# SIMPLICIAL SETS & THE BAR-COBAR CONSTRUCTIONS

Contents



## 1. Simplicial sets

Let $Z$ be a topological space. One way to understand $Z$ is via understanding all the singular chains $S_*(Z)$ of $Z$ and how they relate to each other.

**Remark 1.0.1** (Face & degeneracy maps for $S_*(Z)$)**.** There are natural functions one can define on $X = S_*(Z)$ by using the combinatorics of the standard $n$-simplex $|\Delta^n|$. Recall that

$$|\Delta^n| := \{(e_0, \dots, e_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} e_i = 1, 1 \geq e_i \geq 0\}.$$

Consequently, we have maps

$$\begin{aligned} d^i : \left|\Delta^{n-1}\right| &\longrightarrow |\Delta^n| \\ (e_0, \dots, e_{n-1}) &\longmapsto (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) \end{aligned}$$

and

$$\begin{aligned} \rho^i : \left|\Delta^{n+1}\right| &\longrightarrow |\Delta^n| \\ (e_0, \dots, e_{n+1}) &\longmapsto (e_0, \dots, e_{i-1}, e_i + e_{i+1}, \dots, e_{n+1}). \end{aligned}$$

*Date*: March 14, 2025.

Using these two maps, we may define the following maps on singular chains for each $0 \leq i \leq n$:

$$\begin{aligned} \partial_i : X_n &\longrightarrow X_{n-1} \\ \sigma &\longmapsto \sigma \circ d^i \end{aligned}$$

and

$$\begin{aligned} s_i : X_n &\longrightarrow X_{n+1} \\ \sigma &\longmapsto \sigma \circ \rho^i . \end{aligned}$$

The maps $\partial_i$ and $s_i$ are called the face and degeneracy maps of $X$, respectively.

**Lemma 1.0.2.** *The face and degeneracy maps of $X = S_*(Z)$ satisfies the following identities (called simplicial identities):*

$$\begin{aligned} &\partial_i \partial_j = \partial_{j-1} \partial_i \ \textit{if } i < j \\ &s_i s_j = s_{j+1} s_i \ \textit{if } i \leq j \\ &\partial_i s_j = \begin{cases} s_{j-1} \partial_i & \textit{if } i < j, \\ \mathrm{id} & \textit{if } i = j, j+1, \\ s_j \, \partial_{i-1} & \textit{if } i > j+1. \end{cases} \end{aligned}$$

*Proof.* It follows at once that we have to show the following co-simplicial identities:

$$\begin{aligned} &d^j d^i = d^i d^{j-1} \text{ if } i < j \\ &\rho^j \rho^i = \rho^i \rho^{j+1} \text{ if } i \leq j \\ &\rho^j d^i = \begin{cases} d^i \rho^{j-1} & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j, j+1, \\ d^{i-1} \rho^j & \text{if } i > j+1. \end{cases} \end{aligned}$$

These are immediate from the definitions. □

This motivates the following definition.

**Definition 1.0.3 (Simplicial set).** A simplicial set is a sequence of sets $X = \{X_n\}_{n \geq 0}$ together with maps one for each $0 \leq i \leq n$:

$$\partial_i : X_n \longrightarrow X_{n-1} \ \& \ s_i : X_n \longrightarrow X_{n+1}$$

which satisfies the simplicial identities as stated in Lemma 1.0.2. A simplicial map $f : X \to Y$ is a collection of maps $\{f_n : X_n \to Y_n\}_{n \geq 0}$ which are natural w.r.t. face and degeneracy:

$$f_{n-1} \, \partial_i = \partial_i \, f_n \ \& \ f_{n+1} s_i = s_i f_n .$$

We hence get a category of simplicial sets and simplicial maps, denoted $\mathcal{sSet}$.

**Remark 1.0.4.** By Lemma 1.0.2, $S_*(Z)$ for any space $Z$ is a simplicial set and it thus follows that we have a functor

$$S : \mathcal{T}\mathrm{op} \longrightarrow \mathrm{s}\mathcal{S}\mathrm{et} .$$

One of the main goals of this paper is to establish that upto homotopy, this map loses no further information.

**Remark 1.0.5** (The Kan filler condition)**.** Let $\Delta^n$ be the standard topological $n$-simplex. Note that

$$|\partial_i \Delta^n| = \mathrm{Im}\left(d^i\right) = \{(e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) \mid (e_0, \dots, e_{n-1}) \in \left|\Delta^{n-1}\right|\}.$$

We define the $k^{\text{th}}$-horn of $|\Delta^n|$ for $0 \leq k \leq n$ as

$$|\Lambda_k^n| := \bigcup_{i \neq k}^{n} |\partial_i \Delta^n|,$$

that is, the union of all the faces of $|\Delta^n|$ except the one opposite to $k^{\text{th}}$-vertex. Note that we have $n$ many inclusion maps one for each $0 \leq i \leq n$ and $i \neq k$

$$\begin{aligned} \iota_i : \left|\Delta^{n-1}\right| &\longrightarrow |\Lambda_k^n| \subseteq |\Delta^n| \\ (e_0, \dots, e_{n-1}) &\longmapsto (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}). \end{aligned}$$

An important observation is that there is a retraction

$$r_k : |\Delta^n| \twoheadrightarrow |\Lambda_k^n|.$$

Indeed, consider a line passing through the $k^{\text{th}}$-vertex $v_k = (0, \dots, 0, 1, 0, \dots, 0)$ and pick a point on it outside the simplex $|\Delta^n|$, say $p$. Using $p$, define $r_k$ on $x \in |\Delta^n|$ as that point on $|\Lambda_k^n|$ which is obtained by intersection of the horn with the line joining $p$ and $x$. This map is clearly identity on the horn.

It follows that for any space and any map $\sigma : |\Lambda_k^n| \to Z$, composing with $r_k$ gives a singular $n$-simplex $\sigma \circ r_k : |\Delta^n| \to Z$. If $\tau_0, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n \in S_{n-1}(Z)$ are $n$ many $n-1$-singular simplices such that they can glue to form a map from the $k^{\text{th}}$-horn $|\Lambda_k^n| \to Z$, then by above discussion it would follow that we get an $n$-simplex $\tau \in S_n(Z)$.

We wish to rigorously state the last condition on gluing $n$ many $n-1$-simplices to a horn $\Lambda_k^n$.

**Lemma 1.0.6.** *Let $\tau_0, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n \in S_{n-1}(Z)$ be $n$ many $n-1$-singular simplices of space $Z$ and $0 \leq k \leq n$. Then the following are equivalent:*

1. *The simplices $\tau_i$ glues to a map $\tau : |\Lambda_k^n| \to Z$ where $\tau\iota_i = \tau_i$ for $0 \leq i \leq n$ and $i \neq k$.*
2. *The simplices $\tau_i$ satisfies the following conditions:*

$$\partial_i \tau_j = \partial_{j-1} \tau_i \textit{ for } i < j, i \neq k, j \neq k.$$

*Proof.* (1. $\Rightarrow$ 2.) We observe that $\partial_i \tau_j = \tau\iota_j d^i$ and $\partial_{j-1} \tau_i = \tau\iota_i d^{j-1}$. Hence we need only show that

$$\iota_j d^i = \iota_i d^{j-1}.$$

This is a simple check.

(2. $\Rightarrow$ 1.) Define maps on the image of each $\iota_i$ by $\tau_i$:

$$\begin{aligned} \tilde{\tau}_i : \mathrm{Im}\left(\iota_i\right) &\longrightarrow Z \\ (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-1}) &\longmapsto \tau_i(e_0, \dots, e_{i-1}, e_i, \dots, e_{n-1}). \end{aligned}$$

By pasting lemma, we need only check that for $\tilde{\tau}_i, \tilde{\tau}_j$, $i < j$, we have

$$\tilde{\tau}_i|_{\mathrm{Im}(\iota_j)} = \tilde{\tau}_j|_{\mathrm{Im}(\iota_i)}.$$

Pick $p \in \mathrm{Im}\,(\iota_i) \cap \mathrm{Im}\,(\iota_j)$. Then $p = (p_0, \dots, p_n)$ where $p_i = p_j = 0$. Hence, we have by definitions that

$$\begin{aligned}
\tilde{\tau}_i(p) &= \tau_i(p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_j, p_{j+1}, \dots, p_n) \\
&= \tau_i d^{j-1}(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_{j+1}, \dots, p_n) \\
&= \tau_j d^i(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_{j+1}, \dots, p_n) \\
&= \tau_j(p_0, \dots, p_{i-1}, p_i, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n) \\
&= \tilde{\tau}_j(p_0, \dots, p_{i-1}, p_{i+1}, p_{j-1}, p_j, p_{j+1}, \dots, p_n) \\
&= \tilde{\tau}_j(p),
\end{aligned}$$

as required. □

This motivates the following condition.

**Definition 1.0.7** (**Horns, Kan extension condition & Kan complexes**). Let $X$ be a simplicial set. An $(n,k)$-horn for $0 \le k \le n$ is a collection of $n$ many $n-1$-simplices $x_0, \dots, x_{k-1}, x_k, \dots, x_n \in X_{n-1}$ such that for all $i < j$, $i \neq k$, $j \neq k$, we have

$$\partial_i\, x_j = \partial_{j-1}\, x_i.$$

The simplicial set $X$ is said to satisfy the Kan extension condition if for all $(n,k)$-horns $\{x_i\}$ of $X$, there exists an $n$-simplex $x \in X_n$ such that for all $i \neq k$,

$$\partial_i x = x_i.$$

A simplicial set satisfying Kan extension condition is called a Kan complex, or sometimes an $\infty$-groupoid.

The following result follows at once from Remark 1.0.5.

**Corollary 1.0.8.** *For any space $Z$, the simplicial set $S_*(Z)$ is a Kan complex.* □

**Remark 1.0.9.** By the above result, one may consequently study Kan complexes in themselves, thinking of them as a generalization of spaces. This is a fruitful endeavour, which ends with one establishing that homotopy theory of Kan complexes is "same" as that of CW-complexes.

We next wish to establish a more functorial way of constructing simplicial sets.

**Definition 1.0.10** (**Ordinal category**). Let $\mathbf{\Delta}$ be the category whose objects are defined as

$$[n] := 0 < 1 < 2 < \cdots < n$$

the toset of first $n$ non-negative integers and maps $f : [n] \to [m]$ are defined to be monotone non-decreasing maps. There are two distinguished classes of maps for each $n$ and $0 \le i \le n$:

$$d^i : [n-1] \longrightarrow [n] \;\&\; \rho^i : [n+1] \longrightarrow [n]$$

where

$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \ge i \end{cases} \;\&\; \rho^i(k) = \begin{cases} k & \text{if } k \le i \\ k-1 & \text{if } k > i. \end{cases}$$

These maps are called coface and codegeneracy maps, respectively.

An important aspect of the category $\mathbf{\Delta}$ is that all monotone maps can be generated by coface and codegeneracy maps.

**Remark 1.0.11.** Let $f : [n] \to [m]$ be a monotone map. We claim that $f = d_f \rho_f$ where $d_f$ is composite of certain cofaces $d^i$ and $\rho_f$ is composite of certain codegeneracies $\rho^j$. Observe that if $i \in [m]$ is such that $f^{-1}(i)$ is of size $l$, then by monotonicity, we must have $f(k) = f(k+1) = \cdots = f(k+l-1) = i$. Observe that $f$ partitions $n$ via its fibers. Let $\{i_0, \dots, i_k\}$ be the ordered image of $f$ and let $n_p = \left|f^{-1}(i_p)\right|$. Consequently, we may consider the monotone map $g : [n] \to [k]$ where $g(f^{-1}(i_p)) = \{p\}$ for each $0 \le p \le k$. Clearly, a composition of certain cofaces $d^i$ will give a map $d_f : [k] \to [m]$ such that $d_f g = f$. It hence suffices to show that $g$ can be written as a composite of certain codegeneracies $\rho^i$. To this end, by induction it suffices to show that the map $a : [n] \to [0]$ is a composite of codegeneracies. Such a composite is given by $a = \rho^0 \dots \rho^{n-2}\rho^{n-1}$.

Now if one wishes to define a functor $F : \mathbf{\Delta} \longrightarrow \mathcal{C}$, then by Remark 1.0.11, it is sufficient to define $F$ only on the cofaces and codegeneracies. The following is a simple observation from the definitions.

**Lemma 1.0.12.** *The coface and codegeneracy maps $d^i$ and $\rho^j$ satisfies the cosimplicial identities of Lemma 1.0.2.* □

**Lemma 1.0.13.** *The following are equivalent:*

(1) *$X$ is a simplicial set.*
(2) *$X$ is a presheaf*

$$X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$$

*Consequently,* $\mathrm{s}\mathcal{S}\mathrm{et}$ *is isomorphic to the category of presheaves of sets over* $\mathbf{\Delta}$.

*Proof.* (1. ⇒ 2.) We define a functor

$$[n] \mapsto X_n$$
$$d^i \mapsto \partial_i$$
$$\rho^i \mapsto s_i.$$

The fact that is indeed a functor follows from the decomposition of a map in $\mathbf{\Delta}$ into composition of cofaces followed by codegeneracies as in Remark 1.0.11.

(2. ⇒ 1.) Define $X_n = X([n])$ and $\partial_i = X(d^i)$ and $s_i = X(\rho^i)$. Then, $\{X_n, \partial_i, s_i\}$ is a simplicial set by Lemma 1.0.12. □

**Definition 1.0.14 (Simplicial object).** Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a presheaf $X : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{C}$. Equivalently, its a sequence of objects $\{X_n\}$ of $\mathcal{C}$ together with arrows $\partial_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying the simplicial identities. The category of simplicial objects in $\mathcal{C}$ is denoted by $s\mathcal{C}$.

**Remark 1.0.15** (Fiber products and quotients)**.** Fiber product of simplicial sets exists. Indeed, recall that in a presheaf category, the limits and colimits exists and are defined pointwise. Consequently, if $f : X \to Z$ and $g : Y \to Z$ are two simplicial maps, then the fiber product $X \times_Z Y$ is a simplicial set whose set of $n$-simplices is $X_n \times_{Z_n} Y_n$ via the maps $f_n : X_n \to Z_n$ and $g_n : Y_n \to Z_n$. Furthermore, the face and degeneracy maps are given by those of $X$ and $Y$ applied componentwise.

If $Z \subseteq X$ is a sub-simplicial set, then we can define a simplicial set $X/Z$ whose set of $n$-simplices is $X_n/Z_n$ together with the face and degeneracies which descends from $X$. Note that there is a simplicial map $q : X \to X/Z$.

Our main goal in the rest of this section is to develop basic homotopy theory as for topological spaces, but for Kan complexes. In particular, one of our aim is to define and study homotopy groups of a Kan complex. Also recall that studying homotopy theory in topological spaces amounts to studying fibrations and cofibrations of spaces. We will introduce those notions in the simplicial setting, using which we will establish classical results on homotopy theory in the simplicial setting.

1.1. **Homotopy of simplices in a Kan complex.** Let $Z$ be a space and $X = S_*(Z)$ be its singular simplicial set, which we now know is a Kan-complex by Corollary 1.0.8. We want a notion of homotopy of two $n$-simplices purely in terms of simplices of $X$. The following definition is thus made.

**Definition 1.1.1 (Homotopy of simplices).** Let $X$ be a simplicial set and $x, x' \in X_n$ be two $n$-simplices. Further suppose that they satisfy the compatibility condition: $\partial_i x = \partial_i x'$ for each $0 \leq i \leq n$. Then, $x$ and $x'$ are said to be homotopic if there exists an $n+1$-simplex $y \in X_n$ such that

$$\begin{aligned}
\partial_n y &= x, \\
\partial_{n+1} y &= x', \\
\partial_i y &= s_{n-1} \partial_i x = s_{n-1} \partial_i x' \;\forall\, 0 \leq i \leq n-1.
\end{aligned}$$

Here is the lemma which recovers the usual notion of homotopy when $X = S_*(Z)$.

**Proposition 1.1.2.** *Let $Z$ be a space and $X = S_*(Z)$. If two $n$-simplices $x, x' \in X_n$ are homotopic, then there exists $H : \Delta^n \times I \to Z$ such that*

$$\begin{aligned}
H_0 &= x \\
H_1 &= x' \\
\partial_i H_t &= \partial_i x = \partial_i x' \;\forall\, 0 \leq i \leq n.
\end{aligned}$$

*Proof.* There exists an $n+1$-simplex $y : \Delta^{n+1} \to Z$ such that the conditions of Definition 1.1.1 holds. We define $H$ as follows:

$$\begin{aligned}
H : \Delta^n \times I &\longrightarrow Z \\
((e_0, \dots, e_n), t) &\longmapsto y(e_0, \dots, e_{n-1}, te_n, e_n - te_n).
\end{aligned}$$

This is clearly a continuous map. Indeed, we have that $H_0 = \partial_n y = x$ and $H_1 = \partial_{n+1} y = x'$. Furthermore, for any $0 \leq i \leq n$, we have

$$\begin{aligned}
\partial_i H_t(e_0, \dots, e_{n-1}) &= H_t d^i(e_0, \dots, e_{n-1}) \\
&= y(e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) \\
&= yd^i(e_0, \dots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) = \partial_i y(e_0, \dots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) \\
&= s_{n-1} \partial_i x(e_0, \dots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) \\
&= xd^i \rho^{n-1}(e_0, \dots, e_{n-2}, te_{n-1}, e_{n-1} - te_{n-1}) \\
&= xd^i(e_0, \dots, e_{n-2}, e_{n-1}) = \partial_i x(e_0, \dots, e_{n-1})
\end{aligned}$$

and similarly for $x'$. Finally, to see that $\partial_n H_t = \partial_n x = \partial_n x'$, we simply observe the following:

$$\begin{aligned}
\partial_n H_t(e_0, \dots, e_{n-1}) &= H_t(e_0, \dots, e_{n-1}, 0) = y(e_0, \dots, e_{n-1}, 0, 0) \\
&= \partial_n y(e_0, \dots, e_{n-1}, 0) = x(e_0, \dots, e_{n-1}, 0) \\
&= \partial_{n+1} y(e_0, \dots, e_{n-1}, 0) = x'(e_0, \dots, e_{n-1}, 0),
\end{aligned}$$

as required. $\square$

**Proposition 1.1.3.** *Let $X$ be a Kan complex. Then the relation of homotopy of pairs of compatible $n$-simplices is an equivalence relation for all $n \geq 0$.*

*Proof.* Reflexivity is clear as for $x \in X_n$, we may take the homotopy to be $y = s_n x$ and verify by the simplicial identities that $y$ is indeed a homotopy $x \sim x$. We will show symmetry and transitivity in one go by showing the following: $x \sim x'$ and $x \sim x''$ implies $x' \sim x''$. It is easy to see that both symmetry and transitivity follows from this. We now prove this implication.

Let $y' : x \sim x'$ and $y'' : x \sim x''$. Thus,

$$\partial_n y' = x,\ \partial_{n+1} y' = x',\ \partial_i y' = s_{n-1} \partial_i x = s_{n-1} \partial_i x' \ \forall\ 0 \leq i \leq n-1$$
$$\partial_n y'' = x,\ \partial_{n+1} y'' = x'',\ \partial_i y'' = s_{n-1} \partial_i x = s_{n-1} \partial_i x'' \ \forall\ 0 \leq i \leq n-1.$$

As $X$ is a Kan complex, we will employ the Kan condition to obtain the required homotopy. Consider the $n+2$ many $n+1$-simplices $\{z_i\}_{i=0}^{n+1}$ where

$$z_i = \begin{cases} \partial_i s_n s_n x' & \text{if } 0 \leq i \leq n-1, \\ y' & \text{if } i = n, \\ y'' & \text{if } i = n+1. \end{cases}$$

We claim that $\{z_i\}$ forms an $(n+2, n+2)$-horn. Indeed, we need only check $\partial_i z_j = \partial_{j-1} z_i$ for $i < j < n+2$. We may check this case by case.

(1) If $0 \leq i < j \leq n-1$, then we have
$$\partial_i z_j = \partial_i \partial_j s_n s_n x' = \partial_{j-1} \partial_i s_n s_n x' = \partial_{j-1} z_i.$$

(2) If $0 \leq i \leq n-1$ and $n \leq j \leq n+1$, then we have (we show for $j = n$, for $j = n+1$, same observation will work)
$$\partial_i z_j = \partial_i y' = s_{n-1} \partial_i x'$$
and
$$\partial_{j-1} z_i = \partial_{n-1} \partial_i s_n s_n x' = \partial_{n-1} s_{n-1} \partial_i s_n x' = \partial_i s_n x' = s_{n-1} \partial_i x',$$
as required.

(3) If $i = n$ and $j = n+1$, then
$$\partial_n z_{n+1} = \partial_n y'' = x = \partial_n z_n.$$

Hence by Kan condition on $X$, there exists an $n+2$-simplex $z$ such that $\partial_i z = z_i$ for all $i \neq n+2$. Let $h = \partial_{n+2} z$. We claim that $h : x' \sim x''$. Indeed, observe that for $0 \leq i \leq n-1$, we have

$$\partial_n h = \partial_n \partial_{n+2} z = \partial_{n+1} \partial_n z = \partial_{n+1} z_n = \partial_{n+1} y' = x'$$
$$\partial_{n+1} h = \partial_{n+1} \partial_{n+2} z = \partial_{n+1} \partial_{n+1} z = \partial_{n+1} z_{n+1} = \partial_{n+1} y'' = x''$$
$$\partial_i h = \partial_i \partial_{n+2} z = \partial_{n+1} \partial_i z = \partial_{n+1} \partial_i s_n s_n x' = \partial_{n+1} s_{n-1} s_{n-1} \partial_i x' = s_{n-1} \partial_i x' = s_{n-1} \partial_i x''.$$

This completes the proof. $\square$

Recall that for a space $Z$, the notion of homotopy relative to a subspace $L \subseteq Z$ is needed to define relative homotopy groups. We hence define homotopy of two $n$-simplices relative to a sub-simplicial set $L$ of $X$.

**Definition 1.1.4** (**Relative homotopy**)**.** Let $X$ be a simplicial set and $L \subseteq X$ be a sub-simplicial set. Two $n$-simplices $x, x' \in X_n$ are said to be homotopic rel $L$ if $\partial_i x = \partial_i x'$ for all $1 \leq i \leq n$, $\partial_0 x, \partial_0 x' \in L_{n-1}$ and there is an $n+1$-simplex $w \in K_{n+1}$ such that

$$\partial_n w = x,\ \partial_{n+1} w = x',\ \partial_i w = s_{n-1} \partial_i x = s_{n-1} \partial_i x' \ \forall\ 0 \leq i \leq n-1$$

where furthermore $\partial_0 w \in L_n$ and is a homotopy between $\partial_0 x$ and $\partial_0 x'$.

By the same technique of Proposition 1.1.3 (constructing an appropriate $(n+2, n+2)$-horn in $X$), one can show the following (for more details, see [1], Chapter 1, Proposition 3.4).

**Proposition 1.1.5.** *Let $X$ be a Kan complex and $L \subseteq X$ be a sub-Kan complex. The relation of homotopic rel $L$ is an equivalence relation.*

Any subset of simplices $S_n \subseteq X_n$ of a simplicial set $X$ defines a unique sub-simplicial set of $X$.

**Definition 1.1.6** (**Generated sub-simplicial set**)**.** Let $X$ be a simplicial set and $S_n \subseteq X_n$ be a subset for each $n \geq 0$. The simplicial set generated by $\{S_n\}_{n\geq 0}$ is the smallest sub-simplicial set $\tilde{S}$ of $X$ such that $\tilde{S}_n \supseteq S_n$ for each $n \geq 0$.

We next show that this relation is well-behaved with respect to maps of simplicial sets.

**Lemma 1.1.7.** *Let $f : (X, K) \to (Y, L)$ be a simplicial map of pairs and $x, x' \in K_n$ for some $n \geq 0$. If $x \sim x'$ rel $K$, then $f(x) \sim f(x')$ rel $L$.*

*Proof.* Let $w\ in X_{n+1}$ be a homotopy rel $K$ between $x$ and $x'$. We claim that $f(w) \in Y_{n+1}$ is a homotopy rel $L$ for $f(x)$ and $f(x')$. Indeed, $\partial_n f_{n+1}(w) = f_n \partial_n(w) = f_n(x)$, $\partial_{n+1} f_{n+1}(w) = f_n \partial_{n+1}(w) = f_n(x')$ and for $0 \leq i \leq n-1$, we have

$$\partial_i f_{n+1}(w) = f_n \partial_i(w) = f_n(s_{n-1} \partial_i x) = s_{n-1} f_{n-1}(\partial_i x) = s_{n-1} \partial_i f_n(x).$$

Similarly, one shows that $\partial_i f_{n+1}(w) = s_{n-1} \partial_i f_n(x')$. We need only show that $\partial_0 f(w) \in L_n$ and is a homotopy between $\partial_0 f(x)$ and $\partial_0 f(x')$. This also follows from similar steps as above. ☐

1.2. **Homotopy classes of simplices.** We now define homotopy groups of a Kan complex. To be able to derive the homotopy long exact sequence, we will define relative homotopy groups as well.

**Definition 1.2.1** (**Kan triples & homotopy classes**)**.** Let $K$ be a Kan complex and $\phi \in K_0$ be a vertex. Then $(K, \phi)$ is called a pointed Kan complex. We identify $\phi$ with $\tilde{\phi}$, the sub-simplicial set generated by $\phi$. Note that $\tilde{\phi}$ will have exactly one simplex in each dimension. A Kan triple is a tuple $(K, L, \phi)$ where $L \subseteq K$ is a sub-Kan complex and $\phi \in L_0$ is a vertex. Define $\partial(K, \phi)_n := \{x \in K_n \mid \partial_i x = \phi,\ 0 \leq i \leq n\}$ and

$$\pi_n(K, \phi) := \partial(K, \phi)_n / \sim$$

where $x \sim x'$ is the homotopy equivalence relation. Similarly, we may define $\partial(K, L, \phi)_n = \{x \in K_n \mid \partial_0 x \in L_{n-1}, \partial_i x = \phi,\ 1 \leq i \leq n\}$ and thus

$$\pi_n(K, L, \phi) := \partial(K, L, \phi)_n / \sim$$

where $x \sim x'$ is the homotopy rel $L$ equivalence relation. Note that $[\phi] \in \pi_n(K, L, \phi)$ is a distinguished element, making $\pi_n(K, L, \phi)$ a pointed set. By Lemma 1.1.7, a simplicial map gives rise to a map on $\pi_n$.

**Remark 1.2.2.** Recall that in the case of usual spaces, a pair $(X, A)$ is well-behaved homologically if it is a cofibration; $i : A \to X$ satisfies homotopy extension property. It turns out that in $\mathrm{s}\mathbb{S}\mathrm{et}$, notion of subcomplex is sufficient for pairs $(K, L)$ to be well-behaved. In-fact, we will see that subcomplexes exactly forms the subcategory of cofibrations for a model category structure on $\mathrm{s}\mathbb{S}\mathrm{et}$. The hard part will be to study fibrations in $\mathrm{s}\mathbb{S}\mathrm{et}$.

Our goal now is to derive the analog of homotopy long exact sequence of pairs for a Kan triple.

**Theorem 1.2.3.** *Let $(K, L, \phi)$ be a Kan triple with inclusions $i : (L, \phi) \hookrightarrow (K, \phi)$ and $j : (K, \phi, \phi) \hookrightarrow (K, L, \phi)$. Then there is a long exact sequence of sets induced by inclusions:*

$$\cdots \to \pi_{n+1}(K, L, \phi) \xrightarrow{\partial} \pi_n(L, \phi) \xrightarrow{i} \pi_n(K, \phi) \xrightarrow{j} \pi_n(K, L, \phi) \to \cdots$$

*where $\partial$ is defined as*

$$\begin{aligned} \partial : \pi_{n+1}(K, L, \phi) &\to \pi_n(L, \phi) \\ [x] &\mapsto [\partial_0 x]. \end{aligned}$$

*Proof.* As all other proofs use similar ideas (finding the right horn to fill using Kan condition), we show the exactness at $\pi_n(L, \phi)$. We first show $i\,\partial = \phi$. Pick $x \in \partial(K, L, \phi)_{n+1}$. To show $\partial_0 x \in L_n$ is null-homotopic in $K_n$, i.e. there is a homotopy $w : \partial_0 x \sim \phi$ rel $\phi$ in $K$. We construct a $\Lambda_0^{n+2}$-horn, whose $0^{\text{th}}$-face will be the required homotopy. Indeed, it is easy to see that the $n+2$ many $n+1$-simplices

$$\{\phi, \phi, \dots, \phi, x\}$$

satisfy the Kan condition for $K$ and thus gives a $z \in K_{n+2}$ such that $\partial_i z = \phi$ for $1 \le i \le n+1$ and $\partial_{n+2} z = x$. Let $w = \partial_0 z$. We claim that it is the required homotopy. Indeed, we have for $0 \le i \le n-1$ the following

$$\begin{aligned} \partial_n w = \partial_n \partial_0 z &= \partial_0 \partial_{n+1} z = \phi \\ \partial_{n+1} w = \partial_{n+1} \partial_0 z &= \partial_0 \partial_{n+2} z = \partial_0 x \\ \partial_i w = \partial_i \partial_0 z &= \partial_0 \partial_{i+1} z = \phi, \end{aligned}$$

as required.

Next, we show that $\mathrm{Ker}\,(i) \subseteq \mathrm{Im}\,(\partial)$. Pick $x \in \partial(L, \phi)_n$ such that $x \in K_n$ is null homotopic rel $\phi$. This gives a homotopy $w \in K_{n+1}$ such that $w : x \sim \phi$ rel $\phi$. We wish to construct $y \in \partial(K, L, \phi)_{n+1}$ such that $\partial_0 y \sim x$ rel $\phi$ in $L$. Consider the following $n+2$ many $n+1$-simplices of $K$

$$\{w, \phi, \phi, \dots, \phi\}$$

where, it is easily established that they form a $\Lambda_{n+2}^{n+2}$, giving rise to $z \in K_{n+2}$. Let $y = \partial_{n+2} z$. We claim that $y \in \partial(K, L, \phi)_{n+1}$ and $\partial_0 y \sim x$ rel $\phi$ in $L$. Indeed, we see that for $1 \le i \le n+1$

$$\begin{aligned} \partial_0 y &= \partial_0 \partial_{n+2} z = \partial_{n+1} \partial_0 z = \partial_{n+1} w = x \\ \partial_i y &= \partial_i \partial_{n+2} z = \partial_{n+1} \partial_i z = \phi, \end{aligned}$$

as required. □

In the definition of homotopy of simplices, one may ease out the condition that the top two boundaries of the homotopy yields the simplices between which it is the boundary.

**Proposition 1.2.4.** *Let $(K, \phi)$ be a Kan pair and $h \in K_{n+1}$ be an $n+1$-simplex such that $\partial_i h = \phi$ for all $i \neq k, k+1$. Then there is a homotopy $\hat{h} : \partial_k h \sim \partial_{k+1} h$.*

*Proof.* Consider the following sequence of $n+2$ many $n+1$-simplices ($i \neq k$)

$$z_i = \begin{cases} \phi & \text{if } i \neq k+1, k+2, k+3 \\ s_{k+1}\,\partial_{k+1}\,h & \text{if } i = k+1 \\ h & \text{if } i = k+2 \\ s_k\,\partial_{k+1}\,h & \text{if } i = k+3. \end{cases}$$

We claim that this is a $\Lambda^{n+2}_k$-horn for $K$. This is a simple check. Consequently, there exists an $n+2$-simplex $z \in K_{n+2}$ such that $\partial_i\,z = z_i$ for $i \neq k$. Let $\hat{h}_1 = \partial_k\,z$. Observe that for $i \neq k, k+1$, we have

$$\begin{aligned} \partial_{k+1}\,\hat{h}_1 &= \partial_{k+1}\,\partial_k\,z = \partial_k\,\partial_{k+2}\,z = \partial_k\,h \\ \partial_{k+2}\,\hat{h}_1 &= \partial_{k+2}\,\partial_k\,z = \partial_k\,\partial_{k+3}\,z = \partial_k\,s_k\,\partial_{k+1}\,h = \partial_{k+1}\,h \\ \partial_i\,\hat{h}_1 &= \partial_i\,\partial_k\,h = \begin{cases} \partial_k\,\partial_{i-1}\,h = \phi & \text{if } i \leq k-1 \\ \partial_{k+1}\,\partial_i\,h = \phi & \text{if } i \geq k+2. \end{cases} \end{aligned}$$

Thus, $\hat{h}_1 \in K_{n+1}$ is an $n+1$-simpliex satisfying the same hypotheses as $h$ but for $k$ replaced by $k+1$. Inducting over $k$ gives the required homotopy $\partial_k\,h \sim \partial_{k+1}\,h$. □

1.3. **The simplicial sphere.** We construct analogs of important spaces in topology but in simplicial sets.

**Construction 1.3.1** (Standard simplicial sets)**.** Recall that if $\Delta^n$ is the topological $n$-simplex, then $|\Delta^n| \cong D^n$ and $|\partial\,\Delta^n| \cong S^{n-1}$. Consequently, $(|\Delta^n|\,, |\partial\,\Delta^n|) \cong (D^n, S^{n-1})$ as pairs. We first generalize the notion of $D^n$ to simplicial sets. Indeed, consider the simplicial set given by

$$\Delta^n := h_{[n]},$$

where $h_{[n]} \in \mathrm{s}\mathcal{S}\mathrm{et}$ is the representable functor on $\mathbf{\Delta}$ determined by $[n]$

$$\begin{aligned} h_{[n]} : \mathbf{\Delta}^{\mathrm{op}} &\longrightarrow \mathcal{S}\mathrm{et} \\ [m] &\longmapsto \mathrm{Hom}_{\mathbf{\Delta}}\,([m],[n]). \end{aligned}$$

The face and degeneracy maps are clear from the definition; they are hom-duals of coface and codegeneracy maps. We call $\Delta^n$ the standard $n$-simplicial set and they play the role of $n$-disc in simplicial sets.

There is a very important description of simplices of $\Delta^n$, which is often very useful.

**Lemma 1.3.2.** *Let us denote*

$$\mathrm{Inc}(m,n) := \{(a_0, \ldots, a_m) \mid 0 \leq a_0 \leq \cdots \leq a_m \leq n\}.$$

*Then there is a bijection*

$$\mathrm{Inc}(m,n) \cong \Delta^n(m).$$

*Under this identification, we have*

$$\begin{aligned} \partial_i(a_0, \ldots, a_m) &= (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) \\ s_i(a_0, \ldots, a_m) &= (a_0, \ldots, a_{i-1}, a_i, a_i, a_{i+1}, \ldots, a_m). \end{aligned}$$

*Proof.* Take any $m$-simplex of $\Delta^n$, say $x$. Then $x \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$, i.e. $x$ is a monotone map $x : [m] \to [n]$. Let $a_i = x(i)$ for each $0 \le i \le m$. Then $(a_0, \dots, a_m)$ forms the required element of $\mathrm{Inc}(m,n)$. Converse is also easy. As $\partial_i x = x \circ d^i$ and $s_i(x) = x \circ \rho^i$, the other statement follows at once. □

**Corollary 1.3.3.** *For each $n \ge 0$ and $m > n$, every $m$-simplex of $\Delta^n$ is degenerate.*

*Proof.* By Lemma 1.3.2, any $m$-simplex of $\Delta^n$ is of form $(a_0, \dots, a_m)$ where $0 \le a_0 \le \dots \le a_m \le n$. As $m > n$, thus some $a_i = a_{i+1}$, immediately leading to the simplex being degenerate. □

We would like to identify $\mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(\Delta^n, X)$ for a simplicial set $X$. Yoneda lemma tells us what we need.

**Lemma 1.3.4.** *Let $X$ be a simplicial set. Then the following data are equivalent:*

(1) *$x \in X_n$ is an $n$-simplex of $X$.*
(2) *$\bar{x} : \Delta^n \to X$ is a simplicial map.*

*Proof.* It follows from Yoneda lemma that the map

$$\begin{aligned} \varphi : \mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(\Delta^n, X) &\longrightarrow X_n \\ \bar{x} &\longmapsto \bar{x}_n(\mathrm{id}) \end{aligned}$$

is a bijection. More explicitly, if $x \in X_n$, we may define a map $f : \Delta^n \to X$ by

$$\begin{aligned} f_m : \Delta^n(m) &\longrightarrow X_m \\ \vec{a} = (a_0, \dots, a_m) &\longmapsto X(\vec{a})(x) \end{aligned}$$

where $X(\vec{a}) = X_n \to X_m$, the map induced by $X$ on $\vec{a} : [m] \to [n]$. □

**Remark 1.3.5** ($\Delta^n$ is generated by id)**.** In computations with the standard $n$-simplices, its an important step to reduce to dealing with only one simplex of $\Delta^n$. To this end, one observes that $\Delta^n$ is generated by $(0, 1, \dots, n) \in \Delta^n(n)$, i.e. the $\mathrm{id} : [n] \to [n]$. This is simple to establish as by Remark 1.0.11, any $f \in \Delta^n(m)$ is composite of $d^i$ and $\rho^j$. As face and degeneracies of $\Delta^n$ are just composites with $d^i$ and $\rho^j$, it follows at once that any simplex of $\Delta^n$ is generated by id. Hence for purposes of many constructions with $\Delta^n$, it is sufficient to work with the simplex $\mathrm{id} \in \Delta^n(n)$.

An obvious question is the following: Is $\Delta^n$ a Kan complex? A small amount of thought tells no.

**Remark 1.3.6** ($\Delta^n$ is not a Kan complex)**.** Choose $a_0 < b_0$ in $[n]$ and consider 1-simplices of $\Delta^n$ given by

$$z_0 = (a_0, a),\ z_1 = (b_0, a).$$

It is immediate to see that $\{z_0, z_1\}$ satisfies the $(2,2)$-horn condition. If $\Delta^n$ is a Kan complex, then there exists a 2-simplex $z$ of $\Delta^n$ such that $\partial_i z = z_i$ for $i = 0, 1$. Denoting $z = (p_0, p_1, p_2)$ for $0 \le p_0 \le p_1 \le p_2 \le n$, we deduce that we must have

$$(p_1, p_2) = (a_0, a),\ (p_0, p_2) = (b_0, a).$$

It follows that $p_1 = a_0$ and $p_0 = b_0$. As $p_0 \le p_1$, we must have $b_0 \le a_0$, a contradiction to our beginning assumption. This shows that $\Delta^n$ is not a Kan complex.

There are many important sub simplicial sets of $\Delta^n$, just like there are many important subspaces of the disc $D^n$. We discuss few of them now.

**Construction 1.3.7** (Boundaries and horns)**.** For $0 \leq p \leq n$, define $\partial_p \Delta^n$ to be the sub-simplicial set of $\Delta^n$ such that for $m \geq 0$, the $m$-simplices of $\partial_p \Delta^n$ is given by

$$(\partial_p \Delta^n)(m) := \{(a_0, \dots, a_m) \mid 0 \leq a_0 \leq \dots \leq a_m \leq n,\ a_i \neq p\ \forall i\},$$

Define $\partial \Delta^n$ to be the sub-simplicial set of $\Delta^n$ such that for $m \geq 0$, the $m$-simplices of $\partial \Delta^n$ is given by

$$\begin{aligned}(\partial \Delta^n)(m) &:= \{(a_0, \dots, a_m) \mid 0 \leq a_0 \leq \dots \leq a_m \leq n, \exists 0 \leq p \leq n \text{ s.t. } a_i \neq p\ \forall i\},\\ &= \bigcup_{p=0}^{n} (\partial_p \Delta^n)(m).\end{aligned}$$

i.e. the set of non-decreasing maps $[m] \to [n]$ which are not surjective. We hence write $\partial \Delta^n = \cup_{0 \leq p \leq n} \partial_p \Delta^n$ and call it the boundary simplicial set of $\Delta^n$ and $\partial_p \Delta^n$ the $p^{\text{th}}$-boundary simplicial set of $\Delta^n$.

We similarly define horn $\Lambda^n_k$ for $0 \leq k \leq n$ as a sub-simplicial set of $\Delta^n$ as follows

$$(\Lambda^n_k)(m) := \{(a_0, \dots, a_m) \mid 0 \leq a_0 \leq \dots \leq a_m \leq n,\ a_i = k \text{ for some } i\},$$

i.e. the set of non-decreasing maps $[m] \to [n]$ which always have $k$ in the image. Note that there are $n$ many $n-1$ non-degenerate simplices of $\Lambda^n_k$, each corresponding to the unique coface map $d^i : [n-1] \to [n]$ for $0 \leq i \leq n$, $i \neq k$. The pair $(\Delta^n, \partial \Delta^n)$ acts similar to the pair $(D^n, S^{n-1})$ in topology.

**Remark 1.3.8.** An important aspect of dealing with boundaries and horns is to be able to construct maps into simplicial sets from these special simplicial sets. To this end, an important role is played by results which identifies these simplicial sets as a universal construction. Indeed, we have the following results to this end.

**Theorem 1.3.9** (Universal properties)**.** *The simplicial sets $\partial \Delta^n$ and $\Lambda^n_k$ are coequalizers of the following diagrams:*

$$\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \underset{\iota_i d^{j-1}}{\overset{\iota_j d^i}{\rightrightarrows}} \coprod_{0 \leq p \leq n} \Delta^{n-1} \xrightarrow{g} \partial \Delta^n$$

$$\coprod_{0 \leq i < j \leq n, i \neq k, j \neq k} \Delta^{n-2} \underset{\iota_i d^{j-1}}{\overset{\iota_j d^i}{\rightrightarrows}} \coprod_{0 \leq p \leq n, p \neq k} \Delta^{n-1} \xrightarrow{g} \Lambda^n_k$$

*where $g$ on $(\Delta^{n-1}, p)$ is $d^p$ and $\iota_j d^i$ is the map which on $(\Delta^{n-2}, (i,j))$ maps as $d^i : \Delta^{n-2} \to \Delta^{n-1}$ where the $\Delta^{n-1}$ is in the $j^{\text{th}}$-component of the disjoint union. Similarly for others.*

*Proof.* We prove this for $\partial \Delta^n$ since the proof is similar for $\Lambda^n_k$. Take any $m \geq 0$. As a simplicial set is a presheaf and small colimit of presheaves is computed pointwise, we need only find the coequalizer of the following diagram

$$\coprod_{1 \leq i < j \leq n} \Delta^{n-2}_m \underset{\iota_i d^{j-1}}{\overset{\iota_j d^i}{\rightrightarrows}} \coprod_{0 \leq p \leq n} \Delta^{n-2}_m .$$

Recall that coequalizer of $a, b : X \to Y$ is constructed as the quotient $Y/\sim$ where $\sim$ is generated by $a(x) \sim b(x)$ for $x \in X$. Consequently, we get that the coequalizer of this diagram is the following set

$$X_m = \coprod_{0\le p\le n} \Delta_m^{n-1} \Big/ \sim$$

where $\sim$ is generated by $(d^i k_m^{n-2}, j) \sim (d^{j-1} k_m^{n-2}, i)$ for all $0 \le i < j \le n$ and $k_m^{n-2} \in \Delta_m^{n-2}$. This defines a simplicial set $X$. To complete the proof, we need only construct a map $f : X \to \partial\,\Delta^n$ which is an isomorphism, i.e. maps on simplices of each degree is a bijection. We define $f_m : X_m \to (\partial\,\Delta^n)_m$ as follows. First consider the map

$$\bar{f}_m : \coprod_{0\le p\le n} \Delta_m^{n-1} \longrightarrow (\partial\,\Delta^n)_m$$

such that $\bar{f}_m$ on $(\Delta_m^{n-1}, p)$ is given by $d^p$. One checks easily that $\bar{f}_m$ preserves the relation $\sim$ and thus descends to a map

$$f_m : X_m \to (\partial\,\Delta^n)_m.$$

The surjectivity is easily established. For injectivity, it suffices to show that if $z_m^{n-2}, w_m^{n-2} \in \Delta_m^{n-2}$ are such that $d^i w_m^{n-1} = d^j z_m^{n-1}$ for $i < j$, then $w_m^{n-1} = d^{j-1} k_m^{n-2}$ for some $k_m^{n-2} \in \Delta_m^{n-2}$. Indeed, this can be seen by expanding the equality $d^i w_m^{n-1} = d^j z_m^{n-1}$. □

For example, we expect that an $(n,k)$-horn in a simplicial set $X$ is equivalent to a map $\Lambda_k^n \to X$. Indeed, this is what we show.

**Lemma 1.3.10.** *Let $X$ be a simplicial set and $0 \le k \le n$. Then the following are equivalent:*

1. *The collection $\{z_0, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n\} \subseteq X_{n-1}$ forms a $(n,k)$-horn of $X$.*
2. *There is a simplicial map $z : \Lambda_k^n \to X$ such that $z(d^i) = z_i$ for each $0 \le i \le n$, $i \ne k$.*

*Proof.* It is immediate that 1. $\Rightarrow$ 2. from the easy observation that $\Lambda_k^n$ is generated by the $n-1$-simplices $\{d^i\}_{0\le i\le n, i\ne k}$ and the Theorem 1.3.9. For 2. $\Rightarrow$ 1., observe that for $z_i = z(d^i)$, we need only check the horn condition. This is immediate from simplicial identities. □

In a similar way, one has the following result.

**Lemma 1.3.11.** *Let $X$ be a simplicial set. Then the following are equivalent:*

1. *The collection $\{z_0, \ldots, z_n\} \subseteq X_{n-1}$ satisfies $\partial_i\, z_j = \partial_{j-1}\, z_i$ for all $0 \le i < j \le n$.*
2. *There exists a map $z : \partial\,\Delta^n \to X$ such that $z(d^i) = z_i$ for each $0 \le i \le n$.*

□

We may define simplicial $n$-sphere now as follows.

**Definition 1.3.12 (Simplicial $n$-sphere).** For each $n \ge 0$, the simplicial $n$-sphere is defined to be the quotient $\mathbb{S}^n := \Delta^n / \partial\,\Delta^n$. Observe that $\mathbb{S}^n(m) = \{\text{pt.}\}$ for $m < n$.

1.4. **Homotopy of maps.** Our goal in this section is to revisit the notion of homotopy of simplices and show a type of coherence result that will yield that homotopy classes of maps from the simplicial $n$-sphere to a Kan complex $X$ is indeed in bijection with homotopy classes of $n$-simplices. Let us begin by homotopy between two simplicial maps.

**Definition 1.4.1 (Simplicial homotopy).** Let $K, L$ be two simplicial sets and $f, g : K \to L$ be two simplicial maps. A homotopy $h$ from $f$ to $g$ is a collection of maps $\{h_i^q : K_q \to L_{q+1}\}_{0\le i\le q, q\ge 0}$

which satisfy the following identities:

$$\partial_0 \, h_0 = f, \ \partial_{q+1} \, h_q = g$$

$$\partial_i \, h_j = \begin{cases} h_{j-1} \, \partial_i & \text{if } i < j \\ \partial_i \, h_{i-1} & \text{if } i = j \\ \partial_i \, h_i & \text{if } i = j+1 \\ h_j \, \partial_{i-1} & \text{if } i > j+1 \end{cases}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j \\ h_j s_{i-1} & \text{if } i > j. \end{cases}$$

The usual notions of homotopy of pairs and deformation retracts are immediate.

While this definition is more useful, the following shows how it was arrived at.

**Proposition 1.4.2.** *Let $f, g : X \to Y$ be two simplicial maps. Then the following are equivalent:*

(1) *Maps $f$ and $g$ are homotopic.*
(2) *There exists a simplicial map $H : X \times \Delta^1 \to Y$ such that $H_0 = g$ and $H_1 = f$, where $H_0, H_1 : X \to Y$ are given on $q$-simplices by $x \mapsto H_q(x,0), H_q(x,1)$ respectively where $0, 1 \in \Delta^1(q)$ denotes the constant sequences.*

*Proof.* (1. $\Rightarrow$ 2.) Let $h_i^q : X_q \to Y_{q+1}$ be a homotopy from $f$ to $g$. We construct $H : X \times \Delta^1 \to Y$ as follows on $q$-simplices:

$$H_q : X_q \times \Delta^1(q) \longrightarrow Y_{q+1}$$

$$(x, i) \longmapsto \begin{cases} \partial_{q+1} \, h_q^q(x) & \text{if } i = q+1, \\ \partial_{i+1} \, h_i^q(x) & \text{if } 0 < i \leq q, \\ \partial_0 \, h_0^q(x) & \text{if } i = 0. \end{cases}$$

We denote $i \in \Delta^1(q)$ to be the sequence $(0, \ldots, 0, 1, \ldots, 1)$ where $0 \leq i \leq q+1$ denotes the no. of 0s in the sequence. Observe that $H_q$ maps $(x, q+1) \mapsto \partial_{q+1} \, h_q^q(x) = g_q(x)$ and $(x, 0) \mapsto \partial_0 \, h_0^q(x) = f_q(x)$, as required.

(2. $\Rightarrow$ 1.) Define $h_i^q : X_q \to Y_{q+1}$ as $x \mapsto H_{q+1}(s_i x, i{+}1)$ for $0 \leq i \leq q$. We now establish the relevant identities. First, observe that we have $\partial_0 \, h_0^q(x) = \partial_0 \, H_{q+1}(s_0 x, 0) = H_q(\partial_0 \, s_0 x, \partial_0 \, 1) = H_q(x, 0) = f_q(x)$ and $\partial_{q+1} \, h_q^q(x) = \partial_{q+1} \, H_{q+1}(s_q x, q+1) = H_q(\partial_{q+1} \, s_q x, \partial_{q+1} \, q+1) = H_q(x, q) = f_q(x)$. The remaining identities are straightforward to establish. It also is straightforward to establish that both these constructions are invertible. ☐

The following is now immediate.

**Corollary 1.4.3.** *Let $K, L$ be simplicial sets and $f, g : K \to L$ be two simplicial maps. A homotopy $\{h_i^q : K_q \to L_{q+1}\}_{0 \leq i \leq q, q \geq 0}$ from $f$ to $g$ is equivalent to a 1-simplex $h$ in $L^K$ such that $\partial_0 \, h = f$ and $\partial_1 \, h = g$.* ☐

**Remark 1.4.4** (On homotopy being an equivalence relation)**.** An important aspect of this notion is the question whether it is an equivalence relation on set of simplicial maps $K \to L$. One can understand this best by trying to interpret a homotopy as a 1-simplex in the hypothetic simplicial set of all simplicial maps, $L^K$. Indeed, we can define this simplicial set quite easily. Denote

$$(L^K)_q := \mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}} \left( K \times \Delta^q, L \right)$$

together with $\partial_i(f) := f \circ (\mathrm{id} \times d^i)$ and $s_i(f) := f \circ (\mathrm{id} \times \rho^i)$ for some $f : K \times \Delta^q \to L$. Clearly, this makes $L^K$ a simplicial set. What does homotopy being an equivalence relation translates to $L^K$?

**Lemma 1.4.5.** *Let $K, L$ be simplicial sets and $f, g : K \to L$ be two simplicial maps. If $L^K$ is a Kan complex, then simplicial homotopy is an equivalence relation.*

*Proof.* By Corollary 1.4.3, we know that $f \simeq f$ by the 1-simplex $s_0(f) \in (L^K)_1$. Next, suppose $h : f \simeq g$ and $h' : f \simeq g'$, then we wish to construct $h'' : g \simeq g'$. Consider the collection of 2 many 1-simplices $\{h', h\}$ of $L^K$. Observe that $\{h', h\}$ forms a $\Lambda^2_2$-horn. As $L^K$ is a Kan complex, therefore there exists a 2-simplex $z$ of $L^K$ extending the horn. Let $h'' = \partial_2 z$. Then, $\partial_0 \partial_2 z = \partial_1 \partial_0 z = \partial_1 h' = g'$ and $\partial_1 \partial_2 z = \partial_1 \partial_1 z = \partial_1 h = g$. Thus $h''$ is a homotopy from $g'$ to $g$, as required. ☐

Our main goal now is to give an equivalent formulation of homotopy classes of maps as classes of simplicial homotopy from simplicial sphere to the given Kan complex. To this end, we first need to establish that the homotopy classes of simplices in both ways is same.

**Lemma 1.4.6.** *For a Kan pair $(K, \phi)$ and $x, x' \in \partial(K, \phi)_n$ two compatible $n$-simplices with corresponding maps being $\bar{x}, \bar{x'} : (\Delta^n, \partial \Delta^n) \to (K, \phi)$, the following are equivalent:*

(1) *The simplices $x$,$x'$ are homotopic.*
(2) *The simplicial maps $\bar{x}, \bar{x'}$ are homotopic rel $\partial \Delta^n$.*

*Proof.* (1. $\Rightarrow$ 2.) Let $h \in K_{n+1}$ be a homotopy $x \sim x'$. Define a homotopy $H^q_i : \Delta^n(q) \to K_{q+1}$ for $0 \le i \le q$ as follows. First, observe that since $\Delta^n$ is generated by the identity $n$-simplex, thus it suffices to define the map $H^n_i : \Delta^n(n) \to K_{n+1}$ on $\mathrm{id} = (0, 1, \dots, n)$ which should satisfy the identities for homotopy for id. We define it as follows

$$H^n_i(\mathrm{id}) := \begin{cases} s_i x & \text{if } 0 \le i \le n-1 \\ h & \text{if } i = n. \end{cases}$$

It is then easy to see that $H^n_i$ satisfies the earlier relations.

(2. $\Rightarrow$ 1.) Let $H^q_i : \Delta^n(q) \to K_{q+1}$, $0 \le i \le q$ be a homotopy from $\bar{x}$ to $\bar{x'}$. We wish to construct a homotopy $h \in K_{n+1}$ from $x$ to $x'$. Recall that $\bar{x}_n(\mathrm{id}) = x$ and $\bar{x'}_n(\mathrm{id}) = x'$. Denote for each $0 \le i \le n$ the following $n+1$-simplex

$$z_i = H^n_i(\mathrm{id}).$$

Observe that since $H$ is a homotopy rel $\partial \Delta^n$, therefore we have $\partial_i z_j = \phi$ for $i \ne j, j+1$. It follows from Proposition 1.2.4 that $\partial_i z_i \sim \partial_{i+1} z_i$ for $0 \le i \le n$. As $\partial_0 z_0 = \partial_0 H^n_0(\mathrm{id}) = \bar{x}(\mathrm{id}) = x$ and $\partial_{n+1} H^n_n(\mathrm{id}) = \bar{x'}(\mathrm{id}) = x'$, it follows that $x \sim x'$, as needed. ☐

By Lemma 1.4.6, it is now immediate to see that the two notions of homotopy are equivalent.

**Theorem 1.4.7.** *Let $(K, \phi)$ be a Kan pair. Then we have a bijection*

$$\pi_n(K, \phi) \cong [(\Delta^n, \partial \Delta^n), (K, \phi)].$$

*established by the map $[x] \mapsto [\bar{x}]$.* ☐

1.5. **Homotopy groups.** The set of homotopy classes of simplices $\pi_n(K,\phi)$ for a Kan pair $(K,\phi)$ forms a group, which we will call the homotopy group of the Kan pair.

**Construction 1.5.1** (The group operation)**.** Let $(K,\phi)$ be a Kan pair. We construct a group operation on each $\pi_n(K,\phi)$ for $n \geq 1$ as follows. For any two simplices $x, y \in \partial(K,\phi)_n$, we may consider the $\Lambda^{n+1}_n$-horn given by $(i \neq n)$

$$z_i = \begin{cases} \phi & \text{if } i \neq n-1, n+1 \\ x & \text{if } i = n-1 \\ y & \text{if } i = n+1. \end{cases}$$

This satisfies the horn condition and thus glues to an $n+1$-simplex $z \in X_{n+1}$. We thus define

$$\begin{aligned} \pi_n(K,\phi) \times \pi_n(K,\phi) &\longrightarrow \pi_n(K,\phi) \\ ([x],[y]) &\longmapsto [x]\cdot[y] := [\partial_n z] \end{aligned}$$

**Theorem 1.5.2** (Simplicial homotopy group)**.** *Let $(K,\phi)$ be a Kan pair. Then,*

(1) *The product defined in Construction 1.5.1 is well-defined.*
(2) *Under this product, $\pi_n(K,\phi)$ is a group.*
(3) *For $n \geq 2$, $\pi_n(K,\phi)$ is abelian.*

1.6. **Geometric realization.** From previous discussion, it is clear that a simplicial set (more specifically, a Kan complex), encodes combinatorially the homotopy theory of spaces. In this section we wish to establish more rigorously the relation between spaces and simplicial sets. We will give a functorial construction which will yield a space out of a simplicial set. We already have a functorial construction of a simplicial set (actually a Kan complex) out of a space, by the singular simplicial set construction. The question which now remains is of the relation between these two functors. We will show that they form an adjoint pair

$$\begin{tikzcd} \mathrm{s}\mathcal{S}\mathrm{et} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathcal{T}\mathrm{op} \arrow[l, bend left, "S"] \end{tikzcd}.$$

We begin by constructing the realization of a simplicial set.

**Construction 1.6.1** (Realizing a simplicial set)**.** Let $K \in \mathrm{s}\mathcal{S}\mathrm{et}$ be a simplicial set. We denote by $|\Delta^n|$ the topological $n$-simplex as in Remark 1.0.1. We define the following space

$$T(K) := \coprod_{n\geq 0} K_n \times |\Delta^n| \Big/ \sim$$

where $\sim$ is generated by the following relations $(0 \leq i \leq n)$

$$\begin{aligned} (\partial_i\, k_n, u_{n-1}) &\sim (k_n, d^i u_{n-1}),\ k_n \in K_n, u_{n-1} \in \left|\Delta^{n-1}\right|, \\ (s_i k_n, u_{n+1}) &\sim (k_n, \rho^i u_{n+1}),\ k_n \in K_n, u_{n+1} \in \left|\Delta^{n+1}\right|. \end{aligned}$$

A class of $(k_n, u_n) \in T(K)$ is denoted by $[k_n, u_n]$. If $f : K \to L$ is a simplicial map, then we define a map $T(f) : T(K) \to T(L)$ as follows

$$T(f)[k_n, u_n] := [f_n(k_n), u_n].$$

By definition of simplicial maps, it is easy to see that $T(f)$ takes maps an equivalence class into an equivalence, so that it is well-defined. Furthermore, $T(f)$ is continuous by the universal property of quotient maps

$$\begin{tikzcd} \coprod_{n\geq 0} K_n \times |\Delta^n| \arrow[r, "f\times \mathrm{id}"] \arrow[d, two heads] & \coprod_{n\geq 0} L_n \times |\Delta^n| \arrow[d, two heads] \\ T(K) \arrow[r, dashed, "T(f)"'] & T(L) \end{tikzcd}.$$

In-fact, $T(f)$ is also a cellular map, as is clear from diagram above. Consequently, we get a functor $T : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$.

We now prove the main result about $T(K)$.

**Theorem 1.6.2.** *Let $K$ be a simplicial set. Then $T(K)$ is a CW-complex whose $n$-cells are in bijection with non-degenerate $n$-simplices of $K$.*

Before proving this, we first establish the following.

**Proposition 1.6.3** (Milnor)**.** *Let $K$ be a simplicial set and $[k_n, u_n] \in T(K)$ be a point. Then there is a unique $(k'_m, u'_m) \in K_m \times |\Delta^m|$ such that $[k_n, u_n] = [k'_m, u'_m]$, $k'_m$ is non-degenerate and $u'_m$ is in the interior of $|\Delta^m|$. We call such a pair $(k'_m, u'_m) \in K_m \times |\Delta^m|$ non-degenerate.*

*Proof.* Let $\bar{K} = \coprod_{n\geq 0} K_n \times |\Delta^n|$. We define two maps $\lambda, \rho : \bar{K} \to \bar{K}$ which maps as follows. Pick any $k_n \in K_n$. By Remark 1.0.11, we may write $k_n = s_{j_p} \dots s_{j_1} k_{n-p}$ where $k_{n-p} \in K_{n-p}$ which is non-degenerate and $k_{n-p}$ is unique such. We may thus define the function

$$\lambda(k_n, u_n) = (k_{n-p}, \rho^{j_1} \dots \rho^{j_p} u_n).$$

Similarly, if $u_n \in |\Delta^n|$, then $u_n = d^{i_q} \dots d^{i_1} u_{n-p}$ where $u_{n-p} \in |\Delta^{n-p}|$ is an interior point and is unique. We thus define the map $\rho$ as

$$\rho(k_n, u_n) = (\partial_{i_1} \dots \partial_{i_q} k_n, u_{n-q}).$$

We now prove the following four claims about the map $\lambda \circ \rho$:

(1) $\lambda \circ \rho$ descends to $T(K)$.
(2) $\lambda \circ \rho(k_n, u_n)$ is non-degenerate.
(3) $\lambda \circ \rho(k_n, u_n) \sim (k_n, u_n)$.
(4) $\lambda \circ \rho(k_n, u_n)$ is the unique non-degenerate element in the class $[k_n, u_n]$.

It is clear that proving these four claims will complete the proof. For the statement (1), it suffices to show that $\lambda \circ \rho(\partial_i k_n, u_{n-1}) = \lambda \circ \rho(k_n, d^i u_{n-1})$ and similarly for degeneracy. Indeed, we have

$$\lambda \circ \rho(\partial_i k_n, u_{n-1}) = \lambda(\partial_{i_1} \dots \partial_{i_q} \partial_i k_n, u_{n-1-q})$$
$$\lambda \circ \rho(k_n, d^i u_{n-1}) = \lambda(k_n, d^i d^{i_q} \dots d^{i_1} u_{n-1-q}) = \lambda(\partial_{i_1} \dots \partial_{i_q} \partial_i k_n, u_{n-1-q})$$

so they are same. This proves statement (1). For statement (2), observe that $\rho(k_n, u_n)$ has second coordinate non-degenerate. Furthermore, if $u_{n-q} \in |\Delta^{n-q}|$ is non-degenerate, then so is $\rho^{j_1} \dots \rho^{j_p} u_{n-q} \in |\Delta^{n-q-p}|$. Since the first coordinate of $\lambda(k_n, u_n)$ is non-degenerate, it thus follows that $\lambda \circ \rho(k_n, u_n)$ is non-degenerate, proving statement (2). For statement (3), observe that $\rho(k_n, u_n) \sim (k_n, u_n)$ and $\lambda(k_n, u_n) \sim (k_n, u_n)$, from which it follows immediately. Finally, for statement (4), we first observe that $\lambda \circ \rho(k_n, u_n)$ is non-degenerate by statement (2) and is in the same class as $(k_n, u_n)$ by statement (3). To show uniqueness, take $(k'_n, u'_n) \sim (k_n, u_n)$ such that both are non-degenerate. We wish to show that $(k_n, u_n) = (k'_n, u'_n)$. Observe by statement (1) that $\lambda \circ \rho(k_n, u_n) = \lambda \circ \rho(k'_n, u'_n)$. Since $\lambda \circ \rho$ fixes non-degenerate elements of $K_n \times |\Delta^n|$, it follows that $\lambda \circ \rho(k_n, u_n) = (k_n, u_n)$ and $\lambda \circ \rho(k'_n, u'_n) = (k'_n, u'_n)$. This completes the proof. ∎

We may now alternatively define $T(K)$ as follows by Proposition 1.6.3.

**Corollary 1.6.4.** *Let $K$ be a simplicial set. Then $T(K)$ is homeomorphic to the space*

$$\coprod_{n\geq 0} NK_n \times |\Delta^n| \Big/ \sim$$

*where $\sim$ is generated by $(x_n, d^i u_{n-1}) \sim (\partial_i x_n, u_{n-1})$ for all $x_n \in NK_n$, $u_{n-1} \in \left|\Delta^{n-1}\right|$ and $NK_n \subseteq K_n$ the subset of non-degenerate $n$-simplices of $K$.* □

Using this corollary, we can do some basic computations.

**Example 1.6.5.** Geometric realization of standard simplicial sets $\Delta^n, \partial_p \Delta^n, \partial \Delta^n$ and $\Lambda^n_k$ are exactly what we expect. To begin with, we observe $T(\Delta^n) \cong |\Delta^n|$ as follows. Consider the map

$$\begin{aligned} \varphi : T(\Delta^n) &\longrightarrow |\Delta^n| \\ [(a_0, \dots, a_k), (e_0, \dots, e_k)] &\longmapsto \delta_{a_0} e_0 + \cdots + \delta_{a_k} e_k \end{aligned}$$

where $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$, i.e. 1 is in the index $0 \leq i \leq n$ and $(a_0, \dots, a_k) \in \Delta^n_k$ and $(e_0, \dots, e_k) \in \left|\Delta^k\right|$. This is well-defined since if $u_{k-1} = (e_0, \dots, e_{k-1}) \in \left|\Delta^{k-1}\right|$, then $d^i u_{k-1} = (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{k-1}) \in \left|\Delta^{k-1}\right|$. Consequently,

$$\begin{aligned} \varphi([(a_0, \dots, a_k), (e_0, \dots, e_{i-1}, 0, e_i, \dots, e_{k-1})]) &= \sum_{j=0}^{i-1} \delta_{a_j} e_j + \sum_{j=i+1}^{k} \delta_{a_j} e_{j-1} \\ &= \varphi([\partial_i (a_0, \dots, a_k), (e_0, \dots, e_{k-1})]), \end{aligned}$$

as required. This shows that $\varphi$ is a homeomorphism. Similarly, one can identify all the rest of the spaces. Furthermore, the geometric realization of the simplicial map $d^i : \Delta^{n-1} \to \Delta^n$ is indeed given by the map $d^i : \left|\Delta^{n-1}\right| \to |\Delta^n|$. This immediately follows from the definition of the map $\left|d^i\right|$.

We may now prove the CW-structure on the geometric realization. We begin by the following observation.

**Lemma 1.6.6.** *Let $K$ be a simplicial set and $\mathrm{sk}_n(K)$ be the sub-simplicial set of $K$ generated by all simplices of $K$ of degree $\leq n$, which we call the $n$-skeleton of $K$. Then there is a pushout square*

$$\begin{tikzcd} \mathrm{sk}_n K \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{sk}_{n-1} K \arrow[l, hook'] \\ NK_n \times \Delta^n \arrow[u] & NK_n \times \partial \Delta^n \arrow[l, hook'] \arrow[u, "f_{n-1}"'] \end{tikzcd}$$

*where $NK_n$ is the subset of $K_n$ of non-degenerate $n$-simplices and $f_{n-1}$ maps on $m$-simplices as $(x_n, k^n_m) \mapsto (k^n_m)^*(x_n)$. Furthermore, $K = \bigcup_{n\geq 0} \mathrm{sk}_n K$.*

*Proof.* Consider $P$ to be the pushout of the given map. As the bottom row of the square is given by the inclusion map $\partial \Delta^n \hookrightarrow \Delta^n$, therefore we must have that $P$ is given by

$$P = \frac{(NK_n \times \Delta^n) \amalg \mathrm{sk}_{n-1} K}{\sim}$$

where $\sim$ is generated by $(x_n, d^i k_m^{n-1}) \sim (\partial_i x_n, k_m^{n-1}) \sim (k_m^{n-1})^*(\partial_i x_n)$ for all $x_n \in NK_n$ and $k_m^{n-1} \in \Delta_m^{n-1}$. We identify $P$ with $\mathrm{sk}_n K$ as follows. Define first a map

$$\begin{aligned} \bar{\varphi}_m : ((NK_n \times \Delta^n) \amalg \mathrm{sk}_{n-1} K)_m &\longrightarrow (\mathrm{sk}_n K)_m \\ (x_n, k_m^n) \in NK_n \times \Delta_m^n &\longmapsto (k_m^n)^*(x_n) \\ z \in (\mathrm{sk}_{n-1} K)_m &\longmapsto z. \end{aligned}$$

We claim that this $\bar{\varphi}_m$ descends to $\varphi : P_m \to (\mathrm{sk}_n K)_m$. Indeed, suppose that $\bar{\varphi}_m(x_n, k_m^n) = \bar{\varphi}_m(y_n, l_m^n)$. It follows that $(k_m^n)^*(x_n) = (l_m^n)^*(y_n)$. We now deduce the following

$$(x_n, k_m^n) \sim (k_m^n)^*(x_n) = (l_m^n)^*(y_n) \sim (y_n, l_m^n).$$

Consequently, $\varphi_m$ is injective. For surjectivity, pick $(k_m^p)^*(x_p)$ for some $x_p \in NK_p$ for $0 \le p \le n$. If $p \le n-1$, then its in the image of $\mathrm{sk}_{n-1} K$. If $p = n$, then its in the image of $NK_n \times \Delta^n$. It follows that $\varphi_m : P_m \to (\mathrm{sk}_n K)_m$ is a bijection and $\varphi : P \to \mathrm{sk}_{n-1} K$ is thus an isomorphism, as required. □

The adjunction of $T$ and $S$ is actually simple to establish.

**Proposition 1.6.7.** *The pair of functors $T : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$ and $S : \mathcal{T}\mathrm{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$ establishes an adjunction*

$$\begin{tikzcd} \mathrm{s}\mathcal{S}\mathrm{et} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathcal{T}\mathrm{op} \arrow[l, bend left, "S"] \end{tikzcd}.$$

*Proof.* We wish to show natural isomorphisms

$$\psi : \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(TK, X) \leftrightarrows \mathrm{Hom}_{\mathrm{s}\mathcal{S}\mathrm{et}}(K, SX) : \phi.$$

We first construct $\psi$. Define $\psi(g : TK \to X)$ as follows on $n$-simplices

$$\begin{aligned} \psi(g) : K_n &\to SX_n \\ k_n &\mapsto \psi(g)(k_n) : |\Delta^n| \to X \end{aligned}$$

where $\psi(g)(k_n)(u_n) = g([k_n, u_n])$. Similarly, we define $\phi(f : K \to SX)$ as follows

$$\begin{aligned} \phi(f) : TK &\to X \\ [k_n, u_n] &\mapsto f(k_n)(u_n). \end{aligned}$$

For well-definedness, we need only show that $\psi(g)$ is a simplicial map and $\phi(f)$ is continuous. The former follows from definition and the latter from the universal property of quotients. Finally, the fact that $\phi \circ \psi = \mathrm{id}$ and $\psi \circ \phi = \mathrm{id}$ are immediate. □

*Proof of Theorem 1.6.2.* By Corollary 1.6.4, we may assume $T(K)$ is the space constructed as in the corollary. Applying the functor $T$ on the pushout square of Lemma 1.6.6, we deduce by Proposition 1.6.7 and the fact that left adjoint preserves colimits that we have a pushout square in $\mathcal{T}\mathrm{op}$ given by

$$\begin{tikzcd} {|\mathrm{sk}_n K|} \arrow[dr, phantom, "\ulcorner"] & {|\mathrm{sk}_{n-1} K|} \arrow[l, hook'] \\ NK_n \times |\Delta^n| \arrow[u] & NK_n \times |\partial \Delta^n| \arrow[l, hook'] \arrow[u, "|f_{n-1}|"'] \end{tikzcd}$$

together with $T(K) := |K| = \bigcup_{n \ge 0} |\mathrm{sk}_n K|$. The proof is then complete by calculations in Example 1.6.5. □

**Remark 1.6.8.** The counit of the realization and chain functor given by $\epsilon : TS \to \mathrm{id}_{\mathcal{T}\mathrm{op}}$ is in-fact a weak homotopy equivalence. Using this fact, Milnor showed that any space is weakly equivalent to a CW-complex.

## 2. The cobar construction

We begin our study of bar-cobar construction by studying the categories involved in this functor.

### 2.1. Differential graded coalgebras.

A coalgebra is a formal dual of an algebra. Fix $\boldsymbol{K}$ to be a commutative unital ring.

**Definition 2.1.1** (**Coalgebras & graded coalgebras**)**.** A $\boldsymbol{K}$-coalgebra is a tuple $(A, \Delta, \epsilon)$ where $A$ is a $\boldsymbol{K}$-module, $\Delta$ and $\epsilon$ are $\boldsymbol{K}$-linear maps

$$\Delta : A \to A \times A,\ \epsilon : A \to \boldsymbol{K}$$

for which the following two squares commute (tensor product is over $\boldsymbol{K}$):

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta\otimes\mathrm{id}"] \\ A \otimes A \arrow[r, "\mathrm{id}\otimes\Delta"'] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & A \otimes A \arrow[d, "\mathrm{id}\otimes\epsilon"] \\ A \otimes A \arrow[r, "\epsilon\otimes\mathrm{id}"'] & A \end{tikzcd} .$$

These conditions can be referred to as coassociativity and counitality of the comultiplication $\Delta$. A map of coalgebras $f : A \to B$ is a $\boldsymbol{K}$-linear map of underlying $\boldsymbol{K}$-modules which is compatible with comultiplication and counit in the following sense

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Delta"'] & B \arrow[d, "\Delta"] \\ A \otimes A \arrow[r, "f\otimes f"'] & B \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\epsilon"'] & B \arrow[d, "\epsilon"] \\ & \boldsymbol{K} \end{tikzcd} .$$

Hence we have a category of $\boldsymbol{K}$-coalgebras, which we denote by

$$\mathrm{co}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}} .$$

A graded $\boldsymbol{K}$-coalgebra $(A, \Delta, \epsilon)$ is a $\boldsymbol{K}$-coalgebra whose underlying $\boldsymbol{K}$-module is a graded $\boldsymbol{K}$-module, that is, $A = \bigoplus_{d\in\mathbb{Z}} A_d$, such that the comultiplication and the counit maps $\Delta$ and $\epsilon$ are graded $\boldsymbol{K}$-linear maps together with appropriate signs, where $\boldsymbol{K}$ has the grading concentrated at degree 0.

A map of graded $\boldsymbol{K}$-coalgebras $f : A \to B$ is a graded $\boldsymbol{K}$-linear map of underlying graded $\boldsymbol{K}$-modules which is also a map of coalgebras. Hence we have a category of $\boldsymbol{K}$-coalgebras, which we denote by

$$\mathrm{gco}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}} .$$

**Definition 2.1.2** (**dg-Coalgebras**)**.** A differential graded $\boldsymbol{K}$-coalgebra $(A, \Delta, \epsilon, \partial)$ is a graded $\boldsymbol{K}$-coalgebra $(A, \Delta, \epsilon)$ together with a graded $\boldsymbol{K}$-linear endomorphism on the underlying graded $\boldsymbol{K}$-module $A$

$$\partial : A \to A$$

which is of degree +1, i.e. $\partial : A_n \to A_{n+1}$ if cohomologically graded or degree -1, i.e. $\partial : A_n \to A_{n-1}$ if homologically graded, and satisfies the following two conditions:

(1) $\partial$ is a differential for the comultiplication

$$\Delta \circ \partial = (\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial) \circ \Delta$$

where there is an appropriate sign in the sum on right side,

(2) $\partial^2 = 0$.

A map of dg coalgebras $f : A \to B$ is a map of the underlying graded coalgebras such that $\partial f = f \partial$. Hence we have a category of dg $\boldsymbol{K}$-coalgebras, which we denote by

$$\mathrm{dgco}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}}\,.$$

A dg $\boldsymbol{K}$-coalgebra $(A, \Delta, \epsilon, \partial)$ is said to be connected if $A_d = 0$ for all $d < 0$ and the counit $\epsilon : A_0 \to \boldsymbol{K}$ is an isomorphism.

**Construction 2.1.3** (Alexander-Whitney map)**.** Consider the singular chain complex $(C_*(X), \partial)$, where $\partial : C_n(X) \to C_{n-1}(X)$ and $C_n(X)$ is the free abelian group generated by $S_n(X) = \mathrm{Map}(|\Delta^n|, X)$. Our next goal is to establish a dg coalgebra structure on singular chains $C_*(X)$. To this end, the comultiplication on $C_*(X)$ is given by the Alexander-Whitney map, which is defined as follows. For two spaces $X, Y$, we define the map

$$\alpha : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$$

which maps a singular $n$-simplex $\sigma : |\Delta^n| \to X \times Y$ to the following element in $C_*(X) \otimes C_*(Y)$

$$\alpha(\sigma) = \sum_{k+l=n} \sigma_X \circ \lambda_k^n \otimes \sigma_Y \circ \rho_l^n$$

where $\lambda_k^n : \left|\Delta^k\right| \to |\Delta^n|$ and $\rho_l^n : \left|\Delta^l\right| \to |\Delta^n|$ are given by the following

$$\begin{aligned} \lambda_k^n(e_0, \dots, e_k) &= (e_0, \dots, e_k, 0, \dots, 0) \\ \rho_l^n(e_0, \dots, e_k) &= (0, \dots, 0, e_0, \dots, e_l), \end{aligned}$$

that is, $\lambda_k^n$ maps $\left|\Delta^k\right|$ to the *front $k$-face* of $|\Delta^n|$ and $\rho_l^n$ maps $\left|\Delta^l\right|$ to the *back $l$-face* of $|\Delta^n|$.

**Example 2.1.4** (Singular chain complex is a dg coalgebra)**.** Let $\boldsymbol{K} = \mathbb{Z}$ and $X$ be a space. We define a comultiplication on $C_*(X)$ as follows:

$$\Delta : C_*(X) \longrightarrow C_*(X \times X) \xrightarrow{\alpha} C_*(X) \otimes C_*(X)$$

where the first map is induced by diagonal. Further define the counit to be the following map on $n$-simplices which extends linearly to $n$-chains:

$$\begin{aligned} \epsilon : C_*(X) &\longrightarrow \mathbb{Z} \\ \sigma &\longmapsto 1. \end{aligned}$$

One can verify that $(C_*(X), \Delta, \epsilon)$ is a $\mathbb{Z}$-coalgebra. Further, the grading of $C_*(X)$ furnishes a graded $\mathbb{Z}$-coalgebra structure on it. Finally the (homological) dg structure on it is given by the map

$$\begin{aligned} \partial : C_n(X) &\longrightarrow C_{n-1}(X) \\ \sigma &\longmapsto \sum_{i=0}^{n} (-1)^i \, \partial_i(\sigma). \end{aligned}$$

This clearly satisfies $\partial^2 = 0$. One can further check that $\partial$ is a differential for $\Delta$, thus making $(C_*(X), \Delta, \epsilon, \partial)$ is a connected dg $\mathbb{Z}$-coalgebra.

**Example 2.1.5** (dg coalgebra of based chains)**.** For a based space $(X, x_0)$, we can construct an analogue of above example. Indeed, consider $C'_*(X, x_0)$ to be the subalgebra of $C_*(X)$ generated by those simplices $\sigma : |\Delta^n| \to X$ which maps each vertex of $|\Delta^n|$ to $x_0$. The comultiplication $\Delta$ on $C_*(X)$ restricts to a comultiplication on $C'_*(X, x_0)$. Similarly, the counit. Consider $C_*(X, x_0)$ to be the quotient of $C'_*(X, x_0)$ by the subalgebra generated by all degenerate simplices. The comultiplication on $C'_*(X, x_0)$ then descends to a comultiplication on $C_*(X, x_0)$. We call $(C_*(X, x_0), \Delta, \partial)$ the *dg coalgebra of based chains* on $(X, x_0)$.

**Notation 2.1.6** (Sweedler notation)**.** For a dg coalgebra $(A, \Delta, \epsilon, \partial)$ over $\boldsymbol{K}$ and $c \in A$, Sweedler suggests to write

$$\Delta(c) = \sum_{i=1}^{n} c_{(1)i} \otimes c_{(2)i}$$

which we can further simply write it as

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}.$$

When operations on $A$ are linear, then this notation is sometimes useful.

**Construction 2.1.7.** We begin various operations on the category of dg $\boldsymbol{K}$-coalgebras which would be necessary later on.

(1) Shift functors. Denote $s^i : \mathrm{dgco}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}} \to \mathrm{dgco}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}}$ for $i \in \mathbb{Z}$ to be the functor which maps a dg coalgebra $A$ to $s^i A$ which has the underlying graded $\boldsymbol{K}$-module as $s^i A$ where $(s^i A)_d = A_{d-i}$ for all $d \in \mathbb{Z}$. The algebra structure on $s^i A$ remains the same. The differential picks up a sign $s^i \partial = (-1)^i \partial$. Further if $\varphi : A \to B$ is a map of dg coalgebras, then $s^i \varphi : s^i A \to s^i B$ maps $a \in (s^i A)_d$ to $\varphi_{d-i}(a) \in B_{d-i} = (s^i B)_d$. Sometimes $s^i A$ is denoted by $A[i]$.

**TODO.**

2.2. **Cobar construction.** We will now construct the cobar functor. This is a functor on the category of connected dg-coalgebras over $\boldsymbol{K}$, the category of which we denote by

$$\mathrm{dgco}\mathcal{A}\mathrm{lg}^0_{\boldsymbol{K}}.$$

**Construction 2.2.1** (The cobar functor)**.** Let $(A, \Delta, \partial)$ be a connected dg $\boldsymbol{K}$-coalgebra. We construct a dg $\boldsymbol{K}$-algebra $\Omega(A, \Delta, \partial)$ in the following steps.

(1) (Reducing degree 0 homogeneous terms). From the dg coalgebra $(A, \Delta, \partial)$, consider the dg coalgebra $(A/A_0, \Delta, \partial)$ whose degree 0 homogeneous terms are now 0. We denote $A/A_0$ by $A_{>0}$.

(2) (Shifting degree by -1). Given a dg coalgebra $A$, we may consider the coalgebra $s^{-1}A$, which takes an element of degree $a \in A_d$ to $a \in (s^{-1}A)_{d-1} = A_d$. Note that the differential of $s^{-1}A$ is $-\partial$ where $\partial$ is the differential of $A$.

(3) (Graded tensor algebra). Let $A$ be a dg coalgebra. Then the graded tensor algebra over $A$ is a graded $\boldsymbol{K}$-algebra given by

$$T(A) = \boldsymbol{K} \oplus A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \dots.$$

where degree $d$-term of $T(A)$ is

$$T(A)_d = A_d \oplus (A^{\otimes 2})_d \oplus \dots.$$

(4) (Differential on $T(s^{-1}A)$). Let $(A, \Delta, \partial)$ be a dg coalgebra and consider $s^{-1}A$. On the graded algebra $T(s^{-1}A)$ (multiplication by concatenation), we can further construct a differential to make it a dg algebra as follows. First, define on $s^{-1}A$ the following $\boldsymbol{K}$-linear map

$$D : s^{-1}A \longrightarrow s^{-1}A \oplus (s^{-1}A \otimes s^{-1}A)$$

given by $D = -s^{-1}\,\partial\, s^{+1} + (s^{-1} \otimes s^{-1})\Delta s^{+1}$. We extend this map to $s^{-1}A \otimes s^{-1}A$ by the Leibnitz rule as follows:

$$D(a \otimes b) = D(a) \otimes b + (-1)^{\deg a} a \otimes D(b).$$

Similarly extend $D$ to $(s^{-1}A)^{\otimes 3}$ as

$$D(a \otimes b \otimes c) = D(a \otimes b) \otimes c + (-1)^{\deg(a \otimes b)} a \otimes b \otimes D(c)$$

and so on to all monomials in $T(s^{-1}A)$ and thus to all of $T(s^{-1}A)$. Hence, we have a map

$$D : T(s^{-1}A) \to T(s^{-1}A)$$

which on $T(s^{-1}A)_d$ is a map to $T(s^{-1}A)_{d-1}$ and satisfies $D^2 = 0$. Indeed, this follows from careful tracking of signs, compatibility of $\Delta$ and $\partial$, $\partial^2 = 0$ and graded coassociativity of $\Delta$. As $D$ is extended by the graded Leibnitz rule, it follows that $(T(s^{-1}A), D)$ is a dg $\boldsymbol{K}$-algebra.

Furthermore, if $\varphi : A \to B$ is a map of dg coalgebras, then we get a dg coalgebra map $s^{-1}\varphi : s^{-1}A \to s^{-1}B$. This induces a map graded algebras $T(s^{-1}A) \to T(s^{-1}A)$ which is compatible with the differential $D$ since $\partial$, $\Delta$ and shift functors are so. Hence, we have a functor

$$Ts^{-1} : \mathrm{dgco}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}} \longrightarrow \mathrm{dg}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}},$$

which we call the *shifted tensor dg algebra* of $A \in \mathrm{dgco}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}}$.

The cobar functor is then given by first reducing the degree 0 homogeneous terms of a connected dg-coalgebra and then considering its shifted tensor dg-algebra:

$$\begin{aligned} \boldsymbol{\Omega} : \mathrm{dgco}\mathcal{A}\mathrm{lg}^0_{\boldsymbol{K}} &\longrightarrow \mathrm{dg}\mathcal{A}\mathrm{lg}_{\boldsymbol{K}} \\ (A, \Delta, \partial) &\longmapsto (T(s^{-1}A_{>0}), D). \end{aligned}$$

This is the cobar functor.

2.3. **Adams' theorem on $\Omega_{x_0}X$.** Adams in [2] gave a chain model of based loop space $\Omega_{x_0}X$ by applying cobar functor on the dg coalgebra of based chains on $(X, x_0)$.

**Theorem 2.3.1** (Adams)**.** *Let $(X, x_0)$ be a based space. Then there is a quasi-isomorphism of dg algebras*

$$\boldsymbol{\Omega}(C_*(X, x_0), \Delta, \partial) \simeq C_*(\Omega_{x_0}X).$$

**Remark 2.3.2.** The author of [3] proves this theorem as follows. Given the simplicial set $K = S_*(X, x_0)$ which is furthermore a Kan complex with a unique 0-simplex $x_0$, we first construct a simplicial category $\mathfrak{C}(K)$ called the rigidification of $K$. This will be a category with a single object and will thus be treated as a simplicial monoid. Applying the normalized chains functor

$$C_* : \mathrm{s}\mathcal{S}\mathrm{et} \longrightarrow \mathcal{C}h\,(\mathbb{Z})$$

on $\mathfrak{C}(K)$ one shows that we get a dg algebra $C_*(\mathfrak{C}(K))$. One then shows that there is a natural quasi-isomorphism

$$C_*(\mathfrak{C}(K)) \simeq C_*(\Omega_{x_0}|K|).$$

Note that by Remark 1.6.8, we know that $|K|$ is weakly equivalent to $X$. Using the fact that $C_*$ maps weak equivalences to quasi-isomorphisms, we deduce that

$$C_*(\mathfrak{C}(K)) \simeq C_*(\Omega_{x_0}X).$$

One then shows that there is a natural quasi-isomorphism

$$C_*(\mathfrak{C}(K)) \simeq \mathbf{\Omega}(C_*(X, x_0), \Delta, \partial),$$

completing the proof.

Our goal is to prove Adam's theorem using the method of [3]. To this end, we first study the rigidification functor $\mathfrak{C}$ in detail. This functor is also sometimes called the *coherent realization* for the reason we will see in the next section. In-fact, this functor is an essential part of modern treatment of homotopy coherence.

2.4. **Homotopy coherence.** To rightfully motivate the construction of the rigidification functor, we first go on a slight digression, which will be frutiful later on. Following motivation is taken from [4]. Recall that a *diagram of spaces* is a functor $F : \mathcal{A} \to \mathfrak{T}\mathrm{op}$. A *homotopy commutative diagram* is instead a functor $F : \mathcal{A} \to \mathrm{h}\,\mathfrak{T}\mathrm{op}$, that is, a diagram of spaces commuting in the homotopy category of spaces.

Now suppose if our goal is not to only demand commutativity (i.e. functoriality) of the diagram upto homotopy, but rather to demand to know all of the homotopies involved in making it homotopy commutative. The type of information that we need to store to achieve this is best explained by the following example.

**Example 2.4.1** (Homotopy coherent directed system)**.** Let $\omega$ be the linearly ordered poset of natural numbers treated as a category

$$\omega = \{0 < 1 < 2 < \cdots < n < \dots\}.$$

A *homotopy commutative diagram of shape* $\omega$ is a functor $\omega \to \mathrm{h}\,\mathfrak{T}\mathrm{op}$. A *homotopy coherent diagram of shape* $\omega$ on the other hand is a functor $F : \omega \to \mathfrak{T}\mathrm{op}$ together with the following data (let $F(\{i\}) = X_i$ and $f_{ij} = F(i, j)$):

(1) for all $i < j < k$, a 1-homotopy $h_{i,j,k} : f_{ik} \simeq f_{jk} \circ f_{ij}$. Note this forms a 2-simplex of nerve of $\mathfrak{T}\mathrm{op}$.
(2) for all $i < j < k < l$, 2-homotopy $h_{ijkl}$ filling the square

$$\begin{tikzcd} f_{il} \arrow[r, "h_{ikl}"] \arrow[d, "h_{ijl}"'] \arrow[dr, phantom, "\swarrow\!\!\swarrow"] & f_{kl} \circ f_{ik} \arrow[d, "f_{kl} \circ h_{ijk}"] \\ f_{jl} \circ f_{ij} \arrow[r, "h_{jkl} \circ f_{ij}"'] & f_{kl} \circ f_{jk} \circ f_{ij} \end{tikzcd}.$$

Note this forms a 3-simplex of nerve of $\mathfrak{T}\mathrm{op}$.
(3) for all $i < j < k < l < m$, 3-homotopy $h_{ijklm}$ filling the cube...
(4) ....

We will soon construct a (simplicial) category $\mathfrak{C}(\omega)$, such that a functor $F : \mathfrak{C}(\omega) \to \mathfrak{T}\mathrm{op}$ will automatically encode all these information.

**Construction 2.4.2** (Free resolution of a category)**.** We construct a functor, which is a precursor to rigidification $\mathfrak{C}$, denoted by the same symbol (reason will follow)

$$\mathfrak{C} : \mathcal{C}\mathrm{at} \longrightarrow \mathrm{s}\mathcal{C}\mathrm{at}$$

where $\mathrm{s}\mathcal{C}\mathrm{at}$ is the category of simplicial categories (simplicial objects in $\mathcal{C}\mathrm{at}$), which are equivalent to simplicially enriched categories. For a category $\mathcal{A}$, define the simplicial category $\mathfrak{C}\mathcal{A}$ where

$$\mathfrak{C}\mathcal{A}_n = FU^{n+1}\mathcal{A}$$

where $FU\mathcal{A}$ is the free category on the underlying graph of $\mathcal{A}$; it is a category whose objects are same as $\mathcal{A}$ but morphisms are strings of composable morphisms of $\mathcal{A}$. The category $\mathfrak{C}\mathcal{A}_n$ is called the *category of $n$-arrows* of $\mathcal{A}$ whose arrows are given by strings of morphisms of $\mathcal{A}$ where each morphism in the string is covered in exactly $n$-many pairs of well-formed parantheses. Note that this recovers the description of $\mathfrak{C}\mathcal{A}_0$ given earlier. We define face and degeneracy maps of $\mathfrak{C}\mathcal{A}$ as follows:

$$\partial_{k,j} : \mathfrak{C}\mathcal{A}_{k+j} = (FU)^{k+j+1}\mathcal{A} \longrightarrow \mathfrak{C}\mathcal{A}_{k+j-1} = (FU)^{k+j}\mathcal{A}$$

given by

$$\partial_{k,j} = \begin{cases} \text{remove those pairs of parantheses which are contained in exactly } k \text{ pairs of parantheses,} & \text{if } j \geq 1, \\ \text{compose the morphisms inside the innermost pairs of parantheses,} & \text{if } j = 0. \end{cases}$$

Similarly, one defines the degeneracy maps

$$s_{k,j} : \mathfrak{C}\mathcal{A}_{k+j} = (FU)^{k+j+1}\mathcal{A} \longrightarrow \mathfrak{C}\mathcal{A}_{k+j+1} = (FU)^{k+j+2}\mathcal{A}$$

given by

$$s_{k,j} = \begin{cases} \text{double up the parantheses that are contained in exactly } k \text{ pairs of parantheses,} & \text{if } j \geq 1, \\ \text{add a paranthesis around each morphism,} & \text{if } j = 0. \end{cases}$$

This gives us a simplicial category $\mathfrak{C}\mathcal{A}_*$. As noted earlier, this can also be treated as a category $\mathfrak{C}\mathcal{A}$ whose underlying objects are same as $\mathcal{A}$, but the hom sets are given the structure of a simplicial set as follows

$$\mathrm{Hom}_{\mathfrak{C}\mathcal{A}}(x, y)_n := \mathrm{Hom}_{\mathfrak{C}\mathcal{A}_n}(x, y)$$

with the obvious face and degeneracy maps. We will freely interchange between these two points of view for $\mathfrak{C}\mathcal{A}$.

**Remark 2.4.3.** One may now define a homotopy coherent diagram of shape $\mathcal{A}$ to be a simplicially enriched functor

$$F : \mathfrak{C}\mathcal{A} \longrightarrow \mathcal{T}\mathrm{op},$$

that is, for each object $a \in \mathfrak{C}\mathcal{A}$, we get a space $X_a \in \mathcal{T}\mathrm{op}$ and for each pair of objects $a, b \in \mathfrak{C}\mathcal{A}$, we get a map of simplicial sets

$$\mathfrak{C}\mathcal{A}(a, b) \to \mathrm{Map}(X_a, X_b)$$

which is functorial in each variable. Note that $\mathrm{Map}(X_a, X_b)$ is a simplicial set by the singular simplicial structure on it.

It follows that the notion of homotopy coherent diagram of shape $\mathcal{A}$ is really just needs the mapping spaces to be simplicial sets (which are in above cases also Kan complexes). This motivates the following formal definition of a homotopy coherent diagram.

**Definition 2.4.4** (**Homotopy coherent diagram**)**.** Let $\mathcal{K}$ be a category enriched in Kan complexes and $\mathcal{A}$ be any category. A homotopy coherent diagram of shape $\mathcal{A}$ in $\mathcal{K}$ is a simplicially enriched functor

$$F : \mathfrak{C}\mathcal{A} \longrightarrow \mathcal{K}.$$

We now use this to construct the coherent realization/rigidification of a simplicial set.

2.5. **Rigidification & coherent nerve.** There is an adjunction

$$(*) \qquad \begin{tikzcd} \mathrm{s}\mathcal{S}\mathrm{et} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathrm{s}\mathcal{C}\mathrm{at} \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd}$$

where the left adjoint is the functor which we are after. Recall that any simplicial set $X \in \mathrm{s}\mathcal{S}\mathrm{et}$ is a colimit of simplicial $n$-disc $\Delta^n$. As left adjoints preserves colimits, it is thus first sufficient to understand the simplicial category $\mathfrak{C}\,\Delta^n$, which we call be the following.

**Definition 2.5.1** (**Homotopy coherent simplicial $n$-disc**)**.** Let $[n] \in \mathbf{\Delta}$ be the poset $[n] = \{0 < 1 < \cdots < n\}$. The simplicial category $\mathfrak{C}[n]$ is called (homotopy) coherent simplicial $n$-disc.

There is a description of mapping simplicial sets of coherent $n$-disc.

**Lemma 2.5.2.** *For any $i < j$, consider the category $P_{i,j}$ whose objects are all subsets of $\{i, i+1, \ldots, j\}$ containing both $i$ and $j$ and whose arrows are inclusions. Then,*

$$\mathfrak{C}[n](i,j) \cong N(P_{i,j})$$

*where $N : \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$ is the nerve functor.*

*Proof.* Fix $k \geq 0$. We establish an isomorphism between $k$-simplices of $\mathfrak{C}[n](i,j)$ and $N(P_{i,j})$ as follows. Let $f \in \mathfrak{C}[n](i,j)_k$ be a $k$-arrow. Considering the underlying morphisms of $[n]$ in $f$, we get a sequence of non-negative integers $T^0 = \{i, i_1, \ldots, i_m, j\}$ each $\leq n$. As each bracket in $f$ is well-formed, it follows that if a bracket closes at $p$, then a new starts at $p$. Thus, observing the innermost brackets, we obtain a subset $\{i,j\} \subsetneq T^1 \subsetneq T^0$. Continuing this, we get a sequence of sets $\{i,j\} \subseteq T^k \subsetneq T^{k-1} \subsetneq \cdots \subsetneq T^1 \subsetneq T^0$. This is the required nerve of $P_{i,j}$. Converse is also easy to establish. □

**Theorem 2.5.3** (Rigidification and coherent nerve)**.** *Consider the functor obtained by restricting $\mathfrak{C}$ on $\mathbf{\Delta}$*

$$\mathfrak{C} : \mathbf{\Delta} \longrightarrow \mathrm{s}\mathcal{C}\mathrm{at}\,.$$

*This induces an adjunction by Theorem A.1.1, which we denote by $(*)$.*

1. *The left adjoint $\mathfrak{C}$ is characterized by the property that it maps the simplicial $n$-disc $\Delta^n$ to the coherent $n$-disc $\mathfrak{C}[n]$.*
2. *The right adjoint $\mathfrak{N}$ maps a simplicial category $\mathcal{A}$ to the simplicial set $\mathfrak{N}\mathcal{A}$ where*

$$\mathfrak{N}\mathcal{A}_n = \mathrm{s}\mathcal{C}\mathrm{at}(\mathfrak{C}[n], \mathcal{A}),$$

*that is, the homotopy coherent nerve of $\mathcal{A}$. The face and degeneracy maps of $\mathfrak{N}\mathcal{A}$ are given by those of the simplicial category $\mathfrak{C}[n]$.*

*Proof.* By Theorem A.1.1, we get the adjoint pair of $(*)$ where left adjoint is extension by colimits of $\mathfrak{C} : \mathbf{\Delta} \longrightarrow \mathrm{s}\mathcal{C}\mathrm{at}$. The left adjoint is unique by Lemma A.1.3. The definition of right adjoint follows from Theorem A.1.1. □

# Appendix A. Results from categories

A.1. **Extension by colimits.** We begin with the following foundational theorem of Dan Kan.

**Theorem A.1.1** (Kan). *Let $\mathcal{A}$ be a small category and $\mathcal{C}$ be a locally small category with small colimits. If $u : \mathcal{A} \to \mathcal{C}$ is any functor, then the following functor*

$$\begin{aligned} u^* : \mathcal{C} &\longrightarrow \widehat{\mathcal{A}} \\ Y &\longmapsto \mathrm{Hom}_{\mathcal{C}}(u(-), Y) \end{aligned}$$

*has a left adjoint given by $u_! : \widehat{\mathcal{A}} \to \mathcal{C}$ which maps a presheaf $X$ to the colimit of the diagram $\mathcal{A}/X \to \mathcal{A} \to \mathcal{C}$ by $(a, s) \mapsto a \mapsto u(a)$. Thus we have an adjunction:*

$$\begin{tikzcd} \widehat{\mathcal{A}} \arrow[r, bend left, "u_!"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "u^*"] \end{tikzcd} .$$

*The left adjoint $u_!$ is usually called the realization functor.*

We'll use this theorem many times in what follows.

**Corollary A.1.2.** *In the adjunction of Theorem A.1.1, there exists a unique natural isomorphism*

$$u(a) \cong u_!(h_a)$$

*for any object $a \in \mathcal{A}$. This follows directly from generalized elements and Yoneda lemma.*

So if we have a functor $u : \mathcal{A} \to \mathcal{C}$ satisfying the hypotheses of Theorem A.1.1, then the left adjoint $u_!$ is called the **extension of $u$ by colimits**. There's a converse of the above result, which tells us that if there is a colimit preserving functor $F : \widehat{\mathcal{A}} \to \mathcal{C}$, then there exists a functor $u : \mathcal{A} \to \mathcal{C}$ such that $u_! : \widehat{\mathcal{A}} \to \widehat{\mathcal{C}}$ is isomorphic to $F$. More clearly,

**Lemma A.1.3.** *Let $\mathcal{A}$ be small and $\mathcal{C}$ be a locally small category with small colimits. Let $F : \widehat{\mathcal{A}} \to \mathcal{C}$ be a colimit preserving functor. Then the functor*

$$\begin{aligned} u : \mathcal{A} &\longrightarrow \mathcal{C} \\ a &\longmapsto F(h_a) \end{aligned}$$

*is such that the left adjoint given by $u$ as in Theorem A.1.1, $u_!$, is isomorphic to $F$.*

In particular, above lemma tells us that:

**Corollary A.1.4.** *Let $\mathcal{A}$ and $\mathcal{C}$ be as in Theorem A.1.1. Then a functor $F : \widehat{\mathcal{A}} \to \mathcal{C}$ preserves colimits if and only if $F$ has a right adjoint.*

## References

[1] May, J. P. *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, ISBN 9780226511818, 1967.
[2] Adams, J.F. *On the Cobar Construction*, Proceedings of the National Academy of Sciences of the United States of America, Vol. 42, No. 7, pp. 409-412, 1956.
[3] Rivera, M. *Adams' cobar construction revisited*. Doc. Math. 27, pp. 1213–1223, 2022.
[4] Riehl, E. *Homotopy coherent structures*. arxiv:1801.07404, 2023.
[5] Dugger, D. and Spivak, D. *Rigidification of quasi-categories*. Algebr. Geom. Topol. 11, 225-261, 2011.