Vector bundles & Characteristic Classes

December 12, 2024

Contents

1 Vector bundles

We fix $F = \mathbf{R}$ or $\mathbf{C}$.

1.1 Definitions

Definition 1.1.1 (**Family of vector spaces & vector bundles**)**.** Let X be a space. A family of F-vector spaces is a continuous map $p : E \to X$ such that

1. for all $x \in X$, the fiber $E_x := p^{-1}(x)$ is an F-vector space,
2. the map $F \times E \to E$, $(c, e) \mapsto c \cdot e$ in $E_{p(e)}$ is continuous,
3. the map $E \times_X E \to E$, $(e, e') \mapsto e + e'$ in $E_{p(e)} = E_{p(e')}$ is continuous.

A map of families (E, p) and (E', p') is a map $\varphi : E \to E'$ such that $p'\varphi = p$ and $\varphi_x : E_x \to E'_x$ is an F-linear map.

A family of F-vector spaces is a vector bundle if it is a locally constant family. This means there exists a tuple of data

$$\{U_\alpha, n_\alpha, \varphi_\alpha\}_\alpha$$

where $\{U_\alpha\}_\alpha$ is an open cover of X and $\varphi_\alpha : p^{-1}U_\alpha \to U_\alpha \times F^{n_\alpha}$ is an isomorphism of families such that the following commutes where π_1 is projection onto first factor

$$\begin{tikzcd} p^{-1}U_\alpha \arrow[r, "\varphi_\alpha", "\cong"'] \arrow[d, "p"'] & U_\alpha \times F^{n_\alpha} \arrow[dl, "\pi_1"] \\ U_\alpha & \end{tikzcd} .$$

Hence if $x \in U_\alpha$, then $E_x \cong F^{n_\alpha}$. A map of vector bundles $p : E \to X$ and $p' : E' \to X$ is a map $f : E \to E'$ such that the following commutes:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[dl, "p'"] \\ X & \end{tikzcd} .$$

The category of vector bundles over X is denoted by $\mathcal{V}B_F(X)$. A *global section* of $p : E \to X$ is a continuous map $s : X \to E$ such that $p \circ s = \mathrm{id}_X$.

One useful technical tool to show isomorphism of vector bundles is the following.

Lemma 1.1.2. *Consider vector bundles (E, p, B), (E', p', B) and a map of vector bundles $f : E \to E'$. Then the following are equivalent:*

1. *f is an isomorphism of vector bundles.*
2. *For all $b \in B$, the linear map $f_b : E_b \to E'_b$ is an isomorphism.*

Proof. It is clear that 1. $\Rightarrow$ 2. For the converse, it suffices to show that f is a homeomorphism. By composing by local trivializations, we may assume $f : U_i \times \mathbf{R}^n \to U_i \times \mathbf{R}^n$ is given by $(x, \vec{v}) \mapsto (x, A_x\vec{v})$ where $A_x \in \mathrm{GL}_n(\mathbf{R})$. Mapping $x \mapsto A_x$ is continuous as f is continuous. Composing with $M \mapsto M^{-1}$, we deduce that $x \mapsto A_x^{-1}$ is a continuous map. Thus, we may construct an inverse of f on U_i by the map $(x, \vec{v}) \mapsto (x, A_x^{-1}\vec{v})$. Hence, f is a local homeomorphism. In particular, it is open. It suffices to show that f is a bijection. This is immediate by isomorphism on fibers. $\square$

Example 1.1.3 (Tangent bundle)**.** For any smooth manifold M, we have a tangent manifold TM where an element of TM is a tuple $(m, \vec{v})$ where $m \in M$ and $\vec{v} \in T_m M$ with the projection map $TM \to M$ onto the first factor. If M is of dimension n, then $TM \to M$ is a rank n vector bundle over M, called the tangent bundle.

Example 1.1.4 (Canonical line bundle on $\mathbb{P}^n$)**.** Consider the projective n-space $\mathbb{P}^n$ and the set

$$V_n^1 = \{([x], \lambda x) \in \mathbb{P}^n \times \mathbf{R}^{n+1} \mid x \in \mathbf{R}^{n+1} - 0,\ \lambda \in \mathbf{R}\}.$$

Give V_n^1 the subspace topology coming from $\mathbb{P}^n \times \mathbf{R}^{n+1}$ and define the map

$$\begin{aligned} \pi : V_n^1 &\longrightarrow \mathbb{P}^n \\ ([x], \lambda x) &\longmapsto [x]. \end{aligned}$$

This can be shown to be a line bundle over $\mathbb{P}^n$, called the canonical line bundle of $\mathbb{P}^n$.

How do we know that V_n^1 is non-trivial? If it is so then it must have a nowhere vanishing global section. It does not.

Lemma 1.1.5. *There is no nowhere vanishing global section of the canonical line bundle on $\mathbb{P}^n$. Consequently, $V_n^1 \to \mathbb{P}^n$ is not trivial.*

Proof. Suppose $s : \mathbb{P}^n \to V_n^1$ is a nowhere vanishing global section. We have a quotient map $q : S^n \twoheadrightarrow \mathbb{P}^n$. The composite $s \circ q : S^n \to V_n^1$ maps $x \in S^n$ to $([x], t(x) \cdot x)$. We thus get a continuous map $t : S^n \to \mathbf{R}$. Now, since $s \circ q(-x) = ([-x], t(-x) \cdot -x) = ([x], t(-x) \cdot -x)$, therefore we must have $t(x) = -t(-x)$. It follows by intermediate value theorem that there exists $x_0 \in S^n$ such that $t(x_0) = 0$. Consequently, $s(x_0) = ([x_0], 0)$ which contradicts nowhere vanishing of s. $\square$

Proposition 1.1.6. *Let B be a space and $p : E \to B$ be a rank n-vector bundle. Then, the following are equivalent:*

1. *(E, p, B) is trivial.*
2. *(E, p, B) has global section $s_1, \ldots, s_n$ which are nowhere dependent/everywhere independent*[1]*.*

Proof. It is clear that 1. $\Rightarrow$ 2. For converse, consider the map $B \times \mathbf{R}^n \to E$ given by $(b, \vec{v}) \mapsto v_1 s_1(b) + \ldots v_n s_n(b)$ where the addition is in the fiber E_b. As it is a map over B, we need only show its fiberwise isomorphic (Lemma 1.1.2). As $s_1, \ldots, s_n$ are nowhere dependent, thus the map is indeed a fiberwise isomorphism, as required. $\square$

Example 1.1.7 (S^1 is parallelizable)**.** Recall a manifold is parallelizable if the tangent bundle is trivial. Consider the map

$$\begin{aligned} s : S^1 &\longrightarrow TS^1 \\ z &\longmapsto (z, iz) \end{aligned}$$

where we take $S^1 \subseteq \mathbf{C}$. This is well-defined as the vector $iz \in \mathbf{C}$ is tangent to z after identifying $\mathbf{C}$ as $\mathbf{R}^2$. Thus, we have produced a non-vanishing global section so we may conclude by Proposition 1.1.6.

[1]that is, for all $b \in B$, $\{s_1(b), \ldots, s_n(b)\}$ forms a linearly independent subset of E_b

Euclidean bundles will be an essential player for us. We discuss some basics here.

Definition 1.1.8 (Quadratic forms). A quadratic form over a field K is a two degree homogeneous polynomial with possibly arbitrary, but finite number of coefficients. Hence $x^2 + y^2 - 3xy$, $xy + z^2$ and $z^2 - x^2 - y^2$ are all quadratic forms.

There are many different ways of looking at quadratic forms. We give in the following some of them.

Theorem 1.1.9. *Let K be a field. Then the following are equivalent:*

1. *$q(x_1, \dots, x_n)$ is a quadratic form.*
2. *For an n-dimensional K-vector space V, $q : V \to K$ is a map such that $q(cv) = c^2 q(v)$ and $(u, v) \mapsto q(u + v) - q(u) - q(v)$ is a bilinear map $V \times V \to K$.*
3. *There exists a symmetric matrix $A \in M_n(K)$ such that q is represented as a map $q : K^n \to K$ given by $x \mapsto x^T Ax$.*

Remark 1.1.10. In light of above result, we define the mapping $V \times V \to K$ given by

$$b_q : (u, v) \mapsto \frac{1}{2} (q(u + v) - q(u) - q(v))$$

to be the associated bilinear form of q, called the *associated bilinear form of* q. This is represented by the matrix A in the third item above. We call q *positive definite* if A is a positive definite matrix, i.e. if $q(v) > 0$ for all $v \in V \setminus \{0\}$.

Remark 1.1.11 (Inner product)**.** Consider a positive definite quadratic form $q : V \to \mathbf{R}$ over $\mathbf{R}$ and $A \in M_n(\mathbf{R})$ be the corresponding symmetric matrix. We may define the following map

$$\begin{aligned} \langle -, - \rangle : V \times V &\longrightarrow \mathbf{R} \\ (u, v) &\longmapsto u^T Av. \end{aligned}$$

Note that $\langle u, u \rangle = q(u) \geq 0$ as q is positive definite. One also sees linearity in both entries of $\langle -, - \rangle$. This shows that $\langle -, - \rangle$ is indeed an inner product defined by the positive definite form q.

Definition 1.1.12 (Equivalene of quadratic forms). Two quadratic forms $q, q' : V \to K$ on some n-dimensional K-vector space V are equivalent if there exists an invertible matrix $M \in \mathrm{GL}_n(K)$ such that $q(Mx) = q'(x)$.

Example 1.1.13. The quadratic form $xy : K^2 \to K$ given by $(k_1, k_2) \mapsto k_1 k_2$ is equivalent to the quadratic form $x^2 - y^2 : K^2 \to K$. Indeed, the linear transformation

$$\begin{aligned} K^2 &\longrightarrow K^2 \\ (x, y) &\longmapsto \left(\frac{x + y}{2}, \frac{y - x}{2} \right) \end{aligned}$$

is an element of $\mathrm{GL}_2(K)$, say M, and one can then check that the triangle

$$\begin{tikzcd} K^2 \arrow[r, "M"] \arrow[d, "xy"'] & K^2 \arrow[dl, "x^2 - y^2"] \\ K & \end{tikzcd}$$

indeed commutes. Note that the form xy is not positive definite. This is an example which in particular shows that every quadratic form over a field K, $\mathrm{char}(K) \neq 2$, is equivalent to a diagonal quadratic form.

We next define Euclidean bundles.

Definition 1.1.14 (**Euclidean bundles & Riemannian manifolds**). An Euclidean metric on a vector bundle $p : E \to B$ is a map $\mu : E \to \mathbf{R}$ such that for each $b \in B$, the map $\mu_b : E_b \to \mathbf{R}$ is a positive definite quadratic form. A vector bundle with an Euclidean metric is called an Euclidean bundle. A smooth n-manifold M is said to be Riemannian if the tangent bundle $\pi : TM \to M$ has a smooth Euclidean metric $\mu : TM \to \mathbf{R}$. We denote a Riemannian manifold usually by the tuple (M, μ).

By Remark 1.1.11, we see that every Euclidean bundle has an inner product structure on its fibers which varies continuously. We now generalize Proposition 1.1.6 to Euclidean bundles.

Proposition 1.1.15. *Let $p : E \to B$ be an Euclidean bundle with $\mu : E \to \mathbf{R}$ the Euclidean metric. Then the following are equivalent:*

1. *(E, p, B) is trivial.*
2. *(E, p, B) has global section $s_1, \dots, s_n$ which are everywhere orthonormal w.r.t. μ.*

We next cover the isometry theorem, which says that there can be atmost one Euclidean metrics on a vector bundle upto isomorphism.

Theorem 1.1.16 (Isometry theorem). *Let $p : E \to B$ be a vector bundle and μ, μ' be two Euclidean metrics on E. Then there exists an isomorphism $f : E \to E$ of vector bundles such that for all $b \in B$, the linear map $f_b : (E_b, \mu_b) \to (E_b, \mu'_b)$ is a linear isometric isomorphism.*

Proof. Fix $b \in B$. Observe that for any $\vec{v} \in E_b$, we have $\mu_b(\vec{v}) = \vec{v}^T A_b \vec{v}$ and $\mu'_b(\vec{v}) = \vec{v}^T A'_b \vec{v}$ where A_b, A'_b are positive definite symmetric matrices corresponding to the positive definite quadratic forms $\mu_b, \mu'_b : E_b \to \mathbf{R}$, respectively. Recall that every positive definite symmetric matrix M has a unique square root, that is, a positive definite symmetric matrix $\sqrt{M}$ such that $(\sqrt{M})^2 = M$. Since a positive definite matrix is always invertible as it has all positive eigenvalues, therefore if we write

$$\begin{aligned} A_b &= \sqrt{A_b} \cdot \sqrt{A_b} \\ A'_b &= \sqrt{A'_b} \cdot \sqrt{A'_b}, \end{aligned}$$

then for $B_b = (\sqrt{A'_b})^{-1} \cdot \sqrt{A_b}$ we get

$$B_b^T \cdot A'_b \cdot B_b = A_b.$$

We thus define a map

$$\begin{aligned} f_b : E_b &\longrightarrow E'_b \\ \vec{v} &\longmapsto B_b \vec{v}. \end{aligned}$$

Observe that $\mu'_b(f_b(\vec{v})) = (B_b\vec{v})^T A'_b (B_b \vec{v}) = \vec{v}^T B_b^T A'_b B_b \vec{v} = \vec{v}^T A_b \vec{v} = \mu_b(\vec{v})$, hence f_b is a linear isometric isomorphism. Thus we get a function $f : E \to E'$, which is isomorphism on fibers. To see the continuity of f, we need only show that the mapping $b \mapsto B_b$ is continuous as b varies in B. As the map $b \mapsto B_b$ is the product of $b \mapsto (\sqrt{A'_b})^{-1}$ and $b \mapsto \sqrt{A_b}$, and since the mapping $b \mapsto A_b$, $b \mapsto A'_b$ are continuous by continuity of μ and μ', therefore it is sufficient to show that for the mapping $M \mapsto \sqrt{M}$ for positive definite symmetric matrices M is continuous. This is immediate from power series expansion of $\sqrt{M}$. □

The following lemma is easy, but helps in showing continuity of functions defined on vector bundles.

Lemma 1.1.17. *Let $p : E \to B$ and $p' : E' \to B$ be two vector bundles over B. Let $\varphi : E \to E'$ be a function such that the triangle commutes*

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "p"'] & E' \arrow[dl, "p'"] \\ B & \end{tikzcd}.$$

Then $\varphi : E \to E'$ is continuous if and only if for any common local trivialisation $U \subseteq B$, the horizontal composition $U \times \mathbf{R}^n \to U \times \mathbf{R}^{n'}$

$$\begin{tikzcd} U \times \mathbf{R}^n \arrow[r, "\cong"'] & p^{-1}U \arrow[r, "\varphi"'] \arrow[dr, "p"'] & p'^{-1}U \arrow[d, "p'"] & U \times \mathbf{R}^{n'} \arrow[l, "\cong"] \\ & & U & \end{tikzcd}$$

is continuous. □

1.2 Constructions

We wish to show that the category of vector bundles have many constructions which one does with vector spaces, but albeit they are now of vector bundles.

Construction 1.2.1 (Base change)**.** Let $f : X \to Y$ be a continuous map. We get a base change functor

$$f^* : \mathcal{V}B(Y) \longrightarrow \mathcal{V}B(X)$$

which takes a vector bundle $q : E \to Y$ and considers the pullback

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

so that $f^*E = X \times_Y E = \{(x, e) \mid f(x) = q(e)\}$ and p is projection onto first coordinate. One can then easily check that $p : f^*E \to X$ is a vector bundle over X by verifying first p is a family of F-vector spaces and then showing that if $\{U_\alpha, n_\alpha, \varphi_\alpha : q^{-1}U_\alpha \to U_\alpha \times F^{n_\alpha}\}$ is a vector bundle data for q, then the tuple

$$\{f^{-1}U_\alpha, n_\alpha, \tilde{\varphi}_\alpha : p^{-1}f^{-1}U_\alpha \to f^{-1}U_\alpha \times F^{n_\alpha}\}$$

where $\tilde{\varphi}_\alpha$ maps as $(x, e) \mapsto (x, \pi_2\varphi_\alpha(e))$ with π_2 being the projection on second factor, is a vector bundle data for $p : f^*E \to X$.

Construction 1.2.2 (Whitney sum)**.** We construct coproduct of two vector bundles $p : E \to X$ and $q : E' \to X$ over F by considering $p \oplus q : E \oplus E' \to X$ where $E \oplus E'$ is the pullback

$$\begin{tikzcd} E \oplus E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, "q"] \\ E \arrow[r, "p"'] & X \end{tikzcd} .$$

One can then check that the map $E \oplus E' \to X$ makes $E \oplus E'$ a vector bundle over X, which we call the Whitney sum of E and E'. It is furthermore clear that this is the product in $\mathcal{V}B(X)$.

Construction 1.2.3 ($\mathcal{V}B_F(X)$ is preadditive)**.** For any two $(E,p),(E',p') \in \mathcal{V}B_F(X)$, we claim that the homset $\mathrm{Hom}_X(E,E')$ is an abelian group. Indeed, for $f,g : E \to E'$ two maps, then we can define map $f+g : E \to E'$ which maps $e \mapsto fe + ge$ in E'_{pe}. This is continuous as it factors through fiberwise addition map $E' \times_X E' \to E'$. It is furthermore clear that $f + g = g + f$ and that composition is bilinear.

Lemma 1.2.4. *Let X be a space. The category $\mathcal{V}B_F(X)$ is an additive category.*

Proof. By Construction 1.2.3, $\mathcal{V}B(X)$ is preadditive. By Whitney sum, it has finite products. □

Construction 1.2.5 (Subbundle)**.** Let $(E,p) \in \mathcal{V}B(X)$ with datum $\{U_\alpha, n_\alpha, \varphi_\alpha\}_\alpha$. A vector subbundle of (E,p) is a subspace $E' \subseteq E$ such that $p|_{E'} : E' \to X$ becomes a vector bundle over X. Consequently, $E'_x = E_x \cap E'$ is a subspace of E_x and we have a map

$$E' \to E.$$

Construction 1.2.6 (Quotient bundle)**.** Let $(E,p) \in \mathcal{V}B(X)$ and (E',p) be a subbundle. We define $(E/E', q)$ as follows. We first construct the space E/E'[2]. Indeed, define as a set

$$E/E' := \coprod_{x \in X} E_x/E'_x.$$

We now give a topology on E/E' as follows. **TODO**.

Consequently, the fibers of the quotient bundle $q : E/E' \to X$ is $(E/E')_x = E_x/E'_x$ and we have maps of vector bundles

$$\pi : E \to E/E'.$$

Definition 1.2.7 (s.e.s. of vector bundles). Let $E_1, E_2, E_3 \in \mathcal{V}B(X)$ be vector bundles and $f : E_1 \to E_2$ and $g :: E_2 \to E_3$ be maps of vector bundles. Then

$$0 \to E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3 \to 0$$

is a short exact sequence of vector bundles if for all $x \in X$ the map induced on fibers

$$0 \to E_{1x} \xrightarrow{f_x} E_{2x} \xrightarrow{g_x} E_{3x} \to 0$$

is a short exact sequence of F-vector spaces.

[2]It is important to note that E/E' is not supposed to mean the usual quotient of subspaces!

Remark 1.2.8. Hence, for any subbundle E' of E, we get a short exact sequence

$$0 \to E' \to E \to E/E' \to 0.$$

There is a more unified way of constructing vector bundles out of known algebraic operations on vector spaces.

Definition 1.2.9 (Continuous functor). Let $\mathcal{V}$ be the category of finite dimensional vector spaces with maps being linear isomorophism. Then a functor for $k \geq 1$

$$T : \mathcal{V}^k \longrightarrow \mathcal{V}$$

is said to be continuous if for all V_i, V_i', W_i, W_i' in $\mathcal{V}$ for $1 \leq i \leq k$, the mapping

$$T : \operatorname{Hom}_{\mathcal{V}}\left(V_1, V_1'\right) \times \cdots \times \operatorname{Hom}_{\mathcal{V}}\left(V_k, V_k'\right) \longrightarrow \operatorname{Hom}_{\mathcal{V}}\left(T(V_1, \ldots, V_k), T(V_1', \ldots, V_k')\right)$$

is a continuous map. Note that either $\operatorname{Hom}_{\mathcal{V}}(V, V') \cong \mathrm{GL}_n(\mathbf{R})$ where $\dim V = n = \dim V'$ or is empty.

The main theorem is the following.

Theorem 1.2.10. *Let $T : \mathcal{V}^k \to \mathcal{V}$ be a continuous functor and fix a space B. Consider any k vector bundles $\xi_i = (E_i, \pi_i, B)$ on B of rank n_i and let $E = \coprod_{b \in B} T(E_{1,b}, \ldots, E_{k,b})$ together with the projection map $\pi : E \to B$. Then there exists a topology on E which is unique with respect to the following property:*

* *For a common local trivialization $U \subseteq B$ and isomorphisms $h_i : U \times \mathbf{R}^{n_i} \to \pi_i^{-1}(U)$ for each $1 \leq i \leq k$, the map*

$$\begin{aligned} h : U \times T(\mathbf{R}^{n_1}, \ldots, \mathbf{R}^{n_k}) &\longrightarrow \pi^{-1}(U) \\ (b, \vec{v}) &\longmapsto T(h_{1,b}, \ldots, h_{k,b})(\vec{v}) \end{aligned}$$

  is an isomorphism of families.

Using this notion of continuous functors, we get many global constructions on vector bundles.

Construction 1.2.11 (Global algebra of vector bundles)**.** Let B be a space and $\xi_i = (E_i, \pi_i, B)$ be vector bundles of rank n_i on B for $1 \leq i \leq k$.

1. (Tensor product). Let $T : \mathcal{V}^k \to \mathcal{V}$ be given by $(V_1, \ldots, V_k) \mapsto \bigotimes_{i=1}^k V_i$. This is a continuous functor. By Theorem 1.2.10, we get that there is a bundle

$$\xi_1 \otimes \ldots \otimes \xi_k = (E_1 \otimes \ldots \otimes E_k, \pi, B)$$

   with fiber at $b \in B$ being $E_{1,b} \otimes \ldots \otimes E_{k,b}$.
2. (Hom of bundles). Similarly, taking $T : \mathcal{V}^2 \to \mathcal{V}$ be given by $(V_1, V_2) \mapsto \operatorname{Hom}(V_1, V_2)$, which we again see is continuous, yields for vector bundles ξ_1, ξ_2 the following bundle

$$\mathcal{H}om(\xi_1, \xi_2) = (\mathcal{H}om(E_1, E_2), \pi, B)$$

   with fiber at $b \in B$ being $\operatorname{Hom}(E_{1,b}, E_{2,b})$.

3. (Dual bundle). The functor $\mathcal{V} \to \mathcal{V}$ mapping $V \mapsto \mathrm{Hom}\,(V, F)$ is continuous and hence for a bundle $\xi = (E, \pi, B)$, we get the dual bundle denoted
$$\check{\xi} = (\check{E}, \pi, B)$$
with fiber at $b \in B$ being $\check{E}_b$.
4. (Direct sum/Whitney sum). Let $T : \mathcal{V}^2 \to \mathcal{V}$ be given by $(V, W) \mapsto V \oplus W$. This being continuous, yields for vector bundles ξ_1, ξ_2 the following bundle
$$\xi_1 \oplus \xi_2 = (E_1 \oplus E_2, \pi, B)$$
with fiber at $b \in B$ being $E_{1,b} \oplus E_{2,b}$.

Definition 1.2.12 (**Finite type vector bundles**). A vector bundle $p : E \to X$ with datum $\{U_\alpha, n_\alpha, \varphi_\alpha\}_{\alpha \in I}$ is a finite type vector bundle if I can be taken to be finite.

1.3 Vector bundles & twisting atlas

We now systematically study how to patch together vector bundles each defined on some subspace of X.

Definition 1.3.1 (**Twisting atlas for topological groups**). Let G be a group and X be a space. A twisting atlas on X for G is the tuple $\{U_i, g_{ij}\}_{i,j \in I}$ where U_i is an open cover of X and $g_{ij} : U_i \cap U_j \to G$ a continuous map such that
$$g_{ii} = 1 \; \forall i \in I$$
$$g_{ij} \cdot g_{jk} = g_{ik} \text{ on } U_i \cap U_j \cap U_k.$$

A G-twisting atlas $\{V_k, h_{kl}\}$ is equivalent to $\{U_i, g_{ij}\}$ if $\{V_k\}$ is a refinement of $\{U_i\}$ and $h_{kl} : V_k \cap V_l \to G$ is restriction of $g_{ij} : U_i \cap U_j \to G$ where $U_i \supseteq V_k$ and $U_j \supseteq V_l$.

We now state the main construction which constructs an n-dimensional F-vector bundle from any twisting atlas for $\mathrm{GL}_n(F)$.

Construction 1.3.2 (Twisting atlas to vector bundle). Let $\{U_i, g_{ij}\}_{i,j \in I}$ be a twisting atlas on X for group $\mathrm{GL}_n(F)$. We construct $p : E \to X$ an n-dimensional F-vector bundle over X using this data as follows. Consider the quotient space
$$E := \frac{\coprod_{i \in I} U_i \times F^n}{\sim}$$
where the relation $\sim$ is generated by relations
$$(x, \vec{v}) \sim (x, g_{ij}(x)(\vec{v}))$$
for all $x \in U_i \cap U_j$ for any $i, j \in I$ (the rhs of above is in $U_i \times F^n$ and lhs in $U_j \times F^n$).

Next, consider the map
$$p : E \longrightarrow X$$
$$[(x, \vec{v})] \longmapsto x$$

induced by map

$$\begin{aligned} p' : \coprod_i U_i \times F^n &\longrightarrow X \\ (x_i, \vec{v}_i) &\longmapsto x \end{aligned}$$

and the universal property of quotients. Observe that $p^{-1}x = E_x$ is an n-dim F-vector space compatible with scaling and translating. Moreover, we have linear isomorphisms for any $x \in U_i \cap U_j$

$$\begin{aligned} \theta_{ij} : E_{x,i} &\xrightarrow{\cong} E_{x,j} \\ [(x_i, \vec{v}_i)] &\longmapsto [(x_i, g_{ij}(x_i)(\vec{v}_i))] \end{aligned}$$

where $E_{x,i}$ is the fiber at x while considering $x \in U_i$. **From here** Let us denote

$$\chi_i : E_{x,i} \xrightarrow{\cong} F^n$$

be an identification of fibers with F^n such that the following commutes:

$$\begin{tikzcd} & F^n \\ E_{x,i} \arrow[ur, "\chi_i"] \arrow[r, "\theta_{ij}"'] & E_{x,j} \arrow[u, "\chi_j"'] \end{tikzcd}$$

Note that we have a map

$$\begin{aligned} \varphi_i : p^{-1}U_i &\longrightarrow U_i \times F^n \\ [(x, \vec{v})] &\longmapsto (x, \chi_i([(x, \vec{v})])) \end{aligned}$$

where $(x, \vec{v})$ is the representative in $U_i \times F^n$ **till here**. We claim that this is an isomorphism of families. Indeed, one can first check continuity of φ_i by using basics of quotient maps and hypotheses on g_{ij}. One can then construct an inverse of φ_i, as in $U_i \times F^n \to p^{-1}U_i$ mapping as $(x, \vec{v}) \mapsto [(x, \vec{v})]$. Thus φ is a homeomorphism. Moreover, φ_i on fiber E_x is a linear isomorphism to $\{x\} \times F^n$ essentially by construction, as required. Also note that

$$\begin{aligned} \varphi_i : p^{-1}(U_i \cap U_j) &\longrightarrow U_i \cap U_j \times F^n \\ [(x, \vec{v}_i)] &\longmapsto \chi_i([x, \vec{v}_i]) = \chi_j([x, g_{ij}(x)(\vec{v}_i)]) \end{aligned}$$

We claim that the data

$$\{U_i, n, \varphi_i\}$$

makes $p : E \to X$ a vector bundle. Indeed, we immediately observe that the isomorphisms φ_i for each i fits in the following triangle by construction:

$$\begin{tikzcd} p^{-1}U_i \arrow[r, "\varphi_i", "\cong"'] \arrow[d, "p"'] & U_i \times F^n \arrow[dl, "\pi_1"] \\ U_i & \end{tikzcd} .$$

We call the vector bundle $p : E \to X$ as being obtained by twisting atlas $\{U_i, g_{ij}\}$.

Definition 1.3.3 (Equivalence of twisting atlases)**.**

The converse is also true, and we thus get a bijection.

Theorem 1.3.4. *Let X be a space. Then there is a bijection*

$$\begin{gathered}\{\text{Equivalence classes of } \mathrm{GL}_n(F)\text{-twisting atlases } \{U_i, g_{ij}\} \text{ on } X\} \\ \cong \\ \{\text{Isomorphism classes of } n\text{-dim. vector bundles over } X\}\end{gathered}.$$

Proof. In Construction 1.3.2, we have made a forward map. We next extract a twisting atlas for group $\mathrm{GL}_n(F)$ from an n-dim vector bundle $p : E \to X$. Indeed, let $\{U_i, n, \varphi_i\}$ be the vector bundle data for p. Fix any U_i, U_j. Consider the composite of the the isomorphisms

$$h_{ij} : U_i \cap U_j \times F^n \xrightarrow{\varphi_i^{-1}} p^{-1}U_i \cap U_j \xrightarrow{\varphi_j} U_i \cap U_j \times F^n.$$

We claim that h_{ij} is identity on first factor and on the second factor it "twists"[3] the vectors by some linear automorphism of F^n. Indeed, this follows immediately from definition and the commutativity of the following diagram:

$$\begin{tikzcd} U_i \cap U_j \times F^n \arrow[r, shift right, "\varphi_i^{-1}"'] \arrow[dr, "\pi_1"'] & p^{-1}U_i \cap U_j \arrow[l, shift right, "\varphi_i"'] \arrow[r, "\varphi_j"] \arrow[d, "p"] & U_i \cap U_j \times F^n \arrow[dl, "\pi_1"] \\ & U_i \cap U_j & \end{tikzcd}.$$

Consider the maps

$$\begin{aligned} g_{ij} : U_i \cap U_j &\longrightarrow \mathrm{GL}_n(F) \\ x &\longmapsto g_{ij}(x) := \varphi_j \varphi_i^{-1}(x, -) \end{aligned}$$

One then immediately checks that $\{U_i, g_{ij}\}$ is a $\mathrm{GL}_n(F)$-twisiting atlas on X.

We next check that this construction is inverse to that in Construction 1.3.2. Indeed, let $p : E \to X$ be an n-dim vector bundle and let $\{U_i, g_{ij}\}$ be the $\mathrm{GL}_n(F)$-twisting atlas as obtained. Denote

$$E' = \frac{\coprod_i U_i \times F^n}{\sim}$$

as in Construction 1.3.2 and let $\pi : \coprod_i U_i \times F^n \twoheadrightarrow E'$ be the quotient map. We claim that $E' \cong E$. Indeed, by universal property of quotients, it suffices to construct a map $f : \coprod_i U_i \times F^n \to E$ which identifies fibers of π and is a quotient map. To this end, we consider the map

$$\begin{aligned} f : \coprod_i U_i \times F^n &\longrightarrow E \\ (x_i, \vec{v}_i) &\longmapsto \varphi_i^{-1}(x_i, \vec{v}_i). \end{aligned}$$

[3] this is the reason behind our naming in Definition 1.3.1

We leave it as an exercise in point set topology to verify that f preserves fibers of π and is a quotient map. Hence $E \cong E'$. Moreover, it is clear that this isomorphism is over X, thus giving us an isomorphism of vector bundles, as required.

Conversely, begin from a $\mathrm{GL}_n(F)$-twisting atlas $\{U_i, g_{ij}\}$ on X and construct the n-dim vector bundle $p : E \to X$ as in Construction 1.3.2 with datum $\{U_i, n, \varphi_i\}$. Then applying the above construction yields a $\mathrm{GL}_n(F)$-twisting atlas $\{U_i, g'_{ij}\}$ where $g'_{ij} : U_i \cap U_j \to \mathrm{GL}_n(F)$ maps $x \mapsto \varphi_j\varphi_i^{-1}(x, -)$. Fixing $x \in U_i \cap U_j$, we have

$$g'_{ij}(x)(\vec{v}) = \varphi_j\varphi_i^{-1}((x, \chi_i[(x, \vec{v})])) = \varphi_j([(x, \vec{v})]) = (x, \chi_j[x, g_{ij}(x)(\vec{v})]) = g_{ij}(x)(\vec{v}),$$

which shows that $g_{ij} = g'_{ij}$, as required. □

Using twisting atlases, it is easy to do more algebraic manipulations with vector bundles.

Construction 1.3.5 (Tensor product)**.** Consider two vector bundles $p : E \to X$ and $q : E' \to X$. We can construct the tensor product bundle

$$p \otimes q : E \otimes E' \to X$$

such that $(E \otimes E')_x = E_x \otimes_F E'_x$ and its dimension is the product of the individual dimensions. The construction is quite simple to describe in terms of twisting atlases. Let $\{U_i, g_{ij}\}$ be a twisiting atlas for E and $\{V_k, h_{kl}\}$ be a twisting atlas for E'. We may assume by passing to a component that E is n-dim and E' is m-dim. By taking intersections, we may assume that $\{U_i, g_{ij}\}$ and $\{U_i, h_{ij}\}$ are twisting atlases for E and E' respectively. We may then construct the following atlas:

$$\begin{aligned} t_{ij} := g_{ij} \otimes h_{ij} : U_i \cap U_j &\longrightarrow \mathrm{GL}_n(F) \otimes_F \mathrm{GL}_m(F) \\ x &\longmapsto g_{ij}(x) \otimes h_{ij}(x) \end{aligned}$$

where note that $\mathrm{GL}_n(F) \otimes_F \mathrm{GL}_m(F) \cong \mathrm{GL}_{nm}(F)$. As $t_{ii} = g_{ii} \otimes h_{ii} = 1 \otimes 1 = 1$ and

$$t_{ij} \cdot t_{jk} = (g_{ij} \otimes h_{ij}) \cdot (g_{jk} \otimes h_{jk}) = g_{ik} \otimes h_{ik} = t_{ik},$$

as required. Thus, $\{U_i, t_{ij}\}$ is a $\mathrm{GL}_{nm}(F)$-twisting atlas for a vector bundle which we denote as $p \otimes q : E \otimes E' \to X$.

1.4 Vector bundles & locally free sheaves

1.5 Vector bundles & principal $\mathrm{GL}_n(\mathbf{R})$-bundles

1.6 Homotopy invariance

The following theorem states that the base change along homotopic maps gives isomorphic vector bundles!

Theorem 1.6.1. *Let $f, g : X \to Y$ be homotopic maps where X is paracompact. Then, for any vector bundle $E \in \mathcal{V}B(Y)$, there is an isomorphism in $\mathcal{V}B(X)$*

$$f^*E \cong g^*E.$$

1.7 Direct summands

The following theorem states that any subbundle of vector bundle E on a paracompact space X is a direct summand of E.

Theorem 1.7.1. *Let X be a paracompact space and $p : E \to X$ be a vector bundle. If $p : E' \to X$ is a subbundle of (E, p), then there exists a subbundle $p : E'^{\perp} \to X$ of (E, p) such that*

$$E' \oplus E'^{\perp} \cong E.$$

The following is an even stronger claim, stating that finite type vector bundles are direct summands of a trivial bundle. Thus, we may think of finite type vector bundles as equivalent to finitely generated projective modules.

Theorem 1.7.2. *Let X be a paracompact space and $p : E \to X$ be a vector bundle. Then, the following are equivalent:*

1. *$p : E \to X$ is a finite type vector bundle,*
2. *$p : E \to X$ is a direct summand of a trivial bundle, that is, there exists $q : E' \to X$ such that $p \oplus q : E \oplus E' \to X$ is isomorphic to a trivial bundle $\epsilon^n : X \times F^n \to X$.*

1.8 Orientation of bundles

We study the notion of orientability of bundles.

Definition 1.8.1 (Oriented bundles). Let $\xi = (E, p, B)$ be a vector bundle of rank n. We say ξ is oriented if the determinantal line bundle $\wedge^n \xi$ is a trivial line bundle. A nowhere vanishing section of $\wedge^n \xi$ is called an orientation of ξ.

Recall that an orientation on a vector space V of dimension n is a the choice of a non-zero vector in $\wedge^n V$, or equivalently, choice of a basis of V. A map $f : V \to W$ between oriented vector spaces is said to be orientation preserving if the linear map $\wedge^n f : \wedge^n V \to \wedge^n W$ is of positive determinant. This leads to the following definition.

Definition 1.8.2 (Orientation preserving bundle map). Let $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ be two bundles. A bundle map $(f, g) : \xi \to \xi'$ is said to be orientation preserving if the linear map on each fiber $f_b : E_b \to E'_{g(b)}$ is an orientation preserving linear map of vector spaces.

We next show that the above notion of orientation is equivalent to two other notions, one is geometric and other is cohomological.

Theorem 1.8.3. *Let $\xi = (E, p, B)$ be a rank n vector bundle. Then the following are equivalent:*

1. *The bundle ξ is oriented.*
2. *There is a function s which maps each $b \in B$ to an orientation of E_b such that for $b \in U \subseteq B$ a trivializing neighborhood, the isomorphism*

$$h : U \times \mathbf{R}^n \to p^{-1}(U)$$

at $b \in B$ induces a linear isomorphism $h_b : \mathbf{R}^n \to E_b$ which is an orientation preserving linear map.

3. *There exists a function μ which maps each $b \in B$ to a generator $\mu_b \in H^n(E_b, E_{b0}; \mathbb{Z})$ such that for all $b \in B$, there exists an open set $b \in N \subseteq B$ and $\mu_N \in H^n(p^{-1}(N), p^{-1}(N)_0; \mathbb{Z})$ such that*

$$j_{N,x}(\mu_N) = \mu_x$$

for all $x \in N$ where $j_{N,x} : H^n(p^{-1}(N), p^{-1}(N)_0; \mathbb{Z}) \to H^n(E_x, E_{x0}; \mathbb{Z})$ is the map induced by inclusion $(E_x, E_{x0}) \hookrightarrow (p^{-1}(N), p^{-1}(N)_0)$.

Corollary 1.8.4. *Let M be an n-manifold. Then the following are equivalent:*
1. *M is oriented.*
2. *Tangent bundle $TM \to M$ is oriented.*

□

An important characterization about orientability of bundles is given by vanishing of first Stiefel-Whitney class. This is Theorem **??**, which we use frequently.

2 Cohomology of local systems

We study cohomology with coefficient in a local system. Our main goal is to prove Poincaré duality in a general setting.

2.1 Local systems

Recall that a local system on a space X is a locally constant abelian sheaf on X and denote the subcategory of $\mathcal{S}h(X)$ of all local systems and sheaf morphisms by $\mathrm{LocSys}(X)$. Note that we have a functor

$$\begin{aligned} \mathcal{A}b &\longrightarrow \mathrm{LocSys}(X) \\ A &\longmapsto \underline{A} \end{aligned}$$

where $\underline{A}$ denotes constant sheaf with value A and restrictions being identity. To rightly motivate local systems, we first claim that this is a fully-faithful embedding of abelian groups into abelian sheaves.

Lemma 2.1.1. *The functor $A \mapsto \underline{A}$ as above is a fully-faithful embedding of abelian groups into local systems over X.*

Proof. If $f : A \to B$ is a group homomorphism, we get a sheaf homomorphism $\tilde{f} : \underline{A} \to \underline{B}$ by defining it on open U, $\tilde{f}_U : A \to B$ to be f itself. Conversely, for any map $\varphi : \underline{A} \to \underline{B}$ and any open U, we get from the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_U"] \arrow[d, "\mathrm{id}"'] & B \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "\varphi_X"'] & B \end{tikzcd}$$

that $\varphi_U = \varphi_X$. This shows that $\mathrm{Hom}_{\mathcal{A}b}(A, B) \cong \mathrm{Hom}_{\mathrm{LocSys}(X)}(\underline{A}, \underline{B})$, as required. □

Lemma 2.1.2. *Every local system $\mathcal{L}$ on $[0,1]$ is constant.*

Proof. Let $\mathcal{L}$ be local system on $X = [0,1]$. We may assume $I_i = (s_i, t_i)$ is a finite cover of $[0,1]$ such that $\mathcal{L}$ restricted on each I_i is constant. As each I_i by construction intersects I_{i+1}, it follows that the restriction of $\mathcal{L}$ to each I_i is a constant sheaf with the constant abelian group being the same. Let $t \in \mathcal{L}_x = A$ for any $x \in I_{i_0}$ for some i_0. Then t can be glued to each i to give a constant global section $t \in \Gamma(X, \mathcal{L})$, which is unique by sheaf condition. This shows that $\Gamma(X, \mathcal{L}) = A$. Similarly, one shows that each $\mathcal{L}(U) = A$. We next wish to show that each restriction is identity. It is sufficient to show that $\rho_{X,U} : \mathcal{L}(X) = A \to \mathcal{L}(U) = A$ is identity for any open $U \subseteq X$. Indeed, if $\rho_{X,U}(a) = b \in A$, then for some i, we'll have $\rho_{X,U\cap I_i}(a) = b$ and thus $\rho_{X,I_i}(a) = b$. By construction, we'll have $\rho_{X,I_j}(a) = b$ for all j and hence by unique gluing, we'll have $a = b$, as required. □

Lemma 2.1.3. *If X is path-connected and $\mathcal{L}$ is a local system over X, then all stalks of $\mathcal{L}$ are isomorphic.*

Proof. Pick two points $x \neq y \in X$ and a path $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We wish to show that $\mathcal{L}_x \cong \mathcal{L}_y$. Indeed, consider the sheaf $\gamma^*\mathcal{L}$. As inverse image of a locally constant sheaf is again locally constant, it follows at once from Lemma 2.1.2 that $\gamma^*\mathcal{L}$ is a constant sheaf in say abelian group A. Pick any $t \in I$ and observe that $A \cong (\gamma^*\mathcal{L})_t \cong \mathcal{L}_{\gamma(t)}$. It follows that stalks of $\mathcal{L}$ along the path γ are constant, as needed. □

Construction 2.1.4 (Local systems-Monodromy)**.** Let (X, x_0) be a path-connected, locally path-connected and semi-locally simply connected space. We wish to show the following equivalence between local systems with local groups A and $\mathrm{Aut}\,(A)$-representations of $\pi_1(X)$:

$$\mathrm{LocSys}_A(X) \cong \mathrm{Hom}_{\mathcal{G}rp}\,(\pi_1(X, x_0), \mathrm{Aut}\,(A)).$$

Let $\mathcal{L}$ be a local system of abelian groups over X. By Lemma 2.1.3, it follows that $\mathcal{L}$ is a local system with fiber (stalk) a fixed abelian group A. We will construct a representation of $\pi_1(X, x_0)$ in the group $\mathrm{Aut}\,(A)$. Indeed, consider the map

$$\begin{aligned} \varphi : \pi_1(X, x_0) &\longrightarrow \mathrm{Aut}\,(A) \\ [\gamma] &\longmapsto \gamma^\times : (\gamma^*\mathcal{L})_0 \cong A \cong (\gamma^*\mathcal{L})_1. \end{aligned}$$

We omit the proof that this is well-defined. Conversely, pick any map $\varphi : \pi_1(X, x_0) \to \mathrm{Aut}\,(A)$. We wish to construct a local system $\mathcal{L}$ over X. Let $p : \tilde{X} \to X$ be the universal cover over X, which exists by our hypotheses over X. Recall from covering space theory that $\pi_1(X, x_0) \cong G(\tilde{X}/X)$, the latter being the Deck-group of $(\tilde{X}, p, X)$. Now consider the constant sheaf $\underline{A}$ on $\tilde{X}$. Let $U \subseteq X$ be an evenly covered neighborhood of X. Then, $p^{-1}(U) = \coprod_{\alpha \in \pi_1(X)} V_\alpha$. Now consider the following sheaf $\mathcal{L}$ which on an open set $U \subseteq X$ gives the following set of sections:

$$\mathcal{L}(U) := \{s \in \underline{A}(p^{-1}(U)) \mid s \circ \theta = \varphi(\theta)s \;\forall \theta \in G(\tilde{X}/X) = \pi_1(X, x_0)\}$$

where $s \in \underline{A}(p^{-1}(U))$ is a section of p as in $s : p^{-1}(U) \to \underline{A} = \coprod_{a \in A} \tilde{X}$ and it is in $\mathcal{L}(U)$ if and only if for any deck transformation $\theta \in G(\tilde{X}/X) = \pi_1(X, x_0)$, we must get that the following commutes[4][5]

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\theta", "\cong"'] \arrow[d, "s"'] & p^{-1}(U) \arrow[d, "s"] \\ \underline{A} \arrow[r, "\cong", "\varphi(\theta)"'] & \underline{A} \end{tikzcd}.$$

We wish to show that $\mathcal{L}$ is a locally constant sheaf and its associated monodromy coincides with φ. The local triviality follows from covering property. Monodromy coinciding again follows by simple unravelling of underlying definitions.

The action of $\pi_1(X, x_0)$ on A obtained by a local system $\mathcal{L}$ is called the **monodromy action** of $\mathcal{L}$ on A. Observe that under the above bijection, the trivial action $0 : \pi_1(X, x_0) \to \mathrm{Aut}\,(A)$ which maps $[\gamma]$ to the identity automorphism of A corresponds to the local system $\mathcal{L}$ on X whose local sections are $G(\tilde{X}/X)$-invariant sections over X.

We thus obtain the following result which gives a characterization of local systems.

Proposition 2.1.5. *Let X be a path-connected, locally path-connected and semi-locally simply connected space. The following are equivalent:*

1. *$\mathcal{L}$ is a local system on X of finite dimensional vector spaces,*
2. *$\{\mathcal{L}_x\}_{x\in X}$ is a collection of finite dimensional vector spaces such that for any path $\gamma : I \to X$ we get a linear isomorphism*

$$\gamma^* : \mathcal{L}_{\varphi(0)} \to \mathcal{L}_{\varphi(1)}$$

such that the following two conditions are satisfied:

(a) *for any two paths γ, η homotopic rel end points, the maps γ^* and η^* are same,*

(b) *if $\gamma * \eta$ is the concatenation of two paths, then $(\gamma * \eta)^* = \gamma^* \circ \eta^*$.*

Proof. (1. $\Rightarrow$ 2.) Pick any local system $\mathcal{L}$. By hypothesis on X, we immediately have a collection of isomorphic vector spaces $A \cong \mathcal{L}_x$ for each $x \in X$. Pick any path $\gamma : I \to X$. We get a map

$$\gamma^* : \mathcal{L}_{\varphi(0)} \to \mathcal{L}_{\gamma(1)}$$

by the usual process of taking the inverse image of $\mathcal{L}$ under γ and calculating stalks (see proof of Lemma 2.1.3). Consider the corresponding monodromy (see 2.1.4)

$$\pi_1(X, x_0) \to \mathrm{Aut}\,(A).$$

The two conditions of item 2 now follows from the conditions of monodromy.
(2. $\Rightarrow$ 1.) From the given data, we wish to construct a locally constant sheaf. By 2.1.4, it suffices to obtain an action $\varphi : \pi_1(X, x_0) \to \mathrm{Aut}\,(A)$. Indeed, pick any $[\gamma] \in \pi_1(X, x_0)$. Define $\varphi([\gamma])$ to be the automorphism associated to the loop γ, the γ^*, as provided by the hypothesis. Its well-definedness follows from the first condition of item 2. That it defines a group homomorphism follows from second condition of item 2. ∎

[4] θ by restriction gives a homeomorphism of $p^{-1(U)}$ by definition.

[5] further, any automorphism of A, say $\kappa : A \to A$ gives a homeomorphism of $\underline{A} = \coprod_{a\in A} \tilde{X}$ by permuting the index A by κ.

Corollary 2.1.6. *Let X be a locally path-connected, simply-connected space. Then any local system $\mathcal{L}$ over X is constant.*

Proof. If X is simply-connected, then the deck group of its universal cover is singleton. Hence $\mathrm{Hom}_{\mathcal{G}rp}(\pi_1(X, x_0), \mathrm{Aut}(A))$ consists of only one map, the trivial map. It follows by 2.1.4 that the local system associated to this is the constant sheaf associated to A, $\underline{A}$ over X (which is its own universal cover). □

While Proposition 2.1.5 gives one characterization of local systems using monodromy action, another interpretation of local systems is also sometimes useful.

Proposition 2.1.7. *Let X be a connected and locally path-connected space. The there is an equivalence of categories*

$$\mathcal{C}ov(X) \equiv \mathrm{LocSys}(X)$$

where $\mathcal{C}ov(X)$ is the category of covering spaces over X and $\mathrm{LocSys}(X)$ is the category of locally constant sheaves of sets over X.

Proof. We will show that this equivalence is induced from the well-known equivalence

$$F : \mathcal{E}t(X) \leftrightarrows \mathcal{S}h(X) : G$$

of étale spaces over X and sheaves of sets over X. In particular, the functor F maps $(E, p, X) \mapsto \mathcal{E}$ where $\mathcal{E}$ on U is the set of sections of p on U. On the other hand, G maps $\mathcal{E} \mapsto (E, p, X)$ where $E = \coprod_{x \in X} \mathcal{E}_x$, p being projection and E having the topology generated by basic open sets $B_{U,s} = \{(U, s)_x \in \mathcal{E}_x \mid s \in \mathcal{E}(U)\} \subseteq E$ where $U \subseteq X$ is open and $s \in \mathcal{E}(U)$.

It is sufficient to show that F maps covering spaces to locally constant sheaves and vice-versa for G. Indeed, if (E, p, X) is a covering space and $\mathcal{E}$ is the associated sheaf, then for a connected evenly covered neighborhood $U \subseteq X$ for which $p^{-1}(U) = \amalg_{\alpha \in A_U} V_\alpha$ where $p : V_\alpha \to U$ is a homeomorphism, we get that the set of sections $\mathcal{E}(U)$ is just A_U by connectedness. Moreover, it is clear that $\mathcal{E}(V) = A_U$ again for any connected $V \subseteq U$. This shows that $\mathcal{E}_{|U} = \underline{A_U}$. Hence $\mathcal{E}$ is a local system.

Conversely, if $\mathcal{E}$ is a local system with (E, p, X) its associated étale space, then for $U \subseteq X$ such that $\mathcal{E}_{|U} = \underline{A}$, we get that $p^{-1}(U) = \amalg_{x \in U} \mathcal{E}_x = \amalg_{x \in U} A = \amalg_{\alpha \in A} V_\alpha$ where $V_\alpha = \{\alpha \in A_x \mid x \in U\}$. We first claim that V_α is open. Indeed, it is the basic open set $B_{U,\alpha}$. Next, $V_\alpha \cap V_\beta =$ is clear. Finally, $p : V_\alpha \to U$ being a homeomorphism is also clear as this is a bijection and p is an open map as it is étale. □

2.2 Cohomology

There are many equivalent ways to compute cohomology of a local system. The simplest way to set it up is to define it as the sheaf cohomology of the underlying locally constant sheaf. While it is easy to define,

2.3 Orientation system

We discuss some basics of orientations using the language of local systems. We fix a commutative ring with 1 denoted R.

Remark 2.3.1. Let M be a topological n-manifold. Then for each $x \in M$, we may take an open chart $U \subseteq M$ of x and $x \in B \subseteq U$ such that B is homeomorphic to a finite radius ball in $\mathbf{R}^n$ where $U \cong \mathbf{R}^n$. By excision, we have

$$H_k(X, X - B; R) \cong H_k(U, U - B; R).$$

Note $H^k(U, U - B; R)$ is R if $k = n$ and 0 else. For each $y \in B$, we get a map induced by the inclusion $(X, X - B) \hookrightarrow (X, X - y)$ given as

$$j_{B,y} : H_n(X, X - B; R) \to H_n(X, X - x; R).$$

This is an isomorphism. Indeed, one also sees that if $B \subseteq B'$ are two balls in a chart, then the map induced by inclusion $(X, X - B') \hookrightarrow (X, X - B)$ induces an isomorphism

$$j_{B',B} : H_n(X, X - B'; R) \to H_n(X, X - B; R).$$

Construction 2.3.2 (R-orientation sheaf)**.** Let M be a topological n-manifold. We wish to define a sheaf on M which on ball B of finite radius in a chart U gives $H_n(M, M - B; R)$. To this end, we need only define a sheaf on the basis of M given by all finite radius balls in any chart of M as such balls forms a basis of M, denoted $\mathcal{B}$. Hence, for any chart $U \subseteq M$ and any finite radius ball $B \subseteq U$, we define

$$\mathcal{O}\mathrm{r}(B) := H_n(M, M - B; R)$$

and for $B \subseteq B'$, we define the restriction map of the sheaf by the map $j_{B',B}$ as in Remark 2.3.1.

We claim that this forms a $\mathcal{B}$-sheaf and hence it will extend to a unique sheaf on M. Indeed, we need only show the gluability axiom. To this end, take $U \in \mathcal{B}$ be any ball, $\{B_i\} \subseteq \mathcal{B}$ be a cover of B by balls together with $s_i \in \mathcal{O}\mathrm{r}(B_i) = H_n(M, M - B_i; R)$. For any $i \neq j$, take any cover $B_i \cap B_j = \bigcup_{k \in I_{ij}} B_{ijk}$ for $B_{ijk} \in \mathcal{B}$ such that

$$j_{B_i, B_{ijk}}(s_i) = j_{B_j, B_{ijk}}(s_j).$$

We wish to show that there exists $s \in \mathcal{O}\mathrm{r}(B)$ such that $j_{B,B_i}(s) = s_i$. Indeed, this immediately follows from the commutativity of the following diagram where each map is an isomorphism (Remark 2.3.1):

$$\begin{tikzcd}
& H_n(M, M - B; R) \arrow[dl, "j_{B,B_i}"'] \arrow[d, "j_{B,B_{ijk}}"] \arrow[dr, "j_{B,B_j}"] & \\
H_n(M, M - B_i; R) \arrow[r, "j_{B_i,B_{ijk}}"'] & H_n(M, M - B_{ijk}; R) & H_n(M, M - B_j; R) \arrow[l, "j_{B_j,B_{ijk}}"]
\end{tikzcd}.$$

Hence $\mathcal{O}\mathrm{r}$ is a $\mathcal{B}$-sheaf and thus extends uniquely to a sheaf on M, which we call the R-orientation sheaf.

We see that the stalk of $\mathcal{O}\mathrm{r}$ is the local homology.

Lemma 2.3.3. *Let M be a topological n-manifold and $x \in M$. Then*

$$\mathcal{O}\mathrm{r}_x \cong H_n(M, M-x; R) \cong R.$$

Proof. We have $\mathcal{O}\mathrm{r}_x = \varinjlim_\alpha \mathcal{O}\mathrm{r}(B_\alpha)$ where $\{B_\alpha\}_\alpha$ forms the neighborhood system of $x \in M$ by finite radius open balls in some chart under inclusions. Thus, we need only show that

$$\varinjlim_{\alpha} H_n(M, M - B_\alpha; R) \cong H_n(M, M-x; R).$$

To this end, we show that $H_n(M, M-x; R)$ satisfies the universal property of the said direct limit. Consider the map $j_{B_\alpha,x} : H_n(M, M-B_\alpha; R) \to H_n(M, M-x; R)$ induced by inclusion. By excision, this is an isomorphism and for $B_\alpha \supseteq B_\beta$, the following triangle commutes

$$\begin{tikzcd} & H_n(M, M-x; R) \\ H_n(M, M-B_\alpha; R) \arrow[ur, "j_{B_\alpha,x}"] \arrow[r, "j_{B_\alpha,B_\beta}"'] & H_n(M, M-B_\beta; R) \arrow[u, "j_{B_\beta,x}"'] \end{tikzcd}.$$

It is easy to see that $H_n(M, M-x; R)$ satisfies the said universal property. □

The following tells an alternate way of constructing orientation sheaf.

Lemma 2.3.4. *Let M be a topological n-manifold. Then the R-orientation sheaf $\mathcal{O}\mathrm{r}$ is isomorphic to the sheafification of the presheaf $U \mapsto H_n(M, M-U; R)$.*

Proof. Let F be the presheaf $U \mapsto H_n(M, M-U; R)$. We have a map $\varphi : F \to \mathcal{O}\mathrm{r}$ defined on finite radius balls by identity. By universal property of sheafification, this extends to a map $\tilde{\varphi} : F^{++} \to \mathcal{O}\mathrm{r}$ where F^{++} is the sheafification of F. As φ is a bijection on stalks (Lemma 2.3.3) and finite radius balls on M forms a basis, we deduce that $\tilde{\varphi}$ is bijection on stalks and hence is an isomorphism, as required. □

We next wish to see that $\mathcal{O}\mathrm{r}$ is actually a locally system.

Proposition 2.3.5. *Let M be a topological n-manifold. Then the R-orientation sheaf is a locally constant sheaf of R-modules, i.e. a local system.*

To prove this, we need the following lemma.

Lemma 2.3.6. *The R-orientation sheaf $\mathcal{O}\mathrm{r}$ on $\mathbf{R}^n$ is isomorphic to constant sheaf $\underline{R}$.*

Proof. As finite radius open balls form a basis of $\mathbf{R}^n$ denoted $\mathcal{B}$, hence we produce an isomorphism from $\mathcal{O}\mathrm{r}$ to $\underline{R}$ as $\mathcal{B}$-sheaves, which will extend to a unique isomorphism of sheaves. Indeed, on $B \in \mathcal{B}$, define the following map

$$\varphi_B : \mathcal{O}\mathrm{r}(B) = H_n(\mathbf{R}^n, \mathbf{R}^n - B; R) \longrightarrow R$$

which is the isomorphism $H_n(\mathbf{R}^n, \mathbf{R}^n - B; R) \cong H_n(S^n; R) \cong R$ where the first isomorphism is via the quotient map and the second is a fixed isomorphism for all B. This isomorphism is natural and since the following square commutes for $B' \supseteq B$ in $\mathcal{B}$:

$$\begin{tikzcd} H_n(\mathbf{R}^n, \mathbf{R}^n - B'; R) \arrow[r, "j_{B',B}"] \arrow[d, "\varphi_{B'}"', "\cong"] & H_n(\mathbf{R}^n, \mathbf{R}^n - B; R) \arrow[d, "\cong"', "\varphi_B"] \\ R \arrow[r, "\mathrm{id}"'] & R \end{tikzcd}.$$

Hence φ is an isomorphism as $\mathcal{B}$-sheaves and thus as sheaves on $\mathbf{R}^n$. □

Proof of Proposition 2.3.5. It suffices to show that for each point $x \in M$, there is a neighborhood $x \in U \subseteq M$ such that $o|_U$ is a constant sheaf. Indeed, taking U to be a chart around x in M which is homeomorphic to $\mathbf{R}^n$, we see that $o|_U$ is isomorphic to the R-orientation sheaf of $\mathbf{R}^n$. It follows from Lemma 2.3.6 that $o|_U$ is isomorphic to constant sheaf $\underline{R}$, as required. □

Using the orientation sheaf, we finally define when a topological manifold is orientable.

Definition 2.3.7 (**R-orientation**)**.** Let M be a topological n-manifold. We say that M is R-orientable if the R-orientation sheaf $\mathcal{O}\mathrm{r}$ on M is isomorphic to the constant sheaf $\underline{R}$. An orientation on M is then the data of an open cover $\{B_i\}$ of M by finite radius balls and sections $s_i \in \mathcal{O}\mathrm{r}(B_i)$ such that s_i is a unit of $\mathcal{O}\mathrm{r}(B_i) = H_n(M, M - B_i; R) \cong R$.

We will later give a different definition of orientation in terms of the orientation double cover, which will be much more useful. We would now like to construct a covering space over a manifold M which detects orientability of M. We first recall that local systems and covering spaces are equivalent.

Remark 2.3.8 (Orientation & total orientation cover)**.** Let M be a topological n-manifold. By Remark 2.3.1, local systems correspond to covering spaces. Consequently, we may interpret the orientation sheaf $\mathcal{O}\mathrm{r}$ as a covering space over M, denoted M_R, which we call the *total orientation cover* of M. Consider now the subsheaf of units of $\mathcal{O}\mathrm{r}$ which consists of all the units of $\mathcal{O}\mathrm{r}$, denoted $\mathcal{O}\mathrm{r}^\times$. In particular, for a finite radius ball $B \subseteq M$ in an open chart of M, we have $\mathcal{O}\mathrm{r}^\times(B) = \{\text{units of } H_n(M, M - B; R)\} \cong R^\times$. It is clear that $\mathcal{O}\mathrm{r}^\times$ is a local system, just as in the proof of Proposition 2.3.5.

It follows that $\mathcal{O}\mathrm{r}^\times$ corresponds to a double cover of M which we denote by M_o, which we call the *orientation double cover.*

The following is a remarkable fact about orientation double cover.

Proposition 2.3.9. *Let M be an topological n-manifold. Then the orientation double cover M_o is oriented.*

3 Grassmannians & universal bundles

It is quite interesting to note that even though it doesn't seem like so from the definition, but vector bundles are really "homotopical objects", as has been shown once by homotopy invariance theorem (Theorem 1.6.1). In-fact, much more is true; we can classify vector bundles on a paracompact space X by the homotopy set $[X, BU]$, where BU is a universal space. By Brown's representability theorem, BU will then become an Ω-spectrum. This BU is also called the complex K-theory spectrum.

We begin by discussing an extremely important space, which appears in places well beyond topology. For us, the universality of Grassmannians and its mod 2 cohomology will provide the fundamental constructions which will yield the many characteristic classes which we wish to study in next sections.

3.1 Grassmannians

To define Grassmannians correctly as a topological space, we first need to introduce Stiefel variety. Let $F = \mathbf{R}$ or $\mathbf{C}$.

Definition 3.1.1 ($V_n(V)$)**.** Let V be a finite dimensional F-vector space with the standard inner product induced by F. Then, define

$$V_n(V) := \{(a_1, \ldots, a_n) \mid a_i \in \text{V are orthonormal}\}.$$

This has the subspace topology of V^n. This is the n-Stiefel space over V. This is a compact space as it is a closed subspace of $(S^{\dim V - 1})^n$.

Definition 3.1.2 ($\mathrm{Gr}_n(V)$)**.** Let V be a finite dimensional F-vector space with the standard inner product induced by F. Define $\mathrm{Gr}_n(V)$ to be the set of all n-dimensional F-linear subspaces of V. The topology on $\mathrm{Gr}_n(V)$ is given by the quotient map

$$\begin{aligned} \pi : V_n(V) &\twoheadrightarrow \mathrm{Gr}_n(V) \\ (a_1, \ldots, a_n) &\longmapsto W \end{aligned}$$

where W is the span of $\{a_1, \ldots, a_n\}$ in V. Thus $\mathrm{Gr}_n(V)$ is a compact Hausdorff space. We call this the n-Grassmannian of V.

Remark 3.1.3. Let V be n-dimensional. One can see that $\mathrm{Gr}_k(V)$ is a smooth compact manifold of dimension $k(n-k)$ by the Plücker embedding, which exhibits $\mathrm{Gr}_k(V)$ as a closed submanifold of the projective space of $\wedge^k V$, the k^{th}-exterior power of V. If $V = \mathbf{R}^n$, then we write $\mathrm{Gr}_k(n)$.

Definition 3.1.4 ($BO(n)$ & $BU(n)$)**.** Consider the canonical incusions of vector spaces by adding a 0 in an extra coordinate:

$$F^1 \hookrightarrow F^2 \hookrightarrow \ldots$$

These induce inclusions

$$V_n(F^q) \hookrightarrow V_n(F^{q+1}) \text{ \& } \mathrm{Gr}_n(F^q) \hookrightarrow \mathrm{Gr}_n(F^{q+1})$$

where the latter is induced by universal property of quotients. Taking the direct limit/coherent union, we obtain spaces

$$V_n(F^\infty) \;\&\; \mathrm{Gr}_n(F^\infty).$$

We denote $BO(n) = \mathrm{Gr}_n(\mathbf{R}^\infty)$ and $BU(n) = \mathrm{Gr}_n(\mathbf{C}^\infty)$.

Definition 3.1.5 (BU & BO)**.** Observe that we have inclusions

$$i_n : BU(n) \hookrightarrow BU(n+1)$$

which are induced by the following inclusion map on Stiefel spaces for $\mathbf{C}^\infty$:

$$\begin{aligned} i_n : V_n(\mathbf{C}^\infty) &\longrightarrow V_{n+1}(\mathbf{C}^\infty) \\ (a_1, \dots, a_n) &\longmapsto (a_1, \dots, a_n, e_{q+1}) \end{aligned}$$

where $q \in \mathbf{N}$ is the smallest integer such that $a_i \in \mathbf{C}^q$ for all $1 \le i \le n$, and $e_{q+1} \in \mathbf{C}^{q+1}$ is 1 on $q+1$-entry and 0 on all others[6]. Composing i_n with $\pi : V_{n+1}(\mathbf{C}^\infty) \to \mathrm{Gr}_{n+1}(\mathbf{C}^\infty)$, we get a map which identifies fibers of $\pi : V_n(\mathbf{C}^\infty) \to \mathrm{Gr}_n(\mathbf{C}^\infty)$ since for any $x \in \mathrm{Gr}_n(\mathbf{C}^\infty)$, $\pi^{-1}(x)$ consists of all orthonormal bases of x in $\mathbf{C}^\infty$, which under $\pi \circ i_n$ maps onto the unique $n+1$-dimensional subspace spanned by x and e_{q+1}. Note that for any other orthonormal basis of x, the integer q will be same, so that this map is well-defined. Hence we get the desired inclusion $i_n : BU(n) \hookrightarrow BU(n+1)$, as required. Similarly, we get inclusions in the real case

$$i_n : BO(n) \hookrightarrow BO(n+1).$$

The direct limit/coherent union of $BO(n)$ and $BU(n)$ are called BO and BU respectively.

3.2 Universal bundles

We finally construct the universal bundle $EO(n) \to BO(n)$ and $EU(n) \to BU(n)$.

Construction 3.2.1 (Universal bundles)**.** Consider the trivial bundle over $\mathrm{Gr}_n(F^q)$

$$\pi_q : \mathrm{Gr}_n(F^q) \times F^q \longrightarrow \mathrm{Gr}_n(F^q).$$

Suppose $F = \mathbf{R}$. Let $EO(n)^q$ be the n-dimensional subbundle obtained by the following incidence correspondence:

$$EO(n)^q = \{(X, v) \in \mathrm{Gr}_n(\mathbf{R}^q) \times \mathbf{R}^q \mid v \in X \subseteq \mathbf{R}^q\}.$$

Then, we have a canonical inclusion

$$EO(n)^q \hookrightarrow EO(n)^{q+1}$$

Taking direct limit/coherent union, we obtain

$$EO(n) = \bigcup_q EO(n)^q$$

[6] the integer q is the smallest such that the span of $\{a_1, \dots, a_n\}$ is contained in $\mathbf{C}^q$.

and thus the map

$$\pi : EO(n) \longrightarrow BO(n).$$

Similarly, one constructs for $F = \mathbf{C}$

$$\pi : EU(n) \longrightarrow BU(n).$$

In particular, the universal bundles are the coherent union under canonical inclusions as shown below for real case:

$$\begin{tikzcd}
EO(n)^q \arrow[r, hook] \arrow[d, "\pi_q"'] & EO(n)^{q+1} \arrow[r, dotted, no head] \arrow[d, "\pi_{q+1}"] & EO(n) \arrow[d, "\pi"] \\
\mathrm{Gr}_n(\mathbf{R}^q) \arrow[r, hook] & \mathrm{Gr}_n(\mathbf{R}^{q+1}) \arrow[r, dotted, no head] & BO(n)
\end{tikzcd}.$$

Lemma 3.2.2. *The universal bundles $EO(n) \to BO(n)$ and $EU(n) \to BU(n)$ are real and complex vector bundles of dimension n, respectively.*

The main result is the following.

Theorem 3.2.3 (Homotopy classification)**.** *Let X be a paracompact space and denote $VB_{n,\mathbf{C}}(X)$ and $VB_{n,\mathbf{R}}(X)$ to be isomorphism class of n-dim $\mathbf{C}$ and $\mathbf{R}$-vector bundles on X, respectively. Then, there are natural bijections*

$$VB_{n,\mathbf{C}}(X) \cong [X, BU(n)]$$

and

$$VB_{n,\mathbf{R}}(X) \cong [X, BO(n)]$$

where $[X, Y]$ denotes unbased homotopy classes of maps.

4 Characteristic classes of manifolds

We will study some cohomology classes induced by vector bundles. We will write $\mathbb{Z}_2$ for $\mathbb{Z}/2\mathbb{Z}$.

4.1 Stiefel-Whitney classes

Recall that $H^*(B;\mathbb{Z}_2)$ denotes the graded commutative ring $\bigoplus_{i\geq 0} H^i(B;\mathbb{Z}_2)$ where multiplication is the cup product:

$$\begin{aligned}
\cup : H^p(B;\mathbb{Z}_2) \times H^q(B;\mathbb{Z}_2) &\longrightarrow H^{p+q}(B;\mathbb{Z}_2) \\
([a],[b]) &\longmapsto [a] \cup [b].
\end{aligned}$$

The unit of the ring $H^*(B;\mathbb{Z}_2)$ lies in $H^0(B;\mathbb{Z}_2)$. We begin by the axiomatic system.

Definition 4.1.1 (**Stiefel-Whitney class of a vector bundle**)**.** Let $\xi = (E, \pi, B)$ be a vector bundle of rank n. Then the sequence of elements

$$w_i(\xi) \in H^i(B; \mathbb{Z}_2)$$

for all $i \geq 0$ is said to be Stiefel-Whitney classes (SW-classes for short) of ξ if the following conditions are satisfied.

1. For $i = 0$, we $w_0(\xi) \in H^0(B; \mathbb{Z}_2)$ is the unit of the cohomology ring and $w_i(\xi) = 0$ for $i > n$.
2. If $f : B' \to B$ is a map, then under the pullback

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "f^*\xi"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\xi"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

   we have

$$w_i(f^*\xi) = f^*(w_i(\xi))$$

   where $f^* : H^i(B; \mathbb{Z}_2) \to H^i(B'; \mathbb{Z}_2)$ is the natural map induced on cohomology.
3. We have the following product formula for any two vector bundles ξ, η:

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta).$$

4. For the canonical line bundle $V_n^1 \to \mathbb{P}^n$ on $\mathbb{P}^n$, we have

$$w_1(V_n^1) \neq 0.$$

Before moving on to applications, we first remark how Stiefel-Whitney classes arrange themselves.

Construction 4.1.2 (Total Stiefel-Whitney class)**.** Denote $H^{\Pi}(B; \mathbb{Z}_2)$ to be the direct product group $\Pi_{i \geq 0} H^i(B; \mathbb{Z}_2)$ together with the ring structure given by that of power series multiplication. For any n-plane bundle ξ on B, we then get an element

$$w(\xi) = w_0(\xi) + w_1(\xi) + w_2(\xi) + \cdots + w_n(\xi) \in H^{\Pi}(B; \mathbb{Z}_2).$$

As $w_0(\xi)$ is always 1, therefore $w(\xi)$ is a unit of $H^{\Pi}(B\ Z_2)$. Hence considering $w(\xi)$ as a unit of $H^{\Pi}(B; \mathbb{Z}_2)$, we get that for any other vector bundle η on B we have

$$w(\xi \oplus \eta) = w(\xi) \cdot w(\eta)$$

where the product is in $H^{\Pi}(B; \mathbb{Z}_2)$. The element $w(\xi)$ is called the total SW-class.

The first result on SW-classes is that the trivial bundle ϵ^n has trivial SW-class.

Proposition 4.1.3. *If ϵ^n be a trivial n-plane bundle over B, then $w(\epsilon^n) = 1$.*

Proof. We first observe that the following fibre square for any $k \geq 1$

$$\begin{tikzcd} \epsilon^k_B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \epsilon^k_{\text{pt.}} \arrow[d] \\ B \arrow[r, "f"'] & \text{pt.} \end{tikzcd}.$$

By naturality axiom, we have $f^*(w_l(\epsilon^k_{\text{pt.}})) = w_l(f^*\epsilon^k_{\text{pt.}}) = w_l(\epsilon^k_B)$. As $f^* : H^l(\text{pt.};\mathbb{Z}_2) \to H^l(B;\mathbb{Z}_2)$ and $H^l(\text{pt.};\mathbb{Z}_2) = 0$ for $l \geq 1$, therefore $w_l(\epsilon^k_{\text{pt.}}) = 0$ and hence so is $w_l(\epsilon^1_B)$. This shows that $w_l(\epsilon^k) = 0$ for all $l, k \geq 1$, thus $w(\epsilon^n) = 1$. □

Corollary 4.1.4. *Let M be a smooth n-manifold and TM, NM be tangent, normal bundles over M. Then*

$$w(TM)^{-1} = w(NM)$$

where the inverse is taken in $H^\Pi(M;\mathbb{Z}_2)$.

Proof. As $TM \oplus NM = \epsilon^n$, therefore by Proposition 4.1.3 we have $w(TM) \cdot w(NM) = 1$ in $H^\Pi(M;\mathbb{Z}_2)$, as required. □

The total SW-class of canonical line bundle can be calculated simply.

Lemma 4.1.5. *If $\pi : V^1_n \to \mathbb{P}^n$ is the canonical line bundle over $\mathbb{P}^n$, then*

$$w(V^1_n) = 1 + a.$$

Proof. Indeed, by axiom on canonical bundle, $w_1(V^1_n) \neq 0$. As $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$, therefore $w_1(V^1_n) = a$. By axiom on rank, $w_i(V^1_n) = 0$ for $i \geq 2$, as required. □

4.2 Parallelizability & embeddings of $\mathbb{R}\mathrm{P}^n$

Our first applications of SW-classes are in parallelizability and embedding theorems for $\mathbb{R}\mathrm{P}^n$. Let us begin by recalling the tangent bundle of $\mathbb{R}\mathrm{P}^n$.

Lemma 4.2.1. *Let $V^1_n \to \mathbb{R}\mathrm{P}^n$ be the canonical line bundle on $\mathbb{R}\mathrm{P}^n$. Then*

$$T\,\mathbb{R}\mathrm{P}^n \cong \mathcal{H}om(V^1_n, V^\perp_n)$$

where $V^\perp_n$ is the normal bundle $V^\perp_n = \{([x],\vec{v}) \in \mathbb{R}\mathrm{P}^n \times \mathbf{R}^{n+1} \mid \vec{v} \perp \langle x\rangle$ in $\mathbf{R}^{n+1}\}$.

Proof. Note that by the quotient map $q : S^n \twoheadrightarrow \mathbb{R}\mathrm{P}^n$ we have for point $p \in S^n$

$$Dq_p : T_pS^n \longrightarrow T_{[p]}\,\mathbb{R}\mathrm{P}^n$$

which is surjective. Consequently,

$$T\,\mathbb{R}\mathrm{P}^n = \frac{\{(x,\vec{v}) \in S^n \times \mathbf{R}^n \mid \vec{v} \in T_xS^n\}}{(x,\vec{v}) \sim (-x,-\vec{v})}.$$

Recall that $\vec{v} \in T_x S^n$ is equivalent to $\vec{v} \perp \langle x \rangle$. Consider the map

$$\varphi : T\,\mathbb{RP}^n \longrightarrow \mathrm{Hom}\left(V_n^1, V_n^{\perp}\right)$$

which on fiber at $[x] \in \mathbb{RP}^n$ is given by

$$\begin{aligned} \varphi_{[x]} : T_{[x]}\,\mathbb{RP}^n &\longrightarrow \mathrm{Hom}_{\mathbf{R}}\left(V_{n,x}^1, V_{n,x}^{\perp}\right) \\ ([x], \vec{v}) &\longmapsto \varphi_{[x]}(\vec{v}) \end{aligned}$$

where note that $V_{n,x}^1 = \langle x \rangle$ and $V_{n,x}^{\perp} = \langle x \rangle^{\perp}$ in $\mathbf{R}^{n+1}$, so we define

$$\begin{aligned} \varphi_{[x]}(\vec{v}) : \langle x \rangle &\longrightarrow \langle x \rangle^{\perp} \\ \lambda x &\longmapsto \lambda \vec{v}. \end{aligned}$$

The map φ is continuous as can be checked by going to local trivializations (Lemma 1.1.17). As φ is an isomorphism on fibers, therefore by Lemma 1.1.2, φ is an isomorphism. □

Remark 4.2.2 (Cohomology of $\mathbb{RP}^n$)**.** Recall the homology and cohomology of $\mathbb{RP}^n$ are given as follows.

$$H_i(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}_2 & \text{if } 1 \leq i \leq n-1, i \text{ is odd}, \\ 0 & \text{if } 1 \leq i \leq n-1, i \text{ is even}, \\ \mathbb{Z} & \text{if } i = n, n \text{ is odd}, \\ 0 & \text{if } i = n, n \text{ is even}. \end{cases}$$

By tensoring the cellular complex of $\mathbb{RP}^n$ by $\mathbb{Z}_2$ and taking homology, we get homology with $\mathbb{Z}_2$ coefficient simply as

$$H_i(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2,\ \forall\, 0 \leq i \leq n.$$

Next, integral cohomology of $\mathbb{RP}^n$ is obtained easily by dualizing the cellular chain complex of $\mathbb{RP}^n$ and taking homology, which yields

$$H^i(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } 1 \leq i \leq n-1, i \text{ is odd}, \\ \mathbb{Z}_2 & \text{if } 1 \leq i \leq n-1, i \text{ is even}, \\ \mathbb{Z} & \text{if } i = n, n \text{ is odd}, \\ \mathbb{Z}_2 & \text{if } i = n, n \text{ is even}. \end{cases}$$

As $\mathbb{Z}_2$ is a field, we may compute the mod-2 cohomology of $\mathbb{RP}^n$ by using universal coefficients theorem for fields, so that

$$H^i(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}_2}\left(H_i(\mathbb{RP}^n; \mathbb{Z}_2), \mathbb{Z}_2\right) = \mathbb{Z}_2 \;\forall\, 0 \leq i \leq n.$$

Theorem 4.2.3 (mod-2 cohomology ring of $\mathbb{R}\mathrm{P}^n$)**.** *For any $n \geq 0$, the cohomology ring of $\mathbb{R}\mathrm{P}^n$ is*

$$H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \frac{\mathbb{Z}_2[\alpha]}{\alpha^{n+1}}$$

where $\alpha \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is the generator.

The following important result tells us that $T\,\mathbb{R}\mathrm{P}^n$ is "almost" the n-copies of canonical line bundle.

Proposition 4.2.4. *Let $n \geq 1$ and denote ϵ^1 to be the rank 1 trivial bundle over $\mathbb{R}\mathrm{P}^n$. Then we have an isomorphism*

$$T\,\mathbb{R}\mathrm{P}^n \oplus \epsilon^1 \cong \underbrace{V_n^1 \oplus \cdots \oplus V_n^1}_{n+1\text{-times}}.$$

Proof. By Lemma 4.2.1, we have $T\,\mathbb{R}\mathrm{P}^n \cong \mathcal{H}om(V_n^1, V_n^\perp)$. Moreover, since $\mathcal{H}om(V_n^1, V_n^1) \cong \epsilon^1$, therefore we may write

$$\begin{aligned} T\,\mathbb{R}\mathrm{P}^n \oplus \epsilon^1 &\cong \mathcal{H}om(V_n^1, V_n^\perp) \oplus \mathcal{H}om(V_n^1, V_n^1) \\ &\cong \mathcal{H}om(V_n^1, V_n^\perp \oplus V_n^1) \\ &\cong \mathcal{H}om(V_n^1, \epsilon^{n+1}) \\ &\cong \mathcal{H}om(V_n^1, \epsilon^1)^{\oplus n+1} \\ &\cong V_n^1 \oplus \cdots \oplus V_n^1, \end{aligned}$$

as required. □

Corollary 4.2.5. *For any $n \geq 1$, we have*

$$w(\mathbb{R}\mathrm{P}^n) := w(T\,\mathbb{R}\mathrm{P}^n) = (1+a)^{n+1}$$

in $H^\Pi(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ where $a \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is the generator of the cohomology ring $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$. Hence, $\mathbb{R}\mathrm{P}^n$ has trivial SW-class if and only if $n+1 = 2^m$ for some $m \geq 1$.

Proof. Applying w on Proposition 4.2.4, we have

$$w(T\,\mathbb{R}\mathrm{P}^n \oplus \epsilon^1) = w(T\,\mathbb{R}\mathrm{P}^n) \cdot w(\epsilon^1) = w(V_n^1)^{n+1}.$$

Note $w(\epsilon^1) = 1$ by Proposition 4.1.3. As $w(V_n^1) = 1 + a$ by Lemma 4.1.5, we thus get

$$w(T\,\mathbb{R}\mathrm{P}^n) = (1+a)^{n+1},$$

as required. The other statement is immediate. □

Corollary 4.2.6. *If $\mathbb{R}\mathrm{P}^n$ is parallelizable, then $n+1 = 2^m$.* □

We next find some bounds on $k \geq 0$ such that $\mathbb{R}\mathrm{P}^n$ embeds into $\mathbf{R}^{n+k}$. For this, we first prove the following simple lemma.

Lemma 4.2.7. *Let M be a smooth n-manifold and $f : M \to \mathbf{R}^{n+k}$ be an immersion. Then $\bar{w}_i(M) = 0$ for all $i > k$ where $\bar{w}(M)$ is the inverse of SW-class of TM in $H^{\Pi}(M;\mathbb{Z}_2)$.*

Proof. As we have $TM \oplus NM = \epsilon^{n+k}$ where NM is the normal bundle of the immersion f, therefore applying SW-class yields

$$w(TM) \cdot w(NM) = 1$$

by Proposition 4.1.3. Hence $\bar{w}(M) = w(NM)$ and since NM is a rank k-bundle, we win by rank axiom. □

As an example, lets begin from $n = 9$.

Example 4.2.8 (When does $\mathbb{R}\mathrm{P}^9$ immerse into $\mathbf{R}^{9+k}$?)**.** If $f : \mathbb{R}\mathrm{P}^9 \to \mathbf{R}^{9+k}$ is an immersion, then by Lemma 4.2.7 we yield that $\bar{w}_i(\mathbb{R}\mathrm{P}^9) = 0$ for $i > k$. By Corollary 4.2.5 we have that $w(\mathbb{R}\mathrm{P}^n) = (1+a)^{10} = 1 + a^2 + a^8$. Consequently, $\bar{w}(\mathbb{R}\mathrm{P}^n) = 1 + a^2 + a^4 + a^6$. Thus we get the lower bound on k given as $k \geq 6$. It follows that

$$\text{If } \mathbb{R}\mathrm{P}^9 \to \mathbf{R}^{9+k} \text{ is an immersion, then } k \geq 6.$$

One can generalize this to $\mathbb{R}\mathrm{P}^n$ where $n = 2^r$.

Lemma 4.2.9. *Let $n = 2^r$. If $f : \mathbb{R}\mathrm{P}^n \to \mathbf{R}^{n+k}$ is an immersion, then $k \geq n - 1$.* □

Proof. We have $w(\mathbb{R}\mathrm{P}^n) = (1+a)^{n+1} = (1+a)^n \cdot (1+a) = (1+a^n) \cdot (1+a) = 1 + a + a^n$. One then sees by a simple expansion that $\bar{w}(\mathbb{R}\mathrm{P}^n) = (1+a+a^n)^{-1} = 1 + a + a^2 + \cdots + a^{n-1}$. Hence $k \geq n-1$, as required. □

Remark 4.2.10 (Whitney embedding is best possible for $\mathbb{R}\mathrm{P}^{2^r}$)**.** We claim that the smallest $k \geq 1$ such that $\mathbb{R}\mathrm{P}^n$ immerses into $\mathbf{R}^{n+k}$ for $n = 2^r$ is infact $k = n-1$. Indeed, by above lemma we have $k \geq n-1$. However, Whitney embedding tells us that $\mathbb{R}\mathrm{P}^n$ immerses into $\mathbf{R}^{2n-1}$. Thus, $k \leq n-1$, as required.

4.3 Splitting construction

An important tool in obtaining relations amongst characteristic classes is provided by splitting principle. Using this, we can obtain many relations amnogst characteristic classes of bundles by first assuming that the given bundle is direct sum of line bundles.

Theorem 4.3.1 (Splitting construction)**.** *Let B be a space and $\xi = (E, p, B)$ be a rank k bundle. Then there exists a space X and a map $f : X \to B$ such that*

1. *the map $f^* : H^*(B;\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2)$ is injective,*
2. *the pullback $f^*\xi$ is direct sum of line bundles over X.*

Here's an example of the power of this result.

Theorem 4.3.2. *Let $\xi = (E, p, B)$ be a rank k bundle. Then the following are equivalent:*

1. *ξ is orientable.*
2. *$w_1(\xi) = 0$.*

4.4 Cobordism

We now take the first step towards the construction of cobordism ring. We will see that SW-classes of a compact manifold are all 0 if and only if it is a boundary of some manifold of one higher dimension. Hence this will show that SW-class of a manifold stores important global information about it.

4.5 Cohomology of real Grassmannians

To compute the cohomology of Grassmannian, we will give it a cell structure so that the classes of each cell will generate the cohomology ring. For proofs, refer to Milnor-Stasheff.

Theorem 4.5.1. *Let* Gr_k *be the Grassmannian of* k*-planes in* $\mathbf{R}^\infty$ *and* $V^k \to \mathrm{Gr}_k$ *be the universal* k*-plane bundle. Then:*

1. *The Stiefel-Whitney classes* $w_1(V^k), \dots, w_k(V^k) \in H^*(\mathrm{Gr}_k; \mathbb{Z}_2)$ *are algebraically independent.*
2. *The mod 2 cohomology algebra of* Gr_k *is*

$$H^*(\mathrm{Gr}_k; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(V^k), \dots, w_k(V^k)].$$

4.6 Obstructions & Stiefel-Whitney classes

5 Construction of Stiefel-Whitney classes

We wish to establish the existence and uniqueness of a Stiefel-Whitney classes. It turns out that uniqueness is easy.

Proposition 5.0.1. *There is atmost one assignment* $\xi \mapsto w_i(\xi)$ *which satisfies the axioms of Stiefel-Whitney classes.*

Proof. Idea is simple; for any $n \geq 1$, we will show that the axiom $w_1(V_1^1) \neq 0$ is sufficient to make two such assignments agree on the universal k-plane bundle $V^k \to \mathrm{Gr}_k$, which by naturality axiom and universality of V^k will complete the proof. Suppose v, w are two such assignments. Consider the canonical line bundle $V^1 \to \mathbb{R}\mathrm{P}^\infty = \mathrm{Gr}_1$. Then $v(V^1) = w(V^1) = 1 + a$ by using the fiber square

$$\begin{tikzcd} V^1_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V^1 \arrow[d] \\ \mathbb{R}\mathrm{P}^n \arrow[r, hook, "i"'] & \mathbb{R}\mathrm{P}^\infty \end{tikzcd}$$

where $a \in H^1(\mathrm{Gr}_1; \mathbb{Z}_2)$ is the non-zero element. We will now construct a space X which admits a map $f : X \to \mathrm{Gr}_k$ such that pullback bundle f^*V^k is a direct sum of line bundles on X (the splitting construction for the universal bundle). Indeed, consider $X = \mathbb{R}\mathrm{P}^\infty \times \cdots \times \mathbb{R}\mathrm{P}^\infty$ k-many times. There is a fiber square

$$\begin{tikzcd} V^1 \times \cdots \times V^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V^k \arrow[d] \\ \mathbb{R}\mathrm{P}^\infty \times \cdots \times \mathbb{R}\mathrm{P}^\infty \arrow[r, "f"'] & \mathrm{Gr}_k \end{tikzcd}$$

by the universal property of infinite Grassmannian. To complete the proof, we need only show that $w(V^k) = v(V^k)$. As $f^* : H^i(\mathrm{Gr}_k; \mathbb{Z}_2) \to H^i(\mathbb{R}\mathrm{P}^\infty \times \dots \mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ is injective, it suffices to show that $f^*(w(V^k)) = f^*(v(V^k))$. We have

$$\begin{aligned}
f^*(w(V^k)) &= w(V^1 \times \cdots \times V^1) \\
&= w(p_1^* V^1 \oplus \cdots \oplus p_k^* V^1) \\
&= w(p_1^* V^1) \cdot \cdots \cdot w(p_k^* V^1) \\
&= p_1^*(w(V^1)) \cdot \cdots \cdot p_k^*(w(V^1)) \\
&= \prod_{i=1}^{k} (1 + a_i)
\end{aligned}$$

where $a_i \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ for the i^{th} factor. Similarly we may compute $f^*(v(V^k)) = \prod_{i=1}^{k}(1 + a_i)$. Hence we have the desired equality. $\square$

We now need only show the existence of such classes. To this end, we would need a major result, usually known as mod 2 Thom isomorphism theorem.

5.1 mod 2 Thom isomorphism

Thom's result gives establishes a relation between cohomology of the zero section and the cohomology of the base. Here's the mod 2 version.

Theorem 5.1.1. *Let $p : E \to B$ be a vector bundle of rank n over a space B and let $s : B \to E$ be the zero section. Denote*

$$E_0 = E - s(B) \;\&\; E_{b0} = E_b - s(b).$$

Note that

$$H^i(E_b, E_{b0}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & \text{if } i = n, \\ 0, & \text{else.} \end{cases}$$

Then:

1. (Cohomology of base) *We have $H^*(E, E_0; \mathbb{Z}_2) \cong H^*(B; \mathbb{Z}_2)[-n]$, that is,*

$$H^i(E, E_0; \mathbb{Z}_2) \cong \begin{cases} 0 & \text{if } i < n, \\ H^{i-n}(B; \mathbb{Z}_2) & \text{if } i \geq n. \end{cases}$$

2. (Fundamental class) *The group $H^n(E, E_0; \mathbb{Z}_2)$ contains a class u unique with respect to the following: for all $b \in B$, the restriction*

$$u|_{(E_b, E_{b0})} \in H^n(E_b, E_{b0}; \mathbb{Z}_2)$$

is non-zero.

3. (Cohomology of total space) *The map*

$$\begin{aligned} H^*(E;\mathbb{Z}_2) &\longrightarrow H^*(E,E_0;\mathbb{Z}_2)[n] \\ x &\longmapsto x\cup u \end{aligned}$$

is a $\mathbb{Z}_2$-linear isomorphism.

The isomorphism obtained by the composite

$$\phi_{\mathrm{Th}} : H^*(B;\mathbb{Z}_2) \xrightarrow{p^*} H^*(E;\mathbb{Z}_2) \xrightarrow{-\cup u} H^*(E,E_0;\mathbb{Z}_2)[n]$$

is called the mod 2 Thom isomorphism for the bundle $p : E \to B$.

5.2 Steenrod squares

Another ingredient in defining Stiefel-Whitney classes is the Steenrod squares. We begin with the axiomatics.

Theorem 5.2.1. *Let (X,A) be a pair. Then for all pair of positive integers (m,k), there exists a unique group homomorphism*

$$\mathrm{Sq}^k : H^m(X,A;\mathbb{Z}_2) \longrightarrow H^{m+k}(X,A;\mathbb{Z}_2)$$

satisfying the following properties:

1. (Natural transformation) *If $f : (X,A) \to (Y,B)$ is a map of pairs, then the following commutes*

$$\begin{tikzcd} H^m(X,A;\mathbb{Z}_2) \arrow[r, "f^*"] \arrow[d, "\mathrm{Sq}^k"'] & H^m(Y,B;\mathbb{Z}_2) \arrow[d, "\mathrm{Sq}^k"] \\ H^{m+k}(X,A;\mathbb{Z}_2) \arrow[r, "f^*"'] & H^{m+k}(Y,B;\mathbb{Z}_2) \end{tikzcd}.$$

2. (Edge cases) *We have*

$$\begin{aligned} \mathrm{Sq}^0(a) &= a \\ \mathrm{Sq}^m(a) &= a\cup a \\ \mathrm{Sq}^k(a) &= 0,\ \forall k>m. \end{aligned}$$

3. (Cartan formula) *If $a \in H^p(X,A;\mathbb{Z}_2)$ and $b \in H^q(X,A;\mathbb{Z}_2)$ such that $p+q=m$, then*

$$\mathrm{Sq}^k(a\cup b) = \sum_{i+j=k} \mathrm{Sq}^i(a)\cup \mathrm{Sq}^j(b).$$

For a homogeneous $a \in H^*(X,A;\mathbb{Z}_2)$, we write $\mathrm{Sq}(a) = \mathrm{Sq}^0(a) + \mathrm{Sq}^1(a) + \cdots + \mathrm{Sq}^{\deg a}(a)$ and call it the total Steenrod square.

Remark 5.2.2. Note that $\mathrm{Sq}(a\cup b) = \mathrm{Sq}(a)\cup\mathrm{Sq}(b)$ and $\mathrm{Sq}(1)=1$. Hence $\mathrm{Sq} : H^*(X,A;\mathbb{Z}_2) \to H^*(X,A;\mathbb{Z}_2)$ is a ring homomorphism.

Lemma 5.2.3. *Let $a \in H^p(X, A; \mathbb{Z}_2)$ and $b \in H^q(Y, B; \mathbb{Z}_2)$. Then,*

$$\mathrm{Sq}(a \times b) = \mathrm{Sq}(a) \times \mathrm{Sq}(b)$$

in $H^{p+q}(X \times Y, X \times B \cup A \times Y; \mathbb{Z}_2)$.

Proof. Let $Z = X \times Y$, $p : Z \to X$ and $q : Z \to Y$ be projections and $\Delta : Z \to Z \times Z$ be the diagonal. It follows that $(p \times q) \circ \Delta = \mathrm{id}$. We now have

$$\begin{aligned}
\mathrm{Sq}(a \times b) &= \mathrm{Sq}((a \times 1) \cup (1 \times b)) \\
&= \mathrm{Sq}(a \times 1) \cup \mathrm{Sq}(1 \times b) \\
&= \Delta^* \left(\mathrm{Sq}(a \times 1) \times \mathrm{Sq}(1 \times b)\right) \\
&= \Delta^* (p \times q)^* (\mathrm{Sq}(a) \times \mathrm{Sq}(b)) \\
&= \mathrm{Sq}(a) \times \mathrm{Sq}(b),
\end{aligned}$$

as required. □

5.3 Stiefel-Whitney classes

We now construct SW-classes of a vector bundle.

Construction 5.3.1 (Stiefel-Whitney classes)**.** Let $\xi = (E, p, B)$ be a rank n vector bundle over a base space B. By Thom isomorphism φ_{Th} for bundle ξ and Steenrod squares, we get a map for each $k \geq 0$ as in the diagram below:

$$\begin{tikzcd}
H^n(E, E_0; \mathbb{Z}_2) \arrow[r, "\mathrm{Sq}^k"] & H^{n+k}(E, E_0; \mathbb{Z}_2) \arrow[d, "\phi^{-1}_{\mathrm{Th},k}"', "\cong"] \\
H^0(B; \mathbb{Z}_2) \arrow[u, "\phi_{\mathrm{Th},0}", "\cong"'] \arrow[r, dashed, "\phi^{-1}_{\mathrm{Th},k} \circ \mathrm{Sq}^k \circ \phi_{\mathrm{Th},0}"'] & H^k(B; \mathbb{Z}_2)
\end{tikzcd} .$$

Hence, we define

$$w_k(\xi) := \phi^{-1}_{\mathrm{Th},k} \circ \mathrm{Sq}^k \circ \phi_{\mathrm{Th},0}(1)$$

as the k^{th} Stiefel-Whitney class of ξ.

Lemma 5.3.2. *Let $\xi = (E, p, B)$ be a rank n bundle. Then,*

$$w_n(\xi) = \phi^{-1}_{\mathrm{Th}}(u \cup u)$$

where $u \in H^n(E, E_0; \mathbb{Z}_2)$ is the mod 2 fundamental class of ξ.

Proof. Immediate from the fact that $\mathrm{Sq}^n(a) = a \cup a$ if $a \in H^n(X; \mathbb{Z}_2)$. □

Theorem 5.3.3. *The assignment $\xi \mapsto w_i(\xi)$ in the Construction 5.3.1 satisfies the axioms of Stiefel-Whitney classes.*

An important formula regarding Steenrod squaring operation for Stiefel-Whitney classes is given by Wu.

Proposition 5.3.4 (Wu's formula)**.** *Let $\xi = (E, p, B)$ be a rank k-bundle over B. Let $w_m = w_m(\xi) \in H^m(X; \mathbb{Z}_2)$. Then for all tuples (m, i) of non-negative integers, we have*

$$\mathrm{Sq}^i(w_m) = w_i w_m + \binom{i-m}{1} w_{i-1} w_{m+1} + \cdots + \binom{i-m}{i} w_0 w_{m+i}.$$

Proof. By universal property of Grassmannian, it suffices to show this formula for the universal k-plane bundle $V^k \to \mathrm{Gr}_k$. It further follows by the splitting construction for the universal bundle that it is sufficient to show the above formula for the bundle

$$p_1 \times \cdots \times p_k : V^1 \times \cdots \times V^1 \to \mathbb{R}\mathrm{P}^\infty \times \cdots \times \mathbb{R}\mathrm{P}^\infty = X.$$

To show this, we reduce to showing the formula for two cases by induction: for $\eta \times V^1$ and V^1 where η satisfies the said formula. We first show the former. Observe that

$$w_m(\eta \times V^1) = \sum_{p+q=m} w_p(\eta) \times w_q(V^1) = w_m \times 1 + w_{m-1} \times a$$

where $a \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ is the generator and $w_j = w_j(\eta)$. Moreover, by Lemma 5.2.3, we have

$$\mathrm{Sq}^i(a \times b) = \sum_{p+q=i} \mathrm{Sq}^p(a) \times \mathrm{Sq}^q(b).$$

Hence,

$$\begin{aligned}
\mathrm{Sq}^i(w_m(\eta \times V^1)) &= \mathrm{Sq}^i(w_m \times 1 + w_{m-1} \times a) \\
&= \mathrm{Sq}^i(w_m \times 1) + \mathrm{Sq}^i(w_{m-1} \times a) \\
&= \mathrm{Sq}^i(w_m) \times 1 + \mathrm{Sq}^i w_{m-1} \times a + \mathrm{Sq}^{i-1} w_{m-1} \times a^2 \\
&= \left(w_i w_m + \binom{i-m}{1} w_{i-1} w_{m+1} + \cdots + \binom{i-m}{i} w_0 w_{m+i} \right) \times 1 \\
&\quad + \left(w_i w_{m-1} + \binom{i-m+1}{1} w_{i-1} w_m + \cdots + \binom{i-m+1}{i} w_0 w_{m+i-1} \right) \times a \\
&\quad + \left(w_{i-1} w_{m-1} + \binom{i-m}{1} w_{i-2} w_m + \cdots + \binom{i-m}{i} w_0 w_{m+i} \right) \times a^2.
\end{aligned}$$

One may expand similarly the right hand side of the said formula for $\eta \times V^1$ and use the basic identity that

$$\binom{i-m}{j} + \binom{i-m}{j-1} = \binom{i-m+1}{j}$$

to get the same term as above. The verification of the formula for V^1 is simple. $\square$

6 Oriented Thom isomorphism & Euler class

We wish to now generalize top Stiefel-Whitney class from mod 2 cohomology to integral cohomolgy. To this end, we will follow the same strategy as in §5. Hence, we first cover the integral Thom isomorphism.

6.1 Oriented Thom isomorphism

Here's the theorem. Recall orientation on a bundle as in §1.8.

Theorem 6.1.1. *Let $\xi = (E, p, B)$ be an oriented vector bundle of rank n and $\mu_b \in H^n(E_b, E_{b0}; \mathbb{Z})$ be an orientation (Theorem 1.8.3). Then:*

1. (Cohomology of pair) *We have*
$$H^i(E, E_0; \mathbb{Z}) = 0, \ \forall 0 \leq i < n.$$
2. (Fundamental class) *There is a unique class $u \in H^n(E, E_0; \mathbb{Z})$ such that for all $b \in B$, the map*
$$\begin{aligned} H^n(E, E_0; \mathbb{Z}) &\longrightarrow H^n(E_b, E_{b0}; \mathbb{Z}) \\ u &\longmapsto \mu_b. \end{aligned}$$
3. (Cohomology of total space) *The map*
$$\begin{aligned} H^*(E; \mathbb{Z}) &\longrightarrow H^*(E, E_0; \mathbb{Z})[n] \\ x &\longmapsto x \cup u \end{aligned}$$
is an isomorphism of graded abelian groups.

The isomorphism obtained by the composite
$$\phi_{\mathrm{Th}} : H^*(B; \mathbb{Z}) \xrightarrow{p^*} H^*(E; \mathbb{Z}) \xrightarrow{-\cup u} H^*(E, E_0; \mathbb{Z})[n]$$
is called the oriented Thom isomorphism for the oriented bundle $p : E \to B$.

6.2 Euler class

For an oriented bundle ξ, one can define an analogue of top Steifel-Whitney class in the integral case. The name is due to Corollary 7.4.3.

Definition 6.2.1 (Euler class). Let $\xi = (E, p, B)$ be an oriented rank n bundle. Let $u \in H^n(E, E_0; \mathbb{Z})$ be the fundamental class of ξ and $i : (E, \emptyset) \hookrightarrow (E, E_0)$ be an inclusion of pairs. The Euler class of ξ is defined to be the following element of $H^n(B; \mathbb{Z})$:
$$H^n(E, E_0; \mathbb{Z}) \xrightarrow{i^*} H^n(E; \mathbb{Z}) \xrightarrow{p^{*-1}} H^n(B; \mathbb{Z}).$$
We denote the Euler class of ξ by $e(\xi)$.

We begin by discussing few standard properties of Euler class. The universal property of fundamental class as given by Thom's theorem is essential in what follows.

Lemma 6.2.2 (Naturality)**.** *If we have a fiber square*
$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ B' \arrow[r, "g"'] & B \end{tikzcd}$$
where the map $(f, g) : \xi' \to \xi$ is orientation preserving, then
$$g^*(e(\xi)) = e(\xi').$$

Proof. Let $u \in H^n(E, E_0; \mathbb{Z})$ and $u' \in H^n(E', E'_0; \mathbb{Z})$ be fundamental classes for the bundles ξ and ξ' respectively. We have the following diagram

$$\begin{tikzcd}
H^n(E'_b, E'_{b0}; \mathbb{Z}) & H^n(E_{g(b)}, E_{g(b)0}; \mathbb{Z}) \arrow[l, "f^*"', "\cong"] \\
H^n(E', E'_0; \mathbb{Z}) \arrow[u, "j^*"] \arrow[d, "i^*"'] & H^n(E, E_0; \mathbb{Z}) \arrow[l, "f^*"] \arrow[u, "j^*"'] \arrow[d, "i^*"] \\
H^n(E'; \mathbb{Z}) & H^n(E; \mathbb{Z}) \arrow[l, "f^*"] \\
H^n(B'; \mathbb{Z}) \arrow[u, "p'^*", "\cong"'] & H^n(B; \mathbb{Z}) \arrow[l, "g^*"] \arrow[u, "\cong", "p^*"']
\end{tikzcd}$$

which commutes as it commutes at the space level. As f is orientation preserving, therefore by uniqueness in Theorem 6.1.1, 2, it follows at once from commutativity of top square that $f^*(u) = u'$. By commutativity of the above square, we further have

$$g^*(e(\xi)) = g^*(p^{*-1} i^* u) = p'^{*-1} i^* f^*(u) = p'^{*-1} i^*(u') = e(\xi'),$$

as required. □

By observing that a trivial bundle is pullback of a trivial bundle over a point, we get the following corollary of the above lemma.

Lemma 6.2.3. *If $\xi = (E, p, B)$ is a trivial bundle of positive rank, then its Euler class is 0.* □

The Euler class of a bundle is dependent on the chosen orientation.

Lemma 6.2.4 (Orientation)**.** *Let $\xi = (E, p, B)$ be an oriented rank n bundle and let $\mu_b \in H^n(E_b, E_{b0}; \mathbb{Z})$ be the chosen orientation on ξ. If ξ' is the bundle obtained by changing the orientation to $-\mu_b \in H^n(E_b, E_{b0}; \mathbb{Z})$, then*

$$e(\xi') = -e(\xi).$$

Proof. This is immediate from the definition of Euler class. □

Lemma 6.2.5 (Euler and fundamental class)**.** *Let $\xi = (E, p, B)$ be a rank n oriented bundle and $u \in H^n(E, E_0; \mathbb{Z})$ be its integral fundamental class. Then we have*

$$e(\xi) = \phi_{\mathrm{Th}}^{-1}(i^* u \cup u)$$

where $i : (E, \emptyset) \hookrightarrow (E, E_0)$ is inclusion and $\phi_{\mathrm{Th}} : H^n(B; \mathbb{Z}) \to H^{2n}(E, E_0; \mathbb{Z})$ is the Thom isomorphism at degree n.

Proof. We have following maps

$$\begin{tikzcd}
H^n(B; \mathbb{Z}) \arrow[r, "p^*", "\cong"'] & H^n(E; \mathbb{Z}) \arrow[r, "- \cup u", "\cong"'] & H^{2n}(E, E_0; \mathbb{Z}) \\
& H^n(E, E_0; \mathbb{Z}) \arrow[u, "i^*"'] &
\end{tikzcd}$$

where ϕ_{Th} is the horizontal composite. As $e(\xi) = p^{*-1}i^*(u)$, therefore $\phi_{\mathrm{Th}}(e(\xi)) = i^*u \cup u$, as required. $\square$

Remark 6.2.6. We may sometimes drop i^* in the above equation, since in general cup product is defined as

$$H^p(X, A; \mathbb{Z}) \times H^q(X, B; \mathbb{Z}) \to H^{p+q}(X, A \cup B; \mathbb{Z})$$

for open sets $A, B \subseteq X$.

Lemma 6.2.7. *Let $\xi = (E, p, B)$ be a rank n oriented bundle. If n is odd, then $e(\xi) \in H^n(B; \mathbb{Z})$ is an order 2 element.*

Proof. Follows immediately from graded commutativity of cup product and Lemma 6.2.5. $\square$

Proposition 6.2.8 (Euler & top SW class)**.** *Let $\xi = (E, p, B)$ be a rank n oriented bundle. Let $\rho_2 : H^n(B; \mathbb{Z}) \to H^n(B; \mathbb{Z}_2)$ be the mod 2 coefficient reduction map. Then,*

$$\rho_2(e(\xi)) = w_n(\xi).$$

Proof. Recall the construction of SW classes in Construction 5.3.1. Let $u_{\mathbb{Z}} \in H^n(E, E_0; \mathbb{Z})$ be the integral fundamental class and $u_{\mathbb{Z}_2} \in H^n(E, E_0; \mathbb{Z}_2)$ be the mod 2 fundamental class. We first claim that

$$\rho_2(u_{\mathbb{Z}}) = u_{\mathbb{Z}_2}.$$

We will employ the universal property of fundamental classes to this end. Indeed, we need only show that $\rho_2(u_{\mathbb{Z}})|_{(E_b, E_{b0})} \neq 0$ in $H^n(E_b, E_{b0}; \mathbb{Z}_2)$ by Theorem 5.1.1, 2. This follows from the following commutative square (whose commutativity follows from chain level commutative diagram):

$$\begin{tikzcd} H^n(E, E_0; \mathbb{Z}) \arrow[r, "\rho_2"] \arrow[d] & H^n(E, E_0; \mathbb{Z}_2) \arrow[d] \\ H^n(E_b, E_{b0}; \mathbb{Z}) \arrow[r, "\rho_2"'] & H^n(E_b, E_{b0}; \mathbb{Z}_2) \end{tikzcd}$$

and the fact that the bottom ρ_2 is surjective by the long exact sequence in cohomology induced by coefficient s.e.s.

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

This proves the claim. Observe now that $\phi_{\mathrm{Th}}(e(\xi)) = u_{\mathbb{Z}} \cup u_{\mathbb{Z}}$ by Lemma 6.2.5 and $\phi_{\mathrm{Th}}(w_n(\xi)) = u_{\mathbb{Z}_2} \cup u_{\mathbb{Z}_2}$ by Lemma 5.3.2. The proof is complete by the diagram

$$\begin{tikzcd} H^n(E, E_0; \mathbb{Z}) \arrow[r, "\rho_2"] & H^n(E, E_0; \mathbb{Z}_2) \\ H^n(B; \mathbb{Z}) \arrow[r, "\rho_2"'] \arrow[u, "\phi_{\mathrm{Th}}", "\cong"'] & H^n(B; \mathbb{Z}_2) \arrow[u, "\cong", "\phi_{\mathrm{Th}}"'] \end{tikzcd}$$

which commutes as can be checked on cochain level. $\square$

An important result for Euler class is its behaviour w.r.t. Whitney sum and product bundles. These follow from the general calculation of fundamental class of product bundles

Proposition 6.2.9 (Fundamental class of product). *Let $\xi = (E, p, B), \eta = (E', q, B')$ be bundles of rank n, m respectively. Then*

$$u(\xi \times \eta) = (-1)^{nm} u(\xi) \times u(\eta)$$

where $u(\xi \times \eta) \in H^{n+m}(E \times E', E_0 \times E'_0; \mathbb{Z})$, $u(\xi) \in H^n(E; \mathbb{Z})$ and $u(\eta) \in H^m(E'; \mathbb{Z})$ be fundamental classes of respective bundles.

Proof. This is immediate from the usual fact that if $\mu_b \in H^n(E_b, E_{b0}; \mathbb{Z})$ and $\tau_b \in H^m(E'_b, E'_{b0}; \mathbb{Z})$ are orientations on the bundles ξ and η, then $(-1)^{nm}\mu_b \times \tau_b \in H^{n+m}(E \times E', E \times E'_0 \cup E_0 \times E'; \mathbb{Z})$. □

Lemma 6.2.10. *If ξ, ξ' are oriented bundles, then*

$$e(\xi \times \xi') = e(\xi) \times e(\xi')$$

and if ξ, ξ' has same base, then

$$e(\xi \oplus \xi') = e(\xi) \cup e(\xi').$$

Proof. As by Proposition 6.2.9, we have $e(\xi \times \xi') = (p \times q)^{*-1} i^*(u(\xi \times \xi')) = (-1)^{nm}(p \times q)^{*-1} i^*(u(\xi) \times u(\xi')) = (-1)^{nm} e(\xi) \times e(\xi') = e(\xi) \times e(\xi')$ where last equality follows from Lemma 6.2.7. For the latter, we have $\xi \oplus \xi' = \Delta^*(\xi \times \xi')$. The formula then follows from naturality (Lemma 6.2.2) and definition of cup product as pullback along diagonal of cross product. □

It still remains to be seen what conditions are enforced on a bundle by vanishing of its Euler class.

Lemma 6.2.11. *Let $\xi = (E, p, B)$ be an oriented rank n bundle. If there is a nowhere vanishing cross section of ξ, then $e(\xi) = 0$.*

Proof. We have the following maps

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "\mathrm{id}"'] & E_0 \arrow[d, "j"] \\ B & E \arrow[l, "p"] \end{tikzcd}$$

which yields the following composite in cohomology

$$H^n(B; \mathbb{Z}) \xrightarrow{p^*} H^n(E; \mathbb{Z}) \xrightarrow{j^*} H^n(E_0; \mathbb{Z}) \xrightarrow{s^*} H^n(B; \mathbb{Z})$$

which is just identity. Hence $s^* j^* p^*(e(\xi)) = e(\xi)$. As $p^*(e(\xi)) = i^*(u)$ where $i : (E, \emptyset) \hookrightarrow (E, E_0)$, therefore $s^* j^* p^*(e(\xi)) = s^* j^* i^*(u)$. By long exact sequence in cohomology of the pair (E, E_0), we have $j^* i^* = 0$. This completes the proof. □

Remark 6.2.12. Hence, if $e(\xi) \neq 0$ for an oriented bundle, then ξ has no nowhere vanishing cross section. Moreover by Proposition 6.2.8, it follows that $w_n(\xi) \neq 0$ as well.

7 The embedding problem

We wish to investigate the following question:

Q. Let M be an n-manfiold and A be an $n+k$-Riemannian manifold. When does M embed into A?

Suppose such an embedding $f : M \to A$ exists, then we have

$$TA|_{f(M)} = TM \oplus Nf$$

where Nf is the rank k normal bundle of the embedding f. This question is very difficult in general. We will try to understand the case when A is simplest possible; when $A = \mathbf{R}^{n+k}$. Our first goal is to understand the geometry of normal bundle. An important technical result to this end is the tubular neighborhood theorem.

7.1 Tubular neighborhood of an embedding

Theorem 7.1.1. *Let $i : M \hookrightarrow A$ be an embedding of M in A with normal bundle $Ni \to M$ where A is a Riemannian manifold. Then there is an open neighborhood $U \supseteq M$ and a diffeomorphism*

$$\psi : U \longrightarrow Ni$$

such that for all $x \in M$, $\psi(x) = s(x) \in (Ni)_x$ where s is the zero section of Ni. Moreover, M is a deformation retract of U.

7.2 Normal classes

The following is a simple observation, but with nice consequences.

Lemma 7.2.1. *Let $i : M \hookrightarrow A$ be a closed embedding of M in A with normal bundle $Ni \to M$ where A is a Riemannian manifold and R be a commutative ring with 1. Then there is a natural isomorphism of cohomology rings:*

$$H^*(Ni, Ni_0; R) \cong H^*(A, A - M; R).$$

Proof. Indeed, let $U \supseteq M$ be a tubular neighborhood of M in A. Then by the theorem above, there is a diffeomorphism $\psi : (U, U - M) \to (Ni, Ni_0)$. By excision ($A - M$ is open), the inclusion $i : (U, U - M) \hookrightarrow (A, A - M)$ induces an isomorphism in cohomology:

$$(Ni, Ni_0) \xleftarrow{\psi} (U, U - M) \xrightarrow{i} (A, A - M).$$

Applying cohomology, we get the natural isomorphism. □

We can hence make the following definition.

Definition 7.2.2 (Normal classes). Let $i : M \hookrightarrow A$ be a closed embedding an n-manifold M into an $n+k$-Riemannian manifold A and Ni be the rank k normal bundle over M of this embedding. Using the isomorphism

$$H^k(Ni, Ni_0; \mathbb{Z}_2) \cong H^k(A, A - M; \mathbb{Z}_2),$$

we define the *mod 2 normal class* of $M \subseteq A$ by the image of the mod 2 fundamental class u in $H^k(A, A-M; \mathbb{Z}_2)$, denoted u', of the normal bundle Ni. Similarly, if Ni is oriented, then by the isomorphism

$$H^k(Ni, Ni_0; \mathbb{Z}) \cong H^k(A, A-M; \mathbb{Z})$$

the *integral normal class* of $M \subseteq A$ by the image of the integral fundamental class u in $H^k(A, A-M; \mathbb{Z})$, denoted u' again.

Proposition 7.2.3. *Let $i : M \hookrightarrow A$ be a closed embedding of an n-manifold M into an $n+k$-Riemannian manifold A and Ni be the rank k normal bundle over M of this embedding. Consider the composite induced by inclusions*

$$H^k(A, A-M; R) \to H^k(A; R) \to H^k(M; R).$$

If

1. *$R = \mathbb{Z}_2$, then the composite maps mod 2 normal class u' to top SW class $w_k(Ni)$,*
2. *$R = \mathbb{Z}$ and Ni is oriented, then the composite maps integral normal class u' to the Euler class $e(Ni)$.*

Proof. We have the following commutative diagram where $U \supseteq M$ in A is a tubular neighborhood of M and all unlabelled maps are induced by inclusions:

$$\begin{tikzcd}
H^k(A, A-M; R) \arrow[r] \arrow[d, "\cong"] & H^k(A; R) \arrow[r] \arrow[d] & H^k(M; R) \arrow[d] \\
H^k(U, U-M; R) \arrow[r] & H^k(U; R) \arrow[r] & H^k(M; R) \\
H^k(Ni, Ni_0; R) \arrow[r] \arrow[u, "\psi^*"', "\cong"] & H^k(Ni; R) \arrow[r] \arrow[u, "\psi^*"'] & H^k(s(M); R) \arrow[u, "\cong", "s^*"']
\end{tikzcd}.$$

From the commutativity of the diagram above, it suffices to show that, in each case for R, the following commutes:

$$\begin{tikzcd}
H^k(Ni, Ni_0; R) \arrow[r] \arrow[d, "- \cup u"] & H^k(Ni; R) \arrow[d, "s^*"] \\
H^{2k}(Ni, Ni_0; R) & H^k(M; R) \arrow[l, "\phi_{\mathrm{Th}}"', "\cong"]
\end{tikzcd}.$$

This follows from expanding the Thom isomorphism and $p^* s^* = \mathrm{id}$ ($s \circ p \simeq \mathrm{id}$), completing the proof. $\square$

By the above result, we define the following.

Definition 7.2.4 (Dual normal class). Let $i : M \hookrightarrow A$ be a closed embedding of an n-manifold M into an $n+k$-Riemannian manifold A and Ni be the rank k normal bundle over M of this embedding. We define the dual normal class of M to be the image of normal class u' under the map

$$H^k(A, A-M; R) \to H^k(A; R).$$

We then immediately have the following conclusions.

Corollary 7.2.5. *If the dual normal class of M is zero, then $w_k(Ni) = 0$. Moreover, if Ni is oriented, then $e(Ni) = 0$.* □

Corollary 7.2.6. *If $A = \mathbf{R}^{n+k}$ and M is a closed embedded n-manifold in A, then $w_k(Ni) = 0$. Moreover, if Ni is oriented, then $e(Ni) = 0$. Consequently, if $w_k(Ni) \neq 0$, then M cannot be a closed embedded submanifold of $\mathbf{R}^{n+k}$.*

Proof. The dual normal class of M is an element of $H^k(A; R)$, which is zero since $k < n + k$. □

Example 7.2.7. When can a closed embedding $i : \mathbb{R}\mathrm{P}^n \hookrightarrow \mathbf{R}^{n+k}$ exist? As $w(T\,\mathbb{R}\mathrm{P}^n) = 1 + a + a^n$ for $a \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ being the generator, therefore $w(Ni) = 1 + a + a^2 + \cdots + a^{n-1}$. It follows at once by Corollary 7.2.6 that for such an i to exist, we must have $k \geq n$.

7.3 Diagonal classes

We begin with the following observation relating normal bundle of the diagonal embdding with the tangent bundle of the manifold.

Lemma 7.3.1. *Let M be a Riemannian n-manifold and $\Delta : M \to M \times M$ be the diagonal embedding. Then the normal bundle $N\Delta \to M$ and the tangent bundle $TM \to M$ are isomorphic.*

Proof. Since $N_x\Delta = \{(v, -v) \in T_xM \times T_xM \mid v \in T_xM\}$, therefore the map

$$\begin{aligned} T_xM &\longrightarrow N_x\Delta \\ v &\longmapsto (v, -v) \end{aligned}$$

gives the required isomorphism of bundles. □

The normal class of the diagonal embedding is very special and will be the topic of study of this section. We begin with the universal property of the normal class of diagonal. A proof is given in Lemma 11.7 of cite[Mil].

Proposition 7.3.2. *Let M be a Riemannian n-manifold and $\Delta : M \to M \times M$ be the diagonal embedding with normal bundle $N\Delta$. Denote $u' \in H^n(M \times M, M \times M - \Delta M; R)$ to be the normal class of the embedding Δ, either mod 2 or integral if M oriented by $\{\mu_x\}_{x \in M}$, and*

$$\begin{aligned} j_x : (M, M - x) &\to (M \times M, M \times M - \Delta M) \\ y &\mapsto (x, y). \end{aligned}$$

Then u' is unique w.r.t. the property that for all $x \in M$, the map

$$j_x^* : H^n(M \times M, M \times M - \Delta M; R) \to H^n(M, M - x; R)$$

maps u' to the local orientation μ_x if $R = \mathbb{Z}$ and M oriented or the non-zero element if $R = \mathbb{Z}_2$.

We may now define the diagonal class of a manifold to be the dual normal class of the diagonal embedding.

Definition 7.3.3 (Diagonal classes). Let M be a Riemannian n-manifold and $\Delta : M \to M \times M$ be the diagonal embedding with normal bundle $N\Delta$. The diagonal class of M is defined to be the element $u'' \in H^n(M \times M; R)$ which is the image of the normal class u' under the map induced by inclusion:

$$H^n(M \times M, M \times M - \Delta M; R) \to H^n(M \times M; R).$$

If $R = \mathbb{Z}$ and M oriented, we call u'' the *integral diagonal class* and if $R = \mathbb{Z}_2$ then we call u'' the *mod 2 diagonal class*.

In the remaining section, we wish to give an important relation between the diagonal class and the fundamental class of the manifold.

Lemma 7.3.4 (Concentration on diagonal). *Let $a \in H^*(M; R)$ and u'' be the diagonal class of M. Then*

$$(1 \times a) \cup u'' = (a \times 1) \cup u''.$$

Proof. Let $p, q : M \times M \to M$ be two projections. For $a \in H^k(M; R)$, we have

$$p^*a = a \times 1, \ q^*a = 1 \times a.$$

We first show that $(a \times 1) \cup u' = (1 \times a) \cup u'$ where u' is the normal class of diagonal. Indeed, by tubular neighborhood, we have a neighborhood $U \supseteq \Delta M$ in $M \times M$. As p, q agree on ΔM, it follows that $p|_U \simeq q|_U$. Hence

$$p|_U^* = q|_U^* : H^*(M; R) \to H^*(U; R).$$

Note that the following square commutes by naturality of cup:

$$\begin{tikzcd}
H^k(M \times M; R) \arrow[r] \arrow[d, "- \cup u'"'] & H^k(U; R) \arrow[d, "- \cup u'|_{(U, U - \Delta M)}"] \\
H^{k+n}(M \times M, M \times M - \Delta M; R) \arrow[r, "\text{exc.}", "\cong"'] & H^{k+n}(U, U - \Delta M; R)
\end{tikzcd}.$$

It follows that $(a \times 1) \cup u' = (1 \times a) \cup u'$ are equal in $H^{k+n}(M \times M, M \times M - \Delta M; R)$. Restricting to $M \times M$, we get the desired equality. $\square$

We now cover the main result, relating the fundamental class of a closed oriented manifold with that of the diagonal class via the slant product. Recall the slant product for spaces X, Y and field F is given by

$$\begin{aligned} H^{p+q}(X \times Y; F) \times H_q(Y; F) &\longrightarrow H^p(X; F) \\ (a \times b, \beta) &\longmapsto (a \times b)/\beta := a \cdot \langle b, \beta \rangle. \end{aligned}$$

For a fixed $\beta \in H_q(Y; F)$, the resulting map $H^{p+q}(X \times Y; F) \to H^p(X; F)$ is left $H^*(X; F)$ linear. That is, if $a, b \in H^{p+q}(X \times Y; F)$, then

$$(a \cup b)/\beta = (a/\beta) \cup (b/\beta).$$

Theorem 7.3.5. *Let M be a compact connected orientable n-manifold, $\mu \in H_n(M; \mathbb{Z})$ be its fundamental class and $u'' \in H^n(M \times M; \mathbb{Z})$ be the diagonal class of M. Then*

$$u''/\mu = 1$$

in $H^0(M; \mathbb{Z})$.

7.4 Poincaré duality

Recall that if M is compact, then $H^*(M;F)$ is a finite dimensional F-algebra as each $H^i(M;F)$ is finite dimensional.

Theorem 7.4.1 (Poincaré duality). *Let M be a compact connected oriented n-manifold and $\mu \in H^n(M;F)$ be its fundamental class where F is a field. Let $b_1, \dots, b_r \in H^*(M;F)$ be an F-basis of the cohomology algebra. Then there exists a basis $\check{b}_1, \dots, \check{b}_r \in H^*(M;F)$ such that*

1. *they are in complementary degree:*
$$\deg \check{b}_i = n - \deg b_i,$$
2. *they satisfy*
$$\langle b_i \cup \check{b}_j, \mu \rangle = \delta_{ij},$$
3. *the diagonal class $u'' \in H^n(M \times M;F)$ is given by*
$$u'' = \sum_{i=1}^{r} (-1)^{\deg b_i} b_i \times \check{b}_i.$$

Remark 7.4.2. Consider the map

$$\begin{aligned} H^k(M;F) &\longrightarrow \mathrm{Hom}_F\left(H^{n-k}(M;F), F\right) \\ a &\longmapsto b \mapsto \langle a \cup b, \mu \rangle. \end{aligned}$$

This map is an F-linear isomorphism by duality. Hence $\mathrm{rank} H^k(M;F) = \mathrm{rank} H^{n-k}(M;F)$.

An important application justifies the name of the Euler class.

Corollary 7.4.3. *Let M be a compact connected oriented n-manifold and $\mu \in H_n(M;\mathbb{Z})$ be its fundamental class. Then*

$$\langle e(TM), \mu \rangle = \chi(M).$$

Moreover if M is not oriented and $\mu \in H^n(M;\mathbb{Z}_2)$ is the non-zero element, then

$$\langle w_n(TM), \mu \rangle = \chi(M) \mod 2.$$

Proof. Consider the inclusion $\mathbb{Z} \hookrightarrow \mathbf{Q}$. The following square commutes

$$\begin{array}{ccc} H^*(M \times M;\mathbb{Z}) & \xrightarrow{\Delta^*} & H^*(M;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^*(M \times M;\mathbf{Q}) & \xrightarrow[\Delta^*]{} & H^*(M;\mathbf{Q}) \end{array}.$$

Hence, we may assume $\mathbf{Q}$-coefficients. Now by duality, we have $b_1, \dots, b_r \in H^*(M;\mathbf{Q})$ basis of rational cohomology ring such that the diagonal class

$$u'' = \sum_{i=1}^{r} (-1)^{\deg b_i} b_i \times \check{b}_i.$$

As $\Delta^*(u'') = e(N\Delta) = e(TM)$, therefore we have

$$e(TM) = \sum_{i=1}^{r}(-1)^{\deg b_i} b_i \cup \check{b}_i = \sum_{i=1}^{r}(-1)^{\deg b_i} = \chi(M),$$

as required. □

For another corollary of duality, we will study how to represent SW classes via Steenrod squares of diagonal class. We begin with the following lemma.

Lemma 7.4.4. *Let M be a compact n-manifold, $\mu \in H_n(M;\mathbb{Z}_2)$ be its fundamental class and $u'' \in H^n(M \times M;\mathbb{Z}_2)$ be its mod 2 diagonal class. Then*

$$w_i(M) = \mathrm{Sq}^i(u'')/\mu.$$

Proof. Let $u \in H^n(TM, TM_0;\mathbb{Z}_2)$ be the fundamental class of tangent bundle TM. By Thom, we first have

$$\mathrm{Sq}^i(u) = \phi_{\mathrm{Th}}(w_i) = p^* w_i \cup u.$$

Note $N\Delta = TM$ by Lemma 7.3.1. Let $U \supseteq \Delta M$ be a tubular neighborhood containing ΔM in $M \times M$ and $\psi : U \to N\Delta$ be the diffeomorphism. Using the natural isomorphisms

$$H^*(TM, TM_0;\mathbb{Z}_2) \xrightarrow{\psi^*} H^*(U, U - M;\mathbb{Z}_2) \xleftarrow{i^*} H^*(M \times M, M \times M - \Delta M;\mathbb{Z}_2)$$

which maps $u \mapsto u'$, the diagonal class of $\Delta : M \to M \times M$, we get

$$\mathrm{Sq}^i(u') = (w_i \times 1) \cup u'$$

and on further restricting to $M \times M$,

$$\mathrm{Sq}^i(u'') = (w_i \times 1) \cup u''.$$

Applying linearity of slant, we get

$$\begin{aligned}\mathrm{Sq}^i(u'')/\mu &= \left((w_i \times 1) \cup u''\right)/\mu \\ &= w_i \cdot \langle 1, \mu\rangle \cup (u''/\mu) \\ &= w_i,\end{aligned}$$

as required. □

Next, we construct an element in $H^*(M;\mathbb{Z}_2)$ whose total square gives the total SW class of the manifold.

Definition 7.4.5 (**Wu class**)**.** Let M be compact manifold of dimension n with fundamental class $\mu \in H_n(M;\mathbb{Z}_2)$ and consider the following $\mathbb{Z}_2$-linear map for each $0 \leq k \leq n$:

$$\begin{aligned}\langle \mathrm{Sq}^k(-), \mu\rangle : H^{n-k}(M;\mathbb{Z}_2) &\longrightarrow \mathbb{Z}_2 \\ x &\longmapsto \langle \mathrm{Sq}^k(x), \mu\rangle.\end{aligned}$$

Hence $\langle \mathrm{Sq}^k(-), \mu\rangle \in \mathrm{Hom}_{\mathbb{Z}_2}\left(H^{n-k}(M;\mathbb{Z}_2), \mathbb{Z}_2\right)$. By Remark 7.4.2, we have that there is a unique element $v_k \in H^k(M;\mathbb{Z}_2)$ such that for all $x \in H^{n-k}(M;\mathbb{Z}_2)$, we have

$$\langle \mathrm{Sq}^k(x), \mu\rangle = \langle x \cup v_k, \mu\rangle.$$

If X is connected, then by the similar isomorphism

$$H^0(M;\mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}_2}\left(H^n(M;\mathbb{Z}_2), \mathbb{Z}_2\right),$$

we further get that $\mathrm{Sq}^k(x) = x \cup v_k$. The class

$$v = v_0 + \cdots + v_n \in H^*(M;\mathbb{Z}_2)$$

is called the Wu class of the manifold M and is uniquely determined by the property that

$$\langle x \cup v, \mu\rangle = \langle \mathrm{Sq}(v), \mu\rangle.$$

The following result connects the Steenrod square of Wu class with the SW class.

Theorem 7.4.6. *Let M be a compact manifold of dimension n with the fundamental class $\mu \in H_n(M;\mathbb{Z}_2)$. If $v \in H^*(M;Z_2)$ is the Wu class of M and w the total SW class, then*

$$w(M) = \mathrm{Sq}(v).$$

Proof. By duality (Theorem 7.4.1), there is a basis $b_1, \ldots, b_r \in H^*(M;\mathbb{Z}_2)$ of the mod 2 cohomology ring and $\check{b}_1, \ldots, \check{b}_r$ be the corresponding dual basis. Hence we may write each $x \in H^*(M;\mathbb{Z}_2)$ as $x = \sum_i c_i b_i$, $c_i \in \mathbb{Z}_2$. Applying $\langle - \cup \check{b}_j, \mu\rangle$ on this equation yields that $c_i = \langle x \cup \check{b}_i, \mu\rangle$ so we may write

$$x = \sum_{i=1}^r \langle x \cup \check{b}_i, \mu\rangle b_i.$$

Hence we may write

$$v = \sum_{i=1}^r \langle v \cup \check{b}_i, \mu\rangle b_i = \sum_{i=1}^r \langle \mathrm{Sq}(\check{b}_i), \mu\rangle b_i.$$

Applying the total squaring Sq, we get

$$\begin{aligned}
\mathrm{Sq}(v) &= \sum_{i=1}^r \mathrm{Sq}(b_i)\langle \mathrm{Sq}(\check{b}_i), \mu\rangle \\
&= \sum_{i=1}^r \left(\mathrm{Sq}(b_i) \times \mathrm{Sq}(\check{b}_i)\right)/\mu \\
&= \sum_{i=1}^r \mathrm{Sq}(b_i \times \check{b}_i)/\mu \\
&= \mathrm{Sq}\left(\sum_{i=1}^r b_i \times \check{b}_i\right)/\mu \\
&= \mathrm{Sq}(u'')/\mu \\
&= w(M)
\end{aligned}$$

where in the end we are using Theorem 7.4.1, 3 and Lemma 7.4.4. $\square$

A special class of manifolds which includes S^n and $\mathbb{RP}^n$ is the following.

Lemma 7.4.7. *Let M be a compact connected manifold such that $H^*(M;\mathbb{Z}_2)$ is generated as a $\mathbb{Z}_2$-algebra by $a \in H^k(M;\mathbb{Z}_2)$. If*

$$H^*(M;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cdot a \oplus \ldots \mathbb{Z}_2 \cdot a^m,$$

then $\dim M = km$ *and*

$$w(M) = (1+a)^{m+1}.$$

8 Cobordism ring & Thom spectrum

9 Topological K-theory

We construct the spectrum KU and KO, called the complex and real K-theory spectrum respectively. For the entirety of this section, we fix X to be a compact Hausdorff space (so that it is paracompact in particular).

9.1 KU & BU

Let $VB(X)$ denote the set of isomorphism classes of vector bundles over X. This is a semiring under the addition of Whitney sum $\oplus$ and multiplication of tensor product $\otimes$ of vector bundles. Indeed, the additive identity is the identity vector bundle $\mathrm{id} : X \to X$ with 0-dimensional fibers and multiplicative identity is $\epsilon : X \times F \to X$, the trivial line bundle.

Definition 9.1.1 (**K-group of a space**)**.** The group $KO(X)$ and $KU(X)$ are defined to be the Grothendieck ring of the semirings $(VB_{\mathbf{R}}(X), \oplus, \otimes)$ and $(VB_{\mathbf{C}}(X), \oplus, \otimes)$, respectively. Let us assume to be working only in complex case. In particular, it is the following abelian group

$$KU(X) := \frac{\bigoplus_{p \in VB_{\mathbf{C}}(X)} \mathbb{Z}}{p \oplus q - p - q}.$$

The map

$$\begin{aligned} d : KU(X) &\longrightarrow \mathbb{Z} \\ [p] &\longmapsto \dim p \end{aligned}$$

is a ring homomorphism. We define the reduced K-theory of X to be

$$\widetilde{KU}(X) = \mathrm{Ker}\,(d).$$

Remark 9.1.2. For simplicity, we will only work with complex K-theory KU in these notes.

Remark 9.1.3. As the short exact sequence

$$0 \to \widetilde{KU}(X) \to KU(X) \xrightarrow{d} \mathbb{Z} \to 0$$

is split on the right by the ring map $\mathbb{Z} \to KU(X)$ mapping $n \mapsto [\epsilon^n]$, where $\epsilon^n : X \times F^n \to X$ is the trivial n-dim vector bundle, thus we have the isomorphism

$$KU(X) \cong \widetilde{KU}(X) \oplus \mathbb{Z}.$$

The main theorem is the following.

Theorem 9.1.4 (K-theory as generalized cohomology)**.** *Let X be a compact space. Then, there are natural isomorphisms*

$$KU(X) \cong [X_+, BU \times \mathbb{Z}]_*$$
$$\widetilde{KU}(X) \cong [X, BU \times \mathbb{Z}].$$