Riemann Surfaces

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Contents

1 Introduction

These are some notes on Riemann surfaces. We wish to prove three main results here: monodromy, Riemann-Hurwitz formula and the infamous Riemann-Roch theorem. We also wish to portray example uses of them. A philosophical goal in our mind is also to see how analytic world behaves in comparison to algebraic world. We do this in part so that we can get more insights into the latter, before going into an involved study of it.

We make references to our notes "Facets of Geometry" by writing Theorem FoG.23.2.1.4.

1.1 Definitions and basic properties

After defining Riemann surfaces and giving basic examples, we will cover some basic lemmas some of which generalizes results which we have seen in complex analysis of one variable.

Definition 1.1.1 (Riemann surface). A conformal atlas $\mathcal{A} = (\{U_i\}_i, \{z_i\}_i)$ on a second countable Hausdorff space X is the data of an open cover $\{U_i\}_{i\in I}$ of X together with open embeddings $z_i : U_i \to \mathbb{C}$ such that if $U_i \cap U_j \neq \emptyset$ then the composite

$$z_j \circ z_i^{-1} : z_i(U_i \cap U_j) \to z_j(U_i \cap U_j)$$

is a holomorphic map between two open subsets of $\mathbb{C}$. Two conformal atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent if for any U, z in $\mathcal{A}_1$ and any V, w in $\mathcal{A}_2$, the transition map

$$w \circ z^{-1} : z(U \cap V) \to w(U \cap V)$$

is a conformal map. A Riemann surface X is a connected Hausdorff space with an equivalence class of conformal atlas. We usually fix one atlas in a class which is maximal in that it is the union of all atlases in that class.

Example 1.1.2. Here are few examples of Riemann surfaces.

1. Any open subset $U \subseteq \mathbb{C}$ is a Riemann surface. Indeed, consider $\mathrm{id} : U \to U$, this defines a conformal atlas on U. Thus $\mathbb{C}$ and the open unit disc $\mathbb{D}$ are Riemann surfaces.
2. The Riemann sphere $\bar{\mathbb{C}}$ or usually called complex projective line $\mathbb{P}^1_{\mathbb{C}}$ (see Proposition 1.3.2) is a Riemann surface. Topologically, $\mathbb{P}^1_{\mathbb{C}}$ is S^2. We give a conformal structure on S^2 as follows. Consider the open sets $U_+ = S^2 - p$ and $U_- = S^2 - q$ where p and q are north and south poles respectively. Consider

$$\begin{aligned} z_+ : U_+ &\longrightarrow \mathbb{C} \\ (x_1, x_2, x_3) &\longmapsto \frac{x_1 + ix_2}{1 - x_3} \\ z_- : U_- &\longrightarrow \mathbb{C} \\ (x_1, x_2, x_3) &\longmapsto \frac{x_1 - ix_2}{1 + x_3}. \end{aligned}$$

   These are obtained by usual stereographic projection from north pole p. One can observe that

$$\begin{aligned} z_+(U_+) &= \mathbb{C} \\ z_-(U_-) &= \mathbb{C} - \{0\} \end{aligned}$$

and are thus homeomorphisms. Furthermore $U_+ \cap U_- = S^2 - \{p, q\}$. It follows that $z_+(U_+ \cap U_-) = \mathbb{C}^\times = z_-(U_+ \cap U_-)$, the punctured complex plane. The transition map can be checked to be

$$\begin{aligned} z_+ \circ z_-^{-1} : z_-(U_+ \cap U_-) &\longrightarrow z_+(U_+ \cap U_-) \\ w &\longmapsto \frac{1}{w} \end{aligned}$$

which as a map $\mathbb{C}^\times \to \mathbb{C}^\times$ is conformal.

Here is how we can define maps of Riemann surfaces.

Definition 1.1.3 (**Holomorphic maps of Riemann surfaces**). Let X and Y be two Riemann surfaces with atlases (U_i, z_i) and (V_i, w_i) on X and Y respectively and $f : X \to Y$ be a continuous map. Then, f is said to be holomorphic if for each $x \in X$ and charts $U_i \ni x$ and $V_j \ni f(x)$, the composite

$$w_j \circ f \circ z_i^{-1} : z_i(U_i) \to w_j(V_j)$$

is a holomorphic map between two open sets of $\mathbb{C}$. We denote by $\mathcal{O}(X) = \{f : X \to \mathbb{C} \mid f \text{ is holomorphic}\}$. This is a $\mathbb{C}$-algebra under pointwise addition and multiplication.

Lemma 1.1.4. *Let $f : X \to Y$ and $g : Y \to Z$ be a holomorphic map of Riemann surfaces. Then $g \circ f : X \to Z$ is a holomorphic map.*

Proof. Denote $h = g \circ f : X \to Z$. Pick any $x \in X$ and pick any coordinate charts $(U_x, \varphi_x) \ni x$ and $(W_{h(x)}, \varphi_{h(x)}) \ni h(x)$. We wish to show that $\varphi_{h(x)} \circ h \circ \varphi_x^{-1} : \varphi_x(U_x) \to \varphi_{h(x)}(W_{h(x)})$ is holomorphic. Pick any chart $V_{f(x)} \ni f(x)$. Then we have

$$\varphi_{h(x)} \circ h \circ \varphi_x^{-1} = \varphi_{h(x)} \circ g \circ \varphi_{f(x)}^{-1} \circ \varphi_{f(x)} \circ f \circ \varphi_x^{-1}$$

where $\varphi_{h(x)} \circ g \circ \varphi_{f(x)}^{-1}$ and $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ are holomorphic as f and g are holomorphic. This completes the proof. $\square$

Remark 1.1.5. We get a category of Riemann surfaces, denoted $\mathcal{R}S$.

Definition 1.1.6 (**Subsurface**). Let X be a Riemann surface and $U \subseteq X$ be an open set. Then U is also a Riemann surface with the charts obtained by restrictions of that of X.

There is an identity principle for Riemann surfaces, which would be used quite often.

Lemma 1.1.7 (Identity principle)**.** *Let X, Y be Riemann surface and X be connected. If $f, g : X \to Y$ are holomorphic and there exists $A \subseteq X$ which has a limit point in X such that $f|_A = g|_A$, then $f = g$.*

Proof. Let $a \in X$ be a limit point of A and let (U, z) be a chart of a. Then $f|_U = g|_U$ by usual identity principle of $\mathbb{C}$. Now pick any point $a \neq b \in X$. As X is locally path-connected and connected, therefore it is path-connected. Let $\gamma : a \to b$ be a path joining a and b in X. We claim that f is constant along this path. Indeed, cover $\mathrm{Im}\,(\gamma)$ by finitely many charts of X denoted U_i such that $U_i \cap U_{i+1} \neq \emptyset$ with $U_1 = U$. As f and g agree on an open subset of U_2, therefore by identity principle of $\mathbb{C}$, it follows that $f|_{U_2} = g|_{U_2}$. Continuing this, we conclude that $f = g$ on $\mathrm{Im}\,(\gamma)$ and thus $f(b) = g(b)$, as required. $\square$

Corollary 1.1.8. *Let $f : X \to \mathbb{C}$ be a non-zero holomorphic where X is a Riemann surface. Then $D(f) := \{x \in X \mid f(x) = 0\}$ is a discrete set in X.*

Proof. If $D(f)$ is not discrete, then it has a limit point and thus by Lemma 1.1.7, $f = 0$, a contradiction. $\square$

We can define meromorphic maps between Riemann surfaces as well.

Definition 1.1.9 (**Meromorphic maps**)**.** Let X be a Riemann surface. A meromorphic map on X is a holomorphic map $f : X \to \mathbb{P}^1_{\mathbb{C}}$ such that $f \neq c_\infty$, c_∞ being the constant infinity map. By identity principle (Lemma 1.1.7), thus, $f^{-1}(\infty)$ has to be a discrete set. We denote the set of all meromorphic functions on X as $\mathcal{M}(X)$. Clearly, $\mathcal{M}(X)$ is a $\mathbb{C}$-algebra.

Meromorphic maps form a field!

Lemma 1.1.10. *Let X be a connected Riemann surface. Then $\mathcal{M}(X)$ is a field.*

Proof. Let $f : X \to \mathbb{P}^1_{\mathbb{C}}$ be a non-zero meromorphic map. Then consider $g := 1/f$ on $D(f) = \{x \in X \mid f(x) \neq 0\}$ and $g := \infty$ on $X \setminus D(f)$. Clearly, g is holomorphic on $D(f)$, which is open. Since $D(f)$ is discrete by Corollary 1.1.8. Thus, g is indeed meromorphic. Observe that $f \cdot g = 1$ on X as it is one on $D(f)$ and then we may apply identity principle (Lemma 1.1.7). This completes the proof. $\square$

Remark 1.1.11. As there is a natural inclusion $\mathcal{O}(X) \hookrightarrow \mathcal{M}(X)$, thus it follows $\mathcal{O}(X)$ is a domain. By universal property of fraction fields, $Q(\mathcal{O}(X)) \subseteq \mathcal{M}(X)$.

We now see when $\mathcal{O}(X)$ itself is a field.

Lemma 1.1.12 (General Liouville)**.** *Let X be a compact connected Riemann surface*[1]*, then $\mathcal{O}(X)$ is isomorphic to $\mathbb{C}$ as the only elements in $\mathcal{O}(X)$ are constants.*

Proof. Pick any $f \in \mathcal{O}(X)$. We wish to show that f is a constant. Consider the composite $X \to \mathbb{C} \to \mathbb{R}$ given by $|f|$. As X is compact, thus $|f|$ achieves maxima, say at $x_0 \in X$ and $a = |f(x_0)|$. For a chart $(U, z) \ni x_0$, we have by maximum-modulus for $\mathbb{C}$ that $|f|$ is constant and thus f is constant c_a on U. By identity principle (Lemma 1.1.7), it follows that f is constant c_a on the entire X. $\square$

Open mapping theorem is also true for maps of Riemann surfaces.

Lemma 1.1.13 (Open mapping theorem)**.** *Let X be a connected Riemann surface and $f : X \to Y$ be a holomorphic map. Then f is an open map.*

Proof. Pick any open set $U \subseteq X$ and consider $f(U) \subseteq Y$. We wish to show that $f(U)$ is open. Pick any point $f(x) \in f(U)$ where $x \in U$. Pick any chart $(V, z) \ni x$ and $(W, w) \ni f(x)$ such that $V \subseteq U$. Thus the map $w \circ f \circ z^{-1} : z(V) \to w(W)$ is a holomorphic map. By open mapping theorem for $\mathbb{C}$, it follows that $w \circ f \circ z^{-1}$ is an open map. Thus, let $x \in V' \subseteq V$ be an open set. Then $w \circ f \circ z^{-1}(z(V')) = w(f(V')) \subseteq w(W)$ is open and thus $f(x) \in f(V') \subseteq W$ is open, as required. $\square$

[1] say, for example, $\mathbb{P}^1_{\mathbb{C}}$!

There is an intimate connection between covering spaces and Riemann surfaces, whose first piece we explain as follows. We first need a small lemma.

Lemma 1.1.14. *Let $p_i : X_i \to Y$ for $i = 1, 2$ be a holomorphic map where X_i, Y are Riemann surfaces and X_i are connected. If there exists a continuous $f : X_1 \to X_2$ such that*

$$\begin{tikzcd} & X_2 \arrow[d, "p_2"] \\ X_1 \arrow[ur, "f"] \arrow[r, "p_1"'] & Y \end{tikzcd}$$

commutes, then f is holomorphic.

Proof. Pick any point $x \in X_1$, charts $(U_1, z_1) \ni x$ and $(U_2, z_2) \ni f(x)$. We wish to show that $z_2 \circ f \circ z_1^{-1} : z_1(U_1) \to z_2(U_2)$ is holomorphic. Indeed, we may first assume by continuity of f and p_i that U_i is a sheet of an evenly covered neighborhood $V \subseteq Y$ under p_i for $i = 1, 2$. Now, the restricted maps $p_i : U_i \to z_i(U_i)$ are biholomorphic maps. Now, $z_i \circ p_i^{-1} : V \to z_i(U_i)$ are two charts in Y. As transition maps has to be holomorphic, therefore we get

$$(z_2 \circ p_2^{-1}) \circ (z_1 \circ p_1^{-1})^{-1} = z_2 \circ f \circ z_1^{-1}$$

is holomorphic, as required. □

Proposition 1.1.15. *Let $p : X \to Y$ be a covering map where Y is a Riemann surface. Then there exists a unique conformal structure on X such that p is holomorphic.*

Proof. We first do uniqueness, as it is easy by Lemma 1.1.14. Indeed, if there are two non-equivalent conformal structures on X, then we get two holomorphic covering maps $p_i : X_i \to Y$. As underlying space and maps of each (X_i, p) is same, therefore by lifting criterion for covering maps, we deduce that there is a continuous map $f : X_1 \to X_2$ which is furthermore holomorphic by above lemma and $p \circ f = p$. Now, we may similarly get $g : X_2 \to X_1$ holomorphic such that $p \circ g = p$. As these lifts are based lifts, we get that $f \circ g$ is unique with respect to the fact that it fixes a point, thus it is id, similarly for the other side. Hence $X_1 \cong X_2$, that is, they are biholomorphic and thus have equivalent conformal structure.

We thus need only construct a conformal structure on X via p. Indeed, we may first assume that Y has an atlas (V_i, z_i) fine enough that each $V_i \subseteq Y$ is an evenly covered neighborhood. Hence for each V_i, the map $p : W_{i,j} \to V_i$ is a homeomorphism where $p^{-1}(V_i) = \coprod_j W_{i,j}$. Define an open cover of X by $(W_{i,j}, z_i \circ p)$. We claim that this is an atlas. Indeed, $z_i \circ p : W_{i,j} \to z_i(V_i)$ is a homeomorphism and for any $(i, j), (k, l)$, we have $(z_i \circ p) \circ (z_j \circ p)^{-1} = z_i \circ z_j^{-1}$, which is a holomorphic map. This completes the proof. □

1.2 Structure sheaf and modules

We wish to show that the structure sheaf of a Riemann surface $\mathcal{O}_X$ is such that the meromorphic sheaf $\mathcal{M}$ is an $\mathcal{O}_X$-module. So we first define the structure sheaf.

Remark 1.2.1 (Riemann surface as a locally ringed space)**.** Let X be a Riemann surface with an atlas (U_i, z_i). As discussed in Chapter 8, §8.1.2 on "Sheaves and atlases" in FoG, by Theorem FoG.8.1.2.4, it follows that we get an atlas sheaf (Definition FoG.8.1.2.1) $\mathcal{O}_X$ on X w.r.t which $(X, \mathcal{O}_X)$ is a locally ringed space which is a complex manifold (Definition FoG.8.1.1.3) of dimension 1. Recall that in particular for an open subset $U \subseteq X$, $\mathcal{O}_X(U)$ is defined by

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid f \circ x_i^{-1} : x_i(U \cap U_i) \to \mathbb{C} \text{ is holomorphic}\},$$

that is, $\mathcal{O}_X$ is the sheaf of holomorphic maps on X. The $\mathcal{O}_X$ is also called the structure sheaf of X. Thus, giving a conformal structure on X is equivalent to giving an atlas sheaf.

We will be using this sheaf very frequently, as it will be of fundamental importance to us to translate over working working knowledge of algebraic geometry to this analytic language[2].

Remark 1.2.2. There might be apparent addition of complexity to think of a Riemann surface as a locally ringed space with a sheaf of holomorphic maps without any reference to a chart. But we wish to portray that one can prove results similar to that in previous section from this point of view as well, as this allows us to reduce to *local affine patch* (i.e. local chart) quite immediately.

For example, general Liouville (Lemma 1.1.12) can also be seen by the following argument. Considering that $|f| : X \to \mathbb{R}$ achieves maximum at $x_0 \in X$, for an affine open set containing x_0 say U, the restriction $f|_U : U \to \mathbb{R}$ can be thought of as a map on an open subset of $\mathbb{C}$ which achieves maximum on interior, so $f|_U$ is constant. Thus by identity priniciple, we are done.

An important and crucial observation from complex analysis of one variable is the following:

Proposition 1.2.3. *Let $(X, \mathcal{O}_X)$ be a Riemann surface. Then for any $x \in X$, the stalk is isomorphic to power series ring over $\mathbb{C}$:*

$$\mathcal{O}_{X,x} \cong \mathbb{C}[[z]].$$

Proof. Let $U \ni x$ be an affine open subset of x. Then, $\mathcal{O}_{X,x} = \mathcal{O}_{U,x}$. Let $\varphi : U \to \mathbb{C}$ be a chart. As it is an open embedding, therefore, $\mathcal{O}_{U,x} \cong \mathcal{O}_{\mathbb{C},\varphi(x)}$, where $\mathcal{O}_{\mathbb{C}}$ is the sheaf of holomorphic maps on $\mathbb{C}$. As any homolomorphic map has a power series representation at each point, thus, power series forms a cofinal system in the representation of a holomorphic map in the stalk. The result now follows. $\square$

Remark 1.2.4. This proposition immediately tells us what type of information is stored in the stalk. That is, it tells you how a function locally around a point looks like.

[2]Note that explicit charts are rarely used in schemes, whereas in geometry, one uses it quite frequently.

We next see that meromorphic maps form a sheaf as well.

Definition 1.2.5. Let $(X, \mathcal{O}_X)$ be a Riemann surface. The assignment for each open $U \subseteq X$

$$\mathcal{M}_X(U) = \{f : U \to \mathbb{P}^1_{\mathbb{C}} \mid f \neq c_\infty \text{ holomorphic}\}$$

forms a presheaf under restrictions. This is called the sheaf of meromorphic maps on X.

We first see that $\mathcal{M}_X$ is a constant sheaf!

Proposition 1.2.6. *Let X be a Riemann surface and let $K = \mathcal{M}_X(X)$ the field of global meromorphic maps. Then*

$$\mathcal{M}_X \cong \underline{K},$$

where the latter is the constant sheaf on field K.

We'll see its proof later. An important property is that the stalks of $\mathcal{M}_X$ are again quite simple.

Proposition 1.2.7. *Let $(X, \mathcal{O}_X)$ be a Riemann surface with meromorphic sheaf $\mathcal{M}_X$. Then for any $x \in X$,*

$$\mathcal{M}_{X,x} \cong \mathbb{C}((z)).$$

Proof. Same as Proposition 1.2.3 except that in the end we use the fact that any meromorphic function locally has a Laurent series expansion at each point. □

We now study some important class of Riemann surfaces, those coming from non-singular projective plane curves.

1.3 Smooth algebraic plane curves

We wish to study a class of examples of Riemann surfaces coming from algebra. This will give us a tight intuition about algebraic curves which will guide further development.

We begin by giving an alternate construction of Riemann surface.

Example 1.3.1 (Complex projective line $\mathbb{P}^1_{\mathbb{C}}$)**.** Topologically, we first define $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^2/\sim$ where $(z_0, z_1) \sim (\lambda z_0, \lambda z_1)$ for all $\lambda \in \mathbb{C}$. Denote any point in $\mathbb{P}^1_{\mathbb{C}}$ by $[z_0 : z_1]$ where $z_i \in \mathbb{C}$. We now give a conformal structure on $\mathbb{P}^1_{\mathbb{C}}$. Consider $U_0 = \{[1 : z] \mid z \in \mathbb{C}\}$ and $U_1 = \{[z : 1] \mid z \in \mathbb{C}\}$. These are open subspaces of $\mathbb{P}^1_{\mathbb{C}}$ since under the quotient map $\pi : \mathbb{C}^2 \twoheadrightarrow \mathbb{P}^1_{\mathbb{C}}$, $\pi^{-1}(U_0) = \{(z_0, z_1) \in \mathbb{C}^2 \mid z_0 \neq 0\} = D(z_0)$, the plane minus the z_1-axis, which is open. Similarly for U_1.

Now consider the maps which we will show makes (U_i, φ_i) into an affine chart

$$\begin{aligned} \varphi_0 : U_0 &\longrightarrow \mathbb{C} \\ [z_0 : z_1] &\longmapsto \frac{z_1}{z_0} \\ \varphi_1 : U_1 &\longrightarrow \mathbb{C} \\ [z_0 : z_1] &\longmapsto \frac{z_0}{z_1}. \end{aligned}$$

Note that these are homeomorphisms as the image the whole complex plane which is open and φ_i are homeomorphisms onto it. Indeed, φ_i can be seen to be bijective to $\mathbb{C}$ quite easily and an inverse of φ_0, say, can be constructed by defining $\psi_0 : \mathbb{C} \to U_0$ given by $z \mapsto [1 : z]$. This is continuous and an inverse of φ_0.

Now observe that $U_0 \cap U_1 = \{[z_0 : z_1] \mid z_0, z_1 \neq 0\} = U$. Observe that $\varphi_i(U) = \mathbb{C}^\times$. The transition maps then are

$$\begin{aligned}\varphi_1 \circ \varphi_0^{-1} : \varphi_0(U) &\longrightarrow \varphi_1(U)\\ z &\longmapsto \frac{1}{z},\end{aligned}$$

which is a holomorphic map $\mathbb{C}^\times \to \mathbb{C}^\times$. Thus, we have obtained a Riemann surface $\mathbb{P}^1_{\mathbb{C}}$ with structure sheaf $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$ whose sections on an open subset $U \subseteq \mathbb{P}^1_{\mathbb{C}}$ are those functions $f : U \to \mathbb{C}$ which are holomorphic with respect to the chart $(U_i, \varphi_i)_{i=1,2}$. The Riemann surface $(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}})$ is called the *projective line* over $\mathbb{C}$.

Proposition 1.3.2. *Let $\overline{\mathbb{C}}$ be the Riemann sphere. Then $\overline{\mathbb{C}}$ is biholomorphic to $\mathbb{P}^1_{\mathbb{C}}$*

Proof. Indeed, consider the map

$$\begin{aligned}f : \overline{\mathbb{C}} &\longrightarrow \mathbb{P}^1_{\mathbb{C}}\\ z &\longmapsto \begin{cases}[1 : z] & \text{if } z \neq \infty\\ [0 : 1] & \text{if } z = \infty.\end{cases}\end{aligned}$$

Indeed, this is continuous since on any neighborhood of 0, this is the inverse of the chart map φ_0 and on any neighborhood of ∞ it is the inverse of the chart φ_1. As $\bar{\mathbb{C}}$ is compact and $\mathbb{P}^1_{\mathbb{C}}$ Hausdorff, it follows that f is a homeomorphism.

Using charts of Example 1.1.2, it is immediate to see that this is holomorphic. The inverse of this map is $[z_0 : z_1] \mapsto \frac{z_1}{z_0}$. Again this is continuous and holomorphic by same reasons. ∎

We now introduce a space where most of our geometry will take place.

Construction 1.3.3 ($\mathbb{CP}^2$, the projective plane[3]). Topologically, $\mathbb{CP}^2$ is $\mathbb{C}^3/\sim$ where $(z_0, z_1, z_2) \sim (\lambda z_0, \lambda z_1, \lambda z_2)$. This can be given the structure of a complex 2-manifold by giving an atlas consisting of three charts $(U_i, \varphi_i)_{i=0,1,2}$ where $U_i = \{[z_0 : z_1 : z_2] \mid z_i \neq 0\}$. The maps are given by

$$\begin{aligned}\varphi_0 : U_0 &\longrightarrow \mathbb{C}^2\\ [z_0 : z_1 : z_2] &\longmapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)\\ \varphi_1 : U_1 &\longrightarrow \mathbb{C}^2\\ [z_0 : z_1 : z_2] &\longmapsto \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right)\\ \varphi_2 : U_2 &\longrightarrow \mathbb{C}^2\\ [z_0 : z_1 : z_2] &\longmapsto \left(\frac{z_0}{z_2}, \frac{z_1}{z_2}\right).\end{aligned}$$

[3]We would freely interchange between $\mathbb{CP}^2$ and $\mathbb{P}^2_{\mathbb{C}}$, depending on the temperature outside.

One can check that this makes $\mathbb{CP}^2$ a complex 2-manifold by showing all transitions are holomorphic maps from open subsets of $\mathbb{C}^2$ to $\mathbb{C}^2$ (which would require a knowledge of several complex variables, but we skip over that as really don't require that here).

We would like to know a class of closed (thus compact) subsets of $\mathbb{CP}^2$ formed by polynomials in two variables. These will motivate algebraic counterparts of the analytic geometry that we are consider currently.

Definition 1.3.4 (**Projective algebraic plane curves**)**.** Let $\bar{p}(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ be a polynomial and let $p(z_1, z_2, z_3) \in \mathbb{C}[z_1, z_2, z_3]$ be its homogenization so that p is homogeneous of degree $d \geq 1$. Consider the set

$$V(p) = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid p(z_1, z_2, z_3) = 0\}.$$

This defines a closed subset of $\mathbb{CP}^2$ since it is $\mathbb{CP}^2 \setminus V(p)$ is the image of $\mathbb{C}^3 \setminus V(\bar{p})$ under the quotient map $\pi : \mathbb{C}^3 \to \mathbb{CP}^2$. We call $V(p) \subseteq \mathbb{CP}^2$ a projective algebraic plane curve.

We see that a projective algebraic plane curve Z is formed by three pieces of affine algebraic plane curves.

Lemma 1.3.5. *Let $\bar{p} \in \mathbb{C}[z_0, z_1]$, $p \in \mathbb{C}[z_0, z_1, z_2]$ be its homogenization and $Z = V(p)$ be the projective algebraic curve. Let $(U_i, \varphi_i)_{i=0,1,2}$ be the standard chart of $\mathbb{P}^2_{\mathbb{C}}$ (see Construction 1.3.3). Then the image of $Z \cap U_i$ under φ_i in $\mathbb{C}$ is $V(\bar{p}_i)$ where $\bar{p}_0 = p(1, z_1, z_2)$, $\bar{p}_1 = p(z_0, 1, z_2)$ and $\bar{p}_2 = p(z_0, z_1, 1)$.*

Proof. Indeed, since, say $Z \cap U_0 = \{[1 : z_1 : z_2] \mid p(1, z_1, z_2) = 0\}$, therefore

$$\varphi_0(Z \cap U_0) = \{(z_1, z_2) \mid p(1, z_1, z_2) = 0\} = V(\bar{p}_0).$$

The other cases are same. □

We now show that a certain type of algebraic plane curves define Riemann surfaces.

Definition 1.3.6 (**Smooth algebraic plane curves**)**.** Let $f \in \mathbb{C}[z_1, z_2, z_3]$ be a homogeneous polynomial. Then, the polynomial f is called non-singular or smooth if for all points $p \in V(f) \subseteq \mathbb{CP}^2$, we have that $\left.\frac{\partial f}{\partial z_i}\right|_p \neq 0$ for atleast one i from $0, 1, 2$. In this case, the projective plane curve $V(f)$ that it defines is called the smooth projective algebraic plane curve. A similar definition gives smooth affine algebraic plane curves in $\mathbb{C}^2$.

We now show that every smooth projective plane curve defined by an irreducible smooth homogeneous polynomial in three variables gives a Riemann surface. For that we need following two preliminary results.

Theorem 1.3.7. *Let $p \in \mathbb{C}[z_0, z_1, z_2]$ be a homogeneous polynomial.*
1. *If p is non-singular, then $V(p) \subseteq \mathbb{P}^2_{\mathbb{C}}$ is irreducible.*
2. *If p is irreducible, then $V(p) \subseteq \mathbb{P}^2_{\mathbb{C}}$ is connected.*

We now state the main theorem. Its proof can be seen by implicit function theorem for $\mathbb{C}$, but we omit all such checks.

Theorem 1.3.8. *Let $p \in \mathbb{C}[z_0, z_1, z_2]$ be a non-singular homogeneous polynomial. Then, $V(p) \subseteq \mathbb{P}^2_{\mathbb{C}}$ is a compact connected Riemann surface.*

2 Ramified coverings & Riemann-Hurwitz formula

3 Monodromy & analytic continuation

4 Holomorphic & meromorphic forms

Having differentials on a given geometric object gives us a sense of direction of each point. Exploiting this, one can define very many types of forms (differentiable, holomorphic, meromorphic...) and their interrelations which allows us to study the object in question more deeply.

4.1 Differentials

We will We first construct the sheaf of differentiable maps on a Riemann surface.

Definition 4.1.1 (Sheaf of differentiable maps). Let X be a Riemann surface. Consider the assignment for each open $U \subseteq X$

$$\mathcal{E}_X(U) := \{f : U \to \mathbb{C} \mid \forall \text{ charts } (U_i, z_i),\ f \circ z_i^{-1} : z_i(U \cap U_i) \to \mathbb{C} \text{ is differentiable.}\}$$

This assignments with restrictions naturally forms a sheaf, called the sheaf of differentiable maps on X. This is a sheaf of $\mathbb{C}$-algebras. Moreover, this is an $\mathcal{O}_X$-algebra as well since pointwise product of holomorphic and differentiable map is again differentiable.

We will use the sheaf $\mathcal{E}_X$ to build many other sheaves which will be of prime importance to us. Let us first introduce few operators on the seaf $\mathcal{E}_X$.

Construction 4.1.2 (Operators on $\mathcal{E}_X$)**.** Define $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ as two operators on $\mathcal{E}_X$ as follows. For any open $U \subseteq X$, define

$$\begin{aligned} \frac{\partial}{\partial x} : \mathcal{E}_X(U) &\longrightarrow \mathcal{E}_X(U) \\ f : U \to \mathbb{C} &\longmapsto \frac{\partial f}{\partial x} : U \to \mathbb{C} \end{aligned}$$

where $\frac{\partial f}{\partial x} : U \to \mathbb{C}$ is defined as follows. Let (U_i, z_i) be a chart. As f is differentiable, therefore $f \circ z_i^{-1} : z_i(U \cap U_i) \to \mathbb{C}$ is differentiable. Define

$$\frac{\partial f}{\partial x} \circ z_i^{-1} = \frac{\partial}{\partial x}\left(f \circ z_i^{-1}\right)$$

for each chart (U_i, z_i). Similarly, one defines $\frac{\partial}{\partial y}$. Note that these maps commutes with restrictions. Hence we get sheaf maps $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} : \mathcal{E}_X \to \mathcal{E}_X$. Note that both of these are $\mathbb{C}$-linear.

Consider the two operators

$$\begin{aligned} \frac{\partial}{\partial z} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \end{aligned}$$

These two also define $\mathbb{C}$-linear operators on $\mathcal{E}_X$.

We observe some of the immediate consequences.

Lemma 4.1.3. *Let $(X, \mathcal{O}_X)$ be a Riemann surface. Then,*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}_X \xrightarrow{\frac{\partial}{\partial \bar{z}}} \mathcal{E}_X$$

is exact.

Proof. Note that the map $\mathcal{O}_X \to \mathcal{E}$ is obtained by thinking of a holomorphic map as a real differentiable map. By Cauchy-Riemann, $f : V \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$. It follows that on an open $U \subseteq X$, we have $\mathrm{Ker}\left(\frac{\partial}{\partial \bar{z}}\right) = \{f \in \mathcal{O}_X(U) \mid f \text{ is holomorphic}\} = \mathcal{O}_X(U)$, as required. $\square$

Remark 4.1.4 ($\mathcal{E}_X$ is locally ringed)**.** Consider the sheaf of differentiable maps $\mathcal{E}_X$ on a Riemann surface X. We observe that for any point $x \in X$, the stalk $\mathcal{E}_{X,x}$ is a local ring where the maximal ideal $\mathfrak{m}_x$ consists of those germs which vanishes at point x. Thus, $\mathcal{E}_X$ is a locally ringed $\mathcal{O}_X$-algebra.

Definition 4.1.5 (Cotangent space at a point). Let $(X, \mathcal{O}_X)$ be a Riemann surface and $\mathcal{O}_X$ be the $\mathcal{O}_X$-algebra of differentiable maps. Then, the cotangent space at point $x \in X$ is the $\mathbb{C}$-vector space given by

$$T_x^{(1)} = \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$$

where $(\mathcal{E}_{X,x}, \mathfrak{m}_x)$ is the local ring at point x of sheaf $\mathcal{E}_X$. A point in $T_x^{(1)}$ is referred to as a cotangent vector at $x \in X$.

Remark 4.1.6 (Cotangent vectors and "direction")**.** For a point $a \in X$, pick a covector $(U, f)_a \in T_x^{(1)}$. As $(U, f)_a \in T_x^{(1)}$ and (U_i, z_i) is a chart containing $a \in X$, therefore $f \circ z_i^{-1} : z_i(U \cap U_i) \to \mathbb{C}$ is differentiable. We may write the Taylor expansion of $f \circ z_i^{-1}$ at the point $z_i(a) = (a_1, a_2)$ to get that

$$f \circ z_i^{-1}(x, y) = f(a) + \frac{\partial f}{\partial x}(a)(x - a_1) + \frac{\partial f}{\partial y}(a)(y - a_2) + \text{terms of degree} \geq 2.$$

As $\mathfrak{m}_x^2$ consists of products of those germs vanishing at a and "terms of degree ≥ 2" vanishes at a, therefore, we get that

$$(U, f)_a = \frac{\partial f}{\partial x}(a)(x - a_1) + \frac{\partial f}{\partial y}(a)(y - a_2) + \mathfrak{m}_a^2.$$

The above motivates the following definition.

Definition 4.1.7 (Differential of a map). Let X be a Riemann surface and $f \in \mathcal{O}_X(U)$ be a differentiable map on open $U \subseteq X$. Let $a \in U$. Define the following $\mathbb{C}$-linear transformation

$$\begin{aligned} d_a : \mathcal{E}_{X,a} &\longrightarrow T_a^{(1)} \\ (U, f)_a &\longmapsto (f - f(a)) + \mathfrak{m}_a^2. \end{aligned}$$

As is evident from Remark 4.1.6, the differentials of maps $x, y : U \to \mathbb{C}$ which on a chart (U_i, z_i) is defined by $x \circ z_i^{-1} : z_i(U \cap U)i \to \mathbb{C}$ mapping $(x, y) \mapsto x$ and similarly for y, holds special position amongst all differentials.

Proposition 4.1.8. *Let X be a Riemann surface and $a \in X$ contained in open U. Then,*

1. *$T_a^{(1)}$ has $\{d_a x, d_a y\}$ as basis.*
2. *$T_a^{(1)}$ has $\{d_a z, d_a \bar{z}\}$ as a basis.*
3. *For any $f \in \mathcal{E}_X(U)$,*

$$
\begin{aligned}
d_a f &= \frac{\partial f}{\partial x}(a) d_a x + \frac{\partial f}{\partial y}(a) d_a y \\
&= \frac{\partial f}{\partial z}(a) d_a z + \frac{\partial f}{\partial \bar{z}}(a) d_a \bar{z}.
\end{aligned}
$$

Proof. A simple exercise in reduction to affine charts and using properties of it (in this case, Taylor series). □

Notation 4.1.9 (Decomposition of cotangent space)**.** By Proposition 4.1.8, it follows that we can write

$$
\begin{aligned}
T_a^{(1)} &= \mathbb{C} d_a z \oplus \mathbb{C} d_a \bar{z} \\
&=: T_a^{1,0} \oplus T_a^{0,1}.
\end{aligned}
$$

Elements of $T_a^{1,0}$ are called covectors of type (1,0), same for the other case. For any $f \in \mathcal{E}_X(U)$, we further denote

$$
d_a f = d_a' f + d_a'' f
$$

for unique $d_a' f \in T_a^{1,0}$ and $d_a'' f \in T_a^{0,1}$, where

$$
\begin{aligned}
d_a' f &= \frac{\partial f}{\partial z}(a) d_a z, \\
d_a'' f &= \frac{\partial f}{\partial \bar{z}}(a) d_a \bar{z}.
\end{aligned}
$$

Taking exterior powers of $T_a^{(1)}$ gives us other vector spaces which we will use to define differential k-forms.

Definition 4.1.10 (**Differential k-forms**)**.** Let X be a Riemann surface and $U \subseteq X$ be an open subset. Let $T_a^{(k)} = \wedge^k T_a^{(1)}$ be the k^{th}-exterior power of $T_a^{(1)}$. Note that $\dim_{\mathbb{C}} \wedge^k T_a^{(1)} =$. A differential k-form is a section

$$
\omega : U \to \coprod_{a \in U} T_a^{(k)}
$$

where $\omega(a) \in T_a^{(k)}$ (that is, a differential k-form is a section of k^{th}-exterior power of the cotangent bundle). A differential 1-form ω over U is of type (1,0) if for all $a \in U$, $\omega(a) \in T_a^{1,0}$. Similarly for differential 1-form of type $(0, 1)$.

Using differential forms, we can define differentiable, holomorphic and meromorphic 1-forms. Before that we quickly define Laurent expansion and residuce of holomorphic maps.

Remark 4.1.11 (Laurent expansion & residue)**.** Let X be a Riemann surface and $a, b \in X$. Let (U, z) be a chart containing a, where we may assume $z(a) = 0$. Let $f \in \mathcal{O}_X(U \setminus \{a\})$. Then $f \circ z^{-1} : z(U) \setminus \{0\} \to \mathbb{C}$ is holomorphic. Thus, around point $0 \in z(U)$, there is a Laurent series representation of $f \circ z^{-1}$:

$$(f \circ z^{-1})(x) = \sum_{n=-\infty}^{\infty} c_n x^n,$$

which we may then write in terms of coordinates z as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

Thus, f has a removable singularity or pole of order k at a if and only if so does $f \circ z^{-1}$ at $z(a) = 0$.

Let $\omega = f dz \in \Omega^1_X(U \setminus \{a\})$ be a holomorphic 1-form. Then $f = \sum_n a_n z^n$, we define residue of f at a as $\mathrm{res}_a f = c_{-1}$.

Definition 4.1.12 (**Differentiable, holomorphic and meromorphic** 1**-forms**)**.** Let X be a Riemann surface and and $U \subseteq X$ open. Let (U_i, z_i) be any chart. A differential 1-form ω is said to be:

1. differentiable if on $U \cap U_i$ we have
$$\omega = f dz + g d\bar{z}$$
where $f, g \in \mathcal{E}_X(U \cap U_i)$, denote $\omega \in \mathcal{E}^{(1)}_X(U)$. If $\omega = f dz$, then we say that ω is a differentiable 1-form of type (1,0), denoted $\omega \in \mathcal{E}^{1,0}_X(U)$. Similarly, if $\omega = g d\bar{z}$, then ω is a differentiable 1-form of type (0,1), denoted $\omega \in \mathcal{E}^{0,1}_X(U)$;
2. holomorphic if on $U \cap U_i$ we have
$$\omega = f dz$$
where $f \in \mathcal{O}_X(U \cap U_i)$, denote $\omega \in \Omega^{(1)}_X(U)$,
3. meromorphic if there exists an open subseteq $V \subseteq U$ such that ω on V is a holomorphic 1-form, $U \setminus V$ contains isolated points and ω has a pole at each point in $U \setminus V$. Denote $\omega \in \mathcal{M}^{(1)}_X(U)$.

One can also define a differential 2-form ω to be differentiable if $\omega = f dz \wedge d\bar{z}$ where $f \in \mathcal{E}(U \cap U_i)$. Differentiable 2-forms on $U \subseteq X$ are denoted $\mathcal{E}^{(2)}_X(U)$. Note in all of the above, say in differentiable 2-forms, when we wrote $\omega = f dz \wedge d\bar{z}$, we meant that for any $a \in U$, we have

$$\omega(a) = f(a) d_a z \wedge d_a \bar{z} \in T^{(2)}_a = T^{(1)}_a \wedge T^{(1)}_a.$$

Finally, all $\mathcal{E}^{(1)}_X$, $\mathcal{E}^{(2)}_X$, $\Omega^{(1)}_X$ and $\mathcal{M}^{(1)}_X$ are sheaves of $\mathbb{C}$-vector spaces. One also calls $\mathcal{M}^{(1)}_X$ sheaf of abelian differentials.

Construction 4.1.13 (Exterior derivative)**.** We now construct the following two maps:

$$\mathcal{E}_X \xrightarrow{d} \mathcal{E}_X^{(1)} \xrightarrow{d} \mathcal{E}_X^{(2)}.$$

Indeed, on an open set $U \subseteq X$, define

$$\begin{aligned} d : \mathcal{E}_X(U) &\longrightarrow \mathcal{E}_X^{(1)}(U) \\ f &\longmapsto df \end{aligned}$$

where $df : U \to \coprod_{a \in U} T_a^{(1)}$ is given by $a \mapsto d_a f$. Next, define for

$$\begin{aligned} d : \mathcal{E}_X^{(1)}(U) &\longrightarrow \mathcal{E}_X^{(2)}(U) \\ \omega &\longmapsto d\omega \end{aligned}$$

where if $\omega = \sum_k f_k dg_k$ for $f_k, g_k \in \mathcal{E}_X(U \cap U_i)$ for some chart (U_i, z_i), then $d\omega$ is defined as

$$d\omega = \sum_k df_k \wedge dg_k.$$

Definining for any $f \in \mathcal{E}_X(U)$ elements $d'f \in \mathcal{E}_X^{1,0}(U)$ given by $a \mapsto d'_a f$ and $d''f \in \mathcal{E}_X^{0,1}(U)$ given by $a \mapsto d''_a f$ and similarly the maps $d', d'' : \mathcal{E}_X^{(1)} \to \mathcal{E}_X^{(2)}$, we thus have the following two chains as well:

$$\mathcal{E}_X \xrightarrow{d'} \mathcal{E}_X^{(1)} \xrightarrow{d'} \mathcal{E}_X^{(2)}$$

and

$$\mathcal{E}_X \xrightarrow{d''} \mathcal{E}_X^{(1)} \xrightarrow{d''} \mathcal{E}_X^{(2)}.$$

Construction 4.1.14 (Pullback of forms)**. TODO.**

4.2 Examples

Let us now give some examples of holomorphic and meromorphic 1-forms on Riemann surfaces.

Example 4.2.1. We first begin by a standard example, that of $\mathbb{C}^\times$ and $\omega = \frac{1}{z}dz \in \Omega_{\mathbb{C}}^{(1)}(\mathbb{C}^\times)$. **TODO**

4.3 Dolbeault's lemma

Theorem 4.3.1 (Dolbeault's lemma for $\mathbb{C}$)**.** *Let $X = \{z \in \mathbb{C} \mid |z| < R\}$ for $0 < R \leq \infty$. If $f : X \to \mathbb{C}$ is differentiable, then there exists $g : X \to \mathbb{C}$ differentiable such that*

$$\frac{\partial g}{\partial \bar{z}} = f.$$

Theorem 4.3.2 (Dolbeault's lemma for Riemann surfaces)**.** *Let X be a Riemann surface and $U \subseteq X$ be an open set. Then for any $f \in \mathcal{E}_X(U)$, there exists $g \in \mathcal{E}_X(U)$ such that*

$$\frac{\partial g}{\partial \bar{z}} = f.$$

Proof. Let f be as above. Pick any chart (U_i, z_i) of X. Then $f_i := f \circ z_i^{-1} : z_i(U \cap U_i) \to \mathbb{C}$ is a differentiable map where we may assume $z_i(U \cap U_i)$ to be an open disc by considering finer charts. By Dolbeault's lemma for $\mathbb{C}$ (Theorem 4.3.1), we get that there exists differentiable $g_i : z_i(U \cap U_i) \to \mathbb{C}$ such that $\frac{\partial g_i}{\partial \bar{z}} = f_i$. Thus, we get a differentiable $g : U \to \mathbb{C}$ which on chart (U_i, z_i) is given by $g \circ z_i^{-1} = g_i$ so that $\frac{\partial g}{\partial \bar{z}} \circ z_i^{-1} = \frac{\partial}{\partial \bar{z}}(g \circ z_i^{-1}) = \frac{\partial g_i}{\partial \bar{z}} = f_i$. Thus, $\frac{\partial g}{\partial \bar{z}}$ agrees with f on each chart, hence $\frac{\partial g}{\partial \bar{z}} = f$. $\square$

5 Riemann-Roch theorem

Our goal is to prove and showcase the uses of the following theorem.

Theorem 5.0.1 (Riemann-Roch theorem)**.** *Let X be a compact Riemann surface of genus g and let D be a divisor on X. Then:*

1. *The cohomology groups $H^0(X, \mathcal{O}(D))$ and $H^1(X, \mathcal{O}(D))$ are finite-dimensional $\mathbb{C}$-vector spaces.*
2. *The dimensions of the 0^{th} and 1^{st} cohomology groups satisfy*

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}(D)) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}(D)) = 1 - g + \deg D.$$

Remark 5.0.2. Note that $H^0(X, \mathcal{O}(D)) = \Gamma(X, \mathcal{O}(D))$, so one can interpret $\dim_{\mathbb{C}} H^1(X, \mathcal{O}(D))$ as the correction term to the inequality $\dim_{\mathbb{C}} \Gamma(X, \mathcal{O}(D)) \geq 1-g+\deg D$ so that it becomes an equality.

In the process of proving the above statement, we have to understand the following notions on a Riemann surface : cohomology of sheaves, divisors and genus. We undertake the last two, as we have covered cohomology of sheaves in detail in Chapter FoG.27. However, as we need some results on cohomology of sheaves of differentials and some important long exact sequences, we spend a section setting up the results which we will use.

5.1 Cohomology

Recall from §FoG.27.7 that for a given space X and a sheaf $\mathcal{F}$ on X, we can define its Čech-cohomology groups $H^i(X, \mathcal{F})$. For us the most important is the first cohomology, as is evident in Theorem 5.0.1.

Remark 5.1.1. We first recollect the sheaves that we have so far constructed on any Riemann surface X.

1. $\mathcal{O}_X$ of holomorphic maps on X.
2. $\mathcal{M}_X$ of meromorphic maps on X.
3. $\mathcal{E}_X$ of differentiable maps on X.
4. $\mathcal{E}_X^{(k)}$ of differentiable k-forms on X, $k = 1, 2$.
5. $\mathcal{E}_X^{1,0}$ and $\mathcal{E}_X^{0,1}$ of differentiable 1-forms of type (1,0) and (0,1), repsectively.
6. $\Omega_X^{(1)}$ of holomorphic 1-forms on X.
7. $\mathcal{M}_X^{(1)}$ of meromorphic 1-forms on X.

Every sheaf from 2-7 is an $\mathcal{O}_X$-module. Using these seven sheaves, we can extract quite a bit of geometric information about Riemann surfaces.

We first explore the many maps that one has amongst the above seven sheaves.

Example 5.1.2. Let $(X, \mathcal{O}_X)$ be a Riemann surface. Here are some maps between above sheaves.

1. [Exterior derivative] We have maps

$$\mathcal{E}_X \xrightarrow{d} \mathcal{E}_X^{(1)} \xrightarrow{d} \mathcal{E}_X^{(2)},$$

$$\mathcal{E}_X \xrightarrow{d'} \mathcal{E}_X^{(1)} \xrightarrow{d'} \mathcal{E}_X^{(2)}$$

and

$$\mathcal{E}_X \xrightarrow{d''} \mathcal{E}_X^{(1)} \xrightarrow{d''} \mathcal{E}_X^{(2)}.$$

as constructed in Construction 4.1.13.

2. [Natural inclusions] We have following inclusions:

$$\mathcal{O}_X \hookrightarrow \mathcal{E}_X$$
$$\Omega_X^{(1)} \hookrightarrow \mathcal{E}_X^{1,0}$$

3. [Exponential map] Let $\mathcal{O}_X^\times$ be a sheaf of abelian groups obtained as follows. For each open $U \subseteq X$, define

$$\mathcal{O}_X^\times(U) := \{f \in \mathcal{O}_X(U) \mid f : U \to \mathbb{C}^\times\}.$$

That is, $\mathcal{O}_X^\times$ is the multiplicative abelian group of units of sheaf of $\mathbb{C}$-algebra $\mathcal{O}_X$. We then define the following map

$$\exp : \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times$$

which on an open $U \subseteq X$ is

$$\begin{aligned} \exp_U : \mathcal{O}_X(U) &\longrightarrow \mathcal{O}_X^\times(U) \\ f &\longmapsto e^{2\pi i f}. \end{aligned}$$

This is clearly a map of sheaves. This is called the exponential map and it plays an important role in geometry.

We have an example of a situation where the image presheaf is not a sheaf (hence justifies why we need to sheafify to get the image preseheaf).

Example 5.1.3 (Image presheaf may not be a sheaf)**.** For $X = \mathbb{C}$, consider the open cover $U = \mathbb{C} \setminus (-\infty, 0]$ and $V = \mathbb{C} \setminus [0, \infty)$. Consider the image presheaf of the exponential map $\exp : \mathcal{O}_X \to \mathcal{O}_X^\times$, denoted F. Let $\mathrm{id} \in \mathcal{O}_X^\times(U)$ and $\mathrm{id} \in \mathcal{O}_X^\times(V)$. Observe that they agree on intersection $U \cap V$. Observe further that U and V are simply connected, therefore they have an analytic branch of log, that is, $\mathrm{id} \in \mathrm{Im}\,(\exp_U), \mathrm{Im}\,(\exp_V)$. We claim that there is no section in $F(U \cup V)$ whose restriction to U and V are id. Indeed, since $U \cup V = \mathbb{C}^\times$, therefore if the above two sections glue, then we will have an analytic branch of log on $\mathbb{C}^\times$, not possible.

5.1.1 The cohomology long exact sequence

We now state the main theorem, after proving two lemmas which are nice exercises in general sheaf theory.

Recall that a sheaf map is injective, surjective, if it is so at the level of stalks.

Lemma 5.1.4. [4] *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an injective map of sheaves of abelian groups. Then $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.*

Proof. Indeed, if $\varphi_U(f) = 0$ for some $f \in \mathcal{F}(U)$, then at stalks at each point $x \in U$, we get $\varphi_x((U,f)_x) = (U, \varphi_U(f))_x = 0$. Thus, by injectivity, $(U,f)_x = 0$ in $\mathcal{F}_x$ for all $x \in U$. Consequently, f is locally zero at each point of U, thus it is zero at each point of U. $\square$

Lemma 5.1.5. [5] *Let X be a space and $0 \to \mathcal{F} \overset{\varphi}{\to} \mathcal{G} \overset{\psi}{\to} \mathcal{H}$ be an exact sequence of sheaves of abelian groups on X. If $U \subseteq X$ is any open, then*

$$0 \to \mathcal{F}(U) \overset{\varphi_U}{\to} \mathcal{G}(U) \overset{\psi_U}{\to} \mathcal{H}(U)$$

is exact.

Proof. By Lemma 5.1.4, φ_U is injective. We first show that $\mathrm{Ker}\,(\psi_U) \supseteq \mathrm{Im}\,(\varphi_U)$. Pick any $f \in \mathcal{F}(U)$. Need only show that $\psi_U(\varphi_U(f)) = 0$. Indeed, it suffices to show that $(U, \psi_U(\varphi_U(f)))_x = 0$ in $\mathcal{H}_x$ for all $x \in U$. Pick any $x \in U$. Observe that $(U, \psi_U(\varphi_U(f)))_x = \psi_x \circ \phi_x((U,f)_x)$. The latter is zero by exactness, as needed.

Next, we wish to show that $\mathrm{Ker}\,(\psi_U) \subseteq \mathrm{Im}\,(\varphi_U)$. Indeed, pick $g \in \mathrm{Ker}\,(\psi_U)$ and consider the germ $(U,g)_x \in \mathcal{G}_x$ for any $x \in U$. Observe that $(U,g)_x \in \mathrm{Ker}\,(\psi_x) = \mathrm{Im}\,(\varphi_x)$. Thus, there exists $(V_x, f_x)_x \in \mathcal{F}_x$ such that

$$\varphi_x\left((V_x, f_x)_x\right) = (V_x, \varphi_{V_x}(f_x))_x = (U, g)_x.$$

By definition of germs, we may assume that $\varphi_{V_x}(f_x) = g$ on $V_x \subseteq U$ for all $x \in U$. Hence we have an open covering $\{V_x\}_{x \in U}$ of U and $\varphi_{V_x}(f_x) = g$ on $\mathcal{G}(V_x)$. We claim that $\{f_x\}_{x \in U}$ can be glued. To this end, we wish to show that on $V_x \cap V_y$, we have an equality $f_x = f_y$ in $\mathcal{F}(V_x \cap V_y)$. By Lemma 5.1.4, it suffices to show that $\varphi_{V_x \cap V_y}(f_x) = \varphi_{V_x \cap V_y}(f_y)$ in $\mathcal{G}(V_x \cap V_y)$. Observe that the element $\varphi_{V_x \cap V_y}(f_x) = g|_{V_x \cap V_y} = \varphi_{V_x \cap V_y}(f_y)$ in $\mathcal{G}(V_x \cap V_y)$. Thus, we have the required equality.

It follows that $\{f_x\}_{x \in U}$ can be glued to $f \in \mathcal{F}(U)$ such that $\varphi_U(f)$ in $\mathcal{G}(U)$ is such that its restriction to each V_x is g, thus by sheaf axioms, $\varphi_U(f) = g$, that is, $g \in \mathrm{Im}\,(\varphi_U)$, as needed. $\square$

Remark 5.1.6 (Surjective maps of sheaves)**.** Recall that if $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective on sections, then it is a surjective map, but the converse is not true. Indeed for $X = \mathbb{C}$, the map of sheaves $\exp : \mathcal{O}_X \to \mathcal{O}_X^\times$ is surjective as any germ in the latter locally has a logarithm, but $\exp_{\mathbb{C}^\times}$ is not surjective on sections as the constant map $\mathrm{id} \in \mathcal{O}_X^\times(\mathbb{C}^\times)$ does not have a logarithm.

However, we do have the following "local surjectivity": φ is surjective if and only if for any open $U \subseteq X$ and any $s \in \mathcal{G}(U)$, there exists an open cover $\{U_i\}_{i \in I}$ of U and $t_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(t_i) = s|_{U_i}$.

Moreover, some of the above sheaves are obtained by kernels and gives us several short exact sequences, which will be used later.

[4] To remove before addition to FoG and add relevant reference.

[5] To remove before addition to FoG and add relevant reference.

Example 5.1.7. [Important short exact sequences] Let $(X, \mathcal{O}_X)$ be a Riemann surface. Some of the sheaves in Remark 5.1.1 are kernels of some other map of sheaves and they give rise to some important short exact sequences.

1. The sheaf of holomorphic maps $\mathcal{O}_X$ is obtained as the kernel
$$\mathcal{O}_X = \mathrm{Ker}\left(d'' : \mathcal{E}_X \to \mathcal{E}_X^{0,1}\right).$$
Thus, we have a s.e.s.
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}_X \xrightarrow{d''} \mathcal{E}_X^{0,1} \longrightarrow 0$$
where d'' is surjective by Dolbeault's lemma (Theorem 4.3.2).
2. The sheaf of holomorphic 1-forms $\Omega_X^{(1)}$ is obtained from the kernel
$$\Omega_X^{(1)} = \mathrm{Ker}\left(d : \mathcal{E}_X^{1,0} \to \mathcal{E}_X^{(2)}\right).$$
Thus we have a s.e.s.
$$0 \longrightarrow \Omega_X^{(1)} \longrightarrow \mathcal{E}_X^{1,0} \xrightarrow{d} \mathcal{E}_X^{(2)} \longrightarrow 0$$
where d is surjective as follows. For any $\omega = f dz \wedge d\bar{z}$ in $\mathcal{E}_X^{(2)}(U \cap U_i)$ for (U_i, z_i) a chart and $f \in \mathcal{E}_X(U \cap U_i)$, we get by Dolbeault's lemma (Theorem 4.3.2) that there exists $g \in \mathcal{E}_X(U \cap U_i)$ such that $\frac{\partial g}{\partial \bar{z}} = f$. Thus, $d(-g dz) = -\left(\frac{\partial g}{\partial z} dz + \frac{\partial g}{\partial \bar{z}} d\bar{z}\right) \wedge dz = -\frac{\partial g}{\partial \bar{z}} d\bar{z} \wedge dz = f dz \wedge d\bar{z}$. This shows that d is surjective on sections, as required.
3. Let $\mathcal{L}_X = \mathrm{Ker}\left(d : \mathcal{E}_X^{(1)} \to \mathcal{E}_X^{(2)}\right)$ be the sheaf of closed 1-forms. Then we claim that the following is a s.e.s.
$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{E}_X \xrightarrow{d} \mathcal{L}_X \longrightarrow 0.$$
Indeed, $\mathrm{Ker}\,(d : \mathcal{E}_X \to \mathcal{L}_X)$ is given on an open-connected $U \subseteq X$ by those differentiable maps $f : U \to \mathbb{C}$ such that $df = d'f dz + d''f d\bar{z} = 0$, that is, $\partial f/\partial z = 0$ and $\partial f/\partial \bar{z} = 0$ on U. It follows that f is holomorphic with zero derivative, that is, f is constant (U is connected). Hence, we get the inclusion $\underline{\mathbb{C}} \hookrightarrow \mathcal{E}_X$ whose image is $\mathrm{Ker}\,(d)$.
Now, d is surjective as locally any closed form is exact by local existence of primitives from one variable complex analysis.
4. For the exponential map, observe that we have a map $\underline{\mathbb{Z}} \to \mathcal{O}_X$ which on an open-connected $U \subseteq X$ is given by
$$\begin{aligned} \mathbb{Z} = \underline{\mathbb{Z}}(U) &\longrightarrow \mathcal{O}_X(U) \\ c_n &\longmapsto c_n. \end{aligned}$$
For some arbitrary open set $U \subseteq X$, $\underline{\mathbb{Z}}(U)$ is given by functions which are constant on each open connected component (any Riemann surface is locally connected), so they are in particular also holomorphic. We thus get a s.e.s.
$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$
where exp is surjective as locally any non-zero holomorphic map has an analytic branch of log.

We will now discuss the map in cohomology induced by a map of sheaves and how the connecting homomorphism works.

Construction 5.1.8 (Map in cohomology)**.** [6] Any map of abelian sheaves over X yields a map in the cohomology as well. Indeed, let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves. Then we get a map

$$\begin{aligned} \varphi^q : C^q(\mathcal{U}, \mathcal{F}) &\longrightarrow C^q(\mathcal{U}, \mathcal{G}) \\ s = (s(\alpha_0, \dots, \alpha_q)) &\longmapsto \varphi^q(s) = \big(\varphi_{\alpha_0 \dots \alpha_q}(s(\alpha_0, \dots, \alpha_q))\big) \end{aligned}$$

where $\varphi_{\alpha_0 \dots \alpha_q} = \varphi_{U_{\alpha_0} \cap \dots \cap U_{\alpha_q}}$.

It then follows quite immediately from the fact that each $\varphi_{\alpha_0 \dots \alpha_q}$ is a group homomorphism that $d\varphi^q = \varphi^{q+1} d$. It follows that we get a map of chain complexes

$$\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{G}).$$

Hence, we get a map in cohomology

$$\varphi^q : H^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(\mathcal{U}, \mathcal{G}).$$

Finally, this gives by universal property of direct limits a unique map

$$\varphi^q : \check{H}^q(X, \mathcal{F}) \longrightarrow \check{H}^q(X, \mathcal{G})$$

such that for every open cover $\mathcal{U}$, the following diagram commutes:

$$\begin{tikzcd} \check{H}^q(X, \mathcal{F}) \arrow[r, dashed, "\varphi^q"] & \check{H}^q(X, \mathcal{G}) \\ H^q(\mathcal{U}, \mathcal{F}) \arrow[u] \arrow[r, "\varphi^q"'] & H^q(\mathcal{U}, \mathcal{G}) \arrow[u] \end{tikzcd}$$

where vertical maps are the maps into direct limits.

Construction 5.1.9 (Connecting homomorphism)**.** [7] Let X be a topological space and

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$$

be an exact sequence of sheaves on X. We define the connecting homomorphism

$$H^0(X, \mathcal{H}) \xrightarrow{\delta} H^1(X, \mathcal{F})$$

as follows. First, pick any $h \in H^0(X, \mathcal{H}) = \Gamma(X, \mathcal{H})$. As ψ is surjective therefore there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X and $g_i \in \mathcal{G}(U_i)$ such that $\psi_{U_i}(g_i) = h|_{U_i}$. Using (g_i) and (U_i) we construct a 1-cocycle for $\mathcal{F}$ as follows. Observe that for each $i, j \in I$, we have $\psi_{U_i \cap U_j}(g_i - g_j) = 0$ in $\mathcal{H}(U_i \cap U_j)$. Thus, $g_i - g_j \in \operatorname{Ker}\big(\psi_{U_i \cap U_j}\big)$. By exactness

[6] To remove before addition to FoG and add relevant reference.

[7] To remove before addition to FoG and add relevant reference.

guaranteed by Lemma 5.1.5, it follows that there exists $f_{\alpha_0\alpha_1} \in \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1})$ such that $\varphi_{U_{\alpha_0} \cap U_{\alpha_0}}(f_{\alpha_0\alpha_1}) = g_{\alpha_0} - g_{\alpha_1}$, for each $\alpha_0, \alpha_1 \in I$. We claim that the element

$$f := (f_{\alpha_0\alpha_1})_{\alpha_0,\alpha_1} \in \prod_{(\alpha_0,\alpha_1)\in I^2} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1}) = C^1(\mathcal{U}, \mathcal{F})$$

is a 1-cocycle. Indeed, we need only check that $df = 0$ in $C^2(\mathcal{U}, \mathcal{F})$. Pick any $(\alpha_0, \alpha_1\alpha_2) \in I^3$. We wish to show that $df(\alpha_0, \alpha_1\alpha_2) = 0$. Indeed,

$$\begin{aligned} df(\alpha_0, \alpha_1\alpha_2) &= \sum_{j=0}^{2} (-1)^j \rho_j \left(f_{\alpha_0\hat{\alpha}_j\alpha_2}\right) \\ &= f_{\alpha_1\alpha_2} - f_{\alpha_0\alpha_2} + f_{\alpha_0\alpha_1} \end{aligned}$$

in $\mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2})$. We claim the above is zero. Indeed, By Lemma 5.1.5 on $V := U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}$ we get that φ_V is injective. But since

$$\begin{aligned} \varphi_V(f_{\alpha_1\alpha_2} - f_{\alpha_0\alpha_2} + f_{\alpha_0\alpha_1}) &= \varphi_V(f_{\alpha_1\alpha_2}) - \varphi_V(f_{\alpha_0\alpha_2}) + \varphi_V(f_{\alpha_0\alpha_1}) \\ &= g_{\alpha_1} - g_{\alpha_2} - (g_{\alpha_0} - g_{\alpha_2}) + g_{\alpha_0} - g_{\alpha_1} \\ &= 0, \end{aligned}$$

hence it follows that $df(\alpha_0, \alpha_1\alpha_2) = 0$, as required. Hence $f \in C^1(\mathcal{U}, \mathcal{F})$ is a 1-cocycle. Thus we get an element $[f] \in H^1(\mathcal{U}, \mathcal{F})$. This defines a group homomorphism $H^0(X, \mathcal{H}) \to H^1(\mathcal{U}, \mathcal{F})$. Further by passing to direct limit, we get an element $[f] \in H^1(X, \mathcal{F})$. We thus define

$$\delta(f) := [f] \in H^1(X, \mathcal{F}).$$

This defines the required group homomorphism δ.

Theorem 5.1.10 (Long exact cohomology sequence)**.** [8] *Let X be a topological space and*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$$

be an exact sequence of sheaves on X. Then there exists a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r, "\varphi^0"] & H^0(X,\mathcal{G}) \arrow[r, "\psi^0"] & H^0(X,\mathcal{H}) \arrow[dll, "\delta"] \\ & H^1(X,\mathcal{F}) \arrow[r, "\varphi^1"'] & H^1(X,\mathcal{G}) \arrow[r, "\psi^1"'] & H^1(X,\mathcal{H}) \end{tikzcd}$$

where δ is as in Construction 5.1.9.

[8] To remove before addition to FoG and add relevant reference.

5.1.2 Applications

We now state and prove three big results, which follows from cohomology l.e.s. quite naturally.

The first result is an immediate corollary of the cohomology l.e.s. good to get the muscles moving, which states what happens when the middle sheaf has no first cohomology.

Proposition 5.1.11. *Let X be a topological space and*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$$

be an exact sequence of sheaves on X. If $H^1(X, \mathcal{G}) = 0$, then

$$H^1(X, \mathcal{F}) \cong \frac{\Gamma(X, \mathcal{H})}{\psi_X(\Gamma(X, \mathcal{G}))}$$

Proof. Write the cohomology l.e.s. (Theorem 5.1.10) and use first isomorphism theorem. □

We next give Dolbeault's theorem which for a Riemann surface calculates first cohomology of structure sheaf and holomorphic 1-forms purely in terms of differentiable functions and differentiable 1 and 2-forms. This is essentially already clear from the first two s.e.s. in Example 5.1.7 and above result.

Theorem 5.1.12 (Dolbeault's theorem)**.** *Let X be a Riemann surface.*

1. *All sheaves $\mathcal{E}_X, \mathcal{E}_X^{1,0}, \mathcal{E}_X^{0,1}$, $\mathcal{E}_X^{(1)}$ and $\mathcal{E}_X^{(2)}$ have first cohomology group 0.*
2. *We have isomorphisms*

$$H^1(X, \mathcal{O}_X) \cong \frac{\Gamma(X, \mathcal{E}_X^{0,1})}{d''_X(\Gamma(X, \mathcal{E}_X))},$$

$$H^1(X, \Omega_X^{(1)}) \cong \frac{\Gamma(X, \mathcal{E}_X^{(2)})}{d_X(\Gamma(X, \mathcal{E}_X^{1,0}))}.$$

Proof. We omit the proof of item 1, for it can be found in Forster, Theorem 12.6 cite[Forster].

By Proposition 5.1.11 and the first two s.e.s. in Example 5.1.7, we need only show that $H^1(X, \mathcal{E}_X) = 0 = H^1(X, \mathcal{E}_X^{1,0})$, which we know to be true from item 1. □

Corollary 5.1.13. *Let $X = B_R(0) \subseteq \mathbb{C}$ be an open ball considered as a Riemann surface. Then $H^1(X, \mathcal{O}_X) = 0$.*

Proof. Need only show that $d''_X : \Gamma(X, \mathcal{E}_X) \to \Gamma(X, \mathcal{E}_X^{0,1})$ is surjective. That is, for any differentiable 1-form of type $(0, 1)$, i.e. $\omega = f d\bar{z}$ on X, we wish to find a differentiable map g on X such that $d''g = \frac{\partial g}{\partial \bar{z}} d\bar{z} = f d\bar{z}$. Indeed, by Dolbeault's lemma for $\mathbb{C}$ (Theorem 4.3.1), we get such a g. □

Remark 5.1.14 (deRham cohomology)**.** Let X be a Riemann surface and denote $H^1_{\mathrm{dR}}(X)$ the deRham cohomology of X, that is,

$$H^1_{\mathrm{dR}}(X) = \frac{\operatorname{Ker}\left(d_X : \mathcal{E}_X^{(1)}(X) \to \mathcal{E}_X^{(2)}(X)\right)}{\operatorname{Im}\left(d_X : \mathcal{E}_X(X) \to \mathcal{E}_X^{(1)}(X)\right)}.$$

We can now easily see by cohomology l.e.s. that $H^1_{\mathrm{dR}}(X)$ is same as $H^1(X, \underline{\mathbb{C}})$.

Theorem 5.1.15 (deRham isomorphism for Riemann surfaces)**.** *Let X be a Riemann surface and $\underline{\mathbb{C}}$ be the constant sheaf associated to field $\mathbb{C}$. Then we have an isomorphism*

$$H^1_{\mathrm{dR}}(X) \cong H^1(X, \underline{\mathbb{C}}).$$

Proof. By Example 5.1.7, 3, we have a s.e.s.

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{E}_X \xrightarrow{d} \mathcal{L}_X \longrightarrow 0.$$

where $H^1(X, \mathcal{E}_X) = 0$ by Dolbeault's theorem (Theorem 5.1.12). By Proposition 5.1.11, it follows that

$$H^1(X, \underline{\mathbb{C}}) \cong \frac{\Gamma(X, \mathcal{L}_X)}{d_X \Gamma(X, \mathcal{E}_X)}$$

where $\Gamma(X, \mathcal{L}_X)$ is the set of all global closed differentiable 1-forms and $d_X \Gamma(X, \mathcal{E}_X)$ is the image of all differentiable functions, that is $\frac{\Gamma(X, \mathcal{L}_X)}{d_X \Gamma(X, \mathcal{E}_X)} =: H^1_{\mathrm{dR}}(X)$, as required. $\square$

5.2 Divisors

5.3 Proof of Riemann-Roch theorem

5.4 Applications