The Facets of Geometry

Algebraic

(Under heavy construction!!)

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Contents

Part I

The Algebraic Viewpoint

Chapter 1

Foundational Algebraic Geometry

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1.1 A guiding example

Let X be a compact Hausdorff topological space. In this section we would like to portray the main point of scheme theory in the case of space X, that is, one can study the geometry over "base" space completely by studying the algebra of ring of suitable functions over it. In particular, we would like to establish the following result.

Proposition 1.1.0.1. *Let X be a compact Hausdorff topological space. Denote R to be the ring of continuous real-valued functions on X under pointwise addition and multiplication and denote* $\mathrm{mSpec}\,(R)$ *to be the set of maximal ideals of R. Then,*

1. *We have a set bijection:*
$$\mathrm{mSpec}\,(R) \cong X.$$
2. *We have that* $\mathrm{mSpec}\,(R)$ *and X are isomorphic as topological spaces:*
$$\mathrm{mSpec}\,(R) \cong X$$
where $\mathrm{mSpec}\,(R)$ *is given its Zariski topology.*

Proof. 1. Let $x \in X$ be an arbitrary point. Denote $\mathfrak{m}_x := \{f \in R \mid f(x) = 0\}$ to be the vanishing ideal of point x. This ideal is maximal because the quotient $R/\mathfrak{m}_x \cong \mathbb{R}$ via the map $f + \mathfrak{m}_x \mapsto f(x)$. Indeed, it is a valid ring homomorphism and is surjective by virtue of the continuous map constant at a point in $\mathbb{R}$. Moreover, if $f(x) = g(x)$ for $f, g \in R$, then $f - g \in \mathfrak{m}_x$ and hence $f + \mathfrak{m}_x = g + \mathfrak{m}_x$, so it is injective as well. Now consider the function:

$$\begin{aligned} \varphi : X &\to \mathrm{mSpec}\,(R) \\ x &\mapsto \mathfrak{m}_x. \end{aligned}$$

We claim that φ is bijective. To see injectivity, suppose $\mathfrak{m}_x = \mathfrak{m}_y$ for $x, y \in X$. Then, we have that $R/\mathfrak{m}_x = R/\mathfrak{m}_y \cong \mathbb{R}$. This tells us that for each $f \in R$, $f(x) = f(y) \in \mathbb{R}$. Now assume that $x \neq y$. Since X is T_1, therefore $\{x\}, \{y\}$ are two disjoint closed subspaces of X. Then, by Urysohn's lemma (we have that X is compact Hausdorff), we get that there exists a continuous $\mathbb{R}$-valued function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$, a contradiction. Hence $x = y$.

Next, we wish to establish the surjectivity of φ. For that, take any maximal ideal $\mathfrak{m} \in \mathrm{mSpec}\,(R)$. We wish to show that there exists $x \in X$ such that $\mathfrak{m} = \mathfrak{m}_x$. Assume to the contrary that for all $x \in X$, there exists $f_x \in \mathfrak{m}$ such that $f_x \notin \mathfrak{m}_x$. Indeed getting a contradiction here will do the job by maximality of $\mathfrak{m}_x$. Now, let $U_x \subseteq X$ be the open set such that $f_x(y) \neq 0$ for each $y \in U_x$. We thus have an open cover $\{U_x\}$ of X and by compactness we get $U_{x_1}, \dots, U_{x_n}$ covers X. Now, since X is compact Hausdorff, therefore there exists a partition of unity subordinate to $\{U_{x_i}\}$. That is, we have $g_i : X \to \mathbb{R}$ where $\mathrm{Supp}\,(g_i) \subseteq U_{x_i}$ such that $1 = g_i f_{x_i} + \dots + g_n f_{x_n}$. Since $f_{x_i} \in \mathfrak{m}$ for each i, therefore $1 = g_i f_{x_i} + \dots + g_n f_{x_n} \in \mathfrak{m}$, hence $\mathfrak{m} = R$, a contradiction.

2. Let us first establish that φ as in item 1 above is continuous. Indeed, let $I \leq R$ be an ideal and $V(I) = \{\mathfrak{m} \in \mathrm{mSpec}\,(R) \mid \mathfrak{m} \supseteq I\}$. A closed set of $\mathrm{mSpec}\,(R)$ looks exactly like above. We wish to show that $\varphi^{-1}(V(I))$ is closed in X. It is immediate to observe by item 1 that

$$\varphi^{-1}(V(I)) = \bigcap_{f \in I} \{x \in X \mid f(x) = 0\}.$$

Since $f : X \to \mathbb{R}$ is continuous, so it follows that $\varphi^{-1}(V(I))$ is closed. This shows the continuity of $\varphi : X \to \mathrm{mSpec}\,(R)$. As X is compact and φ a bijective homeomorphism, it is thus sufficient to show that $\mathrm{mSpec}\,(R)$ is Hausdorff.

Fix two points $\mathfrak{m}_x \neq \mathfrak{m}_y$ in $\mathrm{mSpec}\,(R)$ for $x \neq y \in X$. Fix two opens U, V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Let $C = X \setminus U$ and $D = X \setminus V$. Note that $C \cup D = X$. Now applying Urysohn's lemma on C, D yields $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ such that $f(C) = 0$, $f(D) = 1$ and $g(D) = 0$, $g(C) = 1$. Consequently, $fg = 0$ over X. Now consider the basic opens $D(f), D(g) \subseteq \mathrm{mSpec}\,(R)$. As $f(x) \neq 0$ since $x \in D$, therefore $D(f) \ni x$. Similarly, $D(g) \ni y$. Since $D(f) \cap D(g) = D(fg) = D(0) = \emptyset$, therefore x and y can be separated, as required. $\square$

Remark 1.1.0.2. An important corollary of the above result is that we can actually distinguish between the points of X by looking at maximal ideals of R; for $x, y \in X$, $x \neq y$ if and only if $\mathfrak{m}_x \neq \mathfrak{m}_y$. This is interesting because a fundamental goal of algebraic geometry is to study geometric properties of varieties over an algebraically closed field k and dominant maps between them. A fundamental equivalence tells that this is equivalent to studying the ring of regular functions over such a variety. Moreover, this ring recovers the important topology on the variety (there can be atleast two topologies on the variety if we are in, say $\mathbb{C}$). Hence one motivation to undergo this switch of viewpoint, where we try to do everything algebraically is that 1) we can completely recover the points of the variety and the relevant topology on it and that 2) we have a broad generalization of algebro-geometric techniques and constructions to an arbitrary commutative and unital ring R.

Caution 1.1.0.3. While in the sequel we will encounter spaces which are compact, it would rarely (unless you are interested in Boolean rings) be the case that the spaces will be Hausdorff. However, if one notices the way Hausdorff property is used in the above result, then one can see that if we somehow makes sure that the space X constructed out of a ring R is such that every point of X can be "distinguished" by functions on X in R, then you don't need Hausdorff property. This is precisely what will happen.

1.2 Affine schemes and basic properties

Let us first swiftly give an account of basic global constructions in scheme theory. The foundational philosophy of scheme theory is to handle a space completely by the ring of globally defined *nice* functions on it. This is taken to an unprecedented extreme by the definition of an affine scheme, which tells us that one can even do geometry on the base space by the knowledge of globally defined functions on the base space alone; you can indeed *reconstruct* the base space! So, we begin with a general ring R and construct a topological space $\mathrm{Spec}\,(R)$. The way we will define its points is by thinking of each point of this base space $\mathrm{Spec}\,(R)$ as that subset of R, each of whose function becomes zero at a common point. One then sees that these are exactly the prime ideals of R. Hence, the base space $\mathrm{Spec}\,(R)$ is:

$$\mathrm{Spec}\,(R) := \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal of } R\}.$$

Next thing we wish to do is to actually get a *space* structure on this constructed base space, that is, a topology on $\mathrm{Spec}\,(R)$. This is, again, given with the help of the ring R. In particular, we give a topology on $\mathrm{Spec}\,(R)$ where every closed set is given by the zero locus of collections of functions $S \subseteq R$, that is, $V(S) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid \mathfrak{p} \supseteq S\} = \{x \in \mathrm{Spec}\,(R) \mid f(x) = 0\ \forall f \in S\}$ where the last equation tells one how to think about the definition of $V(S)$. This is known as Zariski topology on $\mathrm{Spec}\,(R)$ and is defined by the following:

$$A \subseteq \mathrm{Spec}\,(R) \text{ is closed } \iff A = V(S) \text{ for some } S \subseteq R.$$

After defining the topology on $\mathrm{Spec}\,(R)$, one is interested in interested in understanding the set of all *germs* of functions at a point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. What are germs of functions at a point? Well, heuristically, they are all possible ways a function can *look* different at the given point. So for this, we have to atleast gather all those functions in R which takes different values at point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. Clearly this is given by the quotient domain $R/\mathfrak{p}$. Now from this, we construct the *residue field* of $\mathrm{Spec}\,(R)$ at point $\mathfrak{p}$, denoted $\kappa(\mathfrak{p}) := (R/\mathfrak{p})_{\langle 0 \rangle}$, that is, the fraction field of domain $R/\mathfrak{p}$. What does this $\kappa(\mathfrak{p})$ denotes geometrically? Well, it denotes the field of all different values a function can take at point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. Now, if that is the case, then one sees that if one takes any function $f \in R$, then "evaluating" f at $\mathfrak{p}$ should yield a point $f(\mathfrak{p})$ in $\kappa(\mathfrak{p})$. Indeed, we have the natural quotient maps:

$$R \to R/\mathfrak{p} \to \kappa(\mathfrak{p}).$$

So one should see

$$\kappa(\mathfrak{p}) \textit{ as the field of possible values that a function } f \in R \textit{ can take at point } \mathfrak{p}.$$

However, we have not yet made the set of germs at a point $\mathfrak{p}$. The relation between two functions of having equal germs on R at a point $\mathfrak{p}$ is given by the heuristic that $f, g \in R$ should become equal in some open neighborhood around $\mathfrak{p}$. Since we have a topology on $\mathrm{Spec}\,(R)$, so one can actually do this formally. One will then see this that the set of all germs at point $\mathfrak{p}$ are actually all rational functions of R definable at $\mathfrak{p}$, that is, heuristically, f/g with $g(\mathfrak{p}) \neq 0$ for $f, g \in R$. This in our language turns out to be all the symbols of the form f/g with $g \notin \mathfrak{p}$. This is exactly the local ring $R_{\mathfrak{p}}$, the localization of the ring R (seen as ring of functions over $\mathrm{Spec}\,(R)$) at the point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. So

$$\textit{germs of functions of } R \textit{ at } \mathfrak{p} \textit{ is } R_{\mathfrak{p}}.$$

We will expand more on this when we will talk about the structure sheaf of $\mathrm{Spec}\,(R)$.

Let us now see a basic but important dictionary between the topology of space $\mathrm{Spec}\,(R)$ and the algebra of ideals of R:

Lemma 1.2.0.1. *Let R be a ring. We then have the following:*
1. *If $\mathfrak{a}$, $\mathfrak{b}$ are two ideals of R, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*
2. *If $\{\mathfrak{a}_n\}$ is a collection of ideals of R, then $V\left(\sum_n \mathfrak{a}_n\right) = \bigcap_n V(\mathfrak{a}_n)$.*
3. *If $\mathfrak{a}, \mathfrak{b}$ are two ideals of R, then $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{a} \supseteq \sqrt{b}$.*

Proof. 1. First, let us see that $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Take any $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. Suppose $p \notin V(\mathfrak{a})$ and $\mathfrak{p} \notin V(\mathfrak{b})$. Then there exists $f \in \mathfrak{a}$, $g \in \mathfrak{b}$ such that $fg \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Thus, $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, a contradiction in both cases. Second, it is easy to see that $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ as if either $p \supseteq \mathfrak{a}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, then since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}$, therefore $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$.
2. Let $\mathfrak{p} \supseteq \sum_n \mathfrak{a}_n$. Since ideals are abelian groups so the sum contains each $\mathfrak{a}_n$, hence $\mathfrak{p} \supseteq \mathfrak{a}_n$ for

each n, and so $\mathfrak{p} \in \bigcap_n V(\mathfrak{a}_n)$. Conversely, if $\mathfrak{p} \supseteq \mathfrak{a}_n$ for each n, then $\mathfrak{p} = \sum_n \mathfrak{p} \supseteq \sum_n \mathfrak{a}_n$.
3. (L $\implies$ R) Since each prime ideal containing $\mathfrak{a}$ also contains $\mathfrak{b}$, therefore the intersection of all prime ideals containing $\mathfrak{a}$ will contain the intersection of all prime ideals containing $\mathfrak{b}$.
(R $\implies$ L) Take any prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$. Since $\sqrt{a} \supseteq \sqrt{b}$, therefore $\mathfrak{p} \supseteq \mathfrak{b}$. ☐

1.2.1 Topological properties of $\operatorname{Spec}(R)$

Let us begin by an algebraic characterization of irreducible closed subspaces of $\operatorname{Spec}(R)$.

Lemma 1.2.1.1. *Let R be a ring and $X \hookrightarrow \operatorname{Spec}(R)$ be a closed subspace. Then the following are equivalent:*

1. *X is irreducible.*
2. *There is a unique point $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $X = V(\mathfrak{p})$.*

One calls the point $\mathfrak{p}$ the generic point of the irreducible closed subspace X[1].

Proof. (1. $\Rightarrow$ 2.) Since X is closed therefore $X = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R. If we assume that $X \neq V(\mathfrak{p})$ for each prime $\mathfrak{p} \subseteq R$, then this holds true for points $\mathfrak{p} \in X$ as well. Hence take $\mathfrak{p} \in X$ and consider the proper closed subset $V(\mathfrak{p}) \subsetneq X$. Let $\mathfrak{q} \notin V(\mathfrak{p})$. Then, $V(\mathfrak{q}) \subsetneq X$ as well. Hence we get that $V(\mathfrak{p}) \cup V(\mathfrak{q}) = V(\mathfrak{a})$, which stands in contradiction to the fact that X is irreducible. Hence there exists a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $X = V(\mathfrak{p})$. Uniqueness is quite clear.
(2. $\Rightarrow$ 1.) Suppose $Y = V(\mathfrak{a})$ and $Z = V(\mathfrak{b})$ are two closed subspaces of $X = V(\mathfrak{p})$ such that $X = Y \cup Z = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$ (Lemma 1.2.0.1). Assume that Y, Z are proper inside X. Then, there are two points $\mathfrak{q}_1 \in Y \setminus Z$ and $\mathfrak{q}_2 \in Z \setminus Y$. Algebraically, this is equivalent to saying that $\mathfrak{q}_1 \supseteq \mathfrak{a}$, $\mathfrak{q}_1 \not\supseteq \mathfrak{b}$ and $\mathfrak{q}_2 \supseteq \mathfrak{b}$, $\mathfrak{q}_2 \not\supseteq \mathfrak{a}$. It follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2$ is also a prime ideal which contains $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. Since $X = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{p}) \ni \mathfrak{p}$, hence it follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2 \supseteq \mathfrak{p}$ as it already contains $\mathfrak{a}\mathfrak{b}$. Thus $\mathfrak{q}_1 \cap \mathfrak{q}_2 \in V(\mathfrak{a}) \cap V(\mathfrak{b}) \subseteq V(\mathfrak{p})$. Since $V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b})$, hence it follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2 \supseteq \mathfrak{a}, \mathfrak{b}$, which implies in particular that $\mathfrak{q}_1 \supseteq \mathfrak{a}, \mathfrak{b}$, a contradiction. ☐

Remark 1.2.1.2. The main idea of the above proof has been to first translate the topological condition to algebraic, and then using the critical observation that the closed subspace $V(\mathfrak{p})$ contains point $\mathfrak{p}$ itself.

A simple corollary of above gives all closed points of an affine scheme.

Lemma 1.2.1.3. *Let R be a ring. Then*

$$\{\textit{Closed points of } \operatorname{Spec}(R)\} \cong \{\textit{Maximal ideals of } R.\}$$

Proof. Follows immediately from Lemma 1.2.1.1. ☐

Let us next observe a simple but important observation about topology of $\operatorname{Spec}(R)$.

Lemma 1.2.1.4. *Let R be a ring. For $f \in R$, define $\operatorname{Spec}(R)_f := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \notin \mathfrak{p}\}$. Then,*

1. *$\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)$ is an open set and such open sets form a basis of the Zariski topology on $\operatorname{Spec}(R)$.*
2. *$\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)_g$ if and only if $f \in \sqrt{Rg}$.*

[1]Such spaces where every irreducible closed set has a unique generic point are called sober spaces.

Proof. 1. Clearly $\mathrm{Spec}\,(R)_f = \mathrm{Spec}\,(R) \setminus V(f)$ where we know that $V(f) = \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid f \in \mathfrak{p}\}$. Hence X_f is open. It is also clear that if $U \subseteq \mathrm{Spec}\,(R)$ is open, then $\mathrm{Spec}\,(R) \setminus U = V(\mathfrak{a})$ is closed and hence $U = \bigcup_{f\in\mathfrak{a}} \mathrm{Spec}\,(R)_f$. Further, $\mathrm{Spec}\,(R) = \mathrm{Spec}\,(R)_1$ and $\emptyset = \mathrm{Spec}\,(R)_0$.

2. This follows from the following equivalences. Let $\mathrm{Spec}\,(R)_f \hookrightarrow \mathrm{Spec}\,(R)_g$, then we get the following (we implicitly use Hilbert Nullstellensatz)

$$\mathrm{Spec}\,(R)_f \hookrightarrow \mathrm{Spec}\,(R)_g \iff f(\mathfrak{p}) \neq 0 \implies g(\mathfrak{p}) \neq 0 \iff g(\mathfrak{p}) = 0 \implies f(\mathfrak{p}) = 0 \iff V(g) \subseteq V(f)$$
$$\iff \sqrt{Rg} \supseteq \sqrt{Rf} \supseteq Rf \iff f \in \sqrt{Rg}.$$

This completes the proof. □

Next we observe the equivalent formulation of partitions of unity in the context of algebra.

Lemma 1.2.1.5. *Let R be a ring. Then,*

1. *If $U \hookrightarrow \mathrm{Spec}\,(R)$ is any open set given by $U = \bigcup_{f\in S} \mathrm{Spec}\,(R)_f$ for some subset $S \subseteq R$, then*

$$\mathrm{Spec}\,(R) \setminus U = V\left(\sum_{f\in S} Rf\right).$$

2. *$\mathrm{Spec}\,(R) = \bigcup_{f\in S} \mathrm{Spec}\,(R)_f$ for some $S \subseteq R$ if and only if the ideal of R generated by S is the whole of R.*

Proof. 1. Let $U \hookrightarrow \mathrm{Spec}\,(R)$ be an open set. Then, $\mathfrak{p} \in \mathrm{Spec}\,(R) \setminus U \iff \mathfrak{p} \notin U \iff \forall f \in S,\ \mathfrak{p} \notin \mathrm{Spec}\,(R)_f \iff \forall f \in S,\ f \in \mathfrak{p} \iff \mathfrak{p} \supseteq S \iff \mathfrak{p} \in V(S)$.
2. Follows from 1. □

We next have an interesting observation that $\mathrm{Spec}\,(R)$ are always quasicompact[2].

Lemma 1.2.1.6. *Let R be a ring. Then $\mathrm{Spec}\,(R)$ is quasicompact.*

Proof. Take any arbitrary basic open cover $\bigcup_{f\in S} \mathrm{Spec}\,(R)_f$ for some $S \subseteq R$. Then by Lemma 1.2.1.5, 2, we get that $\sum_{f\in S} Rf \ni 1$ and hence there are $f_1, \dots, f_n \in S$ such that $g_1 f_1 + \dots g_n f_n = 1$ for some $g_i \in R$. Hence $\mathrm{Spec}\,(R) \setminus \bigcup_{i=1}^n = V(f_1, \dots, f_n) = V(R) = \emptyset$. □

Next, we see the topological effects on space $\mathrm{Spec}\,(R)$ of Noetherian hypothesis on ring R. In particular, we see that the space $\mathrm{Spec}\,(R)$ itself becomes *noetherian topological space*, that is, it's closed sets satisfies descending chain condition.

Lemma 1.2.1.7. *Let R be a ring. If R is noetherian, then $\mathrm{Spec}\,(R)$ is noetherian.*

Proof. Use $V(-)$ and $I(-)$, where $I(Y) = \{f \in R \mid f \in \mathfrak{p} \forall \mathfrak{p} \in Y\}$. Rest is trivial. □

We next discuss few things about the irreducible subsets of a closed set of $\mathrm{Spec}\,(R)$. Let $F \hookrightarrow \mathrm{Spec}\,(R)$ be a closed subset. Then we can contemplate irreducible subsets of F. Clearly, each irreducible subset has to be in a maximal irreducible subset, which are called *irreducible components* of $\mathrm{Spec}\,(R)$. We have few basic observations about irreducible components.

[2]it is customary in algebraic geometry to call the topological compactness as quasi-compactness; compactness in algebraic geometry historically means Hausdorff *and* topological compactness.

Lemma 1.2.1.8. *Let R be a ring and F be a closed subset of* $\operatorname{Spec}(R)$*. Then,*

1. *Each irreducible component of F is closed.*
2. *If R is noetherian, then there are only finitely many irreducible components of* $\operatorname{Spec}(R)$*.*
3. *We have that*

$$\{\textit{Irreducible components of } \operatorname{Spec}(R)\} = \{\textit{Closed sets } V(\mathfrak{p}),\ \mathfrak{p} \textit{ is minimal prime}\}.$$

Proof. Statement 1. follows from Lemma 1.2.1.1. Statement 2. follows from Lemma 1.2.1.7 and the fact that a noetherian topological space has only finitely many irreducible components. We now show statement 3. If Z is an irreducible component, then it is closed and $Z = V(\mathfrak{p})$ by Lemma 1.2.1.1. We claim that $\mathfrak{p}$ is a minimal prime. If not, then as every prime has a minimal prime, we will have $\mathfrak{p}' \subsetneq \mathfrak{p}$ such that $\mathfrak{p}'$ is minimal. Consequently, we get $V(\mathfrak{p}') \supsetneq V(\mathfrak{p})$. An another use of Lemma 1.2.1.1 yields that $V(\mathfrak{p}')$ is irreducible. But $V(\mathfrak{p})$ was irreducible component, giving a contradiction. We deduce that $\mathfrak{p}$ is a minimal prime, as required.

Conversely, if $\mathfrak{p}$ is minimal, then $V(\mathfrak{p})$ is an irreducible closed set which cannot be contained in a larger irreducible closed set as otherwise we will have $V(\mathfrak{p}') \supsetneq V(\mathfrak{p})$ and thus, $\sqrt{\mathfrak{p}'} \subsetneq \sqrt{\mathfrak{p}}$ (Lemma 1.2.0.1), but as the ideals are prime, so $\mathfrak{p}' \supsetneq \mathfrak{p}$, a contradiction to minimality. □

Note that we are already in a position to prove some algebraic statements using topological arguments, as the following lemma shows.

Lemma 1.2.1.9. *Let A be a ring and let $a_1, \dots, a_n \in A$ generate the unit ideal in A. Then for all $m > 0$, the collection $a_1^m, \dots, a_n^m \in A$ also generates the unit ideal in A.*

Proof. From Lemma 1.2.1.6, 2, it follows that $\{D(a_i)\}_{i=1,\dots,n}$ covers $\operatorname{Spec}(A)$. Since for any $a \in A$, the basic open $D(a) \subseteq \operatorname{Spec}(A)$ is equal to $D(a^m)$ as a prime $\mathfrak{p}$ doesn't contain a if and only if it doesn't contain any of its power. Consequently, we get that $\{D(a_i^m)\}_{i=1,\dots,n}$ also forms a basic open cover of $\operatorname{Spec}(A)$. An application of Lemma 1.2.1.6, 2 again proves the result. □

1.2.2 The structure sheaf $\mathcal{O}_{\operatorname{Spec}(R)}$

The next important thing we want to consider on $\operatorname{Spec}(R)$ is a sheaf of suitable nice functions over it. This sheaf will be of utmost importance as it will not be treated as an additional structure, but will be an integral part (in-fact, the most important part) of the definition of an affine scheme.

The question now is, *what are* nice functions over $\operatorname{Spec}(R)$ whose sheaf we should take. We turn to classical algebraic varieties for that (one may skip the following if he/she find himself/herself to be brave enough to face the abstraction of the structure sheaf). See Section 1.5 for more details.

Example 1.2.2.1. (*Structure sheaf of an algebraic variety*) Let k be an algebraically closed field. An important aspect of varieties is their morphism. We will display this only in the affine case. Let X, Y be two affine varieties. To define a morphism between X and Y, we would first need to understand the notion of *regular functions* over any variety X. A function $\varphi : X \to k$ is said to be regular if it is locally rational. That is, for each $p \in X$, there exists an open set $U \ni p$ of X and there exists two polynomials $f, g \in k[x_1, \dots, x_n]$ such that $g(q) \neq 0 \forall q \in U$ and $\varphi|_U = f/g$. It then follows that a regular function is continuous when X and k are equipped with its Zariski topology (Lemma 3.1, [**??**] [Hartshorne]). We now define morphism of affine varieties.

A function $\varphi : X \to Y$ is said to be a morphism of varieties if

1. $\varphi : X \to Y$ is continuous,
2. for each open set $V \subseteq Y$ and a regular map $f : V \to k$, the map $f \circ \varphi$ as below

$$\begin{tikzcd} & V \arrow[dr, "f"] & \\ \varphi^{-1}(V) \arrow[ur, "\varphi"] \arrow[rr, "f\circ\varphi"'] & & k \end{tikzcd}$$

   is also a regular map.

Hence the main part of the data of a variety is the locally defined regular maps. This is what we will take as our motivation in defining the structure sheaf over Spec (R), as this example tells us to take care of these local functions to the base field. A question that may arise from this discussion is how are we going to define a regular map from an open set $U \hookrightarrow$ Spec (R) when we don't even have a field. The answer is, as we discussed previously, to work with residue field at a point instead.

We now start to define the structure sheaf of Spec (R). First, let us give the following lemma, which reduces the burden of construction only to basis elements of Spec (R).

Lemma 1.2.2.2. *Let X be a topological space and $\mathcal{B}$ be a basis. Let F be an assignment over sets of $\mathcal{B}$ which satisfies sheaf conditions for it. Then, F extends to a sheaf $\mathcal{F}$ over X.*

Proof. The main observation here is that we can find the stalk of $\mathcal{F}$ at each point x just by the knowledge of F, because of the basis $\mathcal{B}$. Take any point $x \in X$. We see that we can get the stalk $\mathcal{F}_x$ as follows:

$$\mathcal{F}_x := \varinjlim_{x\in B\in\mathcal{B}} F(B).$$

Once we have the stalks, we can define the sections of $\mathcal{F}$ quite easily as follows. Let $U \subseteq X$ be an open set. Then $\mathcal{F}(U)$ is defined to be the subset of $\prod_{x\in U} \mathcal{F}_x$ of those elements (s_x) where there exists a basic open cover $\{B_i\}$ of U and there exists elements $s_i \in F(B_i)$ such that $s_x = (s_i)_x$ for each $x \in B_i$. One can check that this satisfies the conditions of a sheaf. □

Construction 1.2.2.3. (*The* $\mathcal{O}_{\mathrm{Spec}(R)}$) Let R be a ring. By virtue of Lemma 1.2.2.2, we will define $\mathcal{O}_{\mathrm{Spec}(R)}$ only on basic open sets of the form Spec $(R)_f$. Let $X :=$ Spec (R). Motivated by Example 1.2.2.1, take a basic open set $X_f \hookrightarrow X$ for some $f \in R$ and then we wish to consider *rational functions* over X_f. This means those functions of the form g/h for $g, h \in R$ such that $h(\mathfrak{p}) \neq 0 \forall \mathfrak{p} \in X_f$. This is equivalent to demanding that $h \notin \mathfrak{p} \forall \mathfrak{p} \in X_f$, that is, $X_h \supseteq X_f$. This is again equivalent to stating that $f \in \sqrt{Rh}$ by Lemma 1.2.1.4, 2. Hence $f^n = ah$ for some $n \in \mathbb{N}$ and $a \in R$. Thus, we see that the notion of rational functions over X_f is equivalent to all functions of the form g/f^n where $g \in R$ and $n \in \mathbb{N}$. Commutative algebra has an apt name for this, that is, the localization of R at f denoted by $R_f := \{a/f^n \mid a \in R,\ n \in \mathbb{N}\}$ which is again a ring by natural operation on fractions (see Special Topics, **??**). Thus, we should define the sections over X_f as:

$$\mathcal{O}_X(X_f) := R_f.$$

We would not verify the sheaf axioms here as it is a tedious but straightforward calculation. The sheaf $\mathcal{O}_X$ thus formed is called the structure sheaf on the space X. One should think of the sheaf $\mathcal{O}_X$

as natural as the ring R itself. In particular we will see in the next section that it indeed is the case.

Next, we would like to see the stalks of this sheaf $\mathcal{O}_X$. To understand this, we would have to understand the maps on sections induced by $X_f \hookrightarrow X_g$. As we saw earlier, this is equivalent to stating that $f^n = ag$ for some $n \in \mathbb{N}$ and $a \in R$. Hence, the induced map on sections are the restriction maps of the sheaf and is given by

$$\begin{aligned} \rho_{X_g,X_f} : R_g = \mathcal{O}_X(X_g) &\longrightarrow \mathcal{O}_X(X_f) = R_f \\ b/g^m &\longmapsto ba^m/a^m g^m = ba^m/f^{nm}. \end{aligned}$$

We are now ready to calculate the stalk. Take any point $x \in X$. The stalk becomes:

$$\begin{aligned} \mathcal{O}_{X,x} &:= \varinjlim_{x \in X_f} \mathcal{O}_X(X_f) \\ &= \varinjlim_{x \in X_f} R_f \\ &= \varinjlim_{f \notin x} R_f \\ &= R_x \end{aligned}$$

where the last equality follows from a small colimit calculation (which should really be thought of as a definition). Hence $\mathcal{O}_X$ is a sheaf whose stalks are local rings. So we have a complete description of the sheaf $\mathcal{O}_X$ when $X = \operatorname{Spec}(R)$.

We finally define an affine scheme.

Definition 1.2.2.4. (**Affine scheme**) Let R be a ring. Then the pair $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ is called an affine scheme.

Remark 1.2.2.5. (*Evaluation of functions*) Let $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ be an affine scheme. As noted earlier, we now see how all rational functions over $\operatorname{Spec}(R)$ are exactly the elements of R. In particular, since $\Gamma(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}) = R_1 = R$. Hence if we interpret $\mathcal{O}_{\operatorname{Spec}(R)}$ as the sheaf of regular maps over $\operatorname{Spec}(R)$, then R itself appears as the globally defined regular maps.

Now take global map $f \in R$ and any point $\mathfrak{p} \in \operatorname{Spec}(R)$. We can "evaluate" f at $\mathfrak{p}$ via the following composite (note that $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong (R_\mathfrak{p})_\mathfrak{o}$, the last one is the fraction field of $R_\mathfrak{p}$ obtained by localizing at 0 ideal):

$$\Gamma(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}) \longrightarrow \mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}} \longrightarrow \kappa(\mathfrak{p})$$

where the first map on the left is the inclusion into the direct limit and the map on right is the natural quotient map. Algebraically, we have the following maps

$$R \longrightarrow R_\mathfrak{p} \longrightarrow R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$$

given by

$$f \longmapsto \frac{f}{1} \longmapsto \frac{f}{1} + \mathfrak{p}R_\mathfrak{p},$$

where $f/1 + \mathfrak{p}R_\mathfrak{p}$ denotes the class of all those functions in the stalk $\mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}} = R_\mathfrak{p}$ which takes same value at $\mathfrak{p}$ as f does.

For completeness' sake, we give a description of the section of the sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ on any open set $U \subseteq \mathrm{Spec}\,(R)$.

Lemma 1.2.2.6. *Let R be a ring and $(\mathrm{Spec}\,(R), \mathcal{O}_{\mathrm{Spec}(R)})$ the associated affine scheme. Let $U \subseteq \mathrm{Spec}\,(R) =: X$ be an open set. Then,*

$$\mathcal{O}_X(U) = \left\{ (s_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in U} R_{\mathfrak{p}} \mid \forall \mathfrak{p} \in U, \exists \text{ basic open } X_g \ni \mathfrak{p} \ \&\ f/g^n \in R_g \text{ s.t. } s_{\mathfrak{q}} = f/g^n \forall \mathfrak{q} \in X_g \right\}.$$

More concretely, we have

$$\mathcal{O}_X(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} R_{\mathfrak{p}} \mid \forall \mathfrak{p} \in U,\ s(\mathfrak{p}) \in R_{\mathfrak{p}} \ \&\ \exists \text{ open } \mathfrak{p} \in V \subseteq U \ \&\ f, g \in R \text{ s.t. } \forall \mathfrak{q} \in V,\ g \notin \mathfrak{q} \ \&\ s(\mathfrak{q}) = f/g \right\}.$$

Proof. Follows from Lemma 1.2.2.2 and Construction 1.2.2.3. □

Ring morphisms and $\mathrm{Spec}\,(-)$

We now discuss some properties of ring morphisms and the associated map of affine schemes.

Lemma 1.2.2.7. [3] *Let A be a ring and $f \in A$. Then, $D(f) \subseteq \mathrm{Spec}\,(A)$ is empty if and only if f is nilpotent.*

Proof. Both sides follow immediately from the Lemma 23.1.2.9. □

We further obtain the following two results which corresponds to what happens on the level of sheaves.

Proposition 1.2.2.8. [4] *Let $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ be two affine schemes and $\varphi : A \to B$ be a morphism of rings.*

1. *The ring map $\varphi : A \to B$ is injective if and only if the corresponding map of schemes $f : Y \to X$ yields injective map of structure sheaves, that is, $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective.*
2. *If $\varphi : A \to B$ is injective, then $f : Y \to X$ is dominant*[5].
3. *The ring map $\varphi : A \to B$ is surjective if and only if the corresponding map of schemes $f : Y \to X$ is a closed immersion.*

Proof. 1. (L $\Rightarrow$ R) It suffices to show that $f^\flat$ is an injective map over basic opens of X. Pick any $g \in A$ and consider the basic open $D(g) \subseteq X$. We wish to show that the map

$$f^\flat_{D(g)} : \mathcal{O}_X(D(g)) \longrightarrow \mathcal{O}_Y(f^{-1}(D(g)))$$

is an injective homomorphism. Indeed, we first observe that $\mathcal{O}_X(D(g)) \cong A_g$ and $f^{-1}(D(g)) = D(\varphi(g))$, so that $\mathcal{O}_Y(D(\varphi(g))) \cong B_{\varphi(g)}$. It follows that the map $f^\flat_{D(g)} : A_g \to B_{\varphi(g)}$ is the localization map

$$\begin{aligned} \varphi_g : A_g &\longrightarrow B_{\varphi(g)} \\ \frac{a}{g^n} &\longmapsto \frac{\varphi(a)}{\varphi(g)^n}. \end{aligned}$$

[3] Exercise II.2.18, a of Hartshorne.
[4] Exercise II.2.18 b,c,d of Hartshorne.
[5] that is, f has dense image.

We wish to show that the above map is injective. If $\varphi(a)/\varphi(g)^n = 0$, then for some $k \in \mathbb{N}$ we have $\varphi(g)^k\varphi(a) = 0$. It follows by injectivity of φ that $g^k a = 0$ in A. Consequently, we can write

$$\frac{a}{g^n} = \frac{ag^k}{g^{n+k}} = 0.$$

(R $\Rightarrow$ L) As a sheaf map is injective if and only if the kernel sheaf is zero (Theorem 27.3.0.7), where the latter is equivalent to the fact that every map on sections is injective. Consequently, over X, we get

$$f^\flat_X : \Gamma(\mathcal{O}_X, X) \longrightarrow \Gamma(\mathcal{O}_Y, Y)$$

Since $\Gamma(\mathcal{O}_X, X) \cong A$ and $\Gamma(\mathcal{O}_Y, Y) \cong B$, and the map $f^\flat_X : A \to B$ is just φ itself, therefore we are done.

2. We wish to show that for any basic non-empty open $D(g) \subseteq X$ for $g \in A$, the intersection $D(g) \cap f(Y)$ is non-empty. We have the following equalities:

$$\begin{aligned} D(g) \cap f(Y) &= \{\mathfrak{p} \in X \mid \mathfrak{p} \in f(Y) \ \&\ g \notin \mathfrak{p}\} \\ &= \{\varphi^{-1}(\mathfrak{q}) \in X \mid \mathfrak{q} \in Y,\ g \notin \varphi^{-1}(\mathfrak{q})\} \\ &= \{\varphi^{-1}(\mathfrak{q}) \in X \mid \mathfrak{q} \in Y,\ \varphi(g) \notin \mathfrak{q}\} \\ &= f(D(\varphi(g))). \end{aligned}$$

Conequently, $D(g) \cap f(Y)$ is non-empty if and only if $D(\varphi(g))$ is non-empty, which in turn implies by Lemma 1.2.2.7 that $D(g) \cap f(Y)$ is non-empty if and only if $\varphi(g)$ is not nilpotent. As g is not nilpotent because $D(g)$ is not empty, therefore $\varphi(g)$ is not nilpotent as φ is injective.

3. (L $\Rightarrow$ R) Let $\varphi : A \to B$ be surjective and $I \leq A$ be the kernel. We wish to show that $f : Y \to X$ is a closed immersion. For that, we first need to show that f is a topological closed immersion, that is its image is closed and is homeomorphic to it. We claim that $f(Y) = V(I) \subseteq X$. Indeed, for any $\varphi^{-1}(\mathfrak{q}) \in f(Y)$, we have that $I \subseteq \varphi^{-1}(\mathfrak{q})$. Thus, $f(Y) \subseteq V(I)$. Conversely, for any $\mathfrak{p} \in V(I)$, as φ is surjective and $\mathfrak{p}$ contains I, therefore $\varphi(\mathfrak{p}) \in Y$ is a prime ideal such that $\varphi^{-1}(\varphi(\mathfrak{p})) = \mathfrak{p}$, so that $\mathfrak{q} = \varphi(\mathfrak{p}) \in Y$ is such that $f(\varphi(\mathfrak{p})) = \mathfrak{p}$, hence $\mathfrak{p} \in f(Y)$.

Next, we wish to show that f is homeomorphic to its image. It suffices to show that $f : Y \to f(Y)$ is a closed mapping. But this is immediate by the fact that a surjective map $\varphi : A \to B$ with kernel I induces an order preserving isomorphism of ideals of A containing I and ideals of B by mapping ideals of B to those of A containing I via φ^{-1}. Alternatively, one can see that $A/I \cong B$ and $\operatorname{Spec}(A/I) \cong V(I) = f(Y)$, therefore application of $\operatorname{Spec}(-)$ functor would do the job.

Next, we wish to show that $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. We can check this on a basis of X. Let $D(g) \subseteq X$ for some $g \in A$. Indeed, for $t \in (f_*\mathcal{O}_Y)(D(g)) = \mathcal{O}_Y(D(\varphi(g)) \cong B_{\varphi(g)}$, we wish to find an open covering of $D(g)$ say U_i and $s_i \in \mathcal{O}_X(U_i)$ such that $f^\flat_{U_i}(s_i) = t|_{U_i}$ for each i. Indeed, the open set $D(g)$ as its own covering will suffice here as $\mathcal{O}_X(D(g)) \cong A_g$ and the map $f^\flat_{D(g)} = \varphi_g : A_g \to B_{\varphi(g)}$. As φ is surjective, therefore for $t = b/\varphi(g)^n \in B_{\varphi(g)}$, we obtain $a \in A$ such that $\varphi(a) = b$ and thus a/g^n is mapped by φ_g to $b/\varphi(g)^n$, as required.

(R $\Rightarrow$ L) Let $f : Y \to X$ be a closed immersion. We wish to show that $\varphi : A \to B$ is surjective. Pick $b \in B$. We wish to show that there exists $a \in A$ such that $\varphi(a) = b$. As the sheaf map

$f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, therefore there exists a basic open covering (which will be finite by quasi-compactness of affine schemes, Lemma 1.2.1.6) namely $\{D(a_i)\}_{i=1,\dots,n}$ of X together with sections $s_i \in \mathcal{O}_X(D(a_i))$ such that $f^\flat_{D(a_i)}(s_i) \in \mathcal{O}_Y(f^{-1}(D(a_i)))$ is the restriction of $b \in \Gamma(\mathcal{O}_Y, Y)$ to $D(\varphi(a_i))$, namely $\rho_{X,D(\varphi(a_i))}(b)$. As we have $\mathcal{O}_X(D(a_i)) \cong A_{a_i}$, $\mathcal{O}_Y(f^{-1}(D(a_i))) = \mathcal{O}_Y(D(\varphi(a_i))) \cong B_{\varphi(a_i)}$ and that the restriction $\rho_{Y,D(\varphi(a_i))} : \Gamma(\mathcal{O}_X, X) \to \mathcal{O}_X(D(a_i))$ is just the natural localization map $A \to A_{a_i}$, therefore we may identify $s_i = \frac{c_i}{a_i^{k_i}} \in A_{a_i}$ and $\rho_{X,D(a_i)}(b) = \frac{b}{1} \in B_{\varphi(a_i)}$. Consequently, we have for each $i = 1, \dots, n$ the following equation in $B_{\varphi(a_i)}$

$$\frac{b}{1} = \frac{\varphi(c_i)}{\varphi(a_i)^{k_i}}.$$

It follows that we obtain an equation of the form

$$\varphi(a_i^{m_i})b = \varphi(c_i a^{l_i})$$

for some $m_i, l_i \geq 0$. Taking $M = \max_i m_i$, we obtain

$$\varphi(a_i^m)b = \varphi(d_i) \tag{$*$}$$

for some $d_i \in A$.

, 2, the collection $\{a_i\}_{i=1,\dots,n}$ generates the unit ideal in A. By Lemma 1.2.1.9, it follows that the collection $\{a_i^m\}_{i=1,\dots,n}$ also generates the unit ideal in A. Consequently, we have $r_1 a_1^m + \cdots + r_n a_n^m = 1$ for some $r_i \in A$. Using this in $(*)$, we yield

$$b = \varphi\left(\sum_{i=1}^n r_i d_i\right),$$

as required[6]. □

1.2.3 $\mathcal{O}_{\mathrm{Spec}(R)}$-modules

As we pointed out in Construction 1.2.2.3, the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ should really be thought of as natural as the ring R itself. This way of thought will be justified in this section, where we will see that, just like we can understand a ring by understanding the category of R-modules, we can understand the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ by understanding the category of soon to be constructed $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

Let R be a ring and M be an R-module. Just like we underwent a "geometrification" to go from ring R (algebra) to the locally ringed space $\mathrm{Spec}\,(R)$ (geometry), we will also "geometrify" the notion of an R-module. This will yield us a sheaf $\widetilde{M}$ over $\mathrm{Spec}\,(R)$.

Definition 1.2.3.1. ($\widetilde{M}$) Let R be a ring and M be an R-module. The following presheaf on $X := \mathrm{Spec}\,(R)$ generated by the following definition on basic opens

$$X_f \longmapsto \widetilde{M}(X_f) := M_f = M \otimes_R R_f$$

[6] Note that in the whole proof, we didn't even required the fact that $f : Y \to X$ is also a topological closed immersion!

and restrictions given by

$$(X_f \hookrightarrow X_g) \longmapsto M \otimes_R R_g \stackrel{\mathrm{id}\otimes\rho_{X_g,X_f}}{\to} M \otimes_R R_f$$

defines a unique sheaf on Spec (R) corresponding to R-module M denoted $\widetilde{M}$.

The above construction gives the sheaf $\widetilde{M}$ over R a structure of an $\mathcal{O}_{\mathrm{Spec}(R)}$-module, that is, a sheaf $\mathcal{F}$ of abelian groups where for each open $U \subseteq \mathrm{Spec}\,(R)$ the group $\mathcal{F}(U)$ is a $\mathcal{O}_{\mathrm{Spec}(R)}(U)$-module. Since $\widetilde{M}(X_f) = M \otimes_R R_f$ is an $\mathcal{O}_X(X_f) = R_f$-module, therefore $\widetilde{M}$ are basic examples of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

A map $\eta : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules is just a sheaf morphism where for each inclusion $U \hookrightarrow V$ of Spec (R), we get that the following commutes

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\eta_V} & \mathcal{G}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \xrightarrow[\eta_U]{} & \mathcal{G}(U) \end{array}$$

where the top horizontal map is a $\mathcal{O}_{\mathrm{Spec}(R)}(V)$-module homomorphism, bottom horizontal is a $\mathcal{O}_{\mathrm{Spec}(R)}(U)$-module homomorphism and the verticals are the restriction map of sheaves $\mathcal{F}$ and $\mathcal{G}$, which are also module homomorphisms w.r.t. $\mathcal{O}_{\mathrm{Spec}(R)}(V) \to \mathcal{O}_{\mathrm{Spec}(R)}(U)$. The latter has the following meaning. If M is an R-module and N is an S-module, then a map $\phi : M \to N$ is a module homomorphism w.r.t $f : R \to S$ if $\phi(r \cdot m) = f(r) \cdot \phi(m)$.

We thus get a functor

$$\begin{aligned} \widetilde{-} : \mathbf{Mod}(R) &\longrightarrow \mathbf{Mod}(\mathcal{O}_{\mathrm{Spec}(R)}) \\ M &\longmapsto \widetilde{M} \\ f : M \to N &\longmapsto \tilde{f} : \widetilde{M} \to \widetilde{N} \end{aligned}$$

where $\tilde{f}_{X_f} : M_f \to N_f$ is given by localization. We may denote $\widetilde{\mathbf{Mod}}(\mathcal{O}_{\mathrm{Spec}(R)}) \hookrightarrow \mathbf{Mod}(\mathcal{O}_{\mathrm{Spec}(R)})$ to be the full subcategory of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules of the form $\widetilde{M}$.

An explicit form of the sheaf $\widetilde{M}$ can be obtained by expanding the definition of the sheaf we obtain from it's definition on the basis.

Lemma 1.2.3.2. *Let M be an R-module and consider the associated $\mathcal{O}_{\mathrm{Spec}(R)}$-module $\widetilde{M}$. For any open $U \subseteq \mathrm{Spec}\,(R)$, we have*

$$\widetilde{M}(U) \cong \left\{ s : U \to \textstyle\coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}} \mid \forall \mathfrak{p}\in U,\ s(\mathfrak{p})\in M_{\mathfrak{p}}\ \&\ \exists \text{ open } \mathfrak{p}\in V \subseteq U\ \&\ \exists m\in M, f\in R \ \text{ s.t. } \forall \mathfrak{q}\in V,\ f\notin\mathfrak{q}\ \&\ s(\mathfrak{q}) = m/f \right\}$$

Proof. Follows from Remark 27.2.0.4. □

We now collect properties of $\widetilde{M}$ below.

Proposition 1.2.3.3. *Let R be a ring and M, N, M_i be R-modules for $i \in I$,*

1. $\left(\widetilde{M}\right)_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$,
2. $\widetilde{M}(\operatorname{Spec}(R)_f) \cong M_f$ *for all* $f \in R$,
3. $\Gamma(\widetilde{M}, \operatorname{Spec}(R)) \cong M$.

Proof. Statement 1 follows from the alternate definition given in Lemma 1.2.3.2. Indeed one considers the function

$$\begin{aligned} \varphi : \left(\widetilde{M}\right)_{\mathfrak{p}} &\longrightarrow M_{\mathfrak{p}} \\ (U, s)_{\mathfrak{p}} &\longmapsto s(\mathfrak{p}). \end{aligned}$$

One immediately sees this is R-linear. Injectivity and surjectivity is then also trivially checked by the above cited lemma.

Statements 3 follows from statement 2 by setting $f = 1$ and statement is just the Definition 1.2.3.1. ☐

We can also understand how $\mathcal{O}_{\operatorname{Spec}(R)}$-modules behave under morphism of affine schemes (see direct and inverse image of modules at Section 8.5)

Lemma 1.2.3.4. [7] *Let $f : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ be a morphism of affine schemes associated to map $\varphi : R \to S$ of rings. Then,*

1. *if N is an S-module, then $f_*\widetilde{N} \cong \widetilde{{}_RN}$ where ${}_RN$ is the R-module obtained by restriction of scalars by φ,*
2. *if M is an R-module, then $f^*\widetilde{M} \cong \widetilde{(S \otimes_R M)}$ where $S \otimes_R M$ is the S-module obtained by extension of scalars by φ.*

Proof. The proof is routine with main observation being the facts that for $g \in R$, we have $({}_RN)_g \cong N_{\varphi(g)}$ and for $\mathfrak{q} \in \operatorname{Spec}(S)$, we get the natural isomorphism $(f^*\widetilde{M})_{\mathfrak{q}} \cong \widetilde{(S \otimes_R M)}_{\mathfrak{q}}$. ☐

Theorem 1.2.3.5. *Let R be a ring. There is an equivalence of categories between those of R-modules and $\mathcal{O}_{\operatorname{Spec}(R)}$-modules of the form $\widetilde{M}$:*

$$\mathbf{Mod}(R) \underset{\Gamma(X,-)}{\overset{\widetilde{(-)}}{\underset{\equiv}{\rightleftarrows}}} \widetilde{\mathbf{Mod}}(\mathcal{O}_{\operatorname{Spec}(R)})$$

which moreover satisfies the following properties

1. $\widetilde{(-)}$ *is an exact functor; if* $0 \to M' \to M \to M'' \to 0$ *is exact, then* $0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$ *is exact,*
2. $\widetilde{(-)}$ *preserves tensor product;* $\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$,
3. $\widetilde{(-)}$ *preserves coproducts;* $\widetilde{\bigoplus_{i \in I} M_i} \cong \bigoplus_{i \in I} \widetilde{M_i}$.

[7] We will call it the *globalized* extension and restriction of scalars.

Proof. Let $X = \operatorname{Spec}(R)$. Consider the following map

$$\begin{aligned} \operatorname{Hom}_R(M, N) &\to \operatorname{Hom}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right) \\ f : M \to N &\mapsto \widetilde{f} : \widetilde{M} \to \widetilde{N} \\ \eta_X : M \to N &\hookleftarrow \eta : \widetilde{M} \to \widetilde{N} \end{aligned}$$

Now, beginning from η, we may show that $(\widetilde{\eta_X})_{X_g} = \eta_{X_g}$ for some basic open $X_g \hookrightarrow X$. The result follows from the fact that $\eta : \widetilde{M} \to \widetilde{N}$ is completely characterized by the map on global sections $\eta_X : M \to N$ from the following square

$$\begin{tikzcd} M_g \arrow[r, "\eta_{X_g}"] & N_g \\ M \arrow[u] \arrow[r, "\eta_X"'] & N \arrow[u] \end{tikzcd}$$

where the verticals are restrictions morphisms w.r.t $R \to R_g$ and the top horizontal is R_g-module homomorphism and bottom is R-module homomorphism.

For statement 1, by Theorem 27.3.0.8, the question is local in nature. We deduce the result then from Lemma 23.1.2.2.

For statement 2, we proceed as follows. To define an isomorphism

$$\varphi : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_R N}$$

we need only define a map from the presheaf F given by $U \mapsto \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U)$ to $\widetilde{M \otimes_R N}$ such that on basic open sets, we have an isomorphism. Indeed, let $D(f) \subseteq \operatorname{Spec}(R)$ be an open set for some $f \in R$. We define

$$\varphi_U : M_f \otimes_{R_f} N_f \xrightarrow{\cong} (M \otimes_R N)_f$$

as the obvious natural isomorphism. One checks that this does define φ to be a sheaf map.

For statement 3, as $\widetilde{(-)}$ is a left adjoint, therefore it preserves all colimits. $\square$

Remark 1.2.3.6. We will later see that on affine schemes $\operatorname{Spec}(R)$, the category $\widetilde{\mathbf{Mod}}(\mathcal{O}_{\operatorname{Spec}(R)})$ is precisely the category of quasicoherent $\mathcal{O}_{\operatorname{Spec}(R)}$-modules, which is a class of modules of utmost importance in algebraic geometry.

1.3 Schemes and basic properties

We can now define scheme to be a locally ringed space (see Foundational Geometry, 8) with an affine open covering.

Definition 1.3.0.1. (**Schemes**) A locally ringed space $(X, \mathcal{O}_X)$ is a scheme if there exists an open affine cover $\{(\operatorname{Spec}(R_i), \mathcal{O}_{\operatorname{Spec}(R_i)})\}$ of $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X|\operatorname{Spec}(R_i)} \cong \mathcal{O}_{\operatorname{Spec}(R_i)}$.

As we go along in understanding schemes, it will be more and more apparent the need of sheaf language to talk about the "generalized functions" over the scheme X. Indeed, there is a fine interrelationship between the *space structure* of the scheme $(X, \mathcal{O}_X)$ (that is, the topological space X) and the *function structure* on the scheme (that is, the sheaf of functions $\mathcal{O}_X$). A big part of learning scheme theory is to understand and use this relationship between them.

We will now bring some global topological properties of schemes which reflect their affine origins. An analogue of Lemma 1.2.1.1 holds in the general case of schemes.

Lemma 1.3.0.2. [8] *Let X be a scheme. The following are equivalent.*

1. *$S \subseteq X$ is a closed irreducible subset.*
2. *There exists a point $x \in S$ such that $\overline{\{x\}} = S$.*

Proof. (1. $\Rightarrow$ 2.) Let U be an affine open in X intersecting S. Then $U \cap S$ is an open subset of S. As open subsets of irreducibles are dense, therefore $U \cap S$ is dense in S. Consequently, it suffices to show that there exists a point $x \in U \cap S$ such that $\overline{\{x\}} = U \cap S$. As open subsets of irreducibles are irreducible, therefore $U \cap S$ is irreducible. Replacing X by U, we may assume X is affine. The result then follows by Lemma 1.2.1.1.

(2. $\Rightarrow$ 1.) Since $x \in U$ for some open affine $U \subset X$, thus, $x \in U \cap S$. $\overline{\text{Since } U \cap S \subseteq U}$ and U is open, therefore closure of $\{x\}$ in U is same as closure of $\{x\}$ in X. Now, $\overline{\{x\}} = S$ but $\overline{\{x\}} \subseteq U$. It thus follows that $S \subseteq U$ and hence S is in an open affine. The result follows by Lemma 1.2.1.1. $\square$

Every open subspace of a scheme is a scheme.

Lemma 1.3.0.3. *Let X be a scheme and $U \subseteq X$ be an open subspace. Then $(U, \mathcal{O}_{X|U})$ is a scheme.*

Proof. Since for an affine scheme $\operatorname{Spec}(R)$, the basic open $\operatorname{Spec}(R)_f \cong \operatorname{Spec}(R_f)$ for $f \in R$, therefore for an open subspace $U \subseteq X$ and an affine open cover $\{U_i\}$ of X, $U_i \cap U$ is open in U_i and thus covered by affines of the form $\operatorname{Spec}(R_f)$. $\square$

Write **Sch** to be the category of schemes and **Sch**$/S$ to be the category of schemes over S. Morphisms of schemes is merely the same concept as that of morphism of locally ringed spaces (see Foundational Geometry, Chapter 8).

Definition 1.3.0.4. (**Map of schemes**) Let X and Y be two schemes. A map of underlying locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a map of schemes. In a more expanded form, $f : X \to Y$ is a continuous map and $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a map of sheaves such that the induced map (see Topics in Sheaf Theory, Chapter 27) on stalks for each $x \in X$

$$f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a map of local rings, i.e., $(f^\sharp_x)^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$.

An important theorem in global study of schemes is a complete characterization of schemes over $\operatorname{Spec}(R)$, which is of-course of paramount importance.

[8] Exercise II.2.9 of Hartshorne.

Theorem 1.3.0.5. *Let X be a scheme and R be a ring. Then, there's a natural bijection*

$$\mathrm{Hom}_{\mathbf{Sch}}(X, \mathrm{Spec}(R)) \cong \mathrm{Hom}_{\mathbf{Ring}}(R, \Gamma(X, \mathcal{O}_X)).$$

In other words, we have the following adjunction[9]

$$\begin{tikzcd} \mathbf{Sch} \arrow[r, shift left, "\Gamma(-)"] \arrow[r, phantom, "\perp"] & \mathbf{Ring}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{Spec}(-)"] \end{tikzcd} .$$

Proof. The proof will be played out in two steps. In the first one we will show the candidates for the unit and counit of this adjunction. In the second play we will show that they indeed satisfy the required triangle identities.

Act 1 : *The units and counits.*

Let us first define the simpler one of them, the counit. For any $R \in \mathbf{Ring}$, we define a natural transformation $\epsilon : \mathrm{id}_{\mathbf{Ring}} \to \Gamma \circ \mathrm{Spec}()$ given by (note how we adjusted for the contravariant nature of $\mathrm{Spec}(-)$ and $\Gamma(-)$)

$$\begin{aligned} \epsilon_R : R &\longrightarrow \Gamma(\mathrm{Spec}(R)) \cong R \\ f &\longmapsto f. \end{aligned}$$

Thus, $\epsilon_R = \mathrm{id}_R$. Hence, $\epsilon = \mathrm{id}_{\mathbf{Ring}^{\mathrm{op}}}$.

Next, we define the more intricate part, which is the unit. Take any scheme $X \in \mathbf{Sch}$. We define $\eta : \mathrm{id}_{\mathbf{Sch}} \to \mathrm{Spec}(\Gamma)$ on X by

$$\begin{aligned} \eta_X : X &\longrightarrow \mathrm{Spec}(\Gamma(X)) \\ x &\longmapsto \mathfrak{p} = \eta_X(x) := \{f \in \Gamma(X) \mid f_x \in \mathfrak{m}_x\}. \end{aligned}$$

Moreover, the map on structure sheaves is given by

$$(\eta_X)^\flat : \mathcal{O}_{\mathrm{Spec}(\Gamma(X))} \longrightarrow (\eta_X)_* \mathcal{O}_X$$

where as the map on global sections we keep it id and on a basic open $\mathrm{Spec}((\Gamma(X))_f)$ this is defined on sections by

$$(\eta_X)^\flat_{\mathrm{Spec}((\Gamma(X))_f)} : \Gamma(X)_f \cong \mathcal{O}_{\mathrm{Spec}(\Gamma(X))}((\mathrm{Spec}(\Gamma(X)))_f) \longrightarrow \mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}((\Gamma(X))))_f))$$

by the unique map that is obtained in the following diagram

$$\begin{tikzcd} \Gamma(X)_f \arrow[r, dashed] & \mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}(\Gamma(X)))_f)) \\ \Gamma(X) \arrow[u] \arrow[ur, "\rho"'] & \end{tikzcd} ,$$

where, indeed, $f \in \Gamma(X)$ is mapped to to an unit element in $\mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}(\Gamma(X)))_f$ because of the following simple lemma:

[9] This is also sometimes called the *algebra-geometry duality* or the *fundamental duality of algebraic geometry.*

(*) For a locally ringed space $(X, \mathcal{O}_X)$ and an open subspace $U \subseteq X$, $f \in \mathcal{O}_X(U)$ is a unit if and only if $f_x \notin \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ for all $x \in U$.

This construction has the following properties and we give the main idea which drives each one of them.

1. $\eta_X(x)$ *is a prime ideal of* $\Gamma(X)$: This follows from $\mathfrak{m}_x$ being a maximal (hence prime) ideal of $\mathcal{O}_{X,x}$.
2. η_X *is continuous* : Working with basis and reducing to assumption that $X = \operatorname{Spec}(S)$ is affine, we reduce to showing that $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid f_\mathfrak{p} \notin \mathfrak{m}_\mathfrak{p}\}$ is open, which is true as it is equal to $(\operatorname{Spec}(S))_f$.
3. $\eta : \mathrm{id}_{\mathbf{Sch}} \to \operatorname{Spec}() \circ \Gamma$ *is a natural transformation* : We wish to show that commutativity of the natural square. For a map of schemes $f : X \to Y$, this reduces to showing that

$$\forall x \in X,\ \eta_Y(f(x)) = (f_Y^\flat)^{-1}(\eta_X(x)).$$

This further follows from the observation that for $g \in \Gamma(Y)$, $f_Y^\flat(g) \in \mathfrak{m}_x \iff f_x(g_{f(x)}) \in \mathfrak{m}_x$ and the latter is clearly true by the definition of maps of locally ringed spaces, where $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is the map on stalks.

Hence, we have obtained a map of schemes $(\eta_X, \eta_X^\flat) : X \to \operatorname{Spec}(\Gamma(X))$. This is our candidate for the unit of the adjunction.

Act 2 : *η and ϵ satisfies the triangle identities.*

It follows that we wish to show that the following two diagrams commute:

$$\begin{tikzcd} \Gamma(X) & \Gamma(\operatorname{Spec}(\Gamma(X))) \cong \Gamma(X) \arrow[l, "\Gamma\eta_X"'] \\ & \Gamma(X) \arrow[ul, "\mathrm{id}_{\Gamma(X)}"] \arrow[u, "\epsilon_{\Gamma(X)}"'] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}(R) \arrow[r, "\eta_{\operatorname{Spec}(R)}"] \arrow[dr, "\mathrm{id}_{\operatorname{Spec}(R)}"'] & \operatorname{Spec}(\Gamma(\operatorname{Spec}(R))) \cong \operatorname{Spec}(R) \arrow[d, "\operatorname{Spec}(\epsilon_R)"] \\ & \operatorname{Spec}(R) \end{tikzcd}.$$

in **Ring** in **Sch**

This follows from a simple unraveling of the maps involved in the diagram as defined in Act 1.

□

Corollary 1.3.0.6. *The above adjunction restricts to the following equivalence of categories:*

$$\begin{tikzcd} \mathbf{AfSch} \arrow[r, shift left, "\Gamma(-)"] \arrow[r, phantom, "\cong"] & \mathbf{Ring}^{\mathrm{op}} \arrow[l, shift left, "\operatorname{Spec}(-)"] \end{tikzcd}.$$

□

Corollary 1.3.0.7. *Let X be a scheme over* $\operatorname{Spec}(R)$ *for a ring R. Then, for any open affine* $\operatorname{Spec}(S) \subseteq X$*, S is an R-algebra. Consequently, all stalks $\mathcal{O}_{X,p}$ are R-algebras.* □

1.3.1 Basic properties

We can now observe some more basic properties. First, the behaviour of maps with respect to schemes over k and residue fields.

Lemma 1.3.1.1. *Let $f : X \to Y$ be a map of schemes over k. If $p \in X$ is such that $\kappa(p) = k$, then $\kappa(f(p)) = k$.*

Proof. On the stalks, we get the map

$$\begin{tikzcd} \mathcal{O}_{Y,f(p)} \arrow[rr, "f_p^\sharp"] & & \mathcal{O}_{X,p} \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}.$$

As $f_p^\sharp$ is a local k-algebra homomorphism, therefore by quotienting with the respective ideals, we obtain

$$\begin{tikzcd} \kappa(f(p)) \arrow[rr] & & k \\ & k \arrow[ul] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}.$$

The result then follows. ∎

Local rings at non-closed points

Let X be an arbitrary scheme and $p \in X$ be a non-closed point. One can show that the local ring $\mathcal{O}_{X,p}$ is obtained by localizing local rings at closed points. Indeed, we have the following simple observation in this direction.

Lemma 1.3.1.2. *Let X be a scheme and $p \in X$ be a non-closed point. Then, $\mathcal{O}_{X,p}$ is isomorphic to localization of a local ring $\mathcal{O}_{X,x}$ at a prime ideal, where $x \in X$ is a closed point.*

Proof. Let $p \in X$ be a non-closed point and $U = \operatorname{Spec}(A)$ be an open affine containing p. Consequently, p corresponds to a prime ideal $\mathfrak{p} \lneq A$ which is not maximal. Let $\mathfrak{m} \lneq A$ be a maximal ideal containing $\mathfrak{p}$ and let $m \in U$ be the corresponding closed point in X. As $\mathcal{O}_{X,p} \cong A_\mathfrak{p}$ and $\mathcal{O}_{X,m} \cong A_\mathfrak{m}$, and since $(A_\mathfrak{m})_{\mathfrak{p}_\mathfrak{m}} \cong A_\mathfrak{p}$, therefore we have that $\mathcal{O}_{X,p}$ is obtained by localizing $\mathcal{O}_{X,m}$ at a prime ideal, as required. ∎

Using ideas similar to above, we can also prove the following simple result.

Lemma 1.3.1.3. *Let X be an integral scheme and $\eta \in X$ be a non-closed point. Then the fraction field of $\mathcal{O}_{X,\eta} \cong K(X)$ where $K(X)$ is the function field of X.* ∎

Non-vanishing locus of a global section

We next see that how a global section of a scheme defines an open set which is the set of those points where that element, when treated as a function, is non-zero. One then finds what the ring of functions over this open set looks like. First, for any scheme X and any $f \in \Gamma(\mathcal{O}_X, X)$, define the *non-vanishing locus* of f by

$$X_f := \{x \in X \mid f \notin \mathfrak{m}_{X,x}\}.$$

We first have the following simple result about non-vanishing locus.

Lemma 1.3.1.4. *Let $f : X \to \operatorname{Spec}(B)$ be a scheme over a ring B and let $g \in B$. Let $\varphi : B \to \Gamma(\mathcal{O}_X, X)$ be the map induced on the global sections. Then,*

$$f^{-1}(D(g)) = X_{\varphi(g)}.$$

Proof. Observe that $x \in X_{\varphi(g)}$ if and only if $\varphi(g)_x \notin \mathfrak{m}_{X,x}$. As we have the following commutative square

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & \Gamma(\mathcal{O}_X, X) \arrow[d] \\ \mathcal{O}_{\operatorname{Spec}(B),f(x)} \arrow[r, "f^\sharp_x"'] & \mathcal{O}_{X,x} \end{tikzcd}$$

where vertical arrows are image into the stalk, therefore we deduce that $\varphi(g)_x \notin \mathfrak{m}_{X,x}$ if and only if $f^\sharp_x(g_x) \notin \mathfrak{m}_{X,x}$. As $f^\sharp_x$ is a local ring homomorphism, therefore $f^\sharp_x(g_x) \notin \mathfrak{m}_{X,x}$ if and only if $g_x \notin \mathfrak{m}_{\operatorname{Spec}(B),f(x)} = f(x)B_{f(x)}$. As $B \to \mathcal{O}_{\operatorname{Spec}(B),f(x)}$ is just localization map $B \to B_{f(x)}$, therefore $g_x \notin f(x)B_{f(x)}$ if and only if $g \notin f(x)$, that is $f(x) \in D(g)$. This completes the proof. □

Proposition 1.3.1.5. [10] *Let X be a scheme and $f \in \Gamma(\mathcal{O}_X, X)$.*

1. *Let $U = \operatorname{Spec}(A)$ be an affine open subset of X and denote $\bar{f} = \rho_{X,U}(f)$. Then, $U \cap X_f = D(\bar{f})$. Consequently, $X_f \subseteq X$ is an open subscheme.*
2. *Let X be quasicompact and $a \in \Gamma(\mathcal{O}_X, X)$ such that $\rho_{X,X_f}(a) = 0$. Then, $f^n a = 0$ in $\Gamma(\mathcal{O}_X, X)$ for some $n > 0$.*
3. *Let X admit an affine open cover U_i such that $U_i \cap U_j$ is quasicompact. If $b \in \mathcal{O}_X(X_f)$, then there exists $a \in \Gamma(\mathcal{O}_X, X)$ and $n > 0$ such that $f^n b = \rho_{X,X_f}(a)$ in $\mathcal{O}_X(X_f)$.*
4. *There is an isomorphism of rings $\Gamma(\mathcal{O}_{X_f}, X_f) \cong (\Gamma(\mathcal{O}_X, X))_f$.*

Proof. 1. We wish to show that $\{x \in U \mid \bar{f}_x \notin \mathfrak{m}_{X,x}\} = \{x \in U \mid \bar{f} \notin x\}$, where $x \in U$ in latter is treated as a prime ideal of A. The side " $\subseteq$ " follows from the fact that for $x \in U$, we have $\mathcal{O}_{X,x} \cong A_x$, $\mathfrak{m}_{X,x} \cong xA_x$ and the fact that the map into stalks $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is given by the canonical map $A \to A_x$, $a \mapsto a/1$. One further would need the commutativity of the following diagram:

$$\begin{tikzcd} & \mathcal{O}_{X,x} \\ \mathcal{O}_X(U) \arrow[ur] & \Gamma(\mathcal{O}_X, X) \arrow[l] \arrow[u] \end{tikzcd}.$$

The side " $\supseteq$ " also follows from the commutativity of the above triangle together with the canonical isomorphisms of the local ring and its maximal ideal.

2. **TODO from notebook.** □

[10] Exercise II.2.16 of Hartshorne.

Locality of isomorphism on target

We now show a rather simple result on locality of isomorphism on target, but it is quite useful in scenarios where one understands the map well on individual opens of target but not on the global level.

Proposition 1.3.1.6. *Let $f : X \to Y$ be a map of schemes and $Y = \bigcup_{i \in I} U_i$ be an open cover of Y such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is an isomorphism. Then, f is an isomorphism.*

Proof. **TODO from notes.** □

Criterion for affineness

We now show a useful criterion for a scheme to be affine. This also portrays the power of previous result on locality of isomorphism.

Proposition 1.3.1.7. *Let X be a scheme and denote $A = \Gamma(\mathcal{O}_X, X)$. Then the following are equivalent:*

1. *X is affine,*
2. *there exists $f_1, \dots, f_r \in A$ such that X_{f_i} are open affine subsets of X and $\langle f_1, \dots, f_r \rangle = A$.*

Proof. **TODO from notes.** □

1.4 First notions on schemes

Having defined schemes, our next goal is to bring to light some of the obvious definitions that one can make on them. In some sense, having made the general definition of schemes, we are now trying to go back to try and find where does varieties lie in this big world of **Sch**. Indeed, we will see that the definitions introduced in the following few sections are bringing us ever closer to define varieties as certain type of schemes, which will thus enable us to bring to light the most important geometric notions on varieties.

Write the no and example ter .

1.4.1 Noetherian schemes

Definition 1.4.1.1. (**Noetherian schemes**) A scheme X is called *locally noetherian* if there exists an affine open cover $X = \cup_{i\in I} U_i$ where each $U_i = \mathrm{Spec}\,(A_i)$ where A_i is a noetherian ring. If moreover, X is quasicompact, then X is called *noetherian*.

Remark 1.4.1.2. Since $X = \mathrm{Spec}\,(A)$ is already quasi-compact (Lemma 1.2.1.6), therefore for affine schemes X, the notion of locally noetherian and noetherian are equal.

The only immediately important result about such schemes that one needs is that an affine scheme is noetherian if and only if the obvious thing happens.

Lemma 1.4.1.3. *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme. Then, the following are equivalent:*

1. *X is a noetherian scheme.*
2. *A is a noetherian ring.*

Proof. (2. $\Rightarrow$ 1.) This follows from Remark 1.4.1.2 and the fact that localization of noetherian rings are noetherian (Proposition 23.3.0.7).
(1. $\Rightarrow$ 2.) Let X be noetherian. Then there is an affine open cover of X by spectra of noetherian rings. Pick any ideal $I \leq A$. We shall show it is finitely generated. There is a finite cover $\{\mathrm{Spec}\,(A_{f_i})\}_{i=1}^n$ of $\mathrm{Spec}\,(A)$ where A_{f_i} are noetherian and $f_i \in A$. Hence we have that the ideal IA_{f_i} of A_{f_i} is finitely generated for all $i = 1, \dots, n$. By Lemma 1.2.1.5, 2, we see that $f_1, \dots, f_n$ generate the whole ring A. The result then follows by Lemma 23.1.2.10. $\square$

Example 1.4.1.4. By the Lemma 1.4.1.3, we observe that any of the variety over a field is a noetherian scheme (technically, we are identifying the affine variety with its associated scheme, see Section **??**, Schemes associated to varieties). So any of your favorite variety

$$\mathrm{Spec}\left(\frac{k[x,y,z]}{x^2+y^2-z^3-1}\right), k \text{ is algebraically closed}$$

gives a (*is* a) noetherian scheme.

Local rings of a locally noetherian scheme are noetherian.

Lemma 1.4.1.5. *If X is locally noetherian, then $\mathcal{O}_{X,x}$ is a noetherian ring.*

Proof. Since localization of a noetherian ring at a prime is again noetherian by Proposition 23.3.0.7, therefore $\mathcal{O}_{X,x}$ is noetherian. $\square$

Being locally noetherian is a local property.

Proposition 1.4.1.6. *Let X be a locally noetherian scheme. If* $\operatorname{Spec}(A) \subseteq X$ *is an open affine, then* $\operatorname{Spec}(A)$ *is noetherian and thus A is a noetherian ring.*

Proof. Let $U_i = \operatorname{Spec}(A_i)$ be an open cover by noetherian affine schemes (A_i are noetherian). Then, a finitely many of U_i will cover $\operatorname{Spec}(A)$ by quasi-compactness of $\operatorname{Spec}(A)$, say $U_1, \dots, U_n$. Thus we obtain a finite basic open cover $D(f_i)$ of $\operatorname{Spec}(A)$ for $f_i \in A$ where each $D(f_i) \subseteq U_j$ for some j such that $D(f_i)$ is also basic in U_j (Lemma 1.4.4.3). As U_j is noetherian, therefore if we can show that $\mathcal{O}_{U_j}(D(f_i))$ is noetherian, then we would have shown that A_{f_i} is noetherian, which would complete the proof by Lemma 23.3.0.8. We thus reduce to assuming $X = \operatorname{Spec}(A)$ noetherian affine and to show that $U = D(f) \subseteq X$ is noetherian for $f \in A$.

In this case, as A is noetherian, therefore by Corollary 23.3.0.9, the ring A_f is noetherian, as required. □

1.4.2 Reduced, integral schemes and function field

The following are the definitions required, which are clearly geometric in nature.

Definition 1.4.2.1. (**Reduced and integral schemes**) A scheme X is said to be *reduced* if local rings $\mathcal{O}_{X,x}$ for all $x \in X$ is a reduced ring; have no nilpotents. A scheme X is said to be *integral* if it is reduced and irreducible as a topological space.

The one basic result that must be seen about these two types of schemes is that they are characterized by algebraic properties of local sections. Thus being reduced or integral, while defined geometrically, is concretely controlled by the algebraic properties of the structure sheaf.

Lemma 1.4.2.2. *Let X be a scheme. Then,*

1. *X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for each open set $U \subseteq X$* [11].
2. *X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open set $U \subseteq X$.*

Proof. 1. (L ⇒ R) Suppose for some open $U \subseteq X$ there exists a section $f \in \mathcal{O}_X(U)$ which is nilpotent. Using the homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ given by $s \mapsto s_x$, we see that $f_x \in \mathcal{O}_{X,x}$ is a nilpotent element.
(R ⇒ L) Suppose X is not reduced. Hence for some germ $f_x \in \mathcal{O}_{X,x}$ at some point $x \in X$ is a nilpotent where $f \in \mathcal{O}_X(U)$ for some open $x \in U \subseteq X$. Since $f_x^n = 0$ for some $n \in \mathbb{N}$, we get that $f^n = 0$ for some open $W \subseteq U$. Thus $\rho_{U,W}(f) \in \mathcal{O}_X(W)$ is a nilpotent element[12].

2. (L ⇒ R) Pick any open $U \subseteq X$. We wish to show that $\mathcal{O}_X(U)$ is an integral domain. In other words, we wish to show the proposition for the open subscheme $(U, \mathcal{O}_{X|U})$. Replacing X by U, we reduce to showing $\mathcal{O}_X(X)$ is an integral domain. So let $f, g \in \mathcal{O}_X(X)$ be such that $fg = 0$. We wish to show that either $f = 0$ or $g = 0$. Suppose neither f nor g is 0 but $fg = 0$. It follows from Lemma 1.2.0.1, 1, that $V(f)$ and $V(g)$ covers X and hence by irreducibility of X, either $V(f) = 0$ or $V(g) = 0$, that is, $f = 0$ or $g = 0$.
(R ⇒ L) We first need to show that X is reduced. Indeed, by 1. it follows immediately as integral

[11] Exercise II.2.3.a of Hartshorne.

[12] This is a very inefficient way of using the equality on stalks. Indeed, two germs are equal if and only if the representatives are equal on some common shrinking of their domains. This is how usually people work with stalks without being overly full of symbols.

domains are reduced. We then wish to show that X is irreducible. Indeed, if there are two open subsets of X say $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \emptyset$, then we claim that $\mathcal{O}_X(U_1 \cup U_2) \cong \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$. Since both $\mathcal{O}_X(U_1), \mathcal{O}_X(U_2)$ have 0 and 1, thus $\mathcal{O}_X(U_1 \cup U_2)$ will have a zero-divisor, a contradiction. Indeed, consider the following homomorphism, denoting $U := U_1 \cup U_2$

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \\ s &\longmapsto (\rho_{U,U_1}(s), \rho_{U,U_2}(s)). \end{aligned}$$

This is injective by locality axiom and surjective by gluing axiom of sheaves. □

Corollary 1.4.2.3. *Let X be a scheme. If X is integral, then all local rings $\mathcal{O}_{X,x}$ are integral domains.*

Proof. Use Lemma 1.4.2.2, 2 together with the fact that localization of integral domains is an integral domain. □

Corollary 1.4.2.4. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then X is integral if and only if A is an integral domain.*

Proof. Use Lemma 1.4.2.2, 2 on global sections together to get one side. For the "only if" side, stalks are reduced as they are integral (localizations of A) and X is irreducible as for any $V(\mathfrak{a}) \cup V(\mathfrak{b}) = X$, we have $V(\mathfrak{ab}) = X$ and thus $\mathfrak{ab} \subseteq \mathfrak{n}$ where $\mathfrak{n}$ is the intersection of all prime ideals, the nilradical (Lemma 23.1.2.9). Since A is integral, therefore $\mathfrak{o}$ is prime as well and hence $\mathfrak{n} = 0$, making $\mathfrak{ab} = 0$. Since A is integral, hence $\mathfrak{a} = \mathfrak{o}$ or $\mathfrak{b} = \mathfrak{o}$. □

Remark 1.4.2.5. (*Function field of an integral scheme*) Let X be an integral scheme. Since X is irreducible as a topological space, therefore there is a generic point η in X, i.e. a point whose closure is the whole of X (Lemma 1.3.0.2). Now let $\operatorname{Spec}(A) \subseteq X$ be an affine open such that $\eta \in \operatorname{Spec}(A)$. Thus, η is a generic point of $\operatorname{Spec}(A)$ as well. Hence η corresponds to the zero ideal of A, which is indeed an integral domain from Lemma 1.4.2.2, 2. Since $\mathcal{O}_{X,\eta} \cong \mathcal{O}_{\operatorname{Spec}(A),\eta} = A_{\mathfrak{o}}$, therefore $\mathcal{O}_{X,\eta}$ is a field, called the *function field of the integral scheme* X and is in particular given by field of fractions of any domain A such that open $\operatorname{Spec}(A)$ contains η. We denote the function field of X as $K(X)$[13].

Using the fact that the generic point of an integral scheme X will be in every non-empty open set, we can make some fascinating observations about the function field $K(X)$, which thus justifies its name.

Lemma 1.4.2.6. *Let X be an integral scheme with function field $K(X)$. Then for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is contained in $K(X)$.*

Proof. Let $x \in X$, $\eta \in X$ be the generic point and $U = \operatorname{Spec}(A)$ be an open affine in X. By Lemma 1.4.2.2, 2, A is a domain. Clearly, $\eta \in U$ and it corresponds to the zero ideal $\mathfrak{o} \lneq A$. Further we have $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$, $\mathfrak{p} \in U$ is equal to the point $x \in U$. By definition $K(X) = A_{\mathfrak{o}}$. The result follows by observing that $A_{\mathfrak{p}} \subseteq A_{\mathfrak{o}}$. □

The following lemma shows that restriction of functions in an integral scheme is injective.

[13] Exercise II.3.6 of Hartshorne.

Lemma 1.4.2.7. *Let X be an integral scheme and $U \hookrightarrow V$ be an inclusion of open sets. Then, the restriction maps $\rho : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is an injective ring homomorphism.*

Proof. By Lemma 27.3.0.2, we need only show that for any $x \in V$ and any $s \in \mathcal{O}_X(V)$, we have $(V, s)_x = 0$ in $\mathcal{O}_{X,x}$. Let $W = \operatorname{Spec}(A)$ be an open affine containing x. As U is open in X and X is irreducible, therefore it is dense. Consequently, $U \cap W$ is an open non-empty set in X. We may write $\rho_{V,W}(s) = a \in A$. Let $D(f) \subseteq U \cap W$ be a basic open set of W. Since taking germs commutes with restrictions, therefore we have the restriction map $\mathcal{O}_X(W) \to \mathcal{O}_X(D(f))$ which is the localization map $A \to A_f$, which takes $a \mapsto \frac{a}{1}$. As s on U is 0, therefore, s is 0 on $W \cap U$ and thus on $D(f)$. Consequently, we have $\frac{a}{1} = 0$ in A_f. As A is a domain by Lemma 1.4.2.2, it follows that $a = 0$ in A. Thus, $\rho_{V,W}(s) = 0$, hence, $(V, s)_x = 0$ in $\mathcal{O}_{X,x}$, as required. $\square$

Example 1.4.2.8. ($\operatorname{Spec}(\mathbb{Z})$) Since $\mathbb{Z}$ is an integral domain, therefore by Corollary 1.4.2.4, $X = \operatorname{Spec}(\mathbb{Z})$ is an integral scheme. Clearly, X as a topological space consists of all prime numbers and a generic point given by the zero ideal $\mathfrak{o}$. Further, the topology is thus given by cofinite topology. At the level of stalks, we have that for a prime $\mathfrak{p} \in X$, $\mathcal{O}_{X,\mathfrak{p}} \cong \mathbb{Z}_\mathfrak{p}$ and we can describe $\mathbb{Z}_\mathfrak{p}$ as all those rationals whose denominator is not a multiple of prime p where $\mathfrak{p} = \langle p \rangle$ as $\mathbb{Z}$ is a PID (it's ED). Clearly, localizing X at the generic point $\mathfrak{o}$ would yield $\mathcal{O}_{X,\mathfrak{o}} \cong \mathbb{Q}$. More fascinatingly, for a prime $\mathfrak{p} = \langle p \rangle$ in X, the residue field at point $\mathfrak{p}$ is $\kappa(\mathfrak{p}) = \mathbb{Z}_\mathfrak{p}/\mathfrak{p}\mathbb{Z}_\mathfrak{p} \cong \mathbb{F}_p$, the finite field with p elements!

Now for any affine scheme $\operatorname{Spec}(A)$, consider a map $f : X \to \operatorname{Spec}(\mathbb{Z})$. By the fact that $\mathbb{Z}$ is initial in category of rings, therefore $\operatorname{Spec}(\mathbb{Z})$ is terminal in the category of affine schemes (Corollary 1.3.0.6). Since any scheme is locally affine, it further follows that $\operatorname{Spec}(\mathbb{Z})$ is terminal in the category of schemes.

We now introduce a concept which will be used while discussing divisors.

Definition 1.4.2.9. (**Center of a valuation**) Let X be an integral scheme with function field K and $v : K \to G$ be a valuation over K with valuation ring $R \subset K$. A center of v is defined to be a point $x \in X$ such that R dominates $\mathcal{O}_{X,x}$ in K (see Definition 23.10.1.5).

1.4.3 (Locally) finite type schemes over k

This section is the beginning of a theme which we would like to understand intimately, schemes over a field. This is because most of the schemes we will encounter in nature will be varieties whose coordinate rings would be algebras over a field. Here we first understand in scheme language the first thing about coordinate rings of varieties over k, the fact that they are finitely generated as an k-algebra. Indeed, this is what we seek from the following definition.

Definition 1.4.3.1. (**Finite and locally finite type schemes over a field**) Let k be a field and let $X \to \operatorname{Spec}(k)$ be a scheme over k. Then X is said to be locally finite type if there exists an affine open covering $\{\operatorname{Spec}(A_i)\}_{i\in I}$ of X such that each A_i is a finitely generated k-algebra. Moreover, X is said to be finite type if X is locally finite type and quasi-compact.

Example 1.4.3.2. Our $\mathfrak{h}$yperboloid of one sheet (introduced in Example 1.5.1.3) has the following coordinate ring:

$$\frac{k[x,y,z]}{I(V(p))}$$

where $p(x,y,z) = x^2+y^2-z^2-1$, where we have chosen $a=b=c=1$ for simplicity. Let $\mathfrak{h} := I(V(p))$. Clearly $\operatorname{Spec}(k[x,y,z]/\mathfrak{h})$ is a finite type k-scheme.

Great thing about the above definition is that it really doesn't depend on the affine open cover that is chosen.

Lemma 1.4.3.3. *Let k be a field and X be a k-scheme. Then the following are equivalent.*

1. *X is of locally finite type over k.*
2. *For all open affine $U \hookrightarrow X$, the ring $\mathcal{O}_X(U)$ is finitely generated k-algebra.*

Proof. (2. $\Rightarrow$ 1.) Immediate.
(1. $\Rightarrow$ 2.) We shall use Lemma 23.1.2.11 for this. ☐

Complete it? Chapter 1.

1.4.4 Subschemes and immersions

These notions are important in what is to come next.

Definition 1.4.4.1. (**Open subscheme**) Let X be a scheme. An open set $U \subseteq X$ has a canonical scheme structure, given by $(U, \mathcal{O}_{X|U})$. We call $(U, \mathcal{O}_{X|U})$ an open subscheme of X.

Indeed, locally U will look affine via the open affine cover of X. We can relativize this notion to define open immersions.

Definition 1.4.4.2. (**Open immersion**) A map $f : X \to Y$ of schemes is said to be an open immersion if $f : X \to f(X)$ is a homeomorphism, $f(X) \subseteq Y$ is open and $f^\flat_{|f(X)} : \mathcal{O}_{Y|f(X)} \to (f_*\mathcal{O}_X)_{|f(X)}$ is an isomorphism.

We observe that for any point in an intersection of open subschemes is contained in some special open subscheme. This is a very important result as this will be used as a technical tool to allow passage from one open affine with certain properties to another open affine, all the time while handling only basic open sets.

Lemma 1.4.4.3. *Let $U = \mathrm{Spec}\,(A), V = \mathrm{Spec}\,(B) \hookrightarrow X$ be two affine open subsets. For each $x \in U \cap V$, there exists an affine open subset $x \in W \hookrightarrow U \cap V$ such that $W = \mathrm{Spec}\,(A_f)$ and $W = \mathrm{Spec}\,(B_g)$ for some $f \in A$ and $g \in B$. Moreover, under the isomorphism $A_f \cong B_g$, the element $f \in A_f$ maps to $g \in B_g$.*

Proof. By replacing B by B_g for some $g \in B$, we may assume that $x \in V \subseteq U$. Consequently, let $f \in A$ be such that $D_U(f) \subseteq V$ and contains x, where $D_U(f) = \{\mathfrak{p} \in U \mid f \notin \mathfrak{p}\}$. We thus have $x \in D_U(f) \subseteq V \subseteq U$. Consider the restriction $h = \rho_{U,V}(f) \in \mathcal{O}_X(V) = B$. We claim that $D_V(h) = D_U(f)$. Denote $\varphi : A \to B$ obtained by $V \subseteq U$. We then have that $\rho_{U,V} = \varphi$ and $h = \varphi(f)$. Thus $\mathfrak{q} \in D_V(h) \iff h \notin \mathfrak{q} \iff \varphi(f) \notin \mathfrak{q} \iff f \notin \varphi^{-1}(\mathfrak{q})$. As each $\mathfrak{p} \in D_U(f)$ is $\varphi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in V$, therefore we are done. The last statement is immediate from above. □

Closed subschemes are defined in not that obvious way in which we have defined open subschemes, but at any rate, they are natural. We motivate the need for ideal sheaves as follows. Let X be a scheme. Suppose a closed subset $C \hookrightarrow X$ intersects some collection of affine opens $\{\mathrm{Spec}\,(A_i)\}$ and moreover it happens that $C \cap \mathrm{Spec}\,(A_i) = C \cap \mathrm{Spec}\,(A_j)$ for some $i \neq j$. Now by Corollary 1.4.4.14 we may write $C \cap \mathrm{Spec}\,(A_i) = \mathrm{Spec}\,(A_i/\mathfrak{a}_i)$ and $C \cap \mathrm{Spec}\,(A_j) = \mathrm{Spec}\,(A_j/\mathfrak{a}_j)$ for some ideals $\mathfrak{a}_i \subseteq A_i$ and $\mathfrak{a}_j \subseteq A_j$. Hence, we get two different structure sheaves $\mathcal{O}_{\mathrm{Spec}(A_i/\mathfrak{a}_i)}$ and $\mathcal{O}_{\mathrm{Spec}(A_j/\mathfrak{a}_j)}$ on on open subset of C. Thus we have to systematically track such identifications in order to define a unique scheme structure on the closed set C. Indeed, we take the help of the rich amount of constructions that we can make on the category of sheaves over a space (for more information, see Section 8.5).

We first define closed immersions.

Definition 1.4.4.4. (**Closed immersions**) A map $f : X \to Y$ of schemes is a closed immersion if $f : X \to f(X)$ is a homeomorphism, $f(X) \subseteq Y$ is closed and $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a surjective map.

Remark 1.4.4.5. Let $f : X \to Y$ be a closed immersion, so that $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. This is equivalent to saying that for each point $x \in X$, the map on stalks (see Theorem 27.3.0.6 and Lemma 27.5.0.5)

$$f^\flat_{f(x)} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is surjective. Observe that the above map is NOT the usual map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Further observe that since $f^\flat$ is surjective, therefore we have an ideal (see Section 8.5, Global algebra for more details) $\mathcal{I} = \mathrm{Ker}\left(f^\flat\right) \leq \mathcal{O}_Y$. We will later see that a closed subscheme is completely determined by this ideal sheaf and in-fact these ideal sheaves gives us a family of good examples of what will later be called quasicoherent modules over a scheme.

Remark 1.4.4.6. Let $f : X \to Y$ be a closed immersion. Then, the map $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. Pick any $x \in X$. Since we have the following commutative square for any open set $V \ni f(x)$ in Y

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "f^\flat_V"] \arrow[d] & \mathcal{O}_X(f^{-1}(V)) \arrow[d] \\ \mathcal{O}_{Y,f(x)} \arrow[r, "f^\sharp_x"'] & \mathcal{O}_{X,x} \end{tikzcd} .$$

It then follows from surjectivity of $f^\flat$ and $f : X \to f(X)$ being a homeomorphism that the local homomorphiosm $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective. It is also a simple exercise to see that surjectivity of $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ for all $x \in X$ implies surjectivity of $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Consequently, $f : X \to Y$ is a closed immersion if and only if f is a topological closed immersion and for all $x \in X$, the local homomorphism $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective.

A closed subscheme is then defined to be an isomorphism class of closed immersions.

Definition 1.4.4.7. (**Closed subscheme & ideal sheaf**) Let Y be a scheme. A closed subscheme of Y is an isomorphism class of closed immersions over Y. That is, a closed subscheme is the class $[f : X \to Y]$ of closed immersions where two closed immersions $f : X \to Y$ and $f' : X' \to Y$ are identified if there is an isomorphism $X \overset{\cong}{\to} X'$ such that the following commutes

$$\begin{tikzcd} X \arrow[rr, "\cong"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}.$$

For a closed subscheme $f : X \to Y$, we define kernel of $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ to be the ideal sheaf corresponding to the closed subscheme f.

Remark 1.4.4.8. Note that this definition is not "unnatural" as every closed immersion $f : X \to Y$ defines a closed set $f(X) \subseteq Y$ and a scheme structure over it. We then just define a closed subscheme to be the data of this closed set together with its scheme structure that is given by f. Clearly to make such a definition via immersions, we would need to identify those immersions which give same scheme structure on $f(X) \subseteq Y$.

We define an immersion as follows.

Definition 1.4.4.9 (**Immersion**). A map $f : X \to Z$ is said to be an immersion if f is an open immersion into a closed subscheme of Z.

We first understand closed subscheme structures in affine schemes.

Lemma 1.4.4.10. *Let $X = \mathrm{Spec}\,(R)$ be an affine scheme. Then every ideal $\mathfrak{a} \leq R$ defines a closed subscheme of X.*

Proof. Consider the closed set $Y = V(\mathfrak{a}) \subseteq X$. We endow Y with a scheme structure given by the isomorphism $Y \cong \mathrm{Spec}\,(R/\mathfrak{a})$. Now the inclusion map $i : (Y, \mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a})}) \to X$ is clearly a topological closed immersion. Further, $i^\flat : \mathcal{O}_{\mathrm{Spec}(R)} \to i_*\mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a})}$ is given on stalks (see Lemma 27.5.0.5) at point $x \in Y$ as $\mathcal{O}_{\mathrm{Spec}(R),x} \to \mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a}),x}$ which is just $R_x \to (R/\mathfrak{a})_x$ which is surjective. Thus, $\mathfrak{a}$ defines a closed subscheme structure on Y. $\square$

It is important to note that any other ideal $\mathfrak{b} \leq R$ such that $V(\mathfrak{a}) = V(\mathfrak{b})$ will define a possibly different closed subscheme structure on the underlying topological space. This is another example of the phenomenon that algebra has much more finer control over the geometric situation at hand. For example, for $X = \mathrm{Spec}\,(k[x])$, we have $\mathfrak{a}_n = \langle x^n \rangle$ and note that $V(\mathfrak{a}_n) = \{\langle x \rangle\} \subseteq X$. But each ideal $\mathfrak{a}_n$ defines a new closed subscheme structure on the same point $\langle x \rangle \in X$.

Properties of closed immersions

We discuss some general properties of closed immersions. We begin by observing that closed immersions are local on target.

Proposition 1.4.4.11. *Let $f : X \to Y$ be a morphism of schemes. Then the following are equivalent:*

1. *f is a closed immersion.*
2. *There is an affine open cover $\{V_i\}$ of Y such that $f : f^{-1}(V_i) \to V_i$ is a closed immersion for each i.*

Proof. (1. $\Rightarrow$ 2.) As f is a closed immersion, then $f(X) \subseteq Y$ is a closed subset and $f : X \to f(X)$ is a homeomorphism. Pick any open affine $V = \operatorname{Spec}(B) \subseteq Y$. Then, we wish to show that $f : f^{-1}(V) \to V$ is a closed immersion. Indeed, as f is a closed immersion, therefore $f : f^{-1}(V) \to V \cap f(X)$ is a homeomorphism. As $f(X)$ is closed in Y, therefore $V \cap f(X)$ is closed in V. This shows that $g := f|_{f^{-1}(V)}$ is a topological closed immersion.

Next, we wish to show that the map $g^\flat : \mathcal{O}_V \to g_*\mathcal{O}_{f^{-1}(V)}$ is a surjection. By Remark 1.4.4.6, it suffices to show that for any $x \in f^{-1}(V)$, the local morphism $g_x^\sharp : \mathcal{O}_{V,f(x)} \to \mathcal{O}_{f^{-1}(V),x}$ is a surjection. Since $g = f|_{f^{-1}(V)}$, therefore $g_x^\sharp = f_x^\sharp$ because stalks commute with restrictions. Consequently, we wish to show that $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a surjection, but this is true by Remark 1.4.4.6 and the fact that f is a closed immersion.

(2. $\Rightarrow$ 1.) We first wish to show that f is a topological closed immersion. We first establish that f is a homeomorphism onto its image. Indeed, we have $f_i = f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i \cap f(X)$ a homeomorphism for each i. Consequently, we have a map $g_i : V_i \cap f(X) \to f^{-1}(V_i)$ which is a continuous inverse of f_i. Clearly g_i forms a matching family for $f(X) = \bigcup_i V_i \cap f(X)$ and thus can be glued to form a global inverse $g : f(X) \to X$ of f. Consequently, $f : X \to f(X)$ is a homeomorphism.

We wish to show that $f(X)$ is closed in Y. As being a closed set is a local property, therefore we need only check that $V_i \cap f(X)$ is a closed set in V_i, but this is exactly what our hypothesis that $f_i : f^{-1}(V_i) \to V_i$ a closed immersion guarantees.

Finally, we wish to show, by Remark 1.4.4.6, that $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a surjection for each $x \in X$. Indeed, as taking germs commute with restrictions, therefore $f_x^\sharp$ is the same local homomorphism as $(f_i)_x^\sharp : \mathcal{O}_{V_i,f(x)} \to \mathcal{O}_{f^{-1}(V_i),x}$ where $f(x) \in V_i$, which is surjective as f_i is a closed immersion. □

The following shows that closed immersions are stable under base change.

Proposition 1.4.4.12. [14] *Let $f : X \to Y$ be a closed immersion and $g : Y' \to Y$ be any other map. Then, the map $p : X \times_Y Y' \to Y'$ is a closed immersion.*

Proof. As $f : X \to Y$ is a closed immersion, therefore by Proposition 1.4.4.11, there is an affine open cover $\{V_i = \operatorname{Spec}(B_i)\}$ of Y such that $f : f^{-1}(V_i) \to V_i$ is a closed immersion. Consequently, $f^{-1}(V_i) \cong f(f^{-1}(V_i)) \subseteq V_i$ is a closed subscheme, thus $f^{-1}(V_i) \cong \operatorname{Spec}(B_i/\mathfrak{b}_i)$ (see Corollary 1.4.4.14). Consider $g^{-1}(V_i) \subseteq Y'$ and cover it by open affines U_{ij}. Hence, we obtain an affine

[14] Exercise II.3.11, a of Hartshorne.

open cover of Y' given by $\{U_{ij} = \mathrm{Spec}\left(B'_{ij}\right)\}_{i,j}$. We claim that $p^{-1}(U_{ij}) \to U_{ij}$ is a closed immersion. Indeed, by Lemma 1.6.4.8, we have $p^{-1}(U_{ij}) \cong U_{ij} \times_{V_i} f^{-1}(V_i) \cong \mathrm{Spec}\left(B'_{ij} \otimes_{B_i} B_i/\mathfrak{b}_i\right) \cong \mathrm{Spec}\left(B'_{ij}/\mathfrak{b}_i B'_{ij}\right)$, which thus makes $p : p^{-1}(U_{ij}) \to U_{ij}$ equivalent to the scheme morphism $\mathrm{Spec}\left(B'_{ij}/\mathfrak{b}_i B'_{ij}\right) \to \mathrm{Spec}\left(B'_{ij}\right)$ obtained by the natural quotient homomorphism (this follows from the tensor product square obtained by the fiber product $U_{ij} \times_{V_i} f^{-1}(V_i)$). Consequently, it is a closed immersion by Proposition 1.2.2.8, 3, as required. $\square$

Closed subschemes and ideal sheaves

We now study closed subschemes of arbitrary schemes. To read the following results, see Section 1.9 on quasicoherent modules.

Proposition 1.4.4.13. *Let X be a scheme.*

1. *If $\mathcal{I} \leq \mathcal{O}_X$ is the ideal sheaf of a closed subscheme $Y \hookrightarrow X$, then $\mathcal{I}$ is a quasicoherent $\mathcal{O}_X$-module. If further X is Noetherian, then $\mathcal{I}$ is coherent.*
2. *If $\mathcal{I} \leq \mathcal{O}_X$ is an ideal of $\mathcal{O}_X$ such that it is quasicoherent, then $\mathcal{I}$ determines a unique closed subscheme $Y \hookrightarrow X$ where Y is given by* $\mathrm{Supp}\,(\mathcal{O}_X/\mathcal{I})$.
3. *Consequently, we have a correspondence*

$$\left\{\begin{matrix}\textit{Quasicoherent ideal}\\ \textit{sheaves } \mathcal{I} \leq \mathcal{O}_X \textit{ upto}\\ \textit{isomorphism}\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{Closed subschemes}\\ Y \hookrightarrow X\end{matrix}\right\}.$$

Proof. 1. This follows from the following facts; closed subschemes are quasicompact separated maps, that direct image of quasicoherent is quasicoherent for such maps and that kernels of maps of quasicoherent modules is quasicoherent. The second statement follows from reducing to affine and using the fact that we know all quasicoherent modules over affine.
2. Pick an ideal sheaf $\mathcal{I} \leq \mathcal{O}_X$ which is quasicoherent and let $Y = \mathrm{Supp}\,(\mathcal{O}_X/\mathcal{I}) := \{x \in X \mid \mathcal{O}_{X,x}/\mathcal{I}_x \neq 0\}$. Then consider $i : (Y, \mathcal{O}_X/\mathcal{I}) \hookrightarrow (X, \mathcal{O}_X)$. It is straightforward to see that the kernel of $i^\flat$ is exactly $\mathcal{I}$. We wish to show that this is a topological closed immersion and that the map $i^\flat$ is surjective. Clearly i is homeomorphic to its image, thus we need only show that its image is a closed set. This is a local property, so let $X = \mathrm{Spec}\,(R)$, so that $\mathcal{I} = \widetilde{\mathfrak{a}}$ for an ideal $\mathfrak{a} \leq R$. Now $Y = \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid (R/\mathfrak{a})_\mathfrak{p} \neq 0\} = \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\} = V(\mathfrak{a})$. Thus i is a topological closed immersion. Now the surjectivity of the map $i^\flat : \mathcal{O}_X \to i_*\mathcal{O}_X/\mathcal{I}$ follows from going to stalks via Lemma 27.5.0.5. The uniqueness of $(Y, \mathcal{O}_X/\mathcal{I})$ w.r.t. $\mathcal{I}$ is clear. $\square$

Note that the main use of quasicoherence of $\mathcal{I}$ in statement 2 was to make sure that the support of $\mathcal{O}_X/\mathcal{I}$ is indeed closed. We have a straightforward, but important corollary.

Corollary 1.4.4.14. *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme. We have the following bijection*

$$\{\textit{Closed subschemes } Y \hookrightarrow X\} \underset{(\mathrm{Spec}(A/\mathfrak{a}),\widetilde{A/\mathfrak{a}})\leftarrow\!\shortmid\,\mathfrak{a}}{\overset{\mathcal{I}\mapsto\Gamma(\mathcal{I},X)}{\rightleftarrows}} \{\textit{Ideals } \mathfrak{a} \leq A\}/\cong.$$

Note that $\widetilde{A/\mathfrak{a}} \cong \mathcal{O}_{\mathrm{Spec}(A/\mathfrak{a})}$.

Proof. Follows immediately from Proposition 1.4.4.13 and Corollary 1.9.1.12. $\square$

1.5 Varieties

Most examples of schemes that we will encounter in the wild are quasi-projective/affine varieties. Therefore, we first cover them in a semi-classical setting not involving schemes. We will then show how to interpret them as finite type separated integral schemes over the base field. This will enable us to use the machinery we will be developing for schemes in the study of varieties. Indeed, by the end of this section, we will comfortably replace the definition of a variety to mean a separated, integral finite type scheme over an algebraically closed field.

1.5.1 Varieties over an algebraically closed field-I

We define varieties as zero sets of certain polynomials over an algebraically closed field k. We assume that the reader is aware of the Zariski topology that is present over $\mathbb{A}^n_k$. Let us first give the classical version of affine varieties.

Definition 1.5.1.1. (**Affine algebraic variety**) Let k be an algebraically closed field and let $\mathbb{A}^n_k$ be the affine n-space. An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n_k$.

We recall that the Hilbert Nullstellensatz further tells us that for any ideal $\mathfrak{a} \leq k[x_1, \dots, x_n]$, the zero set of the ideal $Z(\mathfrak{a}) \subseteq \mathbb{A}^n_k$ is such that the ideal it generates is equal to the radical of the ideal, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Let $A \subseteq \mathbb{A}^n_k$ be an affine algebraic set. Then, the *affine coordinate ring* of A is defined to be the following finitely generated k-algebra

$$k[A] := \frac{k[x_1, \dots, x_n]}{I(A)}$$

where $I(A) \leq k[x_1, \dots, x_n]$ is the ideal generated by A. An important simple lemma to keep in mind for future is the following.

Lemma 1.5.1.2. *Let k be an algebraically closed field. Then B is a finitely generated k-algebra without nilpotent elements if and only if B is an affine coordinate ring of an algebraic set.*

Proof. One side is trivial and the other uses Nullstellensatz. □

Example 1.5.1.3. (*Hyperboloid of one sheet*) A recurring example that we choose to study in this notebook, amongst the others, is the hyperboloid of one sheet. This is given by the following equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

In the affine space over $\mathbb{R}$, $\mathbb{A}^3_{\mathbb{R}}$, we can draw it as shown in Figure 1.1.

We may simply call it a hyperboloid. This hyperboloid determines an affine variety given by the zero set of the polynomial

$$p(x, y, z) = x^2/a^2 + y^2/b^2 - z^2/c^2 - 1 \in k[x, y, z]$$

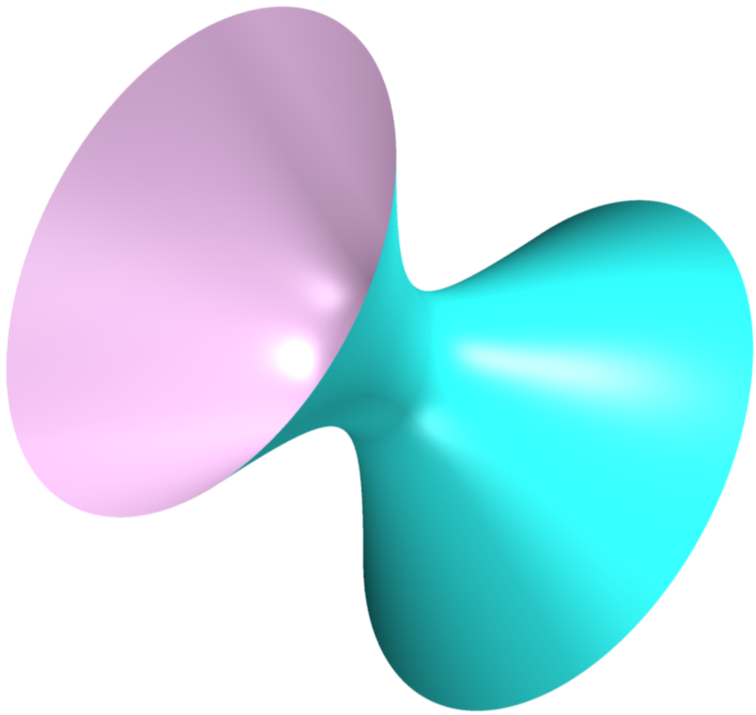

Figure 1.1: A hyperboloid of one sheet as a subvariety of $\mathbb{A}^3_{\mathbb{R}}$. The parameters are $a = 1.05, b = 1.05, c = 1$.

for any field k. Let $X = V(p) \subseteq \mathbb{A}^3_k$. The coordinate ring is given by

$$k[X] = \frac{k[x,y,z]}{I(V(p))}.$$

As we shall see, we will associate to the above variety $(X, \mathcal{O}_X)$ a scheme by considering the spectrum of the coordinate ring, Spec $(k[X])$.

We will understand this fantastic example in much more detail as we develop more tools to handle it.

We now define projective varieties. Consider an algebraically closed field. Then the *projective n-space* is defined to be the quotient $\mathbb{P}^n_k := \mathbb{A}^{n+1}_k / \sim$ where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if and only if there exists $\lambda \in k^\times$ such that $a_i = \lambda b_i$ for all $i = 0, 1 \dots, n$. A point of $\mathbb{P}^n_k$ is denoted by $[a_0 : \cdots : a_n]$ and this presentation of the point is called the homogeneous coordinates of the point. Assuming that the reader is aware about graded rings and the natural grading of $k[x_0, \dots, x_n]$, we observe that we can talk about the *zeroes of a homogeneous polynomial* $p(X) \in k[x_0, \dot{;} x_n]$ as follows:

$$Z(p) := \{P \in \mathbb{P}^n_k \mid p(P) = 0\}.$$

Indeed, one observes that a homogeneous polynomial is zero at a point $P \in P^n_k$ in a manner which is independent of the choice of representation of P in terms of the homogeneous coordinates of P.

With this in our hand, we further define the zero set of a homogeneous ideal $\mathfrak{a} \leq k[x_0, \ldots, x_n]$ as

$$Z(\mathfrak{a}) := \{P \in \mathbb{P}^n_k \mid f(P) = 0 \forall f \in T_{\mathfrak{a}}\}$$

where $T_{\mathfrak{a}}$ is the set of all homogeneous elements of $\mathfrak{a}$. Remember that an ideal in a graded ring is homogeneous if and only if it is generated by the set of all of its homogeneous elements.

Lemma 1.5.1.4. *Let k be a field. Then*

1. *For any two homogeneous ideals $\mathfrak{a}, \mathfrak{b} \leq k[x_0, \ldots, x_n]$, we have $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$.*
2. *For any family of homogeneous ideals $\{\mathfrak{a}_{\mathfrak{i}}\}_{\{}i \in I\}$, we have $\cap_{i\in I} Z(\mathfrak{a}_{\mathfrak{i}}) = Z(\sum_{i\in I} \mathfrak{a}_{\mathfrak{i}})$.*

Proof. Straightforward unravelling of definitions. □

Therefore we obtain a topology on $\mathbb{P}^n_k$ where a set $Y \subseteq \mathbb{P}^n_k$ is closed if and only if $Y = Z(\mathfrak{a}_{\mathfrak{i}})$ for a homogeneous ideal $\mathfrak{a}_{\mathfrak{i}}$ of $k[x_0, \ldots, x_n]$. This is called the Zariski topology of $\mathbb{P}^n_k$.

Definition 1.5.1.5. (**Projective algebraic variety**) Let k be an algebraically closed field. An irreducible algebraic set of $\mathbb{P}^n_k$ is said to be a projective algebraic variety in $\mathbb{P}^n_k$.

Let $V \subseteq \mathbb{P}^n_k$ be a projective algebraic variety. Then the *ideal generated by V* in $k[x_0, \ldots, x_n]$ is $I(V)$ which is the ideal generated by the following set of homogeneous polynomials: $\{f \in k[x_0, \ldots, x_n] \mid f$ is homogeneous and $f(P) = 0\}$.

For a projective algebraic set $Y \subseteq \mathbb{P}^n_k$, we define its *homogeneous coordinate ring* to be the following k-algebra

$$k[Y] := \frac{k[x_0, \ldots, x_n]}{I(Y)}$$

where $I(Y)$ is the homogeneous ideal of Y.

Definition 1.5.1.6 (**Zero set and ideal of an algebraic set**)**.** Define for any set $T \subseteq k[x_0, ;x_n]$ of homogeneous elements the zero set of T as $Z(T) = \{p \in \mathbb{P}^n_k \mid f(p) = 0 \ \forall \ f \in T\}$. For any $Y \subseteq \mathbb{P}^n_k$, define $I(Y)$ as the ideal in $k[x_0, \ldots, x_n]$ generated by $\{f \in k[x_0, \ldots, x_n] \mid f$ is homogeneous & $f(p) = 0 \ \forall p \in Y\}$.

To distinguish between affine and projective cases, we will reserve $Z(\mathfrak{a})$ for zero set of a homogeneous ideal in projective space and $V(\mathfrak{a})$ as the zero set of an ideal in the affine space.

We now show that how the projective space $\mathbb{P}^n_k$ is covered by $n+1$ copies of affine space $\mathbb{A}^n_k$. Before that we discuss few maps which allows us to treat affine case projectively.

Homogenization and dehomogenization

One way to move back and from affine to projective setting is to use to fundamental functions between $k[y_1, \ldots, y_i, \ldots, y_n]$ and $k[x_0, \ldots, x_n]_h$.

Definition 1.5.1.7. (**(De)homogenization**) Let k be an algebraically closed field and let $A := k[y_1, \dots, y_n]$ and $B := k[x_0, \dots, x_n]_h$, the set of all homogeneous polynomials in $k[x_0, \dots, x_n]$. Consider the following two functions

$$\begin{aligned} d_i : B &\longrightarrow A \\ f(x_0, \dots, x_n) &\longmapsto f(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \\ h_i : A &\longrightarrow B \\ g(y_1, \dots, y_n) &\longmapsto x_i^e g\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) \end{aligned}$$

where e is the degree of g and $i = 0, \dots, n$. The map h_i is called the i^{th}-homogenization map and d_i is called the i^{th}-dehomogenization map.

Using this, we can establish the result in question.

Proposition 1.5.1.8. *Let k be an algebraically closed field and consider the projective n-space over k, $\mathbb{P}^n_k$. Then, there exists $n+1$ open subspaces say $U_i \subseteq \mathbb{P}^n_k$, such that $\mathbb{P}^n_k = \bigcup_{i=0}^n U_i$ and for each i, U_i is homeomorphic to $\mathbb{A}^n_k$.*

Proof. Consider the $n+1$ open subspaces of $\mathbb{P}^n_k$ as follows:

$$U_i := \mathbb{P}^n_k \setminus H_i$$

where $H_i = Z(\langle x_i \rangle)$ is the algebraic set obtained by all those points whose i^{th} homogeneous coordinate is zero. Now consider the map

$$\begin{aligned} \varphi_i : U_i &\longrightarrow \mathbb{A}^n_k \\ [a_0 : \dots : a_n] &\longmapsto \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i}\right). \end{aligned}$$

One can check that this pulls closed sets to closed sets by using the i^{th}-homogenization map. Conversely, one can define the map

$$\begin{aligned} \theta_i : \mathbb{A}^n_k &\longrightarrow U_i \\ (a_1, \dots, a_n) &\longmapsto (a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) \end{aligned}$$

and this can again be checked to be continuous by an application of i^{th} dehomogenization map. □

Corollary 1.5.1.9. *Let k be an algebraically closed field and $Y \subseteq \mathbb{P}^n_k$ be a projective algebraic variety. Then, in the notation of Proposition 1.5.1.8, for each $i = 0, \dots, n$, $Y \cap U_i$ is an affine algebraic variety.*

Proof. This follows from the observation that $Y \cap U_i$ is a closed set of $U_i \cong \mathbb{A}^n_k$. The irreducibility follows from the fact that open subsets of irreducible spaces are irreducible. □

Properties of algebraic sets in $\mathbb{P}^n_k$

We now present some basic properties of algebraic sets in $\mathbb{P}^n_k$.

Lemma 1.5.1.10. [15] *(Homogeneous Nullstellensatz) Let k be an algebraically closed field and let $\mathfrak{a} \leq k[x_0, \dots, x_n]$ be a homogeneous ideal. Then,*

$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Proof. Denote by $V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$ to be the vanishing set of $\mathfrak{a}$ in the affine $n+1$-space. This is called the *affine cone* of the ideal $\mathfrak{a}$ in $\mathbb{A}^{n+1}$. We claim that $I(Z(\mathfrak{a})) \hookrightarrow I(V(\mathfrak{a}))$ since if $f \in I(Z(\mathfrak{a}))$ is homogeneous, then $f(P) = 0$ for all $P \in Z(\mathfrak{a}) = \{P \in \mathbb{P}^n_k \mid g(P) = 0 \ \forall g \in \mathfrak{a}\}$. Pick any point $Q \in V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}_k$. We see that $g(Q) = 0$ for all $g \in \mathfrak{a}$. We wish to show that $f(Q) = 0$. As any point $Q \in V(\mathfrak{a})$ determines a point $P \in Z(\mathfrak{a})$ by scaling, that is $P = \lambda Q$, we get by homogeneity of f that $f(Q) = f(\lambda P) = \lambda^d f(P) = 0$, that is, $f \in I(V(\mathfrak{a}))$, as required. By affine Nullstellensatz, it follows that $I(Z(\mathfrak{a})) \subseteq \sqrt{a}$. The converse is straightforward. $\square$

The following tells us when is a projective algebraic set is empty.

Lemma 1.5.1.11. [16] *Let $\mathfrak{a} \leq k[x_0, \dots, x_n] = S$ be a homogeneous ideal. Then, the following are equivalent:*

1. *$Z(\mathfrak{a}) = \emptyset$ in $\mathbb{P}^n_k$,*
2. *$\sqrt{\mathfrak{a}}$ is either S or S_+,*
3. *$\mathfrak{a} \supseteq S_d$ for some $d > 0$.*

Proof. (1. $\Rightarrow$ 2.) The main idea here is again to reduce to affine case by considering the affine cone. Observe that if $Z(\mathfrak{a}) = \emptyset$, then $V(\mathfrak{a}) \subseteq \{0\}$ (where $V(\mathfrak{a})$ is the vanishing in $\mathbb{A}^{n+1}_k$ as in the proof of Lemma 1.5.1.13). Indeed, if not then there exists $p = (p_0, \dots, p_n) \in V(\mathfrak{a})$ such that $p \neq 0$. It follows that $[p_0 : \cdots : p_n] \in Z(\mathfrak{a})$ since any homogeneous element f of $\mathfrak{a}$ vanishes at p in $\mathbb{A}^{n+1}_k$. Now if $V(\mathfrak{a}) = \emptyset$, then by the affine nullstellensatz, we get $\sqrt{\mathfrak{a}} = S$. If $V(\mathfrak{a}) = 0$, then $\sqrt{a} = I(0) = \langle x_0, \dots, x_n \rangle = S_+$.

(2. $\Rightarrow$ 1.) As $\sqrt{a} = I(V(\mathfrak{a})) = S$ or S_+, therefore $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a}) = \emptyset$ or 0. It follows again that $Z(\mathfrak{a}) = \emptyset$.

(2. $\Rightarrow$ 3.) **TODO.** $\square$

Akin to affine varieties, we also have some basic results in projective algebraic sets.

Lemma 1.5.1.12. [17] *Let $\mathbb{P}^n_k$ be the projective n-space over k and let $S = k[x_0, \dots, x_n]$*

1. *If $Y_1 \subseteq Y_2$ in $\mathbb{P}^n_k$, then $I(Y_1) \supseteq I(Y_2)$.*
2. *If $T_1 \subseteq T_2$ in S be subsets of homegeneous elements, then $Z(T_1) \supseteq Z(Y_2)$.*
3. *If $Y_1, Y_2 \subseteq \mathbb{P}^n_k$, then $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*
4. *If $Y \subseteq \mathbb{P}^n_k$, then $Z(I(Y)) = \overline{Y}$.*

Proof. content... $\square$

[15]Exercise I.2.1 of Hartshorne.
[16]Exercise I.2.2 of Hartshorne.
[17]Exercise I.2.3 of Hartshorne.

Some consequences of the homogeneous nullstellensatz yields us the familiar results as in the affine case.

Lemma 1.5.1.13. [18] *Let k be an algebraically closed field and consider the projective n-space $\mathbb{P}^n_k$. Then,*

1. *There is a bijection*

$$\{\text{All algebraic sets } Y \subseteq \mathbb{P}^n_k\} \underset{Z}{\overset{I}{\rightleftarrows}} \{\text{All homogeneous radical ideals of } k[x_0, \dots, x_n]\} .$$

2. *An algebraic set $Y \subseteq \mathbb{P}^n_k$ is irreducible if and only if $I(Y)$ is a prime ideal in $k[x_0, \dots, x_n]$.*
3. *$\mathbb{P}^n_k$ is a projective algebraic variety.*

Remark 1.5.1.14. A corollary of the above lemma is that one can look at projective algebraic varieties in $\mathbb{P}^n_k$ akin to homogeneous prime ideals in $k[x_0, \dots, x_n]$, thus telling us another hint at how the idea of schemes might have looked back in the days.

Proof of Lemma 1.5.1.13. 1. This is a direct consequence of homogeneous nullstellensatz (Lemma 1.5.1.10) and the fact that $Z(I(Y)) = \bar{Y}$ for any $Y \subseteq \mathbb{P}^n_k$.

2. (L $\Rightarrow$ R) Suppose $Y = Z(\mathfrak{a})$ is irreducible and $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ is not prime. Then there exists $f, g \notin \mathfrak{a}$ such that $fg \in \sqrt{\mathfrak{a}}$. Consider the ideals $\mathfrak{b} := \langle f, \sqrt{\mathfrak{a}}\rangle$ and $\mathfrak{c} := \langle g, \sqrt{\mathfrak{a}}\rangle$. We then observe that $Z(\mathfrak{b}), Z(\mathfrak{c}) \subseteq Z(\mathfrak{a})$ and $Z(\mathfrak{b}) \cup Z(\mathfrak{c}) = Z(\mathfrak{b}\mathfrak{c}) = Z(\sqrt{\mathfrak{a}}) = Z(\mathfrak{a})$, where we have used Lemma 1.5.1.10 in the second last equation and the fact that $fg \in \sqrt{a}$ in third last. This yields a contradiction to the irreducibility of Y.
(R $\Rightarrow$ L) Suppose $I(Y)$ is prime but Y is not irreducible. Consequently, there are proper closed sets $Y_1, Y_2 \subseteq Y$ such that $Y_1 \cup Y_2 = Y$. Further, we obtain that $I(Y_i) \gneq I(Y)$ for each $i = 1, 2$. It then follows that there exists $f_i \in I(Y_i) \setminus I(Y)$ such that $f_i \notin I(Y_j)$, $j \neq i$. Consequently, we have $f_1 f_2 \in k[x_0, \dots, x_n]$ such that $f_1 f_2(P) = f_1(P) f_2(P) = 0$ for all $P \in Y$, as $Y = Y_1 \cup Y_2$. We thus have a contradiction to primality of $I(Y)$.

3. Since $I(\mathbb{P}^n_k) = I(Z(\mathfrak{o})) = \sqrt{\mathfrak{o}} = \mathfrak{o}$, then by 2., $\mathbb{P}^n_k$ is irreducible. Note we have used the fact that $k[x_0, \dots, x_n]$ is an integral domain. ☐

One of the reasons that one might be interested in projective varieties is that they "compactify" the question at hand, that is, there are no "missing points" in the ambient space. We will see more into this when we will see projective morphisms and invertible modules, but for now, it is good to keep in mind that reframing your question in the projective spaces/varieties may give you more handle (and of-course, machines) to solve the question at hand. In the same vein, we now see that every affine variety can be embedded compactly into a projective space, and this embedding is called the projective closure of the affine variety.

Definition 1.5.1.15. (**Projective closure of affine varieties**) Let k be an algebraically closed field and consider an affine variety $X \subseteq \mathbb{A}^n_k$. For any $i = 0, \dots, n$, consider the homeomorphism

$$\begin{aligned} \theta_i : \mathbb{A}^n_k &\longmapsto U_i \\ (a_1, \dots, a_n) &\longmapsto [1 : a_1 : \dots : a_n], \end{aligned}$$

[18] Exercise I.2.4 of Hartshorne.

as we considered in Proposition 1.5.1.8. Then, the i^{th}-projective closure of X into $\mathbb{P}^n_k$ is given by the closure $\overline{\theta_i(Y)} \subseteq \mathbb{P}^n_k$ as a subspace in $\mathbb{P}^n_k$. We will usually say the 0th projective closure of X to be simply the *projective closure of* X.

Consider an affine variety $X \subseteq \mathbb{A}^n_k$ and consider $\overline{X} \subseteq \mathbb{P}^n_k$ to be the projective closure of X. Let $I(X) \leq k[y_1, \dots, y_n]$ be the affine ideal of X and let $I(\overline{X}) \leq k[x_0, \dots, x_n]$ be the homogeneous ideal of projective closure. A natural question is that how the homogeneous ideal $I(\overline{X})$ is connected to the affine ideal $I(X)$. The following proposition answers that.

Proposition 1.5.1.16. *Let k be an algebraically closed field and $X \subseteq \mathbb{A}^n_k$ be an affine variety. Let $I(X) \leq k[y_1, \dots, y_n]$ be the affine ideal of X and let $I(\overline{X}) \leq k[x_0, \dots, x_n]$ be the homogeneous ideal of projective closure. Then,*

$$I(\overline{X}) = \langle h_0(I(X)) \rangle$$

where $h_0 : k[y_1, \dots, y_n] \to k[x_0, \dots, x_n]$ is the 0th homogenization function (Definition 1.5.1.7).

Proof. Since $X \subseteq \mathbb{A}^n_k$ is irreducible and closure of irreducible is irreducible, therefore $\overline{X} \subseteq \mathbb{P}^n_k$ is irreducible. It would thus suffice to show that

$$\overline{X} = Z(h_0(I(X))).$$

Indeed, this would imply that $h_0(I(X))$ is a homogeneous prime ideal by Lemma 1.5.1.13, 1, thus applying $I(-)$ would yield the result. We therefore show the above equality. Consider any closed set $Y \supseteq X$ in $\mathbb{P}^n_k$. We then wish to show that $Y \supseteq Z(h_0(I(X)))$. Since $Y \subseteq \mathbb{P}^n_k$ is closed, therefore $Y = Z(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a}$ in $k[x_0, \dots, x_n]$. It would thus suffice to show that

$$\mathfrak{a} \hookrightarrow h_0(I(X)).$$

It would further suffice to show the above inclusion only for homogeneous elements, as $\mathfrak{a}$ is generated by homogeneous elements. Consequently, pick any homogeneous polynomial $f \in \mathfrak{a}$. We can write

$$f(x_0, \dots, x_n) = x_0^e g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

for some $g \in k[y_1, \dots, y_n]$ and $e = \deg f$. In other words, $f = h_0(g)$. Now since $Y \supseteq X$, therefore $f(P) = 0 \forall P \in X \subseteq \mathbb{P}^n_k$, that is, if $P = [1 : a_1, \dots, a_n] \in X$, then $f(1, a_1, \dots, a_n) = 0$ and thus $g(a_1, \dots, a_n) = 0$. Hence $g \in I(X) \leq k[y_1, \dots, y_n]$. Thus $f = h_0(g)$ where $g \in I(X)$, that is, $f \in h_0(I(X))$, as required. $\square$

Dimension, hypersurfaces and complete intersections

Let us first understand how the notion of dimension plays out with the Krull dimension of homogeneous coordinate ring of a projective variety.

Proposition 1.5.1.17. *Let k be an algebraically closed field and $X \subseteq \mathbb{P}^n_k$ be a projective k-variety. Then,*

1. $\dim k[X] = \dim X + 1$,
2. $\dim X = \dim U_i \cap X$ *where $U_i \subseteq X$ is an affine open subset as in Proposition 1.5.1.8, for all $i = 0, \dots, n$.*

Proof. We will prove the two statements together. The main technique here is, as usual, to reduce the computations to one of the affine patches. Let $U_i \subseteq \mathbb{P}^n_k$ be the hyperplane where $x_i \neq 0$. We know that U_is covers $\mathbb{P}^n_k$ and each U_i is isomorphic to $\mathbb{A}^n_k$. Denote $X_i = U_i \cap X$ so that X_i is an open subvariety of X. Further denote $k[X]^h$ to be the homogeneous coordinate ring of X and $k[X_i]^a$ the affine coordinate ring of X_i. Note that $k[X_i]^a = k[x_0, \dots, \hat{x}_i, \dots, x_n]/d_i I(X)$ where d_i is the i^{th} dehomogenisation map. We would now like to note two things to move forward:

1. $\dim X = \dim X_j$ for some $j = 0, \dots, n$,
2. $k[X]^h_{x_i} \cong k[X_i]^a[x_i, 1/x_i]$[19].

The first statement is immediate from the fact that $\dim Y = \sup_i \dim U_i$ for any space Y with U_i an open covering. The second statement is the heart of the proof. Indeed, consider the map $k[X]^h_{x_i} \to k[X_i]^a[x_i, 1/x_i]$ which takes an element f/x_i^n and treats it as a polynomial in $x_i, 1/x_i$ with coefficients in $k[X_i]^a$. One immediately checks all the necessary conditions to ensure that this is an isomorphism.

Observe that if K/k is algebraic, then $K(x)/k(x)$ is algebraic. It follows that trdeg $k[X_i]^a[x_i, 1/x_i] = 1 +$ trdeg $k[X_i]^a$. We now complete the proof. We may assume $\dim X = \dim X_0$. Consequently, via Proposition 1.5.3.10, 6 and Theorem 23.8.2.1, we obtain the following equalities:

$$\begin{aligned}\dim k[X]^h &= \operatorname{trdeg} k[X]^h = \operatorname{trdeg} k[X]^h_{x_0} = \operatorname{trdeg} k[X_0]^a[x_0, 1/x_0] = 1 + \operatorname{trdeg} k[X_0]^a \\ &= 1 + \dim k[X_0]^a = 1 + \dim X_0 = 1 + \dim X.\end{aligned}$$

The statement 2. follows from the following equalities:

$$\begin{aligned}\dim X_i = \dim k[X_i]^a &= \operatorname{trdeg} k[X_i]^a_{x_i} = \operatorname{trdeg} k[X_i]^a[x_i, 1/x_i] - 1 = \operatorname{trdeg} k[X]^h_{x_0} - 1 \\ &= \operatorname{trdeg} k[X]^h - 1 = \dim X + 1 - 1 = \dim X.\end{aligned}$$

□

We would now like to establish the following result, which will later motivate the definition of Weil divisors and of complete intersections.

Lemma 1.5.1.18. *Let k be an algebraically closed field and $X \subseteq \mathbb{P}^n_k$ be a projective k-variety. Then, the following are equivalent*

1. $\dim X = n - 1$.
2. *The homogeneous ideal $I(X) \leq k[x_1, \dots, x_n]$ is generated by a single irreducible homogeneous polynomial.*

Proof. (1. $\Rightarrow$ 2.) By Proposition 1.5.1.17, 1, we have $\dim k[X] = n$, where $k[X] = k[x_0, \dots, x_n]/I(X)$. By Theorem 23.8.2.2, we have ht $I(X) = 1$. Since any height 1 prime ideal of a UFD is principal, therefore $I(X)$ is principal. Since $I(X)$ is homogeneous, therefore the statement 2. follows.
(2. $\Rightarrow$ 1.) By Proposition 1.5.1.17, 2 and Theorem 23.8.2.2, we have

$$\dim X = \dim X_0 = \dim k[X_0]^a = n - \operatorname{ht} d_0(I(X)).$$

We need only show that ht $d_0(I(X)) = 1$. Since $I(X) = \langle p(x_0, \dots, x_n) \rangle$, therefore $d_o(I(X)) = \langle p(1, x_1, \dots, x_n) \rangle$. Since $k[x_0, \dots, x_n]$ is a UFD and an easy observation about UFDs yields that height 1 prime ideals are exactly principal prime ideals, therefore the result follows. □

[19]This statement can be seen as a generalization of Lemma 1.5.3.11.

Cones

d-uple embedding

Veronese surface

Segre embedding

ete cones, d-
onese, Segre
ics, Chap-

1.5.2 Morphism of varieties

We have defined affine and projective varieties so far. One would often, however, would like to know whether a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$ is an open subspace of some affine or projective variety. Due to to this need, we define the following.

Definition 1.5.2.1. (**Quasi-affine/projective variety**) A subset X of $\mathbb{A}^n$ or $\mathbb{P}^n$ is said to be quasi-affine or quasi-projective if X is an open subset of an affine or projective variety, respectively.

Let X be a quasi-affine or projective variety. From our knowledge of geometry, we know that in a real C^α-manifold M, the right type of functions are those which are defined on open subsets of M as C^α-maps to $\mathbb{R}$, where the latter is treated as a C^α-manifold. Consequently, we are interested in the same type of maps to the affine line $\mathbb{A}^1_k$.

Definition 1.5.2.2. (**Regular maps**) This notion is defined differently for quasi-affine and quasi-projective varieties.

1. Let X be a quasi-affine variety. A function
$$\varphi : X \to \mathbb{A}^1_k$$
is said to be a *regular function* if for all $P \in X$, there exists an open subset $U \subseteq X$ such that $\varphi|_U = g/h$ where $g, h \in k[x_1, \dots, x_n]$ and $h(P) \neq 0\ \forall P \in U$.
2. Let X be a quasi-projective variety. A function
$$\varphi : X \to \mathbb{A}^1_k$$
is said to be a *regular function* if for all $P \in X$, there exists an open subset $U \subseteq X$ such that $\varphi|_U = g/h$ where $g, h \in k[x_0, \dots, x_n]$ are homogeneous polynomials of same degree and $h(P) \neq 0\ \forall P \in U$. Note that this defines a valid function to the affine line.

Indeed, regular maps are continuous.

Lemma 1.5.2.3. *Let X be a quasi-affine or quasi-projective variety and $\varphi : X \to \mathbb{A}^1_k$ be a regular function. Then φ is continuous.*

Proof. The Zariski topology on $\mathbb{A}^1_k$ is the cofinite topology, hence any closed set in $\mathbb{A}^1_k$ is a finite union of points of k. It thus suffices to show that for any $a \in k$, $Y := \varphi^{-1}(a) \subseteq X$ is closed. Since checking a set is closed is local in X, that is, $Y \subseteq X$ is closed if and only if there exists an open covering of X, say $\{U_\alpha\}$ such that $U_\alpha \cap Y$ is closed in U_α. We may thus replace X by an open subset of X where φ is represented as g/h for $g, h \in k[y_1, \dots, y_n]$ (in $k[x_0, \dots, x_n]$, homogeneous and of same degree in the projective case). Consequently, $\varphi^{-1}(a) \subseteq X$ is given by $\{P \in X \mid (g - ah)(P) = 0\}$ which in other words is $Z(g - ah)$ ($g - ah$ is homogeneous in the projective case). Thus $\varphi^{-1}(a)$ is closed, as required. $\square$

A simple corollary of above is the first striking result one learns in complex analysis for holomorphic maps (see Proposition 15.2.3.10).

Lemma 1.5.2.4. *(Identity principle) Let $\varphi, \xi : X \to \mathbb{A}^1_k$ be two regular maps over a quasi-affine or quasi-projective variety X. Then, $\varphi = \xi$ if and only if there exists an open set $U \subseteq X$ such that $\varphi = \xi$ over U.*

Proof. L $\Rightarrow$ R is easy. For R $\Rightarrow$ L, observe that for $\phi := \varphi - \xi$ is continuous by Lemma 1.5.2.3. Further, the set $\phi^{-1}(0) \subseteq X$ is closed and contains U. Since $\phi^{-1}(0) \supseteq U$ and U is an open set of an irreducible space, therefore U is dense in X. Consequently, $\phi^{-1}(0)$ is a closed and dense in X, hence is equal to X. □

We now define varieties in general.

Definition 1.5.2.5. (**Varieties**) Let k be an algebraically closed field. A variety over k is defined to be a quasi-affine or a quasi-projective variety in $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$, respectively.

The notion of morphism of varieties is then given by functions which pulls regular functions back by pre-composition.

Definition 1.5.2.6. (**Map of varieties**) Let k be an algebraically closed field and let X, Y be two varieties over k. A map of varieties is a continuous function $f : X \to Y$ such that for any open set $V \subseteq Y$ and any regular function $\varphi : Y \to \mathbb{A}^1_k$, the function

$$\varphi \circ f : \varphi^{-1}(V) \to \mathbb{A}^1_k$$

is a regular function on the open set $\varphi^{-1}(V)$ of X. We may also call a map of varieties a *morphism of varieties*.

We therefore obtain the category of varieties over k, whose objects are varieties over k and arrows are maps of varieties. We will denote this category by

$$\mathbf{Var}_k.$$

Just like in topological spaces, it is not true in general that a bijective continuous map is a homeomorphism, similarly it is not true in general that a bijective map of varieties is an isomorphism of varieties, as the following example shows.

Example 1.5.2.7. Consider the affine line $\mathbb{A}^1_k$ and consider the affine variety $X := Z(y^2 - x^3) \subseteq \mathbb{A}^2_k$. The function

$$\begin{aligned} f : \mathbb{A}^1_k &\longrightarrow X \\ t &\longmapsto (t^2, t^3) \end{aligned}$$

is a map of varieties as for any open set $U \subseteq X$ and regular map $\varphi : X \to \mathbb{A}^1_k$, the composite $\varphi \circ f : \varphi^{-1}(U) \to \mathbb{A}^1_k$ is given by $t \mapsto \varphi(t^2, t^3)$ and then the regularity of this composite can be seen to be a result of regularity of φ. Further note that f induces an inverse continuous function

$$\begin{aligned} f^{-1} : X &\longrightarrow \mathbb{A}^1_k \\ (a, b) &\longmapsto ba^{-1}. \end{aligned}$$

Thus, $\mathbb{A}^1_k$ and X are homeomorphic as topological spaces. However, as varieties, they can not be isomorphic. Indeed, we shall soon see that coordinate rings are invariant of affine varieties and in our case $\mathbb{A}^1_k$ has $k[x]$ as its coordinate ring whereas X has $k[x, y]/\langle y^2 - x^3\rangle$ as its coordinate ring. These are not isomorphic as one is PID and the other is not.

We now construct some more algebraic gadgets on top of varieties and will prove how they will turn out to be invariants of the varieties under question. We have already seen one, the coordinate ring. We will now see the construction of others and we shall do it in a manner so that it is amenable to generalization to schemes, as is studied elsewhere in this chapter.

1.5.3 Varieties as locally ringed spaces

See Chapter 8, Foundational Geometry, for background on locally ringed spaces and basic global algebra. In this section, we would like to interpret varieties as locally ringed spaces, so that we can understand later that how a variety can be interpreted as a scheme. Clearly, for a variety X, we already have an underlying topological space X itself. To give X the structure of a locally ringed space, we need to consider a sheaf over X. We shall use regular functions over open sets of X for that.

Definition 1.5.3.1. (**Structure sheaf of a variety**) Let k be an algebraically closed field and X be a variety over k. For each open set $U \subseteq X$, consider the following set

$$\mathcal{O}_X(U) := \{f : U \to \mathbb{A}^1_k \mid f \text{ is regular}\}.$$

Further, for open $V \subseteq U$ in X, consider the function

$$\begin{aligned} \rho_{U,V} : \mathcal{O}_X(U) &\longrightarrow \mathcal{O}_X(V) \\ f &\longmapsto f|_V . \end{aligned}$$

This defines a sheaf of sets, as the following lemma shows.

Lemma 1.5.3.2. *The assignment $\mathcal{O}_X$ on open sets of a k-variety X as defined in Definition 1.5.3.1 defines a sheaf of sets over X.*

Proof. The locality axiom is straightforward as $\mathcal{O}_X(U)$ is a collection of functions, which thus can be checked locally for equality. It thus suffices to show that $\mathcal{O}_X$ satisfies the gluing axiom. Pick any open set U, an open covering $\{U_i\}_{i\in I}$ of U and a matching family $f_i \in \mathcal{O}_X(U_i)$ for each $i \in I$, that is $\rho_{U_i,U_i\cap U_j}(f_i) = \rho_{U_j,U_i\cap U_j}(f_j)$ for each $i, j \in I$. Consequently, we define $f : U \to \mathbb{A}^1_k$ given by $x \mapsto f_i(x)$ if $x \in U_i$. This is a well-defined function by the matching condition and further f is a regular function as for each point $x \in U$, f can be written as a rational function in some open neighborhood around x (essentially by regularity of f_is). Consequently, $\mathcal{O}_X$ is a sheaf. □

Further, $\mathcal{O}_X$ is a sheaf of k-algebras if X is a k-variety.

Lemma 1.5.3.3. *Let k be an algebraically closed field and consider a k-variety X. The structure sheaf $\mathcal{O}_X$ of X is a sheaf of k-algebras.*

Proof. Indeed, $\mathcal{O}_X$ is a ring by point-wise addition and multiplication. Further, its a k-algebra via the injective ring homomorphism

$$\begin{aligned} k &\hookrightarrow \mathcal{O}_X(U) \\ c &\mapsto c : U \to \mathbb{A}^1_k \end{aligned}$$

where c is treated as the constant rational map. □

Hence, $(X, \mathcal{O}_X)$ is a k-ringed space. We now show that it is locally k-ringed.

Lemma 1.5.3.4. *Let k be an algebraically closed field and let X be a k-variety. Then, for all points $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.*

Proof. We wish to show that $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x \leq \mathcal{O}_{X,x}$. Consider the set

$$\mathfrak{m}_x := \{(U, f) \in \mathcal{O}_{X,x} \mid f(x) = 0\}.$$

It then easily follows that $\mathfrak{m}_x$ an ideal and consequently is a maximal ideal because $\mathcal{O}_{X,x} \setminus \mathfrak{m}_x$ is jut the set of all units of $\mathcal{O}_{X,x}$. □

Remark 1.5.3.5. We have thus established that for any k-variety X we obtain a locally k-ringed space $(X, \mathcal{O}_X)$. We now observe how the data of a morphism of varieties can be represented as data of a morphism of underlying locally ringed spaces.

The notion of morphism of locally ringed spaces is elucidated in Definition 8.1.0.2.

Lemma 1.5.3.6. *Let k be an algebraically closed field and X, Y be two k-varieties. Then, there is an injective inclusion*

$$\mathrm{Hom}_{\mathbf{Var}_k}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathbf{LRSpace}}(X, Y).$$

Proof. Indeed, consider the map

$$\begin{aligned} \theta : \mathrm{Hom}_{\mathbf{Var}_k}(X, Y) &\hookrightarrow \mathrm{Hom}_{\mathbf{LRSpace}}(X, Y) \\ f : X \to Y &\longmapsto (f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \end{aligned}$$

where $\theta(f)$ has the underlying continuous map same as f but the map on sheaves, $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$, is given on sections as follows: let $V \subseteq Y$ be an open set, then the map on sections over V is

$$\begin{aligned} f^\flat_V : \mathcal{O}_Y(V) &\longrightarrow \mathcal{O}_X(f^{-1}(V)) \\ (V, \varphi) &\longmapsto (f^{-1}(V), \varphi \circ f). \end{aligned}$$

The fact that $f^\flat$ as defined above is indeed a sheaf morphism is straightforward. We thus need only show that the adjoint map $f^\sharp$ of the above defines a map on stalks which is local. For this, we need only observe how the comorphism, $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$, as defined in Definition 8.1.0.2, in this case turns out to be the following mapping

$$(V, \varphi)_x \longmapsto (f^{-1}(V), \varphi \circ f)_x.$$

Now if $(V, \varphi)_x \in \mathfrak{m}_{Y,f(x)}$, then $\varphi(f(x)) = 0$ by definition. Thus $(f^{-1}(V), \varphi \circ f) \in \mathfrak{m}_{X,x}$. With this, the fact that θ is injective is straightforward. □

Remark 1.5.3.7. We therefore have an inclusion

$$\mathbf{Var}_k \hookrightarrow \mathbf{LRSpace}.$$

Indeed, we now show that the notion of isomorphisms coincide here.

We will now define various algebraic gadgets out of the structure sheaf $\mathcal{O}_X$ of a variety X. Indeed, to some extent, that's the goal of algebraic geometry in general.

We now define an important field corresponding to each variety X, called its function field.

Definition 1.5.3.8. (**Function field of a variety**) Let k be an algebraically closed field and X be a k-variety. The function field of X, denoted $K(X)$, is obtained as the quotient of the set $\cup_{U\supseteq X,\text{ open}} \cup_{(U,\varphi)\in\mathcal{O}_X(U)} (U,f)$ by the following relation

$$(U,\varphi) \sim (V,\phi) \iff \exists \text{ open } W \subseteq U\cap V \text{ s.t. } \rho_{U,W}(\varphi) = \rho_{V,W}(\phi).$$

Indeed, this has an addition and a multiplication given by restriction to the open sets where they agree. This is further a field as any non-zero element $[(U,f)]$ can be inverted in a small enough open set $W \subseteq U$ (which will be non-empty as X is irreducible) where f is non-zero (otherwise the class $[(U,f)]$ is identically zero).

Remark 1.5.3.9. Note that we have the following ring homomorphisms for any k-variety X and $x \in X$

$$\begin{aligned} \Gamma(\mathcal{O}_X, X) &\longrightarrow \mathcal{O}_{X,x} \longrightarrow K(X) \\ (X,\varphi) &\longmapsto (X,\varphi)_x \longmapsto [(X,\varphi)]. \end{aligned}$$

In-fact, both these are injective by a simple use of the identity principle (Lemma 1.5.2.4). In this way, algebraic gadgets start taking a hold onto the geometry of varieties, which we will see further in this chapter.

We now give two results; one for affine and one for projective; which shows how the three algebraic gadgets introduced in Remark 1.5.3.9 can be realized more algebraically.

Proposition 1.5.3.10. *Let k be an algebraically closed field and let X be an affine k-variety. Let $\mathfrak{m}_p = \{f \in k[X] \mid f(p) = 0 \text{ as a regular function}\}$. Then,*

1. *$\mathfrak{m}_p$ is a maximal ideal of $k[X]$ for every point $p \in X$,*
2. *$\operatorname{mSpec}(k[X]) \cong X$ as sets,*
3. *$k[X]_{\mathfrak{m}_p} \cong \mathcal{O}_{X,p}$,*
4. *$k[X]_{\langle 0\rangle} \cong K(X)$,*
5. *$\Gamma(\mathcal{O}_X, X) \cong k[X]$,*
6. *$\dim X = \operatorname{trdeg} K(X)/k$[20],*
7. *$\dim X = \dim \mathcal{O}_{X,p}$ for all $p \in X$[21].*

[20]Thus the function field $K(X)/k$ holds important global information about the algebra and geometry of X.

[21]Thus the notion of dimension of varieties is detectable at the level of stalks. This is because, as the proof and the statement 3 shows, the local ring $\mathcal{O}_{X,p}$ holds almost all relevant information about the coordinate ring.

Proof. We give the main ideas of each. The main idea in the latter parts is to embed all the relevant rings inside the function field and do the relevant algebra there.

1. Since there is a correspondence between radical ideals of $k[X]$ and algebraic sets of X and since the correspondence is antitone, therefore minimal algebraic sets (point $p \in X$) of X correspond to maximal ideals of $k[X]$ vanishing at p. The result then follows.
2. This follows from 1. Explicitly, one considers the mapping $p \in X \mapsto \mathfrak{m}_p$.
3. Consider the canonical mapping
$$\begin{aligned} k[X]_{\mathfrak{m}_p} &\longrightarrow \mathcal{O}_{X,p} \\ \frac{f}{g} &\longmapsto (X \setminus Z(g), f/g)_p \end{aligned}$$
where $g(p) \neq 0$ (so $g \notin \mathfrak{m}_p$). This is a homomorphism by the fact that $Z(f) \cup Z(g) = Z(fg)$. This is injective because if $f/g = 0$, then $f = 0$ on some open subset $W \subseteq X \setminus Z(g)$. By an application of identity principle (Lemma 1.5.2.4), the injectivity follows. For surjectivity, observe that for any $(U, f)_p \in \mathcal{O}_{X,p}$, we can represent it by the rational function that f looks like around p, so $(U, f)_p = (W, g/h)_p$ where g/h is a rational function. Consequently, $g/h \mapsto (X \setminus Z(h), g/h)_p = (W, g/h)_p$. The result follows.
4. Observe first that if R is a domain and $\mathfrak{p} \leq R$ is an prime ideal of R, then $(R/\mathfrak{p})_{\langle 0 \rangle}$ is isomorphic to $R_{\langle 0 \rangle}$. Now, by 3, we obtain that $k[X]_{\langle 0 \rangle} \cong (k[X]_{\mathfrak{m}_p})\langle 0 \rangle \cong (\mathcal{O}_{X,p})_{\langle 0 \rangle}$. The map
$$\begin{aligned} (\mathcal{O}_{X,p})_{\langle 0 \rangle} &\longrightarrow K(X) \\ \frac{(U, f/g)_p}{(V, h/l)_p} &\longmapsto [(U \cap V, fl/gh)] \end{aligned}$$
can be seen to be a well-defined (use Lemma 1.5.2.4) isomorphism.
5. By Lemma 23.1.2.12, we have that $\bigcap_{\mathfrak{m} < k[X]} k[X]_{\mathfrak{m}} \cong k[X]$. By Remark 1.5.3.9, we have $\Gamma(\mathcal{O}_X, X) \hookrightarrow \mathcal{O}_{X,p}$ (in $K(X)$). We further have $k[X] \hookrightarrow \Gamma(\mathcal{O}_X, X)$. Consequently, we obtain via 3. the following
$$k[X] \hookrightarrow \Gamma(\mathcal{O}_X, X) \hookrightarrow \bigcap_{p \in X} \mathcal{O}_{X,p} \cong \bigcap_{p \in X} k[X]_{\mathfrak{m}_p} \hookrightarrow \bigcap_{\mathfrak{m} < k[X]} k[X]_{\mathfrak{m}} \cong k[X].$$
The result then follows.
6. We have $\dim X = \dim k[X]$ as any irreducible closed subset of X corresponds in a contravariant manner to a prime ideal of $k[X]$. By Theorem 23.8.2.1, we have $\dim k[X] = \operatorname{trdeg} K(X)/k$.
7. By 3, $\dim \mathcal{O}_{X,p} = \operatorname{ht} \mathfrak{m}_p$. By Theorem 23.8.2.2, we have $\operatorname{ht} \mathfrak{m}_p + \dim k[X]/\mathfrak{m}_p = \dim k[X]$. But since $k[X]/\mathfrak{m}_p \cong k$ by Nullstellensatz, therefore the above equation reduces to $\operatorname{ht} \mathfrak{m}_p = \dim k[X]$ and the right side is just $\dim X$.

□

We next do the projective case. See Chapter 23, Section 23.1.2 for homogeneous localization of graded rings.

Lemma 1.5.3.11. *Let k be an algebraically closed field and X be a projective k-variety in $\mathbb{P}^n_k$. Let $U_i = \mathbb{P}^n_k \setminus Z(x_i)$ and $X_i := X \cap U_i$. Then*
$$\varphi_i : k[X_i]^a \cong k[X]^h_{(x_i)}$$

where $k[X_i]^a$ denotes the affine coordinate ring of $X_i \subseteq \mathbb{A}_k^n$ and $k[X]^h$ denotes the homogeneous coordinate ring of $X \subseteq \mathbb{P}_k^n$. Further the localization above is homogeneous.

Proof. Consider the map $k[y_1, \dots, y_n] \to k[x_0, \dots, x_n]$ mapping as $f(y_1, \dots, y_n) \mapsto f\left(\frac{x_0}{x_i}, \dots, \hat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right)$. This can easily be seen to be a well-defined ring isomorphism mapping the ideal $I(X_i) \mapsto I(X_i)^h = I(X)^h_{(x_i)}$. The result follows by quotienting. □

Proposition 1.5.3.12. *Let k be an algebraically closed field and X be a projective k-variety. Let $\mathfrak{m}_p = \langle \{f \in k[X] \mid f \text{ is homogeneous } \& \ f(p) = 0\} \rangle$ for any $p \in X$ and $k[X]$ be the homogeneous coordinate ring of X. Then,*

1. *$\mathfrak{m}_p$ is a maximal ideal of $k[X]$ for every element $p \in X$,*
2. *$k[X]_{(\mathfrak{m}_p)} \cong \mathcal{O}_{X,p}$,*
3. *$k[X]_{(\langle 0 \rangle)} \cong K(X)$,*
4. *$\Gamma(\mathcal{O}_X, X) \cong k$.*

Proof. Denote by $k[X]^h$ the homogeneous coordinate ring and $X_i := X \cap U_i$ where $U_i = \mathbb{P}_k^n \setminus Z(x_i)$. By Lemma **??**, $U_i \cong \mathbb{A}_k^n$ as varieties, therefore denote X_i^a to be the affine variety corresponding to $X_i \subseteq U_i$. We thus denote $k[X_i]^h$ for the homogeneous coordinate ring when $X_i \subseteq U_i$ and $k[X_i]^a$ to be the affine coordinate ring when $X_i \subseteq \mathbb{A}_k^n$. Let $R := k[X]^h$. The main idea of the last part is to use the theory of integral dependence together with algebraic closure of k.

1. Let $P \in X$, so $P \in X_i$ for some $i = 0, \dots, n$. Thus, let $P^a \in X_i^a$ and by Lemma 1.5.3.11 and Proposition 1.5.3.10, we obtain that $\mathfrak{m}_{P^a}$ is a maximal ideal of $k[X_i]^a$. Thus, $\varphi_i(\mathfrak{m}_{P^a}) = \mathfrak{m}_P k[X]^h_{x_i}$ is a maximal ideal of $k[X]^h_{x_i}$.
2. We simply have the following for any $p \in X$ by irreducibility of X, by Lemma 1.5.3.11 and by Proposition 1.5.3.10:
$$\mathcal{O}_{X,p} \cong \mathcal{O}_{X_i,p} \cong \mathcal{O}_{X_i^a,p^a} \cong k[X_i]^a_{\mathfrak{m}_{p^a}} \cong \left(k[X]^h_{x_i}\right)_{\mathfrak{m}_{p^a}} \cong k[X]^h_{\mathfrak{m}_{p^a}}.$$
3. By irreducibility of X, by Lemma 1.5.3.11 and by Proposition 1.5.3.10, we have the following identifications
$$K(X) \cong K(X_i) \cong K(X_i^a) \cong k[X_i]^a_{\langle 0 \rangle} \cong \left(k[X]^h_{x_i}\right)_{\langle 0 \rangle} \cong k[X]^h_{\langle 0 \rangle}.$$
4. First note that $k \hookrightarrow \Gamma(\mathcal{O}_X, X)$. It would thus suffice to show that $\Gamma(\mathcal{O}_X, X) \hookrightarrow k$. Pick any $f \in \Gamma(\mathcal{O}_X, X)$. We wish to show that $f \in k$. Let $R = k[X]^h$. Note that we can embed $\Gamma(X, \mathcal{O}_X)$ inside the (non-homogeneous) fraction field $L = k[X]^h_{\langle 0 \rangle}$. Consequently, by algebraic closure of k, it would suffice to show that $f \in L$ satisfies a polynomial with coefficients in k. Since f is a regular function on each of the X_i, therefore $f \in k[X_i]^a \cong k[X]^h_{x_i}$. Consequently, $f = g_i/x_i^{n_i}$ in L where $\deg g_i = n_i$ and thus $x_i^{n_i} f \in R_{n_i}$ for each $i = 0, \dots, n$. It thus follows that $\deg f = 0$ in L. Consequently, it would suffice to show that $f \in L$ is integral over R (as we can then obtain a polynomial in $k[x]$ whose zero is f by restricting to 0 degree coefficients). By Corollary **??**, it would thus suffice to show that $R[f]$ is a finitely generated R-module.

   It would thus suffice if we show that $\exists M \in \mathbb{N}$ such that $\forall N \geq M$, $R_N f^m \subseteq R_N$ for all $m \geq 0$. Indeed, for $M = \sum_i n_i$, we see that $R_N f \subseteq R_N$ as for any $g \in R_N$, we have that each term of g will have to have one x_i whose power is $\geq n_i$. Repeatedly applying $R_N f \subseteq R_N$ yields $R_N f^m \subseteq R_N$ for all $m \geq 0$, as needed.

□

Remark 1.5.3.13. Note that in Proposition 1.5.3.12, 1, the maximal ideal $\mathfrak{m}_P$ does not contain all of non-constant polynomials in $k[X]$ because $\mathfrak{m}_p$ is generated by homogeneous polynomials vanishing at $p \in X$ and a polynomial with non-zero constant terms cannot be in such an ideal, thus such an $\mathfrak{m}_p$ will exactly be the ideal of all non-constant polynomials in $k[X]$, but then $p \in \bigcap_{f \in k[X], f(0)=0} Z(f) = \emptyset$.

We now show that affine varieties are completely determined by their coordinate rings in the following sense

Theorem 1.5.3.14. *Let k be a algebraically closed field. Then the following*

$$\begin{aligned} k[-] : \mathbf{AfVar^{op}}_k &\longrightarrow \mathbf{FGIAlg}_k \\ X &\longmapsto k[X] \\ X \xrightarrow{\varphi} Y &\longmapsto k[Y] \xrightarrow{k[\varphi]} k[X] \end{aligned}$$

is a functor[22] *which induces an equivalence between the opposite category of affine varieties over k and finitely generated integral domains over k.*

Proof. **TODO.** □

We now show some examples of the machinery developed so far. We first show that any affine plane conic is isomorphic as a variety to either the parabola $y - x^2$ or the hyperbola $xy - 1$. Indeed, we use here the familiar high-school topic that one classifies conics on the basis of discriminant(!) This will further show that the usual substitutions that we so used to do in school days to reduce an algebraic equation into a simpler form can equivalently be stated in algebraic language as finding a correct automorphism of the corresponding ring in question.

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Subvarieties

[22]Note that by Proposition 1.5.3.10, this is just the global sections functor.

1.5.4 Varieties as schemes

In this section we show how to realize a k-variety (see Definition 1.5.2.5) as a scheme. This will be essential as it fulfill all the reasons to work with schemes as they generalize the concept of varieties to just the right level where all algebro-geometric questions can be asked and be attempted to be solved.

We first show a fully-faithful functor which embeds the category of k-varieties into the category of k-schemes (that is, schemes over k). This will hence show how to obtain a scheme from a variety because, as the following construction of the relevant functor will show, it is not straightforward how should one begin defining it[23].

Definition 1.5.4.1. (**Spectral space of** X) For every topological space X, we can associate a topological space

$$t(X) := \{\text{All non-empty closed irreducible subsets of } X\}$$

where any closed set is given by $t(Y) \subseteq t(X)$ for a closed set $Y \subseteq X$. The following lemma shows that this indeed defines a topology on $t(X)$. We will call $t(X)$ the spectral space of X.

Lemma 1.5.4.2. *Let X be a space and $Y, Z, Y_i \subseteq X$ be closed subsets of X. Then,*

1. $t(Y) \subseteq t(X)$,
2. $t(Y \cup Z) = t(Y) \cup t(Z)$,
3. $t\left(\bigcap_i Y_i\right) = \bigcap_i t(Y_i)$.

Proof. 1. Any closed irreducible subset of Y, where Y is closed in X, will again be closed and irreducible in X.
2. Any irreducible subset of $Y \cup Z$ cannot have non-empty intersection with both of them.
3. Follows from 1. □

Indeed, our main idea is to show that for a variety V, the space $t(V)$ will eventually become a scheme. We have few observations about spectral spaces, before we realize that idea.

Lemma 1.5.4.3. *Let X, X_1, X_2 be spaces and $f : X_1 \to X_2$ be a continuous map. Then,*

1. *there is a one-to-one correspondence between closed subsets of X and closed subsets of $t(X)$,*
2. *the following is a continuous map*

$$\begin{aligned} t(f) : t(X_1) &\longrightarrow t(X_2) \\ Y_1 &\longmapsto \overline{f(Y_1)}, \end{aligned}$$

3. *the following is a functor*

$$\begin{aligned} t : \mathbf{Top} &\longrightarrow \mathbf{Top} \\ X &\longmapsto t(X), \end{aligned}$$

4. *the following is a continuous map*

$$\begin{aligned} \alpha : X &\longrightarrow t(X) \\ x &\longmapsto \overline{\{x\}}. \end{aligned}$$

[23] However, one may take a hint (albeit quite vague) from Lemma 1.3.0.2 in the following construction.

Proof. 1. Follows from the definition of topology on the spectral space.
2. Let $Y_2 \subseteq X_2$ be closed so that $t(Y_2) \subseteq t(X_2)$ is closed. We wish to show that $(t(f))^{-1}(t(Y_2)) \subseteq t(X_1)$ is closed. This follows from the observation that for $Y_1 \in t(X_1)$, we have $\overline{f(Y_1)} \in t(Y_2) \iff Y_1 \in t(f^{-1}(Y_2))$.
3. Follows from 2.
4. Pick any closed $Y \subseteq X$ to thus obtain a closed $t(Y) \subseteq t(X)$. Then $\alpha^{-1}(t(Y)) = \{x \in X \mid \overline{\{x\}} \in t(Y)\} = \{x \in X \mid x \in Y\} = Y$. □

We now give scheme structure to the space $t(X)$. But first, we need a small lemma.

Lemma 1.5.4.4. *Let $A = k[V]$ be the coordinate ring of an affine k-variety V over an algebraically closed field k. Then, for any open set $U \subseteq \operatorname{Spec}(A)$, the set of all closed points of U are dense in U.*

Proof. Since all closed points of $\operatorname{Spec}(A)$ are its maximal ideals by Nullstellensatz, thus, any closed point of U is a maximal ideal of A as well. Consequently, we may assume $U = D(f)$ is a basic open set for $f \in A$. But since $D(f) \cong \operatorname{Spec}(A_f)$ and closed points of any affine scheme are always dense, the result follows. □

Theorem 1.5.4.5. *Let k be an algebraically closed field and $(V, \mathcal{O}_V)$ be a k-variety. Let $\alpha : V \to t(V)$ be the continuous map as defined in Lemma 1.5.4.3, 4. Then, $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme over k which admits an affine open cover by $\operatorname{Spec}(A)$ for $A = k[W]$ where W is an affine open subvariety of V.*

Proof. For better clarity of this important proof, we break it in multiple acts.

Act 1 : *We may assume V is an affine k-variety.*

Since we wish to show that $t(V)$ is a scheme, hence we need to produce an open cover of $t(V)$ by affine schemes. Since V is covered by open affine k-varieties, thus if we can show that for an affine k-variety W, the space $t(W)$ is a scheme, then we would be done. Hence we may assume V is affine with coordinate ring $k[V] =: A$.

Act 2 : *$t(V) \cong \operatorname{Spec}(A)$ as topological spaces.*

Consider the usual maps that we know from our study of varieties:

$$t(V) \underset{Z(-)}{\overset{I(-)}{\rightleftarrows}} \operatorname{Spec}(A)$$

These are easily seen to be continuous inverses of each other by the correspondence between closed irreducible subsets of an affine variety and prime ideals of its coordinate ring (Lemma 1.2.1.1).

Act 3 : *The closed points of $\operatorname{Spec}(A)$ are points of V.*

We first construct the following map

$$\begin{aligned} \varphi : V &\longrightarrow \operatorname{Spec}(A) \\ p &\longmapsto \mathfrak{m}_p \end{aligned}$$

where $\mathfrak{m}_p$ is defined together with some properties in Proposition 1.5.3.10. This is continuous by a small check on closed sets. Moreover, this is injective. Now, we claim that $\varphi(V) \subseteq \mathrm{Spec}\,(A)$ are all closed points of $\mathrm{Spec}\,(A)$. Indeed, this follows from the correspondence between closed points of $\mathrm{Spec}\,(A)$ and maximal prime ideals of A (Lemma 1.2.1.3). We will thus denote $\varphi(V)$ as the set of closed points of $\mathrm{Spec}\,(A)$.

Act 4 : *It is enough to show that* $\varphi_*\mathcal{O}_V \cong \mathcal{O}_{\mathrm{Spec}(A)}$.

Since we have the following commutative triangle

$$\begin{tikzcd} & V \arrow[dl, "\varphi"'] \arrow[dr, "\alpha"] & \\ \mathrm{Spec}\,(A) \arrow[rr, shift left, "Z(-)"] \arrow[rr, phantom, "\cong"] & & t(V) \arrow[ll, shift left, "I(-)"] \end{tikzcd},$$

thus $\alpha_*\mathcal{O}_V \cong (Z \circ \varphi)_*\mathcal{O}_V = Z_*\varphi_*\mathcal{O}_V$. Since Z is an isomorphism, thus the reduction is justified.

Act 5 : $\varphi_*\mathcal{O}_V \cong \mathcal{O}_{\mathrm{Spec}(A)}$.

Let $U \subseteq \mathrm{Spec}\,(A)$ be an open set. We will construct an isomorphism between $\mathcal{O}_{\mathrm{Spec}(A)}(U)$ and $\mathcal{O}_V(\varphi^{-1}(U))$. Consider the map

$$\begin{aligned} \eta_U : \mathcal{O}_{\mathrm{Spec}(A)}(U) &\longrightarrow \mathcal{O}_V(\varphi^{-1}(U)) \\ s : U \to \amalg_{\mathfrak{p}\in U} A_{\mathfrak{p}} &\longmapsto \eta_U(s) : \varphi^{-1}(U) \to k \end{aligned}$$

where for any $q \in \varphi^{-1}(U)$, we define $\eta_U(s)(q) = s(\mathfrak{m}_q)(q)$. It clearly is a ring homomorphism which commutes with appropriate restriction maps. Thus, we need to show the following three statements in order to conclude.

1. $\eta_U(s)$ is regular,
2. η_U has zero kernel,
3. η_U is surjective.

In-fact, the above three statements are at the technical heart of the proof. The main driving force behind this is the density of closed points of open sets in $\mathrm{Spec}\,(A)$ (Lemma 1.5.4.4) and the identity principle of regular maps on a variety (Lemma 1.5.2.4).

Statement 1. is immediate as s is regular. For statement 2., suppose that $\eta_U(s) = 0$ over $\varphi^{-1}(U)$. Thus $s(\mathfrak{m}_q)(q) = f_q(q)/g_q(q) = 0$ for all $q \in \varphi^{-1}(U)$. Thus, $\eta_U(s)$ around q is represented by rational function f_q/g_q. By Lemma 1.5.2.4 on $\eta_U(s)$, we obtain that $f_q = 0$ for all $q \in \varphi^{-1}(U)$. Thus s is zero at all closed points of U, which are exactly $\varphi(\varphi^{-1}(U))$. But since closed points of U are dense by Lemma 1.5.4.4 and s is a locally constant function, hence $s = 0$.
Finally, to see statement 3., pick any $f \in \mathcal{O}_V(\varphi^{-1}(U))$ and notice that $W := \varphi(\varphi^{-1}(U))$ is a dense subset of U (set of all closed points, Lemma 1.5.4.4). Thus, it is enough to define a locally constant function s over W whose extension $\tilde{s}$ over U is such that $\eta_U(\tilde{s}) = f$. Indeed, consider

$$\begin{aligned} s : W &\longrightarrow \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}} \\ \mathfrak{m}_q &\longmapsto g_q/h_q \end{aligned}$$

where g_q/h_q is the rational function representing f at the point $q \in V$. Clearly, the extension $\tilde{s}$ is in $\mathcal{O}_{\mathrm{Spec}(A)}(U)$ and it is mapped by η_U to f.

Act 6 : $(t(V), \alpha_* \mathcal{O}_V)$ *is a scheme over* k.

Now let V be a k-variety. We wish to show that $t(V)$ is a scheme over $\mathrm{Spec}\,(k)$. Thus we need to produce a map $t(V) \to \mathrm{Spec}\,(k)$, which is equivalent to a map $k \to \Gamma(\alpha_* \mathcal{O}_V, t(V))$ via the Theorem 1.3.0.5. Since $\Gamma(\alpha_* \mathcal{O}_V, t(V)) = \Gamma(\mathcal{O}_V, V) = A$ via Proposition 1.5.3.10, 5, the result follows.

This completes the proof. □

Remark 1.5.4.6. Theorem 1.5.4.5 yields that the functor t restricts to the following

$$\begin{aligned} t : \mathbf{Var}_k &\longrightarrow \mathbf{Sch}_k \\ (V, \mathcal{O}_V) &\longmapsto (t(V), \alpha_* \mathcal{O}_V). \end{aligned}$$

We will now show that this is a fully-faithful embedding. In other words, any map of $t(V_1) \to t(V_2)$ as of schemes over k is equivalent to a map $V_1 \to V_2$ of k-varieties.

Let us begin with some elementary properties of the residue fields of the k-scheme $t(V)$ attached to a k-variety V.

Lemma 1.5.4.7. *Let k be an algebraically closed field and let V be a k-variety. A point $p \in t(V)$ is closed if and only if $\kappa(p) = k$.*

Proof. (L $\Rightarrow$ R) Since $p \in t(V)$ is closed and closed points of $t(V)$ are exactly points of V, therefore $p \in V \subseteq t(V)$. Consequently, for an affine k-variety $X \subseteq V$ containing p, we obtain the following by Proposition 1.5.3.10, 3 and Nullstellensatz:

$$\kappa(p) = \mathcal{O}_{t(V),p}/\mathfrak{m}_{t(V),p} \cong \mathcal{O}_{V,p}/\mathfrak{m}_{V,p} \cong \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} \cong k[X]_{\mathfrak{m}_p}/\mathfrak{m}_p k[X]_{\mathfrak{m}_p} \cong \left(k[X]/\mathfrak{m}_p k[X]\right)_0 \cong k.$$

(R $\Rightarrow$ L) By Theorem 1.5.4.5, we have that for some open affine k-variety $X \subseteq V$, $p \in \mathrm{Spec}\,(k)[X]$. Consequently, $\kappa(p) = \left(k[X]/pk[X]\right)_0 = k$ where p is treated as a prime ideal of $k[X]$. Consequently, we have that the domain $k[X]/pk[X] = k$ as we have inclusions $k \hookrightarrow k[X]/pk[X] \hookrightarrow \left(k[X]/pk[X]\right)_0$. Thus $p \lneq k[X]$ is maximal. □

Proposition 1.5.4.8. *Let k be an algebraically closed field. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathbf{Var}_k}(V_1, V_2) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(t(V_1), t(V_2)).$$

That is, the functor t is a fully-faithful embedding of k-varieties into schemes over k.

Proof. Exercise 2.15 of Hartshorne Chapter 2. □

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Chapter 1.

Let us now spell out all the properties that the scheme $t(V)$ satisfies for a k-variety V.

Proposition 1.5.4.9. *Let k be an algebraically closed field and V be a k-variety. Then, the scheme $t(V)$ over k is (for * properties, see Section 1.12)*

1. *integral,*
2. *noetherian,*
3. *finite type over k,*

4. *quasi-projective**,
5. *separated**.

Proof. 1. to 3. are immediate from the open covering by $\mathrm{Spec}\,(k[W])$ of $t(V)$ where $W \subseteq V$ is an open affine subvariety (Theorem 1.5.4.5). Consequently $t(V)$ is covered by spectrum of finite type k-algebras.
4. is also immediate as any k-variety is an open subset of an affine or a projective k-variety by definition. Since any affine k-variety can be seen as a projective k-variety, consequently, we have an open immersion of V into a closed subvariety of some projective space over k. This extends to an open immersion of $t(V)$ into a closed subscheme of $\mathbb{P}^n_k$.
5. Follows from 4. and Theorem 1.12.7.2. □

We now state an important rectification result which precisely shows what type of schemes are those which are in the image of functor t as in Remark 1.5.4.6.

Corollary 1.5.4.10. *Let k be an algebraically closed field. Then, the functor of Remark 1.5.4.6*

$$t : \mathbf{Var}_k \longrightarrow \mathbf{QPISch}_k$$

establishes an equivalence between varieties over k and quasi-projective integral schemes over k. Further, the image of projective varieties under this functor is exactly the projective integral schemes over k.

Proof. By Proposition 1.5.4.8, we reduce to showing that t lands into quasi-projective schemes and is essentially surjective. Indeed, for a k-variety V, the scheme $t(V)$ is quasi-projective by Proposition 1.5.4.9, 4. Now, to show essential surjection, we first observe that open subschemes of $t(V)$ is in one-to-one bijection with open subsets of V. Consequently, it would suffice to show that any projective integral k-scheme X is in the essential image of t. Indeed, let V denote the closed points of X as a closed subscheme of some $\mathbb{P}^n_k$. Consequently, as closed points of a finite type k-scheme is dense (Lemma 1.12.2.6), therefore V is irreducible (note we are using irreducibility of X here), thus a projective variety in $\mathbb{P}^n_k$. Now, $t(V)$ and X have same underlying space. As a subspace of $\mathbb{P}^n_k$, $t(V)$ and X have both have the structure of a reduced scheme over the common underlying space. By uniqueness of reduced induced closed subscheme structure on a closed subset, we have that $t(V) \cong X$ (see Section 1.6.3). □

We now redefine varieties as schemes and use them as such for the remainder of the sections.

Definition 1.5.4.11. (**Abstract and classical varieties**) Let k be an algebraically closed field. An *abstract variety* or simply a variety, is a separated, integral finite type k-scheme. Those varieties which are furthermore quasi-projective are exactly the varieties we defined earlier by Corollary 1.5.4.10. We will further call the notion of varieties we defined earlier in Definition 1.5.2.5 by referring to them as *classical varieties.*

1.6 Fundamental constructions on schemes

In this section, we would like to understand some of the basic constructions which one can perform with a collection of schemes.

1.6.1 Points of a scheme

Let X be a scheme. Pick any point $x \in X$. We then have the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Hence we have a projection map

$$\mathcal{O}_{X,x} \to \kappa(x).$$

Consider now an open affine $x \in \mathrm{Spec}\,(A) \subseteq X$. Consequently, we have $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathrm{Spec}(A),x} \cong A_x$. Thus, denoting the inclusion $j_x : \mathrm{Spec}\,(A_x) \hookrightarrow \mathrm{Spec}\,(A)$, we obtain the following composition:

$$i_x : \mathrm{Spec}\,(\kappa(x)) \to \mathrm{Spec}\,(\mathcal{O}_{X,x}) = \mathrm{Spec}\,(A_x) \overset{j_x}{\hookrightarrow} \mathrm{Spec}\,(A) \hookrightarrow X.$$

Remember that $\mathrm{Spec}\,(A_x)$ can be interpreted as the affine subset in $\mathrm{Spec}\,(A)$ which is "very close" to $x \in \mathrm{Spec}\,(A)$. The map j_x takes the singleton point in $\mathrm{Spec}\,(\kappa(x))$ to $x \in X$. This map is usually called the *canonical map of point* $x \in X$. The map on stalks that i_x yields is the natural projection $\mathcal{O}_{X,x} \to \kappa(x)$. This map is quite unique as it is universal amongst all those maps $\mathrm{Spec}\,(K) \to X$ which maps to x. Indeed, we have the following.

Lemma 1.6.1.1. *Let X be a scheme and let $x \in X$ be a point. If K is a field and $f : \mathrm{Spec}\,(K) \to X$ is a map, then*

1. *If $f(\star) = x$, then $\kappa(x) \hookrightarrow K$.*
2. *If $f(\star) = x$, then f factors via the canonical map i_x at point $x \in X$*

$$\begin{tikzcd} \mathrm{Spec}\,(\kappa(x)) \arrow[r, "i_x"] & X \\ \mathrm{Spec}\,(K) \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}.$$

3. $\mathrm{Hom}_{\mathbf{Sch}}\,(\mathrm{Spec}\,(K), X) \cong \{x \in X \mid \kappa(x) \hookrightarrow K\}$.

Proof. 1. At the stalk, we have a local ring homomorphism $\varphi : \mathcal{O}_{X,x} \to K$. Consequently, $\mathrm{Ker}\,(\varphi) = \mathfrak{m}_{X,x}$. It then follows that $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \hookrightarrow K$.
2. Clearly f factors as above as a continuous map. To check the commutativity of sheaf maps, we need only check at stalks (Theorem 27.3.0.7). This is straightforward, as we get on stalks the following commutative diagram:

$$\begin{tikzcd} \kappa(x) \arrow[d, hook] & \mathcal{O}_{X,x} \arrow[l] \arrow[dl, "f^\sharp_\star"] \\ K & \end{tikzcd}.$$

3. It suffices to show that a morphism $f : \mathrm{Spec}\,(K) \to X$ is equivalent to the data of a point $x \in X$ such that $\kappa(x) \hookrightarrow K$. By 1, one side is immediate. Now consider a point $x \in X$ and a field extension $\kappa(x) \hookrightarrow K$. We wish to construct a map $f : \mathrm{Spec}\,(K) \to X$ such that the above

data is obtained via the construction in 1 applied on f. Indeed, the map f on topological spaces is straightforward, $f(\star) = x$. On sheaves, it reduces to define a natural local ring homomorphism $\mathcal{O}_{X,x} \to K$. This is immediate, as we need only define this as $\mathcal{O}_{X,x} \to \kappa(x) \hookrightarrow K$. $\square$

The above lemma shows that defining a map $\mathrm{Spec}\,(K) \to X$ is equivalent to taking a point $x \in X$ such that K extends $\kappa(x)$. There is another similar important characterization of maps from $\mathrm{Spec}\left(\frac{k[x]}{x^2}\right)$ into X, which characterizes all rational points of X together with "direction" (that is, together with an element of the tangent space). We first define a rational point of a k-scheme. Recall that by Corollary 1.3.0.7, $\kappa(x)$ is a field extension of k. Further observe the definition of Zariski tangent space T_xX of a scheme as defined in Definition 1.11.1.10.

Definition 1.6.1.2. (**Rational points**) Let X be a k-scheme. Then a point $x \in X$ is said to be rational if $\kappa(x) = k$.

Let us denote $k[\epsilon] = k[x]/x^2$. The ring $k[\epsilon]$ is usually called the ring of *dual numbers*.

Proposition 1.6.1.3. [24] *Let X be a scheme over a field k. Then, we have a bijection*

$$\mathrm{Hom}_{\mathbf{Sch}/k}(\mathrm{Spec}\,(k[\epsilon]), X) \cong \{(x, \xi) \mid x \in X \text{ is a rational point } \& \ \xi \in T_xX\}$$

Proof. ($\Rightarrow$) Take a scheme homomorphism $f : \mathrm{Spec}\,(k[\epsilon]) \to X$. Note that we have a map

$$k[\epsilon] \to k[\epsilon]/\epsilon \cong k.$$

Consequently, we get a map $g : \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k[\epsilon])$ which by composing by f, we get

$$\mathrm{Spec}\,(k) \xrightarrow{g} \mathrm{Spec}\,(k[\epsilon]) \xrightarrow{f} X.$$

Observe that $\mathrm{Spec}\,(k[\epsilon])$ is a one point scheme, therefore $f(\mathrm{pt.}) = f \circ g(\mathrm{pt.}) =: x$. We wish to show that x is a rational point. By Lemma 1.6.1.1, 3, we have $\kappa(x) \hookrightarrow k$. But since X is a scheme over k, therefore $k \hookrightarrow \kappa(x)$. We further deduce from the fact that X is a k-scheme that we have a triangle

$$\begin{tikzcd} \kappa(x) \arrow[r, hook] & k \\ k \arrow[u, hook] \arrow[ur, "\cong"'] & \end{tikzcd}.$$

This shows that horizontal arrow above is an isomorphism. Thus, $\kappa(x) = k$. We now wish to obtain an element of T_xX.

At the point $x \in X$, we have a map $f : \mathrm{Spec}\,(k[\epsilon]) \to X$. This yields a map on stalks given by

$$\varphi : A \to k[\epsilon]$$

where $A = \mathcal{O}_{X,x}$ is the local ring at point $x \in X$ and φ is furthermore a local k-algebra homomorphism. Let $\mathfrak{m}$ be the maximal ideal of the local ring A. Then, $A/\mathfrak{m} = \kappa(x)$, which is equal to k as x is a rational point. Thus, A is a rational local k-algebra (Definition 23.1.2.17). It follows from Proposition 23.1.2.18 that φ is equivalent to an element of the tangent space $\xi \in TA$ and by definition, $TA = T_xX$. This completes the proof. $\square$

[24] Exercise II.2.8 of Hartshorne.

We now see that closed points of a finite-type k-scheme are those whose residue extension of k is algebraic.

Proposition 1.6.1.4. *Let X be a finite-type k-scheme. Then the following are equivalent:*
1. *$x \in X$ is a closed point.*
2. *$x \in X$ is such that $\kappa(x)/k$ is an algebraic (equivalently, finite).*

Proof. (1. $\Rightarrow$ 2.) Clearly, $\kappa(x)$ is a finitely type field extension of k. By essential Nullstellensatz, $\kappa(x)/k$ is algebraic.

(2. $\Rightarrow$ 1.) Pick an affine open $\mathrm{Spec}\,(A)$ containing x so that $\mathfrak{p} \in \mathrm{Spec}\,(A)$ corresponds to x. We wish to show that $\mathfrak{p}$ is maximal. As $\kappa(x) = Q(A/\mathfrak{p})$ and

$$k \hookrightarrow A \twoheadrightarrow A/\mathfrak{p} \hookrightarrow Q(A/\mathfrak{p}) = \kappa(x),$$

thus, as $\kappa(x)/k$ is algebra, we deduce that $\kappa(x)$ is integral over $A/\mathfrak{p}$. Let B be a finite type k-domain such that $Q(B)$ is integral over B. One can check by writing down the relevant polynomials that this implies for any element $b \in B$, the inverse $b^{-1} \in Q(B)$ is in B by integrality. Using this for $B = A/\mathfrak{p}$, we deduce that $A/\mathfrak{p}$ is a filed, so $\mathfrak{p}$ is maximal, as required. □

1.6.2 Gluing schemes & strongly local constructions

We now show how to obtain new schemes from old by the gluing construction. Indeed, the idea is simple, glue the underlying topological spaces of a certain collection of schemes and identifications and define a new structure sheaf over the resultant space which canonically makes it into a scheme. We will further see that there is a universal property that is satisfied by such a glue. We suggest that the reader make a diagram of *blobs* and draw the corresponding maps in order to see the naturality of the following.

Definition 1.6.2.1. (**Gluing datum**) A tuple of data $(I, \{X_i\}_{i\in I}, \{U_{ij}\}_{i,j\in I}, \{\varphi_{ij}\}_{i,j\in I})$ of an index set I, schemes X_i for each $i \in I$, open subschemes $U_{ij} \subseteq X_i$ for each $i, j \in I$ and scheme isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ for each $i, j \in I$ is a gluing datum if it satisfies the following:
1. $U_{ii} = X_i$ for all $i \in I$,
2. $\varphi_{ji} = \varphi_{ij}^{-1}$,
3. $\varphi_{ii} = \mathrm{id}_{U_{ii}} = \mathrm{id}_{X_i}$,
4. the cocycle condition,

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \text{ on } U_{ij} \cap U_{ik}\ \forall i, j, k \in I.$$

We then have that there is a unique glue of the above.

Proposition 1.6.2.2. *For a gluing datum $(I, \{X_i\}_{i\in I}, \{U_{ij}\}_{i,j\in I}, \{\varphi_{ij}\}_{i,j\in I})$ of schemes, there exists a unique scheme X with the following properties:*
1. *there exists an open embedding of schemes*

$$\phi_i : X_i \to X \text{ for each } i \in I,$$

2. *$\phi_j \circ \varphi_{ij} = \phi_i$ on U_{ij} for all $i, j \in I$,*

3. $X = \bigcup_{i \in I} \phi_i(X_i)$,

4. $\phi_i(X_i) \cap \phi_j(X_j) = \phi_i(U_{ij}) = \phi_j(U_{ji})$ *for all* $i, j \in I$.

Proof. The underlying space of X is obtained by gluing the underlying spaces of X_i in the usual manner;

$$X := \coprod_{i \in I} X_i / \sim$$

where $x_i \sim \varphi_{ij}(x_i)$ for all $x_i \in U_{ij}$ and $i, j \in I$. Let $\phi_i : X_i \to X$ be the canonical inclusion map. The topology is given on X via the quotient topology; $U \subseteq X$ is open if and only if $\phi_i^{-1}(U) \subseteq X_i$ is open for each $i \in I$. Then to define the sheaf $\mathcal{O}_X$, pick any open $U \subseteq X$ and define the sections over it as follows (let us write $\varphi_{ij} : \mathcal{O}_{U_{ij}} \overset{\cong}{\to} \mathcal{O}_{U_{ji}}$ as well):

$$\mathcal{O}_X(U) = \left\{ [(\phi_i^{-1}(U), s_i)] \mid \forall i,\ s_i \in \mathcal{O}_{X_i}(\phi_i^{-1}(U)) \text{ s.t. } \varphi_{ij}(\rho_{\phi_i^{-1}(U), \phi_i^{-1}(U) \cap U_{ij}}(s_i)) = \rho_{\phi_j^{-1}(U), \phi_j^{-1}(U) \cap U_{ji}}(s_j) \right\}.$$

By local nature, this is again a sheaf (also called the glued sheaf). Now, ϕ_i is an open embedding as for any open $U \subseteq X$, it follows that $\mathcal{O}_X(\varphi_i(X_i) \cap U) \cong \mathcal{O}_{X_i}(\phi_i^{-1}(U))$. Thus, X is a scheme as for each $x \in X$, $x \in \phi_i(X_i)$ which is a scheme. $\square$

The uniqueness of the glue follows from the fact the universal property that glued scheme satisfies. **TODO**. Gluing allows us to construct many non-affine schemes, like the projective n-scheme over k.

nice ex- schemes a gluing niqueness, .

A lot of times we have the situation that a certain construction on a ring A leads to a map $\varphi : A \to \tilde{A}$. Consequently, we obtain maps $f : \operatorname{Spec}\left(\tilde{A}\right) \to \operatorname{Spec}(A)$. If X is a scheme, then for each open affine $V_i = \operatorname{Spec}(A_i)$, we get a map $X_i \to V_i$ given by $\operatorname{Spec}\left(\tilde{A}_i\right) \to \operatorname{Spec}(A_i)$. Consequently, we are interested in the conditions that the construction $A \to \tilde{A}$ must satisfy so that X_i glue together to give a scheme $\tilde{X}$ which represents the construction globally.

Definition 1.6.2.3 (**Construction on rings**)**.** A construction on rings is a collection of maps $\{\varphi_A : A \to \tilde{A}\}$ one for each ring A such that for any isomorphism $\eta_{AB} : A \overset{\cong}{\to} B$, we have an isomorphism $\tilde{\eta}_{AB} : \tilde{A} \to \tilde{B}$ which is id if η is id, the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "\eta_{AB}"', "\cong"] & \tilde{A} \arrow[d, "\tilde{\eta}_{AB}"', "\cong"] \\ B \arrow[r, "\varphi_B"'] & \tilde{B} \end{tikzcd}$$

commutes and if $\eta_{BC} \circ \eta_{AB} = \eta_{AC}$, then $\tilde{\eta}_{BC} \circ \tilde{\eta}_{AB} = \tilde{\eta}_{AC}$. That is, constructions are functorial on isomorphisms.

Definition 1.6.2.4 (**Strongly local constructions**)**.** A construction on rings $\{\varphi_A : A \to \tilde{A}\}$ is said to be strongly local if it naturally commutes with localization. That is, for each $g \in A$ not in nilradical, there exists an isomorphism $\widetilde{A_g} \cong \tilde{A}_g$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi_A"'] & A_g \arrow[d, "(\varphi_A)_g"'] \arrow[dr, "\varphi_{A_g}"] & \\ \tilde{A} \arrow[r] & \tilde{A}_g \arrow[r, "\cong"'] & \widetilde{A_g} \end{tikzcd}$$

commutes where $(\varphi_A)_g : A_g \to \tilde{A}_g$ is the localization of map $\varphi_A : A \to \tilde{A}$ at the element $g \in A$ and the horizontal arrows of the square are localization maps.

Remark 1.6.2.5. Let $\eta : A_f \cong B_g$ be an isomorhism where $f \in A$ and $g \in B$. Then we get an isomorphism $\hat{\eta} : \tilde{A}_f \cong \tilde{B}_g$ as in the following commutative diagram:

$$\begin{tikzcd}
\tilde{A}_f \arrow[r, dashed, "\hat{\eta}"] & \tilde{B}_g \\
\widetilde{A_f} \arrow[u, "\cong"] \arrow[r, "\cong", "\tilde{\eta}"'] & \widetilde{B_g} \arrow[u, "\cong"'] \\
A_f \arrow[u, "\varphi_{A_f}"'] \arrow[r, "\cong", "\eta"'] \arrow[uu, bend left=60, "(\varphi_A)_f"] & B_g \arrow[u, "\varphi_{B_g}"] \arrow[uu, bend right=60, "(\varphi_B)_g"']
\end{tikzcd}.$$

Let X be a scheme. Our main goal is to show that strongly local constructions done on each affine open subset of X can be glued to give a scheme $\tilde{X}$ admitting a map $\tilde{X} \to X$.

We will achieve this in steps. We first translate strongly local property more geometrically.

Lemma 1.6.2.6. *Let $\{\varphi_A : A \to \tilde{A}\}$ be a strongly local construction on rings. For any ring A denote $\phi_A : \operatorname{Spec}\left(\tilde{A}\right) \to \operatorname{Spec}(A)$ to be the map corresponding to φ_A. Then, for any $f \in A$, the following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Spec}(A_f) & & \operatorname{Spec}\left(\tilde{A}_f\right) \arrow[ll, "\phi_A|_{\operatorname{Spec}\left(\tilde{A}_f\right)}"'] \arrow[dl, "\cong"] \\
& \operatorname{Spec}\left(\widetilde{A_f}\right) \arrow[ul, "\phi_{A_f}"] &
\end{tikzcd}.$$

Proof. This is the translation of Definition 1.6.2.3 in $\operatorname{Spec}(-)$ where localization amounts to restricting to the corresponding open subscheme. □

The following is an important observation which will help in checking the cocycle condition.

Lemma 1.6.2.7. *Let $\{\varphi_A : A \to \tilde{A}\}$ be a strongly local construction on rings and the following be a commutative triangle of isomorphisms*

$$\begin{tikzcd}
R_f \arrow[r] \arrow[dr] & S_g \arrow[d] \\
& T_h
\end{tikzcd}$$

for $f \in R$, $g \in S$ and $h \in T$. Then, the following triangle of isomorphisms as constructed in Remark 1.6.2.5 also commutes

$$\begin{tikzcd}
\tilde{R}_f \arrow[r] \arrow[dr] & \tilde{S}_g \arrow[d] \\
& \tilde{T}_h
\end{tikzcd}.$$

Proof. By definition of a construction, we get that the following triangle commutes

$$\begin{tikzcd} \widetilde{R_f} \arrow[r] \arrow[dr] & \widetilde{S_g} \arrow[d] \\ & \widetilde{T_h} \end{tikzcd} .$$

By the construction of isomorphism $\tilde{R}_f \to \tilde{S}_g$ and others as in Remark 1.6.2.5, we immediately get that the required triangle commutes. $\square$

Lemma 1.6.2.8. *Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ be two affine schemes. Let R be a ring with isomorphisms $A_f \cong R \cong B_g$ for some $f \in A$ and $g \in B$. Let $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction on rings. Then there are open immersions $\operatorname{Spec}\left(\tilde{R}\right) \hookrightarrow \operatorname{Spec}\left(\tilde{A}\right)$ and $\operatorname{Spec}\left(\tilde{R}\right) \hookrightarrow \operatorname{Spec}\left(\tilde{B}\right)$ so that the following commutes*

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\phi_A"'] & \operatorname{Spec}\left(\tilde{R}\right) \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\phi_R"'] & \operatorname{Spec}\left(\tilde{B}\right) \arrow[d, "\phi_B"] \\ \operatorname{Spec}(A) & \operatorname{Spec}(R) \arrow[l, hook'] \arrow[r, hook] & \operatorname{Spec}(B) \end{tikzcd} .$$

Proof. This follows from the following diagram

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\phi_A"'] & \operatorname{Spec}\left(\tilde{A}_f\right) \cong \operatorname{Spec}\left(\widetilde{A_f}\right) \arrow[l, hook'] \arrow[d, "\phi_A|_{\operatorname{Spec}\left(\tilde{A}_f\right)}"] & \operatorname{Spec}\left(\tilde{R}\right) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\phi_R"] & \operatorname{Spec}\left(\widetilde{B_g}\right) \cong \operatorname{Spec}\left(\tilde{B}_g\right) \arrow[r, hook] \arrow[d, "\phi_B|_{\operatorname{Spec}(\tilde{B}_g)}"] & \operatorname{Spec}\left(\tilde{B}\right) \arrow[d, "\phi_B"] \\ \operatorname{Spec}(A) & \operatorname{Spec}(A_f) \arrow[l, hook'] & \operatorname{Spec}(R) \arrow[l, "\cong"] \arrow[r, "\cong"'] & \operatorname{Spec}(B_g) \arrow[r, hook] & \operatorname{Spec}(B) \end{tikzcd}$$

the commutativity of which follows from Lemma 1.6.2.6 and the definition of a construction. $\square$

Let X be a scheme and $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ be two open affines. We can now glue $\operatorname{Spec}\left(\tilde{A}\right)$ and $\operatorname{Spec}\left(\tilde{B}\right)$ along the intersection $U \cap V$ as follows.

Proposition 1.6.2.9. *Let X be a scheme and $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ be two open affines. Let $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction on rings. Let $\phi_A : \tilde{U} = \operatorname{Spec}\left(\tilde{A}\right) \to \operatorname{Spec}(A)$ and $\phi_B : \tilde{V} = \operatorname{Spec}\left(\tilde{B}\right) \to \operatorname{Spec}(B)$ be the maps corresponding to φ_A and φ_B. Then, there exists an isomorphism of schemes*

$$\Theta : \phi_A^{-1}(U \cap V) \xrightarrow{\cong} \phi_B^{-1}(U \cap V)$$

such that the following commutes for any affine open $\operatorname{Spec}(R) \subseteq U \cap V$ which is basic in both U and V by the isomorphisms $A_f \cong R \cong B_g$ (see Lemma 1.4.4.3)

$$\begin{tikzcd} \phi_A^{-1}(U \cap V) \arrow[r, "\Theta", "\cong"'] & \phi_B^{-1}(U \cap V) \\ \operatorname{Spec}\left(\tilde{A}_f\right) \arrow[u, hook] \arrow[r, "\cong", "\Theta_f"'] & \operatorname{Spec}\left(\tilde{B}_g\right) \arrow[u, hook] \end{tikzcd}$$

where Θ_f is obtained from $\theta : A_f \cong B_g$ via $\tilde{}$ construction (Remark 1.6.2.5).

Proof. Cover $U \cap V$ by open affines which are basic in both U and V (Lemma 1.4.4.3) and write $U \cap V = \bigcup_{i \in I} \mathrm{Spec}\,(A_{f_i}) = \bigcup_{i\in I} \mathrm{Spec}\,(B_{g_i})$ where $f_i \in A$ and $g_i \in B$. Consequently we may write

$$\phi_A^{-1}(U \cap V) = \bigcup_{i\in I} \phi_A^{-1}(\mathrm{Spec}\,(A_{f_i})) = \bigcup_{i \in I} \mathrm{Spec}\left(\tilde{A}_{f_i}\right)$$

and thus similarly,

$$\phi_B^{-1}(U \cap V) = \bigcup_{i\in I} \mathrm{Spec}\left(\tilde{B}_{g_i}\right).$$

For each $i \in I$, Lemma 1.6.2.8 provides us with an isomorphism

$$\Theta_i : \mathrm{Spec}\left(\tilde{A}_{f_i}\right) \xrightarrow{\cong} \mathrm{Spec}\left(\tilde{B}_{g_i}\right) \hookrightarrow \tilde{V}.$$

We claim that Θ_i can be glued. Indeed, for $i \neq j$, we have $\mathrm{Spec}\left(\tilde{A}_{f_i}\right) \cap \mathrm{Spec}\left(\tilde{A}_{f_j}\right) = \mathrm{Spec}\left(\tilde{A}_{f_i f_j}\right)$, therefore we reduce to showing that Θ_i and Θ_j are equal when restricted to $\mathrm{Spec}\left(\tilde{A}_{f_i f_j}\right)$. Observe from Lemma 1.4.4.3 that for each $i \in I$, the isomorphism $A_{f_i} \cong B_{g_i}$ takes $f_i \mapsto g_i$. The above is now equivalent to showing that the isomorphisms $\theta_i : \tilde{A}_{f_i} \cong \tilde{B}_{g_i}$ and $\theta_j : \tilde{A}_{f_j} \cong \tilde{B}_{g_j}$ obtained from $A_{f_i} \cong B_{g_i}$ and $A_{f_j} \cong B_{g_j}$ fit in the following commutative diagram

$$\begin{tikzcd}
\tilde{A}_{f_i f_j} \arrow[r, "(\theta_i)_{f_j}"] \arrow[d, equal, "\mathrm{id}"'] & \tilde{B}_{g_i f_j} \arrow[d, equal, "\mathrm{id}"] \\
\tilde{A}_{f_j f_i} \arrow[r, "(\theta_j)_{f_i}"'] & \tilde{B}_{g_j f_i}
\end{tikzcd} .$$

But $\theta_i(f_j) = g_j$ and $\theta_j(f_i) = g_i$, as mentioned above. Therefore $\tilde{B}_{g_i f_j} = \tilde{B}_{g_i g_j} = \tilde{B}_{g_j g_i} = \tilde{B}_{g_j f_i}$ and the above square commutes, showing that Θ_i glues to give a map $\Theta : \phi_A^{-1}(U \cap V) \to \phi_B^{-1}(U \cap V)$, which is an isomorphism as locally it is an isomorphism (Proposition 1.3.1.6). $\square$

Using Proposition 1.6.2.9, we can now globalize a strongly local construction.

Theorem 1.6.2.10. *Let X be a scheme and $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction on rings. Then there exists a scheme $\alpha : \tilde{X} \to X$ such that for any affine open $\mathrm{Spec}\,(A) \hookrightarrow X$, the following square commutes*

$$\begin{tikzcd}
\mathrm{Spec}\left(\tilde{A}\right) \arrow[r, hook] \arrow[d, "\phi_A"'] & \tilde{X} \arrow[d, "\alpha"] \\
\mathrm{Spec}\,(A) \arrow[r, hook] & X
\end{tikzcd} .$$

Proof. We first construct $\tilde{X}$ by gluing each $\mathrm{Spec}\left(\tilde{A}\right)$. Indeed, let $\{V_i = \mathrm{Spec}\,(A_i)\}_{i\in I}$ be the collection of affine opens in X and let $\{\tilde{X}_i = \mathrm{Spec}\left(\tilde{A}_i\right)\}$ be the collection of corresponding $\tilde{-}$-constructions. Let $\phi_i : \tilde{X}_i \to V_i$ be the maps corresponding to φ_{A_i}.

For each $i \neq j \in I$ we wish to construct open subschemes $U_{ij} \subseteq \tilde{X}_i$ and isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ satisfying the gluing conditions of Definition 1.6.2.1. We let

$$U_{ij} = \phi_i^{-1}(V_i \cap V_j).$$

Then Proposition 1.6.2.9 provides us with an isomorphism

$$\varphi_{ij} : U_{ij} \xrightarrow{\cong} U_{ji}.$$

It is immediate that $U_{ii} = \tilde{X}_i$ and $\varphi_{ii} = \mathrm{id}_{U_{ii}}$. Moreover, $\varphi_{ji} = \varphi_{ij}^{-1}$ by construction. We now check the cocycle condition. Indeed, pick $i, j, k \in I$ and pick an open affine $\operatorname{Spec}(R) \subseteq V_i \cap V_j \cap V_k$ in X which is basic open in V_i, V_j and V_k (Lemma 1.4.4.3 such that we have isomorphisms $A_{i,f_i} \cong A_{j,f_j} \cong A_{k,f_k} \cong R$ so that the following triangle commutes

$$\begin{tikzcd} A_{i,f_i} \arrow[r, "\cong"] \arrow[dr, "\cong"'] & A_{j,f_j} \arrow[d, "\cong"] \\ & A_{k,f_k} \end{tikzcd} . \tag{$*$}$$

By taking inverse images under ϕ_i, it follows that $\operatorname{Spec}\left(\tilde{A}_{i,f_i}\right) \subseteq U_{ij} \cap U_{ik}$ is basic open in both $\tilde{X}_i$ and $\tilde{X}_j$. We wish to show that φ_{ik} restricted to $\operatorname{Spec}\left(\tilde{A}_{i,f_i}\right)$ is the composition $\varphi_{jk} \circ \varphi_{ij}$. By Proposition 1.6.2.9, we get that φ_{ik} on this open affine is an isomorphism to $\operatorname{Spec}\left(\tilde{A}_{k,f_k}\right)$ and φ_{ij} is an isomorphism to $\operatorname{Spec}\left(\tilde{A}_{j,f_j}\right)$. Consequently, we wish to show that the following triangle of isomorphisms commute

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}_{i,f_i}\right) \arrow[r, "\varphi_{ij}"] \arrow[dr, "\varphi_{ik}"'] & \operatorname{Spec}\left(\tilde{A}_{j,f_j}\right) \arrow[d, "\varphi_{jk}"] \\ & \operatorname{Spec}\left(\tilde{A}_{k,f_k}\right) \end{tikzcd} .$$

But these isomorphisms are obtained by the following isomorphisms on the localizations (Proposition 1.6.2.9):

$$\begin{tikzcd} \tilde{A}_{i,f_i} \arrow[r, "\cong"] \arrow[dr, "\cong"'] & \tilde{A}_{j,f_j} \arrow[d, "\cong"] \\ & \tilde{A}_{k,f_k} \end{tikzcd} .$$

Hence it suffices to show that the above triangle commutes. The Lemma 1.6.2.7 applied on $(*)$ yields the required commutativity. $\square$

Definition 1.6.2.11 ($\tilde{\ }$**-fication**)**.** Let $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction of rings and let X be a scheme. The scheme $\tilde{X} \to X$ obtained in Theorem 1.6.2.10 is called the $\tilde{\ }$-fication of X.

1.6.3 Reduced scheme of a scheme

For any scheme X, we can obtain a scheme with the same underlying space but with reduced structure sheaf. This procedure is called *reducing a scheme* to a reduced scheme.

Construction 1.6.3.1. Let X be a scheme. Consider the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)/\mathfrak{n}_U$ where $\mathfrak{n}_U$ is the nilradical of $\mathcal{O}_X(U)$ and denote this sheaf by $\mathcal{O}_X^{\text{red}}$. The pair $(X, \mathcal{O}_X^{\text{red}})$ will be called the *associated reduced scheme* of the scheme $(X, \mathcal{O}_X)$, usually denoted by X_{red}. Indeed, $(X, \mathcal{O}_X^{\text{red}})$ is a scheme as the following result shows.

Remark 1.6.3.2 (*Reducing a ring is a strongly local construction*)**.** It is easy to see that $A \to A/\mathfrak{n}$ for each ring A defines a strongly local construction on rings as in Definition 1.6.2.4. Consequently, by Theorem 1.6.2.10, we immediately get a scheme $\tilde{X}$ obtained by reducing each open affine by dividing by nilradical. Indeed, one checks that we get the same scheme as $(X, \mathcal{O}_X^{\text{red}})$. However, we still give a proof of $(X, \mathcal{O}_X^{\text{red}})$ being a scheme without appealing to Theorem 1.6.2.10.

Lemma 1.6.3.3. [25] *Let X be a scheme. Then,*

1. *the pair $(X, \mathcal{O}_X^{\text{red}})$ is a scheme,*
2. *there exists a map of schemes $\varphi : (X, \mathcal{O}_X^{\text{red}}) \to (X, \mathcal{O}_X)$ which is a homeomorphism on the spaces.*

Proof. 1. Let $(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$ be an open affine of X. We shall show that $(\operatorname{Spec}(A), \mathcal{O}^{\text{red}}_{\operatorname{Spec}(A)})$ is isomorphic to $(\operatorname{Spec}(A_{\text{red}}), \mathcal{O}_{\operatorname{Spec}(A_{\text{red}})})$. First, the isomorphism on spaces is straightforward as every prime ideal contains nilradical (nilradical is the intersection of all prime ideals, Lemma 23.1.2.9). We thus need to produce a sheaf morphism $\mathcal{O}^{\text{red}}_{\operatorname{Spec}(A)} \to \mathcal{O}_{\operatorname{Spec}(A_{\text{red}})}$ which is an isomorphism. Let us denote the presheaf $U \mapsto \mathcal{O}_X(U)/\mathfrak{n}_U$ by F. We first immediately reduce to showing the existence of a map $F \to \mathcal{O}_{\operatorname{Spec}(A_{\text{red}})}$ which is an isomorphism on basic open sets, as we then obtain a map of sheaves $\mathcal{O}^{\text{red}}_{\operatorname{Spec}(A)} \to \mathcal{O}_{\operatorname{Spec}(A_{\text{red}})}$ by the universal property of sheafification (Theorem 27.2.0.1) which is an isomorphism on stalks (Theorem 27.3.0.6, 4).

Since sheaves and sheaf morphisms are uniquely determined by defining them on a basis, thus we further reduce to defining a presheaf map $F \to \mathcal{O}_{\operatorname{Spec}(A_{\text{red}})}$ with above properties on a basis. Since $\operatorname{Spec}(A)$ has a canonical basis, namely, $\mathcal{B} = \{\operatorname{Spec}(A)_f\}_{f \in A}$, consequently one sees that isomorphism $A_f/\mathfrak{n}_f \cong (A/\mathfrak{n})_f$ can be naturally extended to a presheaf map $F \to \mathcal{O}_{\operatorname{Spec}(A_{\text{red}})}$, which is an isomorphism on the basis $\mathcal{B}$.

2. Consider the map $f : (X, \mathcal{O}_X^{\text{red}}) \to (X, \mathcal{O}_X)$ which is given by id_X on spaces but by the following quotient map $\mathcal{O}_X(U) \to \mathcal{O}_X(U)/\mathfrak{n}_U \to \mathcal{O}_X^{\text{red}}(U)$. □

There is a universal property of reduced schemes which says that a map out of a reduced scheme necessarily has to factor through the reduction of the codomain.

Proposition 1.6.3.4. [26] *Let $f : X \to Y$ be a map of schemes with X being a reduced scheme. Then there exists a unique map of schemes $g : X \to Y_{\text{red}}$ such that the triangle commutes:*

$$\begin{tikzcd} Y & Y_{\text{red}} \arrow[l, "\varphi"'] \\ X \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}.$$

[25] Exercise II.2.3.b of Hartshorne.

[26] Exercise II.2.3.c of Hartshorne.

Proof. The map g on spaces is immediate; it shoould be identical to f as φ is identity on spaces. The map $g^\flat$ on the other hand can be constructed as follows. First observe that if A and B are rings with B being reduced, then any ring map $\eta : A \to B$ extends to a unique map $\tilde{\eta} : A_{\text{red}} \to B$ given by $a + \mathfrak{n} \mapsto \eta(a)$ which makes the triangle commute:

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, two heads] & B \\ A_{\text{red}} \arrow[ur, "\tilde{\eta}"'] & \end{tikzcd}.$$

In our case, we therefore get a unique map $\tilde{f^\flat_U}$ as below for any $U \subseteq Y$, which further gives us the required unique map $g^\flat_U : \mathcal{O}^{\text{red}}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ which we need (by universality of sheafification, Theorem 27.2.0.1):

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r, "\varphi_U"] \arrow[rr, bend left, "f^\flat_U"] \arrow[dr, two heads] & \mathcal{O}^{\text{red}}_Y(U) \arrow[r, "=:g^\flat_U"] & \mathcal{O}_X(f^{-1}(U)) \\ & \mathcal{O}_Y(U)/\mathfrak{n}_U \arrow[u] \arrow[ur, "\tilde{f^\flat_U}"'] & \end{tikzcd}.$$

One can then easily check that g as given above makes the triangle commute. $\square$

For each closed set $Z \subseteq X$ of a scheme, we construct a unique closed reduced subscheme structure over it.

Construction 1.6.3.5 (*Reduced induced subscheme*)**.** Let X be a scheme and $Z \subseteq X$ be a closed set. We wish to define a natural scheme structure on the subspace Z. Indeed, if $X = \operatorname{Spec}(A)$ is affine and $Z \subseteq X$ is closed, then let $\mathfrak{a} = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$ so that $Z = V(\mathfrak{a})$. Then we define the reduced induced subscheme structure on Z as that of $\operatorname{Spec}(A/\mathfrak{a})$. Observe that $(Z, \mathcal{O}_{\operatorname{Spec}(A/\mathfrak{a})})$ is a reduced scheme as $\mathfrak{a} \supseteq \mathfrak{n}$ where $\mathfrak{n} \leq A$ is the nilradical.

For an arbitrary scheme X and a closed subset $Z \subseteq X$, we proceed as follows. Let $\{U_i\}_{i \in I}$ be the collection of all open affines in X. Consider the intersections $Z_i = U_i \cap Z$ for each $i \in I$. As $Z_i \subseteq U_i$ are closed subsets in an affine scheme U_i, so by definition they carry the reduced induced subscheme structure on Z_i. We claim that the sheaves on each Z_i can be glued. Indeed, by usual argument involving Lemma 1.4.4.3, we reduce to checking that if $U = \operatorname{Spec}(A)$ is an open affine, $V = D(f) \subseteq U$ a basic open subset, $\mathcal{R}_U$ and $\mathcal{R}_V$ denote the sheaves obtained by reduced induced subscheme structures on $Z \cap U$ and $Z \cap V$ respectively, then

$$(\mathcal{R}_U)_{|Z \cap V} \cong \mathcal{R}_V.$$

Let $\mathfrak{a} = \bigcap_{\mathfrak{p} \in Z \cap U} \mathfrak{p}$ which gives the required structure on $Z \cap U$. Similarly, we have $\mathfrak{b} = \bigcap_{\mathfrak{p} \in Z \cap V} \mathfrak{p}$. We claim that $\mathfrak{b} = \mathfrak{a}A_f$. This would establish the required isomorphism between $A/\mathfrak{a}$ and $A_f/\mathfrak{b}$. Indeed, by definition, it is clear that $\mathfrak{b} \supseteq \mathfrak{a}A_f$. Conversely, pick $x/f^n \in \mathfrak{a}A_f$ where $x \in \mathfrak{a}$. We wish to show that $x/f^n \in \mathfrak{b}$. Pick any prime ideal $\mathfrak{q} \in Z \cap V$. We wish to show that $x/f^n \in \mathfrak{q}$. As $x \in \mathfrak{a}$, therefore $x \in \mathfrak{p}$ for each $\mathfrak{p} \in Z \cap U$. Thus, for $\mathfrak{p} \in D(f)$, $x \in \mathfrak{p}$. As each $\mathfrak{q} \in Z \cap V$ comes from $\mathfrak{p} \in Z \cap D(f)$, therefore $x/1 \in \mathfrak{b}$ and thus $x/f^n \in \mathfrak{b}$.

This completes the gluing procedure, to yield a subscheme structure on Z which we call the reduced induced subscheme structure on Z.

We now show the universal property of the above construction.

Proposition 1.6.3.6 (Universal property of reduced induced subscheme)**.** ***TODO.***

1.6.4 Fiber product of schemes

One of the most important tool in scheme theory is that of fiber product of schemes. This is essential as this is exactly the right notion using which one can define intersection of subschemes, which is one of the fundamental goals of this book.

Existence of fiber products is equivalent to saying that the category of schemes **Sch** have all pullbacks. In particular, it is equivalent to saying that for any two S-schemes X and Y, their product in $\mathbf{Sch}/S$ exists, called the *fiber product* denoted $X \times_S Y$.

However, we need to be more explicit than this abstract definition; we have to show that $X \times_S Y$ actually exists. Since we know how pushouts are constructed in the category of rings, their tensor products, therefore we can define it for affine schemes without much effort using the functor $\mathrm{Spec}\,(-) : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ of Theorem 1.3.0.5.

Definition 1.6.4.1. (**Fiber product of affine schemes**) Let the following be a coCartesian[27] diagram of rings (or of R-algebras)

$$\begin{tikzcd} A \otimes_R B & B \arrow[l] \\ A \arrow[u] & R \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} .$$

Since the $\mathrm{Spec}\,(-) : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ of Theorem 1.3.0.5 is right adjoint to global sections, therefore it preserves all limits of $\mathbf{Ring}^{\mathrm{op}}$, and thus, takes the above pushout diagram of R-algebras to a pullback diagram of affine schemes over $\mathrm{Spec}\,(R)$:

$$\begin{tikzcd} \mathrm{Spec}\,(A \otimes_R B) \arrow[r] \arrow[d] & \mathrm{Spec}\,(B) \arrow[d, "\mathrm{Spec}(g)"] \\ \mathrm{Spec}\,(A) \arrow[r, "\mathrm{Spec}(f)"'] & \mathrm{Spec}\,(R) \end{tikzcd} .$$

We hence define $\mathrm{Spec}\,(A \otimes_R B)$ to be the fiber product of affine schemes $\mathrm{Spec}\,(A)$ and $\mathrm{Spec}\,(B)$ over $\mathrm{Spec}\,(R)$.

Definition 1.6.4.2 (**Fiber product of schemes**)**.** Fiber product of S-schemes X and$Y$ is an S-scheme $X \times_S Y$ such that for any other S-scheme Z with map $f : Z \to X$ and $g : Z \to Y$ over

[27] another name for pushout diagrams.

S, there exists a unique map $u : Z \to X \times_S Y$ such that the diagram commutes

$$\begin{tikzcd} Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd} .$$

The most important part in this construction is the description of the structure sheaf of $X \times_S Y$. We now show how to construct fiber products of arbitrary S-schemes. In the process, we give a rather explicit description of fiber products and its structure sheaf, which we may think of as an explicit definition of fiber product. We begin with the affine case. Recall the notion of compositum of fields in Definition **??**.

Proposition 1.6.4.3. *Let A, B be two R-algebras and let $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(R)$. Then, as a set, we have the following bijection*

$$X \times_S Y \cong \left\{ \begin{matrix} \text{Tuples } (\mathfrak{p}_A, \mathfrak{p}_B, L, \alpha, \beta) \text{ where } \mathfrak{p}_A \in X,\ \mathfrak{p}_B \in Y \\ \text{such that both have same inverse image } \mathfrak{p}_R \text{ in } S \\ \text{and } (L, \alpha, \beta) \text{ is the compositum of fields } \kappa(\mathfrak{p}_A) \\ \text{and } \kappa(\mathfrak{p}_B) \text{ over } \kappa(\mathfrak{p}_R). \end{matrix} \right\}$$

Proof. Pick any prime ideal $\mathfrak{p} \in X \times_S Y = \operatorname{Spec}(A \otimes_R B)$. We wish to construct the datum $(\mathfrak{p}_A, \mathfrak{p}_B, L, \alpha, \beta)$. **TODO**. □

We now construct the fiber product of two schemes. This is more of an exercise in gluing techniques rather than anything else, so is ommited.

Theorem 1.6.4.4. *Let X, Y be two S-schemes. The fiber product $X \times_S Y$ exists.* □

Remark 1.6.4.5. While working with fiber products, one of the most important tool is its universal property. Most of the results about fiber products rarely uses the point-set construction as laid out above, just like the construction of tensor product is rarely used. Consequently, one should/must prove results about fiber products only using universal properties.

We now portray some easy applications of the universal property of fiber products.

Lemma 1.6.4.6. *Let $f : X \to Y$ and $g : Z \to Y$ be scheme morphisms and $U \subseteq X$ be an open subscheme. If $p : X \times_Y Z \to X$ is the scheme over X obtained by base change under f, then $p^{-1}(U) \cong U \times_Y Z$.*

Proof. We claim that the open subscheme $p^{-1}(U)$ of $X \times_Y Z$ is isomorphic to $U \times_Y Z$ by showing that it satisfies the same universal property. Indeed, suppose we have the following diagram

$$\begin{tikzcd} T \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\ & p^{-1}(U) \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ & U \arrow[r, "f"'] & Y \end{tikzcd}$$

where $f \circ h = g \circ k$. By the universal property of fiber product $X \times_Y Z$, we get a unique map $\varphi : T \to X \times_Y Z$ such that $p \circ \varphi = h$ and $q \circ \varphi = k$. As $\mathrm{Im}\,(h) \subseteq U$, therefore $\mathrm{Im}\,(p \circ \varphi) \subseteq U$. Consequently, we have $\mathrm{Im}\,(\varphi) \subseteq p^{-1}(U)$, hence we may write $\varphi : T \to p^{-1}(U)$, where $p^{-1}(U)$ is an open subscheme of $X \times_Y Z$. Thus, we get a unique map $\varphi : T \to p^{-1}(U)$ which makes the above diagram a fiber product diagram, thus completing the proof. □

The following is an important technical result.

Lemma 1.6.4.7. *Let $X = \bigcup_\alpha U_\alpha$ be an open cover of the scheme X. Let $f : X \to Y$ and $g : Z \to Y$ be scheme morphisms. Then,*

$$X \times_Y Z \cong \bigcup_\alpha U_\alpha \times_Y Z.$$

Proof. Let $p : X \times_Y Z \to X$ be the fiber product scheme over X obtained by base change along f. Then,

$$p^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha p^{-1}(U_\alpha).$$

By Lemma 1.6.4.6, we see that $p^{-1}(U_\alpha) \cong U_\alpha \times_Y Z$. It follows that

$$X \times_Y Z = p^{-1}(X) = \bigcup_\alpha p^{-1}(U_\alpha) \cong \bigcup_\alpha U_\alpha \times_Y Z,$$

as needed. □

Lemma 1.6.4.8. *Let $f : X \to Y$ and $g : Z \to Y$ be scheme morphisms and $U \subseteq X$ be an open subscheme such that $f(U) \subseteq V$ for some open subscheme $V \subseteq Y$ and let $W = g^{-1}(V)$ be an open subscheme in Z. If $p : X \times_Y Z \to X$ is the fiber product over X obtained by base change along f, then $p^{-1}(U) \cong U \times_Y Z \cong U \times_V W$.*

Proof. The first isomorphism is the content of Lemma 1.6.4.6. The second isomorphism follows from the simple observation that $U \times_Y Z$ satisfies the same universal property as that of $U \times_V W$. □

We portray some pathologies of fiber product in the following examples.

Example 1.6.4.9. We show that fiber product of one point schemes may have more than one point(!) Indeed, consider the schemes $X = Y = \mathrm{Spec}\,(\mathbb{C})$ over $\mathrm{Spec}\,(\mathbb{R})$. Observe that $X \times_{\mathrm{Spec}(\mathbb{R})} Y \cong \mathrm{Spec}\,(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$. But since we have

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \frac{\mathbb{R}[x]}{x^2+1} \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong \frac{\mathbb{C}[x]}{x^2+1} \\ &\cong \mathbb{C} \times \mathbb{C} \end{aligned}$$

by Chinese remainder theorem. Consequently, $\mathrm{Spec}\,(\mathbb{C} \times \mathbb{C}) \cong \mathrm{Spec}\,(\mathbb{C}) \amalg \mathrm{Spec}\,(\mathbb{C})$ which has 2 points.

1.6.5 Applications of fiber product

We would now like to portray some of the applications of fiber products, especially in endowing the fibers of a morphism with a scheme structure.

Inverse image of a closed subscheme

TODO.

Fibers of a map

Keep in mind the Lemma 27.5.0.3 and the surrounding remarks about stalks of sheaves for the remainder of this discussion. Let $f : X \to Y$ be a map and $y \in Y$ be a point. We endow $f^{-1}(y) \hookrightarrow X$ with a scheme structure. Define the *fiber* of f at y to be the following fiber product:

$$X_y := X \times_Y \operatorname{Spec}(\kappa(y)).$$

We at times denote it by $X \times_Y y$. Note that by natural map onto second factor, X_y is a scheme over $\kappa(y)$.

We now show that fiber of a scheme morphism as defined above matches with the usual notion of fiber in the sense that both spaces are homeomorphic. We first do this for affine schemes.

Proposition 1.6.5.1. *Let $X = \operatorname{Spec}(S)$, $Y = \operatorname{Spec}(R)$ and $f : X \to Y$ be the map associated to a ring homomorphism $\varphi : R \to S$. Let $y = \mathfrak{p} \in Y$ be a prime ideal of R. Then, X_y is homeomorphic to the subspace $f^{-1}(y)$ of Y.*

Proof. We have that $X_y = \operatorname{Spec}(S \otimes_R \kappa(\mathfrak{p}))$, that is, the fiber of φ at prime ideal $\mathfrak{p}$ (Definition 23.5.1.4). We now calculate $S \otimes_R \kappa(\mathfrak{p})$. Indeed, we have

$$\begin{aligned} S \otimes_R \kappa(\mathfrak{p}) = S \otimes_R F(R/\mathfrak{p}) &\cong S \otimes_R (R/\mathfrak{p} \otimes_R R_{\mathfrak{p}}) \\ &\cong S/\mathfrak{p}S \otimes_R R_{\mathfrak{p}} \\ &\cong (S/\mathfrak{p}S)_{\varphi(R\setminus\mathfrak{p})}. \end{aligned}$$

It follows from Lemma 23.1.2.3 that $\operatorname{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ is exactly the subspace of X consisting of those primes $\mathfrak{q}$ such that $\mathfrak{q} \supseteq \varphi(\mathfrak{p})$ and does not intersects $\varphi(R \setminus \mathfrak{p})$. This is equivalent to saying that $\varphi^{-1}(\mathfrak{q}) \supseteq \mathfrak{p}$ and $\varphi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$, that is, $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, as needed. □

We now do the general case. The main idea is just to reduce to the affine case as above.

Lemma 1.6.5.2. *Let $f : X \to Y$ be a scheme morphism and $y \in Y$. Then, $f^{-1}(y)$ as a subspace of X is homeomorphic to X_y.*

Proof. Let $V = \operatorname{Spec}(B)$ be an open affine of Y containing y. Then, by definition of fiber products, we immediately see that $f^{-1}(V) \cong X \times_Y V$. Clearly, $f^{-1}(y) \subseteq f^{-1}(V)$. Cover $f^{-1}(V)$ by open affines $\{U_\alpha = \operatorname{Spec}(R_\alpha)\}$. By Proposition 1.6.5.1, we see that $f^{-1}(y) \cap U_\alpha \cong \operatorname{Spec}(R_\alpha \otimes_B \kappa(y)) =$

$U_\alpha \times_V \mathrm{Spec}\,(\kappa(y))$. Since

$$\begin{aligned} X_y = X \times_Y \mathrm{Spec}\,(\kappa(y)) &\cong f^{-1}(V) \times_V \mathrm{Spec}\,(\kappa(y)) \\ &= \left(\bigcup_\alpha U_\alpha\right) \times_V \mathrm{Spec}\,(\kappa(y)) \\ &\cong \bigcup_\alpha (U_\alpha \times_V \mathrm{Spec}\,(\kappa(y))) \\ &\cong \bigcup_\alpha f^{-1}(y) \cap U_\alpha \\ &= f^{-1}(y), \end{aligned}$$

as needed. □

Example 1.6.5.3. We calculate explicit fibers of a map at every point of a familiar map. *Write solution of Exercise 3.10 of Hartshorne Chapter 2, written in notebook.*

The fibers of $\mathrm{Spec}\,(\mathbb{Z}[x]) \to \mathrm{Spec}\,(\mathbb{Z})$

We know that $\mathrm{Spec}\,(\mathbb{Z})$ is the final object in the category of schemes **Sch**. We also know that there is the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$. This induces a map

$$\varphi : \mathrm{Spec}\,(\mathbb{Z}[x]) \longrightarrow \mathrm{Spec}\,(\mathbb{Z}).$$

Understanding the fibers of this map will allow us to understand the affine arithmetic surface $\mathrm{Spec}\,(\mathbb{Z})$ (as $\mathbb{Z}[x]$ is a 2-dimensional ring). Note that we can already understand $\mathrm{Spec}\,(\mathbb{Z}[x])$ by the results surrounding Gauss' lemma as done in Theorem 23.1.5.3, but the following is a more geometric way of understadning this.

Proposition 1.6.5.4. *The prime ideals of $\mathbb{Z}[x]$ can be categorized into following three types.*

1. *$\langle p\rangle$ where $p \in \mathbb{Z}$ is a prime,*
2. *$\langle f(x)\rangle$ where $f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial,*
3. *$\langle p, f(x)\rangle$ where $p \in \mathbb{Z}$ is a prime and $f(x) \in \mathbb{Z}[x]$ irreducible in $\mathbb{Z}[x]$ which remains irreducible in $\mathbb{Z}/p\mathbb{Z}$,*

Proof. We will prove this by analyzing the fibers of $f : \mathrm{Spec}\,(\mathbb{Z}[x]) \to \mathrm{Spec}\,(\mathbb{Z})$. Pick a prime $p \in \mathbb{Z}$ and denote $X = \mathrm{Spec}\,(\mathbb{Z}[x])$. The fiber $X_p = \mathrm{Spec}\,(\mathbb{Z}[x]) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}\,(\kappa(p))$. As $\kappa(p) = \mathbb{F}_p$, finite field with p elements, therefore we have that $X_p = \mathrm{Spec}\,(\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{F}_p) = \mathrm{Spec}\,(\mathbb{F}_p[x])$. Note that for same reasons we have $X_{\mathfrak{o}} = \mathrm{Spec}\,(\mathbb{Q}[x])$.

As fibers of f covers the whole scheme, it follows that any point in $\mathbb{Z}[x]$ looks like one of the following:

1. a prime ideal in $\mathbb{Q}[x]$,
2. a prime ideal in $\mathbb{F}_p[x]$.

Moreover, we have the following diagrams

$$\begin{tikzcd}
\mathrm{Spec}\left(\mathbb{F}_p[x]\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\left(\mathbb{Z}[x]\right) \arrow[d] \arrow[r, phantom, "\leftsquigarrow\!\!\rightsquigarrow"] & \mathbb{F}_p[x] & \mathbb{Z}[x] \arrow[l] \arrow[dl, phantom, "\ulcorner", very near start] \\
\mathrm{Spec}\left(\mathbb{F}_p\right) \arrow[r] & \mathrm{Spec}\left(\mathbb{Z}\right) & \mathbb{F}_p \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}.$$

$$\begin{tikzcd}
\mathrm{Spec}\left(\mathbb{Q}[x]\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\left(\mathbb{Z}[x]\right) \arrow[d] \arrow[r, phantom, "\leftsquigarrow\!\!\rightsquigarrow"] & \mathbb{Q}[x] & \mathbb{Z}[x] \arrow[l] \arrow[dl, phantom, "\ulcorner", very near start] \\
\mathrm{Spec}\left(\mathbb{Q}\right) \arrow[r] & \mathrm{Spec}\left(\mathbb{Z}\right) & \mathbb{Q} \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

Observe that $\mathbb{Z}[x] \to \mathbb{F}_p[x]$ is the mod-p map. Since every prime ideal of $\mathbb{Z}[x]$ now is a inverse image of a prime ideal by $\mathbb{Z}[x] \to \mathbb{F}_p[x]$ and $\mathbb{Z}[x] \to \mathbb{Q}[x]$, we get the desired result. □

Geometric properties

Cover geometric reducibility and etc etc from Hartshorne exercises.

1.6.6 Normal schemes and normalization

Do mainly Exercise 3.7, 3.8 of Chapter 2 of Hartshorne. Also do Exercise 3.17, 3.18 of Chapter 1 of Hartshorne.

We now study a class of schemes which globalizes the notion of integral closure from algebra (Definition 23.7.1.10). These will find its main use in arithmetic where normal domains fundamental.

Definition 1.6.6.1 (**Normal schemes**)**.** A scheme X is said to be normal if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain.

The following is immediate from local nature of normal domains (Proposition 23.7.2.10).

Lemma 1.6.6.2. *Let X be an integral scheme. Then the following are equivalent:*

1. *X is a normal scheme.*
2. *For all open affine* $\mathrm{Spec}\left(A\right) \subseteq X$*, the ring A is a normal domain.*

Proof. As X is integral, therefore for every open affine $\mathrm{Spec}\left(A\right)$ of X, A is a domain by Lemma 1.4.2.2. As X is normal iff $\mathcal{O}_{X,x}$ is a normal domain for all $x \in X$, the result follows from Proposition 23.7.2.10. □

The main result in normal schemes is that any integral scheme induces a unique normal scheme obtained by normalizing each open affine.

Theorem 1.6.6.3. [28] *Let X be an integral scheme. Then there exists a scheme $\tilde{X} \to X$ over X where $\tilde{X}$ is a normal integral scheme such that for any normal integral scheme Z and a dominant*

[28] Exercise II.3.8 of Hartshorne.

map $f : Z \to X$, there exists a unique map $\tilde{f} : Z \to \tilde{X}$ such that the following commutes

$$\begin{tikzcd} \tilde{X} \arrow[d] & Z \arrow[l, dashed, "\tilde{f}"'] \arrow[dl, "f"] \\ X & \end{tikzcd}.$$

The scheme $\tilde{X} \to X$ is called the normalization of X and is unique upto isomorphism.

We first see this for affine domains.

Lemma 1.6.6.4. *Let $X = \operatorname{Spec}(A)$ be an integral affine scheme and $Z = \operatorname{Spec}(B)$ be a normal integral affine scheme. Let $\tilde{X} = \operatorname{Spec}\left(\tilde{A}\right)$ be the normalization of X and denote the natural map $\pi : \tilde{X} \to X$. If $f : Z \to X$ is any dominant map, then there exists a map $\tilde{f} : Z \to \tilde{X}$ such that $\pi \circ \tilde{f} = f$.*

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\pi"'] & \operatorname{Spec}(B) \arrow[l, dashed, "\tilde{f}"'] \arrow[dl, "f"] \\ \operatorname{Spec}(A) & \end{tikzcd}.$$

Proof. Indeed, by Proposition 23.7.2.12, this follows immediately. □

Remark 1.6.6.5. By Remark 23.7.2.11, it follows that normalization is a strongly local property. Thus Theorem 1.6.6.3 holds.

Proof of Theorem 1.6.6.3. By Remark 23.7.2.11, it follows that normalization is a strongly local construction for domains. Let $A \hookrightarrow \tilde{A}$ be the normalization map for any domain A. Therefore by Theorem 1.6.2.10, we have a scheme $\alpha : \tilde{X} \to X$ such that for any open affine $\operatorname{Spec}(A) \hookrightarrow X$, the following diagram commutes

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d, "\alpha"] \\ \operatorname{Spec}(A) \arrow[r, hook] & X \end{tikzcd}$$

where the left vertical map is the map corresponding to normalization $A \hookrightarrow \tilde{A}$. This shows the construction of $\alpha : \tilde{X} \to X$.

Now let Z be an arbitrary normal integral scheme and $f : Z \to X$ be a dominant map. Pick any open affine $\operatorname{Spec}(A) \subseteq X$ and consider the non-empty (f is dominant) open subset $f^{-1}(\operatorname{Spec}(A))$. Write

$$f^{-1}(\operatorname{Spec}(A)) = \bigcup_{i \in I} \operatorname{Spec}(B_i)$$

where $\operatorname{Spec}(B_i) \subseteq Z$ are open affine. As Z is normal integral, therefore B_i are normal domains from Lemma 1.6.6.2. By restriction we thus have the map

$$f|_{\operatorname{Spec}(B_i)} : \operatorname{Spec}(B_i) \to \operatorname{Spec}(A)$$

for each $i \in I$. Observe that $\alpha^{-1}(\mathrm{Spec}\,(A)) \supseteq \mathrm{Spec}\left(\tilde{A}\right)$. By Lemma 1.6.6.4, it follows that we have a unique map $\tilde{f}_i : \mathrm{Spec}\,(B_i) \to \mathrm{Spec}\left(\tilde{A}\right)$ such that the following commutes

$$\begin{tikzcd} \mathrm{Spec}\left(\tilde{A}\right) \arrow[d, "\alpha|_{\mathrm{Spec}(\tilde{A})}"'] & \mathrm{Spec}\,(B_i) \arrow[l, dashed, "\tilde{f}_i"'] \arrow[dl, "f|_{\mathrm{Spec}(B_i)}"] \\ \mathrm{Spec}\,(A) & \end{tikzcd}.$$

It thus follows that for every open affine $\mathrm{Spec}\,(B_{ij}) \subseteq \mathrm{Spec}\,(B_i)$, we have a map $\tilde{f}_i : \mathrm{Spec}\,(B_i) \to \mathrm{Spec}\left(\tilde{A}\right)$ by restriction. Hence by Lemma 1.6.6.4, we have that this is unique. As $\mathrm{Spec}\,(A) \subseteq X$ is arbitrary open affine, therefore we have an open affine covering $\{\mathrm{Spec}\,(A_i)\}_{i\in I}$ of X which by inverse image gives an open affine covering $\{\mathrm{Spec}\,(B_{ij})\}$ of Z and a collection of open affines $\{\mathrm{Spec}\left(\tilde{A}_i\right)\}$ of $\tilde{X}$ such that for each i, we have a unique map $\tilde{f}_{ij} : \mathrm{Spec}\,(B_{ij}) \to \tilde{X}$ such that

$$\begin{tikzcd} \tilde{X} \arrow[d, "\alpha"'] & \mathrm{Spec}\left(\tilde{A}_i\right) \arrow[l, hook'] \arrow[d, "\alpha"'] & \mathrm{Spec}\,(B_{ij}) \arrow[l, dashed, "\tilde{f}_{ij}"'] \arrow[dl, "f"] \\ X & \mathrm{Spec}\,(A_i) \arrow[l, hook'] & \end{tikzcd}$$

commutes. We claim that $\tilde{f}_{ij}$ can be glued to a unique map $\tilde{f} : Z \to \tilde{X}$, which would complete the proof. First, for a fixed i, we glue $\tilde{f}_{ij}$ and $\tilde{f}_{il}$. Indeed, covering the intersection $\mathrm{Spec}\,(B_{ij}) \cap \mathrm{Spec}\,(B)_{il}$ by open affines $\mathrm{Spec}\,(C_p)$, we immediately by restriction get maps $\tilde{f}_{ij} : \mathrm{Spec}\,(C_p) \to \mathrm{Spec}\left(\tilde{A}_i\right)$ and $\tilde{f}_{il} : \mathrm{Spec}\,(C_p) \to \mathrm{Spec}\left(\tilde{A}_i\right)$ which are thus equal by uniqueness. Hence, for each i, we may glue the maps $\{\tilde{f}_{ij}\}_j$ to obtain a unique map $\tilde{f}_i : Z_i = f^{-1}(\mathrm{Spec}\,(A_i)) \to \mathrm{Spec}\left(\tilde{A}_i\right)$ as in

$$\begin{tikzcd} \mathrm{Spec}\left(\tilde{A}_i\right) \arrow[d, "\alpha"'] & Z_i \arrow[l, dashed, "\tilde{f}_i"'] \arrow[dl, "f"] \\ \mathrm{Spec}\,(A_i) & \end{tikzcd}.$$

We now wish to glue these $\tilde{f}_i$. To this end, pick an affine open $\mathrm{Spec}\,(C) \subseteq Z_i \cap Z_k = f^{-1}(\mathrm{Spec}\,(A_i) \cap \mathrm{Spec}\,(A_k))$ and observe $\alpha^{-1}(\mathrm{Spec}\,(A_i) \cap \mathrm{Spec}\,(A_k)) \supseteq \mathrm{Spec}\left(\tilde{A}_i\right) \cap \mathrm{Spec}\left(\tilde{A}_k\right)$. We thus have the following diagram

$$\begin{tikzcd} \mathrm{Spec}\left(\tilde{A}_i\right) \arrow[d, "\alpha"'] & \mathrm{Spec}\,(C) \arrow[l, "\tilde{f}_i"'] \arrow[r, "\tilde{f}_k"] \arrow[d, "f"] & \mathrm{Spec}\left(\tilde{A}_k\right) \arrow[d, "\alpha"] \\ \mathrm{Spec}\,(A_i) & \mathrm{Spec}\,(A_i) \cap \mathrm{Spec}\,(A_k) \arrow[l, hook'] \arrow[r, hook] & \mathrm{Spec}\,(A_k) \end{tikzcd}.$$

By Lemma 1.6.6.4, it then suffices to show that $\tilde{f}_i(\mathrm{Spec}\,(C)), \tilde{f}_k(\mathrm{Spec}\,(C)) \subseteq \mathrm{Spec}\left(\tilde{A}_i\right) \cap \mathrm{Spec}\left(\tilde{A}_k\right)$, as then uniqueness would imply $\tilde{f}_i$ and $\tilde{f}_k$ are equal over $\mathrm{Spec}\,(C)$. By symmetry, it suffices to show this for $\tilde{f}_i$. Since $\alpha \circ \tilde{f}_i(\operatorname{Spec}(C)) \subseteq \operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k)$, therefore $\tilde{f}_i(\operatorname{Spec}(C)) \subseteq \alpha^{-1}(\operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k)) \cap \operatorname{Spec}\left(\tilde{A}_i\right) \subseteq \operatorname{Spec}\left(\tilde{A}_i\right) \cap \operatorname{Spec}\left(\tilde{A}_k\right)$, as required. Hence $\tilde{f}_i$ can be glued to a unique map $\tilde{f} : Z \to \tilde{X}$, thus completing the proof. □

The following is the globalization of the fact that normalization of a finite type algebra is again a finite type algebra, over a field (Noether's Theorem **??**).

Corollary 1.6.6.6. *If X is a finite type integral scheme, then the the normalization $\tilde{X} \to X$ is a finite map.*

Proof. **TODO**. □

1.7 Dimension of schemes

Do from Vakil, Hartshorne Exercise 3.20, 3.21, 3.22 of Chapter 2.
The notion of dimension of a geometric object serves as an essential tool for any attempt at its understanding. Schemes are no different and we have a notion of dimension for them. However, we also have a notion of dimension of rings. This section explores how these two interrelates and thus facilitate understanding of geometry of schemes.

1.7.1 General properties

Before moving to schemes that we will encounter the most, let us first give a general review of the notion of dimension of topological spaces and some general properties of dimension of schemes. Recall that the dimension of a topological space is the supremum of the length of the strictly decreasing chains of finite length of closed irreducible subsets of the space. Further for a space X and a closed irreducible subset $Z \subseteq X$, the codimension of Z in X is defined to be the supremum of the length of strictly increasing chains of closed irreducible subsets starting from Z. For an arbitrary closed subset $Y \subseteq X$, we define $\text{codim}\,(Y, X) = \inf_{Z \subseteq Y} \text{codim}\,(Z, X)$ where Z varies over all closed irreducible subsets of Y. For any closed set $Y \subseteq X$, if $\dim X < \infty$, we always have $\text{codim}\,(Y, X) \leq \dim X$.

Proposition 1.7.1.1. *Let X be a topological space. Then,*

1. *If $Y \subseteq X$ is a subspace, then $\dim Y \leq \dim X$.*
2. *If $\{U_i\}_{i\in I}$ is an open covering of X, then $\dim X = \sup_i \dim U_i$.*
3. *Let $Y \subseteq X$ be a closed subspace and X be of finite dimension. If X is irreducible and $\dim Y = \dim X$, then $Y = X$.*

Proof. The main tool in all of them is just a clear understanding of the definition of dimension and of closed irreducible sets. We establish some terminologies to work with in this proof. For any space X a strictly decreasing chain of finite length of closed irreducible subsets will be called a *finite chain* of X and set of all finite chains will be denoted by $FC(X)$. We denote a chain by $Z_\bullet \in FC(X)$ and its length by $l(Z_\bullet)$. Consequently, $\dim X = \sup_{Z_\bullet \in FC(X)} l(Z_\bullet)$.

1. First observe that if Y is closed then the result is immediate as any finite chain of Y will be a finite chain of X. Consequently, we reduce to showing that $\dim Y \leq \dim \bar{Y}$. In particular, we reduce to showing that if Y is dense in X, then $\dim Y \leq \dim X$. It further suffices to show existence of a length preserving map $FC(Y) \to FC(X)$. Indeed, for any $Z_\bullet \in FC(Y)$, one observes that $\text{Cl}_X(Z_i)$ is a closed subset of X which is further irreducible in X. Consequently, $\text{Cl}(Z_\bullet)$ is a finite chain of X of same length as of $Z_\bullet$[29].
2. By 1. we already have $\sup_i \dim U_i \leq \dim X$ so we need only show that $\dim X \leq \sup_i \dim U_i$. It suffices to show that for each $Z_\bullet \in FC(X)$, there exists $i \in I$ and $W_\bullet \in FC(U_i)$ such that $l(Z_\bullet) \leq l(W_\bullet)$. Let $r = l(Z_\bullet)$ and $i \in I$ be such that $U_i \cap Z_r \neq \emptyset$. Then, $W_\bullet = U_i \cap Z_\bullet$ forms a finite chain of U_i of same length as $Z_\bullet$. To see this, observe that if $U_i \cap Z_a = U_i \cap Z_b$ where we may assume $Z_a \supsetneq Z_b$, then the open set $U_i \cap Z_a$ of Z_a is contained in the closed set Z_b of Z_a, hence the closure of $U_i \cap Z_a$ in Z_a is inside Z_b. But since Z_a is irreducible so $U_i \cap Z_a$ must be dense in Z_a, a contradiction.

[29] Actually we didn't needed the reduction to Y being dense in X.

3. Let $r = \dim X = \dim Y$. Suppose $Y \subsetneq X$. Let $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ be a maximal finite chain of Y. Then the chain $X \supsetneq Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ is a finite chain in X as Y is closed. Thus $\dim X \geq 1 + r$, therefore $r = \dim Y \geq r + 1$, a contradiction.

$\square$

The following technical lemma was employed in proving the statement 2 of above, but is good to keep in handy.

Lemma 1.7.1.2. *Let X be a topological space and $Z_\bullet \in FC(X)$ a finite chain (in the terminology of Proposition 1.7.1.1) of length $l(Z_\bullet) = r$. If $U \subseteq X$ is an open set such that $U \cap Z_r \neq \emptyset$, then $U \cap Z_\bullet$ is a finite chain of length r in U.* $\square$

The following result gives a connection between all closed irreducible containing a given point and prime ideals of the local ring at that point.

Proposition 1.7.1.3. *Let X be a scheme and $x \in X$ be a point. We obtain an order reversing bijection*

$$\{\textit{Closed irreducibles } Y \textit{ of } X \textit{ containing } x\} \cong \operatorname{Spec}(\mathcal{O}_{X,x}).$$

Proof. Denote the collection of all closed irreducibles of X containing x as I. Let $U = \operatorname{Spec}(A)$ be an affine open containing $x \in X$ so that $\mathcal{O}_{X,x} \cong A_x$. Consequently, we wish to show a bijection $I \cong \operatorname{Spec}(A_x)$, which is further equivalent to showing that I is bijective to all prime ideals of A contained in x. As all prime ideals of A contained in x is further bijective to all closed irreducible of U containing x by Lemma 1.2.1.1, we thus reduce to showing existence of a bijection between I and closed irreducibles of U containing x, denoted J.

Consider the following function

$$\begin{aligned} \varphi : I &\longrightarrow J \\ Y &\longmapsto Y \cap U. \end{aligned}$$

Indeed, this map is well-defined as for any $Y \in I$, $\varphi(Y) = Y \cap U$ is first irreducible as any open subset of an irreducible set is irreducible. Further, it is closed in U as Y is closed. In order to show injectivity, we need only recall that any open subset of an irreducible set is dense. Finally, for surjectivity, take any $Z \in J$ so that Z is a closed irreducible in U containing x. Now let Y to be the closure of Z in X. We thus need only show that Y is irreducible in X. That follows immediately from the fact that closure of irreducible is again an irreducible, which in turn follows immediately from a simple observation on open subsets of the closure. $\square$

One then observes the following general result which will be used heavily in the future.

Lemma 1.7.1.4. *Let X be a scheme and Y be an irreducible closed subscheme of X with $\eta \in Y$ being its generic point. Then,*

$$\operatorname{codim}(Y, X) = \dim \mathcal{O}_{X,\eta}.$$

Proof. This is immediate from Proposition 1.7.1.3 as Y is the smallest closed irreducible containing η. $\square$

1.7.2 Dimension of finite type k-schemes

In this section, we prove various results surrounding the relationship between dimension of a given integral finite type k-scheme as a topological space and Krull dimension of various local rings.

Theorem 1.7.2.1. [30] *Let k be a field and X be a finite type integral k-scheme.*

1. *If $U, V \subseteq X$ are two open affines which are spectra of finite type k-domains, then* $\dim U = \dim V$.
2. *If $\{U_i\}_{i=1}^n$ is any finite open affine covering by spectra of finite type k-domains, then* $\dim U_i = \dim X$ *for all* $i = 1, \dots, n$.
3. *If $p \in X$ is a closed point, then* $\dim X = \dim \mathcal{O}_{X,p}$.
4. *Let $K(X)$ be the function field of X. Then* $\dim X = \operatorname{trdeg} K(X)/k$.
5. *If Y is a closed subset of X, then* $\operatorname{codim}(Y, X) = \inf_{p \in Y} \dim \mathcal{O}_{X,p}$.
6. *If Y is a closed subset of X, then* $\dim Y + \operatorname{codim}(Y, X) = \dim X$.

Proof. The main tools are the Theorems 23.8.2.1 and 23.8.2.2.

1. Observe that since X is irreducible, therefore U and V are dense open subsets of X, so $U \cap V \neq \emptyset$. Consequently, it will suffice to show that any dense affine open subset $W \subseteq U$ has same dimension as U. Indeed, U is spectra of finite type k-domain, so it is a separated finite type integral affine scheme, that is, an abstract affine variety. Consequently, by Proposition I.1.10 of cite[Hartshorne], $\dim W = \dim \overline{W} = \dim U$.
2. Follows from Proposition 1.7.1.1, 2 and statement 1.
3. As X is finite type, it admits a finite open affine covering by spectra of finite type k-domains. Let $U = \operatorname{Spec}(A)$ be one such open affine such that $p \in U$. Consequently, $p = \mathfrak{m} \in \operatorname{Spec}(A)$ represents a maximal ideal of R (Lemma 1.2.1.3). Thus, $\mathcal{O}_{X,p} \cong A_{\mathfrak{m}}$ and so $\dim \mathcal{O}_{X,p} = \dim A_{\mathfrak{m}}$. Note that A is a finite type k-algebra which is an integral domain. It thus follows by Theorem 23.8.2.2 that we have $\operatorname{ht} \mathfrak{m} + \dim A/\mathfrak{m} = \dim A$ and since $\dim A/\mathfrak{m} = 0$, therefore $\operatorname{ht} \mathfrak{m} = \dim A$. Further, since $\dim A_{\mathfrak{m}} = \operatorname{ht} \mathfrak{m}$, therefore we have $\dim \mathcal{O}_{X,p} = \dim A_{\mathfrak{m}} = \dim A = \dim U$. By statement 2, $\dim U = \dim X$ and the result follows.
4. Function field is defined to be the local ring at the generic point of X, say $\eta \in X$ (Remark 1.4.2.5). Let $\eta \in \operatorname{Spec}(A)$ where $\operatorname{Spec}(A)$ is a member of an open affine cover of X by spectra of finite type k-domains. Observe that $\operatorname{Spec}(A)$ has η as its generic point as well. Consequently, $\dim \operatorname{Spec}(A) = \dim A = \operatorname{trdeg} K(A)/k$ and since $K(A) = \mathcal{O}_{\operatorname{Spec}(A),\eta} \cong \mathcal{O}_{X,\eta} = K(X)$, therefore $\dim \operatorname{Spec}(A) = \operatorname{trdeg} K(X)/k$. By statement 2, $\dim \operatorname{Spec}(A) = \dim X$ and the result follows.
5. First observe that for any closed irreducible $Z \subseteq X$, we have $\operatorname{codim}(Z, X) \leq \dim X$. By statement 3, therefore, we have $\inf_{p \in Y} \dim \mathcal{O}_{X,p} = \inf_{p \in Y \text{ non-closed}} \mathcal{O}_{X,p}$. We will now show that for any closed irreducible subset $Z \subseteq X$ with $\eta \in Z$ its generic paint (schemes are *sober*[31]), we have $\dim \mathcal{O}_{X,\eta} = \operatorname{codim}(Z, X)$. By taking infimum, the result would then follow, so it would suffice to show the above claim.

   Let $\{\operatorname{Spec}(A_\alpha)\}$ be a finite open affine cover of X where A_α is a finite type k-domain. Observe that if $Z \cap \operatorname{Spec}(A_\alpha) \neq \emptyset$, then $\eta \in \operatorname{Spec}(A_\alpha)$. Now, $\eta \in \operatorname{Spec}(A_\alpha)$ is a point whose closure in $\operatorname{Spec}(A_\alpha)$ is $Z \cap \operatorname{Spec}(A_\alpha)$ so $Z \cap \operatorname{Spec}(A_\alpha)$ is a closed irreducible subspace of $\operatorname{Spec}(A_\alpha)$ whose generic point is η and thus $Z \cap \operatorname{Spec}(A_\alpha) \cong \operatorname{Spec}(A_\alpha/\eta)$, where we treat

[30] Exercise II.3.20 of Hartshorne.

[31] a space where all closed irreducibles have a unique generic point.

$\eta \lneq A_\alpha$ as a prime ideal of A_α. Consequently, $\dim \mathcal{O}_{X,\eta} = \dim \mathcal{O}_{\mathrm{Spec}(A_\alpha),\eta} = \dim(A_\alpha)_\eta = \mathrm{ht}\,\eta$. Since A_α is a finite type k-domain, therefore by Theorem 23.8.2.2, we obtain that $\mathrm{ht}\,\eta + \dim A_\alpha/\eta = \dim A_\alpha$, which thus yields $\mathrm{ht}\,\eta = \dim X - \dim A_\alpha/\eta$ by statement 2. It thus suffices to show that for some index α we get $\dim A_\alpha/\eta = \dim Z$ as then we would obtain $\dim \mathcal{O}_{X,\eta} = \dim X - \dim Z = \mathrm{codim}\,(Z, X)$.
Indeed, since $\{\mathrm{Spec}\,(A_\alpha/\eta)\}$ forms a finite open affine cover of Z, therefore by Proposition 1.7.1.1, 2 we get such an index α.

6. Observe that since $\mathrm{codim}\,(Y, X) < \infty$, therefore there exists a maximal closed irreducible $Z \subseteq Y$ such that $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(Z, X)$. Consequently, we have a finite chain of X, say $Z_\bullet$, ending at Z such that $l(Z_\bullet) = \mathrm{codim}\,(Y, X)$.
Let $U = \mathrm{Spec}\,(A)$ be an open affine where A is a finite type k-domain such that $U \cap Z \neq \emptyset$. Further, $\dim U \cap Y = \dim Y$. Consequently, by Lemma 1.7.1.2, we have $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(Z \cap U, U)$. Since $U \cap Y$ is a closed subscheme of U, therefore we may write $U \cap Y = \mathrm{Spec}\,(A/I)$ for an ideal $I \leq A$. Consequently, $\mathrm{codim}\,(Y, X) = \mathrm{codim}\,(\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A))$.
It is immediate from first definitions that

$$\begin{aligned}\mathrm{codim}\,(\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A)) &= \inf_{\mathfrak{p} \supseteq I} \mathrm{codim}\,(\mathrm{Spec}\,(A/\mathfrak{p}), \mathrm{Spec}\,(A)) \\ &= \inf_{\mathfrak{p} \supseteq I} \mathrm{ht}\,\mathfrak{p}.\end{aligned}$$

Now by Theorem 23.8.2.2 and above, we further obtain that

$$\begin{aligned}\mathrm{codim}\,(\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A)) &= \inf_{\mathfrak{p} \supseteq I} (\dim A - \dim A/\mathfrak{p}) \\ &= \dim A - \sup_{\mathfrak{p} \supseteq I} \dim A/\mathfrak{p} \\ &= \dim X - \dim U \cap Y \\ &= \dim X - \dim Y\end{aligned}$$

where $\dim A = \dim X$ because of statement 2.

□

Corollary 1.7.2.2. *Let X be a variety over a field k. Then* $\dim X < \infty$.

Proof. As X is a finite type integral k-scheme, therefore by Theorem 1.7.2.1, 3, $\dim X = \dim \mathcal{O}_{X,p}$ for any closed point $p \in X$. Fixing a closed point $p \in X$ in an open affine $\mathrm{Spec}\,(A)$ of X, we first deduce that A is a finite type k-domain. Let p be the maximal ideal $\mathfrak{m} \leq A$. Hence, $\mathcal{O}_{X,p} \cong A_\mathfrak{m}$. Hence $\dim \mathcal{O}_{X,p} = \mathrm{ht}\,\mathfrak{m}$ in ring A. By Theorem 23.8.2.2, $\mathrm{ht}\,\mathfrak{m} = \dim A - \dim A/\mathfrak{m} = \dim A$ as $A/\mathfrak{m}$ is a field. As A is a finite type k-domain, therefore its dimension is finite, as required. □

Corollary 1.7.2.3. *Let k be a field $\mathbb{A}^n_k$ be the affine n-space over k. Let H be a hyperplane in $\mathbb{A}^n_k$, that is $H = V(f)$ where $f \in k[x_1, \dots, x_n]$ is a linear polynomial. Then* $\dim H = n - 1$.

Proof. As $H = \mathrm{Spec}\,(A/\langle f \rangle)$, and $\langle f \rangle$ is a prime ideal as any linear polynomial is irreducible in $k[x_1, \dots, x_n]$ and since the latter is a UFD, therefore f prime as well. By Theorem 23.8.2.2, we have $\dim H = \dim k[x_1, \dots, x_n] - \mathrm{ht}\,\langle f \rangle = n - 1$, as required. □

1.7.3 Dimension of fibers

In this section, we discuss the question of how the dimension of fibers of a morphism varies. We'll see that certain nice geometric situations are encoded in the maps for which the dimension of fibers is not too erratic.

1.8 Projective schemes

The most important type of examples that we will encounter in our study of algebraic geometry are subvarieties of projective space $\mathbb{P}^n_k$. Indeed, this is a construction which is fundamental because of the many nice properties enjoyed by realizing familiar constructions in it. One of them being this classical observation that any two straight lines are bound to intersect at atleast one point in the projective space. We shall see more equally nice results, not to mention the quadrics with which we wish to spend some considerable time as the main motivating example for us (Example 1.5.1.3) is itself realized as a quadric in projective space.

We recall that the notion of projective varieties, whose generalization we shall embark now on, has been covered in Section 1.5.

We first begin by defining the space $\mathrm{Proj}(S)$ of a graded ring $S = \bigoplus_{d\geq 0} S_d$.

Definition 1.8.0.1. (**Projective spectrum of a graded ring**) Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring and let $S_+ = \bigoplus_{d>0} S_d$ be the ideal generated by non-zero degree elements. Denote

$$\mathrm{Proj}(S) := \{\mathfrak{p} \lneq S \mid \mathfrak{p} \text{ is homogeneous prime ideal } \& \ \mathfrak{p} \not\supseteq S_+\}.$$

The set $\mathrm{Proj}(S)$ is called the projective spectrum of the graded ring S.

Note that the latter condition is motivated by Remark 1.5.3.13. This is also used in a technical manner to show existence of a nice basis over $\mathrm{Proj}(S)$ in Lemma 1.8.1.3 and in other proofs as well. We now show that there is a natural topology over $\mathrm{Proj}(S)$, akin to the affine case.

Lemma 1.8.0.2. *Let S be a graded ring and denote for a homogeneous ideal $\mathfrak{a} \leq S$, the following subset of* $\mathrm{Proj}(S)$*:*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Then, for any homogeneous ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}_i$ of S, we obtain

1. $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$,
2. $\bigcap_i V(\mathfrak{a}_i) = V\left(\sum_i \mathfrak{a}_i\right)$.

Proof. Same as Lemma 1.2.0.1. □

We thus obtain a topological space $\mathrm{Proj}(S)$ where a set is closed if and only if it is of the form $V(\mathfrak{a})$ for a homogeneous ideal $\mathfrak{a} \leq S$. This is called the Zariski topology over the $\mathrm{Proj}(S)$.

We now give some more topological properties of $\mathrm{Proj}(S)$.

1.8.1 Topological properties of $\mathrm{Proj}(S)$

The first obvious question is how does the inclusion $\mathrm{Proj}(S) \hookrightarrow \mathrm{Spec}\,(S)$ looks topologically?

Lemma 1.8.1.1. *Let S be a graded ring. The topology of* $\mathrm{Proj}(S)$ *is obtained by subspace topology of* $\mathrm{Spec}\,(S)$*. Thus, there is a continuous inclusion*

$$\mathrm{Proj}(S) \hookrightarrow \mathrm{Spec}\,(S).$$

Proof. Immediate from definitions. □

We further note that for a graded ring S, the degree zero elements S_0 form a subring of S by the virtue of the fact that $S_d \cdot S_e \subseteq S_{d+e}$. Thus, we obtain a continuous map as the following shows.

Lemma 1.8.1.2. *Let S be a graded ring. Then the following is a continuous map*

$$\begin{aligned} \varphi : \mathrm{Proj}(S) &\longrightarrow \mathrm{Spec}\,(S_0) \\ \mathfrak{p} &\longmapsto \mathfrak{p} \cap S_0. \end{aligned}$$

Proof. Pick any ideal $\mathfrak{a} \leq S_0$ and notice that it is already homogeneous in S. Consequently, $\varphi^{-1}(V(\mathfrak{a})_a) = V(\mathfrak{a})_h$ where $V(\mathfrak{a})_a \subseteq \mathrm{Spec}\,(S_0)$ and $V(\mathfrak{a})_h \subseteq \mathrm{Proj}(S)$. □

We now find a collection of open sets which forms a basis for $\mathrm{Proj}(S)$. This is akin to Lemma 1.2.1.4.

Lemma 1.8.1.3. *Let S be a graded ring and $f, g \in S_d$ for some $d > 0$ be homogeneous elements. Denote*

$$D_+(f) := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

Then,

1. *$D_+(f)$ is an open subset of* $\mathrm{Proj}(S)$,
2. *$D_+(f) \cap D_+(g) = D_+(fg)$,*
3. *$\{D_+(f)\}_{f \in S_d, d>0}$ forms a basis of* $\mathrm{Proj}(S)$.

Proof. 1. Since $D_+(f) = \mathrm{Proj}(S) \setminus V(f)$, thus $D_+(f)$ is open.
2. Straightforward.
3. Since for any $\mathfrak{p} \in \mathrm{Proj}(S)$, there exists $f \in S_d$ for some $d > 0$ such that $f \notin \mathfrak{p}$ as $\mathfrak{p}$ does not contain all of S_+, thus $\bigcup_{f \in S_d, d>0} D_+(f) = \mathrm{Proj}(S)$. The rest follows by 2. □

Remark 1.8.1.4. As tempting as it might be to think, but not all projective schemes are quasi-compact. An example is given by the graded ring $S = \mathbb{Z}[x_1, x_2, \dots]$, the polynomial ring over $\mathbb{Z}$ with countably infinitely many indeterminates. Then one observes that $\mathrm{Proj}(S) = \bigcup_{n=1}^{\infty} D_+(x_n)$. Moreover, as for any $\mathfrak{p} \in \mathrm{Proj}(S)$ can not contain S_+, therefore $\mathfrak{p}$ necessarily has to not contain some x_i, otherwise it contains S_+. Consequently, we cannot form a finite subcover of the above cover, showing that $\mathrm{Proj}(S)$ is not quasi-compact.

However, the following lemma might be helpful in checking when a projective scheme has a finite cover by basic open sets.

Lemma 1.8.1.5. *Let S be a graded ring and consider $X = \mathrm{Proj}(S)$. Let $f = f_0 + \cdots + f_n$ be a decomposition of $f \in S$ into homogeneous elements $f_d \in S_d$. Then,*

$$D(f) \cap X = (D(f_0) \cap X) \cup \bigcup_{d=1}^{n} D_+(f_d)$$

where we view $X \subseteq \mathrm{Spec}\,(S)$ and $D(f), D(f_0) \subseteq \mathrm{Spec}\,(S)$.

Proof. This is a rather straightforward proof. To show ($\subseteq$), consider a point $\mathfrak{p} \in D(f) \cap X$ so that $f \notin \mathfrak{p}$. It follows from $f = f_0 + \cdots + f_n$ that for some $d = 0, \ldots, n$, $f_d \notin \mathfrak{p}$, which is in turn equivalent to stating that $\mathfrak{p} \in D_+(f_i)$ if $i \geq 1$ or $\mathfrak{p} \in D(f_0) \cap X$ if $d = 0$.

Conversely, pick $\mathfrak{p} \in (D(f_0) \cap X) \cup \bigcup_{d=1}^{n} D_+(f_d)$. We obtain that for some $d = 0, \ldots, n$, $f_d \notin \mathfrak{p}$. It follows from $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{p} \cap S_i$ that if $f \in \mathfrak{p}$, then we get by uniqueness of representatives of the direct sum that $f_d \in \mathfrak{p}$, a contradiction. □

1.8.2 The structure sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$ and projective schemes

We have studied some basic properties of the topological space $\mathrm{Proj}(S)$ so far, we now construct a structure sheaf over it and make it, first, into a locally ringed space and, second, into a scheme. We first define the structure sheaf of projective spectrum, in which there is nothing new in comparison to projective varieties (see Definition 1.5.3.1).

Definition 1.8.2.1. (**The structure sheaf** $\mathcal{O}_{\mathrm{Proj}(S)}$) Let S be a graded ring. Let $U \subseteq \mathrm{Proj}(S)$ be an open set of the projective spectrum of S. Define the following set

$$\mathcal{O}_{\mathrm{Proj}(S)}(U) := \left\{ s : U \to \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \mid \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})} \text{ \& } \exists \text{ open } \mathfrak{p} \in V \subseteq U \text{ \& } f, g \in S_d,\ d \geq 0 \text{ s.t. } \forall \mathfrak{q} \in V,\ g \notin \mathfrak{q} \text{ \& } s(\mathfrak{q}) = f/g \right\}.$$

From the fact that its elements are functions locally defined, one immediately obtains that $\mathcal{O}_{\mathrm{Proj}(S)}$ is a sheaf with obvious restriction maps. By appropriate restrictions on the domain, one further sees that under pointwise addition and multiplication, $\mathcal{O}_X(U)$ forms a commutative ring with 1.

Let us now show that $\mathrm{Proj}(S)$ is a scheme over $\mathrm{Spec}\,(S_0)$ in a natural manner.

Lemma 1.8.2.2. *Let S be a graded ring. Then $\mathrm{Proj}(S)$ is a scheme over $\mathrm{Spec}\,(S_0)$.*

Proof. We need only define a map $\mathrm{Proj}(S) \to \mathrm{Spec}\,(S_0)$. By Theorem 1.3.0.5, we need only construct a homomorphism $S_0 \to \Gamma(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. This is straightforward, as we can interpret each $a \in S_0$ as a homogeneous regular function $s : \mathrm{Proj}(S) \to \amalg_{\mathfrak{p} \in \mathrm{Proj}(S)} S_{(\mathfrak{p})}$ mapping as $\mathfrak{p} \mapsto a/1$. □

Thus, $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a ringed space. We now see that the stalk of this sheaf is isomorphic to the homogeneous localization. This will thus show that $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a locally ringed space (Lemma 23.2.1.1).

Lemma 1.8.2.3. *Let S be a graded ring and consider the ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. For each $\mathfrak{p} \in \mathrm{Proj}(S)$, we have*

$$\mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}} \cong S_{(\mathfrak{p})}.$$

Proof. Consider the following map

$$\begin{aligned} \varphi : \mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}} &\longrightarrow S_{(\mathfrak{p})} \\ (U, s)_{\mathfrak{p}} &\longmapsto s(\mathfrak{p}). \end{aligned}$$

It is straightforward to see that φ is a well-defined ring homomorphism. To see injectivity, suppose $(U, s)_{\mathfrak{p}} \mapsto 0$. Thus $s(\mathfrak{p}) = 0$. Consequently, for some open $V \subseteq U$ containing $\mathfrak{p}$ where s is given by f/g for $f, g \in S_d$, $d \geq 0$, we obtain $s(\mathfrak{q}) = f/g = 0$ for all $\mathfrak{q} \in V$. Thus $s = 0$ on V and hence $(U, s)_{\mathfrak{p}} = (V, \rho_{U,V}(s))_{\mathfrak{p}} = 0$. To see surjectivity, pick any $f/g \in S_{(\mathfrak{p})}$. Observe that $g \notin \mathfrak{p}$. Thus consider $(D_+(g), f/g)_{\mathfrak{p}} \in \mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}}$. □

We now show that the locally ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a scheme. For this purpose we would need to show that $\mathrm{Proj}(S)$ is covered by affine opens. Indeed, we have the following lemma.

Lemma 1.8.2.4. *Let S be a graded ring and consider the locally ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. For each $f \in S_d$, $d > 0$, we have the following isomorphism of locally ringed spaces*

$$\left(D_+(f), \mathcal{O}_{\mathrm{Proj}(S)|D_+(f)}\right) \cong \left(\mathrm{Spec}\left(S_{(f)}\right), \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)}\right).$$

Proof. Consider the map

$$\begin{aligned} \varphi : D_+(f) &\longrightarrow \mathrm{Spec}\left(S_{(f)}\right) \\ \mathfrak{p} &\longmapsto (\mathfrak{p} \cdot S_f)_0. \end{aligned}$$

By Lemma 23.2.1.4, it follows that φ is a bijection. To show that φ is an isomorphism it is sufficient to show that φ is a closed map. This is immediate as $\mathfrak{p} \supseteq \mathfrak{a}$ in $D_+(f)$ if and only if $(\mathfrak{p} \cdot S_f)_0 \supseteq (\mathfrak{a} \cdot S_f)_0$ in $\mathrm{Spec}\left(S_{(f)}\right)$.

We now wish to show isomorphism of corresponding sheaves. For this, we construct a map

$$\varphi^\flat : \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)} \longrightarrow \varphi_* \mathcal{O}_{D_+(f)}$$

and show that this is an isomorphism. Indeed, we first observe a canonical isomorphism on stalks

$$\mathcal{O}_{D_+(f),\mathfrak{p}} \cong S_{(\mathfrak{p})} \xrightarrow{\eta_\mathfrak{p}} \left(S_{(f)}\right)_{(\mathfrak{p}\cdot S_f)_0} \cong \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right),\varphi(\mathfrak{p})}.$$

Then one can construct the above isomorphism $\varphi^\flat$ by observing the following square for sections of the relevant sheaves over open $U \subseteq \mathrm{Spec}\left(S_{(f)}\right)$ and the corresponding $\varphi^{-1}(U) \subseteq D_+(f)$:

$$\begin{tikzcd} U \arrow[d, "s"'] & & \varphi^{-1}(U) \arrow[ll, "\varphi"', "\cong"] \arrow[d, "t"] \\ \coprod_{\mathfrak{p}\in\varphi^{-1}(U)}\left(\left(S_{(f)}\right)_{\varphi(\mathfrak{p})}\right) & & \coprod_{\mathfrak{p}\in\varphi^{-1}(U)} S_\mathfrak{p} \arrow[ll, "\amalg_\mathfrak{p}\eta_\mathfrak{p}"', "\cong"] \end{tikzcd},$$

where $s \in \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)}(U)$ and $t \in \mathcal{O}_{D_+(f)}(\varphi^{-1}(U))$ its image under $\varphi^\flat$ (which is defined by the above square). One can indeed check that $\varphi^\flat$ as defined is natural w.r.t restrictions. □

Remark 1.8.2.5. Thus, for a graded ring S, we obtain a scheme $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$, which is called the projective scheme associated to a graded ring S.

We now give some more properties of $\mathrm{Proj}(S)$.

Proposition 1.8.2.6. [32] *Let S be a graded ring. Then,*

1. $\mathrm{Proj}(S) = \emptyset$ *if and only if* $\forall s \in S_+$, *s is a nilpotent element of S.*

[32]Exercise II.2.14 of Hartshorne.

2. *Let $\varphi : S \to T$ be a graded map of graded rings. Then $U = \{\mathfrak{q} \in \operatorname{Proj}(T) \mid \mathfrak{q} \not\supseteq \varphi(S_+)\}$ is an open set and the natural map*

$$\begin{aligned} f : U &\longrightarrow \operatorname{Proj}(S) \\ \mathfrak{q} &\longmapsto \varphi^{-1}(\mathfrak{q}) \end{aligned}$$

defines a map of schemes.

3. *Let $\varphi : S \to T$ be a graded map of graded rings for which there exists $d_0 \in \mathbb{N}$ such that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$. Then, $U = \operatorname{Proj}(T)$ and $f : \operatorname{Proj}(T) \to \operatorname{Proj}(S)$ is an isomorphism.*

Proof. 1. The R $\implies$ L is immediate. Otherwise take an element $s \in S_d$. By Lemmas 1.8.1.3, 3 and 1.8.2.4, we obtain that $\operatorname{Spec}\left(S_{(s)}\right) = \emptyset$. Consequently, any prime ideal of $\operatorname{Spec}(S_s)$ has no zero degree terms, which can be seen to be not true. Consequently, $D(s) = \operatorname{Spec}(S_s) = \emptyset$. It follows from Lemma 1.2.2.7 that s is nilpotent.

2. The fact that U is open depends on φ being graded, i.e. $\varphi(S_d) \subseteq T_d$ for all $d \geq 0$. The continuity of f follows from the same observation. The map on sheaves is given by extending the natural map on stalks $\varphi_{(\mathfrak{q})} : S_{(\varphi^{-1}(\mathfrak{q}))} \to T_{(\mathfrak{q})}$, whose well-definedness, again, uses the fact that φ is graded.

3. The main trick here is to observe that if $s \in S_d$ for $d < d_0$, then raising some high enough power of s will make $s^n \in S_e$ where $\deg s^n \geq d_0$. For showing isomorphism on stalks $\varphi_{(\mathfrak{q})} : S_{(\varphi^{-1}(\mathfrak{q}))} \to T_{(\mathfrak{q})}$, it comes down to observing the following: let $s/t \in S_{(\varphi^{-1}(\mathfrak{q}))}$, then $s/t = st^n/t^{n+1}$ for any $n \in \mathbb{N}$. Then use the trick above. □

Remark 1.8.2.7. The above Proposition 1.8.2.6 shows that the mapping $S \mapsto \operatorname{Proj}(S)$ is NOT functorial! However the statement 3. might give some hint how to fix this.

Next, we understand all closed subschemes of $\operatorname{Proj}(S)$ in the following two results (Corollary 1.8.2.10)

Proposition 1.8.2.8. *Let S, T be a graded rings.*

1. *If $\varphi : S \to T$ is a surjective graded map, then the open set $U = \operatorname{Proj}(T)$ and $f : \operatorname{Proj}(T) \to \operatorname{Proj}(S)$ is a closed immersion (see Proposition 1.8.2.6, 2).*
2. *Let $I \leq S$ be a homogeneous ideal and consider the ideal $I' = \bigoplus_{d \geq d_0} I_d$. Then, I, I' defines the same closed subscheme of $\operatorname{Proj}(S)$.*
3. *Let $I \leq S$ be a homogeneous ideal and let $\pi : S \to S/I$ be the natural projection. Then the closed subscheme $f : \operatorname{Proj}(S/I) \to \operatorname{Proj}(S)$ (as in 1.) has the ideal sheaf given by $\widetilde{I} \leq \mathcal{O}_{\operatorname{Proj}(S)}$.*

Proof. 1. $U = \operatorname{Proj}(T)$ because $\varphi(S_+) = T_+$. The fact that f is a topological immersion follows from the observations that $f(\operatorname{Proj}(T)) = V(\operatorname{Ker}(\varphi))$ where $\operatorname{Ker}(\varphi)$ is homogeneous and that for any ideal $\mathfrak{q} \leq T$, it follows from surjectivity that $\varphi(\varphi^{-1}(\mathfrak{q})) = \mathfrak{q}$. To show surjectivity of sheaves, it reduces to showing surjectivity of localization maps $S_{\varphi^{-1}(\mathfrak{q})} \xrightarrow{\varphi(\mathfrak{q})} T_{(\mathfrak{q})}$, which is immediate from surjectivity of φ.

2. We wish to show an isomorphism as in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Proj}(S/I) \arrow[rr, "\cong", no head] \arrow[dr] & & \mathrm{Proj}(S/I') \arrow[dl] \\ & \mathrm{Proj}(S) & \end{tikzcd}.$$

Now since $(S/I)_d = S_d/I_d$ for all $d \geq 0$, therefore we have an isomorphism $\varphi : (S/I)_d \to (S/I')_d$ given by $s_d + I_d \mapsto s_d + I'_d$. The result follows from Proposition 1.8.2.6, 3.
3. We wish to show that Ker $\left(f^\flat : \mathcal{O}_{\mathrm{Proj}(S)} \to f_* \mathcal{O}_{\mathrm{Proj}(S/I)}\right)$ is given by $\widetilde{I}$. It suffices to check this on basic open sets $D_+(g)$, $g \in S_d, d > 0$, by uniqueness of the sheaf defined on a basis. Indeed it follows that $f^\flat$ on $D_+(g)$ is given by the localisation map $S_{(g)} \to S_{(g)}/I_{(g)}$, whose kernel is $I_{(g)} = \widetilde{I}(D_+(g))$. □

Proposition 1.8.2.9. *Let $S = A[x_0, \dots, x_r]$ for a ring A and let $X = \mathrm{Proj}(S)$.*

1. *Let $I \leq S$ be a homogeneous ideal and denote $\bar{I} = \{s \in S \mid \forall i = 0, \dots, r,\ \exists n_i \text{ s.t. } x_i^{n_i} s \in I\}$ to be the saturation of I. Then, $\bar{I}$ is homogeneous.*
2. *Let $I, J \leq S$ be two homogeneous ideals. Then $\mathrm{Proj}(S/I) \cong \mathrm{Proj}(S/J)$ if and only if $\bar{I} = \bar{J}$.*
3. *Let $Y \hookrightarrow \mathrm{Proj}(S)$ be a closed subscheme. Then, $\Gamma_*(\mathcal{I}_Y)$ is a saturated ideal of S.*

Proof. 1. This follows from a simple consideration of the uniqueness of homogeneous decomposition of each element in a graded ring.
2. We may reduce to showing that I and $\bar{I}$ defines the same closed subscheme. We already have $I \hookrightarrow \bar{I}$ which translates to $V(\bar{I}) \hookrightarrow V(I)$. Conversely, pick $\mathfrak{p} \in V(I) \subseteq \mathrm{Proj}(S)$. We wish to show $\mathfrak{p} \supseteq \bar{I}$. Pick any $s \in \bar{I}$. Assume that $s \notin \mathfrak{p}$. It then follows that $\mathfrak{p} = \langle x_0, \dots, x_r \rangle$ which is a prime ideal which contains S_+, thus $\mathfrak{p} \notin \mathrm{Proj}(S)$, a contradiction.
We then wish to show isomorphism of sheaves. Going to basic opens, this reduces to showing surjection is an injection:

$$\begin{aligned} (S/I)_{(f)} &\longrightarrow (S/\bar{I})_{(f)} \\ \frac{s+I}{f^n} &\longmapsto \frac{s+\bar{I}}{f^n}. \end{aligned}$$

This follows from the fact that $\bar{I}$ is saturated[33].
3. Pick a homogeneous element $s \in S_d$ such that for each $i = 0, \dots, r$, there exists $n_i \in \mathbb{N}$ such that $x_i^{n_i} s \in \Gamma(\mathcal{I}_Y(d+n_i), X)$. We wish to show that $s \in \Gamma_*(\mathcal{I}_Y)$. Note that $s \in \Gamma(\mathcal{O}_X(d), X)$. Cover X by $D_+(x_i)$ and consider the restrictions $x_i^{n_i} s \in \mathcal{I}_Y(d+n_i)(D_+(x_i))$. Multiplying (tensoring) $x_i^{n_i} s$ with $x_i^{-n_i} \in \mathcal{O}_X(-n_i)(D_+(x_i))$ yields $s \in \mathcal{I}_Y(d+n_i) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-n_i) \cong \mathcal{I}_Y(d)$ over $D_+(x_i)$. Thus, gluing these sections up from each $D_+(x_i)$, we get $s \in \Gamma(\mathcal{I}_Y(d), X) \subseteq \Gamma_*(\mathcal{I}_Y)$, as required. □

Using the above result, it is possible to find a characterization of closed subschemes of $\mathrm{Proj}(S)$ in terms of algebraic data.

Corollary 1.8.2.10. *Let $S = A[x_0, \dots, x_r]$ be a graded ring for a ring A. Then there is a correspondence:*

$$\{\textit{All closed subschemes of } \mathrm{Proj}(S)\} \quad \cong \quad \{\textit{All saturated ideals of } S\}\ .$$

[33] In-fact, this step shows exactly why the definition of saturation would've been made!

Proof. Follows from Proposition 1.8.2.9. □

Next, let us show how projective n-spaces over a ring changes with extension of scalars.

Definition 1.8.2.11. (**Projective n-space over a ring**) Let A be a ring. The projective n-space over A is defined to be $\mathbb{P}^n_A := \operatorname{Proj}(A[x_0, \dots, n])$. By Lemma 1.8.2.2, $\mathbb{P}^n_A$ is a scheme over $\operatorname{Spec}(A)$.

We now see how $\mathbb{P}^n_A$ behaves under extension of scalars.

Lemma 1.8.2.12. *Let $A \to B$ be a map of rings and $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the corresponding map of affine schemes. Then,*

$$\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B).$$

Proof. Observe that $D_+(x_i) \subseteq \mathbb{P}^n_A$ for $i = 0, \dots, n$ covers $\mathbb{P}^n_A$ as $A[x_0, \dots, x_n]$ is finitely generated by x_i as an A-algebra. By Lemma 1.6.4.7 together with Lemma 1.8.2.4 we obtain the following:

$$\begin{aligned}
\mathbb{P}^n_A \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B) &= \left(\bigcup_{i=0}^n D_+(x_i)\right) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B) \\
&\cong \bigcup_{i=0}^n D_+(x_i) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B) \\
&\cong \bigcup_{i=0}^n \operatorname{Spec}\left(A[x_0, \dots, x_n]_{(x_i)} \otimes_A B\right) \\
&\cong \bigcup_{i=0}^n \operatorname{Spec}\left(A[x_0/_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i] \otimes_A B\right) \\
&\cong \bigcup_{i=0}^n \operatorname{Spec}\left(B[x_0, \dots, x_n]_{(x_i)}\right) \\
&\cong \mathbb{P}^n_B.
\end{aligned}$$

□

Remark 1.8.2.13. Since any ring A is a $\mathbb{Z}$-algebra and $\mathbb{P}^n_A$ is naturally a $\mathbb{Z}$-scheme, therefore $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(A)$, where the projection map $\mathbb{P}^n_A \to \operatorname{Spec}(A)$ is the usual structure map. This further motivates the construction of a projective space over any scheme.

Definition 1.8.2.14. (**Projective n-space over a scheme**) Let X be a scheme. The projective n-space over X is defined to be

$$\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} X.$$

The natural projection map thus makes $\mathbb{P}^n_X$ a scheme over X.

1.8.3 Blowups

Do from Chapter 3 of Mumford and Hartshorne.

1.9 $\mathcal{O}_X$-modules

We will now cover certain types of important $\mathcal{O}_X$-modules that we will need in our study. Note that we defined $\mathcal{O}_X$-modules and various other algebraic constructions on them in Chapter 8, thus we assume the basic notion of $\mathcal{O}_X$-modules and its global algebra being known and we will thus specialize to the case of X being a scheme. The main goal is to define and study coherent and quasi-coherent modules over a scheme X. Its importance will manifest later in our study of projective schemes and their cohomology, the latter of which is an extremely powerful and versatile tool for doing geometry over schemes.

1.9.1 Coherent and quasi-coherent modules on schemes

Quasi-coherent sheaves form an integral part of the backbone of an attempt at doing geometry on schemes. Even though the definitions here makes sense in the setting of locally ringed spaces, but this theory is much more better behaved in the setting of schemes; for schemes, such sheaves have nice description on affine opens. This is the reason it is not included in Foundational Geometry, Chapter 8.

We first define the notion of quasicoherent modules on schemes.

Definition 1.9.1.1. (**Quasicoherent and coherent $\mathcal{O}_X$-modules**) Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is called quasicoherent if there exists an affine open cover $\{U_i := \operatorname{Spec}(R_i)\}_{i\in I}$ of X and $\{M_i\}_{i\in I}$ where M_i is an R_i-module such that $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$ for all $i \in I$. Further, $\mathcal{F}$ is said to be a coherent module if each M_i is a finitely generated R_i-module for each $i \in I$.

Remark 1.9.1.2. There are five basic properties of quasi-coherent sheaves on a scheme, which we point out now.

1. Quasicoherence of a module can be checked locally.
2. The global sections functor of a quasicoherent module over an affine scheme is exact[34].
3. The image of the functor $\widetilde{(-)} : \mathbf{Mod}(R) \to \mathbf{Mod}(\mathcal{O}_{\operatorname{Spec}(R)})$ (see Definition 1.2.3.1 and remarks surrounding it) is precisely all quasicoherent modules over $\operatorname{Spec}(R)$.
4. Quasicoherence is preserved under inverse image. It is further preserved under direct image if domain is a Noetherian scheme or if the map is quasi-compact and separated.
5. The category of all quasicoherent modules
$$\mathbf{QCoh}(\mathcal{O}_X)$$
is a Grothendieck-abelian category.

We will come to these results one by one. We first discuss some basic properties and examples.

Examples of quasicoherent modules

Lemma 1.9.1.3. *Let $X = \operatorname{Spec}(A)$ be an affine scheme and $\mathfrak{a} \leq A$ be an ideal. Consider the corresponding closed immersion*
$$i : \operatorname{Spec}(A/\mathfrak{a}) = Y \hookrightarrow \operatorname{Spec}(A) = X.$$

[34]This in cohomological language means that the first cohomology group $H^1(X, \mathcal{F}) = 0$, as we shall see after few sections.

Then,

1. *$i_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module,*
2. *$i_*\mathcal{O}_Y \cong \widetilde{A/\mathfrak{a}}$.*

Proof. 1. Consider the following

$$\varphi : \mathcal{O}_X \times i_*\mathcal{O}_Y \longrightarrow i_*\mathcal{O}_Y$$

which on a basic open $D(f) \subseteq X$ for $f \in A$ is given by

$$\varphi_{D(f)} : A_f \times (A/\mathfrak{a})_{\bar{f}} \to (A/\mathfrak{a})_{\bar{f}}$$

as the usual A_f-module structure over $(A/\mathfrak{a})_{\bar{f}}$. Indeed, as the above maps are natural w.r.t. restrictions, this suffices by StacksProject Lemma 009O. Thus, $i_*\mathcal{O}_Y$ is an $\mathcal{O}_X$-module. Further, $i_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module as the open cover $\{D(f)\}_{f\in A}$ of X is such that $i_*\mathcal{O}_Y(D(f)) \cong (A/\mathfrak{a})_{\bar{f}}$ is an $\mathcal{O}_X(D(f)) \cong A_f$-module generated by $\bar{1}$ (in the case when $f \in A$, we have $i_*\mathcal{O}_Y(D(f)) = 0$ and so trivially a finitely generated A_f-module).

2. Again, by the use of above mentioned lemma, we may reduce to working over a basis of X. Choosing $\{D(f)\}_{f\in A}$ to be such a basis, we see that $i_*\mathcal{O}_Y(D(f)) \cong (A/\mathfrak{a})_f$ and $\widetilde{A/\mathfrak{a}}(D(f)) \cong (A/\mathfrak{a})_f$. Hence, we may define a map $i_*\mathcal{O}_Y \to \widetilde{A/\mathfrak{a}}$ which on basic opens is identity. Consequently, this map on stalks is identity. This by above used lemma again yields a unique sheaf morphism $\varphi : i_*\mathcal{O}_Y \to \widetilde{A/\mathfrak{a}}$ which is an isomorphism as at stalks it is an isomorphism. $\square$

Example 1.9.1.4. Let X be an integral noetherian scheme and let K be its function field. Let $\mathcal{K}$ be the constant sheaf of field K over X. Then $\mathcal{K}$ is a quasi-coherent $\mathcal{O}_X$-module.

As X is noetherian, therefore let $X = \bigcup_{i=1}^n \operatorname{Spec}(A_i)$ where A_i are noetherian rings. As X is integral, therefore by Lemma 1.4.2.2, each A_i is a noetherian domain. Thus we deduce that $K \cong Q(A_i)$ for each i, where $Q(A_i)$ is the fraction field of A_i. This is because $\operatorname{Spec}(A_i)$ are open and X irreducible. We now show that $\mathcal{K}$ is an $\mathcal{O}_X$-module.

Pick any open $U \subseteq X$. Recall from Chapter 27 that a section of $\mathcal{K}(U)$ is a continuous map $U \to K$ with K in discrete topology. For any point $p \in \operatorname{Spec}(A_i)$, as A_i is a domain, we see that $(A_i)_p \hookrightarrow Q(A_i) \cong K$ for each i. We thus deduce that K is an $\mathcal{O}_{X,x}$-algebra for each $x \in X$ in a natural way and $\mathcal{O}_{X,x} \subseteq K$. So we may now define

$$\begin{aligned} \mathcal{O}_X(U) \times \mathcal{K}(U) &\to \mathcal{K}(U) \\ (c, s) &\mapsto c \cdot s \end{aligned}$$

where $c \cdot s : U \to K$ is defined by $c(x)s(x) \in K$, $c(x) \in \mathcal{O}_{X,x} \subseteq K$. This is continuous as each $c \in \mathcal{O}_X(U)$ is seen to be a continuous map $c : U \to \coprod_{x\in U} \mathcal{O}_{X,x} \subseteq K$ as it is locally constant (Remark 27.2.0.4 and that locally around each point we have an affine open inside every open). This is automatically compatible with restrictions. Consequently, $\mathcal{K}$ is an $\mathcal{O}_X$-module.

Next, to see this is quasi-coherent, we claim that the affine open cover $\{U_i = \operatorname{Spec}(A_i)\}_{i=1,\dots,n}$ is such that $\mathcal{K}_{|U_i}$ is isomorphic to $\widetilde{K}$. Consequently, we reduce to proving the following claim : Let $X = \operatorname{Spec}(A)$ be an affine scheme where A is a noetherian domain and let $K = Q(A)$ be its fraction field. Then, the constant sheaf $\mathcal{K}$ associated to K is isomorphic to the $\mathcal{O}_X$-module $\widetilde{K}$.

It suffices to construct a map $\varphi : \mathcal{K} \to \widetilde{K}$ defined only on a basis such that on basics it is an isomorphism. For this, we notice that since localization of a domain at an element is again a domain, therefore for each $g \in R$, the open $D(g) \subseteq X$ is connected. Hence, $\mathcal{K}(D(g)) = K$ and $\widetilde{K}(D(g)) \cong K_g = K$. Thus, we may define $\varphi_{D(g)} : K \to K$ to be identity which is then easily seen to be a sheaf morphism. Hence, these sheaves are isomorphic as $\mathcal{O}_X$-modules.

Example 1.9.1.5. We now discuss a specific example of quasi-coherent modules over $\mathrm{Spec}\,(\mathbb{Z})$, which brings to light the constraints put on by quasi-coherence on an $\mathcal{O}_X$-module. We ask the following question : *What are all quasicoherent skyscraper $\mathcal{O}_\mathbb{Z}$-modules over* $\mathrm{Spec}\,(\mathbb{Z})$ *supported at non-zero prime $p \in \mathbb{Z}$?* We claim that these are in bijection with all p^∞-torsion $\mathbb{Z}$-modules, that is, every element of the module is annihilated by some power of p:

$$\left\{\begin{matrix}\text{Quasicoherent} \quad \mathcal{O}_\mathbb{Z}\text{−modules} \quad \mathcal{F} \\ \text{skyscraper at } p \in \mathbb{Z}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Abelian groups } M \text{ which are } p^\infty\text{-} \\ \text{torsion.}\end{matrix}\right\}$$

Indeed, let $\mathcal{F}$ be a quasicoherent skyscraper module at prime $p \in \mathbb{Z}$. Let us invoke the Corollary 1.9.1.12, to conclude that $\mathcal{F} = \widetilde{M}$ for some $\mathbb{Z}$-module M. As it is skyscraper, therefore for any open $U \ni p$ in $\mathrm{Spec}\,(\mathbb{Z})$, we have $\mathcal{F}(U) = G$ where G is a fixed abelian group and $\mathcal{F}(U) = 0$ if $p \notin U$. Consequently, we have that $\mathcal{F}_x = 0$ if $x \neq p$ and $\mathcal{F}_p = G$. As $\mathcal{F} = \widetilde{M}$, therefore we have

$$\Gamma(\mathcal{F}, X) = G \cong M.$$

Further, for any basic open $D(f) \subseteq \mathrm{Spec}\,(\mathbb{Z})$ containing prime p, we deduce that $\mathcal{F}(D(f)) \cong M_f \cong G \cong M$. This, when unravelled, yields that for any integer $f \in \mathbb{Z}$ such that $f \notin \langle p \rangle \iff p \not| f$, we have $M_f \cong M$. Further, if $\langle p \rangle \notin D(f) \iff p|f$, then $M_f = 0$. Now fix any $m \in M$. We claim that some power of p annihilates m. Indeed, consider $D(p)$ which does not contain $\langle p \rangle$ as $p \in \langle p \rangle$. Thus, by above, we have that $\frac{m}{1} = 0$ in M_p. Consequently, for some $k \in \mathbb{N}$, we have $p^k m = 0$, as required. Hence, $T_{p^\infty}(M) = M$.

Conversely, consider a p^∞-torsion abelian group M. We wish to show that the quasicoherent module associated to M, $\widetilde{M}$, is skyscraper at $p \in \mathbb{Z}$. Let $D(f) \subseteq \mathrm{Spec}\,(\mathbb{Z})$ be a basic open not containing $\langle p \rangle$, equivalently, $p|f$. Then, we see that $\widetilde{M}(D(f)) \cong M_f$. Now pick any $\frac{m}{f^k} \in M_f$. Let $p^n m = 0$. Thus, $f^n m = 0$ as $p|f$. Consequently, we may write $\frac{m}{f^k} = \frac{1}{f^k}\frac{m}{1} = \frac{1}{f^k}\frac{f^n m}{f^n} = \frac{1}{f^k}\frac{0}{f^n} = 0$. Thus, $M_f = 0$.

Let $D(f)$ now be a basic open set which contains $\langle p \rangle$, equivalently, $p \not| f$. Then, $\widetilde{M}(D(f)) \cong M_f$ and we wish to show that $M_f \cong M$. Indeed, observe that since $p \not| f$, therefore $\gcd(p^k, f^l) = 1$ for all $k, l \geq 1$. It follows that there exists $a_k, b_l \in \mathbb{Z}$ such that

$$a_k p^k + b_l f^l = 1.$$

Thus, for any $\frac{m}{f^n} \in M_f$, where $p^k m = 0$, we obtain $a_k, b_n \in \mathbb{Z}$ such that $a_k p^k + b_n f^n = 1$. Using this on module M, we yield $a_k p^k m + b_n f^n m = m$, that is, $b_n f^n m = m$. Consequently, we may write

$$\frac{m}{f^n} = \frac{b_n f^n m}{f^n} = \frac{b_n m}{1}$$

in M_f. It follows that the localization map $\varphi : M \to M_f$ is surjective. We thus need only establish the injectivity of φ. Indeed, if $\varphi(m) = \frac{m}{1} = 0$ in M_f, then $f^n m = 0$ for some $n \in \mathbb{N}$. By above, we have $b_n \in \mathbb{Z}$ such that $b_n f^n m = m$. Consequently, $m = b_n f^n m = 0$, that is, $\mathrm{Ker}\,(\varphi) = 0$, as required. Thus, $\varphi : M \to M_f$ is the required isomorphism. This completes the proof.

Locality of quasicoherence

We now discuss some more results which would culminate in the proofs of statement 1 in Remark 1.9.1.2.

Lemma 1.9.1.6. *Let $\mathcal{F}$ be a quasicoherent module over an affine scheme $X = \mathrm{Spec}\,(R)$. Then X admits a finite open affine cover $\{D(g_i)\}_{i=1}^n$ such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$ where M_i is an R_{g_i}-module.*

Proof. Since $\mathcal{F}$ is quasicoherent, therefore there exists an open affine cover $\{U_i = \mathrm{Spec}\,(S_i)\}_i$ of X such that $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$ where M_i is an S_i-module. Since subsets of the form $D(g)$ forms a basis of X therefore for $D(g) \subseteq U_i$ we obtain via Lemma 1.2.3.4, 2, that $\mathcal{F}_{|D(g)} \cong \widetilde{R_g \otimes_{S_i} M_i}$ as $D(g) \cong \mathrm{Spec}\,(R_g)$. Since $N_i := R_g \otimes_{S_i} M_i$ is an R_g-module, so we have a cover of X by finitely many $D(g_i)$ by Lemma 1.2.1.6 such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{N_i}$ where N_i is an R_{g_i}-module. $\square$

Using the above, we first show a technical lemma, which will be generalized later on, which will be used to show locality of quasi-coherent modules[35].

Lemma 1.9.1.7. *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme and $\mathcal{F} \in \mathbf{QCoh}(X)$ be a quasi-coherent module. Let $D(f) \subseteq X$ be a basic open set for some $f \in A$.*

1. *If $s \in \Gamma(\mathcal{F}, X)$ is a global section of the module $\mathcal{F}$ such that s restricted on $D(f)$ is 0, then there exists $n > 0$ such that $f^n s = 0$ over X.*
2. *If $t \in \mathcal{F}(D(f))$, then there exists $n > 0$ such that $f^n t \in \mathcal{F}(D(f))$ extends to a global section of the module $\mathcal{F}$.*

Proof. 1. By Lemma 1.9.1.6, there exists a finite open cover $D(g_i)$ of X such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$. Denoting the restriction of s to $D(g_i)$ as $s_i \in M_i$, we see that the image of s_i is zero in $(M_i)_f$ when restricted to $D(fg_i) = D(f) \cap D(g_i)$. Consequently, for some $n_i > 0$, we have $f^{n_i} s_i = 0$ over $D(g_i)$. As g_i are finitely many, taking large enough n, we obtain $f^n s_i = 0$ over each $D(g_i)$. It follows that the global section $f^n s$ of the module $\mathcal{F}$ is such that it's restriction to each open set of an open cover of X is 0. By sheaf axioms, it follows that $f^n s = 0$ over X.

2. Fix the finite open affine cover $\{D(g_i)\}_{i=1}^n$ of X coming from Lemma 1.9.1.6. Consider all the finitely many intersections $D(g) \cap D(g_i) = D(fg_i)$. Restricting t from $D(f)$ to $D(fg_i)$, we obtain $t_i \in (M_i)_f$ for each i. Hence, for each i, there is some $n_i > 0$ such that $f^{n_i} t_i \in M_i = \mathcal{F}(D(g_i))$. By multiplying by large f^k to each $f^{n_i} t_i$ which are finitely many, we may arrange that $f^n t_i \in \mathcal{F}(D(g_i))$.

We now form a matching family for the module $\mathcal{F}$ over the open cover $\{D(g_i)\}$ which would glue up to give the required global section. Indeed, fix two $D(g_i)$ and $D(g_j)$. Restrict $f^n t_i$ and $f^n t_j$ to $D(g_i) \cap D(g_j) = D(g_i g_j)$. Observe that over the even smaller open $D(fg_i g_j)$, the section $f^n t_i - f^n t_j$ is zero as $t_i = t_j = t$ over $D(fg_i g_j) \subseteq D(f)$. Hence by item 1 applied over $D(g_i g_j)$, there exists $m_{ij} > 0$ such that $f^{m_{ij}}(f^n t_i - f^n t_j) = 0$, hence $f^{n+m_{ij}}(t_i - t_j) = 0$ over $D(fg_i g_j)$. As i and j are finitely many, so taking m large enough, we obtain $f^{n+m} t_i = f^{n+m} t_j$ over $\mathcal{F}(D(g_i g_j))$ for each i and j. Thus, the family $\{f^{n+m} t_i\}$ is a matching family which glues up to give $s \in \Gamma(\mathcal{F}, X)$ such that its restriction over $D(f)$ is $f^{n+m} t$[36]. $\square$

[35] The result is similar in flavour to Proposition 1.3.1.5.

[36] Note that we have implicitly used the fact the restriction maps of $\mathcal{F}$ preserves the respective module structures (see remarks surrounding Definition 8.5.0.1)

Remark 1.9.1.8. Let $X = \mathrm{Spec}\,(R)$ be an affine scheme and $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module over X. Then, we obtain a map

$$\alpha : \widetilde{\Gamma(\mathcal{F}, X)} \longrightarrow \mathcal{F}$$

which on a basic open set $D(f) \subseteq X$, $f \in R$ is given by $\Gamma(\mathcal{F}, X)_f \to \mathcal{F}(D(f))$ mapping as $m/f^n \mapsto \rho_{X,D(f)}(m)/f^n$. Indeed, this is a $\mathcal{O}_X$-linear homomorphism which on stalks yields the $R_{\mathfrak{p}}$-linear map

$$\Gamma(\mathcal{F}, X)_{\mathfrak{p}} \longrightarrow \mathcal{F}_{\mathfrak{p}}$$

which is given by

$$\Gamma(\mathcal{F}, X) \otimes_R R_{\mathfrak{p}} \cong \Gamma(\mathcal{F}, X) \otimes_R \varinjlim_{f \notin \mathfrak{p}} R_f \cong \varinjlim_{f \notin \mathfrak{p}} \Gamma(\mathcal{F}, X) \otimes_R R_f = \varinjlim_{D(f) \ni \mathfrak{p}} \widetilde{\Gamma(\mathcal{F}, X)}(D(f)) \to \varinjlim_{D(f) \ni \mathfrak{p}} \mathcal{F}(D(f)).$$

We will see that this map α would become an isomorphism, especially due to the fact that quasicoherent modules behave very nicely on open affines of the form $D(f)$, as the Lemma 1.9.1.6 shows.

Corollary 1.9.1.9. *Let $\mathcal{F}$ be a quasicoherent sheaf over an affine scheme $X = \mathrm{Spec}\,(A)$. Then, there is a natural isomorphism of A_f-modules for each $f \in A$*

$$\Gamma(\mathcal{F}, X)_f \xrightarrow{\cong} \mathcal{F}(D(f))$$

given by $m/f^n \mapsto \rho(m)/f^n$ where ρ is the restriction map of $\mathcal{F}$ from X to $D(f)$

Proof. Follows from Lemma 1.9.1.7. □

Using the above, one proves the local nature of quasicoherence.

Proposition 1.9.1.10. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module over a scheme X. Then, $\mathcal{F}$ is quasicoherent if and only if for all open affine $U = \mathrm{Spec}\,(A) \subseteq X$ we have $\mathcal{F}_{|U} \cong \widetilde{M}$ where M is an A-module.*

Proof. We need to only show $R \Rightarrow L$. Let $\mathcal{F}$ be quasicoherent and $U = \mathrm{Spec}\,(A)$ open affine. We may assume $X = \mathrm{Spec}\,(A)$. Thus we need to show $\mathcal{F} \cong \widetilde{M}$ for an A-module M. By Lemma 1.9.1.6, we obtain an open affine cover $D(g_i)$ of X where $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$ for an A_{g_i}-module M_i. Let $M = \Gamma(\mathcal{F}, X)$, which is an A-module. By Corollary 1.9.1.9, we obtain a natural isomorphism $M_i \cong M_{g_i}$. Thus we have the required result. □

A similar result is true for coherent modules.

Proposition 1.9.1.11. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module over a Noetherian scheme X. Then, $\mathcal{F}$ is coherent if and only if for all open affine $U = \mathrm{Spec}\,(A) \subseteq X$ we have $\mathcal{F}_{|U} \cong \widetilde{M}$ where M is a finitely generated A-module.*

Proof. See Proposition 5.4, Chapter 2 [Hartshorne]. □

Corollary 1.9.1.12. *The image of the functor $\widetilde{(-)}$ of Definition 1.2.3.1 is exactly all quasicoherent modules over $\mathrm{Spec}\,(R)$. In other words, $\mathbf{Mod}(R) \equiv \widetilde{\mathbf{Mod}}(\mathcal{O}_{\mathrm{Spec}(R)}) = \mathbf{QCoh}(\mathcal{O}_{\mathrm{Spec}(R)})$. Further, if R is noetherian, then this restricts to $\mathbf{Mod}(R)^{f.g.} \equiv \mathbf{Coh}(\mathcal{O}_{\mathrm{Spec}(R)})$.* □

Quasicoherence and exactness of global sections

We next see the exactness of global sections functor.

Proposition 1.9.1.13. *Let X be an affine scheme and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ is quasicoherent, then*

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$

is exact.

Proof. Proposition 5.6, Chapter 2 [Hartshorne]. □

The category Qcoh(X)

The category of quasicoherent modules is further a Grothendieck-abelian category.

Theorem 1.9.1.14. *Let X be a scheme. The category $\mathbf{QCoh}(\mathcal{O}_X)$ is a Grothendieck-abelian category. Consequently, it is an abelian category which has all coproducts.*

Proof. Tag 077P, [Stacksproject]. □

Quasicoherence, direct and inverse images

We now see behavior of quasicoherence and coherence under inverse and direct images.

Lemma 1.9.1.15. *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{G}$ be a quasicoherent $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a quasicoherent $\mathcal{O}_X$-module. If X, Y are Noetherian schemes and $\mathcal{G}$ is coherent, then $f^*\mathcal{G}$ is coherent.*

Proof. The first question is local in both X and Y by Proposition 1.9.1.10. Indeed, pick $x \in X$ and an open affine $V \ni f(x)$ in Y. Then by continuity there is an open affine $U \ni x$ in X such that $f(U) \subset V$. This shows that we may assume X and Y to be affine. The result is now immediate by Corollary 1.9.1.12 and Lemma 1.2.3.4, 2. The same technique works for coherent case. □

For stability under direct image, we need some conditions on the map if noetherian conditions need to be dropped.

Lemma 1.9.1.16. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Then $f_*\mathcal{F}$ is a quasicoherent $\mathcal{O}_Y$-module if any of the following holds:*

1. *X is noetherian,*
2. *f is quasi-compact and separated.*

Proof. **TODO.** □

More properties

As promised earlier, we state a general result about invertible modules and quasicoherent modules over schemes. This is a fundamental result and will be used to portray the simplicity of the techniques developed so far. Moreover, its proof showcases the simplicity of the sheaf language and is thus a good exercise.

Lemma 1.9.1.17. *Let X be a scheme, $\mathcal{L} \in \mathrm{Pic}(X)$, $\mathcal{F} \in \mathbf{QCoh}(X)$, $f \in \Gamma(\mathcal{L}, X)$ and $s \in \Gamma(X, \mathcal{F})$. Denote by $X_f \subseteq X$ the open subset $X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x\mathcal{L}_x\}$.*

1. *If X is quasicompact and s is such that $s|_{X_f} = 0$, then there exists $n \in \mathbb{N}$ such that $f^n s = 0$ in $\Gamma(\mathcal{L}^{\otimes n} \otimes \mathcal{F}, X)$.*
2. *If X admits a finite affine open cover $\{U_i\}$ where $\mathcal{L}_{|U_i}$ is free (of rank 1) and $U_i \cap U_j$ is quasicompact, then for any $t \in \mathcal{F}(X_f)$, there exists $n \in \mathbb{N}$ such that $f^n t \in (\mathcal{L}^{\otimes n} \otimes \mathcal{F})(X_f)$ extends to a global section $s \in \Gamma(\mathcal{L}^{\otimes n} \otimes \mathcal{F}, X)$.*

Proof. 1. Cover X by finitely many affine open sets $U = \mathrm{Spec}\,(A)$ which satisfies $\varphi : \mathcal{L}_{|U} \cong \mathcal{O}_{X|U} \cong \mathcal{O}_{\mathrm{Spec}(A)}$. Further, denote $\mathcal{F}_{|U} = \widetilde{M}$ where M is an A-module (Corollary 1.9.1.12). By restricting f to U, we may write $g = \varphi_U(f) \in A$ and by restricting s to U, we may write $s \in M$. Since $s|_{X_f} = 0$ and $X_f \cap U = D(g)$, therefore $s/1 = 0$ in M_g. Consequently, there exists $n \in \mathbb{N}$ such that $g^n s = 0$ in $M = \Gamma(U, \mathcal{F}_{|U})$. We then observe the following isomorphisms (see Lemma 27.2.0.5):

$$(\mathcal{L}^{\otimes n} \otimes \mathcal{F})_{|U} \cong \mathcal{O}_{X|U}^{\otimes n} \otimes \mathcal{F}_{|U} \cong \mathcal{O}_{\mathrm{Spec}(A)}^{\otimes n} \otimes \widetilde{M} \cong \widetilde{M}.$$

Consequently, we get isomorphisms in sections over U which yields that $f^{\otimes n} \otimes s \mapsto g^n s = 0$. hence $f^{\otimes n} \otimes s = 0$ in $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ over U. Since this happens for all finitely many Us, therefore taking large enough n, we observe that $f^{\otimes n} \otimes s = 0$ in $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ over X.

2. Pick $t \in \mathcal{F}(X_f)$. For each of the finitely many i, let $U_i = \mathrm{Spec}\,(A_i)$. As $\mathcal{L}_{|U_i} \cong \mathcal{O}_{X|U_i}$, therefore $X_f \cap U_i = \{\mathfrak{p} \in \mathrm{Spec}\,(A_i) \mid f_{\mathfrak{p}} \notin \mathfrak{p}A_{\mathfrak{p}}\} = D(f)$ where we interpret $f \in \mathcal{L}(U_i)$ by restricting the global section f. By locality of quasicoherence (Proposition 1.9.1.10), we have an A_i-module M_i such that $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$. As $t \in \mathcal{F}(X_f)$, therefore by restriction, we have $t_i \in \mathcal{F}(U_i \cap X_f) = \mathcal{F}(D(f)) \cong M_f$ (Proposition 1.2.3.3). It follows that for some n_i, we have $f^{n_i} t \in M_i = \mathcal{F}(U_i)$. Since U_i are atmost finite, so we may take a large enough n so that $f^n t \in M_i = \mathcal{F}(U_i)$.

Observe that

$$(\mathcal{L}^{\otimes n} \otimes \mathcal{F})_{|U_i} \cong \mathcal{O}_{\mathrm{Spec}(A_i)}^{\otimes n} \otimes \mathcal{F}_{|U_i} \cong \mathcal{F}_{|U_i} \cong \widetilde{M_i}$$

where $f^n t \in \mathcal{F}(U_i)$ corresponds to $f^{\otimes n} \otimes t \in \mathcal{L}^{\otimes n} \otimes \mathcal{F}(U_i)$. As $t_i = t = t_j \in \mathcal{F}(U_i \cap U_j \cap X_f)$, therefore $f^n(t_i - t_j) = 0$ in $\mathcal{F}(U_i \cap U_j \cap X_f)$. Our hypothesis that $U_i \cap U_j$ is quasicompact ensures by item 1 that there exists $k > 0$ such that $f^{n+k}(t_i - t_j) = 0$ in $\mathcal{F}(U_i \cap U_j)$, for all i, j. It follows that $f^{n+k} t_i \in \mathcal{F}(U_i) = \mathcal{L}^{\otimes n+k} \otimes \mathcal{F}(U_i)$ is a matching family. It follows that there exists $s \in \Gamma(\mathcal{L}^{\otimes n+k} \otimes \mathcal{F}, X)$ which on X_f is $f^{n+k} t$, as required. □

These were some of the basic results on quasicoherent modules. We now do perhaps the most important application of $\mathcal{O}_X$-modules, that when X is a projective scheme.

1.9.2 $\mathcal{O}_{\mathrm{Proj}(S)}$-modules

It is extremely important to do all exercises from Chapter 2 of Hartshorne from 5.9 to 5.14, as they deal with geometry coming out of modules over projective schemes.

Let S be a graded ring and M a graded S-module. We attach a sheaf $\widetilde{M}$ to M over Proj$S$.

Definition 1.9.2.1. ($\widetilde{M}$) Let S be a graded ring and M be a graded S-module. Then we define a sheaf $\widetilde{M}$ over $\mathrm{Proj}(S)$ given on an open set $U \subseteq \mathrm{Proj}(S)$ by

$$\widetilde{M}(U) := \left\{ s : U \to \coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})} \mid \forall \mathfrak{p} \in U,\ s(\mathfrak{p}) \in M_{(\mathfrak{p})} \& \exists \text{ open } \mathfrak{p} \in V \subseteq U \ \&\ m \in M_d, f \in S_d \text{ s.t. } f \notin \mathfrak{q} \ \&\ s(\mathfrak{q}) = m/f \forall \mathfrak{q} \in V \right\}.$$

The restrictions are the obvious ones. It is clear that if we treat S as a graded S-module, then $\widetilde{S} \cong \mathcal{O}_{\mathrm{Proj}(S)}$ where we treat $\mathcal{O}_{\mathrm{Proj}(S)}$ as an $\mathcal{O}_{\mathrm{Proj}(S)}$-module.

Remark 1.9.2.2. Over a projective scheme $X = \mathrm{Proj}(S)$, the theory of quasi-coherent modules is the most useful. In particular, we will have the following observations to make about them:
1. Any graded S-module gives an $\mathcal{O}_X$-module $\widetilde{M}$ which is furthermore quasicoherent.
2. Any $\mathcal{O}_X$-module $\mathcal{F}$ gives a graded S-module $\Gamma_*(\mathcal{F})$.
3. For X being the projective n-space over a ring A, we have $\Gamma_*(\mathcal{O}_X) \cong A[x_0, \dots, x_n]$.
4. Assume S is furthermore finitely generated by degree 1 elements. If $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module, then $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$.
5. All projective schemes over Spec$\,(A)$ is of the form $\mathrm{Proj}(S)$ where $S_0 = A$ and S is finitely generated as by S_1 as an S_0-algebra.

These are the main takeaways from the general theory of quasicoherent $\mathcal{O}_{\mathrm{Proj}(S)}$-modules.

We now attend to these results one-by-one. We first have analogous results to the affine case on the behaviour of $\widetilde{M}$ on basis, on stalks and its quasicoherence.

Proposition 1.9.2.3. *Let S be a graded ring, M be a graded S-module, $X = \mathrm{Proj}(S)$ be the projective scheme over S and $\widetilde{M}$ to be the associated sheaf of M over X. Then,*
1. *for any $\mathfrak{p} \in X$,*
$$(\widetilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})},$$
2. *for any $f \in S_d$, $d > 0$ and basic open $D_+(f)$,*
$$\widetilde{M}_{|D_+(f)} \cong \widetilde{M_{(f)}},$$
3. *the sheaf $\widetilde{M}$ is an $\mathcal{O}_X$-module which is furthermore quasicoherent,*
4. *if S is a noetherian ring and M is finitely generated, then $\widetilde{M}$ is coherent.*

Proof. 1. and 2. follows from repeating Lemma 1.8.2.4. Statement 3. follows from local property of quasicoherence (Proposition 1.9.1.10), the fact that sets of the form $D_+(f)$ for $f \in S_d$, $d > 0$ forms a basis of X (Lemma 1.8.1.3) and statement 2 above. Statement 4 follows from coherence being a local property for Noetherian schemes (Proposition 1.9.1.11) and statement 2 above. $\square$

Remark 1.9.2.4. The theory of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules is rich because of various constructions which interrelates the category $\mathbf{grMod}(S)$ of graded S-modules and graded maps and the category $\mathbf{Mod}(\mathcal{O}_{\mathrm{Proj}(S)})$. Indeed, these constructions is what we will study now, and these will be absolutely indispensable to do geometry in projective spaces $\mathrm{Proj}(k[x_0, \dots, x_n]/f)$ for a homogeneous polynomial f.

Remark 1.9.2.5. The construction of $\mathcal{O}_X$-modules is functorial ($X = \mathrm{Proj}(S)$):

$$\begin{aligned} \widetilde{(-)} : \mathbf{grMod}(S) &\longrightarrow \mathbf{QCoh}(\mathcal{O}_X) \\ M &\longmapsto \widetilde{M} \\ M \stackrel{\varphi}{\to} N &\longmapsto \widetilde{M} \stackrel{\eta}{\to} \widetilde{N} \end{aligned}$$

where η on a basic open $D_+(f)$ is given by the localization maps $\eta_{D_+(f)} : M_{(f)} \to N_{(f)}$.

We first begin by twisting each $\mathcal{O}_{\mathrm{Proj}(S)}$-module.

Twists and Serre twists

Definition 1.9.2.6. (**Twists**) Let S be a graded ring and $X = \mathrm{Proj}(S)$. For each $n \in \mathbb{Z}$, we define the n^{th}-*Serre twist* to be $\mathcal{O}_X(n)$ which is defined to be $\widetilde{S(n)}$, the sheaf associated to the n-th twisted graded S-module $S(n)$ (Definition 23.2.1.3). For each $\mathcal{O}_X$-module $\mathcal{F}$, we then define the n^{th}-*twist of* $\mathcal{F}$ to be $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Some obvious questions are: what happens to n^{th}-twist of $\widetilde{M}$ for a graded S-module M, what is so special about $\mathcal{O}_X(n)$ in relation to $\mathcal{O}_X$? We answer these in the following result.

Proposition 1.9.2.7. *Let S be a graded ring generated by S_1 as an S_0-algebra and $X = \mathrm{Proj}(S)$. Then,*

1. *$\mathcal{O}_X(n)$ is an invertible module for all $n \in \mathbb{Z}$,*
2. *for any graded S-modules M, N and $n \in \mathbb{Z}$,*
   (a) $\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$,
   (b) $\widetilde{M}(n) \cong \widetilde{M(n)}$.
3. *Let T be a graded ring generated by T_1 as a T_0-algebra. Let M be a graded S-module and N a graded T-module. Let $\varphi : S \to T$ be a graded map and $f : U \to \mathrm{Proj}(S)$ be the corresponding map (Proposition 1.8.2.6). Then,*
   (a) $f_*(\widetilde{N}_{|U}) \cong \widetilde{{}_S N}$,
   (b) $f^*(\widetilde{M}) \cong \widetilde{(M \otimes_S T)}_{|U}$,
   (c) $f_*(\mathcal{O}_{\mathrm{Proj}(T)|U}) \cong \widetilde{T}$, *where T is treated to be an S-module via φ.*
4. *Let φ and f as in 3 and let $Y = \mathrm{Proj}(T)$. Then,*
   (a) $f_*(\mathcal{O}_Y(n)_{|U}) \cong f_*(\mathcal{O}_{Y|U})(n)$,
   (b) $f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)_{|U}$.
5. *For all $n, m \in \mathbb{Z}$,*

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m).$$

Proof. 1. Cover X by basic open sets of the form $D_+(f)$ for $f \in S_1$. One then easily reduces to showing that $S(n)_{(f)}$ is a free $S_{(f)}$-module of rank 1. Indeed, one shows that the following map is an $S_{(f)}$-linear map which is an isomorphism as $S_{(f)}$-modules: $S_{(f)} \to S(n)_{(f)}$ given by $s/f^k \mapsto sf^n/f^k$. One really needs f to be of degree 1 to be able to show that this is an isomorphism.
2. This reduces to finding natural isomorphism $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \cong (M \otimes_S N)_{(f)}$ which one can do constructing two sided inverses. One of these maps well-definedness will use the fact that degree of f is 1.
3. See Lemma 1.2.3.4 for $a)$ and $b)$ and observe that $f^{-1}(D_+(g)) = D_+(\varphi(g))$ for $g \in S$ homogeneous by a simple unravelling of definition of $U \subseteq \mathrm{Proj}(T)$. The statement $c)$ is immediate by looking the respective sections on a basic open set $D_+(g)$.
4. Statement $a)$ follows from 3.$a)$ and 3.$c)$ is immediate from 3.$b)$.
5. Follows from 2.$a)$. □

Remark 1.9.2.8. The twisting functor given by

$$\begin{aligned} \mathbf{Mod}(\mathcal{O}_X) &\longrightarrow \mathbf{Mod}(\mathcal{O}_X) \\ \mathcal{F} \xrightarrow{f} \mathcal{G} &\longmapsto \mathcal{F}(n) \xrightarrow{f\otimes \mathrm{id}} \mathcal{G}(n) \end{aligned}$$

is exact. This is immediate as $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ and thus localizing at a point x, we get $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_X(n)_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}$ where the latter isomorphism follows from Proposition 1.9.2.7, 1.
In general this tells us also that the stalks of all twisted sheaves $\mathcal{F}(n)$ is identical to that of $\mathcal{F}$.

Remark 1.9.2.9. Let S be a graded ring and $X = \mathrm{Proj}(S)$ be the corresponding projective scheme with $\mathcal{F} \in \mathbf{QCoh}(X)$. Our goal in the next few pages is to understand how we can recover $\mathcal{F}$ by the global sections of all the twisted sheaves $\mathcal{F}(n)$. This is recorded in Propositions 1.9.2.12 and 1.9.2.13.

Associated graded S-module

We now associate to each $\mathcal{O}_X$-module $\mathcal{F}$ a graded S-module, where $X = \mathrm{Proj}(S)$.

Definition 1.9.2.10. (**Associated graded S-module**) Let S be a graded ring, $X = \mathrm{Proj}(S)$ and $\mathcal{F}$ an $\mathcal{O}_X$-module. Define the associated graded S-module to be

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathcal{F}(n), X)$$

where the S-module structure is given as follows: we need only define the scalar multiplication for homogeneous elements, so let $s_d \in S_d$ and $t_n \in \Gamma(\mathcal{F}(n), X)$. Then define $s_d \cdot t_n$ to be the image of $s_d \otimes t_n \in \Gamma(\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(n), X)$ under the isomorphism $\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(n) \cong \mathcal{F}(n+d)$ via Proposition 1.9.2.7, 5, in order to obtain an element of $\Gamma(\mathcal{F}(n+d), X)$, as needed.

Remark 1.9.2.11. There are two main results about associated graded S-modules.

1. Let $S = A[x_0, \dots, x_r]$ for a ring A and $r \geq 1$. Then $\Gamma_*(\mathcal{O}_X) \cong S$ for $X = \mathrm{Proj}(S)$. The relevance of this result is as follows. We know that the global sections of the structure sheaf over a projective scheme doesn't recover the homogeneous coordinate ring back. This

result tells us that looking only at global sections of structure sheaf won't suffice (hopefully obvious by now), we need to instead look at global sections of all twists of the structure sheaf in order to recover the coordinate ring. For example, consider the quadric $xy - wz$ in $\mathbb{P}^5_k$. The corresponding coordinate ring is $S = k[w,x,y,z,a,b]/xy - wz \cong \frac{k[w,x,y,z]}{xy-wz}[a,b]$ and the corresponding scheme is $X = \text{Proj}\,(S)$. Consequently, we can write $S = A[a,b]$ for $A = \frac{k[w,x,y,z]}{xy-wz}$ and thus this result would yield that S is isomorphic to $\Gamma_*(\mathcal{O}_X)$. Note that to use this result, we have to force ourselves to go 2 dimensions up.

2. Let S be a graded ring which is *finitely* generated by S_1 as an S_0-algebra and let $X = \text{Proj}(S)$. Then, for any quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, we obtain a natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$. This result therefore tells us that the functor $M \mapsto \widetilde{M}$ of Remark 1.9.2.5 from graded S-modules to quasicoherent $\mathcal{O}_X$-modules is essentially surjective.

We'll later see that these results will allow us to obtain an equivalent criterion of when is a scheme over an affine scheme projective.

We now state these results and sketch their proofs.

Proposition 1.9.2.12. *Let $S = A[x_0, \dots, x_r]$ for a ring A and $r \geq 1$ and denote $X = \text{Proj}(S)$. Then $\Gamma_*(\mathcal{O}_X) \cong S$.*

Proof. The main idea is to keep reducing the problem to a problem about graded ring S. Since S is generated by x_is as an A-algebra, therefore $D_+(x_i)$ for $i = 0, \dots, r$ covers X. An element in $\Gamma_*(\mathcal{O}_X)$ is given by a sum of elements $t_n \in \Gamma(\mathcal{O}_X(n), X)$. Let $t_n \in \Gamma(\mathcal{O}_X(n), X)$. The data of t_n is equivalently represented by the $(t_{n,0}, \dots, t_{n,r})$ where $t_{n,i} \in \mathcal{O}_X(n)(D_+(x_i)) = S(n)_{(x_i)}$ are the corresponding restrictions. Thus, $t_{n,i} \in S_{x_i}$ is a homogeneous element of degree n. Thus, $t = \sum_n t_n$ is equivalently represented by the tuple $(t_0, \dots, t_r)$ where $t_i = \sum_n t_{n,i}$ such that the image of t_i under $S_{x_i} \to S_{x_i,x_j}$ is same as the image of t_j under $S_{x_j} \to S_{x_j,x_i}$ for all $i, j = 0, \dots, r$. Note each of these S_{x_i,x_j} for varying i, j are contained in $R = S_{x_0,\dots,x_r}$. Now, we have injective maps $S \to S_{x_i} \to S_{x_i,x_j} \to R$ and thus $t = (t_0, \dots, t_r)$ as above is contained in $\bigcap_{i=0}^r S_{x_i} \hookrightarrow R$. In fact, any element of this intersection also corresponds to an element of $\Gamma_*(\mathcal{O}_X)$. Consequently, $\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^r S_{x_i}$. It is straightforward to see that this intersection is exactly S by writing a general homogeneous element of R and observing what it needs to satisfy to be in the intersection. □

We now show the essential surjectivity of $\widetilde{(-)}$. The proof of this result nicely shows the elegance of the techniques developed so far.

Proposition 1.9.2.13. *Let S be a graded ring which is finitely generated by S_1 as an S_0-algebra and $X = \text{Proj}S$.*

1. *For each $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural map*

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}.$$

2. *For each quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, the above map β is an isomorphism, that is,*

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\cong} \mathcal{F}.$$

Proof. 1. Since $D_+(f)$ for $f \in S_1$ covers X, we may define β naturally only on $D_+(f)$. This is done as follows:

$$\begin{tikzcd} \Gamma_*(\mathcal{F})_{(f)} \arrow[r, "\beta_{D_+(f)}"] \arrow[dr] & \mathcal{F}(D_+(f)) \\ & \Gamma(\mathcal{F}(d)_{|D_+(f)} \otimes \mathcal{O}_X(-d), X) \arrow[u, "\cong"', "\varphi"] \end{tikzcd}$$

where the diagonal map is given by

$$\frac{m}{f^d} \mapsto m \otimes \frac{1}{f^d}$$

and the isomorphism φ is given by restrictions.
2. In the above, we need to show that the diagonal map is an isomorphism. Suppose for some $m/f^d \in \Gamma_*(\mathcal{F})_{(f)}$, we have that $m \otimes 1/f^d = 0$ in $\Gamma(\mathcal{F}(d)_{|D_+(f)} \otimes \mathcal{O}_X(-d), X)$. Denote $\mathcal{G} = \mathcal{F}(d) \otimes \mathcal{O}_X(-d)$, which is quasicoherent. Note that $m \otimes 1/f^d = 0$ as an element in $\mathcal{G}(D_+(f))$ and also note that $D_+(f) = X_f$. Consequently from Lemma 1.9.1.17, 1, there exists $n \in \mathbb{N}$ such that $f^n \otimes m \otimes 1/f^d$ is zero as a global section of $\mathcal{O}_X(1)^{\otimes n} \otimes \mathcal{G} \cong \mathcal{F}(n)$. Hence, $f^{n-d}m = 0$ in $\Gamma(\mathcal{F}(n), X)$ and thus $\frac{m}{f^d} = \frac{f^{n-d}m}{f^n}$ is zero in $\Gamma_*(\mathcal{F})_{(f)}$. This shows injectivity. We now show surjectivity. Pick $t \in \mathcal{F}(D_+(f))$. By Lemma 1.9.1.17, 2, (which applies here as $D_+(f)$s are affine and finitely many whose intersection is again affine), we obtain a section $f^n t$ of $\mathcal{O}_X(1)^{\otimes n} \otimes \mathcal{F} \cong \mathcal{O}_X(n) \otimes \mathcal{F} \cong \mathcal{F}(n)$ over $D_+(f)$ which extends to a global section of $\mathcal{F}(n)$, say s. Consider s/f^n in $\Gamma_*(\mathcal{F})_{(f)}$, which maps to $s \otimes 1/f^n = t$ in $\mathcal{F}(D_+(f))$, as needed. □

Closed subschemes of $\mathbb{P}^n_A$

We can use these results to obtain a nice characterization of closed subschemes of projective schemes and an equivalent characterization of projective schemes over affine schemes. Denote by $\mathbb{P}^r_A = \mathrm{Proj}(A[x_0, \dots, x_r])$ for a ring A.

Proposition 1.9.2.14. *Let $Y \hookrightarrow \mathbb{P}^r_A$ be a closed subscheme with ideal sheaf $\mathcal{I}_Y$ of the projective r-space over a ring A. Then $I = \Gamma_*(\mathcal{I}_Y)$ is a homogeneous ideal of $A[x_0, \dots, x_r]$ and we have*

$$Y \cong \mathrm{Proj}\left(A[x_0, \dots, x_r]/I\right).$$

Proof. Let $S = A[x_0, \dots, x_r]$. The fact that I is a homogeneous ideal of S follows from exactness twisting functor (Remark 1.9.2.8), left exactness of global sections and $\Gamma_*(\mathcal{O}_X) = S$ of Proposition 1.9.2.12. In order to show that $Y \cong \mathrm{Proj}(S/I)$, it is enough to show that they both define isomorphic ideal sheaves (Proposition 1.4.4.13, 3). The ideal sheaf of $\mathrm{Proj}(S/I)$ is $\widetilde{I}$ by Proposition 1.8.2.8, 3 and the ideal sheaf of Y is $\mathcal{I}_Y$. Since $I = \Gamma_*(\mathcal{I}_Y)$, therefore the result follows from Proposition 1.9.2.13, 2. □

Proposition 1.9.2.15. *Let A be a ring. A scheme $Y \to \mathrm{Spec}\,(A)$ is projective if and only if $Y \cong \mathrm{Proj}(S)$ for a graded ring S with $S_0 = A$ and which is finitely generated by S_1 as an S_0-algebra.*

Proof. (L $\Rightarrow$ R) We have a closed immersion $Y \to \mathbb{P}^r_A$. From Proposition 1.9.2.14, it follows that $Y \cong \mathrm{Proj}(S)$ where $S = A[x_0, \dots, x_r]/I$, but S_0 might not be A. By Proposition 1.8.2.8, 2, we may assume I to not have any degree 0 component. Thus, S as defined will satisfy the necessary criterion.
(R $\Rightarrow$ L) We have $S \cong A[x_0, \dots, x_r]/I$, so by Proposition 1.8.2.8, 1, we have a closed immersion $\mathrm{Proj}(S) \to \mathbb{P}^r_A$. $\square$

Very ample invertible modules

We now study modules which determine when a scheme is projective.

Definition 1.9.2.16 (Twisting modules). Let X be a scheme and consider $\mathbb{P}^n_X \to X$ to be the projective n-scheme over X. Consider the projection $p : \mathbb{P}^n_X \to \mathbb{P}^n_{\mathbb{Z}}$. The k^{th}-Serre twist sheaf over $\mathbb{P}^n_X$ are defined to be $p^*(\mathcal{O}(k))$ where $\mathcal{O}(k)$ is the k^{th}-Serre twist sheaf over the projective scheme $\mathbb{P}^n_{\mathbb{Z}}$.

What we have defined above is indeed a generalization of usual twisted sheaves available on projective schemes, as the following lemma shows.

Lemma 1.9.2.17. *Let $X = \mathrm{Spec}\,(A)$. Denote $p : \mathbb{P}^n_X \to \mathbb{P}^n_{\mathbb{Z}}$ the projection map. Then,*
1. $\mathbb{P}^n_X \cong \mathbb{P}^n_A$,
2. *The twisting module $p^*(\mathcal{O}(k)) \cong \mathcal{O}_{\mathbb{P}^n_A}(k)$ under the above isomorphism.*

Proof. Item 1 follows from Lemma 1.8.2.12. For item 2, observe that the map p is obtained by composing with the isomorphism $\mathbb{P}^n_X \cong \mathbb{P}^n_A$ the canonical map $q : \mathbb{P}^n_A \to \mathbb{P}^n_{\mathbb{Z}}$, which is induced from the canonical map $\varphi : \mathbb{Z}[x_0, \dots, x_n] \to A[x_0, \dots, x_n]$ (see Proposition 1.8.2.6). Thus we wish to show that $q^*(\mathcal{O}(k)) \cong \mathcal{O}_{\mathbb{P}^n_A}(k)$. Denote $S = \mathbb{Z}[x_0, \dots, x_n]$ so that $\mathcal{O}(k) = \widetilde{S(k)}$. Hence, by Proposition 1.9.2.7, 4, we have

$$q^*(\mathcal{O}(k)) \cong \mathcal{O}_{\mathbb{P}^n_A}(k),$$

as needed. $\square$

Definition 1.9.2.18 (Very ample invertible module). Let $X \to Y$ be a scheme over Y. An invertible module $\mathcal{L}$ over X is said to be very ample over Y if there is an immersion (Definition 1.4.4.9) $i : X \to \mathbb{P}^n_Y$ such that $i^*(\mathcal{O}(1)) \cong \mathcal{L}$.

Proposition 1.9.2.19. *Let Y be a Noetherian scheme. Then the following are equivalent:*
1. *Scheme $f : X \to Y$ is projective.*
2. *Scheme $f : X \to Y$ is proper and there exists a very ample invertible sheaf over X relative to Y.*

1.9.3 Vector bundles

We study vector bundles over schemes. Recall that if X is a scheme then $\mathbb{A}^n_X$ is the fiber product of X with $\mathbb{A}^n_{\mathbb{Z}}$ over $\mathbb{Z}$. It is also the global spec $\mathbf{Spec}(\mathcal{O}_X[x_1, \dots, x_n])$.

Definition 1.9.3.1 (Vector bundle). Let Y be a scheme. A rank n geometric vector bundle over Y is the datum $(f : X \to Y, \{V_i\}_{i\in I}, \psi_i : f^{-1}(V_i) \xrightarrow{\cong} \mathbb{A}^n_{V_i})$ where $\{V_i\}_{i\in I}$ is an open cover of Y such that for any open affine $V = \mathrm{Spec}\,(A) \subseteq V_i \cap V_j$, the restriction of $\psi_j \circ \psi_i^{-1} : \mathbb{A}^n_{U_i\cap U_j} \to \mathbb{A}^n_{U_i\cap U_j}$ to V gives an isomorphism of $\mathbb{A}^n_V = \mathrm{Spec}\,(A[x_1, \dots, x_n])$ which is induced by a linear isomorphism of $A[x_1, \dots, x_n]$.

An isomorphism of vector bundles $(f : X \to Y, \{V_i\}, \psi_i)$ and $(f' : X' \to Y, \{V_i'\}, \psi_i')$ is given by a map $g : X \to X'$ such that $f' \circ g = f$ and $(f : X \to Y, \{V_i\} \cup \{V_i'\}, \psi_i \cup \psi_j')$ is another vector bundle structure on X.

The first theorem in this regard is the following classification.

Theorem 1.9.3.2. *Let X be a scheme. There is a bijection*

$$\{\textit{Locally free modules of rank } n\} / \cong \;\leftrightarrow \{\textit{Geometric vector bundles of rank } n\} / \cong .$$

We construct both the maps in the following.

Construction 1.9.3.3 (Vector bundle from a locally free module).

Construction 1.9.3.4 (Locally free module from a vector bundle).

1.10 Divisors

The notion of divisors is one of the central tools for understanding the geometrical properties of a given scheme. Indeed, in the special case of curves in projective plane, a (Weil) divisor is just a formal linear combination of points of the curve. From this data, one can in-fact recover the embedding of the curve in the projective plane. Hence the data that divisors of a scheme stores is rich in geometric information.

We will first define the notion of Weil divisors in those schemes in which the points lying on codimension 1 subset of the scheme are regular (see Lemma 1.7.1.4 for a motivation behind the definition).

Definition 1.10.0.1. (**Regular in codimension 1**) A scheme X is said to be regular in codimension 1 if the local rings $\mathcal{O}_{X,p}$ which are of dimension 1 are regular.

Remark 1.10.0.2. All non-singular abstract varieties are regular in codimension 1 as all local rings are regular. In this section, we will be working with schemes which are noetherian integral separated and regular in codimension 1. We call them *Weil schemes*. All non-singular abstract varieties are Weil schemes.

1.10.1 Weil divisors & Weil divisor class group

We will define the notion of Weil divisors and divisor class group on Weil schemes.

Definition 1.10.1.1. (**Weil divisors**) Let X be a Weil scheme. A *prime divisor* is an integral closed subscheme of codimension 1. A *Weil divisor* is an element of the free abelian group generated by the set of all prime divisors, denoted Div (X). A Weil divisor is denoted $\sum_{i=1}^{k} n_i Y_i \in \text{Div } X$. A Weil divisor $\sum_i n_i Y_i$ is *effective* if $n_i \geq 0$ for all i.

We now look at a foundational result which will guide the further development. Its proof is important as it combines a lot of our previous knowledge.

Proposition 1.10.1.2. *Let X be a Weil scheme and $Y \subseteq X$ be a prime divisor with $\eta \in Y$ be its generic point. Then there is an injective map*

$$\text{PDiv}(X) \to \text{DVal}(K(X))$$

where PDiv *is the set of all prime divisors of X, $K(X)$ is the function field of X and* $\text{DVal}(K(X))$ *is the set of all discrete valuations over $K(X)$.*

Proof. Note that if Y is a prime divisor, then there is a generic point $\eta \in Y$. Since codim $(Y, X) = \dim \mathcal{O}_{X,\eta}$ by Lemma 1.7.1.4, therefore we obtain that $\mathcal{O}_{X,\eta}$ is a regular local ring of dimension 1. Now there is a special result for such rings, which in particular establishes equivalences of such rings with a lot of other type of rings. Indeed, by Theorem 23.10.1.8 we obtain that in our case $\mathcal{O}_{X,\eta}$ is a DVR. As the fraction field of $\mathcal{O}_{X,\eta}$ (it is a domain as X is integral) is $K(X)$, the function field of X (Lemma 1.3.1.3), therefore we have a valuation $v : K(X) \to \mathbb{Z}$ whose valuation ring is $\mathcal{O}_{X,\eta}$. By Lemma 1.12.4.8 (which holds for Y as Y is separated by Corollary 1.12.4.5, 2), the valuation v uniquely determines the point $\eta \in X$ as v has center η because the valuation ring $\mathcal{O}_{X,\eta}$ of v dominates the local ring $\mathcal{O}_{X,\eta}$. As the information of point $\eta \in X$ yields the closed set $Y \subseteq X$, therefore the valuation $v : K(X) \to \mathbb{Z}$ uniquely determines the prime divisor Y. $\square$

Remark 1.10.1.3. As a consequence, we can study a prime divisor via the valuation that comes through the Proposition 1.10.1.2. Indeed, for a prime divisor $Y \subseteq X$ and the associated discrete valuation $v_Y : K(X) \to \mathbb{Z}$, we can think of the value $v_Y(f)$ for some $f/g \in K(X) \setminus \{0\}$ to be telling us the number of poles that f/g has along Y. We can justify this via the proof of Proposition 1.10.1.2 as follows. For a prime divisor $Y \subseteq X$ with generic point $\eta \in Y$, the corresponding valuation is obtained by the fact that $\mathcal{O}_{X,\eta}$ is a DVR in our case. For a DVR R with fraction field K, the corresponding discrete valuation $v : K \to \mathbb{Z}$ can be thought of as an abstraction of the idea that we want to know how many poles a fraction $f/g \in K$ has and v provides that data to us. In particular, we think that if $v(f/g)$ is positive, then that tells us f/g has that many zeros in Y and if $v(f/g)$ is negative then that many poles in Y. We mostly have only this idea in mind when dealing with valuations.

As we know from complex analysis that a meromorphic function can only have discretely many singularities, along the similar lines we have that an arbitrary $f \in K(X)$ can atmost have some poles or some zeros at only finitely many prime divisors.

Proposition 1.10.1.4. ***TODO.***

1.10.2 Weil divisors on affine schemes

We will study Weil divisors over affine Weil schemes (see Remark 1.10.0.2), to portray the type of information that they contain. However, we will introduce something more general (Krull domains) but will show later that with noetherian hypothesis, we have done no extra work (see Corollary 1.10.2.5).

Definition 1.10.2.1 (Weil & Krull domains). Let R be a domain. We call R to be a Weil domain if $R_\mathfrak{p}$ is a DVR (equivalently regular, or Dedekind by Theorem 23.10.1.8) for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{ht}(\mathfrak{p}) = 1$ (equivalently, $\dim R_\mathfrak{p} = 1$ by Lemma 23.8.1.4). A Weil domain is a Krull domain if moreover $R = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} R_\mathfrak{p} = 1$ in $F = Q(R)$ and every non-zero $r \in R$ is in only finitely many prime ideals of height 1. Hence any Krull domain is in particular a Weil domain.

The first observation is that noetherian normal domains are Krull domains.

Proposition 1.10.2.2. *Let R be a ring.*

1. *If R is a noetherian normal domain, then R is a Weil domain.*
2. *If R is a noetherian Weil domain, then R is a Krull domain.*

Proof. 1. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be a height 1 prime ideal. As localization of normal domains is normal (Proposition 23.7.2.8), therefore $R_\mathfrak{p}$ is a noetherian local domain of dimension 1 which is normal. By Theorem 23.10.1.8, we deduce that $R_\mathfrak{p}$ is a DVR, as required.

2. We first wish to show that $R = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} R_\mathfrak{p}$. We need only show ($\supseteq$). Indeed, let $\frac{r}{s} \in R_\mathfrak{p}$ for all $\mathfrak{p}$ of height 1. It follows that $s \notin \mathfrak{p}$ for all height 1 prime ideals. We claim that any non-zero element of R is contained in a height 1 prime ideal. Indeed, this is the content of Krull's principal ideal theorem (Theorem 23.8.3.2). Thus $s \in R$ is a unit, hence $\frac{r}{s} = rs^{-1} \in R$.

Finally, we wish to show that any non-zero element $r \in R$ is in only finitely many height 1 primes. By going modulo rR and recalling that R is a domain, we need only show that there are finitely many height 0 primes in $S = R/rR$. As S is noetherian and minimal primes are equivalent to height 0 primes, so we reduce to showing that a noetherian ring has finitely many minimal primes. Indeed, this is the content of Lemma 1.2.1.8. ☐

Example 1.10.2.3. A Dedekind domain is therefore a Krull domain of dimension 1.

Any Krull domain is normal.

Lemma 1.10.2.4. *If R is a Krull domain, then R is a normal domain.*

Proof. As $R = \bigcap_{\mathrm{ht}\,(\mathfrak{p})=1} R_{\mathfrak{p}}$ where each $R_{\mathfrak{p}}$ is a normal domain in particular, therefore the integral closure of R in $F = Q(R)$ is contained in each $R_{\mathfrak{p}}$, thus is contained in $\bigcap_{\mathrm{ht}\,(\mathfrak{p})=1} R_{\mathfrak{p}}$ which is R, as required. ☐

In summary, we have now shown the following important equivalences.

Corollary 1.10.2.5. *Let R be a ring. Then the following are equivalent:*
1. *R is a noetherian normal domain.*
2. *R is a noetherian Weil domain.*
3. *R is a noetherian Krull domain.*

Proof. Follows from Proposition 1.10.2.2 and Lemma 1.10.2.4 ☐

Remark 1.10.2.6. Consequently, a Krull domain can be thought of as an abstraction of all the nice "divisorial properties" that we expect from noetherian normal domains, so that we can talk about it in non-noetherian settings.

The following is what we expect and the following is indeed true.

Proposition 1.10.2.7. *Let $X = \mathrm{Spec}\,(R)$ be an affine scheme. Then the following are equivalent:*
1. *X is an affine Weil scheme.*
2. *R is a noetherian Weil domain.*

Proof. Suppose X is an affine Weil scheme. Then X is in particular noetherian and integral. By Lemmas 1.4.1.3 and 1.4.2.2, we deduce that R is a noetherian domain. As X is regular in codimension 1, $R_{\mathfrak{p}}$ is regular if $\mathrm{ht}\,(\mathfrak{p}) = 1$. Consequently, $R_{\mathfrak{p}}$ is a noetherian local domain of dimension 1. Hence by Theorem 23.10.1.8 we deduce that $R_{\mathfrak{p}}$ is a DVR, as required.

Now if R is a noetherian Weil domain, then X is a noetherian integral scheme regular in codimension 1. As affine schemes are separated, so we conclude the proof. ☐

As for most purposes we do enforce noetherian hypothesis, thus by above result it is noetherian normal domains, i.e. noetherian Weil domains, which are most important for us. We now define Weil divisors on Weil domains just as we did in Definition 1.10.1.1.

Definition 1.10.2.8 (Weil divisors on Weil domains). Let R be a Weil domain. A *prime divisor* is a height 1 prime ideal. A *Weil divisor* is an element of the free abelian group generated by all prime divisors. A Weil divisor is denoted by $D = \sum_i n_i[\mathfrak{p}_i]$. The group of all Weil divisors over R is denoted by $\mathrm{Div}\,(R)$. An *effective Weil divisor* is $D = \sum_i n_i[\mathfrak{p}_i]$ where $n_i \geq 0$ for all i. We denote by $\mathrm{PDiv}(R)$ the set of prime divisors, that is, the generating set of $\mathrm{Div}\,(R)$.

Every Weil domain comes equipped with a discrete valuation at height 1 prime.

Definition 1.10.2.9 (**$\mathfrak{p}$-adic valuation**)**.** Let R be a Weil domain and $\mathfrak{p}$ be any height 1 prime ideal of R. Then $R_\mathfrak{p}$ is a DVR and thus has a function

$$\begin{aligned}\nu_\mathfrak{p} : \mathrm{Cart}(R) &\longrightarrow \mathbb{Z}\\ I &\longmapsto \nu_\mathfrak{p}(I)\end{aligned}$$

where since $R_\mathfrak{p}$ is a PID in particular (Proposition 23.10.1.9), so we can write the invertible ideal $I_\mathfrak{p}$ as $I_\mathfrak{p} = \mathfrak{p}^{\nu_\mathfrak{p}(I)} R_\mathfrak{p}$. We call $\nu_\mathfrak{p}$ as the $\mathfrak{p}$-adic valuation of R for height 1 prime ideal $\mathfrak{p}$.

Remark 1.10.2.10. Indeed, the name is justified by the simple observation that if we consider the usual valuation $\nu : Q(R_\mathfrak{p}) \to \mathbb{Z}$ given by $f \mapsto \nu(f)$ such that $ut^{\nu(f)} = f$ where $t \in R_\mathfrak{p}$ is the local parameter of the DVR, then $\nu_\mathfrak{p}$ is the function $\mathrm{Cart}(R) \to \mathrm{Cart}(R_\mathfrak{p}) = Q(R_\mathfrak{p})^\times \xrightarrow{\nu} \mathbb{Z}$ which is given by $I \mapsto I_\mathfrak{p} = fR_\mathfrak{p} \mapsto \nu(f)$.

Construction 1.10.2.11 (The Cart-Div homomorphism)**.** Let R be a Krull domain. We define a group homomorphism

$$\begin{aligned}\nu : \mathrm{Cart}(R) &\longrightarrow \mathrm{Div}\ (R)\\ I &\longmapsto \sum_{\mathrm{ht}\ (\mathfrak{p})=1} \nu_\mathfrak{p}(I)[\mathfrak{p}]\end{aligned}$$

which is well defined as $\nu_\mathfrak{p}(I) \neq 0$ only for finitely many $\mathfrak{p}$ of height 1 by the third axiom of Krull domains[37]. This is a group homomorphism as $\nu(IJ) = \sum \nu_\mathfrak{p}(IJ)[\mathfrak{p}] = \sum(\nu_\mathfrak{p}(I) + \nu_\mathfrak{p}(J))[\mathfrak{p}] = \nu(I) + \nu(J)$. We call this the Cart-Div homomorphism. We also call the divisor $\nu(I)$ corresponding to an invertible ideal to be the *associated divisor of* I.

Definition 1.10.2.12 (**Principal divisors**)**.** Let R be a Krull domain. For any invertible ideal $I \in \mathrm{Cart}(R)$, the divisor $\nu(I) = \sum \nu_\mathfrak{p}(I)[\mathfrak{p}]$ is called the principal divisor of I. Any divisor in the image of ν will be called a principal divisor.

An immediate question will be to see what do effective principal divisors correspond to. Indeed, they exactly correspond to ideals of R.

Lemma 1.10.2.13. *Let R be a Krull domain. Then there is a bijection*

$$\{\textit{Effective principal divisors on } R\} \leftrightsquigarrow \{\textit{Ideals of } R\}.$$

Proof. Indeed, if $D = \nu(I)$ is an effective principal divisor, then $\nu_\mathfrak{p}(I) \geq 0$ for each $\mathfrak{p}$ of height 1. Consequently, $I_\mathfrak{p}$ is an ideal of $R_\mathfrak{p}$ for each $\mathfrak{p}$ of height 1. Thus, $I \subseteq \bigcap_{\mathrm{ht}\ (\mathfrak{p})=1} I_\mathfrak{p} \subseteq \bigcap_{\mathrm{ht}\ (\mathfrak{p})=1} R_\mathfrak{p} = R$, as required. For the converse, repeat the same in reverse. □

Lemma 1.10.2.14. *Let R be a Krull domain. Then the Cart-Div homomorphism*

$$\nu : \mathrm{Cart}(R) \to \mathrm{Div}\ (R)$$

is injective.

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[37] One may see this as follows. As $I \subseteq \frac{a}{b}R$, $a \neq b$, therefore for any height 1 prime $\mathfrak{p}$, we get $I_\mathfrak{p} = t_\mathfrak{p}^{\nu_\mathfrak{p}} R_\mathfrak{p} \subseteq \frac{a}{b} R_\mathfrak{p}$ where $t_\mathfrak{p} \in R_\mathfrak{p}$ is the local parameter of the DVR. Consequently, $\frac{a}{b} t_\mathfrak{p}^{\mu_\mathfrak{p}} = t_\mathfrak{p}^{\nu_\mathfrak{p}}$. Note that $\nu_\mathfrak{p} = 0$ if and only if $\frac{a}{b} \cdot R_\mathfrak{p} = R_\mathfrak{p}$ which in turn happens if and only if $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$, i.e. if and only if $\frac{a}{b}$ is a unit of $R_\mathfrak{p}$. Hence, $\nu_\mathfrak{p} \neq 0$ if and only if $\frac{a}{b}$ or $\frac{b}{a}$ is a non-unit of $R_\mathfrak{p}$. As $\frac{a}{b}$ is a non-unit of $R_\mathfrak{p}$ if and only if $a \notin \mathfrak{p}$ and there are only finitely many height 1 primes containing a, thus there are only finitely many height 1 primes $\mathfrak{p}$ such that $\frac{a}{b} \in R_\mathfrak{p}$ is non-unit. Similarly for $\frac{b}{a}$. This shows that only for finitely many height 1 primes do we have that $\nu_\mathfrak{p}(I) \neq 0$.

Proof. Indeed, if $\nu(I) = 0$, then $I_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all height 1 primes $\mathfrak{p}$. As there is an invertible ideal J such that $IJ = R$, therefore $\nu(I) + \nu(J) = 0$, from which it follows that $\nu(J) = 0$ as well. As $I \subseteq \bigcap_{\text{ht }(\mathfrak{p})=1} I_{\mathfrak{p}} = \bigcap_{\text{ht }(\mathfrak{p})=1} R_{\mathfrak{p}} = R$ and similarly for J, thus, I, J are ideals of R such that $IJ = R$. Consequently, as $R = IJ \subseteq I \cap J \subseteq I, J$, thus we get that $I = J = R$, as required. □

We now define the divisor class group of a Krull domain.

Definition 1.10.2.15 (**Divisor class group & Pic-Cl map**). Let R be a Krull domain. The Weil divisor class group $\mathrm{Cl}(R)$ of R is defined to be the cokernel of the composite

$$F^\times \xrightarrow{\text{div}} \mathrm{Cart}(R) \xrightarrow{\nu} \mathrm{Div}\ (R).$$

That is,

$$\mathrm{Cl}(R) = \frac{\mathrm{Div}\ (R)}{\mathrm{Im}\,(\nu \circ \mathrm{div})}.$$

We write $\nu \circ \mathrm{div} : F^\times \to \mathrm{Div}\ (R)$ as div as well.

As $\mathrm{Pic}(R) = \mathrm{CoKer}\,(\mathrm{div} : F^\times \to \mathrm{Cart}(R))$, $\mathrm{Cl}(R) = \mathrm{CoKer}\,(\mathrm{div} : F^\times \to \mathrm{Div}\ (R))$ and we have a map $\nu : \mathrm{Cart}(R) \to \mathrm{Div}\ (R)$, thus by universal property of cokernels, we get a map

$$\tilde{\nu} : \mathrm{Pic}(R) \longrightarrow \mathrm{Cl}(R)$$

which we call the Pic-Cl map.

Remark 1.10.2.16 (Div-Cl sequence). We summarize the whole discussion by observing the following exact sequence:

$$1 \to R^\times \to F^\times \xrightarrow{\text{div}} \mathrm{Div}\ (R) \to \mathrm{Cl}(R) \to 0.$$

The exactness at $F^\times$ follows from the second axiom of Krull domains.

Its an easy observation by 4-lemma that the Pic-Cl map is injective as well.

Lemma 1.10.2.17 (Cart-Pic to Div-Cl). *Let R be a Krull domain. Then the Pic-Cl map $\tilde{\nu}$: $\mathrm{Pic}(R) \to \mathrm{Cl}(R)$ is injective. Moreover, the following is commutative:*

$$\begin{tikzcd}
1 \arrow[r] & R^\times \arrow[r] \arrow[d, "\mathrm{id}"] & F^\times \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{id}"] & \mathrm{Cart}(R) \arrow[r] \arrow[d, hook, "\nu"] & \mathrm{Pic}(R) \arrow[r] \arrow[d, hook, "\tilde{\nu}"] & 0 \\
1 \arrow[r] & R^\times \arrow[r] & F^\times \arrow[r, "\mathrm{div}"] & \mathrm{Div}\ (R) \arrow[r] & \mathrm{Cl}(R) \arrow[r] & 0
\end{tikzcd}.$$

Proof. Commutativity is clear. Injectivity of $\tilde{\nu}$ is from 4-lemma. □

A simple corollary yields when Picard group and Weil divisor class groups are same.

Corollary 1.10.2.18. *Let R be a Krull domain. Then the following are equivalent:*

1. $\nu : \mathrm{Cart}(R) \to \mathrm{Div}\ (R)$ *is an isomorphism.*
2. $\tilde{\nu} : \mathrm{Pic}(R) \to \mathrm{Cl}(R)$ *is an isomorphism.*

Proof. Both sides are immediate from 5-lemma. □

Relative Weil divisors

As in homology, it is necessary at times to study *relative invariants*. Indeed, same is true for Weil divisors, as some results mentioned later will show.

Definition 1.10.2.19 (Relative Weil divisors). Let R be a Krull domain and $S \subseteq R$ be a multiplicative set. Define Div $(R, S^{-1}R)$ to be the free abelian group generated by height 1 primes $\mathfrak{p}$ such that $\mathfrak{p} \cap S \neq \emptyset$. We call Div $(R, S^{-1}R)$ Weil divisors on R relative to S.

Remark 1.10.2.20. It is immediate that

$$\text{Div } (R) = \text{Div } (S^{-1}R) \oplus \text{Div } (R, S^{-1}R).$$

Furthermore, we may define the map div as usual ($F = Q(R)$):

$$\begin{aligned} \text{div} : F^\times &\longrightarrow \text{Div } (R, S^{-1}R) \\ f &\longmapsto \sum_{\text{ht } \mathfrak{p}=1, \mathfrak{p}\cap S \neq \emptyset} \nu_{\mathfrak{p}}(fR)[\mathfrak{p}]. \end{aligned}$$

The main result here is the following relative version of Div-Cl exact sequence.

Proposition 1.10.2.21. *Let R be a Krull domain and $S \subseteq R$ be a multiplicative set. Then, the following sequence is exact:*

$$1 \to R^\times \to (S^{-1}R)^\times \stackrel{\text{div}}{\to} \text{Div } (R, S^{-1}R) \to \text{Cl}(R) \to \text{Cl}(S^{-1}R) \to 0.$$

Proof. To check exactness at $(S^{-1}R)^\times$, we see that if $\text{div}(f) = 0$ for some $f \in S^{-1}R$, then $\nu_{\mathfrak{p}}(fR) = 0$ for all $\mathfrak{p}$ of height 1 and $\mathfrak{p} \cap S \neq \emptyset$. Thus, $fR_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all such primes. As $f \in (S^{-1}R)^\times$, consequently if $f = \frac{a}{b}$, then $a, b \in S$. Hence if $\mathfrak{p} \cap S = \emptyset$, then $S \subseteq R \setminus \mathfrak{p}$ and thus $a, b \in R \setminus \mathfrak{p}$ so that $f \in R_{\mathfrak{p}}$ is a unit. It follows that

$$fR = \bigcap_{\text{ht }(\mathfrak{p})=1} fR_{\mathfrak{p}} = \bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S=\emptyset} fR_{\mathfrak{p}} \cap \bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S\neq\emptyset} fR_{\mathfrak{p}}.$$

By above, $\bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S=\emptyset} fR_{\mathfrak{p}} = \bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S=\emptyset} R_{\mathfrak{p}}$ and since $f \in (S^{-1}R)^\times$, so $\bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S\neq\emptyset} fR_{\mathfrak{p}} = \bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S\neq\emptyset} R_{\mathfrak{p}}$. Finally, we get $fR = \bigcap_{\text{ht }(\mathfrak{p})=1} R_{\mathfrak{p}} = R$, so that $f \in R^\times$, as required. Conversely, if $u \in R^\times$, then $\text{div}(u) = 0$.

To see exactness at Div $(R, S^{-1}R)$, observe that if $D_S \in \text{Div } (R, S^{-1}R)$ is a relative divisor such that $[D_S] = 0$ in $\text{Cl}(R)$, then $D_S = \nu(fR)$ for some $f \in F^\times$, i.e. D_S is a principal divisor. Then, since $D_S = \nu(fR) \in \text{Div } (R, S^{-1}R)$ and is a principal divisor as a divisor on R, thus we deduce that $\nu_{\mathfrak{p}}(fR) = 0$ for all $\mathfrak{p}$ of height 1 such that $\mathfrak{p} \cap S = \emptyset$, that is, $f \in R_{\mathfrak{p}}$ is a unit for all such primes. Consequently, $fR_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all $\mathfrak{p} \cap S = \emptyset$ of height 1. Now, as $S^{-1}R$ is a Krull domain, we get

$$f \cdot (S^{-1}R) = f \cdot \bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S=\emptyset} S^{-1}R_{\mathfrak{p}} = \bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S=\emptyset} S^{-1}fR_{\mathfrak{p}} = \bigcap_{\text{ht }(\mathfrak{p})=1, \mathfrak{p}\cap S=\emptyset} S^{-1}R_{\mathfrak{p}} = S^{-1}R,$$

as required. Conversely, if $f \in (S^{-1}R)^\times$, then $\text{div}(f)$ is by construction a principal divisor on R, so that its image in $\text{Cl}(R)$ will be 0.

Exactness at $\mathrm{Cl}(R)$ is clear as if $[D]$ is 0 in $\mathrm{Cl}(S^{-1}R)$, then D can be written as a sum of two divisors, one on $S^{-1}R$, say D_1, and other a relative on R, say D_2, where D_1 is principal. Hence, $[D] = [D_2]$ and since D_2 is a relative divisor on R, thus its in image of $\mathrm{Div}\ (R, S^{-1}R) \to \mathrm{Cl}(R)$.

Finally, exactness at $\mathrm{Cl}(S^{-1}R)$ is clear as any $[D_S] \in \mathrm{Cl}(S^{-1}R)$ such that $D_S = \sum_{\mathrm{ht}\ (\mathfrak{p})=1, \mathfrak{p}\cap S \neq \emptyset} n_{\mathfrak{p}}[\mathfrak{p}]$ is also a divisor on R, so that $D_S \in \mathrm{Div}\ (R)$ and thus defines the class $[D_S] \in \mathrm{Cl}(R)$, whose image in $\mathrm{Cl}(S^{-1}R)$ is $[D_S]$. This completes the proof. □

We can now prove an important result.

Proposition 1.10.2.22. *Let R be a Krull domain and $f \in R$ be a prime element (that is, fR is a prime ideal). Then*

$$\mathrm{Cl}(R) \cong \mathrm{Cl}(R_f).$$

Proof. By relative Weil divisors exact sequence of Propsosition 1.10.2.21, we get that the map

$$\mathrm{Div}\ (R, R_f) \to \mathrm{Cl}(R) \to \mathrm{Cl}(R_f) \to 0$$

is exact. We need only show that the image of $\mathrm{Div}\ (R, R_f)$ in $\mathrm{Cl}(R)$ is 0. Indeed, let $D \in \mathrm{Div}\ (R, R_f)$ be a relative divisor. Thus

$$D = \sum_{\mathrm{ht}\ (\mathfrak{p})=1, \mathfrak{p}\ni f} n_{\mathfrak{p}}[\mathfrak{p}].$$

Now $fR \subseteq \mathfrak{p}$ and fR is prime by hypothesis. As $\mathrm{ht}\ (\mathfrak{p}) = 1$ and R is a domain, thus $fR = \mathfrak{p}$. Hence, only height 1 prime containing f is fR. Hence $\mathrm{Div}\ (R, R_f) = \mathbb{Z}([fR]) \cong \mathbb{Z}$. As fR is a principal ideal, thus it can be shown that its image in $\mathrm{Cl}(R)$ is 0 as $[fR] = \nu(fR)$ (fR is prime), and thus the image of $\mathrm{Div}\ (R, R_f) \to \mathrm{Cl}(R)$ is 0, as required. □

1.10.3 Cartier divisors & Cartier divisor class group

1.10.4 Cartier divisors on affine schemes

We discuss Cartier divisors first on affine schemes. Here, we will see that a Cartier divisor is nothing but an invertible ideal.

Definition 1.10.4.1 (**Fractional & invertible ideals**)**.** Let R be a domain and $F = Q(R)$. A fractional ideal is a non-zero R-module $I \subseteq F$ such that there exists $f \in F$ for which $I \subseteq f \cdot R$ in F. That is, I consists of some R-multiples of a fraction $f \in F$. Note that if I and J are fractional, then so is IJ. A fractional ideal I is said to be invertible, if there exists a fractional ideal J such that $IJ = R$. The set of all fractional ideals form an abelian group with identity being R which we denote by $\mathrm{Cart}(R)$ defined to be the abelian group of Cartier divisors on $\mathrm{Spec}\,(R)$ (or just R).

Remark 1.10.4.2. An invertible ideal over domain R can equivalently be defined to be an R-module $I \subseteq F$ such that there exists $b \in R$ for which $bI \leq \langle a \rangle$ for some $a \in R$. That is, bI is an ideal of R which is contained in some principal ideal.

Example 1.10.4.3. Let R be a domain and $n \in \mathbb{Z} \setminus \{0\}$ with $\mathrm{char}(R) \neq n$. Denote $(\frac{1}{n}) = \frac{1}{n}R \subseteq F$ and $(n) = nR \subseteq F$ be two fractional ideals (where $n = 1 + \cdots + 1$, n-times). Clearly $(\frac{1}{n}) \cdot (n) = R$. Thus, $(\frac{1}{n})$ is a Cartier divisor on R.

Remark 1.10.4.4 (Divisor map)**.** For any domain R with fraction field F, we have a group homomorphism

$$\begin{aligned} \mathrm{div} : F^\times &\longrightarrow \mathrm{Cart}(R) \\ f &\longmapsto fR. \end{aligned}$$

It is interesting to note when is this an isomorphism. An immediate calculation shows that it is so when R is a PID. Thus, $\mathrm{Cart}(R)$ has information about factorization in R.

By analyzing kernel and cokernel of the divisor map, we get a useful exact sequence.

Theorem 1.10.4.5 (Cart-Pic sequence)**.** *Let R be a domain.*

1. *The map*

$$\begin{aligned} \mathrm{Cart}(R) &\longrightarrow \mathrm{Pic}(R) \\ I &\longmapsto [I] \end{aligned}$$

   is a group homomorphism. That is, $I \otimes_R J \cong IJ$, for any two $I, J \in \mathrm{Cart}(R)$. This is also true for any two line bundles $I, J \in \mathrm{Pic}(R)$ such that $I, J \subseteq F$.
2. *We have $\mathrm{Ker}\,(\mathrm{div} : F^\times \to \mathrm{Cart}(R)) = R^\times$.*
3. *The following is an exact sequence*

$$1 \to R^\times \to F^\times \overset{\mathrm{div}}{\to} \mathrm{Cart}(R) \to \mathrm{Pic}(R) \to 0.$$

Proof. 1. We first show well-definedness of $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$. To this end, we need to show that any invertible ideal is a rank 1 projective module. Indeed, as there is an invertible ideal J such that $IJ = R$, thus there exists $\{x_i\} \subseteq I$ and $\{y_i\} \subseteq I$ finitely many such that $1 = x_1y_1 + \cdots + x_ny_n$.

Using this, we immediately get maps $I \to R^n \to I$ whose composite is identity. Consequently, we get that I is a direct summand of R^n, as required.

As R is a domain, so $\mathrm{Spec}\,(R)$ is in particular connected. Consequently, we need only find $\dim_F(I \otimes_R F)$. As $I \otimes_R F = I \otimes_R R_\mathfrak{o} = I_\mathfrak{o} = F$, thus, $\mathrm{rank}_\mathfrak{o}(F) = 1$ and by connectedness, $\mathrm{rank}(I)$ is a constant map to 1. This shows that I is a line bundle, hence $[I] \in \mathrm{Pic}(R)$.

Next, we wish to show that for any $I, J \in \mathrm{Cart}(R)$, we get $I \otimes_R J \cong IJ$. This will show that the above map is a group homomorphism, as required. To this end, observe that we have a map

$$\begin{aligned} \varphi : I \otimes_R J &\longrightarrow IJ \\ x \otimes y &\longmapsto xy. \end{aligned}$$

We claim that φ is an isomorphism. Indeed, as I is a line bundle, therefore I is a projective R-module. Consequently it is flat and thus $I \otimes_R -$ is exact. As $J \hookrightarrow F$ is the inclusion map, therefore $I \otimes_R J \hookrightarrow I \otimes_R F \cong F$ is also an inclusion. Note that $I \otimes_R F \cong F$ is the map given by $x \otimes y \mapsto xy$. Hence, we have shown that φ is injective. Surjectivity of φ is immediate, so that φ is an isomorphism.

2. Let $f \in F^\times$ be such that $\mathrm{div}(f) = fR = R$, then f is a unit of R, as required.

3. We need only show exactness at $\mathrm{Cart}(R)$ and surjectivity of $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$. We first show the former. An invertible ideal $I \in \mathrm{Cart}(R)$ is in the kernel iff $I \cong R$ as an R-module. If $\varphi : R \to I$ is the isomorphism, then $I \cong fR$ where $f = \varphi(1)$, as required.

Next we show surjectivity of $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$. To this end, we have to show that any line bundle L over R is isomorphic to an invertible module I on R. Indeed, as L is rank 1, therefore $L \otimes_R F \cong F$. As $R \hookrightarrow F$ and L is projective hence flat, thus $L \cong L \otimes_R R \to L \otimes_R F \cong F$ is injective. Let the image of L in F be I. We claim that $I \subseteq F$ is an invertible module. Indeed, as I is finitely generated, therefore $I = f_1 R + \cdots + f_n R$ for $\frac{a_i}{b_i} = f_i \in F$, which we may write as $I \subseteq \frac{1}{b_1 \dots b_n} R$, so I is fractional. To see that I is invertible, let $J \subseteq F$ be the fractional ideal corresponding to $\check{L}$. As $L \otimes_R \check{L} \cong R$ in $\mathrm{Pic}(R)$, it follows that $L \otimes_R \check{L} \cong I \otimes_R J \cong IJ$ where the last isomorphism is obtained from item 1. Hence $IJ \cong R$, where $I, J \subseteq F$ so that $IJ \subseteq F$. Consequently, IJ is a free R-module of rank 1 in F. It follows that $IJ = uR$ for some $u \in R^\times$, so that $I(u^{-1}J) = R$, as required. $\square$

1.10.5 Divisors and invertible modules

1.10.6 Divisors on curves

1.11 Smoothness & differential forms

In this section, we would like to understand the notion of smoothness in algebraic geometry. We will first begin by defining a non-singular point of a variety over an algebraically closed field, which would be an extrinsic definition. However, by a fundamental observation of Zariski, we can have an intrinsic definition of non-singular points, which would be in terms of regular local rings. The main thrust behind this latter definition will be the expectation that over non-signular points, the dimension of the tangent space is equal to the dimension of the variety (which is true in the case of, say manifolds). We would further see that for a variety over an algebraically closed field, the set of singular points is closed and proper.

We would then introduce the important notion of sheaf of differentials over a scheme. This will again allow us to characterize non-singular points of a variety, and much more.

1.11.1 Non-singular varieties

To start investigating the notion of non-singularity, we first investigate it in the setting of classical affine varieties (Definition 1.5.4.11). We will then proceed to abstract varieties.

Definition 1.11.1.1. (**Non-singular points of a classical affine variety**) Let k be an algebraically closed field and X be a classical affine k-variety with $I(X) = \langle f_1, \dots, f_m \rangle \lneq k[x_1, \dots, x_n]$. A point $p \in X$ is said to be non-singular if the $n \times m$ Jacobian matrix

$$[J_p]_{n\times m} = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{ij}$$

is of rank $n - \dim X$.

The first obvious question is whether the above definition is independent of the choice of the generators of prime ideal $I(X)$. The following lemma says yes.

Lemma 1.11.1.2. *Let k be an algebraically closed field and X be a classical affine k-variety. The definition of a non-singular point $p \in X$ is independent of the choice of the generating set of $I(X)$.*

Proof. Let $I(X) = \langle f_1, \dots, f_m \rangle = \langle g_1, \dots, g_l \rangle$. We wish to show that

$$\operatorname{rank}\left[\frac{\partial f_i}{\partial x_j}(p)\right]_{ij} = \operatorname{rank}\left[\frac{\partial g_i}{\partial x_j}(p)\right]_{ij}.$$

This follows immediately after writing $f_i = \sum_{a=1}^{l} c_{i_a} g_a$, $c_{i_a} \in k[x_1, \dots, x_n]$, differentiating it and observing that $g_a(p) = 0$ for all $a = 1, \dots, l$. □

Remark 1.11.1.3. In geometry, one notes that at a smooth point, the dimension of tangent space equals the dimension of the manifold itself. We would like to do a similar construction here. Indeed, if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and 0 is regular for f, then we know by implicit function theorem that $M = Z(f) \subseteq \mathbb{R}^n$ is a smooth manifold with normal vector field $(\nabla f) : \mathbb{R}^n \to \mathbb{R}^n$. Consequently, one can define the tangent space $T_x M$ for $x \in M$ to be the set of all those vectors which are normal to $(\nabla f)_x$. We mimic this definition for classical affine varieties.

Definition 1.11.1.4. (**Tangent space of a classical affine variety**) Let k be an algebraically closed field and let X be a classical affine k-variety in $\mathbb{A}^n_k$ with $I(X) = \langle f_1, \dots, f_m \rangle$. Denote for each $f \in k[x_1, \dots, x_n]$ and $p \in \mathbb{A}^n_k$ the following linear functional

$$\begin{aligned}(df)_p : k^n &\longrightarrow k \\ v &\longmapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) v_i.\end{aligned}$$

For a point $p \in X$, we define the tangent space T_pX as the following k-vector space

$$\begin{aligned}T_pX &:= \{v \in k^n \mid (df_i)_p(v) = 0 \; i = 1, \dots, m\} \\ &= \{v \in k^n \mid (df)_p(v) = 0 \; \forall f \in I(X)\}.\end{aligned}$$

We now show that this definition of tangent space is intrinsic. Indeed, we will show that the $T_pX = T\mathcal{O}_{X,p} := \operatorname{Hom}_k\left(\mathfrak{m}/\mathfrak{m}^2, k\right)$, where $(\mathcal{O}_{X,p}, \mathfrak{m})$ is the local ring at $p \in X$ and $\kappa(p) = k$ in this case (see Definition 23.1.2.14). Let us begin with a series of observations.

Lemma 1.11.1.5. *Let k be an algebraically closed field and $p \in \mathbb{A}^n_k$. Then the k-linear map*

$$\begin{aligned}\theta_p : k[x_1, \dots, x_n] &\longrightarrow k^n \\ f &\longmapsto \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p)\right)\end{aligned}$$

induces a k-linear isomorphism

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \cong k^n$$

where $\mathfrak{m}_p = \langle x_1 - p_1, \dots, x_n - p_n \rangle$ is the maximal ideal of $k[x_1, \dots, x_n]$ corresponding to the point p.

Proof. Let $p = (p_1, \dots, p_n)$. Observe that $\{\theta_p(x_i - p_i)\}_{i=1,\dots,n}$ forms a basis of k^n. Consequently, θ_p restricts to a surjective k-linear map $\hat{\theta}_p : \mathfrak{m}_p \to k^n$. Now one observes that $f \in \operatorname{Ker}(\theta)_p$ if and only if $f \in \mathfrak{m}_p^2$. Thus we have by first isomorphism theorem that $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong k^n$. □

Lemma 1.11.1.6. *Let k be an algebraically closed field and X be a classical affine k-variety in $\mathbb{A}^n_k$ with $p \in X$. Let $(\mathcal{O}_{X,p}, \mathfrak{m})$ denote the local ring of X at p and $I \leq k[x_1, \dots, x_n]$ be the ideal of X. Then,*

$$\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}_p/(\mathfrak{m}_p^2 + I).$$

Proof. Let $A = k[x_1, \dots, x_n]$. By Proposition 1.5.3.10, 3, we have $\mathfrak{m} = (\mathfrak{m}_p)_{\mathfrak{m}_p}/I_{\mathfrak{m}_p}$ and $\mathfrak{m}^2 = \left((\mathfrak{m}_p^2)_{\mathfrak{m}_p} + I_{\mathfrak{m}_p}\right)/I_{\mathfrak{m}_p}$. By quotienting, we obtain

$$\begin{aligned}\mathfrak{m}/\mathfrak{m}^2 &\cong \frac{(\mathfrak{m}_p)_{\mathfrak{m}_p}}{(\mathfrak{m}_p^2 + I)_{\mathfrak{m}_p}} \\ &\cong \left(\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I}\right)_{\mathfrak{m}_p/(\mathfrak{m}_p^2+I)} \\ &\cong \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I}.\end{aligned}$$

□

Recall the notion of regular local ring from Definition 23.1.2.16. We now see that non-singular points are classified by the local ring being regular.

Theorem 1.11.1.7. *Let k be an algebraically closed field and X be a classical affine k-variety and let $p \in X$. The following are equivalent:*
1. *The point $p \in X$ is non-singular.*
2. *The local ring $\mathcal{O}_{X,p}$ is regular.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{X,p}$. By definition, we have $\mathcal{O}_{X,p}$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}_{X,p}$. By Proposition 1.5.3.10, 7, we further have that $\mathcal{O}_{X,p}$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim X$. Whereas by Lemmas 1.11.1.6 and 1.11.1.5, we observe

$$\begin{aligned}\dim_k \mathfrak{m}/\mathfrak{m}^2 &= \dim_k \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I} \\ &= \dim_k \left(\frac{\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}}{\frac{\mathfrak{m}_p^2+I}{\mathfrak{m}_p^2}}\right) \\ &= \dim_k \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} - \dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2} \\ &= n - \dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2}.\end{aligned}$$

With these two observations, we thus reduce to proving that

$$\dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2} = \text{rank } J_p$$

where $J_p = \left[\frac{\partial f_i}{\partial x_j}\right]$ for $I = \langle f_1, \dots, f_m\rangle$. This now follows by the following two rather straightforward observations; in the notations of Lemma 1.11.1.5 and its proof, one observes
1. $\hat{\theta}_p^{-1}(\theta_p(I))$ is isomorphic as k-vector space to $I + \mathfrak{m}_p^2$,
2. $\dim_k \theta_p(I) = \text{rank } J_p$.

The result now follows. □

With the above result, we formulate the following definition of non-singular abstract varieties.

Definition 1.11.1.8. (**Non-singular abstract variety**) Let k be an algebraically closed field. A variety X over k is said to be non-singular if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a regular local ring.

Remark 1.11.1.9. Note that in the definition of non-singular varieties, it is sufficient to demand that $\mathcal{O}_{X,x}$ is a regular local ring for all closed points $x \in X$ only. Indeed, by Lemma 1.3.1.2, local ring at a non-closed point is obtained by localizing the local ring at a closed point at a prime ideal. As the localization of a regular local ring at a prime ideal is again a regular local ring by Theorem **??**, the result follows.

We now define the Zariski (co)tangent space of a scheme at a point.

Definition 1.11.1.10. (**Zariski (co)tangent space**) Let X be a scheme and $x \in X$ be a point and let κ be the residue field at point x. Then

1. the Zariski cotangent space at x is defined to be the κ-vector space
$$T_x^* X := \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2,$$
2. the Zariski tangent space at x is defined to be the κ-vector space
$$T_x X := \operatorname{Hom}_\kappa\left(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, \kappa\right).$$

These are the analogues to the case in algebra (see Definition 23.1.2.14).

TODO : State how this is related to usual definition of tangent spaces

1.11.2 Regular schemes

"*.... Of what use is it to know the definition of a scheme if one does not realize that a ring of integers in an algebraic number field, an algebraic curve, and a compact Riemann surface are all examples of a 'regular scheme of dimension 1'?*"- Hartshorne.

After defining and discussing basic properties of regular schemes, we will prove the above result stated by Hartshorne.

Definition 1.11.2.1 (**Regular schemes).** A locally noetherian scheme X is said to be regular if the local rings $\mathcal{O}_{X,x}$ for all $x \in X$ is a regular local ring.

Observe that any smooth affine curve over a field is spectrum of a Dedekind domain, so is in particular a regular scheme (as local rings of Dedekind domains are regular, as noted in Theorem 23.10.1.8).

Proposition 1.11.2.2. *Let C be a smooth affine plane plane curve over a field k. Then the coordinate ring of C is a Dedekind domain.*

Proof. Let $C = \operatorname{Spec}(R)$. As C is a curve, therefore C is an integral finite type scheme of dimension 1 over k. We thus deduce that R is a finite type k-domain of dimension 1. By Hilbert basis theorem (Theorem 23.3.0.6), we deduce that R is noetherian. Smoothness of C yields that $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in C$. As $\dim R = 1$, we deduce that $R_\mathfrak{p}$ is a noetherian local domain of dimension 1 which is also regular. By Theorem 23.10.1.8, we deduce that $R_\mathfrak{p}$ is normal for all $\mathfrak{p} \in C$. By local criterion of normal domains (Proposition 23.7.2.10), we deduce that R is normal. Hence R is noetherian normal domain of dimension 1, as required. $\square$

1.12 Morphism of schemes

The main use of schemes to answer geometric questions begin with defining various types of situations that one usually finds himself/herself in algebraic geometry. We discuss them here one by one by giving examples. In the first section we discuss some basic type of maps among schemes.

1.12.1 Elementary types of morphism

We first cover some basic type of maps between schemes.

Definition 1.12.1.1. (**Quasi-compact maps**) A map $f : X \to Y$ of schemes is said to be quasi-compact if there exists an affine open cover $\{V_i\}$ of Y such that the space $f^{-1}(V_i) \subseteq X$ is quasi-compact for each i.

Remark 1.12.1.2. Observe that a scheme X over k has a quasi-compact structure map $X \to \operatorname{Spec}(k)$ if and only if X is quasi-compact.

We now see that quasi-compact maps are local on target.

Proposition 1.12.1.3. [38] *A map $f : X \to Y$ is quasi-compact if and only if for each open affine $V \subseteq Y$, the space $f^{-1}(V) \subseteq X$ is quasi-compact.*

Proof. The ($\Leftarrow$) is immediate. For ($\Rightarrow$), pick any open affine $V \subseteq Y$. We wish to show that $f^{-1}(V)$ is quasi-compact. Let $V_i = \operatorname{Spec}(B_i)$ be the collection of open affines covering Y such that $f^{-1}(V_i)$ is quasi-compact. We now obtain a finite covering of V by affine opens which are affine open in V_i for some i as well. Indeed, by Lemma 1.4.4.3, we may cover $V \cap V_i$ by open affines which are basic open in both V and V_i. Doing this for each i, we obtain a cover of V by basic opens. As V is affine, so by Lemma 1.2.1.6 we obtain a finite collection of basic opens $\{D(g_i)\}_{i=1}^n$ where $g_i \in B_i$ such that $D(g_i)$ is a basic open in V as well.

We now have that $f^{-1}(V) = \bigcup_{i=1}^n f^{-1}(D(g_i))$. Hence it suffices to show that $f^{-1}(D(g_i))$ is a quasi-compact subspace. To this end, we first immediately reduce to assuming that X is quasicompact (by replacing X by $f^{-1}(V_i)$) and $Y = \operatorname{Spec}(B)$ is affine (by replacing Y by V_i). We now wish to prove that for any $g \in B$, $f^{-1}(D(g))$ is quasi-compact.

As X is quasi-compact, therefore there exists a finite affine open cover of X by $\operatorname{Spec}(A_i)$. It suffices to show that $\operatorname{Spec}(A_i) \cap f^{-1}(D(g))$ is a quasicompact space. Observe that $f|_{\operatorname{Spec}(A_i)} : \operatorname{Spec}(A_i) \to \operatorname{Spec}(B)$ is a morphism of affine schemes. It follows from Corollary 1.3.0.6 that $f|_{\operatorname{Spec}(A_i)}$ is induced from a ring map $\varphi_i : B \to A_i$. As $\operatorname{Spec}(A_i) \cap f^{-1}(D(g)) = (f|_{\operatorname{Spec}(A_i)})^{-1}(D(g)) = D(\varphi_i(g))$, which is an affine open, therefore by Lemma 1.2.1.6, we deduce that $\operatorname{Spec}(A_i) \cap f^{-1}(D(g))$ is quasi-compact, as required. □

Definition 1.12.1.4. (**Quasi-finite maps**) A map $f : X \to Y$ of schemes is quasi-finite if for each $y \in Y$, the fiber X_y is a finite set.

Example 1.12.1.5. Let k be an algebraically closed field. Consider the map

$$f : X = \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3}\right) \longrightarrow \mathbb{A}_k^1$$

[38] Exercise II.3.2 of Hartshorne.

obtained by the map $k[x] \to \frac{k[x,y]}{y^2-x^3}$ given by $x \mapsto x + \langle y^2 - x^3 \rangle$. Take any point $\mathfrak{p} = \langle p(x) \rangle \in \mathbb{A}^1_k$. Hence the fiber is

$$\begin{aligned} X_{\mathfrak{p}} &= \operatorname{Spec}\left(\frac{k[x,y]}{y^2-x^3} \otimes_k \kappa(\mathfrak{p})\right) \\ &= \operatorname{Spec}\left(\frac{k[x,y]}{y^2-x^3} \otimes_k \frac{k[x]_{\mathfrak{p}}}{\mathfrak{p}k[x]_{\mathfrak{p}}}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2-x^3} \otimes_k \left(\frac{k[x]}{\mathfrak{p}}\right)_{\mathfrak{p}}\right). \end{aligned}$$

Let $\mathfrak{p} \neq \mathfrak{o}$. As $k[x]$ is a PID, therefore $\mathfrak{p}$ is a maximal ideal. Consequently, we have $\kappa(\mathfrak{p}) = k[x]/\mathfrak{p}$. Hence,

$$\begin{aligned} X_{\mathfrak{p}} &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2-x^3} \otimes_k \frac{k[x]}{p(x)}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2-x^3, p(x)}\right). \end{aligned}$$

As k is algebraically closed, therefore by weak Nullstellensatz, we obtain that $p(x) = x - a$ for some $a \in k$. Consequently, if we have $a \neq 0$ then

$$\begin{aligned} X_{\mathfrak{p}} &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2-x^3, x-a}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[y]}{y^2-a^3}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[y]}{(y+a^{3/2})(y-a^{3/2})}\right) \\ &\cong \operatorname{Spec}(k \times k) \\ &\cong \operatorname{Spec}(k) \amalg \operatorname{Spec}(k). \end{aligned}$$

If $a = 0$, then

$$X_{\mathfrak{p}} \cong \operatorname{Spec}\left(\frac{k[y]}{y^2}\right)$$

and we know that $k[y]/y^2$ has only one prime ideal, the one generated by $y + \langle y^2 \rangle \in k[y]/y^2$. Hence $X_{\mathfrak{p}}$ consists of two points at all non-zero closed points and of a single point at the origin, showing that f has finite fibers at all closed points. However, at the generic point $\mathfrak{p} = \mathfrak{o}$, we have a more complicated story:

$$\begin{aligned} X_{\mathfrak{o}} &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2-x^3} \otimes_k k(x)\right) \\ &\cong \operatorname{Spec}\left(\frac{k(x)[x,y]}{y^2-x^3}\right) \\ &\cong \operatorname{Spec}\left(\frac{k(x)[y]}{y^2-x^3}\right). \end{aligned}$$

As $k(x)[y]$ is a PID, therefore points of $X_{\mathfrak{o}}$ are thus in bijective correspondence with prime ideals of $k(x)[y]$ containing $y^2 - x^3$, which in turn is in bijection with the set of irreducible factors of $y^2 - x^3$ in $k(x)[y]$. As $k(x)[y]$ is a UFD, therefore there can atmost be finitely many such irreducible factors. Hence, $X_{\mathfrak{o}}$ is finite.

Hence all fibers are finite, making f quasi-finite.

1.12.2 Finite type

We already considered one example of such maps in the case of schemes over a field in Section 1.4.3

Definition 1.12.2.1. (**Locally finite type**) Let $f : X \to Y$ be a map of schemes. Then f is said to be locally of finite type if there is an affine open cover $V_i = \operatorname{Spec}(B_i)$, $i \in I$ of Y such that for each $i \in I$, $f^{-1}(V_i)$ has an open affine cover $U_{ij} = \operatorname{Spec}(A_{ij})$, $j \in J$ such that for each j, the ring A_{ij} is finite type[39] B_i-algebra.

Definition 1.12.2.2. (**Finite type**) Let $f : X \to Y$ be a map of schemes. Then f is said to be of finite type if there is an open affine cover $V_i = \operatorname{Spec}(B_i)$, $i \in I$ of Y such that for each $i \in I$, $f^{-1}(V_i)$ has a finite open affine cover $U_{ij} = \operatorname{Spec}(A_{ij})$, $j = 1, \dots, n$ such that for each j, A_{ij} is a finite type B_i-algebra.

It is an important observation that both the above definitions are local on target.

Proposition 1.12.2.3. [40] *A map $f : X \to Y$ is locally of finite type if and only if for all open affine $V = \operatorname{Spec}(B)$ in Y, there is an open affine cover $U_i = \operatorname{Spec}(A_i)$ of $f^{-1}(V)$ in X such that each A_i is a finite type B-algebra.*

Proof. The R $\Rightarrow$ L follows immediately. Let $V_i = \operatorname{Spec}(B_i)$ be an open affine cover of Y such that $f^{-1}(V_i)$ is covered by open affines $U_{ij} = \operatorname{Spec}(A_{ij})$ where each A_{ij} is a finite type B_i-algebra. Pick any affine open $V = \operatorname{Spec}(B)$ in Y and a point $x \in f^{-1}(V)$. We wish to find an open affine $x \in U = \operatorname{Spec}(A)$ inside $f^{-1}(V)$ such that A is a finite type B-algebra.

Consider $f(x) \in V$ and let $f(x) \in V \cap V_i$. Consequently, $x \in f^{-1}(V)$ will be contained in some U_{ij}, so $x \in f^{-1}(V) \cap U_{ij}$. By continuity of f, there exists a basic open $D(g) \subseteq V \cap V_i$ for some $g \in B_i$ which contains $f(x)$ such that $f^{-1}(D(g)) \subseteq f^{-1}(V) \cap U_{ij}$ is open. Restricting f to U_{ij}, we have $f : U_{ij} \to V_i$ which induces a map $\varphi : B_i \to A_{ij}$ which is of finite type. Denote $U = f^{-1}(D(g)) = D(\varphi(g)) = \operatorname{Spec}\left((A_{ij})_{\varphi(g)}\right) \subseteq f^{-1}(V) \cap U_{ij}$. We therefore get that the restriction of f on U, which is given by $f : U \to D(g)$, induces the localization map on algebras $\varphi_g : (B_i)_g \to (A_{ij})_{\varphi(g)}$. As localization of algebras are finite type, therefore φ_g makes $(A_{ij})_{\varphi(g)}$ a finite type $(B_i)_g$-algebra.

By Lemma 1.4.4.3, we have an isomorphism $B_h \to (B_i)_g$. Thus, we have

$$B \to B_h \overset{\cong}{\to} (B_i)_g \to (A_{ij})_{\varphi(g)}$$

where each map is of finite type. Since composite of finite type maps is of finite type, therefore $(A_{ij})_{\varphi(g)}$ is a finite type B-algebra, as required.

$\square$

[39] finite type algebra := finitely generated as an algebra.

[40] Exercise II.3.1 of Hartshorne.

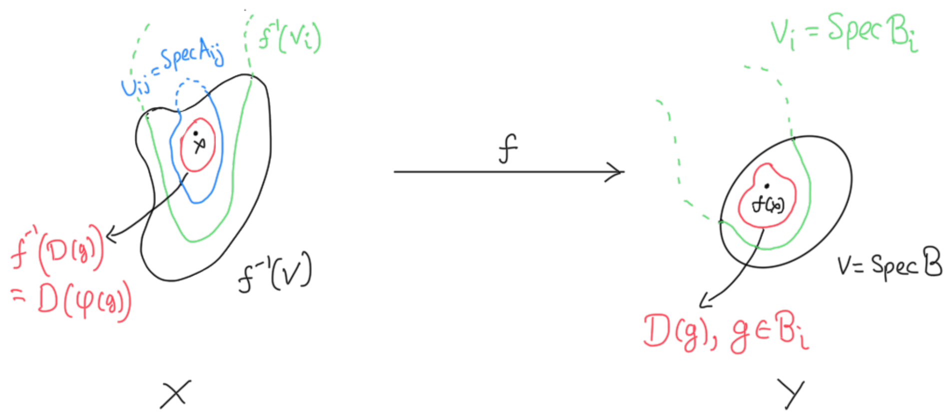

Figure 1.2: Sketch of the proof of Proposition 1.12.2.3

We next see that finite type maps are also local on target and a nice property that they satisfy which says that finite type property descends to every open affine inside the inverse image of an open affine.

Theorem 1.12.2.4. [41] *Let $f : X \to Y$ be a map of schemes. Then,*

1. *f is of finite type if and only if f is locally of finite type and quasi-compact,*
2. *f is of finite type if and only if for every open affine $V = \mathrm{Spec}\,(B)$, the space $f^{-1}(V)$ can be covered by finitely many open affines $U_i = \mathrm{Spec}\,(A_i)$ where each A_i is a finite type B-algebra,*
3. *if f is of finite type, then for any open affine $V = \mathrm{Spec}\,(B) \subseteq Y$ and any open affine $U = \mathrm{Spec}\,(A) \subseteq f^{-1}(V)$, A is a finite type B-algebra.*

Proof. 1. (L $\Rightarrow$ R) As f is of finite type, therefore there exists an open affine cover $\{V_i = \mathrm{Spec}\,(B_i)\}$ of Y such that $f^{-1}(V)$ can be covered by finitely many $U_{ij} = \mathrm{Spec}\,(A_{ij})$ where A_{ij} is a finite type B_i-algebra. Consequently, f is locally finite type. As each affine scheme is quasi-compact (Lemma 1.2.1.6) and finite union of quasi-compact spaces is quasi-compact, therefore we deduce that $f^{-1}(V)$ is quasi-compact.

(R $\Rightarrow$ L) As f is locally of finite type, therefore there exists an open affine cover $\{V_i = \mathrm{Spec}\,(B_i)\}$ of Y such that $f^{-1}(V_i)$ is covered by open affines $U_{ij} = \mathrm{Spec}\,(A_{ij})$ where each A_{ij} is a finite type B_i-algebra. As f is quasicompact, therefore we have a finite sub-cover $U_{i1}, \dots, U_{in}$ covering $f^{-1}(V)$, as required.

2. (R $\Rightarrow$ L) Immediate from definition.

(L $\Rightarrow$ R) Pick an open affine $V = \mathrm{Spec}\,(B)$ in Y. We wish to show that $f^{-1}(V)$ is covered by finitely many open affines each of which is spectrum of a finite type B-algebra. Indeed, as f is quasi-compact by statement 1 above, therefore by Proposition 1.12.1.3, we see that $f^{-1}(V)$ is quasi-compact. Also by statement 1, f is of locally finite type. Hence by Proposition 1.12.2.3, $f^{-1}(V)$ is covered by spectra of finite type B-algebras. As $f^{-1}(V)$ is quasi-compact, we get a finite subcover, as required.

3. Pick any open affine $V = \mathrm{Spec}\,(B)$ in Y and an open affine $U = \mathrm{Spec}\,(A) \subseteq f^{-1}(V)$. As

[41] Exercise II.3.3 of Hartshorne.

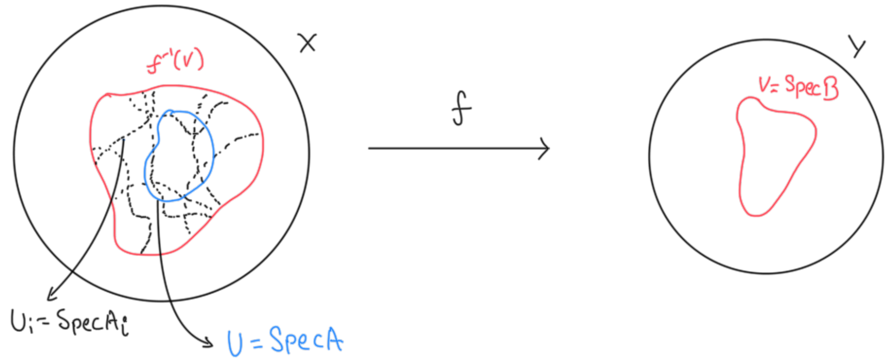

Figure 1.3: Sketch of the proof of Theorem 1.12.2.4, 3.

f is of finite type, therefore by statement 2 above, we obtain a finite collection $U_i = \text{Spec}\,(A_i)$ of open affines covering $f^{-1}(V)$. Observe that $U \cap U_i$ is an open set of U_i. By virtue of Lemma 1.4.4.3, we may cover $U \cap U_i$ by basic open sets of U_i which are basic open in U as well. Doing this for each i furnishes us with an open cover of U. As U is quasi-compact as it is affine (Lemma 1.2.1.6), consequently we get a finitely many elements $h_1, \dots, h_n \in A$ such that $D(h_i) \subseteq U$ covers U and furthermore for each $i = 1, \dots, n$, $D(h_i) \cong D(g_i)$ where $D(g_i) \subseteq U_i$ and $g_i \in A_i$. In particular, we have $A_{h_i} \cong (A_i)_{g_i}$. Now for each $i = 1, \dots, n$, we have

$$B \to A_i \to (A_i)_{g_i} \cong A_{h_i}$$

where each of the arrows makes the codomain a finite type algebra over the domain. Hence, A_{h_i} is a finite type B-algebra. Consequently, we have $h_1, \dots, h_n \in A$ such that $\cup_{i=1}^n D(h_i) = U$ (which is equivalent to saying that h_is generate the unit ideal of A by Lemma 1.2.1.5, 2) and A_{h_i} is a finite type B-algebra. It follows from Lemma 23.1.2.11 that A is a finite type B-algebra. This completes the proof. □

We now list out some properties of finite type maps as we shall encounter them quite frequently.

Proposition 1.12.2.5. [42] *Properties of finite type maps.*

1. *A closed immersion $X \to Y$ is of finite type.*
2. *A quasicompact open immersion $X \to Y$ is of finite type.*
3. *Composition of finite type maps $X \to Y \to Z$ is of finite type.*
4. *Product of finite type schemes $X \to S$ and $Y \to S$ in* $\mathbf{Sch}/S$ *denoted $X \times_S Y \to S$ is of finite type.*
5. *Maps of finite type are stable under base extensions.*
6. *If $X \to Y$ is quasicompact and the composite $X \to Y \to Z$ is of finite type, then $X \to Y$ is of finite type.*
7. *If $X \to Y$ is of finite type and Y is noetherian, then X is noetherian.*

Proof. **TODO!** □

The following is something we all expect, which indeed holds true for finite type schemes.

[42] Exercise II.3.13 of Hartshorne.

Lemma 1.12.2.6. *Let k be a field and X be a finite type k-scheme. The set of all closed points of X is dense in X.*

Proof. **TODO**. $\square$

Example 1.12.2.7. We give a number of examples of finite type maps.

1. Let k be a field. Consider the projection map $\pi : \mathbb{A}^2_k \to \mathbb{A}^1_k$ defined by the k-algebra map $k[x] \to k[x,y]$ mapping as $x \mapsto x$. Note that π is a finite type map of schemes as the open covering of $\mathbb{A}^1_k$ as itself yields that $\pi^{-1}(\mathbb{A}^1_k) = \mathbb{A}^2_k$ and $\mathbb{A}^2_k$ is spectra of $k[x,y]$ which is a finite type $k[x]$ algebra via the above map. Indeed, $k[x,y]$ is generated by $\{y\}$ as a $k[x]$-algebra. We deduce that projection maps $\mathbb{A}^n_k \to \mathbb{A}^1_k$ are finite type maps for any $n \in \mathbb{N}$.
2. We next consider a family of curves parameterized by a parameter t. Consider the map
$$\mathbb{C}[t] \longrightarrow \mathbb{C}[t][x,y]\langle y^2 - x^3 - t\rangle.$$
This yields the following map at the level of schemes
$$X := \operatorname{Spec}\left(\frac{\mathbb{C}[t][x,y]}{\langle y^2 - x^3 - t\rangle}\right) \to \operatorname{Spec}(\mathbb{C}[t]).$$
Pick the closed point corresponding to $a \in \mathbb{C}$ in $\operatorname{Spec}(\mathbb{C}[t])$. As $\mathbb{C}[t][x,y]/\langle y^2 - x^3 - t\rangle$ is a finite type $\mathbb{C}[t]$-algebra, therefore the above map of schemes is of finite type.
Observe that the fiber of X at $a \in \operatorname{Spec}(\mathbb{C}[t])$ (by abuse of notation) is given by
$$X_a = X \times_{\operatorname{Spec}(\mathbb{C}[t])} \operatorname{Spec}(\kappa(a)).$$
As $\kappa(a)$ is the fraction field of $\mathbb{C}[t]/\langle t - a\rangle$, which is $\mathbb{C}[a] = \mathbb{C}$, therefore we get the following
$$\begin{aligned} X_a &= \operatorname{Spec}\left(\frac{\mathbb{C}[t][x,y]}{\langle y^2 - x^3 - t\rangle} \otimes_{\mathbb{C}[t]} \mathbb{C}[a]\right) \\ &\cong \operatorname{Spec}\left(\frac{\mathbb{C}[x,y]}{\langle y^2 - x^3 - a\rangle}\right). \end{aligned}$$
Hence, we get the curve $y^2 - x^3 - a$ back as the fiber at the point $a \in \operatorname{Spec}(\mathbb{C}[t])$.
3. Consider the map
$$k[t] \longrightarrow \frac{k[t][w,x,y,z]}{\langle (w-y)^2 + (x-z)^2 - t^2\rangle}$$
which yields the map on geometric level as
$$X := \operatorname{Spec}\left(\frac{k[t][w,x,y,z]}{\langle (w-y)^2 + (x-z)^2 - t^2\rangle}\right) \longrightarrow \operatorname{Spec}(k[t]).$$
Again, this is a finite type map and for a closed point $a \in k$ corresponding to the ideal $\langle t - a\rangle \lneq k[t]$, the fiber is
$$\begin{aligned} X_a &\cong \operatorname{Spec}\left(\frac{k[t][w,x,y,z]}{\langle (w-y)^2 + (x-z)^2 - t^2\rangle} \otimes_{k[t]} k[a]\right) \\ &\cong \operatorname{Spec}\left(\frac{k[w,x,y,z]}{\langle (w-y)^2 + (x-z)^2 - a^2\rangle}\right) \end{aligned}$$
on $\mathbb{A}^2_{\mathbb{R}}$).

4. Any projective variety $X \to \mathbb{P}^n_k$ will by definition be finite type over k (Theorem 1.12.7.2, 1). For example, the projective parabola $X = \text{Proj}\left(\frac{k[x,y,z]}{y^2-xz}\right)$ is a finite type scheme over k. Indeed, by Proposition 1.8.2.8, 1, we get that the natural map $X \to \text{Proj}(k[x,y,z]) = \mathbb{P}^3_k$ coming from the quotient $k[x,y,z] \twoheadrightarrow \frac{k[x,y,z]}{y^2-xz}$ (which is a graded map) is a closed immersion. Hence, it defines a closed subscheme of projective 3-space $\mathbb{P}^3_k$ over k.

1.12.3 Finite

This is a more stronger version of finite type maps discussed in previous section.

Definition 1.12.3.1. (**Finite**) Let $f : X \to Y$ be a map of schemes. Then f is said to be finite if there is an open affine covering $V_i = \text{Spec}\,(B_i)$, $i \in I$ of Y such that $f^{-1}(V_i)$ is equal to an open affine $\text{Spec}\,(A_i)$ where A_i is a finite B_i-algebra[43].

We see that finite maps are local on target.

Proposition 1.12.3.2. *A map $f : X \to Y$ of schemes is finite if and only if for each open affine $V = \text{Spec}\,(B)$, we have $f^{-1}(V)$ is an open affine* $\text{Spec}\,(A)$ *in X such that $B \to A$ makes A a finite B-algebra.*

Proof. (R $\Rightarrow$ L) is immediate from definitions.

(L $\Rightarrow$ R) Pick any open affine $V = \text{Spec}\,(B)$ in Y. We first wish to show that $U = f^{-1}(V)$ is an affine scheme. We may employ criterion for affineness, Proposition 1.3.1.7, for this purpose. Hence for showing that U is affine, we reduce to finding $g_1, \dots, g_n \in \Gamma(\mathcal{O}_{X|U}, U) = \mathcal{O}_X(U)$ such that U_{g_i} is affine and $\langle g_1, \dots, g_n \rangle = \mathcal{O}_X(U)$.

As f is finite, therefore there exists an open affine covering $V_i = \text{Spec}\,(B_i)$ of Y such that $f^{-1}(V_i) = \text{Spec}\,(A_i) = U_i$ is affine and A_i is a finite B_i-algebra. Observe that $V \cap V_i$ forms an open covering of V. As V is affine, so it is quasi-compact (Lemma 1.2.1.6). Consequeently, we obtain a finite cover of V by V_is. Now cover each $V \cap V_i$ by basic opens which are basic in both V and V_i (Lemma 1.4.4.3). Doing this for each of the finitely many i, we obtain a cover of V by basic open sets. As V is quasi-compact (Lemma 1.2.1.6), therefore we have obtained a cover of V by finitely many basics $D(k_i)$ for $k_i \in B$ such that $D(k_i) \cong D(l_i)$ where $D(l_i) \subseteq V_i$ and $l_i \in B_i$ for $i = 1, \dots, n$. Consequently by Lemma 1.2.1.5, the ideal generated by $k_1, \dots, k_n$ in B is the unit ideal.

As we have

$$U = f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^n D(k_i)\right) = \bigcup_{i=1}^n f^{-1}(D(k_i)),$$

therefore by Lemma 1.3.1.4, we may write

$$U = \bigcup_{i=1}^n U_{\varphi(k_i)}$$

where $\varphi : B \to \mathcal{O}_X(U)$ is the map induced by the restricted map $f : U \to V$ on the global sections. Furthermore, as $\sum_{i=1}^n k_i B = B$, therefore $\sum_{i=1}^n \varphi(k_i)\mathcal{O}_X(U) = \mathcal{O}_X(U)$. Hence, it now suffices to

[43] finite algebra := finitely generated as a module.

show that each $U_{\varphi(k_i)}$ is affine.

We have $U_{\varphi(k_i)} = f^{-1}(D(k_i)) \cong f^{-1}(D(l_i)) = D(\varphi_i(l_i))$ where $\varphi_i : B_i \to A_i$ is the map on global sections obtained by the restriction $f : U_i \to V_i$. As $D(\varphi_i(l_i))$ is affine, thus, so is $U_{\varphi(k_i)}$. This shows that indeed, $f^{-1}(V)$ is an open affine.

We may now write $U = \mathrm{Spec}\,(A)$. We reduce now to showing that A is a finite B-algebra. For this observe that in the above, we obtained a finite open cover of U given by $U_{\varphi(k_i)} \cong D(\varphi_i(l_i))$ where $D(\varphi_i(l_i)) \subseteq U_i$. As $U = \mathrm{Spec}\,(A)$, therefore $\mathcal{O}_X(U) = A$, so we may let $\varphi(k_i) = g_i$ for $i = 1, \dots, n$. Now, since $U_{\varphi(k_i)} = U_{g_i} = D(g_i) \cong D(\varphi_i(l_i))$, therefore we have $A_{g_i} \cong (A_i)_{\varphi_i(l_i)}$. As A_i is a finite B_i-algebra, therefore by Lemma 23.1.2.10, $(A_i)_{\varphi_i(l_i)}$ is a finite $(B_i)_{l_i}$-algebra. Further, as we saw in the beginning that $D(k_i) \cong D(l_i)$, hence we get $B_{k_i} \cong B_{l_i}$. We thus obtain a map $B_{k_i} \to A_{g_i}$ as in

$$\begin{tikzcd}
(A_i)_{\varphi_i(l_i)} & (B_i)_{l_i} \arrow[l, "(\varphi_i)_{l_i}"'] \\
A_{g_i} \arrow[u, phantom, "\cong" sloped] & B_{k_i} \arrow[l, dashed] \arrow[u, phantom, "\cong" sloped]
\end{tikzcd}$$

which thus makes A_{g_i} a finite B_{k_i}-algebra, in particular, a finitely generated B_{k_i}-module. This is for each of the $i = 1, \dots, n$, and since we have that $k_1, \dots, k_n$ generates the unit ideal in B, hence by another application of Lemma 23.1.2.10, we deduce that A is a finite B-algebra, as required. □

Base change preserves finiteness.

Proposition 1.12.3.3. *Let $f : X \to S$ be a finite map of schemes. If $g : S' \to S$ is any map, then the map $f' : X' \to S'$ as in the base change*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
S' \arrow[r, "g"'] & S
\end{tikzcd}$$

is finite.

One important property of finite maps is that their fibers are finite.

Proposition 1.12.3.4. [44] *Let $f : X \to Y$ be a finite morphism. Then f is quasi-finite.*

Proof. Pick any point $y \in Y$ and an affine open $V = \mathrm{Spec}\,(B) \ni y$ in Y. As f is finite, therefore by restriction we have map $f : f^{-1}(V) \to V$ where $f^{-1}(V) = U = \mathrm{Spec}\,(A)$ and A is a finite B-algebra. Thus $f^{-1}(y) \subseteq U$ and hence we reduce to the affine case $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$.

Fix $\mathfrak{q} \in Y$. As the fiber $X_{\mathfrak{q}}$ is the base change of $f : X \to Y$ under the inclusion $\mathrm{Spec}\,(\kappa(\mathfrak{q})) \hookrightarrow Y$, thus by Proposition 1.12.3.3 we deduce that $X_{\mathfrak{q}} = \mathrm{Spec}\,(A \otimes_B \kappa(\mathfrak{q}))$ is finite over $\mathrm{Spec}\,(\kappa(\mathfrak{q}))$. In particular, $C = A \otimes_B \kappa(\mathfrak{q})$ is a finite $\kappa(q)$-algebra. By cite[AMD], Exercise 8.3, C is an Artinian ring, as required. □

Another nice property enjoyed by finite maps is that they are closed.

[44] Exercise II.3.5, a) of Hartshorne.

Proposition 1.12.3.5. [45] *Let $f : X \to Y$ be a finite morphism. Then f is a closed map.*

Proof. Our goal is to reduce to the affine case as much as possible, where we have many algebraic results to use. Let $Z \subseteq X$ be a closed subset of X. We wish to show that $f(Z)$ is closed in Y. It first suffices to show that for every open affine $V \subseteq Y$, $V \cap f(Z)$ is closed in V. By definition, $U = f^{-1}(V)$ is an open affine. Consider then the restricted map

$$f : U \cap Z \longrightarrow V \cap f(Z).$$

As $U \cap Z$ is closed in U and $f(U \cap Z) = V \cap f(Z)$, we hence reduce to the assumption that $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ are affine. Let $Z \subseteq X$ be a closed subscheme. Then $Z = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \leq A$. Considering the restriction $f : V(I) \cong \operatorname{Spec}(A/I) \to Y$ and the fact that $A \twoheadrightarrow A/I$ is a finite map, we immediate reduce to the further assumption that $Z = X$.

Consider $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ and $f : X \to Y$ a finite map corresponding to $\varphi : B \to A$. As required, we claim that f has a closed image. Indeed, consider $I = \operatorname{Ann}_B(A)$ to be the annihilator ideal of B-module A. We claim that $\operatorname{Im}(f) = V(I)$. For $(\subseteq)$, pick any $\mathfrak{p} \in \operatorname{Spec}(A)$. We wish to show that $\varphi^{-1}\mathfrak{p} \supseteq I$. It suffices to show that $\mathfrak{p} \supseteq \varphi(I)$. Indeed, for any $b \in I$, we must show $\varphi(b) \in \mathfrak{p}$. As $\varphi(b) \cdot A = 0$, therefore $\varphi(b) \cdot \varphi(b) = 0 \in \mathfrak{p}$ and thus $\varphi(b) \in \mathfrak{p}$, as required. Conversely for $(\supseteq)$, fix a prime $\mathfrak{q} \in V(I)$. We wish to find $\mathfrak{p} \in X$ such that $\mathfrak{q} = \varphi^{-1}\mathfrak{p}$. Indeed, consider the map

$$B \overset{\varphi}{\twoheadrightarrow} \operatorname{Im}(\varphi) =: B' \subseteq A.$$

Note that as φ is finite, therefore φ is integral (Proposition 23.7.1.8). Note that $B' \cong B/\operatorname{Ker}(\varphi)$, induced by $B \twoheadrightarrow B/\operatorname{Ker}(\varphi)$. Observe that as $\operatorname{Ker}(\varphi) \subseteq I$, therefore $\bar{\mathfrak{q}} \leq B'$ is a prime containing $\bar{I} \leq B'$. It follows by Cohen-Seidenberg theorems (Theorem **??**) that there exists $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\mathfrak{p} \cap B' = \bar{\mathfrak{q}}$. Hence it follows at once that $\varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$, as required. ☐

Remark 1.12.3.6. [46] As tempting it might be to say that, but it is not true that a surjective, finite type, quasi-finite map is finite.

Indeed, let k be an algebraically closed field. Consider the map

$$f : \operatorname{Spec}\left(\frac{k[x,y]}{xy-1}\right) \amalg \operatorname{Spec}(k) \longrightarrow \mathbb{A}^1_k$$

induced by $k[x] \to \frac{k[x,y]}{xy-1} \times k$ given by $x \mapsto (x + \langle xy - 1\rangle, 0)$. As a nice exercise, one checks that (we write $a \in \mathbb{A}^1_k$ to mean $\langle x - a\rangle \in \mathbb{A}^1_k$)

1. $f^{-1}(0)$ is a singleton ($\operatorname{Spec}(k)$),
2. $f^{-1}(a)$ is a singleton, given by point (a, a^{-1}) (in particular, the point $\langle x - a, xy - 1\rangle$),
3. the generic fiber $f^{-1}(\mathfrak{o})$ is isomorphic to $\operatorname{Spec}(k(x))$ in $\operatorname{Spec}\left(\frac{k[x,y]}{xy-1}\right)$, hence a singleton.

Consequently, f is surjective, quasi-finite and furthermore of finite type. But still, $\frac{k[x,y]}{xy-1} \times k$ is not a finite $k[x]$-algebra.

Remark 1.12.3.7. Let k be a field. Observe that $\mathbb{A}^1_k \times_k \mathbb{A}^1_k \cong \mathbb{A}^2_k$. However, the underlying set in $\mathbb{A}^2_k$ is not the product of underlying set of $\mathbb{A}^1_k$ with itself. Indeed, this is essentially due to the fact that every prime ideal of $k[x,y]$ is not of form $\mathfrak{p}_1 \times \mathfrak{p}_2$ where $\mathfrak{p}_1 \in k[x]$ and $\mathfrak{p}_2 \in k[y]$, as the prime ideal $xy - 1$ in $k[x,y]$ shows.

[45] Exercise II.3.5, b) of Hartshorne.

[46] Exercise II.3.5, c) of Hartshorne.

Example 1.12.3.8. Consider the canonical map $k[t] \to \frac{k[t,x]}{x^n-t}$ and the corresponding map $X = \mathrm{Spec}\left(\frac{k[t,x]}{x^n-t}\right) \to \mathrm{Spec}\,(k[t]) = \mathbb{A}^1_k$. As $\frac{k[t,x]}{x^n-t}$ is a finite $k[t]$-algebra of rank n, therefore $X \to \mathbb{A}^1_k$ is a finite map. Note that for each closed point $a \in \mathbb{A}^1_k$, the fiber $X_a \cong \mathrm{Spec}\left(\frac{k[x]}{x^n-a}\right)$, which has n closed points if k is algebraically closed and $a \neq 0$.

Any closed immersion is a finite map.

Proposition 1.12.3.9. *Let $i : Z \hookrightarrow X$ be a closed immersion. Then i is a finite map.*

Proof. By Proposition 1.4.4.11, we have an open affine cover $\{V_k\}$ of X such that $i : i^{-1}(V_k) \to V_k$ is a closed immersion. Write $V_k = \mathrm{Spec}\,(A_k)$. Since $i^{-1}(V_k) = Z \cap V_k$ and $Z \cap V_k$ is a closed subscheme of V_k, therefore $Z \cap V_k = \mathrm{Spec}\,(A_k/I_k)$ and the map $\mathrm{Spec}\,(A_k/I_k) \to \mathrm{Spec}\,(A_k)$ is induced from the quotient map $\pi : A_k \twoheadrightarrow A_k/I_k$, which is finite. Hence i is a finite map, as required. □

Generic finiteness

Definition 1.12.3.10 (Generically finite map). Let $f : X \to Y$ be a map of schemes such that Y is irreducible. The map f is said to be generically finite if $f^{-1}(\eta)$ for $\eta \in Y$ the generic point is a finite set.

The following is an important result in this regard, which says, like many statements about generic points, that a generically finite dominant map is *almost* like a finite map.

Theorem 1.12.3.11. [47] *Let X, Y be integral schemes and $f : X \to Y$ be a dominant, generically finite and finite type map. Then there exists a dense open $V \subseteq Y$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is a finite map.*

Proof. We first prove this for X and Y affine integral schemes. We will later reduce to this case. Let $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ be affine schemes where A, B are domains. Let $f : \mathrm{Spec}\,(A) \to \mathrm{Spec}\,(B)$ be a finite type, dominant, generically finite map so that A is a finite type B-algebra. Let this be induced by a finite type ring homomotphism $\varphi : B \to A$. Our first goal is to show that the generic point of X is mapped to generic point of Y and that the induced map of function fields $K(Y) \hookrightarrow K(X)$ is a finite extension.

Indeed, let $\xi \in X$ and $\eta \in Y$ be the generic point of X and Y respectively. By continuity of f, we have $f(\bar{\xi}) \subseteq \overline{f(\xi)}$. As $\bar{\xi} = X$, we have $f(X) \subseteq \overline{f(\xi)}$. As $f(X)$ is dense in Y by dominance of f, we deduce that $Y \subseteq \overline{f(\xi)}$, that is, $f(\xi)$ is a generic point of Y. As schemes are sober and in our case Y is irreducible, therefore Y has a unique generic point which is η. It follows that $f(\xi) = \eta$. Dominance of f further shows that φ is injective since $\xi = \mathfrak{o} \in f^{-1}(\eta) = f^{-1}(\mathfrak{o})$. As $f^{-1}(\mathfrak{o}) = \{\mathfrak{p} \in \mathrm{Spec}\,(A) \mid \varphi^{-1}(\mathfrak{p}) = \mathfrak{o}\}$ therefore if $\mathfrak{o} \in f^{-1}(\mathfrak{o})$, then it follows that $\mathrm{Ker}\,(\varphi) = 0$, that is, φ is injective.

Thus, by considering the comorphism at stalks, we get a map

$$\varphi_{\mathfrak{o}} = f^\sharp_\xi : \mathcal{O}_{Y,f(\xi)} = K(Y) = Q(B) \longrightarrow \mathcal{O}_{X,\eta} = K(X) = Q(A).$$

Note that this map is the field homomorphism induced by $\varphi : B \hookrightarrow A$ on the fraction fields. As this is a map of fields, therefore $\varphi : K(Y) \to K(X)$ is injective. By replacing $K(Y)$ by the image

[47] Exercise II.3.7 of Hartshorne.

of φ, we may assume φ is an inclusion. We wish to show that $K(X)/K(Y)$ is a finite extension.

To this end, we first observe the following by generic finiteness. Let $A = B[\alpha_1, \dots, \alpha_n]$. The fiber at η is

$$\begin{aligned} f^{-1}(\eta) &= \operatorname{Spec}(A \otimes_B \kappa(\eta)) \\ &= \operatorname{Spec}(A \otimes_B Q(B)) \\ &= \operatorname{Spec}(B[\alpha_1, \dots, \alpha_n] \otimes_B Q(B)) \\ &= \operatorname{Spec}(Q(B)[\alpha_1, \dots, \alpha_n]) \end{aligned}$$

Thus by generic finiteness, $\operatorname{Spec}(Q(B)[\alpha_1, \dots, \alpha_n])$ is a finite set. We wish to show that it is discrete so that $f^{-1}(\eta)$ is a finite discrete affine scheme, that is, $Q(B)[\alpha_1, \dots, \alpha_n]$ is an artinian ring. It would thus follow that $Q(B)[\alpha_1, \dots, \alpha_n]$ is an artinian finite type $Q(B)$-algebra and is thus a finite $Q(B)$-algebra. Now, finiteness is preserved under going to fraction fields (Lemma 23.6.1.4) thus $Q(Q(B)[\alpha_1, \dots, \alpha_n])$ is a finite extension of $Q(B)$. But $Q(Q(B)[\alpha_1, \dots, \alpha_n]) = Q(A)$. Hence, $Q(A)$ is a finite extension of $Q(B)$, as required. We thus reduce to proving that the finite spectrum $\operatorname{Spec}(Q(B)[\alpha_1, \dots, \alpha_n])$ is discrete. We wish to show that all finitely many points of it are open. To this end it suffices to show that all finitely many primes of $Q(B)[\alpha_1, \dots, \alpha_n]$ are incomparable. **TODO.**

Thus we have shown that $K(X)/K(Y)$ is a finite extension. Using this, we now find the required open subset $V \subseteq Y$. Indeed, we find a basic open $V = D(b) \subseteq Y$ where $b \in B \subseteq A$ and $f^{-1}(D(b)) = D(b) \subseteq X$ is such that $f : f^{-1}(D(b)) \to D(b)$ is a finite map. That is, we wish to show that there exists $b \in B$ such that $\varphi_b : B_b \hookrightarrow A_b$ is a finite map, using the fact that $Q(B) \hookrightarrow Q(A)$ is a finite extension. Indeed, let $\frac{a_1}{a_1'}, \dots, \frac{a_n}{a_n'}$ be a $Q(B)$-basis of $Q(A)$. Observe that we have

$$Q(A) = Q(B)\frac{a_1}{a_1'} + \dots + Q(B)\frac{a_n}{a_n'}.$$

Thus, multiplying both sides by a_i', we get that there exists $a_1, \dots, a_N \in A$ such that $Q(A)$ is a $Q(B)$-span of $a_1, \dots, a_N$. Denote $A = B[\alpha_1, \dots, \alpha_n]$. Observe that for any $\alpha_i \in A$, the set $\{1, \alpha_i, \dots, \alpha_i^{N-1}, \alpha_i^N\}$ is linearly dependent as its size is greater than the degree $N = [Q(A) : Q(B)]$. Consequently, we see that every α_i^k for $k \geq N$ is a linear combination of $\{1, \alpha_i, \dots, \alpha_i^{N-1}\}$. Now consider any $0 \leq i_1, \dots, i_n$ and the term $\alpha_1^{i_1} \dots \alpha_n^{i_n}$. Then this can be written as linear combination of various $\alpha_1^{j_1} \dots \alpha_n^{j_n}$ where $0 \leq j_1, \dots, j_n \leq N$.

Thus we have a finite collection of terms $\{\alpha_1^{i_1} \dots \alpha_n^{i_n}\}_{0 \leq i_1, \dots, i_n \leq N-1}$ in A. In $Q(A)$, we thus get the following expression for each of them:

$$\alpha_1^{i_1} \dots \alpha_n^{i_n} = \sum_{k=1}^{N} \frac{b_{i_1 \dots i_N, k}}{b'_{i_1 \dots i_N, k}} a_k$$

where $b_{i_1 \dots i_N, k}, b'_{i_1 \dots i_N, k} \in B$. Collect all the finitely many denominators $\{b'_{i_1 \dots i_N, k}\}_{i_1, \dots, i_N, k}$ and consider their product $b \in B$. We claim that the induced map $\varphi_b : B_b \hookrightarrow A_b$ is a finite map.

Indeed, pick any $\frac{a}{b^p} \in A_b$. Then, $a = \sum c_{i_1\dots i_n}\alpha_1^{i_1}\dots\alpha_n^{i_n}$ for $c_{i_1\dots i_n} \in B$. Consequently, we have

$$\begin{aligned} a &= \sum_{i_1,\dots,i_n} c_{i_1\dots i_n}\alpha_1^{i_1}\dots\alpha_n^{i_n} \\ &= \sum_{i_1,\dots,i_n} c_{i_1\dots i_n}\left(\sum_{k=1}^{N}\frac{b_{i_1\dots i_N,k}}{b'_{i_1\dots i_N,k}}a_k\right) \\ &= \sum_{k=1}^{N}\underbrace{\left(\sum_{i_1,\dots,i_n} c_{i_1\dots i_n}\frac{b_{i_1\dots i_N,k}}{b'_{i_1\dots i_N,k}}\right)}_{d_k}a_k \\ &= \sum_{k=1}^{N} d_k a_k \end{aligned}$$

where $d_k \in Q(B)$. Observe that denominator of d_k is some product of elements of $\{b'_{i_1\dots i_N,k}\}_{i_1,\dots,i_N,k}$. Consequently, we get that in A_b, we will have

$$\frac{a}{b^p} = \sum_{k=1}^{N}\frac{d_k}{b^p}a_k$$

where d_k/b^p is an element of B_b since $d_k \in B_b$. Hence, we have shown that there exists elements $a_1,\dots,a_N \in A$ such that A_b is finite over B_b. This completes the proof for affine case.

TODO : General case. □

1.12.4 Separated

This notion corresponds to the Hausdorff property for topological spaces. Recall that a space X is Hausdorff if and only if the diagonal $\Delta : X \to X \times X$ is closed. We shall mimic this in the category of schemes.

Definition 1.12.4.1. (**Separated**) A map $f : X \to Y$ of schemes is said to be separated if the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. A scheme X is said to be separated if $X \to \operatorname{Spec}(\mathbb{Z})$ is separated.

It follows that any map of affine schemes is separated.

Lemma 1.12.4.2. *Let $f : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ be a map of affine schemes. Then f is separated.*

Proof. By Corollary 1.3.0.6, f corresponds to a map of rings $\varphi : B \to A$. Similarly, the diagonal map $\Delta : \operatorname{Spec}(A) \to \operatorname{Spec}(A) \times_{\operatorname{Spec}(B)} \operatorname{Spec}(A)$ corresponds to the B-algebra structure map over A, given by $m : A \otimes_B A \to A$, which is surjective. Consequently, by Corollary 1.4.4.14, Δ is a closed immersion. □

Since any scheme locally is affine, we get a nice consequence of the above lemma.

Lemma 1.12.4.3. *Let $f : X \to Y$ be a map of schemes. Then the following are equivalent.*

1. *f is separated.*
2. *The diagonal $\Delta : X \to X \times_Y X$ has closed image.*

Proof. (1. $\Rightarrow$ 2.) Immediate.
(2. $\Rightarrow$ 1.) By the definition of diagonal, it is immediate that $\Delta : X \to X \times_Y X$ is a homeomorphism onto its image, which is further closed by the given hypothesis. Thus, we need only show that $\Delta^\flat : \mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is a surjective map. By Theorem 27.3.0.6, 3, this is a local property. Consequently, we further reduce to showing that for any point $x \in X$ there is an open set $f(x) \in V \subseteq X \times_Y X$ such that $\Delta^\flat_{|V} : \mathcal{O}_{V,f(x)} \to (\Delta_* \mathcal{O}_{\Delta^{-1}(V)})_{f(x)}$ is surjective. Now we may choose by continuity of f a small affine open $x \in U$ such that $f(U)$ is contained in an affine open V in Y. Consequently, $U \times_V U$ is an affine open subset of $X \times_Y X$ containing $f(x)$. We thus reduce to showing that $\mathcal{O}_{U \times_V U, f(x)} \to (\Delta_* \mathcal{O}_U)_x$ is surjective, which follows immediately from Lemma 1.12.4.2. □

Next, we state an important characterization of separatedness which allows us to derive some very important and convenient results about it.

Theorem 1.12.4.4. *(Valuative criterion of separatedness) Let $f : X \to Y$ be a map of schemes where X is noetherian. Then the following are equivalent,*

1. *f is separated.*
2. *Pick any field K and any valuation ring R with fraction field K (see Section 23.10). Let $i : \operatorname{Spec}(K) \to \operatorname{Spec}(R)$ be the map corresponding to $R \hookrightarrow K$. For all $g : \operatorname{Spec}(R) \to Y$ and $h : \operatorname{Spec}(K) \to X$ such that the square commutes, there exists atmost one lift of g along f as to make the following diagram commute:*

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "h"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}.$$

Proof. See Theorem 4.3, Chapter 2 of cite[Hartshorne]. ☐

The following important corollaries can now easily be derived from this characterization.

Corollary 1.12.4.5. *Let us work in the category of noetherian schemes. Then,*

1. *separated maps are stable under base extension,*
2. *open and closed immersions are separated*[48]*,*
3. *composition of separated maps is separated,*
4. *for a base scheme S, product of any two separated maps is separated in* $\mathbf{Sch}/S$*,*
5. *if the composite $X \to Y \to Z$ is separated, then $X \to Y$ is separated,*
6. *a map $f : X \to Y$ is separated if and only if there is an open cover V_i of Y such that the restricted maps $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is separated*[49]*.*

Proof. **TODO: From notebook.** ☐

An important result about separate schemes is that the intersection of any two open affines is again an open affine.

Lemma 1.12.4.6. [50] *Let X be a separated scheme. If $U, V \subseteq X$ are two open affines then $U \cap V$ is again an open affine.*

Proof. Let $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$. We may replace X by $U \cup V$ and X would still be separated by Corollary 1.12.4.5, 6. Now let $W = U \cap V$. Then again by Corollary 1.12.4.5, 6, we have that W is separated. Consequently, we get that $\Delta : W \to W \times_{\mathbb{Z}} W$ is a closed immersion. We now claim that $W \times_{\mathbb{Z}} W \cong U \times_{\mathbb{Z}} V$. Indeed, this follows immediately from the universal property of fiber product. It follows that $\Delta : W \to \operatorname{Spec}(A \otimes_{\mathbb{Z}} B)$ is a closed immersion. By Corollary 1.4.4.14, W is spectrum of a quotient of $A \otimes_{\mathbb{Z}} B$. Consequently, W is affine, as needed. ☐

Separatedness of projective schemes

We next see that any projective scheme is separated.

Lemma 1.12.4.7. *Let S be a graded ring. Then,* $\operatorname{Proj}(S) \to \operatorname{Spec}(\mathbb{Z})$ *is separated.*

Proof. We need only check that the diagonal $\Delta : \operatorname{Proj}(S) \to \operatorname{Proj}(S) \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Proj}(S)$ has closed image (Lemma 1.12.4.3). Since one can check a closed set locally and sets of the form $D_+(f) \times D_+(g)$ forms an open cover of $\operatorname{Proj}(S) \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Proj}(S)$ for $f, g \in S_+$ homogeneous, therefore we reduce to checking that for $C = \Delta^{-1}(D_+(f) \times D_+(g))$, the restriction $\Delta|_C : C \to D_+(f) \times D_+(g)$ has closed image.

Since $C = D_+(fg) \cong \operatorname{Spec}\left(S_{(fg)}\right)$ and $D_+(f) \times D_+(g) \cong \operatorname{Spec}\left(S_{(f)} \otimes_{\mathbb{Z}} S_{(g)}\right)$, therefore we reduce to showing that the induced map $S_{(f)} \otimes_{\mathbb{Z}} S_{(g)} \to S_{(fg)}$ is surjective. This is clear, as for any $u/f^n g^n \in S_{(fg)}$ where let us denote $k = \deg f, l = \deg g$, for any m large enough such that all exponents in the below are positive, we obtain that

$$\frac{ug^{mk-n}}{f^{ml+n}} \otimes \frac{f^{ml}}{g^{mk}} \mapsto \frac{u}{f^n g^n}.$$

[48] in-fact, any topological immersion is separated, as is clear from the proof.
[49] This doesn't require the noetherian hypothesis.
[50] Exercise II.4.3 of Hartshorne.

Thus, the image of Δ is closed[51]. $\square$

Uniqueness of centers of valuations for varieties

We show a curious property for abstract varieties that any valuation defined over its function field has a unique *center*, if it exists[52]. See Definition 1.4.2.9 for definition of center points of a valuation over function field of an integral scheme.

Lemma 1.12.4.8. [53] *Let X be an integral scheme of finite type over k with function field K. If X is separated, then any valuation over K has a unique center if it exists.*

Proof. We will use the valuative criterion for this. Let $v : K \to G$ be a valuation over K with valuation ring $R \subseteq K$. Let $x, y \in X$ be two centers of v. As $K \subseteq K$, therefore by Lemma 1.6.1.1, 3, there exists a unique map $\operatorname{Spec}(K) \to X$ mapping $\star \mapsto \eta$, where η is the generic point of X. It follows that we have the following commutative square

$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R) & \longrightarrow & \operatorname{Spec}(\mathbb{Z}) \end{array} . \tag{*}$$

As R is a local ring, therefore by Lemma 01J6 of StacksProject, we have a bijection between maps $\operatorname{Spec}(R) \to X$ and tuples (z, φ) where $z \in X$ and $\varphi : \mathcal{O}_{X,z} \to R$ is a local ring homomorphism. Consequently, as $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ are dominated by R, we obtain two tuples (x, ι_x) and (y, ι_y) where $\iota_x : \mathcal{O}_{X,x} \hookrightarrow R$ and $\iota_y : \mathcal{O}_{X,y} \hookrightarrow R$ are the two domination maps. Note that by the definition of domination, these two maps are local ring homomorphisms. Consequently, we get two maps $\operatorname{Spec}(R) \to X$ which makes the (*) commute. By the valuative crieterion of Theorem 1.12.4.4, the two maps $\operatorname{Spec}(R) \to X$ are same, and thus so are the tuples (x, ι_x) and (y, ι_y), proving that $x = y$. $\square$

[51] in-fact we have also shown in the process that Δ is a closed immersion, thus we may not use Lemma 1.12.4.3

[52] It will always exist (and thus be unique) if the variety is proper, as is shown in the next section.

[53] Exericse II.4.5 a) of Hartshorne.

1.12.5 Affine morphisms and global Spec

In this section, we cover important global generalization of $\mathrm{Spec}\,(-)$. In particular, let X be a scheme and $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-algebra, that is, an $\mathcal{O}_X$-module which is a sheaf of rings as well. Then we will construct a scheme $\mathbf{Spec}(\mathcal{F})$ over X which will behave as if it is constructed out of open affine subschemes U of X and the corresponding algebras $\mathcal{F}(U)$.

This construction will be used to show how locally free sheaf of constant rank actually corresponds to vector bundles. They are used elsewhere as well.

Definition 1.12.5.1 (**Affine morphism**)**.** A map $f : X \to Y$ of schemes is called an affine morphism if there is an affine open cover $\{V_\alpha\}$ of Y such that $f^{-1}(V_\alpha)$ is an open affine scheme.

Remark 1.12.5.2. It follows from definition that any finite morphism is affine.

The first major property of affine maps is that they are local on target.

Proposition 1.12.5.3. *Let $f : X \to Y$ be a map. Then the following are equivalent.*

1. *f is affine.*
2. *For any open affine $V \subseteq Y$, $f^{-1}(V)$ is an open affine in X.*

Proof. We need only do $1 \Rightarrow 2$. This has been done in the proof of Proposition 1.12.3.2. □

Lemma 1.12.5.4. *Let $f : X \to Y$ be an affine morphism. Then f is quasicompact and separated.*

Proof. The fact that f is quasicompact is immediate by definition. Separatedness follows from Corollary 1.12.4.5, 6 and Lemma 1.12.4.2. □

The main theorem for affine maps is that they all come from quasicoherent algebras over the structure sheaf. Indeed, we have the following construction to obtain a scheme over Y by a quasicoherent $\mathcal{O}_Y$-algebra.

Theorem 1.12.5.5. [54] *Let Y be a scheme and $\mathcal{A}$ be a quasicoherent $\mathcal{O}_Y$-algebra over Y. Then there exists a scheme*

$$f : \mathbf{Spec}(\mathcal{A}) \to Y$$

unique with respect to the property that for any open affine $V \subseteq Y$, we have $f^{-1}(V) \cong \mathrm{Spec}\,(\mathcal{A}(V))$ and for any inclusion $U \hookrightarrow V$ of open affines, the map $f^{-1}(U) \to f^{-1}(V)$ is induced by the restriction map $\rho : \mathcal{A}(V) \to \mathcal{A}(U)$.

Proof. Let $V = \mathrm{Spec}\,(B) \subseteq Y$ be an open affine in Y. Then we have a ring homomorphism $B \to \mathcal{A}(V)$ as $\mathcal{A}(V)$ is a B-algebra. Consequently, we get the map $\pi_V : \mathrm{Spec}\,(\mathcal{A}(V)) \to Y$ factoring through V. Observe that for any open affine $U \hookrightarrow V$, we have the following commutative triangle

$$\begin{tikzcd} \mathcal{A}(V) \arrow[dr, "\rho"] & \\ B \arrow[u] \arrow[r] & \mathcal{A}(U) \end{tikzcd}.$$

[54] Exercise II.5.17, c) of Harthsorne.

We now wish to glue the affine schemes $\pi_i : \mathrm{Spec}\,(\mathcal{A}(V_i)) \to Y$ where V_i varies over open affines of Y. Indeed, let $X_i = \mathrm{Spec}\,(\mathcal{A}(V_i))$ and $U_{ij} = \pi_i^{-1}(V_i \cap V_j)$ an open subscheme of X_i. We claim that there is a natural isomorphism $\varphi_{ij} : U_{ij} \to U_{ji}$ which satisfies the cocycle condition, so that we can glue these schemes together by Proposition 1.6.2.2 to get the desired scheme unique with the given properties. Indeed, to find φ_{ij}, we first observe that for $\pi_V : \mathrm{Spec}\,(\mathcal{A}(V)) \to Y$, we have that $\pi_{V*}\mathcal{O}_{\mathrm{Spec}(\mathcal{A}(V))} \cong \mathcal{A}_{|V}$. This is where quasicoherence is used and follows from checking on basis and using globalized restriction of scalars (Lemma 1.2.3.4). Using this isomorphism, we see that $\mathcal{O}_{X_i}(\pi_i^{-1}(V_i \cap V_j)) \cong \mathcal{A}(V_i \cap V_j) \cong \mathcal{O}_{X_j}(\pi_j^{-1}(V_j \cap V_i))$. Consequently, we have a commutative triangle where $V_i = \mathrm{Spec}\,(B_i)$

$$\begin{tikzcd} \mathcal{A}(V_i) \arrow[dr, "\rho"] & \\ B_i \arrow[u] \arrow[r] & \mathcal{O}_{X_j}(\pi_j^{-1}(V_i \cap V_j)) \end{tikzcd}.$$

By Theorem 1.3.0.5, we get the following commutative triangle

$$\begin{tikzcd} X_i \arrow[d, "\pi_i"'] & \\ V_i & \pi_j^{-1}(V_i \cap V_j) \arrow[l] \arrow[ul, "\varphi_{ji}"'] \end{tikzcd}$$

By commutativity of this triangle, it follows that the unique morphism φ_{ji} factors through $\pi_i^{-1}(V_i \cap V_j)$. Interchanging i and j we get that φ_{ji} is an isomorphism. By uniqueness of φ_{ij}, we further get the the cocycle condition, as required. $\square$

We see from the proof the following.

Corollary 1.12.5.6. *Let Y be a scheme, $\mathcal{A}$ a quasicoherent $\mathcal{O}_Y$-algebra and $f : \mathbf{Spec}(\mathcal{A}) \to Y$ the global spec. Then, $f_*\mathcal{O}_{\mathbf{Spec}(\mathcal{A})} \cong \mathcal{A}$.*

Proof. In the proof, we showed that for any open affine $V \subseteq Y$, we have $f_*\mathcal{O}_{\mathbf{Spec}(\mathcal{A})|f^{-1}(V)} \cong \pi_{V*}\mathcal{O}_{\mathrm{Spec}(\mathcal{A}(V))} \cong \mathcal{A}_{|V}$ and this isomorphism is compatible with restrictions. Consequently, we have an isomorphism between $f_*\mathcal{O}_{\mathbf{Spec}(\mathcal{A})}$ and $\mathcal{A}$ over a base, which gives the required isomorphism as sheaves over Y. $\square$

It is immediate to see by above theorem that global spec is always affine over the base.

Corollary 1.12.5.7. *Let Y be a scheme and $\mathcal{A}$ a quasicoherent $\mathcal{O}_Y$-algebra. Then the morphism*

$$f : \mathbf{Spec}(\mathcal{A}) \to Y$$

is affine. $\square$

We now prove the converse of the above corollary.

Proposition 1.12.5.8. *Let $f : X \to Y$ be an affine morphism. Then,*

1. *$f_*\mathcal{O}_X$ is a quasicoherent $\mathcal{O}_Y$-algebra,*

2. there is an isomorphism

$$X \cong \mathbf{Spec}(f_* \mathcal{O}_X).$$

Proof. 1. This is immediate from the fact that the morphism f is quasicompact and separated by Lemma 1.12.5.4 (Lemma 1.9.1.16).

2. Let $\{V_\alpha\}$ be a basis consisting of open affines of Y. Then, $\{f^{-1}(V_\alpha)\}$ is an open affine basis of X by Proposition 1.12.5.3. Then, we have a canonical isomorphism $f^{-1}(V_\alpha) \cong \operatorname{Spec}\left(\mathcal{O}_X(f^{-1}(V_\alpha))\right)$. Moreover, for $V_\alpha \hookrightarrow V_\beta$, we have $f^{-1}(V_\alpha) \hookrightarrow f^{-1}(V_\beta)$ obtained by restriction $\rho : \mathcal{O}_X(f^{-1}(V_\beta)) \to \mathcal{O}_X(f^{-1}(V_\alpha))$. Hence by uniqueness of Theorem 1.12.5.5, we conclude the proof. □

We may sum this up in the following bijection.

Corollary 1.12.5.9. [55] *Let Y be a scheme. We have the following bijection*

$$\left\{\text{Affine morphisms } X \xrightarrow{f} Y\right\} \cong \left\{\begin{matrix}\text{Quasicoherent} & \mathcal{O}_Y\text{-} \\ \text{algebras } \mathcal{A} & \end{matrix}\right\}$$

established by $f \mapsto f_*\mathcal{O}_X$ and $\mathbf{Spec}(\mathcal{A}) \leftarrow \mathcal{A}$. □

We see that any closed immersion is an affine map.

Proposition 1.12.5.10. *Let $i : Z \to X$ be a closed immersion. Then i is an affine map.*

Proof. By Proposition 1.4.4.11, there is an open affine cover $\{V_k\}$ of X such that $i : Z \cap V_k \hookrightarrow V_k$ is a closed immersion. Denote $V_k = \operatorname{Spec}(A_k)$. Thus, $Z \cap V_k = \operatorname{Spec}(A_k/I_k)$ and hence $i^{-1}(V_k) = \operatorname{Spec}(A_k/I_k)$, as required. Alternatively, it follows from the fact that any closed immersion is a finite map (Proposition 1.12.3.9). □

Example 1.12.5.11 (A non-affine map)**.** Consider the map $\mathbb{A}^2 \setminus \{0\} \hookrightarrow \mathbb{A}^2$. We claim that this is not an affine map. Indeed, assuming to the contrary, there is a basic open affine $U = D(f)$ for $f \in k[x, y]$ of $\mathbb{A}^2$ containing 0 such that $U \setminus 0$ is open affine. But as can be checked, the coordinate rings of U and $U \setminus 0$ are isomorphic. It follows at once that $U \cong \operatorname{Spec}(k[x, y]_f) \cong U \setminus 0$, a contradiction to the fact that $U \ncong U \setminus 0$.

[55] Exercise II.5.17, d) of Hartshorne.

1.12.6 Proper

This and the next section brings us closer to detecting when a scheme is projective (i.e. is a subscheme of projective scheme). Proper maps corresponds roughly to the intuition that the scheme $X \to Y$ should not have any *missing points*.

Definition 1.12.6.1. (**Universally closed and proper maps**) A map $f : X \to Y$ is said to be universally closed if f is closed and for any base extension $Y' \to Y$, the base extension of X, denoted $f' : X' \to Y'$ is also closed as in the diagram below:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}.$$

Consequently, f is said to be proper if it is separated, finite type and is universally closed.

The main result here is again a valuative criterion which allows a lot of properties of such maps to be derived quite easily.

Theorem 1.12.6.2. *(Valuative criterion of properness) Let $f : X \to Y$ be a finite type map of schemes where X is noetherian. Then the following are equivalent.*

1. *f is proper.*
2. *Pick any field K and any valuation ring R with fraction field K (see Section 23.10). Let $i : \mathrm{Spec}\,(K) \to \mathrm{Spec}\,(R)$ be the map corresponding to $R \hookrightarrow K$. For all $g : \mathrm{Spec}\,(R) \to Y$ and $h : \mathrm{Spec}\,(K) \to X$ such that the square commutes, there exists a unique lift of g along f as to make the following diagram commute:*

$$\begin{tikzcd} \mathrm{Spec}\,(K) \arrow[r, "h"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \mathrm{Spec}\,(R) \arrow[r, "g"'] \arrow[ur, dashed, "!"] & Y \end{tikzcd}.$$

Note that whereas in Theorem 1.12.4.4 we had that there exists *atmost one* lift, here we have that there exists *unique* lift (it exists and there is only one).

Corollary 1.12.6.3. *Let us work in the category of noetherian schemes. Then,*

1. *if $X \to Y \to Z$ is proper and $Y \to Z$ is separated, then $X \to Y$ is proper,*
2. *closed immersion are proper,*
3. *proper maps are stable under base extensions,*
4. *composite of proper maps is proper,*
5. *for two proper schemes $X \to S, Y \to S$ in $\mathbf{Sch}/S$, their product $X \times_S Y \to S$ is proper,*
6. *a map $f : X \to Y$ is proper if and only if there exists an open cover V_i of Y such that the restriction $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is proper.*

Proof. **TODO : From notebook.** ∎

1.12.7 Projective

We now define maps of schemes which factors through a projective space over the target. This will be fundamental, as the most natural type of schemes we find in nature are projective varieties appearing as closed subschemes of the projective space over a field. Though we will work more generally, but this will pay off in some of the later discussions. See Definition 1.8.2.14 for projective spaces over a scheme.

Definition 1.12.7.1. (**Projective and quasi-projective maps**) Let $f : X \to Y$ be a map of schemes. We say f is projective if there exists an $n \in \mathbb{N}$ such that f factors as a closed immersion $X \to \mathbb{P}^n_Y$ followed by the struture map $\mathbb{P}^n_Y \to Y$ as in

$$\begin{tikzcd} X \arrow[r, "\text{cl. imm.}"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}.$$

Further, a map $f : X \to Y$ is said to be *quasi-projective* if f factors first into an open immersion $X \to X'$ and then a projective map $X' \to Y$ as in

$$\begin{tikzcd} X' \arrow[r, "\text{cl. imm.}"] & \mathbb{P}^n_Y \arrow[d] \\ X \arrow[u, "\text{op. imm.}"] \arrow[r, "f"'] & Y \end{tikzcd}.$$

Thus quasi-projective maps corresponds to the usual notion of quasi-projective varieties (open subsets of projective varieties in a projective n-space).

The important point to keep in mind about projective maps is that they are proper.

Theorem 1.12.7.2. *Let X and Y be noetherian schemes.*

1. *If $f : X \to Y$ is projective, then f is proper.*
2. *If $f : X \to Y$ is quasi-projective, then f is finite type and separated.*

Proof. 1. Since closed immersions are proper and proper maps are stable under base change (Corollary 1.12.6.3), we may reduce to showing that for each $n \in \mathbb{N}$, the scheme $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ is proper. It is clear that $\mathbb{P}^n_{\mathbb{Z}}$ is finite type $\mathbb{Z}$-scheme which is furthermore separated by Corollary 1.12.4.7.

In order to show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper, we will proceed by induction over n. For $n = 0$, we have $\mathbb{P}^0_{\mathbb{Z}} \cong \operatorname{Spec}(\mathbb{Z})$, which is trivially proper over $\operatorname{Spec}(\mathbb{Z})$. Now suppose $\mathbb{P}^{n-1}_{\mathbb{Z}}$ is proper over $\operatorname{Spec}(\mathbb{Z})$. We wish to show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper. We will use valuative criterion for this (Theorem 1.12.6.2). Consider a valuation ring R with fraction field K such that we have maps g, h making the following commute:

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "h"] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ \operatorname{Spec}(R) \arrow[r, "g"'] & \operatorname{Spec}(\mathbb{Z}) \end{tikzcd}.$$

Consequently, we wish to define a unique map $\mathrm{Spec}\,(R) \to \mathbb{P}^n_{\mathbb{Z}}$ which makes everything commute.

Denote $\mathrm{Spec}\,(K) = \{\star\}$ and $\xi = h(\star) \in \mathbb{P}^n_{\mathbb{Z}}$. We now observe that if $\xi \in V(x_{i_0})$ for any $i_0 = 0, \dots, n$, then by the natural isomorphism $V(x_{i_0}) \cong \mathbb{P}^{n-1}_{\mathbb{Z}}$ and obvious restrictions, we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spec}\,(K) \arrow[r, "h"] \arrow[d] & \mathbb{P}^{n-1}_{\mathbb{Z}} \arrow[d] \\
\mathrm{Spec}\,(R) \arrow[r, "g"'] & \mathrm{Spec}\,(\mathbb{Z})
\end{tikzcd}.$$

Consequently, by inductive hypothesis, we have a unique lift $\mathrm{Spec}\,(R) \to \mathbb{P}^{n-1}_{\mathbb{Z}}$ and thus a map $\mathrm{Spec}\,(R) \to \mathbb{P}^n_{\mathbb{Z}}$ making the diagram commutative. This is sufficient by the fact that $\mathbb{P}^n_{\mathbb{Z}}$ is separated (Lemma 1.12.4.7) and by valuative criterion (Theorem 1.12.4.4).

We next need to cover the case when ξ is not in any hyperplane $V(x_i)$, that is, when $\xi \in \bigcap_{i=0}^n D_+(x_i)$. We will construct a map $\mathrm{Spec}\,(R) \to \mathbb{P}^n_{\mathbb{Z}}$ which makes everything commute and we will be done by separatedness of $\mathbb{P}^n_{\mathbb{Z}}$ (Lemma 1.12.4.7). In this case, we obtain that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},\xi} \cong \mathbb{Z}[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i]$ for all $i = 0, \dots, n$ as $D_+(x_i) \cong \mathrm{Spec}\,\left(\mathbb{Z}[x_0, \dots, x_n]_{(x_i)}\right)$, (Lemma 1.8.2.4). Consequently, $x_i/x_j \in \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},\xi}$ is invertible for all $i, j = 0, \dots, n$, hence $x_i/x_j \notin \mathfrak{m}_{\mathbb{P}^n_{\mathbb{Z}},\xi}$. Denote further $f_{ij} \in \kappa(\xi)$ to be the image of x_i/x_j under the map $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},\xi} \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},\xi}/\mathfrak{m}_{\mathbb{P}^n_{\mathbb{Z}},\xi} = \kappa(\xi)$.

The map $h : \mathrm{Spec}\,(K) \to \mathbb{P}^n_{\mathbb{Z}}$ is equivalent to the data of the point $\xi \in \mathbb{P}^n_{\mathbb{Z}}$ and $\kappa(\xi) \hookrightarrow K$ (Lemma 1.6.1.1). Thus we have $f_{ij} \in K$ for all $i, j = 0, \dots, n$. In order to define the map $j : \mathrm{Spec}\,(R) \to \mathbb{P}^n_{\mathbb{Z}}$ in this case, it is sufficient to obtain a map $\mathbb{Z}[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i] \to R$ such that the following commutes:

$$\begin{tikzcd}
K & Z[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i] \arrow[l] \arrow[dl] \\
R \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}.$$

We will now construct such a map. Let $v : K \to G$ be the valuation corresponding to the valuation ring R (so that R is the value ring of v), where G is a totally ordered abelian group. Consider the collection of elements $f_{10}, \dots, f_{n0} \in K$ and denote $g_i = v(f_{i0}) \in G$. Let $g_m = \min_i g_i$. Consequently, for each $i = 0, \dots, n$ we obtain $0 \le g_i - g_m = v(f_{i0}) - v(f_{m0}) = v(f_{i0} f_{0m}) = v(f_{im})$. Thus, $f_{im} \in R$. Hence, we can construct the following map:

$$\begin{aligned}
Z[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i] &\longrightarrow R \\
\frac{x_i}{x_m} &\longmapsto f_{im}.
\end{aligned}$$

It is immediate that the above map makes the above diagram commute.
2. Since open immersions are separated (Corollary 1.12.4.5) and an open immersion $X \to X'$ where X is noetherian is immediately quasicompact, so by Proposition 1.12.2.5, 2, the result follows. $\square$

1.12.8 Flat

Look at the following MO post for more clarifications. Flat maps of schemes capture the notion of a "continuous family of schemes parameterized by points of a base scheme". However, the notion of flatness is very algebraic, as we shall soon see. We collect the properties of flat modules in the Special Topics, Chapter 23.

1.12.9 Smooth

1.12.10 Unramified

1.12.11 Étale

Étale maps is the place from where one enters the land of algebraic topology via algebraic geometry. Indeed, the fundamental goal here is to capture the notion of local isomorphism but in an algebraic context. The simplest place where one can understand them is a restricted version of this called finite étale maps. This is where we begin from as we shall need this in our discussion of Galois theory of schemes.

Finite étale

We refer to Algebra, Chapter 23 for background on separable algebras, in particular, to Definition 23.22.2.2 for free separable algebras.

We now define finite étale maps.

Definition 1.12.11.1. (**Finite étale scheme**) Let X be a base scheme. An X-scheme $p : Y \to X$ is said to be finite étale if there is an open affine covering of X given by $\{\mathrm{Spec}\,(A_i)\}_{i\in I}$ such that $p^{-1}(\mathrm{Spec}\,(A_i))$ is an open affine subscheme of Y given by $\mathrm{Spec}\,(B_i)$ such that the induced map $A_i \to B_i$ makes B_i a free separable A_i-algebra, for all $i \in I$. In such a situation, one calls Y a finite étale covering of X. Denote the category $\mathbf{Et}_{\mathrm{fin}}(X)$ to be the full subcategory of $\mathbf{Sch}/X$ consisting of finite étale coverings of X.

Let us now give an example of finite étale scheme.

Example 1.12.11.2.

1.13 Coherent and quasicoherent sheaf cohomology

All schemes in this section are Noetherian. Cohomology will serve as an important tool to derive invariants on a given scheme. We would need the cohomology of abelian sheaves over a space (Chapter 27) and the notion of Noetherian schemes (Section 1.4) in this section. *Apart from Mumford, you may like to give Hida a visit.*

We refer to Topics in Sheaf Theory, Chapter 27 for classical Čech cohomology, derived functor cohomology and relations between them on topological spaces.

Since we are only dealing with noetherian schemes and the most important such schemes would be those which are closed subvarieties of projective space, so finite dimensional, therefore the following theorem of Grothendieck is of particular importance.

Theorem 1.13.0.1 (Grothendieck)**.** *Let X be a noetherian topological space of dimension n and $\mathcal{F}$ be an abelian sheaf over X. Then*

$$H^i(X, \mathcal{F}) = 0$$

for all $i > n$.

We now show some basic theorems in cohomology of sheaves over schemes which allows us to use Čech-cohomology for calculations instead of derived functor cohomology, because both becomes isomorphic.

1.13.1 Quasicoherent sheaf cohomology

Do from Hartshorne and Bruzzo.

1.13.2 Application : Serre-Grothendieck duality

Do from Hida and Hartshorne.

1.13.3 Application : Riemann-Roch theorem for curves

Do from Hida and Hartshorne.