & H_q(X;\pi) \arrow[r] & H_q(X,A;\pi) \arrow[dll, dashed, "\partial"'] \\ H_{q-1}(A;\pi) \arrow[r] & H_{q-1}(X;\pi) \arrow[r] & H_{q-1}(X,A;\pi) \arrow[dll, dashed, "\partial"'] \\ \cdots & & \end{tikzcd}.$$

5. *Singular homology is excision invariant; for an excisive triple (X, A, B), that is $A, B \hookrightarrow X$ and $X = A^\circ \cup B^\circ$, the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism at all degree $q \in \mathbb{Z}$:*
$$H_q(A, A \cap B; \mathbb{Z}) \xrightarrow{\cong} H_q(X, B; \mathbb{Z}) .$$

An equivalent restatement is that if $A \supseteq B$ such that $\bar{B} \subseteq A^\circ$, then the inclusion $(X-B, A-B) \hookrightarrow (X, A)$ induces isomorphism in homology

$$H_q(X-B, A-B;\mathbb{Z}) \xrightarrow{\cong} H_q(X, A;\mathbb{Z}).$$

6. *Singular homology preserves coproducts, that is, if $\{(X_i, A_i)\}_{i\in I}$ is a collection of pairs of spaces, then*

$$\bigoplus_i H_q(X_i, A_i;\pi) \xrightarrow{\cong} H_q\left(\coprod_i (X_i, A_i);\pi\right)$$

where the maps are induced by the inclusions $(X_{i_0}, A_{i_0}) \hookrightarrow \coprod_i (X_i, A_i)$.

7. *Singular homology satisfies strong π_*-insensitivity, that is, if $f, g : X \to Y$ are two homotopic maps, then $f_* = g_* : H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$.*

Proof. 1. Observe that $S_i(X) = \bigoplus_k S_i(X_k)$ by path-connectedness of each X_k. Moreover, $Z_i(X) = \bigoplus_k Z_i(X_k)$ and $B_i(X) = \bigoplus_k B_i(X_k)$. The result follows.

2. First observe that every $S_i(X)$ is isomorphic to $\mathbb{Z}$ as there is only one i-simplex, namely $c_{\text{pt.}}$, the constant map. We have for $c_{\text{pt.}} \in Z_{i+1}(X)$ its boundary as

$$\partial(c_{pt}) = \sum_{j=0}^{i+1} (-1)^j \partial_j(c_{\text{pt.}})$$

where note that the j^{th}-boundary of $c_{\text{pt.}}$ is still $c_{\text{pt.}}$. Thus, if $i+2$ is even, then $\partial : S_{i+1}(X) \to S_i(X)$ is zero and if $i+2$ is odd, then $\partial : S_{i+1}(X) \to S_i(X)$ is an isomorphism. Hence, we get that

$$d_p : S_p(X) \to S_{p-1}(X)$$

is 0 if p is odd and an isomorphism if p is even. From this, it immediately follows that $H_p(\text{pt.};\mathbb{Z}) = 0$ if $p > 0$ and $H_0(\text{pt.};\mathbb{Z}) \cong \mathbb{Z}$.

3. (L $\Rightarrow$ R) Let X be a path-connected space. Recall that $H_0(X;\mathbb{Z}) = S_0(X)/\text{Im}(\partial_1)$. Consider the following map

$$\begin{aligned} \epsilon : S_0(X) &\longrightarrow \mathbb{Z} \\ \sum_j n_j x_j &\longmapsto \sum_j n_j. \end{aligned}$$

Clearly this is surjective. We claim that $\text{Ker}(\epsilon) = \text{Im}(\partial_1)$. Suppose $\sum_j n_j x_j \in S_0(X)$ and each x_j is distinct with $\sum_j n_j = 0$. We wish to find a 1-chain $\sigma = \sum_j m_j \sigma_j$ such that $\partial_1 \sigma = \sum_j n_j x_j$. Fix $x_0 \in X$ a point different from x_j and let $\gamma_j : I \to X$ be a path joining x_0 to x_j. Consider $\sigma = \sum_j n_j \gamma_j$. We claim that $\partial\sigma = \sum_j n_j x_j$. Indeed, we have

$$\partial\sigma = \sum_j n_j(\gamma_j(1) - \gamma_j(0)) = \sum_j n_j(x_j - x_0) = \sum_j n_j x_j - \left(\sum_j n_j\right) x_0 = \sum_j n_j x_j,$$

as required.

TODO □

Corollary 10.4.6.3. *The construction of the sequence of functors $H_k(-,-;\mathbb{Z}) : \mathbf{Top_2} \to \mathbf{AbGrp}$ is a homology theory.* $\square$

Remark 10.4.6.4 (Mayer-Vietoris sequence for singular homology)**.** Consider a space X and an excisive triple (X, A, B). Then since singular homology is a homology theory, hence we have the Mayer-Vietoris sequence as in Theorem 10.4.3.1. After long exact sequence for pairs, this is the second most important long exact sequence in homology:

$$\begin{tikzcd} H_q(A\cap B) \arrow[r, "\begin{bmatrix} i_* \\ j_* \end{bmatrix}"] & H_q(A)\oplus H_q(B) \arrow[r, "\begin{bmatrix} k_* - l_* \end{bmatrix}"] & H_q(X) \arrow[dll, "\bar{\partial}"] \\ H_{q-1}(A\cap B) \arrow[r] & H_{q-1}(A)\oplus H_{q-1}(B) \arrow[r] & H_{q-1}(X) \arrow[dll, dashed, no head] \\ {} & & \end{tikzcd} .$$

This also holds for reduced homology.

Remark 10.4.6.5 (Triplet long exact sequence for singular homology)**.** Consider a triplet (X, A, B) where $X \supseteq A \supseteq B$. Then since singular homology is a homology theory, hence we get a triplet long exact sequence induced by inclusions as in Theorem 10.4.1.5. This is the third long exact sequence that one derives in singular homology, after l.e.s. of pairs and Mayer-Vietoris. This also holds for reduced homology.

We now showcase a result which we will meet again later, which relates fundamental group and first homology group.

Theorem 10.4.6.6 (Hurewicz for π_1)**.** *Let X be a path-connected space and $x_0 \in X$. The canonical map*

$$\begin{aligned} \varphi : \pi_1(X, x_0) &\longrightarrow H_1(X;\mathbb{Z}) \\ \langle\alpha\rangle &\longmapsto [\alpha] \end{aligned}$$

is surjective with $\mathrm{Ker}\,(\varphi) = [\pi_1(X, x_0) : \pi_1(X, x_0)]$.

Corollary 10.4.6.7. *Let (X, x_0) be a path-connected space and such that $\pi_1(X, x_0)$ is abelian. Then $\pi_1(X, x_0) \cong H_1(X;\mathbb{Z})$.* $\square$

Remark 10.4.6.8 (Suspension isomorphism)**.** Let X be a space and SX be unreduces suspension. Then we have an isomorphism as in Theorem 10.4.4.2:

$$H_q(SX;\mathbb{Z}) \cong \tilde{H}_{q-1}(X;\mathbb{Z}).$$

One can also directly prove this by analyzing the Mayer-Vietoris for the $X_1 = SX - [x, 1]$ and $X_2 = SX - [x, 0]$.

10.4.7 Results & computations for singular homology

We now present many computations for singular homology theory, which showcases the strength of the tools available.

For this section, we may assume that our spaces are not compactly generated.

Remark 10.4.7.1. We begin with the list of topics that we cover here, for mental clarity and quick reference.

- Path components & relative homology.
- Map of long exact sequence of pairs.
- Immediate applications of Mayer-Vietoris.
- Degree of a map $f : S^n \to S^n$.
- Antipode preserving maps $f : S^n \to S^1$.
- Jordan-Brower separation theorem.

Path components & relative homology

Lemma 10.4.7.2. *Let $A \subseteq X$ be a non-empty subspace and X be path-connected. Then*

$$H_0(X, A; \mathbb{Z}) = 0.$$

Proof. Consider $\bar{d} : S_1(X, A) \to S_0(X, A)$. We claim that $\operatorname{Im}\left(\bar{d}\right) = S_0(X, A)$. Suffices to show that $\operatorname{Im}\left(\bar{d}\right)$ contains the class of generators $x : \Delta_0 \to X$. Pick any x as given. To show that there exists $\sigma + S_1(A) \in S_1(X, A)$ whose boundary is x. Indeed, as X is path-connected, so for any fixed point $x_0 \in A$, we may consider a path σ joinging x_0 to x. This defines an element $\sigma + S_1(A)$ whose boundary is $x - x_0 + S_0(A) = x + S_0(A)$, as needed. □

Lemma 10.4.7.3. *Let $\{X_k\}$ be path-components of X and $A \subseteq X$ be non-empty. Then*

$$H_n(X, A; \mathbb{Z}) \cong \bigoplus_k H_n(X_k, A \cap X_k; \mathbb{Z}).$$

Proof. As $S_n(X, A) = \bigoplus_k S_n(X_k, A \cap X_k)$, the result then follows by quotienting. □

Lemma 10.4.7.4. *Let $A \subseteq X$ be a non-empty subspace, then*

$$\operatorname{rank}(H_0(X, A; \mathbb{Z})) = \#\ \textit{path components of } X \textit{ not intersecting } A.$$

Proof. By Theorem 10.4.6.2, 3 and Lemmas 10.4.7.2 and 10.4.7.3, the result is immediate. □

Lemma 10.4.7.5. *Let X have r-path components. Then,*

$$H_0(X, \mathrm{pt.}; \mathbb{Z}) \cong \mathbb{Z}^{\oplus r-1}$$

Proof. Use Lemma 10.4.7.4. □

Example 10.4.7.6 (Homology of (D^n, S^{n-1}))**.** We claim that

$$\tilde{H}_i(D^n, S^{n-1}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

Indeed, this follows immediately from les of a pair and Lemma 10.4.7.4.

The following is an important observation in geometry.

Proposition 10.4.7.7 (Künneth formula-1)**.** *Let X be a T_1-space and $x \in X$. If $U \subseteq X$ is an open set containing x, then we have*

$$H_i(X, X-x;\mathbb{Z}) \cong H_i(U, U-x;\mathbb{Z}).$$

Proof. For $A = U$ and $B = X - x$, we see that both of them are open (B is open as $\{x\}$ is closed). Then, (X, A, B) forms an excisive triple. Performing excision, we observe (as $A \cap B = U - x$) that

$$H_i(U, U-x;\mathbb{Z}) \cong H_i(X, X-x;\mathbb{Z}),$$

as required. □

Remark 10.4.7.8. It is really necessary for U in Künneth formula above to be open, for $(S^2 - x, I - x) \hookrightarrow (S^2, I)$ for some path $I \hookrightarrow X$ does not induces isomorphism in homology, as is readily visible a small computation in the associated les of pairs.

Map of long exact sequence of pairs

Proposition 10.4.7.9. *Let $f : (X, A) \to (Y, B)$ be a map of pairs. Then, we get a map in the long exact sequences of the corresponding pairs. That is, the following commutes*[12]

$$\begin{tikzcd}
{} \arrow[r, dashed] & H_n(A) \arrow[r] \arrow[d, "f_*"'] & H_n(X) \arrow[r] \arrow[d, "f_*"'] & H_n(X,A) \arrow[r, "\Delta"] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[d, "f_*"'] \arrow[r, dashed, dash] & {} \\
{} \arrow[r, dashed] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\Delta"'] & H_{n-1}(B) \arrow[r, dashed, dash] & {}
\end{tikzcd}.$$

Proof. Since all maps in the long exact sequence of a pair except the connecting homomorphism are induced by inclusions, therefore we need only check the commutativity of the rightmost square. This follows from unravelling the definition of connecting homomorphism as constructed from the chain level. □

We also have a map in Mayer-Vietoris.

Proposition 10.4.7.10. *Let $f : (X, A, B) \to (Y, C, D)$ be a map of triples, where each is an excisive triple. Then we get a map in the Mayer-Vietoris sequences of the corresponding pairs. That is, the following commutes*

$$\begin{tikzcd}
{} \arrow[r, dashed] & H_n(A\cap B) \arrow[r] \arrow[d, "f_*"'] & H_n(A)\oplus H_n(B) \arrow[r] \arrow[d, "f_*"'] & H_n(X) \arrow[r, "\Delta"] \arrow[d, "f_*"'] & H_{n-1}(A\cap B) \arrow[d, "f_*"'] \arrow[r, dashed, dash] & {} \\
{} \arrow[r, dashed] & H_n(C\cap D) \arrow[r] & H_n(C)\oplus H_n(D) \arrow[r] & H_n(Y) \arrow[r, "\Delta"'] & H_{n-1}(C\cap D) \arrow[r, dashed, dash] & {}
\end{tikzcd}.$$

Proof. Follows directly from Proposition 10.4.7.9 and the proof of original Mayer-Vietoris (in which we show that Mayer-Vietoris is obtained by les of a pair and excision). □

[12] we drop the group $\mathbb{Z}$ in the following diagram.

Lemma 10.4.7.11. *If $f : (X, A) \to (Y, B)$ is a homotopy equivalence of pairs, that is, there exists $g : (Y, B) \to (X, A)$ such that $f : X \rightleftarrows Y : g$ and $f : A \rightleftarrows B : g$ are both homotopy equivalences, then*

$$f_* : H_n(X, A) \xrightarrow{\cong} H_n(Y, B)$$

is an isomorphism.

Proof. Use 5-lemma on the diagram in Proposition 10.4.7.9. □

Immediate applications of Mayer-Vietoris

Example 10.4.7.12 (Homology of spheres)**.** We wish to show that

$$\tilde{H}_i(S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

Indeed, let $U = S^n - p$ and $V = S^n - q$ where p, q are north and south poles respectively. Note $U \cap V \simeq S^{n-1}$. Then (S^n, U, V) is an excisive triple and thus by Mayer-Vietoris (Remark 10.4.6.4), we deduce that the connecting homomorphism $H_q(S^n) \cong \tilde{H}_{q-1}(S^{n-1})$. We conclude by induction.

Example 10.4.7.13 (Homology of wedge of spheres)**.** We wish to show that for each $i \geq 0$, we have

$$\tilde{H}_i(S^m \vee S^n) \cong \tilde{H}_i(S^m) \oplus \tilde{H}_i(S^n)$$

Indeed this follows by considering U to be S^m with some open part of S^n and V to be S^n with some open part of S^m. We get that $U \cap V \simeq \text{pt.}$, $U \simeq S^m$, $V \simeq S^n$ and (X, U, V) an excisive triple. The result now follows by Mayer-Vietoris (Remark 10.4.6.4).

Using Example 10.4.7.12, we can prove the following seemingly obvious, but otherwise hard to prove statement.

Theorem 10.4.7.14. *Let $n, m \in \mathbb{N}$.*
1. *S^n is homeomorphic to S^m if and only if $n = m$.*
2. *$\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$.*

Proof. Item 1 is immediate application of computation in Example 10.4.7.12. Item 2 can be obtained from removing a point from the given homeomorphism $\varphi : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m$ to get a homotopy equivalence $S^{n-1} \to S^{m-1}$. Thus, they have same homology. Invoking Example 10.4.7.12, we win. □

Degree of a map $f : S^n \to S^n$

For a map $f : S^n \to S^n$, consider the map $f_* : \mathbb{Z} \to \mathbb{Z}$ obtained by $H_n(S^n) \to H_n(S^n)$. Thus, f_* takes a generator a to $k \cdot a$, $k \in \mathbb{Z}$. We define $\deg(f) = k$. We begin with some basics.

Lemma 10.4.7.15. *Let $f : S^n \to S^n$ be a map.*

1. *If* $f : S^n \to S^n$ *and* $g : S^n \to S^n$*,*
$$\deg(g \circ f) = \deg(g) \cdot \deg f.$$
2. *If* $f, g : S^n \to S^n$ *are homotopy equivalent, then* $\deg(f) = \deg(g)$*.*

Proof. Immediate. □

The main theorem is the following, which computes the degree of reflections.

Theorem 10.4.7.16 (Degree of reflections)**.** *Define the following map*

$$\begin{aligned} f : S^n &\longrightarrow S^n \\ (x_1, x_2 \dots, x_{n+1}) &\longmapsto (-x_1, x_2, \dots, x_{n+1}). \end{aligned}$$

Then,

$$\deg(f) = -1.$$

Proof. Use induction on n and observe that for $X_1 = S^n - p$ and $X_2 = S^n - q$, we get a map induced in Mayer-Vietoris (Proposition 10.4.7.10). This yields the following commutative square where connecting homomorphism is an isomorphism:

$$\begin{tikzcd} H_n(S^n) \arrow[r, "\Delta"] \arrow[d, "f_*"'] & \tilde{H}_{n-1}(S^{n-1}) \arrow[d, "f_*"] \\ H_n(S^n) \arrow[r, "\Delta"'] & \tilde{H}_{n-1}(S^{n-1}) \end{tikzcd}.$$

The result now follows by inductive hypothesis. □

Corollary 10.4.7.17. *Define the following map*

$$\begin{aligned} f : S^n &\longrightarrow S^n \\ (x_1, x_2 \dots, x_{n+1}) &\longmapsto (-x_1, -x_2, \dots, -x_{n+1}). \end{aligned}$$

Then,

$$\deg(f) = (-1)^{n+1}.$$

Proof. Immediate from Theorem 10.4.7.16. □

Remark 10.4.7.18 (Fixed points and degree)**.** It is an easy observation that if $f : S^n \to S^n$ has no fixed points, then f is homotopic to $a : S^n \to S^n$ which is the antipodal map. Thus the degree of a map $f : S^n \to S^n$ which has no fixed points is $(-1)^{n+1}$.

An easy corollary of this observation is that if $f : S^n \to S^n$ is null homotopic, then f has a fixed point. Indeed as $\deg f = 0$, therefore by contrapositive of above, we deduce that f must have a fixed point.

A simple use of above remark yields the following fact for maps $f : S^{2n} \to S^{2n}$.

Proposition 10.4.7.19. *Let $f : S^{2n} \to S^{2n}$ be a map. Then, there exists $x \in S^{2n}$ such that either $f(x) = x$ or $f(x) = -x$.*

Proof. Suppose f has no fixed points. Then by Remark 10.4.7.18, it follows that $f \simeq a$, where $a : S^{2n} \to S^{2n}$ is the antipodal map. Thus, $\deg f = \deg a = (-1)^{2n+1} = -1$. It follows that $\deg(-f) = 1$. Hence, $-f$ must have a fixed point by Remark 10.4.7.18. Consequently, there exists $x \in S^{2n}$ such that $-f(x) = x$, as required. □

We also have the following conclusion.

Proposition 10.4.7.20. *Let $f : S^n \to S^n$ be a degree 0 map. Then there exists $x, y \in S^n$ such that $f(x) = x$ and $f(y) = -y$.*

Proof. Indeed, by above we immediately conclude that both f and $-f$ has degree 0, thus have fixed points. □

A more non-trivial application of ideas surrounding degree is the following.

Lemma 10.4.7.21. *Any linear map $T : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ has an eigenvector.*

Proof. We may assume that T is a bijection. Thus, T takes one dimensional linear subspaces to one dimensional linear subspaces. We get in particular a map $g : S^{2n} \to S^{2n}$ given by $\frac{v}{\|v\|} \mapsto \frac{Tv}{\|Tv\|}$. Use Proposition 10.4.7.19 to conclude. □

Antipode preserving maps $f : S^n \to S^1$

Another interesting application of singular homology is to show that if $n > 1$, then there is no antipode preserving map $f : S^n \to S^1$, where a map $f : S^m \to S^n$ is antipode preserving if for all $x \in S^m$, we have $-f(x) = f(-x)$.

Theorem 10.4.7.22. *If $n > 1$, then there is no antipode preserving map $f : S^n \to S^1$.*

Remark 10.4.7.23. One can deduce Borsuk-Ulam theorem, that for any map $f : S^2 \to \mathbb{R}^2$ there exists $x \in S^2$ such that $f(x) = f(-x)$, from Theorem 10.4.7.22 as follows. By composing by linear shift, we may assume $\mathrm{Im}\,(f)$ does not contain origin. Composing with the map $y \mapsto \frac{y}{\|y\|}$, we obtain the map $g : S^2 \to S^1$ mapping as $x \mapsto \frac{f(x)}{\|f(x)\|}$. Applying the above theorem, Borsuk-Ulam follows.

Jordan-Brower separation theorem

We wish to show the following result.

Theorem 10.4.7.24 (JBST)**.** *Suppose $C \subseteq S^n$ is a subspace of S^n homeomorphic to S^{n-1}. Then $S^n - C$ has two components and has boundary C.*

More important for us is the two homological results which will be used to prove the above theorem.

Definition 10.4.7.25 (Cells in a space). A k-cell in a space X is a subspace $A \subseteq X$ homeomorphic to D^k.

Theorem 10.4.7.26. *Let A be a k-cell in S^n. Then,*

$$\tilde{H}_i(S^n - A; \mathbb{Z}) = 0$$

for every $i \geq 0$.

Using above theorem, we have the following result.

Proposition 10.4.7.27. *Let $h : S^k \hookrightarrow S^n$ be an embedding where $n > k \geq 0$. Then*

$$\tilde{H}_i(S^n - h(S^k); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = n - k - 1, \\ 0 & \text{else.} \end{cases}$$

Proof. This follows from Mayer-Vietoris and induction on k, where we define $X_1, X_2 \subseteq S^n - h(S^k) = X$ as follows. Let $E_k^+ = S^k - q$ and $E_k^- = S^k - p$, p, q are north, south poles, respectively. Then define $X_1 = S^n - h(E_k^+)$ and $X_2 = S^n - h(E_k^-)$. Then $X_1 \cap X_2 = S^n - h(S^k)$ and $X_1 \cup X_2 = S^n - h(S^{k-1})$. Using Theorem 10.4.7.26 will yield the isomorphism $\tilde{H}_q(S^n - h(S^{k-1})) \cong \tilde{H}_{q-1}(S^n - h(S^{k-1}))$. We conclude by inductive hypothesis. □

Remark 10.4.7.28. Note that Proposition 10.4.7.27 already shows the first statement of Theorem 10.4.7.24. Indeed, Using the result, we get for $k = n - 1$, that $\tilde{H}_0(S^n - h(S^{n-1}); \mathbb{Z}) = \mathbb{Z}$, that is, there are two path-components of $S^n - h(S^{n-1})$. As S^n is locally path-connected, so number of components and path-components are same.

An important application is the invariance of domain.

Theorem 10.4.7.29 (Invariance of domain)**.** *Let $U \subseteq \mathbb{R}^n$ be a n open set and consider a map $f : U \to \mathbb{R}^n$ which is a continuous bijection. Then,*

1. *$f(U)$ is open in $\mathbb{R}^n$,*
2. *$f : U \to f(U)$ is a homeomorphism.*

That is, f is an open embedding.

Proof. Pick any open ball $B \subseteq U$ such that $\bar{B} \subseteq U$. Observe $S^{n-1} \cong \bar{B} - B = \partial B$. Consider the composite $f : \partial B \to f(U) \hookrightarrow S^n$ where we consider $\mathbb{R}^n \hookrightarrow S^n$. By JBST, $f : S^{n-1} \to S^n$ separates S^n into two components, say $S^n - f(S^{n-1}) = W_1 \amalg W_2$. If $f(B) \subseteq W_1$, we claim that $f(B) = W_1$. Indeed, this follows from Theorem 10.4.7.26 which says that removing a k-cell still keeps S^n path-connected. □

10.4.8 Homology with local coefficients

10.5 Cohomology theories

10.6 Cohomology products and duality

10.7 CW-complexes & CW homotopy types

One of the important properties of compactly generated spaces is that any such space can be approximated upto homotopy by a class of spaces constructed in a rather simple manner. These are precisely the CW complexes. Once the above approximation theorems are set up, we can safely reduce a lot of computation in homology to such a CW-approximation. Moreover, the reductions run so deep that in-fact any homology theory E_q on general compactly generated spaces necessarily induces and comes from the restriction of E_q to CW-complexes. An application of Hurewicz theorem will then tell us that upto natural isomorphism, there is a unique homology theory over CW-complexes. Moreover, the fundamental result of Whitehead would allow us to interpret homotopy groups as a complete set of homotopical invariants for CW-complexes

10.7.1 Basic theory

10.7.2 Approximation theorems

10.7.3 CW homotopy types

We wish to prove some foundational results on homotopy equivalences of CW-complexes.

Whitehead's theorem

We wish to see the following important result.

Theorem 10.7.3.1 (Whitehead)**.** *Let X and Y be weakly equivalent CW-complexes. Then X and Y are homotopy equivalent.*

Applications of Whitehead's theorem

Lemma 10.7.3.2 (Weak uniqueness of universal covers)**.** *Let X be a CW-complex. If E is a CW-complex and $f : E \to X$ is such that*

$$f_* : \pi_k(E) \to \pi_k(X)$$

is an isomorphism for all $k \geq 2$ and $\pi_k(E) = 0$ for $k = 0, 1$, then E is homotopy equivalent to the universal cover $\tilde{X}$ of X.

Proof. As $\pi_0(E) = 0$, therefore E is connected. It follows by unique lifting (which is possible as $\pi_1(E) = 0$) that we have a commutative diagram of spaces:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ E \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}.$$

Applying π_k for any $k \geq 2$, we deduce from our hypothesis that $\tilde{f}_* : \pi_k(E) \to \pi_k(\tilde{X})$ is an isomorphism. As $\pi_0(\tilde{X}) = \pi_1(\tilde{X}) = 0$, therefore $\tilde{f}$ is a weak equivalence. It follows by Whitehead's theorem (Theorem 10.7.3.1) that $\tilde{f}$ is a homotopy equivalence, as required. □

10.8 Homotopy and homology

10.8.1 Hurewicz's theorem

Theorem 10.8.1.1 (Hurewicz-1). *Let X be an $(n-1)$-connected based space. Then the Hurewicz map*

$$h_n : \pi_n(X) \to H_n(X;\mathbb{Z})$$

is an isomorphism and

$$h_{n+1} : \pi_{n+1}(X) \to H_{n+1}(X;\mathbb{Z})$$

is a surjection.

It is also very beneficial to keep the following version of Hurewicz in mind as it is usually used to deduce conclusion about homology groups from some information about homotopy groups and vice-versa. The second item is often used after passing to universal covers.

Theorem 10.8.1.2 (Hurewicz-2). *Let X, Y be path-connected based spaces and $f : X \to Y$ be a based map. Let $n \in \mathbb{N}$.*

1. *If $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and a surjection for $k = n$, then $f_* : H_k(X;\mathbb{Z}) \to H_k(Y;\mathbb{Z})$ is an isomorphism for $k < n$ and a surjection for $k = n$.*
2. *If X, Y are simply connected and $f_* : H_k(X;\mathbb{Z}) \to H_k(Y;\mathbb{Z})$ is an isomorphism for $k < n$ and a surjection for $k = n$, then $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and a surjection for $k = n$.*

10.9 Homotopy & algebraic structures

10.9.1 H-spaces

Definition 10.9.1.1 (**H-spaces & groups**). Let (X, e) be a based space. Then X is said to be a an H-space if there exists a continuous map

$$\begin{aligned} m : X \times X &\longrightarrow X \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

such that

1. $e \cdot e = e$,
2. $m_e : X \to X$, $x \mapsto x \cdot e$ and $m^e : X \to X$, $x \mapsto e \cdot x$ are both homotopy equivalent to id_X rel $\{e\}$.

An H-space $(X, e, \cdot)$ is said to be an H-group if moreover the map m satisfies the following:

1. the two associativity maps $X \times X \times X \rightrightarrows X$ are homotopic to each other rel $\{(e, e, e)\}$,
2. there exists an inverse map $(-)^{-1} : X \to X$ such that $e^{-1} = e$ and that the two left/right multiplication by inverse maps $X \rightrightarrows X$, $x \mapsto x \cdot x^{-1}$ and $x \mapsto x^{-1} \cdot x$ is homotopic to constant map c_e rel $\{e\}$.

Example 10.9.1.2. Every topological group is a strict H-group.

Example 10.9.1.3. Every loop space ΩX is an H-group where the product of two loops is the concatenation and inverse is the inverse of the loop. The required conditions for ΩX to be an H-group is then immediate.

The following is one of the most important result for H-spaces. It says that the contravariant hom functor that they represent is group valued.

Theorem 10.9.1.4. *Let Y be an H-group. Then for any based space X, the based homotopy classes of maps $[X, Y]$ forms a group whose operation is*

$$(f \cdot g)(x) := f(x) \cdot g(x).$$

10.10 Model categories & abstract homotopy

10.11 Classifying spaces

10.11.1 Eilenberg-Maclane spaces

Remark 10.11.1.1 (The canonical map). Let X be a connected space and $G = \pi_1(X)$. Then there is a natural map $i : X \to BG$ which identifies X as a subcomplex of BG. Moreover $i_* : \pi_1(X) \to \pi_1(BG) = G$ is an isomorphism.

10.12 Spectra

Spectra are objects which generalizes both the notion of cohomology theories and spaces, in that there are mappings from cohomology theories and spaces into the homotopy category of spectra. Thus, one needs to construct a good category of spectra, give a model structure on it and thus by Quillen's theory obtain this absolutely wonderful homotopy category of spectra, which unites the viewpoint of cohomology and spaces. However, we are getting ahead of ourselves, as finding the right homotopy category and giving a construction of category of spectra is easier said than done. We will meet this topic later in our discussion of ∞-categories (they will form a prototypical example of stable ∞-categories).

10.13 Lifting & extension problems