Existence and Uniqueness of homotopy type X^+

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Contents

1 The +-construction & its uniqueness

Let X be a connected CW-complex and $P \leq \pi_1(X)$ be a perfect normal subgroup of $\pi_1(X)$. Consider the problem of constructing a CW-complex X^+ such that $\pi_1(X^+) = \pi_1(X)/P$ and that it has same homology as X. This is an important problem as construction X^+ applied on $\mathrm{BGL}(R)$ for some associative unital ring R, can give us a space $\mathrm{BGL}(R)^+$ whose fundamental group is $K_1(R)$ (it can also be further shown that $\pi_2(\mathrm{BGL}(R)^+) \cong K_2(R)$ using characterizations of $K_2(R)$ done earlier, see Theorem 5.1.7 of main notes). Thus, one can define higher K-theory of R as homotopy groups of $\mathrm{BGL}(R)^+$. In this section, we construct such a space X^+ and prove the uniqueness of its homotopy type.

Recall that a map $f : X \to Y$ is acyclic if its homotopy fiber has homology of a point.

Definition 1.0.1 (**+-construction**)**.** Let X be a based connected CW-complex and G be a perfect normal subgroup of $\pi_1(X)$. Then a map of CW-complexes $f : X \to Y$ is called a +-construction on X w.r.t. G if f is acyclic and $\mathrm{Ker}\,(f_* : \pi_1(X) \to \pi_1(Y)) = G$.

Remark 1.0.2. Let $f : X \to Y$ be a +-construction w.r.t. $P \leq \pi_1(X)$ perfect normal subgroup. By homotopy long exact sequence corresponding to map $Ff \to X \xrightarrow{f} Y$, we can immediately get following exact sequence:

$$\pi_1(Ff) \to \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \to \pi_0(Ff).$$

By Theorem A.0.1, Ff is acyclic and thus $\pi_0(Ff) = 0$. Thus we have the exact sequence:

$$0 \to G \to \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \to 0.$$

The following construction of X^+ is taken from Theorem 2.1 of [Sri95].

Construction 1.0.3 (The construction of X^+)**.** Let X be a based connected CW-complex and $G \leq \pi_1(X)$ a perfect normal subgroup. We construct an inclusion $i : X \to X^+$ which is a +-construction of X w.r.t. G. To this end, the main strategy is as follows:

1. First attach 2-cells to X to kill G in $\pi_1(X)$.
2. Then attach 3-cells to remove the extra homology classes added by step 1.

Let us denote G in generators as follows:

$$G = \langle g_\alpha \mid \alpha \in I \rangle.$$

As $g_\alpha \in \pi_1(X)$, therefore we may interpret them as loops

$$g_\alpha : S^1 \to X.$$

Now attach 2-cells to X along each of the g_α:

$$\begin{tikzcd} X' \arrow[dr, phantom, "\ulcorner", very near start] & \amalg_\alpha D^2 \arrow[l] \\ X \arrow[u, hook, "j_0"] & \amalg_\alpha S^1 \arrow[l, "\amalg_\alpha g_\alpha"'] \arrow[u, hook, "i_0"'] \end{tikzcd} \quad . \tag{A1}$$

We first claim that $\pi_1(X')$ is $\pi_1(X)/G$ via j_0. Indeed, the map

$$j_{0*} : \pi_1(X) \longrightarrow \pi_1(X')$$

is surjective since any element $h : S^1 \to X'$ in $\pi_1(X')$ by cellular approximation theorem factors through the inclusion j_0. In particular, the 1-skeleton of X' is same as that of X. Consequently to prove our claim, we need only show that $\mathrm{Ker}\,(j_{0*}) = G$. Clearly, $\mathrm{Ker}\,(j_{0*}) \supseteq G$ by construction. Furthermore, if $k : S^1 \to X$ is null-homotopic in X', then k extends to $k' : D^2 \to X'$. By cellular approximation, we may assume that k' is a cellular map, so that k' is mapping in the 2-skeleton of X'. It follows at once that if k is not in G, then k (which we assume, by cellular approximation, that it is in 1-skeleton of X) on composition with j_0 gives a non-contractible loop as X' only trivializes all loops in G, a contradiction.

This shows that

$$\pi_1(X') = \pi_1(X)/G.$$

To complete the proof, we have to now kill all "new" homology classes of X' with an arbitrary choice of coefficient system $\mathcal{L}$ whose groups are isomorphic to L. To this end, we will attach 3-cells to X' to obtain the space X^+.

To illustrate the idea, suppose we have constructed X^+ by attaching 3-cells to X'. Our goal is then to show that $H_k(X^+;\mathcal{L}) \cong H_k(X;\mathcal{L})$. We thus have a triplet (X^+, X', X). By homology l.e.s. for the pair (X^+, X) , it suffices to show that

$$H_k(X^+, X; \mathcal{L}) = 0$$

for all $k \geq 0$. Recall that the homology of pair (X^+, X') with coefficient $\mathcal{L}$ is given by the homology of complex $L \otimes_{\mathbb{Z}[\pi_1(X)/G]} C_\bullet(\widetilde{X^+}, \hat{X})$ where $\hat{X}$ is the pullback of $\widetilde{X^+}$ along $X \to X^+$. It is thus sufficient to show that $C_\bullet(\widetilde{X^+}, \hat{X})$ is an acyclic complex (whose homology in every degree is 0). As $\widetilde{X^+}/\hat{X}$ will be a 3-dimensional CW-complex with no 1-cells, it is thus sufficient to show that the differential

$$d : C_3(\widetilde{X^+}, \hat{X}) \to C_2(\widetilde{X^+}, \hat{X})$$

is an isomorphism.

Now since we have isomorphisms $C_3(\widetilde{X^+}, \hat{X}) \cong C_3(\widetilde{X^+}, \widetilde{X'}) \cong H_3(\widetilde{X^+}, \widetilde{X'})$ and $C_2(\widetilde{X^+}, \hat{X}) \cong C_2(\widetilde{X'}, \hat{X}) \cong H_2(\widetilde{X'}, \hat{X})$ by the fact that cells of universal cover are obtained by lifting, therefore we have to show that the boundary map obtained by the triplet l.e.s. for $(\widetilde{X^+}, \widetilde{X'}, \hat{X})$ is an isomorphism. This is how we construct X^+ and then show that for this construction the above actually holds.

In order to construct X^+, we need maps $S^2 \to X'$ through which we can attach 3-cells. In particular, these are elements of $\pi_2(X')$. Consider the following pullback square

$$\begin{tikzcd}
\hat{X} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X}' \arrow[d, "\pi"] \\
X \arrow[r, hook, "j_0"'] & X'
\end{tikzcd}$$

where $\tilde{X}' \to X'$ is the universal cover. As pullback of covering is a covering, thus the map $\hat{X} \to X$ is a covering. Now, it is clear that $\hat{X} = \pi^{-1}(X)$, thus the inclusion $\hat{X} \hookrightarrow \tilde{X}'$ is also induced by attaching 2-cells to $\hat{X}$. It follows that $\pi_1(\hat{X}) \cong G$.

Next, observe that in the homology l.e.s. of $(\tilde{X}', \hat{X})$, we get the following isomorphism by Hurewicz (as $\tilde{X}'$ is 1-connected)

$$\pi_2(\tilde{X}') \xrightarrow{\cong} H_2(\tilde{X}') \xrightarrow{j_*} H_2(\tilde{X}', \hat{X}) \longrightarrow H_1(\hat{X}).$$

Again, by Hurewicz, we have

$$H_1(\hat{X}) \cong \pi_1(\hat{X})^{ab} = G^{ab} = 0$$

as G is perfect. Hence the above sequence becomes

$$\pi_2(\tilde{X}') \xrightarrow{\cong} H_2(\tilde{X}') \overset{j_*}{\twoheadrightarrow} H_2(\tilde{X}', \hat{X}).$$

Using the above, we have a surjection $\pi_2(\tilde{X}') \twoheadrightarrow H_2(\tilde{X}', \hat{X})$. For each homology class $[c_\beta] \in H_2(\tilde{X}', \hat{X})$ in a fixed generating set, choose one and only element in the fiber $[\tilde{h}_\beta] \in \pi_2(\tilde{X}')$. We thus have a collection of maps $\{\tilde{h}_\beta : S^2 \to \tilde{X}'\}_\beta$. Composing them with $\pi : \tilde{X}' \to X'$ yields maps $\{h_\beta : S^2 \to X'\}_\beta$. We use these maps to attach 3-cells to X'. Indeed, consider the pushout space:

$$\begin{tikzcd} X^+ \arrow[dr, phantom, "\ulcorner", very near start] & \amalg_\beta D^3 \arrow[l, hook', "\amalg_\beta d_\beta"'] \\ X' \arrow[u, hook, "k_0"] & \amalg_\beta S^2 \arrow[l, hook', "\amalg_\beta h_\beta"] \arrow[u, hook] \end{tikzcd} . \tag{A2}$$

We thus have the following inclusions of subcomplexes of X^+:

$$X \overset{j_0}{\hookrightarrow} X' \overset{k_0}{\hookrightarrow} X^+.$$

We again pass to universal cover of X^+ in order and take pullback along $X' \hookrightarrow X^+$ to have better algebraic control via Hurewicz:

$$\begin{tikzcd} \hat{X}' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X^+} \arrow[d, "\pi"] \\ X' \arrow[r, hook, "k_0"'] & X^+ \end{tikzcd} .$$

But $\pi_1(\hat{X}') = 0$ since k_{0*} is an isomorphism on π_1 and $\pi_1(\widetilde{X^+}) = 0$. Hence, we deduce that

$$\hat{X}' \cong \tilde{X}',$$

that is, $\hat{X}'$ is the universal cover of X'.

By naturality of Hurewicz, we have a map between the long exact sequences of homotopy groups induced by the map $\hat{X}' \hookrightarrow \widetilde{X^+}$ to that of homology groups

$$\begin{tikzcd} \cdots & \pi_{n+1}(\widetilde{X^+}, \hat{X}') \arrow[r] \arrow[d] & \pi_n(\hat{X}') \arrow[r] \arrow[d] & \pi_n(\widetilde{X^+}) \arrow[r] \arrow[d] & \pi_n(\widetilde{X^+}, \hat{X}') \arrow[d] & \cdots \\ \cdots & H_{n+1}(\widetilde{X^+}, \hat{X}') \arrow[r] & H_n(\hat{X}') \arrow[r] & H_n(\widetilde{X^+}) \arrow[r] & H_n(\widetilde{X^+}, \hat{X}') & \cdots \end{tikzcd} .$$

For $n = 3$, we get the following sequence from the above

$$\begin{tikzcd}
\pi_3(\widetilde{X^+}, \widetilde{X'}) \arrow[r] \arrow[d] & \pi_2(\widetilde{X'}) \arrow[d, "\cong"] & \\
H_3(\widetilde{X^+}, \widetilde{X'}) \arrow[r, "\tilde{\partial}"'] & H_2(\widetilde{X'}) \arrow[r, two heads, "j_*"'] & H_2(\widetilde{X'}, \hat{X})
\end{tikzcd}.$$

We claim that $j_* \circ \tilde{\partial}$ is an isomorphism. Note that this is isomorphic to the required boundary map $d : C_3(\widetilde{X^+}, \hat{X}) \to C_2(\widetilde{X^+}, \hat{X})$, as discussed earlier. This will hence complete the proof. Indeed, observe that $H_3(\widetilde{X^+}, \widetilde{X'})$ is a free abelian group generated by the lift of 3-cells attached by $\tilde{h}_\beta$. We thus need only show that $j_* \circ \tilde{\partial}$ maps this bijectively onto the generators of $H_2(\widetilde{X'}, \hat{X})$ which we know are $[c_\beta]$. We know that the lifted map $\tilde{h}_\beta : S^2 \to \widetilde{X'}$ determines an element in $\pi_3(\widetilde{X^+}, \widetilde{X'})$ by definition of relative homotopy, whose image in $\pi_2(\widetilde{X'})$ is exactly $[\tilde{h}_\beta]$. Moreover, the class determined by $\tilde{h}_\beta$ in $\pi_3(\widetilde{X^+}, \widetilde{X'})$, under the Hurewicz map, determines a class $[l_\beta] \in H_3(\widetilde{X^+}, \widetilde{X'})$. By commutativity of above, it follows that $j_* \circ \tilde{\partial}$ maps $[l_\beta] \mapsto [c_\beta]$. As for each generator $[c_\beta] \in H_2(\widetilde{X'}, \hat{X})$, the element $[l_\beta]$ is unique by construction, we get that $j_* \circ \tilde{\partial}$ is an isomorphism, as required.

Remark 1.0.4. While it is rarely that we will use the explicit construction above, it is still good to keep in mind the precise way in which we found the 3-cells to attach to X' to get X^+. In particular, the attaching steps (A1) and (A2) are good to keep in mind.

Example 1.0.5 (+-construction of homology spheres)**.** Let X be a based connected CW-complex which is a homology n-sphere for $n > 1$ so that $\pi_1(X)$ is perfect. For $P = \pi_1(X)$, we claim that any +-construction of X w.r.t. P, $f : X \to X^+$, is such that $S^n \simeq X^+$.

Indeed, observe that $\pi_1(X)$ is perfect as X is a homology n-sphere. As f is a +-construction, therefore $\pi_1(X^+)$ is $\pi_1(X)/\pi_1(X) = 0$ by Remark 1.0.2. Moreover, X^+ itself is a homology n-sphere as $f : X \to X^+$ is acyclic. We now find a map $g : S^n \to X$ such that g is a weak equivalence, so that by Whitehead's theorem we will conclude that g is a homotopy equivalence, as required.

Indeed, observe that since X^+ is 1-connected, therefore by Hurewicz's theorem, we have $\pi_2(X^+) \cong H_2(X^+)$. If $n \neq 2$, then $\pi_2(X^+) = 0$ as X^+ is also a homology n-sphere. By induction and using Hurewicz repeatedly, we get that $\pi_k(X^+) = 0$ for all $0 \leq k \leq n-1$, so that X^+ is $n-1$-connected and thus by another application of Hurewicz, we have $\pi_n(X^+) \cong H_n(X^+) = \mathbb{Z}$. We thus have a non-trivial map $g : S^n \to X^+$ whose homology class is the generator. We finally claim that g induces an isomorphism in integral homology, which will complete the proof by Theorem 7.5.9 of [Spa66] (Whitehead's theorem). To this end, as X^+ is a also a homology n-sphere, thus we need only show that $g_* : H_n(S^n) = \mathbb{Z} \to H_n(X^+) = \mathbb{Z}$ takes $[\mathrm{id}] \mapsto [g]$. Indeed, we have $g_*([\mathrm{id}]) = [g \circ \mathrm{id}] = [g] \in H_n(X^+)$, as needed.

Proposition 1.0.6. *Let $i : X \to X^+$ and $j : Y \to Y^+$ be +-constructions w.r.t. perfect normal subgroups $G \leq \pi_1(X)$ and $H \leq \pi_1(Y)$. Then*

$$i \times j : X \times Y \to X^+ \times Y^+$$

is a +-construction of $X \times Y$ w.r.t. the perfect normal subgroup $G \times H \leq \pi_1(X \times Y)$.

Proof. We first show acyclicity of $i \times j$. By unravelling definitions, one reduces to showing that $F(i \times j) \cong F(i) \times F(j)$ is acyclic. To this end, use Künneth formula to deduce that if X, Y are acyclic, then so is $X \times Y$. The fact that kernel of $(i \times j)_*$ is $G \times H$ follows from $(i \times j)_* = i_* \times j_* : \pi_1(X) \times \pi_1(Y) \to \pi_1(X^+) \times \pi_1(Y^+)$, as required. □

The following universal property tells us what we need, and then some more[1].

Theorem 1.0.7 (Universal property of X^+)**.** *Let X be a CW-complex and P be a perfect normal subgroup of $\pi_1(X)$. Let $f : X \to Y$ be a +-construction on X w.r.t. P. If $g : X \to Z$ is a map such that*

$$P \subseteq \operatorname{Ker}(g_* : \pi_1(X) \to \pi_1(Z)),$$

then there exists a map $h : Y \to Z$ such that the following diagram of spaces commutes

$$\begin{tikzcd} Y \arrow[r, dashed, "h"] & Z \\ X \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

and h is unique upto homotopy.

An immediate corollary is what we seek.

Corollary 1.0.8 (Uniqueness of +-construction)**.** *Let X be a CW-complex and P be a perfect normal subgroup of $\pi_1(X)$. If $f : X \to Y$ and $g : X \to Z$ are two +-constructions, then there is a homotopy equivalence $h : Y \xrightarrow{\simeq} Z$.* □

Another important consequence is that we have maps in +-construction.

Lemma 1.0.9. *Let X, Y be two connected CW-complexes and $i : X \to X^+$ and $j : Y \to Y^+$ be +-constructions w.r.t. perfect normal subgroups $G \leq \pi_1(X)$ and $H \leq \pi_1(Y)$ respectively. If $f : X \to Y$ is a map such that $f_* : \pi_1(X) \to \pi_1(Y)$ maps G into H, then there exists a map $\tilde{f} : X^+ \to Y^+$ unique upto homotopy w.r.t. the commutativity of the following square of spaces:*

$$\begin{tikzcd} X^+ \arrow[r, dashed, "\tilde{f}"] & Y^+ \\ X \arrow[u, "i"] \arrow[r, "f"'] & Y \arrow[u, "j"'] \end{tikzcd}.$$

Proof. The map $j \circ f$ on π_1 takes G to 0, so by Theorem 1.0.7 gives the required map unique upto homotopy. □

We shall prove Theorem 1.0.7 by using obstruction theory as developed in [Whi78], Chapter VI.

[1] This is usually attriuted to Quillen, who mentioned this in his ICM report [Qui70] without proof.

Proof of Theorem 1.0.7. Consider the based connected CW-complex X^+ obtained by Construction 1.0.3. Let $g : X \to Z$ be a map such that

$$P \subseteq \operatorname{Ker}(g_* : \pi_1(X) \to \pi_1(Z)).$$

We wish to extend g to $\tilde{g} : X^+ \to Z$. Consider the map $\theta : \pi_1(X)/P \to \pi_1(Z)$ as in the triangle below which exists by hypothesis on g_*:

$$\begin{tikzcd} \pi_1(X)/P \arrow[r, dashed, "\theta"] & \pi_1(Z) \\ \pi_1(X) \arrow[u, "i_*"] \arrow[ur, "g_*"'] & \end{tikzcd}.$$

We wish to show that g extends to $\tilde{g} : X^+ \to Z$ such that $\tilde{g}_* = \theta$. To this end, by obstruction theory, it is sufficient to show that

$$H^q(X^+, X; \mathcal{L}) = 0$$

for all $q \geq 3$ and all local coefficient systems $\mathcal{L}$ on X^+. Fix a local coefficient system $\mathcal{L}$ with group G. Note that we have

$$H^q(X^+, X; \mathcal{L}) \cong H^q\left(\operatorname{Hom}_{\mathbb{Z}[\pi_1(X^+)]}\left(C_\bullet(\widetilde{X^+}, \hat{X}), G\right)\right)$$

where we have the following pullback of the universal cover of X^+:

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X^+} \arrow[d] \\ X \arrow[r, hook] & X^+ \end{tikzcd}.$$

Now note from the Construction 1.0.3 that

$$C_k(\widetilde{X^+}, \hat{X}) = 0$$

for all $k \neq 2, 3$ and $d : C_3(\widetilde{X^+}, \hat{X}) \to C_2(\widetilde{X^+}, \hat{X})$ is an isomorphism. It follows at once that $H^q(X^+, X; \mathcal{L}) = 0$ for all $q \geq 0$, as required.

For uniqueness upto homotopy, obstruction theory further gives us a sufficient criterion that $H^2(X^+, X; \mathcal{L}) = 0$. Hence we are done. Moreover, by the long exact sequence of pairs for cohomology with local coefficients, we deduce that the map $i : X \hookrightarrow X^+$ induces isomorphism

$$i^* : H^q(X^+; \mathcal{L}) \to H^q(X; i^*\mathcal{L}),$$

that is, $i : X \to X^+$ is cohomologically acyclic as well. This shows the universal property for the explicit construction. We now show that any +-construction on X w.r.t. P is homotopy equivalent to the explicit one. This will then complete the proof.

Let $f : X \to Y$ be a +-construction w.r.t. P. Then by above there exists a map $\tilde{f} : X^+ \to Y$ as in the following triangle

$$\begin{tikzcd} X^+ \arrow[r, dashed, "\tilde{f}"] & Y \\ X \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}.$$

We claim that the map $\tilde{f}$ is a homotopy equivalence. By Whitehead's theorem, it is sufficient to show that $\tilde{f}$ is a weak-equivalence. Observe that as i and f are homologically acyclic, it follows at once that $\tilde{f}$ is also acyclic. Moreover, $\tilde{f}$ induces isomorphism in fundamental groups. By acyclic fiber theorem (Theorem A.0.1), it follows that the homotopy fiber $F\tilde{f}$ is acyclic. We further claim that $F\tilde{f}$ is 1-connected. Indeed, from the long exact sequence for homotopy groups for $\tilde{f}$ and that $\tilde{f}_* : \pi_1(X^+) \to \pi_1(Y)$ is an isomorphism, it follows that the map $\pi_1(F\tilde{f}) \to \pi_1(X^+)$ is the zero map. It suffices to show that the transgression $\pi_2(Y) \to \pi_1(F\tilde{f})$, which is surjective by exactness, is the zero map as well. As $F\tilde{f}$ is acyclic, therefore $\pi_1(F\tilde{f})$ is a perfect group. By above, it is also abelian, and thus the zero group, as required.

Hence $F\tilde{f}$ is a 1-connected acyclic space, so that by Hurewicz's theorem, all homotopy groups of $F\tilde{f}$ are 0. By homotopy long exact sequence of $\tilde{f}$, it follows that $\tilde{f}$ is a weak-equivalence, as required. This also proves Corollary 1.0.8. □

A Acyclic fiber theorem

The following is an important characterization of acyclicity in terms of homotopy fiber.

Theorem A.0.1 (Acyclic fiber theorem)**.** *Let $f : X \to Y$ be a based map of connected CW-complexes. Then the following are equivalent:*

1. *For all $k \geq 0$, we have*
$$f_* : H_k(X; M) \xrightarrow{\cong} H_k(Y; M)$$
for every $\pi_1(Y)$-module M[2]*.*
2. *The homotopy fiber Ff of f is acyclic*[3]*.*

Proof. (1. ⇒ 2.) By replacing X by the fibration replacement of f (see Construction 10.2.1.11 of [FoG]), we may assume that we have a fibration $Ff \xrightarrow{i} X \xrightarrow{f} Y$. Assume that $\pi_1(Y) = 0$, so that we have a Serre spectral sequence $E^2_{pq} = H_p(Y; H_q(Ff)) \Rightarrow H_{p+q}(X)$ and for the trivial fibration pt. $\to Y \xrightarrow{\mathrm{id}} Y$ which gives another Serre spectral sequence ${}'E^2_{pq} = H_p(Y; H_q(\mathrm{pt.})) \Rightarrow H_{p+q}(Y)$. We have a commutative diagram:

$$\begin{tikzcd} Ff \arrow[r, "i"] \arrow[d] & X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\mathrm{id}"'] \\ \mathrm{pt.} \arrow[r] & Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd} .$$

By comparison theorem (Proposition 5.13 of [Hat04]), we deduce that Ff is acyclic. It follows that if Y is simply connected and f induces isomorphism on integral homology, then homotopy fiber of f is acyclic.

Now suppose $\pi_1(Y) \neq 0$. The main idea is to reduce to the simply connected case by going to universal cover of Y. Indeed, if $\tilde{Y}$ is the universal cover of Y, then we have the following pullback

[2] That is, M is a left $\mathbb{Z}[\pi_1(Y)]$-module.
[3] that is, Ff has homology of a point.

diagrams (by Lemma 10.2.1.2 of [FoG], we have that $\tilde{f}$ is a fibration):

$$\begin{tikzcd} F\tilde{f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times_Y \tilde{Y} \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{Y} \arrow[d, "p"] \\ Ff \arrow[r] & X \arrow[r, "f"'] & Y \end{tikzcd} .$$

Denote $\tilde{X} = X \times_Y \tilde{Y}$. It then follows by maps constructed by unique path lifting that $F\tilde{f} \cong Ff$. It thus suffices to show that $F\tilde{f}$ is acyclic. To this end, by above, we reduce to showing that we have an isomorphism $\tilde{f}_* : H_k(\tilde{X};\mathbb{Z}) \to H_k(\tilde{Y};\mathbb{Z})$ for all $k \geq 0$. This follows from the following comutative square with vertical maps being isomorphisms:

$$\begin{tikzcd} H_k(\tilde{X};\mathbb{Z}) \arrow[r, "\tilde{f}_*"] \arrow[d, "\cong"'] & H_k(\tilde{Y};\mathbb{Z}) \arrow[d, "\cong"] \\ H_k(X;\mathbb{Z}[\pi_1(Y)]) \arrow[r, "f_*"'] & H_k(X;\mathbb{Z}[\pi_1(Y)]) \end{tikzcd} .$$

As f_* is an isomorphism by hypothesis, we win.

(2. $\Rightarrow$ 1.) As before, we may assume that $Ff \xrightarrow{i} X \xrightarrow{f} Y$ is a fibration. Observe that the E^2-page of Serre spectral sequence $E^2_{pq} = H_p(Y; H_q(Ff)) \Rightarrow H_{p+q}(X;\mathbb{Z})$ is all 0 except possibly the bottom row (which consists of $H_q(Y;\mathbb{Z})$) since $H_q(Ff) = 0$ forall $q \geq 1$ and $H_0(Ff) = \mathbb{Z}$. It follows that E collapses on the E^2-page, so that $H_n(X;\mathbb{Z}) \cong H_n(Y;\mathbb{Z})$. In particular, this isomorphism comes from f_* as the above isomorphim is by the edge homomorphism which we know in Serre spectral sequence is via the map $f : X \to Y$ (see Addendum 2, Theorem 5.3.2 of [Wei94]). $\square$

References

[Wei13] Weibel, C.A. (2013). The K-book: An Introduction to Algebraic K-theory. Graduate Studies in Mathematics, American Mathematical Society, ISBN 9780821891322.

[AM69] Atiyah, M.F., MacDonald I.G. (1969). Introduction to commutative algebra. Addison-Wesley-Longman, ISBN 978-0-201-40751-8.

[Wei94] Weibel, C.A. (1994). An Introduction to Homological Algebra. Cambridge Studies in Advanced Mathematics, Cambridge University Press, ISBN 9780521559874.

[Mil71] Milnor, J.W. (1971). Introduction to Algebraic K-theory, Annals of Mathematics Studies, Princeton University Press, ISBN 9780691081014.

[Eis95] Eisenbud, D. (1995). Commutative Algebra: With a View Toward Algebraic Geometry, Graduate Texts in Mathematics, Springer, ISBN 9780387942698.

[McC01] McCleary, J. (2001). A User's Guide to Spectral Sequences, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, ISBN 0-521-56759-9.

[Qui70] Quillen, D. (1970). Cohomology of groups, Actes du Congrès International des Mathèmaticiens, Tome 2, Gauthier-Villars, Paris, 1971, pp. 47–51. MR MR0488054.

[Hat04] Hatcher, A. (2004). Spectral Sequences in Algebraic Topology. Available online.

[Whi78] Whitehead, G.W. (1978). Elements of Homotopy Theory, Graduate Texts in Mathematics, Springer-Verlag, New York, ISBN 0-387-90336-4.

[Spa66] Spanier, E.H. (1966). Algebraic Topology, McGraw-Hill Book Company. ISBN 0387906460.

[Nei80] Neisendorfer, J. (1980). Primary homotopy theory, Memoirs of the American Mathematical Society, Number 232, ISBN 978-1-4704-0636-3.

[GH19] Greenberg, M.J., Harper, J.R. (2019). Algebraic Topology (reprint), Mathematics Lecture Note Series, Taylor & Francis Group, ISBN 9780367091880.

[Sri95] Srinivas, V. (1995). Algebraic K-Theory, Modern Birkhäuser Classics, Birkhäuser Boston, ISBN 9780817647360.

[FoG] A. Renanse. Facets of Geometry (author's notes). Available online & under construction.