The Facets of Geometry

Algebraic, Arithmetical, Topological, Analytic & Categorical

(Under heavy construction!!)

April 3, 2024

I follow here the footing of thy feet,
That with thy meaning, so I may the rather meet.
-Edmund Spenser

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Part I

The Algebraic Viewpoint

Chapter 1

Foundational Algebraic Geometry

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1.1 A guiding example

Let X be a compact Hausdorff topological space. In this section we would like to portray the main point of scheme theory in the case of space X, that is, one can study the geometry over "base" space completely by studying the algebra of ring of suitable functions over it. In particular, we would like to establish the following result.

Proposition 1.1.0.1. *Let X be a compact Hausdorff topological space. Denote R to be the ring of continuous real-valued functions on X under pointwise addition and multiplication and denote* $\mathrm{mSpec}\,(R)$ *to be the set of maximal ideals of R. Then,*

1. *We have a set bijection:*

$$\mathrm{mSpec}\,(R) \cong X.$$

2. *We have that* $\mathrm{mSpec}\,(R)$ *and X are isomorphic as topological spaces:*

$$\mathrm{mSpec}\,(R) \cong X$$

where $\mathrm{mSpec}\,(R)$ *is given its Zariski topology.*

Proof. 1. Let $x \in X$ be an arbitrary point. Denote $\mathfrak{m}_x := \{f \in R \mid f(x) = 0\}$ to be the vanishing ideal of point x. This ideal is maximal because the quotient $R/\mathfrak{m}_x \cong \mathbb{R}$ via the map $f + \mathfrak{m}_x \mapsto f(x)$. Indeed, it is a valid ring homomorphism and is surjective by virtue of the continuous map constant at a point in $\mathbb{R}$. Moreover, if $f(x) = g(x)$ for $f, g \in R$, then $f - g \in \mathfrak{m}_x$ and hence $f + \mathfrak{m}_x = g + \mathfrak{m}_x$, so it is injective as well. Now consider the function:

$$\begin{aligned} \varphi : X &\to \mathrm{mSpec}\,(R) \\ x &\mapsto \mathfrak{m}_x. \end{aligned}$$

We claim that φ is bijective. To see injectivity, suppose $\mathfrak{m}_x = \mathfrak{m}_y$ for $x, y \in X$. Then, we have that $R/\mathfrak{m}_x = R/\mathfrak{m}_y \cong \mathbb{R}$. This tells us that for each $f \in R$, $f(x) = f(y) \in \mathbb{R}$. Now assume that $x \neq y$. Since X is T_1, therefore $\{x\}, \{y\}$ are two disjoint closed subspaces of X. Then, by **Urysohn's lemma** (we have that X is compact Hausdorff), we get that there exists a continuous $\mathbb{R}$-valued function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$, a contradiction. Hence $x = y$.

Next, we wish to establish the surjectivity of φ. For that, take any maximal ideal $\mathfrak{m} \in \mathrm{mSpec}\,(R)$. We wish to show that there exists $x \in X$ such that $\mathfrak{m} = \mathfrak{m}_x$. Assume to the contrary that for all $x \in X$, there exists $f_x \in \mathfrak{m}$ such that $f_x \notin \mathfrak{m}_x$. Indeed getting a contradiction here will do the job by maximality of $\mathfrak{m}_x$. Now, let $U_x \subseteq X$ be the open set such that $f_x(y) \neq 0$ for each $y \in U_x$. We thus have an open cover $\{U_x\}$ of X and by compactness we get $U_{x_1}, \ldots, U_{x_n}$ covers X. Now, since X is compact Hausdorff, therefore there exists a **partition of unity** subordinate to $\{U_{x_i}\}$. That is, we have $g_i : X \to \mathbb{R}$ where $\mathrm{Supp}\,(g_i) \subseteq U_{x_i}$ such that $1 = g_i f_{x_i} + \cdots + g_n f_{x_n}$. Since $f_{x_i} \in \mathfrak{m}$ for each i, therefore $1 = g_i f_{x_i} + \cdots + g_n f_{x_n} \in \mathfrak{m}$, hence $\mathfrak{m} = R$, a contradiction.

2. □

Complete the pr of Theorem 1.1. Chapter 1.

Remark 1.1.0.2. An important corollary of the above result is that we can actually distinguish between the points of X by looking at maximal ideals of R; for $x, y \in X$, $x \neq y$ if and only if $\mathfrak{m}_x \neq \mathfrak{m}_y$. This is interesting because a fundamental goal of algebraic geometry is to study geometric properties of varieties over an algebraically closed field k and dominant maps between them. A fundamental equivalence tells that this is equivalent to studying the ring of regular functions over such a variety. Moreover, this ring recovers the important topology on the variety (there can be atleast two topologies on the variety if we are in, say $\mathbb{C}$). Hence one motivation to undergo this switch of viewpoint, where we try to do everything algebraically is that 1) we can completely recover the points of the variety and the relevant topology on it and that 2) we have a broad generalization of algebro-geometric techniques and constructions to an arbitrary commutative and unital ring R.

Caution 1.1.0.3. While in the sequel we will encounter spaces which are compact, it would rarely (unless you are interested in Boolean rings) be the case that the spaces will be Hausdorff. However, if one notices the way Hausdorff property is used in the above result, then one can see that if we somehow makes sure that the space X constructed out of a ring R is such that every point of X can be "distinguished" by functions on X in R, then you don't need Hausdorff property. This is precisely what will happen.

1.2 Affine schemes and basic properties

Let us first swiftly give an account of basic global constructions in scheme theory. The foundational philosophy of scheme theory is to handle a space completely by the ring of globally defined *nice* functions on it. This is taken to an unprecedented extreme by the definition of an affine scheme, which tells us that one can even do geometry on the base space by the knowledge of globally defined functions on the base space alone; you can indeed *reconstruct* the base space! So, we begin with a general ring R and construct a topological space $\operatorname{Spec}(R)$. The way we will define its points is by thinking of each point of this base space $\operatorname{Spec}(R)$ as that subset of R, each of whose function becomes zero at a common point. One then sees that these are exactly the prime ideals of R. Hence, the base space $\operatorname{Spec}(R)$ is:

$$\operatorname{Spec}(R) := \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal of } R\}.$$

Next thing we wish to do is to actually get a *space* structure on this constructed base space, that is, a topology on $\operatorname{Spec}(R)$. This is, again, given with the help of the ring R. In particular, we give a topology on $\operatorname{Spec}(R)$ where every closed set is given by the zero locus of collections of functions $S \subseteq R$, that is, $V(S) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \supseteq S\} = \{x \in \operatorname{Spec}(R) \mid f(x) = 0 \ \forall f \in S\}$ where the last equation tells one how to think about the definition of $V(S)$. This is known as Zariski topology on $\operatorname{Spec}(R)$ and is defined by the following:

$$A \subseteq \operatorname{Spec}(R) \text{ is closed } \iff A = V(S) \text{ for some } S \subseteq R.$$

After defining the topology on $\operatorname{Spec}(R)$, one is interested in interested in understanding the set of all *germs* of functions at a point $\mathfrak{p} \in \operatorname{Spec}(R)$. What are germs of functions at a point? Well, heuristically, they are all possible ways a function can *look* different at the given point. So for this, we have to atleast gather all those functions in R which takes different values at point $\mathfrak{p} \in \operatorname{Spec}(R)$. Clearly this is given by the quotient domain $R/\mathfrak{p}$. Now from this, we construct the *residue field* of $\operatorname{Spec}(R)$ at point $\mathfrak{p}$, denoted $\kappa(\mathfrak{p}) := (R/\mathfrak{p})_{\langle 0 \rangle}$, that is, the fraction field of domain $R/\mathfrak{p}$. What does this $\kappa(\mathfrak{p})$ denotes geometrically? Well, it denotes the field of all different values a function can take at point $\mathfrak{p} \in \operatorname{Spec}(R)$. Now, if that is the case, then one sees that if one takes any function $f \in R$, then "evaluating" f at $\mathfrak{p}$ should yield a point $f(\mathfrak{p})$ in $\kappa(\mathfrak{p})$. Indeed, we have the natural quotient maps:

$$R \to R/\mathfrak{p} \to \kappa(\mathfrak{p}).$$

So one should see

$\kappa(\mathfrak{p})$ *as the field of possible values that a function* $f \in R$ *can take at point* $\mathfrak{p}$.

However, we have not yet made the set of germs at a point $\mathfrak{p}$. The relation between two functions of having equal germs on R at a point $\mathfrak{p}$ is given by the heuristic that $f, g \in R$ should become equal in some open neighborhood around $\mathfrak{p}$. Since we have a topology on $\mathrm{Spec}\,(R)$, so one can actually do this formally. One will then see this that the set of all germs at point $\mathfrak{p}$ are actually all rational functions of R definable at $\mathfrak{p}$, that is, heuristically, f/g with $g(\mathfrak{p}) \neq 0$ for $f, g \in R$. This in our language turns out to be all the symbols of the form f/g with $g \notin \mathfrak{p}$. This is exactly the local ring $R_{\mathfrak{p}}$, the localization of the ring R (seen as ring of functions over $\mathrm{Spec}\,(R)$) at the point $\mathfrak{p} \in \mathrm{Spec}\,(R)$. So

germs of functions of R *at* $\mathfrak{p}$ *is* $R_{\mathfrak{p}}$.

We will expand more on this when we will talk about the structure sheaf of $\mathrm{Spec}\,(R)$.

Let us now see a basic but important dictionary between the topology of space $\mathrm{Spec}\,(R)$ and the algebra of ideals of R:

Lemma 1.2.0.1. *Let R be a ring. We then have the following:*

1. *If $\mathfrak{a}$, $\mathfrak{b}$ are two ideals of R, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*
2. *If $\{\mathfrak{a}_n\}$ is a collection of ideals of R, then $V\left(\sum_n \mathfrak{a}_n\right) = \bigcap_n V(\mathfrak{a}_n)$.*
3. *If $\mathfrak{a}, \mathfrak{b}$ are two ideals of R, then $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{a} \supseteq \sqrt{b}$.*

Proof. 1. First, let us see that $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$. Take any $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. Suppose $p \notin V(\mathfrak{a})$ and $\mathfrak{p} \notin V(\mathfrak{b})$. Then there exists $f \in \mathfrak{a}$, $g \in \mathfrak{b}$ such that $fg \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Thus, $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, a contradiction in both cases. Second, it is easy to see that $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$ as if either $p \supseteq \mathfrak{a}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, then since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}$, therefore $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$.
2. Let $\mathfrak{p} \supseteq \sum_n \mathfrak{a}_n$. Since ideals are abelian groups so the sum contains each $\mathfrak{a}_n$, hence $\mathfrak{p} \supseteq \mathfrak{a}_n$ for each n, and so $\mathfrak{p} \in \bigcap_n V(\mathfrak{a}_n)$. Conversely, if $\mathfrak{p} \supseteq \mathfrak{a}_n$ for each n, then $\mathfrak{p} = \sum_n \mathfrak{p} \supseteq \sum_n \mathfrak{a}_n$.
3. (L $\implies$ R) Since each prime ideal containing $\mathfrak{a}$ also contains $\mathfrak{b}$, therefore the intersection of all prime ideals containing $\mathfrak{a}$ will contain the intersection of all prime ideals containing $\mathfrak{b}$. (R $\implies$ L) Take any prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$. Since $\sqrt{a} \supseteq \sqrt{b}$, therefore $\mathfrak{p} \supseteq \mathfrak{b}$. □

1.2.1 Topological properties of $\mathrm{Spec}\,(R)$

Let us begin by an algebraic characterization of irreducible closed subspaces of $\mathrm{Spec}\,(R)$.

Lemma 1.2.1.1. *Let R be a ring and $X \hookrightarrow \mathrm{Spec}\,(R)$ be a closed subspace. Then the following are equivalent:*

1. *X is irreducible.*
2. *There is a unique point $\mathfrak{p} \in \mathrm{Spec}\,(R)$ such that $X = V(\mathfrak{p})$.*

One calls the point $\mathfrak{p}$ the generic point of the irreducible closed subspace X[1].

Proof. (1. $\Rightarrow$ 2.) Since X is closed therefore $X = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of R. If we assume that $X \neq V(\mathfrak{p})$ for each prime $\mathfrak{p} \subseteq R$, then this holds true for points $\mathfrak{p} \in X$ as well. Hence take $\mathfrak{p} \in X$ and consider the proper closed subset $V(\mathfrak{p}) \subsetneq X$. Let $\mathfrak{q} \notin V(\mathfrak{p})$. Then, $V(\mathfrak{q}) \subsetneq X$ as well. Hence we get that $V(\mathfrak{p}) \cup V(\mathfrak{q}) = V(\mathfrak{a})$, which stands in contradiction to the fact that X is irreducible. Hence there exists a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $X = V(\mathfrak{p})$. Uniqueness is quite clear.
(2. $\Rightarrow$ 1.) Suppose $Y = V(\mathfrak{a})$ and $Z = V(\mathfrak{b})$ are two closed subspaces of $X = V(\mathfrak{p})$ such that $X = Y \cup Z = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$ (Lemma 1.2.0.1). Assume that Y, Z are proper inside X. Then, there are two points $\mathfrak{q}_1 \in Y \setminus Z$ and $\mathfrak{q}_2 \in Z \setminus Y$. Algebraically, this is equivalent to saying that $\mathfrak{q}_1 \supseteq \mathfrak{a}$, $\mathfrak{q}_1 \not\supseteq \mathfrak{b}$ and $\mathfrak{q}_2 \supseteq \mathfrak{b}$, $\mathfrak{q}_2 \not\supseteq \mathfrak{a}$. It follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2$ is also a prime ideal which contains $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. Since $X = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{p}) \ni \mathfrak{p}$, hence it follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2 \supseteq \mathfrak{p}$ as it already contains $\mathfrak{a}\mathfrak{b}$. Thus $\mathfrak{q}_1 \cap \mathfrak{q}_2 \in V(\mathfrak{a}) \cap V(\mathfrak{b}) \subseteq V(\mathfrak{p})$. Since $V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b})$, hence it follows that $\mathfrak{q}_1 \cap \mathfrak{q}_2 \supseteq \mathfrak{a}, \mathfrak{b}$, which implies in particular that $\mathfrak{q}_1 \supseteq \mathfrak{a}, \mathfrak{b}$, a contradiction. $\square$

Remark 1.2.1.2. The main idea of the above proof has been to first translate the topological condition to algebraic, and then using the critical observation that the closed subspace $V(\mathfrak{p})$ contains point $\mathfrak{p}$ itself.

A simple corollary of above gives all closed points of an affine scheme.

Lemma 1.2.1.3. *Let R be a ring. Then*

$$\{\textit{Closed points of } \operatorname{Spec}(R)\} \cong \{\textit{Maximal ideals of } R.\}$$

Proof. Follows immediately from Lemma 1.2.1.1. $\square$

Let us next observe a simple but important observation about topology of $\operatorname{Spec}(R)$.

Lemma 1.2.1.4. *Let R be a ring. For $f \in R$, define $\operatorname{Spec}(R)_f := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \notin \mathfrak{p}\}$. Then,*

1. *$\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)$ is an open set and such open sets form a basis of the Zariski topology on $\operatorname{Spec}(R)$.*
2. *$\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)_g$ if and only if $f \in \sqrt{Rg}$.*

Proof. 1. Clearly $\operatorname{Spec}(R)_f = \operatorname{Spec}(R) \setminus V(f)$ where we know that $V(f) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid f \in \mathfrak{p}\}$. Hence X_f is open. It is also clear that if $U \subseteq \operatorname{Spec}(R)$ is open, then $\operatorname{Spec}(R) \setminus U = V(\mathfrak{a})$ is closed and hence $U = \bigcup_{f \in \mathfrak{a}} \operatorname{Spec}(R)_f$. Further, $\operatorname{Spec}(R) = \operatorname{Spec}(R)_1$ and $\emptyset = \operatorname{Spec}(R)_0$.

2. This follows from the following equivalences. Let $\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)_g$, then we get the following (we implicitly use Hilbert Nullstellensatz)

$$\operatorname{Spec}(R)_f \hookrightarrow \operatorname{Spec}(R)_g \iff f(\mathfrak{p}) \neq 0 \implies g(\mathfrak{p}) \neq 0 \iff g(\mathfrak{p}) = 0 \implies f(\mathfrak{p}) = 0 \iff V(g) \subseteq V(f)$$
$$\iff \sqrt{Rg} \supseteq \sqrt{Rf} \supseteq Rf \iff f \in \sqrt{Rg}.$$

This completes the proof. $\square$

[1] Such spaces where every irreducible closed set has a unique generic point are called sober spaces.

Next we observe the equivalent formulation of partitions of unity in the context of algebra.

Lemma 1.2.1.5. *Let R be a ring. Then,*

1. *If $U \hookrightarrow \operatorname{Spec}(R)$ is any open set given by $U = \bigcup_{f\in S} \operatorname{Spec}(R)_f$ for some subset $S \subseteq R$, then*
$$\operatorname{Spec}(R) \setminus U = V\left(\sum_{f\in S} Rf\right).$$
2. *$\operatorname{Spec}(R) = \bigcup_{f\in S} \operatorname{Spec}(R)_f$ for some $S \subseteq R$ if and only if the ideal of R generated by S is the whole of R.*

Proof. 1. Let $U \hookrightarrow \operatorname{Spec}(R)$ be an open set. Then, $\mathfrak{p} \in \operatorname{Spec}(R) \setminus U \iff \mathfrak{p} \notin U \iff \forall f \in S,\ \mathfrak{p} \notin \operatorname{Spec}(R)_f \iff \forall f \in S,\ f \in \mathfrak{p} \iff \mathfrak{p} \supseteq S \iff \mathfrak{p} \in V(S)$.
2. Follows from 1. □

We next have an interesting observation that $\operatorname{Spec}(R)$ are always quasicompact[2].

Lemma 1.2.1.6. *Let R be a ring. Then $\operatorname{Spec}(R)$ is quasicompact.*

Proof. Take any arbitrary basic open cover $\bigcup_{f\in S} \operatorname{Spec}(R)_f$ for some $S \subseteq R$. Then by Lemma 1.2.1.5, 2, we get that $\sum_{f\in S} Rf \ni 1$ and hence there are $f_1, \dots, f_n \in S$ such that $g_1 f_1 + \dots g_n f_n = 1$ for some $g_i \in R$. Hence $\operatorname{Spec}(R) \setminus \bigcup_{i=1}^n = V(f_1, \dots, f_n) = V(R) = \emptyset$. □

Next, we see the topological effects on space $\operatorname{Spec}(R)$ of Noetherian hypothesis on ring R. In particular, we see that the space $\operatorname{Spec}(R)$ itself becomes *noetherian topological space*, that is, it's closed sets satisfies descending chain condition.

Lemma 1.2.1.7. *Let R be a ring. If R is noetherian, then $\operatorname{Spec}(R)$ is noetherian.*

Proof. Use $V(-)$ and $I(-)$, where $I(Y) = \{f \in R \mid f \in \mathfrak{p} \forall \mathfrak{p} \in Y\}$. Rest is trivial. □

We next discuss few things about the irreducible subsets of a closed set of $\operatorname{Spec}(R)$. Let $F \hookrightarrow \operatorname{Spec}(R)$ be a closed subset. Then we can contemplate irreducible subsets of F. Clearly, each irreducible subset has to be in a maximal irreducible subset, which are called *irreducible components* of $\operatorname{Spec}(R)$. We have few basic observations about irreducible components

Lemma 1.2.1.8. *Let R be a ring and F be a closed subset of $\operatorname{Spec}(R)$. Then,*

1. *Each irreducible component of F is closed.*
2. *There are only finitely many irreducible components of F.*
3. $$\{\textit{Irreducible components of } \operatorname{Spec}(R)\} \cong \{\textit{Closed sets } V(\mathfrak{p}),\ \mathfrak{p} \textit{ is minimal prime}\}.$$

[2] it is customary in algebraic geometry to call the topological compactness as quasi-compactness; compactness in algebraic geometry historically means Hausdorff *and* topological compactness.

Proof. Statements 1. follows from Lemma 1.2.1.1. Statement 2. follows from Lemma 1.2.1.6. For 3. we see that if there exists a prime $\mathfrak{p}' \subsetneq \mathfrak{p}$, then $V(\mathfrak{p}') \supsetneq V(\mathfrak{p})$. But by Lemma 1.2.1.1, $V(\mathfrak{p})$ is no longer an irreducible component, a contradiction. □

Note that we are already in a position to prove some algebraic statements using topological arguments, as the following lemma shows.

Lemma 1.2.1.9. *Let A be a ring and let $a_1, \dots, a_n \in A$ generate the unit ideal in A. Then for all $m > 0$, the collection $a_1^m, \dots, a_n^m \in A$ also generates the unit ideal in A.*

Proof. From Lemma 1.2.1.6, 2, it follows that $\{D(a_i)\}_{i=1,\dots,n}$ covers $\mathrm{Spec}\,(A)$. Since for any $a \in A$, the basic open $D(a) \subseteq \mathrm{Spec}\,(A)$ is equal to $D(a^m)$ as a prime $\mathfrak{p}$ doesn't contain a if and only if it doesn't contain any of its power. Consequently, we get that $\{D(a_i^m)\}_{i=1,\dots,n}$ also forms a basic open cover of $\mathrm{Spec}\,(A)$. An application of Lemma 1.2.1.6, 2 again proves the result. □

1.2.2 The structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$

The next important thing we want to consider on $\mathrm{Spec}\,(R)$ is a sheaf of suitable nice functions over it. This sheaf will be of utmost importance as it will not be treated as an additional structure, but will be an integral part (in-fact, the most important part) of the definition of an affine scheme.

The question now is, *what are* nice functions over $\mathrm{Spec}\,(R)$ whose sheaf we should take. We turn to classical algebraic varieties for that (one may skip the following if he/she find himself/herself to be brave enough to face the abstraction of the structure sheaf). See Section 1.5 for more details.

Example 1.2.2.1. (*Structure sheaf of an algebraic variety*) Let k be an algebraically closed field. An important aspect of varieties is their morphism. We will display this only in the affine case. Let X, Y be two affine varieties. To define a morphism between X and Y, we would first need to understand the notion of *regular functions* over any variety X. A function $\varphi : X \to k$ is said to be regular if it is locally rational. That is, for each $p \in X$, there exists an open set $U \ni p$ of X and there exists two polynomials $f, g \in k[x_1, \dots, x_n]$ such that $g(q) \neq 0 \forall q \in U$ and $\varphi|_U = f/g$. It then follows that a regular function is continuous when X and k are equipped with its Zariski topology (Lemma 3.1, [**??**] [Hartshorne]). We now define morphism of affine varieties.

A function $\varphi : X \to Y$ is said to be a morphism of varieties if
1. $\varphi : X \to Y$ is continuous,
2. for each open set $V \subseteq Y$ and a regular map $f : V \to k$, the map $f \circ \varphi$ as below

$$\begin{tikzcd} & V \arrow[dr, "f"] & \\ \varphi^{-1}(V) \arrow[ur, "\varphi"] \arrow[rr, "f\circ\varphi"'] & & k \end{tikzcd}$$

is also a regular map.

Hence the main part of the data of a variety is the locally defined regular maps. This is what we will take as our motivation in defining the structure sheaf over $\operatorname{Spec}(R)$, as this example tells us to take care of these local functions to the base field. A question that may arise from this discussion is how are we going to define a regular map from an open set $U \hookrightarrow \operatorname{Spec}(R)$ when we don't even have a field. The answer is, as we discussed previously, to work with residue field at a point instead.

We now start to define the structure sheaf of $\operatorname{Spec}(R)$. First, let us give the following lemma, which reduces the burden of construction only to basis elements of $\operatorname{Spec}(R)$.

Lemma 1.2.2.2. *Let X be a topological space and $\mathcal{B}$ be a basis. Let F be an assignment over sets of $\mathcal{B}$ which satisfies sheaf conditions for it. Then, F extends to a sheaf $\mathcal{F}$ over X.*

Proof. The main observation here is that we can find the stalk of $\mathcal{F}$ at each point x just by the knowledge of F, because of the basis $\mathcal{B}$. Take any point $x \in X$. We see that we can get the stalk $\mathcal{F}_x$ as follows:

$$\mathcal{F}_x := \varinjlim_{x \in B \in \mathcal{B}} F(B).$$

Once we have the stalks, we can define the sections of $\mathcal{F}$ quite easily as follows. Let $U \subseteq X$ be an open set. Then $\mathcal{F}(U)$ is defined to be the subset of $\prod_{x \in U} \mathcal{F}_x$ of those elements (s_x) where there exists a basic open cover $\{B_i\}$ of U and there exists elements $s_i \in F(B_i)$ such that $s_x = (s_i)_x$ for each $x \in B_i$. One can check that this satisfies the conditions of a sheaf. $\square$

Construction 1.2.2.3. (*The* $\mathcal{O}_{\operatorname{Spec}(R)}$) Let R be a ring. By virtue of Lemma 1.2.2.2, we will define $\mathcal{O}_{\operatorname{Spec}(R)}$ only on basic open sets of the form $\operatorname{Spec}(R)_f$. Let $X := \operatorname{Spec}(R)$. Motivated by Example 1.2.2.1, take a basic open set $X_f \hookrightarrow X$ for some $f \in R$ and then we wish to consider *rational functions* over X_f. This means those functions of the form g/h for $g, h \in R$ such that $h(\mathfrak{p}) \neq 0 \forall \mathfrak{p} \in X_f$. This is equivalent to demanding that $h \notin \mathfrak{p} \forall \mathfrak{p} \in X_f$, that is, $X_h \supseteq X_f$. This is again equivalent to stating that $f \in \sqrt{Rh}$ by Lemma 1.2.1.4, 2. Hence $f^n = ah$ for some $n \in \mathbb{N}$ and $a \in R$. Thus, we see that the notion of rational functions over X_f is equivalent to all functions of the form g/f^n where $g \in R$ and $n \in \mathbb{N}$. Commutative algebra has an apt name for this, that is, the localization of R at f denoted by $R_f := \{a/f^n \mid a \in R,\ n \in \mathbb{N}\}$ which is again a ring by natural operation on fractions (see Special Topics, **??**). Thus, we should define the sections over X_f as:

$$\mathcal{O}_X(X_f) := R_f.$$

We would not verify the sheaf axioms here as it is a tedious but straightforward calculation. The sheaf $\mathcal{O}_X$ thus formed is called the structure sheaf on the space X. One should think of the sheaf $\mathcal{O}_X$ as natural as the ring R itself. In particular we will see in the next section that it indeed is the case.

Next, we would like to see the stalks of this sheaf $\mathcal{O}_X$. To understand this, we would have to understand the maps on sections induced by $X_f \hookrightarrow X_g$. As we saw earlier, this is equivalent to stating that $f^n = ag$ for some $n \in \mathbb{N}$ and $a \in R$. Hence, the induced map on sections are the restriction maps of the sheaf and is given by

$$\begin{aligned} \rho_{X_g, X_f} : R_g = \mathcal{O}_X(X_g) &\longrightarrow \mathcal{O}_X(X_f) = R_f \\ b/g^m &\longmapsto ba^m/a^m g^m = ba^m/f^{nm}. \end{aligned}$$

We are now ready to calculate the stalk. Take any point $x \in X$. The stalk becomes:

$$\begin{aligned} \mathcal{O}_{X,x} &:= \varinjlim_{x \in X_f} \mathcal{O}_X(X_f) \\ &= \varinjlim_{x \in X_f} R_f \\ &= \varinjlim_{f \notin x} R_f \\ &= R_x \end{aligned}$$

where the last equality follows from a small colimit calculation (which should really be thought of as a definition). Hence $\mathcal{O}_X$ is a sheaf whose stalks are local rings. So we have a complete description of the sheaf $\mathcal{O}_X$ when $X = \operatorname{Spec}(R)$.

We finally define an affine scheme.

Definition 1.2.2.4. (**Affine scheme**) Let R be a ring. Then the pair $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ is called an affine scheme.

Remark 1.2.2.5. (*Evaluation of functions*) Let $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ be an affine scheme. As noted earlier, we now see how all rational functions over $\operatorname{Spec}(R)$ are exactly the elements of R. In particular, since $\Gamma(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}) = R_1 = R$. Hence if we interpret $\mathcal{O}_{\operatorname{Spec}(R)}$ as the sheaf of regular maps over $\operatorname{Spec}(R)$, then R itself appears as the globally defined regular maps.

Now take global map $f \in R$ and any point $\mathfrak{p} \in \operatorname{Spec}(R)$. We can "evaluate" f at $\mathfrak{p}$ via the following composite (note that $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong (R_\mathfrak{p})_\mathfrak{o}$, the last one is the fraction field of $R_\mathfrak{p}$ obtained by localizing at 0 ideal):

$$\Gamma(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)}) \longrightarrow \mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}} \longrightarrow \kappa(\mathfrak{p})$$

where the first map on the left is the inclusion into the direct limit and the map on right is the natural quotient map. Algebraically, we have the following maps

$$R \longrightarrow R_\mathfrak{p} \longrightarrow R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$$

given by

$$f \longmapsto \frac{f}{1} \longmapsto \frac{f}{1} + \mathfrak{p}R_\mathfrak{p},$$

where $f/1 + \mathfrak{p}R_\mathfrak{p}$ denotes the class of all those functions in the stalk $\mathcal{O}_{\operatorname{Spec}(R),\mathfrak{p}} = R_\mathfrak{p}$ which takes same value at $\mathfrak{p}$ as f does.

For completeness' sake, we give a description of the section of the sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ on any open set $U \subseteq \mathrm{Spec}\,(R)$.

Lemma 1.2.2.6. *Let R be a ring and $(\mathrm{Spec}\,(R), \mathcal{O}_{\mathrm{Spec}(R)})$ the associated affine scheme. Let $U \subseteq \mathrm{Spec}\,(R) =: X$ be an open set. Then,*

$$\mathcal{O}_X(U) = \left\{ (s_\mathfrak{p}) \in \prod_{\mathfrak{p} \in U} R_\mathfrak{p} \mid \forall \mathfrak{p} \in U, \exists \textit{ basic open } X_g \ni \mathfrak{p} \;\&\; f/g^n \in R_g \textit{ s.t. } s_\mathfrak{q} = f/g^n \forall \mathfrak{q} \in X_g \right\}.$$

More concretely, we have

$$\mathcal{O}_X(U) = \left\{ s : U \to \textstyle\coprod_{\mathfrak{p} \in U} R_\mathfrak{p} \mid \forall \mathfrak{p} \in U,\, s(\mathfrak{p}) \in R_\mathfrak{p} \;\&\; \exists \textit{ open } \mathfrak{p} \in V \subseteq U \;\&\; f,g \in R \textit{ s.t. } \forall \mathfrak{q} \in V,\, g \notin \mathfrak{q} \;\&\; s(\mathfrak{q}) = f/g \right\}.$$

Proof. Follows from Lemma 1.2.2.2 and Construction 1.2.2.3. □

Ring morphisms and $\mathrm{Spec}\,(-)$

We now discuss some properties of ring morphisms and the associated map of affine schemes.

Lemma 1.2.2.7. [3] *Let A be a ring and $f \in A$. Then, $D(f) \subseteq \mathrm{Spec}\,(A)$ is empty if and only if f is nilpotent.*

Proof. Both sides follow immediately from the Lemma 23.1.2.9. □

We further obtain the following two results which corresponds to what happens on the level of sheaves.

Proposition 1.2.2.8. [4] *Let $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ be two affine schemes and $\varphi : A \to B$ be a morphism of rings.*

1. *The ring map $\varphi : A \to B$ is injective if and only if the corresponding map of schemes $f : Y \to X$ yields injective map of structure sheaves, that is, $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective.*
2. *If $\varphi : A \to B$ is injective, then $f : Y \to X$ is dominant*[5]*.*
3. *The ring map $\varphi : A \to B$ is surjective if and only if the corresponding map of schemes $f : Y \to X$ is a closed immersion.*

Proof. 1. (L $\Rightarrow$ R) It suffices to show that $f^\flat$ is an injective map over basic opens of X. Pick any $g \in A$ and consider the basic open $D(g) \subseteq X$. We wish to show that the map

$$f^\flat_{D(g)} : \mathcal{O}_X(D(g)) \longrightarrow \mathcal{O}_Y(f^{-1}(D(g)))$$

is an injective homomorphism. Indeed, we first observe that $\mathcal{O}_X(D(g)) \cong A_g$ and $f^{-1}(D(g)) = D(\varphi(g))$, so that $\mathcal{O}_Y(D(\varphi(g))) \cong B_{\varphi(g)}$. It follows that the map $f^\flat_{D(g)} : A_g \to B_{\varphi(g)}$ is the localization map

$$\begin{aligned} \varphi_g : A_g &\longrightarrow B_{\varphi(g)} \\ \frac{a}{g^n} &\longmapsto \frac{\varphi(a)}{\varphi(g)^n}. \end{aligned}$$

[3] Exercise II.2.18, a of Hartshorne.
[4] Exercise II.2.18 b,c,d of Hartshorne.
[5] that is, f has dense image.

We wish to show that the above map is injective. If $\varphi(a)/\varphi(g)^n = 0$, then for some $k \in \mathbb{N}$ we have $\varphi(g)^k\varphi(a) = 0$. It follows by injectivity of φ that $g^k a = 0$ in A. Consequently, we can write

$$\frac{a}{g^n} = \frac{ag^k}{g^{n+k}} = 0.$$

(R $\Rightarrow$ L) As a sheaf map is injective if and only if the kernel sheaf is zero (Theorem 27.3.0.7), where the latter is equivalent to the fact that every map on sections is injective. Consequently, over X, we get

$$f^\flat_X : \Gamma(\mathcal{O}_X, X) \longrightarrow \Gamma(\mathcal{O}_Y, Y)$$

Since $\Gamma(\mathcal{O}_X, X) \cong A$ and $\Gamma(\mathcal{O}_Y, Y) \cong B$, and the map $f^\flat_X : A \to B$ is just φ itself, therefore we are done.

2. We wish to show that for any basic non-empty open $D(g) \subseteq X$ for $g \in A$, the intersection $D(g) \cap f(Y)$ is non-empty. We have the following equalities:

$$\begin{aligned} D(g) \cap f(Y) &= \{\mathfrak{p} \in X \mid \mathfrak{p} \in f(Y) \;\&\; g \notin \mathfrak{p}\} \\ &= \{\varphi^{-1}(\mathfrak{q}) \in X \mid \mathfrak{q} \in Y,\ g \notin \varphi^{-1}(\mathfrak{q})\} \\ &= \{\varphi^{-1}(\mathfrak{q}) \in X \mid \mathfrak{q} \in Y,\ \varphi(g) \notin \mathfrak{q}\} \\ &= f(D(\varphi(g))). \end{aligned}$$

Conequently, $D(g) \cap f(Y)$ is non-empty if and only if $D(\varphi(g))$ is non-empty, which in turn implies by Lemma 1.2.2.7 that $D(g) \cap f(Y)$ is non-empty if and only if $\varphi(g)$ is not nilpotent. As g is not nilpotent because $D(g)$ is not empty, therefore $\varphi(g)$ is not nilpotent as φ is injective.

3. (L $\Rightarrow$ R) Let $\varphi : A \to B$ be surjective and $I \leq A$ be the kernel. We wish to show that $f : Y \to X$ is a closed immersion. For that, we first need to show that f is a topological closed immersion, that is its image is closed and is homeomorphic to it. We claim that $f(Y) = V(I) \subseteq X$. Indeed, for any $\varphi^{-1}(\mathfrak{q}) \in f(Y)$, we have that $I \subseteq \varphi^{-1}(\mathfrak{q})$. Thus, $f(Y) \subseteq V(I)$. Conversely, for any $\mathfrak{p} \in V(I)$, as φ is surjective and $\mathfrak{p}$ contains I, therefore $\varphi(\mathfrak{p}) \in Y$ is a prime ideal such that $\varphi^{-1}(\varphi(\mathfrak{p})) = \mathfrak{p}$, so that $\mathfrak{q} = \varphi(\mathfrak{p}) \in Y$ is such that $f(\varphi(\mathfrak{p})) = \mathfrak{p}$, hence $\mathfrak{p} \in f(Y)$.

Next, we wish to show that f is homeomorphic to its image. It suffices to show that $f : Y \to f(Y)$ is a closed mapping. But this is immediate by the fact that a surjective map $\varphi : A \to B$ with kernel I induces an order preserving isomorphism of ideals of A containing I and ideals of B by mapping ideals of B to those of A containing I via φ^{-1}. Alternatively, one can see that $A/I \cong B$ and $\operatorname{Spec}(A/I) \cong V(I) = f(Y)$, therefore application of $\operatorname{Spec}(-)$ functor would do the job.

Next, we wish to show that $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. We can check this on a basis of X. Let $D(g) \subseteq X$ for some $g \in A$. Indeed, for $t \in (f_*\mathcal{O}_Y)(D(g)) = \mathcal{O}_Y(D(\varphi(g)) \cong B_{\varphi(g)}$, we wish to find an open covering of $D(g)$ say U_i and $s_i \in \mathcal{O}_X(U_i)$ such that $f^\flat_{U_i}(s_i) = t|_{U_i}$ for each i. Indeed, the open set $D(g)$ as its own covering will suffice here as $\mathcal{O}_X(D(g)) \cong A_g$

and the map $f^\flat_{D(g)} = \varphi_g : A_g \to B_{\varphi(g)}$. As φ is surjective, therefore for $t = b/\varphi(g)^n \in B_{\varphi(g)}$, we obtain $a \in A$ such that $\varphi(a) = b$ and thus a/g^n is mapped by φ_g to $b/\varphi(g)^n$, as required.

(R $\Rightarrow$ L) Let $f : Y \to X$ be a closed immersion. We wish to show that $\varphi : A \to B$ is surjective. Pick $b \in B$. We wish to show that there exists $a \in A$ such that $\varphi(a) = b$. As the sheaf map $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, therefore there exists a basic open covering (which will be finite by quasi-compactness of affine schemes, Lemma 1.2.1.6) namely $\{D(a_i)\}_{i=1,\dots,n}$ of X together with sections $s_i \in \mathcal{O}_X(D(a_i))$ such that $f^\flat_{D(a_i)}(s_i) \in \mathcal{O}_Y(f^{-1}(D(a_i)))$ is the restriction of $b \in \Gamma(\mathcal{O}_Y, Y)$ to $D(\varphi(a_i))$, namely $\rho_{X,D(\varphi(a_i))}(b)$. As we have $\mathcal{O}_X(D(a_i)) \cong A_{a_i}$, $\mathcal{O}_Y(f^{-1}(D(a_i))) = \mathcal{O}_Y(D(\varphi(a_i))) \cong B_{\varphi(a_i)}$ and that the restriction $\rho_{Y,D(\varphi(a_i))} : \Gamma(\mathcal{O}_X, X) \to \mathcal{O}_X(D(a_i))$ is just the natural localization map $A \to A_{a_i}$, therefore we may identify $s_i = \frac{c_i}{a_i^{k_i}} \in A_{a_i}$ and $\rho_{X,D(a_i)}(b) = \frac{b}{1} \in B_{\varphi(a_i)}$. Consequently, we have for each $i = 1, \dots, n$ the following equation in $B_{\varphi(a_i)}$

$$\frac{b}{1} = \frac{\varphi(c_i)}{\varphi(a_i)^{k_i}}.$$

It follows that we obtain an equation of the form

$$\varphi(a_i^{m_i})b = \varphi(c_i a^{l_i})$$

for some $m_i, l_i \geq 0$. Taking $M = \max_i m_i$, we obtain

$$\varphi(a_i^m)b = \varphi(d_i) \tag{$*$}$$

for some $d_i \in A$.

, 2, the collection $\{a_i\}_{i=1,\dots,n}$ generates the unit ideal in A. By Lemma 1.2.1.9, it follows that the collection $\{a_i^m\}_{i=1,\dots,n}$ also generates the unit ideal in A. Consequently, we have $r_1 a_1^m + \dots + r_n a_n^m = 1$ for some $r_i \in A$. Using this in $(*)$, we yield

$$b = \varphi\left(\sum_{i=1}^n r_i d_i\right),$$

as required[6]. $\square$

1.2.3 $\mathcal{O}_{\mathrm{Spec}(R)}$-modules

As we pointed out in Construction 1.2.2.3, the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ should really be thought of as natural as the ring R itself. This way of thought will be justified in this section, where we will see that, just like we can understand a ring by understanding the category of R-modules, we can understand the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ by understanding the category of soon to be constructed $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

Let R be a ring and M be an R-module. Just like we underwent a "geometrification" to go from ring R (algebra) to the locally ringed space $\mathrm{Spec}\,(R)$ (geometry), we will also "geometrify" the notion of an R-module. This will yield us a sheaf $\widetilde{M}$ over $\mathrm{Spec}\,(R)$.

[6]Note that in the whole proof, we didn't even required the fact that $f : Y \to X$ is also a topological closed immersion!

Definition 1.2.3.1. ($\widetilde{M}$) Let R be a ring and M be an R-module. The following presheaf on $X := \operatorname{Spec}(R)$ generated by the following definition on basic opens

$$X_f \longmapsto \widetilde{M}(X_f) := M_f = M \otimes_R R_f$$

and restrictions given by

$$(X_f \hookrightarrow X_g) \longmapsto M \otimes_R R_g \overset{\mathrm{id}\otimes\rho_{X_g,X_f}}{\to} M \otimes_R R_f$$

defines a unique sheaf on $\operatorname{Spec}(R)$ corresponding to R-module M denoted $\widetilde{M}$.

The above construction gives the sheaf $\widetilde{M}$ over R a structure of an $\mathcal{O}_{\operatorname{Spec}(R)}$-module, that is, a sheaf $\mathcal{F}$ of abelian groups where for each open $U \subseteq \operatorname{Spec}(R)$ the group $\mathcal{F}(U)$ is a $\mathcal{O}_{\operatorname{Spec}(R)}(U)$-module. Since $\widetilde{M}(X_f) = M \otimes_R R_f$ is an $\mathcal{O}_X(X_f) = R_f$-module, therefore $\widetilde{M}$ are basic examples of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules.

A map $\eta : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules is just a sheaf morphism where for each inclusion $U \hookrightarrow V$ of $\operatorname{Spec}(R)$, we get that the following commutes

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\eta_V"] \arrow[d] & \mathcal{G}(V) \arrow[d] \\
\mathcal{F}(U) \arrow[r, "\eta_U"'] & \mathcal{G}(U)
\end{tikzcd}$$

where the top horizontal map is a $\mathcal{O}_{\operatorname{Spec}(R)}(V)$-module homomorphism, bottom horizontal is a $\mathcal{O}_{\operatorname{Spec}(R)}(U)$-module homomorphism and the verticals are the restriction map of sheaves $\mathcal{F}$ and $\mathcal{G}$, which are also module homomorphisms w.r.t. $\mathcal{O}_{\operatorname{Spec}(R)}(V) \to \mathcal{O}_{\operatorname{Spec}(R)}(U)$. The latter has the following meaning. If M is an R-module and N is an S-module, then a map $\phi : M \to N$ is a module homomorphism w.r.t $f : R \to S$ if $\phi(r \cdot m) = f(r) \cdot \phi(m)$.

We thus get a functor

$$\begin{aligned}
\widetilde{-} : \mathbf{Mod}(R) &\longrightarrow \mathbf{Mod}(\mathcal{O}_{\operatorname{Spec}(R)}) \\
M &\longmapsto \widetilde{M} \\
f : M \to N &\longmapsto \tilde{f} : \widetilde{M} \to \widetilde{N}
\end{aligned}$$

where $\tilde{f}_{X_f} : M_f \to N_f$ is given by localization. We may denote $\widetilde{\mathbf{Mod}}(\mathcal{O}_{\operatorname{Spec}(R)}) \hookrightarrow \mathbf{Mod}(\mathcal{O}_{\operatorname{Spec}(R)})$ to be the full subcategory of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules of the form $\widetilde{M}$.

An explicit form of the sheaf $\widetilde{M}$ can be obtained by expanding the definition of the sheaf we obtain from it's definition on the basis.

Lemma 1.2.3.2. *Let M be an R-module and consider the associated $\mathcal{O}_{\operatorname{Spec}(R)}$-module $\widetilde{M}$. For any open $U \subseteq \operatorname{Spec}(R)$, we have*

$$\widetilde{M}(U) \cong \left\{ s : U \to \coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}} \mid \forall \mathfrak{p}\in U,\ s(\mathfrak{p})\in M_{\mathfrak{p}}\ \&\ \exists \text{ open } \mathfrak{p}\in V\subseteq U\ \&\ \exists m\in M, f\in R \text{ s.t. } \forall \mathfrak{q}\in V,\ f\notin\mathfrak{q}\ \&\ s(\mathfrak{q}) = m/f \right\}$$

Proof. Follows from Remark 27.2.0.4. $\square$

We now collect properties of $\widetilde{M}$ below.

Proposition 1.2.3.3. *Let R be a ring and M, N, M_i be R-modules for $i \in I$,*
1. $\left(\widetilde{M}\right)_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$,
2. $\widetilde{M}(\operatorname{Spec}(R)_f) \cong M_f$ *for all* $f \in R$,
3. $\Gamma(\widetilde{M}, \operatorname{Spec}(R)) \cong M$.

Proof. Statement 1 follows from the alternate definition given in Lemma 1.2.3.2. Indeed one considers the function

$$\begin{aligned} \varphi : \left(\widetilde{M}\right)_{\mathfrak{p}} &\longrightarrow M_{\mathfrak{p}} \\ (U, s)_{\mathfrak{p}} &\longmapsto s(\mathfrak{p}). \end{aligned}$$

One immediately sees this is R-linear. Injectivity and surjectivity is then also trivially checked by the above cited lemma.

Statements 3 follows from statement 2 by setting $f = 1$ and statement is just the Definition 1.2.3.1. $\square$

We can also understand how $\mathcal{O}_{\operatorname{Spec}(R)}$-modules behave under morphism of affine schemes (see direct and inverse image of modules at Section 8.5)

Lemma 1.2.3.4. [7] *Let $f : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ be a morphism of affine schemes associated to map $\varphi : R \to S$ of rings. Then,*
1. *if N is an S-module, then $f_* \widetilde{N} \cong \widetilde{{}_R N}$ where ${}_R N$ is the R-module obtained by restriction of scalars by φ,*
2. *if M is an R-module, then $f^* \widetilde{M} \cong \widetilde{(S \otimes_R M)}$ where $S \otimes_R M$ is the S-module obtained by extension of scalars by φ.*

Proof. The proof is routine with main observation being the facts that for $g \in R$, we have $({}_R N)_g \cong N_{\varphi(g)}$ and for $\mathfrak{q} \in \operatorname{Spec}(S)$, we get the natural isomorphism $(f^* \widetilde{M})_{\mathfrak{q}} \cong \widetilde{(S \otimes_R M)}_{\mathfrak{q}}$. $\square$

Theorem 1.2.3.5. *Let R be a ring. There is an equivalence of categories between those of R-modules and $\mathcal{O}_{\operatorname{Spec}(R)}$-modules of the form $\widetilde{M}$:*

$$\mathbf{Mod}(R) \underset{\Gamma(X,-)}{\overset{\widetilde{(-)}}{\rightleftarrows}} \widetilde{\mathbf{Mod}}(\mathcal{O}_{\operatorname{Spec}(R)})$$

which moreover satisfies the following properties
1. $\widetilde{(-)}$ *is an exact functor; if $0 \to M' \to M \to M'' \to 0$ is exact, then $0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$ is exact,*
2. $\widetilde{(-)}$ *preserves tensor product;* $\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$,

[7] We will call it the *globalized* extension and restriction of scalars.

3. $\widetilde{(-)}$ *preserves coproducts;* $\widetilde{\bigoplus_{i\in I} M_i} \cong \bigoplus_{i\in I} \widetilde{M_i}$.

Proof. Let $X = \operatorname{Spec}(R)$. Consider the following map

$$\begin{aligned}\operatorname{Hom}_R(M,N) &\to \operatorname{Hom}_{\mathcal{O}_X}\left(\widetilde{M},\widetilde{N}\right)\\ f: M\to N &\mapsto \widetilde{f}:\widetilde{M}\to\widetilde{N}\\ \eta_X: M\to N &\hookleftarrow \eta:\widetilde{M}\to\widetilde{N}\end{aligned}$$

Now, beginning from η, we may show that $(\widetilde{\eta_X})_{X_g} = \eta_{X_g}$ for some basic open $X_g \hookrightarrow X$. The result follows from the fact that $\eta : \widetilde{M} \to \widetilde{N}$ is completely characterized by the map on global sections $\eta_X : M \to N$ from the following square

$$\begin{array}{ccc} M_g & \xrightarrow{\eta_{X_g}} & N_g \\ \uparrow & & \uparrow \\ M & \xrightarrow[\eta_X]{} & N \end{array}$$

where the verticals are restrictions morphisms w.r.t $R \to R_g$ and the top horizontal is R_g-module homomorphism and bottom is R-module homomorphism.

For statement 1, by Theorem 27.3.0.8, the question is local in nature. We deduce the result then from Lemma 23.1.2.2.

For statement 2, we proceed as follows. To define an isomorphism

$$\varphi : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_R N}$$

we need only define a map from the presheaf F given by $U \mapsto \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U)$ to $\widetilde{M \otimes_R N}$ such that on basic open sets, we have an isomorphism. Indeed, let $D(f) \subseteq \operatorname{Spec}(R)$ be an open set for some $f \in R$. We define

$$\varphi_U : M_f \otimes_{R_f} N_f \xrightarrow{\cong} (M \otimes_R N)_f$$

as the obvious natural isomorphism. One checks that this does define φ to be a sheaf map.

For statement 3, as $\widetilde{(-)}$ is a left adjoint, therefore it preserves all colimits. □

Remark 1.2.3.6. We will later see that on affine schemes $\operatorname{Spec}(R)$, the category $\widetilde{\mathbf{Mod}}(\mathcal{O}_{\operatorname{Spec}(R)})$ is precisely the category of quasicoherent $\mathcal{O}_{\operatorname{Spec}(R)}$-modules, which is a class of modules of utmost importance in algebraic geometry.

1.3 Schemes and basic properties

We can now define scheme to be a locally ringed space (see Foundational Geometry, 8) with an affine open covering.

Definition 1.3.0.1. (**Schemes**) A locally ringed space $(X, \mathcal{O}_X)$ is a scheme if there exists an open affine cover $\{(\operatorname{Spec}(R_i), \mathcal{O}_{\operatorname{Spec}(R_i)})\}$ of $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X|\operatorname{Spec}(R_i)} \cong \mathcal{O}_{\operatorname{Spec}(R_i)}$.

As we go along in understanding schemes, it will be more and more apparent the need of sheaf language to talk about the "generalized functions" over the scheme X. Indeed, there is a fine interrelationship between the *space structure* of the scheme $(X, \mathcal{O}_X)$ (that is, the topological space X) and the *function structure* on the scheme (that is, the sheaf of functions $\mathcal{O}_X$). A big part of learning scheme theory is to understand and use this relationship between them.

We will now bring some global topological properties of schemes which reflect their affine origins. An analogue of Lemma 1.2.1.1 holds in the general case of schemes.

Lemma 1.3.0.2. [8] *Let X be a scheme. The following are equivalent.*
1. *$S \subseteq X$ is a closed irreducible subset.*
2. *There exists a point $x \in S$ such that $\overline{\{x\}} = S$.*

Proof. (1. $\Rightarrow$ 2.) Let U be an affine open in X intersecting S. Then $U \cap S$ is an open subset of S. As open subsets of irreducibles are dense, therefore $U \cap S$ is dense in S. Consequently, it suffices to show that there exists a point $x \in U \cap S$ such that $\overline{\{x\}} = U \cap S$. As open subsets of irreducibles are irreducible, therefore $U \cap S$ is irreducible. Replacing X by U, we may assume X is affine. The result then follows by Lemma 1.2.1.1.

(2. $\Rightarrow$ 1.) Since $x \in U$ for some open affine $U \subset X$, thus, $x \in U \cap S$. Since $U \cap S \subseteq U$ and U is open, therefore closure of $\{x\}$ in U is same as closure of $\{x\}$ in X. Now, $\overline{\{x\}} = S$ but $\overline{\{x\}} \subseteq U$. It thus follows that $S \subseteq U$ and hence S is in an open affine. The result follows by Lemma 1.2.1.1. □

Every open subspace of a scheme is a scheme.

Lemma 1.3.0.3. *Let X be a scheme and $U \subseteq X$ be an open subspace. Then $(U, \mathcal{O}_{X|U})$ is a scheme.*

Proof. Since for an affine scheme $\operatorname{Spec}(R)$, the basic open $\operatorname{Spec}(R)_f \cong \operatorname{Spec}(R_f)$ for $f \in R$, therefore for an open subspace $U \subseteq X$ and an affine open cover $\{U_i\}$ of X, $U_i \cap U$ is open in U_i and thus covered by affines of the form $\operatorname{Spec}(R_f)$. □

Write **Sch** to be the category of schemes and **Sch**$/S$ to be the category of schemes over S. Morphisms of schemes is merely the same concept as that of morphism of locally ringed spaces (see Foundational Geometry, Chapter 8).

Definition 1.3.0.4. (**Map of schemes**) Let X and Y be two schemes. A map of underlying locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a map of schemes. In a more expanded form, $f : X \to Y$ is a continuous map and $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a map of sheaves such that the induced map (see Topics in Sheaf Theory, Chapter 27) on stalks for each $x \in X$

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a map of local rings, i.e., $(f_x^\sharp)^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$.

[8] Exercise II.2.9 of Hartshorne.

An important theorem in global study of schemes is a complete characterization of schemes over $\mathrm{Spec}\,(R)$, which is of-course of paramount importance.

Theorem 1.3.0.5. *Let X be a scheme and R be a ring. Then, there's a natural bijection*

$$\mathrm{Hom}_{\mathbf{Sch}}\,(X, \mathrm{Spec}\,(R)) \cong \mathrm{Hom}_{\mathbf{Ring}}\,(R, \Gamma(X, \mathcal{O}_X)).$$

In other words, we have the following adjunction[9]

$$\begin{tikzcd} \mathbf{Sch} \arrow[r, shift left, "\Gamma(-)"] \arrow[r, phantom, "\perp"] & \mathbf{Ring}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{Spec}(-)"] \end{tikzcd} .$$

Proof. The proof will be played out in two steps. In the first one we will show the candidates for the unit and counit of this adjunction. In the second play we will show that they indeed satisfy the required triangle identities.

Act 1 : *The units and counits.*

Let us first define the simpler one of them, the counit. For any $R \in \mathbf{Ring}$, we define a natural transformation $\epsilon : \mathrm{id}_{\mathbf{Ring}} \to \Gamma \circ \mathrm{Spec}\,()$ given by (note how we adjusted for the contravariant nature of $\mathrm{Spec}\,(-)$ and $\Gamma(-)$)

$$\begin{aligned} \epsilon_R : R &\longrightarrow \Gamma(\mathrm{Spec}\,(R)) \cong R \\ f &\longmapsto f. \end{aligned}$$

Thus, $\epsilon_R = \mathrm{id}_R$. Hence, $\epsilon = \mathrm{id}_{\mathbf{Ring}^{\mathrm{op}}}$.

Next, we define the more intricate part, which is the unit. Take any scheme $X \in \mathbf{Sch}$. We define $\eta : \mathrm{id}_{\mathbf{Sch}} \to \mathrm{Spec}\,(\Gamma)$ on X by

$$\begin{aligned} \eta_X : X &\longrightarrow \mathrm{Spec}\,(\Gamma(X)) \\ x &\longmapsto \mathfrak{p} = \eta_X(x) := \{f \in \Gamma(X) \mid f_x \in \mathfrak{m}_x\}. \end{aligned}$$

Moreover, the map on structure sheaves is given by

$$(\eta_X)^\flat : \mathcal{O}_{\mathrm{Spec}(\Gamma(X))} \longrightarrow (\eta_X)_* \mathcal{O}_X$$

where as the map on global sections we keep it id and on a basic open $\mathrm{Spec}\,((\Gamma(X))_f)$ this is defined on sections by

$$(\eta_X)^\flat_{\mathrm{Spec}\left((\Gamma(X))_f\right)} : \Gamma(X)_f \cong \mathcal{O}_{\mathrm{Spec}(\Gamma(X))}((\mathrm{Spec}\,(\Gamma(X)))_f) \longrightarrow \mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}\,((\Gamma(X))))_f))$$

by the unique map that is obtained in the following diagram

$$\begin{tikzcd} \Gamma(X)_f \arrow[r, dashed] & \mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}\,(\Gamma(X)))_f)) \\ \Gamma(X) \arrow[u] \arrow[ur, "\rho"'] & \end{tikzcd} ,$$

where, indeed, $f \in \Gamma(X)$ is mapped to to an unit element in $\mathcal{O}_X(\eta_X^{-1}((\mathrm{Spec}\,(\Gamma(X)))_f$ because of the following simple lemma:

[9] This is also sometimes called the *algebra-geometry duality* or the *fundamental duality of algebraic geometry.*

(*) For a locally ringed space $(X, \mathcal{O}_X)$ and an open subspace $U \subseteq X$, $f \in \mathcal{O}_X(U)$ is a unit if and only if $f_x \notin \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ for all $x \in U$.

This construction has the following properties and we give the main idea which drives each one of them.

1. *$\eta_X(x)$ is a prime ideal of $\Gamma(X)$* : This follows from $\mathfrak{m}_x$ being a maximal (hence prime) ideal of $\mathcal{O}_{X,x}$.
2. *η_X is continuous* : Working with basis and reducing to assumption that $X = \operatorname{Spec}(S)$ is affine, we reduce to showing that $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid f_\mathfrak{p} \notin \mathfrak{m}_\mathfrak{p}\}$ is open, which is true as it is equal to $(\operatorname{Spec}(S))_f$.
3. *$\eta : \mathrm{id}_{\mathbf{Sch}} \to \operatorname{Spec}() \circ \Gamma$ is a natural transformation* : We wish to show that commutativity of the natural square. For a map of schemes $f : X \to Y$, this reduces to showing that
$$\forall x \in X,\ \eta_Y(f(x)) = (f_Y^\flat)^{-1}(\eta_X(x)).$$
This further follows from the observation that for $g \in \Gamma(Y)$, $f_Y^\flat(g) \in \mathfrak{m}_x \iff f_x(g_{f(x)}) \in \mathfrak{m}_x$ and the latter is clearly true by the definition of maps of locally ringed spaces, where $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is the map on stalks.

Hence, we have obtained a map of schemes $(\eta_X, \eta_X^\flat) : X \to \operatorname{Spec}(\Gamma(X))$. This is our candidate for the unit of the adjunction.

Act 2 : *η and ϵ satisfies the triangle identities.*

It follows that we wish to show that the following two diagrams commute:

$$\begin{tikzcd} \Gamma(X) & \Gamma(\operatorname{Spec}(\Gamma(X))) \cong \Gamma(X) \arrow[l, "\Gamma\eta_X"'] \\ & \Gamma(X) \arrow[u, "\epsilon_{\Gamma(X)}"'] \arrow[ul, "\mathrm{id}_{\Gamma(X)}"] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec}(R) \arrow[r, "\eta_{\operatorname{Spec}(R)}"] \arrow[dr, "\mathrm{id}_{\operatorname{Spec}(R)}"'] & \operatorname{Spec}(\Gamma(\operatorname{Spec}(R))) \cong \operatorname{Spec}(R) \arrow[d, "\operatorname{Spec}(\epsilon_R)"] \\ & \operatorname{Spec}(R) \end{tikzcd} .$$

in **Ring** in **Sch**

This follows from a simple unraveling of the maps involved in the diagram as defined in Act 1.

□

Corollary 1.3.0.6. *The above adjunction restricts to the following equivalence of categories:*

$$\mathbf{AfSch} \underset{\operatorname{Spec}(-)}{\overset{\Gamma(-)}{\rightleftarrows}} \mathbf{Ring}^{\mathrm{op}} .$$

□

Corollary 1.3.0.7. *Let X be a scheme over $\operatorname{Spec}(R)$ for a ring R. Then, for any open affine $\operatorname{Spec}(S) \subseteq X$, S is an R-algebra. Consequently, all stalks $\mathcal{O}_{X,p}$ are R-algebras.* □

1.3.1 Basic properties

We can now observe some more basic properties. First, the behaviour of maps with respect to schemes over k and residue fields.

Lemma 1.3.1.1. *Let $f : X \to Y$ be a map of schemes over k. If $p \in X$ is such that $\kappa(p) = k$, then $\kappa(f(p)) = k$.*

Proof. On the stalks, we get the map

$$\begin{tikzcd} \mathcal{O}_{Y,f(p)} \arrow[rr, "f_p^\sharp"] & & \mathcal{O}_{X,p} \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}.$$

As $f_p^\sharp$ is a local k-algebra homomorphism, therefore by quotienting with the respective ideals, we obtain

$$\begin{tikzcd} \kappa(f(p)) \arrow[rr] & & k \\ & k \arrow[ul] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}.$$

The result then follows. $\square$

Local rings at non-closed points

Let X be an arbitrary scheme and $p \in X$ be a non-closed point. One can show that the local ring $\mathcal{O}_{X,p}$ is obtained by localizing local rings at closed points. Indeed, we have the following simple observation in this direction.

Lemma 1.3.1.2. *Let X be a scheme and $p \in X$ be a non-closed point. Then, $\mathcal{O}_{X,p}$ is isomorphic to localization of a local ring $\mathcal{O}_{X,x}$ at a prime ideal, where $x \in X$ is a closed point.*

Proof. Let $p \in X$ be a non-closed point and $U = \mathrm{Spec}\,(A)$ be an open affine containing p. Consequently, p corresponds to a prime ideal $\mathfrak{p} \lneq A$ which is not maximal. Let $\mathfrak{m} \lneq A$ be a maximal ideal containing $\mathfrak{p}$ and let $m \in U$ be the corresponding closed point in X. As $\mathcal{O}_{X,p} \cong A_\mathfrak{p}$ and $\mathcal{O}_{X,m} \cong A_\mathfrak{m}$, and since $(A_\mathfrak{m})_{\mathfrak{p}_\mathfrak{m}} \cong A_\mathfrak{p}$, therefore we have that $\mathcal{O}_{X,p}$ is obtained by localizing $\mathcal{O}_{X,m}$ at a prime ideal, as required. $\square$

Using ideas similar to above, we can also prove the following simple result.

Lemma 1.3.1.3. *Let X be an integral scheme and $\eta \in X$ be a non-closed point. Then the fraction field of $\mathcal{O}_{X,\eta} \cong K(X)$ where $K(X)$ is the function field of X.* $\square$

Non-vanishing locus of a global section

We next see that how a global section of a scheme defines an open set which is the set of those points where that element, when treated as a function, is non-zero. One then finds what the ring of functions over this open set looks like. First, for any scheme X and any $f \in \Gamma(\mathcal{O}_X, X)$, define the *non-vanishing locus* of f by

$$X_f := \{x \in X \mid f \notin \mathfrak{m}_{X,x}\}.$$

We first have the following simple result about non-vanishing locus.

Lemma 1.3.1.4. *Let $f : X \to \mathrm{Spec}\,(B)$ be a scheme over a ring B and let $g \in B$. Let $\varphi : B \to \Gamma(\mathcal{O}_X, X)$ be the map induced on the global sections. Then,*

$$f^{-1}(D(g)) = X_{\varphi(g)}.$$

Proof. Observe that $x \in X_{\varphi(g)}$ if and only if $\varphi(g)_x \notin \mathfrak{m}_{X,x}$. As we have the following commutative square

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & \Gamma(\mathcal{O}_X, X) \arrow[d] \\ \mathcal{O}_{\mathrm{Spec}(B), f(x)} \arrow[r, "f_x^\sharp"'] & \mathcal{O}_{X,x} \end{tikzcd}$$

where vertical arrows are image into the stalk, therefore we deduce that $\varphi(g)_x \notin \mathfrak{m}_{X,x}$ if and only if $f_x^\sharp(g_x) \notin \mathfrak{m}_{X,x}$. As $f_x^\sharp$ is a local ring homomorphism, therefore $f_x^\sharp(g_x) \notin \mathfrak{m}_{X,x}$ if and only if $g_x \notin \mathfrak{m}_{\mathrm{Spec}(B),f(x)} = f(x)B_{f(x)}$. As $B \to \mathcal{O}_{\mathrm{Spec}(B),f(x)}$ is just localization map $B \to B_{f(x)}$, therefore $g_x \notin f(x)B_{f(x)}$ if and only if $g \notin f(x)$, that is $f(x) \in D(g)$. This completes the proof. $\square$

Proposition 1.3.1.5. [10] *Let X be a scheme and $f \in \Gamma(\mathcal{O}_X, X)$.*

1. *Let $U = \mathrm{Spec}\,(A)$ be an affine open subset of X and denote $\bar{f} = \rho_{X,U}(f)$. Then, $U \cap X_f = D(\bar{f})$. Consequently, $X_f \subseteq X$ is an open subscheme.*
2. *Let X be quasicompact and $a \in \Gamma(\mathcal{O}_X, X)$ such that $\rho_{X,X_f}(a) = 0$. Then, $f^n a = 0$ in $\Gamma(\mathcal{O}_X, X)$ for some $n > 0$.*
3. *Let X admit an affine open cover U_i such that $U_i \cap U_j$ is quasicompact. If $b \in \mathcal{O}_X(X_f)$, then there exists $a \in \Gamma(\mathcal{O}_X, X)$ and $n > 0$ such that $f^n b = \rho_{X,X_f}(a)$ in $\mathcal{O}_X(X_f)$.*
4. *There is an isomorphism of rings $\Gamma(\mathcal{O}_{X_f}, X_f) \cong (\Gamma(\mathcal{O}_X, X))_f$.*

Proof. 1. We wish to show that $\{x \in U \mid \bar{f}_x \notin \mathfrak{m}_{X,x}\} = \{x \in U \mid \bar{f} \notin x\}$, where $x \in U$ in latter is treated as a prime ideal of A. The side " $\subseteq$ " follows from the fact that for $x \in U$, we have $\mathcal{O}_{X,x} \cong A_x$, $\mathfrak{m}_{X,x} \cong xA_x$ and the fact that the map into stalks $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ is given by the canonical map $A \to A_x$, $a \mapsto a/1$. One further would need the commutativity of the following diagram:

$$\begin{tikzcd} & \mathcal{O}_{X,x} \\ \mathcal{O}_X(U) \arrow[ur] & \Gamma(\mathcal{O}_X, X) \arrow[l] \arrow[u] \end{tikzcd}.$$

[10] Exercise II.2.16 of Hartshorne.

The side " $\supseteq$ " also follows from the commutativity of the above triangle together with the canonical isomorphisms of the local ring and its maximal ideal.

2. **TODO from notebook.** □

Locality of isomorphism on target

We now show a rather simple result on locality of isomorphism on target, but it is quite useful in scenarios where one understands the map well on individual opens of target but not on the global level.

Proposition 1.3.1.6. *Let $f : X \to Y$ be a map of schemes and $Y = \bigcup_{i\in I} U_i$ be an open cover of Y such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is an isomorphism. Then, f is an isomorphism.*

Proof. **TODO from notes.** □

Criterion for affineness

We now show a useful criterion for a scheme to be affine. This also portrays the power of previous result on locality of isomorphism.

Proposition 1.3.1.7. *Let X be a scheme and denote $A = \Gamma(\mathcal{O}_X, X)$. Then the following are equivalent:*

1. *X is affine,*
2. *there exists $f_1, \dots, f_r \in A$ such that X_{f_i} are open affine subsets of X and $\langle f_1, \dots, f_r\rangle = A$.*

Proof. **TODO from notes.** □

1.4 First notions on schemes

Having defined schemes, our next goal is to bring to light some of the obvious definitions that one can make on them. In some sense, having made the general definition of schemes, we are now trying to go back to try and find where does varieties lie in this big world of **Sch**. Indeed, we will see that the definitions introduced in the following few sections are bringing us ever closer to define varieties as certain type of schemes, which will thus enable us to bring to light the most important geometric notions on varieties.

Write the notion and examples, C ter 1.

1.4.1 Noetherian schemes

Definition 1.4.1.1. (**Noetherian schemes**) A scheme X is called *locally noetherian* if there exists an affine open cover $X = \cup_{i\in I} U_i$ where each $U_i = \mathrm{Spec}\,(A_i)$ where A_i is a noetherian ring. If moreover, X is quasicompact, then X is called *noetherian*.

Remark 1.4.1.2. Since $X = \mathrm{Spec}\,(A)$ is already quasi-compact (Lemma 1.2.1.6), therefore for affine schemes X, the notion of locally noetherian and noetherian are equal.

The only immediately important result about such schemes that one needs is that an affine scheme is noetherian if and only if the obvious thing happens.

Lemma 1.4.1.3. *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme. Then, the following are equivalent:*

1. *X is a noetherian scheme.*
2. *A is a noetherian ring.*

Proof. (2. $\Rightarrow$ 1.) This follows from Remark 1.4.1.2 and the fact that localization of noetherian rings are noetherian (Proposition 23.3.0.7).
(1. $\Rightarrow$ 2.) Let X be noetherian. Then there is an affine open cover of X by spectra of noetherian rings. Pick any ideal $I \leq A$. We shall show it is finitely generated. There is a finite cover $\{\mathrm{Spec}\,(A_{f_i})\}_{i=1}^n$ of $\mathrm{Spec}\,(A)$ where A_{f_i} are noetherian and $f_i \in A$. Hence we have that the ideal IA_{f_i} of A_{f_i} is finitely generated for all $i = 1, \dots, n$. By Lemma 1.2.1.5, 2, we see that $f_1, \dots, f_n$ generate the whole ring A. The result then follows by Lemma 23.1.2.10. □

Example 1.4.1.4. By the Lemma 1.4.1.3, we observe that any of the variety over a field is a noetherian scheme (technically, we are identifying the affine variety with its associated scheme, see Section **??**, Schemes associated to varieties). So any of your favorite variety

$$\mathrm{Spec}\left(\frac{k[x,y,z]}{x^2+y^2-z^3-1}\right), k \text{ is algebraically closed}$$

gives a (*is* a) noetherian scheme.

Local rings of a locally noetherian scheme are noetherian.

Lemma 1.4.1.5. *If X is locally noetherian, then $\mathcal{O}_{X,x}$ is a noetherian ring.*

Proof. Since localization of a noetherian ring at a prime is again noetherian by Proposition 23.3.0.7, therefore $\mathcal{O}_{X,x}$ is noetherian. □

Being locally noetherian is a local property.

Proposition 1.4.1.6. *Let X be a locally noetherian scheme. If* $\operatorname{Spec}(A) \subseteq X$ *is an open affine, then* $\operatorname{Spec}(A)$ *is noetherian and thus A is a noetherian ring.*

Proof. Let $U_i = \operatorname{Spec}(A_i)$ be an open cover by noetherian affine schemes (A_i are noetherian). Then, a finitely many of U_i will cover $\operatorname{Spec}(A)$ by quasi-compactness of $\operatorname{Spec}(A)$, say $U_1, \ldots, U_n$. Thus we obtain a finite basic open cover $D(f_i)$ of $\operatorname{Spec}(A)$ for $f_i \in A$ where each $D(f_i) \subseteq U_j$ for some j such that $D(f_i)$ is also basic in U_j (Lemma 1.4.4.3). As U_j is noetherian, therefore if we can show that $\mathcal{O}_{U_j}(D(f_i))$ is noetherian, then we would have shown that A_{f_i} is noetherian, which would complete the proof by Lemma 23.3.0.8. We thus reduce to assuming $X = \operatorname{Spec}(A)$ noetherian affine and to show that $U = D(f) \subseteq X$ is noetherian for $f \in A$.

In this case, as A is noetherian, therefore by Corollary 23.3.0.9, the ring A_f is noetherian, as required. □

1.4.2 Reduced, integral schemes and function field

The following are the definitions required, which are clearly geometric in nature.

Definition 1.4.2.1. (**Reduced and integral schemes**) A scheme X is said to be *reduced* if local rings $\mathcal{O}_{X,x}$ for all $x \in X$ is a reduced ring; have no nilpotents. A scheme X is said to be *integral* if it is reduced and irreducible as a topological space.

The one basic result that must be seen about these two types of schemes is that they are characterized by algebraic properties of local sections. Thus being reduced or integral, while defined geometrically, is concretely controlled by the algebraic properties of the structure sheaf.

Lemma 1.4.2.2. *Let X be a scheme. Then,*

1. *X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for each open set $U \subseteq X$* [11].
2. *X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open set $U \subseteq X$.*

Proof. 1. (L $\Rightarrow$ R) Suppose for some open $U \subseteq X$ there exists a section $f \in \mathcal{O}_X(U)$ which is nilpotent. Using the homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ given by $s \mapsto s_x$, we see that $f_x \in \mathcal{O}_{X,x}$ is a nilpotent element.
(R $\Rightarrow$ L) Suppose X is not reduced. Hence for some germ $f_x \in \mathcal{O}_{X,x}$ at some point $x \in X$ is a nilpotent where $f \in \mathcal{O}_X(U)$ for some open $x \in U \subseteq X$. Since $f_x^n = 0$ for some $n \in \mathbb{N}$, we get that $f^n = 0$ for some open $W \subseteq U$. Thus $\rho_{U,W}(f) \in \mathcal{O}_X(W)$ is a nilpotent element[12].

2. (L $\Rightarrow$ R) Pick any open $U \subseteq X$. We wish to show that $\mathcal{O}_X(U)$ is an integral domain. In other words, we wish to show the proposition for the open subscheme $(U, \mathcal{O}_{X|U})$. Replacing X by U, we reduce to showing $\mathcal{O}_X(X)$ is an integral domain. So let $f, g \in \mathcal{O}_X(X)$ be such that $fg = 0$. We wish to show that either $f = 0$ or $g = 0$. Suppose neither f nor g

[11] Exercise II.2.3.a of Hartshorne.

[12] This is a very inefficient way of using the equality on stalks. Indeed, two germs are equal if and only if the representatives are equal on some common shrinking of their domains. This is how usually people work with stalks without being overly full of symbols.

is 0 but $fg = 0$. It follows from Lemma 1.2.0.1, 1, that $V(f)$ and $V(g)$ covers X and hence by irreducibility of X, either $V(f) = 0$ or $V(g) = 0$, that is, $f = 0$ or $g = 0$.
(R $\Rightarrow$ L) We first need to show that X is reduced. Indeed, by 1. it follows immediately as integral domains are reduced. We then wish to show that X is irreducible. Indeed, if there are two open subsets of X say $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \emptyset$, then we claim that $\mathcal{O}_X(U_1 \cup U_2) \cong \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$. Since both $\mathcal{O}_X(U_1), \mathcal{O}_X(U_2)$ have 0 and 1, thus $\mathcal{O}_X(U_1 \cup U_2)$ will have a zero-divisor, a contradiction. Indeed, consider the following homomorphism, denoting $U := U_1 \cup U_2$

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \\ s &\longmapsto (\rho_{U,U_1}(s), \rho_{U,U_2}(s)). \end{aligned}$$

This is injective by locality axiom and surjective by gluing axiom of sheaves. □

Corollary 1.4.2.3. *Let X be a scheme. If X is integral, then all local rings $\mathcal{O}_{X,x}$ are integral domains.*

Proof. Use Lemma 1.4.2.2, 2 together with the fact that localization of integral domains is an integral domain. □

Corollary 1.4.2.4. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then X is integral if and only if A is an integral domain.*

Proof. Use Lemma 1.4.2.2, 2 on global sections together to get one side. For the "only if" side, stalks are reduced as they are integral (localizations of A) and X is irreducible as for any $V(\mathfrak{a}) \cup V(\mathfrak{b}) = X$, we have $V(\mathfrak{a}\mathfrak{b}) = X$ and thus $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{n}$ where $\mathfrak{n}$ is the intersection of all prime ideals, the nilradical (Lemma 23.1.2.9). Since A is integral, therefore $\mathfrak{o}$ is prime as well and hence $\mathfrak{n} = 0$, making $\mathfrak{a}\mathfrak{b} = 0$. Since A is integral, hence $\mathfrak{a} = \mathfrak{o}$ or $\mathfrak{b} = \mathfrak{o}$. □

Remark 1.4.2.5. (*Function field of an integral scheme*) Let X be an integral scheme. Since X is irreducible as a topological space, therefore there is a generic point η in X, i.e. a point whose closure is the whole of X (Lemma 1.3.0.2). Now let $\operatorname{Spec}(A) \subseteq X$ be an affine open such that $\eta \in \operatorname{Spec}(A)$. Thus, η is a generic point of $\operatorname{Spec}(A)$ as well. Hence η corresponds to the zero ideal of A, which is indeed an integral domain from Lemma 1.4.2.2, 2. Since $\mathcal{O}_{X,\eta} \cong \mathcal{O}_{\operatorname{Spec}(A),\eta} = A_{\mathfrak{o}}$, therefore $\mathcal{O}_{X,\eta}$ is a field, called the *function field of the integral scheme* X and is in particular given by field of fractions of any domain A such that open $\operatorname{Spec}(A)$ contains η. We denote the function field of X as $K(X)$[13].

Using the fact that the generic point of an integral scheme X will be in every non-empty open set, we can make some fascinating observations about the function field $K(X)$, which thus justifies its name.

Lemma 1.4.2.6. *Let X be an integral scheme with function field $K(X)$. Then for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is contained in $K(X)$.*

[13] Exercise II.3.6 of Hartshorne.

Proof. Let $x \in X$, $\eta \in X$ be the generic point and $U = \mathrm{Spec}\,(A)$ be an open affine in X. By Lemma 1.4.2.2, 2, A is a domain. Clearly, $\eta \in U$ and it corresponds to the zero ideal $\mathfrak{o} \lneq A$. Further we have $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$, $\mathfrak{p} \in U$ is equal to the point $x \in U$. By definition $K(X) = A_{\mathfrak{o}}$. The result follows by observing that $A_{\mathfrak{p}} \subseteq A_{\mathfrak{o}}$. □

The following lemma shows that restriction of functions in an integral scheme is injective.

Lemma 1.4.2.7. *Let X be an integral scheme and $U \hookrightarrow V$ be an inclusion of open sets. Then, the restriction maps $\rho : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is an injective ring homomorphism.*

Proof. By Lemma 27.3.0.2, we need only show that for any $x \in V$ and any $s \in \mathcal{O}_X(V)$, we have $(V,s)_x = 0$ in $\mathcal{O}_{X,x}$. Let $W = \mathrm{Spec}\,(A)$ be an open affine containing x. As U is open in X and X is irreducible, therefore it is dense. Consequently, $U \cap W$ is an open non-empty set in X. We may write $\rho_{V,W}(s) = a \in A$. Let $D(f) \subseteq U \cap W$ be a basic open set of W. Since taking germs commutes with restrictions, therefore we have the restriction map $\mathcal{O}_X(W) \to \mathcal{O}_X(D(f))$ which is the localization map $A \to A_f$, which takes $a \mapsto \frac{a}{1}$. As s on U is 0, therefore, s is 0 on $W \cap U$ and thus on $D(f)$. Consequently, we have $\frac{a}{1} = 0$ in A_f. As A is a domain by Lemma 1.4.2.2, it follows that $a = 0$ in A. Thus, $\rho_{V,W}(s) = 0$, hence, $(V,s)_x = 0$ in $\mathcal{O}_{X,x}$, as required. □

Example 1.4.2.8. (Spec $(\mathbb{Z})$) Since $\mathbb{Z}$ is an integral domain, therefore by Corollary 1.4.2.4, $X = \mathrm{Spec}\,(\mathbb{Z})$ is an integral scheme. Clearly, X as a topological space consists of all prime numbers and a generic point given by the zero ideal $\mathfrak{o}$. Further, the topology is thus given by cofinite topology. At the level of stalks, we have that for a prime $\mathfrak{p} \in X$, $\mathcal{O}_{X,\mathfrak{p}} \cong \mathbb{Z}_{\mathfrak{p}}$ and we can describe $\mathbb{Z}_{\mathfrak{p}}$ as all those rationals whose denominator is not a multiple of prime p where $\mathfrak{p} = \langle p \rangle$ as $\mathbb{Z}$ is a PID (it's ED). Clearly, localizing X at the generic point $\mathfrak{o}$ would yield $\mathcal{O}_{X,\mathfrak{o}} \cong \mathbb{Q}$. More fascinatingly, for a prime $\mathfrak{p} = \langle p \rangle$ in X, the residue field at point $\mathfrak{p}$ is $\kappa(\mathfrak{p}) = \mathbb{Z}_{\mathfrak{p}}/\mathfrak{p}\mathbb{Z}_{\mathfrak{p}} \cong \mathbb{F}_p$, the finite field with p elements!

Now for any affine scheme Spec (A), consider a map $f : X \to \mathrm{Spec}\,(\mathbb{Z})$. By the fact that $\mathbb{Z}$ is initial in category of rings, therefore Spec $(\mathbb{Z})$ is terminal in the category of affine schemes (Corollary 1.3.0.6). Since any scheme is locally affine, it further follows that Spec $(\mathbb{Z})$ is terminal in the category of schemes.

We now introduce a concept which will be used while discussing divisors.

Definition 1.4.2.9. (**Center of a valuation**) Let X be an integral scheme with function field K and $v : K \to G$ be a valuation over K with valuation ring $R \subset K$. A center of v is defined to be a point $x \in X$ such that R dominates $\mathcal{O}_{X,x}$ in K (see Definition 23.10.1.5).

1.4.3 (Locally) finite type schemes over k

This section is the beginning of a theme which we would like to understand intimately, schemes over a field. This is because most of the schemes we will encounter in nature will be varieties whose coordinate rings would be algebras over a field. Here we first understand in scheme language the first thing about coordinate rings of varieties over k, the fact that they are finitely generated as an k-algebra. Indeed, this is what we seek from the following definition.

Definition 1.4.3.1. (**Finite and locally finite type schemes over a field**) Let k be a field and let $X \to \operatorname{Spec}(k)$ be a scheme over k. Then X is said to be locally finite type if there exists an affine open covering $\{\operatorname{Spec}(A_i)\}_{i\in I}$ of X such that each A_i is a finitely generated k-algebra. Moreover, X is said to be finite type if X is locally finite type and quasi-compact.

Example 1.4.3.2. Our $\mathfrak{h}$yperboloid of one sheet (introduced in Example 1.5.1.3) has the following coordinate ring:

$$\frac{k[x,y,z]}{I(V(p))}$$

where $p(x,y,z) = x^2 + y^2 - z^2 - 1$, where we have chosen $a = b = c = 1$ for simplicity. Let $\mathfrak{h} := I(V(p))$. Clearly $\operatorname{Spec}(k[x,y,z]/\mathfrak{h})$ is a finite type k-scheme.

Great thing about the above definition is that it really doesn't depend on the affine open cover that is chosen.

Lemma 1.4.3.3. *Let k be a field and X be a k-scheme. Then the following are equivalent.*

1. *X is of locally finite type over k.*
2. *For all open affine $U \hookrightarrow X$, the ring $\mathcal{O}_X(U)$ is finitely generated k-algebra.*

Proof. (2. $\Rightarrow$ 1.) Immediate.
(1. $\Rightarrow$ 2.) We shall use Lemma 23.1.2.11 for this. $\square$

Complete it's pr
Chapter 1.

1.4.4 Subschemes and immersions

These notions are important in what is to come next.

Definition 1.4.4.1. (**Open subscheme**) Let X be a scheme. An open set $U \subseteq X$ has a canonical scheme structure, given by $(U, \mathcal{O}_{X|U})$. We call $(U, \mathcal{O}_{X|U})$ an open subscheme of X.

Indeed, locally U will look affine via the open affine cover of X. We can relativize this notion to define open immersions.

Definition 1.4.4.2. (**Open immersion**) A map $f : X \to Y$ of schemes is said to be an open immersion if $f : X \to f(X)$ is a homeomorphism, $f(X) \subseteq Y$ is open and $f^\flat_{|f(X)} : \mathcal{O}_{Y|f(X)} \to (f_*\mathcal{O}_X)_{|f(X)}$ is an isomorphism.

We observe that for any point in an intersection of open subschemes is contained in some special open subscheme. This is a very important result as this will be used as a technical tool to allow passage from one open affine with certain properties to another open affine, all the time while handling only basic open sets.

Lemma 1.4.4.3. *Let $U = \mathrm{Spec}\,(A), V = \mathrm{Spec}\,(B) \hookrightarrow X$ be two affine open subsets. For each $x \in U \cap V$, there exists an affine open subset $x \in W \hookrightarrow U \cap V$ such that $W = \mathrm{Spec}\,(A_f)$ and $W = \mathrm{Spec}\,(B_g)$ for some $f \in A$ and $g \in B$. Moreover, under the isomorphism $A_f \cong B_g$, the element $f \in A_f$ maps to $g \in B_g$.*

Proof. By replacing B by B_g for some $g \in B$, we may assume that $x \in V \subseteq U$. Consequently, let $f \in A$ be such that $D_U(f) \subseteq V$ and contains x, where $D_U(f) = \{\mathfrak{p} \in U \mid f \notin \mathfrak{p}\}$. We thus have $x \in D_U(f) \subseteq V \subseteq U$. Consider the restriction $h = \rho_{U,V}(f) \in \mathcal{O}_X(V) = B$. We claim that $D_V(h) = D_U(f)$. Denote $\varphi : A \to B$ obtained by $V \subseteq U$. We then have that $\rho_{U,V} = \varphi$ and $h = \varphi(f)$. Thus $\mathfrak{q} \in D_V(h) \iff h \notin \mathfrak{q} \iff \varphi(f) \notin \mathfrak{q} \iff f \notin \varphi^{-1}(\mathfrak{q})$. As each $\mathfrak{p} \in D_U(f)$ is $\varphi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in V$, therefore we are done. The last statement is immediate from above. $\square$

Closed subschemes are defined in not that obvious way in which we have defined open subschemes, but at any rate, they are natural. We motivate the need for ideal sheaves as follows. Let X be a scheme. Suppose a closed subset $C \hookrightarrow X$ intersects some collection of affine opens $\{\mathrm{Spec}\,(A_i)\}$ and moreover it happens that $C \cap \mathrm{Spec}\,(A_i) = C \cap \mathrm{Spec}\,(A_j)$ for some $i \neq j$. Now by Corollary 1.4.4.14 we may write $C \cap \mathrm{Spec}\,(A_i) = \mathrm{Spec}\,(A_i/\mathfrak{a}_i)$ and $C \cap \mathrm{Spec}\,(A_j) = \mathrm{Spec}\,(A_j/\mathfrak{a}_j)$ for some ideals $\mathfrak{a}_i \subseteq A_i$ and $\mathfrak{a}_j \subseteq A_j$. Hence, we get two different structure sheaves $\mathcal{O}_{\mathrm{Spec}(A_i/\mathfrak{a}_i)}$ and $\mathcal{O}_{\mathrm{Spec}(A_j/\mathfrak{a}_j)}$ on on open subset of C. Thus we have to systematically track such identifications in order to define a unique scheme structure on the closed set C. Indeed, we take the help of the rich amount of constructions that we can make on the category of sheaves over a space (for more information, see Section 8.5).

We first define closed immersions.

Definition 1.4.4.4. (**Closed immersions**) A map $f : X \to Y$ of schemes is a closed immersion if $f : X \to f(X)$ is a homeomorphism, $f(X) \subseteq Y$ is closed and $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a surjective map.

Remark 1.4.4.5. Let $f : X \to Y$ be a closed immersion, so that $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. This is equivalent to saying that for each point $x \in X$, the map on stalks (see Theorem 27.3.0.6 and Lemma 27.5.0.5)

$$f^\flat_{f(x)} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is surjective. Observe that the above map is NOT the usual map on stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Further observe that since $f^\flat$ is surjective, therefore we have an ideal (see Section 8.5, Global algebra for more details) $\mathcal{I} = \mathrm{Ker}\left(f^\flat\right) \leq \mathcal{O}_Y$. We will later see that a closed subscheme is completely determined by this ideal sheaf and in-fact these ideal sheaves gives us a family of good examples of what will later be called quasicoherent modules over a scheme.

Remark 1.4.4.6. Let $f : X \to Y$ be a closed immersion. Then, the map $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. Pick any $x \in X$. Since we have the following commutative square for any open set $V \ni f(x)$ in Y

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "f^\flat_V"] \arrow[d] & \mathcal{O}_X(f^{-1}(V)) \arrow[d] \\ \mathcal{O}_{Y,f(x)} \arrow[r, "f^\sharp_x"'] & \mathcal{O}_{X,x} \end{tikzcd} .$$

It then follows from surjectivity of $f^\flat$ and $f : X \to f(X)$ being a homeomorphism that the local homomorphiosm $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective. It is also a simple exercise to see that surjectivity of $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ for all $x \in X$ implies surjectivity of $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Consequently, $f : X \to Y$ is a closed immersion if and only if f is a topological closed immersion and for all $x \in X$, the local homomorphism $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective.

A closed subscheme is then defined to be an isomorphism class of closed immersions.

Definition 1.4.4.7. (**Closed subscheme & ideal sheaf**) Let Y be a scheme. A closed subscheme of Y is an isomorphism class of closed immersions over Y. That is, a closed subscheme is the class $[f : X \to Y]$ of closed immersions where two closed immersions $f : X \to Y$ and $f' : X' \to Y$ are identified if there is an isomorphism $X \xrightarrow{\cong} X'$ such that the following commutes

$$\begin{tikzcd} X \arrow[rr, "\cong"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd} .$$

For a closed subscheme $f : X \to Y$, we define kernel of $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ to be the ideal sheaf corresponding to the closed subscheme f.

Remark 1.4.4.8. Note that this definition is not "unnatural" as every closed immersion $f : X \to Y$ defines a closed set $f(X) \subseteq Y$ and a scheme structure over it. We then just define a closed subscheme to be the data of this closed set together with its scheme structure that is given by f. Clearly to make such a definition via immersions, we would need to identify those immersions which give same scheme structure on $f(X) \subseteq Y$.

We define an immersion as follows.

Definition 1.4.4.9 (**Immersion**)**.** A map $f : X \to Z$ is said to be an immersion if f is an open immersion into a closed subscheme of Z.

We first understand closed subscheme structures in affine schemes.

Lemma 1.4.4.10. *Let $X = \mathrm{Spec}\,(R)$ be an affine scheme. Then every ideal $\mathfrak{a} \leq R$ defines a closed subscheme of X.*

Proof. Consider the closed set $Y = V(\mathfrak{a}) \subseteq X$. We endow Y with a scheme structure given by the isomorphism $Y \cong \mathrm{Spec}\,(R/\mathfrak{a})$. Now the inclusion map $i : (Y, \mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a})}) \to X$ is clearly a topological closed immersion. Further, $i^\flat : \mathcal{O}_{\mathrm{Spec}(R)} \to i_*\mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a})}$ is given on stalks (see Lemma 27.5.0.5) at point $x \in Y$ as $\mathcal{O}_{\mathrm{Spec}(R),x} \to \mathcal{O}_{\mathrm{Spec}(R/\mathfrak{a}),x}$ which is just $R_x \to (R/\mathfrak{a})_x$ which is surjective. Thus, $\mathfrak{a}$ defines a closed subscheme structure on Y. $\square$

It is important to note that any other ideal $\mathfrak{b} \leq R$ such that $V(\mathfrak{a}) = V(\mathfrak{b})$ will define a possibly different closed subscheme structure on the underlying topological space. This is another example of the phenomenon that algebra has much more finer control over the geometric situation at hand. For example, for $X = \mathrm{Spec}\,(k[x])$, we have $\mathfrak{a}_n = \langle x^n \rangle$ and note that $V(\mathfrak{a}_n) = \{\langle x \rangle\} \subseteq X$. But each ideal $\mathfrak{a}_n$ defines a new closed subscheme structure on the same point $\langle x \rangle \in X$.

Properties of closed immersions

We discuss some general properties of closed immersions. We begin by observing that closed immersions are local on target.

Proposition 1.4.4.11. *Let $f : X \to Y$ be a morphism of schemes. Then the following are equivalent:*

1. *f is a closed immersion.*
2. *There is an affine open cover $\{V_i\}$ of Y such that $f : f^{-1}(V_i) \to V_i$ is a closed immersion for each i.*

Proof. (1. $\Rightarrow$ 2.) As f is a closed immersion, then $f(X) \subseteq Y$ is a closed subset and $f : X \to f(X)$ is a homeomorphism. Pick any open affine $V = \mathrm{Spec}\,(B) \subseteq Y$. Then, we wish to show that $f : f^{-1}(V) \to V$ is a closed immersion. Indeed, as f is a closed immersion, therefore $f : f^{-1}(V) \to V \cap f(X)$ is a homeomorphism. As $f(X)$ is closed in Y, therefore $V \cap f(X)$ is closed in V. This shows that $g := f|_{f^{-1}(V)}$ is a topological closed immersion.

Next, we wish to show that the map $g^\flat : \mathcal{O}_V \to g_*\mathcal{O}_{f^{-1}(V)}$ is a surjection. By Remark 1.4.4.6, it suffices to show that for any $x \in f^{-1}(V)$, the local morphism $g^\sharp_x : \mathcal{O}_{V,f(x)} \to \mathcal{O}_{f^{-1}(V),x}$ is a surjection. Since $g = f|_{f^{-1}(V)}$, therefore $g^\sharp_x = f^\sharp_x$ because stalks commute with restrictions. Consequently, we wish to show that $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a surjection, but this is true by Remark 1.4.4.6 and the fact that f is a closed immersion.

(2. $\Rightarrow$ 1.) We first wish to show that f is a topological closed immersion. We first establish that f is a homeomorphism onto its image. Indeed, we have $f_i = f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i \cap f(X)$ a homeomorphism for each i. Consequently, we have a map $g_i : V_i \cap f(X) \to f^{-1}(V_i)$ which is a continuous inverse of f_i. Clearly g_i forms a matching family for $f(X) = \bigcup_i V_i \cap f(X)$ and thus can be glued to form a global inverse $g : f(X) \to X$ of f. Consequently, $f : X \to f(X)$ is a homeomorphism.

We wish to show that $f(X)$ is closed in Y. As being a closed set is a local property, therefore we need only check that $V_i \cap f(X)$ is a closed set in V_i, but this is exactly what our hypothesis that $f_i : f^{-1}(V_i) \to V_i$ a closed immersion guarantees.

Finally, we wish to show, by Remark 1.4.4.6, that $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a surjection for each $x \in X$. Indeed, as taking germs commute with restrictions, therefore $f^\sharp_x$ is the same local homomorphism as $(f_i)^\sharp_x : \mathcal{O}_{V_i,f(x)} \to \mathcal{O}_{f^{-1}(V_i),x}$ where $f(x) \in V_i$, which is surjective as f_i is a closed immersion. $\square$

The following shows that closed immersions are stable under base change.

Proposition 1.4.4.12. [14] *Let $f : X \to Y$ be a closed immersion and $g : Y' \to Y$ be any other map. Then, the map $p : X \times_Y Y' \to Y'$ is a closed immersion.*

Proof. As $f : X \to Y$ is a closed immersion, therefore by Proposition 1.4.4.11, there is an affine open cover $\{V_i = \operatorname{Spec}(B_i)\}$ of Y such that $f : f^{-1}(V_i) \to V_i$ is a closed immersion. Consequently, $f^{-1}(V_i) \cong f(f^{-1}(V_i)) \subseteq V_i$ is a closed subscheme, thus $f^{-1}(V_i) \cong \operatorname{Spec}(B_i/\mathfrak{b}_i)$ (see Corollary 1.4.4.14). Consider $g^{-1}(V_i) \subseteq Y'$ and cover it by open affines U_{ij}. Hence, we obtain an affine open cover of Y' given by $\{U_{ij} = \operatorname{Spec}\left(B'_{ij}\right)\}_{i,j}$. We claim that $p^{-1}(U_{ij}) \to U_{ij}$ is a closed immersion. Indeed, by Lemma 1.6.4.8, we have $p^{-1}(U_{ij}) \cong U_{ij} \times_{V_i} f^{-1}(V_i) \cong \operatorname{Spec}\left(B'_{ij} \otimes_{B_i} B_i/\mathfrak{b}_i\right) \cong \operatorname{Spec}\left(B'_{ij}/\mathfrak{b}_i B'_{ij}\right)$, which thus makes $p : p^{-1}(U_{ij}) \to U_{ij}$ equivalent to the scheme morphism $\operatorname{Spec}\left(B'_{ij}/\mathfrak{b}_i B'_{ij}\right) \to \operatorname{Spec}\left(B'_{ij}\right)$ obtained by the natural quotient homomorphism (this follows from the tensor product square obtained by the fiber product $U_{ij} \times_{V_i} f^{-1}(V_i)$). Consequently, it is a closed immersion by Proposition 1.2.2.8, 3, as required. $\square$

Closed subschemes and ideal sheaves

We now study closed subschemes of arbitrary schemes. To read the following results, see Section 1.9 on quasicoherent modules.

Proposition 1.4.4.13. *Let X be a scheme.*

1. *If $\mathcal{I} \leq \mathcal{O}_X$ is the ideal sheaf of a closed subscheme $Y \hookrightarrow X$, then $\mathcal{I}$ is a quasicoherent $\mathcal{O}_X$-module. If further X is Noetherian, then $\mathcal{I}$ is coherent.*
2. *If $\mathcal{I} \leq \mathcal{O}_X$ is an ideal of $\mathcal{O}_X$ such that it is quasicoherent, then $\mathcal{I}$ determines a unique closed subscheme $Y \hookrightarrow X$ where Y is given by* $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$.
3. *Consequently, we have a correspondence*

$$\left\{\begin{matrix}\textit{Quasicoherent ideal}\\ \textit{sheaves } \mathcal{I} \leq \mathcal{O}_X \textit{ upto}\\ \textit{isomorphism}\end{matrix}\right\} \cong \left\{\begin{matrix}\textit{Closed subschemes}\\ Y \hookrightarrow X\end{matrix}\right\}.$$

Proof. 1. This follows from the following facts; closed subschemes are quasicompact separated maps, that direct image of quasicoherent is quasicoherent for such maps and that kernels of maps of quasicoherent modules is quasicoherent. The second statement follows from reducing to affine and using the fact that we know all quasicoherent modules over affine.
2. Pick an ideal sheaf $\mathcal{I} \leq \mathcal{O}_X$ which is quasicoherent and let $Y = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I}) := \{x \in X \mid \mathcal{O}_{X,x}/\mathcal{I}_x \neq 0\}$. Then consider $i : (Y, \mathcal{O}_X/\mathcal{I}) \hookrightarrow (X, \mathcal{O}_X)$. It is straightforward to see that the kernel of $i^\flat$ is exactly $\mathcal{I}$. We wish to show that this is a topological closed immersion and that the map $i^\flat$ is surjective. Clearly i is homeomorphic to its image, thus we need only show that its image is a closed set. This is a local property, so let $X = \operatorname{Spec}(R)$, so that $\mathcal{I} = \widetilde{\mathfrak{a}}$ for an ideal $\mathfrak{a} \leq R$. Now $Y = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid (R/\mathfrak{a})_\mathfrak{p} \neq 0\} = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\} = V(\mathfrak{a})$. Thus i is a topological closed immersion. Now the surjectivity of the map $i^\flat : \mathcal{O}_X \to i_*\mathcal{O}_X/\mathcal{I}$ follows from going to stalks via Lemma 27.5.0.5. The uniqueness of $(Y, \mathcal{O}_X/\mathcal{I})$ w.r.t. $\mathcal{I}$ is clear. $\square$

[14] Exercise II.3.11, a of Hartshorne.

Note that the main use of quasicoherence of $\mathcal{I}$ in statement 2 was to make sure that the support of $\mathcal{O}_X/\mathcal{I}$ is indeed closed. We have a straightforward, but important corollary.

Corollary 1.4.4.14. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. We have the following bijection*

$$\{\textit{Closed subschemes } Y \hookrightarrow X\} \underset{(\operatorname{Spec}(A/\mathfrak{a}),\widetilde{A/\mathfrak{a}})\leftarrow\!\shortmid\mathfrak{a}}{\overset{\mathcal{I}\mapsto\Gamma(\mathcal{I},X)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} \{\textit{Ideals } \mathfrak{a} \le A\}/\cong\ .$$

Note that $\widetilde{A/\mathfrak{a}} \cong \mathcal{O}_{\operatorname{Spec}(A/\mathfrak{a})}$.

Proof. Follows immediately from Proposition 1.4.4.13 and Corollary 1.9.1.12. □

1.5 Varieties

Most examples of schemes that we will encounter in the wild are quasi-projective/affine varieties. Therefore, we first cover them in a semi-classical setting not involving schemes. We will then show how to interpret them as finite type separated integral schemes over the base field. This will enable us to use the machinery we will be developing for schemes in the study of varieties. Indeed, by the end of this section, we will comfortably replace the definition of a variety to mean a separated, integral finite type scheme over an algebraically closed field.

1.5.1 Varieties over an algebraically closed field-I

We define varieties as zero sets of certain polynomials over an algebraically closed field k. We assume that the reader is aware of the Zariski topology that is present over $\mathbb{A}^n_k$. Let us first give the classical version of affine varieties.

Definition 1.5.1.1. (**Affine algebraic variety**) Let k be an algebraically closed field and let $\mathbb{A}^n_k$ be the affine n-space. An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n_k$.

We recall that the Hilbert Nullstellensatz further tells us that for any ideal $\mathfrak{a} \leq k[x_1, \dots, x_n]$, the zero set of the ideal $Z(\mathfrak{a}) \subseteq \mathbb{A}^n_k$ is such that the ideal it generates is equal to the radical of the ideal, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Let $A \subseteq \mathbb{A}^n_k$ be an affine algebraic set. Then, the *affine coordinate ring* of A is defined to be the following finitely generated k-algebra

$$k[A] := \frac{k[x_1, \dots, x_n]}{I(A)}$$

where $I(A) \leq k[x_1, \dots, x_n]$ is the ideal generated by A. An important simple lemma to keep in mind for future is the following.

Lemma 1.5.1.2. *Let k be an algebraically closed field. Then B is a finitely generated k-algebra without nilpotent elements if and only if B is an affine coordinate ring of an algebraic set.*

Proof. One side is trivial and the other uses Nullstellensatz. □

Example 1.5.1.3. (*Hyperboloid of one sheet*) A recurring example that we choose to study in this notebook, amongst the others, is the hyperboloid of one sheet. This is given by the following equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

In the affine space over $\mathbb{R}$, $\mathbb{A}^3_{\mathbb{R}}$, we can draw it as shown in Figure 1.1.
We may simply call it a hyperboloid. This hyperboloid determines an affine variety given

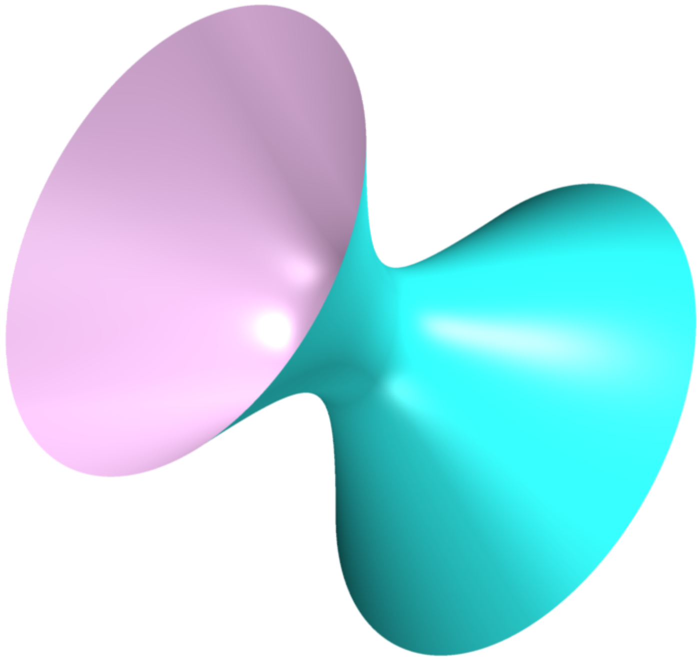

Figure 1.1: A hyperboloid of one sheet as a subvariety of $\mathbb{A}^3_{\mathbb{R}}$. The parameters are $a = 1.05, b = 1.05, c = 1$.

by the zero set of the polynomial

$$p(x, y, z) = x^2/a^2 + y^2/b^2 - z^2/c^2 - 1 \in k[x, y, z]$$

for any field k. Let $X = V(p) \subseteq \mathbb{A}^3_k$. The coordinate ring is given by

$$k[X] = \frac{k[x, y, z]}{I(V(p))}.$$

As we shall see, we will associate to the above variety $(X, \mathcal{O}_X)$ a scheme by considering the spectrum of the coordinate ring, $\operatorname{Spec}(k[X])$.

We will understand this fantastic example in much more detail as we develop more tools to handle it.

We now define projective varieties. Consider an algebraically closed field. Then the *projective n-space* is defined to be the quotient $\mathbb{P}^n_k := \mathbb{A}^{n+1}_k / \sim$ where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if and only if there exists $\lambda \in k^\times$ such that $a_i = \lambda b_i$ for all $i = 0, 1 \dots, n$. A point of $\mathbb{P}^n_k$ is denoted by $[a_0 : \dots : a_n]$ and this presentation of the point is called the homogeneous coordinates of the point. Assuming that the reader is aware about graded rings and the natural grading of $k[x_0, \dots, x_n]$, we observe that we can talk about the *zeroes of a homogeneous polynomial* $p(X) \in k[x_0, ;x_n]$ as follows:

$$Z(p) := \{P \in \mathbb{P}^n_k \mid p(P) = 0\}.$$

Indeed, one observes that a homogeneous polynomial is zero at a point $P \in P_k^n$ in a manner which is independent of the choice of representation of P in terms of the homogeneous coordinates of P. With this in our hand, we further define the zero set of a homogeneous ideal $\mathfrak{a} \leq k[x_0, \ldots, x_n]$ as

$$Z(\mathfrak{a}) := \{P \in \mathbb{P}_k^n \mid f(P) = 0 \forall f \in T_\mathfrak{a}\}$$

where $T_\mathfrak{a}$ is the set of all homogeneous elements of $\mathfrak{a}$. Remember that an ideal in a graded ring is homogeneous if and only if it is generated by the set of all of its homogeneous elements.

Lemma 1.5.1.4. *Let k be a field. Then*

1. *For any two homogeneous ideals $\mathfrak{a}, \mathfrak{b} \leq k[x_0, \ldots, x_n]$, we have $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$.*
2. *For any family of homogeneous ideals $\{\mathfrak{a}_\mathfrak{i}\}_{\{}i \in I\}$, we have $\cap_{i\in I} Z(\mathfrak{a}_\mathfrak{i}) = Z(\sum_{i\in I} \mathfrak{a}_\mathfrak{i})$.*

Proof. Straightforward unravelling of definitions. □

Therefore we obtain a topology on $\mathbb{P}_k^n$ where a set $Y \subseteq \mathbb{P}_k^n$ is closed if and only if $Y = Z(\mathfrak{a}_\mathfrak{i})$ for a homogeneous ideal $\mathfrak{a}_\mathfrak{i}$ of $k[x_0, \ldots, x_n]$. This is called the Zariski topology of $\mathbb{P}_k^n$.

Definition 1.5.1.5. (**Projective algebraic variety**) Let k be an algebraically closed field. An irreducible algebraic set of $\mathbb{P}_k^n$ is said to be a projective algebraic variety in $\mathbb{P}_k^n$.

Let $V \subseteq \mathbb{P}_k^n$ be a projective algebraic variety. Then the *ideal generated by V* in $k[x_0, \ldots, x_n]$ is $I(V)$ which is the ideal generated by the following set of homogeneous polynomials: $\{f \in k[x_0, \ldots, x_n] \mid f$ is homogeneous and $f(P) = 0\}$.

For a projective algebraic set $Y \subseteq \mathbb{P}_k^n$, we define its *homogeneous coordinate ring* to be the following k-algebra

$$k[Y] := \frac{k[x_0, \ldots, x_n]}{I(Y)}$$

where $I(Y)$ is the homogeneous ideal of Y.

Definition 1.5.1.6 (**Zero set and ideal of an algebraic set**)**.** Define for any set $T \subseteq k[x_0, \dot{;} x_n]$ of homogeneous elements the zero set of T as $Z(T) = \{p \in \mathbb{P}_k^n \mid f(p) = 0 \ \forall \ f \in T\}$. For any $Y \subseteq \mathbb{P}_k^n$, define $I(Y)$ as the ideal in $k[x_0, \ldots, x_n]$ generated by $\{f \in k[x_0, \ldots, x_n] \mid f$ is homogeneous & $f(p) = 0 \ \forall p \in Y\}$.

To distinguish between affine and projective cases, we will reserve $Z(\mathfrak{a})$ for zero set of a homogeneous ideal in projective space and $V(\mathfrak{a})$ as the zero set of an ideal in the affine space.

We now show that how the projective space $\mathbb{P}_k^n$ is covered by $n+1$ copies of affine space $\mathbb{A}_k^n$. Before that we discuss few maps which allows us to treat affine case projectively.

Homogenization and dehomogenization

One way to move back and from affine to projective setting is to use to fundamental functions between $k[y_1, \dots, y_i, \dots, y_n]$ and $k[x_0, \dots, x_n]_h$.

Definition 1.5.1.7. (**(De)homogenization**) Let k be an algebraically closed field and let $A := k[y_1, \dots, y_n]$ and $B := k[x_0, \dots, x_n]_h$, the set of all homogeneous polynomials in $k[x_0, \dots, x_n]$. Consider the following two functions

$$\begin{aligned} d_i : B &\longrightarrow A \\ f(x_0, \dots, x_n) &\longmapsto f(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \\ h_i : A &\longrightarrow B \\ g(y_1, \dots, y_n) &\longmapsto x_i^e g\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) \end{aligned}$$

where e is the degree of g and $i = 0, \dots, n$. The map h_i is called the i^{th}-homogenization map and d_i is called the i^{th}-dehomogenization map.

Using this, we can establish the result in question.

Proposition 1.5.1.8. *Let k be an algebraically closed field and consider the projective n-space over k, $\mathbb{P}^n_k$. Then, there exists $n+1$ open subspaces say $U_i \subseteq \mathbb{P}^n_k$, such that $\mathbb{P}^n_k = \bigcup_{i=0}^n U_i$ and for each i, U_i is homeomorphic to $\mathbb{A}^n_k$.*

Proof. Consider the $n+1$ open subspaces of $\mathbb{P}^n_k$ as follows:

$$U_i := \mathbb{P}^n_k \setminus H_i$$

where $H_i = Z(\langle x_i \rangle)$ is the algebraic set obtained by all those points whose i^{th} homogeneous coordinate is zero. Now consider the map

$$\begin{aligned} \varphi_i : U_i &\longrightarrow \mathbb{A}^n_k \\ [a_0 : \dots : a_n] &\longmapsto \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i}\right). \end{aligned}$$

One can check that this pulls closed sets to closed sets by using the i^{th}-homogenization map. Conversely, one can define the map

$$\begin{aligned} \theta_i : \mathbb{A}^n_k &\longrightarrow U_i \\ (a_1, \dots, a_n) &\longmapsto (a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) \end{aligned}$$

and this can again be checked to be continuous by an application of i^{th} dehomogenization map. $\square$

Corollary 1.5.1.9. *Let k be an algebraically closed field and $Y \subseteq \mathbb{P}^n_k$ be a projective algebraic variety. Then, in the notation of Proposition 1.5.1.8, for each $i = 0, \dots, n$, $Y \cap U_i$ is an affine algebraic variety.*

Proof. This follows from the observation that $Y \cap U_i$ is a closed set of $U_i \cong \mathbb{A}^n_k$. The irreducibility follows from the fact that open subsets of irreducible spaces are irreducible. $\square$

Properties of algebraic sets in $\mathbb{P}^n_k$

We now present some basic properties of algebraic sets in $\mathbb{P}^n_k$.

Lemma 1.5.1.10. [15] *(Homogeneous Nullstellensatz) Let k be an algebraically closed field and let $\mathfrak{a} \leq k[x_0, \dots, x_n]$ be a homogeneous ideal. Then,*

$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Proof. Denote by $V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$ to be the vanishing set of $\mathfrak{a}$ in the affine $n+1$-space. This is called the *affine cone* of the ideal $\mathfrak{a}$ in $\mathbb{A}^{n+1}$. We claim that $I(Z(\mathfrak{a})) \hookrightarrow I(V(\mathfrak{a}))$ since if $f \in I(Z(\mathfrak{a}))$ is homogeneous, then $f(P) = 0$ for all $P \in Z(\mathfrak{a}) = \{P \in \mathbb{P}^n_k \mid g(P) = 0 \, \forall g \in \mathfrak{a}\}$. Pick any point $Q \in V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}_k$. We see that $g(Q) = 0$ for all $g \in \mathfrak{a}$. We wish to show that $f(Q) = 0$. As any point $Q \in V(\mathfrak{a})$ determines a point $P \in Z(\mathfrak{a})$ by scaling, that is $P = \lambda Q$, we get by homogeneity of f that $f(Q) = f(\lambda P) = \lambda^d f(P) = 0$, that is, $f \in I(V(\mathfrak{a}))$, as required. By affine Nullstellensatz, it follows that $I(Z(\mathfrak{a})) \subseteq \sqrt{a}$. The converse is straightforward. $\square$

The following tells us when is a projective algebraic set is empty.

Lemma 1.5.1.11. [16] *Let $\mathfrak{a} \leq k[x_0, \dots, x_n] = S$ be a homogeneous ideal. Then, the following are equivalent:*
1. *$Z(\mathfrak{a}) = \emptyset$ in $\mathbb{P}^n_k$,*
2. *$\sqrt{\mathfrak{a}}$ is either S or S_+,*
3. *$\mathfrak{a} \supseteq S_d$ for some $d > 0$.*

Proof. (1. $\Rightarrow$ 2.) The main idea here is again to reduce to affine case by considering the affine cone. Observe that if $Z(\mathfrak{a}) = \emptyset$, then $V(\mathfrak{a}) \subseteq \{0\}$ (where $V(\mathfrak{a})$ is the vanishing in $\mathbb{A}^{n+1}_k$ as in the proof of Lemma 1.5.1.13). Indeed, if not then there exists $p = (p_0, \dots, p_n) \in V(\mathfrak{a})$ such that $p \neq 0$. It follows that $[p_0 : \dots : p_n] \in Z(\mathfrak{a})$ since any homogeneous element f of $\mathfrak{a}$ vanishes at p in $\mathbb{A}^{n+1}_k$. Now if $V(\mathfrak{a}) = \emptyset$, then by the affine nullstellensatz, we get $\sqrt{\mathfrak{a}} = S$. If $V(\mathfrak{a}) = 0$, then $\sqrt{a} = I(0) = \langle x_0, \dots, x_n \rangle = S_+$.

(2. $\Rightarrow$ 1.) As $\sqrt{a} = I(V(\mathfrak{a})) = S$ or S_+, therefore $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a}) = \emptyset$ or 0. It follows again that $Z(\mathfrak{a}) = \emptyset$.

(2. $\Rightarrow$ 3.) **TODO.** $\square$

Akin to affine varieties, we also have some basic results in projective algebraic sets.

Lemma 1.5.1.12. [17] *Let $\mathbb{P}^n_k$ be the projective n-space over k and let $S = k[x_0, \dots, x_n]$*
1. *If $Y_1 \subseteq Y_2$ in $\mathbb{P}^n_k$, then $I(Y_1) \supseteq I(Y_2)$.*
2. *If $T_1 \subseteq T_2$ in S be subsets of homegeneous elements, then $Z(T_1) \supseteq Z(Y_2)$.*
3. *If $Y_1, Y_2 \subseteq \mathbb{P}^n_k$, then $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*
4. *If $Y \subseteq \mathbb{P}^n_k$, then $Z(I(Y)) = \overline{Y}$.*

Proof. content... $\square$

[15] Exercise I.2.1 of Hartshorne.
[16] Exercise I.2.2 of Hartshorne.
[17] Exercise I.2.3 of Hartshorne.

Some consequences of the homogeneous nullstellensatz yields us the familiar results as in the affine case.

Lemma 1.5.1.13. [18] *Let k be an algebraically closed field and consider the projective n-space $\mathbb{P}^n_k$. Then,*

1. *There is a bijection*

$$\{All\ algebraic\ sets\ Y \subseteq \mathbb{P}^n_k\} \underset{Z}{\overset{I}{\rightleftarrows}} \{All\ homogeneous\ radical\ ideals\ of\ k[x_0, \dots, x_n]\}\ .$$

2. *An algebraic set $Y \subseteq \mathbb{P}^n_k$ is irreducible if and only if $I(Y)$ is a prime ideal in $k[x_0, \dots, x_n]$.*
3. *$\mathbb{P}^n_k$ is a projective algebraic variety.*

Remark 1.5.1.14. A corollary of the above lemma is that one can look at projective algebraic varieties in $\mathbb{P}^n_k$ akin to homogeneous prime ideals in $k[x_0, \dots, x_n]$, thus telling us another hint at how the idea of schemes might have looked back in the days.

Proof of Lemma 1.5.1.13. 1. This is a direct consequence of homogeneous nullstellensatz (Lemma 1.5.1.10) and the fact that $Z(I(Y)) = \bar{Y}$ for any $Y \subseteq \mathbb{P}^n_k$.

2. (L $\Rightarrow$ R) Suppose $Y = Z(\mathfrak{a})$ is irreducible and $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ is not prime. Then there exists $f, g \notin \mathfrak{a}$ such that $fg \in \sqrt{\mathfrak{a}}$. Consider the ideals $\mathfrak{b} := \langle f, \sqrt{\mathfrak{a}} \rangle$ and $\mathfrak{c} := \langle g, \sqrt{\mathfrak{a}} \rangle$. We then observe that $Z(\mathfrak{b}), Z(\mathfrak{c}) \subseteq Z(\mathfrak{a})$ and $Z(\mathfrak{b}) \cup Z(\mathfrak{c}) = Z(\mathfrak{b}\mathfrak{c}) = Z(\sqrt{\mathfrak{a}}) = Z(\mathfrak{a})$, where we have used Lemma 1.5.1.10 in the second last equation and the fact that $fg \in \sqrt{a}$ in third last. This yields a contradiction to the irreducibility of Y.
(R $\Rightarrow$ L) Suppose $I(Y)$ is prime but Y is not irreducible. Consequently, there are proper closed sets $Y_1, Y_2 \subseteq Y$ such that $Y_1 \cup Y_2 = Y$. Further, we obtain that $I(Y_i) \gneq I(Y)$ for each $i = 1, 2$. It then follows that there exists $f_i \in I(Y_i) \setminus I(Y)$ such that $f_i \notin I(Y_j)$, $j \neq i$. Consequently, we have $f_1 f_2 \in k[x_0, \dots, x_n]$ such that $f_1 f_2(P) = f_1(P) f_2(P) = 0$ for all $P \in Y$, as $Y = Y_1 \cup Y_2$. We thus have a contradiction to primality of $I(Y)$.

3. Since $I(\mathbb{P}^n_k) = I(Z(\mathfrak{o})) = \sqrt{\mathfrak{o}} = \mathfrak{o}$, then by 2., $\mathbb{P}^n_k$ is irreducible. Note we have used the fact that $k[x_0, \dots, x_n]$ is an integral domain. □

One of the reasons that one might be interested in projective varieties is that they "compactify" the question at hand, that is, there are no "missing points" in the ambient space. We will see more into this when we will see projective morphisms and invertible modules, but for now, it is good to keep in mind that reframing your question in the projective spaces/varieties may give you more handle (and of-course, machines) to solve the question at hand. In the same vein, we now see that every affine variety can be embedded compactly into a projective space, and this embedding is called the projective closure of the affine variety.

Definition 1.5.1.15. (**Projective closure of affine varieties**) Let k be an algebraically closed field and consider an affine variety $X \subseteq \mathbb{A}^n_k$. For any $i = 0, \dots, n$, consider the

[18] Exercise I.2.4 of Hartshorne.

homeomorphism

$$\begin{aligned}\theta_i : \mathbb{A}^n_k &\longmapsto U_i\\ (a_1, \dots, a_n) &\longmapsto [1 : a_1 : \dots : a_n],\end{aligned}$$

as we considered in Proposition 1.5.1.8. Then, the i^{th}-projective closure of X into $\mathbb{P}^n_k$ is given by the closure $\overline{\theta_i(Y)} \subseteq \mathbb{P}^n_k$ as a subspace in $\mathbb{P}^n_k$. We will usually say the 0th projective closure of X to be simply the *projective closure of* X.

Consider an affine variety $X \subseteq \mathbb{A}^n_k$ and consider $\overline{X} \subseteq \mathbb{P}^n_k$ to be the projective closure of X. Let $I(X) \leq k[y_1, \dots, y_n]$ be the affine ideal of X and let $I(\overline{X}) \leq k[x_0, \dots, x_n]$ be the homogeneous ideal of projective closure. A natural question is that how the homogeneous ideal $I(\overline{X})$ is connected to the affine ideal $I(X)$. The following proposition answers that.

Proposition 1.5.1.16. *Let k be an algebraically closed field and $X \subseteq \mathbb{A}^n_k$ be an affine variety. Let $I(X) \leq k[y_1, \dots, y_n]$ be the affine ideal of X and let $I(\overline{X}) \leq k[x_0, \dots, x_n]$ be the homogeneous ideal of projective closure. Then,*

$$I(\overline{X}) = \langle h_0(I(X))\rangle$$

where $h_0 : k[y_1, \dots, y_n] \to k[x_0, \dots, x_n]$ is the 0th homogenization function (Definition 1.5.1.7).

Proof. Since $X \subseteq \mathbb{A}^n_k$ is irreducible and closure of irreducible is irreducible, therefore $\overline{X} \subseteq \mathbb{P}^n_k$ is irreducible. It would thus suffice to show that

$$\overline{X} = Z(h_0(I(X))).$$

Indeed, this would imply that $h_0(I(X))$ is a homogeneous prime ideal by Lemma 1.5.1.13, 1, thus applying $I(-)$ would yield the result. We therefore show the above equality. Consider any closed set $Y \supseteq X$ in $\mathbb{P}^n_k$. We then wish to show that $Y \supseteq Z(h_0(I(X)))$. Since $Y \subseteq \mathbb{P}^n_k$ is closed, therefore $Y = Z(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a}$ in $k[x_0, \dots, x_n]$. It would thus suffice to show that

$$\mathfrak{a} \hookrightarrow h_0(I(X)).$$

It would further suffice to show the above inclusion only for homogeneous elements, as $\mathfrak{a}$ is generated by homogeneous elements. Consequently, pick any homogeneous polynomial $f \in \mathfrak{a}$. We can write

$$f(x_0, \dots, x_n) = x_0^e g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

for some $g \in k[y_1, \dots, y_n]$ and $e = \deg f$. In other words, $f = h_0(g)$. Now since $Y \supseteq X$, therefore $f(P) = 0 \forall P \in X \subseteq \mathbb{P}^n_k$, that is, if $P = [1 : a_1, \dots, a_n] \in X$, then $f(1, a_1, \dots, a_n) = 0$ and thus $g(a_1, \dots, a_n) = 0$. Hence $g \in I(X) \leq k[y_1, \dots, y_n]$. Thus $f = h_0(g)$ where $g \in I(X)$, that is, $f \in h_0(I(X))$, as required. $\square$

Dimension, hypersurfaces and complete intersections

Let us first understand how the notion of dimension plays out with the Krull dimension of homogeneous coordinate ring of a projective variety.

Proposition 1.5.1.17. *Let k be an algebraically closed field and $X \subseteq \mathbb{P}^n_k$ be a projective k-variety. Then,*

1. $\dim k[X] = \dim X + 1$,
2. $\dim X = \dim U_i \cap X$ *where $U_i \subseteq X$ is an affine open subset as in Proposition 1.5.1.8, for all $i = 0, \dots, n$.*

Proof. We will prove the two statements together. The main technique here is, as usual, to reduce the computations to one of the affine patches. Let $U_i \subseteq \mathbb{P}^n_k$ be the hyperplane where $x_i \neq 0$. We know that U_is covers $\mathbb{P}^n_k$ and each U_i is isomorphic to $\mathbb{A}^n_k$. Denote $X_i = U_i \cap X$ so that X_i is an open subvariety of X. Further denote $k[X]^h$ to be the homogeneous coordinate ring of X and $k[X_i]^a$ the affine coordinate ring of X_i. Note that $k[X_i]^a = k[x_0, \dots, \hat{x}_i, \dots, x_n]/d_i I(X)$ where d_i is the i^{th} dehomogenisation map. We would now like to note two things to move forward:

1. $\dim X = \dim X_j$ for some $j = 0, \dots, n$,
2. $k[X]^h_{x_i} \cong k[X_i]^a[x_i, 1/x_i]$[19].

The first statement is immediate from the fact that $\dim Y = \sup_i \dim U_i$ for any space Y with U_i an open covering. The second statement is the heart of the proof. Indeed, consider the map $k[X]^h_{x_i} \to k[X_i]^a[x_i, 1/x_i]$ which takes an element f/x_i^n and treats it as a polynomial in $x_i, 1/x_i$ with coefficients in $k[X_i]^a$. One immediately checks all the necessary conditions to ensure that this is an isomorphism.

Observe that if K/k is algebraic, then $K(x)/k(x)$ is algebraic. It follows that trdeg $k[X_i]^a[x_i, 1/x_i] = 1 + \text{trdeg } k[X_i]^a$. We now complete the proof. We may assume $\dim X = \dim X_0$. Consequently, via Proposition 1.5.3.10, 6 and Theorem 23.8.2.1, we obtain the following equalities:

$$\begin{aligned}\dim k[X]^h &= \text{trdeg } k[X]^h = \text{trdeg } k[X]^h_{x_0} = \text{trdeg } k[X_0]^a[x_0, 1/x_0] = 1 + \text{trdeg } k[X_0]^a \\ &= 1 + \dim k[X_0]^a = 1 + \dim X_0 = 1 + \dim X.\end{aligned}$$

The statement 2. follows from the following equalities:

$$\begin{aligned}\dim X_i = \dim k[X_i]^a &= \text{trdeg } k[X_i]^a_{x_i} = \text{trdeg } k[X_i]^a[x_i, 1/x_i] - 1 = \text{trdeg } k[X]^h_{x_0} - 1 \\ &= \text{trdeg } k[X]^h - 1 = \dim X + 1 - 1 = \dim X.\end{aligned}$$

□

We would now like to establish the following result, which will later motivate the definition of Weil divisors and of complete intersections.

Lemma 1.5.1.18. *Let k be an algebraically closed field and $X \subseteq \mathbb{P}^n_k$ be a projective k-variety. Then, the following are equivalent*

1. $\dim X = n - 1$.

[19] This statement can be seen as a generalization of Lemma 1.5.3.11.

2. *The homogeneous ideal $I(X) \leq k[x_1, \dots, x_n]$ is generated by a single irreducible homogeneous polynomial.*

Proof. (1. $\Rightarrow$ 2.) By Proposition 1.5.1.17, 1, we have $\dim k[X] = n$, where $k[X] = k[x_0, \dots, x_n]/I(X)$. By Theorem 23.8.2.2, we have ht $I(X) = 1$. Since any height 1 prime ideal of a UFD is principal, therefore $I(X)$ is principal. Since $I(X)$ is homogeneous, therefore the statement 2. follows.
(2. $\Rightarrow$ 1.) By Proposition 1.5.1.17, 2 and Theorem 23.8.2.2, we have

$$\dim X = \dim X_0 = \dim k[X_0]^a = n - \text{ht } d_0(I(X)).$$

We need only show that ht $d_0(I(X)) = 1$. Since $I(X) = \langle p(x_0, \dots, x_n) \rangle$, therefore $d_o(I(X)) = \langle p(1, x_1, \dots, x_n) \rangle$. Since $k[x_0, \dots, x_n]$ is a UFD and an easy observation about UFDs yields that height 1 prime ideals are exactly principal prime ideals, therefore the result follows. □

Cones

d-uple embedding

Veronese surface

Segre embedding

To complete cor uple, Veronese, and quadrics, C ter 1.

1.5.2 Morphism of varieties

We have defined affine and projective varieties so far. One would often, however, would like to know whether a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$ is an open subspace of some affine or projective variety. Due to to this need, we define the following.

Definition 1.5.2.1. (**Quasi-affine/projective variety**) A subset X of $\mathbb{A}^n$ or $\mathbb{P}^n$ is said to be quasi-affine or quasi-projective if X is an open subset of an affine or projective variety, respectively.

Let X be a quasi-affine or projective variety. From our knowledge of geometry, we know that in a real C^α-manifold M, the right type of functions are those which are defined on open subsets of M as C^α-maps to $\mathbb{R}$, where the latter is treated as a C^α-manifold. Consequently, we are interested in the same type of maps to the affine line $\mathbb{A}^1_k$.

Definition 1.5.2.2. (**Regular maps**) This notion is defined differently for quasi-affine and quasi-projective varieties.

1. Let X be a quasi-affine variety. A function

$$\varphi : X \to \mathbb{A}^1_k$$

is said to be a *regular function* if for all $P \in X$, there exists an open subset $U \subseteq X$ such that $\varphi|_U = g/h$ where $g, h \in k[x_1, \dots, x_n]$ and $h(P) \neq 0 \ \forall P \in U$.

2. Let X be a quasi-projective variety. A function
$$\varphi : X \to \mathbb{A}^1_k$$
is said to be a *regular function* if for all $P \in X$, there exists an open subset $U \subseteq X$ such that $\varphi|_U = g/h$ where $g, h \in k[x_0, \dots, x_n]$ are homogeneous polynomials of same degree and $h(P) \neq 0\ \forall P \in U$. Note that this defines a valid function to the affine line.

Indeed, regular maps are continuous.

Lemma 1.5.2.3. *Let X be a quasi-affine or quasi-projective variety and $\varphi : X \to \mathbb{A}^1_k$ be a regular function. Then φ is continuous.*

Proof. The Zariski topology on $\mathbb{A}^1_k$ is the cofinite topology, hence any closed set in $\mathbb{A}^1_k$ is a finite union of points of k. It thus suffices to show that for any $a \in k$, $Y := \varphi^{-1}(a) \subseteq X$ is closed. Since checking a set is closed is local in X, that is, $Y \subseteq X$ is closed if and only if there exists an open covering of X, say $\{U_\alpha\}$ such that $U_\alpha \cap Y$ is closed in U_α. We may thus replace X by an open subset of X where φ is represented as g/h for $g, h \in k[y_1, \dots, y_n]$ (in $k[x_0, \dots, x_n]$, homogeneous and of same degree in the projective case). Consequently, $\varphi^{-1}(a) \subseteq X$ is given by $\{P \in X \mid (g - ah)(P) = 0\}$ which in other words is $Z(g - ah)$ ($g - ah$ is homogeneous in the projective case). Thus $\varphi^{-1}(a)$ is closed, as required. □

A simple corollary of above is the first striking result one learns in complex analysis for holomorphic maps (see Proposition 15.2.3.10).

Lemma 1.5.2.4. *(Identity principle) Let $\varphi, \xi : X \to \mathbb{A}^1_k$ be two regular maps over a quasi-affine or quasi-projective variety X. Then, $\varphi = \xi$ if and only if there exists an open set $U \subseteq X$ such that $\varphi = \xi$ over U.*

Proof. L $\Rightarrow$ R is easy. For R $\Rightarrow$ L, observe that for $\phi := \varphi - \xi$ is continuous by Lemma 1.5.2.3. Further, the set $\phi^{-1}(0) \subseteq X$ is closed and contains U. Since $\phi^{-1}(0) \supseteq U$ and U is an open set of an irreducible space, therefore U is dense in X. Consequently, $\phi^{-1}(0)$ is a closed and dense in X, hence is equal to X. □

We now define varieties in general.

Definition 1.5.2.5. (**Varieties**) Let k be an algebraically closed field. A variety over k is defined to be a quasi-affine or a quasi-projective variety in $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$, respectively.

The notion of morphism of varieties is then given by functions which pulls regular functions back by pre-composition.

Definition 1.5.2.6. (**Map of varieties**) Let k be an algebraically closed field and let X, Y be two varieties over k. A map of varieties is a continuous function $f : X \to Y$ such that for any open set $V \subseteq Y$ and any regular function $\varphi : Y \to \mathbb{A}^1_k$, the function
$$\varphi \circ f : \varphi^{-1}(V) \to \mathbb{A}^1_k$$
is a regular function on the open set $\varphi^{-1}(V)$ of X. We may also call a map of varieties a *morphism of varieties*.

We therefore obtain the category of varieties over k, whose objects are varieties over k and arrows are maps of varieties. We will denote this category by

$$\mathbf{Var}_k.$$

Just like in topological spaces, it is not true in general that a bijective continuous map is a homeomorphism, similarly it is not true in general that a bijective map of varieties is an isomorphism of varieties, as the following example shows.

Example 1.5.2.7. Consider the affine line $\mathbb{A}^1_k$ and consider the affine variety $X := Z(y^2 - x^3) \subseteq \mathbb{A}^2_k$. The function

$$\begin{aligned} f : \mathbb{A}^1_k &\longrightarrow X \\ t &\longmapsto (t^2, t^3) \end{aligned}$$

is a map of varieties as for any open set $U \subseteq X$ and regular map $\varphi : X \to \mathbb{A}^1_k$, the composite $\varphi \circ f : \varphi^{-1}(U) \to \mathbb{A}^1_k$ is given by $t \mapsto \varphi(t^2, t^3)$ and then the regularity of this composite can be seen to be a result of regularity of φ. Further note that f induces an inverse continuous function

$$\begin{aligned} f^{-1} : X &\longrightarrow \mathbb{A}^1_k \\ (a, b) &\longmapsto ba^{-1}. \end{aligned}$$

Thus, $\mathbb{A}^1_k$ and X are homeomorphic as topological spaces. However, as varieties, they can not be isomorphic. Indeed, we shall soon see that coordinate rings are invariant of affine varieties and in our case $\mathbb{A}^1_k$ has $k[x]$ as its coordinate ring whereas X has $k[x, y]/\langle y^2 - x^3\rangle$ as its coordinate ring. These are not isomorphic as one is PID and the other is not.

We now construct some more algebraic gadgets on top of varieties and will prove how they will turn out to be invariants of the varieties under question. We have already seen one, the coordinate ring. We will now see the construction of others and we shall do it in a manner so that it is amenable to generalization to schemes, as is studied elsewhere in this chapter.

1.5.3 Varieties as locally ringed spaces

See Chapter 8, Foundational Geometry, for background on locally ringed spaces and basic global algebra. In this section, we would like to interpret varieties as locally ringed spaces, so that we can understand later that how a variety can be interpreted as a scheme. Clearly, for a variety X, we already have an underlying topological space X itself. To give X the structure of a locally ringed space, we need to consider a sheaf over X. We shall use regular functions over open sets of X for that.

Definition 1.5.3.1. (**Structure sheaf of a variety**) Let k be an algebraically closed field and X be a variety over k. For each open set $U \subseteq X$, consider the following set

$$\mathcal{O}_X(U) := \{f : U \to \mathbb{A}^1_k \mid f \text{ is regular}\}.$$

Further, for open $V \subseteq U$ in X, consider the function

$$\begin{aligned} \rho_{U,V} : \mathcal{O}_X(U) &\longrightarrow \mathcal{O}_X(V) \\ f &\longmapsto f|_V . \end{aligned}$$

This defines a sheaf of sets, as the following lemma shows.

Lemma 1.5.3.2. *The assignment $\mathcal{O}_X$ on open sets of a k-variety X as defined in Definition 1.5.3.1 defines a sheaf of sets over X.*

Proof. The locality axiom is straightforward as $\mathcal{O}_X(U)$ is a collection of functions, which thus can be checked locally for equality. It thus suffices to show that $\mathcal{O}_X$ satisfies the gluing axiom. Pick any open set U, an open covering $\{U_i\}_{i\in I}$ of U and a matching family $f_i \in \mathcal{O}_X(U_i)$ for each $i \in I$, that is $\rho_{U_i,U_i\cap U_j}(f_i) = \rho_{U_j,U_i\cap U_j}(f_j)$ for each $i, j \in I$. Consequently, we define $f : U \to \mathbb{A}^1_k$ given by $x \mapsto f_i(x)$ if $x \in U_i$. This is a well-defined function by the matching condition and further f is a regular function as for each point $x \in U$, f can be written as a rational function in some open neighborhood around x (essentially by regularity of f_is). Consequently, $\mathcal{O}_X$ is a sheaf. $\square$

Further, $\mathcal{O}_X$ is a sheaf of k-algebras if X is a k-variety.

Lemma 1.5.3.3. *Let k be an algebraically closed field and consider a k-variety X. The structure sheaf $\mathcal{O}_X$ of X is a sheaf of k-algebras.*

Proof. Indeed, $\mathcal{O}_X$ is a ring by point-wise addition and multiplication. Further, its a k-algebra via the injective ring homomorphism

$$\begin{aligned} k &\hookrightarrow \mathcal{O}_X(U) \\ c &\mapsto c : U \to \mathbb{A}^1_k \end{aligned}$$

where c is treated as the constant rational map. $\square$

Hence, $(X, \mathcal{O}_X)$ is a k-ringed space. We now show that it is locally k-ringed.

Lemma 1.5.3.4. *Let k be an algebraically closed field and let X be a k-variety. Then, for all points $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.*

Proof. We wish to show that $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x \leq \mathcal{O}_{X,x}$. Consider the set

$$\mathfrak{m}_x := \{(U, f) \in \mathcal{O}_{X,x} \mid f(x) = 0\}.$$

It then easily follows that $\mathfrak{m}_x$ an ideal and consequently is a maximal ideal because $\mathcal{O}_{X,x} \setminus \mathfrak{m}_x$ is jut the set of all units of $\mathcal{O}_{X,x}$. $\square$

Remark 1.5.3.5. We have thus established that for any k-variety X we obtain a locally k-ringed space $(X, \mathcal{O}_X)$. We now observe how the data of a morphism of varieties can be represented as data of a morphism of underlying locally ringed spaces.

The notion of morphism of locally ringed spaces is elucidated in Definition 8.1.0.2.

Lemma 1.5.3.6. *Let k be an algebraically closed field and X, Y be two k-varieties. Then, there is an injective inclusion*

$$\mathrm{Hom}_{\mathbf{Var}_k}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathbf{LRSpace}}(X,Y).$$

Proof. Indeed, consider the map

$$\begin{aligned}\theta : \mathrm{Hom}_{\mathbf{Var}_k}(X,Y) &\hookrightarrow \mathrm{Hom}_{\mathbf{LRSpace}}(X,Y)\\ f : X \to Y &\longmapsto (f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\end{aligned}$$

where $\theta(f)$ has the underlying continuous map same as f but the map on sheaves, $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$, is given on sections as follows: let $V \subseteq Y$ be an open set, then the map on sections over V is

$$\begin{aligned}f^\flat_V : \mathcal{O}_Y(V) &\longrightarrow \mathcal{O}_X(f^{-1}(V))\\ (V,\varphi) &\longmapsto (f^{-1}(V), \varphi \circ f).\end{aligned}$$

The fact that $f^\flat$ as defined above is indeed a sheaf morphism is straightforward. We thus need only show that the adjoint map $f^\sharp$ of the above defines a map on stalks which is local. For this, we need only observe how the comorphism, $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$, as defined in Definition 8.1.0.2, in this case turns out to be the following mapping

$$(V,\varphi)_x \longmapsto (f^{-1}(V), \varphi \circ f)_x.$$

Now if $(V,\varphi)_x \in \mathfrak{m}_{Y,f(x)}$, then $\varphi(f(x)) = 0$ by definition. Thus $(f^{-1}(V), \varphi \circ f) \in \mathfrak{m}_{X,x}$. With this, the fact that θ is injective is straightforward. $\square$

Remark 1.5.3.7. We therefore have an inclusion

$$\mathbf{Var}_k \hookrightarrow \mathbf{LRSpace}.$$

Indeed, we now show that the notion of isomorphisms coincide here.

We will now define various algebraic gadgets out of the structure sheaf $\mathcal{O}_X$ of a variety X. Indeed, to some extent, that's the goal of algebraic geometry in general.

We now define an important field corresponding to each variety X, called its function field.

Definition 1.5.3.8. (**Function field of a variety**) Let k be an algebraically closed field and X be a k-variety. The function field of X, denoted $K(X)$, is obtained as the quotient of the set $\cup_{U \supseteq X,\text{ open}} \cup_{(U,\varphi)\in\mathcal{O}_X(U)} (U,f)$ by the following relation

$$(U,\varphi) \sim (V,\phi) \iff \exists \text{ open } W \subseteq U \cap V \text{ s.t. } \rho_{U,W}(\varphi) = \rho_{V,W}(\phi).$$

Indeed, this has an addition and a multiplication given by restriction to the open sets where they agree. This is further a field as any non-zero element $[(U,f)]$ can be inverted in a small enough open set $W \subseteq U$ (which will be non-empty as X is irreducible) where f is non-zero (otherwise the class $[(U,f)]$ is identically zero).

Remark 1.5.3.9. Note that we have the following ring homomorphisms for any k-variety X and $x \in X$

$$\begin{aligned}\Gamma(\mathcal{O}_X, X) &\longrightarrow \mathcal{O}_{X,x} \longrightarrow K(X)\\ (X, \varphi) &\longmapsto (X, \varphi)_x \longmapsto [(X, \varphi)].\end{aligned}$$

In-fact, both these are injective by a simple use of the identity principle (Lemma 1.5.2.4). In this way, algebraic gadgets start taking a hold onto the geometry of varieties, which we will see further in this chapter.

We now give two results; one for affine and one for projective; which shows how the three algebraic gadgets introduced in Remark 1.5.3.9 can be realized more algebraically.

Proposition 1.5.3.10. *Let k be an algebraically closed field and let X be an affine k-variety. Let $\mathfrak{m}_p = \{f \in k[X] \mid f(p) = 0$ as a regular function$\}$. Then,*

1. *$\mathfrak{m}_p$ is a maximal ideal of $k[X]$ for every point $p \in X$,*
2. *$\mathrm{mSpec}\,(k[X]) \cong X$ as sets,*
3. *$k[X]_{\mathfrak{m}_p} \cong \mathcal{O}_{X,p}$,*
4. *$k[X]_{\langle 0\rangle} \cong K(X)$,*
5. *$\Gamma(\mathcal{O}_X, X) \cong k[X]$,*
6. *$\dim X = \mathrm{trdeg}\, K(X)/k$[20],*
7. *$\dim X = \dim \mathcal{O}_{X,p}$ for all $p \in X$[21].*

Proof. We give the main ideas of each. The main idea in the latter parts is to embed all the relevant rings inside the function field and do the relevant algebra there.

1. Since there is a correspondence between radical ideals of $k[X]$ and algebraic sets of X and since the correspondence is antitone, therefore minimal algebraic sets (point $p \in X$) of X correspond to maximal ideals of $k[X]$ vanishing at p. The result then follows.
2. This follows from 1. Explicitly, one considers the mapping $p \in X \mapsto \mathfrak{m}_p$.
3. Consider the canonical mapping

$$\begin{aligned}k[X]_{\mathfrak{m}_p} &\longrightarrow \mathcal{O}_{X,p}\\ \frac{f}{g} &\longmapsto (X \setminus Z(g), f/g)_p\end{aligned}$$

where $g(p) \neq 0$ (so $g \notin \mathfrak{m}_p$). This is a homomorphism by the fact that $Z(f) \cup Z(g) = Z(fg)$. This is injective because if $f/g = 0$, then $f = 0$ on some open subset $W \subseteq X \setminus Z(g)$. By an application of identity principle (Lemma 1.5.2.4), the injectivity follows. For surjectivity, observe that for any $(U, f)_p \in \mathcal{O}_{X,p}$, we can represent it by the rational function that f looks like around p, so $(U, f)_p = (W, g/h)_p$ where g/h is a rational function. Consequently, $g/h \mapsto (X \setminus Z(h), g/h)_p = (W, g/h)_p$. The result follows.

[20] Thus the function field $K(X)/k$ holds important global information about the algebra and geometry of X.

[21] Thus the notion of dimension of varieties is detectable at the level of stalks. This is because, as the proof and the statement 3 shows, the local ring $\mathcal{O}_{X,p}$ holds almost all relevant information about the coordinate ring.

4. Observe first that if R is a domain and $\mathfrak{p} \leq R$ is an prime ideal of R, then $(R/\mathfrak{p})_{\langle 0\rangle}$ is isomorphic to $R_{\langle 0\rangle}$. Now, by 3, we obtain that $k[X]_{\langle 0\rangle} \cong (k[X]_{\mathfrak{m}_p})_{\langle} 0\rangle \cong (\mathcal{O}_{X,p})_{\langle 0\rangle}$. The map
$$\begin{aligned}(\mathcal{O}_{X,p})_{\langle 0\rangle} &\longrightarrow K(X)\\ \frac{(U, f/g)_p}{(V, h/l)_p} &\longmapsto [(U\cap V, fl/gh)]\end{aligned}$$
can be seen to be a well-defined (use Lemma 1.5.2.4) isomorphism.
5. By Lemma 23.1.2.12, we have that $\bigcap_{\mathfrak{m}<k[X]} k[X]_{\mathfrak{m}} \cong k[X]$. By Remark 1.5.3.9, we have $\Gamma(\mathcal{O}_X, X) \hookrightarrow \mathcal{O}_{X,p}$ (in $K(X)$). We further have $k[X] \hookrightarrow \Gamma(\mathcal{O}_X, X)$. Consequently, we obtain via 3. the following
$$k[X] \hookrightarrow \Gamma(\mathcal{O}_X, X) \hookrightarrow \bigcap_{p\in X} \mathcal{O}_{X,p} \cong \bigcap_{p\in X} k[X]_{\mathfrak{m}_p} \hookrightarrow \bigcap_{\mathfrak{m}<k[X]} k[X]_{\mathfrak{m}} \cong k[X].$$
The result then follows.
6. We have $\dim X = \dim k[X]$ as any irreducible closed subset of X corresponds in a contravariant manner to a prime ideal of $k[X]$. By Theorem 23.8.2.1, we have $\dim k[X] = \operatorname{trdeg} K(X)/k$.
7. By 3, $\dim \mathcal{O}_{X,p} = \operatorname{ht} \mathfrak{m}_p$. By Theorem 23.8.2.2, we have $\operatorname{ht} \mathfrak{m}_p + \dim k[X]/\mathfrak{m}_p = \dim k[X]$. But since $k[X]/\mathfrak{m}_p \cong k$ by Nullstellensatz, therefore the above equation reduces to $\operatorname{ht} \mathfrak{m}_p = \dim k[X]$ and the right side is just $\dim X$.

$\square$

We next do the projective case. See Chapter 23, Section 23.1.2 for homogeneous localization of graded rings.

Lemma 1.5.3.11. *Let k be an algebraically closed field and X be a projective k-variety in $\mathbb{P}^n_k$. Let $U_i = \mathbb{P}^n_k \setminus Z(x_i)$ and $X_i := X \cap U_i$. Then*
$$\varphi_i : k[X_i]^a \cong k[X]^h_{(x_i)}$$
where $k[X_i]^a$ denotes the affine coordinate ring of $X_i \subseteq \mathbb{A}^n_k$ and $k[X]^h$ denotes the homogeneous coordinate ring of $X \subseteq \mathbb{P}^n_k$. Further the localization above is homogeneous.

Proof. Consider the map $k[y_1, \dots, y_n] \to k[x_0, \dots, x_n]$ mapping as $f(y_1, \dots, y_n) \mapsto f\left(\frac{x_0}{x_i}, \dots, \frac{\hat{x_i}}{x_i}, \dots, \frac{x_n}{x_i}\right)$. This can easily be seen to be a well-defined ring isomorphism mapping the ideal $I(X_i) \mapsto I(X_i)^h = I(X)^h_{(x_i)}$. The result follows by quotienting. $\square$

Proposition 1.5.3.12. *Let k be an algebraically closed field and X be a projective k-variety. Let $\mathfrak{m}_p = \langle\{f \in k[X] \mid f$ is homogeneous & $f(p) = 0\}\rangle$ for any $p \in X$ and $k[X]$ be the homogeneous coordinate ring of X. Then,*
1. *$\mathfrak{m}_p$ is a maximal ideal of $k[X]$ for every element $p \in X$,*
2. *$k[X]_{(\mathfrak{m}_p)} \cong \mathcal{O}_{X,p}$,*
3. *$k[X]_{(\langle 0\rangle)} \cong K(X)$,*
4. *$\Gamma(\mathcal{O}_X, X) \cong k$.*

Proof. Denote by $k[X]^h$ the homogeneous coordinate ring and $X_i := X \cap U_i$ where $U_i = \mathbb{P}^n_k \setminus Z(x_i)$. By Lemma **??**, $U_i \cong \mathbb{A}^n_k$ as varieties, therefore denote X_i^a to be the affine variety corresponding to $X_i \subseteq U_i$. We thus denote $k[X_i]^h$ for the homogeneous coordinate ring when $X_i \subseteq U_i$ and $k[X_i]^a$ to be the affine coordinate ring when $X_i \subseteq \mathbb{A}^n_k$. Let $R := k[X]^h$. The main idea of the last part is to use the theory of integral dependence together with algebraic closure of k.

1. Let $P \in X$, so $P \in X_i$ for some $i = 0, \dots, n$. Thus, let $P^a \in X_i^a$ and by Lemma 1.5.3.11 and Proposition 1.5.3.10, we obtain that $\mathfrak{m}_{P^a}$ is a maximal ideal of $k[X_i]^a$. Thus, $\varphi_i(\mathfrak{m}_{P^a}) = \mathfrak{m}_P k[X]^h_{x_i}$ is a maximal ideal of $k[X]^h_{x_i}$.
2. We simply have the following for any $p \in X$ by irreducibility of X, by Lemma 1.5.3.11 and by Proposition 1.5.3.10:
$$\mathcal{O}_{X,p} \cong \mathcal{O}_{X_i,p} \cong \mathcal{O}_{X_i^a,p^a} \cong k[X_i]^a_{\mathfrak{m}_{p^a}} \cong \left(k[X]^h_{x_i}\right)_{\mathfrak{m}_{p^a}} \cong k[X]^h_{\mathfrak{m}_{p^a}}.$$
3. By irreducibility of X, by Lemma 1.5.3.11 and by Proposition 1.5.3.10, we have the following identifications
$$K(X) \cong K(X_i) \cong K(X_i^a) \cong k[X_i]^a_{\langle 0\rangle} \cong \left(k[X]^h_{x_i}\right)_{\langle 0\rangle} \cong k[X]^h_{\langle 0\rangle}.$$
4. First note that $k \hookrightarrow \Gamma(\mathcal{O}_X, X)$. It would thus suffice to show that $\Gamma(\mathcal{O}_X, X) \hookrightarrow k$. Pick any $f \in \Gamma(\mathcal{O}_X, X)$. We wish to show that $f \in k$. Let $R = k[X]^h$. Note that we can embed $\Gamma(X, \mathcal{O}_X)$ inside the (non-homogeneous) fraction field $L = k[X]^h_{\langle 0\rangle}$. Consequently, by algebraic closure of k, it would suffice to show that $f \in L$ satisfies a polynomial with coefficients in k. Since f is a regular function on each of the X_i, therefore $f \in k[X_i]^a \cong k[X]^h_{x_i}$. Consequently, $f = g_i/x_i^{n_i}$ in L where $\deg g_i = n_i$ and thus $x_i^{n_i} f \in R_{n_i}$ for each $i = 0, \dots, n$. It thus follows that $\deg f = 0$ in L. Consequently, it would suffice to show that $f \in L$ is integral over R (as we can then obtain a polynomial in $k[x]$ whose zero is f by restricting to 0 degree coefficients). By Corollary **??**, it would thus suffice to show that $R[f]$ is a finitely generated R-module.

   It would thus suffice if we show that $\exists M \in \mathbb{N}$ such that $\forall N \geq M$, $R_N f^m \subseteq R_N$ for all $m \geq 0$. Indeed, for $M = \sum_i n_i$, we see that $R_N f \subseteq R_N$ as for any $g \in R_N$, we have that each term of g will have to have one x_i whose power is $\geq n_i$. Repeatedly applying $R_N f \subseteq R_N$ yields $R_N f^m \subseteq R_N$ for all $m \geq 0$, as needed.

$\square$

Remark 1.5.3.13. Note that in Proposition 1.5.3.12, 1, the maximal ideal $\mathfrak{m}_P$ does not contain all of non-constant polynomials in $k[X]$ because $\mathfrak{m}_p$ is generated by homogeneous polynomials vanishing at $p \in X$ and a polynomial with non-zero constant terms cannot be in such an ideal, thus such an $\mathfrak{m}_p$ will exactly be the ideal of all non-constant polynomials in $k[X]$, but then $p \in \bigcap_{f \in k[X], f(0)=0} Z(f) = \emptyset$.

We now show that affine varieties are completely determined by their coordinate rings in the following sense

Theorem 1.5.3.14. *Let k be a algebraically closed field. Then the following*

$$\begin{aligned} k[-] : \mathbf{AfVar^{op}}_k &\longrightarrow \mathbf{FGIAlg}_k \\ X &\longmapsto k[X] \\ X \stackrel{\varphi}{\to} Y &\longmapsto k[Y] \stackrel{k[\varphi]}{\to} k[X] \end{aligned}$$

is a functor[22] *which induces an equivalence between the opposite category of affine varieties over k and finitely generated integral domains over k.*

Proof. **TODO.** □

We now show some examples of the machinery developed so far. We first show that any affine plane conic is isomorphic as a variety to either the parabola $y - x^2$ or the hyperbola $xy - 1$. Indeed, we use here the familiar high-school topic that one classifies conics on the basis of discriminant(!) This will further show that the usual substitutions that we so used to do in school days to reduce an algebraic equation into a simpler form can equivalently be stated in algebraic language as finding a correct automorphism of the corresponding ring in question.

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Subvarieties

[22]Note that by Proposition 1.5.3.10, this is just the global sections functor.

1.5.4 Varieties as schemes

In this section we show how to realize a k-variety (see Definition 1.5.2.5) as a scheme. This will be essential as it fulfill all the reasons to work with schemes as they generalize the concept of varieties to just the right level where all algebro-geometric questions can be asked and be attempted to be solved.

We first show a fully-faithful functor which embeds the category of k-varieties into the category of k-schemes (that is, schemes over k). This will hence show how to obtain a scheme from a variety because, as the following construction of the relevant functor will show, it is not straightforward how should one begin defining it[23].

Definition 1.5.4.1. (**Spectral space of** X) For every topological space X, we can associate a topological space

$$t(X) := \{\text{All non-empty closed irreducible subsets of } X\}$$

where any closed set is given by $t(Y) \subseteq t(X)$ for a closed set $Y \subseteq X$. The following lemma shows that this indeed defines a topology on $t(X)$. We will call $t(X)$ the spectral space of X.

Lemma 1.5.4.2. *Let X be a space and $Y, Z, Y_i \subseteq X$ be closed subsets of X. Then,*

1. $t(Y) \subseteq t(X)$,
2. $t(Y \cup Z) = t(Y) \cup t(Z)$,
3. $t\left(\bigcap_i Y_i\right) = \bigcap_i t(Y_i)$.

Proof. 1. Any closed irreducible subset of Y, where Y is closed in X, will again be closed and irreducible in X.
2. Any irreducible subset of $Y \cup Z$ cannot have non-empty intersection with both of them.
3. Follows from 1. $\square$

Indeed, our main idea is to show that for a variety V, the space $t(V)$ will eventually become a scheme. We have few observations about spectral spaces, before we realize that idea.

Lemma 1.5.4.3. *Let X, X_1, X_2 be spaces and $f : X_1 \to X_2$ be a continuous map. Then,*

1. *there is a one-to-one correspondence between closed subsets of X and closed subsets of $t(X)$,*
2. *the following is a continuous map*

$$\begin{aligned} t(f) : t(X_1) &\longrightarrow t(X_2) \\ Y_1 &\longmapsto \overline{f(Y_1)}, \end{aligned}$$

3. *the following is a functor*

$$\begin{aligned} t : \mathbf{Top} &\longrightarrow \mathbf{Top} \\ X &\longmapsto t(X), \end{aligned}$$

[23] However, one may take a hint (albeit quite vague) from Lemma 1.3.0.2 in the following construction.

4. *the following is a continuous map*

$$\begin{aligned} \alpha : X &\longrightarrow t(X) \\ x &\longmapsto \overline{\{x\}}. \end{aligned}$$

Proof. 1. Follows from the definition of topology on the spectral space.
2. Let $Y_2 \subseteq X_2$ be closed so that $t(Y_2) \subseteq t(X_2)$ is closed. We wish to show that $(t(f))^{-1}(t(Y_2)) \subseteq t(X_1)$ is closed. This follows from the observation that for $Y_1 \in t(X_1)$, we have $\overline{f(Y_1)} \in t(Y_2) \iff Y_1 \in t(f^{-1}(Y_2))$.
3. Follows from 2.
4. Pick any closed $Y \subseteq X$ to thus obtain a closed $t(Y) \subseteq t(X)$. Then $\alpha^{-1}(t(Y)) = \{x \in X \mid \overline{\{x\}} \in t(Y)\} = \{x \in X \mid x \in Y\} = Y$. □

We now give scheme structure to the space $t(X)$. But first, we need a small lemma.

Lemma 1.5.4.4. *Let $A = k[V]$ be the coordinate ring of an affine k-variety V over an algebraically closed field k. Then, for any open set $U \subseteq \operatorname{Spec}(A)$, the set of all closed points of U are dense in U.*

Proof. Since all closed points of $\operatorname{Spec}(A)$ are its maximal ideals by Nullstellensatz, thus, any closed point of U is a maximal ideal of A as well. Consequently, we may assume $U = D(f)$ is a basic open set for $f \in A$. But since $D(f) \cong \operatorname{Spec}(A_f)$ and closed points of any affine scheme are always dense, the result follows. □

Theorem 1.5.4.5. *Let k be an algebraically closed field and $(V, \mathcal{O}_V)$ be a k-variety. Let $\alpha : V \to t(V)$ be the continuous map as defined in Lemma 1.5.4.3, 4. Then, $(t(V), \alpha_*\mathcal{O}_V)$ is a scheme over k which admits an affine open cover by* $\operatorname{Spec}(A)$ *for $A = k[W]$ where W is an affine open subvariety of V.*

Proof. For better clarity of this important proof, we break it in multiple acts.

Act 1 : *We may assume V is an affine k-variety.*

Since we wish to show that $t(V)$ is a scheme, hence we need to produce an open cover of $t(V)$ by affine schemes. Since V is covered by open affine k-varieties, thus if we can show that for an affine k-variety W, the space $t(W)$ is a scheme, then we would be done. Hence we may assume V is affine with coordinate ring $k[V] =: A$.

Act 2 : $t(V) \cong \operatorname{Spec}(A)$ *as topological spaces.*

Consider the usual maps that we know from our study of varieties:

$$t(V) \underset{Z(-)}{\overset{I(-)}{\rightleftarrows}} \operatorname{Spec}(A)$$

These are easily seen to be continuous inverses of each other by the correspondence between closed irreducible subsets of an affine variety and prime ideals of its coordinate ring (Lemma 1.2.1.1).

Act 3 : *The closed points of* Spec (A) *are points of* V.

We first construct the following map

$$\begin{aligned}\varphi : V &\longrightarrow \mathrm{Spec}\,(A)\\ p &\longmapsto \mathfrak{m}_p\end{aligned}$$

where $\mathfrak{m}_p$ is defined together with some properties in Proposition 1.5.3.10. This is continuous by a small check on closed sets. Moreover, this is injective. Now, we claim that $\varphi(V) \subseteq$ Spec (A) are all closed points of Spec (A). Indeed, this follows from the correspondence between closed points of Spec (A) and maximal prime ideals of A (Lemma 1.2.1.3). We will thus denote $\varphi(V)$ as the set of closed points of Spec (A).

Act 4 : *It is enough to show that* $\varphi_*\mathcal{O}_V \cong \mathcal{O}_{\mathrm{Spec}(A)}$.

Since we have the following commutative triangle

$$\begin{tikzcd} & V \arrow[dl, "\varphi"'] \arrow[dr, "\alpha"] & \\ \mathrm{Spec}\,(A) \arrow[rr, shift left, "Z(-)"] \arrow[rr, phantom, "\cong"] & & t(V) \arrow[ll, shift left, "I(-)"] \end{tikzcd},$$

thus $\alpha_*\mathcal{O}_V \cong (Z \circ \varphi)_*\mathcal{O}_V = Z_*\varphi_*\mathcal{O}_V$. Since Z is an isomorphism, thus the reduction is justified.

Act 5 : $\varphi_*\mathcal{O}_V \cong \mathcal{O}_{\mathrm{Spec}(A)}$.

Let $U \subseteq$ Spec (A) be an open set. We will construct an isomorphism between $\mathcal{O}_{\mathrm{Spec}(A)}(U)$ and $\mathcal{O}_V(\varphi^{-1}(U))$. Consider the map

$$\begin{aligned}\eta_U : \mathcal{O}_{\mathrm{Spec}(A)}(U) &\longrightarrow \mathcal{O}_V(\varphi^{-1}(U))\\ s : U \to \amalg_{\mathfrak{p}\in U} A_{\mathfrak{p}} &\longmapsto \eta_U(s) : \varphi^{-1}(U) \to k\end{aligned}$$

where for any $q \in \varphi^{-1}(U)$, we define $\eta_U(s)(q) = s(\mathfrak{m}_q)(q)$. It clearly is a ring homomorphism which commutes with appropriate restriction maps. Thus, we need to show the following three statements in order to conclude.

1. $\eta_U(s)$ is regular,
2. η_U has zero kernel,
3. η_U is surjective.

In-fact, the above three statements are at the technical heart of the proof. The main driving force behind this is the density of closed points of open sets in Spec (A) (Lemma 1.5.4.4) and the identity principle of regular maps on a variety (Lemma 1.5.2.4).

Statement 1. is immediate as s is regular. For statement 2., suppose that $\eta_U(s) = 0$ over $\varphi^{-1}(U)$. Thus $s(\mathfrak{m}_q)(q) = f_q(q)/g_q(q) = 0$ for all $q \in \varphi^{-1}(U)$. Thus, $\eta_U(s)$ around q is represented by rational function f_q/g_q. By Lemma 1.5.2.4 on $\eta_U(s)$, we obtain that $f_q = 0$ for all $q \in \varphi^{-1}(U)$. Thus s is zero at all closed points of U, which are exactly $\varphi(\varphi^{-1}(U))$.

But since closed points of U are dense by Lemma 1.5.4.4 and s is a locally constant function, hence $s = 0$.
Finally, to see statement 3., pick any $f \in \mathcal{O}_V(\varphi^{-1}(U))$ and notice that $W := \varphi(\varphi^{-1}(U))$ is a dense subset of U (set of all closed points, Lemma 1.5.4.4). Thus, it is enough to define a locally constant function s over W whose extension $\tilde{s}$ over U is such that $\eta_U(\tilde{s}) = f$. Indeed, consider

$$\begin{aligned} s : W &\longrightarrow \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}} \\ \mathfrak{m}_q &\longmapsto g_q/h_q \end{aligned}$$

where g_q/h_q is the rational function representing f at the point $q \in V$. Clearly, the extension $\tilde{s}$ is in $\mathcal{O}_{\mathrm{Spec}(A)}(U)$ and it is mapped by η_U to f.

Act 6 : *$(t(V), \alpha_*\mathcal{O}_V)$ is a scheme over k.*

Now let V be a k-variety. We wish to show that $t(V)$ is a scheme over $\mathrm{Spec}\,(k)$. Thus we need to produce a map $t(V) \to \mathrm{Spec}\,(k)$, which is equivalent to a map $k \to \Gamma(\alpha_*\mathcal{O}_V, t(V))$ via the Theorem 1.3.0.5. Since $\Gamma(\alpha_*\mathcal{O}_V, t(V)) = \Gamma(\mathcal{O}_V, V) = A$ via Proposition 1.5.3.10, 5, the result follows.

This completes the proof. □

Remark 1.5.4.6. Theorem 1.5.4.5 yields that the functor t restricts to the following

$$\begin{aligned} t : \mathbf{Var}_k &\longrightarrow \mathbf{Sch}_k \\ (V, \mathcal{O}_V) &\longmapsto (t(V), \alpha_*\mathcal{O}_V). \end{aligned}$$

We will now show that this is a fully-faithful embedding. In other words, any map of $t(V_1) \to t(V_2)$ as of schemes over k is equivalent to a map $V_1 \to V_2$ of k-varieties.

Let us begin with some elementary properties of the residue fields of the k-scheme $t(V)$ attached to a k-variety V.

Lemma 1.5.4.7. *Let k be an algebraically closed field and let V be a k-variety. A point $p \in t(V)$ is closed if and only if $\kappa(p) = k$.*

Proof. (L $\Rightarrow$ R) Since $p \in t(V)$ is closed and closed points of $t(V)$ are exactly points of V, therefore $p \in V \subseteq t(V)$. Consequently, for an affine k-variety $X \subseteq V$ containing p, we obtain the following by Proposition 1.5.3.10, 3 and Nullstellensatz:

$$\kappa(p) = \mathcal{O}_{t(V),p}/\mathfrak{m}_{t(V),p} \cong \mathcal{O}_{V,p}/\mathfrak{m}_{V,p} \cong \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} \cong k[X]_{\mathfrak{m}_p}/\mathfrak{m}_p k[X]_{\mathfrak{m}_p} \cong (k[X]/\mathfrak{m}_p k[X])_0 \cong k.$$

(R $\Rightarrow$ L) By Theorem 1.5.4.5, we have that for some open affine k-variety $X \subseteq V$, $p \in \mathrm{Spec}\,(k)[X]$. Consequently, $\kappa(p) = (k[X]/pk[X])_0 = k$ where p is treated as a prime ideal of $k[X]$. Consequently, we have that the domain $k[X]/pk[X] = k$ as we have inclusions $k \hookrightarrow k[X]/pk[X] \hookrightarrow (k[X]/pk[X])_0$. Thus $p \lneq k[X]$ is maximal. □

Proposition 1.5.4.8. *Let k be an algebraically closed field. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathbf{Var}_k}(V_1, V_2) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(t(V_1), t(V_2)).$$

That is, the functor t is a fully-faithful embedding of k-varieties into schemes over k.

Proof. Exercise 2.15 of Hartshorne Chapter 2. □

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Chapter 1.

Let us now spell out all the properties that the scheme $t(V)$ satisfies for a k-variety V.

Proposition 1.5.4.9. *Let k be an algebraically closed field and V be a k-variety. Then, the scheme $t(V)$ over k is (for * properties, see Section 1.12)*

1. *integral,*
2. *noetherian,*
3. *finite type over k,*
4. *quasi-projective*,*
5. *separated*.*

Proof. 1. to 3. are immediate from the open covering by $\mathrm{Spec}\,(k[W])$ of $t(V)$ where $W \subseteq V$ is an open affine subvariety (Theorem 1.5.4.5). Consequently $t(V)$ is covered by spectrum of finite type k-algebras.
4. is also immediate as any k-variety is an open subset of an affine or a projective k-variety by definition. Since any affine k-variety can be seen as a projective k-variety, consequently, we have an open immersion of V into a closed subvariety of some projective space over k. This extends to an open immersion of $t(V)$ into a closed subscheme of $\mathbb{P}^n_k$.
5. Follows from 4. and Theorem 1.12.7.2. □

We now state an important rectification result which precisely shows what type of schemes are those which are in the image of functor t as in Remark 1.5.4.6.

Corollary 1.5.4.10. *Let k be an algebraically closed field. Then, the functor of Remark 1.5.4.6*

$$t : \mathbf{Var}_k \longrightarrow \mathbf{QPISch}_k$$

establishes an equivalence between varieties over k and quasi-projective integral schemes over k. Further, the image of projective varieties under this functor is exactly the projective integral schemes over k.

Proof. By Proposition 1.5.4.8, we reduce to showing that t lands into quasi-projective schemes and is essentially surjective. Indeed, for a k-variety V, the scheme $t(V)$ is quasi-projective by Proposition 1.5.4.9, 4. Now, to show essential surjection, we first observe that open subschemes of $t(V)$ is in one-to-one bijection with open subsets of V. Consequently, it would suffice to show that any projective integral k-scheme X is in the essential image of t. Indeed, let V denote the closed points of X as a closed subscheme of some $\mathbb{P}^n_k$. Consequently, as closed points of a finite type k-scheme is dense (Lemma 1.12.2.6), therefore V is irreducible (note we are using irreducibility of X here), thus a projective variety in $\mathbb{P}^n_k$. Now, $t(V)$ and X have same underlying space. As a subspace of $\mathbb{P}^n_k$, $t(V)$ and X have both have the structure of a reduced scheme over the common underlying space. By uniqueness of reduced induced closed subscheme structure on a closed subset, we have that $t(V) \cong X$ (see Section 1.6.3). □

We now redefine varieties as schemes and use them as such for the remainder of the sections.

Definition 1.5.4.11. (**Abstract and classical varieties**) Let k be an algebraically closed field. An *abstract variety* or simply a variety, is a separated, integral finite type k-scheme. Those varieties which are furthermore quasi-projective are exactly the varieties we defined earlier by Corollary 1.5.4.10. We will further call the notion of varieties we defined earlier in Definition 1.5.2.5 by referring to them as *classical varieties*.

1.6 Fundamental constructions on schemes

In this section, we would like to understand some of the basic constructions which one can perform with a collection of schemes.

1.6.1 Points of a scheme

Let X be a scheme. Pick any point $x \in X$. We then have the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. Hence we have a projection map

$$\mathcal{O}_{X,x} \to \kappa(x).$$

Consider now an open affine $x \in \operatorname{Spec}(A) \subseteq X$. Consequently, we have $\mathcal{O}_{X,x} \cong \mathcal{O}_{\operatorname{Spec}(A),x} \cong A_x$. Thus, denoting the inclusion $j_x : \operatorname{Spec}(A_x) \hookrightarrow \operatorname{Spec}(A)$, we obtain the following composition:

$$i_x : \operatorname{Spec}(\kappa(x)) \to \operatorname{Spec}(\mathcal{O}_{X,x}) = \operatorname{Spec}(A_x) \stackrel{j_x}{\hookrightarrow} \operatorname{Spec}(A) \hookrightarrow X.$$

Remember that $\operatorname{Spec}(A_x)$ can be interpreted as the affine subset in $\operatorname{Spec}(A)$ which is "very close" to $x \in \operatorname{Spec}(A)$. The map j_x takes the singleton point in $\operatorname{Spec}(\kappa(x))$ to $x \in X$. This map is usually called the *canonical map of point $x \in X$*. The map on stalks that i_x yields is the natural projection $\mathcal{O}_{X,x} \to \kappa(x)$. This map is quite unique as it is universal amongst all those maps $\operatorname{Spec}(K) \to X$ which maps to x. Indeed, we have the following.

Lemma 1.6.1.1. *Let X be a scheme and let $x \in X$ be a point. If K is a field and $f : \operatorname{Spec}(K) \to X$ is a map, then*

1. *If $f(\star) = x$, then $\kappa(x) \hookrightarrow K$.*
2. *If $f(\star) = x$, then f factors via the canonical map i_x at point $x \in X$*

$$\begin{tikzcd} \operatorname{Spec}(\kappa(x)) \arrow[r, "i_x"] & X \\ \operatorname{Spec}(K) \arrow[u] \arrow[ur, "f"'] & \end{tikzcd}.$$

3. $\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec}(K), X) \cong \{x \in X \mid \kappa(x) \hookrightarrow K\}$.

Proof. 1. At the stalk, we have a local ring homomorphism $\varphi : \mathcal{O}_{X,x} \to K$. Consequently, $\operatorname{Ker}(\varphi) = \mathfrak{m}_{X,x}$. It then follows that $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \hookrightarrow K$.
2. Clearly f factors as above as a continuous map. To check the commutativity of sheaf maps, we need only check at stalks (Theorem 27.3.0.7). This is straightforward, as we get on stalks the following commutative diagram:

$$\begin{tikzcd} \kappa(x) \arrow[d, hook] & \mathcal{O}_{X,x} \arrow[l] \arrow[dl, "f^\sharp_\star"] \\ K & \end{tikzcd}.$$

3. It suffices to show that a morphism $f : \operatorname{Spec}(K) \to X$ is equivalent to the data of a point $x \in X$ such that $\kappa(x) \hookrightarrow K$. By 1, one side is immediate. Now consider a point $x \in X$

and a field extension $\kappa(x) \hookrightarrow K$. We wish to construct a map $f : \operatorname{Spec}(K) \to X$ such that the above data is obtained via the construction in 1 applied on f. Indeed, the map f on topological spaces is straightforward, $f(\star) = x$. On sheaves, it reduces to define a natural local ring homomorphism $\mathcal{O}_{X,x} \to K$. This is immediate, as we need only define this as $\mathcal{O}_{X,x} \to \kappa(x) \hookrightarrow K$. □

The above lemma shows that defining a map $\operatorname{Spec}(K) \to X$ is equivalent to taking a point $x \in X$ such that K extends $\kappa(x)$. There is another similar important characterization of maps from $\operatorname{Spec}\left(\frac{k[x]}{x^2}\right)$ into X, which characterizes all rational points of X together with "direction" (that is, together with an element of the tangent space). We first define a rational point of a k-scheme. Recall that by Corollary 1.3.0.7, $\kappa(x)$ is a field extension of k. Further observe the definition of Zariski tangent space T_xX of a scheme as defined in Definition 1.11.1.10.

Definition 1.6.1.2. (**Rational points**) Let X be a k-scheme. Then a point $x \in X$ is said to be rational if $\kappa(x) = k$.

Let us denote $k[\epsilon] = k[x]/x^2$. The ring $k[\epsilon]$ is usually called the ring of *dual numbers*.

Proposition 1.6.1.3. [24] *Let X be a scheme over a field k. Then, we have a bijection*

$$\operatorname{Hom}_{\mathbf{Sch}/k}(\operatorname{Spec}(k[\epsilon]), X) \cong \{(x, \xi) \mid x \in X \textit{ is a rational point } \& \ \xi \in T_xX\}$$

Proof. ($\Rightarrow$) Take a scheme homomorphism $f : \operatorname{Spec}(k[\epsilon]) \to X$. Note that we have a map

$$k[\epsilon] \to k[\epsilon]/\epsilon \cong k.$$

Consequently, we get a map $g : \operatorname{Spec}(k) \to \operatorname{Spec}(k[\epsilon])$ which by composing by f, we get

$$\operatorname{Spec}(k) \xrightarrow{g} \operatorname{Spec}(k[\epsilon]) \xrightarrow{f} X.$$

Observe that $\operatorname{Spec}(k[\epsilon])$ is a one point scheme, therefore $f(\text{pt.}) = f \circ g(\text{pt.}) =: x$. We wish to show that x is a rational point. By Lemma 1.6.1.1, 3, we have $\kappa(x) \hookrightarrow k$. But since X is a scheme over k, therefore $k \hookrightarrow \kappa(x)$. We further deduce from the fact that X is a k-scheme that we have a triangle

$$\begin{tikzcd} \kappa(x) \arrow[r, hook] & k \\ k \arrow[u, hook] \arrow[ur, "\cong"'] & \end{tikzcd}.$$

This shows that horizontal arrow above is an isomorphism. Thus, $\kappa(x) = k$. We now wish to obtain an element of T_xX.

At the point $x \in X$, we have a map $f : \operatorname{Spec}(k[\epsilon]) \to X$. This yields a map on stalks given by

$$\varphi : A \to k[\epsilon]$$

[24] Exercise II.2.8 of Hartshorne.

where $A = \mathcal{O}_{X,x}$ is the local ring at point $x \in X$ and φ is furthermore a local k-algebra homomorphism. Let $\mathfrak{m}$ be the maximal ideal of the local ring A. Then, $A/\mathfrak{m} = \kappa(x)$, which is equal to k as x is a rational point. Thus, A is a rational local k-algebra (Definition 23.1.2.16). It follows from Proposition 23.1.2.17 that φ is equivalent to an element of the tangent space $\xi \in TA$ and by definition, $TA = T_xX$. This completes the proof. □

We now see that closed points of a finite-type k-scheme are those whose residue extension of k is algebraic.

Proposition 1.6.1.4. *Let X be a finite-type k-scheme. Then the following are equivalent:*
1. *$x \in X$ is a closed point.*
2. *$x \in X$ is such that $\kappa(x)/k$ is an algebraic (equivalently, finite).*

Proof. (1. $\Rightarrow$ 2.) Clearly, $\kappa(x)$ is a finitely type field extension of k. By essential Nullstellensatz, $\kappa(x)/k$ is algebraic.

(2. $\Rightarrow$ 1.) Pick an affine open $\operatorname{Spec}(A)$ containing x so that $\mathfrak{p} \in \operatorname{Spec}(A)$ corresponds to x. We wish to show that $\mathfrak{p}$ is maximal. As $\kappa(x) = Q(A/\mathfrak{p})$ and

$$k \hookrightarrow A \twoheadrightarrow A/\mathfrak{p} \hookrightarrow Q(A/\mathfrak{p}) = \kappa(x),$$

thus, as $\kappa(x)/k$ is algebra, we deduce that $\kappa(x)$ is integral over $A/\mathfrak{p}$. Let B be a finite type k-domain such that $Q(B)$ is integral over B. One can check by writing down the relevant polynomials that this implies for any element $b \in B$, the inverse $b^{-1} \in Q(B)$ is in B by integrality. Using this for $B = A/\mathfrak{p}$, we deduce that $A/\mathfrak{p}$ is a filed, so $\mathfrak{p}$ is maximal, as required. □

1.6.2 Gluing schemes & strongly local constructions

We now show how to obtain new schemes from old by the gluing construction. Indeed, the idea is simple, glue the underlying topological spaces of a certain collection of schemes and identifications and define a new structure sheaf over the resultant space which canonically makes it into a scheme. We will further see that there is a universal property that is satisfied by such a glue. We suggest that the reader make a diagram of *blobs* and draw the corresponding maps in order to see the naturality of the following.

Definition 1.6.2.1. (**Gluing datum**) A tuple of data $(I, \{X_i\}_{i\in I}, \{U_{ij}\}_{i,j\in I}, \{\varphi_{ij}\}_{i,j\in I})$ of an index set I, schemes X_i for each $i \in I$, open subschemes $U_{ij} \subseteq X_i$ for each $i, j \in I$ and scheme isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ for each $i, j \in I$ is a gluing datum if it satisfies the following:
1. $U_{ii} = X_i$ for all $i \in I$,
2. $\varphi_{ji} = \varphi_{ij}^{-1}$,
3. $\varphi_{ii} = \mathrm{id}_{U_{ii}} = \mathrm{id}_{X_i}$,
4. the cocycle condition,

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \text{ on } U_{ij} \cap U_{ik}\ \forall i, j, k \in I.$$

We then have that there is a unique glue of the above.

Proposition 1.6.2.2. *For a gluing datum $(I, \{X_i\}_{i\in I}, \{U_{ij}\}_{i,j\in I}, \{\varphi_{ij}\}_{i,j\in I})$ of schemes, there exists a unique scheme X with the following properties:*

1. *there exists an open embedding of schemes*
$$\phi_i : X_i \to X \text{ for each } i \in I,$$
2. *$\phi_j \circ \varphi_{ij} = \phi_i$ on U_{ij} for all $i, j \in I$,*
3. *$X = \bigcup_{i\in I} \phi_i(X_i)$,*
4. *$\phi_i(X_i) \cap \phi_j(X_j) = \phi_i(U_{ij}) = \phi_j(U_{ji})$ for all $i, j \in I$.*

Proof. The underlying space of X is obtained by gluing the underlying spaces of X_i in the usual manner;
$$X := \coprod_{i\in I} X_i / \sim$$
where $x_i \sim \varphi_{ij}(x_i)$ for all $x_i \in U_{ij}$ and $i, j \in I$. Let $\phi_i : X_i \to X$ be the canonical inclusion map. The topology is given on X via the quotient topology; $U \subseteq X$ is open if and only if $\phi_i^{-1}(U) \subseteq X_i$ is open for each $i \in I$. Then to define the sheaf $\mathcal{O}_X$, pick any open $U \subseteq X$ and define the sections over it as follows (let us write $\varphi_{ij} : \mathcal{O}_{U_{ij}} \overset{\cong}{\to} \mathcal{O}_{U_{ji}}$ as well):
$$\mathcal{O}_X(U) = \left\{ [(\phi_i^{-1}(U), s_i)] \mid \forall i,\ s_i \in \mathcal{O}_{X_i}(\phi_i^{-1}(U)) \text{ s.t. } \varphi_{ij}(\rho_{\phi_i^{-1}(U), \phi_i^{-1}(U)\cap U_{ij}}(s_i)) = \rho_{\phi_j^{-1}(U), \phi_j^{-1}(U)\cap U_{ji}}(s_j) \right\}.$$
By local nature, this is again a sheaf (also called the glued sheaf). Now, ϕ_i is an open embedding as for any open $U \subseteq X$, it follows that $\mathcal{O}_X(\varphi_i(X_i) \cap U) \cong \mathcal{O}_{X_i}(\phi_i^{-1}(U))$. Thus, X is a scheme as for each $x \in X$, $x \in \phi_i(X_i)$ which is a scheme. □

The uniqueness of the glue follows from the fact the universal property that glued scheme satisfies. **TODO**. Gluing allows us to construct many non-affine schemes, like the projective n-scheme over k.

Put some nice e amples of schem formed via gluin and do uniquene Chapter 1.

A lot of times we have the situation that a certain construction on a ring A leads to a map $\varphi : A \to \tilde{A}$. Consequently, we obtain maps $f : \text{Spec}\left(\tilde{A}\right) \to \text{Spec}\,(A)$. If X is a scheme, then for each open affine $V_i = \text{Spec}\,(A_i)$, we get a map $X_i \to V_i$ given by $\text{Spec}\left(\tilde{A}_i\right) \to \text{Spec}\,(A_i)$. Consequently, we are interested in the conditions that the construction $A \to \tilde{A}$ must satisfy so that X_i glue together to give a scheme $\tilde{X}$ which represents the construction globally.

Definition 1.6.2.3 (**Construction on rings**)**.** A construction on rings is a collection of maps $\{\varphi_A : A \to \tilde{A}\}$ one for each ring A such that for any isomorphism $\eta_{AB} : A \overset{\cong}{\to} B$, we have an isomorphism $\tilde{\eta}_{AB} : \tilde{A} \to \tilde{B}$ which is id if η is id, the diagram
$$\begin{array}{ccc} A & \xrightarrow{\varphi_A} & \tilde{A} \\ {\scriptstyle \eta_{AB}}\downarrow{\scriptstyle\cong} & & {\scriptstyle \tilde{\eta}_{AB}}\downarrow{\scriptstyle\cong} \\ B & \xrightarrow[\varphi_B]{} & \tilde{B} \end{array}$$
commutes and if $\eta_{BC} \circ \eta_{AB} = \eta_{AC}$, then $\tilde{\eta}_{BC} \circ \tilde{\eta}_{AB} = \tilde{\eta}_{AC}$. That is, constructions are functorial on isomorphisms.

Definition 1.6.2.4 (Strongly local constructions). A construction on rings $\{\varphi_A : A \to \tilde{A}\}$ is said to be strongly local if it naturally commutes with localization. That is, for each $g \in A$ not in nilradical, there exists an isomorphism $\widetilde{A_g} \cong \tilde{A}_g$ such that

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varphi_A"'] & A_g \arrow[d, "(\varphi_A)_g"'] \arrow[dr, "\varphi_{A_g}"] & \\
\tilde{A} \arrow[r] & \tilde{A}_g \arrow[r, "\cong"'] & \widetilde{A_g}
\end{tikzcd}$$

commutes where $(\varphi_A)_g : A_g \to \tilde{A}_g$ is the localization of map $\varphi_A : A \to \tilde{A}$ at the element $g \in A$ and the horizontal arrows of the square are localization maps.

Remark 1.6.2.5. Let $\eta : A_f \cong B_g$ be an isomorhism where $f \in A$ and $g \in B$. Then we get an isomorphism $\hat{\eta} : \tilde{A}_f \cong \tilde{B}_g$ as in the following commutative diagram:

$$\begin{tikzcd}
\tilde{A}_f \arrow[r, dashed, "\hat{\eta}"] & \tilde{B}_g \\
\widetilde{A_f} \arrow[u, "\cong"] \arrow[r, "\cong", "\tilde{\eta}"'] & \widetilde{B_g} \arrow[u, "\cong"'] \\
A_f \arrow[u, "\varphi_{A_f}"'] \arrow[r, "\cong", "\eta"'] \arrow[uu, bend left=60, "(\varphi_A)_f"] & B_g \arrow[u, "\varphi_{B_g}"] \arrow[uu, bend right=60, "(\varphi_B)_g"']
\end{tikzcd}.$$

Let X be a scheme. Our main goal is to show that strongly local constructions done on each affine open subset of X can be glued to give a scheme $\tilde{X}$ admitting a map $\tilde{X} \to X$.

We will achieve this in steps. We first translate strongly local property more geometrically.

Lemma 1.6.2.6. *Let $\{\varphi_A : A \to \tilde{A}\}$ be a strongly local construction on rings. For any ring A denote $\phi_A : \operatorname{Spec}\left(\tilde{A}\right) \to \operatorname{Spec}(A)$ to be the map corresponding to φ_A. Then, for any $f \in A$, the following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Spec}(A_f) & & \operatorname{Spec}\left(\tilde{A}_f\right) \arrow[ll, "\phi_A|_{\operatorname{Spec}\left(\tilde{A}_f\right)}"'] \arrow[dl, "\cong"] \\
& \operatorname{Spec}\left(\widetilde{A_f}\right) \arrow[ul, "\phi_{A_f}"] &
\end{tikzcd}.$$

Proof. This is the translation of Definition 1.6.2.3 in $\operatorname{Spec}(-)$ where localization amounts to restricting to the corresponding open subscheme. $\square$

The following is an important observation which will help in checking the cocycle condition.

Lemma 1.6.2.7. *Let $\{\varphi_A : A \to \tilde{A}\}$ be a strongly local construction on rings and the following be a commutative triangle of isomorphisms*

$$\begin{tikzcd} R_f \arrow[r] \arrow[dr] & S_g \arrow[d] \\ & T_h \end{tikzcd}$$

for $f \in R$, $g \in S$ and $h \in T$. Then, the following triangle of isomorphisms as constructed in Remark 1.6.2.5 also commutes

$$\begin{tikzcd} \tilde{R}_f \arrow[r] \arrow[dr] & \tilde{S}_g \arrow[d] \\ & \tilde{T}_h \end{tikzcd}.$$

Proof. By definition of a construction, we get that the following triangle commutes

$$\begin{tikzcd} \widetilde{R_f} \arrow[r] \arrow[dr] & \widetilde{S_g} \arrow[d] \\ & \widetilde{T_h} \end{tikzcd}.$$

By the construction of isomorphism $\tilde{R}_f \to \tilde{S}_g$ and others as in Remark 1.6.2.5, we immediately get that the required triangle commutes. $\square$

Lemma 1.6.2.8. *Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ be two affine schemes. Let R be a ring with isomorphisms $A_f \cong R \cong B_g$ for some $f \in A$ and $g \in B$. Let $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction on rings. Then there are open immersions $\operatorname{Spec}\left(\tilde{R}\right) \hookrightarrow \operatorname{Spec}\left(\tilde{A}\right)$ and $\operatorname{Spec}\left(\tilde{R}\right) \hookrightarrow \operatorname{Spec}\left(\tilde{B}\right)$ so that the following commutes*

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\phi_A"'] & \operatorname{Spec}\left(\tilde{R}\right) \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\phi_R"'] & \operatorname{Spec}\left(\tilde{B}\right) \arrow[d, "\phi_B"] \\ \operatorname{Spec}(A) & \operatorname{Spec}(R) \arrow[l, hook'] \arrow[r, hook] & \operatorname{Spec}(B) \end{tikzcd}.$$

Proof. This follows from the following diagram

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\phi_A"'] & \operatorname{Spec}\left(\tilde{A}_f\right) \cong \operatorname{Spec}\left(\widetilde{A_f}\right) \arrow[l, hook'] \arrow[d, "\phi_A|_{\operatorname{Spec}\left(\tilde{A}_f\right)}"] & \operatorname{Spec}\left(\tilde{R}\right) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\phi_R"] & \operatorname{Spec}\left(\widetilde{B_g}\right) \cong \operatorname{Spec}\left(\tilde{B}_g\right) \arrow[r, hook] \arrow[d, "\phi_B|_{\operatorname{Spec}(\tilde{B}_g)}"] & \operatorname{Spec}\left(\tilde{B}\right) \arrow[d, "\phi_B"] \\ \operatorname{Spec}(A) & \operatorname{Spec}(A_f) \arrow[l, hook'] & \operatorname{Spec}(R) \arrow[l, "\cong"] \arrow[r, "\cong"'] & \operatorname{Spec}(B_g) \arrow[r, hook] & \operatorname{Spec}(B) \end{tikzcd}$$

the commutativity of which follows from Lemma 1.6.2.6 and the definition of a construction. $\square$

Let X be a scheme and $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ be two open affines. We can now glue $\operatorname{Spec}\left(\tilde{A}\right)$ and $\operatorname{Spec}\left(\tilde{B}\right)$ along the intersection $U \cap V$ as follows.

Proposition 1.6.2.9. *Let X be a scheme and $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ be two open affines. Let $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction on rings. Let $\phi_A : \tilde{U} = \operatorname{Spec}\left(\tilde{A}\right) \to \operatorname{Spec}(A)$ and $\phi_B : \tilde{V} = \operatorname{Spec}\left(\tilde{B}\right) \to \operatorname{Spec}(B)$ be the maps corresponding to φ_A and φ_B. Then, there exists an isomorphism of schemes*

$$\Theta : \phi_A^{-1}(U \cap V) \xrightarrow{\cong} \phi_B^{-1}(U \cap V)$$

such that the following commutes for any affine open $\operatorname{Spec}(R) \subseteq U \cap V$ which is basic in both U and V by the isomorphisms $A_f \cong R \cong B_g$ (see Lemma 1.4.4.3)

$$\begin{tikzcd}
\phi_A^{-1}(U \cap V) \arrow[r, "\Theta", "\cong"'] & \phi_B^{-1}(U \cap V) \\
\operatorname{Spec}\left(\tilde{A}_f\right) \arrow[u, hook] \arrow[r, "\cong", "\Theta_f"'] & \operatorname{Spec}\left(\tilde{B}_g\right) \arrow[u, hook]
\end{tikzcd}$$

where Θ_f is obtained from $\theta : A_f \cong B_g$ via $\tilde{}$ construction (Remark 1.6.2.5).

Proof. Cover $U \cap V$ by open affines which are basic in both U and V (Lemma 1.4.4.3) and write $U \cap V = \bigcup_{i \in I} \operatorname{Spec}(A_{f_i}) = \bigcup_{i \in I} \operatorname{Spec}(B_{g_i})$ where $f_i \in A$ and $g_i \in B$. Consequently we may write

$$\phi_A^{-1}(U \cap V) = \bigcup_{i \in I} \phi_A^{-1}(\operatorname{Spec}(A_{f_i})) = \bigcup_{i \in I} \operatorname{Spec}\left(\tilde{A}_{f_i}\right)$$

and thus similarly,

$$\phi_B^{-1}(U \cap V) = \bigcup_{i \in I} \operatorname{Spec}\left(\tilde{B}_{g_i}\right).$$

For each $i \in I$, Lemma 1.6.2.8 provides us with an isomorphism

$$\Theta_i : \operatorname{Spec}\left(\tilde{A}_{f_i}\right) \xrightarrow{\cong} \operatorname{Spec}\left(\tilde{B}_{g_i}\right) \hookrightarrow \tilde{V}.$$

We claim that Θ_i can be glued. Indeed, for $i \neq j$, we have $\operatorname{Spec}\left(\tilde{A}_{f_i}\right) \cap \operatorname{Spec}\left(\tilde{A}_{f_j}\right) = \operatorname{Spec}\left(\tilde{A}_{f_i f_j}\right)$, therefore we reduce to showing that Θ_i and Θ_j are equal when restricted to $\operatorname{Spec}\left(\tilde{A}_{f_i f_j}\right)$. Observe from Lemma 1.4.4.3 that for each $i \in I$, the isomorphism $A_{f_i} \cong B_{g_i}$ takes $f_i \mapsto g_i$. The above is now equivalent to showing that the isomorphisms $\theta_i : \tilde{A}_{f_i} \cong \tilde{B}_{g_i}$ and $\theta_j : \tilde{A}_{f_j} \cong \tilde{B}_{g_j}$ obtained from $A_{f_i} \cong B_{g_i}$ and $A_{f_j} \cong B_{g_j}$ fit in the following commutative diagram

$$\begin{tikzcd}
\tilde{A}_{f_i f_j} \arrow[r, "(\theta_i)_{f_j}"] \arrow[d, equal, "\mathrm{id}"'] & \tilde{B}_{g_i f_j} \arrow[d, equal, "\mathrm{id}"] \\
\tilde{A}_{f_j f_i} \arrow[r, "(\theta_j)_{f_i}"'] & \tilde{B}_{g_j f_i}
\end{tikzcd}.$$

But $\theta_i(f_j) = g_j$ and $\theta_j(f_i) = g_i$, as mentioned above. Therefore $\tilde{B}_{g_i f_j} = \tilde{B}_{g_i g_j} = \tilde{B}_{g_j g_i} = \tilde{B}_{g_j f_i}$ and the above square commutes, showing that Θ_i glues to give a map $\Theta : \phi_A^{-1}(U \cap V) \to \phi_B^{-1}(U \cap V)$, which is an isomorphism as locally it is an isomorphism (Proposition 1.3.1.6). ☐

Using Proposition 1.6.2.9, we can now globalize a strongly local construction.

Theorem 1.6.2.10. *Let X be a scheme and $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction on rings. Then there exists a scheme $\alpha : \tilde{X} \to X$ such that for any affine open* $\operatorname{Spec}(A) \hookrightarrow X$*, the following square commutes*

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[r, hook] \arrow[d, "\phi_A"'] & \tilde{X} \arrow[d, "\alpha"] \\ \operatorname{Spec}(A) \arrow[r, hook] & X \end{tikzcd} .$$

Proof. We first construct $\tilde{X}$ by gluing each $\operatorname{Spec}\left(\tilde{A}\right)$. Indeed, let $\{V_i = \operatorname{Spec}(A_i)\}_{i \in I}$ be the collection of affine opens in X and let $\{\tilde{X}_i = \operatorname{Spec}\left(\tilde{A}_i\right)\}$ be the collection of corresponding $\tilde{-}$-constructions. Let $\phi_i : \tilde{X}_i \to V_i$ be the maps corresponding to φ_{A_i}.

For each $i \neq j \in I$ we wish to construct open subschemes $U_{ij} \subseteq \tilde{X}_i$ and isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ satisfying the gluing conditions of Definition 1.6.2.1. We let

$$U_{ij} = \phi_i^{-1}(V_i \cap V_j).$$

Then Proposition 1.6.2.9 provides us with an isomorphism

$$\varphi_{ij} : U_{ij} \xrightarrow{\cong} U_{ji}.$$

It is immediate that $U_{ii} = \tilde{X}_i$ and $\varphi_{ii} = \mathrm{id}_{U_{ii}}$. Moreover, $\varphi_{ji} = \varphi_{ij}^{-1}$ by construction. We now check the cocycle condition. Indeed, pick $i, j, k \in I$ and pick an open affine $\operatorname{Spec}(R) \subseteq V_i \cap V_j \cap V_k$ in X which is basic open in V_i, V_j and V_k (Lemma 1.4.4.3 such that we have isomorphisms $A_{i,f_i} \cong A_{j,f_j} \cong A_{k,f_k} \cong R$ so that the following triangle commutes

$$\begin{tikzcd} A_{i,f_i} \arrow[r, "\cong"] \arrow[dr, "\cong"'] & A_{j,f_j} \arrow[d, "\cong"] \\ & A_{k,f_k} \end{tikzcd} . \qquad (*)$$

By taking inverse images under ϕ_i, it follows that $\operatorname{Spec}\left(\tilde{A}_{i,f_i}\right) \subseteq U_{ij} \cap U_{ik}$ is basic open in both $\tilde{X}_i$ and $\tilde{X}_j$. We wish to show that φ_{ik} restricted to $\operatorname{Spec}\left(\tilde{A}_{i,f_i}\right)$ is the composition $\varphi_{jk} \circ \varphi_{ij}$. By Proposition 1.6.2.9, we get that φ_{ik} on this open affine is an isomorphism to $\operatorname{Spec}\left(\tilde{A}_{k,f_k}\right)$ and φ_{ij} is an isomorphism to $\operatorname{Spec}\left(\tilde{A}_{j,f_j}\right)$. Consequently, we wish to show that the following triangle of isomorphisms commute

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}_{i,f_i}\right) \arrow[r, "\varphi_{ij}"] \arrow[dr, "\varphi_{ik}"'] & \operatorname{Spec}\left(\tilde{A}_{j,f_j}\right) \arrow[d, "\varphi_{jk}"] \\ & \operatorname{Spec}\left(\tilde{A}_{k,f_k}\right) \end{tikzcd} .$$

But these isomorphisms are obtained by the following isomorphisms on the localizations (Proposition 1.6.2.9):

$$\begin{tikzcd} \tilde{A}_{i,f_i} \arrow[r, "\cong"] \arrow[dr, "\cong"'] & \tilde{A}_{j,f_j} \arrow[d, "\cong"] \\ & \tilde{A}_{k,f_k} \end{tikzcd}.$$

Hence it suffices to show that the above triangle commutes. The Lemma 1.6.2.7 applied on $(*)$ yields the required commutativity. $\square$

Definition 1.6.2.11 ($\tilde{}$**-fication**)**.** Let $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction of rings and let X be a scheme. The scheme $\tilde{X} \to X$ obtained in Theorem 1.6.2.10 is called the $\tilde{}$-fication of X.

1.6.3 Reduced scheme of a scheme

For any scheme X, we can obtain a scheme with the same underlying space but with reduced structure sheaf. This procedure is called *reducing a scheme* to a reduced scheme.

Construction 1.6.3.1. Let X be a scheme. Consider the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)/\mathfrak{n}_U$ where $\mathfrak{n}_U$ is the nilradical of $\mathcal{O}_X(U)$ and denote this sheaf by $\mathcal{O}_X^{\text{red}}$. The pair $(X, \mathcal{O}_X^{\text{red}})$ will be called the *associated reduced scheme* of the scheme $(X, \mathcal{O}_X)$, usually denoted by X_{red}. Indeed, $(X, \mathcal{O}_X^{\text{red}})$ is a scheme as the following result shows.

Remark 1.6.3.2 (*Reducing a ring is a strongly local construction*)**.** It is easy to see that $A \to A/\mathfrak{n}$ for each ring A defines a strongly local construction on rings as in Definition 1.6.2.4. Consequently, by Theorem 1.6.2.10, we immediately get a scheme $\tilde{X}$ obtained by reducing each open affine by dividing by nilradical. Indeed, one checks that we get the same scheme as $(X, \mathcal{O}_X^{\text{red}})$. However, we still give a proof of $(X, \mathcal{O}_X^{\text{red}})$ being a scheme without appealing to Theorem 1.6.2.10.

Lemma 1.6.3.3. [25] *Let X be a scheme. Then,*

1. *the pair $(X, \mathcal{O}_X^{\text{red}})$ is a scheme,*
2. *there exists a map of schemes $\varphi : (X, \mathcal{O}_X^{\text{red}}) \to (X, \mathcal{O}_X)$ which is a homeomorphism on the spaces.*

Proof. 1. Let $(\text{Spec}\,(A), \mathcal{O}_{\text{Spec}(A)})$ be an open affine of X. We shall show that $(\text{Spec}\,(A), \mathcal{O}_{\text{Spec}(A)}^{\text{red}})$ is isomorphic to $(\text{Spec}\,(A_{\text{red}}), \mathcal{O}_{\text{Spec}(A_{\text{red}})})$. First, the isomorphism on spaces is straightforward as every prime ideal contains nilradical (nilradical is the intersection of all prime ideals, Lemma 23.1.2.9). We thus need to produce a sheaf morphism $\mathcal{O}_{\text{Spec}(A)}^{\text{red}} \to \mathcal{O}_{\text{Spec}(A_{\text{red}})}$ which is an isomorphism. Let us denote the presheaf $U \mapsto \mathcal{O}_X(U)/\mathfrak{n}_U$ by F. We first immediately reduce to showing the existence of a map $F \to \mathcal{O}_{\text{Spec}(A_{\text{red}})}$ which is an isomorphism on basic open sets, as we then obtain a map of sheaves $\mathcal{O}_{\text{Spec}(A)}^{\text{red}} \to \mathcal{O}_{\text{Spec}(A_{\text{red}})}$ by the universal property of sheafification (Theorem 27.2.0.1) which is an isomorphism on stalks (Theorem 27.3.0.6, 4).

[25] Exercise II.2.3.b of Hartshorne.

Since sheaves and sheaf morphisms are uniquely determined by defining them on a basis, thus we further reduce to defining a presheaf map $F \to \mathcal{O}_{\operatorname{Spec}(A_{\mathrm{red}})}$ with above properties on a basis. Since $\operatorname{Spec}(A)$ has a canonical basis, namely, $\mathcal{B} = \{\operatorname{Spec}(A)_f\}_{f \in A}$, consequently one sees that isomorphism $A_f/\mathfrak{n}_f \cong (A/\mathfrak{n})_f$ can be naturally extended to a presheaf map $F \to \mathcal{O}_{\operatorname{Spec}(A_{\mathrm{red}})}$, which is an isomorphism on the basis $\mathcal{B}$.

2. Consider the map $f : (X, \mathcal{O}_X^{\mathrm{red}}) \to (X, \mathcal{O}_X)$ which is given by id_X on spaces but by the following quotient map $\mathcal{O}_X(U) \to \mathcal{O}_X(U)/\mathfrak{n}_U \to \mathcal{O}_X^{\mathrm{red}}(U)$. □

There is a universal property of reduced schemes which says that a map out of a reduced scheme necessarily has to factor through the reduction of the codomain.

Proposition 1.6.3.4. [26] *Let $f : X \to Y$ be a map of schemes with X being a reduced scheme. Then there exists a unique map of schemes $g : X \to Y_{\mathrm{red}}$ such that the triangle commutes:*

$$\begin{tikzcd} Y & Y_{\mathrm{red}} \arrow[l, "\varphi"'] \\ X \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd} .$$

Proof. The map g on spaces is immediate; it shoould be identical to f as φ is identity on spaces. The map $g^\flat$ on the other hand can be constructed as follows. First observe that if A and B are rings with B being reduced, then any ring map $\eta : A \to B$ extends to a unique map $\tilde{\eta} : A_{\mathrm{red}} \to B$ given by $a + \mathfrak{n} \mapsto \eta(a)$ which makes the triangle commute:

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, two heads] & B \\ A_{\mathrm{red}} \arrow[ur, "\tilde{\eta}"'] & \end{tikzcd} .$$

In our case, we therefore get a unique map $\tilde{f^\flat_U}$ as below for any $U \subseteq Y$, which further gives us the required unique map $g^\flat_U : \mathcal{O}_Y^{\mathrm{red}}(U) \to \mathcal{O}_X(f^{-1}(U))$ which we need (by universality of sheafification, Theorem 27.2.0.1):

$$\begin{tikzcd} \mathcal{O}_Y(U) \arrow[r, "\varphi_U"] \arrow[rr, bend left, "f^\flat_U"] \arrow[dr, two heads] & \mathcal{O}_Y^{\mathrm{red}}(U) \arrow[r, "=:g^\flat_U"] & \mathcal{O}_X(f^{-1}(U)) \\ & \mathcal{O}_Y(U)/\mathfrak{n}_U \arrow[u] \arrow[ur, "\tilde{f^\flat_U}"'] & \end{tikzcd} .$$

One can then easily check that g as given above makes the triangle commute. □

For each closed set $Z \subseteq X$ of a scheme, we construct a unique closed reduced subscheme structure over it.

[26] Exercise II.2.3.c of Hartshorne.

Construction 1.6.3.5 (*Reduced induced subscheme*)**.** Let X be a scheme and $Z \subseteq X$ be a closed set. We wish to define a natural scheme structure on the subspace Z. Indeed, if $X = \operatorname{Spec}(A)$ is affine and $Z \subseteq X$ is closed, then let $\mathfrak{a} = \bigcap_{\mathfrak{p}\in Z} \mathfrak{p}$ so that $Z = V(\mathfrak{a})$. Then we define the reduced induced subscheme structure on Z as that of $\operatorname{Spec}(A/\mathfrak{a})$. Observe that $(Z, \mathcal{O}_{\operatorname{Spec}(A/\mathfrak{a})})$ is a reduced scheme as $\mathfrak{a} \supseteq \mathfrak{n}$ where $\mathfrak{n} \leq A$ is the nilradical.

For an arbitrary scheme X and a closed subset $Z \subseteq X$, we proceed as follows. Let $\{U_i\}_{i\in I}$ be the collection of all open affines in X. Consider the intersections $Z_i = U_i \cap Z$ for each $i \in I$. As $Z_i \subseteq U_i$ are closed subsets in an affine scheme U_i, so by definition they carry the reduced induced subscheme structure on Z_i. We claim that the sheaves on each Z_i can be glued. Indeed, by usual argument involving Lemma 1.4.4.3, we reduce to checking that if $U = \operatorname{Spec}(A)$ is an open affine, $V = D(f) \subseteq U$ a basic open subset, $\mathcal{R}_U$ and $\mathcal{R}_V$ denote the sheaves obtained by reduced induced subscheme structures on $Z \cap U$ and $Z \cap V$ respectively, then

$$(\mathcal{R}_U)_{|Z\cap V} \cong \mathcal{R}_V.$$

Let $\mathfrak{a} = \bigcap_{\mathfrak{p}\in Z\cap U} \mathfrak{p}$ which gives the required structure on $Z \cap U$. Similarly, we have $\mathfrak{b} = \bigcap_{\mathfrak{p}\in Z\cap V} \mathfrak{p}$. We claim that $\mathfrak{b} = \mathfrak{a}A_f$. This would establish the required isomorphism between $A/\mathfrak{a}$ and $A_f/\mathfrak{b}$. Indeed, by definition, it is clear that $\mathfrak{b} \supseteq \mathfrak{a}A_f$. Conversely, pick $x/f^n \in \mathfrak{a}A_f$ where $x \in \mathfrak{a}$. We wish to show that $x/f^n \in \mathfrak{b}$. Pick any prime ideal $\mathfrak{q} \in Z \cap V$. We wish to show that $x/f^n \in \mathfrak{q}$. As $x \in \mathfrak{a}$, therefore $x \in \mathfrak{p}$ for each $\mathfrak{p} \in Z \cap U$. Thus, for $\mathfrak{p} \in D(f)$, $x \in \mathfrak{p}$. As each $\mathfrak{q} \in Z \cap V$ comes from $\mathfrak{p} \in Z \cap D(f)$, therefore $x/1 \in \mathfrak{b}$ and thus $x/f^n \in \mathfrak{b}$.

This completes the gluing procedure, to yield a subscheme structure on Z which we call the reduced induced subscheme structure on Z.

We now show the universal property of the above construction.

Proposition 1.6.3.6 (Universal property of reduced induced subscheme)**.** ***TODO.***

1.6.4 Fiber product of schemes

One of the most important tool in scheme theory is that of fiber product of schemes. This is essential as this is exactly the right notion using which one can define intersection of subschemes, which is one of the fundamental goals of this book.

Existence of fiber products is equivalent to saying that the category of schemes **Sch** have all pullbacks. In particular, it is equivalent to saying that for any two S-schemes X and Y, their product in **Sch**$/S$ exists, called the *fiber product* denoted $X \times_S Y$.

However, we need to be more explicit than this abstract definition; we have to show that $X \times_S Y$ actually exists. Since we know how pushouts are constructed in the category of rings, their tensor products, therefore we can define it for affine schemes without much effort using the functor $\operatorname{Spec}(-) : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ of Theorem 1.3.0.5.

Definition 1.6.4.1. (**Fiber product of affine schemes**) Let the following be a coCartesian[27] diagram of rings (or of R-algebras)

$$\begin{tikzcd} A \otimes_R B & B \arrow[l] \\ A \arrow[u] & R \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}.$$

Since the $\mathrm{Spec}\,(-) : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ of Theorem 1.3.0.5 is right adjoint to global sections, therefore it preserves all limits of $\mathbf{Ring}^{\mathrm{op}}$, and thus, takes the above pushout diagram of R-algebras to a pullback diagram of affine schemes over $\mathrm{Spec}\,(R)$:

$$\begin{tikzcd} \mathrm{Spec}\,(A \otimes_R B) \arrow[r] \arrow[d] & \mathrm{Spec}\,(B) \arrow[d, "\mathrm{Spec}(g)"] \\ \mathrm{Spec}\,(A) \arrow[r, "\mathrm{Spec}(f)"'] & \mathrm{Spec}\,(R) \end{tikzcd}.$$

We hence define $\mathrm{Spec}\,(A \otimes_R B)$ to be the fiber product of affine schemes $\mathrm{Spec}\,(A)$ and $\mathrm{Spec}\,(B)$ over $\mathrm{Spec}\,(R)$.

Definition 1.6.4.2 (**Fiber product of schemes**)**.** Fiber product of S-schemes X and$Y$ is an S-scheme $X \times_S Y$ such that for any other S-scheme Z with map $f : Z \to X$ and $g : Z \to Y$ over S, there exists a unique map $u : Z \to X \times_S Y$ such that the diagram commutes

$$\begin{tikzcd} Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}.$$

The most important part in this construction is the description of the structure sheaf of $X \times_S Y$. We now show how to construct fiber products of arbitrary S-schemes. In the process, we give a rather explicit description of fiber products and its structure sheaf, which we may think of as an explicit definition of fiber product. We begin with the affine case. Recall the notion of compositum of fields in Definition **??**.

Proposition 1.6.4.3. *Let A, B be two R-algebras and let $X = \mathrm{Spec}\,(A), Y = \mathrm{Spec}\,(B)$ and $S = \mathrm{Spec}\,(R)$. Then, as a set, we have the following bijection*

$$X \times_S Y \cong \left\{ \begin{array}{l} \textit{Tuples } (\mathfrak{p}_A, \mathfrak{p}_B, L, \alpha, \beta) \textit{ where } \mathfrak{p}_A \in X,\ \mathfrak{p}_B \in Y \\ \textit{such that both have same inverse image } \mathfrak{p}_R \textit{ in } S \\ \textit{and } (L, \alpha, \beta) \textit{ is the compositum of fields } \kappa(\mathfrak{p}_A) \\ \textit{and } \kappa(\mathfrak{p}_B) \textit{ over } \kappa(\mathfrak{p}_R). \end{array} \right\}$$

Proof. Pick any prime ideal $\mathfrak{p} \in X \times_S Y = \mathrm{Spec}\,(A \otimes_R B)$. We wish to construct the datum $(\mathfrak{p}_A, \mathfrak{p}_B, L, \alpha, \beta)$. **TODO**. □

[27] another name for pushout diagrams.

We now construct the fiber product of two schemes. This is more of an exercise in gluing techniques rather than anything else, so is ommited.

Theorem 1.6.4.4. *Let X, Y be two S-schemes. The fiber product $X \times_S Y$ exists.* □

Remark 1.6.4.5. While working with fiber products, one of the most important tool is its universal property. Most of the results about fiber products rarely uses the point-set construction as laid out above, just like the construction of tensor product is rarely used. Consequently, one should/must prove results about fiber products only using universal properties.

We now portray some easy applications of the universal property of fiber products.

Lemma 1.6.4.6. *Let $f : X \to Y$ and $g : Z \to Y$ be scheme morphisms and $U \subseteq X$ be an open subscheme. If $p : X \times_Y Z \to X$ is the scheme over X obtained by base change under f, then $p^{-1}(U) \cong U \times_Y Z$.*

Proof. We claim that the open subscheme $p^{-1}(U)$ of $X \times_Y Z$ is isomorphic to $U \times_Y Z$ by showing that it satisfies the same universal property. Indeed, suppose we have the following diagram

$$\begin{tikzcd} T \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\ & p^{-1}(U) \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ & U \arrow[r, "f"'] & Y \end{tikzcd}$$

where $f \circ h = g \circ k$. By the universal property of fiber product $X \times_Y Z$, we get a unique map $\varphi : T \to X \times_Y Z$ such that $p \circ \varphi = h$ and $q \circ \varphi = k$. As $\operatorname{Im}(h) \subseteq U$, therefore $\operatorname{Im}(p \circ \varphi) \subseteq U$. Consequently, we have $\operatorname{Im}(\varphi) \subseteq p^{-1}(U)$, hence we may write $\varphi : T \to p^{-1}(U)$, where $p^{-1}(U)$ is an open subscheme of $X \times_Y Z$. Thus, we get a unique map $\varphi : T \to p^{-1}(U)$ which makes the above diagram a fiber product diagram, thus completing the proof. □

The following is an important technical result.

Lemma 1.6.4.7. *Let $X = \bigcup_\alpha U_\alpha$ be an open cover of the scheme X. Let $f : X \to Y$ and $g : Z \to Y$ be scheme morphisms. Then,*

$$X \times_Y Z \cong \bigcup_\alpha U_\alpha \times_Y Z.$$

Proof. Let $p : X \times_Y Z \to X$ be the fiber product scheme over X obtained by base change along f. Then,

$$p^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha p^{-1}(U_\alpha).$$

By Lemma 1.6.4.6, we see that $p^{-1}(U_\alpha) \cong U_\alpha \times_Y Z$. It follows that

$$X \times_Y Z = p^{-1}(X) = \bigcup_\alpha p^{-1}(U_\alpha) \cong \bigcup_\alpha U_\alpha \times_Y Z,$$

as needed. □

Lemma 1.6.4.8. *Let $f : X \to Y$ and $g : Z \to Y$ be scheme morphisms and $U \subseteq X$ be an open subscheme such that $f(U) \subseteq V$ for some open subscheme $V \subseteq Y$ and let $W = g^{-1}(V)$ be an open subscheme in Z. If $p : X \times_Y Z \to X$ is the fiber product over X obtained by base change along f, then $p^{-1}(U) \cong U \times_Y Z \cong U \times_V W$.*

Proof. The first isomorphism is the content of Lemma 1.6.4.6. The second isomorphism follows from the simple observation that $U \times_Y Z$ satisfies the same universal property as that of $U \times_V W$. □

We portray some pathologies of fiber product in the following examples.

Example 1.6.4.9. We show that fiber product of one point schemes may have more than one point(!) Indeed, consider the schemes $X = Y = \mathrm{Spec}\,(\mathbb{C})$ over $\mathrm{Spec}\,(\mathbb{R})$. Observe that $X \times_{\mathrm{Spec}(\mathbb{R})} Y \cong \mathrm{Spec}\,(\mathbb{C} \otimes_\mathbb{R} \mathbb{C})$. But since we have

$$\begin{aligned}\mathbb{C} \otimes_\mathbb{R} \mathbb{C} &\cong \frac{\mathbb{R}[x]}{x^2+1} \otimes_\mathbb{R} \mathbb{C} \\ &\cong \frac{\mathbb{C}[x]}{x^2+1} \\ &\cong \mathbb{C} \times \mathbb{C}\end{aligned}$$

by Chinese remainder theorem. Consequently, $\mathrm{Spec}\,(\mathbb{C} \times \mathbb{C}) \cong \mathrm{Spec}\,(\mathbb{C}) \amalg \mathrm{Spec}\,(\mathbb{C})$ which has 2 points.

1.6.5 Applications of fiber product

We would now like to portray some of the applications of fiber products, especially in endowing the fibers of a morphism with a scheme structure.

Inverse image of a closed subscheme

TODO.

Fibers of a map

Keep in mind the Lemma 27.5.0.3 and the surrounding remarks about stalks of sheaves for the remainder of this discussion. Let $f : X \to Y$ be a map and $y \in Y$ be a point. We endow $f^{-1}(y) \hookrightarrow X$ with a scheme structure. Define the *fiber* of f at y to be the following fiber product:

$$X_y := X \times_Y \mathrm{Spec}\,(\kappa(y)).$$

We at times denote it by $X \times_Y y$. Note that by natural map onto second factor, X_y is a scheme over $\kappa(y)$.

We now show that fiber of a scheme morphism as defined above matches with the usual notion of fiber in the sense that both spaces are homeomorphic. We first do this for affine schemes.

Proposition 1.6.5.1. *Let $X = \operatorname{Spec}(S)$, $Y = \operatorname{Spec}(R)$ and $f : X \to Y$ be the map associated to a ring homomorphism $\varphi : R \to S$. Let $y = \mathfrak{p} \in Y$ be a prime ideal of R. Then, X_y is homeomorphic to the subspace $f^{-1}(y)$ of Y.*

Proof. We have that $X_y = \operatorname{Spec}(S \otimes_R \kappa(\mathfrak{p}))$, that is, the fiber of φ at prime ideal $\mathfrak{p}$ (Definition 23.5.1.4). We now calculate $S \otimes_R \kappa(\mathfrak{p})$. Indeed, we have

$$\begin{aligned} S \otimes_R \kappa(\mathfrak{p}) &= S \otimes_R F(R/\mathfrak{p}) \cong S \otimes_R (R/\mathfrak{p} \otimes_R R_{\mathfrak{p}}) \\ &\cong S/\mathfrak{p}S \otimes_R R_{\mathfrak{p}} \\ &\cong (S/\mathfrak{p}S)_{\varphi(R\setminus\mathfrak{p})}. \end{aligned}$$

It follows from Lemma 23.1.2.3 that $\operatorname{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ is exactly the subspace of X consisting of those primes $\mathfrak{q}$ such that $\mathfrak{q} \supseteq \varphi(\mathfrak{p})$ and does not intersects $\varphi(R \setminus \mathfrak{p})$. This is equivalent to saying that $\varphi^{-1}(\mathfrak{q}) \supseteq \mathfrak{p}$ and $\varphi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$, that is, $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, as needed. □

We now do the general case. The main idea is just to reduce to the affine case as above.

Lemma 1.6.5.2. *Let $f : X \to Y$ be a scheme morphism and $y \in Y$. Then, $f^{-1}(y)$ as a subspace of X is homeomorphic to X_y.*

Proof. Let $V = \operatorname{Spec}(B)$ be an open affine of Y containing y. Then, by definition of fiber products, we immediately see that $f^{-1}(V) \cong X \times_Y V$. Clearly, $f^{-1}(y) \subseteq f^{-1}(V)$. Cover $f^{-1}(V)$ by open affines $\{U_\alpha = \operatorname{Spec}(R_\alpha)\}$. By Proposition 1.6.5.1, we see that $f^{-1}(y) \cap U_\alpha \cong \operatorname{Spec}(R_\alpha \otimes_B \kappa(y)) = U_\alpha \times_V \operatorname{Spec}(\kappa(y))$. Since

$$\begin{aligned} X_y = X \times_Y \operatorname{Spec}(\kappa(y)) &\cong f^{-1}(V) \times_V \operatorname{Spec}(\kappa(y)) \\ &= \left(\bigcup_\alpha U_\alpha\right) \times_V \operatorname{Spec}(\kappa(y)) \\ &\cong \bigcup_\alpha (U_\alpha \times_V \operatorname{Spec}(\kappa(y))) \\ &\cong \bigcup_\alpha f^{-1}(y) \cap U_\alpha \\ &= f^{-1}(y), \end{aligned}$$

as needed. □

Example 1.6.5.3. We calculate explicit fibers of a map at every point of a familiar map. *Write solution of Exercise 3.10 of Hartshorne Chapter 2, written in notebook.*

The fibers of $\mathrm{Spec}\,(\mathbb{Z}[x]) \to \mathrm{Spec}\,(\mathbb{Z})$

We know that $\mathrm{Spec}\,(\mathbb{Z})$ is the final object in the category of schemes **Sch**. We also know that there is the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$. This induces a map

$$\varphi : \mathrm{Spec}\,(\mathbb{Z}[x]) \longrightarrow \mathrm{Spec}\,(\mathbb{Z}).$$

Understanding the fibers of this map will allow us to understand the affine arithmetic surface $\mathrm{Spec}\,(\mathbb{Z})$ (as $\mathbb{Z}[x]$ is a 2-dimensional ring). Note that we can already understand $\mathrm{Spec}\,(\mathbb{Z}[x])$ by the results surrounding Gauss' lemma as done in Theorem 23.1.5.3, but the following is a more geometric way of understadning this.

Proposition 1.6.5.4. *The prime ideals of $\mathbb{Z}[x]$ can be categorized into following three types.*

1. *$\langle p\rangle$ where $p \in \mathbb{Z}$ is a prime,*
2. *$\langle f(x)\rangle$ where $f(x) \in \mathbb{Z}[x]$ is an irreducible polynomial,*
3. *$\langle p, f(x)\rangle$ where $p \in \mathbb{Z}$ is a prime and $f(x) \in \mathbb{Z}[x]$ irreducible in $\mathbb{Z}[x]$ which remains irreducible in $\mathbb{Z}/p\mathbb{Z}$,*

Proof. We will prove this by analyzing the fibers of $f : \mathrm{Spec}\,(\mathbb{Z}[x]) \to \mathrm{Spec}\,(\mathbb{Z})$. Pick a prime $p \in \mathbb{Z}$ and denote $X = \mathrm{Spec}\,(\mathbb{Z}[x])$. The fiber $X_p = \mathrm{Spec}\,(\mathbb{Z}[x]) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}\,(\kappa(p))$. As $\kappa(p) = \mathbb{F}_p$, finite field with p elements, therefore we have that $X_p = \mathrm{Spec}\,(\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{F}_p) = \mathrm{Spec}\,(\mathbb{F}_p[x])$. Note that for same reasons we have $X_{\mathfrak{o}} = \mathrm{Spec}\,(\mathbb{Q}[x])$.

As fibers of f covers the whole scheme, it follows that any point in $\mathbb{Z}[x]$ looks like one of the following:

1. a prime ideal in $\mathbb{Q}[x]$,
2. a prime ideal in $\mathbb{F}_p[x]$.

Moreover, we have the following diagrams

$$\begin{tikzcd}
\mathrm{Spec}\,(\mathbb{F}_p[x]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\,(\mathbb{Z}[x]) \arrow[d] \arrow[dr, phantom, "\leftrightsquigarrow"] & \mathbb{F}_p[x] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{Z}[x] \arrow[l] \\
\mathrm{Spec}\,(\mathbb{F}_p) \arrow[r] & \mathrm{Spec}\,(\mathbb{Z}) & \mathbb{F}_p \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

.

$$\begin{tikzcd}
\mathrm{Spec}\,(\mathbb{Q}[x]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\,(\mathbb{Z}[x]) \arrow[d] \arrow[dr, phantom, "\leftrightsquigarrow"] & \mathbb{Q}[x] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{Z}[x] \arrow[l] \\
\mathrm{Spec}\,(\mathbb{Q}) \arrow[r] & \mathrm{Spec}\,(\mathbb{Z}) & \mathbb{Q} \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

Observe that $\mathbb{Z}[x] \to \mathbb{F}_p[x]$ is the mod-p map. Since every prime ideal of $\mathbb{Z}[x]$ now is a inverse image of a prime ideal by $\mathbb{Z}[x] \to \mathbb{F}_p[x]$ and $\mathbb{Z}[x] \to \mathbb{Q}[x]$, we get the desired result. $\square$

Geometric properties

Cover geometric reducibility and etc etc from Hartshorne exercises.

1.6.6 Normal schemes and normalization

Do mainly Exercise 3.7, 3.8 of Chapter 2 of Hartshorne. Also do Exercise 3.17, 3.18 of Chapter 1 of Hartshorne.
We now study a class of schemes which globalizes the notion of integral closure from algebra (Definition 23.7.1.10). These will find its main use in arithmetic where normal domains fundamental.

Definition 1.6.6.1 (**Normal schemes**)**.** A scheme X is said to be normal if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain.

The following is immediate from local nature of normal domains (Proposition 23.7.2.10).

Lemma 1.6.6.2. *Let X be an integral scheme. Then the following are equivalent:*
1. *X is a normal scheme.*
2. *For all open affine $\mathrm{Spec}\,(A) \subseteq X$, the ring A is a normal domain.*

Proof. As X is integral, therefore for every open affine $\mathrm{Spec}\,(A)$ of X, A is a domain by Lemma 1.4.2.2. As X is normal iff $\mathcal{O}_{X,x}$ is a normal domain for all $x \in X$, the result follows from Proposition 23.7.2.10. $\square$

The main result in normal schemes is that any integral scheme induces a unique normal scheme obtained by normalizing each open affine.

Theorem 1.6.6.3. [28] *Let X be an integral scheme. Then there exists a scheme $\tilde{X} \to X$ over X where $\tilde{X}$ is a normal integral scheme such that for any normal integral scheme Z and a dominant map $f : Z \to X$, there exists a unique map $\tilde{f} : Z \to \tilde{X}$ such that the following commutes*

$$\begin{tikzcd} \tilde{X} \arrow[d] & Z \arrow[l, dashed, "\tilde{f}"'] \arrow[dl, "f"] \\ X & \end{tikzcd}.$$

The scheme $\tilde{X} \to X$ is called the normalization of X and is unique upto isomorphism.

We first see this for affine domains.

Lemma 1.6.6.4. *Let $X = \mathrm{Spec}\,(A)$ be an integral affine scheme and $Z = \mathrm{Spec}\,(B)$ be a normal integral affine scheme. Let $\tilde{X} = \mathrm{Spec}\left(\tilde{A}\right)$ be the normalization of X and denote the natural map $\pi : \tilde{X} \to X$. If $f : Z \to X$ is any dominant map, then there exists a map $\tilde{f} : Z \to \tilde{X}$ such that $\pi \circ \tilde{f} = f$.*

$$\begin{tikzcd} \mathrm{Spec}\left(\tilde{A}\right) \arrow[d, "\pi"'] & \mathrm{Spec}\,(B) \arrow[l, dashed, "\tilde{f}"'] \arrow[dl, "f"] \\ \mathrm{Spec}\,(A) & \end{tikzcd}.$$

[28] Exercise II.3.8 of Hartshorne.

Proof. Indeed, by Proposition 23.7.2.12, this follows immediately. □

Remark 1.6.6.5. By Remark 23.7.2.11, it follows that normalization is a strongly local property. Thus Theorem 1.6.6.3 holds.

Proof of Theorem 1.6.6.3. By Remark 23.7.2.11, it follows that normalization is a strongly local construction for domains. Let $A \hookrightarrow \tilde{A}$ be the normalization map for any domain A. Therefore by Theorem 1.6.2.10, we have a scheme $\alpha : \tilde{X} \to X$ such that for any open affine $\operatorname{Spec}(A) \hookrightarrow X$, the following diagram commutes

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d, "\alpha"] \\ \operatorname{Spec}(A) \arrow[r, hook] & X \end{tikzcd}$$

where the left vertical map is the map corresponding to normalization $A \hookrightarrow \tilde{A}$. This shows the construction of $\alpha : \tilde{X} \to X$.

Now let Z be an arbitrary normal integral scheme and $f : Z \to X$ be a dominant map. Pick any open affine $\operatorname{Spec}(A) \subseteq X$ and consider the non-empty (f is dominant) open subset $f^{-1}(\operatorname{Spec}(A))$. Write

$$f^{-1}(\operatorname{Spec}(A)) = \bigcup_{i \in I} \operatorname{Spec}(B_i)$$

where $\operatorname{Spec}(B_i) \subseteq Z$ are open affine. As Z is normal integral, therefore B_i are normal domains from Lemma 1.6.6.2. By restriction we thus have the map

$$f|_{\operatorname{Spec}(B_i)} : \operatorname{Spec}(B_i) \to \operatorname{Spec}(A)$$

for each $i \in I$. Observe that $\alpha^{-1}(\operatorname{Spec}(A)) \supseteq \operatorname{Spec}\left(\tilde{A}\right)$. By Lemma 1.6.6.4, it follows that we have a unique map $\tilde{f}_i : \operatorname{Spec}(B_i) \to \operatorname{Spec}\left(\tilde{A}\right)$ such that the following commutes

$$\begin{tikzcd} \operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\alpha|_{\operatorname{Spec}(\tilde{A})}"'] & \operatorname{Spec}(B_i) \arrow[l, dashed, "\tilde{f}_i"'] \arrow[dl, "f|_{\operatorname{Spec}(B_i)}"] \\ \operatorname{Spec}(A) & \end{tikzcd}.$$

It thus follows that for every open affine $\operatorname{Spec}(B_{ij}) \subseteq \operatorname{Spec}(B_i)$, we have a map $\tilde{f}_i$: $\operatorname{Spec}(B_i) \to \operatorname{Spec}\left(\tilde{A}\right)$ by restriction. Hence by Lemma 1.6.6.4, we have that this is unique. As $\operatorname{Spec}(A) \subseteq X$ is arbitrary open affine, therefore we have an open affine covering $\{\operatorname{Spec}(A_i)\}_{i \in I}$ of X which by inverse image gives an open affine covering $\{\operatorname{Spec}(B_{ij})\}$ of Z and a collection of open affines $\{\operatorname{Spec}\left(\tilde{A}_i\right)\}$ of $\tilde{X}$ such that for each i, we have a unique map $\tilde{f}_{ij} : \operatorname{Spec}(B_{ij}) \to \tilde{X}$ such that

$$\begin{tikzcd} \tilde{X} \arrow[d, "\alpha"'] & \operatorname{Spec}\left(\tilde{A}_i\right) \arrow[l, hook'] \arrow[d, "\alpha"'] & \operatorname{Spec}(B_{ij}) \arrow[l, dashed, "\tilde{f}_{ij}"'] \arrow[dl, "f"] \\ X & \operatorname{Spec}(A_i) \arrow[l, hook'] & \end{tikzcd}$$

commutes. We claim that $\tilde{f}_{ij}$ can be glued to a unique map $\tilde{f} : Z \to \tilde{X}$, which would complete the proof. First, for a fixed i, we glue $\tilde{f}_{ij}$ and $\tilde{f}_{il}$. Indeed, covering the intersection $\mathrm{Spec}\,(B_{ij}) \cap \mathrm{Spec}\,(B)_{il}$ by open affines $\mathrm{Spec}\,(C_p)$, we immediately by restriction get maps $\tilde{f}_{ij} : \mathrm{Spec}\,(C_p) \to \mathrm{Spec}\left(\tilde{A}_i\right)$ and $\tilde{f}_{il} : \mathrm{Spec}\,(C_p) \to \mathrm{Spec}\left(\tilde{A}_i\right)$ which are thus equal by uniqueness. Hence, for each i, we may glue the maps $\{\tilde{f}_{ij}\}_j$ to obtain a unique map $\tilde{f}_i : Z_i = f^{-1}(\mathrm{Spec}\,(A_i)) \to \mathrm{Spec}\left(\tilde{A}_i\right)$ as in

$$\begin{tikzcd} \mathrm{Spec}\left(\tilde{A}_i\right) \arrow[d, "\alpha"'] & Z_i \arrow[l, dashed, "\tilde{f}_i"'] \arrow[dl, "f"] \\ \mathrm{Spec}\,(A_i) & \end{tikzcd} .$$

We now wish to glue these $\tilde{f}_i$. To this end, pick an affine open $\mathrm{Spec}\,(C) \subseteq Z_i \cap Z_k = f^{-1}(\mathrm{Spec}\,(A_i) \cap \mathrm{Spec}\,(A_k))$ and observe $\alpha^{-1}(\mathrm{Spec}\,(A_i) \cap \mathrm{Spec}\,(A_k)) \supseteq \mathrm{Spec}\left(\tilde{A}_i\right) \cap \mathrm{Spec}\left(\tilde{A}_k\right)$. We thus have the following diagram

$$\begin{tikzcd} \mathrm{Spec}\left(\tilde{A}_i\right) \arrow[d, "\alpha"'] & \mathrm{Spec}\,(C) \arrow[l, "\tilde{f}_i"'] \arrow[r, "\tilde{f}_k"] \arrow[d, "f"] & \mathrm{Spec}\left(\tilde{A}_k\right) \arrow[d, "\alpha"] \\ \mathrm{Spec}\,(A_i) & \mathrm{Spec}\,(A_i) \cap \mathrm{Spec}\,(A_k) \arrow[l, hook'] \arrow[r, hook] & \mathrm{Spec}\,(A_k) \end{tikzcd} .$$

By Lemma 1.6.6.4, it then suffices to show that $\tilde{f}_i(\mathrm{Spec}\,(C)), \tilde{f}_k(\mathrm{Spec}\,(C)) \subseteq \mathrm{Spec}\left(\tilde{A}_i\right) \cap \mathrm{Spec}\left(\tilde{A}_k\right)$, as then uniqueness would imply $\tilde{f}_i$ and $\tilde{f}_k$ are equal over $\mathrm{Spec}\,(C)$. By symmetry, it suffices to show this for $\tilde{f}_i$. Since $\alpha \circ \tilde{f}_i(\mathrm{Spec}\,(C)) \subseteq \mathrm{Spec}\,(A_i) \cap \mathrm{Spec}\,(A_k)$, therefore $\tilde{f}_i(\mathrm{Spec}\,(C)) \subseteq \alpha^{-1}(\mathrm{Spec}\,(A_i) \cap \mathrm{Spec}\,(A_k)) \cap \mathrm{Spec}\left(\tilde{A}_i\right) \subseteq \mathrm{Spec}\left(\tilde{A}_i\right) \cap \mathrm{Spec}\left(\tilde{A}_k\right)$, as required. Hence $\tilde{f}_i$ can be glued to a unique map $\tilde{f} : Z \to \tilde{X}$, thus completing the proof. □

The following is the globalization of the fact that normalization of a finite type algebra is again a finite type algebra, over a field (Noether's Theorem **??**).

Corollary 1.6.6.6. *If X is a finite type integral scheme, then the the normalization $\tilde{X} \to X$ is a finite map.*

Proof. **TODO**. □

1.7 Dimension of schemes

Do from Vakil, Hartshorne Exercise 3.20, 3.21, 3.22 of Chapter 2.
The notion of dimension of a geometric object serves as an essential tool for any attempt at its understanding. Schemes are no different and we have a notion of dimension for them. However, we also have a notion of dimension of rings. This section explores how these two interrelates and thus facilitate understanding of geometry of schemes.

1.7.1 General properties

Before moving to schemes that we will encounter the most, let us first give a general review of the notion of dimension of topological spaces and some general properties of dimension of schemes. Recall that the dimension of a topological space is the supremum of the length of the strictly decreasing chains of finite length of closed irreducible subsets of the space. Further for a space X and a closed irreducible subset $Z \subseteq X$, the codimension of Z in X is defined to be the supremum of the length of strictly increasing chains of closed irreducible subsets starting from Z. For an arbitrary closed subset $Y \subseteq X$, we define codim $(Y, X) = \inf_{Z \subseteq Y}$ codim (Z, X) where Z varies over all closed irreducible subsets of Y. For any closed set $Y \subseteq X$, if $\dim X < \infty$, we always have codim $(Y, X) \leq \dim X$.

Proposition 1.7.1.1. *Let X be a topological space. Then,*

1. *If $Y \subseteq X$ is a subspace, then $\dim Y \leq \dim X$.*
2. *If $\{U_i\}_{i \in I}$ is an open covering of X, then $\dim X = \sup_i \dim U_i$.*
3. *Let $Y \subseteq X$ be a closed subspace and X be of finite dimension. If X is irreducible and $\dim Y = \dim X$, then $Y = X$.*

Proof. The main tool in all of them is just a clear understanding of the definition of dimension and of closed irreducible sets. We establish some terminologies to work with in this proof. For any space X a strictly decreasing chain of finite length of closed irreducible subsets will be called a *finite chain* of X and set of all finite chains will be denoted by $FC(X)$. We denote a chain by $Z_\bullet \in FC(X)$ and its length by $l(Z_\bullet)$. Consequently, $\dim X = \sup_{Z_\bullet \in FC(X)} l(Z_\bullet)$.

1. First observe that if Y is closed then the result is immediate as any finite chain of Y will be a finite chain of X. Consequently, we reduce to showing that $\dim Y \leq \dim \bar{Y}$. In particular, we reduce to showing that if Y is dense in X, then $\dim Y \leq \dim X$. It further suffices to show existence of a length preserving map $FC(Y) \to FC(X)$. Indeed, for any $Z_\bullet \in FC(Y)$, one observes that $\mathrm{Cl}_X(Z_i)$ is a closed subset of X which is further irreducible in X. Consequently, $\mathrm{Cl}(Z_\bullet)$ is a finite chain of X of same length as of $Z_\bullet$[29].
2. By 1. we already have $\sup_i \dim U_i \leq \dim X$ so we need only show that $\dim X \leq \sup_i \dim U_i$. It suffices to show that for each $Z_\bullet \in FC(X)$, there exists $i \in I$ and $W_\bullet \in FC(U_i)$ such that $l(Z_\bullet) \leq l(W_\bullet)$. Let $r = l(Z_\bullet)$ and $i \in I$ be such that $U_i \cap Z_r \neq \emptyset$. Then, $W_\bullet = U_i \cap Z_\bullet$ forms a finite chain of U_i of same length as $Z_\bullet$. To see this, observe that if $U_i \cap Z_a = U_i \cap Z_b$ where we may assume $Z_a \supsetneq Z_b$, then the open set $U_i \cap Z_a$ of Z_a is contained in the closed set Z_b of Z_a, hence the closure of

[29] Actually we didn't needed the reduction to Y being dense in X.

$U_i \cap Z_a$ in Z_a is inside Z_b. But since Z_a is irreducible so $U_i \cap Z_a$ must be dense in Z_a, a contradiction.

3. Let $r = \dim X = \dim Y$. Suppose $Y \subsetneq X$. Let $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ be a maximal finite chain of Y. Then the chain $X \supsetneq Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ is a finite chain in X as Y is closed. Thus $\dim X \geq 1 + r$, therefore $r = \dim Y \geq r + 1$, a contradiction.

□

The following technical lemma was employed in proving the statement 2 of above, but is good to keep in handy.

Lemma 1.7.1.2. *Let X be a topological space and $Z_\bullet \in FC(X)$ a finite chain (in the terminology of Proposition 1.7.1.1) of length $l(Z_\bullet) = r$. If $U \subseteq X$ is an open set such that $U \cap Z_r \neq \emptyset$, then $U \cap Z_\bullet$ is a finite chain of length r in U.* □

The following result gives a connection between all closed irreducible containing a given point and prime ideals of the local ring at that point.

Proposition 1.7.1.3. *Let X be a scheme and $x \in X$ be a point. We obtain an order reversing bijection*

$$\{\textit{Closed irreducibles } Y \textit{ of } X \textit{ containing } x\} \cong \operatorname{Spec}(\mathcal{O}_{X,x}).$$

Proof. Denote the collection of all closed irreducibles of X containing x as I. Let $U = \operatorname{Spec}(A)$ be an affine open containing $x \in X$ so that $\mathcal{O}_{X,x} \cong A_x$. Consequently, we wish to show a bijection $I \cong \operatorname{Spec}(A_x)$, which is further equivalent to showing that I is bijective to all prime ideals of A contained in x. As all prime ideals of A contained in x is further bijective to all closed irreducible of U containing x by Lemma 1.2.1.1, we thus reduce to showing existence of a bijection between I and closed irreducibles of U containing x, denoted J.

Consider the following function

$$\begin{aligned} \varphi : I &\longrightarrow J \\ Y &\longmapsto Y \cap U. \end{aligned}$$

Indeed, this map is well-defined as for any $Y \in I$, $\varphi(Y) = Y \cap U$ is first irreducible as any open subset of an irreducible set is irreducible. Further, it is closed in U as Y is closed. In order to show injectivity, we need only recall that any open subset of an irreducible set is dense. Finally, for surjectivity, take any $Z \in J$ so that Z is a closed irreducible in U containing x. Now let Y to be the closure of Z in X. We thus need only show that Y is irreducible in X. That follows immediately from the fact that closure of irreducible is again an irreducible, which in turn follows immediately from a simple observation on open subsets of the closure. □

One then observes the following general result which will be used heavily in the future.

Lemma 1.7.1.4. *Let X be a scheme and Y be an irreducible closed subscheme of X with $\eta \in Y$ being its generic point. Then,*

$$\operatorname{codim}(Y, X) = \dim \mathcal{O}_{X,\eta}.$$

Proof. This is immediate from Proposition 1.7.1.3 as Y is the smallest closed irreducible containing η. □

1.7.2 Dimension of finite type k-schemes

In this section, we prove various results surrounding the relationship between dimension of a given integral finite type k-scheme as a topological space and Krull dimension of various local rings.

Theorem 1.7.2.1. [30] *Let k be a field and X be a finite type integral k-scheme.*

1. *If $U, V \subseteq X$ are two open affines which are spectra of finite type k-domains, then* $\dim U = \dim V$.
2. *If $\{U_i\}_{i=1}^n$ is any finite open affine covering by spectra of finite type k-domains, then* $\dim U_i = \dim X$ *for all $i = 1, \dots, n$.*
3. *If $p \in X$ is a closed point, then* $\dim X = \dim \mathcal{O}_{X,p}$.
4. *Let $K(X)$ be the function field of X. Then* $\dim X = \text{trdeg } K(X)/k$.
5. *If Y is a closed subset of X, then* $\text{codim }(Y, X) = \inf_{p \in Y} \dim \mathcal{O}_{X,p}$.
6. *If Y is a closed subset of X, then* $\dim Y + \text{codim }(Y, X) = \dim X$.

Proof. The main tools are the Theorems 23.8.2.1 and 23.8.2.2.

1. Observe that since X is irreducible, therefore U and V are dense open subsets of X, so $U \cap V \neq \emptyset$. Consequently, it will suffice to show that any dense affine open subset $W \subseteq U$ has same dimension as U. Indeed, U is spectra of finite type k-domain, so it is a separated finite type integral affine scheme, that is, an abstract affine variety. Consequently, by Proposition I.1.10 of cite[Hartshorne], $\dim W = \dim \overline{W} = \dim U$.
2. Follows from Proposition 1.7.1.1, 2 and statement 1.
3. As X is finite type, it admits a finite open affine covering by spectra of finite type k-domains. Let $U = \text{Spec}\,(A)$ be one such open affine such that $p \in U$. Consequently, $p = \mathfrak{m} \in \text{Spec}\,(A)$ represents a maximal ideal of R (Lemma 1.2.1.3). Thus, $\mathcal{O}_{X,p} \cong A_{\mathfrak{m}}$ and so $\dim \mathcal{O}_{X,p} = \dim A_{\mathfrak{m}}$. Note that A is a finite type k-algebra which is an integral domain. It thus follows by Theorem 23.8.2.2 that we have $\text{ht } \mathfrak{m} + \dim A/\mathfrak{m} = \dim A$ and since $\dim A/\mathfrak{m} = 0$, therefore $\text{ht } \mathfrak{m} = \dim A$. Further, since $\dim A_{\mathfrak{m}} = \text{ht } \mathfrak{m}$, therefore we have $\dim \mathcal{O}_{X,p} = \dim A_{\mathfrak{m}} = \dim A = \dim U$. By statement 2, $\dim U = \dim X$ and the result follows.
4. Function field is defined to be the local ring at the generic point of X, say $\eta \in X$ (Remark 1.4.2.5). Let $\eta \in \text{Spec}\,(A)$ where $\text{Spec}\,(A)$ is a member of an open affine cover of X by spectra of finite type k-domains. Observe that $\text{Spec}\,(A)$ has η as its generic point as well. Consequently, $\dim \text{Spec}\,(A) = \dim A = \text{trdeg } K(A)/k$ and since $K(A) = \mathcal{O}_{\text{Spec}(A),\eta} \cong \mathcal{O}_{X,\eta} = K(X)$, therefore $\dim \text{Spec}\,(A) = \text{trdeg } K(X)/k$. By statement 2, $\dim \text{Spec}\,(A) = \dim X$ and the result follows.
5. First observe that for any closed irreducible $Z \subseteq X$, we have $\text{codim }(Z, X) \leq \dim X$. By statement 3, therefore, we have $\inf_{p \in Y} \dim \mathcal{O}_{X,p} = \inf_{p \in Y \text{ non-closed}} \mathcal{O}_{X,p}$. We will now show that for any closed irreducible subset $Z \subseteq X$ with $\eta \in Z$ its generic paint (schemes are *sober*[31]), we have $\dim \mathcal{O}_{X,\eta} = \text{codim }(Z, X)$. By taking infimum, the result would then follow, so it would suffice to show the above claim.
   Let $\{\text{Spec}\,(A_\alpha)\}$ be a finite open affine cover of X where A_α is a finite type k-domain. Observe that if $Z \cap \text{Spec}\,(A_\alpha) \neq \emptyset$, then $\eta \in \text{Spec}\,(A_\alpha)$. Now, $\eta \in \text{Spec}\,(A_\alpha)$ is a

[30] Exercise II.3.20 of Hartshorne.

[31] a space where all closed irreducibles have a unique generic point.

point whose closure in $\mathrm{Spec}\,(A_\alpha)$ is $Z \cap \mathrm{Spec}\,(A_\alpha)$ so $Z \cap \mathrm{Spec}\,(A_\alpha)$ is a closed irreducible subspace of $\mathrm{Spec}\,(A_\alpha)$ whose generic point is η and thus $Z \cap \mathrm{Spec}\,(A_\alpha) \cong \mathrm{Spec}\,(A_\alpha/\eta)$, where we treat $\eta \lneq A_\alpha$ as a prime ideal of A_α. Consequently, $\dim \mathcal{O}_{X,\eta} = \dim \mathcal{O}_{\mathrm{Spec}(A_\alpha),\eta} = \dim(A_\alpha)_\eta = \mathrm{ht}\ \eta$. Since A_α is a finite type k-domain, therefore by Theorem 23.8.2.2, we obtain that $\mathrm{ht}\ \eta + \dim A_\alpha/\eta = \dim A_\alpha$, which thus yields $\mathrm{ht}\ \eta = \dim X - \dim A_\alpha/\eta$ by statement 2. It thus suffices to show that for some index α we get $\dim A_\alpha/\eta = \dim Z$ as then we would obtain $\dim \mathcal{O}_{X,\eta} = \dim X - \dim Z = \mathrm{codim}\ (Z, X)$.
Indeed, since $\{\mathrm{Spec}\,(A_\alpha/\eta)\}$ forms a finite open affine cover of Z, therefore by Proposition 1.7.1.1, 2 we get such an index α.

6. Observe that since $\mathrm{codim}\ (Y, X) < \infty$, therefore there exists a maximal closed irreducible $Z \subseteq Y$ such that $\mathrm{codim}\ (Y, X) = \mathrm{codim}\ (Z, X)$. Consequently, we have a finite chain of X, say $Z_\bullet$, ending at Z such that $l(Z_\bullet) = \mathrm{codim}\ (Y, X)$.
Let $U = \mathrm{Spec}\,(A)$ be an open affine where A is a finite type k-domain such that $U \cap Z \neq \emptyset$. Further, $\dim U \cap Y = \dim Y$. Consequently, by Lemma 1.7.1.2, we have $\mathrm{codim}\ (Y, X) = \mathrm{codim}\ (Z \cap U, U)$. Since $U \cap Y$ is a closed subscheme of U, therefore we may write $U \cap Y = \mathrm{Spec}\,(A/I)$ for an ideal $I \leq A$. Consequently, $\mathrm{codim}\ (Y, X) = \mathrm{codim}\ (\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A))$.
It is immediate from first definitions that
$$\begin{aligned}\mathrm{codim}\ (\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A)) &= \inf_{\mathfrak{p} \supseteq I} \mathrm{codim}\ (\mathrm{Spec}\,(A/\mathfrak{p}), \mathrm{Spec}\,(A)) \\ &= \inf_{\mathfrak{p} \supseteq I} \mathrm{ht}\ \mathfrak{p}.\end{aligned}$$
Now by Theorem 23.8.2.2 and above, we further obtain that
$$\begin{aligned}\mathrm{codim}\ (\mathrm{Spec}\,(A/I), \mathrm{Spec}\,(A)) &= \inf_{\mathfrak{p} \supseteq I} (\dim A - \dim A/\mathfrak{p}) \\ &= \dim A - \sup_{\mathfrak{p} \supseteq I} \dim A/\mathfrak{p} \\ &= \dim X - \dim U \cap Y \\ &= \dim X - \dim Y\end{aligned}$$
where $\dim A = \dim X$ because of statement 2.

□

1.7.3 Dimension of fibers

In this section, we discuss the question of how the dimension of fibers of a morphism varies. We'll see that certain nice geometric situations are encoded in the maps for which the dimension of fibers is not too erratic.

1.8 Projective schemes

The most important type of examples that we will encounter in our study of algebraic geometry are subvarieties of projective space $\mathbb{P}^n_k$. Indeed, this is a construction which is fundamental because of the many nice properties enjoyed by realizing familiar constructions in it. One of them being this classical observation that any two straight lines are bound to intersect at atleast one point in the projective space. We shall see more equally nice results, not to mention the quadrics with which we wish to spend some considerable time as the main motivating example for us (Example 1.5.1.3) is itself realized as a quadric in projective space.

We recall that the notion of projective varieties, whose generalization we shall embark now on, has been covered in Section 1.5.

We first begin by defining the space $\mathrm{Proj}(S)$ of a graded ring $S = \bigoplus_{d\geq 0} S_d$.

Definition 1.8.0.1. (**Projective spectrum of a graded ring**) Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring and let $S_+ = \bigoplus_{d>0} S_d$ be the ideal generated by non-zero degree elements. Denote

$$\mathrm{Proj}(S) := \{\mathfrak{p} \lneq S \mid \mathfrak{p} \text{ is homogeneous prime ideal } \& \ \mathfrak{p} \not\supseteq S_+\}.$$

The set $\mathrm{Proj}(S)$ is called the projective spectrum of the graded ring S.

Note that the latter condition is motivated by Remark 1.5.3.13. This is also used in a technical manner to show existence of a nice basis over $\mathrm{Proj}(S)$ in Lemma 1.8.1.3 and in other proofs as well. We now show that there is a natural topology over $\mathrm{Proj}(S)$, akin to the affine case.

Lemma 1.8.0.2. *Let S be a graded ring and denote for a homogeneous ideal $\mathfrak{a} \leq S$, the following subset of* $\mathrm{Proj}(S)$*:*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Then, for any homogeneous ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{a}_i$ of S, we obtain

1. $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$,
2. $\bigcap_i V(\mathfrak{a}_i) = V\left(\sum_i \mathfrak{a}_i\right)$.

Proof. Same as Lemma 1.2.0.1. □

We thus obtain a topological space $\mathrm{Proj}(S)$ where a set is closed if and only if it is of the form $V(\mathfrak{a})$ for a homogeneous ideal $\mathfrak{a} \leq S$. This is called the Zariski topology over the $\mathrm{Proj}(S)$.

We now give some more topological properties of $\mathrm{Proj}(S)$.

1.8.1 Topological properties of $\mathrm{Proj}(S)$

The first obvious question is how does the inclusion $\mathrm{Proj}(S) \hookrightarrow \mathrm{Spec}\,(S)$ looks topologically?

Lemma 1.8.1.1. *Let S be a graded ring. The topology of $\mathrm{Proj}(S)$ is obtained by subspace topology of $\mathrm{Spec}\,(S)$. Thus, there is a continuous inclusion*

$$\mathrm{Proj}(S) \hookrightarrow \mathrm{Spec}\,(S).$$

Proof. Immediate from definitions. $\square$

We further note that for a graded ring S, the degree zero elements S_0 form a subring of S by the virtue of the fact that $S_d \cdot S_e \subseteq S_{d+e}$. Thus, we obtain a continuous map as the following shows.

Lemma 1.8.1.2. *Let S be a graded ring. Then the following is a continuous map*

$$\begin{aligned} \varphi : \mathrm{Proj}(S) &\longrightarrow \mathrm{Spec}\,(S_0) \\ \mathfrak{p} &\longmapsto \mathfrak{p} \cap S_0. \end{aligned}$$

Proof. Pick any ideal $\mathfrak{a} \leq S_0$ and notice that it is already homogeneous in S. Consequently, $\varphi^{-1}(V(\mathfrak{a})_a) = V(\mathfrak{a})_h$ where $V(\mathfrak{a})_a \subseteq \mathrm{Spec}\,(S_0)$ and $V(\mathfrak{a})_h \subseteq \mathrm{Proj}(S)$. $\square$

We now find a collection of open sets which forms a basis for $\mathrm{Proj}(S)$. This is akin to Lemma 1.2.1.4.

Lemma 1.8.1.3. *Let S be a graded ring and $f, g \in S_d$ for some $d > 0$ be homogeneous elements. Denote*

$$D_+(f) := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

Then,

1. *$D_+(f)$ is an open subset of $\mathrm{Proj}(S)$,*
2. *$D_+(f) \cap D_+(g) = D_+(fg)$,*
3. *$\{D_+(f)\}_{f \in S_d, d>0}$ forms a basis of $\mathrm{Proj}(S)$.*

Proof. 1. Since $D_+(f) = \mathrm{Proj}(S) \setminus V(f)$, thus $D_+(f)$ is open.
2. Straightforward.
3. Since for any $\mathfrak{p} \in \mathrm{Proj}(S)$, there exists $f \in S_d$ for some $d > 0$ such that $f \notin \mathfrak{p}$ as $\mathfrak{p}$ does not contain all of S_+, thus $\bigcup_{f \in S_d, d>0} D_+(f) = \mathrm{Proj}(S)$. The rest follows by 2. $\square$

Remark 1.8.1.4. As tempting as it might be to think, but not all projective schemes are quasi-compact. An example is given by the graded ring $S = \mathbb{Z}[x_1, x_2, \dots]$, the polynomial ring over $\mathbb{Z}$ with countably infinitely many indeterminates. Then one observes that $\mathrm{Proj}(S) = \bigcup_{n=1}^{\infty} D_+(x_n)$. Moreover, as for any $\mathfrak{p} \in \mathrm{Proj}(S)$ can not contain S_+, therefore $\mathfrak{p}$ necessarily has to not contain some x_i, otherwise it contains S_+. Consequently, we cannot form a finite subcover of the above cover, showing that $\mathrm{Proj}(S)$ is not quasi-compact.

However, the following lemma might be helpful in checking when a projective scheme has a finite cover by basic open sets.

Lemma 1.8.1.5. *Let S be a graded ring and consider $X = \mathrm{Proj}(S)$. Let $f = f_0 + \cdots + f_n$ be a decomposition of $f \in S$ into homogeneous elements $f_d \in S_d$. Then,*

$$D(f) \cap X = (D(f_0) \cap X) \cup \bigcup_{d=1}^{n} D_+(f_d)$$

where we view $X \subseteq \mathrm{Spec}\,(S)$ and $D(f), D(f_0) \subseteq \mathrm{Spec}\,(S)$.

Proof. This is a rather straightforward proof. To show ($\subseteq$), consider a point $\mathfrak{p} \in D(f) \cap X$ so that $f \notin \mathfrak{p}$. It follows from $f = f_0 + \cdots + f_n$ that for some $d = 0, \ldots, n$, $f_d \notin \mathfrak{p}$, which is in turn equivalent to stating that $\mathfrak{p} \in D_+(f_i)$ if $i \geq 1$ or $\mathfrak{p} \in D(f_0) \cap X$ if $d = 0$.

Conversely, pick $\mathfrak{p} \in (D(f_0) \cap X) \cup \bigcup_{d=1}^{n} D_+(f_d)$. We obtain that for some $d = 0, \ldots, n$, $f_d \notin \mathfrak{p}$. It follows from $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{p} \cap S_i$ that if $f \in \mathfrak{p}$, then we get by uniqueness of representatives of the direct sum that $f_d \in \mathfrak{p}$, a contradiction. $\square$

1.8.2 The structure sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$ and projective schemes

We have studied some basic properties of the topological space $\mathrm{Proj}(S)$ so far, we now construct a structure sheaf over it and make it, first, into a locally ringed space and, second, into a scheme. We first define the structure sheaf of projective spectrum, in which there is nothing new in comparison to projective varieties (see Definition 1.5.3.1).

Definition 1.8.2.1. (**The structure sheaf $\mathcal{O}_{\mathrm{Proj}(S)}$**) Let S be a graded ring. Let $U \subseteq \mathrm{Proj}(S)$ be an open set of the projective spectrum of S. Define the following set

$$\mathcal{O}_{\mathrm{Proj}(S)}(U) := \left\{ s : U \to \textstyle\coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \mid \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})} \;\&\; \exists \text{ open } \mathfrak{p} \in V \subseteq U \;\&\; f, g \in S_d,\, d \geq 0 \text{ s.t. } \forall \mathfrak{q} \in V,\, g \notin \mathfrak{q} \;\&\; s(\mathfrak{q}) = f/g \right\}.$$

From the fact that its elements are functions locally defined, one immediately obtains that $\mathcal{O}_{\mathrm{Proj}(S)}$ is a sheaf with obvious restriction maps. By appropriate restrictions on the domain, one further sees that under pointwise addition and multiplication, $\mathcal{O}_X(U)$ forms a commutative ring with 1.

Let us now show that $\mathrm{Proj}(S)$ is a scheme over $\mathrm{Spec}\,(S_0)$ in a natural manner.

Lemma 1.8.2.2. *Let S be a graded ring. Then $\mathrm{Proj}(S)$ is a scheme over $\mathrm{Spec}\,(S_0)$.*

Proof. We need only define a map $\mathrm{Proj}(S) \to \mathrm{Spec}\,(S_0)$. By Theorem 1.3.0.5, we need only construct a homomorphism $S_0 \to \Gamma(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. This is straightforward, as we can interpret each $a \in S_0$ as a homogeneous regular function $s : \mathrm{Proj}(S) \to \amalg_{\mathfrak{p} \in \mathrm{Proj}(S)} S_{(\mathfrak{p})}$ mapping as $\mathfrak{p} \mapsto a/1$. $\square$

Thus, $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a ringed space. We now see that the stalk of this sheaf is isomorphic to the homogeneous localization. This will thus show that $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a locally ringed space (Lemma 23.2.1.1).

Lemma 1.8.2.3. *Let S be a graded ring and consider the ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. For each $\mathfrak{p} \in \mathrm{Proj}(S)$, we have*

$$\mathcal{O}_{\mathrm{Proj}(S), \mathfrak{p}} \cong S_{(\mathfrak{p})}.$$

Proof. Consider the following map

$$\begin{aligned} \varphi : \mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}} &\longrightarrow S_{(\mathfrak{p})} \\ (U,s)_{\mathfrak{p}} &\longmapsto s(\mathfrak{p}). \end{aligned}$$

It is straightforward to see that φ is a well-defined ring homomorphism. To see injectivity, suppose $(U,s)_{\mathfrak{p}} \mapsto 0$. Thus $s(\mathfrak{p}) = 0$. Consequently, for some open $V \subseteq U$ containing $\mathfrak{p}$ where s is given by f/g for $f, g \in S_d$, $d \geq 0$, we obtain $s(\mathfrak{q}) = f/g = 0$ for all $\mathfrak{q} \in V$. Thus $s = 0$ on V and hence $(U,s)_{\mathfrak{p}} = (V, \rho_{U,V}(s))_{\mathfrak{p}} = 0$. To see surjectivity, pick any $f/g \in S_{(\mathfrak{p})}$. Observe that $g \notin \mathfrak{p}$. Thus consider $(D_+(g), f/g)_{\mathfrak{p}} \in \mathcal{O}_{\mathrm{Proj}(S),\mathfrak{p}}$. □

We now show that the locally ringed space $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ is a scheme. For this purpose we would need to show that $\mathrm{Proj}(S)$ is covered by affine opens. Indeed, we have the following lemma.

Lemma 1.8.2.4. *Let S be a graded ring and consider the locally ringed space* $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$. *For each $f \in S_d$, $d > 0$, we have the following isomorphism of locally ringed spaces*

$$\left(D_+(f), \mathcal{O}_{\mathrm{Proj}(S)|D_+(f)}\right) \cong \left(\mathrm{Spec}\left(S_{(f)}\right), \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)}\right).$$

Proof. Consider the map

$$\begin{aligned} \varphi : D_+(f) &\longrightarrow \mathrm{Spec}\left(S_{(f)}\right) \\ \mathfrak{p} &\longmapsto (\mathfrak{p} \cdot S_f)_0. \end{aligned}$$

By Lemma 23.2.1.4, it follows that φ is a bijection. To show that φ is an isomorphism it is sufficient to show that φ is a closed map. This is immediate as $\mathfrak{p} \supseteq \mathfrak{a}$ in $D_+(f)$ if and only if $(\mathfrak{p} \cdot S_f)_0 \supseteq (\mathfrak{a} \cdot S_f)_0$ in $\mathrm{Spec}\left(S_{(f)}\right)$.

We now wish to show isomorphism of corresponding sheaves. For this, we construct a map

$$\varphi^\flat : \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)} \longrightarrow \varphi_* \mathcal{O}_{D_+(f)}$$

and show that this is an isomorphism. Indeed, we first observe a canonical isomorphism on stalks

$$\mathcal{O}_{D_+(f),\mathfrak{p}} \cong S_{(\mathfrak{p})} \xrightarrow{\eta_{\mathfrak{p}}} (S_{(f)})_{(\mathfrak{p}\cdot S_f)_0} \cong \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right),\varphi(\mathfrak{p})}.$$

Then one can construct the above isomorphism $\varphi^\flat$ by observing the following square for sections of the relevant sheaves over open $U \subseteq \mathrm{Spec}\left(S_{(f)}\right)$ and the corresponding $\varphi^{-1}(U) \subseteq D_+(f)$:

$$\begin{tikzcd} U \arrow[d, "s"'] & & \varphi^{-1}(U) \arrow[ll, "\varphi"', "\cong"] \arrow[d, "t"] \\ \coprod_{\mathfrak{p}\in\varphi^{-1}(U)}((S_{(f)})_{\varphi(\mathfrak{p})}) & & \coprod_{\mathfrak{p}\in\varphi^{-1}(U)} S_{\mathfrak{p}} \arrow[ll, "\amalg_{\mathfrak{p}}\eta_{\mathfrak{p}}", "\cong"'] \end{tikzcd},$$

where $s \in \mathcal{O}_{\mathrm{Spec}\left(S_{(f)}\right)}(U)$ and $t \in \mathcal{O}_{D_+(f)}(\varphi^{-1}(U))$ its image under $\varphi^\flat$ (which is defined by the above square). One can indeed check that $\varphi^\flat$ as defined is natural w.r.t restrictions. □

Remark 1.8.2.5. Thus, for a graded ring S, we obtain a scheme $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$, which is called the projective scheme associated to a graded ring S.

We now give some more properties of $\mathrm{Proj}(S)$.

Proposition 1.8.2.6. [32] *Let S be a graded ring. Then,*

1. $\mathrm{Proj}(S) = \emptyset$ *if and only if* $\forall s \in S_+$, *s is a nilpotent element of S.*
2. *Let $\varphi : S \to T$ be a graded map of graded rings. Then $U = \{\mathfrak{q} \in \mathrm{Proj}(T) \mid \mathfrak{q} \not\supseteq \varphi(S_+)\}$ is an open set and the natural map*
$$f : U \longrightarrow \mathrm{Proj}(S)$$
$$\mathfrak{q} \longmapsto \varphi^{-1}(\mathfrak{q})$$
defines a map of schemes.
3. *Let $\varphi : S \to T$ be a graded map of graded rings for which there exists $d_0 \in \mathbb{N}$ such that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$. Then, $U = \mathrm{Proj}(T)$ and $f : \mathrm{Proj}(T) \to \mathrm{Proj}(S)$ is an isomorphism.*

Proof. 1. The R $\implies$ L is immediate. Otherwise take an element $s \in S_d$. By Lemmas 1.8.1.3, 3 and 1.8.2.4, we obtain that $\mathrm{Spec}\left(S_{(s)}\right) = \emptyset$. Consequently, any prime ideal of $\mathrm{Spec}\,(S_s)$ has no zero degree terms, which can be seen to be not true. Consequently, $D(s) = \mathrm{Spec}\,(S_s) = \emptyset$. It follows from Lemma 1.2.2.7 that s is nilpotent.

2. The fact that U is open depends on φ being graded, i.e. $\varphi(S_d) \subseteq T_d$ for all $d \geq 0$. The continuity of f follows from the same observation. The map on sheaves is given by extending the natural map on stalks $\varphi_{(\mathfrak{q})} : S_{(\varphi^{-1}(\mathfrak{q}))} \to T_{(\mathfrak{q})}$, whose well-definedness, again, uses the fact that φ is graded.

3. The main trick here is to observe that if $s \in S_d$ for $d < d_0$, then raising some high enough power of s will make $s^n \in S_e$ where $\deg s^n \geq d_0$. For showing isomorphism on stalks $\varphi_{(\mathfrak{q})} : S_{(\varphi^{-1}(\mathfrak{q}))} \to T_{(\mathfrak{q})}$, it comes down to observing the following: let $s/t \in S_{(\varphi^{-1}(\mathfrak{q}))}$, then $s/t = st^n/t^{n+1}$ for any $n \in \mathbb{N}$. Then use the trick above. $\square$

Remark 1.8.2.7. The above Proposition 1.8.2.6 shows that the mapping $S \mapsto \mathrm{Proj}(S)$ is NOT functorial! However the statement 3. might give some hint how to fix this.

Next, we understand all closed subschemes of $\mathrm{Proj}(S)$ in the following two results (Corollary 1.8.2.10)

Proposition 1.8.2.8. *Let S, T be a graded rings.*

1. *If $\varphi : S \to T$ is a surjective graded map, then the open set $U = \mathrm{Proj}(T)$ and $f : \mathrm{Proj}(T) \to \mathrm{Proj}(S)$ is a closed immersion (see Proposition 1.8.2.6, 2).*
2. *Let $I \leq S$ be a homogeneous ideal and consider the ideal $I' = \bigoplus_{d \geq d_0} I_d$. Then, I, I' defines the same closed subscheme of* $\mathrm{Proj}(S)$.
3. *Let $I \leq S$ be a homogeneous ideal and let $\pi : S \to S/I$ be the natural projection. Then the closed subscheme $f : \mathrm{Proj}(S/I) \to \mathrm{Proj}(S)$ (as in 1.) has the ideal sheaf given by $\widetilde{I} \leq \mathcal{O}_{\mathrm{Proj}(S)}$.*

[32] Exercise II.2.14 of Hartshorne.

Proof. 1. $U = \mathrm{Proj}(T)$ because $\varphi(S_+) = T_+$. The fact that f is a topological immersion follows from the observations that $f(\mathrm{Proj}(T)) = V(\mathrm{Ker}\,(\varphi))$ where $\mathrm{Ker}\,(\varphi)$ is homogeneous and that for any ideal $\mathfrak{q} \leq T$, it follows from surjectivity that $\varphi(\varphi^{-1}(\mathfrak{q})) = \mathfrak{q}$. To show surjectivity of sheaves, it reduces to showing surjectivity of localization maps $S_{\varphi^{-1}(\mathfrak{q})} \xrightarrow{\varphi_{(\mathfrak{q})}} T_{(\mathfrak{q})}$, which is immediate from surjectivity of φ.
2. We wish to show an isomorphism as in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Proj}(S/I) \arrow[rr, dash, "\cong"] \arrow[dr] & & \mathrm{Proj}(S/I') \arrow[dl] \\
& \mathrm{Proj}(S) &
\end{tikzcd}.$$

Now since $(S/I)_d = S_d/I_d$ for all $d \geq 0$, therefore we have an isomorphism $\varphi : (S/I)_d \to (S/I')_d$ given by $s_d + I_d \mapsto s_d + I'_d$. The result follows from Proposition 1.8.2.6, 3.
3. We wish to show that $\mathrm{Ker}\left(f^\flat : \mathcal{O}_{\mathrm{Proj}(S)} \to f_*\mathcal{O}_{\mathrm{Proj}(S/I)}\right)$ is given by $\widetilde{I}$. It suffices to check this on basic open sets $D_+(g)$, $g \in S_d, d > 0$, by uniqueness of the sheaf defined on a basis. Indeed it follows that $f^\flat$ on $D_+(g)$ is given by the localisation map $S_{(g)} \to S_{(g)}/I_{(g)}$, whose kernel is $I_{(g)} = \widetilde{I}(D_+(g))$. □

Proposition 1.8.2.9. *Let $S = A[x_0, \dots, x_r]$ for a ring A and let $X = \mathrm{Proj}(S)$.*

1. *Let $I \leq S$ be a homogeneous ideal and denote $\bar{I} = \{s \in S \mid \forall i = 0, \dots, r,\ \exists n_i \text{ s.t. } x_i^{n_i} s \in I\}$ to be the saturation of I. Then, $\bar{I}$ is homogeneous.*
2. *Let $I, J \leq S$ be two homogeneous ideals. Then $\mathrm{Proj}(S/I) \cong \mathrm{Proj}(S/J)$ if and only if $\bar{I} = \bar{J}$.*
3. *Let $Y \hookrightarrow \mathrm{Proj}(S)$ be a closed subscheme. Then, $\Gamma_*(\mathcal{I}_Y)$ is a saturated ideal of S.*

Proof. 1. This follows from a simple consideration of the uniqueness of homogeneous decomposition of each element in a graded ring.
2. We may reduce to showing that I and $\bar{I}$ defines the same closed subscheme. We already have $I \hookrightarrow \bar{I}$ which translates to $V(\bar{I}) \hookrightarrow V(I)$. Conversely, pick $\mathfrak{p} \in V(I) \subseteq \mathrm{Proj}(S)$. We wish to show $\mathfrak{p} \supseteq \bar{I}$. Pick any $s \in \bar{I}$. Assume that $s \notin \mathfrak{p}$. It then follows that $\mathfrak{p} = \langle x_0, \dots, x_r \rangle$ which is a prime ideal which contains S_+, thus $\mathfrak{p} \notin \mathrm{Proj}(S)$, a contradiction.
We then wish to show isomorphism of sheaves. Going to basic opens, this reduces to showing surjection is an injection:

$$\begin{aligned} (S/I)_{(f)} &\longrightarrow (S/\bar{I})_{(f)} \\ \frac{s+I}{f^n} &\longmapsto \frac{s+\bar{I}}{f^n}. \end{aligned}$$

This follows from the fact that $\bar{I}$ is saturated[33].
3. Pick a homogeneous element $s \in S_d$ such that for each $i = 0, \dots, r$, there exists $n_i \in \mathbb{N}$ such that $x_i^{n_i} s \in \Gamma(\mathcal{I}_Y(d + n_i), X)$. We wish to show that $s \in \Gamma_*(\mathcal{I}_Y)$. Note that $s \in \Gamma(\mathcal{O}_X(d), X)$. Cover X by $D_+(x_i)$ and consider the restrictions $x_i^{n_i} s \in \mathcal{I}_Y(d+n_i)(D_+(x_i))$. Multiplying (tensoring) $x_i^{n_i} s$ with $x_i^{-n_i} \in \mathcal{O}_X(-n_i)(D_+(x_i))$ yields $s \in \mathcal{I}_Y(d + n_i) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-n_i) \cong \mathcal{I}_Y(d)$ over $D_+(x_i)$. Thus, gluing these sections up from each $D_+(x_i)$, we get $s \in \Gamma(\mathcal{I}_Y(d), X) \subseteq \Gamma_*(\mathcal{I}_Y)$, as required. □

[33] In-fact, this step shows exactly why the definition of saturation would've been made!

Using the above result, it is possible to find a characterization of closed subschemes of $\mathrm{Proj}(S)$ in terms of algebraic data.

Corollary 1.8.2.10. *Let $S = A[x_0, \dots, x_r]$ be a graded ring for a ring A. Then there is a correspondence:*

$$\{\textit{All closed subschemes of } \mathrm{Proj}(S)\} \quad \cong \quad \{\textit{All saturated ideals of } S\}\ .$$

Proof. Follows from Proposition 1.8.2.9. $\square$

Next, let us show how projective n-spaces over a ring changes with extension of scalars.

Definition 1.8.2.11. (**Projective n-space over a ring**) Let A be a ring. The projective n-space over A is defined to be $\mathbb{P}^n_A := \mathrm{Proj}(A[x_0, \dots, n])$. By Lemma 1.8.2.2, $\mathbb{P}^n_A$ is a scheme over $\mathrm{Spec}\,(A)$.

We now see how $\mathbb{P}^n_A$ behaves under extension of scalars.

Lemma 1.8.2.12. *Let $A \to B$ be a map of rings and $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ be the corresponding map of affine schemes. Then,*

$$\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(B).$$

Proof. Observe that $D_+(x_i) \subseteq \mathbb{P}^n_A$ for $i = 0, \dots, n$ covers $\mathbb{P}^n_A$ as $A[x_0, \dots, x_n]$ is finitely generated by x_i as an A-algebra. By Lemma 1.6.4.7 together with Lemma 1.8.2.4 we obtain the following:

$$\begin{aligned}
\mathbb{P}^n_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(B) &= \left(\bigcup_{i=0}^{n} D_+(x_i)\right) \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(B) \\
&\cong \bigcup_{i=0}^{n} D_+(x_i) \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(B) \\
&\cong \bigcup_{i=0}^{n} \mathrm{Spec}\left(A[x_0, \dots, x_n]_{(x_i)} \otimes_A B\right) \\
&\cong \bigcup_{i=0}^{n} \mathrm{Spec}\left(A[x_0/_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i] \otimes_A B\right) \\
&\cong \bigcup_{i=0}^{n} \mathrm{Spec}\left(B[x_0, \dots, x_n]_{(x_i)}\right) \\
&\cong \mathbb{P}^n_B.
\end{aligned}$$

$\square$

Remark 1.8.2.13. Since any ring A is a $\mathbb{Z}$-algebra and $\mathbb{P}^n_A$ is naturally a $\mathbb{Z}$-scheme, therefore $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}\,(A)$, where the projection map $\mathbb{P}^n_A \to \mathrm{Spec}\,(A)$ is the usual structure map. This further motivates the construction of a projective space over any scheme.

Definition 1.8.2.14. (**Projective n-space over a scheme**) Let X be a scheme. The projective n-space over X is defined to be

$$\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} X.$$

The natural projection map thus makes $\mathbb{P}^n_X$ a scheme over X.

1.8.3 Blowups

Do from Chapter 3 of Mumford and Hartshorne.

1.9 $\mathcal{O}_X$-modules

We will now cover certain types of important $\mathcal{O}_X$-modules that we will need in our study. Note that we defined $\mathcal{O}_X$-modules and various other algebraic constructions on them in Chapter 8, thus we assume the basic notion of $\mathcal{O}_X$-modules and its global algebra being known and we will thus specialize to the case of X being a scheme. The main goal is to define and study coherent and quasi-coherent modules over a scheme X. Its importance will manifest later in our study of projective schemes and their cohomology, the latter of which is an extremely powerful and versatile tool for doing geometry over schemes.

1.9.1 Coherent and quasi-coherent modules on schemes

Quasi-coherent sheaves form an integral part of the backbone of an attempt at doing geometry on schemes. Even though the definitions here makes sense in the setting of locally ringed spaces, but this theory is much more better behaved in the setting of schemes; for schemes, such sheaves have nice description on affine opens. This is the reason it is not included in Foundational Geometry, Chapter 8.

We first define the notion of quasicoherent modules on schemes.

Definition 1.9.1.1. (**Quasicoherent and coherent $\mathcal{O}_X$-modules**) Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is called quasicoherent if there exists an affine open cover $\{U_i := \operatorname{Spec}(R_i)\}_{i\in I}$ of X and $\{M_i\}_{i\in I}$ where M_i is an R_i-module such that $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$ for all $i \in I$. Further, $\mathcal{F}$ is said to be a coherent module if each M_i is a finitely generated R_i-module for each $i \in I$.

Remark 1.9.1.2. There are five basic properties of quasi-coherent sheaves on a scheme, which we point out now.

1. Quasicoherence of a module can be checked locally.
2. The global sections functor of a quasicoherent module over an affine scheme is exact[34].
3. The image of the functor $\widetilde{(-)} : \mathbf{Mod}(R) \to \mathbf{Mod}(\mathcal{O}_{\operatorname{Spec}(R)})$ (see Definition 1.2.3.1 and remarks surrounding it) is precisely all quasicoherent modules over $\operatorname{Spec}(R)$.
4. Quasicoherence is preserved under inverse image. It is further preserved under direct image if domain is a Noetherian scheme or if the map is quasi-compact and separated.
5. The category of all quasicoherent modules

$$\mathbf{QCoh}(\mathcal{O}_X)$$

is a Grothendieck-abelian category.

We will come to these results one by one. We first discuss some basic properties and examples.

[34]This in cohomological language means that the first cohomology group $H^1(X, \mathcal{F}) = 0$, as we shall see after few sections.

Examples of quasicoherent modules

Lemma 1.9.1.3. *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme and $\mathfrak{a} \leq A$ be an ideal. Consider the corresponding closed immersion*

$$i : \mathrm{Spec}\,(A/\mathfrak{a}) = Y \hookrightarrow \mathrm{Spec}\,(A) = X.$$

Then,

1. *$i_* \mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module,*
2. *$i_* \mathcal{O}_Y \cong \widetilde{A/\mathfrak{a}}$.*

Proof. 1. Consider the following

$$\varphi : \mathcal{O}_X \times i_* \mathcal{O}_Y \longrightarrow i_* \mathcal{O}_Y$$

which on a basic open $D(f) \subseteq X$ for $f \in A$ is given by

$$\varphi_{D(f)} : A_f \times (A/\mathfrak{a})_{\bar{f}} \to (A/\mathfrak{a})_{\bar{f}}$$

as the usual A_f-module structure over $(A/\mathfrak{a})_{\bar{f}}$. Indeed, as the above maps are natural w.r.t. restrictions, this suffices by StacksProject Lemma 009O. Thus, $i_* \mathcal{O}_Y$ is an $\mathcal{O}_X$-module. Further, $i_* \mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module as the open cover $\{D(f)\}_{f \in A}$ of X is such that $i_* \mathcal{O}_Y(D(f)) \cong (A/\mathfrak{a})_{\bar{f}}$ is an $\mathcal{O}_X(D(f)) \cong A_f$-module generated by $\bar{1}$ (in the case when $f \in A$, we have $i_* \mathcal{O}_Y(D(f)) = 0$ and so trivially a finitely generated A_f-module).

2. Again, by the use of above mentioned lemma, we may reduce to working over a basis of X. Choosing $\{D(f)\}_{f \in A}$ to be such a basis, we see that $i_* \mathcal{O}_Y(D(f)) \cong (A/\mathfrak{a})_f$ and $\widetilde{A/\mathfrak{a}}(D(f)) \cong (A/\mathfrak{a})_f$. Hence, we may define a map $i_* \mathcal{O}_Y \to \widetilde{A/\mathfrak{a}}$ which on basic opens is identity. Consequently, this map on stalks is identity. This by above used lemma again yields a unique sheaf morphism $\varphi : i_* \mathcal{O}_Y \to \widetilde{A/\mathfrak{a}}$ which is an isomorphism as at stalks it is an isomorphism. ∎

Example 1.9.1.4. Let X be an integral noetherian scheme and let K be its function field. Let $\mathcal{K}$ be the constant sheaf of field K over X. Then $\mathcal{K}$ is a quasi-coherent $\mathcal{O}_X$-module.

As X is noetherian, therefore let $X = \bigcup_{i=1}^n \mathrm{Spec}\,(A_i)$ where A_i are noetherian rings. As X is integral, therefore by Lemma 1.4.2.2, each A_i is a noetherian domain. Thus we deduce that $K \cong Q(A_i)$ for each i, where $Q(A_i)$ is the fraction field of A_i. This is because $\mathrm{Spec}\,(A_i)$ are open and X irreducible. We now show that $\mathcal{K}$ is an $\mathcal{O}_X$-module.

Pick any open $U \subseteq X$. Recall from Chapter 27 that a section of $\mathcal{K}(U)$ is a continuous map $U \to K$ with K in discrete topology. For any point $p \in \mathrm{Spec}\,(A_i)$, as A_i is a domain, we see that $(A_i)_p \hookrightarrow Q(A_i) \cong K$ for each i. We thus deduce that K is an $\mathcal{O}_{X,x}$-algebra for each $x \in X$ in a natural way and $\mathcal{O}_{X,x} \subseteq K$. So we may now define

$$\begin{aligned} \mathcal{O}_X(U) \times \mathcal{K}(U) &\to \mathcal{K}(U) \\ (c, s) &\mapsto c \cdot s \end{aligned}$$

where $c \cdot s : U \to K$ is defined by $c(x)s(x) \in K$, $c(x) \in \mathcal{O}_{X,x} \subseteq K$. This is continuous as each $c \in \mathcal{O}_X(U)$ is seen to be a continuous map $c : U \to \coprod_{x \in U} \mathcal{O}_{X,x} \subseteq K$ as it is locally constant (Remark 27.2.0.4 and that locally around each point we have an affine open inside every open). This is automatically compatible with restrictions. Consequently, $\mathcal{K}$ is an $\mathcal{O}_X$-module.

Next, to see this is quasi-coherent, we claim that the affine open cover $\{U_i = \mathrm{Spec}\,(A_i)\}_{i=1,\dots,n}$ is such that $\mathcal{K}_{|U_i}$ is isomorphic to $\widetilde{K}$. Consequently, we reduce to proving the following claim : Let $X = \mathrm{Spec}\,(A)$ be an affine scheme where A is a noetherian domain and let $K = Q(A)$ be its fraction field. Then, the constant sheaf $\mathcal{K}$ associated to K is isomorphic to the $\mathcal{O}_X$-module $\widetilde{K}$.

It suffices to construct a map $\varphi : \mathcal{K} \to \widetilde{K}$ defined only on a basis such that on basics it is an isomorphism. For this, we notice that since localization of a domain at an element is again a domain, therefore for each $g \in R$, the open $D(g) \subseteq X$ is connected. Hence, $\mathcal{K}(D(g)) = K$ and $\widetilde{K}(D(g)) \cong K_g = K$. Thus, we may define $\varphi_{D(g)} : K \to K$ to be identity which is then easily seen to be a sheaf morphism. Hence, these sheaves are isomorphic as $\mathcal{O}_X$-modules.

Example 1.9.1.5. We now discuss a specific example of quasi-coherent modules over $\mathrm{Spec}\,(\mathbb{Z})$, which brings to light the constraints put on by quasi-coherence on an $\mathcal{O}_X$-module. We ask the following question : *What are all quasicoherent skyscraper $\mathcal{O}_{\mathbb{Z}}$-modules over* $\mathrm{Spec}\,(\mathbb{Z})$ *supported at non-zero prime $p \in \mathbb{Z}$?* We claim that these are in bijection with all p^∞-torsion $\mathbb{Z}$-modules, that is, every element of the module is annihilated by some power of p:

$$\left\{\begin{matrix}\text{Quasicoherent} \quad \mathcal{O}_{\mathbb{Z}}-\text{modules} \quad \mathcal{F} \\ \text{skyscraper at } p \in \mathbb{Z}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Abelian groups } M \text{ which are } p^\infty\text{-} \\ \text{torsion.}\end{matrix}\right\}$$

Indeed, let $\mathcal{F}$ be a quasicoherent skyscraper module at prime $p \in \mathbb{Z}$. Let us invoke the Corollary 1.9.1.12, to conclude that $\mathcal{F} = \widetilde{M}$ for some $\mathbb{Z}$-module M. As it is skyscraper, therefore for any open $U \ni p$ in $\mathrm{Spec}\,(\mathbb{Z})$, we have $\mathcal{F}(U) = G$ where G is a fixed abelian group and $\mathcal{F}(U) = 0$ if $p \notin U$. Consequently, we have that $\mathcal{F}_x = 0$ if $x \neq p$ and $\mathcal{F}_p = G$. As $\mathcal{F} = \widetilde{M}$, therefore we have

$$\Gamma(\mathcal{F}, X) = G \cong M.$$

Further, for any basic open $D(f) \subseteq \mathrm{Spec}\,(\mathbb{Z})$ containing prime p, we deduce that $\mathcal{F}(D(f)) \cong M_f \cong G \cong M$. This, when unravelled, yields that for any integer $f \in \mathbb{Z}$ such that $f \notin \langle p \rangle \iff p \not| f$, we have $M_f \cong M$. Further, if $\langle p \rangle \notin D(f) \iff p|f$, then $M_f = 0$. Now fix any $m \in M$. We claim that some power of p annihilates m. Indeed, consider $D(p)$ which does not contain $\langle p \rangle$ as $p \in \langle p \rangle$. Thus, by above, we have that $\frac{m}{1} = 0$ in M_p. Consequently, for some $k \in \mathbb{N}$, we have $p^k m = 0$, as required. Hence, $T_{p^\infty}(M) = M$.

Conversely, consider a p^∞-torsion abelian group M. We wish to show that the quasi-coherent module associated to M, $\widetilde{M}$, is skyscraper at $p \in \mathbb{Z}$. Let $D(f) \subseteq \mathrm{Spec}\,(\mathbb{Z})$ be a basic open not containing $\langle p \rangle$, equivalently, $p|f$. Then, we see that $\widetilde{M}(D(f)) \cong M_f$. Now pick any $\frac{m}{f^k} \in M_f$. Let $p^n m = 0$. Thus, $f^n m = 0$ as $p|f$. Consequently, we may write $\frac{m}{f^k} = \frac{1}{f^k}\frac{m}{1} = \frac{1}{f^k}\frac{f^n m}{f^n} = \frac{1}{f^k}\frac{0}{f^n} = 0$. Thus, $M_f = 0$.

Let $D(f)$ now be a basic open set which contains $\langle p \rangle$, equivalently, $p \not| f$. Then, $\widetilde{M}(D(f)) \cong M_f$ and we wish to show that $M_f \cong M$. Indeed, observe that since $p \not| f$, therefore $\gcd(p^k, f^l) = 1$ for all $k, l \geq 1$. It follows that there exists $a_k, b_l \in \mathbb{Z}$ such that

$$a_k p^k + b_l f^l = 1.$$

Thus, for any $\frac{m}{f^n} \in M_f$, where $p^k m = 0$, we obtain $a_k, b_n \in \mathbb{Z}$ such that $a_k p^k + b_n f^n = 1$. Using this on module M, we yield $a_k p^k m + b_n f^n m = m$, that is, $b_n f^n m = m$. Consequently, we may write

$$\frac{m}{f^n} = \frac{b_n f^n m}{f^n} = \frac{b_n m}{1}$$

in M_f. It follows that the localization map $\varphi : M \to M_f$ is surjective. We thus need only establish the injectivity of φ. Indeed, if $\varphi(m) = \frac{m}{1} = 0$ in M_f, then $f^n m = 0$ for some $n \in \mathbb{N}$. By above, we have $b_n \in \mathbb{Z}$ such that $b_n f^n m = m$. Consequently, $m = b_n f^n m = 0$, that is, $\mathrm{Ker}\,(\varphi) = 0$, as required. Thus, $\varphi : M \to M_f$ is the required isomorphism. This completes the proof.

Locality of quasicoherence

We now discuss some more results which would culminate in the proofs of statement 1 in Remark 1.9.1.2.

Lemma 1.9.1.6. *Let $\mathcal{F}$ be a quasicoherent module over an affine scheme $X = \mathrm{Spec}\,(R)$. Then X admits a finite open affine cover $\{D(g_i)\}_{i=1}^n$ such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$ where M_i is an R_{g_i}-module.*

Proof. Since $\mathcal{F}$ is quasicoherent, therefore there exists an open affine cover $\{U_i = \mathrm{Spec}\,(S_i)\}_i$ of X such that $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$ where M_i is an S_i-module. Since subsets of the form $D(g)$ forms a basis of X therefore for $D(g) \subseteq U_i$ we obtain via Lemma 1.2.3.4, 2, that $\mathcal{F}_{|D(g)} \cong \widetilde{R_g \otimes_{S_i} M_i}$ as $D(g) \cong \mathrm{Spec}\,(R_g)$. Since $N_i := R_g \otimes_{S_i} M_i$ is an R_g-module, so we have a cover of X by finitely many $D(g_i)$ by Lemma 1.2.1.6 such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{N_i}$ where N_i is an R_{g_i}-module. $\square$

Using the above, we first show a technical lemma, which will be generalized later on, which will be used to show locality of quasi-coherent modules[35].

Lemma 1.9.1.7. *Let $X = \mathrm{Spec}\,(A)$ be an affine scheme and $\mathcal{F} \in \mathbf{QCoh}(X)$ be a quasi-coherent module. Let $D(f) \subseteq X$ be a basic open set for some $f \in A$.*

1. *If $s \in \Gamma(\mathcal{F}, X)$ is a global section of the module $\mathcal{F}$ such that s restricted on $D(f)$ is 0, then there exists $n > 0$ such that $f^n s = 0$ over X.*
2. *If $t \in \mathcal{F}(D(f))$, then there exists $n > 0$ such that $f^n t \in \mathcal{F}(D(f))$ extends to a global section of the module $\mathcal{F}$.*

Proof. 1. By Lemma 1.9.1.6, there exists a finite open cover $D(g_i)$ of X such that $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$. Denoting the restriction of s to $D(g_i)$ as $s_i \in M_i$, we see that the image of s_i is zero in $(M_i)_f$ when restricted to $D(fg_i) = D(f) \cap D(g_i)$. Consequently, for some $n_i > 0$, we have $f^{n_i} s_i = 0$ over $D(g_i)$. As g_i are finitely many, taking large enough n, we obtain $f^n s_i = 0$ over

[35] The result is similar in flavour to Proposition 1.3.1.5.

each $D(g_i)$. It follows that the global section $f^n s$ of the module $\mathcal{F}$ is such that it's restriction to each open set of an open cover of X is 0. By sheaf axioms, it follows that $f^n s = 0$ over X.

2. Fix the finite open affine cover $\{D(g_i)\}_{i=1}^n$ of X coming from Lemma 1.9.1.6. Consider all the finitely many intersections $D(g) \cap D(g_i) = D(fg_i)$. Restricting t from $D(f)$ to $D(fg_i)$, we obtain $t_i \in (M_i)_f$ for each i. Hence, for each i, there is some $n_i > 0$ such that $f^{n_i} t_i \in M_i = \mathcal{F}(D(g_i))$. By multiplying by large f^k to each $f^{n_i} t_i$ which are finitely many, we may arrange that $f^n t_i \in \mathcal{F}(D(g_i))$.

We now form a matching family for the module $\mathcal{F}$ over the open cover $\{D(g_i)\}$ which would glue up to give the required global section. Indeed, fix two $D(g_i)$ and $D(g_j)$. Restrict $f^n t_i$ and $f^n t_j$ to $D(g_i) \cap D(g_j) = D(g_i g_j)$. Observe that over the even smaller open $D(fg_ig_j)$, the section $f^n t_i - f^n t_j$ is zero as $t_i = t_j = t$ over $D(fg_ig_j) \subseteq D(f)$. Hence by item 1 applied over $D(g_ig_j)$, there exists $m_{ij} > 0$ such that $f^{m_{ij}}(f^n t_i - f^n t_j) = 0$, hence $f^{n+m_{ij}}(t_i - t_j) = 0$ over $D(fg_ig_j)$. As i and j are finitely many, so taking m large enough, we obtain $f^{n+m} t_i = f^{n+m} t_j$ over $\mathcal{F}(D(g_ig_j))$ for each i and j. Thus, the family $\{f^{n+m} t_i\}$ is a matching family which glues up to give $s \in \Gamma(\mathcal{F}, X)$ such that its restriction over $D(f)$ is $f^{n+m} t$[36]. □

Remark 1.9.1.8. Let $X = \operatorname{Spec}(R)$ be an affine scheme and $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module over X. Then, we obtain a map

$$\alpha : \widetilde{\Gamma(\mathcal{F}, X)} \longrightarrow \mathcal{F}$$

which on a basic open set $D(f) \subseteq X,\ f \in R$ is given by $\Gamma(\mathcal{F}, X)_f \to \mathcal{F}(D(f))$ mapping as $m/f^n \mapsto \rho_{X,D(f)}(m)/f^n$. Indeed, this is a $\mathcal{O}_X$-linear homomorphism which on stalks yields the $R_\mathfrak{p}$-linear map

$$\Gamma(\mathcal{F}, X)_\mathfrak{p} \longrightarrow \mathcal{F}_\mathfrak{p}$$

which is given by

$$\Gamma(\mathcal{F}, X) \otimes_R R_\mathfrak{p} \cong \Gamma(\mathcal{F}, X) \otimes_R \varinjlim_{f \notin \mathfrak{p}} R_f \cong \varinjlim_{f \notin \mathfrak{p}} \Gamma(\mathcal{F}, X) \otimes_R R_f = \varinjlim_{D(f) \ni \mathfrak{p}} \widetilde{\Gamma(\mathcal{F}, X)}(D(f)) \to \varinjlim_{D(f) \ni \mathfrak{p}} \mathcal{F}(D(f)).$$

We will see that this map α would become an isomorphism, especially due to the fact that quasicoherent modules behave very nicely on open affines of the form $D(f)$, as the Lemma 1.9.1.6 shows.

Corollary 1.9.1.9. *Let $\mathcal{F}$ be a quasicoherent sheaf over an affine scheme $X = \operatorname{Spec}(A)$. Then, there is a natural isomorphism of A_f-modules for each $f \in A$*

$$\Gamma(\mathcal{F}, X)_f \xrightarrow{\cong} \mathcal{F}(D(f))$$

given by $m/f^n \mapsto \rho(m)/f^n$ where ρ is the restriction map of $\mathcal{F}$ from X to $D(f)$

Proof. Follows from Lemma 1.9.1.7. □

[36]Note that we have implicitly used the fact the restriction maps of $\mathcal{F}$ preserves the respective module structures (see remarks surrounding Definition 8.5.0.1)

Using the above, one proves the local nature of quasicoherence.

Proposition 1.9.1.10. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module over a scheme X. Then, $\mathcal{F}$ is quasicoherent if and only if for all open affine $U = \operatorname{Spec}(A) \subseteq X$ we have $\mathcal{F}_{|U} \cong \widetilde{M}$ where M is an A-module.*

Proof. We need to only show $R \Rightarrow L$. Let $\mathcal{F}$ be quasicoherent and $U = \operatorname{Spec}(A)$ open affine. We may assume $X = \operatorname{Spec}(A)$. Thus we need to show $\mathcal{F} \cong \widetilde{M}$ for an A-module M. By Lemma 1.9.1.6, we obtain an open affine cover $D(g_i)$ of X where $\mathcal{F}_{|D(g_i)} \cong \widetilde{M_i}$ for an A_{g_i}-module M_i. Let $M = \Gamma(\mathcal{F}, X)$, which is an A-module. By Corollary 1.9.1.9, we obtain a natural isomorphism $M_i \cong M_{g_i}$. Thus we have the required result. □

A similar result is true for coherent modules.

Proposition 1.9.1.11. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module over a Noetherian scheme X. Then, $\mathcal{F}$ is coherent if and only if for all open affine $U = \operatorname{Spec}(A) \subseteq X$ we have $\mathcal{F}_{|U} \cong \widetilde{M}$ where M is a finitely generated A-module.*

Proof. See Proposition 5.4, Chapter 2 [Hartshorne]. □

Corollary 1.9.1.12. *The image of the functor $\widetilde{(-)}$ of Definition 1.2.3.1 is exactly all quasi-coherent modules over $\operatorname{Spec}(R)$. In other words, $\mathbf{Mod}(R) \equiv \widetilde{\mathbf{Mod}}(\mathcal{O}_{\operatorname{Spec}(R)}) = \mathbf{QCoh}(\mathcal{O}_{\operatorname{Spec}(R)})$. Further, if R is noetherian, then this restricts to $\mathbf{Mod}(R)^{f.g.} \equiv \mathbf{Coh}(\mathcal{O}_{\operatorname{Spec}(R)})$.* □

Quasicoherence and exactness of global sections

We next see the exactness of global sections functor.

Proposition 1.9.1.13. *Let X be an affine scheme and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ is quasicoherent, then*

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$

is exact.

Proof. Proposition 5.6, Chapter 2 [Hartshorne]. □

The category $\mathbf{Qcoh}(X)$

The category of quasicoherent modules is further a Grothendieck-abelian category.

Theorem 1.9.1.14. *Let X be a scheme. The category $\mathbf{QCoh}(\mathcal{O}_X)$ is a Grothendieck-abelian category. Consequently, it is an abelian category which has all coproducts.*

Proof. Tag 077P, [Stacksproject]. □

Quasicoherence, direct and inverse images

We now see behavior of quasicoherence and coherence under inverse and direct images.

Lemma 1.9.1.15. *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{G}$ be a quasicoherent $\mathcal{O}_Y$-module. Then $f^*\mathcal{G}$ is a quasicoherent $\mathcal{O}_X$-module. If X, Y are Noetherian schemes and $\mathcal{G}$ is coherent, then $f^*\mathcal{G}$ is coherent.*

Proof. The first question is local in both X and Y by Proposition 1.9.1.10. Indeed, pick $x \in X$ and an open affine $V \ni f(x)$ in Y. Then by continuity there is an open affine $U \ni x$ in X such that $f(U) \subset V$. This shows that we may assume X and Y to be affine. The result is now immediate by Corollary 1.9.1.12 and Lemma 1.2.3.4, 2. The same technique works for coherent case. □

For stability under direct image, we need some conditions on the map if noetherian conditions need to be dropped.

Lemma 1.9.1.16. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Then $f_*\mathcal{F}$ is a quasicoherent $\mathcal{O}_Y$-module if any of the following holds:*

1. *X is noetherian,*
2. *f is quasi-compact and separated.*

Proof. **TODO.** □

More properties

As promised earlier, we state a general result about invertible modules and quasicoherent modules over schemes. This is a fundamental result and will be used to portray the simplicity of the techniques developed so far. Moreover, its proof showcases the simplicity of the sheaf language and is thus a good exercise.

Lemma 1.9.1.17. *Let X be a scheme, $\mathcal{L} \in \mathrm{Pic}(X)$, $\mathcal{F} \in \mathbf{QCoh}(X)$, $f \in \Gamma(\mathcal{L}, X)$ and $s \in \Gamma(X, \mathcal{F})$. Denote by $X_f \subseteq X$ the open subset $X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x\mathcal{L}_x\}$.*

1. *If X is quasicompact and s is such that $s|_{X_f} = 0$, then there exists $n \in \mathbb{N}$ such that $f^n s = 0$ in $\Gamma(\mathcal{L}^{\otimes n} \otimes \mathcal{F}, X)$.*
2. *If X admits a finite affine open cover $\{U_i\}$ where $\mathcal{L}_{|U_i}$ is free (of rank 1) and $U_i \cap U_j$ is quasicompact, then for any $t \in \mathcal{F}(X_f)$, there exists $n \in \mathbb{N}$ such that $f^n t \in (\mathcal{L}^{\otimes n} \otimes \mathcal{F})(X_f)$ extends to a global section $s \in \Gamma(\mathcal{L}^{\otimes n} \otimes \mathcal{F}, X)$.*

Proof. 1. Cover X by finitely many affine open sets $U = \mathrm{Spec}\,(A)$ which satisfies $\varphi : \mathcal{L}_{|U} \cong \mathcal{O}_{X|U} \cong \mathcal{O}_{\mathrm{Spec}(A)}$. Further, denote $\mathcal{F}_{|U} = \widetilde{M}$ where M is an A-module (Corollary 1.9.1.12). By restricting f to U, we may write $g = \varphi_U(f) \in A$ and by restricting s to U, we may write $s \in M$. Since $s|_{X_f} = 0$ and $X_f \cap U = D(g)$, therefore $s/1 = 0$ in M_g. Consequently, there exists $n \in \mathbb{N}$ such that $g^n s = 0$ in $M = \Gamma(U, \mathcal{F}_{|U})$. We then observe the following isomorphisms (see Lemma 27.2.0.5):

$$(\mathcal{L}^{\otimes n} \otimes \mathcal{F})_{|U} \cong \mathcal{O}_{X|U}^{\otimes n} \otimes \mathcal{F}_{|U} \cong \mathcal{O}_{\mathrm{Spec}(A)}^{\otimes n} \otimes \widetilde{M} \cong \widetilde{M}.$$

Consequently, we get isomorphisms in sections over U which yields that $f^{\otimes n} \otimes s \mapsto g^n s = 0$. hence $f^{\otimes n} \otimes s = 0$ in $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ over U. Since this happens for all finitely many Us, therefore taking large enough n, we observe that $f^{\otimes n} \otimes s = 0$ in $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ over X.

2. Pick $t \in \mathcal{F}(X_f)$. For each of the finitely many i, let $U_i = \operatorname{Spec}(A_i)$. As $\mathcal{L}_{|U_i} \cong \mathcal{O}_{X|U_i}$, therefore $X_f \cap U_i = \{\mathfrak{p} \in \operatorname{Spec}(A_i) \mid f_\mathfrak{p} \notin \mathfrak{p}A_\mathfrak{p}\} = D(f)$ where we interpret $f \in \mathcal{L}(U_i)$ by restricting the global section f. By locality of quasicoherence (Proposition 1.9.1.10), we have an A_i-module M_i such that $\mathcal{F}_{|U_i} \cong \widetilde{M_i}$. As $t \in \mathcal{F}(X_f)$, therefore by restriction, we have $t_i \in \mathcal{F}(U_i \cap X_f) = \mathcal{F}(D(f)) \cong M_f$ (Proposition 1.2.3.3). It follows that for some n_i, we have $f^{n_i} t \in M_i = \mathcal{F}(U_i)$. Since U_i are atmost finite, so we may take a large enough n so that $f^n t \in M_i = \mathcal{F}(U_i)$.

Observe that

$$(\mathcal{L}^{\otimes n} \otimes \mathcal{F})_{|U_i} \cong \mathcal{O}^{\otimes n}_{\operatorname{Spec}(A_i)} \otimes \mathcal{F}_{|U_i} \cong \mathcal{F}_{|U_i} \cong \widetilde{M_i}$$

where $f^n t \in \mathcal{F}(U_i)$ corresponds to $f^{\otimes n} \otimes t \in \mathcal{L}^{\otimes n} \otimes \mathcal{F}(U_i)$. As $t_i = t = t_j \in \mathcal{F}(U_i \cap U_j \cap X_f)$, therefore $f^n(t_i - t_j) = 0$ in $\mathcal{F}(U_i \cap U_j \cap X_f)$. Our hypothesis that $U_i \cap U_j$ is quasicompact ensures by item 1 that there exists $k > 0$ such that $f^{n+k}(t_i - t_j) = 0$ in $\mathcal{F}(U_i \cap U_j)$, for all i, j. It follows that $f^{n+k} t_i \in \mathcal{F}(U_i) = \mathcal{L}^{\otimes n+k} \otimes \mathcal{F}(U_i)$ is a matching family. It follows that there exists $s \in \Gamma(\mathcal{L}^{\otimes n+k} \otimes \mathcal{F}, X)$ which on X_f is $f^{n+k} t$, as required. $\square$

These were some of the basic results on quasicoherent modules. We now do perhaps the most important application of $\mathcal{O}_X$-modules, that when X is a projective scheme.

1.9.2 $\mathcal{O}_{\text{Proj}(S)}$-modules

It is extremely important to do all exercises from Chapter 2 of Hartshorne from 5.9 to 5.14, as they deal with geometry coming out of modules over projective schemes.

Let S be a graded ring and M a graded S-module. We attach a sheaf $\widetilde{M}$ to M over Proj$S$.

Definition 1.9.2.1. ($\widetilde{M}$) Let S be a graded ring and M be a graded S-module. Then we define a sheaf $\widetilde{M}$ over $\text{Proj}(S)$ given on an open set $U \subseteq \text{Proj}(S)$ by

$$\widetilde{M}(U):=\left\{s:U\to\coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})} \mid \forall \mathfrak{p}\in U,\ s(\mathfrak{p})\in M_{(\mathfrak{p})} \& \exists \text{ open } \mathfrak{p}\in V\subseteq U \ \&\ m\in M_d, f\in S_d \text{ s.t. } f\notin \mathfrak{q} \ \&\ s(\mathfrak{q})=m/f \forall \mathfrak{q}\in V\right\}.$$

The restrictions are the obvious ones. It is clear that if we treat S as a graded S-module, then $\widetilde{S} \cong \mathcal{O}_{\text{Proj}(S)}$ where we treat $\mathcal{O}_{\text{Proj}(S)}$ as an $\mathcal{O}_{\text{Proj}(S)}$-module.

Remark 1.9.2.2. Over a projective scheme $X = \text{Proj}(S)$, the theory of quasi-coherent modules is the most useful. In particular, we will have the following observations to make about them:

1. Any graded S-module gives an $\mathcal{O}_X$-module $\widetilde{M}$ which is furthermore quasicoherent.
2. Any $\mathcal{O}_X$-module $\mathcal{F}$ gives a graded S-module $\Gamma_*(\mathcal{F})$.
3. For X being the projective n-space over a ring A, we have $\Gamma_*(\mathcal{O}_X) \cong A[x_0, \dots, x_n]$.
4. Assume S is furthermore finitely generated by degree 1 elements. If $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module, then $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$.
5. All projective schemes over Spec(A) is of the form $\text{Proj}(S)$ where $S_0 = A$ and S is finitely generated as by S_1 as an S_0-algebra.

These are the main takeaways from the general theory of quasicoherent $\mathcal{O}_{\text{Proj}(S)}$-modules.

We now attend to these results one-by-one. We first have analogous results to the affine case on the behaviour of $\widetilde{M}$ on basis, on stalks and its quasicoherence.

Proposition 1.9.2.3. *Let S be a graded ring, M be a graded S-module, $X = \text{Proj}(S)$ be the projective scheme over S and $\widetilde{M}$ to be the associated sheaf of M over X. Then,*

1. *for any $\mathfrak{p} \in X$,*
$$(\widetilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})},$$
2. *for any $f \in S_d$, $d > 0$ and basic open $D_+(f)$,*
$$\widetilde{M}_{|D_+(f)} \cong \widetilde{M_{(f)}},$$
3. *the sheaf $\widetilde{M}$ is an $\mathcal{O}_X$-module which is furthermore quasicoherent,*
4. *if S is a noetherian ring and M is finitely generated, then $\widetilde{M}$ is coherent.*

Proof. 1. and 2. follows from repeating Lemma 1.8.2.4. Statement 3. follows from local property of quasicoherence (Proposition 1.9.1.10), the fact that sets of the form $D_+(f)$ for $f \in S_d$, $d > 0$ forms a basis of X (Lemma 1.8.1.3) and statement 2 above. Statement 4 follows from coherence being a local property for Noetherian schemes (Proposition 1.9.1.11) and statement 2 above. $\square$

Remark 1.9.2.4. The theory of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules is rich because of various constructions which interrelates the category $\mathbf{grMod}(S)$ of graded S-modules and graded maps and the category $\mathbf{Mod}(\mathcal{O}_{\mathrm{Proj}(S)})$. Indeed, these constructions is what we will study now, and these will be absolutely indispensable to do geometry in projective spaces $\mathrm{Proj}(k[x_0,\dots,x_n]/f)$ for a homogeneous polynomial f.

Remark 1.9.2.5. The construction of $\mathcal{O}_X$-modules is functorial ($X = \mathrm{Proj}(S)$):

$$\widetilde{(-)} : \mathbf{grMod}(S) \longrightarrow \mathbf{QCoh}(\mathcal{O}_X)$$
$$M \longmapsto \widetilde{M}$$
$$M \xrightarrow{\varphi} N \longmapsto \widetilde{M} \xrightarrow{\eta} \widetilde{N}$$

where η on a basic open $D_+(f)$ is given by the localization maps $\eta_{D_+(f)} : M_{(f)} \to N_{(f)}$.

We first begin by twisting each $\mathcal{O}_{\mathrm{Proj}(S)}$-module.

Twists and Serre twists

Definition 1.9.2.6. (**Twists**) Let S be a graded ring and $X = \mathrm{Proj}(S)$. For each $n \in \mathbb{Z}$, we define the n^{th}-*Serre twist* to be $\mathcal{O}_X(n)$ which is defined to be $\widetilde{S(n)}$, the sheaf associated to the n-th twisted graded S-module $S(n)$ (Definition 23.2.1.3). For each $\mathcal{O}_X$-module $\mathcal{F}$, we then define the n^{th}-*twist of* $\mathcal{F}$ to be $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Some obvious questions are: what happens to n^{th}-twist of $\widetilde{M}$ for a graded S-module M, what is so special about $\mathcal{O}_X(n)$ in relation to $\mathcal{O}_X$? We answer these in the following result.

Proposition 1.9.2.7. *Let S be a graded ring generated by S_1 as an S_0-algebra and $X = \mathrm{Proj}(S)$. Then,*

1. *$\mathcal{O}_X(n)$ is an invertible module for all $n \in \mathbb{Z}$,*
2. *for any graded S-modules M, N and $n \in \mathbb{Z}$,*
   (a) *$\widetilde{M \otimes_S N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$,*
   (b) *$\widetilde{M}(n) \cong \widetilde{M(n)}$.*
3. *Let T be a graded ring generated by T_1 as a T_0-algebra. Let M be a graded S-module and N a graded T-module. Let $\varphi : S \to T$ be a graded map and $f : U \to \mathrm{Proj}(S)$ be the corresponding map (Proposition 1.8.2.6). Then,*
   (a) *$f_*(\widetilde{N}_{|U}) \cong \widetilde{{}_S N}$,*
   (b) *$f^*(\widetilde{M}) \cong \widetilde{(M \otimes_S T)}_{|U}$,*
   (c) *$f_*(\mathcal{O}_{\mathrm{Proj}(T)|U}) \cong \widetilde{T}$, where T is treated to be an S-module via φ.*
4. *Let φ and f as in 3 and let $Y = \mathrm{Proj}(T)$. Then,*
   (a) *$f_*(\mathcal{O}_Y(n)_{|U}) \cong f_*(\mathcal{O}_{Y|U})(n)$,*
   (b) *$f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)_{|U}$.*
5. *For all $n, m \in \mathbb{Z}$,*

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m).$$

Proof. 1. Cover X by basic open sets of the form $D_+(f)$ for $f \in S_1$. One then easily reduces to showing that $S(n)_{(f)}$ is a free $S_{(f)}$-module of rank 1. Indeed, one shows that the following map is an $S_{(f)}$-linear map which is an isomorphism as $S_{(f)}$-modules: $S_{(f)} \to S(n)_{(f)}$ given by $s/f^k \mapsto sf^n/f^k$. One really needs f to be of degree 1 to be able to show that this is an isomorphism.
2. This reduces to finding natural isomorphism $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \cong (M \otimes_S N)_{(f)}$ which one can do constructing two sided inverses. One of these maps well-definedness will use the fact that degree of f is 1.
3. See Lemma 1.2.3.4 for a) and b) and observe that $f^{-1}(D_+(g)) = D_+(\varphi(g))$ for $g \in S$ homogeneous by a simple unravelling of definition of $U \subseteq \mathrm{Proj}(T)$. The statement c) is immediate by looking the respective sections on a basic open set $D_+(g)$.
4. Statement a) follows from 3.a) and 3.c) is immediate from 3.b).
5. Follows from 2.a). □

Remark 1.9.2.8. The twisting functor given by

$$\begin{aligned} \mathbf{Mod}(\mathcal{O}_X) &\longrightarrow \mathbf{Mod}(\mathcal{O}_X) \\ \mathcal{F} \xrightarrow{f} \mathcal{G} &\longmapsto \mathcal{F}(n) \xrightarrow{f\otimes \mathrm{id}} \mathcal{G}(n) \end{aligned}$$

is exact. This is immediate as $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ and thus localizing at a point x, we get $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_X(n)_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}$ where the latter isomorphism follows from Proposition 1.9.2.7, 1.
In general this tells us also that the stalks of all twisted sheaves $\mathcal{F}(n)$ is identical to that of $\mathcal{F}$.

Remark 1.9.2.9. Let S be a graded ring and $X = \mathrm{Proj}(S)$ be the corresponding projective scheme with $\mathcal{F} \in \mathbf{QCoh}(X)$. Our goal in the next few pages is to understand how we can recover $\mathcal{F}$ by the global sections of all the twisted sheaves $\mathcal{F}(n)$. This is recorded in Propositions 1.9.2.12 and 1.9.2.13.

Associated graded S-module

We now associate to each $\mathcal{O}_X$-module $\mathcal{F}$ a graded S-module, where $X = \mathrm{Proj}(S)$.

Definition 1.9.2.10. (**Associated graded S-module**) Let S be a graded ring, $X = \mathrm{Proj}(S)$ and $\mathcal{F}$ an $\mathcal{O}_X$-module. Define the associated graded S-module to be

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathcal{F}(n), X)$$

where the S-module structure is given as follows: we need only define the scalar multiplication for homogeneous elements, so let $s_d \in S_d$ and $t_n \in \Gamma(\mathcal{F}(n), X)$. Then define $s_d \cdot t_n$ to be the image of $s_d \otimes t_n \in \Gamma(\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(n), X)$ under the isomorphism $\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(n) \cong \mathcal{F}(n + d)$ via Proposition 1.9.2.7, 5, in order to obtain an element of $\Gamma(\mathcal{F}(n + d), X)$, as needed.

Remark 1.9.2.11. There are two main results about associated graded S-modules.

1. Let $S = A[x_0, \dots, x_r]$ for a ring A and $r \geq 1$. Then $\Gamma_*(\mathcal{O}_X) \cong S$ for $X = \mathrm{Proj}(S)$. The relevance of this result is as follows. We know that the global sections of the structure sheaf over a projective scheme doesn't recover the homogeneous coordinate ring back. This result tells us that looking only at global sections of structure sheaf won't suffice (hopefully obvious by now), we need to instead look at global sections of all twists of the structure sheaf in order to recover the coordinate ring. For example, consider the quadric $xy - wz$ in $\mathbb{P}^5_k$. The corresponding coordinate ring is $S = k[w,x,y,z,a,b]/xy - wz \cong \frac{k[w,x,y,z]}{xy-wz}[a,b]$ and the corresponding scheme is $X = \mathrm{Proj}\,(S)$. Consequently, we can write $S = A[a,b]$ for $A = \frac{k[w,x,y,z]}{xy-wz}$ and thus this result would yield that S is isomorphic to $\Gamma_*(\mathcal{O}_X)$. Note that to use this result, we have to force ourselves to go 2 dimensions up.
2. Let S be a graded ring which is *finitely* generated by S_1 as an S_0-algebra and let $X = \mathrm{Proj}(S)$. Then, for any quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, we obtain a natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$. This result therefore tells us that the functor $M \mapsto \widetilde{M}$ of Remark 1.9.2.5 from graded S-modules to quasicoherent $\mathcal{O}_X$-modules is essentially surjective.

We'll later see that these results will allow us to obtain an equivalent criterion of when is a scheme over an affine scheme projective.

We now state these results and sketch their proofs.

Proposition 1.9.2.12. *Let $S = A[x_0, \dots, x_r]$ for a ring A and $r \geq 1$ and denote $X = \mathrm{Proj}(S)$. Then $\Gamma_*(\mathcal{O}_X) \cong S$.*

Proof. The main idea is to keep reducing the problem to a problem about graded ring S. Since S is generated by x_is as an A-algebra, therefore $D_+(x_i)$ for $i = 0, \dots, r$ covers X. An element in $\Gamma_*(\mathcal{O}_X)$ is given by a sum of elements $t_n \in \Gamma(\mathcal{O}_X(n), X)$. Let $t_n \in \Gamma(\mathcal{O}_X(n), X)$. The data of t_n is equivalently represented by the $(t_{n,0}, \dots, t_{n,r})$ where $t_{n,i} \in \mathcal{O}_X(n)(D_+(x_i)) = S(n)_{(x_i)}$ are the corresponding restrictions. Thus, $t_{n,i} \in S_{x_i}$ is a homogeneous element of degree n. Thus, $t = \sum_n t_n$ is equivalently represented by the tuple $(t_0, \dots, t_r)$ where $t_i = \sum_n t_{n,i}$ such that the image of t_i under $S_{x_i} \to S_{x_i, x_j}$ is same as the image of t_j under $S_{x_j} \to S_{x_j, x_i}$ for all $i, j = 0, \dots, r$. Note each of these S_{x_i, x_j} for varying i, j are contained in $R = S_{x_0, \dots, x_r}$. Now, we have injective maps $S \to S_{x_i} \to S_{x_i, x_j} \to R$ and thus $t = (t_0, \dots, t_r)$ as above is contained in $\bigcap_{i=0}^r S_{x_i} \hookrightarrow R$. In fact, any element of this intersection also corresponds to an element of $\Gamma_*(\mathcal{O}_X)$. Consequently, $\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^r S_{x_i}$. It is straightforward to see that this intersection is exactly S by writing a general homogeneous element of R and observing what it needs to satisfy to be in the intersection. $\square$

We now show the essential surjectivity of $\widetilde{(-)}$. The proof of this result nicely shows the elegance of the techniques developed so far.

Proposition 1.9.2.13. *Let S be a graded ring which is finitely generated by S_1 as an S_0-algebra and $X = \mathrm{Proj}S$.*

1. *For each $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural map*

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}.$$

2. For each quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, the above map β is an isomorphism, that is,

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\cong} \mathcal{F}.$$

Proof. 1. Since $D_+(f)$ for $f \in S_1$ covers X, we may define β naturally only on $D_+(f)$. This is done as follows:

$$\begin{tikzcd}
\Gamma_*(\mathcal{F})_{(f)} \arrow[r, "\beta_{D_+(f)}"] \arrow[dr] & \mathcal{F}(D_+(f)) \\
& \Gamma(\mathcal{F}(d)_{|D_+(f)} \otimes \mathcal{O}_X(-d), X) \arrow[u, "\varphi"', "\cong"]
\end{tikzcd}$$

where the diagonal map is given by

$$\frac{m}{f^d} \mapsto m \otimes \frac{1}{f^d}$$

and the isomorphism φ is given by restrictions.
2. In the above, we need to show that the diagonal map is an isomorphism. Suppose for some $m/f^d \in \Gamma_*(\mathcal{F})_{(f)}$, we have that $m \otimes 1/f^d = 0$ in $\Gamma(\mathcal{F}(d)_{|D_+(f)} \otimes \mathcal{O}_X(-d), X)$. Denote $\mathcal{G} = \mathcal{F}(d) \otimes \mathcal{O}_X(-d)$, which is quasicoherent. Note that $m \otimes 1/f^d = 0$ as an element in $\mathcal{G}(D_+(f))$ and also note that $D_+(f) = X_f$. Consequently from Lemma 1.9.1.17, 1, there exists $n \in \mathbb{N}$ such that $f^n \otimes m \otimes 1/f^d$ is zero as a global section of $\mathcal{O}_X(1)^{\otimes n} \otimes \mathcal{G} \cong \mathcal{F}(n)$. Hence, $f^{n-d}m = 0$ in $\Gamma(\mathcal{F}(n), X)$ and thus $\frac{m}{f^d} = \frac{f^{n-d}m}{f^n}$ is zero in $\Gamma_*(\mathcal{F})_{(f)}$. This shows injectivity. We now show surjectivity. Pick $t \in \mathcal{F}(D_+(f))$. By Lemma 1.9.1.17, 2, (which applies here as $D_+(f)$s are affine and finitely many whose intersection is again affine), we obtain a section $f^n t$ of $\mathcal{O}_X(1)^{\otimes n} \otimes \mathcal{F} \cong \mathcal{O}_X(n) \otimes \mathcal{F} \cong \mathcal{F}(n)$ over $D_+(f)$ which extends to a global section of $\mathcal{F}(n)$, say s. Consider s/f^n in $\Gamma_*(\mathcal{F})_{(f)}$, which maps to $s \otimes 1/f^n = t$ in $\mathcal{F}(D_+(f))$, as needed. □

Closed subschemes of $\mathbb{P}^n_A$

We can use these results to obtain a nice characterization of closed subschemes of projective schemes and an equivalent characterization of projective schemes over affine schemes. Denote by $\mathbb{P}^r_A = \operatorname{Proj}(A[x_0, \dots, x_r])$ for a ring A.

Proposition 1.9.2.14. *Let $Y \hookrightarrow \mathbb{P}^r_A$ be a closed subscheme with ideal sheaf $\mathcal{I}_Y$ of the projective r-space over a ring A. Then $I = \Gamma_*(\mathcal{I}_Y)$ is a homogeneous ideal of $A[x_0, \dots, x_r]$ and we have*

$$Y \cong \operatorname{Proj}\left(A[x_0, \dots, x_r]/I\right).$$

Proof. Let $S = A[x_0, \dots, x_r]$. The fact that I is a homogeneous ideal of S follows from exactness twisting functor (Remark 1.9.2.8), left exactness of global sections and $\Gamma_*(\mathcal{O}_X) = S$ of Proposition 1.9.2.12. In order to show that $Y \cong \operatorname{Proj}(S/I)$, it is enough to show that they both define isomorphic ideal sheaves (Proposition 1.4.4.13, 3). The ideal sheaf of $\operatorname{Proj}(S/I)$ is $\widetilde{I}$ by Proposition 1.8.2.8, 3 and the ideal sheaf of Y is $\mathcal{I}_Y$. Since $I = \Gamma_*(\mathcal{I}_Y)$, therefore the result follows from Proposition 1.9.2.13, 2. □

Proposition 1.9.2.15. *Let A be a ring. A scheme $Y \to \operatorname{Spec}(A)$ is projective if and only if $Y \cong \operatorname{Proj}(S)$ for a graded ring S with $S_0 = A$ and which is finitely generated by S_1 as an S_0-algebra.*

Proof. (L $\Rightarrow$ R) We have a closed immersion $Y \to \mathbb{P}^r_A$. From Proposition 1.9.2.14, it follows that $Y \cong \operatorname{Proj}(S)$ where $S = A[x_0, \dots, x_r]/I$, but S_0 might not be A. By Proposition 1.8.2.8, 2, we may assume I to not have any degree 0 component. Thus, S as defined will satisfy the necessary criterion.
(R $\Rightarrow$ L) We have $S \cong A[x_0, \dots, x_r]/I$, so by Proposition 1.8.2.8, 1, we have a closed immersion $\operatorname{Proj}(S) \to \mathbb{P}^r_A$. $\square$

Very ample invertible modules

We now study modules which determine when a scheme is projective.

Definition 1.9.2.16 (Twisting modules). Let X be a scheme and consider $\mathbb{P}^n_X \to X$ to be the projective n-scheme over X. Consider the projection $p : \mathbb{P}^n_X \to \mathbb{P}^n_{\mathbb{Z}}$. The k^{th}-Serre twist sheaf over $\mathbb{P}^n_X$ are defined to be $p^*(\mathcal{O}(k))$ where $\mathcal{O}(k)$ is the k^{th}-Serre twist sheaf over the projective scheme $\mathbb{P}^n_{\mathbb{Z}}$.

What we have defined above is indeed a generalization of usual twisted sheaves available on projective schemes, as the following lemma shows.

Lemma 1.9.2.17. *Let $X = \operatorname{Spec}(A)$. Denote $p : \mathbb{P}^n_X \to \mathbb{P}^n_{\mathbb{Z}}$ the projection map. Then,*
1. *$\mathbb{P}^n_X \cong \mathbb{P}^n_A$,*
2. *The twisting module $p^*(\mathcal{O}(k)) \cong \mathcal{O}_{\mathbb{P}^n_A}(k)$ under the above isomorphism.*

Proof. Item 1 follows from Lemma 1.8.2.12. For item 2, observe that the map p is obtained by composing with the isomorphism $\mathbb{P}^n_X \cong \mathbb{P}^n_A$ the canonical map $q : \mathbb{P}^n_A \to \mathbb{P}^n_{\mathbb{Z}}$, which is induced from the canonical map $\varphi : \mathbb{Z}[x_0, \dots, x_n] \to A[x_0, \dots, x_n]$ (see Proposition 1.8.2.6). Thus we wish to show that $q^*(\mathcal{O}(k)) \cong \mathcal{O}_{\mathbb{P}^n_A}(k)$. Denote $S = \mathbb{Z}[x_0, \dots, x_n]$ so that $\mathcal{O}(k) = \widetilde{S(k)}$. Hence, by Proposition 1.9.2.7, 4, we have

$$q^*(\mathcal{O}(k)) \cong \mathcal{O}_{\mathbb{P}^n_A}(k),$$

as needed. $\square$

Definition 1.9.2.18 (Very ample invertible module). Let $X \to Y$ be a scheme over Y. An invertible module $\mathcal{L}$ over X is said to be very ample over Y if there is an immersion (Definition 1.4.4.9) $i : X \to \mathbb{P}^n_Y$ such that $i^*(\mathcal{O}(1)) \cong \mathcal{L}$.

Proposition 1.9.2.19. *Let Y be a Noetherian scheme. Then the following are equivalent:*
1. *Scheme $f : X \to Y$ is projective.*
2. *Scheme $f : X \to Y$ is proper and there exists a very ample invertible sheaf over X relative to Y.*

1.9.3 Vector bundles

Do Exercise 5.18 from Chapter 2.

1.10 Divisors

The notion of divisors is one of the central tools for understanding the geometrical properties of a given scheme. Indeed, in the special case of curves in projective plane, a (Weil) divisor is just a formal linear combination of points of the curve. From this data, one can in-fact recover the embedding of the curve in the projective plane. Hence the data that divisors of a scheme stores is rich in geometric information.

We will first define the notion of Weil divisors in those schemes in which the points lying on codimension 1 subset of the scheme are regular (see Lemma 1.7.1.4 for a motivation behind the definition).

Definition 1.10.0.1. (**Regular in codimension 1**) A scheme X is said to be regular in codimension 1 if the local rings $\mathcal{O}_{X,p}$ which are of dimension 1 are regular.

Remark 1.10.0.2. All non-singular abstract varieties are regular in codimension 1 as all local rings are regular. In this section, we will be working with schemes which are noetherian integral separated and regular in codimension 1. We call them *Weil schemes*. All non-singular abstract varieties are Weil schemes.

1.10.1 Weil divisors and divisor class group

We will define the notion of Weil divisors and divisor class group on Weil schemes.

Definition 1.10.1.1. (**Weil divisors**) Let X be a Weil scheme. A *prime divisor* is an integral closed subscheme of codimension 1. A *Weil divisor* is an element of the free abelian group generated by the set of all prime divisors, denoted $\text{Div } X$. A Weil divisor is denoted $\sum_{i=1}^{k} n_i Y_i \in \text{Div } X$. A Weil divisor $\sum_i n_i Y_i$ is *effective* if $n_i \geq 0$ for all i.

We now look at a foundational result which will guide the further development. Its proof is important as it combines a lot of our previous knowledge.

Proposition 1.10.1.2. *Let X be a Weil scheme and $Y \subseteq X$ be a prime divisor with $\eta \in Y$ be its generic point. Then there is an injective map*

$$\text{PDiv}(X) \to \text{DVal}(K(X))$$

where PDiv *is the set of all prime divisors of X, $K(X)$ is the function field of X and* $\text{DVal}(K(X))$ *is the set of all discrete valuations over $K(X)$.*

Proof. Note that if Y is a prime divisor, then there is a generic point $\eta \in Y$. Since $\text{codim}\,(Y, X) = \dim \mathcal{O}_{X,\eta}$ by Lemma 1.7.1.4, therefore we obtain that $\mathcal{O}_{X,\eta}$ is a regular local ring of dimension 1. Now there is a special result for such rings, which in particular establishes equivalences of such rings with a lot of other type of rings. Indeed, by Theorem 23.10.1.8 we obtain that in our case $\mathcal{O}_{X,\eta}$ is a DVR. As the fraction field of $\mathcal{O}_{X,\eta}$ (it is a domain as X is integral) is $K(X)$, the function field of X (Lemma 1.3.1.3), therefore we have a valuation $v : K(X) \to \mathbb{Z}$ whose valuation ring is $\mathcal{O}_{X,\eta}$. By Lemma 1.12.4.8 (which holds for Y as Y is separated by Corollary 1.12.4.5, 2), the valuation v uniquely determines the point $\eta \in X$ as v has center η because the valuation ring $\mathcal{O}_{X,\eta}$ of v dominates the local ring $\mathcal{O}_{X,\eta}$. As the information of point $\eta \in X$ yields the closed set $Y \subseteq X$, therefore the valuation $v : K(X) \to \mathbb{Z}$ uniquely determines the prime divisor Y. $\square$

Remark 1.10.1.3. As a consequence, we can study a prime divisor via the valuation that comes through the Proposition 1.10.1.2. Indeed, for a prime divisor $Y \subseteq X$ and the associated discrete valuation $v_Y : K(X) \to \mathbb{Z}$, we can think of the value $v_Y(f)$ for some $f/g \in K(X) \setminus \{0\}$ to be telling us the number of poles that f/g has along Y. We can justify this via the proof of Proposition 1.10.1.2 as follows. For a prime divisor $Y \subseteq X$ with generic point $\eta \in Y$, the corresponding valuation is obtained by the fact that $\mathcal{O}_{X,\eta}$ is a DVR in our case. For a DVR R with fraction field K, the corresponding discrete valuation $v : K \to \mathbb{Z}$ can be thought of as an abstraction of the idea that we want to know how many poles a fraction $f/g \in K$ has and v provides that data to us. In particular, we think that if $v(f/g)$ is positive, then that tells us f/g has that many zeros in Y and if $v(f/g)$ is negative then that many poles in Y. We mostly have only this idea in mind when dealing with valuations.

As we know from complex analysis that a meromorphic function can only have discretely many singularities, along the similar lines we have that an arbitrary $f \in K(X)$ can atmost have some poles or some zeros at only finitely many prime divisors.

Proposition 1.10.1.4. ***TODO***.

1.10.2 Cartier divisors

1.10.3 Divisors and invertible modules

1.11 Smoothness & differential forms

In this section, we would like to understand the notion of smoothness in algebraic geometry. We will first begin by defining a non-singular point of a variety over an algebraically closed field, which would be an extrinsic definition. However, by a fundamental observation of Zariski, we can have an intrinsic definition of non-singular points, which would be in terms of regular local rings. The main thrust behind this latter definition will be the expectation that over non-signular points, the dimension of the tangent space is equal to the dimension of the variety (which is true in the case of, say manifolds). We would further see that for a variety over an algebraically closed field, the set of singular points is closed and proper.

We would then introduce the important notion of sheaf of differentials over a scheme. This will again allow us to characterize non-singular points of a variety, and much more.

1.11.1 Non-singular varieties

To start investigating the notion of non-singularity, we first investigate it in the setting of classical affine varieties (Definition 1.5.4.11). We will then proceed to abstract varieties.

Definition 1.11.1.1. (**Non-singular points of a classical affine variety**) Let k be an algebraically closed field and X be a classical affine k-variety with $I(X) = \langle f_1, \dots, f_m \rangle \lneq k[x_1, \dots, x_n]$. A point $p \in X$ is said to be non-singular if the $n \times m$ Jacobian matrix

$$[J_p]_{n\times m} = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{ij}$$

is of rank $n - \dim X$.

The first obvious question is whether the above definition is independent of the choice of the generators of prime ideal $I(X)$. The following lemma says yes.

Lemma 1.11.1.2. *Let k be an algebraically closed field and X be a classical affine k-variety. The definition of a non-singular point $p \in X$ is independent of the choice of the generating set of $I(X)$.*

Proof. Let $I(X) = \langle f_1, \dots, f_m \rangle = \langle g_1, \dots, g_l \rangle$. We wish to show that

$$\operatorname{rank}\left[\frac{\partial f_i}{\partial x_j}(p)\right]_{ij} = \operatorname{rank}\left[\frac{\partial g_i}{\partial x_j}(p)\right]_{ij}.$$

This follows immediately after writing $f_i = \sum_{a=1}^{l} c_{i_a} g_a$, $c_{i_a} \in k[x_1, \dots, x_n]$, differentiating it and observing that $g_a(p) = 0$ for all $a = 1, \dots, l$. □

Remark 1.11.1.3. In geometry, one notes that at a smooth point, the dimension of tangent space equals the dimension of the manifold itself. We would like to do a similar construction here. Indeed, if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and 0 is regular for f, then we know by implicit function theorem that $M = Z(f) \subseteq \mathbb{R}^n$ is a smooth manifold with normal vector field $(\nabla f) : \mathbb{R}^n \to \mathbb{R}^n$. Consequently, one can define the tangent space T_xM for $x \in M$ to be the set of all those vectors which are normal to $(\nabla f)_x$. We mimic this definition for classical affine varieties.

Definition 1.11.1.4. (**Tangent space of a classical affine variety**) Let k be an algebraically closed field and let X be a classical affine k-variety in $\mathbb{A}^n_k$ with $I(X) = \langle f_1, \dots, f_m \rangle$. Denote for each $f \in k[x_1, \dots, x_n]$ and $p \in \mathbb{A}^n_k$ the following linear functional

$$\begin{aligned}(df)_p : k^n &\longrightarrow k \\ v &\longmapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) v_i.\end{aligned}$$

For a point $p \in X$, we define the tangent space T_pX as the following k-vector space

$$\begin{aligned}T_pX &:= \{v \in k^n \mid (df_i)_p(v) = 0 \; i = 1, \dots, m\} \\ &= \{v \in k^n \mid (df)_p(v) = 0 \; \forall f \in I(X)\}.\end{aligned}$$

We now show that this definition of tangent space is intrinsic. Indeed, we will show that the $T_pX = T\mathcal{O}_{X,p} := \mathrm{Hom}_k\left(\mathfrak{m}/\mathfrak{m}^2, k\right)$, where $(\mathcal{O}_{X,p}, \mathfrak{m})$ is the local ring at $p \in X$ and $\kappa(p) = k$ in this case (see Definition 23.1.2.13). Let us begin with a series of observations.

Lemma 1.11.1.5. *Let k be an algebraically closed field and $p \in \mathbb{A}^n_k$. Then the k-linear map*

$$\begin{aligned}\theta_p : k[x_1, \dots, x_n] &\longrightarrow k^n \\ f &\longmapsto \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p)\right)\end{aligned}$$

induces a k-linear isomorphism

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \cong k^n$$

where $\mathfrak{m}_p = \langle x_1 - p_1, \dots, x_n - p_n \rangle$ is the maximal ideal of $k[x_1, \dots, x_n]$ corresponding to the point p.

Proof. Let $p = (p_1, \dots, p_n)$. Observe that $\{\theta_p(x_i - p_i)\}_{i=1,\dots,n}$ forms a basis of k^n. Consequently, θ_p restricts to a surjective k-linear map $\hat{\theta}_p : \mathfrak{m}_p \to k^n$. Now one observes that $f \in \mathrm{Ker}\,(\theta)_p$ if and only if $f \in \mathfrak{m}_p^2$. Thus we have by first isomorphism theorem that $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong k^n$. ☐

Lemma 1.11.1.6. *Let k be an algebraically closed field and X be a classical affine k-variety in $\mathbb{A}^n_k$ with $p \in X$. Let $(\mathcal{O}_{X,p}, \mathfrak{m})$ denote the local ring of X at p and $I \leq k[x_1, \dots, x_n]$ be the ideal of X. Then,*

$$\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}_p/(\mathfrak{m}_p^2 + I).$$

Proof. Let $A = k[x_1, \dots, x_n]$. By Proposition 1.5.3.10, 3, we have $\mathfrak{m} = (\mathfrak{m}_p)_{\mathfrak{m}_p}/I_{\mathfrak{m}_p}$ and $\mathfrak{m}^2 = \left((\mathfrak{m}_p^2)_{\mathfrak{m}_p} + I_{\mathfrak{m}_p}\right)/I_{\mathfrak{m}_p}$. By quotienting, we obtain

$$\begin{aligned}\mathfrak{m}/\mathfrak{m}^2 &\cong \frac{(\mathfrak{m}_p)_{\mathfrak{m}_p}}{(\mathfrak{m}_p^2 + I)_{\mathfrak{m}_p}} \\ &\cong \left(\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I}\right)_{\mathfrak{m}_p/(\mathfrak{m}_p^2+I)} \\ &\cong \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I}.\end{aligned}$$

☐

Recall the notion of regular local ring from Definition 23.1.2.15. We now see that non-singular points are classified by the local ring being regular.

Theorem 1.11.1.7. *Let k be an algebraically closed field and X be a classical affine k-variety and let $p \in X$. The following are equivalent:*
1. *The point $p \in X$ is non-singular.*
2. *The local ring $\mathcal{O}_{X,p}$ is regular.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{X,p}$. By definition, we have $\mathcal{O}_{X,p}$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}_{X,p}$. By Proposition 1.5.3.10, 7, we further have that $\mathcal{O}_{X,p}$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim X$. Whereas by Lemmas 1.11.1.6 and 1.11.1.5, we observe

$$\begin{aligned}\dim_k \mathfrak{m}/\mathfrak{m}^2 &= \dim_k \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2 + I}\\ &= \dim_k \left(\frac{\frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}}{\frac{\mathfrak{m}_p^2+I}{\mathfrak{m}_p^2}}\right)\\ &= \dim_k \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} - \dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2}\\ &= n - \dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2}.\end{aligned}$$

With these two observations, we thus reduce to proving that

$$\dim_k \frac{\mathfrak{m}_p^2 + I}{\mathfrak{m}_p^2} = \text{rank } J_p$$

where $J_p = \left[\frac{\partial f_i}{\partial x_j}\right]$ for $I = \langle f_1, \dots, f_m\rangle$. This now follows by the following two rather straightforward observations; in the notations of Lemma 1.11.1.5 and its proof, one observes
1. $\hat{\theta}_p^{-1}(\theta_p(I))$ is isomorphic as k-vector space to $I + \mathfrak{m}_p^2$,
2. $\dim_k \theta_p(I) = \text{rank } J_p$.

The result now follows. □

With the above result, we formulate the following definition of non-singular abstract varieties.

Definition 1.11.1.8. (**Non-singular abstract variety**) Let k be an algebraically closed field. A variety X over k is said to be non-singular if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a regular local ring.

Remark 1.11.1.9. Note that in the definition of non-singular varieties, it is sufficient to demand that $\mathcal{O}_{X,x}$ is a regular local ring for all closed points $x \in X$ only. Indeed, by Lemma 1.3.1.2, local ring at a non-closed point is obtained by localizing the local ring at a closed point at a prime ideal. As the localization of a regular local ring at a prime ideal is again a regular local ring by Theorem **??**, the result follows.

We now define the Zariski (co)tangent space of a scheme at a point.

Definition 1.11.1.10. (**Zariski (co)tangent space**) Let X be a scheme and $x \in X$ be a point and let κ be the residue field at point x. Then

1. the Zariski cotangent space at x is defined to be the κ-vector space
$$T_x^* X := \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2,$$
2. the Zariski tangent space at x is defined to be the κ-vector space
$$T_x X := \operatorname{Hom}_\kappa\left(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, \kappa\right).$$

These are the analogues to the case in algebra (see Definition 23.1.2.13).

TODO : State how this is related to usual definition of tangent spaces

1.11.2 Regular schemes

"*.... Of what use is it to know the definition of a scheme if one does not realize that a ring of integers in an algebraic number field, an algebraic curve, and a compact Riemann surface are all examples of a 'regular scheme of dimension 1'?*"- Hartshorne.

After defining and discussing basic properties of regular schemes, we will prove the above result stated by Hartshorne.

1.12 Morphism of schemes

The main use of schemes to answer geometric questions begin with defining various types of situations that one usually finds himself/herself in algebraic geometry. We discuss them here one by one by giving examples. In the first section we discuss some basic type of maps among schemes.

1.12.1 Elementary types of morphism

We first cover some basic type of maps between schemes.

Definition 1.12.1.1. (**Quasi-compact maps**) A map $f : X \to Y$ of schemes is said to be quasi-compact if there exists an affine open cover $\{V_i\}$ of Y such that the space $f^{-1}(V_i) \subseteq X$ is quasi-compact for each i.

Remark 1.12.1.2. Observe that a scheme X over k has a quasi-compact structure map $X \to \operatorname{Spec}(k)$ if and only if X is quasi-compact.

We now see that quasi-compact maps are local on target.

Proposition 1.12.1.3. [37] *A map $f : X \to Y$ is quasi-compact if and only if for each open affine $V \subseteq Y$, the space $f^{-1}(V) \subseteq X$ is quasi-compact.*

Proof. The ($\Leftarrow$) is immediate. For ($\Rightarrow$), pick any open affine $V \subseteq Y$. We wish to show that $f^{-1}(V)$ is quasi-compact. Let $V_i = \operatorname{Spec}(B_i)$ be the collection of open affines covering Y such that $f^{-1}(V_i)$ is quasi-compact. We now obtain a finite covering of V by affine opens which are affine open in V_i for some i as well. Indeed, by Lemma 1.4.4.3, we may cover $V \cap V_i$ by open affines which are basic open in both V and V_i. Doing this for each i, we obtain a cover of V by basic opens. As V is affine, so by Lemma 1.2.1.6 we obtain a finite collection of basic opens $\{D(g_i)\}_{i=1}^n$ where $g_i \in B_i$ such that $D(g_i)$ is a basic open in V as well.

We now have that $f^{-1}(V) = \bigcup_{i=1}^n f^{-1}(D(g_i))$. Hence it suffices to show that $f^{-1}(D(g_i))$ is a quasi-compact subspace. To this end, we first immediately reduce to assuming that X is quasicompact (by replacing X by $f^{-1}(V_i)$) and $Y = \operatorname{Spec}(B)$ is affine (by replacing Y by V_i). We now wish to prove that for any $g \in B$, $f^{-1}(D(g))$ is quasi-compact.

As X is quasi-compact, therefore there exists a finite affine open cover of X by $\operatorname{Spec}(A_i)$. It suffices to show that $\operatorname{Spec}(A_i) \cap f^{-1}(D(g))$ is a quasicompact space. Observe that $f|_{\operatorname{Spec}(A_i)} : \operatorname{Spec}(A_i) \to \operatorname{Spec}(B)$ is a morphism of affine schemes. It follows from Corollary 1.3.0.6 that $f|_{\operatorname{Spec}(A_i)}$ is induced from a ring map $\varphi_i : B \to A_i$. As $\operatorname{Spec}(A_i) \cap f^{-1}(D(g)) = (f|_{\operatorname{Spec}(A_i)})^{-1}(D(g)) = D(\varphi_i(g))$, which is an affine open, therefore by Lemma 1.2.1.6, we deduce that $\operatorname{Spec}(A_i) \cap f^{-1}(D(g))$ is quasi-compact, as required. □

Definition 1.12.1.4. (**Quasi-finite maps**) A map $f : X \to Y$ of schemes is quasi-finite if for each $y \in Y$, the fiber X_y is a finite set.

[37] Exercise II.3.2 of Hartshorne.

Example 1.12.1.5. Let k be an algebraically closed field. Consider the map

$$f : X = \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3}\right) \longrightarrow \mathbb{A}^1_k$$

obtained by the map $k[x] \to \frac{k[x,y]}{y^2-x^3}$ given by $x \mapsto x + \langle y^2 - x^3\rangle$. Take any point $\mathfrak{p} = \langle p(x)\rangle \in \mathbb{A}^1_k$. Hence the fiber is

$$\begin{aligned} X_{\mathfrak{p}} &= \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k \kappa(\mathfrak{p})\right) \\ &= \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k \frac{k[x]_{\mathfrak{p}}}{\mathfrak{p}k[x]_{\mathfrak{p}}}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k \left(\frac{k[x]}{\mathfrak{p}}\right)_{\mathfrak{p}}\right). \end{aligned}$$

Let $\mathfrak{p} \neq \mathfrak{o}$. As $k[x]$ is a PID, therefore $\mathfrak{p}$ is a maximal ideal. Consequently, we have $\kappa(\mathfrak{p}) = k[x]/\mathfrak{p}$. Hence,

$$\begin{aligned} X_{\mathfrak{p}} &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k \frac{k[x]}{p(x)}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3, p(x)}\right). \end{aligned}$$

As k is algebraically closed, therefore by weak Nullstellensatz, we obtain that $p(x) = x - a$ for some $a \in k$. Consequently, if we have $a \neq 0$ then

$$\begin{aligned} X_{\mathfrak{p}} &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3, x - a}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[y]}{y^2 - a^3}\right) \\ &\cong \operatorname{Spec}\left(\frac{k[y]}{(y + a^{3/2})(y - a^{3/2})}\right) \\ &\cong \operatorname{Spec}(k \times k) \\ &\cong \operatorname{Spec}(k) \amalg \operatorname{Spec}(k). \end{aligned}$$

If $a = 0$, then

$$X_{\mathfrak{p}} \cong \operatorname{Spec}\left(\frac{k[y]}{y^2}\right)$$

and we know that $k[y]/y^2$ has only one prime ideal, the one generated by $y + \langle y^2\rangle \in k[y]/y^2$. Hence $X_{\mathfrak{p}}$ consists of two points at all non-zero closed points and of a single point at the origin, showing that f has finite fibers at all closed points. However, at the generic point

$\mathfrak{p} = \mathfrak{o}$, we have a more complicated story:

$$\begin{aligned} X_{\mathfrak{o}} &\cong \operatorname{Spec}\left(\frac{k[x,y]}{y^2 - x^3} \otimes_k k(x)\right) \\ &\cong \operatorname{Spec}\left(\frac{k(x)[x,y]}{y^2 - x^3}\right) \\ &\cong \operatorname{Spec}\left(\frac{k(x)[y]}{y^2 - x^3}\right). \end{aligned}$$

As $k(x)[y]$ is a PID, therefore points of $X_{\mathfrak{o}}$ are thus in bijective correspondence with prime ideals of $k(x)[y]$ containing $y^2 - x^3$, which in turn is in bijection with the set of irreducible factors of $y^2 - x^3$ in $k(x)[y]$. As $k(x)[y]$ is a UFD, therefore there can atmost be finitely many such irreducible factors. Hence, $X_{\mathfrak{o}}$ is finite.

Hence all fibers are finite, making f quasi-finite.

1.12.2 Finite type

We already considered one example of such maps in the case of schemes over a field in Section 1.4.3

Definition 1.12.2.1. (**Locally finite type**) Let $f : X \to Y$ be a map of schemes. Then f is said to be locally of finite type if there is an affine open cover $V_i = \operatorname{Spec}(B_i)$, $i \in I$ of Y such that for each $i \in I$, $f^{-1}(V_i)$ has an open affine cover $U_{ij} = \operatorname{Spec}(A_{ij})$, $j \in J$ such that for each j, the ring A_{ij} is finite type[38] B_i-algebra.

Definition 1.12.2.2. (**Finite type**) Let $f : X \to Y$ be a map of schemes. Then f is said to be of finite type if there is an open affine cover $V_i = \operatorname{Spec}(B_i)$, $i \in I$ of Y such that for each $i \in I$, $f^{-1}(V_i)$ has a finite open affine cover $U_{ij} = \operatorname{Spec}(A_{ij})$, $j = 1, \dots, n$ such that for each j, A_{ij} is a finite type B_i-algebra.

It is an important observation that both the above definitions are local on target.

Proposition 1.12.2.3. [39] *A map $f : X \to Y$ is locally of finite type if and only if for all open affine $V = \operatorname{Spec}(B)$ in Y, there is an open affine cover $U_i = \operatorname{Spec}(A_i)$ of $f^{-1}(V)$ in X such that each A_i is a finite type B-algebra.*

Proof. The R $\Rightarrow$ L follows immediately. Let $V_i = \operatorname{Spec}(B_i)$ be an open affine cover of Y such that $f^{-1}(V_i)$ is covered by open affines $U_{ij} = \operatorname{Spec}(A_{ij})$ where each A_{ij} is a finite type B_i-algebra. Pick any affine open $V = \operatorname{Spec}(B)$ in Y and a point $x \in f^{-1}(V)$. We wish to find an open affine $x \in U = \operatorname{Spec}(A)$ inside $f^{-1}(V)$ such that A is a finite type B-algebra.

Consider $f(x) \in V$ and let $f(x) \in V \cap V_i$. Consequently, $x \in f^{-1}(V)$ will be contained in some U_{ij}, so $x \in f^{-1}(V) \cap U_{ij}$. By continuity of f, there exists a basic open $D(g) \subseteq V \cap V_i$ for some $g \in B_i$ which contains $f(x)$ such that $f^{-1}(D(g)) \subseteq f^{-1}(V) \cap U_{ij}$ is open. Restricting f to U_{ij}, we have $f : U_{ij} \to V_i$ which induces a map $\varphi : B_i \to A_{ij}$ which is of finite type. Denote $U = f^{-1}(D(g)) = D(\varphi(g)) = \operatorname{Spec}\left((A_{ij})_{\varphi(g)}\right) \subseteq f^{-1}(V) \cap U_{ij}$. We therefore get

[38] finite type algebra := finitely generated as an algebra.

[39] Exercise II.3.1 of Hartshorne.

that the restriction of f on U, which is given by $f : U \to D(g)$, induces the localization map on algebras $\varphi_g : (B_i)_g \to (A_{ij})_{\varphi(g)}$. As localization of algebras are finite type, therefore φ_g makes $(A_{ij})_{\varphi(g)}$ a finite type $(B_i)_g$-algebra.

By Lemma 1.4.4.3, we have an isomorphism $B_h \to (B_i)_g$. Thus, we have

$$B \to B_h \overset{\cong}{\to} (B_i)_g \to (A_{ij})_{\varphi(g)}$$

where each map is of finite type. Since composite of finite type maps is of finite type, therefore $(A_{ij})_{\varphi(g)}$ is a finite type B-algebra, as required.

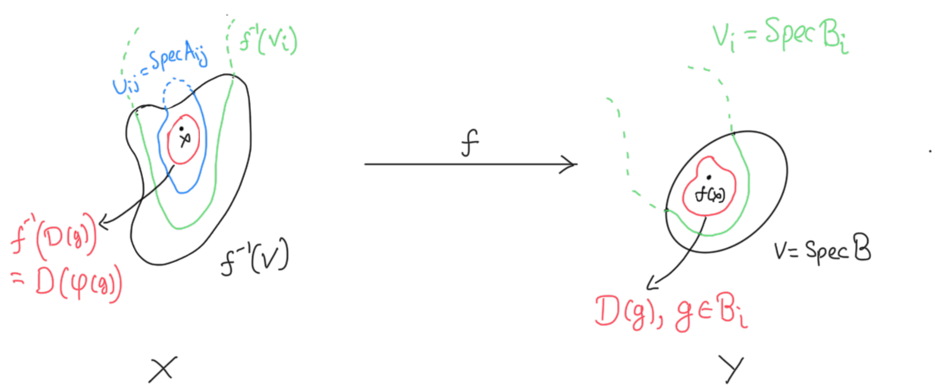


Figure 1.2: Sketch of the proof of Proposition 1.12.2.3

□

We next see that finite type maps are also local on target and a nice property that they satisfy which says that finite type property descends to every open affine inside the inverse image of an open affine.

Theorem 1.12.2.4. [40] *Let $f : X \to Y$ be a map of schemes. Then,*

1. *f is of finite type if and only if f is locally of finite type and quasi-compact,*
2. *f is of finite type if and only if for every open affine $V = \mathrm{Spec}\,(B)$, the space $f^{-1}(V)$ can be covered by finitely many open affines $U_i = \mathrm{Spec}\,(A_i)$ where each A_i is a finite type B-algebra,*
3. *if f is of finite type, then for any open affine $V = \mathrm{Spec}\,(B) \subseteq Y$ and any open affine $U = \mathrm{Spec}\,(A) \subseteq f^{-1}(V)$, A is a finite type B-algebra.*

Proof. 1. (L $\Rightarrow$ R) As f is of finite type, therefore there exists an open affine cover $\{V_i = \mathrm{Spec}\,(B_i)\}$ of Y such that $f^{-1}(V)$ can be covered by finitely many $U_{ij} = \mathrm{Spec}\,(A_{ij})$ where A_{ij} is a finite type B_i-algebra. Consequently, f is locally finite type. As each affine scheme is quasi-compact (Lemma 1.2.1.6) and finite union of quasi-compact spaces is quasi-compact, therefore we deduce that $f^{-1}(V)$ is quasi-compact.

(R $\Rightarrow$ L) As f is locally of finite type, therefore there exists an open affine cover $\{V_i = \mathrm{Spec}\,(B_i)\}$ of Y such that $f^{-1}(V_i)$ is covered by open affines $U_{ij} = \mathrm{Spec}\,(A_{ij})$ where each A_{ij} is a finite type B_i-algebra. As f is quasicompact, therefore we have a finite subcover $U_{i1}, \dots, U_{in}$ covering $f^{-1}(V)$, as required.

[40] Exercise II.3.3 of Hartshorne.

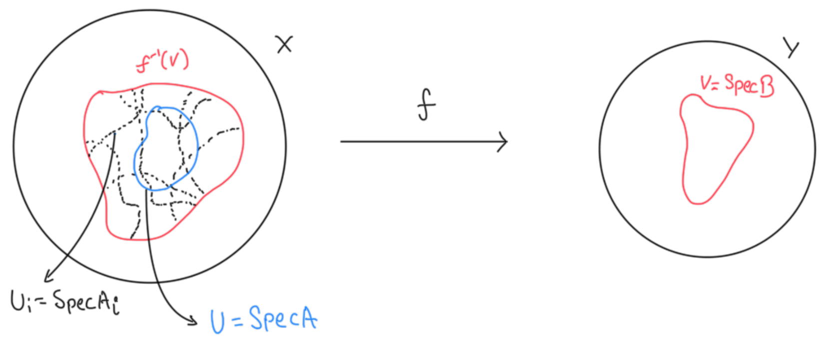


Figure 1.3: Sketch of the proof of Theorem 1.12.2.4, 3.

2. (R $\Rightarrow$ L) Immediate from definition.
(L $\Rightarrow$ R) Pick an open affine $V = \operatorname{Spec}(B)$ in Y. We wish to show that $f^{-1}(V)$ is covered by finitely many open affines each of which is spectrum of a finite type B-algebra. Indeed, as f is quasi-compact by statement 1 above, therefore by Proposition 1.12.1.3, we see that $f^{-1}(V)$ is quasi-compact. Also by statement 1, f is of locally finite type. Hence by Proposition 1.12.2.3, $f^{-1}(V)$ is covered by spectra of finite type B-algebras. As $f^{-1}(V)$ is quasi-compact, we get a finite subcover, as required.

3. Pick any open affine $V = \operatorname{Spec}(B)$ in Y and an open affine $U = \operatorname{Spec}(A) \subseteq f^{-1}(V)$. As f is of finite type, therefore by statement 2 above, we obtain a finite collection $U_i = \operatorname{Spec}(A_i)$ of open affines covering $f^{-1}(V)$. Observe that $U \cap U_i$ is an open set of U_i. By virtue of Lemma 1.4.4.3, we may cover $U \cap U_i$ by basic open sets of U_i which are basic open in U as well. Doing this for each i furnishes us with an open cover of U. As U is quasi-compact as it is affine (Lemma 1.2.1.6), consequently we get a finitely many elements $h_1, \dots, h_n \in A$ such that $D(h_i) \subseteq U$ covers U and furthermore for each $i = 1, \dots, n$, $D(h_i) \cong D(g_i)$ where $D(g_i) \subseteq U_i$ and $g_i \in A_i$. In particular, we have $A_{h_i} \cong (A_i)_{g_i}$. Now for each $i = 1, \dots, n$, we have

$$B \to A_i \to (A_i)_{g_i} \cong A_{h_i}$$

where each of the arrows makes the codomain a finite type algebra over the domain. Hence, A_{h_i} is a finite type B-algebra. Consequently, we have $h_1, \dots, h_n \in A$ such that $\cup_{i=1}^n D(h_i) = U$ (which is equivalent to saying that h_is generate the unit ideal of A by Lemma 1.2.1.5, 2) and A_{h_i} is a finite type B-algebra. It follows from Lemma 23.1.2.11 that A is a finite type B-algebra. This completes the proof. $\square$

We now list out some properties of finite type maps as we shall encounter them quite frequently.

Proposition 1.12.2.5. [41] *Properties of finite type maps.*
1. *A closed immersion $X \to Y$ is of finite type.*

[41] Exercise II.3.13 of Hartshorne.

2. *A quasicompact open immersion $X \to Y$ is of finite type.*
3. *Composition of finite type maps $X \to Y \to Z$ is of finite type.*
4. *Product of finite type schemes $X \to S$ and $Y \to S$ in* $\mathbf{Sch}/S$ *denoted $X \times_S Y \to S$ is of finite type.*
5. *Maps of finite type are stable under base extensions.*
6. *If $X \to Y$ is quasicompact and the composite $X \to Y \to Z$ is of finite type, then $X \to Y$ is of finite type.*
7. *If $X \to Y$ is of finite type and Y is noetherian, then X is noetherian.*

Proof. **TODO!** □

The following is something we all expect, which indeed holds true for finite type schemes.

Lemma 1.12.2.6. *Let k be a field and X be a finite type k-scheme. The set of all closed points of X is dense in X.*

Proof. **TODO**. □

Example 1.12.2.7. We give a number of examples of finite type maps.

1. Let k be a field. Consider the projection map $\pi : \mathbb{A}^2_k \to \mathbb{A}^1_k$ defined by the k-algebra map $k[x] \to k[x, y]$ mapping as $x \mapsto x$. Note that π is a finite type map of schemes as the open covering of $\mathbb{A}^1_k$ as itself yields that $\pi^{-1}(\mathbb{A}^1_k) = \mathbb{A}^2_k$ and $\mathbb{A}^2_k$ is spectra of $k[x, y]$ which is a finite type $k[x]$ algebra via the above map. Indeed, $k[x, y]$ is generated by $\{y\}$ as a $k[x]$-algebra.
We deduce that projection maps $\mathbb{A}^n_k \to \mathbb{A}^1_k$ are finite type maps for any $n \in \mathbb{N}$.
2. We next consider a family of curves parameterized by a parameter t. Consider the map
$$\mathbb{C}[t] \longrightarrow \mathbb{C}[t][x, y]\langle y^2 - x^3 - t\rangle.$$
This yields the following map at the level of schemes
$$X := \mathrm{Spec}\left(\frac{\mathbb{C}[t][x, y]}{\langle y^2 - x^3 - t\rangle}\right) \to \mathrm{Spec}\,(\mathbb{C}[t]).$$
Pick the closed point corresponding to $a \in \mathbb{C}$ in $\mathrm{Spec}\,(\mathbb{C}[t])$. As $\mathbb{C}[t][x, y]/\langle y^2 - x^3 - t\rangle$ is a finite type $\mathbb{C}[t]$-algebra, therefore the above map of schemes is of finite type.
Observe that the fiber of X at $a \in \mathrm{Spec}\,(\mathbb{C}[t])$ (by abuse of notation) is given by
$$X_a = X \times_{\mathrm{Spec}(\mathbb{C}[t])} \mathrm{Spec}\,(\kappa(a)).$$
As $\kappa(a)$ is the fraction field of $\mathbb{C}[t]/\langle t - a\rangle$, which is $\mathbb{C}[a] = \mathbb{C}$, therefore we get the following
$$\begin{aligned} X_a &= \mathrm{Spec}\left(\frac{\mathbb{C}[t][x, y]}{\langle y^2 - x^3 - t\rangle} \otimes_{\mathbb{C}[t]} \mathbb{C}[a]\right) \\ &\cong \mathrm{Spec}\left(\frac{\mathbb{C}[x, y]}{\langle y^2 - x^3 - a\rangle}\right). \end{aligned}$$
Hence, we get the curve $y^2 - x^3 - a$ back as the fiber at the point $a \in \mathrm{Spec}\,(\mathbb{C}[t])$.

3. Consider the map
$$k[t] \longrightarrow \frac{k[t][w,x,y,z]}{\langle (w-y)^2+(x-z)^2-t^2\rangle}$$
which yields the map on geometric level as
$$X := \mathrm{Spec}\left(\frac{k[t][w,x,y,z]}{\langle (w-y)^2+(x-z)^2-t^2\rangle}\right) \longrightarrow \mathrm{Spec}\,(k[t]).$$
Again, this is a finite type map and for a closed point $a \in k$ corresponding to the ideal $\langle t-a\rangle \lneq k[t]$, the fiber is
$$\begin{aligned} X_a &\cong \mathrm{Spec}\left(\frac{k[t][w,x,y,z]}{\langle (w-y)^2+(x-z)^2-t^2\rangle} \otimes_{k[t]} k[a]\right) \\ &\cong \mathrm{Spec}\left(\frac{k[w,x,y,z]}{\langle (w-y)^2+(x-z)^2-a^2\rangle}\right) \end{aligned}$$
on $\mathbb{A}^2_{\mathbb{R}}$).
4. Any projective variety $X \to \mathbb{P}^n_k$ will by definition be finite type over k (Theorem 1.12.7.2, 1). For example, the projective parabola $X = \mathrm{Proj}\left(\frac{k[x,y,z]}{y^2-xz}\right)$ is a finite type scheme over k. Indeed, by Proposition 1.8.2.8, 1, we get that the natural map $X \to \mathrm{Proj}(k[x,y,z]) = \mathbb{P}^3_k$ coming from the quotient $k[x,y,z] \twoheadrightarrow \frac{k[x,y,z]}{y^2-xz}$ (which is a graded map) is a closed immersion. Hence, it defines a closed subscheme of projective 3-space $\mathbb{P}^3_k$ over k.
5. **TODO**: *Add more examples from maps of projective schemes. This is important.*

1.12.3 Finite

This is a more stronger version of finite type maps discussed in previous section.

Definition 1.12.3.1. (**Finite**) Let $f : X \to Y$ be a map of schemes. Then f is said to be finite if there is an open affine covering $V_i = \mathrm{Spec}\,(B_i)$, $i \in I$ of Y such that $f^{-1}(V_i)$ is equal to an open affine $\mathrm{Spec}\,(A_i)$ where A_i is a finite B_i-algebra[42].

We see that finite maps are local on target.

Proposition 1.12.3.2. *A map $f : X \to Y$ of schemes is finite if and only if for each open affine $V = \mathrm{Spec}\,(B)$, we have $f^{-1}(V)$ is an open affine $\mathrm{Spec}\,(A)$ in X such that $B \to A$ makes A a finite B-algebra.*

Proof. (R $\Rightarrow$ L) is immediate from definitions.

(L $\Rightarrow$ R) Pick any open affine $V = \mathrm{Spec}\,(B)$ in Y. We first wish to show that $U = f^{-1}(V)$ is an affine scheme. We may employ criterion for affineness, Proposition 1.3.1.7, for this purpose. Hence for showing that U is affine, we reduce to finding $g_1, \dots, g_n \in \Gamma(\mathcal{O}_{X|U}, U) = \mathcal{O}_X(U)$ such that U_{g_i} is affine and $\langle g_1, \dots, g_n\rangle = \mathcal{O}_X(U)$.

As f is finite, therefore there exists an open affine covering $V_i = \mathrm{Spec}\,(B_i)$ of Y such that

[42] finite algebra := finitely generated as a module.

$f^{-1}(V_i) = \mathrm{Spec}\,(A_i) = U_i$ is affine and A_i is a finite B_i-algebra. Observe that $V \cap V_i$ forms an open covering of V. As V is affine, so it is quasi-compact (Lemma 1.2.1.6). Consequeently, we obtain a finite cover of V by V_is. Now cover each $V \cap V_i$ by basic opens which are basic in both V and V_i (Lemma 1.4.4.3). Doing this for each of the finitely many i, we obtain a cover of V by basic open sets. As V is quasi-compact (Lemma 1.2.1.6), therefore we have obtained a cover of V by finitely many basics $D(k_i)$ for $k_i \in B$ such that $D(k_i) \cong D(l_i)$ where $D(l_i) \subseteq V_i$ and $l_i \in B_i$ for $i = 1, \dots, n$. Consequently by Lemma 1.2.1.5, the ideal generated by $k_1, \dots, k_n$ in B is the unit ideal.

As we have

$$U = f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{n} D(k_i)\right) = \bigcup_{i=1}^{n} f^{-1}(D(k_i)),$$

therefore by Lemma 1.3.1.4, we may write

$$U = \bigcup_{i=1}^{n} U_{\varphi(k_i)}$$

where $\varphi : B \to \mathcal{O}_X(U)$ is the map induced by the restricted map $f : U \to V$ on the global sections. Furthermore, as $\sum_{i=1}^{n} k_i B = B$, therefore $\sum_{i=1}^{n} \varphi(k_i)\mathcal{O}_X(U) = \mathcal{O}_X(U)$. Hence, it now suffices to show that each $U_{\varphi(k_i)}$ is affine.

We have $U_{\varphi(k_i)} = f^{-1}(D(k_i)) \cong f^{-1}(D(l_i)) = D(\varphi_i(l_i))$ where $\varphi_i : B_i \to A_i$ is the map on global sections obtained by the restriction $f : U_i \to V_i$. As $D(\varphi_i(l_i))$ is affine, thus, so is $U_{\varphi(k_i)}$. This shows that indeed, $f^{-1}(V)$ is an open affine.

We may now write $U = \mathrm{Spec}\,(A)$. We reduce now to showing that A is a finite B-algebra. For this observe that in the above, we obtained a finite open cover of U given by $U_{\varphi(k_i)} \cong D(\varphi_i(l_i))$ where $D(\varphi_i(l_i)) \subseteq U_i$. As $U = \mathrm{Spec}\,(A)$, therefore $\mathcal{O}_X(U) = A$, so we may let $\varphi(k_i) = g_i$ for $i = 1, \dots, n$. Now, since $U_{\varphi(k_i)} = U_{g_i} = D(g_i) \cong D(\varphi_i(l_i))$, therefore we have $A_{g_i} \cong (A_i)_{\varphi_i(l_i)}$. As A_i is a finite B_i-algebra, therefore by Lemma 23.1.2.10, $(A_i)_{\varphi_i(l_i)}$ is a finite $(B_i)_{l_i}$-algebra. Further, as we saw in the beginning that $D(k_i) \cong D(l_i)$, hence we get $B_{k_i} \cong B_{l_i}$. We thus obtain a map $B_{k_i} \to A_{g_i}$ as in

$$\begin{tikzcd}
(A_i)_{\varphi_i(l_i)} & (B_i)_{l_i} \arrow[l, "(\varphi_i)_{l_i}"'] \\
A_{g_i} \arrow[u, phantom, "\cong" rotate=90] & B_{k_i} \arrow[u, phantom, "\cong" rotate=90] \arrow[l, dashed]
\end{tikzcd}$$

which thus makes A_{g_i} a finite B_{k_i}-algebra, in particular, a finitely generated B_{k_i}-module. This is for each of the $i = 1, \dots, n$, and since we have that $k_1, \dots, k_n$ generates the unit ideal in B, hence by another application of Lemma 23.1.2.10, we deduce that A is a finite B-algebra, as required. $\square$

One important property of finite maps is that their fibers are finite.

Proposition 1.12.3.3. [43] *Let $f : X \to Y$ be a finite morphism. Then f is quasi-finite.*

Proof is WRON

Proof. Pick any $y \in Y$. We wish to show that $X_y = X \times_Y \mathrm{Spec}\,(\kappa(y))$ is a finite set. Let $V = \mathrm{Spec}\,(B)$ be an open affine containing y. Then since $f^{-1}(y) \subseteq f^{-1}(V)$, therefore we have $X_y = f^{-1}(V) \times_Y \mathrm{Spec}\,(\kappa(y))$. As f is finite, therefore $f^{-1}(V) = \mathrm{Spec}\,(A)$ for some finite B-algebra A. Hence we may replace X by $\mathrm{Spec}\,(A)$, Y by $\mathrm{Spec}\,(B)$ and point $y \in Y$ by a prime ideal $\mathfrak{q} \in \mathrm{Spec}\,(B)$. Consequently, we obtain

$$\begin{aligned} X_y &= \mathrm{Spec}\,(A) \times_{\mathrm{Spec}(B)} \mathrm{Spec}\,(\kappa(\mathfrak{q})) \\ &\cong \mathrm{Spec}\left(A \otimes_B \frac{B_\mathfrak{q}}{\mathfrak{q}B_\mathfrak{q}}\right). \end{aligned}$$

As A is a finite B-algebra, therefore $A \cong B^n/I$ where $n \in \mathbb{N}$ and $I \leq B^n$ is an ideal. Further, any ideal of product of rings is product of ideals, therefore $A \cong B^n/I_1 \times \cdots \times I_n \cong B/I_1 \times \cdots \times B/I_n$ where $I_i \leq B$ is an ideal.

As tensor product is a filtered colimit and they commute over finite limits, therefore

$$\begin{aligned} A \otimes_B \frac{B_\mathfrak{q}}{\mathfrak{q}B_\mathfrak{q}} &\cong \prod_{i=1}^n \frac{B}{I_i} \otimes_B \frac{B_\mathfrak{q}}{\mathfrak{q}B_\mathfrak{q}} \\ &\cong \prod_{i=1}^n \frac{B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}}{I_i \cdot (B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q})}. \end{aligned}$$

As $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ is a field, therefore $I_i \cdot (B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q})$ is either 0 or $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} = \kappa(\mathfrak{q})$. Hence,

$$\mathrm{Spec}\left(A \otimes_B \frac{B_\mathfrak{q}}{\mathfrak{q}B_\mathfrak{q}}\right) \cong \coprod_{i=1}^n \mathrm{Spec}\,(R)$$

where R is either 0 or $\kappa(\mathfrak{q})$. Hence $\mathrm{Spec}\,(R)$ is either $\emptyset$ or $\{\text{pt.}\}$. Hence the above is atmost a finite set. ☐

Another nice property enjoyed by finite maps is that they are closed.

Proposition 1.12.3.4. [44] *Let $f : X \to Y$ be a finite morphism. Then f is a closed map.*

is WRONG!

Proof. Let $C \subseteq X$ be a closed set. We wish to show that $f(C)$ is a closed set. Recall that a set is closed if and only if its intersection with any open set is closed in that open set. Consequently, we reduce to showing that the intersection $f(C) \cap V$ where $V \subseteq Y$ is an open affine is closed in V. As $f^{-1}(f(C) \cap V) = C \cap f^{-1}(V)$ and f is finite, therefore we may write $f^{-1}(V) = U = \mathrm{Spec}\,(A)$ where A is a finite B-algebra. Consequently, we reduce to assuming $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ and $C \subseteq X$ given by $C = V(J)$ is a closed set where $J \leq A$ is an ideal. Denote by $\varphi : B \to A$ the corresponding map of rings.

As A is a finite B-algebra, therefore we may write $A \cong B/I_1 \times \cdots \times B/I_n$ where $I_i \leq B$ are ideals. Hence the ideal $J \leq A$ corresponds to $J = \bar{J}_1 \times \cdots \times \bar{J}_n \leq B/I_1 \times \cdots \times B/I_n$ where $J_i \leq B$ is an ideal containing I_i. Now, $C = V(J)$ is the set of prime ideals of A containing J. As $\mathrm{Spec}\,(A) \cong \coprod_{i=1}^n \mathrm{Spec}\,(B/I_i)$, hence C in $\mathrm{Spec}\,(A)$ is given by $\coprod_{i=1}^n V(\bar{J}_i)$. As the

[43] Exercise II.3.5, a) of Hartshorne.
[44] Exercise II.3.5, b) of Hartshorne.

$\varphi : B \to \prod_{i=1}^{n} B/I_i$ takes $b \mapsto (b + I_i)_{i=1,\dots,n}$ is the unique map obtained by projections $B \twoheadrightarrow B/I_i$, therefore

$$\begin{aligned} f(C) &= f\left(\coprod_{i=1}^{n} V(\bar{J}_i)\right) \\ &= \bigcup_{i=1}^{n} f(V(\bar{J}_i)) \\ &= \bigcup_{i=1}^{n} V(J_i) \end{aligned}$$

where $V(J_i)$ is closed in Y. As finite union of closed sets is closed, hence we are through. $\square$

Remark 1.12.3.5. [45] As tempting it might be to say that, but it is not true that a surjective, finite type, quasi-finite map is finite.

Indeed, let k be an algebraically closed field. Consider the map

$$f : \operatorname{Spec}\left(\frac{k[x,y]}{xy-1}\right) \amalg \operatorname{Spec}(k) \longrightarrow \mathbb{A}^1_k$$

induced by $k[x] \to \frac{k[x,y]}{xy-1} \times k$ given by $x \mapsto (x + \langle xy - 1\rangle, 0)$. As a nice exercise, one checks that (we write $a \in \mathbb{A}^1_k$ to mean $\langle x - a\rangle \in \mathbb{A}^1_k$)

1. $f^{-1}(0)$ is a singleton ($\operatorname{Spec}(k)$),
2. $f^{-1}(a)$ is a singleton, given by point (a, a^{-1}) (in particular, the point $\langle x - a, xy - 1\rangle$),
3. the generic fiber $f^{-1}(\mathfrak{o})$ is isomorphic to $\operatorname{Spec}(k(x))$ in $\operatorname{Spec}\left(\frac{k[x,y]}{xy-1}\right)$, hence a singleton.

Consequently, f is surjective, quasi-finite and furthermore of finite type. But still, $\frac{k[x,y]}{xy-1} \times k$ is not a finite $k[x]$-algebra.

Remark 1.12.3.6. Let k be a field. Observe that $\mathbb{A}^1_k \times_k \mathbb{A}^1_k \cong \mathbb{A}^2_k$. However, the underlying set in $\mathbb{A}^2_k$ is not the product of underlying set of $\mathbb{A}^1_k$ with itself. Indeed, this is essentially due to the fact that every prime ideal of $k[x,y]$ is not of form $\mathfrak{p}_1 \times \mathfrak{p}_2$ where $\mathfrak{p}_1 \in k[x]$ and $\mathfrak{p}_2 \in k[y]$, as the prime ideal $xy - 1$ in $k[x,y]$ shows.

Example 1.12.3.7. We give some examples of finite maps.

1. Consider the canonical map $k[t] \to \frac{k[t,x]}{x^n-t}$ and the corresponding map $X = \operatorname{Spec}\left(\frac{k[t,x]}{x^n-t}\right) \to \operatorname{Spec}(k[t]) = \mathbb{A}^1_k$. As $\frac{k[t,x]}{x^n-t}$ is a finite $k[t]$-algebra of rank n, therefore $X \to \mathbb{A}^1_k$ is a finite map. Note that for each closed point $a \in \mathbb{A}^1_k$, the fiber $X_a \cong \operatorname{Spec}\left(\frac{k[x]}{x^n-a}\right)$, which has n closed points if k is algebraically closed and $a \neq 0$.
2. **TODO :** *Add more as you create or find.*

Generic finiteness

Definition 1.12.3.8 (Generically finite map). Let $f : X \to Y$ be a map of schemes such that Y is irreducible. The map f is said to be generically finite if $f^{-1}(\eta)$ for $\eta \in Y$ the generic point is a finite set.

[45] Exercise II.3.5, c) of Hartshorne.

The following is an important result in this regard, which says, like many statements about generic points, that a generically finite dominant map is *almost* like a finite map.

Theorem 1.12.3.9. [46] *Let X, Y be integral schemes and $f : X \to Y$ be a dominant, generically finite and finite type map. Then there exists a dense open $V \subseteq Y$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is a finite map.*

Proof. We first prove this for X and Y affine integral schemes. We will later reduce to this case. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$ be affine schemes where A, B are domains. Let $f : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ be a finite type, dominant, generically finite map so that A is a finite type B-algebra. Let this be induced by a finite type ring homomotphism $\varphi : B \to A$. Our first goal is to show that the generic point of X is mapped to generic point of Y and that the induced map of function fields $K(Y) \hookrightarrow K(X)$ is a finite extension.

Indeed, let $\xi \in X$ and $\eta \in Y$ be the generic point of X and Y respectively. By continuity of f, we have $f(\bar{\xi}) \subseteq \overline{f(\xi)}$. As $\bar{\xi} = X$, we have $f(X) \subseteq \overline{f(\xi)}$. As $f(X)$ is dense in Y by dominance of f, we deduce that $Y \subseteq \overline{f(\xi)}$, that is, $f(\xi)$ is a generic point of Y. As schemes are sober and in our case Y is irreducible, therefore Y has a unique generic point which is η. It follows that $f(\xi) = \eta$. Dominance of f further shows that φ is injective since $\xi = \mathfrak{o} \in f^{-1}(\eta) = f^{-1}(\mathfrak{o})$. As $f^{-1}(\mathfrak{o}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \varphi^{-1}(\mathfrak{p}) = \mathfrak{o}\}$ therefore if $\mathfrak{o} \in f^{-1}(\mathfrak{o})$, then it follows that $\operatorname{Ker}(\varphi) = 0$, that is, φ is injective.

Thus, by considering the comorphism at stalks, we get a map

$$\varphi_{\mathfrak{o}} = f^{\sharp}_{\xi} : \mathcal{O}_{Y, f(\xi)} = K(Y) = Q(B) \longrightarrow \mathcal{O}_{X,\eta} = K(X) = Q(A).$$

Note that this map is the field homomorphism induced by $\varphi : B \hookrightarrow A$ on the fraction fields. As this is a map of fields, therefore $\varphi : K(Y) \to K(X)$ is injective. By replacing $K(Y)$ by the image of φ, we may assume φ is an inclusion. We wish to show that $K(X)/K(Y)$ is a finite extension.

To this end, we first observe the following by generic finiteness. Let $A = B[\alpha_1, \dots, \alpha_n]$. The fiber at η is

$$\begin{aligned} f^{-1}(\eta) &= \operatorname{Spec}(A \otimes_B \kappa(\eta)) \\ &= \operatorname{Spec}(A \otimes_B Q(B)) \\ &= \operatorname{Spec}(B[\alpha_1, \dots, \alpha_n] \otimes_B Q(B)) \\ &= \operatorname{Spec}(Q(B)[\alpha_1, \dots, \alpha_n]) \end{aligned}$$

Thus by generic finiteness, $\operatorname{Spec}(Q(B)[\alpha_1, \dots, \alpha_n])$ is a finite set. We wish to show that it is discrete so that $f^{-1}(\eta)$ is a finite discrete affine scheme, that is, $Q(B)[\alpha_1, \dots, \alpha_n]$ is an artinian ring. It would thus follow that $Q(B)[\alpha_1, \dots, \alpha_n]$ is an artinian finite type $Q(B)$-algebra and is thus a finite $Q(B)$-algebra. Now, finiteness is preserved under going to fraction fields (Lemma 23.6.1.4) thus $Q(Q(B)[\alpha_1, \dots, \alpha_n])$ is a finite extension of $Q(B)$. But $Q(Q(B)[\alpha_1, \dots, \alpha_n]) = Q(A)$. Hence, $Q(A)$ is a finite extension of $Q(B)$, as required. We thus reduce to proving that the finite spectrum $\operatorname{Spec}(Q(B)[\alpha_1, \dots, \alpha_n])$ is discrete. We wish to show that all finitely many points of it are open. To this end it suffices to show that all finitely many primes of $Q(B)[\alpha_1, \dots, \alpha_n]$ are incomparable. **TODO.**

[46] Exercise II.3.7 of Hartshorne.

Thus we have shown that $K(X)/K(Y)$ is a finite extension. Using this, we now find the required open subset $V \subseteq Y$. Indeed, we find a basic open $V = D(b) \subseteq Y$ where $b \in B \subseteq A$ and $f^{-1}(D(b)) = D(b) \subseteq X$ is such that $f : f^{-1}(D(b)) \to D(b)$ is a finite map. That is, we wish to show that there exists $b \in B$ such that $\varphi_b : B_b \hookrightarrow A_b$ is a finite map, using the fact that $Q(B) \hookrightarrow Q(A)$ is a finite extension. Indeed, let $\frac{a_1}{a_1'}, \dots, \frac{a_n}{a_n'}$ be a $Q(B)$-basis of $Q(A)$. Observe that we have

$$Q(A) = Q(B)\frac{a_1}{a_1'} + \cdots + Q(B)\frac{a_n}{a_n'}.$$

Thus, multiplying both sides by a_i', we get that there exists $a_1, \dots, a_N \in A$ such that $Q(A)$ is a $Q(B)$-span of $a_1, \dots, a_N$. Denote $A = B[\alpha_1, \dots, \alpha_n]$. Observe that for any $\alpha_i \in A$, the set $\{1, \alpha_i, \dots, \alpha_i^{N-1}, \alpha_i^N\}$ is linearly dependent as its size is greater than the degree $N = [Q(A) : Q(B)]$. Consequently, we see that every α_i^k for $k \geq N$ is a linear combination of $\{1, \alpha_i, \dots, \alpha_i^{N-1}\}$. Now consider any $0 \leq i_1, \dots, i_n$ and the term $\alpha_1^{i_1} \dots \alpha_n^{i_n}$. Then this can be written as linear combination of various $\alpha_1^{j_1} \dots \alpha_n^{j_n}$ where $0 \leq j_1, \dots, j_n \leq N$.

Thus we have a finite collection of terms $\{\alpha_1^{i_1} \dots \alpha_n^{i_n}\}_{0 \leq i_1, \dots, i_n \leq N-1}$ in A. In $Q(A)$, we thus get the following expression for each of them:

$$\alpha_1^{i_1} \dots \alpha_n^{i_n} = \sum_{k=1}^{N} \frac{b_{i_1 \dots i_N, k}}{b'_{i_1 \dots i_N, k}} a_k$$

where $b_{i_1 \dots i_N, k}, b'_{i_1 \dots i_N, k} \in B$. Collect all the finitely many denominators $\{b'_{i_1 \dots i_N, k}\}_{i_1, \dots, i_N, k}$ and consider their product $b \in B$. We claim that the induced map $\varphi_b : B_b \hookrightarrow A_b$ is a finite map.

Indeed, pick any $\frac{a}{b^p} \in A_b$. Then, $a = \sum c_{i_1 \dots i_n} \alpha_1^{i_1} \dots \alpha_n^{i_n}$ for $c_{i_1 \dots i_n} \in B$. Consequently, we have

$$\begin{aligned}
a &= \sum_{i_1, \dots, i_n} c_{i_1 \dots i_n} \alpha_1^{i_1} \dots \alpha_n^{i_n} \\
&= \sum_{i_1, \dots, i_n} c_{i_1 \dots i_n} \left(\sum_{k=1}^{N} \frac{b_{i_1 \dots i_N, k}}{b'_{i_1 \dots i_N, k}} a_k \right) \\
&= \sum_{k=1}^{N} \underbrace{\left(\sum_{i_1, \dots, i_n} c_{i_1 \dots i_n} \frac{b_{i_1 \dots i_N, k}}{b'_{i_1 \dots i_N, k}} \right)}_{d_k} a_k \\
&= \sum_{k=1}^{N} d_k a_k
\end{aligned}$$

where $d_k \in Q(B)$. Observe that denominator of d_k is some product of elements of $\{b'_{i_1 \dots i_N, k}\}_{i_1, \dots, i_N, k}$. Consequently, we get that in A_b, we will have

$$\frac{a}{b^p} = \sum_{k=1}^{N} \frac{d_k}{b^p} a_k$$

where d_k/b^p is an element of B_b since $d_k \in B_b$. Hence, we have shown that there exists elements $a_1, \dots, a_N \in A$ such that A_b is finite over B_b. This completes the proof for affine case.

TODO : General case. $\square$

1.12.4 Separated

This notion corresponds to the Hausdorff property for topological spaces. Recall that a space X is Hausdorff if and only if the diagonal $\Delta : X \to X \times X$ is closed. We shall mimic this in the category of schemes.

Definition 1.12.4.1. (**Separated**) A map $f : X \to Y$ of schemes is said to be separated if the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion. A scheme X is said to be separated if $X \to \operatorname{Spec}(\mathbb{Z})$ is separated.

It follows that any map of affine schemes is separated.

Lemma 1.12.4.2. *Let $f : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ be a map of affine schemes. Then f is separated.*

Proof. By Corollary 1.3.0.6, f corresponds to a map of rings $\varphi : B \to A$. Similarly, the diagonal map $\Delta : \operatorname{Spec}(A) \to \operatorname{Spec}(A) \times_{\operatorname{Spec}(B)} \operatorname{Spec}(A)$ corresponds to the B-algebra structure map over A, given by $m : A \otimes_B A \to A$, which is surjective. Consequently, by Corollary 1.4.4.14, Δ is a closed immersion. $\square$

Since any scheme locally is affine, we get a nice consequence of the above lemma.

Lemma 1.12.4.3. *Let $f : X \to Y$ be a map of schemes. Then the following are equivalent.*

1. *f is separated.*
2. *The diagonal $\Delta : X \to X \times_Y X$ has closed image.*

Proof. (1. $\Rightarrow$ 2.) Immediate.
(2. $\Rightarrow$ 1.) By the definition of diagonal, it is immediate that $\Delta : X \to X \times_Y X$ is a homeomorphism onto its image, which is further closed by the given hypothesis. Thus, we need only show that $\Delta^\flat : \mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is a surjective map. By Theorem 27.3.0.6, 3, this is a local property. Consequently, we further reduce to showing that for any point $x \in X$ there is an open set $f(x) \in V \subseteq X \times_Y X$ such that $\Delta^\flat_{|V} : \mathcal{O}_{V,f(x)} \to (\Delta_* \mathcal{O}_{\Delta^{-1}(V)})_{f(x)}$ is surjective. Now we may choose by continuity of f a small affine open $x \in U$ such that $f(U)$ is contained in an affine open V in Y. Consequently, $U \times_V U$ is an affine open subset of $X \times_Y X$ containing $f(x)$. We thus reduce to showing that $\mathcal{O}_{U \times_V U, f(x)} \to (\Delta_* \mathcal{O}_U)_x$ is surjective, which follows immediately from Lemma 1.12.4.2. $\square$

Next, we state an important characterization of separatedness which allows us to derive some very important and convenient results about it.

Theorem 1.12.4.4. *(Valuative criterion of separatedness) Let $f : X \to Y$ be a map of schemes where X is noetherian. Then the following are equivalent,*

1. *f is separated.*
2. *Pick any field K and any valuation ring R with fraction field K (see Section 23.10). Let $i : \operatorname{Spec}(K) \to \operatorname{Spec}(R)$ be the map corresponding to $R \hookrightarrow K$. For all $g : \operatorname{Spec}(R) \to Y$ and $h : \operatorname{Spec}(K) \to X$ such that the square commutes, there exists atmost one lift of g along f as to make the following diagram commute:*

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "h"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}.$$

Proof. See Theorem 4.3, Chapter 2 of cite[Hartshorne]. □

The following important corollaries can now easily be derived from this characterization.

Corollary 1.12.4.5. *Let us work in the category of noetherian schemes. Then,*

1. *separated maps are stable under base extension,*
2. *open and closed immersions are separated*[47]*,*
3. *composition of separated maps is separated,*
4. *for a base scheme S, product of any two separated maps is separated in $\mathbf{Sch}/S$,*
5. *if the composite $X \to Y \to Z$ is separated, then $X \to Y$ is separated,*
6. *a map $f : X \to Y$ is separated if and only if there is an open cover V_i of Y such that the restricted maps $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is separated*[48]*.*

Proof. **TODO: From notebook.** □

An important result about separate schemes is that the intersection of any two open affines is again an open affine.

Lemma 1.12.4.6. [49] *Let X be a separated scheme. If $U, V \subseteq X$ are two open affines then $U \cap V$ is again an open affine.*

Proof. Let $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$. We may replace X by $U \cup V$ and X would still be separated by Corollary 1.12.4.5, 6. Now let $W = U \cap V$. Then again by Corollary 1.12.4.5, 6, we have that W is separated. Consequently, we get that $\Delta : W \to W \times_{\mathbb{Z}} W$ is a closed immersion. We now claim that $W \times_{\mathbb{Z}} W \cong U \times_{\mathbb{Z}} V$. Indeed, this follows immediately from the universal property of fiber product. It follows that $\Delta : W \to \operatorname{Spec}(A \otimes_{\mathbb{Z}} B)$ is a closed immersion. By Corollary 1.4.4.14, W is spectrum of a quotient of $A \otimes_{\mathbb{Z}} B$. Consequently, W is affine, as needed. □

Separatedness of projective schemes

We next see that any projective scheme is separated.

Lemma 1.12.4.7. *Let S be a graded ring. Then,* $\operatorname{Proj}(S) \to \operatorname{Spec}(\mathbb{Z})$ *is separated.*

Proof. We need only check that the diagonal $\Delta : \operatorname{Proj}(S) \to \operatorname{Proj}(S) \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Proj}(S)$ has closed image (Lemma 1.12.4.3). Since one can check a closed set locally and sets of the form $D_+(f) \times D_+(g)$ forms an open cover of $\operatorname{Proj}(S) \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Proj}(S)$ for $f, g \in S_+$ homogeneous, therefore we reduce to checking that for $C = \Delta^{-1}(D_+(f) \times D_+(g))$, the restriction $\Delta|_C : C \to D_+(f) \times D_+(g)$ has closed image.

Since $C = D_+(fg) \cong \operatorname{Spec}\left(S_{(fg)}\right)$ and $D_+(f) \times D_+(g) \cong \operatorname{Spec}\left(S_{(f)} \otimes_{\mathbb{Z}} S_{(g)}\right)$, therefore we reduce to showing that the induced map $S_{(f)} \otimes_{\mathbb{Z}} S_{(g)} \to S_{(fg)}$ is surjective. This is clear,

[47] in-fact, any topological immersion is separated, as is clear from the proof.
[48] This doesn't require the noetherian hypothesis.
[49] Exercise II.4.3 of Hartshorne.

as for any $u/f^n g^n \in S_{(fg)}$ where let us denote $k = \deg f, l = \deg g$, for any m large enough such that all exponents in the below are positive, we obtain that

$$\frac{ug^{mk-n}}{f^{ml+n}} \otimes \frac{f^{ml}}{g^{mk}} \mapsto \frac{u}{f^n g^n}.$$

Thus, the image of Δ is closed[50]. $\square$

Uniqueness of centers of valuations for varieties

We show a curious property for abstract varieties that any valuation defined over its function field has a unique *center*, if it exists[51]. See Definition 1.4.2.9 for definition of center points of a valuation over function field of an integral scheme.

Lemma 1.12.4.8. [52] *Let X be an integral scheme of finite type over k with function field K. If X is separated, then any valuation over K has a unique center if it exists.*

Proof. We will use the valuative criterion for this. Let $v : K \to G$ be a valuation over K with valuation ring $R \subseteq K$. Let $x, y \in X$ be two centers of v. As $K \subseteq K$, therefore by Lemma 1.6.1.1, 3, there exists a unique map $\operatorname{Spec}(K) \to X$ mapping $\star \mapsto \eta$, where η is the generic point of X. It follows that we have the following commutative square

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(R) \arrow[r] & \operatorname{Spec}(\mathbb{Z}) \end{tikzcd} . \tag{*}$$

As R is a local ring, therefore by Lemma 01J6 of StacksProject, we have a bijection between maps $\operatorname{Spec}(R) \to X$ and tuples (z, φ) where $z \in X$ and $\varphi : \mathcal{O}_{X,z} \to R$ is a local ring homomorphism. Consequently, as $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ are dominated by R, we obtain two tuples (x, ι_x) and (y, ι_y) where $\iota_x : \mathcal{O}_{X,x} \hookrightarrow R$ and $\iota_y : \mathcal{O}_{X,y} \hookrightarrow R$ are the two domination maps. Note that by the definition of domination, these two maps are local ring homomorphisms. Consequently, we get two maps $\operatorname{Spec}(R) \to X$ which makes the (*) commute. By the valuative crieterion of Theorem 1.12.4.4, the two maps $\operatorname{Spec}(R) \to X$ are same, and thus so are the tuples (x, ι_x) and (y, ι_y), proving that $x = y$. $\square$

[50]in-fact we have also shown in the process that Δ is a closed immersion, thus we may not use Lemma 1.12.4.3

[51]It will always exist (and thus be unique) if the variety is proper, as is shown in the next section.

[52]Exericse II.4.5 a) of Hartshorne.

1.12.5 Affine morphisms and global Spec

In this section, we cover important global generalization of $\mathrm{Spec}\,(-)$. In particular, let X be a scheme and $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-algebra, that is, an $\mathcal{O}_X$-module which is a sheaf of rings as well. Then we will construct a scheme $\mathbf{Spec}(\mathcal{F})$ over X which will behave as if it is constructed out of open affine subschemes U of X and the corresponding algebras $\mathcal{F}(U)$.

This construction will be used to show how locally free sheaf of constant rank actually corresponds to vector bundles. They are used elsewhere as well.

Definition 1.12.5.1 (**Affine morphism**)**.** A map $f : X \to Y$ of schemes is called an affine morphism if there is an affine open cover $\{V_\alpha\}$ of Y such that $f^{-1}(V_\alpha)$ is an open affine scheme.

Remark 1.12.5.2. It follows from definition that any finite morphism is affine.

The first major property of affine maps is that they are local on target.

Proposition 1.12.5.3. *Let $f : X \to Y$ be a map. Then the following are equivalent.*

1. *f is affine.*
2. *For any open affine $V \subseteq Y$, $f^{-1}(V)$ is an open affine in X.*

Proof. We need only do $1 \Rightarrow 2$. This has been done in the proof of Proposition 1.12.3.2. □

Lemma 1.12.5.4. *Let $f : X \to Y$ be an affine morphism. Then f is quasicompact and separated.*

Proof. The fact that f is quasicompact is immediate by definition. Separatedness follows from Corollary 1.12.4.5, 6 and Lemma 1.12.4.2. □

The main theorem for affine maps is that they all come from quasicoherent algebras over the structure sheaf. Indeed, we have the following construction to obtain a scheme over Y by a quasicoherent $\mathcal{O}_Y$-algebra.

Theorem 1.12.5.5. [53] *Let Y be a scheme and $\mathcal{A}$ be a quasicoherent $\mathcal{O}_Y$-algebra over Y. Then there exists a scheme*

$$f : \mathbf{Spec}(\mathcal{A}) \to Y$$

unique with respect to the property that for any open affine $V \subseteq Y$, we have $f^{-1}(V) \cong$ $\mathrm{Spec}\,(\mathcal{A}(V))$ and for any inclusion $U \hookrightarrow V$ of open affines, the map $f^{-1}(U) \to f^{-1}(V)$ is induced by the restriction map $\rho : \mathcal{A}(V) \to \mathcal{A}(U)$.

Proof. Let $V = \mathrm{Spec}\,(B) \subseteq Y$ be an open affine in Y. Then we have a ring homomorphism $B \to \mathcal{A}(V)$ as $\mathcal{A}(V)$ is a B-algebra. Consequently, we get the map $\pi_V : \mathrm{Spec}\,(\mathcal{A}(V)) \to Y$ factoring through V. Observe that for any open affine $U \hookrightarrow V$, we have the following commutative triangle

$$\begin{tikzcd} \mathcal{A}(V) \arrow[dr, "\rho"] & \\ B \arrow[u] \arrow[r] & \mathcal{A}(U) \end{tikzcd}.$$

[53] Exercise II.5.17, c) of Harthsorne.

We now wish to glue the affine schemes $\pi_i : \mathrm{Spec}\,(\mathcal{A}(V_i)) \to Y$ where V_i varies over open affines of Y. Indeed, let $X_i = \mathrm{Spec}\,(\mathcal{A}(V_i))$ and $U_{ij} = \pi_i^{-1}(V_i \cap V_j)$ an open subscheme of X_i. We claim that there is a natural isomorphism $\varphi_{ij} : U_{ij} \to U_{ji}$ which satisfies the cocycle condition, so that we can glue these schemes together by Proposition 1.6.2.2 to get the desired scheme unique with the given properties. Indeed, to find φ_{ij}, we first observe that for $\pi_V : \mathrm{Spec}\,(\mathcal{A}(V)) \to Y$, we have that $\pi_{V*}\mathcal{O}_{\mathrm{Spec}(\mathcal{A}(V))} \cong \mathcal{A}_{|V}$. This is where quasicoherence is used and follows from checking on basis and using globalized restriction of scalars (Lemma 1.2.3.4). Using this isomorphism, we see that $\mathcal{O}_{X_i}(\pi_i^{-1}(V_i \cap V_j)) \cong \mathcal{A}(V_i \cap V_j) \cong \mathcal{O}_{X_j}(\pi_j^{-1}(V_j \cap V_i))$. Consequently, we have a commutative triangle where $V_i = \mathrm{Spec}\,(B_i)$

$$\begin{tikzcd}
\mathcal{A}(V_i) \arrow[dr, "\rho"] & \\
B_i \arrow[u] \arrow[r] & \mathcal{O}_{X_j}(\pi_j^{-1}(V_i \cap V_j))
\end{tikzcd}.$$

By Theorem 1.3.0.5, we get the following commutative triangle

$$\begin{tikzcd}
X_i \arrow[d, "\pi_i"'] & \\
V_i & \pi_j^{-1}(V_i \cap V_j) \arrow[l] \arrow[ul, "\varphi_{ji}"']
\end{tikzcd}$$

By commutativity of this triangle, it follows that the unique morphism φ_{ji} factors through $\pi_i^{-1}(V_i \cap V_j)$. Interchanging i and j we get that φ_{ji} is an isomorphism. By uniqueness of φ_{ij}, we further get the the cocycle condition, as required. $\square$

We see from the proof that

Corollary 1.12.5.6. *Let Y be a scheme, $\mathcal{A}$ a quasicoherent $\mathcal{O}_Y$-algebra and $f : \mathbf{Spec}(\mathcal{A}) \to Y$ the global spec. Then, $f_*\mathcal{O}_{\mathbf{Spec}(\mathcal{A})} \cong \mathcal{A}$.*

Proof. In the proof, we showed that for any open affine $V \subseteq Y$, we have $f_*\mathcal{O}_{\mathbf{Spec}(\mathcal{A})|f^{-1}(V)} \cong \pi_{V*}\mathcal{O}_{\mathrm{Spec}(\mathcal{A}(V))} \cong \mathcal{A}_{|V}$ and this isomorphism is compatible with restrictions. Consequently, we have an isomorphism between $f_*\mathcal{O}_{\mathbf{Spec}(\mathcal{A})}$ and $\mathcal{A}$ over a base, which gives the required isomorphism as sheaves over Y. $\square$

It is immediate to see by above theorem that global spec is always affine over the base.

Corollary 1.12.5.7. *Let Y be a scheme and $\mathcal{A}$ a quasicoherent $\mathcal{O}_Y$-algebra. Then the morphism*

$$f : \mathbf{Spec}(\mathcal{A}) \to Y$$

is affine. $\square$

We now prove the converse of the above corollary.

Proposition 1.12.5.8. *Let $f : X \to Y$ be an affine morphism. Then,*

1. *$f_*\mathcal{O}_X$ is a quasicoherent $\mathcal{O}_Y$-algebra,*

2. there is an isomorphism

$$X \cong \mathbf{Spec}(f_*\mathcal{O}_X).$$

Proof. 1. This is immediate from the fact that the morphism f is quasicompact and separated by Lemma 1.12.5.4 (Lemma 1.9.1.16).

2. Let $\{V_\alpha\}$ be a basis consisting of open affines of Y. Then, $\{f^{-1}(V_\alpha)\}$ is an open affine basis of X by Proposition 1.12.5.3. Then, we have a canonical isomorphism $f^{-1}(V_\alpha) \cong \operatorname{Spec}\left(\mathcal{O}_X(f^{-1}(V_\alpha))\right)$. Moreover, for $V_\alpha \hookrightarrow V_\beta$, we have $f^{-1}(V_\alpha) \hookrightarrow f^{-1}(V_\beta)$ obtained by restriction $\rho : \mathcal{O}_X(f^{-1}(V_\beta)) \to \mathcal{O}_X(f^{-1}(V_\alpha))$. Hence by uniqueness of Theorem 1.12.5.5, we conclude the proof. $\square$

We may sum this up in the following bijection.

Corollary 1.12.5.9. [54] *Let Y be a scheme. We have the following bijection*

$$\left\{\text{Affine morphisms } X \xrightarrow{f} Y\right\} \cong \begin{Bmatrix}\text{Quasicoherent} & \mathcal{O}_Y\text{-} \\ \text{algebras } \mathcal{A} & \end{Bmatrix}$$

established by $f \mapsto f_*\mathcal{O}_X$ and $\mathbf{Spec}(\mathcal{A}) \leftarrow \mathcal{A}$. $\square$

[54] Exercise II.5.17, d) of Hartshorne.

1.12.6 Proper

This and the next section brings us closer to detecting when a scheme is projective (i.e. is a subscheme of projective scheme). Proper maps corresponds roughly to the intuition that the scheme $X \to Y$ should not have any *missing points*.

Definition 1.12.6.1. (**Universally closed and proper maps**) A map $f : X \to Y$ is said to be universally closed if f is closed and for any base extension $Y' \to Y$, the base extension of X, denoted $f' : X' \to Y'$ is also closed as in the diagram below:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}.$$

Consequently, f is said to be proper if it is separated, finite type and is universally closed.

The main result here is again a valuative criterion which allows a lot of properties of such maps to be derived quite easily.

Theorem 1.12.6.2. *(Valuative criterion of properness) Let $f : X \to Y$ be a finite type map of schemes where X is noetherian. Then the following are equivalent.*

1. *f is proper.*
2. *Pick any field K and any valuation ring R with fraction field K (see Section 23.10). Let $i : \mathrm{Spec}\,(K) \to \mathrm{Spec}\,(R)$ be the map corresponding to $R \hookrightarrow K$. For all $g : \mathrm{Spec}\,(R) \to Y$ and $h : \mathrm{Spec}\,(K) \to X$ such that the square commutes, there exists a unique lift of g along f as to make the following diagram commute:*

$$\begin{tikzcd} \mathrm{Spec}\,(K) \arrow[r, "h"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \mathrm{Spec}\,(R) \arrow[r, "g"'] \arrow[ur, dashed, "!"] & Y \end{tikzcd}.$$

Note that whereas in Theorem 1.12.4.4 we had that there exists *atmost one* lift, here we have that there exists *unique* lift (it exists and there is only one).

Corollary 1.12.6.3. *Let us work in the category of noetherian schemes. Then,*

1. *if $X \to Y \to Z$ is proper and $Y \to Z$ is separated, then $X \to Y$ is proper,*
2. *closed immersion are proper,*
3. *proper maps are stable under base extensions,*
4. *composite of proper maps is proper,*
5. *for two proper schemes $X \to S, Y \to S$ in $\mathbf{Sch}/S$, their product $X \times_S Y \to S$ is proper,*
6. *a map $f : X \to Y$ is proper if and only if there exists an open cover V_i of Y such that the restriction $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is proper.*

Proof. **TODO : From notebook.** $\square$

1.12.7 Projective

We now define maps of schemes which factors through a projective space over the target. This will be fundamental, as the most natural type of schemes we find in nature are projective varieties appearing as closed subschemes of the projective space over a field. Though we will work more generally, but this will pay off in some of the later discussions. See Definition 1.8.2.14 for projective spaces over a scheme.

Definition 1.12.7.1. (**Projective and quasi-projective maps**) Let $f : X \to Y$ be a map of schemes. We say f is projective if there exists an $n \in \mathbb{N}$ such that f factors as a closed immersion $X \to \mathbb{P}^n_Y$ followed by the struture map $\mathbb{P}^n_Y \to Y$ as in

$$\begin{tikzcd} X \arrow[r, "\text{cl. imm.}"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd} .$$

Further, a map $f : X \to Y$ is said to be *quasi-projective* if f factors first into an open immersion $X \to X'$ and then a projective map $X' \to Y$ as in

$$\begin{tikzcd} X' \arrow[r, "\text{cl. imm.}"] & \mathbb{P}^n_Y \arrow[d] \\ X \arrow[u, "\text{op. imm.}"] \arrow[r, "f"'] & Y \end{tikzcd} .$$

Thus quasi-projective maps corresponds to the usual notion of quasi-projective varieties (open subsets of projective varieties in a projective n-space).

The important point to keep in mind about projective maps is that they are proper.

Theorem 1.12.7.2. *Let X and Y be noetherian schemes.*
1. *If $f : X \to Y$ is projective, then f is proper.*
2. *If $f : X \to Y$ is quasi-projective, then f is finite type and separated.*

Proof. 1. Since closed immersions are proper and proper maps are stable under base change (Corollary 1.12.6.3), we may reduce to showing that for each $n \in \mathbb{N}$, the scheme $\mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}\,(\mathbb{Z})$ is proper. It is clear that $\mathbb{P}^n_{\mathbb{Z}}$ is finite type $\mathbb{Z}$-scheme which is furthermore separated by Corollary 1.12.4.7.

In order to show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper, we will proceed by induction over n. For $n = 0$, we have $\mathbb{P}^0_{\mathbb{Z}} \cong \mathrm{Spec}\,(\mathbb{Z})$, which is trivially proper over $\mathrm{Spec}\,(\mathbb{Z})$. Now suppose $\mathbb{P}^{n-1}_{\mathbb{Z}}$ is proper over $\mathrm{Spec}\,(\mathbb{Z})$. We wish to show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper. We will use valuative criterion for this (Theorem 1.12.6.2). Consider a valuation ring R with fraction field K such that we have maps g, h making the following commute:

$$\begin{tikzcd} \mathrm{Spec}\,(K) \arrow[r, "h"] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ \mathrm{Spec}\,(R) \arrow[r, "g"'] & \mathrm{Spec}\,(\mathbb{Z}) \end{tikzcd} .$$

Consequently, we wish to define a unique map $\mathrm{Spec}(R) \to \mathbb{P}^n_{\mathbb{Z}}$ which makes everything commute.

Denote $\mathrm{Spec}(K) = \{\star\}$ and $\xi = h(\star) \in \mathbb{P}^n_{\mathbb{Z}}$. We now observe that if $\xi \in V(x_{i_0})$ for any $i_0 = 0, \dots, n$, then by the natural isomorphism $V(x_{i_0}) \cong \mathbb{P}^{n-1}_{\mathbb{Z}}$ and obvious restrictions, we get the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "h"] \arrow[d] & \mathbb{P}^{n-1}_{\mathbb{Z}} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r, "g"'] & \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}.$$

Consequently, by inductive hypothesis, we have a unique lift $\mathrm{Spec}(R) \to \mathbb{P}^{n-1}_{\mathbb{Z}}$ and thus a map $\mathrm{Spec}(R) \to \mathbb{P}^n_{\mathbb{Z}}$ making the diagram commutative. This is sufficient by the fact that $\mathbb{P}^n_{\mathbb{Z}}$ is separated (Lemma 1.12.4.7) and by valuative criterion (Theorem 1.12.4.4).

We next need to cover the case when ξ is not in any hyperplane $V(x_i)$, that is, when $\xi \in \bigcap_{i=0}^n D_+(x_i)$. We will construct a map $\mathrm{Spec}(R) \to \mathbb{P}^n_{\mathbb{Z}}$ which makes everything commute and we will be done by separatedness of $\mathbb{P}^n_{\mathbb{Z}}$ (Lemma 1.12.4.7). In this case, we obtain that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},\xi} \cong \mathbb{Z}[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i]$ for all $i = 0, \dots, n$ as $D_+(x_i) \cong \mathrm{Spec}\left(\mathbb{Z}[x_0, \dots, x_n]_{(x_i)}\right)$, (Lemma 1.8.2.4). Consequently, $x_i/x_j \in \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},\xi}$ is invertible for all $i, j = 0, \dots, n$, hence $x_i/x_j \notin \mathfrak{m}_{\mathbb{P}^n_{\mathbb{Z}},\xi}$. Denote further $f_{ij} \in \kappa(\xi)$ to be the image of x_i/x_j under the map $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},\xi} \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},\xi}/\mathfrak{m}_{\mathbb{P}^n_{\mathbb{Z}},\xi} = \kappa(\xi)$.

The map $h : \mathrm{Spec}(K) \to \mathbb{P}^n_{\mathbb{Z}}$ is equivalent to the data of the point $\xi \in \mathbb{P}^n_{\mathbb{Z}}$ and $\kappa(\xi) \hookrightarrow K$ (Lemma 1.6.1.1). Thus we have $f_{ij} \in K$ for all $i, j = 0, \dots, n$. In order to define the map $j : \mathrm{Spec}(R) \to \mathbb{P}^n_{\mathbb{Z}}$ in this case, it is sufficient to obtain a map $\mathbb{Z}[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i] \to R$ such that the following commutes:

$$\begin{tikzcd} K & Z[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i] \arrow[l] \arrow[dl] \\ R \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u] \end{tikzcd}.$$

We will now construct such a map. Let $v : K \to G$ be the valuation corresponding to the valuation ring R (so that R is the value ring of v), where G is a totally ordered abelian group. Consider the collection of elements $f_{10}, \dots, f_{n0} \in K$ and denote $g_i = v(f_{i0}) \in G$. Let $g_m = \min_i g_i$. Consequently, for each $i = 0, \dots, n$ we obtain $0 \le g_i - g_m = v(f_{i0}) - v(f_{m0}) = v(f_{i0}f_{0m}) = v(f_{im})$. Thus, $f_{im} \in R$. Hence, we can construct the following map:

$$\begin{aligned} Z[x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i] &\longrightarrow R \\ \frac{x_i}{x_m} &\longmapsto f_{im}. \end{aligned}$$

It is immediate that the above map makes the above diagram commute.
2. Since open immersions are separated (Corollary 1.12.4.5) and an open immersion $X \to X'$

where X is noetherian is immediately quasicompact, so by Proposition 1.12.2.5, 2, the result follows. $\square$

1.12.8 Flat

Look at the following MO post for more clarifications. Flat maps of schemes capture the notion of a "continuous family of schemes parameterized by points of a base scheme". However, the notion of flatness is very algebraic, as we shall soon see. We collect the properties of flat modules in the Special Topics, Chapter 23.

1.12.9 Smooth

1.12.10 Unramified

1.12.11 Étale

Étale maps is the place from where one enters the land of algebraic topology via algebraic geometry. Indeed, the fundamental goal here is to capture the notion of local isomorphism but in an algebraic context. The simplest place where one can understand them is a restricted version of this called finite étale maps. This is where we begin from as we shall need this in our discussion of Galois theory of schemes.

Finite étale

We refer to Algebra, Chapter 23 for background on separable algebras, in particular, to Definition 23.22.2.2 for free separable algebras.

We now define finite étale maps.

Definition 1.12.11.1. (**Finite étale scheme**) Let X be a base scheme. An X-scheme $p : Y \to X$ is said to be finite étale if there is an open affine covering of X given by $\{\text{Spec}\,(A_i)\}_{i \in I}$ such that $p^{-1}(\text{Spec}\,(A_i))$ is an open affine subscheme of Y given by $\text{Spec}\,(B_i)$ such that the induced map $A_i \to B_i$ makes B_i a free separable A_i-algebra, for all $i \in I$. In such a situation, one calls Y a finite étale covering of X. Denote the category $\mathbf{Et}_{\text{fin}}(X)$ to be the full subcategory of $\mathbf{Sch}/X$ consisting of finite étale coverings of X.

Let us now give an example of finite étale scheme.

Example 1.12.11.2.

1.13 Coherent and quasicoherent sheaf cohomology

All schemes in this section are Noetherian. Cohomology will serve as an important tool to derive invariants on a given scheme. We would need the cohomology of abelian sheaves over a space (Chapter 27) and the notion of Noetherian schemes (Section 1.4) in this section. *Apart from Mumford, you may like to give Hida a visit.*

We refer to Topics in Sheaf Theory, Chapter 27 for classical Čech cohomology, derived functor cohomology and relations between them on topological spaces.

Since we are only dealing with noetherian schemes and the most important such schemes would be those which are closed subvarieties of projective space, so finite dimensional, therefore the following theorem of Grothendieck is of particular importance.

Theorem 1.13.0.1 (Grothendieck)**.** *Let X be a noetherian topological space of dimension n and $\mathcal{F}$ be an abelian sheaf over X. Then*

$$H^i(X, \mathcal{F}) = 0$$

for all $i > n$.

We now show some basic theorems in cohomology of sheaves over schemes which allows us to use Čech-cohomology for calculations instead of derived functor cohomology, because both becomes isomorphic.

1.13.1 Quasicoherent sheaf cohomology

Do from Hartshorne and Bruzzo.

1.13.2 Application : Serre-Grothendieck duality

Do from Hida and Hartshorne.

1.13.3 Application : Riemann-Roch theorem for curves

Do from Hida and Hartshorne.

Chapter 2

Varieties over an algebraically closed field

We develop here results in the theory of affine and projective varieties using the language of schemes and cohomology as developed in Chapter 1. We begin by discussing the important assumptions on the schemes that we will use in this chapter. A lot of claims below follows from results of Chapter 1, but we don't mention that, leaving it as an exercise for the pedantic. Reading this simultaneously with Chapter 17 might be beneficial.

Notation 2.0.0.1. Here are our conventions for this chapter.

1. The base field k will be algebraically closed unless otherwise stated.
2. An *affine algebraic variety X over k* for us is Spec (A) where A is a finitely generated reduced k-domain.
3. An *algebraic variety over k* for us is a noetherian separated finite type integral scheme over k.
4. A *point in an algebraic variety* is a closed point unless otherwise stated.

Remark 2.0.0.2. There are salient features of our definition of affine algebraic varieties and algebraic varieties:

1. An affine algebraic variety is an abstract variety over k in the sense of Definition 1.5.4.11 as it is separated (because affine), integral (reduced as A has no nilpotents and irreducible as A is a domain), finite type (A is finitely generated) k-scheme.
2. The category of affine algebraic varieties is dual to the category of finitely generated reduced k-domains by Corollary 1.3.0.6.
3. An algebraic variety over k is thus equivalent to the data of a quasi-compact irreducible locally ringed space with a finite open cover by affine algebraic varieties over k. We will mostly think of this while working with an algebraic variety over k.
4. Every open subset U of an algebraic variety X over k is an algebraic variety over k. Indeed, any open subset of an irreducible space is irreducible and thus quasi-compact since X is. As any basic open subset of Spec (A) where A is a finitely generated reduced k-domain is Spec (A_f) for $f \in A$ and A_f is also a finitely generated reduced k-domain, therefore U is also an algebraic variety over k.
5. Any point in an algebraic variety over k has its residue extension a finite (equivalently algebraic) extension of k. Indeed this is what Proposition 1.6.1.4 says together with

essential nullstellensatz ().

Example 2.0.0.3. Note that $X = (\mathbb{A}^2_k - V(x)) \cup \{0\}$ is not an algebraic variety over k as it is not quasi-compact since any basic open set $D(f)$ containing 0 (point 0, so the maximal ideal $\langle x-0, y-0\rangle$) for $f \in k[x,y]$ has to intersect the y-axis at some $(0,t)$, otherwise if $\langle x, y-t\rangle \notin D(f)$ for all $t \neq 0$, then $f \in \langle x, y-t\rangle$ for all $t \neq 0$ and thus $f(0,y)$ has infinitely many zeroes as a polynomial in $k[y]$, a contradiction.

Thus, no open affine subvariety contains 0 and not any other point in y-axis, thus X cannot be covered affine open subvarieties and hence is not an algebraic variety over k.

We next fix our conventions for projective spaces $\mathbb{P}^n$.

1. Denote by $\mathbb{A}^{n+1}_k$ to be the subspace of closed points of $\operatorname{Spec}(k[x_0, \dots, x_n])$. Consider the topological space obtained by quotienting $\mathbb{A}^{n+1}_k - \{0\}$ via the equivalence relation $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$ for $\lambda \in K^\times$. The topological space $\mathbb{A}^{n+1}_k - \{0\}/\sim$ is defined to be the *projective n-space* $\mathbb{P}^n_k$. The sheaf $\mathcal{O}_{\mathbb{P}^n_k}$ of regular functions is as defined in Definition 1.5.2.2.
2. Projective n-space is an algebraic variety as we have a finite open covering by affine varieties. Indeed, consider the subspace
$$U_i = \{[x_0 : \dots : x_n] \mid x_i \neq 0\}$$
for $0 \leq i \leq n$. These are open subsets of $\mathbb{P}^n_k$ since as their inverse image under the quotient map $\mathbb{A}^{n+1}_k - \{0\} \twoheadrightarrow \mathbb{P}^n_k$ yields the subset $\{(x_0, \dots, x_n) \mid x_i \neq 0\}$, which is open.
Moreover, U_i is isomorphic to the closed point of an affine algebraic variety as shown below:
$$\begin{aligned} U_i &\longrightarrow \operatorname{mSpec}(k[x_0, \dots, \hat{x}_i, \dots, x_n]) \\ [a_0 : \dots : 1 : \dots : a_n] &\longmapsto \langle x_0 - a_0, \dots, \widehat{x_i - 1}, \dots, x_n - a_n\rangle. \end{aligned}$$
This is evident from the proof of Proposition 1.5.1.8.
3. We will later prove that projective variety $\mathbb{P}^n_k$ defined above over an algebraically closed field k is exactly the subspace of closed points of $\operatorname{Proj}(k[x_0, \dots, x_n])$. In this section, whenever we say a projective variety, we always mean the subspace of closed points of proj, or equivalently, $\mathbb{P}^n_k$ as defined above.

Chapter 3

Elliptic Curves

Start writing from Hida's Geometric modular forms and elliptic curves. Need to complete divisors, Riemann-Roch, . Main topics are : definition and basic properties, GMF of level 1, A bit of classical Weierstrass theory of elliptic curves over $\mathbb{C}$, Put background on p-adic fields and then you may cover some bit of elliptic curves over them. The important part is level structure of elliptic curves, L-functions and moduli problems. For deformation, Drinfeld's theorem might do for now.

Chapter 4

Étale topology

See Bhatt's notes on the topic and Stacksproject.

After all this time, we finally do Galois theory of schemes...

One of the goals of this chapter is to prove the following theorem.

Theorem 4.0.0.1. *Let X be a connected scheme and let $\mathbf{Et}_{\text{fin}}(X)$ be the category of finite étale coverings of X and $\pi(X)$-$\mathbf{Set}$ be the category of sets with $\pi(X)$-action where $\pi(X)$ is the étale fundamental group of X. Then,*

$$\mathbf{Et}_{\text{fin}}(X) \equiv \pi(X)\text{-}\mathbf{Set}.$$

This is a statement about general (connected) schemes X. When we instantiate X into some special cases, we will, to our surprise, recover the classical Galois theories of fields, of commutative algebras and of infinite dimensional commutative algebras. Hence, the importance of this theorem is paramount.

We will use the results and definitions of Chapter 1 quite freely. One of the final goal of this chapter is to see what the the Theorem 4.0.0.1 says about an elliptic curve. Hence we would be computing the étale fundamental group of an elliptic curve as an example.

Chapter 5

Deformation Theory

In this chapter, we work out the classical deformation theory of schemes.

Chapter 6

Algebraic Geometry

Contents

We would like to discuss and give an overview of some distinguished topics of interest in contemporary algebraic geometry.

6.1 Functor of points

Cover in french from Demazure-Gabriel.

6.2 Analytification and GAGA

Cover from Neeman.

6.3 Intersection theory

Let X be an algebraic variety over an algebraically closed field k. Suppose $U, V \subseteq X$ are irreducible subvarieties of X. Suppose that, set theoretically, $U \cap V \neq \emptyset$. Then, for an element $x \in U \cap V$, one can ask whether the varieties U and V intersect at x only once, or twice, or thrice or.... This is a very intuition specific question, for example the tangent to parabola $y - x^2$ at $(0, 0)$ might seem to meet it only at one point, but as soon we perturb the tangent line, we see that the single point suddenly devolves into two points on the parabola. One may then intuitively say then that $(0, 0) \in (y - x^2) \cap (y)$ has *intersection multiplicity*

2. The goal of this section is to illuminate a definition of intersection multiplicity that will be helpful in elucidating the above intuition.

6.3.1 Intersection in projective space

Do from Hartshorne Chapter 1, Section 7.

6.4 Overview of K-theory of schemes

Do several exercises from Chapter 2 and Chapter 3 of Hartshorne on Grothendieck group K_0 and K_1.

6.4.1 Overview

6.5 The Riemann-Hilbert correspondence

6.6 Serre's intersection formula

Part II

The Arithmetic Viewpoint

Chapter 7

Foundational Arithmetic

Contents

In this foundational chapter, we will discuss some topics in classical number theory.

7.1 Fundamental properties of $\mathbb{Z}$

We wish to see the following list of properties of integers, all of which are immediate, but good to keep in mind.

Theorem 7.1.0.1. *Consider the ring of integers $\mathbb{Z}$. Then, $\mathbb{Z}$ is*

1. *an Euclidean domain,*
2. *a gcd domain,*
3. *a principal ideal domain,*
4. *a unique factorization domain,*
5. *a noetherian ring,*
6. *a normal domain,*
7. *a dimension 1 ring.*

7.2 Algebraic number fields

The main objects of study in number theory are number fields. We will define them and will discuss some of their basic properties, before doing a more involved study of real and imaginary quadratic number fields.

Definition 7.2.0.1 (Algebraic number fields and ring of integers). A field K is an algebraic number field if it is a finite extension of $\mathbb{Q}$. The ring of integers or the integral ring of an algebraic number field K is the integral closure of the inclusion $\mathbb{Z} \hookrightarrow K$, and is denoted by $\mathcal{O}_K$.

Remark 7.2.0.2. The Proposition 23.7.1.9 guarantees that $\mathcal{O}_K$ is indeed a ring. Observe that $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$ as $\mathbb{Z}$ is a normal domain. Indeed, this can be generalized.

It is clear that every algebraic number field is of characteristic 0, as it contains $\mathbb{Q}$. Some of the first (but important) properties of $\mathcal{O}_K$ are as follows.

Proposition 7.2.0.3. *Let K be an algebraic number field.*
1. *The integral ring $\mathcal{O}_K$ is a domain.*
2. *Let R be a normal domain and K be its fraction field. Then $\mathcal{O}_K$ is a subring of R.*
3. *The fraction field of $\mathcal{O}_K$ is K.*
4. *We have $K = \mathbb{Q} \cdot \mathcal{O}_K$.*
5. *The integral ring $\mathcal{O}_K$ is a normal domain.*
6. *The integral ring $\mathcal{O}_K$ is of dimension 1.*
7. *The integral ring $\mathcal{O}_K$ is noetherian.*

Proof. 1. As $\mathcal{O}_K \subseteq K$, therefore it has no zero-divisors.
2. As R is a normal domain, therefore its integral closure in K is R itself. Thus, $\mathcal{O}_K \subseteq R$.
3. Pick any $s \in K$. As K is algebraic over $\mathbb{Q}$, therefore there exists $p(x) \in \mathbb{Q}[x]$ such that $p(s) = 0$ in K. Multiplying by common denominators of coefficients of p, it follows that

$$d_n s^n + d_{n-1}s^{n-1} + \cdots + d_1 s + d_0 = 0$$

in K where $d_i \in \mathbb{Z}$. Multiplying this by d_n^{n-1} and writing $t = sd_n$, we get

$$t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0 = 0$$

where $c_i \in \mathbb{Z}$. It follows that $t \in \mathcal{O}_K$ and thus $s = a/m$ where $a \in \mathcal{O}_K$ and $m \in \mathbb{Z}$. Thus, $s \in L$ where L is the fraction field of $\mathcal{O}_K$.
4. Follows from the proof of item 3.
5. Let $C \subseteq K$ be normalization of $\mathcal{O}_K$ in K. Thus we have $\mathbb{Z} \hookrightarrow \mathcal{O}_K \hookrightarrow C$ where both are integral maps. It follows from Lemma 23.7.1.11 that the composite $\mathbb{Z} \hookrightarrow C$ is integral. Consequently, $C \subseteq \mathcal{O}_K$, yielding that $\mathcal{O}_K$ is a normal domain.
6. This follows from the corollary of Cohen-Seidenberg theorems (Corollary **??**) and that $\mathcal{O}_K$ is integral over $\mathbb{Z}$.
7. **TODO.** □

A direct corollary of this is that $\mathcal{O}_K$ is a very special type of ring.

Corollary 7.2.0.4. *Let K be an algebraic number field and $\mathcal{O}_K$ be its integral ring. Then $\mathcal{O}_K$ is a Dedekind domain (see Definition 23.11.0.1).*

Proof. From Proposition 7.2.0.3, 5,6,7, the result follows. □

The following is an important example of ring of integers.

Theorem 7.2.0.5. *Let $L = \mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is a square-free integer (i.e. product of distinct primes). Then, $\mathcal{O}_L$ is given as follows:*

$$\mathcal{O}_L = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d = 2,3 \mod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d = 1 \mod 4. \end{cases}$$

Proof. □

We now wish to study units and irreducibles in $\mathcal{O}_L$. To this end we require norm and trace of a finite extension as discussed in §**??**.

7.2.1 Quadratic number fields

We study imaginary quadratic number fields obtained by taking square root of some integer. A key tool for studying the relation between arithmetic and algebra of the situation is that of the norm and trace. We will also state for which quadratic number fields is the ring of integers a UFD, thus solving the fundamental problem of algebraic number theory in this restricted case; when is a ring of integers of a number field a UFD?

Part III

The Topological Viewpoint

Goals:

1. Define real and complex manifolds from locally ringed spaces, examples (Wedhorn).
2. Basic constructions like linearization, product, fiber products, submanifolds, quotients (Wedhorn for theory and Bredon for applications).
3. $\mathcal{O}_X$-modules and global algebra.
4. Lie groups (Wedhorn and Taubes).
5. Torsors and 1^{st}-Cech cohomology group (Wedhorn and Mumford's chapter on cohomology of sheaves).
6. Bundles and applications (Taubes Ch.3,4,5,6,7,10, Wedhorn and Bredon both for theory and their exercises for applications).
7. Singular homology and cohomology as ES-axioms. Properties, applications and results (Bredon and May). Singular cohomology as sheaf cohomology (Wedhorn Chapter 11).
8. Fundamental group and covering maps (classification) as etale spaces of certain sheaves (Bredon Chapter 3 and my algebraic topology notes).
9. Differential forms and de-Rham cohomology (Wedhorn's Section 8.6 and Bredon's Chapter 5, with examples and exercises).
10. ($\star$ **Geometric milestone**) *Covariant derivative, connections, classes and curvature* (Taubes Ch. 11,12,13,14,15,16).
11. ($\star$ **Algebraic milestone**) *Cohomological methods in geometry* (Bredon's Chapter 6 full).
12. ($\star$ **Homotopical milestone**) *Homotopical methods* (Bredon's Chapter 7 and May).

I have to rearrange the following chapters to suit the above outline.

These chapters need not be filled with unwarranted details. They should provide the point of the construction clearly and all minute details can be safely skipped over after understanding them.

Chapter 8

Foundational Geometry

Contents

Complete this chapter from Wedhorn's manifolds, sheaves and cohomology, and by Bredon's topology and geometry.
We will define the notion of a real and complex manifold. Some foundational constructions are made on them. We will take a rather modern viewpoint on the matter. We will further discuss ....

8.1 Locally ringed spaces and manifolds

We will make very fluid use of sheaves (see Chapter 27). Let us begin by the foundational structure in all of geometry, a (locally)ringed space.

Definition 8.1.0.1. (**Ringed and locally ringed spaces**) A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative R-algebras.

The space $(X, \mathcal{O}_X)$ is locally ringed if the stalk $\mathcal{O}_{X,x}$ at each point $x \in X$ is a local ring. The sheaf $\mathcal{O}_X$ is called the structure sheaf of X.

In order to understand the relation between two such spaces, we next have to understand the morphism of (locally)ringed spaces. For a motivation, see Example 1.2.2.1.

Definition 8.1.0.2. (**Morphism of ringed and locally ringed spaces**) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and a map of sheaves over X denoted $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed, then for $(f, f^\sharp)$ to be morphism of locally ringed spaces has to satisfy an additional condition that the induced map on stalks is a map of local rings. That is, for each $x \in X$, the induced map on stalks

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is such that $(f_x^\sharp)^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$ (see Special Topics, Remark 27.5.0.6). We call this map the comorphism at $x \in X$. In particular, this map is given by the unique map obtained by universality of direct limits under question: consider any open $V \ni f(x)$ in Y, we then obtain the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{X,x} & \mathcal{O}_{Y,f(x)} \arrow[l, dashed, "=:f_x^\sharp"'] \\
\mathcal{O}_X(f^{-1}(V)) \arrow[u, "\iota_{f^{-1}(V)}"] & \mathcal{O}_Y(V) \arrow[l, "f_V^\flat"] \arrow[u, "\iota_V"'] \arrow[ul, "\iota_{f^{-1}(V)} \circ f_V^\flat"']
\end{tikzcd}.$$

In most of our purposes, the map $f^\flat$ will be given on sections by composing with f. In such situations, the map on stalks being local corresponds to the geometric intuition that all non-invertible functions around some open subset of $f(x)$ comes from non-invertible maps around x. This in some sense makes sure that the local data around $f(x)$ is completely available via f.

Definition 8.1.0.3. (**Composition**) Composition of two maps of locally ringed spaces is defined in the obvious manner. For $X \xrightarrow{g} Y \xrightarrow{f} Z$, we get maps $g^\flat : \mathcal{O}_Y \to g_*\mathcal{O}_X$ and $f^\sharp : f^{-1}\mathcal{O}_Z \to \mathcal{O}_Y$. Then, the map $f \circ g : X \to Z$ is defined on space level by just the composite $f \circ g$ of the continuous maps and on the sheaf level as the corresponding flat and sharp maps of $f \circ g : X \to Z$:

$$h^\flat : \mathcal{O}_Z \longrightarrow (f \circ g)_*\mathcal{O}_X$$
$$h^\sharp : (f \circ g)^{-1}\mathcal{O}_Z \longrightarrow \mathcal{O}_X.$$

In particular, for an open set $U \subseteq Z$, the corresponding map on local sections $h_U^\flat$ is given by the following composite:

$$\begin{tikzcd}
\mathcal{O}_Z(U) \arrow[r, dashed, "h_U^\flat"] \arrow[d, "f_U^\flat"] & (f_*g_*\mathcal{O}_X)(U) \arrow[r, equal] & \mathcal{O}_X(g^{-1}f^{-1}(U)) \\
(f_*\mathcal{O}_Y)(U) \arrow[rr, equal] & & \mathcal{O}_Y(f^{-1}(U)) \arrow[u, "g_{f^{-1}(U)}^\flat"']
\end{tikzcd}.$$

Similarly, the corresponding morphism of stalks given by $h_x^\sharp$ is given by the usual

$$h_x^\sharp : (g^{-1}f^{-1}\mathcal{O}_Z)_x \cong \mathcal{O}_{Z,f(g(x))} \longrightarrow \mathcal{O}_{X,x}$$

which is the composite

$$\begin{tikzcd}
\mathcal{O}_{Z,h(x)} \arrow[d, "f^\sharp_{g(x)}"'] & \mathcal{O}_Z(W) \arrow[l] \arrow[d, "f^\flat_W"] \\
\mathcal{O}_{Y,g(x)} \arrow[d, "g^\sharp_x"'] & \mathcal{O}_Y(f^{-1}(W)) \arrow[l] \arrow[d, "g^\flat_{f^{-1}(W)}"] \\
\mathcal{O}_{X,x} & \mathcal{O}_X(g^{-1}(f^{-1}(W))) \arrow[l]
\end{tikzcd} .$$

Lemma 8.1.0.4. *Let $h : X \stackrel{g}{\to} Y \stackrel{f}{\to} Z$ be a morphism of ringed spaces. Consider the base change functors corresponding to maps g and f:*

$$g^{-1} : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X)$$
$$f_* : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(Z).$$

and consider the following composite in $\mathbf{Sh}(Y)$

$$f^{-1}\mathcal{O}_Z \xrightarrow{f^\sharp} \mathcal{O}_Y \xrightarrow{g^\flat} g_*\mathcal{O}_X \ .$$

Then,
1. *$g^{-1}(g^\flat \circ f^\sharp) \cong h^\sharp$,*
2. *$f_*(g^\flat \circ f^\sharp) \cong h^\flat$.*

Proof. These are cumbersome but straightforward identities. For example, one has to observe that $f_*(f^\sharp) \cong f^\flat$ and that for an open set $U \subseteq Z$, we have $(f_*(g^\flat))_U = g^\flat_{f^{-1}(U)}$. □

We have a simple lemma for isomorphism of ringed spaces.

Lemma 8.1.0.5. *Let $f : X \to Y$ be a morphism of ringed spaces. Then, f is an isomorphism if and only if $f : X \to Y$ is a homeomorphism and $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

Proof. (L ⇒ R) Use Theorem 27.3.0.6, 3 and 4.
(R ⇒ L) One can explicitly construct a map of sheaves in the other direction in a straightforward manner. □

An open subspace of a ringed space also inherits the structure of a ringed space.

Definition 8.1.0.6. (**Open subspace and embedding**) Let $(X, \mathcal{O}_X)$ be a (locally) ringed space. An open subspace of $(X, \mathcal{O}_X)$ is an open subset $i : U \hookrightarrow X$ together with the inverse image sheaf $i^{-1}\mathcal{O}_X = \mathcal{O}_{X|U}$[1]. The pair $(U, \mathcal{O}_{X|U})$ is called an open subspace, $(U, \mathcal{O}_{X|U}) \hookrightarrow (X, \mathcal{O}_X)$. A map $(j, j^\sharp) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is an open embedding if $U := j(Z) \hookrightarrow X$ is open and $(j, j^\sharp) : (Z, \mathcal{O}_Z) \to (U, \mathcal{O}_{X|U})$ is an isomorphism of ringed spaces.

[1]It's a trivial matter to observe that inverse image of a sheaf to an open inclusion will be the restriction sheaf (see Lemma 27.5.0.3).

An important concept is of local isomorphism of ringed spaces, which will prove it's worth while defining manifolds.

Definition 8.1.0.7. (**Local isomorphism**) Let $f : X \to Y$ be a morphism of ringed spaces. One calls f to be a local isomorphism if there exists an open cover $\{U_i\}_{i\in I}$ of X such that $f|_{U_i} : U_i \to Y$ is an open embedding for all $i \in I$.

8.1.1 Local models and manifolds

Before we proceed further, we have to clearly state some of our local model spaces that we are going to use while defining the manifolds. Therefore the following example of ringed spaces are foundational.

Example 8.1.1.1. (*Sheaf of C^α-maps*) Let $X \subseteq \mathbb{R}^n$ be an open set and $\alpha \in \mathbb{N}^\infty$. One defines the following presheaf

$$\mathcal{C}^{\alpha}_{X;\mathbb{R}^m} := \{f : X \to \mathbb{R}^m \mid f \text{ is } C^\alpha\}$$

where the restriction maps are usual functional restrictions. Then, $\mathcal{C}^{\alpha}_{X;\mathbb{R}^m}$ forms a sheaf, called the sheaf of C^α maps on X. This sheaf has stalks as local rings which can be seen quite easily (set of all functions defined in *some* neighborhood of $x \in X$ has a ring structure with maximal ideal being all those functions taking value 0 at x). Hence, $(X, \mathcal{C}^{\alpha}_{\mathbb{R}^m})$ is a locally ringed space, where we dropped the subscript X for notational convenience.

Example 8.1.1.2. (*Sheaf of holomorphic maps*) Let $X \subseteq \mathbb{C}^n$ be an open set. One defines the following presheaf

$$\mathcal{C}^{\text{hol}}_{X;\mathbb{C}^m} := \{f : X \to \mathbb{C}^m \mid f \text{ is holomorphic}\}$$

where the restriction maps are the usual functional restriction. This is easily seen to be a sheaf, called the sheaf of holomorphic functions over X. This endows $(X, \mathcal{C}^{\text{hol}}_{\mathbb{C}^m})$ with the structure of a locally ringed space.

With these two examples, we can come to the notion of real and complex manifolds as follows.

Definition 8.1.1.3. (**Real and complex manifolds**) Let X be a Hausdorff and second-countable topological space. Then,

1. A locally $\mathbb{R}$-ringed space $(X, \mathcal{O}_X)$ is a real C^α-manifold if there exists an open covering $\{U_i\}_{i\in I}$ of X and for each $i \in I$, there exists a positive integer $n_i \in \mathbb{N}$ and an isomorphism of locally $\mathbb{R}$-ringed spaces $\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^{\alpha}_{\mathbb{R}})$ for some open $Y_i \subseteq \mathbb{R}^{n_i}$. Hence a real C^α-manifold structure on X is the following tuple of data:
$$\left(X, \mathcal{O}_X, \{U_i\}_{i\in I}, \{Y_i \subseteq \mathbb{R}^{n_i}\}_{i\in I}, \{\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \overset{\cong}{\to} (Y_i, \mathcal{C}^{\alpha}_{\mathbb{R}})\}_{i\in I}\right)$$
2. A locally $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ is a complex manifold if there exists an open covering $\{U_i\}_{i\in I}$ of X and for each $i \in I$ there exists $n_i \in \mathbb{N}$ and an isomorphism of locally $\mathbb{C}$-ringed spaces $\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^{\text{hol}}_{\mathbb{C}})$ for some open $Y_i \subseteq \mathbb{C}^{n_i}$. Hence a complex manifold structure on X is the following tuple of data:
$$\left(X, \mathcal{O}_X, \{U_i\}_{i\in I}, \{Y_i \subseteq \mathbb{C}^{n_i}\}_{i\in I}, \{\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \overset{\cong}{\to} (Y_i, \mathcal{C}^{\text{hol}}_{\mathbb{C}})\}_{i\in I}\right)$$

In both of these, the isomorphisms $\{\varphi_i\}$ are called *charts* of the manifold and the sheaf $\mathcal{O}_X$ the structure sheaf of the manifold. Also, we can rather consider $\{\varphi_i\}_{i\in I}$ to be open embeddings. A map of manifolds is just defined to be a map of locally ringed spaces. Let $\mathbf{Mfd}_\alpha^{\mathbb{R}}$ and $\mathbf{Mfd}^{\mathbb{C}}$ denote the category of real C^α and complex manifolds respectively. A map of manifolds are just locally ringed maps between them. Isomorphisms in them are called C^α-diffeomorphism and biholomorphic maps respectively.

Let us now dwell into some of the immediate observations and remarks coming out of this definition. Let us first ease some notations. Let $(X, \mathcal{O}_X)$ be a real or complex manifold. The local chart (U_i, φ_i) is usually denoted by (U_i, x) where $x : U_i \to \mathbb{R}^n$ is a local embedding of locally ($\mathbb{R}$ or $\mathbb{C}$)-ringed spaces, where n depends on U_i. We usually suppress all the sheaves and their morphisms unless necessary (we will soon see why that's the case). For a local chart (U_i, x), the n component maps $\pi_j \circ x : U_i \to \mathbb{R}$ are denoted by x^j. Moreover, since $x : U \to x(U)$ is an isomorphism, therefore we denote $x^{-1} : x(U) \to U$ to be its inverse. All this will come in handy when we will start doing geometry over $(X, \mathcal{O}_X)$.

Let $(X, \mathcal{O}_X)$ be a real or complex manifold. We call an open subspace $(U, \mathcal{O}_{X|U}) \hookrightarrow (X, \mathcal{O}_X)$ an open submanifold.

One now sees that any morphism of manifolds as locally ringed spaces is completely determined by what happens at the level of points. In-fact, the sheaf allowed on X is also restricted if its a manifold. This is why we usually completely suppress the map of sheaves from our notation as that will be vacuous as long as we are working with map of manifolds. Let $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ be two manifolds ($\mathbb{R}$ or $\mathbb{C}$, but both of same type). We can define a sheaf $\mathcal{O}_{M;N}$ on M given by following sections: for some open $U \subseteq M$, we have a sheaf

$$\mathcal{O}_{M;N}(U) := \{f : (U, \mathcal{O}_{X|U}) \to (N, \mathcal{O}_N) \mid f \text{ is a map of manifolds}\}.$$

Now we show a foundational result which says that the notion of morphism of locally ringed spaces are nothing new in the classical world of $\mathbb{R}^n$ or $\mathbb{C}^n$. We place high importance on the following result as it becomes our point of departure (and thus a point of motivation) as to why the notion of a morphism of locally ringed spaces is defined as what it is; because it is the right notion of a "geometric map" in more abstract situations.

Theorem 8.1.1.4. *Let K be either $\mathbb{R}$ or $\mathbb{C}$, $X \subseteq K^n$ and $Y \subseteq K^m$ be two open subsets of the standard spaces. If $f : (X, \mathcal{C}_X^\alpha) \to (Y, \mathcal{C}_Y^\alpha)$ is a map of locally ringed spaces, then*

1. *$f^\flat : \mathcal{C}_Y^\alpha \to f_*\mathcal{C}_X^\alpha$ is given on an open set $V \subseteq Y$ by the standard composition map*

$$\begin{aligned} f_V^\flat : \mathcal{C}_Y^\alpha(V) &\longrightarrow \mathcal{C}_X^\alpha(f^{-1}(V)) \\ V \xrightarrow{t} K &\longmapsto f^{-1}(V) \xrightarrow{f} V \xrightarrow{t} K, \end{aligned}$$

2. *f is a C^α-map.*

Remark 8.1.1.5. As a slogan, we may remember the above theorem as the following principle:

In $\mathbb{R}^n$ or $\mathbb{C}^n$, locally ringed maps are exactly real C^α or holomorphic maps.

As a consequence of this, whenever we would like to consider C^α maps from, say $\mathbb{R}^n$ to $\mathbb{R}^m$, we might as well ask to produce a map of locally ringed spaces $(\mathbb{R}^n, \mathcal{C}^\alpha_{\mathbb{R}^n})$ to $(\mathbb{R}^m, \mathcal{C}^\alpha_{\mathbb{R}^m})$, which again shows how much geometric information is hidden in the notion of sheaves.

Proof of Theorem 8.1.1.4. [2] Pick any open $V \subseteq Y$ and any $t \in \mathcal{C}^\alpha_Y(V)$. We wish to show that $f^\flat_V(t) = t \circ f$ as a map $f^{-1}(V) \to K$. Consequently, pick any point $p \in f^{-1}(V)$. We wish to show that $f^\flat_V(t)(p) = t(f(p))$. To this end, we consider the evaluation homomorphism which are available at stalks. Observe that we have the following commutative square of K-algebras:

$$\begin{tikzcd} \mathcal{C}^\alpha_Y(V) \arrow[r, "f^\flat_V"] \arrow[d] & \mathcal{C}^\alpha_X(f^{-1}(V)) \arrow[d] \\ \mathcal{C}^\alpha_{Y,f(p)} \arrow[r, "f^\sharp_p"'] & \mathcal{C}^\alpha_{X,p} \end{tikzcd} .$$

In order to show $f^\flat_V(t)(p) = t(f(p))$, it is sufficient to show that the following triangle commutes:

$$\begin{tikzcd} \mathcal{C}^\alpha_{Y,f(p)} \arrow[rr, "f^\sharp_p"] \arrow[dr, "\mathrm{ev}_{f(p)}"'] & & \mathcal{C}^\alpha_{X,p} \arrow[dl, "\mathrm{ev}_p"] \\ & K & \end{tikzcd} .$$

But this is immediate from the fact that the K-algebra homomorphism $f^\sharp_p$ is a local ring homomorphism and the kernels of the evaluation maps are exactly the corresponding unique maximal ideals, so by quotienting by the maximal ideals, we obtain a K-algebra homomorphism $K \to K$ which necessarily is identity as it is a K-algebra homomorphism. Hence the triangle indeed commutes.

In order to show that the map f is a C^α-map, we need only show that the m projection maps $\pi_i : K^m \to K$ when composed with f yields C^α maps given by $X \to K$, but that is immediate from 1. $\square$

Using the above result, one can show that any manifold essentially has a unique structure sheaf of the form $\mathcal{O}_{X;\mathbb{R}}$ or $\mathcal{O}_{X;\mathbb{C}}$.

Proposition 8.1.1.6. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. If $(X, \mathcal{O}_X)$ is a real or complex manifold, then $\mathcal{O}_X \cong \mathcal{O}_{X;\mathbb{R}}$ or $\mathcal{O}_X \cong \mathcal{O}_{X;\mathbb{C}}$.*

Proof. We wish to show that there is an isomorphism of sheaves $\varphi : \mathcal{O}_{X;\mathbb{R}} \to \mathcal{O}_X$. For an open set $U \subseteq X$, we define φ_U as follows:

$$\begin{aligned} \varphi_U : \mathcal{O}_{X;\mathbb{R}}(U) &\longrightarrow \mathcal{O}_X(U) \\ t : (U, \mathcal{O}_{X|U}) \to (\mathbb{R}, \mathcal{C}^\alpha_{\mathbb{R}}) &\longmapsto t^\flat_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}). \end{aligned}$$

We claim that this map of sheaves is an isomorphism. We need only show that the map on stalks $\varphi_x : \mathcal{O}_{X;\mathbb{R},x} \to \mathcal{O}_{X,x}$ is an isomorphism. So we may assume that X has a global

[2] First proof in my new creator of meaning!

chart $\eta : (X, \mathcal{O}_X) \cong (W, \mathcal{C}^{\alpha}_{W;\mathbb{R}})$ where $W \subseteq \mathbb{R}^n$ is an open subset. Consequently, we have $\eta^{\sharp}_x : \mathcal{C}^{\alpha}_{W;\mathbb{R},\eta(x)} \cong \mathcal{O}_{X,x}$. Furthermore, $\mathcal{O}_{X;\mathbb{R},x} \cong \mathcal{C}^{\alpha}_{W;\mathbb{R},\eta(x)}$. Consequently, we wish to show that $\varphi_x : \mathcal{C}^{\alpha}_{W;\mathbb{R},\eta(x)} \to \mathcal{C}^{\alpha}_{W;\mathbb{R},\eta(x)}$ given by $(W, t : W \to \mathbb{R})_{\eta(x)} \mapsto (W, t^{\flat}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}))_{\eta(x)}$ is an isomorphism. Since by Theorem 8.1.1.4, 1, the map $t^{\flat}_{\mathbb{R}}$ is given by precomposition by t, therefore $t^{\flat}_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}})$ is just t. Consequently, φ_x is identity, which proves the result. □

Remark 8.1.1.7. By virtue of Proposition 8.1.1.6, we can assume that any C^{α}-manifold is a locally ringed space of the form $(X, \mathcal{O}_{X;\mathbb{R}})$ (similarly for $\mathbb{C}$-manifolds).

8.1.2 Sheaves & atlases

We have defined a manifold to be a space with an open covering by a model locally ringed spaces. There is a traditional definition, whereas, which is used heavily in traditional geometry because we really care about the charts (which is usually not done in algebraic geometry). This elucidates how one has to undertake a different viewpoint of geometry in algebraic geometry.

We wish to show that giving a manifold structure on a second countable Hausdorff space X as defined above is equivalent to giving an atlas in the classical sense. Indeed, for each atlas on X, we first define a sheaf on X.

Definition 8.1.2.1 (Atlas sheaf). Let X be a second countable Hausdorff space and $\mathcal{A} = (U_i, x_i)_{i\in I}$ be a C^{α}-atlas on X where $x_i : U_i \to \mathbb{C}^{n_i}$ is an open embedding. Consider the following assignment for each open $V \subseteq X$:

$$\mathcal{O}_{\mathcal{A}}(V) := \{f : V \to K \mid f \circ x_i^{-1} : x_i(U_i \cap V) \to K \text{ is } C^{\alpha}\text{-map}\}.$$

Then $\mathcal{O}_{\mathcal{A}}$ is a sheaf of $\mathbb{R}$-algebras, called the sheaf of atlas $\mathcal{A}$. Similarly for the holomorphic case.

We first observe that equivalent atlases give same atlas sheaves.

Lemma 8.1.2.2. *Let X be a second-countable Hausdorff space with $\mathcal{A} = (U_i, x_i)_i$ and $\mathcal{B} = (V_i, y_i)_i$ being two equivalent C^{α} or holomorphic atlases on X. Then the atlas sheaves $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ are isomorphic.*

Proof. Indeed, for each open $W \subseteq X$, define the map

$$\begin{aligned} \varphi_W : \mathcal{O}_{\mathcal{A}}(W) &\longrightarrow \mathcal{O}_{\mathcal{B}}(W) \\ f : W \to K &\longmapsto f : W \to K. \end{aligned}$$

To show that this is well-defined, we have to show that $f \in \mathcal{O}_{\mathcal{B}}(W)$. Indeed, pick any chart $y_i : V_i \to K$ of $\mathcal{B}$. We wish to show that $f \circ y_i^{-1} : y_i(V_i \cap W) \to K$ is C^{α} or holomorphic. As either condition is local on domain, so pick any point in $y_i(V_i \cap W)$. Pick a chart $x_i : U_i \to x_i(U_i)$ containing that point. Note that it is sufficent to show $f \circ y_i^{-1} : y_i(V_i \cap U_i \cap W) \to K$ is C^{α} or holomorphic. Indeed, we can write this as

$$f \circ y_i^{-1} = (f \circ x_i^{-1}) \circ (x_i \circ y_i^{-1}) : y_i(U_i \cap V_i \cap W) \to K.$$

Since $\mathcal{A}$ and $\mathcal{B}$ are equivalent and $f \in \mathcal{O}_{\mathcal{A}}$, it follows repsectively that $(x_i \circ y_i^{-1})$ and $(f \circ x_i^{-1})$ are C^α or holomorphic, as required.

Thus $\varphi : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$ is a sheaf map, which is identity, hence both sheaves are same. $\square$

We next see that a C^α or holomorphic atlas sheaf on a space X gives a C^α or $\mathbb{C}$ manifold structure on X.

Proposition 8.1.2.3. *Let $(X, \mathcal{O}_{X;\mathbb{C}})$ be a locally ringed space and $Y \subseteq \mathbb{C}^n$ be open. If $\varphi : (X, \mathcal{O}_{X;\mathbb{C}}) \to (Y, \mathcal{C}^{\text{hol}}_{Y;\mathbb{C}})$ is a map of locally ringed spaces, then $\varphi^\flat$ on open $V \subseteq Y$ is given by*

$$\begin{aligned} \varphi^\flat_V : \mathcal{C}^{\text{hol}}_{Y;\mathbb{C}}(V) &\longrightarrow \mathcal{O}_{X;\mathbb{C}}(\varphi^{-1}(V)) \\ t : V \to \mathbb{C} &\longmapsto t \circ \varphi : \varphi^{-1}(V) \to \mathbb{C}. \end{aligned}$$

Moreover, the following are equivalent:

1. *$\varphi : (X, \mathcal{O}_{X;\mathbb{C}}) \to (Y, \mathcal{C}^{\text{hol}}_{Y;\mathbb{C}})$ is an isomorphism of locally ringed spaces.*
2. *$\varphi : X \to Y$ is a homeomorphism such that for any open $U \subseteq X$ and any $f : U \to \mathbb{C}$ in $\mathcal{O}_X(U)$, $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is a holomorphic map.*

The same conclusions hold true for C^α-manifolds as well.

Proof. The proof of the first statement is exactly same as that of Theorem 8.1.1.4, hence is omitted. We now show the equivalence of items 1 and 2.

(1. $\Rightarrow$ 2.) This is immediate as the map $\varphi^\flat$ is an isomorphism, so in particular a bijection on sections.

(2. $\Rightarrow$ 1.) Pick any open $V \subseteq Y$. Then $\varphi^\flat_V$ is injective as φ is an isomorphism. It is also surjective by the given hypothesis and homeomorphism φ. This shows that $\varphi^\flat$ is an isomorphism. $\square$

Theorem 8.1.2.4. *Let X be a second-countable Hausdorff space and $(X, \mathcal{O}_X)$ be a locally ringed space. Then the following are equivalent.*

1. *$(X, \mathcal{O}_X)$ is a C^α/complex manifold.*
2. *$\mathcal{O}_X$ is a C^α/complex atlas sheaf.*

To avoid repetitions, we will do the complex case only, as there is no change in the proof for the real case.

Proof. (1. $\Rightarrow$ 2.) By Proposition 8.1.1.6, we may assume that $\mathcal{O}_X$ is just $\mathcal{O}_{X;\mathbb{C}}$, the sheaf of locally ringed maps from X to $\mathbb{C}$. We have an open cover $\{U_i\}_{i \in I}$ of X and isomorphisms of locally ringed spaces $\varphi_i : (U_i, \mathcal{O}_{U_i;\mathbb{C}}) \to (Y_i, \mathcal{C}^{\text{hol}}_{\mathbb{C}})$. This makes (U_i, φ_i) into an usual atlas as follows. For any i, j such that $U_i \cap U_j \neq \emptyset$, we obtain that the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

This is holomorphic since $\varphi_j : U_i \cap U_j \to \mathbb{C}$ is a map of locally ringed spaces in $\mathcal{O}_{X;\mathbb{C}}(U_i \cap U_j)$. Now, $\varphi_i : (U_i, \mathcal{O}_{U_i;\mathbb{C}}) \to (Y_i, \mathcal{C}^{\text{hol}}_{Y_i;\mathbb{C}})$ is an isomorphism, therefore by Proposition 8.1.2.3, it follows that $\varphi_j \circ \varphi_i^{-1}$ is a holomorphic map, as required.

We claim that this makes $\mathcal{O}_X$ into an atlas sheaf. Indeed, observe that $f \in \mathcal{O}_X(V)$ is a locally ringed map $f : (V, \mathcal{O}_{V;\mathbb{C}}) \to (Y, \mathcal{C}^{\text{hol}}_{\mathbb{C}})$. We claim that the data of f is equivalent to saying that $f \circ \varphi_i^{-1} : \varphi_i(V \cap U_i) \to \mathbb{C}$ is holomorphic. Indeed, this is the content of Proposition 8.1.2.3.

(2. $\Rightarrow$ 1.) Let $\mathcal{A} = (U_i, \varphi_i)$ be a complex atlas where $\varphi_i : U_i \to Y_i$ for open $Y_i \subseteq \mathbb{C}^{n_i}$ is a homeomorphism with holomorphic transitions. We need only show the item 2 of Proposition 8.1.2.3 for φ_i as then it would follow that $\varphi_i : (U_i, \mathcal{O}_{U_i;\mathbb{C}}) \to (Y_i, \mathcal{C}^{\mathrm{hol}}_{Y_i;\mathbb{C}})$ is an isomorphism of locally ringed spaces, completing the proof. Indeed, pick any open $U \subseteq X$ and any $f : U \to \mathbb{C}$ in $\mathcal{O}_X(U)$. As $\mathcal{O}_X$ is the atlas sheaf of $\mathcal{A}$, therefore for φ_i in particular, we have that $f \circ \varphi_i^{-1} : \varphi_i(U \cap U_i) \to \mathbb{C}$ is a holomorphic map, as required. This completes the proof. $\square$

8.1.3 Locally ringed spaces & spaces with functions

As observed in previous section, it is apparent that the conclusion of Theorem 8.1.1.4 can be generalized. Indeed, this generalization is useful in the study of varieties over algebraically closed fields as in that case, the sheaf is a sheaf of regular functions, which has more refined properties.

TODOs

1. Equivalence of chart Φ definition and structure sheaf $\mathcal{O}_X$ definition.
2. Example of projective space.
3. Topological properties (Lindelof, paracompact, normal).
4. Product of manifolds.
5. Covering of manifolds.

8.2 Linearization

till fiber prod-
Chapter 8.

8.3 Constructions on manifolds

8.4 Lie groups

8.5 Global algebra

Let $(X, \mathcal{O}_X)$ be a locally ringed space. We will discuss here the operations on and properties of $\mathbf{Mod}(\mathcal{O}_X)$, the category of $\mathcal{O}_X$-modules [3]. An $\mathcal{O}_X$-module is a sheaf $\mathcal{M}$ on X such that $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module and the restriction maps of $\mathcal{M}$ are given as module homomorphism w.r.t the corresponding restriction map of $\mathcal{O}_X$ (more precisely below). There are several important constructions and properties that one can make with these. In-fact, just like one understands a ring R by understanding R-modules, one can understand $\mathcal{O}_X$ by understanding $\mathcal{O}_X$-modules. The similarity runs deeper as we can also define in certain cases the very same constructions we do in module, but in the case of $\mathcal{O}_X$-modules, and these constructions and operations becomes indispensable in doing geometry over locally ringed spaces of special kind, like schemes. A lot of such phenomenon is merely due to the fact that $\mathbf{Mod}(\mathcal{O}_X)$ is an abelian category. In-fact, notice that for each singleton space $X = \{\text{pt.}\}$, a ring R can be seen as the structure sheaf $\mathcal{O}_X$ over X and any R-module as a $\mathcal{O}_X$-module. Hence one may also think of the concept of $\mathcal{O}_X$-modules as the global version of classical commutative algebra.

Needless to say, this is an indispensable section for the purposes of geometry in general.

Let us first observe that over any topological space X, the product of two sheaves $\mathcal{F}, \mathcal{G}$ over X defined by $(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$ is indeed a sheaf with restriction maps as products of the restrictions. This allows us to define $\mathcal{O}_X$-modules very naturally.

For the rest of this section, we fix a ringed space $(X, \mathcal{O}_X)$.

Definition 8.5.0.1. ($\mathcal{O}_X$**-modules**) An abelian sheaf $\mathcal{F}$ over X is an $\mathcal{O}_X$-module if there is a map of sheaves

$$\begin{aligned} \mathcal{O}_X \times \mathcal{F} &\longrightarrow \mathcal{F} \\ (c, s) &\longmapsto cs \end{aligned}$$

where $c \in \mathcal{O}_X(U), s \in \mathcal{F}(U)$ for all open $U \subseteq X$ which endows $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module structure.

An $\mathcal{O}_X$-linear map of $\mathcal{O}_X$-modules is defined as a sheaf map $\varphi : \mathcal{F} \to \mathcal{G}$ between $\mathcal{O}_X$-modules such that for each open $U \subseteq X$, the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-linear map and that the restrictions preserves the respective module structures.

The above definition, when unravelled, yields that the scalar multiplication of each $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ commutes with restrictions; for $c \in \mathcal{O}_X(U)$, $s \in \mathcal{F}(U)$ and an open subset $V \subseteq U$, we have $(c \cdot s)|_V = c|_V \cdot s|_V$.

Remark 8.5.0.2. For an $\mathcal{O}_X$-module $\mathcal{F}$ we have the following easy observations:

1. $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module for all $x \in X$. Indeed, this follows from the universal property of direct limits and the fact that direct limits commutes with product; we have the

[3] we will give some general constructions for arbitrary sheaves over a topological case at times, before specializing to $\mathcal{O}_X$-module case.

following diagram

$$\begin{tikzcd}
\mathcal{O}_{X,x} \times \mathcal{F}_x \arrow[ddrr, dashed, "\tilde{m}"] & & \\
\varinjlim_{x \in V} \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[u, "\cong"] & & \\
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[u] \arrow[r, "m_U"] & \mathcal{F}(U) \arrow[r] & \mathcal{F}_x
\end{tikzcd}.$$

Explicitly, the $\mathcal{O}_{X,x}$-module structure on $\mathcal{F}_x$ is given by

$$\begin{aligned}
\mathcal{O}_{X,x} \times \mathcal{F}_x &\longrightarrow \mathcal{F}_x \\
((U,c)_x, (U,s)_x) &\longmapsto (U, c \cdot s)_x
\end{aligned}$$

where we may assume c and s are defined on same open neighborhood of x by appropriately restricting.

2. For a homomorphism $f : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules, we get a $\mathcal{O}_{X,x}$-module homomorphism $f_x : \mathcal{F}_x \to \mathcal{G}_x$ mapping as $(U,s)_x \mapsto (U, f_U(s))_x$ for each $x \in X$,
3. Let X be locally ringed space. Then, $\mathcal{F}_x/\mathfrak{m}_{X,x}\mathcal{F}_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a $\kappa(x)$-vector space. This is called the *fiber of module $\mathcal{F}$ over x*, denoted by $\mathcal{F}(x)$. Recall this is how the fiber of a module over a prime ideal of the ring is defined.

We first give few basic constructions, which is useful to keep in mind.

Definition 8.5.0.3. (**Support of a sheaf**) Let X be a topological space and $\mathcal{F}$ be an abelian sheaf over X. Let $U \subseteq X$ be an open set. For $s \in \mathcal{F}(U)$, we define the support of s as the subset

$$\operatorname{Supp}(s) := \{x \in U \mid (U,s)_x \neq 0 \text{ in } \mathcal{F}_x\}.$$

We further define the support of the sheaf as

$$\operatorname{Supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Support of a section is always a closed subset, but the support of a sheaf may not be closed.

Lemma 8.5.0.4. [4] *Let X be a space and $\mathcal{F}$ be a sheaf over X with $s \in \mathcal{F}(U)$ for an open set $U \subseteq X$. Then $\operatorname{Supp}(s) \subseteq U$ is a closed subset of U.*

Proof. Take any point $y \in U \setminus \operatorname{Supp}(s)$. We will find an open set $W \subseteq U \setminus \operatorname{Supp}(s)$ with $W \ni y$. Indeed, as $(U,s)_y = 0$, therefore we get a $W \subseteq U$ with $s|_W = 0$. For any $z \in W$, one further checks that $(U,s)_z = (W, s|_W)_z = 0$. Thus, $z \notin \operatorname{Supp}(s)$ and consequently, $W \subseteq U \setminus \operatorname{Supp}(s)$. ◻

Do skyscraper and subsheaf with support (Exercises 1.17 and 1.20 in Hartshorne.)

[4] Exercise II.1.14 of Hartshorne.

8.5.1 Global algebra : The algebra of $\mathcal{O}_X$-modules

In our quest to do geometry over schemes, we will make heavy use of the algebra of sheaves, especially that of exact sequences, so we give a lot of constructions that we may have to make out in the wild. We will make heavy use of sheafification (Theorem 27.2.0.1) in the sequel. An important question that arises is whether sheafification of an algebraic construction over collection of $\mathcal{O}_X$-modules actually is again an $\mathcal{O}_X$-module or not? The answer is yes, as can be easily checked by explicitly looking at sections of sheafification directly (see Remark 27.2.0.4 to observe that its not difficult, anyways we will show the explicit checks consistently).

Caution 8.5.1.1. The following pages might seem to be filled with *unnecessary details* about checking whether a given construction on $\mathcal{O}_X$-modules results in an $\mathcal{O}_X$-module or not. While for some this might be unnecessary, but working this out in experience has been satisfying and tends to give a deeper understanding of the various module structures (*algebraic* modules) that gets associated with an $\mathcal{O}_X$-module $\mathcal{F}$ and how they interrelate. Indeed, we will see that with more elaborate constructions, we get more and more module structures to handle with. Thus it is necessary to work some details out of this. At any rate, we will be using notions presented in the sequel quite frequently in algebraic geometry and in particular while doing cohomology (Cĕch cohomology in particular!) so we need a good knowledge of the $\mathcal{O}_X$-modules and their internal technicalities.

Remark 8.5.1.2. Since there are a lot of constructions in the sequel, so to have a sense of mental clarity, let us list them here:

- Submodules and ideals of $\mathcal{O}_X$.✓
- Quotient of modules.✓
- Image and kernel modules.✓
- Exact sequences of modules.✓
- The $\Gamma(\mathcal{O}_X, X)$-module $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.✓
- $\mathcal{H}om_{\mathcal{O}_X}$ module.✓
- Direct sum of modules.✓
- Direct product of modules.✓
- Tensor product of modules.✓
- Free, locally free & finite locally free $\mathcal{O}_X$-modules.✓
- Invertible modules and the Picard group.✓
- Direct and inverse image modules.✓
- Sums & intersections of submodules.
- Modules generated by sections.
- Inverse limit.
- Direct limit.
- Tensor, symmetric & exterior algebras.
- $\mathcal{E}xt$ module.
- $\mathcal{T}or$ module.

Submodules and ideals of $\mathcal{O}_X$

Definition 8.5.1.3. (**Submodules and ideals**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. A *submodule* of $\mathcal{F}$ is an $\mathcal{O}_X$-module which is a subsheaf $\mathcal{G} \subseteq \mathcal{F}$ such that for all open $U \subseteq X$, the inclusion

$$\mathcal{G}(U) \hookrightarrow \mathcal{F}(U)$$

is an $\mathcal{O}_X(U)$-module homomorphism. Since $\mathcal{O}_X$ is an $\mathcal{O}_X$-module, thus, to be in line with usual terminology, we define submodules of $\mathcal{O}_X$ as *ideals* of $\mathcal{O}_X$.

Remark 8.5.1.4. Note that for any $\mathcal{O}_X$ submodule $\mathcal{G} \subseteq \mathcal{F}$, we get a submodule $\mathcal{G}_x \subseteq \mathcal{F}_x$ of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.

Quotient of modules

Definition 8.5.1.5. (**Quotient modules**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{G}$ be a submodule of $\mathcal{F}$. The *quotient module* is the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$, denoted by $\mathcal{F}/\mathcal{G}$ (see Definition 27.3.0.4). Indeed, $\mathcal{F}/\mathcal{G}$ is an $\mathcal{O}_X$-module by the following lemma.

Lemma 8.5.1.6. *$\mathcal{F}/\mathcal{G}$ is an $\mathcal{O}_X$-module.*

Proof. We will use the definition of sheafification as given in Remark 27.2.0.4. For each open set $U \subseteq X$, consider the following map:

$$\begin{aligned} \eta_U : \mathcal{O}_X(U) \times (\mathcal{F}/\mathcal{G})(U) &\longrightarrow (\mathcal{F}/\mathcal{G})(U) \\ (c, s) &\longmapsto \eta_U(c, s) : U \to \amalg_{x \in U} \mathcal{F}_x/\mathcal{G}_x \end{aligned}$$

where $\eta_U(c, s)(x) := c_x \cdot s(x)$ where $c_x \in \mathcal{O}_{X,x}$ and $s(x) \in \mathcal{F}_x/\mathcal{G}_x$ and the multiplication $c_x \cdot s(x)$ is coming from the $\mathcal{O}_{X,x}$-module structure that $\mathcal{F}_x/\mathcal{G}_x$ has. We now need to show following two statements:

1. $\eta_U(c, s)$ is indeed in $(\mathcal{F}/\mathcal{G})(U)$,
2. $\eta : \mathcal{O}_X \times \mathcal{F}/\mathcal{G} \to \mathcal{F}/\mathcal{G}$ is a sheaf map.

For statement 1, we need to show that for each $x \in U$, there exists an open set $x \in V \subseteq U$ and there exists $r \in \mathcal{F}(U)/\mathcal{G}(U)$ such that for all $y \in V$ we have the equality $c_y \cdot s(y) = r_y$ in $\mathcal{F}_y/\mathcal{G}_y$. Indeed, this can easily be seen via the fact that $s \in (\mathcal{F}/\mathcal{G})(U)$. Statement 2 is immediate after drawing the relevant square whose commutativity is under investigation. $\square$

Remark 8.5.1.7. Note further that we get a natural map

$$\mathcal{F} \to \mathcal{F}/\mathcal{G}$$

which factors through the inclusion of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$ into the sheaf $\mathcal{F}/\mathcal{G}$.

Image and kernel modules

Definition 8.5.1.8. (**Image and kernel modules**) Let $f : \mathcal{F} \to \mathcal{G}$ be a $\mathcal{O}_X$-module homomorphism. We then get the image sheaf $\mathrm{Im}\,(f)$ and the kernel sheaf $\mathrm{Ker}\,(f)$ by Definition 27.3.0.5. Indeed, both of these are $\mathcal{O}_X$-modules as the following lemma shows.

Lemma 8.5.1.9. $\mathrm{Im}\,(f)$ *and* $\mathrm{Ker}\,(f)$ *are* $\mathcal{O}_X$*-modules.*

Proof. $\mathrm{Ker}\,(f)$ is straightforward. For $\mathrm{Im}\,(f)$, we first observe that if we denote $\mathrm{Im}\,(f) = (\mathrm{im}\,(f))^{++}$, then $(\mathrm{im}\,(f))_x = f_x(\mathcal{F}_x)$. We thus define the $\mathcal{O}_X$-module structure on $\mathrm{Im}\,(f)$ as follows:

$$\begin{aligned} \eta_U : \mathcal{O}_X(U) \times \mathrm{Im}\,(f)(U) &\longrightarrow \mathrm{Im}\,(f)(U) \\ (c, s : U \to \amalg_{x\in U} f_x(\mathcal{F}_x)) &\longmapsto \eta_U(c, s) \end{aligned}$$

where $\eta_U(c,s)(x) = c_x \cdot s(x)$ where $s(x) \in f_x(\mathcal{F}_x) \subseteq \mathcal{G}_x$. One checks like for quotient modules that this defines an $\mathcal{O}_X$-module structure on $\mathrm{Im}\,(f)$. Further, it is clear that $\mathrm{Im}\,(f) \subseteq \mathcal{G}$. □

Corollary 8.5.1.10. *For a* $\mathcal{O}_X$*-module homomorphism* $f : \mathcal{F} \to \mathcal{G}$*, we have* $\mathrm{Ker}\,(f) \leq \mathcal{F}$ *and* $\mathrm{Im}\,(f) \leq \mathcal{G}$ *are submodules.*

Proof. Use Remark 27.2.0.4 to get this immediately. □

We have a "first isomorphism theorem" for modules then.

Lemma 8.5.1.11. *For a map* $f : \mathcal{F} \to \mathcal{G}$ *of* $\mathcal{O}_X$*-modules, we obtain an isomorphism*

$$\mathcal{F}/\mathrm{Ker}\,(f) \cong \mathrm{Im}\,(f).$$

Proof. For each $x \in X$ let $\varphi_x : \mathcal{F}_x/\ker f_x \xrightarrow{\cong} \mathrm{im}\,(f_x)$. Then we define the following for any $U \subseteq X$ open

$$\begin{aligned} (\mathcal{F}/\mathrm{Ker}\,(f))(U) &\longrightarrow \mathrm{Im}\,(f)(U) \\ s : U \to \amalg_{x\in U}\mathcal{F}_x/\ker f_x &\mapsto \varphi \circ s \end{aligned}$$

where $(\varphi \circ s)(x) = \varphi_x(s(x))$. This is immediately an isomorphism by going to stalks (Theorem 27.3.0.6, 3). □

Exact sequences of modules

Definition 8.5.1.12. (**Exact sequences**) A sequence of $\mathcal{O}_X$-modules

$$\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}''$$

is said to be *exact* if $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$.

Remark 8.5.1.13. By Lemma 27.3.0.8, $\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}''$ is exact if and only if $\mathrm{Ker}\,(g_x) = \mathrm{Im}\,(f_x)$ at all points $x \in X$.

The $\Gamma(\mathcal{O}_X, X)$-module $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$

We now consider the set of all $\mathcal{O}_X$-module homomorphisms $f : \mathcal{F} \to \mathcal{G}$ and observe very easily that it has a $\Gamma(\mathcal{O}_X, X)$-module structure. This generalizes the fact that under point-wise addition and scalar multiplication, the set $\mathrm{Hom}_R(M, N)$ for two R-modules M, N is again an R-module.

Definition 8.5.1.14. (**$\Gamma(\mathcal{O}_X, X)$-module $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$**) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then the collection of all $\mathcal{O}_X$-module homomorphisms $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a $\Gamma(X, \mathcal{O}_X)$-module. Indeed, for two $f, g \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $c \in \Gamma(\mathcal{O}_X, X)$, we define $f + g : \mathcal{F} \to \mathcal{G}$ by $s \mapsto f(s) + g(s)$ and we define $c \cdot f : \mathcal{F} \to \mathcal{G}$ by $s \mapsto \rho_{X,U}(s) \cdot f(s)$ for any open set $U \subseteq X$ and $s \in \mathcal{F}(U)$.

We will now globalize the construction of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to obtain an $\mathcal{O}_X$-module out of it.

$\mathcal{H}om_{\mathcal{O}_X}$ module

Definition 8.5.1.15. (**Hom module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$**) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then the following presheaf

$$U \mapsto \mathcal{H}om_{\mathcal{O}_{X|U}}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$$

with restriction given by restriction of sheaf maps, is an $\mathcal{O}_X$-module denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, as the following lemma shows.

Lemma 8.5.1.16. *$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an $\mathcal{O}_X$-module*

Proof. The fact that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf can be seen immediately. The $\mathcal{O}_X$-module structure is defined as follows: pick any open $U \subseteq X$

$$\begin{aligned} \eta_U : \mathcal{O}_X(U) \times \mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{F}_{|U}, \mathcal{G}_{|U}\right) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_U}\left(\mathcal{F}_{|U}, \mathcal{G}_{|U}\right) \\ (c, f) &\longmapsto cf \end{aligned}$$

where $cf : \mathcal{F}_{|U} \to \mathcal{G}_{|U}$ is given on an open set $V \subseteq U$ by

$$\begin{aligned} (cf)_V : \mathcal{F}(V) &\longmapsto \mathcal{G}(V) \\ s &\longmapsto \rho_{U,V}(c) \cdot f_V(s). \end{aligned}$$

One easily check that η is a well-defined natural map of sheaves, thus making $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ into an $\mathcal{O}_X$-module. $\square$

Do Exercise 1.15 Chapter 2 of Hartshorne as well.

Remark 8.5.1.17. It is in general NOT true that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$.

We now define the dual of a module in the obvious manner.

Definition 8.5.1.18. (**Dual module**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The dual of $\mathcal{F}$ is defined to be the module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. We denote the dual by $\mathcal{F}^\vee$.

There are some isomorphisms regarding $\mathcal{H}om$ that is akin to their usual algebraic counterparts. We outline them in the following lemma.

Lemma 8.5.1.19. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,*
1. $\mathcal{H}om(\mathcal{O}_X^n, \mathcal{F}) \cong \mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n$,
2. $\mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$.

Proof. In both cases we construct a map and its inverses and it is straightforward to see that they are well-defined, natural and indeed inverses of each other.
1. Consider the map

$$\mathcal{H}om(\mathcal{O}_X^n, \mathcal{F}) \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n$$

which on an open set $U \subseteq X$ maps as

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}^n, \mathcal{F}_{|U}\right) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right)^n \\ f : \mathcal{O}_{X|U}^n \to \mathcal{F}_{|U} &\longmapsto (f_i)_{i=1,\dots,n} \end{aligned}$$

where for $V \subseteq U$, we have that $f_{i,V} : \mathcal{O}_X(V) \to \mathcal{F}(V)$ maps as $s \mapsto s \cdot f_V(e_i) = f_V(s \cdot e_i)$ where e_i is i^{th} standard vector in $\mathcal{O}_X(V)^n$. Conversely, define the map

$$\mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n \longrightarrow \mathcal{H}om(\mathcal{O}_X^n, \mathcal{F})$$

which on $U \subseteq X$ open maps as

$$\left(g_i : \mathcal{O}_{X|U} \to \mathcal{F}_{|U}\right)_{i=1,\dots,n} \longmapsto g : \mathcal{O}_{X|U}^n \to \mathcal{F}_{|U}$$

where on $V \subseteq U$ open, we define $g_V : \mathcal{O}_X(V)^n \to \mathcal{F}(V)$ as $(s_1, \dots, s_n) \mapsto \sum_{i=1}^n g_{i,V}(s_i) = \sum_{i=1}^n s_i \cdot g_{i,V}(e_i)$.
2. Define the map

$$\mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \longrightarrow \mathcal{F}$$

on open $U \subseteq X$ by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right) &\longrightarrow \mathcal{F}(U) \\ f : \mathcal{O}_{X|U} \to \mathcal{F}_{|U} &\longmapsto f_U(1). \end{aligned}$$

Define the inverse

$$\mathcal{F} \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{F})$$

on an open set $U \subseteq X$ by

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right) \\ s &\longmapsto f : \mathcal{O}_{X|U} \to \mathcal{F}_{|U} \end{aligned}$$

where for an open set $V \subseteq U$, we define $f_V(t) = f_V(t \cdot 1) := t \cdot s$. $\square$

Direct sum of modules

Definition 8.5.1.20. (**Direct sum of modules**) Let $\{\mathcal{F}_i\}_{i\in I}$ be a family of $\mathcal{O}_X$-modules. The direct sum of $\mathcal{F}_i$ is the sheafification of the presheaf

$$U \mapsto \bigoplus_{i\in I} \mathcal{F}_i(U)$$

whose restriction is the direct sum of the corresponding restrictions. We denote this sheaf by $\bigoplus_{i\in I} \mathcal{F}_i$ and it is an $\mathcal{O}_X$-module by the following lemma. If for all $i \in I$, we have $\mathcal{F}_i = \mathcal{F}$, then we write

$$\bigoplus_{i\in I} \mathcal{F} = \mathcal{F}^{\oplus I} = \mathcal{F}^{(I)}$$

as usually is done in algebra.

Lemma 8.5.1.21. *$\bigoplus_{i\in I} \mathcal{F}_i$ is an $\mathcal{O}_X$-module and $\left(\bigoplus_{i\in I} \mathcal{F}_i\right)_x \cong \bigoplus_{i\in I} \mathcal{F}_{i,x}$ for all $x \in X$.*

Proof. Since stalks functor is left adjoint (to skyscraper, we didn't covered this but this is a basic known fact), therefore it preserves all colimits and thus $\left(\bigoplus_{i\in I} \mathcal{F}_i\right)_x \cong \bigoplus_{i\in I} \mathcal{F}_{i,x}$. Now, the $\mathcal{O}_X$-module structure over $\bigoplus_{i\in I} \mathcal{F}_i$ is obtained as follows: pick any $U \subseteq X$ open and consider the map

$$\begin{aligned} \eta_U : \mathcal{O}_X(U) \times \left(\bigoplus_{i\in I} \mathcal{F}_i\right)(U) &\longrightarrow \left(\bigoplus_{i\in I} \mathcal{F}_i\right)(U) \\ (c, s : U \to \amalg_{x\in U} \oplus_{i\in I} \mathcal{F}_{i,x}) &\longmapsto cs \end{aligned}$$

where $cs(x) = c_x \cdot s(x)$ where $s(x) \in \bigoplus_{i\in I} \mathcal{F}_{i,x}$ and $\bigoplus_{i\in I} \mathcal{F}_{i,x}$ is an $\mathcal{O}_{X,x}$-module. By exactly same techniques employed in proving them in earlier cases, it can be observed that the above defines a map $\eta : \mathcal{O}_X \times \oplus_{i\in I}\mathcal{F}_i \to \oplus_{i\in I}\mathcal{F}_i$ which is a sheaf map. $\square$

We now cover the other construction we know from algebra.

Direct product of modules

Definition 8.5.1.22. (**Direct product of modules**) Let $\{\mathcal{F}\}_{i\in I}$ be a family of $\mathcal{O}_X$-modules. The direct product of them is defined to be the sheaf

$$U \mapsto \prod_{i\in I} \mathcal{F}_i(U)$$

with product of restrictions as its restriction. Indeed, it is immediate it is a sheaf and that the canonical map $\eta_U : \mathcal{O}_X(U) \times \prod_{i\in I} \mathcal{F}_i(U) \to \prod_{i\in I} \mathcal{F}_i(U)$ mapping as $(c, (s_i)_{i\in I}) \mapsto (c \cdot s_i)_{i\in I}$ makes $\prod_{i\in I} \mathcal{F}_i$ an $\mathcal{O}_X$-module. If $\mathcal{F}_i = \mathcal{F}$ for all $i \in I$, then we denote

$$\prod_{i\in I} \mathcal{F} = \mathcal{F}^{\prod I} = \mathcal{F}^I$$

as is usually done in algebra.

We now define tensor product of two $\mathcal{O}_X$-modules.

Tensor product of modules

Definition 8.5.1.23. (**Tensor product of modules**) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. The tensor product of $\mathcal{F}$ and $\mathcal{G}$ is given by the sheafification of the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, as the following lemma shows.

Lemma 8.5.1.24. *$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is an $\mathcal{O}_X$-module and $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$ for each $x \in X$.*

Proof. The second statement is immediate from Lemma 23.5.1.2. The $\mathcal{O}_X$-module structure is the obvious one: pick any open $U \subseteq X$ and then consider the map

$$\begin{aligned} \eta_U : \mathcal{O}_X(U) \times (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) &\longrightarrow (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) \\ (a, s : U \to \amalg_{x \in U} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x) &\longmapsto as \end{aligned}$$

where $as(x) = a_x s(x)$. One easily checks that this defines a well-defined natural sheaf map. $\square$

A simple observation also yields the usual identity we know from modules.

Lemma 8.5.1.25. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,*

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}.$$

Proof. Consider the map

$$\eta : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \longrightarrow \mathcal{F}$$

given on an open $U \subseteq X$ by the map corresponding to the following natural isomorphism (Theorem 27.2.0.1)

$$\eta_U : \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \stackrel{\cong}{\to} \mathcal{F}(U).$$

This yields the similar isomorphic map on stalks via Lemma 8.5.1.24 to yield the result via Theorem 27.3.0.6, 3. $\square$

Tensor product of modules is obviously commutative.

Lemma 8.5.1.26. *Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then, $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{G} \otimes \mathcal{F}$.*

Proof. Construct the map $\tilde{\eta} : \mathcal{F} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{F}$ as the unique map corresponding to the following

$$\begin{tikzcd} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \arrow[r, "\eta_U", "\cong"'] \arrow[dr, "j_U \eta_U"'] & \mathcal{G}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \arrow[d, "j_U"] \\ & (\mathcal{G} \otimes \mathcal{F})(U) \end{tikzcd}.$$

This map on the stalks gives the usual twist isomorphism $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \cong \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$. $\square$

Free, locally free & finite locally free $\mathcal{O}_X$-modules

Definition 8.5.1.27. (**Free, locally free and finite locally free modules**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,

1. $\mathcal{F}$ is called *free* if $\mathcal{F} \cong \mathcal{O}_X^{(I)}$ for some index set I,
2. $\mathcal{F}$ is called *locally free* if for all $x \in X$, there exists open $U \ni x$ such that $\mathcal{F}_{|U} \cong \mathcal{O}_{X|U}^{(I_x)}$ where I_x is an indexing set depending on x,
3. $\mathcal{F}$ is called *finite locally free* if $\mathcal{F}$ is locally free and the indexing set I_x is finite for each $x \in X$. If $I_x = I$ and I has size n, then we say that $\mathcal{F}$ is *locally free of rank n*.

We now observe that the hom sheaf of two locally free modules of finite rank is again locally free of finite rank.

Lemma 8.5.1.28. *Let $\mathcal{F}, \mathcal{E}$ be two locally free $\mathcal{O}_X$-modules of ranks n and m respectively. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})$ is a locally free module of rank nm.*

Proof. For each $x \in X$, there exists an open set $U \ni x$ such that $\mathcal{F}_{|U} \cong \mathcal{O}^n_{X|U}$ and $\mathcal{E}_{|U} \cong \mathcal{O}^m_{X|U}$. We then observe the following

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})(U) = \operatorname{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{F}_{|U}, \mathcal{E}_{|U}\right) \cong \operatorname{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}^n_{X|U}, \mathcal{O}^m_{X|U}\right) \cong \mathcal{O}^{nm}_{X|U}$$

where the last isomorphism can be established easily by reducing to the usual module case $(\operatorname{Hom}_R(R^n, R^m) \cong R^{nm})$. □

An important corollary of the above lemma is as follows.

Corollary 8.5.1.29. *Let $\mathcal{F}$ be be a locally free module of rank n. Then the dual $\mathcal{F}^\vee$ is locally free of rank n.*

Proof. By Lemma 8.5.1.28, $\mathcal{F}^\vee$ is locally free of rank n. □

One may think of finite locally free modules as those modules which are locally free in the usual sense. Consequently, these modules satisfy global version of the properties enjoyed by the usual notion of free modules, as the following result shows.

Proposition 8.5.1.30. [5] *Let $\mathcal{E}$ be a finite locally free of rank n. Then,*

1. $\mathcal{E}^{\vee\vee} \cong \mathcal{E}$.
2. *For any $\mathcal{O}_X$-module $\mathcal{F}$, we have*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}.$$

3. *($\otimes$-hom adjunction) For any $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we have*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}_{\mathcal{O}_X}\left(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})\right).$$

[5] Exercise II.5.1 of Hartshorne.

Proof. As $\mathcal{E}$ is locally of free of rank n, therefore there is an open cover $\{U_i\}$ of X such that $\mathcal{E}_{|U_i} \cong \mathcal{O}^n_{X|U_i}$. Let $\{B_j\}$ be a basis of X where each B_j is in some U_i. Consequently, we reduce to constructing an isomorphism in each case only as sheaves over the basis $\{B_j\}$.

1. Indeed, as each B_j is in some U_i, therefore $\mathcal{E}_{|B_j} \cong \mathcal{O}^n_{X|B_j}$. Consequently, we get the following isomorphisms for any $U \in \{B_j\}$

$$\begin{aligned}
\mathcal{E}^{\vee\vee}(U) &= \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)\big|_U, \mathcal{O}_{X|U}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}^n_X, \mathcal{O}_X)\big|_U, \mathcal{O}_{X|U}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}((\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X))^n\big|_U, \mathcal{O}_{X|U}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{O}^n_{X|U}, \mathcal{O}_{X|U}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{O}_{X|U}\right)^n \\
&\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)(U)^n \\
&\cong \mathcal{O}_X(U)^n \\
&\cong \mathcal{E}(U),
\end{aligned}$$

and its naturality with resepect to restrictions is evident.

2. Pick any $U \in \{B_j\}$. We then have

$$\begin{aligned}
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})(U) &\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{E}_{|U}, \mathcal{F}_{|U}\right) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}^n_{X|U}, \mathcal{F}_{|U}\right) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right)^n \\
&\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})(U)^n \\
&\cong \mathcal{F}(U)^n \\
&\cong (\mathcal{O}^n_X \otimes_{\mathcal{O}_X} \mathcal{F})(U)
\end{aligned}$$

by Lemma 8.5.1.25. The fact that this isomorphism is natural with respect to restrictions is immediate.

3. □

Invertible modules and the Picard group

Definition 8.5.1.31. (**Invertible modules**) An $\mathcal{O}_X$-module $\mathcal{L}$ is said to be invertible if it is locally free of rank 1.

The name is justified by the fact that the set of all invertible modules upto isomorphism forms a group under tensor product and is one of the important invariants of a (ringed) space amongst many others. We now show that indeed this forms a group. We will drop the subscript $\mathcal{O}_X$ from the tensor product, for clarity, in the following.

Proposition 8.5.1.32. *Let $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ be invertible $\mathcal{O}_X$-modules. Then,*

1. *$\mathcal{L}_1 \otimes \mathcal{L}_2$ is invertible,*
2. *$(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \mathcal{L}_3 \cong \mathcal{L}_1 \otimes (\mathcal{L}_2 \otimes \mathcal{L}_3)$,*
3. *$\mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{O}_X$.*

Proof. 1. This is a local question, so pick $x \in X$ and an open set $U \ni x$ such that $\mathcal{L}_{1|U} \cong \mathcal{O}_{X|U} \cong \mathcal{L}_{2|U}$. We wish to construct a natural map $(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2)(U) \to \mathcal{O}_X(U)$ which is an isomorphism. By Theorem 27.2.0.1, it suffices to show a natural isomorphism $\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U) \to \mathcal{O}_X(U)$. This is constructed quite easily as $\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U) \cong \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \cong \mathcal{O}_X(U)$. Thus we just need to consider $\mathrm{id}_{\mathcal{O}_X(U)}$.
2. This is again a local question, which can be answered by establishing an isomorphism (by using Theorem 27.2.0.1)

$$(\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_3(U) \cong \mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} (\mathcal{L}_2(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_3(U))$$

for any open $U \subseteq X$, but that is an immediate observation from algebra.
3. By Corollary 8.5.1.29, we have that $\mathcal{L}^\vee$ is invertible. By Theorem 27.3.0.6, 3, the result would follow if we can show that there is a natural $\mathcal{O}_X$-linear map $\varphi : \mathcal{L}^\vee \otimes \mathcal{L} \to \mathcal{O}_X$ such that for each point $x \in X$ there exists an open set $x \in U \subseteq X$ such that on U, φ yields an $\mathcal{O}_X(U)$-linear isomorphism $(\mathcal{L}^\vee \otimes \mathcal{L})(U) \cong \mathcal{O}_X(U)$. We may take U small enough so that $\mathcal{L}^\vee_{|U} \cong \mathcal{O}_{X|U} \cong \mathcal{L}_{|U}$. Thus, after replacing X by U, we may assume $\mathcal{L} = \mathcal{O}_X = \mathcal{L}^\vee$. By Lemmas 8.5.1.19 and 8.5.1.25, we obtain the following isomorphisms

$$\mathcal{L}^\vee \otimes \mathcal{L} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \otimes \mathcal{O}_X \cong \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes \mathcal{O}_X \cong \mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X.$$

This can easily be promoted to a sheaf map. □

Definition 8.5.1.33. (**Picard group of** X) The Picard group of X is defined to be the set of all isomorphism classes of invertible modules with the operation of tensor product. We denote this by

$$\mathrm{Pic}(X)$$

The Proposition 8.5.1.32 and Lemma 8.5.1.26 shows that $\mathrm{Pic}(X)$ is indeed an abelian group.

Direct and inverse image modules

In this and the next sections, we show how the modules behave under map of ringed spaces.

Definition 8.5.1.34. (**Direct image**) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the direct image of $\mathcal{F}$ under f is the direct image sheaf $f_*\mathcal{F}$ which is again an $\mathcal{O}_Y$-module given by the following composition

$$\mathcal{O}_Y \times f_*\mathcal{F} \xrightarrow{f^\flat \times \mathrm{id}} f_*\mathcal{O}_X \times f_*\mathcal{F} \xrightarrow{f_* m} f_*\mathcal{F}$$

where $m : \mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$ is the $\mathcal{O}_X$-module structure on $\mathcal{F}$. Note that f_* commutes with products as f_* is a right-adjoint.

The inverse image of a module, on the other hand, is an involved construction.

Definition 8.5.1.35. (**Inverse image**) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. The inverse image of $\mathcal{G}$ is defined to be the map

$$f^*\mathcal{G} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

which is indeed an $\mathcal{O}_X$-module as the following lemma shows.

Lemma 8.5.1.36. *The sheaf $f^*\mathcal{G}$ is an $\mathcal{O}_Y$-module.*

Proof. We need to show three statements:
1. $\mathcal{O}_X$ is an $f^{-1}\mathcal{O}_Y$-module.
2. $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module.
3. $f^*\mathcal{G}$ is an $\mathcal{O}_X$-module.

Statement 1 follows from the following composition

$$f^{-1}\mathcal{O}_Y \times \mathcal{O}_X \xrightarrow{f^\sharp \times \mathrm{id}} \mathcal{O}_X \times \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

where the latter is just the multiplication structure on $\mathcal{O}_X$. Statement 2 follows from $\mathcal{O}_Y$-module structure on $\mathcal{G}$ and the fact that $f^{-1}(\mathcal{G} \times \mathcal{G}') = f^{-1}\mathcal{G} \times f^{-1}\mathcal{G}'$ for two sheaves $\mathcal{G}, \mathcal{G}'$ over Y. Indeed, the latter follows from the fact that $f^+(\mathcal{G} \times \mathcal{G}') = f^+\mathcal{G} \times f^+\mathcal{G}'$, which in turn follows from the fact that filtered colimit commutes with finite limits. Statement 3 now follows immediately. $\square$

We now state an important result, that is $f_* \vdash f^*$.

Proposition 8.5.1.37. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces. Then,*

$$\begin{tikzcd} \mathbf{Mod}(\mathcal{O}_Y) \arrow[r, shift left=1.5ex, "f^*"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathcal{O}_X) \arrow[l, shift left=1.5ex, "f_*"] \end{tikzcd} .$$

In other words, we have a natural isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Proof. Omitted. $\square$

Sums & intersections of submodules

Modules generated by sections

Inverse limit

Do Hartshorne Exercise 1.12 as well.

Direct limit

Do Hartshorne Exercise 1.11 as well.

Tensor, symmetric & exterior powers

We now define $T(\mathcal{F})$, $S(\mathcal{F})$ and $\wedge(\mathcal{F})$ for a module $\mathcal{F}$.

Definition 8.5.1.38 ($T(\mathcal{F}), S(\mathcal{F})$ **and** $\wedge(\mathcal{F})$)**.** Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The sheafification of presheaf $U \mapsto T(\mathcal{F}(U))$ or $S(\mathcal{F}(U))$ or $\wedge(\mathcal{F}(U))$ is denoted to be $T(\mathcal{F})$ or $S(\mathcal{F})$ or $\wedge(\mathcal{F})$ called the tensor or symmetric or exterior algebra, respectively. This is an $\mathcal{O}_X$-algebra, i.e. a sheaf of rings which is an $\mathcal{O}_X$-module. Moreover, we have

$$T(\mathcal{F}) = \bigoplus_{n\geq 0} T^n(\mathcal{F})$$

where $T^n(\mathcal{F})$ is the sheafification of $U \mapsto T^n(\mathcal{F}(U))$. Note that this makes sense as sheafification is a left adjoint, so it commutes with all colimits. We call $T^n(\mathcal{F})$ the n^{th}-tensor power of $\mathcal{F}$. Similarly, we define $S^n(\mathcal{F})$ and $\wedge^n(\mathcal{F})$.

We now indulge in generalizing some local properties of tensor algebra to this global case. We first have the standard observation of instantiating these definitions on the finite locally free case, which generalizes the usual tensor calculations of free modules.

Proposition 8.5.1.39. [6] *Let $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-module of rank n. Then, $T^r(\mathcal{F}), S^r(\mathcal{F})$ and $\wedge^r(\mathcal{F})$ is a finite locally free $\mathcal{O}_X$-module of rank n^r, ${}^{n+r-1}C_{n-1}$ and ${}^nC_r$ respectively.*

Proof. Let $\{U_\alpha\}$ be an open cover of X where $\mathcal{F}$ is $\mathcal{O}^n_{X|U_\alpha}$ for each α. Let $\mathcal{B}$ be a basis of X such that for any $B \in \mathcal{B}$, we have $B \subseteq U_\alpha$ for some α. Observe that $\mathcal{F}_{|B} \cong \mathcal{O}^n_{X|B}$. Consequently, we obtain that $T^r(\mathcal{F})_{|B} \cong T^r(\mathcal{O}^n_{X|B})$. Since $T^r(\mathcal{O}^n_{X|B})$ is isomorphic to $\mathcal{O}^n_{X|B} \otimes \ldots \otimes \mathcal{O}^n_{X|B}$ r-times, which in turn is isomorphic to $\mathcal{O}^{n^r}_{X|B}$, therefore we have $T^r(\mathcal{F})$ is locally free of rank n^r.

Now for $S^r(\mathcal{F})$, we proceed as follows. We claim that $S^r(\mathcal{O}^n_{X|U_\alpha}) \cong \mathcal{O}^{{}^{n+r-1}C_{n-1}}_{X|U_\alpha}$. For this purpose, we may replace $\mathcal{F}$ by $\mathcal{O}_X$ by replacing X by U_α. Consequently, we wish to show that $S^r(\mathcal{O}^n_X) \cong \mathcal{O}^{{}^{n+r-1}C_{n-1}}_X$. Let F be the presheaf $V \mapsto S^r(\mathcal{O}_X(V)^n)$. Since $S^r(\mathcal{O}_X(V)^n) \cong \mathcal{O}_X(V)^{{}^{n+r-1}C_{n-1}}$ and this isomorphism is compatible with restrictions, therefore we see that we have an isomorphism $F \cong \mathcal{O}^{{}^{n+r-1}C_{n-1}}_X$ of sheaves and thus, $S^r(\mathcal{O}^n_X) \cong \mathcal{O}^{{}^{n+r-1}C_{n-1}}_X$, as required. Exactly similar argument yields $\wedge^r(\mathcal{O}^n_X) \cong \mathcal{O}^{{}^nC_r}_X$. $\square$

Another global phenomenon that is borrowed by tensor calculation of free modules is the perfect pairing of wedge product.

TODO.

The usual $\otimes -$ Hom adjunction has a global analogue.

TODO.

[6] Exercise II.5.16 of Hartshorne.

$\mathcal{E}xt$ module

$\mathcal{T}or$ module

8.5.2 The abelian category of $\mathcal{O}_X$-modules

We now show an important result that category of $\mathcal{O}_X$-modules over any ringed space is an abelian category (thus we can do whole of homological algebra over it!). We have essentially done everything, but we write it here for clear reference.

Theorem 8.5.2.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category* $\mathbf{Mod}(\mathcal{O}_X)$ *of $\mathcal{O}_X$-modules is an abelian category.*

Proof. For any two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, we have $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an abelian group where for any two $f, g \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, the sum $h = f + g$ is defined to as follows: pick any open $U \subseteq X$ and define $h_U = f_U + g_U$. This is an $\mathcal{O}_X$-linear sheaf map because f and g are. Hence $\mathbf{Mod}(\mathcal{O}_X)$ is preadditive. Moreover $\mathbf{Mod}(\mathcal{O}_X)$ is additive. This is what we did in the preceding section while defining finite products of $\mathcal{O}_X$-modules. The preceding section also shows that $\mathbf{Mod}(\mathcal{O}_X)$ has all kernels and cokernels. Consequently, we need only show that the for any $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Mod}(\mathcal{O}_X)$, $\operatorname{CoIm}(f) \cong \operatorname{Im}(f)$. Indeed, this is a local question and can be thus immediately seen by first isomorphism theorem. More precisely, we need only construct this isomorphism on a basis of X, where the canonical map $\operatorname{CoIm}(f) \to \operatorname{Im}(f)$ is an isomorphism by first isomorphism theorem. This completes the proof. □

Theorem 8.5.2.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the abelian category* $\mathbf{Mod}(\mathcal{O}_X)$ *has enough injectives.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We wish to find an injective $\mathcal{O}_X$-module $\mathcal{I}$ such that $\mathcal{F} \hookrightarrow \mathcal{I}$. First note that for each $x \in X$, we have an injective $\mathcal{O}_{X,x}$-module I_x such that $\mathcal{F}_x \hookrightarrow I_x$ by Theorem 26.2.2.7. Observe that I_x is a sheaf over $i : \{x\} \hookrightarrow X$. Let $\mathcal{I} = \prod_{x \in X} i_* I_x$ be the corresponing $\mathcal{O}_X$-module. We claim that $\mathcal{I}$ is an injective $\mathcal{O}_X$-module and there is an injective map $\mathcal{F} \hookrightarrow \mathcal{I}$.

To see that there is an injective map $\mathcal{F} \hookrightarrow \mathcal{I}$, we claim the following three isomorphisms

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) \cong \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_* I_x) \cong \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x).$$

The first isomorphism is immediate from limit preserving property of covariant hom. The second isomorphism is obtained by the following isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_* I_x) \cong \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x) \tag{$*$}$$

for each $x \in X$. Indeed, this follows from the maps $f \mapsto f_x$ and $(\tilde{\kappa} : \mathcal{F} \to i_* I_x) \leftarrow\!\shortmid (\kappa : \mathcal{F}_x \to I_x)$ where $\tilde{\kappa}$ is defined on an open set $U \subseteq X$ as $\tilde{\kappa}_U : \mathcal{F}(U) \to I_x$ mapping as $s \mapsto \kappa((U, s)_x)$. These are clearly inverses of each other. It then follows that a map $\mathcal{F} \to \mathcal{I}$ is equivalent to a collection of maps $\mathcal{F}_x \to I_x$ and since we have $\mathcal{F}_x \hookrightarrow I_x$, therefore we obtain a unique injective map $\mathcal{F} \hookrightarrow \mathcal{I}$.

Finally, we claim that $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact as a functor into the category of abelian groups. To this end, by left exactness of hom, we need only show that this is right exact. This immediately follows from isomorphism $(*)$ and I_x being injective and that product of surjective homomorphisms is surjective. This completes the proof. □

8.6 Torsors and 1^{st}-Čech cohomology group

Once we have understood the constructions of the last section, we can now start doing some serious geometry over our manifolds. Indeed, this is what we start laying out in this section.

8.7 Bundles

We give here the general theory of fiber, principal and vector bundles. When the need arises, we will instantiate this into different areas (like in the chapter on differential geometry). The material in previous chapter will allow a very united way of looking at the notion of bundles, and will start portraying the intimate connection that bundles and cohomology has.

8.7.1 Generalities on twisting atlases

Let $p : E \to B$ be a map of topological spaces/manifolds together with a specified subsheaf of groups $\mathcal{G} \subseteq \mathcal{A}_B(E) \in \mathbf{Sh}(B)$ where $\mathcal{A}_B(E)$ is the sheaf of homeomorphisms/isomorphisms over B; for any open $U \subseteq B$, the group $\mathcal{A}_B(E)(U)$ consists of all homeomorphisms/isomorphisms $\varphi : p^{-1}(U) \to p^{-1}(U)$ such that $p \circ \varphi = p$.

The tuple $(p : E \to B, \mathcal{G})$ is the pre-datum for defining $(p, \mathcal{G})$-twisting atlas for a map $\pi : X \to B$.

Definition 8.7.1.1 ($(p, \mathcal{G})$**-twisting atlas for a map**)**.** Let $p : E \to B$ be a map and $\mathcal{G}$ be a subsheaf of groups $\mathcal{G} \subseteq \mathcal{A}_B(E)$. Let $\pi : X \to B$ be a map. Then, a $(p, \mathcal{G})$-twisting atlas for π is a family $(U_i, h_i)_{i\in I}$ where $\{U_i\}_{i\in I}$ is an open cover of B and $h_i : \pi^{-1}(U_i) \xrightarrow{\cong} p^{-1}(U_i)$ is an isomorphism over U_i such that for any $i, j \in I$, denoting $U_{ij} = U_i \cap U_j$, we have

$$\begin{tikzcd} p^{-1}(U_{ij}) \arrow[rr, shift right, "h_j^{-1}|_{p^{-1}(U_{ij})}"'] \arrow[dr, "p"'] & & \pi^{-1}(U_{ij}) \arrow[ll, shift right, "h_i|_{\pi^{-1}(U_{ij})}"'] \arrow[dl, "\pi"] \\ & U_{ij} & \end{tikzcd}$$

and from which we require that

$$h_{ij} = h_i|_{\pi^{-1}(U_{ij})} \circ h_j^{-1}\Big|_{p^{-1}(U_{ij})}$$

is a section in $\mathcal{G}(U_{ij})$. We then call $\pi : X \to B$ together with (U_i, h_i) *a twist of* $p : E \to B$ *with structure sheaf* $\mathcal{G}$.

Using this, we may define a general notion of a bundle.

Definition 8.7.1.2 (**Bundles**)**.** Let $\pi : X \to B$ be a map, F a space/manifold and $p : B \times F \to B$ be the projection map onto first coordinate. Then π is a bundle with fiber F if there is a $(p, \mathcal{A}_B(B \times F))$-twisting atlas for π. Equivalently, π is a bundle with fiber F if it is a twist of $p : B \times F \to B$ with full structure sheaf $\mathcal{A}_B(B \times F)$.

Remark 8.7.1.3. Let $\pi : X \to B$ be a bundle with fiber F. Consequently we have a $\mathcal{A}_B(B \times F)$-twisting atlas of $p : B \times F \to B$ denoted (U_i, h_i), where $h_i : \pi^{-1}(U_i) \to p^{-1}(U_i)$ is an isomorphism over U_i such that the transition maps $h_{ij} : p^{-1}(U_{ij}) = U_{ij} \times F \to U_{ij} \times F = p^{-1}(U_{ij})$ is just an isomorphism over U_{ij} (i.e. $h_{ij} \in \mathcal{A}_B(B \times F)(U_{ij})$).

8.8 Differential forms and de-Rham cohomology

Do this from Section 8.6 and Section 10.4 of Wedhorn, via sheaf cohomology. Add motivation from courses.

8.8.1 Differential forms on $\mathbb{R}^n$

We first discuss differential forms on $\mathbb{R}^n$ as this provides clear and sufficient motivation for the abstract treatment of differential forms in all other places where it is used. We begin by defining the main ingredients. The material of Section 23.5 is used in the following.

Definition 8.8.1.1. (**Coordinate forms on** $\mathbb{R}^n$) Fix $n \in \mathbb{N}$. Let $V = \mathbb{R}^n$ be the n-dimensional $\mathbb{R}$-module. The functional

$$\begin{aligned} dx_i : V &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\longmapsto x_i \end{aligned}$$

is called the i^{th}-coordinate form on V, for each $i = 1, \dots, n$. Note that dx_i is a 1-form/1-tensor, i.e. $dx_i \in M^1(V) = V^*$. Observe that dx_i is the dual basis of V^* corresponding to standard basis e_i of V.

Next, we define a multilinear map which for each choices of axes, gives the volume of the parallelopiped obtained by the projection along those axes, given a parallelopiped spanned by some vectors.

Definition 8.8.1.2. (**Projection forms on** $\mathbb{R}^n$) Fix $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Let $V = \mathbb{R}^n$ be the n-dimensional $\mathbb{R}$-module. Let $I = (i_1, \dots, i_k)$ be an ordered k-tuple where $1 \le i_j \le n$ for each $j = 1, \dots, k$. Then, we define the I-projection form as

$$dx_I := \pi_k(dx_{i_1} \otimes \dots \otimes dx_{i_k}) = D_I$$

which is an alternating k-form on V, that is $dx_I \in \Lambda^k(V)$ (see Example 23.5.3.11). More explicitly, it is given by the following k-linear form on V

$$\begin{aligned} dx_I : V \times \dots \times V &\longrightarrow \mathbb{R} \\ (v_1, \dots, v_k) &\longmapsto \det \begin{bmatrix} dx_{i_1}(v_1) & dx_{i_2}(v_1) & \dots & dx_{i_k}(v_1) \\ dx_{i_1}(v_2) & dx_{i_2}(v_2) & \dots & dx_{i_k}(v_2) \\ \vdots & \vdots & \dots & \vdots \\ dx_{i_1}(v_k) & dx_{i_2}(v_k) & \dots & dx_{i_k}(v_k) \end{bmatrix}. \end{aligned}$$

Remark 8.8.1.3. Recall from Theorem 23.5.3.14 that $\Lambda^k(V)$ has basis given by dx_I for distinct increasing k-tuples from $1, \dots, n$. Thus, $\{dx_I\}_I$ forms an $\mathbb{R}$-basis of $\Lambda^k(V)$ of size ${}^nC_k$.

Remark 8.8.1.4. Recall that wedge product of forms is given by the following (where one defines them only on the basis elements)

$$\begin{aligned} \Lambda^k(V) \times \Lambda^l(V) &\longrightarrow \Lambda^{k+l}(V) \\ (dx_I, dx_J) &\longmapsto dx_I \wedge dx_J := dx_{(I,J)} \end{aligned}$$

where recall that $dx_{(I,J)}$ will be zero if there is any index common in I and J (see Definition 23.5.4.1), where I, J are increasing tuples of indices from $\{1, \dots, n\}$ of lengths k and l respectively. From the above, we see that for any alternating k-form $\omega = \sum_I a_I dx_I$ and alternating l-form $\eta = \sum_J b_j dx_J$, their wedge product is defined as

$$\omega \wedge \eta = \sum_J \sum_I a_I b_J (dx_I \wedge dx_J).$$

Remark 8.8.1.5. Let $U \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. Observe that $\mathcal{C}^\infty(U)$, the ring of smooth $\mathbb{R}$-valued functions on U, is an $\mathbb{R}$-algebra. In the same vein, we know that alternating k-forms $\Lambda^k(\mathbb{R}^n)$ forms an $\mathbb{R}$-vector space of dimension ${}^nC_k$ (see Theorem 23.5.3.14).

Definition 8.8.1.6. (**Differential k-forms**) Let $U \subseteq \mathbb{R}^n$ be an open set and $0 \leq k \leq n$. The module of differential k-forms is defined to be the following $\mathbb{R}$-vector space

$$\Omega_U^k = \Lambda^k(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathcal{C}^\infty(U).$$

As $\Lambda^k(\mathbb{R}^n)$ is a free $\mathbb{R}$-module with rank ${}^nC_k$, therefore Ω_U^k is a free $\mathcal{C}^\infty(U)$-module of rank ${}^nC_k$.

Remark 8.8.1.7. Observe that $\{\Omega_U^k\}$ obtains the wedge product structure from the wedge product on $\{\Lambda^k(\mathbb{R}^n)\}$ as we may define for $\omega = \sum_I f_I dx_I \in \Lambda^k(\mathbb{R}^n)$ and $\eta = \sum_J g_J dx_J$ the following

$$\begin{aligned} \omega \wedge \eta &:= \left(\sum_I f_I dx_I\right) \wedge \left(\sum_J g_J dx_J\right) \\ &= \sum_I \sum_J f_I g_J dx_I \wedge dx_J. \end{aligned}$$

Thus, $\bigoplus_{k\geq 0} \Omega_U^k$ forms a graded $\mathcal{C}^\infty(U)$-algebra.

Remark 8.8.1.8. An arbitrary element $\omega \in \Omega_U^k$ is called a differential k-form over U and is written as

$$\omega = \sum_{I \in X_k} f_I(x_1, \dots, x_n) dx_I$$

where X_k is the set of size ${}^nC_k$ of all k-combinations in increasing order of $\{1, \dots, n\}$ and $f_I \in \mathcal{C}^\infty(U)$ is a smooth function. Observe that $\Omega_U^0 = \mathcal{C}^\infty(U)$.

We now construct the exterior derivative which will be a differential over the chain complex Ω_U^k, as we will see soon.

Definition 8.8.1.9. (**Exterior derivative**) Let $U \subseteq \mathbb{R}^n$ be an open subset and $\{\Omega_U^k\}_{k\in\mathbb{N}}$ be the modules of differential k-forms. For each $k \in \mathbb{N} \cup \{0\}$, we define a map $d : \Omega_U^k \to \Omega^{k+1_U}$ as follows. Define for $k = 0$ the following

$$\begin{aligned} d : \Omega_U^0 &\longrightarrow \Omega_U^1 \\ f &\longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \end{aligned}$$

where since $f \in \mathcal{C}^\infty(U)$ is smooth, therefore so is $\partial f/\partial x_i$. Further, since $dx_i \in \Lambda^1(\mathbb{R}^n)$, therefore the above is well-defined. For $k \geq 1$, we define d as follows

$$\begin{aligned} d : \Omega_U^k &\longrightarrow \Omega_U^{k+1} \\ \omega = \sum_{I \in X_k} f_I dx_I &\longmapsto d\omega = \sum_{I \in X_k} df_I \wedge dx_I \end{aligned}$$

where $dx_I \in \lambda^k(\mathbb{R}^n)$. Observe that $df_I \in \Omega_U^1$, thus indeed $df_I \wedge dx_I \in \Omega_U^{k+1}$. This map d is called the exterior derivative of differential forms.

The following are immediate but important properties of exterior derivative. **TODO**.

Chapter 9

Foundational Differential Geometry

Contents

The material of previous chapter has already gotten us very close to topics in geometry over C^∞-manifolds.

9.1 Bundles in differential geometry and applications

9.2 Cohomological methods

9.3 Covariant derivative, connections, classes and curvatures

Chapter 10

Foundational Homotopy Theory

Contents

We introduce basic players of homotopy theory.

Let us first engage in a discussion of the type of spaces we would like to work with, that is, compactly generated space.

Definition 10.0.0.1 (**Compactly generated spaces**)**.** A space X is said to be compactly generated if it satisfies

1. (*weak Hausdorff*) for any compact Hausdorff space K and a map $g : K \to X$, the image $g(K)$ is closed,
2. (*k-space*) for any $A \subseteq X$, if $g^{-1}(A)$ is closed in K for any $g : K \to X$ where K is a compact Hausdorff space[1], then A is closed in X.

The following are some immediate observations.

Proposition 10.0.0.2. *Let X be a compactly generated space. Then,*

1. *Every compact subspace of X is closed.*
2. *If K is compact Hausdorff and $g : K \to X$ is a map, then $g(K) \subseteq X$ is compact Hausdorff.*
3. *If X is compactly generated and $f : X \to Y$ is a function, then f is continuous if and only if $f|_K$ is continuous for all compact subspaces $K \subseteq X$.*
4. *Any closed subspace of a compactly generated space is compactly generated.*

Proof. **TODO.** □

Example 10.0.0.3. Following are some examples of compactly generated spaces.
1. Any compact Hausdorff space is compactly generated. Indeed, for any compact Hausdorff K and a map $g : K \to X$, we have $g(K)$ is compact in X which is Hausdorff, so closed. Furthermore, if $A \subseteq X$ and $g^{-1}(A)$ is closed in K for any such g, then letting $K = X$ and $g = \mathrm{id}$, we immediately deduce that A is closed, as required.

2. Any Hausdorff space X which is locally compact is compactly generated. Indeed, for any compact Hausdorff K and a map $g : K \to X$, we have $g(K)$ is compact in X which is Hausdorff, so closed. Furthermore, if $A \subseteq X$ and $g^{-1}(A)$ is closed in K for any such g, then letting $\tilde{X}$ denote the 1-pt. compactification of X, we see that $\tilde{X}$ is compact Hausdorff. Consequently we may consider the map $\mathrm{id} : \tilde{X} \to \tilde{X}$. As any compact Hausdorff space is compactly generated as shown above, therefore $\mathrm{id}^{-1}(A) = A$ is closed by hypothesis, as needed.

Remark 10.0.0.4. The above example in particular shows that any real or complex manifold is a compactly generated space.

Construction 10.0.0.5. (*k-ification*) Let X be a weak-Hausdorff space. Then, X can be made into a compactly generated space. Define kX to have the same set as X but a finer topology obtained by deeming any compactly closed subspace to be closed in kX. It then follows that

1. kX is compactly generated,
2. the function $\mathrm{id} : kX \to X$ is continuous,

[1] we then call A to be *compactly closed*

3. X and kX have same compact subsets,
4. for weak Hausdorff spaces X and Y, we have $k(X \times Y) = kX \times kY$.

We now show why we restrict our gaze to only these spaces. In part because the category of compactly generated spaces is well-behaved.

TODO *Category* $\mathbf{Top}^{cg}$ *has limits, colimits and exponential objects (all after k-ification) and that the dual notion of homotopy as a path in function space is same as that of the usual notion.*

Remark 10.0.0.6. From now on in this chapter, we only work with the category of compactly generated spaces, $\mathbf{Top}^{cg}$. Moreover, any construction on spaces that we do is assumed to be k-ified, i.e. functor k is applied to it to always end up with the category of compactly generated spaces.

Next, we introduce constructions that one can do on based spaces. We denote $\mathbf{Top}_*^{cg}$ to be the category of based compactly generated spaces and based maps between them.

Construction 10.0.0.7 (*Based constructions*)**.** Let X and Y be two based spaces. Then, we denote by

1. $[X, Y]$ the based homotopy classes of based maps from X to Y. This is a based set itself, the basepoint being the homotopy class of $c_* : X \to Y$ mapping $x \mapsto *$. If $X \simeq X'$ and $Y \simeq Y'$, then there is a base point preserving bijection $[X, Y] \cong [X', Y']$.
2. $X \wedge Y$ the smash product given by $X \times Y/X \vee Y$ where $X \vee Y = \{*\} \times Y \cup X \times \{*\}$. This is a based space, the base point being the point corresponding to the subspace $X \vee Y$.
3. $\mathrm{Map}_*(X, Y)$ the collection of based maps from X to Y. This is again a based space in compact-open topology where the basepoint is c_*.
4. X_+ the based space obtained by adjoining a distinct point $*$ to X.
5. $X \wedge I_+$ the reduced cylinder of X where X is based. For any based X and unbased Y the based space $X \wedge Y_+$ is naturally homeomorphic to $X \times Y/\{*\} \times Y$.

There is a natural "$\otimes$-Hom" adjunction in $\mathbf{Top}_*^{cg}$.

Theorem 10.0.0.8. *Let X, Y, Z be based spaces in $\mathbf{Top}_*^{cg}$. Then we have a natural isomorphism*

$$\mathrm{Map}_*(X \wedge Y, Z) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)).$$

Proof. (*Sketch*) Let $f : X \wedge Y \to Z$. Then by universal property of quotients, we get a map $\bar{f} : X \times Y \to Z$ which is constant on $X \vee Y$. Now construct

$$\begin{aligned} \tilde{f} : X &\longrightarrow \mathrm{Map}_*(Y, Z) \\ x &\longmapsto y \mapsto \bar{f}(x, y). \end{aligned}$$

The fact that this is based follows from $\bar{f}$ being constant on $X \vee Y$.

Let $g : X \to \mathrm{Map}_*(Y, Z)$ a based map. Then we get

$$\begin{aligned} \bar{g} : X \times Y &\longrightarrow Z \\ (x, y) &\longmapsto g(x)(y). \end{aligned}$$

This is based immediately. Further, on $X \vee Y$, we see that $\bar{g}$ is constant. By universal property of quotients, we get the required $\tilde{g} : X \wedge Y \to Z$. $\square$

This theorem shows the duality between smash products and mapping space constructions.

Construction 10.0.0.9 (*More based constructions*)**.** We now give two constructions each for smash product and mapping space which complement each other.

1. CX the cone of X obtained by $X \wedge I$ where 1 is the basepoint of I.
2. ΣX the suspension of X obtained by $X \wedge S^1$.
3. PX the path space of X obtained by $\mathrm{Map}_*(I, X)$.
4. ΩX the loop space of X obtained by $\mathrm{Map}_*(S^1, X)$.

It follows from Theorem 10.0.0.8 that we have following natural isomorphisms

$$\mathrm{Map}_*(CX, Y) \cong \mathrm{Map}_*(X, PY)$$

and

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y),$$

the latter being the famous *suspension-loop space* adjunction.

In the next few items, we give results which are simple to see but important as technical tools.

Proposition 10.0.0.10. *Let X, Y be based spaces in $\mathbf{Top}_*^{cg}$. Then*

$$\pi_0(\mathrm{Map}_*(X, Y)) \cong [X, Y].$$

Proof. (*Sketch*) In $\mathbf{Top}^{cg}$, both left and right notions of homotopy are equivalent. Consequently, a path-component in $\mathrm{Map}_*(X, Y)$ is equivalently the set of based maps $X \to Y$ which are homotopic, as required. □

Every space can be *pointified.*

Definition 10.0.0.11 (**Pointification**)**.** The functor $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$ given by $X \mapsto X_+$ and $f : X \to Y$ mapping to $f_+ : X_+ \to Y_+$ is called the pointification functor.

There are important relationships between based and unbased constructions. We first have the following simple observation.

Lemma 10.0.0.12. *Let X be a based space. We have the following bijection*

$$\left\{\begin{matrix}\textit{Based homotopies } h\ : \\ X \times I \to Y\end{matrix}\right\} \cong \mathrm{Map}_*(X \wedge I_+, Y).$$

□

Remark 10.0.0.13. Let X be an unbased space. All the construction of Construction 10.0.0.9 have an unbased counterpart where smash products are replaced by Cartesian product and Map_* are replaced by Map. In particular,

1. CX the unreduced cone of X obtained by $X \times I/X \times \{1\}$.
2. ΣX the unreduced suspension of X obtained by $X \times S^1/X \times \{1\}$.
3. PX the unbased path space of X obtained by $\mathrm{Map}(I, X)$.

4. ΩX the unbased loop space of X obtained by $\mathrm{Map}(S^1, X)$.

We also call them by same name, if it is clearly understood that the space in question is unbased.

The following is an important observation about pointification and cones.

Lemma 10.0.0.14. *Let X be an unbased space. Then, the unreduced cone of X is isomorphic to the reduced cone on X_+. That is,*

$$CX \cong CX_+.$$

□

10.1 Cofibrations and cofiber sequences

Most of the long exact sequences appearing in algebraic topology are derived from the topics that we will cover in this chapter. These should rather be seen as an important conceptual tool in order to do computations. We will begin with cofibrations, closed subspaces from whose homotopies can be extended to the whole space, and then fibrations, which can be thought of as generalizations of covering spaces (more generally, fiber bundles) which one studies in a first course in algebraic topology.

Cofibrations can be treated as an intermediary tool for developing more sophisticated concepts in algebraic topology. In particular, we will be using this to derive an exact sequence of groups out of a map of based spaces.

Note that there is little to no difference in based or unbased cofibrations, so we will prove something for unbased context and will use it as it has been proved for based context as well. We will give some remarks towards the end.

10.1.1 Definition and first properties

Definition 10.1.1.1. (**Cofibrations**) A map $i : A \to X$ is a cofibration if it satisfies the homotopy extension property; if $f : X \to Y$ is a continuous map such that there is a homotopy $h : A \times I \to Y$ where $h(-, 0) = f \circ i$, then that homotopy can be lifted to $\tilde{h} : X \times I \to Y$ where $\tilde{h}(-, 0) = f$. More abstractly, if $h \circ i_0 = f \circ i$ in the following diagram, then there exists $\tilde{h}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"'] \arrow[dr, "i\times\mathrm{id}"] & \\ X \arrow[r, "f"] \arrow[rr, bend right, "i_0"'] & Y & X \times I \arrow[l, dashed, "\tilde{h}"] \end{tikzcd} .$$

One sees that pushout of a cofibration along any map is a cofibration.

Lemma 10.1.1.2. *Let $i : A \to X$ be a cofibration and $f : A \to B$ be any other map. Then, the pushout $j : B \to B \cup_f X$ is a cofibration.*

Proof. Take any map $g : B \cup_f X \to Y$ and a homotopy $h : B \times I \to Y$ where $h \circ i_0 = g \circ j$. We have the following diagram:

$$
\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dddd, "i\ (\text{cofibration})"'] & & & \\
& B \arrow[rr, "i_0"] \arrow[dd, "j"'] \arrow[l, phantom, "\text{p.o.}", near end, shift right=8ex] & & B \times I \arrow[dl, "j\times\mathrm{id}"'] \arrow[dd, "h"] \\
& & (B \cup_f X) \times I & \\
& B \cup_f X \arrow[ur, "i_0"] \arrow[rr, "g"'] & & Y \\
X \arrow[ur] & & &
\end{tikzcd} .
$$

We wish to show that there is a map $\tilde{h} : (B \cup_f X) \times I \to Y$ which commutes with the diagram shown above. Since we have the following pushout square:

$$
\begin{tikzcd}
B \cup_f X & X \arrow[l] \\
B \arrow[u, "j"] \arrow[ur, phantom, "\text{p.o.}"] & A \arrow[l, "f"] \arrow[u, "i"']
\end{tikzcd} ,
$$

therefore after applying functor $- \times I$, which has a right adjoint, so is colimit preserving (we are working in the category of compactly generated spaces which is cartesian closed), we get the following pushout square which is closer to what we have in the first diagram:

$$
\begin{tikzcd}
(B \cup_f X) \times I & X \times I \arrow[l] \\
B \times I \arrow[u, "j\times\mathrm{id}"] \arrow[ur, phantom, "\text{p.o.}"] & A \times I \arrow[l, "f\times\mathrm{id}"] \arrow[u, "i\times\mathrm{id}"']
\end{tikzcd} .
$$

Now, we get a map h' as below by the virtue of i being a cofibration:

$$
\begin{tikzcd}
A \arrow[rrrr, "i_0"] \arrow[dr, "f"] \arrow[dddd, "i\ (\text{cofibration})"'] & & & & A \times I \arrow[dl, "f\times\mathrm{id}"'] \arrow[dddd, "i\times\mathrm{id}"] \\
& B \arrow[rr, "i_0"] \arrow[dd, "j"'] \arrow[l, phantom, "\text{p.o.}", near end, shift right=8ex] & & B \times I \arrow[dl, "i\times\mathrm{id}"'] \arrow[dd, "h"] & \\
& & (B \cup_f X) \times I & & \\
& B \cup_f X \arrow[ur, "i_0"] \arrow[rr, "g"'] & & Y & \\
X \arrow[ur] \arrow[rrrr, "i_0"'] & & & & X \times I \arrow[ul, dashed, "h'"']
\end{tikzcd} .
$$

Next, by the universal property of pushout $(B \cup_f X) \times I$, we get a map $\tilde{h}$

$$\begin{tikzcd}
A \arrow[rrrr, "i_0"] \arrow[dr, "f"] \arrow[dddd, "i \text{ (cofibration)}"'] & & & & A \times I \arrow[dl, "f\times\mathrm{id}"'] \arrow[dddd, "i\times\mathrm{id}"] \\
& B \arrow[rr, "i_0"] \arrow[dd, "j"'] \arrow[ddl, phantom, "\text{p.o.}"] & & B \times I \arrow[dl, "i\times\mathrm{id}"'] \arrow[dd, "h"] & \\
& & (B \cup_f X) \times I \arrow[dr, dashed, "\tilde{h}"] & & \\
& B \cup_f X \arrow[ur, "i_0"] \arrow[rr, "g"'] & & Y & \\
X \arrow[ur] \arrow[rrrr, "i_0"'] & & & & X \times I \arrow[ul, dashed, "h'"']
\end{tikzcd},$$

which satisfies the required commutativity. $\square$

To check that a map $i : A \to X$ is a cofibration, we can reduce to checking the homotopy extension property to the map $X \to Mi$ where Mi is the *mapping cylinder*.

Definition 10.1.1.3 (**Mapping cylinder**). Let $f : X \to Y$ be a map. Then the mapping cylinder of f is the following pushout space

$$\begin{tikzcd}
Mf & X \times I \arrow[l] \\
Y \arrow[u] \arrow[ur, phantom, "\ulcorner", very near start] & X \arrow[l, "f"] \arrow[u, "i_0"']
\end{tikzcd}.$$

More explicitly, it is $((X \times I) \amalg Y)/\sim$ where $(x, 0) \sim f(x)$ for all $x \in X$.

Let $f : X \to Y$ be a map. More pictorially, Mf is formed by gluing cylinder $X \times I$ to Y along f. In mind, one pictures a cylinder "popping out" of Y from where $f(X)$ lived in Y, as shown in the following diagram: A based version of mapping cylinder is as follows.

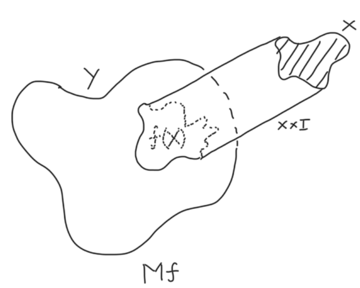


Figure 10.1: Schematic representation of mapping cylinder for $f : X \to Y$.

Definition 10.1.1.4 (**Based mapping cylinder**)**.** Let $f : X \to Y$ be a based map. The based mapping cylinder $M_* f$ is the pushout of reduced cylinder about f:

$$\begin{tikzcd} M_* f & X \wedge I_+ \arrow[l] \\ Y \arrow[u] \arrow[ur, phantom, "\lrcorner"] & X \arrow[l, "f"] \arrow[u, "i_0"'] \end{tikzcd} .$$

Indeed, we have the following lemma:

Proposition 10.1.1.5. *Let $i : A \to X$ be a map. Then the following are equivalent:*

1. *i is a cofibration.*
2. *i satisfies homotopy extension property for any $f : X \to Y$ and for any Y.*
3. *i satisfies homotopy extension property for the natural map $X \to Mi$ and the homotopy $h : A \times I \to Mi$ obtained from pushout.*

Proof. The only non-trivial part is to show $3 \Rightarrow 2$. Take any map $f : X \to Y$ and any homotopy $h : A \times I \to Y$. Consider

$$\begin{tikzcd} A \arrow[rrr, "i_0"] \arrow[dd, "i"'] & & & A \times I \arrow[dll] \arrow[dl, "i \times \mathrm{id}"] \arrow[dd, "h"] \\ & Mi \arrow[drr, dashed, "g"] & X \times I \arrow[l, dashed, "h_1"'] & \\ X \arrow[ur] \arrow[urr, "i_0"] \arrow[rrr, "f"'] & & & Y \end{tikzcd} .$$

The map h_1 is formed by homotopy extension property of i for $X \to Mi$ and g is formed by universal property of pushout which is Mi. The map $gh_1 : X \times I \to Y$ follows the required commutativity relations. $\square$

Consequently, we have the following result.

Proposition 10.1.1.6. *Any cofibration $i : A \to X$ is an inclusion with closed image.*

Proof. Consider the natural maps $j : X \to Mi$ and $h : A \times I \to Mi$ obtained by the pushout square. Since $hi_0 = ji$, therefore by Proposition 10.1.1.5, 3, we obtain a map $\tilde{h} : X \times I \to Mi$ fitting in the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"] \arrow[dr] & \\ X \arrow[r, "j"] \arrow[rr, bend right, "i_0"'] & Mi & X \times I \arrow[l, dashed, "\tilde{h}"] \end{tikzcd} .$$

Let $k : Mi \to X \times I$ be obtained by the following diagram

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"'] \arrow[ddr, "i\times\mathrm{id}"] & \\
X \arrow[r, "j"] \arrow[drr, "i_0"'] & Mi \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed, "k"] & \\
 & & X \times I
\end{tikzcd}.$$

It follows that $\tilde{h} \circ k : Mi \to Mi$ is id, that is, Mi is a retract of $X \times I$. Consequently, restricting onto $i(A)$, we see that $i(A)$ is a retract of $X \times I$, hence closed as $X \times I$ is compactly generated. It also follows from $\tilde{h} \circ k = \mathrm{id}$ that i is injective. $\square$

We see the following from the proof of the above result.

Corollary 10.1.1.7. *Let $i : A \to X$ be a map. Then the following are equivalent:*
1. *Map $i : A \to X$ is a cofibration.*
2. *Mapping cylinder Mi is a retract of $X \times I$.*

Proof. 1. $\Rightarrow$ 2. is immediate from the proof. For 2. $\Rightarrow$ 1. we seee that if $Mi \hookrightarrow X \times I \twoheadrightarrow Mi$ is a retract, then letting $\tilde{h} : X \times I \twoheadrightarrow Mi$, we have $\tilde{h} \circ i_0 = \mathrm{id}_X$ and $\tilde{h}\Big|_{A\times I} = h$, as needed. $\square$

Let $f : X \to Y$ be an arbitrary map of spaces. We can replace f by a cofibration followed by a homotopy equivalence.

Construction 10.1.1.8. (*Replacement by a cofibration and a homotopy equivalence*)
Let $f : X \to Y$ be a map of spaces. Consider the following commutative triangle:

$$\begin{tikzcd}
 & Mf \arrow[dr, "r"] & \\
X \arrow[ur, "j"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

where $Mf = Y \cup_f (X \times I)$ is the mapping cylinder and the other two maps are given as follows:

1. Map $j : X \to Mf$ is given by $x \mapsto (x, 1)$. We claim that j is a cofibration. Indeed, if $g : Mf \to Z$ is any map and we have a diagram as in Definition 10.1.1.1, then we can form the required homotopy $\tilde{h} : Mf \times I \to Z$ by defining
$$\tilde{h}([(x,s)],t) := \begin{cases} g(x) & \text{if } x \in Y \\ h(x,st) & \text{if } [(x,s)] \in X \times I. \end{cases}$$
We then see that $\tilde{h}(j \times \mathrm{id})(x,t) = \tilde{h}([(x,1)],t) = h(x,t)$ and that $\tilde{h}i_0([(x,s)]) = \tilde{h}([(x,s)],0) = h(x,0) = g(x)$. So we have the required extension and hence $j : X \to Mf$ is a cofibration.

2. Map $r : Mf \to Y$ is given by $r|_Y = \mathrm{id}_Y$ and $r|_{X\times I}(x,t) = f(x)$ for $t > 0$. We claim that r is a homotopy equivalence. For this, we have a map $i : Y \to Mf$ taking $y \mapsto [y]$. We then see that $ri = \mathrm{id}_Y$ and $ir \simeq \mathrm{id}_{Mf}$. The former is simple and the latter is established by the following homotopy $h : Mf \times I \to Mf$ mapping as $([(x,s)],t) \mapsto [(x,(1-t)s)]$ on $X \times I$ and $(y,t) \mapsto y$ on Y. This is indeed a homotopy from ir to id_{Mf}. Thus, $r : Mf \to Y$ establishes that Y is a deformation retract of the mapping cylinder Mf.

Hence, one can replace a map of spaces $f : X \to Y$ by a cofibration $j : X \to Mf$ followed by a homotopy equivalence $r : Mf \to Y$.

We now discuss an important characterization of cofibrations. For this we define first the following notion.

Definition 10.1.1.9 (**Neighborhood deformation retract**)**.** A pair (X, A) where $A \subseteq X$ is a neighborhood deformation retract (NDR) if there exists a map $u : X \to I$ such that $u^{-1}(0) = A$ and a homotopy $h : X \times I \to X$ such that $h(x, 0) = \mathrm{id}_X(x) = x$, $h(a,t) = a$ for all $a \in A$ and all $t \in I$ and $h(x, 1) \in A$ if $u(x) < 1$.

Remark 10.1.1.10. Let (X, A) be an NDR-pair. If $u(X) \subseteq [0, 1)$, then $A \hookrightarrow X$ is a closed subspace which is a deformation retract of X.

Theorem 10.1.1.11. *Let A be a closed subsapce of X. Then the following are equivalent:*
1. *(X, A) is an NDR-pair.*
2. *$i : A \to X$ is a cofibration.*

We now define the notion of homotopy equivalence under a space. This will come in handy later. Recall that if $\mathbf{C}$ is a category $c \in \mathbf{C}$ is an object, then $\mathbf{C}_{c/}$ denotes the under category at c, i.e., where objects are $i : c \to a$ and maps are commutative triangles

$$\begin{tikzcd} & c \arrow[dl, "i"'] \arrow[dr, "j"] & \\ a \arrow[rr, "f"'] & & b \end{tikzcd}.$$

Definition 10.1.1.12 (**Relative homotopy**)**.** Let $i : A \to X$ and $j : A \to Y$ be in $\mathbf{Top}^{cg}_{A/}$. Let $f, g : X \rightrightarrows Y$ be maps in $\mathbf{Top}^{cg}_{A/}$. Then $h : X \times I \to Y$ is a homotopy rel A between f and g if $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ and $h(i(a), t) = j(a)$ for all $a \in A$ and $t \in I$.

The notion of homotopy equivalence rel A is special as the Theorem 10.1.1.14 shows, hence we give it the following name.

Definition 10.1.1.13 (**Cofiber homotopy equivalence**)**.** Let $i : A \to X$ and $j : A \to Y$ be two spaces under A in $\mathbf{Top}^{cg}_{A/}$. If i and j homotopy equivalent under A, then X and Y are said to be cofiber homotopy equivalent.

Theorem 10.1.1.14. *Let $i : A \to X$ and $j : A \to Y$ be two cofibrations under A and $f : X \to Y$ be a map under A. If f is a homotopy equivalence, then f is a cofiber homotopy equivalence.*

Example 10.1.1.15. Let $i : A \to X$ be a cofibration. Then by Construction 10.1.1.8, we have

$$\begin{tikzcd} & Mi \arrow[dr, "r"] & \\ A \arrow[ur, "j"] \arrow[rr, "i"'] & & X \end{tikzcd}$$

where j is a cofibration and r is a homotopy equivalence. Since r is a homotopy equivalence under A, therefore by Theorem 10.1.1.14, r is a cofiber homotopy equivalence. Consequently, there is a homotopy inverse $\kappa : X \to Mi$ of r under A.

The following is a mild generalization of Theorem 10.1.1.14 in the sense that we allow mapping between two cofibration pairs now.

Proposition 10.1.1.16. *Let (X, A) and (Y, B) be two cofibration pairs and let $f : X \to Y$ and $d : A \to B$ be maps such that $f|_A = d$. If f and d are homotopy equivalences, then the map of pairs $(f, d) : (X, A) \to (Y, B)$ is a homotopy equivalence of pairs*[2]*.*

We next portray how a cofibration pair (X, A) in some cases behaves homotopically same as the quotient X/A.

Proposition 10.1.1.17. *Let $i : A \to X$ be a cofibration and A be contractible. Then the quotient map $p : X \twoheadrightarrow X/A$ is a homotopy equivalence.*

Proof. As A is contractible, therefore for some $x_0 \in A$, we have a homotopy $h : A \times I \to A$ such that $h_0 = \mathrm{id}_A$ and $h_1 = c_{x_0}$. Consequently, we obtain $\tilde{h}$ as in the commutative square

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"] \arrow[dr, "i\times\mathrm{id}"] & \\ X \arrow[r, "\mathrm{id}"] \arrow[rr, bend right, "i_0"'] & X & X \times I \arrow[l, dashed, "\tilde{h}"'] \end{tikzcd}$$

where we have $\tilde{h}_0 = \mathrm{id}_X$, $\tilde{h}_t(A) \subseteq A$ for all $t \in I$ and $\tilde{h}_1(A) = \{x_0\} \in A$. Consequently, $\tilde{h}_1$ fits in the following diagram

$$\begin{tikzcd} X \arrow[d, two heads, "p"'] \arrow[dr, "\tilde{h}_1"] & \\ X/A \arrow[r, "g"'] & X \end{tikzcd}$$

where $g : X/A \to X$ comes from the universal property of quotients. We claim that g is the required homotopy inverse of p. Indeed, by definition $\tilde{h} : \mathrm{id}_X \simeq g \circ p$. Consequently, we need only show that $\mathrm{id}_{X/A} \simeq p \circ g$. We derive this homotopy from $\tilde{h}$ as well. Indeed, for any $t \in I$, we obtain $\tilde{q}_t$ by universal property of quotients as in

$$\begin{tikzcd} X \arrow[r, "\tilde{h}_t"] \arrow[d, two heads, "p"'] & X \arrow[d, two heads, "p"] \\ X/A \arrow[r, "\tilde{q}_t"'] & X/A \end{tikzcd}.$$

[2]as defined in Definition 10.3.1.1.

It follows that the homotopy $\tilde{q} : X/A \times I \to X/A$ is such that $\tilde{q}_0 = \mathrm{id}_{X/A}$ and $\tilde{q}_1 = p \circ g$, as needed. □

Let us end this section by discussing how we will tell the same story in the based setting.

Remark 10.1.1.18 (*Based cofibration*)**.** A based map $i : A \to X$ is a based cofibration if it satisfies the based version of homotopy extension property. The following are few remarks which are easily verifiable of the situation in the based case.

1. If a based map $i : A \to X$ is an unbased cofibration, then it is a based cofibration.
2. If $A \subseteq X$ is a closed subspace such that $* \to A$ and $* \to X$ are cofibrations and $i : A \to X$ is a based cofibration, then $i : A \to X$ is an unbased cofibration.
3. A based map $i : A \to X$ is a based cofibration if and only if M_*i is a retract of $X \wedge I_+$.

We see the following example of above remark.

Lemma 10.1.1.19. *Let X be a based space. Then the inclusion $X \hookrightarrow CX$ to the base of the cone*

1. *is a deformation retract,*
2. *is a cofibration.*

Proof. The inclusion map is $x \mapsto [x, 0]$. The fact that X is deformation retract is immediate by the based homotopy $h : CX \times I \to CX$ given by $([x, t], s) \mapsto [x, t(1 - s)]$. We will use Remark 10.1.1.18, 3 for showing $i : X \hookrightarrow CX$ is a cofibration. Indeed, consider the map $CX \wedge I_+ \to M_*i$ given by $[[x, t], s] \mapsto [x, s + t]$. The inclusion $M_*i \to Y \wedge I_+$ is the map which on CX is $[x, t] \mapsto [[x, t], 0]$ and on $X \wedge I_+$ is $[x, t] \mapsto [[x, 0], t]$. One checks that this makes M_*i a retract of $CX \wedge I_+$. □

10.1.2 Based cofiber sequences

The main point of cofiber sequences is to obtain an exact sequence of groups, which will prove to be helpful later. All cofibrations in this section are based cofibrations. We first observe that $[\Sigma X, Y]$ is a group.

Proposition 10.1.2.1. *Let X, Y be based spaces. Then*

1. *$[\Sigma X, Y]$ is a group under concatenation,*
2. *$[\Sigma^2 X, Y]$ is an abelian group under the same operation.*

Proof. The concatenation operation here is as follows : for $f, g \in \mathrm{Map}_*(\Sigma X, Y)$, define $f + g$ as

$$(f + g)([(x, t)]) := \begin{cases} f([(x, 2t)]) & \text{if } 0 \leq t \leq 1/2 \\ g([x, 2t - 1]) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

This tells us that $[\Sigma X, Y] \cong [X, \Omega Y]$ is a group. The second statement uses Theorem 10.0.0.8 to observe that a map $\Sigma^2 X \to Y$ is a map $S^1 \wedge S^1 = S^2 \to \mathrm{Map}_*(X, Y)$. □

Definition 10.1.2.2. (**Homotopy cofiber/Mapping cone**) Let $f : X \to Y$ be a based map and let $j : X \to M_*f$, $x \mapsto (x, 1)$ be it's cofibrant replacement. The homotopy cofiber

Cf of f is defined to be the quotient of the based mapping cylinder M_*f of f by the image of the map j taking $x \mapsto (x, 1)$. That is,

$$Cf := M_*f/j(X).$$

Alternatively, it is the pushout $Cf = Y \cup_f CX$.

There is a relationship between unbased cofiber and based cofiber.

Lemma 10.1.2.3. *Let X be an unbased space. Then the unreduced cone of X is isomorphic to the reduced cone of pointification of X. That is,*

$$CX \cong CX_+.$$

Proof. We have

$$\begin{aligned} CX_+ = X_+ \wedge I = \frac{X_+ \times I}{\{\text{pt.}\} \times I \amalg X \times \{1\}} &= \frac{X \times I \amalg \{\text{pt.}\} \times I}{\{\text{pt.}\} \times I \amalg X \times \{1\}} \\ \cong \frac{X \times I}{X \times \{1\}} = CX, \end{aligned}$$

as needed. □

This is an important observation, as it says that unreduced homotopy cofiber is isomorphic to the homotopy cofiber of the poinitification.

Proposition 10.1.2.4. *Let X, Y be unbased spaces and $f : X \to Y$ be an unbased map. Then the unreduced homotopy cofiber of f is isomorphic to the homotopy cofiber of $f_+ : X_+ \to Y_+$. That is,*

$$Cf \cong Cf_+.$$

Proof. By Lemma 10.1.2.3, we can write

$$Cf_+ = Y_+ \cup_{f_+} CX_+ \cong Y_+ \cup_{f_+} CX$$

where $X_+ \to CX$ is the map which takes pt. $\mapsto [x, 1]$ as the basepoint of CX is $[x, 1]$. Consequently, $Y_+ \cup_{f_+} CX$ is isomorphic to $Y \cup_f CX$. □

Remark 10.1.2.5. It follows from Proposition 10.1.2.4 that there is really no difference between reduced and unreduced cofiber as unreduced cofiber is really a special case of reduced cofiber by pointification.

The following result shows that the homotopy cofiber of a based cofibration is is of the same homotopy type as X/A. This is an important property of cofibrations.

Proposition 10.1.2.6. *Let $i : A \to X$ be a based cofibration between based spaces. Then,*

1. *$Ci/CA \cong X/A$,*
2. *$\pi : Ci \to Ci/CA$ is a based homotopy equivalence.*

Proof. **TODO.** □

Pictorially, one sees that the mapping cone Cf of $f : X \to Y$ is obtained by gluing Y to the cone of X at it's base. We are now ready to construct cofiber sequence of a based map $f : X \to Y$.

Construction 10.1.2.7. (*Cofiber sequence*) Let $f : X \to Y$ be a based map and denote Cf to be the mapping cone of f. We have a natural map $i : Y \to Cf$ which is the inclusion of Y into the mapping cone. This is a cofibration because it is the pushout (Lemma 10.1.1.2) of the inclusion $X \to CX$ of X into the 0-th level of the cone CX and this inclusion is a cofibration (Lemma 10.1.1.19). The sequence $X \to Y \to Cf$ is called the *short cofiber sequence of f*.

Consider also the map $-\Sigma f : \Sigma X \to \Sigma Y$ which maps $[(x, t)] \mapsto [(f(x), 1 - t)]$. We have another natural map from the mapping cone to its quotient by Y given by $\pi : Cf \to Cf/Y \cong \Sigma X$. We then get the following sequence of based maps, called the *long cofiber sequence of map f*:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, tail, "i"] & Cf \arrow[r, two heads, "\pi"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \arrow[r, "-\Sigma i"] & \Sigma Cf \arrow[dlllll, "-\Sigma\pi"] \\
\Sigma^2 X \arrow[r, "\Sigma^2 f"'] & \Sigma^2 Y \arrow[r, "\Sigma^2 i"'] & \Sigma^2 Cf \arrow[r, "\Sigma^2 \pi"'] & \Sigma^3 X \arrow[r, "-\Sigma^3 f"'] & \Sigma^3 Y \arrow[r, "-\Sigma^3 i"'] & \Sigma^3 Cf \cdots
\end{tikzcd}.$$

The main theorem that will be used continuously elsewhere is that cofiber sequence of a map gives a long exact sequence in homotopy sets. First, recall that for any based space Z, we have the homotopy classes of maps $[X, Z]$. Moreover, $[-, Z]$ is contravariantly functorial as for any based map $f : X \to Y$, we get

$$\begin{aligned}
[f, Z] : [Y, Z] &\longrightarrow [X, Z] \\
g &\longmapsto g \circ f.
\end{aligned}$$

We are now ready to state the main theorem.

Theorem 10.1.2.8 (*Main theorem of cofiber sequences*)**.** *Let $f : X \to Y$ be a based map and Z be a based space in $\mathbf{Top}_*^{cg}$. Then the functor $[-, Z]$ applied on the long cofiber sequence of f yields a long exact sequence of based sets:*

$$\begin{tikzcd}
\cdots [\Sigma^3 Cf, Z] & {[\Sigma^3 Y, Z]} \arrow[l] & {[\Sigma^3 X, Z]} \arrow[l] & {[\Sigma^2 Cf, Z]} \arrow[l] & {[\Sigma^2 Y, Z]} \arrow[l] & {[\Sigma^2 X, Z]} \arrow[l] \arrow[dlllll] \\
{[\Sigma Cf, Z]} & {[\Sigma Y, Z]} \arrow[l] & {[\Sigma X, Z]} \arrow[l] & {[Cf, Z]} \arrow[l] & {[Y, Z]} \arrow[l] & {[X, Z]} \arrow[l]
\end{tikzcd}.$$

The proof of this theorem relies on the following fundamental observation.

Proposition 10.1.2.9. *Let $f : X \to Y$ be a based map and Z be a based space. Consider the short cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} Cf.$$

Then the sequence of based sets

$$[Cf, Z] \longrightarrow [Y, Z] \longrightarrow [X, Z]$$

is exact.

Proof. Let $g \in [Y, Z]$ such that $gf \simeq c_*$ in $[X, Z]$. We wish to show that there is a map $k \in [Cf, Z]$ such that $ki \simeq g$ in $[Y, Z]$. We first have a based homotopy $h : X \times I \to Z$ between gf and c_*. As h is constant on $X \vee I$, therefore we obtain a map $\bar{h} : CX \to Z$. Note that the following pushout diagram commutes so to give a unique map $k : Cf \to Z$

$$\begin{tikzcd}
Z & & \\
& Cf \arrow[ul, dashed, "k"'] \arrow[dr, phantom, "\ulcorner"] & CX \arrow[l] \arrow[ull, bend right, "\bar{h}"'] \\
& Y \arrow[u, "i"] \arrow[uul, bend left, "g"] & X \arrow[l, "f"] \arrow[u, "i_0"']
\end{tikzcd}$$

Hence we have that $ki = g$, hence we don't even need to construct a homotopy between ki and g. ∎

We will now show that each term in the cofiber sequence is obtained by taking cofiber of the previous map. For that, we would need the following small result.

Lemma 10.1.2.10. *Let $f : X \to Y$ be a based map. Then,*

1. *We have a natural based homeomorphism $\Sigma Cf \cong C\Sigma f$.*
2. *The suspension functor takes the short cofiber sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} Cf$$

to a short cofiber sequence

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma i} \Sigma Cf.$$

Proof. The first one follows from Σ being a left adjoint. The second statement follows from first statement as we have the following isomorphism

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma i"] \arrow[dr, hook] & \Sigma Cf \arrow[d, "\cong"] \\
& & C\Sigma f
\end{tikzcd}.$$

∎

Proposition 10.1.2.11. *Let $f : X \to Y$ be a based map. Then each consecutive pair of maps in the long cofiber sequence of f is a short cofiber sequence.*

Proof. Note that the following square commutes

$$\begin{tikzcd}
\Sigma Cf \arrow[r, "\Sigma\pi"] \arrow[d, "\cong"'] & \Sigma^2 X \arrow[r, "-\Sigma^2 f"] \arrow[d, "\tau"] & \Sigma^2 Y \arrow[d, equal] \\
C\Sigma f \arrow[r, "\pi'"'] & \Sigma^2 X \arrow[r, "\Sigma^2 f"'] & \Sigma^2 Y
\end{tikzcd}$$

where $\tau([x,t,s]) = [x,s,t]$ is a homeomorphism and $\pi' : C\Sigma f \to C\Sigma f/\Sigma Y$ is the quotient map. We claim that τ is homotopic to $-\mathrm{id}$, where $(-\mathrm{id})([x,t,s]) = [x,t,1-s]$. With this claim and Lemma 10.1.2.10, we would reduce to showing that $Y \to Cf \to \Sigma X$ and $Cf \to \Sigma X \to \Sigma Y$ in the cofiber sequence of f are short cofiber sequences.

To see a based homotopy between τ and $-\mathrm{id}$ as based maps $\Sigma^2 X \to \Sigma^2 X$, we see that the following map will work

$$\begin{aligned} h : \Sigma^2 X \times I &\longrightarrow \Sigma^2 X \\ ([x,t,s],r) &\longmapsto [x,(1-r)s+rt,(1-r)t+r(1-s)]. \end{aligned}$$

We now wish to show that the two pairs are short cofiber sequenes. The fact that $Y \to Cf \to \Sigma X$ is a short cofiber sequence is immediate from Proposition 10.1.2.6 as it will yield the following diagram

$$\begin{tikzcd} & & Ci \arrow[d, dash, "\simeq"] \\ Y \arrow[r, "i"'] & Cf \arrow[ur, "\pi'"] \arrow[r, "\pi"'] & \Sigma X \end{tikzcd} .$$

The fact that $Cf \to \Sigma X \to \Sigma Y$ is also a short cofiber sequence follows from the following diagram which we claim to be commutative

$$\begin{tikzcd} Cf \arrow[r, "\pi"] \arrow[d, equal] & \Sigma X \arrow[r, "-\Sigma f"] \arrow[d, dash, "\simeq"] & \Sigma Y \arrow[d, dash, "\cong"] \\ Cf \arrow[r, "\pi'"'] & Ci \arrow[r, "\pi''"'] & Ci/Cf \end{tikzcd} .$$

TODO $\square$

10.2 Fibrations and fiber sequences

We now study fibrations, which is a generalization of covering spaces. Indeed, recall that covering spaces satisfies homotopy lifting property. That *becomes* the definition of a fibration. Indeed, one can have a fruitful time reading about fibrations by keeping the basic results about covering spaces in mind. We'll see that familiar objects from geometry are fibrations (fiber bundles, for example).

10.2.1 Definition and first properties

Definition 10.2.1.1 (**Fibrations**). A surjective map $p : E \to B$ is a fibration if it satisfies homotopy lifting property. That if, for any map $f : Y \to E$ and any homotopy $h : Y \times I \to B$ such that $p \circ f = h \circ i_0$, there exists $\tilde{h} : Y \times I \to E$ such that the following commutes

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd} .$$

Just as pushouts of cofibrations along any map is a cofibration, we have pullback of a fibration along any map is a fibration.

Lemma 10.2.1.2. *Let $p : E \to B$ be a fibration and $g : A \to B$ be any map. Then the pullback of p along g given by $p' : E \times_B A \to A$ is a fibration.*

Proof. Consider the following diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \times_B A \arrow[r, "\pi"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] & A \arrow[r, "g"'] & B \end{tikzcd}.$$

As p is a fibration, we yield a homotopy $\tilde{h}_1 : Y \times I \to E$ as in

$$\begin{tikzcd} Y \arrow[r, "\pi f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "gh"'] \arrow[ur, dashed, "\tilde{h}_1"] & B \end{tikzcd}.$$

Consequently, we get a pullback diagram

$$\begin{tikzcd} Y \times I \arrow[r, dashed, "!\tilde{h}"] \arrow[rr, bend left, "\tilde{h}_1"] \arrow[dr, "h"'] & E \times_B A \arrow[r, "\pi"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ & A \arrow[r, "g"'] & B \end{tikzcd}$$

which yields $\tilde{h} : Y \times I \to E \times_B A$. We claim that this is the required homotopy extension. We immediately have $p'\tilde{h} = h$ from the above diagram. We need only show that $\tilde{h}i_0 = f$. To this end, consider the following pullback square

$$\begin{tikzcd} Y \arrow[r, dashed, "!\kappa"] \arrow[rr, bend left, "\pi f"] \arrow[dr, "hi_0"'] & E \times_B A \arrow[r, "\pi"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ & A \arrow[r, "g"'] & B \end{tikzcd}$$

which yields a unique $\kappa : Y \to E \times_B A$. It follows that both f and $\tilde{h}i_0$ satisfies the same commutation properties as κ. It follows from uniqueness of κ w.r.t. these properties that $\tilde{h}i_0 = f$, as required. □

We now introduce a sort of intermediary space for further studying fibrations.

Definition 10.2.1.3 (Mapping path space). Let $f : X \to Y$ be a map. The mapping path space Nf is defined to be the following pullback

$$\begin{tikzcd} Nf := X \times_Y Y^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y^I \arrow[r, "p_0"'] & Y \end{tikzcd}$$

where $p_0 : Y^I \to Y$ takes $\gamma \mapsto \gamma(0)$.

Remark 10.2.1.4. Consequently, the mapping path space $Nf = \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(0)\}$. Hence a point in Nf is the data of a point $x \in X$ *upstairs* and a path $\gamma \in Y^I$ starting *downstairs* at the image of x under f.

With regards to mapping path spaces, one important type of function Nf is that of a *path lifters*.

Definition 10.2.1.5 (Path lifters). Let $f : X \to Y$ be a map. Let $k : X^I \to Nf$ be the unique map obtained by the following pullback diagram

$$\begin{tikzcd}
X^I \arrow[r, dashed, "k"] \arrow[rr, bend left, "p_0"] \arrow[dr, "f^I"'] & Nf \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& Y^I \arrow[r, "p_0"'] & Y
\end{tikzcd}.$$

A path lifter $s : Nf \to X^I$ is a global section of k, i.e. $k \circ s = \mathrm{id}_{Nf}$.

Remark 10.2.1.6. The main content of a path lifter $s : Nf \to X^I$ is the fact that its a global section of k. That is, if we let $\tilde{\gamma} = s(x, \gamma) \in X^I$, then $k(\tilde{\gamma}) = (p_0(\tilde{\gamma}), f \circ \tilde{\gamma}) = (x, \gamma)$. It follows that $s(x, \gamma) = \tilde{\gamma}$ is a lift of the path $\gamma \in Y^I$ starting at $f(x)$ to a path $\tilde{\gamma} \in X^I$ starting at x. We may keep the following picture in mind (Figure 10.2).

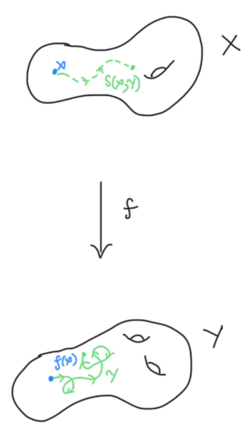


Figure 10.2: Path lifter s taking (x, γ) downstairs to a lift $s(x, \gamma)$ in X upstairs.

Remark 10.2.1.7. (*Covering maps have a unique path lifter*). Recall that a covering space $p : E \to B$ has *unique* homotopy lifting property, hence in particular it is a cofibration. Furthermore recall that a covering space also has *unique* path lifting property, hence in particular it has a unique path-lifter.

We have the following reduction of fibration criterion to mapping path space.

Proposition 10.2.1.8. *Let $p : E \to B$ be a surjective map. Then the following are equivalent:*

1. *p is a fibration.*
2. *p satisfies homotopy lifting property for the natural projection map $Np \to E$.*

Proof. 1. $\Rightarrow$ 2. is definition. For 2. $\Rightarrow$ 1. we proceed as follows. Consider the following diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] & Np \arrow[l, "\pi"'] \arrow[d, "\eta"] \arrow[dl, phantom, "\llcorner"] \\ Y \times I \arrow[r, "h"'] & B & B^I \arrow[l, "p_0"] \end{tikzcd} .$$

We may write $h : Y \times I \to B$ as $h^T : Y \to B^I$. Observe that $p_0 h^T = pf$, leading to the following unique map $\kappa : Y \to Np$ as below

$$\begin{tikzcd} Y \arrow[r, dashed, "\kappa"] \arrow[rr, bend left, "f"] \arrow[dr, "h^T"'] & Np \arrow[r, "\pi"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ & B^I \arrow[r, "p_0"'] & B \end{tikzcd} .$$

Similar to h^T, we also have $\eta^T : Np \times I \to B$. It is immediate from $\eta\kappa = h^T$ that $\eta^T(\kappa \times \mathrm{id}) = h : Y \times I \to B$. Consequently, we have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\kappa"] \arrow[rr, bend left, "f"] \arrow[d, "i_0"'] & Np \arrow[r, "\pi"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "\kappa\times\mathrm{id}"'] \arrow[rr, bend right, "h"'] & Np \times I \arrow[r, "\eta^T"'] \arrow[ur, dashed, "\tilde{\eta}^T"'] & B \end{tikzcd}$$

and composing $\tilde{\eta}^T$ with $\kappa \times \mathrm{id}$ yields the required lift of h. $\square$

Let $f : X \to Y$ be an arbitrary map of spaces. We can replace f by a homotopy equivalence followed by a fibration.

Construction 10.2.1.9. (*Replacement by a homotopy equivalence and a fibration*).
Let $f : X \to Y$ be a map. Consider the following commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\nu"'] & & Y \\ & Nf \arrow[ur, two heads, "\rho"'] & \end{tikzcd}$$

where

$$\begin{aligned} \nu : X &\longrightarrow Nf \\ x &\longmapsto (x, c_{f(x)}) \end{aligned}$$

and

$$\begin{aligned} \rho : Nf &\longrightarrow Y \\ (x,\gamma) &\longmapsto \gamma(1). \end{aligned}$$

We now make the following claims:

1. Map ν is a homotopy equivalence. Indeed, consider the natural projection map $\pi : Nf \to X$ given by $(x,\gamma) \mapsto x$. We claim that π is a homotopy inverse of ν. Indeed, $\pi\nu = \mathrm{id}_X$ is immediate. We claim $\nu\pi \simeq \mathrm{id}_{Nf}$. Indeed, we may consider the homotopy

$$\begin{aligned} h : Nf \times I &\longrightarrow Nf \\ ((x,\gamma),t) &\longmapsto (x,\gamma_t) \end{aligned}$$

where $\gamma_t(s) = \gamma((1-t)s)$.

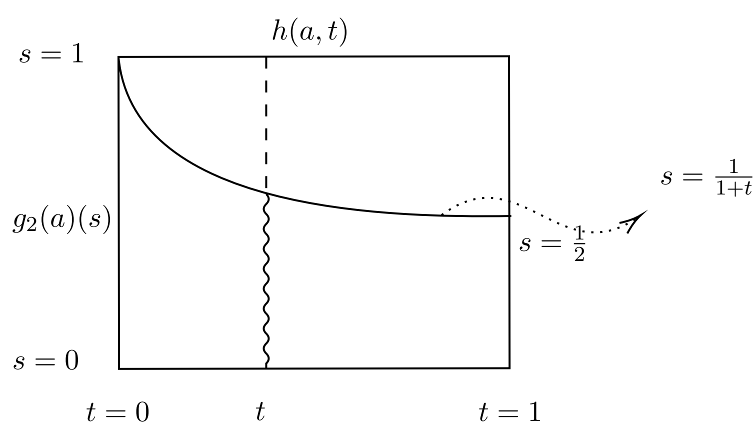
2. Map ρ is a fibration. Let $g : A \to Nf$ be a map such that the following square commutes

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"'] & Nf \arrow[d, "\rho"] \\ A \times I \arrow[r, "h"'] & Y \end{tikzcd} .$$

We wish to construct $\tilde{h} : A \times I \to Nf$ which would lift h. Indeed, let $g(a) = (g_1(a), g_2(a))$ where $g_1 : A \to X$ and $g_2 : A \to Y^I$ are the component functions. In order to construct $\tilde{h}$, we need only construct $\alpha : A \times I \to Y^I$ and $\beta : A \times I \to X$ such that the following holds (these are obtained by unravelling $\rho\tilde{h} = h$, $\tilde{h}i_0 = g$ and the respective pullback square):

(a) $f\beta = p_0\alpha$,
(b) $\beta(a,0) = g_1(a)$,
(c) $\alpha(a,0) = g_2(a)$,
(d) $\alpha(a,t)(1) = h(a,t)$.

We may immediately set $\beta(a,t) = g_1(a)$. For $\alpha : A \times I \to Y^I$, we may dually write α as $\alpha : A \times I \times I \to Y$ (recall we are in compactly generated spaces, where the dual notion of homotopy is equivalent to the usual one). We construct this α as follows. Fix $a \in A$. We then define the following homotopy



which more explicitly is given by

$$\alpha(a,t,s) = \begin{cases} g_2(a)(s\cdot(1+t)) & \text{if } 0 \le s \le \frac{1}{1+t} \\ h(a, s\cdot(1+t)-1) & \text{if } \frac{1}{1+t} \le s \le 1. \end{cases}$$

One can then observe that this α satisfies conditions (a), (c) and (d) mentioned above.

10.2.2 Bundles and change of fibers

We now see that, under some mild hypothesis, fibration is a local property on base. As a consequence, we will show that under some mild hypothesis any bundle (Definition 8.7.1.2) is a fibration.

An open cover $\{U_\alpha\}$ of B will be called *numerable* if for each α, there is a map $f_\alpha : B \to I$ such that $f_\alpha^{-1}((0,1]) = U_\alpha$ and $\{U_\alpha\}$ is a locally finite cover.

Theorem 10.2.2.1. *Let $p : E \to B$ be a map and $\{U_\alpha\}$ be a numerable open cover of B. Then the following are equivalent:*

1. *$p : E \to B$ is a fibration.*
2. *$p : p^{-1}(U_\alpha) \to U_\alpha$ is a fibration for each α.*

Proof. 1. $\Rightarrow$ 2. is immediate from Lemma 10.2.1.2.

(2. $\Rightarrow$ 1.) The main idea is to patch up the lifts of a homotopy that we obtain by virtue of each $p|_{p^{-1}(U_\alpha)}$ being a fibration. **TODO**. □

We claimed in the beginning that fibrations are upto homotopy generalizations of covering spaces/certain bundles. We know that such objects have homeomorphic fibres (say, when base is path-connected). This fact can be generalized to fibrations which would yield that fibres of a fibration may not be homeomorphic, but will be of same homotopy type!

Construction 10.2.2.2. (*Homotopy invariance of path-lifting for fibrations*). We now show that a path γ in the base gives a map of fibers which is invariant under homotopy class of γ.

In particular, let $p : E \to B$ be a fibration and $\gamma : I \to B$ be a path from b to b' in B. Let E_b and $E_{b'}$ be fibers at b and b' respectively under p. We claim that we get a map $\tilde{\gamma} : E_b \to E_{b'}$ whose homotopy class is independent of the path γ upto homotopy.

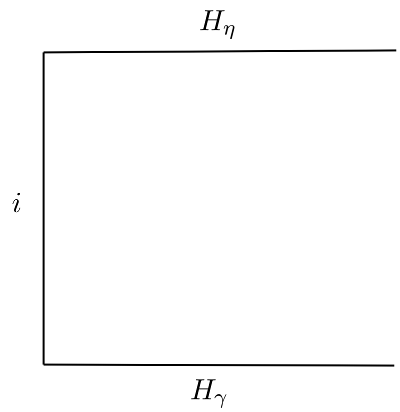
We first construct $\tilde{\gamma} : E_b \to E_{b'}$. Indeed, we have the following diagram

$$\begin{tikzcd} E_b \arrow[rr, "i"] \arrow[d, "i_0"'] & & E \arrow[d, "p"] \\ E_b \times I \arrow[r, "\pi_2"'] \arrow[urr, dashed, "H_\gamma"] & I \arrow[r, "\gamma"'] & B \end{tikzcd}$$

by virtue of fibration p. Observe that $H_{\gamma,1}(e) = H_\gamma(e,1)$ is such that $pH_\gamma(e,1) = \gamma(1) = b'$ for all $e \in E_b$. Consequently, $\tilde{\gamma} = H_{\gamma,1} : E_b \to E_{b'}$ is the required map. This shows the construction of $\tilde{\gamma}$. We now show that its homotopy class is invariant of homotopy class of γ.

Let $\gamma, \eta \in B^I$ be two paths joining b and b' together with a homotopy $h : I \times I \to B$ rel $\{0,1\}$ such that $h_0 = \gamma$ and $h_1 = \eta$, that is h is a homotopy between γ and η through paths joining b and b'. We wish to show that $\tilde{\gamma}$ and $\tilde{\eta}$ are homotopy equivalent as well. To this end, we need to construct a homotopy $\tilde{h} : E_b \times I \to E_{b'}$ satisfying $\tilde{h}_0 = \tilde{\gamma} = H_{\gamma,1}$ and $\tilde{h}_1 = \tilde{\eta} = H_{\eta,1}$.

Fix an $e \in E_b$. Our goal is to fill the right side of this square continuously with $e \in E_b$



where $i : E_b \hookrightarrow E$ the inclusion. To this end, we first observe that there is a homeomorphism of pairs

$$(I \times I, S) \xrightarrow{\alpha} (I \times I, I \times 0)$$

where S is the union of three sides of the square as shown above; $S = I \times \{0,1\} \cup \{0\} \times I$. Using this homeomorphism, we obtain the following square

$$\begin{tikzcd}
E_b \times S \arrow[rr, "f"] \arrow[d, "k"'] & & E \arrow[d, "p"] \\
E_b \times I \times I \arrow[r, "\kappa"'] \arrow[urr, dashed, "l"] & I \times I \arrow[r, "h"'] & B
\end{tikzcd}$$

where $k = \iota(\mathrm{id} \times \alpha)$ where $\iota : E_b \times (I \times 0) \hookrightarrow E_b \times (I \times I)$ and $\kappa(e,t,s) = \alpha^{-1}(t,s)$. Moreover, $f : E_b \times S \to E$ is defined as in the incomplete square above; on $I \times \{0\}$, f is given by H_γ, on $I \times \{1\}$, f is given by H_η and on $0 \times I$, f is given by i. .Observe that $\kappa k(e,t,s) = (t,s)$. The fact that this is a commutative square is immediate. It follows from p being a fibration that there is a lift $l : E_b \times I \times I \to E$ which fits in the above commutative square. Consequently, we have $pl = h\pi_2$ and $lk = f$. By appropriately composing l with α and replacing l with this composition, we get that $l : E_b \times I \times I$ which is given by following schematic homotopy cube

TODO!

Consequently, we get the following map $\tilde{h} : E_b \times I \to E_{b'}$ where

$$\tilde{h}(-,s) := l(-,1,s) : E_b \times I \to E_{b'}$$

where $l(e,1,s) \in E_{b'}$ because $h(1,s) \in b'$ (h is a homotopy through paths joining b and b'). Moreover, $\tilde{h}(e,0) = l(e,1,0) = H_{\gamma,1}(e) = \tilde{\gamma}(e)$ and $\tilde{h}(e,1) = H_{\eta,1}(e) = \tilde{\eta}(e)$. Thus, $\tilde{h}$ is the required homotopy between $\tilde{\gamma}$ and $\tilde{\eta}$.

10.2.3 Based fiber sequences

10.3 Homology, cohomology and properties

Give a remark of viewing singular cohomology as ES-axioms and also for manifolds as sheaf cohomology.

We will begin by introducing (co)homology from an axiomatic point of view and will derive few properties off of it. This will come in handy for discussing the main properties of differential manifolds in (co)homological language, especially characteristic classes and orientations and what not. The main thing that we wish to do is the Hurewicz theorem, which will allow us to connect homotopy groups and homology groups on the one hand, and will allow us to prove the uniqueness of homology theories for CW complexes on the other hand.

All spaces X are assumed to be compactly generated (Definition 10.0.0.1).

We will use the theory of cofibrations and fibrations as developed above quite freely.

10.3.1 Homology theories

We begin with the category of pairs on which homology theories are defined.

Definition 10.3.1.1 ($\mathbf{Top}_2$)**.** The $\mathbf{Top}_2$ denotes the category of pairs (X, A) of spaces where $A \hookrightarrow X$ and maps $(X, A) \to (Y, B)$ which consists of the pair $f : X \to Y$ and $g : A \to B$ such that $g = f|_A$. A map of pairs $(f, d) : (X, A) \to (Y, B)$ is said to be a homotopy equivalence if there is a map of pairs $(g, e) : (Y, B) \to (X, A)$ and there are homotopies $H : g \circ f \simeq \mathrm{id}_X$ and $K : f \circ g \simeq \mathrm{id}_Y$ which extends the homotopies $h : e \circ d \simeq \mathrm{id}_A$ and $k : d \circ e \simeq \mathrm{id}_B$ respectively.

Definition 10.3.1.2. (**Homology theory**) A homology theory for a group π is a sequence of functors

$$H_q(-, -; \pi) : \mathbf{Top}_2 \longrightarrow \mathbf{AbGrp}$$

for $q \in \mathbb{Z}$ equipped with natural transformations

$$\partial : H_q(-, -; \pi) \longrightarrow H_{q-1}(-, -; \pi)$$

whose component at (X, A) is given by $\partial : H_q(X, A; \pi) \to H_{q-1}(A, \emptyset; \pi)$. Denote $H_q(X; \pi) := H_q(X, \emptyset; \pi)$. This data must satisfy the following axioms:

1. *(Homology of a point)* : If $X = \{\text{pt.}\}$, then homology must be concentrated at degree 0:

$$H_q(\{\text{pt.}\}; \pi) = \begin{cases} \pi & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

2. (*Homology long exact sequence*) : The trivial inclusions $A \hookrightarrow X$ and $(X, \emptyset) \hookrightarrow (X, A)$

induces the following long exact sequence:

$$
\begin{tikzcd}
 & & \cdots \arrow[dll, dashed, "\partial"'] \\
H_q(A;\pi) \arrow[r] & H_q(X;\pi) \arrow[r] & H_q(X,A;\pi) \arrow[dll, dashed, "\partial"'] \\
H_{q-1}(A;\pi) \arrow[r] & H_{q-1}(X;\pi) \arrow[r] & H_{q-1}(X,A;\pi) \arrow[dll, dashed, "\partial"'] \\
\cdots & &
\end{tikzcd}
.
$$

3. *(Excision invariance)* : For an excisive triple (X, A, B), that is $A, B \hookrightarrow X$ and $X = A^\circ \cup B^\circ$, the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism at all degree $q \in \mathbb{Z}$:
$$H_q(A, A \cap B; \pi) \xrightarrow{\cong} H_q(X, B; \pi) \ .$$
4. *(Coproduct preserving)* : If (X_i, A_i) is an arbitrary collection of objects in $\mathbf{Top_2}$, then the homology in any degree of their disjoint union is the sum of the corresponding homology groups:
$$\bigoplus_i H_q(X_i, A_i; \pi) \xrightarrow{\cong} H_q\left(\coprod_i (X_i, A_i); \pi\right)$$
where the maps are induced by the inclusions $(X_{i_0}, A_{i_0}) \hookrightarrow \coprod_i (X_i, A_i)$.
5. *(π_*-insensitivity)* : If $f : (X, A) \longrightarrow (Y, B)$ is a weak equivalence, then in all degrees the corresponding homology groups are isomorphic:
$$f_p : H_q(X, A; \pi) \xrightarrow{\cong} H_q(Y, B; \pi).$$

Remark 10.3.1.3. In nature, there are some homology theories which satisfy all of the above axioms except the dimension axiom, that is, the group that they assign to a point is not concentrated in degree 0 (axiom 1. above). A famous example of this is K-theory via the Bott-periodicity theorem. One calls such a homology theory to be a *generalized homology theory*. All results that we will derive here will hold true for a generalized homology theory E_q.

General properties

We now discuss some general properties of homology theories that one can deduce from the axioms.

Proposition 10.3.1.4. *Let π be a group and E_q be a generalized homology theory. Let X be a space.*

1. *If $A \hookrightarrow X \xrightarrow{r} A$ is a retract of X, then the following natural maps form a short-exact sequence of E-homology groups:*
$$0 \to E_q(A) \to E_q(X) \to E_q(X, A) \to 0.$$

2. $E_q(X, X) \cong 0$.

Proof. 1. The fact that $E_q(A) \to E_q(X)$ is injective follows from a set theoretic observation; any factorization of identity is a monic followed by an epic. By homology long-exact sequence, we then have that all boundary maps ∂ are trivial. It follows that maps $E_q(X) \to E_q(X, A)$ is surjective. The exactness at middle is given by the homology long-exact sequence.
2. Since X is always a retract of itself, therefore from item 1, it follows that $E_q(X, X) \cong E_q(X)/E_q(X) \cong 0$. □

The following is a long exact sequence in homology that one obtains from a *triplet* (X, A, B) where $X \supseteq A \supseteq B$.

Proposition 10.3.1.5 (Triplet long-exact sequence)**.** *Let* (X, A, B) *be a triplet and denote* $i : (A, B) \hookrightarrow (X, B)$ *and* $j : (X, B) \hookrightarrow (X, A)$ *to be inclusions. Also denote* $\partial' : E_q(X, A) \to E_{q-1}(A, B)$ *to be the composite* $E_q(X, A) \xrightarrow{\partial} E_{q-1}(A) \to E_{q-1}(A, B)$*. Then there is a long exact sequence*

$$\begin{tikzcd}
E_q(A,B) \arrow[r, "i_*"] & E_q(X,B) \arrow[r, "j_*"] & E_q(X,A) \arrow[dll, "\partial'"'] \\
E_{q-1}(A,B) \arrow[r, "i_*"] & E_{q-1}(X,B) \arrow[r, "j_*"] & E_{q-1}(X,A)
\end{tikzcd} .$$

Proof. This follows from a fairly long diagram chase involving the homology long-exact sequence corresponding to each of the pairs (A, B), (X, B) and (X, A) which one has to expand for degrees q and $q - 1$. From that big diagram, the chase is straightforwad after some reductions and hence is omitted. □

There is an equivalent form of excision which is also quite useful.

Lemma 10.3.1.6 (Excision-II)**.** *Let* $(X, A) \in \mathbf{Top}_2$ *be a pair and* E_q *be a homology theory. If* $B \subseteq A$ *is a subspace such that* $\bar{B} \subseteq A^\circ$*, then* B *can be excised, that is, the inclusion*

$$(X - B, A - B) \hookrightarrow (X, A)$$

induces an isomorphism in homology:

$$E_q(X - B, A - B; \pi) \cong E_q(X, A; \pi).$$

Proof. Consider the triple $(X, A, X - B)$. This is an excisive triple since $A^\circ \cup (X - B)^\circ = X$ since $(X - B)^\circ = X - \bar{B}$. Thus by excision axiom, the inclusion

$$j : (X - B, A \cap X - B) \hookrightarrow (X, A)$$

induces isomorphism in E_q. As $A \cap (X - B) = A - B$, we get the desired result. □

10.3.2 Reduced homology

For each homology theory $E_q(-,-)$, we can construct a based version of the theory denoted $\tilde{E}_q(-,\mathrm{pt.})$. For a based space $(X,\mathrm{pt.})$, define the following

$$\tilde{E}_q(X) := E_q(X,\mathrm{pt.}).$$

This tends to remove the effect of the defining group of the homology theory, so to normalize the theory in the sense of Lemma 10.3.2.1, 1. In particular, if E_q satisfies dimension axiom, it follows that $E_0(\mathrm{pt.}) = \pi$. Thus this lemma will tell that $\tilde{E}_0(X) = \tilde{E}_0(X) \oplus \pi$.

Let us spell out some basic relations of this reduced homology $\tilde{E}_q$ to that of original homology E_q.

Proposition 10.3.2.1. *Let π be a group and E_q be a generalized homology theory. Let $(X,\mathrm{pt.})$ be a based space and $(A,\mathrm{pt.}) \hookrightarrow (X,\mathrm{pt.})$ be a based subspace.*

1. *$E_q(X) = \tilde{E}_q(X) \oplus E_q(\mathrm{pt.})$ and the map $\iota_* : E_q(A) \to E_q(X)$ restricted on $E_q(\mathrm{pt.})$ is the identity map $\iota_* : E_q(\mathrm{pt.}) \to E_q(\mathrm{pt.})$.*
2. *There is a long exact sequence*

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\partial"'] \\
\tilde{E}_q(A) \arrow[r] & \tilde{E}_q(X) \arrow[r] & E_q(X,A) \arrow[dll, "\partial"'] \\
\tilde{E}_{q-1}(A) \arrow[r] & \tilde{E}_{q-1}(X) \arrow[r] & E_{q-1}(X,A) \arrow[dll, "\partial"'] \\
\cdots & &
\end{tikzcd}.$$

3. *If E_q is an ordinary homology theory, then for any $q \geq 2$, we have*

$$\tilde{E}_q(X) \cong E_q(X).$$

Proof. 1. The following is split exact on the left as the map $\mathrm{pt.} \hookrightarrow X$ is a retract (Proposition 10.3.1.4):

$$0 \to E_q(\mathrm{pt.}) \to E_q(X) \to E_q(X,\mathrm{pt.}) \to 0.$$

Note that the left map here is split by the retraction $r_* : E_q(X) \to E_q(\mathrm{pt.})$. The latter statement follows from the fact that $E_q(-,\emptyset)$ is a functor and thus takes $\mathrm{id}_{\mathrm{pt.}}$ to $\mathrm{id} : E_q(\mathrm{pt.}) \to E_q(\mathrm{pt.})$.
2. Consider $i : A \hookrightarrow X$. Then $E_q(A) \to E_q(X)$ takes $E_q(\mathrm{pt.})$ to $E_q(\mathrm{pt.})$ isomorphically as in item 1. Hence we may quotient it out under the exactness to get the desired sequence.
3. This is immediate from long exact sequence of the pair $(X,\mathrm{pt.})$. $\square$

In-fact, one can obtain the unreduced homology back by reduced homology via a simple use of coproduct preservation axiom.

Lemma 10.3.2.2. *Let X be a space and denote X_+ to be the based space obtained by disjoint union of X with a point* pt.. *For any generalized homology theory E_q, we have*

$$E_q(X) \cong \tilde{E}_q(X_+).$$

Proof. As $X_+ = X \amalg \{\text{pt.}\}$, therefore by additivity of homology theories, we obtain

$$\begin{aligned}\tilde{E}_q(X_+) &= E_q(X \amalg \{\text{pt.}\}, \text{pt.}) = E_q((X, \text{pt.}) \amalg (\text{pt.}, \text{pt.})) \cong E_q(X, \text{pt.}) \oplus E_q(\text{pt.}, \text{pt.}) \\ &\cong \tilde{E}_q(X) \oplus E_q(\text{pt.}) \cong E_q(X)\end{aligned}$$

where the second-to-last isomorphism follows from Proposition 10.3.2.1, 1 and the last from 4. □

There are two important results for reduced homology. One showing that one really loses nothing by considering reduced homology as compared to homology of pairs (X, A) where $A \hookrightarrow X$ is a cofibration. The other gives a suspension isomorphism type result akin to that of homotopy groups. **TODO**

10.3.3 Mayer-Vietoris sequence in homology

We now cover an important calculational tool for generaized homology theories, which relates the homology groups of X with those of A, B and $A \cap B$ where (X, A, B) forms an excisive triad.

Theorem 10.3.3.1 (Mayer-Vietoris for homology-I)**.** *Let (X, A, B) be an excisive triple and denote $i : A \cap B \hookrightarrow A$, $j : A \cap B \hookrightarrow B$, $k : A \hookrightarrow X$ and $l : B \hookrightarrow X$. Then there is a long exact sequence*

$$\begin{tikzcd}
E_q(A \cap B) \arrow[r, "\begin{bmatrix} i_* \\ j_* \end{bmatrix}"] & E_q(A) \oplus E_q(B) \arrow[r, "\begin{bmatrix} k_* - l_* \end{bmatrix}"] & E_q(X) \arrow[dll, "\overline{\partial}"'] \\
E_{q-1}(A \cap B) \arrow[r] & E_{q-1}(A) \oplus E_{q-1}(B) \arrow[r] & E_{q-1}(X) \arrow[dll, dashed, no head] \\
{} & &
\end{tikzcd}$$

where $\overline{\partial}$ is obtained as the following composite

$$\begin{tikzcd}
E_q(X) \arrow[r] \arrow[d, dashed, "\overline{\partial}"'] & E_q(X, B) \arrow[d, "\cong"] \\
E_{q-1}(A \cap B) & E_q(A, A \cap B) \arrow[l, "\partial"]
\end{tikzcd}$$

where top horizontal arrow is corresponds to $(X, \emptyset) \hookrightarrow (X, B)$, the right vertical is exicision isomorphism and the bottom horizontal is the boundary map of homology long exact sequence of the pair $(A, A \cap B)$.

Proof. The proof will follow from excision and long exact sequence for homology quite easily. **TODO.** □

10.3.4 Singular homology & applications

We define the usual singular homology groups and will mention that it is a homology theory. Once that's set-up, then with the explicit description of chain complexes in singular homology and the ES-axioms and all the surrounding results, we will have a good toolbox to compute homology groups of very many spaces. In-fact, these applications are important to really showcase that if in any situation we have an invariant of any class of objects which is a homology theory, then we can immediately make this invariant very palpatable to calculations, which is very important in aspects where the objects are abstract entities like rings or schemes.

Definition 10.3.4.1 (**Singular homology**). Let X be a space and fix a field F. Let $S_i(X)$ be the free F-vector space generated by the set of all i-simplices $\{f : \Delta^i \to X \mid f \text{ is continuous}\}$. An element of $S_i(X)$ is called *singular i-chain*. Consider the map $\partial : S_i(X) \to S_{i-1}(X)$ which on an i-simplex σ is given by $\sigma \mapsto \sum_{j=0}^{i}(-1)^j \partial_j \sigma$ where $\partial_j \sigma$ is the σ restricted to the face opposite to j^{th}-vertex. It follows that $\partial^2 = 0$. Thus, we have a chain complex $(S_i(X), \partial)$, called the singular chain complex. The homology of this chain complex is defined to be the singular homology of X, denoted $H_i(X;\mathbb{Z})$ or simply $H_i(X)$. A map $f : X \to Y$ on spaces yields a map on singular complex $f_\sharp : S_\bullet(X) \to S_\bullet(Y)$. As map of complexes induces map on homology, we get $f_* : H_\bullet(X) \to H_\bullet(Y)$.

Let (X, A) be a pair. We define the relative singular i-chains to be

$$S_\bullet(X, A) := S_\bullet(X)/S_\bullet(A).$$

The boundary map of $S_\bullet(X)$ descends to a boundary map on $S_\bullet(X, A)$ by properties of quotients and thus we define the singular homology of a pair (X, A) to be homology of the complex $S_\bullet(X, A)$ denoted $H_i(X, A;\mathbb{Z})$.

In the following result, we state some important first properties of singular homology.

Theorem 10.3.4.2 (Singular homology is a homology theory). *Let X be a space.*

1. *If $\{X_k\}$ is the collection of path-components of X, then*

$$H_i(X;\mathbb{Z}) \cong \bigoplus_k H_i(X_k\ \mathbb{Z}).$$

2. *Singular homology satisfies dimension axiom:*

$$H_i(\{\text{pt.}\};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

3. *X is path-connected if and only if*

$$H_0(X;\mathbb{Z}) \cong \mathbb{Z}.$$

4. *Singular homology has long exact sequence of pairs, that is, if (X, A) is a pair, then there is a long exact sequence obtained by inclusions $A \hookrightarrow X$ and $(X, \emptyset) \hookrightarrow (X, A)$ as*

follows:

$$
\begin{tikzcd}
 & & \cdots \arrow[dll, dashed, "\partial"'] \\
H_q(A;\pi) \arrow[r] & H_q(X;\pi) \arrow[r] & H_q(X,A;\pi) \arrow[dll, dashed, "\partial"'] \\
H_{q-1}(A;\pi) \arrow[r] & H_{q-1}(X;\pi) \arrow[r] & H_{q-1}(X,A;\pi) \arrow[dll, dashed, "\partial"'] \\
\cdots & &
\end{tikzcd}.
$$

5. *Singular homology preserves coproducts, that is, if $\{(X_i, A_i)\}_{i\in I}$ is a collection of pairs of spaces, then*

$$\bigoplus_i H_q(X_i, A_i; \pi) \xrightarrow{\cong} H_q\left(\coprod_i (X_i, A_i); \pi\right)$$

where the maps are induced by the inclusions $(X_{i_0}, A_{i_0}) \hookrightarrow \coprod_i (X_i, A_i)$.

6. *Singular homology satisfies strong π_*-insensitivity, that is, if $f, g : X \to Y$ are two homotopic maps, then $f_* = g_* : H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$.*

Proof. 1. Observe that $S_i(X) = \bigoplus_k S_i(X_k)$ by path-connectedness of each X_k. Moreover, $Z_i(X) = \bigoplus_k Z_i(X_k)$ and $B_i(X) = \bigoplus_k B_i(X_k)$. The result follows.

2. First observe that every $S_i(X)$ is isomorphic to $\mathbb{Z}$ as there is only one i-simplex, namely $c_{\text{pt.}}$, the constant map. We have for $c_{\text{pt.}} \in Z_{i+1}(X)$ its boundary as

$$\partial(c_{pt}) = \sum_{j=0}^{i+1} (-1)^j \partial_j(c_{\text{pt.}})$$

where note that the j^{th}-boundary of $c_{\text{pt.}}$ is still $c_{\text{pt.}}$. Thus, if $i+2$ is even, then $\partial : S_{i+1}(X) \to S_i(X)$ is zero and if $i+2$ is odd, then $\partial : S_{i+1}(X) \to S_i(X)$ is an isomorphism. Hence, we get that

$$d_p : S_p(X) \to S_{p-1}(X)$$

is 0 if p is odd and an isomorphism if p is even. From this, it immediately follows that $H_p(\text{pt.};\mathbb{Z}) = 0$ if $p > 0$ and $H_0(\text{pt.};\mathbb{Z}) S_0(\text{pt.}) \cong \mathbb{Z}$.

3. (L $\Rightarrow$ R) Let X be a path-connected space. Recall that $H_0(X;\mathbb{Z}) = S_0(X)/\mathrm{Im}\,(\partial_1)$. Consider the following map

$$
\begin{aligned}
\epsilon : S_0(X) &\longrightarrow \mathbb{Z} \\
\sum_j n_j x_j &\longmapsto \sum_j n_j.
\end{aligned}
$$

Clearly this is surjective. We claim that $\mathrm{Ker}\,(\epsilon) = \mathrm{Im}\,(\partial_1)$. Suppose $\sum_j n_j x_j \in S_0(X)$ and each x_j is distinct with $\sum_j n_j = 0$. We wish to find a 1-chain $\sigma = \sum_j m_j \sigma_j$ such that

$\partial_1 \sigma = \sum_j n_j x_j$. Fix $x_0 \in X$ a point different from x_j and let $\gamma_j : I \to X$ be a path joining x_0 to x_j. Consider $\sigma = \sum_j n_j \gamma_j$. We claim that $\partial \sigma = \sum_j n_j x_j$. Indeed, we have

$$\partial \sigma = \sum_j n_j(\gamma_j(1) - \gamma_j(0)) = \sum_j n_j(x_j - x_0) = \sum_j n_j x_j - \left(\sum_j n_j \right) x_0 = \sum_j n_j x_j,$$

as required. $\square$

10.4 Fundamental group and covering maps

10.4.1 Examples and applications

The following is an important example of a covering map.

Lemma 10.4.1.1. *The map $\varphi : S^1 \to S^1$ given by $z \mapsto z^n$ is a covering map.*

Proof. Pick any $z_0 = e^{i\theta_0} \in S^1$. We wish to show that there exists an open set $U_0 \ni z_0$ in S^1 such that

$$\varphi^{-1}(U_0) = \coprod_{k=0}^{n-1} V_k$$

where V_k are open in S^1 and $\varphi|_{V_k} : V_k \to U_0$ is a homeomorphism.

Denote by $\gamma : \mathbb{R} \to S^1$ the continuous surjective map given by $t \mapsto e^{it}$. Thus, $z_0 = \gamma(\theta_0)$. Consider the interval $I_0 = \left(\theta_0 - \frac{\pi}{n}, \theta_0 + \frac{\pi}{n}\right)$. As the map $\gamma : \mathbb{R} \to S^1$ is an open map, therefore we have $U_0 = \gamma(I_0)$ which is an open set of S^1 containing z_0. We claim that U_0 is an evenly covered neighborhood for z_0. Indeed, we see that

$$\begin{aligned}
\varphi^{-1}(U_0) &= \{z \in S^1 \mid z^n \in U_0\} \\
&= \left\{e^{i\theta} \in S^1 \mid e^{ni\theta} \in \gamma(I_0)\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } \gamma(\kappa) = e^{i\kappa} = e^{ni\theta}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } n\theta = \kappa + 2k\pi, \text{ for some } k \in \mathbb{Z}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \exists \kappa \in I_0 \text{ s.t. } \theta = \frac{\kappa}{n} + \frac{2\pi k}{n}, \text{ for some } k \in \mathbb{Z}\right\} \\
&= \left\{e^{i\theta} \in S^1 \mid \theta \in \coprod_{k \in \mathbb{Z}} \left(\frac{\theta_0}{n} - \frac{\pi}{n^2} + \frac{2\pi k}{n}, \frac{\theta_0}{n} + \frac{\pi}{n^2} + \frac{2\pi k}{n}\right)\right\} \\
&= \coprod_{k=0}^{n-1} \gamma\left(\left(\frac{\theta_0}{n} - \frac{\pi}{n^2} + \frac{2\pi k}{n}, \frac{\theta_0}{n} + \frac{\pi}{n^2} + \frac{2\pi k}{n}\right)\right).
\end{aligned}$$

This completes the proof. □

We next discuss the notion of mapping torus of a map and how van Kampen can be used to compute its fundamental group.

Definition 10.4.1.2 (Mapping torus). For any map $f : X \to X$ the mapping torus of f is $T_f := X \times I/\sim$ where $(x, 0) \sim (f(x), 1)$.

Example 10.4.1.3. For $\mathrm{id} : X \to X$, we claim that $T_{\mathrm{id}} = X \times S^1$. **TODO**.

10.5 Higher homotopy groups

10.6 CW-complexes : Basic theory

One of the important properties of compactly generated spaces is that any such space can be approximated upto homotopy by a class of spaces constructed in a rather simple manner. These are precisely the CW complexes. Once the above approximation theorems are set up, we can safely reduce a lot of computation in homology to such a CW-approximation. Moreover, the reductions run so deep that in-fact any homology theory E_q on general compactly generated spaces necessarily induces and comes from the restriction of E_q to CW-complexes. An application of Hurewicz theorem will then tell us that upto natural isomorphism, there is a unique homology theory over CW-complexes.

10.7 CW-complexes : Approximation theorems

10.8 CW-complexes : Homotopy theory

10.9 Homotopy groups of Lie groups

10.10 Homotopy and homology : Hurewicz's theorem

10.11 Homotopy and homology : Whitehead's theorem

10.12 Eilenberg-Maclane spaces

10.13 Cohomology products and duality

10.14 Spectra and an introduction to stable homotopy

Spectra are objects which generalizes both the notion of cohomology theories and spaces, in that there are mappings from cohomology theories and spaces into the homotopy category of spectra. Thus, one needs to construct a good category of spectra, give a model structure on it and thus by Quillen's theory obtain this absolutely wonderful homotopy category of spectra, which unites the viewpoint of cohomology and spaces. However, we are getting ahead of ourselves, as finding the right homotopy category and giving a construction of category of spectra is easier said than done. We will meet this topic later in our discussion of ∞-categories (they will form a prototypical example of stable ∞-categories).

10.15 Lifting problems and obstructions

Chapter 11

Stable Homotopy Theory

In this chapter, we give an overview of stable homotopy theory.

Chapter 12

Classical Ordinary Differential Equations

Contents

We will prove some basic existence/uniqueness results about ODEs here, with a classical/analytic viewpoint in mind. Let us first begin by stating what is meant by an *initial value problem* and what is meant by *solving an initial value problem.* A main focus will be on doing analytical proofs, which is always extremely helpful. In particular, we will see how much weird and pathological behaviors can emerge after *passing to limit*, thus justifying why commuting with limits is a sought after property in all over analysis.

12.1 Initial value problems

Let us begin by understanding what is meant by a differential equation. Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set. Consider a continuous function $f : D \to \mathbb{R}^n$ mapping as $(t, x) \mapsto f(t, x)$ where $t \in \mathbb{R}, x \in \mathbb{R}^n$. A fundamental goal that one wishes to achieve is to find a "nice" function $x : I \subseteq \mathbb{R} \to D$ such that the function f can be known upto first derivatives, that is, we want to construct such a function $x : I \to \mathbb{R}^n$ such that it can tell us the following about f:

1. (*Correct domain*)$\forall t \in I$, we shall have $(t, x(t)) \in D$,
2. (*Differential equation*)$\frac{dx}{dt}(t_0) = f(t_0, x(t_0)), \forall t_0 \in I$. That is, the first derivative of x can give us exactly the values that f takes on D.

To find such a function x, the main difficulty is the condition 2 above, for this requires $x : I \to \mathbb{R}^n$ to be continuously differentiable (so of class C^1) and that we necessarily have to construct a function x by the knowledge only of it's first derivative (which is $f(t, x)$).

This problem of constructing a C^1 map $x : I \subseteq \mathbb{R} \to \mathbb{R}^n$ from only the data of it's continuous first derivative is called the process of solving a differential equation.

Clearly, many C^1 maps can have same first derivative (we need only add a scalar in front). So the uniqueness of the above problem is hopeless. However, one can add an extra data to the problem above that x shall satisfy and then we do get uniqueness at times. In particular, we demand the following from x:

3. (*Initial value*) for some fixed $s_0 \in I$ and $x_0 \in \mathbb{R}^n$, we require $x(s_0) = x_0$.

We then define an initial value problem (IVP) as follows:

Definition 12.1.0.1. (**IVP & solutions**) Let $f : D \to \mathbb{R}^n$ be a continuous map on an open set $D \subseteq \mathbb{R} \times \mathbb{R}^n$. An IVP is a construction problem where from the tuple of data $(f, (t_0, x_0))$ for some $(t_0, x_0) \in D$, we have to construct the following:

1. an interval $I \subseteq \mathbb{R}$ containing t_0,
2. a function $x : I \to \mathbb{R}^n$.

This function x should then satisfy the following:

1. $(t, x(t)) \in D \ \forall t \in I$,
2. $\frac{dx}{dt}(t) = f(t, x) \ \forall t \in I$,
3. $x(t_0) = x_0$.

We identify the above IVP with the tuple $(f, (t_0, x_0))$. If such a function $x : I \to \mathbb{R}^n$ exists, then it is called a *solution to the IVP* $(f, (t_0, x_0))$.

12.1.1 Existence: Peano's theorem

We have an elementary result which tells us that, if the solution exists, then what should be its form.

Lemma 12.1.1.1. *Let $f : D \to \mathbb{R}^n$ be a continuous map and $(f, (t_0, x_0))$ be an IVP. Then, a continuous map $x : I \to \mathbb{R}^n$ is a solution to the IVP $(f, (t_0, x_0))$ if and only if $\forall t \in I$, $x(t)$ is the following line integral*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Proof. (L $\Rightarrow$ R) Since x is a solution, therefore $\frac{dx}{dt}(t) = f(t, x(t))\ \forall t \in I$ and $t_0 \in I$. Then use fundamental theorem of calculus to calculate the line integral of the vector field ∇x along the line joining t_0 and t.

(R $\Rightarrow$ L) By continuity of x, we get that $t \mapsto f(t, x(t))$ is continuous. Since $x(t_0) = x_0$, therefore by continuity of $t \mapsto (t, x(t))$, there exists an open interval $I \ni t_0$ of $\mathbb{R}$ such that $(t, x(t)) \in D$ for all $t \in I$. It then follows by an application of fundamental theorem of calculus that $\frac{dx}{dt}(t) = f(t, x(t))$ for each $t \in I$. □

We next do Peano's theorem, which tells us that indeed solutions exists. This when combined with above tells us that solutions to IVP $(f, (t_0, x_0))$ exists and is of same "form". However, it will require a classic result in analysis called Arzela-Ascoli theorem. Let us do that first.

Theorem 12.1.1.2. *(Arzela-Ascoli theorem) Let $x_n : [0, 1] \to \mathbb{R}^n$ be a sequence of continuous functions such that $\{x_n\}$ is a uniformly bounded and equicontinuous family of maps. Then there exists a subsequence of $\{x_n\}$ which is uniformly convergent.*

Proof. We need to define a function on whole of $[0, 1]$. Since $\mathbb{Q} \cap [0, 1]$ is dense, so we might as well define it only for the rational points and then pass to the limit to get a function over whole of $[0, 1]$. This will be the main idea of the proof. □

Complete the pr
of Theorem 12.1
Chapter 12.

We can now approach the existence result.

Theorem 12.1.1.3. *(Peano's theorem) Let $f : D \to \mathbb{R}^n$ be a continuous map where $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open and let $(t_0, x_0) \in D$ so that the tuple $(f, (t_0, x_0))$ forms an IVP. Choose $r > 0$ and $c > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$[1]. Then, denoting $M := \max_{x \in [t_0 - c, t_0 + c] \times \overline{B}_r(x_0)} f(x)$ and $h := \min\{c, \frac{r}{M}\}$, there exists a solution to the IVP $(f, (t_0, x_0))$ given by*

$$x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).$$

Proof. We will construct the solution x in a limiting manner. First, we may replace t_0 by 0 as we can shift the solution to t_0 thus obtained. Second, we may define x on $[0, h]$ as we may translate and scale the solution as desired. Now, consider the sequence of functions defined as follows:

$$\begin{aligned} x_n(t) : [0, h] &\longrightarrow \mathbb{R}^n \\ t &\longmapsto \begin{cases} x_0 \text{ if } t \in [0, \frac{h}{n}], \\ x_0 + \int_0^{t - h/n} f(s, x_n(s)) ds & \text{if } t \in [\frac{h}{n}, h]. \end{cases} \end{aligned}$$

So we obtain a sequence of functions $\{x_n\}$ defined over $[0, h]$. Now, in the limiting case, we will have a function exactly of the form required by Lemma 12.1.1.1, so we reduce to showing that a subsequence of the above converges and converges to a continuous function. We will use the Arzela-Ascoli (Theorem 12.1.1.2) for showing this. We thus need only show that

[1] That is, choose a basic closed set around (t_0, x_0) in D.

the sequence $\{x_n\}$ is uniformly bounded and equicontinuous. For uniform boundedness, we will simply show that $x_n(t) \in \overline{B}_r(x_0)\ \forall t \in [0, h]$. This follows from the following:

$$\begin{aligned}|x_n(t) - x_0| &\le \left|\int_0^{t-h/n} f(s, x_n(s))ds\right| \\ &\le \int_0^{t-h/n} |f(s, x_n(s))|\, ds \\ &\le M(t - \frac{h}{n}) \\ &\le Mh \\ &\le r.\end{aligned}$$

Finally, to see equicontinuity, we may simply observe that for any $\epsilon > 0$ and for any $n \in \mathbb{N}$,

$$\begin{aligned}|x_n(s) - x_n(t)| &\le \left|\int_{t-h/n}^{s-h/n} f(u, x_n(u))\right| du \\ &\le \int_{t-h/n}^{s-h/n} |f(u, x_n(u))|\, du \\ &\le M(s - t).\end{aligned}$$

This shows equicontinuity. □

Remark 12.1.1.4. (*Comments on proof of Theorem 12.1.1.3*) The main idea of the proof was to *find the required function through a limiting procedure*, where to make sure that we do get the limit, we used Arzela-Ascoli. One of the foremost things we did as well was to *reduce to the nicest possible setting*, which will be very necessary to clear things around.

12.1.2 Uniqueness: Picard-Lindelöf theorem

We will now show that for an IVP $(f, (t_0, x_0))$, we may get unique solutions provided some hypotheses on f. In order to understand what this hypothesis on f is, we need to review Lipschitz and contractive functions.

Definition 12.1.2.1. (**(locally)Lipschitz functions**) A map $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function if $\exists L > 0$ such that $\forall\ x, y \in R$, we have

$$\|f(x) - f(y)\| < L\|x - y\|.$$

The function f is called locally Lipschitz if $\forall x \in E$, there exists $r > 0$ such that $f|_{B_r(x)}$ is a Lipschitz map.

Example 12.1.2.2. The map $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^{1/3}$ is not locally Lipschitz at $x = 0$. This is because if it is so, then $\exists \epsilon > 0$ such that on $B_\epsilon(0)$ the map f is Lipschitz. But for $x, y \in B_\epsilon(0)$ we get

$$\begin{aligned}|x - y| &= \left|(x^{1/3})^3 - (y^{1/3})^3\right| \\ &= \left|(x^{1/3} - y^{1/3})(x^{2/3} + y^{2/3} + (xy)^{1/3})\right| \\ &\le 2\epsilon.\end{aligned}$$

Thus,

$$\left|x^{1/3} - y^{1/3}\right| \leq \frac{2\epsilon}{\left|x^{2/3} + y^{2/3} + (xy)^{1/3}\right|},$$

which shows that f can not be Lipschitz on $B_\epsilon(0)$.

We have that all continuously differentiable maps are locally Lipschitz.

Lemma 12.1.2.3. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a C^1-map on an open set E, then f is locally Lipschitz.*

Proof. Take $a \in E$. By Theorem **??**, 2, we reduce to showing that there exists a $\epsilon > 0$ such that $\overline{B}_\epsilon(0) \subset E$ so that the continuous map $Df : E \to L(\mathbb{R}^n, \mathbb{R}^m)$ achieves maxima on the compact set. This follows from the fact that E is open. □

One definition that we will need is that of uniform Lipschitz.

Definition 12.1.2.4. (**Uniform Lipschitz**) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Then f is called uniformly Lipschitz w.r.t. x if there exists $L > 0$ such that

$$\|f(t, x) - f(t, y)\| < L\|x - y\|$$

for all $(t, x), (t, y) \in D$.

A contraction is defined in an obvious manner.

Definition 12.1.2.5. (**Contractive mappings**) Let $f : X \to X$ be a continuous map of metric spaces. Then f is said to be contractive if there exists $0 < \lambda < 1$ such that

$$d(f(x), f(y)) < \lambda d(x, y)$$

for all $x, y \in X$.

Our goal is to find the conditions that one must impose on f for the IVP $(f, (t_0, x_0))$ to have a unique solution. This means we need to find a solution $x : I \to \mathbb{R}^n$ in such a manner that x is the unique solution possible on that interval I. Now a place uniqueness comes into the picture is Banach fixed point theorem. Indeed, we will use it to find such an interval I and map x so that it would be unique for the said IVP.

Theorem 12.1.2.6. *(Banach fixed point theorem) Let X be a complete metric space and $f : X \to X$ be a contractive mapping. Then, f has a unique fixed point.*

Proof. We will first show the existence of such a fixed point. There is an obvious process of doing so. Take any point $x_0 \in X$. We then form the sequence $\{x_n\}$ in X where $x_n = f^n(x_0)$.

We claim that $\{x_n\}$ is Cauchy. Indeed, we have that for any $\epsilon > 0$ (we may take $n \geq m$):

$$\begin{aligned}
d(x_n, x_m) &= d(f^n(x_0), f^m(x_0)) \\
&< \lambda^m d(f^{n-m}(x_0), x_0) \\
&< \lambda^m \left(d(f^{n-m}(x_0), f(x_0)) + d(f(x_0), x_0)\right) \\
&< \lambda^m \left(\lambda d(f^{n-m-1}(x_0), x_0) + d(x_1, x_0)\right) \\
&= \lambda^{m+1} d(f^{n-m-1}(x_0), x_0) + \lambda^m d(x_1, x_0) \\
&< d(x_1, x_0) \left(\lambda^m + \cdots + \lambda^n\right) \\
&= \lambda^m \frac{1 - \lambda^{n-m}}{1 - \lambda} d(x_1, x_0) \\
&< \frac{\lambda^m}{1 - \lambda} d(x_1, x_0).
\end{aligned}$$

Next, by completeness of X, we have that there exists $x = \varinjlim_n x_n$ in X. Now, $f(x) = f(\varinjlim_n x_n) = \varinjlim_n f(x_n)$ by continuity and $\varinjlim_n f(x_n) = x$ by definition of x_n. The uniqueness is simple by contractive property of f. $\square$

We now come to the main result, the uniqueness of solutions of IVP. Before stating it, let us state how we will be proving it, using the following bijection between solutions of $(f, (t_0, x_0))$ and fixed points of certain mapping.

Construction 12.1.2.7. Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping where D is open and let $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Now consider the following space for some $c > 0$

$$X := C^1 \left[[t_0 - c, t_0 + c], \mathbb{R}^n\right]$$

and consider the following map

$$\begin{aligned}
T : X &\longrightarrow X \\
x(t) &\longmapsto T(x)(t) := x_0 + \int_{t_0}^{t} f(s, x(s)) ds.
\end{aligned}$$

Then, by Lemma 12.1.1.1, we see that $x(t) \in X$ is a solution of $(f, (t_0, x_0))$ if and only if $T(x(t)) = x(t)$. Hence

$$\{\textit{Solutions of IVP } (f, (t_0, x_0))\} \cong \{\textit{Fixed points of } T : X \to X\}.$$

Theorem 12.1.2.8. *(Weak Picard-Lindelöf) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ such that $(f, (t_0, x_0))$ forms an IVP. Choose $c > 0$ and $r > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$. Denote $M := \max_{x \in [t_0 - c, t_0 + c] \times \overline{B}_r(x_0)} f(x)$. If the map*

$$f : [t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \longrightarrow \mathbb{R}^n$$

is uniformly Lipschitz w.r.t. x and Lipschitz constant being L, then, denoting $h := \min\{c, \frac{r}{M}, \frac{1}{L}\}$, there exists a unique solution of IVP $(f, (t_0, x_0))$ given by

$$x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).$$

Proof. The main part of the proof will be the idea in Construction 12.1.2.7 and Banach fixed point theorem. Let X denote the following space

$$X := \left\{ y \in C^0\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right] \mid y(t_0) = x_0 \ \& \sup_{x \in [t_0 - h, t_0 + h]} \|x_0 - y(t_0)\| \leq hM \right\}.$$

Consider the following function on X

$$\begin{aligned} T : X &\longrightarrow X \\ y &\longmapsto x_0 + \int_{t_0}^{t} f(s, y(s))ds. \end{aligned}$$

By Theorem 12.1.2.6, we reduce to showing that function X is complete and T is a contraction mapping. Let us first show completeness of X. We will show that $X \hookrightarrow C\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right]$ is a closed subspace and it will suffice since $C\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right]$ is complete and closed subspaces of complete spaces are complete. . $\square$

Complete the pr
of Theorem 12.1
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We will now prove Picard-Lindelöf again but with a weakening of hypotheses as compared to Theorem 12.1.2.8. This is important because most of the time one doesn't has the information of Lipschitz constant L as is required in Theorem 12.1.2.8 while constructing the interval of the solution.

Lemma 12.1.2.9. *Something about Picard iterates: If f is Lipschitz with constant $L > 0$, then the Picard iterates $\{x_n(t)\}$ satisfies*

$$\|x_{n+1}(t) - x_n(t)\| \leq \frac{ML^n(t - t_0)^{n+1}}{(n+1)!}.$$

Theorem 12.1.2.10. *(Strong Picard-Lindelöf) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map on an open set D and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Choose $c > 0$ and $r > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$. Denote $M := \max_{x \in [t_0 - c, t_0 + c] \times \overline{B}_r(x_0)} f(x)$. If the map*

$$f : [t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \longrightarrow \mathbb{R}^n$$

is uniformly Lipschitz w.r.t. x, then, for any $h < \min\{c, \frac{r}{M}\}$, there exists a unique solution of IVP $(f, (t_0, x_0))$ given by

$$x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).$$

Proof. $\square$

Start the proof
Theorem 12.1.2.
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The following corollary tells us an alternate sufficient condition on f for the existence of unique solution to an IVP on f.

Corollary 12.1.2.11. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous map where D is open. If $\frac{\partial f_i}{\partial x_j} : D \to \mathbb{R}$ are continuous maps for all $1 \leq i, j \leq n$, then for each $(t_0, x_0) \in D$ there exists an open neighborhood around $(t_0, x_0) \in D$ in which there is a unique solution to IVP $(f, (t_0, x_0))$.*

Proof. $\square$

Start the proof
Corollary 12.1.2
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Remark 12.1.2.12. In practice, to reduce to an open neighborhood where the solution is unique, the above corollary will be useful.

12.1.3 Continuation of solutions

Consider the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(t, x) \mapsto x^2$ and $(0, 1) \in \mathbb{R} \times \mathbb{R}$. One sees that the IVP $(f, (0, 1))$ has a solution given by

$$\begin{aligned} x : (-1, 1) &\longrightarrow \mathbb{R} \\ t &\longmapsto \frac{1}{1-t}. \end{aligned}$$

However, this solution can be "extended"/"continued" to the following solution of the said IVP

$$\begin{aligned} y : (-\infty, 1) &\longrightarrow \mathbb{R} \\ t &\longmapsto \frac{1}{1-t}. \end{aligned}$$

These two are different solutions but the domain of one is inside the domain of the other. This concept of solutions extending from one domain to a larger domain will be investigated in this section.

The following definition is obvious.

Definition 12.1.3.1. (**Continuation of solutions**) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : I \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$. Then the solution x is said to be continuable if there exists a solution y of $(f, (t_0, x_0))$ given by $y : J \to \mathbb{R}^n$ where $J \supseteq I$ and $y|_I = x$.

The following theorem tells us a sufficient criterion on the solution which would make it continuable to some larger interval.

Theorem 12.1.3.2. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : (a, b) \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$.*

1. *If $\varinjlim_{t\to b^-} x(t)$ exists and $(b, \varinjlim_{t\to b^-} x(t)) \in D$, then there exists $\epsilon > 0$ such that x can be continued to a solution $\tilde{x} : (a, b + \epsilon) \to \mathbb{R}^n$.*
2. *If $\varinjlim_{t\to a^+} x(t)$ exists and $(a, \varinjlim_{t\to a^+} x(t)) \in D$, then there exists $\epsilon > 0$ such that x can be continued to a solution $\tilde{x} : (a - \epsilon, b) \to \mathbb{R}^n$.*

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Proof. □

The following lemma states that for mild conditions on f, the boundary limits might exist for a solution.

Lemma 12.1.3.3. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. If f is bounded, then for any solution $x : (a, b) \to \mathbb{R}^n$, the limits*

$$\varinjlim_{t\to b^-} x(t) \;\&\; \varinjlim_{t\to a^+} x(t) \textit{ exist.}$$

Proof. Use Lemma 12.1.1.1 to get that x is uniformly continuous over (a, b), so it has unique extension to its boundary. □

12.1.4 Maximal interval of solutions

Let $(f,(t_0,x_0))$ be an IVP and let $x : I \to \mathbb{R}^n$ be a solution. A natural question is whether there is a "maximal continuation" of x in the sense of Definition 12.1.3.1. This is what we investigate here. The following definition is clear.

Definition 12.1.4.1. (**Maximal interval of solution**) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f,(t_0,x_0))$ forms an IVP. The maximal interval of solution x is an interval $J \subseteq \mathbb{R}$ such that there exists a continuation of x on J and there is no continuation of $z : L \to \mathbb{R}^n$ of y where $L \supsetneq J$.

Lemma 12.1.4.2. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f,(t_0,x_0))$ forms an IVP. If $x : I \to \mathbb{R}^n$ is a solution of $(f,(t_0,x_0))$, then there exists a maximal interval of solution x.*

Proof. This is a simple application of Zorn's lemma on the poset

$$P = \{y : J \to \mathbb{R}^n \mid y \text{ is a continuation of } x\}$$

where $y \leq z$ iff z is a continuation of x. □

We wish to now find a characterization of maximal intervals of a solution. That is, we wish to know when can we say that a given solution is maximal.

Proposition 12.1.4.3. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f,(t_0,x_0))$ forms an IVP. Let $x : (a,b) \to \mathbb{R}^n$ be a solution of $(f,(t_0,x_0))$. Then,*

1. *The interval $[t_0, b)$ is a right maximal interval of solution x if and only if for any compact subset $K \subseteq D$, there exists $t \in [t_0, b)$ such that $(t, x(t)) \notin K$.*
2. *The interval $(a, t_0]$ is a left maximal interval of solution x if and only if for any compact subset $K \subseteq D$, there exists $t \in (a, t_0]$ such that $(t, x(t)) \notin K$.*

Proof. By symmetry, we reduce to showing 1. The main idea is to use the maximality and the results of previous section. □

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12.1.5 Solution on boundary

In this section, we investigate the limiting cases of solutions of ODEs on a maximal interval (see Lemma 12.1.4.2). We see that if the one-sided limit of a maximal solution exists, then it's graph has to lie on the boundary of the domain.

Theorem 12.1.5.1. *Let $f : D \to \mathbb{R}^n$ be a continuous map where $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open and let $(t_0, x_0) \in D$ so to make $(f,(t_0,x_0))$ an IVP. If $x : I \to \mathbb{R}^n$ is a solution to $(f,(t_0,x_0))$ and $I = (a,b)$ is a maximal interval of solution, then*

1. *If $\partial D \neq \emptyset$, $b < \infty$ and $\varinjlim_{t \to b^-} x(t)$ exists, then*

$$\left(b, \varinjlim_{t \to b^-} x(t)\right) \in \partial D.$$

2. *If $\partial D = \emptyset$, $b < \infty$ then*

$$\limsup_{t \to b^-} x(t) = \infty.$$

A similar statement holds for left sided limit towards a.

Proof. 1. Suppose not. Then $\left(b, \underrightarrow{\lim}_{t\to b^-} x(t)\right) \in D$ as D is open. It follows from Lemma 12.1.3.2 that $[t_0, b)$ is not maximal.

2. Suppose not. Then $\limsup_{t\to b^-} x(t) \neq \infty$. Hence, there exists $M > 0$ such that $\|x(t)\| < M$ for all $t \in [t_0, b)$. Now, construct $K = [t_0, b] \times C$ where C is a compact disc such that $\forall t \in [t_0, b)$, $x(t) \in C$, which can be chosen as an appropriate disc in $B_M(x_0)$. Since $K \subseteq D$, therefore by Proposition 12.1.4.3 we get a contradiction to maximality of $[t_0, b)$. $\square$

That's all we have to say here, so far.

12.1.6 Global solutions

So far we have studied solutions $x(t)$ to IVP defined only on some small enough intervals I such that $(t, x(t)) \in D$. However, we defined $D \subseteq \mathbb{R} \times \mathbb{R}^n$ as an arbitrary open set. In this section we would restrict to certain type of domains D, namely of the form $D = I \times \mathbb{R}^n$ and will try to study whether we can obtain a solution $x(t) : I \to \mathbb{R}^n$ to an IVP $(f, (t_0, x_0))$. If they exists, we call such a solution to be a *global solution* of the IVP $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ with initial values $(t_0, x_0) \in I \times \mathbb{R}^n$.

Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where $I \subseteq \mathbb{R}$ is an open interval and choose $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : J \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$. The main result of this section says that every such solution $x(t)$ can be extended to a global solution on I given some regularity conditions of values of f.

Theorem 12.1.6.1. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where $I \subseteq \mathbb{R}$ is an open interval and choose $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Suppose*

$$\|f(t, x)\| \leq M(t) + \|x\| N(t)$$

where $M, N : I \to \mathbb{R}$ are non-negative continuous maps, $\forall (t, x) \in I \times \mathbb{R}^n$. Then any solution $x : J \to \mathbb{R}^n$ of $(f, (t_0, x_0))$ can be continued to a solution $\tilde{x} : I \to \mathbb{R}^n$.

Proof. Omitted due to unnecessary technicality. See [[**??**] Kelley-Peterson?]. $\square$

We are more interested in the applications of the above theorem, which we now present.

Corollary 12.1.6.2. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be uniformly Lipschitz w.r.t. x. Then, there exists a unique global solution $x : I \to \mathbb{R}^n$ of the IVP $(f, (t_0, x_0))$.*

Proof. We get that $\exists L > 0$ such that

$$\|f(t,x) - f(t,y)\| < L\|x - y\|$$

for all $t \in I$ and $x, y \in \mathbb{R}^n$. In particular for $y = 0$, we get

$$\begin{aligned} \|f(t,x)\| &\leq \|f(t,x) - f(t,0)\| + \|f(t,0)\| \\ &\leq L\|x\| + \|f(t,0)\| \end{aligned}$$

where $N(t) = L$ and $M(t) = \|f(t,0)\|$ in the notation of Theorem 12.1.6.1. Hence, by the same theorem, if there exists a solution of $(f, (t_0, x_0))$, say x on $J \subseteq I$, then it extends to a solution on I. Now by Strong Picard-Lindelöf (Theorem 12.1.2.10), we conclude that there is a unique solution on I; if there are two solutions on I, then by restriction on the interval obtained from Picard-Lindelöf, we would get a contradiction to it's uniqueness. □

For a system of equations linear in x, for $x \in \mathbb{R}^n$, we have the following result.

Corollary 12.1.6.3. *Let $f(t,x) = A(t)x + b(t)$ be a map from $I \times \mathbb{R}^n$ to $\mathbb{R}^n$ where $A(t) \in C(I, \mathbb{R}^{n\times n})$ and $b \in C(I, \mathbb{R}^n)$ for an open interval $I \subseteq \mathbb{R}$ and $x = (x_1, \ldots, x_n)$. For $(t_0, x_0) \in I \times \mathbb{R}^n$, consider the IVP $(f, (t_0, x_0))$. Then there exists a unique solution*

$$x : I \times \mathbb{R}^n \to \mathbb{R}^n.$$

Proof. Using triangle inequality, we obtain

$$\|f(t,x)\| \leq \|A(t)\|\|x\| + \|b\|.$$

The result follows by an application of Theorem 12.1.6.1 and Corollary 12.1.6.2. □

12.2 Linear systems

So far, we covered solutions of ODE of the form

$$\frac{dx}{dt} = f(t, x(t))$$

where $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $x : I \to \mathbb{R}^n$. In particular, $\frac{dx}{dt}$ is given as

$$\frac{dx}{dt}(t) = \begin{bmatrix} \frac{dx_1}{dt} & \frac{dx_2}{dt} & \cdots & \frac{dx_n}{dt} \end{bmatrix}$$

where each $x_i : I \to \mathbb{R}$. On the other hand, the right side consists of $f(t,x)$, which is a continuous function from a subset of $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$.

In this section, we would now study in detail a particular type of IVP in which the aforementioned function $f(t,x)$ is a linear map. In particular, the mapping f is given by

$$\begin{aligned} f : D \subseteq \mathbb{R} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t,x) &\longmapsto Ax \end{aligned}$$

for a real matrix A.

Remark 12.2.0.1. One should keep in mind that these are not new ODEs; a linear system is same as $\frac{dx}{dt} = f(t,x)$ where $f(t,x) = Ax$, so they are special cases of general ODEs and have special properties like uniqueness of solutions. In particular, all the results of the previous section on general ODEs will obviously hold in the linear case.

Remark 12.2.0.2. By Lemma 12.1.1.1, we know that a solution of $\frac{dx}{dt} = Ax$ is necessarily of the form

$$x(t) = x_0 + A\int_0^t x(s)ds.$$

12.2.1 Some properties of matrices

Let us begin by stating some of the properties of matrix algebra, especially of exponential of matrices as it will be used in Theorem 12.2.2.1. Since these are not fancy results so we omit the proof of all except the main observations required in each.

Theorem 12.2.1.1. *Let $A, B \in M_n(\mathbb{R})$. Then,*

1. $\|A + B\| \leq \|A\| + \|B\|$.
2. $\|AB\| \leq \|A\|\|B\|$.
3. *The series e^X defined by*

$$e^X := \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

converges for all $X \in M_n(\mathbb{R})$.

4. $e^0 = I$.
5. $(e^A)^T = e^{A^T}$.
6. e^X *is invertible and* $(e^X)^{-1} = e^{-X}$ *for all* $X \in M_n(\mathbb{R})$.
7. *If* $AB = BA$, *then* $e^{A+B} = e^A e^B = e^B e^A$.
8. *If* $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, *then* $e^A = \mathrm{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.
9. *If P is invertible, then* $e^{PAP^{-1}} = Pe^A P^{-1}$.

Proof. We omit the proof of all but the 3. To show that the series converges, by M-test, we reduce to showing that $\sum_n \frac{\|X\|^n}{n!}$ converges as

$$\|\frac{X^n}{n!}\| \leq \frac{\|X\|^n}{n!}.$$

Indeed, it converges to $e^{\|X\|}$. □

Out of the above, perhaps the most important is the last one, as it tells us that if we have a diagonalizable matrix $A = PDP^{-1}$, then knowing its eigenvalues (that is, knowing D) and the matrix P is enough for us to calculate the e^A. Indeed, one should note that the exponent of a matrix is not easy to compute all the time!

We now give the lemma which will be quite useful for our goals, that the derivative of exponential of matrices is the obvious one.

Lemma 12.2.1.2. *Let $X \in M_n(\mathbb{R})$. Then,*

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Proof. One would need to interchange two limits at one point, which could only be done if the convergences are uniform. This could be shown by M-test. □

12.2.2 Fundamental theorem of linear systems

The most important theorem for linear systems of the form $\frac{dx}{dt} = Ax$ is that that they have a unique solution.

Theorem 12.2.2.1. *Let $A \in M_n(\mathbb{R})$. Then for any $x_0 \in \mathbb{R}^n$, the IVP*

$$\frac{dx}{dt} = Ax(t)$$

with $x(0) = x_0$ has a unique solution given by

$$x(t) = e^{At}x_0.$$

Proof. Suppose $y(t)$ is another solution. Then, define $z(t) = e^{-At}y(t)$. Differentiating this, we get

$$\frac{d}{dt}z(t) = -Ae^{-At}y(t) + e^{-At}\frac{dy}{dt}(t).$$

Since $\frac{dy}{dt} = Ay$, thus the above equation gives $\frac{d}{dt}z(t) = -Ae^{-At}y + e^{-At}Ay = 0$. Hence $z(t) = c$ is constant, therefore $y(t) = ce^{At}$. Since $y(0) = x_0 = c$, therefore $y = x$. □

Non-homogeneous linear systems

A non-homogeneous linear system is a linear IVP with an *offset*; they are of the form:

$$\frac{dx}{dt} = Ax(t) + b(t)$$

with $x(0) = x_0$. Their solution have a peculiar form.

Lemma 12.2.2.2. *Let $\frac{dx}{dt} = Ax(t) + b(t)$ with $x(0) = x_0$ be a non-homogeneous IVP for $A \in M_n(\mathbb{R})$. Then x is a solution if and only if*

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)ds.$$

Proof. We can multiply the IVP by e^{-At} to obtain

$$\begin{aligned} e^{-At}\frac{dx}{dt} &= Ae^{-At}x + e^{-At}b(t) \\ e^{-At}\frac{dx}{dt} - Ae^{-At}x &= e^{-At}b(t) \\ \frac{d}{dt}[e^{-At}x] &= e^{-At}b(t) \\ x(t) &= e^{At}x_0 + e^{At}\int_0^t e^{-As}b(s)ds. \end{aligned}$$

One can easily check that the given form satisfies the IVP, by an application of fundamental theorem of calculus. □

Finding e^A explicitly?

We will show by example how to find e^A by Jordan decomposition $A = PBP^{-1}$ where B is a matrix in Jordan form.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}.$$

One first calculates the eigenvalues of the matrix A. We get that these are $2, 2, 2$, that is there is only on eigenvalue with algebraic multiplicity 3. In order to find the Jordan form B, we need only now find the geometric multiplicity of

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12.3 Stability of linear systems in $\mathbb{R}^2$

Consider the linear IVP given by

$$\frac{dx}{dt} = Ax(t)$$

with $x(0) = x_0$ where $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ and $A \in M_2(\mathbb{R})$. From the fundamental theorem, we know that the solution is of the form $x(t) = e^{At}x_0$. By Jordan form, we know that there exists base change matrix $P \in GL_2(\mathbb{R})$ such that $A = P^{-1}BP$ where B is in Jordan form and hence it is of either of the three forms:

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

By Theorem 12.2.1.1, 9, we get

$$x(t) = e^{At}x_0 = e^{P^{-1}BPt}x_0 = P^{-1}e^{Bt}Px_0$$

so we reduce to understanding the plots of $e^{Bt}x_0$ for the aforementioned three cases, in order to understand the plot of $e^{At}x_0$ as both are related by coordinate transformation by P.

A phase portrait of a linear system

$$\frac{dx}{dt} = Ax(t)$$

is a plot of $x_1(t)$ vs $x_2(t)$ for various choices of initial points. Indeed, the choice of initial points is paramount if one ought to find the behavior of solutions. On the basis of the analysis of the three cases for B, we make the following definitions.

Definition 12.3.0.1. Let $\frac{dx}{dt} = Ax$ be a linear system where $\det A \neq 0$ and $A \in M_2(\mathbb{R})$. Then, the system is said to have

1. **saddle at origin** if $A \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ where $\lambda < 0 < \mu$,
2. **node at origin** if
   (a) $A \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ where λ, μ have same sign,
   (b) $A \sim \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$,
3. **focus at origin** if $A \sim \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$,
4. **center at origin** if $A \sim \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$.

12.4 Autonomous systems

An IVP is said to be *autonomous* if the governing equation

$$\frac{dx}{dt} = f(x(t))$$

is such that the continuous map $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is independent of time parameter t and we further assume that $f \in C^1$. In such a case we write $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and for a fixed initial datum, the maximal interval of existence is unique as well (see Corollary 12.1.2.11)

One calls a point $x_0 \in D$ to be an *equilibrium point* of the $\frac{dx}{dt} = f(x(t))$ if $f(x_0) = 0$.

12.4.1 Flows and Liapunov stability theorem

In our attempt at a better understanding of the autonomous system's dependence on initial point, we develop a basic machinery to handle it. The phase plots were a tool only available for linear systems, but we are not dealing with then in this section. Note however that a linear system is also autonomous.

The first tool we want to make is the notion of flows.

Definition 12.4.1.1. (**Flows**) Consider the following autonomous ODE

$$\frac{dx}{dt} = f(x(t))$$

where $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map. Denote by $\varphi(-, y) : I_y \to E$ to be a solution of the IVP $(f, (0, y))$ defined on **the** maximal interval of existence I_y for $\varphi(-, y)$ (Lemma 12.1.4.2). The map

$$\begin{aligned} \varphi : I \times E &\longrightarrow E \\ (t, y) &\longmapsto \varphi(t, y) \end{aligned}$$

is called the flow of the system and the map $\varphi(t,-) : E \to E$ is called the *flow of the system at time t*. As we argued in the beginning, there is only one maximal interval of existence for each initial datum.

Remark 12.4.1.2. For a pair $(t, y) \in I \times E$, the value of the flow $\varphi(t, y) \in E$ tells us where the solution $\varphi(-, y)$ takes the initial point y at time t.

We have some obvious observations.

Lemma 12.4.1.3. *Consider the following autonomous ODE*

$$\frac{dx}{dt} = f(x(t)).$$

Let $\varphi : I \times E \to E$ be the flow of the system. Then,

1. $\varphi(0, y) = y$.
2. $\varphi(s, \varphi(t, y)) = \varphi(s + t, y)$.
3. $\varphi(-t, \varphi(t, y)) = y$.

Proof. Trivial. □

We now define the important notions surrounding stability.

Definition 12.4.1.4. (**Stability**) Consider the following autonomous ODE

$$\frac{dx}{dt} = f(x(t)).$$

Let $\varphi : I \times E \to E$ be the flow of the system.

1. An equilibrium point $x_0 \in E$ is said to be *(Liapunov)stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \in B_\delta(x_0) \implies \varphi(t, x) \in B_\epsilon(x_0)$ $\forall t \geq 0$.
2. An equilibrium point $x_0 \in E$ is said to be *unstable* if it is not stable.
3. An equilibrium point $x_0 \in E$ is said to be *asymptotically stable* if it is stable and $\exists r > 0$ such that

$$x \in B_r(x_0) \implies \varinjlim_{t\to\infty} \varphi(t, x) = x_0.$$

We are now ready to state one of the most important results in stability theory, the Liapunov stability theorem. This result gives a sufficient condition for stability of a given point in the domain of $f : E \to \mathbb{R}^n$ of an autonomous system.

Theorem 12.4.1.5. *(Liapunov stability theorem) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and*

$$\frac{dx}{dt} = f(x(t))$$

be a given autonomous system with $x_0 \in E$ being an equilibrium point. If there exists a map of class C^1

$$V : E \to \mathbb{R}$$

such that $V(x_0) = 0$ and $V(x) > 0$ for all $x \in E \setminus \{x_0\}$, then

1. *if* $V'(x) \leq 0$ *for all* $x \in E \setminus \{x_0\}$*, then* x_0 *is stable,*
2. *if* $V'(x) < 0$ *for all* $x \in E \setminus \{x_0\}$*, then* x_0 *is asymptotically stable,*
3. *if* $V'(x) > 0$ *for all* $x \in E \setminus \{x_0\}$*, then* x_0 *is unstable.*

Remark 12.4.1.6. It is important to note that for most of the autonomous systems in nature, the function V as above which will do the job will be the energy functional of the physical system, that is, sum of kinetic and potential energy.

Proof of Theorem 12.4.1.5. □

Start the proof o
Theorem 12.4.1.
Chapter 12.

12.5 Linearization and flow analysis

Consider the following *system*:

$$x' = f(x)$$

where $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map and E is an open set. In the terminology of what we have covered so far, we have an autonomous system. In general, the above system may not be linear, as we studied previously. However, we can *linearize* the system at an equilibrium point x_0, as we shall show below. Indeed, this allows us to analyze the general autonomous system around each point as if it were linear.

Construction 12.5.0.1. (*Linearization of system at a point*) Let $E \subseteq \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}^n$ be a C^1 map. Let $x_0 \in E$ be an equilibrium point. For any $x \in E$, by Taylor's theorem, we get

$$\begin{aligned} f(x) &= f(x_0) + Df(x_0)(x - x_0) + \text{higher order terms} \\ &= Df(x_0)(x - x_0) + \text{higher order terms} \\ &= A(x - x_0) + \text{higher order terms.} \end{aligned}$$

We thus call the $x' = Df(x_0)x$ to be the *linearization of the system* f *at point* x_0.

Few definitions are in order.

Definition 12.5.0.2. (**Hyperbolic, sink, source & saddle points**) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. An equilibrium point $x_0 \in E$ is said to be:

1. *hyperbolic* if all eigenvalues of $Df(x_0)$ has non-zero real part,
2. *sink* if all eigenvalues of $Df(x_0)$ has negative real part,
3. *source* if all eigenvalues of $Df(x_0)$ has positive real part,
4. *saddle* if there exists eigenvalues λ, μ of $Df(x_0)$ such that real part of λ is > 0 and real part of μ is < 0.

12.5.1 Stable manifold theorem

"*For a non-linear system, there are stable and unstable submanifolds, so that once you are in either of them, the flow will constrain you to remain there.*"

We will do an important theorem in the theory of linearization of autonomous systems. We shall avoid the proof this theorem. A reference is pp 107, [cite Perko]. Let us first define three important subspaces corresponding to a linear system.

Definition 12.5.1.1. (**Stable, unstable & center subspaces**) Let

$$x' = Ax$$

be a linear system where $A \in M_n(\mathbb{R})$. Let $\lambda_j = a_j + ib_j$ be eigenvalues of A and $w_j = u_j + iv_j$ be a generalized eigenvector of λ_j. Then,

1. the stable subspace E^s is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \ldots, n$ such that $a_j < 0$,
2. the unstable subspace E^u is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \ldots, n$ such that $a_j > 0$,
3. the center subspace E^c is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \ldots, n$ such that $a_j = 0$.

Lemma 12.5.1.2. *Let $x' = Ax$ be a linear system for $A \in M_n(\mathbb{R})$. Then,*

1. $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$,
2. E^s, E^u *and* E^c *are invariant under the flow* $\varphi(t, x)$ *of the linear system, which as we know is given by* $e^{At}x$.

Proof. 1. This is easy, as generalized eigenvectors always span the whole space.
2. We need only show that for a generalized eigenvector w_j corresponding to $\lambda_j = a_j + ib_j$ with $a_j < 0$, the vector $A^k w_j$ is again a genralized eigenvector. Indeed, this follows from definition of a generalized eigenvector as $(A - \lambda_j I)w_j$ is again a generalized eigenvector. □

We now come to the real deal.

Theorem 12.5.1.3. *(Stable manifold theorem) Let $E \subseteq \mathbb{R}^n$ be an open subset with $0 \in E$, consider $f : E \to \mathbb{R}^n$ to be a C^1-map and consider the system that it defines. Denote E^s and E^u to be the stable and unstable subspaces of the system $x' = Df(0)x$. If,*

- $f(0) = 0$,
- $Df(0) : \mathbb{R}^n \to \mathbb{R}^n$ *has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part,*

then:

1. *There exists a k-dimensional differentiable manifold S inside E such that*
   (a) $T_0 S = E^s$,
   (b) *for all $t \geq 0$ and for all $x \in S$, we have*

$$\varphi(t, x) \in S,$$

   (c) *for all $x \in S$, we have*

$$\varinjlim_{t \to \infty} \varphi(t, x) = 0.$$

2. *There exists an $n-k$-dimensional differentiable manifold inside E such that*
   (a) $T_0 U = E^u$,
   (b) *for all $t \leq 0$ and for all $x \in U$, we have*

$$\varphi(t, x) \in S,$$

(c) for all $x \in U$, we have

$$\lim_{\overrightarrow{t\to-\infty}} \varphi(t,x) = 0.$$

Let us explain via an example

Example 12.5.1.4. Consider the system

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}.$$

This is not a linear system as for $f((x_1,x_2,x_3)) = (-x_1, -x_2+x_1^2, x_3+x_1^2)$, the above system is given by

$$x' = f(x) \tag{12.1}$$

and $f(x)$ is clearly not linear in x. However, note that $f(0) = 0$. Thus, linearizing the system (12.1) at 0, we obtain the linear system

$$x' = Ax \tag{12.2}$$

where

$$A := Df(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So A has two eigenvalues with negative real part, namely -1 and -1 and one eigenvalue with positive real part, namely 1. In particular A is diagonalizable, hence E^s and E^u are just span of the eigenvectors (as all generalized eigenvectors in this case are just your regular eigenvectors). Hence we see

$$E^s = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$
$$E^u = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence $E^s = x$-y plane and E^u is the z-axis of $\mathbb{R}^3$.

By an application of stable manifold theorem on this system, the stable manifold S is of dimension 2 and unstable manifold U is of dimension 1. Now, by elementary calculations, we can actually solve the linear system (12.2) and we thus obtain the following solution

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{-t} + c_1^2(e^{-t} - e^{-2t}) \\ x_3(t) &= c_3 e^{t} + \frac{c_1^2}{3}(e^{t} - e^{-2t}). \end{aligned}$$

Hence, the flow of the system is given by

$$\begin{aligned}
\varphi : \mathbb{R} \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\
(t, (c_1, c_2, c_3)) &\longmapsto \begin{pmatrix} cc_1 e^{-t} \\ c_2 e^{-t} + c_1^2 (e^{-t} - e^{-2t}) \\ c_3 e^t + \frac{c_1^2}{3}(e^t - e^{-2t}) \end{pmatrix}.
\end{aligned}$$

Now, notice the following for any $c = (c_1, c_2, c_3) \in \mathbb{R}^3$

$$\begin{aligned}
\varinjlim_{t\to\infty} \varphi(t, c) = 0 &\iff c_3 + \frac{c_1^2}{3} = 0 \\
\varinjlim_{t\to-\infty} \varphi(t, c) = 0 &\iff c_1 = c_2 = 0.
\end{aligned}$$

Notice the fact that the above equivalence is very particular to this example. But this leads us to the following conclusions

$$\begin{aligned}
S &= \{(c_1, c_2, c_3) \in \mathbb{R}^3 \mid c_3 + c_1^2/3\} \\
U &= \{(c_1, c_2, c_3) \in \mathbb{R}^3 \mid c_1 = c_2 = 0\} \cong z\text{-axis}.
\end{aligned}$$

Note that it is indeed true that for all $c \in S$ and any $t \geq 0$, $\varphi(t, c) \in S$. Similarly for U. Finally, one can check that $T_0 S = E^s$ and $T_0 U = E^u$, where the latter is immediate.

12.5.2 Poincaré-Bendixon theorem

So far, for a system we have defined its flow. Flow or integral curves of the system holds important information about the system at hand. However, we have not done any serious analysis with them. We shall begin the analysis of flows of a system now and prove the aforementioned theorem. It's use is predominantly to find closed trajectories of a system, which most of the times appears as a boundary of two differing phenomenon of the system, hence the importance of closed trajectories and of the theorem.

We first set up the terminology to be used in order to define basic objects of analysis of flow of a system.

Definition 12.5.2.1. (ω & α -limit set) Let $E \subseteq \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}^n$ be a C^1 map. Let $\varphi : \mathbb{R} \times E \to \mathbb{R}^n$ be the flow of the system. Then,

1. a point $y \in E$ is said to be a ω-limit point of $x \in E$ if there exists a sequence $t_1 < t_2 < \cdots < t_n < \dots$ in $\mathbb{R}$ such that $\varinjlim_{n\to\infty} t_n = \infty$ and $\varinjlim_{n\to\infty} \varphi(t_n, x) = y$.
2. a point $y \in E$ is said to be an α-limit point of $x \in E$ if there exists a sequence $t_1 > t_2 > \cdots > t_n > \dots$ in $\mathbb{R}$ such that $\varinjlim_{n\to\infty} t_n = -\infty$ and $\varinjlim_{n\to\infty} \varphi(t_n, x) = y$.

Let $x \in E$, the set of all ω and α limit points of x are denoted $L_\omega(x)$ and $L_\alpha(x)$ respectively.

The following are some simple observations from the definition

Lemma 12.5.2.2. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system given by it.*

1. *If $y \in L_\omega(x)$ and $z \in L_\omega(y)$ then $z \in L_\omega(x)$.*
2. *If $y \in L_\omega(x)$ and $z \in L_\alpha(y)$ then $z \in L_\omega(x)$.*
3. *For any $x \in E$, the limit sets $L_\omega(x)$ and $L_\alpha(x)$ are closed in E.*

Using the concept of limit points, we can define certain nice subspaces of E conducive to them.

Definition 12.5.2.3. (**Positively invariant set**) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. A region $D \subseteq E$ is said to be positively invariant if for all $x \in D$, $\varphi(t,x) \in D$ for all $t \geq 0$ where $\varphi : \mathbb{R} \times E \to E$ is the flow.

We then have the following simple result.

Lemma 12.5.2.4. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it.*

1. *If x, z are on same flow line/trajectory, then $L_\omega(x) = L_\omega(z)$.*
2. *For any $x \in E$, the limit set $L_\omega(x)$ is positively invariant.*
3. *If $D \subseteq E$ is a closed positively invariant set, then for all $x \in D$, $L_\omega(x) \subseteq D$.*

We now define another set of tools helpful in doing flow analysis. First is a notion which will come in handy while trying to discuss both the topology of underlying space and the flow together. A hyperplane in $\mathbb{R}^n$ is a codimension 1 linear subspace.

Definition 12.5.2.5. (**Local sections**) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. Let $0 \in E$. A local section S of f is an open connected subset of a linear hyperplane $H \subseteq \mathbb{R}^n$ such that $0 \in S$ and H is transverse to f, that is, $f(x) \notin H$ for all $x \in S$.

The next tool helps to "straighten" out flow around a local section.

Definition 12.5.2.6. (**Flow box**) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. Let $0 \in E$ and S be a local section of f. A flow box around S is a diffeomorphism Φ between $(-\epsilon, \epsilon) \times S \subseteq \mathbb{R} \times E$ and $V_\epsilon \subseteq E$ given by $V_\epsilon := \{\varphi(t,x) \mid t \in (-\epsilon,\epsilon), x \in S\}$:

$$\begin{aligned} \Phi : (-\epsilon,\epsilon) \times S &\longrightarrow V_\epsilon \\ (t,x) &\longmapsto \varphi(t,x). \end{aligned}$$

We identify $(-\epsilon,\epsilon) \times S$ as the flow box around S.

For a flow box, the diffeomorphism is important as it tells us that we can assume WLOG in a flow box that flow line are identical to the orthogonal coordinate system of $(-\epsilon,\epsilon) \times S \subseteq \mathbb{R}^{n+1}$.

We would now like to do flow analysis for the special case of planar systems. Indeed, the main theorem of this section is about the behaviour of certain limit sets of planar systems.

Let us first observe that for a planar system, any local section intersects a flow line at only discretely many points.

Lemma 12.5.2.7. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ and consider a local section S around x. Let*

$$\Sigma := \{\varphi(t, x) \in E \mid t \in [-l, l]\}.$$

Then $\Sigma \cap S$ is discrete.

Next, we see that if a sequence of points in a local section S of a planar system is monotonous in S and those same points appear in a trajectory, then it is monotonous in that trajectory as well. Indeed, a sequence of points $\{\varphi(t_n, x)\}$ along a trajectory is said to be *monotonous* if $\varinjlim_{n\to\infty} t_n = \infty$. Note that for a planar system, a codimension 1 linear subspace is a line, hence it has an inherent order and thus we can talk about monotonous sequences in a local section.

Proposition 12.5.2.8. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let S be a local section of the system. If $x_n = \varphi(t_n, x)$ is a sequence of points monotonous along the trajectory and $x_n \in S$, then $\{x_n\}$ are monotonous in S as well.*

One can use the above proposition to deduce some "eventual" properties of points in a local section by observing their intersection points with a flow line (which are discrete). Further it can be used for replacing a sequence along a trajectory to a sequence along a local section, which might be easier to analyze (as it's behaviour will just be that of monotonous sequences in $\mathbb{R}$).

Next we see an important observation, that trajectories of some special points cannot intersect a local section at more than one point(!)

Lemma 12.5.2.9. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. For some $x \in E$, let $y \in L_\omega(x) \cup L_\alpha(x)$. Then the trajectory of y intersects any local section at not more than single point.*

The next result is interesting, for it says that if the trajectory of a point intersects a local section, then there is a whole neighborhood worth of point around it, each of whose trajectories will intersect the local section(!) In some sense, this corresponds to the continuity of flow.

Proposition 12.5.2.10. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $\varphi : \mathbb{R} \times E \to \mathbb{R}^2$ denote the flow of the system. Let S be a local section around $y \in E$. If there exists $z_0 \in E$ such that for some $t_0 > 0$ we have $\varphi(t_0, z_0) = y$, then*

1. *there exists an open set $U \ni z_0$,*
2. *there exists a unique C^1-map $\tau : U \to \mathbb{R}$,*

where τ has the property that $\tau(z_0) = t_0$ and

$$\varphi(\tau(z), z) \in S \;\forall z \in U.$$

With this, we define the main object of study, a closed orbit.

Definition 12.5.2.11. (**Closed orbits**) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. A closed orbit is a periodic trajectory which doesn't contain an equilibrium point.

Note that if a trajectory contains an equilibrium point, then it will terminate after some finite time, hence the above requirement.

We now come to the main theorem of this section, which tells us a sufficient condition to find a closed orbits of a planar system.

Theorem 12.5.2.12. *(Poincaré-Bendixon theorem) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ be such that $L_\omega(x)$ ($L_\alpha(x)$) is a non-empty compact limit set which doesn't contain an equilibrium point. Then $L_\omega(x)$ ($L_\alpha(x)$) is a closed orbit.*

Let us now give some applications of the above theorem. First, we can classify limit sets $L_\omega(x)$ completely.

Theorem 12.5.2.13. *(Classification of limit sets) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ be such that $L_\omega(x)$*
- *is connected,*
- *is compact,*
- *has finitely many equilibrium points.*

Then one of the following holds
1. *$L_\omega(x)$ is a singleton.*
2. *$L_\omega(x)$ is periodic trajectory with no equilibrium points.*
3. *$L_\omega(x)$ consists of equilibrium points $\{x_j\}$ and a set of non-periodic trajectories $\{\gamma_i\}$ such that for all i, the trajectory γ_i tends to some x_j as $t \to \pm\infty$.*

The main use of Poincaré-Bendixon is to find limit cycles.

Definition 12.5.2.14. (**Limit cycles**) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. A limit cycle is a periodic trajectory γ such that there exists $x \in E$ for which $\gamma \subseteq L_\omega(x)$ or $\gamma \subseteq L_\alpha(x)$.

We now state the corollary of Poincaré-Bendixon which allows us to find the existence of limit cycles.

Corollary 12.5.2.15. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. If there exists a subseteq $D \subseteq E$ such that D*
1. *is compact,*
2. *is positively invariant,*
3. *has no equilibrium points,*

then there exists a limit cycle in D.

Proof. By Poincaré-Bendixon, we need only find $x \in D$ such that $L_\omega(x)$ is compact, as then $L_\omega(x)$ itself will be the limit cycle. This is straightforward, as D is positively invariant and compact, so $L_\omega(x)$ is inside D and is closed (hence compact). $\square$

12.6 Second order ODE

We now discuss some basic theory of second order ordinary differential equations.

Definition 12.6.0.1. (**Second order system and solutions**) Let $I \subseteq \mathbb{R}$ be an interval of $\mathbb{R}$ and consider $a_0, a_1, a_2, g \in C(I)$ to be four continuous maps $I \to \mathbb{R}$ such that $a_0(x) > 0\ \forall x \in I$. Then, a second order system with parameters a_0, a_1, a_2, g is given by

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = g(x). \tag{12.3}$$

Note that $y' := \frac{dy}{dx}$. A *solution* of a second order system (q, r, f) is a $C^2(I)$ map $y(x)$ such that it satisfies (12.3).

Remark 12.6.0.2. A second order ODE can be written in the form

$$y'' + q(x)y' + r(x)y = f(x)$$

where $q, r, f \in C(I)$. This form is the one that we shall use and will identify a second order system by the tuple (q, r, f).

Remark 12.6.0.3. On the $\mathbb{R}$-vector space $C^2(I)$ of twice continuously differentiable functions, every 2nd order system (q, r, f) defines a linear transformation

$$\begin{aligned} L : C^2(I) &\longrightarrow C(I) \\ y(x) &\longmapsto (D^2 + q(x)D + r(x))y \end{aligned}$$

where $D : C^2(I) \to C(I)$ is the derivative transformation $y \mapsto y'$, which is evidently linear. In this notation, we can write a second order system (q, r, f) as

$$Ly = f$$

where $L = D^2 + qD + r$. We call this linear transformation L the *transform associated to* (q, r, f).

Definition 12.6.0.4. (**Solution space**) Let (q, r, f) be a 2nd order system and $L : C^2(I) \to C(I)$ be the associated transform. The solution space of (q, r, f) is defined as the $\mathrm{Ker}\,(L) \subseteq C^2(I)$. Note that the set of all solutions of (q, r, f) in $C^2(I)$ is given by $L^{-1}(f) \subseteq C^2(I)$.

Lemma 12.6.0.5. *Let (q, r, f) be a 2nd order system and L be the associated transform. Then* $\dim_{\mathbb{R}}(\mathrm{Ker}\,(L)) = 2$.

Proof. □

We now observe that one can obtain all solutions of the 2nd order system $S := (q, r, f)$ by obtaining a basis of the solution space of S and one solution of S.

Lemma 12.6.0.6. *Let $S = (q, r, f)$ be a 2nd order system and L be the associated transform. Then, for any $y_p \in L^{-1}(f)$*

$$L^{-1}(f) = y_p + \mathrm{Ker}\,(L).$$

Proof. Observe that $y - y_p \in \operatorname{Ker}(L)$ and a linear transformation has all fibers of same size. $\square$

We define a tool which helps in distinguishing independent or dependent solutions of a homogeneous system.

Definition 12.6.0.7. (**Wronskian**) Let $f, g \in C^1(I)$. The Wronskian of f and g is given by

$$W(f,g) : I \to \mathbb{R}$$

where for any $x \in I$, we have

$$\begin{aligned} W(f,g)(x) : &= \det \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} \\ &= f(x)g'(x) - g(x)f'(x). \end{aligned}$$

Lemma 12.6.0.8. *Let $(q, r, 0)$ be a homogeneous system and let $y_1, y_2 \in C^2(I)$ be two solutions. Then,*

1. *$W(y_1, y_2)$ is either constant 0 for all $x \in I$ or $W(y_1, y_2)(x) \neq 0$ for all $x \in I$.*
2. *y_1, y_2 are linearly independent if and only if $W(y_1, y_2) \neq 0 \; \forall x \in I$.*

Proof. $\square$

12.6.1 Zero set of homogeneous systems

Let (q, r, f) be a 2nd order system and let y be a solution. There are some peculiar properties of the zero set $Z(y) := \{x \in I \mid y(x) = 0\} \subseteq \mathbb{R}$. We first show that the set $Z(y)$ is discrete if the system is homogeneous.

Lemma 12.6.1.1. *Let $(q, r, 0)$ be a 2nd order homogeneous system and let y be a solution. The zeroes of $y(x)$ are isolated, that is, $Z(y)$ is discrete.*

Proof. $\square$

Strum separation and comparison theorems

These theorems are at the heart of the analysis of zeros of homogeneous systems.

Theorem 12.6.1.2. *(Strum separation theorem) Let $(q, r, 0)$ be a 2nd order homogeneous system. Let y_1, y_2 be two distinct linearly independent solutions of the system. Then,*

1. *$Z(y_1)$ and $Z(y_2)$ are disjoint.*
2. *$Z(y_1)$ and $Z(y_2)$ are braided, that is, for any two x_1^1 and x_2^1 in $Z(y_1)$, there exists $x_1^2 \in Z(y_2)$ between them, and vice versa.*

Theorem 12.6.1.3. *(Strum comparison test) Consider two homogeneous 2nd order systems $(0, r_1, 0)$ and $(0, r_2, 0)$. Let y be a solution of $(0, r_1, 0)$ and u be a solution of $(0, r_2, 0)$, both non-trivial. Let $x_1, x_2 \in Z(u)$ such that*

1. *$r_1(x) \geq r_2(x)$ for all $x \in (x_1, x_2)$,*
2. *$\exists \, x_k \in (x_1, x_2)$ such that $r_1(x_k) > r_2(x_k)$.*

Then, there exists $z \in Z(y)$ such that $z \in (x_1, x_2)$.

12.6.2 Boundary value problems

A *boundary value problem* (BVP) is a second order system on an interval $I = [a, b]$ given by

$$y'' + qy' + ry = f$$

for $q, r, f \in C(I)$ such that its solutions has to satisfy certain conditions on the boundary given by

$$\begin{aligned} B_a(y) &:= \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ B_b(y) &:= \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{aligned}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$. This is clearly a different problem than that of IVP. However, with some construction, we can convert this problem into a pair of 2nd order IVPs. It will turn out that the solution of this pair has important consequences for the original IVP at hand.

Reduction to a pair of 2nd order IVPs and criterion for uniqueness of BVP solution

Theorem 12.6.2.1. *Let $I = [a, b]$ and $q, r, f \in C(I)$. Consider the 2nd order system (q, r, f) and denote the associated transform as $L : C^2(I) \to C^2(I)$. From the system (q, r, f) consider the BVP given explicitly by*

$$\begin{aligned} Ly &:= y'' + qy' + ry = f \\ B_a(y) &:= \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ B_b(y) &:= \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{aligned} \tag{12.4}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$. Construct the following two 2nd order IVPs

$$\begin{aligned} Ly &:= y'' + qy' + ry = 0 \\ y(a) &= \beta_1 \\ y'(a) &= -\alpha_1 \end{aligned} \tag{12.5}$$

and

$$\begin{aligned} Ly &:= y'' + qy' + ry = 0 \\ y(b) &= \beta_2 \\ y'(b) &= -\alpha_2. \end{aligned} \tag{12.6}$$

Then the following are equivalent

1. *Let y_1 be a solution of* (12.5) *and y_2 be a solution of* (12.6). *Then y_1 and y_2 are linearly independent in the solution space* $\operatorname{Ker}(L)$.
2. *The homogeneous BVP*

$$\begin{aligned} Ly &:= y'' + qy' + ry = 0 \\ B_a(y) &= 0 \\ B_b(y) &= 0 \end{aligned} \tag{12.7}$$

   has only 0 as solution.
3. *The BVP* (12.4) *has a unique solution.*

Variation of parameters

Variation of parameters can give us a general form of a particular solution of $Ly = f$, in terms of the solutions of IVPs (12.5) and (12.6). Indeed, we have the following theorem.

Theorem 12.6.2.2. *Let y_1 be a solution of* (12.5) *and y_2 be a solution of* (12.6). *Let*

$$\begin{aligned} c_1(x) &= \int_a^x \frac{-f(s)y_2(s)}{W(y_1, y_2)(s)} ds \\ c_2(x) &= \int_a^x \frac{f(s)y_1(s)}{W(y_1, y_2)(s)} ds. \end{aligned} \tag{12.8}$$

Then,

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \tag{12.9}$$

is a particular solution of $Ly = f$ with $y_p(a) = 0$.

Further, we obtain a general form of solution of BVP (12.4).

Theorem 12.6.2.3. *Consider the notations of Theorems 12.6.2.1 and 12.6.2.2.*

1. *Any solution y of BVP* (12.4) *is*

$$y = y_p - c_1(b)y_1$$

   where y_1 is a solution of (12.5) *and $c_1(x)$ is defined in* (12.8).

2. *Any solution of the BVP* (12.4) *is given by the integral*

$$y(x) = \int_a^b G(x, s)f(s)ds \tag{12.10}$$

   for all $x \in I$, where

$$G(x, s) = \begin{cases} \frac{y_1(x)y_2(s)}{W(y_1,y_2)(s)} & \text{if } x \le s \le b \\ \frac{y_1(s)y_2(x)}{W(y_1,y_2)(s)} & \text{if } a \le s \le x. \end{cases} \tag{12.11}$$

   This map G is called the Green's function for the transformation $L : C^2(I) \to C(I)$.

Strum-Liouville system

Let $p, q \in C^2(I)$ and $f \in C(I)$ with $p > 0$. Define the 2nd order system

$$py'' + p'y' + qy = f.$$

We can write it in neater terms as follows

$$(py')' + qy = f. \tag{12.12}$$

We will call this the *Strum-Liouville system*, denoted by (p, q, f), and the associated transform as $L : C^2(I) \to C(I)$ mapping $y \mapsto (py')' + qy$. Consequently, (12.12) can be written as

$$Ly := (py')' + qy = f.$$

We have some basic results about the associated transform L.

Lemma 12.6.2.4. *Let (p, q, f) be a Strum-Liouville system and L be the associated transform.*

1. *(Lagrange's identity) If $y_1, y_2 \in C^2(I)$, then*

$$y_1 L y_2 - y_2 L y_1 = (pW(y_1, y_2))'.$$

2. *(Abel's formula) If y_1, y_2 are solutions of $Ly = 0$, that is, they are solutions of the Strum-Liouville system defined by $(p, q, 0)$, then*

$$W(y_1, y_2) = c/p$$

   for some constant $c \in \mathbb{R}$.

Strum-Liouville Boundary Value Problems (SL-BVPs)

Consider a homogeneous Strum-Liouville system $(p, q, 0)$ and let L be the associated transform. Consider $r \in C(I)$ and $\lambda \in \mathbb{C}$. Then, a *Strum-Liouville boundary value problem* is a following type of 2nd order BVP

$$Ly + \lambda r y = 0 \tag{12.13}$$

with

$$B_a(y) = 0$$
$$B_b(y) = 0.$$

Strum-Liouville EigenValue Problems (SL-EVPs)

An SL-EVP consists of an SL-BVP (12.13) and the following question: find $\lambda \in \mathbb{C}$ such that the SL-BVP (12.13) admits a non-zero solution $y_\lambda \in C^2(I)$. In such a case λ is called the *eigenvalue* and y_λ the *eigenfunction* of the corresponding SL-EVP. We then call the tuple (p, q, r) as the SL-EVP.

Types of SL-EVPs

We further classify an SL-EVP (p, q, r) based on the properties of the underlying functions.

1. **regular** if $p > 0$ and $r > 0$ on $[a, b]$,
2. **singular** if $p > 0$ on (a, b), $p(a) = 0 = p(b)$ and $r \geq 0$ on $[a, b]$,
3. **periodic** if $p > 0$ on $[a, b]$, $p(a) = p(b)$ and $r > 0$ on $[a, b]$.

We next see that any eigenvalue of SL-EVP is always real.

Lemma 12.6.2.5. *Let (p, q, r) be a regular SL-EVP. Then all eigenvalues of (p, q, r) are real.*

Strum-Liouville theorem for regular SL-EVP

Properties of regular SL-EVPs

Examples

Chapter 13

K-Theory of Vector Bundles

Chapter 14

Jet Bundles

It is through the concepts explained in this chapter that we shall begin doing some geometry over our base manifolds. A classical use of differential equations is elaborated in Chapter 12, whereas here we shall be more conceptual in order to elucidate the underlying structure of the notion of "differential equations".

Part IV

The Analytic Viewpoint

Chapter 15

Analysis on Complex Plane

Contents

We will here review some of the classical results of complex function theory of one variable, namely the following four topics:

- Analytic functions; Cauchy-Riemann equations, harmonic functions.
- Complex integration; Zeroes of analytic functions, winding numbers, Cauchy's formula and theorem, Liouville's theorem, Morera's theorem, open-mapping theorem, Schwarz's lemma.
- Singularities; Classification, Laurent series, Casorati-Weierstrass theorem, residues and applications, meromorphic maps, Rouché's theorem.

- Conformal maps; Möbius transformations, normality and compactness, Riemann mapping theorem.

All this is important as it will build one's intuition of geometry in complex case, which is something we don't learn as early in our studies as, say, real geometry. Of-course this would be of immense use in complex algebraic geometry, some if which we shall cover in later chapters. Moreover, a complex manifold by definition locally looks like $\mathbb{C}^n$, hence it is imperative that we understand the geometry and analysis of complex plane and make it as second nature as the usual geometry over $\mathbb{R}^2$ is to us.

15.1 Holomorphic functions

Let $\Omega \subseteq \mathbb{C}$ denote an open subset of the complex plane $\mathbb{C}$ for the rest of this chapter. Consider a function $f : \Omega \to \mathbb{C}$. Motivated by the classical case of real differentiability in one variable, we can define a notion of differentiation for f at $a \in \Omega$.

Definition 15.1.0.1. (**$\mathbb{C}$-differentiable/holomorphic functions**) A function $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable or holomorphic at $a \in \Omega$ if the following limit exists:

$$\varinjlim_{z \to 0} \frac{f(a+z) - f(a)}{z}$$

in which case it's value is said to be the derivative of f at a and is denoted by $\frac{df}{dz}(a) = f'(a) \in \mathbb{C}$.

Remark 15.1.0.2. As we shall soon see, this seemingly innocuous definition for some surprising reason gives the following fantastic results:

1. Theorems 15.1.1.2 and **??** tells us:

$$\begin{gathered}\{\textit{All } \mathbb{C}\textit{-differentiable maps } f : \Omega \to \mathbb{C}\} \\ \text{\rotatebox{90}{$\cong$}} \\ \{\textit{All pairs of differentiable maps } u, v : \Omega \to \mathbb{R}\textit{, related by CR-equations}\}\end{gathered}.$$

2. Corollary 15.1.2.2 and Theorem **??** tells us:

$$\mathbb{C}\textit{-differentiable maps are conformal.}$$

3. Theorem **??** tells us:

$$\mathbb{C}\textit{-differentiable functions are harmonic.}$$

   Moreover, Theorem **??** tells us that if Ω is simply connected, then

$$\begin{gathered}\{\textit{Harmonic functions } \Omega \subseteq \mathbb{R}^2 \cong \mathbb{C}\} \\ \text{\rotatebox{90}{$\cong$}} \\ \{\mathbb{C}\textit{-differentiable functions on } \Omega \subseteq \mathbb{C}\}\end{gathered}.$$

4. Theorem **??** tells us:

$$\textit{Contour integral of a } \mathbb{C}\textit{-differentiable map around a loop is 0.}$$

5. Theorem **??** tells us:
   A $\mathbb{C}$-differentiable function inside a disc is determined by its values on the disc's boundary.
6. Corollary 15.2.3.5 tells us:

$$\begin{array}{c}\{\mathbb{C}\text{-}\textit{differentiable maps } f:\Omega\to\mathbb{C}\}\\ \text{\rotatebox{90}{$\cong$}}\\ \{\textit{Analytic maps } f:\Omega\to\mathbb{C}\}\end{array}.$$

This shows the sheer importance of the notion of $\mathbb{C}$-differentiability, which we will explore later in a more *local* setting. Our goal in the rest of this chapter is to provide rather quick proofs to these results while portraying the main ideas employed in them.

Let us start by analyzing some elementary properties of holomorphic maps.

15.1.1 Cauchy-Riemann equations

Let $f:\Omega\to\mathbb{C}$ be a holomorphic map on an open subset $\Omega\subseteq\mathbb{C}$. Now, there is a homeomorphism $\varphi:\mathbb{R}^2\to\mathbb{C}$ given by $(x,y)\mapsto x+iy$. Composing f with this map, we get that f can equivalently be stated as the data of two real valued maps $u:\mathbb{R}^2\to\mathbb{R}$ and $v:\mathbb{R}^2\to\mathbb{R}$ given by $u(x,y)=\Re f(\varphi(x,y))$ and $v(x,y)=\Im f(\varphi(x,y))$.

Like in the case of $\mathbb{R}$-differentiability, in our case we can also define partial differential operators of f w.r.t. x,y and z.

Definition 15.1.1.1. (**Partial differential operators on** f) Let $f:\Omega\to\mathbb{C}$ be a holomorphic map on an open subset Ω of $\mathbb{C}$. Then, we define the following quantities in an obvious manner:
1. $\frac{\partial f}{\partial x}:=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$.
2. $\frac{\partial f}{\partial y}:=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}$.
3. $\frac{\partial f}{\partial z}:=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)$.
4. $\frac{\partial f}{\partial \bar{z}}:=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)$.

Then the fact that f is holomorphic can be equivalently stated in terms of real differentiability of the maps u and v as the following theorem states.

Theorem 15.1.1.2. *Suppose $f:\Omega\to\mathbb{C}$ is any $\mathbb{C}$-valued function on an open set Ω of $\mathbb{C}$. Then write $f(x+iy)=u(x,y)+iv(x,y)$ where $u,v:\mathbb{R}^2\rightrightarrows\mathbb{R}$.*
1. *$f:\Omega\to\mathbb{C}$ is holomorphic at $z_0\in\Omega$ if and only if u,v are real differentiable and satisfy any of the following equivalent PDEs at z_0:*
   (a) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.
   (b) $\frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}$.

   (c) $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}$.
   (d) $\frac{\partial f}{\partial \bar{z}} = 0$.
2. *If $u, v : \Omega \to \mathbb{R}$ is a pair of C^1-maps satisfying the CR-equations, then $f := u + iv$ is a holomorphic map.*

Proof. Equivalence of the four PDEs is straightforward. Now let $f : \Omega \to \mathbb{C}$ be a holomorphic map. This means that for any $a \in \Omega$, we have

$$\frac{\partial f}{\partial z}(a) = \varinjlim_{z \to 0} \frac{f(a+z) - f(a)}{z}.$$

The required PDEs for u and v follows by letting z approach 0 first from real axis and then from imaginary axis and deeming them equal.

Next, we may write $R(z) = f(a+z) - f(a) - cz$ for some $c = c_1 + ic_2$ and then $R(z) = R_u(z) + iR_v(z)$ where $R_u(z) = u(a+z) - u(a) - c_1 x + c_2 y$ and $R_v(z) = v(a+z) - v(a) - c_2 x - c_1 y$. Then, f is holomorphic at a with $\frac{df}{dz}(a) = c$ if and only if $\varinjlim_{z \to 0} \frac{R(z)}{z} = 0$. But the latter happens if and only if $\varinjlim_{z \to 0} \frac{R_u(z)}{z} = 0 = \varinjlim_{z \to 0} \frac{R_v(z)}{z}$. Now $\frac{R_u(z)}{z} = 0$ if and only if $c_1 = \frac{\partial u}{\partial x}(a)$ and $c_2 = -\frac{\partial u}{\partial y}(a)$. Similarly, $\varinjlim_{z \to 0} \frac{R_v(z)}{z} = 0$ if and only if $c_2 = \frac{\partial v}{\partial x}(a)$ and $c_1 = \frac{\partial v}{\partial y}(a)$. □

15.1.2 Conformal maps

We will now show that holomorphic maps "preserves angles". The meaning of angle is not well-defined a-priori on the complex plane, so we will have to develop that first.

A *curve* in $\mathbb{C}$ is a continuous map $\gamma : I \to \mathbb{C}$. It is said to be *differentiable* if $\Re\gamma : I \to \mathbb{R}$ and $\Im\gamma : I \to \mathbb{R}$ are differentiable $\mathbb{R}$-valued functions. It is said to be regular at $t \in I$ if $\gamma'(t) \neq 0 \in \mathbb{C}$. Now, let $\gamma_1, \gamma_2 : I \to \mathbb{C}$ be two curves which intersect at $\gamma_1(t_1) = \gamma_2(t_2)$ for $t_1, t_2 \in I$ such that γ_i is regular at t_i, $i = 1, 2$. Such an intersection is said to be *regular*. Then, the angle of intersection of γ_1 and γ_2 at a regular point is defined to be:

$$\angle\gamma_1(t_1), \gamma_2(t_2) := \arg \gamma_2'(t_2) - \arg \gamma_1'(t_1).$$

A function $f : \Omega \to \mathbb{C}$ is *conformal* at $z_0 \in \Omega$ if f preserves angles of all regular intersections of two curves at z_0.

It is now an easy observation that holomorphic maps will be conformal.

Lemma 15.1.2.1. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic map on an open set Ω of $\mathbb{C}$. If $z_0 \in \Omega$ such that $f'(z_0) \neq 0$, then for any two curves γ_1, γ_2 such that $\gamma_1(t_1) = z_0 = \gamma_2(t_2)$ and γ_1, γ_2 are regular at t_1, t_2 respectively, then*

$$\angle\gamma_1(t_1), \gamma_2(t_2) = \angle f \circ \gamma_1(t_1), f \circ \gamma_2(t_2).$$

Proof. The result follows from chain rule and the fact that $\arg wz = \arg w + \arg z$. □

A map $f : \Omega \to \mathbb{C}$ is called *conformal* if it preserves angles of all regularly intersecting curves. Thus,

Corollary 15.1.2.2. *All holomorphic functions are conformal except at those points at which derivative is zero.* □

We will now show that even an arbitrary conformal map $f : \Omega \to \mathbb{C}$ is also holomorphic.

Theorem 15.1.2.3. *Let $f : \Omega \to \mathbb{C}$ be a conformal map such that $\Re f$ and $\Im f$ are of class C^1. Then,*

1. *f is holomorphic.*
2. *$f'(z) \neq 0$ for all $z \in \Omega$.*

Proof. **TODO.** □

15.1.3 Harmonic maps

A function $f : \Omega \to \mathbb{C}$ is said to be harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Lemma 15.1.3.1. *Let $f = u + iv : \Omega \to \mathbb{C}$ be a function where $u, v : \Omega \rightrightarrows \mathbb{R}$. Then, f is harmonic if and only if u and v are harmonic (in $\mathbb{R}$-sense).*

Proof. A straightforward application of Cauchy-Riemann. □

Lemma 15.1.3.2. *All holomorphic maps are harmonic.*

Proof. **TODO.** □

Lemma 15.1.3.3. *All conformal maps are harmonic.*

Proof. **TODO.** □

15.1.4 Stereographic projection and the Riemann sphere

TODO.

15.1.5 Linear fractional transformations

A linear fractional transformation is a map

$$\begin{aligned} \varphi : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{az+b}{cz+d} \end{aligned}$$

where $a, b, c, d \in \mathbb{C}$. These are important as they provide a class of workable examples of rational functions, which are pretty much the bread and butter of algebraic geometry. Moreover, these maps arrange themselves in a group and it then follows that it contains as a subgroup the biholomorphic automorphism group of lots of geometric objects of interest (see Lemmas 15.1.5.2, 15.1.5.3). However, these maps makes the most sense on the complex projective line, $\mathbb{C}P^1$, the quotient of $\mathbb{C}^2$ by all lines passing through origin, which is the usual Riemann sphere $\bar{\mathbb{C}}$ (see Lemma **??**).

Let us work out this connection in detail. We have the following maps:

$$\alpha : \mathbb{C}^2 \twoheadrightarrow \mathbb{C}P^1 \xrightarrow{\cong} \bar{\mathbb{C}}$$

$$(w, z) \longmapsto [w, z] \longmapsto \frac{w}{z}$$

Notice that $L_f(\bar{\mathbb{C}})$, the collection of all linear fractional transforms on $\bar{\mathbb{C}}$ forms a group where the identity is given when $a = 0 = c$. The multiplication of two fractional transforms is again a fractional transform, as can be checked easily. Hence, it follows that $L_f(\bar{\mathbb{C}})$ is a subgroup of all biholomorphic maps of $\bar{\mathbb{C}}$, the $\mathrm{Aut}\left(\bar{C}\right)$. Hence we have a hold on one type of global biholomorphic maps of the Riemann sphere(!)

We then have the following result.

Lemma 15.1.5.1. *Let $\bar{\mathbb{C}}$ denote the Riemann sphere. Then,*

$$L_f(\bar{\mathbb{C}}) \cong GL_2(\mathbb{C})/\mathbb{C}^\times I_2$$

Proof. There's a natural map

$$\kappa : GL_2(\mathbb{C}) \longrightarrow L_f(\bar{\mathbb{C}})$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{az+b}{cz+d}.$$

This is a group homomorphism, as can be checked easily. The kernel of this homomorphism consists of matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $\frac{az+b}{cz+d} = z$. Unravelling, we get $c = 0 = b$ and $a = d \neq 0$. □

This group is also known by *projective general linear group*, $PGL_2(\mathbb{C}) := L_f(\bar{\mathbb{C}})$. The group $L_f(\bar{\mathbb{C}})$ also has some special subgroups. For example, it consists of all biholomorphic maps of $D^\circ := \{z \in \mathbb{C} \mid |z| < 1\}$.

Lemma 15.1.5.2. *For the open unit ball D°, we have*

$$\mathrm{Aut}\,(D^\circ) \cong \left\{ \frac{t(z-a)}{1-\bar{a}z} \mid |t| = 1 \;\&\; a \in D^\circ \right\}.$$

Proof. **TODO.** □

Similarly, it also contains an isomorphic copy of all biholomorphic maps of upper half plane $\mathbb{H}$.

Lemma 15.1.5.3. *For the upper half plane $\mathbb{H} \subset \mathbb{C}$, we have*

$$\mathrm{Aut}\,(\mathbb{H}) \cong SL_2(\mathbb{R}) \subset GL_2(\mathbb{C}).$$

Proof. **TODO.** □

Properties

Let us now state some basic properties of fractional transforms.

Lemma 15.1.5.4. *If $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a non-identity fractional transform, then either it has one or two fixed points, but not zero.*

Proof. A non-identity fractional transform $\varphi(z) = \frac{az+b}{cz+d}$ follows that either $b, d \neq 0$ or $a \neq c$. Suppose the former is not the case. Now if $\varphi(z) = z$, then it follows that $cz^2 + (d-a)z - b = 0$ where $b = d = 0$. Thus we obtain $z(cz - a) = 0$, which gives atleast one and atmost two solutions. Similarly, if $a = c$, then $b, d \neq 0$. It then follows that the above quadratic has either one or two solutions. □

Another property of fractional transforms is that they are uniquely determined by how they map on three points.

Lemma 15.1.5.5. *If z_1, z_2, z_3 and w_1, w_2, w_3 are two pair of distinct points in $\bar{\mathbb{C}}$, then there exists a unique fractional transform $\varphi \in L_f(\bar{\mathbb{C}})$ such that*

$$f(z_i) = w_i \; \forall i = 1, 2, 3.$$

Proof. Uniqueness follows from the fact that if $\varphi, \varpi : \bar{\mathbb{C}} \rightrightarrows \bar{\mathbb{C}}$ are two fractional transforms taking $z_i \mapsto w_i$, then the fractional transform $\varphi \circ \varpi^{-1}$ has 3 fixed points. It follows from Lemma 15.1.5.4 that $\varphi \circ \varpi^{-1} = \mathrm{id}$.

To show existence, take any arbitrary triple $v_1, v_2, v_3 \in \bar{\mathbb{C}}$. We will show that one can construct a fractional transform depending on v_i mapping as $z_i \mapsto v_i$. Denote then the map φ, $z_i \mapsto v_i$ and ϖ, $w_i \mapsto v_i$. Then $\varpi^{-1} \circ \varphi$ would be the required map. Since v_i can be arbitrary, therefore we choose it as per our convenience. It is perhaps easier to write it for $\infty, 0, 1$. □

One last basic property that may be observed for fractional transforms is that they are conformal.

Lemma 15.1.5.6. *All fractional transforms $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ are conformal.*

Proof. Since fractional transforms are holomorphic, therefore by Corollary 15.1.2.2, we reduce to showing that $\varphi'(z) \neq 0$ for all $z \in \bar{\mathbb{C}}$. Indeed, we have

$$\phi'(z) = \frac{ad - bc}{(cz + d)^2},$$

where since $ad - bc \neq 0$ by definition, therefore $\phi'(z) \neq 0$. □

Example : The Cayley transform

We will discuss here the properties of the following fractional transform, known by Cayley's name:

$$\begin{aligned} \varphi : \bar{\mathbb{C}} &\longrightarrow \bar{\mathbb{C}} \\ z &\longmapsto \frac{z+i}{z-i}. \end{aligned}$$

15.1.6 e^z, $\log z$, $\arg z$

15.1.7 Branches of $\log z$ and $\arg z$

15.2 La théorie des cartes holomorphes

The theory of holomorphic maps. We now begin another part of complex function theory which is at the heart of a lot of sources of interest in it. We first consider the line integrals.

Complete compl integration, Cha 15.

15.2.1 Line integrals

Let $\gamma : [a, b] \to \mathbb{C}$ be a continuous function. Suppose $G \subseteq \mathbb{C}$ is an open subset containing γ and it's interior and let $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map $f : G \to \mathbb{C}$. Then, the *line integral* of f along γ is defined as

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

where definite integral of a complex valued function $g : [a, b] \to \mathbb{C}$ where $g = u + iv$ is given simply as the Riemann integral on each of the real and imaginary parts:

$$\int_a^b g(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

A continuous map $\gamma : [a, b] \to \mathbb{C}$ is called piecewise C^1 if γ is C^1 at all but finitely many points and where it isn't differentiable, one sided derivative exists.

Few properties of line integrals are in order.

Theorem 15.2.1.1. *Let $\gamma : [a, b] \to \mathbb{C}$ be a curve in $\mathbb{C}$ and let $G \subseteq \mathbb{C}$ be an open subset containing γ. Let $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map over G. Then,*

1. *(FTOC) If γ is piecewise C^1, then*
$$\int_a^b \gamma'(t)dt = \gamma(b) - \gamma(a).$$
2. *If $f \in \mathcal{C}^{\text{hol}}(G)$ where G contains γ, then*
$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a)).$$
So if γ is a closed loop, then integral of f' along it is 0.
3. *If $f \in \mathcal{C}^{\text{hol}}(G)$ and $\tilde{\gamma}$ is a reparametrization of γ, then $\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$.*
4. *(Estimate) If $f \in \mathcal{C}^{\text{hol}}(G)$ and $M = \sup_{t\in[a,b]} |f(\gamma(t))|$, then*
$$\left|\int_\gamma f(z)dz\right| \leq ML(\gamma)$$
where $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$ is the arc-length.

Proof. Assuming 1 by FTOC on each piece, all results follows from basic analysis. □

15.2.2 Cauchy's theorem - I

We will now state the Cauchy's theorems on holomorphic maps and integrals. This will be a special case of the general version, which we shall do later, for we will find almost all of the traditional applications without needing that generality. We will begin with it's infantile version, which is quite simple to state now with line integrals in our pouch.

Theorem 15.2.2.1. *(Cauchy's theorem) Let $\gamma : [a, b] \to \mathbb{C}$ be a closed piecewise C^1 loop in $\mathbb{C}$ and let $G \subseteq \mathbb{C}$ be a convex open set containing γ and it's interior $\mathrm{Int}(\gamma)$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$, then*

$$\int_\gamma f(z)dz = 0.$$

Then there is the Cauchy integral formula.

Theorem 15.2.2.2. *(Cauchy's integral formula) Let C be a circle oriented in the counterclockwise manner and let $G \subseteq \mathbb{C}$ be an open set containing C and its interior $\mathrm{Int}(C)$. Then,*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}dw$$

for all $z \in \mathrm{Int}(C)$.

Remark 15.2.2.3. Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on open $G \subseteq \mathbb{C}$. The integral formula tells us that the value of f at $z \in G$ can be given in terms of line integral of f around a small enough circle C in the CCW orientation centered at z so that $C \subseteq G$. Hence the integral formula tells us that holomorphic maps are pretty much completely determined by taking their line integrals around circles.

We will provide some results which can be derived from them. In particular, using these, we would be able to show that a holomorphic map is analytic (Corollary 15.2.3.5).

Proof of Cauchy's theorem : Holomorphic maps have primitives

A *primitive* of a holomorphic map f is a holomorphic map g such that $g' = f$. We first state the following theorem without proof using which we will prove the Cauchy's theorem.

Theorem 15.2.2.4. *(Cauchy's triangle theorem) Let T be a triangle in $\mathbb{C}$ and $G \subseteq \mathbb{C}$ be an open set containing T and $\mathrm{Int}(T)$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$, then*

$$\int_T f(z)dz = 0.$$

Proof. [**??**] [Sarason]. □

Now, we will prove the following lemma using the above triangle theorem.

Lemma 15.2.2.5. *Let $G \subseteq \mathbb{C}$ be a convex open set and $f \in \mathcal{C}^{\mathrm{hol}}(G)$. Then there exists a map $g \in \mathcal{C}^{\mathrm{hol}}(G)$ such that $g' = f$.*

Proof. For a fixed $z_0 \in G$, define

$$\begin{aligned} g : G &\longrightarrow \mathbb{C} \\ z &\longmapsto \int_{[z_0,z]} f(z)dz \end{aligned}$$

where $[z_0, z]$ denotes the path formed by line joining z_0 and z in G. We claim that for all $z \in G$, $g'(z) = f(z)$. Indeed, pick any $z_1 \in G$ to form triangle $T = (z_0, z_1, z)$ inside G (G is convex). Then, by Theorem 15.2.2.4, we get the following

$$\begin{aligned} 0 &= \int_T f(w)dw \\ &= \int_{[z_0,z_1]} f(w)dw + \int_{[z_1,z]} f(w)dw + \int_{[z,z_0]} f(w)dw \\ g(z) - g(z_1) &= \int_{[z_1,z]} f(w)dw. \end{aligned}$$

We wish to estimate

$$\begin{aligned} \left|\frac{g(z) - g(z_1)}{z - z_1} - f(z_1)\right| &= \left|\frac{1}{z - z_1}\int_{[z_1,z]} f(w)dw - f(z_1)\right| \\ &= \left|\frac{1}{z - z_1}\int_{[z_1,z]} (f(w) - f(z_1))dw\right| \\ &\leq \frac{1}{|z - z_1|}\int_{[z_1,z]} |f(w) - f(z_1)|\, dw. \end{aligned}$$

Since f is continuous, therefore for any $\epsilon > 0$, there is a $\delta > 0$ such that $|w - z_1| < \delta$ implies $|f(w) - f(z_1)| < \epsilon$. Hence, for $|w - z_1| < \delta$, we get

$$\begin{aligned} &\leq \frac{1}{|z - z_1|}\int_{[z_1,z]} \epsilon dw \\ &= \epsilon. \end{aligned}$$

Hence as $z \to z_1$, the above difference $\to 0$. $\square$

Proof of Theorem 15.2.2.1. Since $f \in \mathcal{C}^{\mathrm{hol}}(G)$, therefore by Lemma 15.2.2.5, there exists $g \in \mathcal{C}^{\mathrm{hol}}(G)$ such that $g' = f$. Hence the result follows by Theorem 15.2.1.1, 2. $\square$

Proof of Cauchy's integral formula : Cauchy integrals

We would like to present the proof of Cauchy integral formula as it portrays how to use the fact that integral of holomorphic maps around closed loops are zero (Theorem 15.2.2.1).

Proof of Theorem 15.2.2.2. Pick any $z_0 \in \mathrm{Int}(C)$. We shall show the result for this chosen z_0. We shall use the Cauchy's theorem 15.2.2.1 in an essential manner. Indeed, consider the following figure on the complex plane inside G: Integrating the holomorphic map $\frac{f(w)}{w-z_0}$

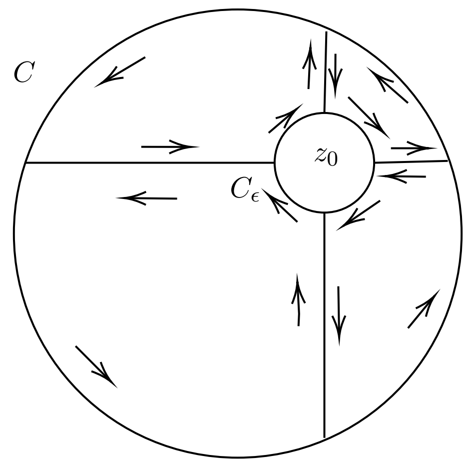

Figure 15.1: Contour over which to integrate $\frac{f(w)}{w-z_0}$.

over the each of the four regions will give zero by Theorem 15.2.2.1. However, summing them up, one can see that we get the difference $\int_C \frac{f(w)}{w-z_0}dw - \int_{C_\epsilon} \frac{f(w)}{w-z_0}dw$, which should thus be zero, yielding us $\int_C \frac{f(w)}{w-z_0}dw = \int_{C_\epsilon} \frac{f(w)}{w-z_0}dw$. Note this is true for all $\epsilon < d(z_0, C)$.

Now recall that $\int_C \frac{1}{z}dz = 2\pi i$. Hence, we get the following estimate for any chosen $\epsilon < d(z_0, C)$

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}dw - f(z_0)\right| &= \left|\frac{1}{2\pi i}\int_{C_\epsilon} \frac{f(w)}{w-z_0}dw - f(z_0)\right| \\ &= \left|\frac{1}{2\pi i}\int_{C_\epsilon} \frac{f(w)-f(z_0)}{w-z_0}dw\right|\end{aligned}$$

Now, by Theorem 15.2.1.1, 4, let $M_\epsilon = \sup_{w\in C_\epsilon}\left|\frac{f(w)-f(z_0)}{w-z_0}\right|$ to obtain the following inequality

$$\begin{aligned}&\leq \frac{M_\epsilon}{2\pi}L(C_\epsilon) \\ &= \frac{M_\epsilon}{2\pi}2\pi\epsilon \\ &= \epsilon M_\epsilon.\end{aligned}$$

Since f is holomorphic, therefore $\varinjlim_{\epsilon\to 0} M_\epsilon = |f'(z_0)|$. Hence, $\varinjlim_{\epsilon\to 0} \epsilon M_\epsilon = 0$, which gives the desired result. $\square$

15.2.3 Theory of holomorphic maps

We now present applications of the two highly useful results of Cauchy (Theorems 15.2.2.1, 15.2.2.2). The results covered here are as follows:

- Mean value property.

- Power series representation of Cauchy integrals.
- Morera's theorem.
- Derivatives.
- Liouville's theorem.
- Identity theorem.
- Maximum modulus theorem.
- Schwarz's lemma.
- Classification of bijective holomorphic maps of open unit ball.
- Open mapping theorem.
- Fundamental theorem of algebra.
- Inverse function theorem.
- Local m^{th} power property.
- Harmonic conjugates.

These results lie at the heart of complex analysis.

Let us begin by understanding the behavior of a holomorphic map around a circle centered at a point.

Mean value property of holomorphic maps

Proposition 15.2.3.1. *Let $G \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{C}^{\text{hol}}(G)$. Then, for all $z_0 \in G$ and C_r a circle of radius r centered at z_0 contained inside G together with its interior* $\text{Int}(C)$*, we have*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt.$$

Proof. Using the integral formula (Theorem 15.2.2.2) and using $\gamma(t) = z_0 + re^{it}$ as a parameterization of C_r, the result follows. $\square$

Power series representation of Cauchy integrals

We will in this section show that functions defined by Cauchy integrals are analytic. Since holomorphic maps are given by Cauchy integrals, thus we would be able to show that holomorphic maps are analytic.

Definition 15.2.3.2. (**Maps given by Cauchy integral**) Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve in $\mathbb{C}$ and $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on an open subset $G \subseteq \mathbb{C}$ where G contains $\text{Im}(\gamma)$. Define the following map

$$\begin{aligned}\tilde{f} : \mathbb{C} \setminus \text{Im}(\gamma) &\longrightarrow \mathbb{C}\\ z &\longmapsto \int_\gamma \frac{f(w)}{w-z}dw.\end{aligned}$$

Then $\tilde{f}$ is called the Cauchy integral associated to $f \in \mathcal{C}^{\text{hol}}(G)$ and $\gamma : [a, b] \to G$.

We first show that holomorphic maps are given by Cauchy integrals.

Lemma 15.2.3.3. *Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on an open set $G \subseteq \mathbb{C}$. Then f is locally given by a Cauchy integral.*

Proof. Indeed, by Theorem 15.2.2.2, we see that for all $z \in G$, choosing a small circle C_z around z and such that C_z and $\mathrm{Int}(C_z)$ are inside G, we can write

$$f(z) = \frac{1}{2\pi i}\int_{C_z} \frac{f(w)}{w-z}dw.$$

Hence locally f looks like a Cauchy integral. □

We now show that Cauchy integrals are analytic, making holomorphic maps analytic by above lemma.

Proposition 15.2.3.4. *Maps defined by Cauchy integrals are analytic.*

Proof. Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ where G is open and let $\gamma : [a,b] \to G$ be a piecewise C^1 curve. We wish to show that $\tilde{f}$ defined on $\mathbb{C} \setminus \mathrm{Im}\,(\gamma)$ is given locally by power series. Indeed, pick any $z \in \mathbb{C} \setminus \mathrm{Im}\,(\gamma)$. Since $\mathrm{Im}\,(\gamma)$ is closed, therefore there exists a ball of radius r, B_r, such that $B_r \subseteq \mathbb{C} \setminus \mathrm{Im}\,(\gamma)$. In order to expand $\tilde{f}(z) = \int_\gamma \frac{f(w)}{w-z}dw$ as a power series, we first focus on $\frac{1}{w-z}$, where $w \in \mathrm{Im}\,(\gamma)$ and z is as above. Indeed, for any $z_0 \in B_r$, we have $|z - z_0| < r$ and $|w - z_0| > r$, thus yielding that $\left|\frac{z-z_0}{w-z_0}\right| < 1$ and hence we can write

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{(w-z_0)(1-\frac{z-z_0}{w-z_0})} \\
&= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}.
\end{aligned}$$

Moreover the convergence is uniform as we are within the radius of convergence. Now, $f(w) \leq M$ for all $w \in \mathrm{Im}\,(\gamma)$ as $\mathrm{Im}\,(\gamma)$ is compact and f is continuous over it. Hence we get that that following holds for all $w \in \mathrm{Im}\,(\gamma)$

$$\frac{f(w)}{w-z} = \sum_{n=0}^{\infty}\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}.$$

Taking integral both sides, it thus follows from uniform convergence of above series that

$$\begin{aligned}
\tilde{f}(z) = \int_\gamma \frac{f(w)}{w-z}dw &= \int_\gamma \sum_{n=0}^{\infty}\frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}dw \\
&= \sum_{n=0}^{\infty}\int_\gamma \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}dw \\
&= \sum_{n=0}^{\infty}\left(\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}dw\right)(z-z_0)^n.
\end{aligned}$$

Hence locally $\tilde{f}$ looks like a power series, i.e. it is analytic. □

Corollary 15.2.3.5. *Holomorphic maps are analytic.*

Proof. By Lemma 15.2.3.3, holomorphic maps are given by Cauchy integrals. By Proposition 15.2.3.4, maps given by Cauchy integrals are analytic. □

Morera's theorem : Converse of Cauchy's triangle theorem

Proposition 15.2.3.6. *If $f : G \to \mathbb{C}$ is a continuous map on an open set $G \subseteq \mathbb{C}$ such that for all triangles $T \subseteq G$ where $\mathrm{Int}(T) \subseteq G$ as well we get*

$$\int_T f(z)dz = 0,$$

then f is holomorphic.

Proof. □

Derivatives of a holomorphic map

Proposition 15.2.3.7. *Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on an open set $G \subseteq \mathbb{C}$. Then, f is differentiable to all orders and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(w)}{(w-z)^{n+1}} dw$$

where C_r is a circle in CCW orientation of radius r such that $C_r \subseteq G$ and $\mathrm{Int}(C_r) \subseteq G$. Moreover, for all $z \in \mathrm{Int}(C_r)$ with z_0 as center, we have that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \left(\int_{C_r} \frac{f(w)}{(w-z)^{n+1}} dw \right) (z-z_0)^n \end{aligned}$$

Proof. □

Liouville's theorem

A holomorphic map f on the entire complex plane, that is $f \in \mathcal{C}^{\mathrm{hol}}(\mathbb{C})$, is said to be *entire*.

Proposition 15.2.3.8. *Any entire bounded function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Proof. □

Zeroes of holomorphic maps

Proposition 15.2.3.9. *Let $G \subseteq \mathbb{C}$ be an open connected subset of $\mathbb{C}$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$ is a holomorphic map on G, then the zero set $V(f) = \{z \in G \mid f(z) = 0\}$ has no limit point in G i.e. either $V(f) = G$ or $V(f)$ is discrete.*

Proof. □

Identity theorem

Proposition 15.2.3.10. *Let $f, g \in \mathcal{C}^{\mathrm{hol}}(G)$ be two holomorphic maps defined on an open connected set $G \subseteq \mathbb{C}$. Then $f = g$ on G if and only if there exists a set $A \subseteq G$ which has a limit point in G such that $f|_A = g|_A$.*

Proof. □

Corollary 15.2.3.11. *Let f, g be two holomorphic maps on open connected subset $G \subseteq \mathbb{C}$ such that there exists an open set $U \subsetneq G$ contained inside of G such that $\partial U \neq \emptyset$ and $\bar{U} \subseteq G$ and $f|_U = g|_U$. Then $f = g$ on G.*

Proof. Indeed, since any element in ∂U is a limit point of U in G and $f|_U = g|_U$, then the result follows by Proposition 15.2.3.10. □

Corollary 15.2.3.12. *Let f, g be two holomorphic maps on open connected subset $G \subseteq \mathbb{C}$ such that there exists a closed ball $B \subset G$ on which $f|_B = g|_B$, then $f = g$ on G.*

Proof. A closed ball has non-empty interior. The result follows by Corollary 15.2.3.11. □

Maximum modulus principle

Proposition 15.2.3.13. *Let $G \subseteq \mathbb{C}$ be an open connected set and $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on G. Then $|f|$ doesn't achieves local maxima in G.*

Proof. □

Schwarz's lemma

Lemma 15.2.3.14. *Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc. If $f \in \mathcal{C}^{\mathrm{hol}}(D)$ is a holomorphic map $f : D \to D$ such that $f(0) = 0$, then*
1. *$|f(z)| \leq |z|$ for all $z \in D$.*
2. *$|f'(0)| \leq 1$.*
3. *If f is not of the form λz for $\lambda \in S^1$, then the inequality in 1. & 2. is strict at all points $z \in D \setminus \{0\}$. In particular, if there exists $z_0 \in D \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, then $f(z) = \lambda z$ for $|\lambda| = 1$ and $\lambda = f'(0)$.*

Proof. Consider the map defined by

$$\begin{aligned} g : D &\longrightarrow \mathbb{C} \\ z &\longmapsto \begin{cases} \frac{f(z)}{z} & \text{if } z \in D \setminus \{0\} \\ f'(0) & \text{if } z = 0. \end{cases} \end{aligned}$$

Clearly g is holomorphic. Now, for any $r \in (0, 1)$, for $C_r \subset D$, by maximum modulus, Proposition 15.2.3.13, we have

$$|g(z)| < \frac{1}{r}$$

for all $z \in \mathrm{Int}(C_r)$. Taking limit as $r \to 1$, we obtain $|g(z)| \leq 1$ for all $z \in D$. Now, if $\exists w \in D$ such that $|f(w)| = |w|$, then $|g(w)| = 1$. Since $|g(z)| < 1$ for all $z \in D$ as shown above, therefore by another use of maximum modulus, Proposition 15.2.3.13, it follows that $g(z) = \lambda$ is a constant where $|\lambda| = 1$. Thus $f(z) = \lambda z$. $\square$

Corollary 15.2.3.15. *(Pick's lemma) Let $f : D \to D$ be a holomorphic map where $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Then, for any two points $z, w \in D$*

$$\left|\frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}}\right| \leq \left|\frac{z - w}{1 - z\bar{w}}\right|$$

except if f is a linear fractional transform mapping disc onto itself.

Proof. Define for each $w \in D$ the following fractional transform

$$\begin{aligned} g_w : D &\longrightarrow D \\ z &\longmapsto \frac{z - w}{1 - z\bar{w}}. \end{aligned}$$

Then apply Schwarz's lemma (Lemma 15.2.3.14) on $g_{f(w)} \circ f \circ g_w^{-1} : D \to D$ as fractional transforms are biholomorphic. $\square$

Classification of bijective holomorphic maps of open unit ball

We shall classify all bijective holomorphic maps $f : D \to D$ for $D := \{z \in \mathbb{C} \mid |z| < 1\}$ and see that in the process that they are biholomorphic as well. For this, we first define the following important map which we encountered in Pick's lemma (Corollary 15.2.3.15). Define the following map for each $\alpha \in D$:

$$\begin{aligned} \varphi_\alpha : \bar{D} &\longrightarrow \bar{D} \\ z &\longmapsto \frac{z - \alpha}{1 - \bar{\alpha} z}. \end{aligned}$$

This is indeed a holomorphic map over $\bar{D}$. We now see that this is biholomorphic.

Theorem 15.2.3.16. *For any $\alpha \in D$, the map $\varphi_\alpha : \bar{D} \to \bar{D}$ is such that*

1. *φ_α takes D to D,*
2. *φ_α takes ∂D to ∂D,*
3. *φ_α is injective,*
4. *φ_α is surjective,*
5. *φ_α has a holomorphic inverse given by $\varphi_{-\alpha}$.*

Proof. Fix an $\alpha \in D$. We first show 2. For any $z \in \partial D$, we can write $z = e^{it}$ for $t \in \mathbb{R}$.

Thus we have

$$\begin{aligned}\left|\varphi_\alpha(e^{it})\right| &= \left|\frac{e^{it}-\alpha}{1-\bar{\alpha}e^{it}}\right| \\ &= \left|\frac{e^{it}-\alpha}{1-\bar{\bar{\alpha}}\bar{e^{it}}}\right| \\ &= \left|\frac{e^{it}-\alpha}{1-\alpha e^{-it}}\right| \\ &= \left|\frac{e^{it}-\alpha}{e^{it}-\alpha}\right| \\ &= 1.\end{aligned}$$

Thus, $\varphi_\alpha(e^{it}) \in \partial D$. This shows 2. Now we show 1. For this, by maximum modulus (Proposition 15.2.3.13), we have that $|\varphi_\alpha|$ achieves maxima on ∂D, and by 1., that maxima is 1, hence at every point of ∂D does $|\varphi_\alpha|$ achieves maxima. Hence $\varphi_\alpha(D) \subseteq D$. This shows 1. Next, it is a matter of simple calculation to see that $\varphi_\alpha \circ \varphi_{-\alpha} = \mathrm{id}_{\bar{D}}$ and thus by symmetry $\mathrm{id}_{\bar{D}} = \varphi_{-\alpha} \circ \varphi_\alpha$. Hence, φ_α is a biholomorphic map taking D onto D and ∂D onto ∂D. $\square$

We would now like to see that all biholomorphic maps of open unit ball are given by some unit modulus scalar multiples of φ_α. However, we need an idea to do so, which is provided by the following result.

Proposition 15.2.3.17. *(Extremality) For fixed $\alpha, \beta \in D$, denote $\mathcal{C}_{\alpha,\beta}$ to be the class of holomorphic maps into the unit disc $f : D \to D$ such that $f(\alpha) = \beta$. Then,*

1. *we have*

$$\sup_{f \in \mathcal{C}_{\alpha,\beta}} \left|f'(\alpha)\right| = \frac{1-|\beta|^2}{1-|\alpha|^2}.$$

2. *The map $f \in \mathcal{C}_{\alpha,\beta}$ achieving the suprema is given by the following rational map*

$$f = \varphi_{-\beta} \circ \lambda \circ \varphi_\alpha$$

where $\lambda \in \partial D$ is a scalar.

Proof. 1. We need only show that for each $f \in \mathcal{C}_{\alpha,\beta}$, we get

$$\left|f'(\alpha)\right| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}.$$

Indeed, this simply follows from a similar idea as used in the proof Pick's lemma (Corollary 15.2.3.15) above; consider the map $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$ and use Schwarz's lemma (Lemma 15.2.3.14) on it to get the bound $|g'(0)| \leq 1$. Now use chain rule while keeping in mind that $\varphi'(0) = 1 - |\alpha|^2$ and $\varphi'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}$.

2. From proof of 1, it follows that the equality is achieved if and only if $|g'(0)| = 1$. By Schwarz's lemma (Lemma 15.2.3.14) this happens only if $g(z) = \lambda z$ for $\lambda \in \partial D$. Rest follows by composing with inverses of φ_β and $\varphi_{-\alpha}$ which we know from Theorem 15.2.3.21, 5. $\square$

We now come to the real deal. The following shows that all bijective holomorphic maps $D \to D$ are biholomorphic and are given by unit modulus scalar multiples of φ_α for some $\alpha \in D$. However we shall need a topic which we will cover in the next few sections, namely the inverse function theorem for one complex variable (see Section **??**, Theorem **??**). Moreover we shall also need another result which we do only in a further section called Rouché's theorem (Section **??**, Theorem **??**).

Theorem 15.2.3.18. *(Bijective holomorphic maps $D \to D$) Let $f : D \to D$ be a bijective holomorphic map. Denote $\alpha \in D$ to be the unique element such that $f(\alpha) = 0$. Then, there exists $\lambda \in \partial D$ such that*

$$f = \lambda\varphi_\alpha.$$

Proof. Consider the set-theoretic inverse of f, denoted $g : D \to D$. By Rouché's theorem (Theorem **??**) and by inverse function theorem (Theorem **??**), we obtain that $g \in \mathcal{C}^{\text{hol}}(D)$. Now by chain rule we obtain $g'(f(\alpha))f'(\alpha) = 1$, that is, $g'(0) = 1/f'(\alpha)$. Now by Proposition 15.2.3.17, we obtain the following inequality for f and g where $f(\alpha) = 0$ and $g(0) = \alpha$:

$$\begin{aligned}|f(z)| &\leq \frac{1}{1 - |\alpha|^2}\\ |g(z)| &\leq 1 - |\alpha|^2 .\end{aligned}$$

In particular, we obtain that $1 - |\alpha|^2 \geq g'(0) = 1/f'(\alpha) \geq 1 - |\alpha|^2$, thus $g'(0) = 1 - |\alpha|^2$. Similarly, $|f'(\alpha)| = \frac{1}{1-|\alpha|^2}$. Hence f achieves the suprema of Proposition 15.2.3.17, 1. By Proposition 15.2.3.17, the result follows. □

Corollary 15.2.3.19. *There is a bijection*

$$\left\{\textit{Bijective holomorphic maps } f : D \to D\right\}$$

$$\cong$$

$$\left\{\textit{Rational functions of the form } \lambda\tfrac{z-\alpha}{1-\bar{\alpha}z},\ \alpha \in D,\ \lambda \in \partial D\right\}.$$

□

Using this and Schwarz's lemma, we can show that a holomorphic map $f : D \to D$ can have atmost one fixed point.

Corollary 15.2.3.20. *Let $f : D \to D$ be a holomorphic map. Then f has atmost one fixed point.*

Proof. The idea is quite simple and we have used it already in the proof of Pick's lemma (Corollary 15.2.3.15). Indeed, we will construct $\varphi_\alpha : D \to D$ in such a manner that Schwarz's lemma can be applied to $\varphi \circ f \circ \varphi^{-1}$ and will use the results about the function φ_α (Theorem 15.2.3.21).

Suppose $z_1 \neq z_2 \in D$ are two fixed points of f. Consider the map $\varphi_{z_1}(z) := \frac{z - z_1}{1 - \bar{z_1} z}$. This is a biholomorphic mapping $\varphi_{-z_1} : D \to D$. Consider

$$h = \varphi_{z_1} \circ f \circ \varphi_{-z_1}.$$

Then $h : D \to D$ is a holomorphic map and $h(0) = 0$. Applying Schwarz's lemma (Lemma 15.2.3.14), we obtain that $|h(z)| \leq |z|$. But notice that $h(\varphi_{z_1}(z_2)) = \varphi_{z_1}(z_2)$. Thus $\varphi_{z_1}(z_2)$ is a fixed point of h. Moreover, $\varphi_{z_1}(z_2) \neq 0$ as other wise $z_2 = z_1$, a contradiction. Thus, there exists $w \in D$ such that $|h(w)| = |w|$ (in particular, for $w = \varphi_{z_1}(z_2)$). Hence by contrapositive of Lemma 15.2.3.14, 3, we obtain that $h(z) = \lambda z$. Since $h(w) = w = \lambda w$, we obtain that $\lambda = 1$. Hence $h = \text{id}$, thus $f = \text{id}$. □

Open mapping theorem

This theorem is quite an important result in the theory of holomorphic maps. It says a very simple thing, *all holomorphic maps on open connected sets are open maps*(!)

Theorem 15.2.3.21. *Let $G \subseteq \mathbb{C}$ be an open connected subset and let $f \in \mathcal{C}^{\text{hol}}(G)$ be a non-constant holomorphic map $f : G \to \mathbb{C}$. Then f is an open map.*

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Proof. □

Fundamental theorem of algebra

Proposition 15.2.3.22. *Every non-constant polynomial $f(x) \in \mathbb{C}[x]$ can be factored into linear factors.*

Proof. Suppose $f(x) \in \mathbb{C}[x]$ is a non-constant polynomial given by

$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$

Suppose to the contrary that f has no zeros in $\mathbb{C}$. Then $g(x) = \frac{1}{f(x)} : \mathbb{C} \to \mathbb{C}$ is an entire map. We wish to use Liouville's theorem (Proposition 15.2.3.8) on $g(x)$ in order to obtain a contradiction. Indeed, to get an upper bound for $|g(x)|$, we need a lower bound for $|f(x)|$. To this end we have

$$\begin{aligned}
|f(x)| &\geq |a_n x^n + \cdots + a_1 x + a_0| \\
&\geq |a_n x^n| \left| \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right) \right| \\
&\geq |a_n x^n| \left(1 - \left| \frac{a_{n-1}}{a_n x} \right| - \cdots - \left| \frac{a_1}{a_n x^{n-1}} \right| - \left| \frac{a_0}{a_n x^n} \right| \right)
\end{aligned}$$

where the last inequality comes from triangle inequality. Now write $h(x) = 1 - \left| \frac{a_{n-1}}{a_n x} \right| - \cdots - \left| \frac{a_1}{a_n x^{n-1}} \right| - \left| \frac{a_0}{a_n x^n} \right|$. In order to get a further lower bound for $|f(x)|$, we need to get an upper bound for $h(x)$. Since $h(x) \to 0$ as $x \to \infty$, therefore for some $R > 0$, we shall have $h(x) \leq \frac{1}{3}$ for $|x| > R$. Thus, we get

$$|f(x)| \geq |a_n R^n| \frac{2}{3}$$

for $|x| > R$. Now on $|x| \leq R$, by continuity of $|f|$ on a compact domain, we get that it achieves a minima, and hence $|f|$ is a lower bounded map and hence $g(x)$ is an upper bounded map. $\square$

Inverse function theorem

Remember that for a differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$, if $x_0 \in \mathbb{R}^n$ is a point such that Df_{x_0} is invertible, the inverse function theorem tells us that f is a diffeomorphism in some neighborhood around x_0. A similar statement is true for holomorphic maps $f : G \subseteq \mathbb{C} \to \mathbb{C}$.

Theorem 15.2.3.23. *(Inverse function theorem) Let $G \subseteq \mathbb{C}$ be an open connected set and $\varphi \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on G. If for $z_0 \in G$ we have that $f'(z_0) \neq 0$, then there exists a neighborhood $z_0 \in V \subseteq G$ such that*

1. *$\varphi|_V : V \to \varphi(V)$ is bijective,*
2. *$\varphi(V) \subseteq G$ is open,*
3. *the map $\psi : \varphi(V) \to V$ given by $\varphi(z) \mapsto z$ is in $\mathcal{C}^{\text{hol}}(\varphi(V))$,*
4. *the map $\varphi|_V : V \to \varphi(V)$ is biholomorphic.*

We will now prove it. Let us begin with the following simple lemma.

Lemma 15.2.3.24. *Let $G \subseteq \mathbb{C}$ be an open connected set. If $f : G \to \mathbb{C}$ is a holomorphic map, then the map defined by*

$$\begin{aligned} g : G \times G &\longrightarrow \mathbb{C} \\ (z, w) &\longmapsto \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w, \\ f'(z) & \text{if } z = w \end{cases} \end{aligned}$$

is continuous.

Proof. Clearly g is continuous for all (z, w) with $z \neq w$. Pick any $a \in G$. We will show that g is continuous at (a, a). For that, we wish to estimate $|g(z, w) - g(a, a)|$. For this, note that we can write $g(z, w)$ as follows where γ is the straight path $\gamma(t) = (1 - t)z + tw$:

$$\begin{aligned} g(z, w) &= \frac{f(z) - f(w)}{z - w} \\ &= \frac{f(\gamma(0)) - f(\gamma(1))}{\gamma(0) - \gamma(1)} \\ &= \frac{1}{w - z} \int_\gamma f'(z) dz \\ &= \int_0^1 f'(\gamma(t)) dt \end{aligned}$$

where the third equality follows from Theorem 15.2.1.1, 2. Thus we can write

$$\begin{aligned}|g(z,w)-g(a,a)| &= \left|\int_0^1 f'(\gamma(t))dt - f'(a)\right| \\ &= \left|\int_0^1 f'(\gamma(t)) - f'(a)dt\right| \\ &\leq \int_0^1 \left|f'(\gamma(t)) - f'(a)\right| dt.\end{aligned}$$

Now by continuity of f', the estimate follows. □

We can now prove the inverse function theorem.

Proof of Theorem 15.2.3.23. 1. The surjectivity is clear. For injectivity, we will show that for two $z_1 \neq z_2 \in V$, $|\varphi(z_1) - \varphi(z_2)| \geq M$ for some $M > 0$ using the lemma just proved. Indeed, using Lemma 15.2.3.24 and triangle inequality, we obtain for $\epsilon = \frac{1}{2}|\varphi'(z_0)|$ an open set $\tilde{V}$ containing (z_0, z_0) such that for all $(z_1, z_2) \in \tilde{V}$ with $z_1 \neq z_2$ we get the following

$$\left|\left|\frac{\varphi(z_1)-\varphi(z_2)}{z_1-z_2}\right| - \left|\varphi'(z_0)\right|\right| \leq \left|\frac{\varphi(z_1)-\varphi(z_2)}{z_1-z_2} - \varphi'(z_0)\right| < \epsilon = \frac{1}{2}\left|\varphi'(z_0)\right|.$$

Using this, we obtain that

$$|\varphi(z_1)-\varphi(z_2)| \geq \frac{1}{2}\left|\varphi'(z_0)\right| |z_1 - z_2|.$$

Thus for $z_1, z_2 \in V \subseteq G$ where V is obtained by projecting a small open set inside $\tilde{V}$ back to G, we see that on that V φ is injective.

2. This is just open mapping theorem, Theorem 15.2.3.21.

3. Shrink V enough so that $\varphi'(z) \neq 0$ for all $z \in V$. Then everything is straightforward using

$$|\varphi(z_1)-\varphi(z_2)| \geq \frac{1}{2}\left|\varphi'(z_0)\right| |z_1 - z_2|$$

which we obtained in 1. □

Local m^{th} power property

Any holomorphic map around a point can be represented by the m^{th} power of some other special holomorphic map. Indeed, this is what the following theorem tells us.

Theorem 15.2.3.25. *Let $G \subseteq \mathbb{C}$ be an open-connected subset of $\mathbb{C}$ and let $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on G. Let $z_0 \in G$ and denote $w_0 = f(z_0)$. Let m be the order of zero that $f - w_0$ has at z_0. Then, there exists an open set $z_0 \in V \subseteq G$ and a holomorphic map*

$$\varphi : V \to \mathbb{C}$$

in $\mathcal{C}^{\text{hol}}(G)$ such that

1. *$f(z) = w_0 + (\varphi(z))^m$ for all $z \in V$,*
2. *φ' is nowhere vanishing in V, i.e. has no zero in V,*
3. *there exists $r > 0$ such that φ is biholomorphic onto $D_r(0)$, the open disc of radius r around 0. Thus, $\varphi : V \to D_r(0)$ is bijective.*

Proof. The main point of the proof is to try to represent the desired φ as $\exp \frac{??}{m}$. We just need to fill ?? correctly. Since $f - w_0$ has zero of order m at z_0, therefore there exists $g \in \mathcal{C}^{\text{hol}}(G)$ such that

$$f(z) - w_0 = (z - z_0)^m g(z).$$

Now, by appropriately shrinking G away from zeros of g, we may assume $g \neq 0 \forall z \in G \backslash \{z_0\}$[1]. Thus we have that $\frac{g'}{g}$ is holomorphic on G (this is our V). By Lemma 15.2.2.5, we get $h \in \mathcal{C}^{\text{hol}}(G)$ such that $h' = \frac{g'}{g}$. We now claim that $g = \exp h$. Indeed, it is a simple matter to see that the derivative of $g \exp -h$ is zero. Thus, by using surjectivity of exp, we can absorb the additive constant into h to obtain the above claim. One then sees that

$$\varphi(z) = (z - z_0) \exp \frac{h(z)}{m}$$

does the job for 1. The rest is straightforward. □

Harmonic conjugates

We will now show that any real valued harmonic map $u : G \subseteq \mathbb{R}^2 \to \mathbb{R}$ defines a unique (upto some constant) holomorphic map $g : G \to \mathbb{C}$ whose real part is u.

Theorem 15.2.3.26. *Let $G \subseteq \mathbb{C}$ be a convex open connected set. Let $u : G \to \mathbb{R}$ be a harmonic real valued function. Then, there exists holomorphic map $g : G \to \mathbb{C}$ unique upto an additive constant such that*

$$\Re g = u.$$

Proof. The main idea is to construct a holomorphic map f on G via the data of partial derivatives of u, and then use the Lemma 15.2.2.5, to get a primitive g, which will do the job. Indeed, we can make f via the following observation: u is harmonic real valued function if and only if $\frac{\partial^2}{\partial \bar{z} \partial z} = 0$. Using this, just define $f = u_x - i u_y$ and to show that f is holomorphic, observe that $\frac{\partial f}{\partial \bar{z}} = 0$. □

In combination with Lemma 15.1.3.2, we get that

Corollary 15.2.3.27. *Let $G \subseteq \mathbb{C}$ be open connected. Then,*

$$\{g : G \to \mathbb{C} \text{ is holomorphic}\} \cong \{u : G \to \mathbb{R} \text{ is harmonic}\}$$

where we identify functions upto additive constant. □

[1] We are implicitly using the isolated zeros theorem (Theorem **??**) which we shall do later.

15.3 Singularities

Consider the map $f(z) = 1/z$ on $\mathbb{C}^\times$. It is holomorphic. However, at $z = 0$, it is not holomorphic. Such points are called singularities of f, as we shall define more clearly later. Our goal is to study this phenomenon more carefully in this section. For this, we first need to develop a tool for local analysis of such "bad" points (some may also call it "the" points).

15.3.1 Laurent series

Definition 15.3.1.1. (**Laurent series**) A Laurent series centered at $z_0 \in \mathbb{C}$, denoted by $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, is a series of functions defined on some annulus $A_{z_0}(R_1, R_2) := \{z \in \mathbb{C} \mid R_1 < |z-z_0| < R_2\}$ centered at z_0 for $0 \le R_1 < R_2$ such that the series converges at all points $z \in A_{z_0}(R_1, R_2)$. That is, the sequence of holomorphic maps $\{\sum_{n=-N}^{n=N} a_n(z-z_0)^n\}_N$ on $A_{z_0}(R_1, R_2)$ converges uniformly and absolutely to a holomorphic function $f : A_{z_0}(R_1, R_2) \to \mathbb{C}$ (by Weierstrass theorem).

For a Laurent series, we can find the coefficents in terms of Cauchy integral of the function it represents.

Lemma 15.3.1.2. *Let $f(z) = \sum_{n=-\infty}^{n=\infty} a_n(z-z_0)^n$ be a Laurent series around $z_0 \in \mathbb{C}$ in an annulus. Then for all $n \in \mathbb{Z}$*

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{(w-z_0)^{n+1}}dw$$

where $R_1 < r < R_2$.

Proof. Use the uniform convergence of the Laurent series on $\frac{f(z)}{(z-z_0)^{n+1}}$ (so to limits out of integrals) and the fact that $\int_{C_r}(z-z_0)^n dz = 2\pi i$. □

By Cauchy-Hadamard theorem for calculation of radius of convergence we also get the parameters for the maximum annulus on which a Laurent series can exist.

Lemma 15.3.1.3. *For a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, the smallest value of R_1 and largest value of R_2 such that $f(z)$ converges on $A_{z_0}(R_1, R_2)$ is given by*

1. $R_1 = \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}}$
2. $R_2 = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$

Proof. Straightforward use of Cauchy-Hadamard. □

Our goal now is to show the following theorem.

Theorem 15.3.1.4. *Consider any $0 < R_1 < R_2$ and any $z_0 \in \mathbb{C}$. If $f : A_{z_0}(R_1, R_2) \to \mathbb{C}$ is holomorphic, then it is represented by a Laurent series.*

We shall see this in the following subsections.

Cauchy integral in a neighborhood of ∞

Cauchy formula for an annulus

Existence of Laurent series

15.3.2 Isolated singularities : Removable, poles and essential

We now come to the main matter of the present study, the notion of singularities. A holomorphic function $f : G \to \mathbb{C}$ is said to have an *isolated singularity* at $z_0 \notin G$ if there exists a punctured disc $A_{z_0}(0,r) \hookrightarrow G$. Consequently, by Theorem 15.3.1.4, we obtain a Laurent series expansion of f in $A_{z_0}(0,r)$. Let us denote it by

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n.$$

We can then classify the isolated singularity z_0 into three types:

1. z_0 is a *removable singularity* if $a_n = 0$ for all $n < 0$,
2. z_0 is a *pole of order m* if $\min\{n < 0 \mid a_n \neq 0\} = m$,
3. z_0 is an *essential singularity* if $\min\{n < 0 \mid a_n \neq 0\} = -\infty$ or unbounded.

There are three characterizing theorems of each of the three kinds of singularities.

Theorem 15.3.2.1. *(Riemann's extension theorem) Let $f : G \to \mathbb{C}$ be a holomorphic map. Then the following are equivalent.*

1. *The point $z_0 \in \mathbb{C} \setminus G$ is a removable singularity of f.*
2. *There exists a punctured disc $A_{z_0}(0,r) \hookrightarrow G$ such that f is bounded on it.*

Theorem 15.3.2.2. *(Criterion for a pole) Let $f : G \to \mathbb{C}$ be a holomorphic map. Then the following are equivalent.*

1. *The point $z_0 \in \mathbb{C} \setminus G$ is a pole of f of some order.*
2. *We have*

$$\varinjlim_{z \to z_0} |f(z)| = \infty.$$

Theorem 15.3.2.3. *(Casorati-Weierstrauss theorem) Let $f : G \to \mathbb{C}$ be a holomorphic map. If the point $z_0 \in \mathbb{C} \setminus G$ is an essential singularity of f, then there exists a punctured disc $A_{z_0}(0,r) \hookrightarrow G$ such that $f(A_{z_0}(0,r))$ is dense in $\mathbb{C}$.*

The last theorem in particular shows the chaotic behaviour of essential singularities. We shall prove them in the remaining part of this section.

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15.4 Cauchy's theorem - II

Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus \mathrm{Im}(\gamma)$. We define the *index of γ* to be the following map over Ω:

$$\begin{aligned} \mathrm{Ind}_\gamma(z) : \Omega &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}dw. \end{aligned}$$

Lemma 15.4.0.1. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus \mathrm{Im}\,(\gamma)$. Then $\mathrm{Ind}_\gamma(z)$ is a holomorphic map on Ω.*

Proof. This follows from Proposition 15.2.3.4, as $\mathrm{Ind}_\gamma(z)$ is the Cauchy integral of the constant function 1. □

The following is the main theorem that we shall use.

Theorem 15.4.0.2. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus \mathrm{Im}\,(\gamma)$. Then,*

1. *$\mathrm{Ind}_\gamma(z)$ is an integer valued map,*
2. *$\mathrm{Ind}_\gamma(z)$ is constant on each connected component of Ω,*
3. *$\mathrm{Ind}_\gamma(z)$ is 0 on unbounded component of Ω.*

We now introduce the main Cauchy's theorem.

15.4.1 General Cauchy's theorem

To state the Cauchy's theorem in full generality, we first need to build the small language of chains, which is just a slight generalization of curves. Let $\{\gamma_i : I_i \to \mathbb{C}\}_{i=1}^n$ be a finite collection of piecewise C^1 curves over $\mathbb{C}$. A *chain generated by* $\{\gamma_i\}$ is a formal sum of the form

$$\Gamma = \gamma_1 + \cdots + \gamma_n.$$

One can be more precise here by treating Γ as an element of the free abelian group of all singular 1-chains, but we don't need that technology right now. We denote

$$\mathrm{Im}\,(\Gamma) := \bigcup_{i=1}^n \mathrm{Im}\,(\gamma_i).$$

Moreover, for a continuous map $f : \mathrm{Im}\,(\Gamma) \to \mathbb{C}$, we further denote

$$\int_\Gamma f(z)dz := \sum_{i=1}^n \int_{\gamma_i} f(z)dz.$$

We can further define the *index of a chain* Γ as simply the sum of indices of individual curves:

$$\mathrm{Ind}_\Gamma(z) := \sum_{i=1}^n \mathrm{Ind}_{\gamma_i}(z)$$

for all $z \in \Omega$, where $\Omega = \mathbb{C}^\times \setminus \mathrm{Im}\,(\Gamma)$. Note that the set Ω here will have multiple components if each element of the cycle is a distinct loop. Indeed, if $\Gamma = \gamma_1 + \cdots + \gamma_n$ is a cycle where each γ_i is a closed loop, then we call Γ a *cycle*. The general Cauchy's theorem is then a statement about integral over cycles.

Theorem 15.4.1.1. *(Cauchy's theorem) Let $\Omega \subseteq \mathbb{C}$ be an open set and $\Gamma \hookrightarrow \Omega$ be a cycle such that*

$$\mathrm{Ind}_\Gamma(z) = 0 \;\; \forall z \in \mathbb{C}^\times \setminus \Omega.$$

Let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then,

1. *(Integral formula)*

$$\mathrm{Ind}_\Gamma(z) f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw$$

   for all $z \in \Omega \setminus \mathrm{Im}\,(\Gamma)$.

2. *(Integral theorem)*

$$\int_\Gamma f(z)dz = 0,$$

3. *if $\Gamma_0, \Gamma_1 \hookrightarrow \Omega$ are two cycles such that $\mathrm{Ind}_{\Gamma_0}(z) = \mathrm{Ind}_{\Gamma_1}(z)$ for all $z \notin \Omega$, then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

The most important of the above triad of conclusions is the first one, which clearly generalizes the known integral formula.

15.4.2 Homotopy & Cauchy's theorem

Theorem 15.4.2.1. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set. If $\gamma_0, \gamma_1 \hookrightarrow \Omega$ are two piecewise C^1 closed loops in Ω such that they are homotopic in Ω, then*

$$\mathrm{Ind}_{\gamma_0}(z) = \mathrm{Ind}_{\gamma_1}(z) \;\; \forall z \notin \Omega.$$

This has some major corollaries in combination with Theorem 15.4.1.1.

Corollary 15.4.2.2. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. If $\gamma_0, \gamma_1 \hookrightarrow \Omega$ are two piecewise C^1 closed loops in Ω such that they are homotopic in Ω, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Corollary 15.4.2.3. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. If $\gamma \hookrightarrow \Omega$ is a piecewise C^1 closed loop in Ω and Ω is simply connected, then*

$$\int_\gamma f(z)dz = 0.$$

15.5 Residues and meromorphic maps

Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and $f : \Omega \to \mathbb{C}$ be holomorphic. Let $z_0 \notin \Omega$ be a point of isolated singularity of f. The *residue of f at z_0* is then defined to be the coefficient a_{-1} of the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

of the map f around z_0. We denote residue of f at z_0 by $\mathrm{res}_{z_0}(f) := a_{-1}$. For example, consider the following integral where C_r is a circle of radius r centered at z_0

$$\int_{C_r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Since all terms $a_n(z - z_0)^n$ are $n \neq -1$ contributes zero integral as the positive parts of holomorphic in the interior of the loop and the negative parts are derivatives of constant 1, which is zero, therefore the only non-zero term is contributed by $n = -1$. Consequently, we have

$$\begin{aligned}\int_{C_r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n &= 2\pi i a_{-1} \\ &= 2\pi i \mathrm{res}_{z_0}(f)\end{aligned}$$

where $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

We now define a class of holomorphic maps which one encounters often in complex analysis.

Definition 15.5.0.1. (**Meromorphic maps**) Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and $f : \Omega \to \mathbb{C}$ be any function. We say that f is meromorphic if

1. there exists a set $A \subset \Omega$ which has no limit points in Ω,
2. $f : \Omega \setminus A \to \mathbb{C}$ is holomorphic,
3. every point of A is a pole of f.

One often calls the set A as the *set of poles of f*.

There are some observations to be made.

Lemma 15.5.0.2. *Let $f : \Omega \to \mathbb{C}$ be a meromorphic map on an open-connected set Ω. Then, the set of poles of f is atmost countable.*

Proof. Let $A \subset \Omega$ be the set of poles of f. Covering Ω by countably many compact sets $\{K_i\}$, we observe that intersection of each of $K_i \cap A$ has to be atmost finite, otherwise there exists a sequence in $K_i \cap A$, which consequently admits a convergent subsequence, that is, a limit point in Ω. Consequently, A is a countable union of finite sets. $\square$

Remark 15.5.0.3. For the purposes of residue of f at $a \in A$, one can replace analysis of f with analysis of f by the analysis of $Q = \sum_{n=-m}^{-1} a_n(z-a)^n$, called the principal part of f at a where m is the order of pole of f at $a \in A$. Clearly, $\mathrm{res}_a Q = \mathrm{res}_a f$. Moreover, one sees that

$$\mathrm{res}_a(f)\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma Q(z)dz$$

where γ is a piecewise C^1-loop centered at a, in $\Omega \setminus A$. This is again a consequence of the fact that all terms inside the integral are zero except the one corresponding to a_{-1}. Indeed, this hints at a general phenomenon, which is clarified by the following theorem.

Theorem 15.5.0.4. *(The residue theorem) Let $\Omega \subseteq \mathbb{C}$ be an open-connected set. If $f : \Omega \to \mathbb{C}$ is a meromorphic map with $A \subseteq \Omega$ its set of poles and Γ a cycle in $\Omega \setminus A$ such that*

$$\mathrm{Ind}_\Gamma(z) = 0 \ \ \forall z \notin \Omega,$$

then

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{a\in A} \mathrm{res}_a(f)\mathrm{Ind}_\Gamma(a).$$

We now an important result, which gives us information of zeroes of holomorphic maps on certain subsets.

Theorem 15.5.0.5. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1-loop in an open-connected set $\Omega \subseteq \mathbb{C}$ such that*

1. $\mathrm{Ind}_\gamma(z) = 0$ *for all* $z \notin \Omega$,
2. $\mathrm{Ind}_\gamma(z) = 0$ *or 1 for all* $z \in \Omega \setminus \mathrm{Im}(\gamma)$.

Then we have that for any holomorphic maps $f, g : \Omega \to \mathbb{C}$, denoting $\Omega_1 := \{z \in \Omega \setminus \mathrm{Im}(\gamma) \mid \mathrm{Ind}_\gamma(z) = 1\}$ and $N_f = \#Z(f) \cap \Omega_1$, we get that

1. *if f has no zeros on $\mathrm{Im}(\gamma) \subseteq \Omega$, then*

$$N_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \mathrm{Ind}_{f\circ\gamma}(0).$$

2. *(Rouché's theorem) if*

$$|f(z) - g(z)| < |g(z)| \ \ \forall z \in \mathrm{Im}(\gamma),$$

then $N_g = N_f$.

15.6 Riemann mapping theorem

Chapter 16

Riemann Surfaces

We discuss the theory of one dimensional complex manifolds, aka Riemann surfaces. The tools which we will use here will yield a good motivation to study them in the scheme theoretic setting, as done in Chapter 1. Our main goal in these notes is to reach Riemann-Roch theorem and discuss some applications.

Let us begin by defining a Riemann surface and then see some examples.

Chapter 17

Foundational Complex Geometry

We here give a quick introduction to complex, Kähler manifolds and vector bundles. This chapter can also be treated as a motivation for abstract algebraic geometry. We follow Huybrechts[cite] for all details we miss.

17.1 Overview of analysis of several variables

17.2 Complex manifolds

17.3 Kähler manifolds

17.4 Vector bundles

Part V

The Categorical Viewpoint

Out of the four, this is the most foundational and the deepest one of them all. It has to be, as the main motive here is to understand some of the foundational notions of geometry, like *intersection* and *deformation*, and to act as their natural mathematical residential address. However, we would need to cover a lot of ground before we start doing geometry in this new world, most of it is due to a fundamental different way of thinking than what is done classically (more categorical than set theoretic, the latter is abound in some of the previous chapters of this book). But the rewards are high, for it will provide us a deeper understanding of fundamental questions raised throughout this book, one of them being the question of *a concrete, robust and complete theory of intersections of manifolds and schemes.*

Chapter 18

Classical Topoi

We give a *general review*[1] of classical topos theory, the main references being the cite[MacMoe], cite[Moe] and cite[Elephant]. *You can add material from your past notes.*

18.1 Topoi, maps of topoi and examples

18.2 Topoi as generalized spaces

18.3 Universal constructions on topoi

18.4 Cohomology and homotopy of topoi

[1] Its important to highlight that this is supposed to be an overview not an encyclopedic reference!

Chapter 19

Language of ∞-Categories

Contents

In this chapter we give an overview of the language of ∞-categories. The ultimate goal of this chapter is to define the ∞-category of ∞-groupoids and state the Yoneda lemma in ∞-categorical setting. Therefore, if you believe that ∞-groupoids are spaces (homotopy hypothesis), then we would construct the ∞-category of spaces by the end of this chapter. In the process, we will learn about the techniques invovled in manipulating ∞-categories. In this chapter, we will only work with the category of compactly generated spaces and by **Top** we mean category of compactly generated spaces.

19.1 Simplicial sets

The goal of simplicial sets is to obtain a combinatorial approximation of topological spaces. We describe simplicial sets as a presheaf over simplex category. One can motivate themselves why this definition is the correct definition for combinatorially handling topological spaces by reading the review paper cite[Bergner's simplicial set paper]. We give some basic properties of such objects. We also give a general important result about presheaves, "every presheaf is a colimit of representable presheaves". This is of fundamental importance in the development of ∞-categories.

Remark 19.1.0.1. (*Notations*)

1. As we will be frequently constructing and dealing with presheaf category over a category $\mathbf{C}$, therefore instead of only denoting it by $\mathbf{PSh}(\mathbf{C})$, we will also be denoting it by $\widehat{\mathbf{C}}$, depending on the convenience of the given situation.
2. We will denote objects of an ordinary 1-category $\mathbf{C}$ by lowercase alphabets like $a, b, c, .., x, y, z,$ morphisms in $\mathbf{C}$ by lowercase letters like $f, g, h, \dots$ and functors $\mathbf{C} \to \mathbf{D}$ by uppercase alphabets like $A, B, C, ..., X, Y, Z$ and also by lowercase alphabets like $f, g, h, \dots$.
3. Let $\mathbf{C}$ be a category and $c \in \mathbf{C}$ be an object. We denote by $h_c : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ the contravariant hom-functor given by $a \mapsto \mathrm{Hom}_{\mathbf{C}}(a, c)$.
4. We will denote ∞-categories by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$.

Let us also as a reminder put the usual Yoneda lemma.

Lemma 19.1.0.2. *(Yoneda lemma) Let $\mathbf{A}$ be a category, $a \in \mathbf{A}$ be an object and F be a presheaf over $\mathbf{A}$. Then, there is a natural isomorphism*

$$\mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F) \xrightarrow{\cong} F(a).$$

Proof. Consider the following maps

$$\begin{aligned} \varphi : \mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F) &\longrightarrow F(a) \\ \alpha &\longmapsto \alpha_a(\mathrm{id}_a). \end{aligned}$$

and

$$\begin{aligned} \xi : F(a) &\longrightarrow \mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, F) \\ x &\longmapsto \beta : h_a \to F \end{aligned}$$

where β is defined as follows. Consider any $a' \in \mathbf{A}$ and any $f \in h_a(a')$. Define $\beta_{a'} : h_a(a') \to F(a')$ by mapping as $f \mapsto F(f)(x)$.

One then sees that $\xi \circ \varphi = \mathrm{id}$ by naturality of α and that $\varphi \circ \xi = \mathrm{id}$ by funtoriality of F. $\square$

Corollary 19.1.0.3. *Let $\mathbf{A}$ be a category and define* $\mathrm{Yon} : A \to \widehat{\mathbf{A}}$ *to be the Yoneda functor given by $a \mapsto h_a$ and $a \to b \mapsto h_a \to h_b$. Then,* Yon *is fully-faithful.*

Proof. We have by Yoneda lemma (Lemma 19.1.0.2) that $\mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, h_b) \cong h_b(a) = \mathrm{Hom}_{\mathbf{A}}(a, b)$. $\square$

19.1.1 Extension of functor by colimits-I

The following is a fundamental result in category theory.

Theorem 19.1.1.1. *Let* $\mathbf{A}$ *be a small category and* $\mathbf{C}$ *be a locally small category with all small colimits. Then, for any functor*

$$f : \mathbf{A} \to \mathbf{C}$$

we get two functors

$$f_! : \widehat{\mathbf{A}} \longrightarrow \mathbf{C}$$

and

$$\begin{aligned} f^* : \mathbf{C} &\longrightarrow \widehat{\mathbf{A}} \\ c &\longmapsto \mathrm{Hom}_{\mathbf{C}}(f(-), c) \end{aligned}$$

such that

1. $f_!$ *is left adjoint of* f^*

$$\begin{tikzcd} \widehat{\mathbf{A}} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathbf{C} \arrow[l, shift left, "f^*"] \end{tikzcd} .$$

   The functor $f_!$ *is called the extension of* f *by colimits.*

2. *Denoting* $\mathrm{Yon} : \mathbf{A} \hookrightarrow \widehat{\mathbf{A}}$ *to be the Yoneda embedding, we get that the following commutes upto a unique natural isomorphism*

$$\begin{tikzcd} \widehat{\mathbf{A}} \arrow[r, "f_!"] \arrow[dr, phantom, "\Searrow"] & \mathbf{C} \\ \mathbf{A} \arrow[u, hook, "\mathrm{Yon}"] \arrow[ur, "f"'] & \end{tikzcd} .$$

   That is, $f_!(h_a) \cong f(a)$ *where* $h_a = \mathrm{Hom}_{\mathbf{A}}(-, a)$.

Proof. The main heart of the proof is the fact that every presheaf is a colimit of representables (Proposition 1.1.8 of cite[Cis]).

1. We define $f_! : \widehat{\mathbf{A}} \to \mathbf{C}$ as follows. Pick any $X \in \widehat{\mathbf{A}}$. Consider the functor $\varphi_X : \mathbf{A}/X \to \widehat{\mathbf{A}}$ from the category of elements of X given by $(a, s) \mapsto h_a$ and on maps by $f : (a, s) \to (b, t)$ by $h_f : h_a \to h_b$. By Proposition 1.1.8 of cite[Cis], we have $\varinjlim_{(a,s)} h_a = X$. Define $f_!(X) = \varinjlim_{(a,s)} f(a)$ which exists in $\mathbf{C}$ as $\mathbf{C}$ has all small colimits. Consequently, we obtain the following natural isomorphisms by Yoneda lemma (Lemma 19.1.0.2) and limit preserving properties of contravariant homs

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(f_!(X), c) &\cong \mathrm{Hom}_{\mathbf{C}}\left(\varinjlim_{(a,s)} f(a), c\right) \cong \varprojlim_{(a,s)} \mathrm{Hom}_{\mathbf{C}}(f(a), c) \\ &\cong \varprojlim_{(a,s)} \mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, f^*(c)) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}}\left(\varinjlim_{(a,s)} h_a, f^*(c)\right) \\ &\cong \mathrm{Hom}_{\widehat{\mathbf{A}}}(X, f^*(c)). \end{aligned}$$

2. Since $\mathrm{Hom}_{\mathbf{C}}(f_!(h_a), c) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}}(h_a, f^*c) \cong f^*(c)(a) = \mathrm{Hom}_{\mathbf{C}}(f(a), c)$ for all objects $c \in \mathbf{C}$, therefore the result follows by Corollary 19.1.0.3. □

The above theorem will take a central place in constructions of this chapter. Indeed, let us point the following applications of this theorem before stating its proof.

Corollary 19.1.1.2. *Let* $\mathbf{A}$ *be a small category and* $\mathbf{C}$ *be a category with small colimits. If* $F : \widehat{\mathbf{A}} \to \mathbf{C}$ *is a colimit preserving functor. Then*

1. *there exists* $f : \mathbf{A} \to \mathbf{C}$ *such that* $f_! \cong F$ *naturally,*
2. F *has a right adjoint.*

Proof. We first define the required f. For any object $a \in \mathbf{A}$, define $f(a) := F(h_a)$. Then by Theorem 19.1.1.1, 1, it follows that $f_!(h_a) \cong f(a) = F(h_a)$. Since we know that every presheaf is presented as a colimit of representable functors indexed by its category of elements, thus we get that for any $X \in \widehat{\mathbf{A}}$, $F(X) \cong f_!(X)$ naturally. The second conclusion also follows from Theorem 19.1.1.1, 2. □

The following result will be important in order to define simplicial mapping spaces.

Proposition 19.1.1.3. *For any small category* $\mathbf{A}$*, the presheaf category* $\widehat{\mathbf{A}}$ *is Cartesian closed where the internal hom object is defined by*

$$\underline{\mathrm{Hom}}(X, Y)(a) := \mathrm{Hom}_{\widehat{\mathbf{A}}}(X \times h_a, Y)$$

and on morphisms as

$$\underline{\mathrm{Hom}}(X, Y)(f) := \mathrm{Hom}_{\widehat{\mathbf{A}}}(X \times h_f, Y).$$

Proof. We wish to show that $\underline{\mathrm{Hom}}(-,-)$ acts as internal hom object in $\widehat{\mathbf{A}}$. This can be seen by establishing following natural bijections

$$\mathrm{Hom}_{\widehat{\mathbf{A}}}(T \times X, Y) \cong \mathrm{Hom}_{\widehat{\mathbf{A}}}(T, \underline{\mathrm{Hom}}(X, Y))$$

This follows from contemplating the functor $f_X : \mathbf{A} \to \widehat{\mathbf{A}}$ for all $X \in \widehat{\mathbf{A}}$ given by $a \mapsto X \times \mathrm{Yon}(-)$, together with Theorem 19.1.1.1. Indeed, first observe that $\widehat{\mathbf{A}}$ is locally small with all small colimits. Second, observe from the proof of Theorem 19.1.1.1 that for any $Z \in \widehat{\mathbf{A}}$, we have that $f_{X!} : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}$ takes any $Z \in \widehat{\mathbf{A}}$ and maps it to $f_{X!}(Z) = \varinjlim_{(a,s)} f_X(a)$ where (a, s) varies over $\mathbf{A}/Z$, the category of elements of Z. Since $f_X(a) = X \times h_a$ and filtered colimits commute with finite limits, therefore we get a natural isomorphism $f_{X!}(Z) \cong X \times Z$. Consequently, the adjunction of Theorem 19.1.1.1 completes the proof. □

19.1.2 Categories $\boldsymbol{\Delta}$ and sSet

Consider the category of all finite sets $[n]$ with $n+1$ elements with linear order $0 < 1 < \cdots < n$ and mappings being the non-decreasing maps. Denote this category by $\boldsymbol{\Delta}$ and call it the *simplex category*. A *simplicial set* is then a presheaf over $\boldsymbol{\Delta}$. Let the category of all simplicial sets be denoted **sSet**.

There are two important class of maps in $\boldsymbol{\Delta}$.

Definition 19.1.2.1 (**Face and degeneracy maps**)**.** For each $n \in \mathbb{N}$, we have $n+1$ face maps

$$\begin{aligned} d^i : [n-1] &\to [n] \\ j &\mapsto \begin{cases} j & \text{if } j \leq i \\ j+1 & \text{if } j > i \end{cases} \end{aligned}$$

where $0 \leq i \leq n$ and n degeneracy maps

$$\begin{aligned} s^i : [n] &\to [n-1] \\ j &\mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases} \end{aligned}$$

where $0 \leq i \leq n-1$.

Remark 19.1.2.2. By Yoneda embedding, we will allow ourselves to abuse the notation by writing $d^i : [n-1] \to [n]$ as the unique map $d^i : \Delta^{n-1} \to \Delta^n$ and $s^i : [n] \to [n-1]$ as the unique map $s^i : \Delta^n \to \Delta^{n-1}$ in **sSet**. For a simplicial set X we may thus interpret $x \in X_n$ as $x : \Delta^n \to X$. The above give maps which we denote as

$$\begin{aligned} d_i : X_n &\longrightarrow X_{n-1} \\ x &\longmapsto x \circ d_i \end{aligned}$$

for each $0 \leq i \leq n$ also called the face maps and

$$\begin{aligned} s_i : X_{n-1} &\longrightarrow X_n \\ x &\longmapsto x \circ s_i \end{aligned}$$

for each $0 \leq i \leq n-1$ also called degeneracy maps.

It is quite easy to observe, but very important for applications, the following relations satisfied by face and degeneracy maps. All these are immediate from definition given above.

Proposition 19.1.2.3. *The following relations hold in* $\boldsymbol{\Delta}$ *(and thus in* **sSet** *):*

1. *If* $i < j$*, then* $d^j d^i = d^i d^{j-1}$.
2. *If* $i \leq j$*, then* $s^j s^i = s^i s^{j+1}$.
3. *We have*

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ d^{i-1} s^j & \text{if } j+1 < i. \end{cases}$$

□

Consequently, for a simplicial set, dual relations hold.

Proposition 19.1.2.4. *Let* X *be a simplicial set. Then the face and degeneracy maps of* X *satisfies the following relations:*

1. *If $i < j$, then $d_i d_j = d_{j-1} d_i$.*
2. *If $i \leq j$, then $s_i s_j = s_{j+1} s_i$.*
3. *We have*

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } j+1 < i. \end{cases}$$

□

Remark 19.1.2.5. One may keep the following picture in mind while working with a simplicial set X:

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2, "d^0"] \arrow[r, "d^1"'] & {[1]} \arrow[l, shift left=2, "s^0"] \arrow[r, shift left=3, "d^0"] \arrow[r, shift left=1, "d^1"] \arrow[r, shift right=1, "d^2"'] & {[2]} \arrow[l, shift left=4, "s^1"] \arrow[l, shift left=6, "s^0"] \arrow[r, shift left=2, "d^i"] & \cdots \arrow[l, shift left=2, "s^i"] \\ X_0 \arrow[r, shift right=2, "s_0"'] & X_1 \arrow[l, shift right=1, "d_0"'] \arrow[l, shift right=3, "d_1"'] \arrow[r, shift right=2, "s_0"'] \arrow[r, shift right=4, "s_1"'] & X_2 \arrow[l, "d_0"'] \arrow[l, shift right=2, "d_1"'] \arrow[l, shift right=4, "d_2"'] \arrow[r, shift right=2, "s_i"'] & \cdots \arrow[l, "d_i"'] \end{tikzcd}.$$

Remark 19.1.2.6. There is a functor

$$\begin{aligned} |-| : \mathbf{\Delta} &\longrightarrow \mathbf{Top} \\ [n] &\longmapsto |\Delta^n| \end{aligned}$$

where $|\Delta^n|$ is the standard topological n-simplex in $\mathbb{R}^{n+1}$ and for $f : [n] \to [m]$ in $\mathbf{\Delta}$, we have

$$\begin{aligned} |f| : |\Delta^n| &\to |\Delta^m| \\ (t_0, \ldots, t_n) &\longmapsto \left(\sum_{i \in f^{-1}(0)} i, \ldots, \sum_{i \in f^{-1}(m)} i \right). \end{aligned}$$

Example 19.1.2.7 (*Singular chains*)**.** An important example of a simplicial set is that of $\mathrm{Sing}(X)$ defined as

$$\begin{aligned} \mathrm{Sing}(X) : \mathbf{\Delta}^{op} &\to \mathbf{Set} \\ [n] &\mapsto \mathrm{Hom}_{\mathbf{Top}}\left(|\Delta^n|, X\right). \end{aligned}$$

The main point is that $\mathrm{Sing}(X)$ as a simplicial set knows all about the homotopy type of space X. Consequently, we will denote $X([n]) := \mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathbf{Top}}\left(|\Delta^n|, X\right)$.

Example 19.1.2.8 (*Nerve of a category*)**.** Let $\mathbf{C}$ be a category. Define the nerve of $\mathbf{C}$ as

$$\begin{aligned} N\mathbf{C} : \mathbf{\Delta}^{op} &\to \mathbf{Set} \\ [n] &\mapsto \mathrm{Hom}_{\mathbf{Cat}}\left([n], \mathbf{C}\right) \end{aligned}$$

where $[n]$ is a category as it is a poset. Consequently, $N\mathbf{C}_n$ is the set of all n-composable arrows of $\mathbf{C}$.

Theorem 19.1.2.9. *Nerve construction is a fully-faithfull embedding of categories into simplicial sets.*

19.1.3 Operations on simplicial sets

Define product, coproducts, subspaces, unions, quotients, limits, colimits and mapping objects. **TODO!!**

Example 19.1.3.1 (*Standard Δ^n, boundaries $\partial_i\Delta^n$, $\partial\Delta^n$ and horn Λ^n_i*)**.** Denote $\Delta^n = N[n]$ to be the nerve of the category $[n]$. That is,

$$\Delta^n([m]) = \Delta^n_m = \mathrm{Hom}_{\mathbf{\Delta}}([m],[n]) = h_{[n]}([m])$$

which are exactly all representable presheaves over $\mathbf{\Delta}$. These are the combinatorial analogues of the topological n-simplex $|\Delta^n|$ and we tend to think about Δ^n using the intuition gained from the topological one. There are some important simplicial subsets of Δ^n.

Let $E \subseteq [n]$ be a totally ordered subset. Define $\Delta^E = NE$ to be a simplicial subset of Δ^n. From this we derive the following simplicial subsets of Δ^n. The first is the i^{th}*-boundary of* Δ^n for $0 \le i \le n$ given by

$$\partial_i\Delta^n = \bigcup_{i\notin E\subsetneq[n]} \Delta^E \cong \Delta^{n-1}.$$

The second is the *boundary of* Δ^n given by

$$\partial\Delta^n = \bigcup_{E\subsetneq[n]} \Delta^E = \bigcup_{i=0}^{n} \partial_i\Delta^n$$

The third is the i^{th}-horn of Δ^n denoted Λ^n_i given by

$$\Lambda^n_i = \bigcup_{i\in E\subsetneq[n]} \Delta^E.$$

Remark 19.1.3.2. Let $n \ge 1$, $0 \le i \le n$ and $m \ge 0$. Note that we have

$$(\partial_i\Delta^n)_m = \left\{\begin{matrix}\text{Order preserving maps } f : [m] \to \\ [n] \text{ which are not surjective and } i \notin \\ \mathrm{Im}\,(f).\end{matrix}\right\}$$

$$(\partial\Delta^n)_m = \left\{\begin{matrix}\text{Order preserving maps } f : [m] \to \\ [n] \text{ which are not surjective.}\end{matrix}\right\}$$

$$(\Lambda^n_i)_m = \left\{\begin{matrix}\text{Order preserving maps } f : [m] \to \\ [n] \text{ which are not surjective and } i \in \\ \mathrm{Im}\,(f).\end{matrix}\right\}.$$

For two simplicial sets, we can define the internal hom using the Proposition 19.1.1.3.

Definition 19.1.3.3 (Homotopy & mapping complex). Let S, T be a simplicial set. Then $\underline{\mathrm{Hom}}(S,T)$ denotes the following simplicial set

$$[n] \mapsto \mathrm{Hom}_{\mathbf{sSet}}(S \times \Delta^n, T).$$

An n-simplex of $\underline{\mathrm{Hom}}(S,T)$ is defined to be an n-homotopy from S to T. A 1-homotopy H is also referred to as a homotopy from $H|_{S\times\{0\}} =: f$ to $H|_{S\times\{1\}} =: g$.

19.1.4 Basic properties

Do results and exercises from various sources to showcase the combinatorial arguments used while working with simplicial sets.

We discuss some properties of simplicial sets which would be useful later on. Some of these might be taken as exercises on combinatorial manipulations with simplicial sets.

We first begin with a simple observation.

Definition 19.1.4.1 (**n-degenerate**). A simplicial set X is said to be n-degenerate if for all $m > n$, all m-simplices in X_m are degenerate.

Example 19.1.4.2. Each standard simplicial sets Δ^n are n-degenerate. Indeed, its m-simplices for $m > n$ are

$$\Delta^n_m = \{\text{Order preserving maps } f : [m] \to [n]\}.$$

But as $m > n$, therefore every such f is necessarily non-injective. It follows that each simplex in Δ^n_m is in the image of $s_i : X_{m-1} \to X_m$ for some $0 \le i \le m-1$.

For similar reasons, the i^{th}-boundary $\partial_i\Delta^n$, boundary $\partial\Delta^n$ and horns Λ^n_i for $0 \le i \le n$ are all $n-1$-degenerate.

Lemma 19.1.4.3. *Let X be an n-degenerate simplicial set and Y be a simplicial set. Then any collection of functions $\{\varphi_m : X_m \to Y_m\}_{0\le m\le n}$ such that for any $f : [k] \to [l]$ in $\mathbf{\Delta}$ with $0 \le k, l \le n$, the following square commutes*

$$\begin{tikzcd} X_l \arrow[r, "\varphi_l"] \arrow[d, "f^*"'] & Y_l \arrow[d, "f^*"] \\ X_k \arrow[r, "\varphi_k"'] & Y_k \end{tikzcd},$$

the collection $\{\varphi_m\}_{0\le m\le n}$ lifts to a unique map of simplicial sets $\varphi : X \to Y$.

Remark 19.1.4.4. As a consequence of Lemma 19.1.4.3, in order to give a map of simplicial sets from an n-degenerate simplicial set S, it suffices to construct the required map only on m-simplices for $0 \le m \le n$.

Proof of Lemma 19.1.4.3. Let $\{\varphi_m\}_{0\le m\le n}$ be as given. We wish to define $\varphi_{m'}$ for $m' > n$. We proceed by induction on m'. Suppose $\varphi_{m'-1}$ is given to us. Since we have the following diagram

$$\begin{tikzcd} X_{m'-1} \arrow[r, "\varphi_{m'-1}"] \arrow[d, "s_i"'] & Y_{m'-1} \arrow[d, "s_i"] \\ X_{m'} \arrow[r, "\varphi_{m'}"'] & Y_{m'} \end{tikzcd}$$

and that every element of $X_{m'}$ is in image of s_i (guaranteed by Proposition 19.1.2.4, 3), it follows that there is a unique choice of $\varphi_{m'}$ to fit in the above diagram, as required. $\square$

Lemma 19.1.4.5. *Let $n \ge 1$ and $0 \le i \le n$. Then,*

1. *$\partial_i\Delta^n$ has exactly 1 non-degenerate $n-1$-simplex,*
2. *$\partial\Delta^n$ has exactly $n+1$ non-degenerate $n-1$-simplices,*
3. *Λ^n_i has exactly n non-degenerate $n-1$-simplices.*

Proof. These three items are immediate from Remark 19.1.3.2. □

The following is an important adjunction which will be consistently used in later sections. Moreover, it is generally good to keep in mind all the time while working with simplicial sets so that one transfer intution from topological spaces to that of simplicial sets, as we would need to do time and time again (for example when dealing with homotopy of simplicial sets).

Theorem 19.1.4.6 (*Geometric realization*)**.** *The singular functor* Sing : **Top** → **sSet** *has a left adjoint* $|-|$: **sSet** → **Top**

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, shift left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, shift left, "\mathrm{Sing}"] \end{tikzcd}.$$

The functor $|-|$ is called the geometric realization and for a simplicial set X, we have

$$|X| = \coprod_{n\geq 0} X_n \times |\Delta^n| / \sim$$

where $\sim$ is generated by $(f^*x, t) \sim (x, |f|\, t)$ for all $f : [n] \to [m]$ in $\mathbf{\Delta}$, $x \in X_m$ and $t \in |\Delta^n|$.

Proof. The main idea is that any map $X \to \mathrm{Sing}(Y)$ is a natural transform of presheaves. One observes that the naturality conditions on this morphism is equivalently represented in terms of a map $|X| \to Y$. **TODO**. □

Example 19.1.4.7. As an example, we calculate $\Delta^1 \times \Delta^1$ and show that

$$\Delta^1 \times \Delta^1 \cong \Delta^2 \cup_{\Delta^1} \Delta^2.$$

TODO.

Observe that even though Δ^1 has all simplices of dimension ≥ 2 as degenerate, yet $\Delta^1 \times \Delta^1$ has two non-degenerate 2-dimensional simplices.

A consequence of the above isomorphism is that the geometric realization of $\Delta^1 \times \Delta^1$ is exactly I^2, the unit square, which is the product of the geometric realization of Δ^1 with itself. Indeed, this is an instantiation of the general result in Theorem 19.1.4.9.

Remark 19.1.4.8. It is immediate to observe from Theorem 19.1.4.6 that geometric realization of Δ^n is exactly the standard topological n-simplex. Similarly, $\partial\Delta^n$ and $|\Lambda^n_i|$ are homeomorphic to exactly the pictures that we used in our mind to understand them.

Finally, the main result is as follows.

Theorem 19.1.4.9. *Let X, Y be simplicial sets. Then the natural map*

$$|X \times Y| \to |X| \times |Y|$$

is a homeomorphism.

Proof. **TODO.** $\square$

We show that nerve of a category satisfies lifting property which would prove to be useful later on while discussing ∞-categories.

Proposition 19.1.4.10. *Let $X = N\mathbf{C}$ be thr nerve of a small category $\mathbf{C}$. Then for any $0 < i < n$, the following lifting problem is uniquely filled:*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "!"'] & \end{tikzcd} .$$

Proof. **TODO.** $\square$

The following is an important result which will find its use later on.

Proposition 19.1.4.11. *Let S be a simplicial set and X a Kan complex. Then $\underline{\mathrm{Hom}}\,(S, X)$ is a Kan complex, denoted $\mathrm{Map}(S, X)$.*

Proof. **TODO.** $\square$

19.1.5 Eilenberg-Zilber categories

This is an abstraction of the type of combinatorial proofs that we would like to make in the simplex category $\mathbf{\Delta}$. Indeed, as we would have to work with surjections and injections in $\mathbf{\Delta}$ primarily, which interacts with the the size of $[n]$ (which we would define to be n in a minute) as an injection would only increase the size and a surjection would only decrease it, therefore we need a systematic toolset to work with these things. In particular, if we denote $\mathbf{\Delta}_+$ to be the subcategory of all injective maps, $\mathbf{\Delta}_-$ to be the subcategory of all surjective maps in $\mathbf{\Delta}$ and $d : \mathrm{Ob}(\mathbf{\Delta}) \to \mathbb{N}$ the size map, then we have the following properties about the tuple $(\mathbf{\Delta}, \mathbf{\Delta}_+, \mathbf{\Delta}_-, d)$:

1. All bijections are in both $\mathbf{\Delta}_+$ and $\mathbf{\Delta}_-$.
2. The dimension map d takes bijective cardinals to the same natural.
3. Let $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$ which is not bijective. If σ is injective (i.e. in $\mathbf{\Delta}_+$), then $d(a) < d(b)$ and if σ is surjective (i.e. in $\mathbf{\Delta}_-$), then $d(a) > d(b)$.
4. Any map $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$ factors as a surjection followed by an injection.
5. For any surjective map $\sigma : [n] \to [m]$ in $\mathbf{\Delta}$, there exists a section $\pi : [m] \to [n]$, i.e. such that $\pi\sigma = \mathrm{id}_{[n]}$.
6. If $\sigma : [n] \to [m]$ is surjective, then the set of sections of σ uniquely determines the map σ.

Remark 19.1.5.1. We need this abstraction of properties of $\mathbf{\Delta}$ so that with the same techniques, we can work with bisimplicial sets, which is important if one wishes to consider simplicial objects in **sSet**.

These considerations about $\mathbf{\Delta}$ motivates the following definition.

Definition 19.1.5.2. (**Eilenberg-Zilber categories**) A category $\mathbf{A}$ is said to be an Eilenberg-Zilber category (or much simply, *EZ cateogry*), if there exists subcategories $\mathbf{A}_+, \mathbf{A}_-$ and a function $d : \mathrm{Ob}(\mathbf{A}) \to \mathbb{N}$ which satisfies the following axioms:

1. All isomorphisms of $\mathbf{A}$ are in both $\mathbf{A}_+$ and $\mathbf{A}_-$.
2. If a, a' are isomorphic objects in $\mathbf{A}$, then $d(a) = d(a')$.
3. Let $\sigma : a \to a'$ not be an isomorphism. If σ is in $\mathbf{A}_+$, then $d(a) < d(a')$. If σ is in $\mathbf{A}_-$, then $d(a) > d(a')$.
4. For any map $\sigma : a \to a'$ in $\mathbf{A}$, there exists unique factorization of σ into maps $p : a \to c$ in $\mathbf{A}_-$ and $i : c \to a'$ in $\mathbf{A}_+$

$$\begin{tikzcd} & c \arrow[dr, "i"] & \\ a \arrow[ur, "p"] \arrow[rr, "\sigma"'] & & a' \end{tikzcd} .$$

5. If $\sigma : a \to a'$ is a map in $\mathbf{A}_-$, then there exists a section $\pi : a' \to a$, i.e. $\pi\sigma = \mathrm{id}_a$.
6. If σ, σ' are two maps in $\mathbf{A}_-$ such both of them has the same collection of sections, then $\sigma = \sigma'$.

The main thrust of this section is to discuss presheaves over an EZ category $\mathbf{A}$, keeping in mind the prototypical case of $\mathbf{A} = \mathbf{\Delta}$. Indeed, the main result and its corollaries will serve first as a practice for the type of arguments we shall need later and also as a tool to be consistently used later in constructions with simplicial sets (which are, presheaves over $\mathbf{\Delta}$).

Example 19.1.5.3. By previous discussion, it is clear that the simplex category $\mathbf{\Delta}$ is an EZ category.

Let $\mathbf{A}$ be an EZ category and X a presheaf over $\mathbf{A}$. Then the category of elements of X, $\mathbf{A}/X$, is an EZ category again. Indeed, define $(\mathbf{A}/X)_+$ exactly as those pairs (a, s) where $a \in \mathbf{A}_+$ and $(\mathbf{A}/X)_-$ exactly as those pairs (a, s) where $a \in \mathbf{A}_-$. Further for an object (a, s), define $d(a, s) = d(a)$, where the latter d is coming from the EZ structure on $\mathbf{A}$.

19.1.6 Kan complexes and homotopy groups

In the beginning section, we showed how simplicial sets can be viewed as combinatorial version of usual spaces. However, in order to do homotopy theory in this combinatorial setting, we need to isolate a class of simplicial sets which are right for this kind. Indeed, these will be Kan complexes.

We wish to define $\pi_n(X, x)$ where $x \in X_0$ is a 0-simplex of a simplicial set. To this end, we immediately run into problems as $\pi_0(X, x)$ should be the equivalence class of all those 0-simplices which are boundaries of a 1-simplex. But it is immediately clear that this is not an equivalence relation! Indeed, consider Δ^1. Then the above relation is not symmetric as we have $0 \to 1$ as a 1-simplex but there is no $1 \to 0$ in Δ^1_1. Similarly, if we try to prove transitivity of the above relation, we land up in the following situation. Let $x \to y, y \to z$ be two 1-simplices in X. Then, we wish to find a 1-simplex $x \to z$ in X. Note that given $x \to y$ and $y \to z$, we have a map $\Lambda^2_1 \to X$ and we wish to wish to fill the following diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd} .$$

So we need those simplicial sets, where the above dotted arrow always exists.

Definition 19.1.6.1 (**Kan complex**). A simplicial set X is a Kan complex if for any $n \geq 0$ and any $0 \leq i \leq n$, the following lifting diagram is filled

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} .$$

Example 19.1.6.2. Clearly, Δ^n are not Kan complexes for $n \geq 0$. Moreover, $N\mathbf{C}$ is a Kan complex if and only if $\mathbf{C}$ is a groupoid. This is immediate.

The prototypical (and in some sense, the only example) of Kan complexes are those obtained by Theorem 19.1.4.6.

Proposition 19.1.6.3. *Let X be a topological space. Then* $\mathrm{Sing}(X)$ *is a Kan complex.*

Proof. Let $n \geq 0$, $0 \leq i \leq n$. Then a map $\Lambda_i^n \to \mathrm{Sing}(X)$ is equivalent to a map $|\Lambda_i^n| \to X$ by adjunction of Theorem 19.1.4.6. Consequently, we wish to fill the following diagram

$$\begin{tikzcd} {|\Lambda_i^n|} \arrow[r] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}.$$

But this is immediate as $|\Lambda_i^n| \hookrightarrow |\Delta^n|$ is a retraction. $\square$

Another example of a Kan complex is the mapping complex.

Definition 19.1.6.4 (Mapping complex). Let X, Y be two spaces. The mapping complex, denoted $\mathrm{Map}(X, Y)$, is the one whose n-simplices are

$$[n] \mapsto \mathrm{Hom}_{\mathbf{Top}}\left(X \times |\Delta^n|, Y\right).$$

Note that $\mathrm{Map}(X, Y)$ has 0-simplices as continuous maps, 1-simplices as homotopies and so on. The name is justified by the following result.

Corollary 19.1.6.5. *Let X, Y be spaces. Then the mapping complex* $\mathrm{Map}(X, Y)$ *is a Kan complex.*

Proof. As all spaces are compactly generated, therefore we have $\mathrm{Hom}_{\mathbf{Top}}\left(X \times |\Delta^n|, Y\right) \cong \mathrm{Hom}_{\mathbf{Top}}\left(|\Delta^n|, Y^X\right)$. Consequently, $\mathrm{Map}(X, Y) \cong \mathrm{Sing}(Y^X)$, which is a Kan complex by Proposition 19.1.6.3. $\square$

One can now check that the relation mentioned in the beginning on X_0 is indeed an equivalence relation and would thus yield the definition of $\pi_0(X, x)$, however, we would like to define all homotopy groups in one go. We first define the notion of two simplicies being homotopic.

Definition 19.1.6.6 (Homotopic rel ∂). Let X be a Kan complex and $\sigma, \tau \in X_n$ be two n-simplicies. Then σ and τ are homotopic rel ∂ if there exists an n+1-simplex $\psi \in X_{n+1}$ such that $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$ and $\partial_n \psi = \sigma$, $\partial_{n+1}\psi = \tau$ and for all $0 \leq i \leq n-1$, $\partial_i \psi = \sigma d^i s^{n-1} = \tau d^i s^{n-1}$. We can diagramatically represent these conditions as the commutativity of the the diagrams

$$\begin{tikzcd} X & \Delta^n \arrow[l, "\tau"'] \\ \Delta^n \arrow[u, "\sigma"] & \partial\Delta^n \arrow[l, hook'] \arrow[u, hook] \end{tikzcd},$$

$$\begin{tikzcd} \Delta^{n+1} \arrow[r, "\psi"] & X & \Delta^{n+1} \arrow[l, "\psi"'] \\ \Delta^n \arrow[u, "d^n"] \arrow[ur, "\sigma"'] & & \Delta^n \arrow[u, "d^{n+1}"'] \arrow[ul, "\tau"] \end{tikzcd}$$

and for $0 \leq i \leq n-1$

$$\begin{tikzcd}
\Delta^{n+1} \arrow[r, "\psi"] & X & \Delta^n \arrow[l, "\tau"'] \\
\Delta^n \arrow[u, "d^i"] \arrow[rr, "s^{n-1}"'] & & \Delta^{n-1} \arrow[u, "d^i"']
\end{tikzcd}$$

in **sSet**.

Indeed, we see that this generalizes our previous notion of homotopy relative to boundary as follows.

Lemma 19.1.6.7. *Let $X = \mathrm{Sing}(Y)$ be the Kan complex associated to a space Y. Then two n-simplices $\sigma, \tau \in X_n$ are homotopic rel ∂ if and only if $\sigma, \tau : |\Delta^n| \to Y$ are homotopic relative to $|\partial\Delta^n| \hookrightarrow |\Delta^n|$ in the classical sense.*

Proof. Let $\sigma, \tau : \Delta^n \to X$ be two n-simplices. These are homotopic rel ∂ if the diagrams in Definition 19.1.6.6 commutes in **sSet**. By the adjunction of Theorem 19.1.4.6, this is equivalent to commutativity of the following diagrams in **Top**

$$\begin{tikzcd}
Y & {|\Delta^n|} \arrow[l, "\tau"'] \\
{|\Delta^n|} \arrow[u, "\sigma"] & {|\partial\Delta^n|} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd} ,$$

$$\begin{tikzcd}
{\left|\Delta^{n+1}\right|} \arrow[r, "\psi"] & X & {\left|\Delta^{n+1}\right|} \arrow[l, "\psi"'] \\
{|\Delta^n|} \arrow[u, "d^n"] \arrow[ur, "\sigma"'] & & {|\Delta^n|} \arrow[u, "d^{n+1}"'] \arrow[ul, "\tau"]
\end{tikzcd}$$

and for $0 \leq i \leq n-1$

$$\begin{tikzcd}
{\left|\Delta^{n+1}\right|} \arrow[r, "\psi"] & Y & {|\Delta^n|} \arrow[l, "\tau"'] \\
{|\Delta^n|} \arrow[u, "d^i"] \arrow[rr, "s^{n-1}"'] & & {\left|\Delta^{n-1}\right|} \arrow[u, "d^i"']
\end{tikzcd} .$$

But this is equivalent to the data of a homotopy $H : |\Delta^n| \times I \to Y$ rel boundary. Indeed, since we want $H_0 = \sigma$, $H_1 = \tau$ and $H_t|_{|\partial\Delta^n|} = \sigma|_{|\partial\Delta^n|} = \tau|_{\partial\Delta^n}$ for all $t \in I$, therefore we naturally get that H factors through $\left|\Delta^{n+1}\right|$. This can be visualized for a 3-simplex immediately. $\square$

Finally, when X is a Kan complex, then this is an equivalence relation. This is where combinatorial relations between face and degeneracy maps of a simplicial set as provided in Proposition 19.1.2.3 becomes handy.

Proposition 19.1.6.8. *Let X be a Kan complex. The homotopy rel ∂ is an equivalence relation on each X_n for $n \geq 1$.*

Proof. We first show reflexivity. Consider $\sigma : \Delta^n \to X$ an n-simplex. We claim that $\psi = \sigma s^n : \Delta^{n+1} \to X$ works as the homotopy from σ to σ. Indeed, we see that $\psi d^n = \sigma s^n d^n = \sigma$ and $\psi d^{n+1} = \sigma s^n d^{n+1} = \sigma$ by Proposition 19.1.2.3. Similarly, for $0 \leq i \leq n-1$, we have $\psi d^i = \sigma s^n d^i = \sigma d^i s^{n-1}$ by the same result. This establishes reflexivity.

We now show symmetry. Let $\sigma \sim \tau$ for some $\sigma, \tau : \Delta^n \to X$ with $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$. Then, there exists $\psi : \Delta^{n+1} \to X$ with $\psi d^n = \sigma$, $\psi d^{n+1} = \tau$ and $\psi d^i = \sigma d^i s^{n-1} = \tau d^i s^{n-1}$ for $0 \leq i \leq n-1$. We wish to show that $\tau \sim \sigma$. We will use the fact that X is a Kan complex (so that all horns can be filled). In particular, we will construct a horn $\kappa : \Lambda^{n+2}_n \to X$, filling which will give us the required homotopy from τ to σ. Indeed, let κ be obtained from ψ by adding degeneracies such that $\kappa d^{n+2} = \psi$ and $\kappa d^{n+1} = \sigma s^n$, that is, the degenerate $n+1$-simplex obtained by repeating the n^{th}-vertex of σ. For $0 \leq i \leq n-1$, we keep $\kappa d^i = \psi d^i s^{n-1}$. Filling the horn κ yields an $n+2$-simplex $\tilde{\kappa} : \Delta^{n+2} \to X$ whose n^{th}-face is exactly a homotopy from τ to σ. Indeed, denote $\phi = \tilde{\kappa} d^n$. Then, $\phi d^n = \tilde{\kappa} d^n d^n = \tilde{\kappa} d^{n+1} d^n = \sigma$. Similarly, $\phi d^{n+1} = \tau$. These follow from the observation that $d^n d^n$ is the unique n-simplex of $\tilde{\kappa}$ not containing the vertices n and $n+1$ in $\tilde{\kappa}$. For $0 \leq i \leq n-1$, by Proposition 19.1.2.3, we have $\phi d^i = \tilde{\kappa} d^n d^i = \tilde{\kappa} d^i d^{n-1}$, which in turn is $\kappa d^i d^{n-1} = \psi d^i s^{n-1} d^{n-1} = \psi d^i = \sigma d^i s^{n-1}$, as needed. This shows symmetry.

Finally, we wish to show transitivity. Let $\psi, \psi' \in X_{n+1}$ be homotopies $\sigma \sim \tau$ and $\tau \sim \eta$ respectively for some $\sigma, \tau, \eta \in X_n$. We wish to construct a homotopy $\psi'' \in X_{n+1}$ between σ and η. Indeed, we obtain a horn $\kappa : \Lambda^{n+2}_{n+2} \to X$ whose n^{th} and $n+1^{\text{th}}$ boundaries are ψ and ψ' respectively and the rest boundaries are required degeneracies. Filling this horn up by the Kan condition gives $\tilde{\kappa}$ and its $n+2^{\text{th}}$-boundary is the required homotopy. $\square$

Consequently, we define homotopy groups of a Kan complex as follows.

Definition 19.1.6.9 (**Homotopy groups of a Kan complex**)**.** Let X be a Kan complex and $x \in X_0$ be a base point. Then for $n \geq 1$ define

$$\pi_n(X, x_0) = \{\sigma \in X_n \mid \sigma|_{\partial\Delta^n} = c_{x_0}\} / \sim$$

where $\sim$ is the homotopy rel ∂. For $n = 0$, define

$$\pi_0(X) = X_0 / \sim$$

where

$$x \sim y \iff \exists \gamma \in X_1 \text{ s.t. } \gamma d^1 = x \;\&\; \gamma d^0 = y.$$

We now show that $\pi_n(X, x)$ is indeed a group.

Construction 19.1.6.10 (*Composition and group operation on* $\pi_n(X, x)$)**.** Let X be a Kan complex and $x \in X$ be a point in it (i.e a 0-simplex). Let $\sigma, \tau \in X_n$ be two n-simplices such that $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n} = c_x$. We construct the composition $\sigma \cdot \tau$ of σ and τ as the following n-simplex. Construct the following horn $\kappa : \Lambda^{n+1}_n \to X$ whose $n-1^{\text{th}}$-boundary is σ, $n+1^{\text{th}}$-boundary is τ[1] and for $0 \leq i \leq n-2$, $\kappa d^i = c_x$. It follows from horn-filling condition

[1]If one is thinking of paths, then it is important to note that $\sigma \cdot \tau$ is the one where τ is traversed first and then σ. One has to let go of the past notation because here simplices are not merely going to be paths, homotopies and so on, but rather more general objects like arrows of a category, 2-arrows and so on, so to be consistent with the notion of composition, it is best that we change the order in which we concatenate paths.

of X that we get an $n+1$-simplex $\psi : \Delta^{n+1} \to X$ extending the horn κ. Consequently, we define the *concatenation* $\sigma \cdot \tau$ as the n^{th}-boundary of ψ, that is,

$$\sigma \cdot \tau = \psi d^n.$$

We claim that the operation

$$\begin{aligned} \cdot : \pi_n(X,x) \times \pi_n(X,x) &\longrightarrow \pi_n(X,x) \\ ([\alpha],[\beta]) &\longmapsto [\alpha \cdot \beta] \end{aligned}$$

is a well-defined function, that is, $[\alpha \cdot \beta]$ only depends on $[\alpha]$ and $[\beta]$.

Indeed, suppose $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$ in $\pi_n(X,x)$. Then, we have a homotopy rel ∂ denoted $\psi \in X_{n+1}$ from α to α' and $\chi \in X_{n+1}$ from β to β'. We wish to construct $\phi \in X_{n+1}$ which is a homotopy rel ∂ from $\alpha \cdot \beta$ to $\alpha' \cdot \beta'$. As usual, we obtain this homotopy by constructing a higher horn and filling it by Kan condition.

To correctly denote the simplex to be constructed, we first observe that if $\delta \sim \epsilon$ is a homotopy rel ∂ of two n-simplices with $\delta|_{\partial\Delta^n} = \epsilon|_{\partial\Delta^n} = x$, then $\delta \cdot x \sim \epsilon$. Indeed, consider an $n+2$-horn whose $n+2$-boundary is the composition simplex $\delta \cdot x$, $n+1$-boundary is ϵs_0 and all i-boundaries for $0 \leq i \leq n-1$ are degeneracies of δ. Filling this yields the n-boundary as the required homotopy.

We now show that if $\beta \sim \beta'$, then $\alpha \cdot \beta \sim \alpha \cdot \beta'$. This would complete the proof. Indeed, this is immediate by considering an $n+2$-horn given by $\kappa : \Lambda^{n+2}_{n+1} \to X$ such that κd^{n+2} is the homotopy $\beta \sim \beta'$, κd^n is the composition $\alpha \cdot \beta$, κd^{n-1} is the composition $\alpha \cdot \beta'$ and the κd^i for $0 \leq i \leq n-2$ are all x.

Consequently, we get that $\cdot$ is a well defined operation on $\pi_n(X,x)$. We will later show that $\cdot$ makes $\pi_n(X,x)$ into a group. Moreover, we will show that $\pi_n(X,x) \cong \pi_n(|X|,x)$ and $\pi_n(\mathrm{Sing}(Y),y) \cong \pi_n(Y,y_0)$!

19.1.7 The fundamental group of a Kan complex

Let X be a Kan complex and $x_0 \in X$ be a 0-simplex. The fundamental group $\pi_1(X, x_0)$ is explicitly given by the following

$$\begin{aligned}\pi_1(X, x_0) &= \{\sigma : \Delta^1 \to X \mid \sigma|_{\partial\Delta^1} = x_0\}/\sim \\ &= \{\sigma \in X_1 \mid d_0(\sigma) = d_1(\sigma) = x_0\}/\sim\end{aligned}$$

where $\sigma \sim \tau$ if and only if there exists a homotopy rel ∂ denoted $H : \Delta^2 \to X$, from σ to τ. That is, $d_1(H) = \sigma$, $d_2(H) = \tau$ and $d_0(H) = s_0(d_0(\sigma)) = s_0(x_0) = \mathrm{id}_{x_0}$. This is a group where the operation is

$$\begin{aligned}\cdot : \pi_1(X, x_0) \times \pi_1(X, x_0) &\longrightarrow \pi_1(X, x_0) \\ ([\sigma], [\tau]) &\longmapsto [\sigma \cdot \tau]\end{aligned}$$

where $\sigma \cdot \tau$ is a 1-simplex which is their composition (Construction 19.1.6.10), obtained by the 1-boundary of the 2-simplex δ obtained by filling the horn

$$\kappa : \Lambda_1^2 \to X$$

whose $\partial_0 \Lambda_1^2 = \sigma$ and $\partial_1 \Lambda_1^2 = \tau$. More precisely, we define κ_1 as follows (this is sufficient by Example 19.1.4.2 and Lemma 19.1.4.3):

$$\begin{aligned}\kappa_1 : (\Lambda_1^2)_1 &\longrightarrow X_1 \\ \{0,1\} &\longmapsto \sigma \\ \{1,2\} &\longmapsto \tau \\ \{1,1\} &\longmapsto s_0(d_0(\sigma)).\end{aligned}$$

In this section, following the usual terminology, we will write $\sigma \cdot \tau$ as $\tau * \sigma$ for $\sigma, \tau \in X_1$ with $d_0(\sigma) = d_1(\sigma) = x_0$.

We now prove some basic results about $\pi_1(X, x_0)$. It is a good exercise to show that $\pi_1(X, x_0)$ is a group.

Theorem 19.1.7.1. *Let X be a Kan complex and $x_0 \in X_0$. Then $\pi_1(X, x_0)$ is a group.*

Proof. We first show that for any three $\alpha, \beta, \gamma \in X_1$ with their boundaries being x_0, we have $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$. Indeed, let H be the witness of composition $\alpha * \beta$, K that of $\beta * \gamma$, L that of $(\alpha * \beta) * \gamma$ and P that of $\alpha * (\beta * \gamma)$. Now consider the following horn $\kappa : \Lambda_2^3 \to X$ given by following on non-degenerate 2-simplices:

$$\begin{aligned}\kappa_2 : (\Lambda_2^3)_2 &\longrightarrow X_2 \\ \{0,1,2\} &\longmapsto H \\ \{1,2,3\} &\longmapsto K \\ \{0,2,3\} &\longmapsto L.\end{aligned}$$

By Kan condition, the above horn is filled and its 2-boundary yields a 2-simplex $\chi \in X_2$ such that $d_0(\chi) = \beta * \gamma$, $d_1(\chi) = (\alpha * \beta) * \gamma$ and $d_2(\chi) = \alpha$. We now construct the required homotopy by filling another 3-horn. Indeed, consider a horn $\lambda : \Lambda^3_1 \to X$ such that $d_0(\lambda) = s_1(\beta * \gamma)$, $d_2(\lambda) = P$ and $d_3(\lambda) = \chi$. This fills to give its 1-boundary as the required homotopy.

The identity element being c_{x_0} and the existence of inverses are also immediate results of horn filling and is thus omitted. $\square$

A Kan complex is path-connected if $\pi_0(X) = 0$.

19.1.8 ∞-categories

19.1.9 Theorem of Boardman-Vogt

19.2 Classical homotopical algebra

19.2.1 Model categories

We discuss now a general setup in which one can "do" homotopy theory.

Definition 19.2.1.1 (**Model categories**)**.** Let $\mathbf{C}$ be a category and $W, C, F \subseteq \mathbf{C}$ be subcategories of $\mathbf{C}$ which are called weak equivalences ($\approx$), cofibrations ($\rightarrowtail$) and fibrations ($\twoheadrightarrow$) respectively. We call $W \cap C$ weak/acyclic cofibrations and $W \cap F$ weak/acyclic fibrations. Then, the tuple $(\mathbf{C}, W, C, F)$ is a model category if it satisfies the following axioms:

1. The category $\mathbf{C}$ has all finite limits and colimits.
2. Weak equivalences satisfies 2 out-of 3 property.
3. For any $f : X \to Y$ in $\mathbf{C}$, we have two factorizations

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "\approx"'] & & Y \\ & Z \arrow[ur, two heads] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail] & & Y \\ & Z \arrow[ur, two heads, "\approx"'] & \end{tikzcd}$$

   one a weak cofibration followed by fibration and another one a cofibration followed by a weak fibration.
4. We have

$$rlp(W \cap C) = F$$
$$C = llp(W \cap F)$$

   where for a subcategory $S \subseteq \mathbf{C}$, the collection $rlp(S)$ denotes the collection of all maps $X \to Y$ in $\mathbf{C}$ satisfying right lifting property wrt S, i.e., such that for *any* commutative square

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\in S"'] & X \arrow[d, "f"] \\ W \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

   where left vertical arrow is in S, there exists a lift $h : W \to X$ as shown which makes all diagrams commute. Similarly, one defines $llp(S)$.

Definition 19.2.1.2 (**Cofibrant/fibrant objects**)**.** An object X in a model category $\mathbf{C}$ is cofibrant (fibrant) if the unique map from initial object $\emptyset \to X$ (to terminal object $X \to \text{pt.}$) is a cofibration (fibration).

19.2.2 Basic constructions

19.2.3 Construction of model structures on $\mathbf{PSh}(\mathbf{C})$

19.2.4 From one model structure to many

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Special Topics

Chapter 23

Commutative Algebra

Contents

In this chapter, we collect topics from contemporary commutative algebra. The most need of all this material comes from algebraic goemetry. In particular, in the following, we list out the topics that we would need for our treatment of basic algebraic geometry.

1. Dimension theory : For dimension of schemes, Hauptidealsatz, local complete intersection, etc.
2. Integral dependence : For proper maps between affine varieties, normalization, finiteness of integral closure, certain DVRs of dimension 1, etc.
3. Field theory : For birational classification of varieties, primitive element theorem, basic algebra in general, etc.
4. Completions : Local analysis of singularities, formal schemes, complete local rings, Cohen structure theorem, Krull's theorem, etc.
5. Valuation rings : For curves and their non-singular points (DVRs) and various equivalences, Dedekind domains, etc.

6. Multiplicities : For intersections in projective spaces, intersection multiplicity, Hilbert polynomials, flat families, etc.
7. Kähler differentials : For differential forms on schemes, this will be used consistently in further topics.
8. Depth and Cohen-Macaulay : For local complete intersections, blowing up, etc.
9. Tor and Ext functors : They are tools for other algebraic notions, generizable to global algebra, tor dimension, etc.
10. Projective modules : For vector bundles, projective dimension and Ext, pd + depth = dim for regular local rings, etc.
11. Flatness : Family of schemes varying continuously, smooth and étalé maps, etc.
12. Lifting properties - Étale, unramified and smooth morphisms : These are used heavily for the corresponding scheme maps, and beyond.

Notation 23.0.0.1. Let R be a ring and $f(x) \in R[x]$ be a polynomial. We will denote $c_n(f) \in R$ to be the coefficient of x^n in $f(x)$. If $f(x, y) \in R[x, y]$, then we will denote $c_{n,m}(f) \in R$ to be the coefficient of $x^n y^m$ in $f(x, y)$. We may also write $c_{x^n}(f)$ for $c_n(f)$ and $c_{x^n y^m}(f)$ for $c_{n,m}(f)$ if it makes statements more clear.

Remark 23.0.0.2. We will consistently keep using the geometric viewpoint given by the theory of schemes (see Chapter 1) in discussing the topics below, as a viewpoint to complement the algebraic viewpoint. This will also showcase the usefulness of scheme language.

23.1 General algebra

We discus here general results about prime ideals, modules and algebras.

23.1.1 Jacobson radical and Nakayama lemma

Let R be a ring. Denote the set of all units of R as $R^\times$. The *Jacobson radical* is an ideal $\mathfrak{r}$ of R formed by the intersection of all maximal ideals of R. A *finitely generated R-module* M is a module which has a finite collection of elements $\{x_1, \ldots, x_n\} \subset M$ such that for any $z \in M$, there are $r_1, \ldots, r_n \in R$ so that $z = r_1x_1 + \cdots + r_nx_n$. More concisely, if there is a surjection R-module homomorphism $R^n \twoheadrightarrow M$. We then have the following results about $\mathfrak{r}$.

Proposition 23.1.1.1. *Let R be a ring and let $\mathfrak{r}$ denotes it Jacobson radical. Then,*

1. *$x \in \mathfrak{r}$ if and only if $1 - xy \in R^\times$ for any $y \in R$.*
2. *(Nakayama lemma) Let M be a finitely generated R-module. If $\mathfrak{q} \subseteq \mathfrak{r}$ is an ideal of R such that $\mathfrak{q}M = M$, then $M = 0$.*
3. *Let M be a finitely generated module and $\mathfrak{q} \subseteq \mathfrak{r}$. Let $N \leq M$ be a submodule of M such that $M = N + \mathfrak{q}M$, then $M = N$.*
4. *If R is a local ring and M, N are two finitely generated modules, then*

$$M \otimes_R N = 0 \iff M = 0 \text{ or } N = 0.$$

Proof. 1. (L ⇒ R) Suppose there is $y \in R$ such that $1 - xy \notin R^\times$. Since each non-unit element is contained in a maximal ideal by Zorn's lemma, therefore $1 - xy \in \mathfrak{m}$ for some maximal ideal. Since $x \in \mathfrak{r}$, therefore $x \in \mathfrak{m}$. Hence $xy, 1 - xy \in \mathfrak{m}$, which means that

$1 \in \mathfrak{m}$, a contradiction.
(R $\Rightarrow$ L) Suppose $1 - xy \in R^\times$ for all $y \in R$ and $x \notin \mathfrak{r}$. Then, again by Zorn's lemma we have $x \in R^\times$. Hence let $y = x^{-1}$ to get that $1 - xy = 1 - 1 = 0 \in R^\times$, a contradiction.

2. Suppose $M \neq 0$. Since M is finitely generated, therefore there is a submodule $N \subset M$ such that M/N is simple (has no proper non-trivial submodule). Simple R-modules are isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R via the map $M' \mapsto R/\mathrm{Ann}(x)$ where $x \neq 0$ in M. Therefore $M/N \cong R/\mathfrak{m}$. Then, $\mathfrak{m}R \neq R$ which is same as $\mathfrak{m}M \neq M$. Since $\mathfrak{q} \subseteq \mathfrak{r} \subseteq \mathfrak{m}$, hence $\mathfrak{q}M \neq M$, a contradiction.

3. Apply 2. on M/N.

4. The only non-trivial part is L $\Rightarrow$ R. Since $(M \otimes_R N)/\mathfrak{m}(M \otimes_R N) = M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N$, therefore we have $M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N = 0$. Since $R/\mathfrak{m}$ is a field therefore $M/\mathfrak{m}M = 0$ WLOG. Hence, $M = \mathfrak{m}M$ and since R is local, therefore $\mathfrak{r} = \mathfrak{m}$. We conclude by Nakayama. $\square$

23.1.2 Localization

We next consider localization of rings and R-modules. Take any multiplicative set $S \subset R$ which contains 1. Then, localizing an R-module M on S is defined as

$$S^{-1}M := \{m/s \mid m \in M, s \in S\}.$$

where $m/s = n/t$ if and only if $\exists u \in S$ such that $u(mt - ns) = 0$. We have that $S^{-1}M$ is an R-module where addition $m/s + n/t = (mt + ns)/st$. In the case when $M = R$, we get a ring structure on $S^{-1}R$ as well where multiplication is given by $m/s \cdot n/t := mn/st$. There is a natural map $M \to S^{-1}M$ which maps $m \mapsto m/1$ and it may not be an injection if $\exists m \in M$ and $s \in S$ such that $s \cdot M = 0$.

Lemma 23.1.2.1. *Let $S \subset R$ be a multiplicative set in a ring R and M be an R-module. Then,*

$$S^{-1}M \cong S^{-1}R \otimes_R M.$$

Proof. One can do this by directly checking the universal property of tensor product of $S^{-1}R$ and M over R for $S^{-1}M$. We have the map $\varphi : S^{-1}R \times M \to S^{-1}M$ given by $(r/s, m) \mapsto rm/s$. Now for any bilinear map $f : S^{-1}R \times M \to N$, we can define the map $\tilde{f} : S^{-1}M \to N$ given by $\tilde{f}(m/s) := f(1/s, m)$. Clearly, $\tilde{f}$ is well-defined and $\tilde{f}\varphi = f$. Moreover, if $g : S^{-1}M \to N$ is such that $g\varphi = f$, then $g(m/s) = f(1/s, m) = \tilde{f}(m/s)$. Hence $\tilde{f}$ is unique with this property. $\square$

Lemma 23.1.2.2. *Localization w.r.t a multiplicative set $S \subset R$ is an exact functor on* $\mathbf{Mod}(R)$.

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Then we have the localized sequence $S^{-1}M' \to S^{-1}M \to S^{-1}M''$. Since $S^{-1}0 = 0$, therefore this is left exact. Exactness at middle follows from exactness at middle of the first sequence. The right exactness can be seen by right exactness of tensor product functor $S^{-1}R \otimes_R -$ and by Lemma 23.1.2.1. □

Lemma 23.1.2.3. *Let R be a ring and $S \subset R$ be a multiplicative set. Then*

$$\{\textit{prime ideals of } R \textit{ not intersecting } S\} \xrightarrow{\cong} \{\textit{prime ideals of } S^{-1}R\}$$
$$\mathfrak{p} \longmapsto S^{-1}\mathfrak{p}$$

Proof. Trivial. □

Next we see an important property of modules, that is their "local characteristic". This means that one can check whether an element of a module is in a submodule by checking it locally at each prime, as the following lemma suggests. This has geometric significance in algebraic geometry (M induces and is induced by a quasi-coherent sheaf over $\mathrm{Spec}\,(R)$, see **??**).

Lemma 23.1.2.4. *Let M be an R-module. Then,*

1. *$M \neq 0$ if and only if there exists a point $\mathfrak{p} \in \mathrm{Spec}\,(R)$ such that $M_\mathfrak{p} \neq 0$.*
2. *If $N \subset M$ is a submodule and $0 \neq x \in M$, then $x \in N$ if and only if $x \in N_\mathfrak{p} \subseteq M_\mathfrak{p}$ for each point $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*

Proof. 1. (L $\Rightarrow$ R) Since $\exists x \in M$ which is non-zero, therefore consider the annihilator ideal $\mathrm{Ann}(x) = \{r \in R \mid rx = 0\}$ of R. Then, this ideal is contained in a maximal ideal $\mathfrak{m}$ of R by Zorn's lemma. Hence consider $M_\mathfrak{m}$, which contains $x/1$. Now if there exists $r \in R \setminus \mathfrak{m}$ such that $rx = 0$, then $r \in \mathrm{Ann}(x)$, but since $\mathfrak{m} \supseteq \mathrm{Ann}(x)$, hence we have a contradiction. (R $\Rightarrow$ L) Let $\mathfrak{p} \in \mathrm{Spec}\,(R)$ be such that $x/r \in M_\mathfrak{p}$ and $x/r \neq 0$. Since $M_\mathfrak{p}$ is an R-module, therefore $r \cdot (x/r)$ is well-defined in $M_\mathfrak{p}$. Hence $(rx)/r = x/1 \in M_\mathfrak{p}$. If $x/1 = 0$ in $M_\mathfrak{p}$, therefore $\varphi_\mathfrak{p}(x) = 0$ and hence $x = 0$ as $\varphi_\mathfrak{p}$ is injective. Thus, $x/r = 0$ in $M_\mathfrak{p}$, a contradiction. Therefore $x/1 \neq 0$ and hence $x \neq 0$ in M.

2. This follows from using 1. on the module $(N + Rx)/N$. We do this by observing the following chain of equivalences, whose key steps are explained below:

$$x \in N \iff N + Rx = N \iff (N + Rx)/N = 0 \iff ((N + Rx)/N)_\mathfrak{p}\, \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff$$
$$(N + Rx)_\mathfrak{p}/N_\mathfrak{p} = 0 \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff (N + Rx)_\mathfrak{p} = N_\mathfrak{p} \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff$$
$$N_\mathfrak{p} + (Rx)_\mathfrak{p} = N_\mathfrak{p} \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff (Rx)_\mathfrak{p} \subseteq N_\mathfrak{p} \forall \mathfrak{p} \in \mathrm{Spec}\,(R) \iff \varphi_\mathfrak{p}(x) = x/1 \in N_\mathfrak{p} \forall \mathfrak{p} \in \mathrm{Spec}\,(R).$$

For two submodules $N, K, L \subset M$ where $L \subseteq N$ and $\mathfrak{p} \in \mathrm{Spec}\,(R)$, we get $(N/L)_\mathfrak{p} = N_\mathfrak{p}/L_\mathfrak{p}$ by exactness of localization (Lemma 23.1.2.2) on the exact sequence

$$0 \to L \to N \to N/L \to 0.$$

Finally $(N+K)_\mathfrak{p} = N_\mathfrak{p} + K_\mathfrak{p}$ in $M_\mathfrak{p}$ is true by direct checking and where we use the primality of $\mathfrak{p}$. □

Remark 23.1.2.5. (*Few life hacks*) The above proof tells us few ways how one can approach the problems in ring theory. Note especially that $x \in N$ if and only if $N + Rx = N$, which quickly turns a set-theoretic relation into an algebraic one, where we can now use various constructions as we did, like localization.

The following is the universal property for localization.

Proposition 23.1.2.6. *Let R be a ring and S be a multiplicative set. If $\varphi : R \to T$ is a ring homomorphism such that $\varphi(S) \subseteq T^\times$ where $T^\times$ is the unit group of T, then there exists a unique map $\tilde{\varphi} : S^{-1}R \to T$ such that the following commutes*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "i"'] & T \\ S^{-1}R \arrow[ur, dashed, "\tilde{\varphi}"'] & \end{tikzcd}.$$

Proof. Pick any ring map $\varphi : R \to T$. Take any map $f : S^{-1}R \to T$ which makes the above commute. We claim that $f(r/s) = \varphi(r)\varphi(s)^{-1}$. Indeed, we have that $f(r/1) = \varphi(r)$ for all $r \in R$. Further, for any $s \in S$, we have $f(1/s) = 1/f(s/1) = 1/\varphi(s) = \varphi(s)^{-1}$. Consequently, we get for any $r/s \in S^{-1}R$ the following

$$f\left(\frac{r}{s}\right) = f\left(\frac{r}{1} \cdot \frac{1}{s}\right) = f\left(\frac{r}{1}\right) \cdot f\left(\frac{1}{s}\right) = \varphi(r)\varphi(s)^{-1}.$$

This proves uniqueness. Clearly, this is a ring homomorphism. This completes the proof. $\square$

Remark 23.1.2.7. As Proposition 23.1.2.6 is the universal property of localization, therefore the construction $S^{-1}R$ is irrelevant; the property above completely characterizes localization upto a unique isomorphism.

Lemma 23.1.2.8. *Let R be a ring and $f \in R \setminus \{0\}$. Then,*

$$R_f \cong \frac{R[x]}{\langle fx - 1\rangle}.$$

In particular, R_f is a finite type R-algebra.

Proof. We shall use Proposition 23.1.2.6. We need only show that $R[x]/\langle fx - 1\rangle$ satisfies the same universal property as stated in Proposition 23.1.2.6. Indeed, we first have the map $i : R \to R[x]/\langle fx - 1\rangle$ given by $r \mapsto r + \langle fx - 1\rangle$. Let $\varphi : R \to T$ be any map such that $\varphi(f) \in T^\times$. We claim that there exists a unique map $\tilde{\varphi} : R[x]/\langle fx - 1\rangle \to T$ such that $\tilde{\varphi} \circ i = \varphi$. Indeed, take any map $g : R[x]/\langle fx - 1\rangle \to T$ such that $g \circ i = \varphi$. Thus, for all $r \in R$, we have $g(r + \langle fx - 1\rangle) = \varphi(r)$. As $fx + \langle fx - 1\rangle = 1 + \langle fx - 1\rangle$, therefore we obtain that $g(f + \langle fx - 1\rangle) \cdot g(x + \langle fx - 1\rangle) = \varphi(f) \cdot g(x + \langle fx - 1\rangle) = 1$. Hence, we see that $g(x + \langle fx - 1\rangle) = \varphi(f)^{-1}$. Hence for any element $p(x) + \langle fx - 1\rangle$, we see that $f(p(x) + \langle fx - 1\rangle) = p(\varphi(f)^{-1})$. This makes g unique well-defined ring homomiorphism. This completes the proof. $\square$

The following is a simple but important application of technique of localization.

Lemma 23.1.2.9. *Let R be a ring. Then the nilradical of R, $\mathfrak{n}$, the ideal consisting of nilpotent elements is equal to the intersection of all prime ideals of R:*

$$\mathfrak{n} = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}.$$

Proof. Take $x \in \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$. We then have $x \in \mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$. Hence if for each $n \in \mathbb{N}$ we have that $x^n \neq 0$, then we get that $S = \{1, x, x^2, \dots\}$ forms a multiplicative system. Considering the localization $S^{-1}R$, we see that it is non-zero. Therefore $S^{-1}R$ has a prime ideal, which corresponds to a prime ideal $\mathfrak{p}$ of R which does not intersects S, by Lemma 23.1.2.3. But this is a contradiction as x is in every prime ideal.
Conversely, take any $x \in \mathfrak{n}$ and any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$. Since $x^n = 0$ for some $n \in \mathbb{N}$, therefore $x^n \in \mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$. Hence it follows from primality of each $\mathfrak{p}$ that $x \in \mathfrak{p}$. □

We next give two results which are of prominent use in algebraic geometry. The first result says that finite generation of a module can be checked locally.

Lemma 23.1.2.10. *Let M be an R-module and suppose $f_i \in R$ are elements such that $\sum_{i=1}^n Rf_i = R$. Then, the following are equivalent:*

1. *M is a finitely generated R-module.*
2. *M_{f_i} is a finitely generated R_{f_i}-module for all $i = 1, \dots, n$.*

Proof. (1. $\Rightarrow$ 2.) This is simple, as finite generation is preserved under localization.
(2. $\Rightarrow$ 1.) Let M_{f_i} be generated by $m_{ij}/(f_i)^{n_{ij}}$ for $j = 1, \dots, n_i$. Let $N \leq M$ be a submodule generated by m_{ij} for each $j = 1, \dots, n_i$ and for each $i = 1, \dots, n$. Clearly, N is a finitely generated R-module. Moreover, N_{f_i} for each $i = 1, \dots, n$ is equal to M_{f_i}. Since localization at a prime ideal $\mathfrak{p} \lneq R$ is given by direct limit of all localization of elements not in $\mathfrak{p}$, therefore $(M/N)_{\mathfrak{p}} \cong \varinjlim_{i=1,\dots,n} (M/N)_{f_i}$ and since $M_{f_i} = N_{f_i}$, therefore $(M/N)_{f_i} = 0$. It follows that $(M/N)_{\mathfrak{p}} = 0$ for all primes $\mathfrak{p}$ and hence $M/N = 0$ by Lemma 23.1.2.4, 1, hence $M = N$ and M is finitely generated. □

The second result gives a partial analogous result as to Lemma 23.1.2.10 did, but for algebras. This is again an important technical tool used often in algebraic geometry.

Lemma 23.1.2.11. *Let A be a ring and B be an A-algebra. Suppose $f_1, \dots, f_n \in B$ are such that $\sum_{i=1}^n Bf_i = B$. If for all $i = 1, \dots, n$, B_{f_i} is a finitely generated A-algebra, then B is a finitely generated A-algebra.*

Proof. Let B_{f_i} be generated by

$$\left\{ \frac{b_{ij}}{f_i^{n_j}} \right\}_{j=1,\dots,M_i}$$

as an A-algebra, for each $i = 1, \dots, n$. Further, we have $c_1, \dots, c_n \in B$ such that $c_1 f_1 + \dots + c_n f_n = 1$. We claim that $S = \{b_{ij}, f_i, c_i\}_{i,j}$ is a finite generating set for B.

Let C be the sub-algebra of B generated by S. Pick any $b \in B$. We wish to show that $b \in C$. Fix an $i = 1, \dots, n$. Observe that the image of b in the localized ring B_{f_i} is generated by some polynomial with coefficients in A and indeterminates replaced by

$$\left\{ \frac{b_{ij}}{f_i^{n_j}} \right\}_{j=1,\dots,M_i}.$$

We may multiply b by $f_i^{N_i}$ for N_i large enough so that $f_i^{N_i} b$ is then represented by a polynomial with coefficients in A evaluated in f_i and b_{ij} for $j = 1, \dots, M_i$. Consequently, $f_i^{N_i} b \in C$, for each $i = 1, \dots, n$. Observe that $f_1, \dots, f_n$ in C generates the unit ideal in C. By Lemma 23.23.0.2, 2, we see that $f_1^{N_1}, \dots, f_n^{N_n}$ also generates the unit ideal in C. Hence, we have $d_1, \dots, d_n \in C$ such that $1 = d_1 f_1^{N_1} + \dots + d_n f_n^{N_n}$. Multiplying by b, we obtain $b = d_1 f_1^{N_1} b + \dots + d_n f_n^{N_n}$ where by above, we now know that each term is in C. This completes the proof. □

An observation which is of importance in the study of varieties is the following.

Lemma 23.1.2.12. *Let R be an integral domain. Then*

$$\bigcap_{\mathfrak{m} < R} R_{\mathfrak{m}} \cong R$$

where the intersection runs over all maximal ideals $\mathfrak{m}$ of R and the intersection is carried out in the fraction field $R_{\langle 0 \rangle}$.

Proof. We already have that

$$R \hookrightarrow R_{\mathfrak{m}}$$

for any maximal ideal $\mathfrak{m} < R$. Thus,

$$R \hookrightarrow \bigcap_{\mathfrak{m} < R} R_{\mathfrak{m}}.$$

Thus it would suffice to show that $\bigcap_{\mathfrak{m} < R} R_{\mathfrak{m}} \hookrightarrow R$. Indeed, consider the following map

$$\begin{aligned} \bigcap_{\mathfrak{m} < R} R_{\mathfrak{m}} &\longrightarrow R \\ [f_{\mathfrak{m}}/g_{\mathfrak{m}}] &\longmapsto f_{\mathfrak{m}} g_{\mathfrak{m}'} \end{aligned}$$

where $f_{\mathfrak{m}}/g_{\mathfrak{m}} = f_{\mathfrak{m}'}/g_{\mathfrak{m}'}$ for two maximal ideals $\mathfrak{m}, \mathfrak{m}'$ in R. Thus, $f_{\mathfrak{m}} g_{\mathfrak{m}'} = f_{\mathfrak{m}'} g_{\mathfrak{m}}$. Hence the above map is well-defined and is injective as $f_{\mathfrak{m}} g_{\mathfrak{m}'} = 0$ implies $f_{\mathfrak{m}} = 0$ as $g_{\mathfrak{m}'} \neq 0$. The result follows. □

Local rings

A ring R is said to be *local* if there is a unique maximal ideal of R. In such a case we denote it by $(R, \mathfrak{m})$.

Definition 23.1.2.13. (**Zariski (co)tangent space**) Let $(R, \mathfrak{m})$ be a local ring. Then, we define the Zariski *cotangent space* of $(R, \mathfrak{m})$ to be $T^*R = \mathfrak{m}/\mathfrak{m}^2$ and the Zariski *tangent space* to be its dual $TR = \mathrm{Hom}_k\left(\mathfrak{m}/\mathfrak{m}^2, k\right)$.

Remark 23.1.2.14. The Zariski cotangent space T^*R is a κ-vector space where $\kappa = R/\mathfrak{m}$ is the residue field. Indeed, the scalar multiplication is given by

$$\kappa \times T^*R \longrightarrow T^*R$$
$$(c + \mathfrak{m}, x + \mathfrak{m}^2) \longmapsto cx + \mathfrak{m}^2$$

where $c \in R$ and $x \in \mathfrak{m}$. Indeed, this is well-defined as can be seen by a simple check. Consequently, the tangent space $TR = \mathrm{Hom}_k\left(\mathfrak{m}/\mathfrak{m}^2, k\right)$ is also a κ-vector space.

Definition 23.1.2.15. (**Regular local ring**) Let $(A, \mathfrak{m})$ be a local ring with $k = A/\mathfrak{m}$ being the residue field. Then A is said to be regular if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

There is an important geometric lemma that one should keep in mind about certain local rings.

Definition 23.1.2.16. (**Rational local k-algebras**) Let k be a field. A local k-algebra $(R, \mathfrak{m})$ is said to be rational if its residue field $\kappa = R/\mathfrak{m}$ is isomorphic to the field k.

Rational local k-algebras have a rather simple tangent space.

Proposition 23.1.2.17. *Let $(A, \mathfrak{m}_A)$ be a rational local k-algebra. Then,*

$$TA \cong \mathrm{Hom}_{k,\mathrm{loc}}\left(A, k[\epsilon]\right)$$

where $k[\epsilon] := k[x]/x^2$ is the ring of dual numbers and $\mathrm{Hom}_{k,\mathrm{loc}}\left(A, k[\epsilon]\right)$ *denotes the set of all local k-algebra homomorphisms.*

Proof. Pick any k-algebra homomorphism $\varphi : A \to k[\epsilon]$. Denote by $\mathfrak{m}_\epsilon = \langle \epsilon \rangle \lneq k[\epsilon]$ the unique maximal ideal of $k[\epsilon]$. Since

$$k[\epsilon]/\mathfrak{m}_\epsilon \cong k,$$

therefore $k[\epsilon]$ is a rational local k-algebra as well. By Lemma 23.23.0.7, we may write $A = k \oplus \mathfrak{m}_A$ and $k[\epsilon] = k \oplus \mathfrak{m}_\epsilon$. We now claim that the datum of a local k-algebra homomorphism $\varphi : A \to k[\epsilon]$ is equivalent to datum of a k-linear map of k-modules $\theta : \mathfrak{m}_A/\mathfrak{m}_A^2 \to k$.

Indeed, we first observe that for any $\varphi : A \to k[\epsilon]$ as above, we have $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_\epsilon$. Thus, $\varphi(\mathfrak{m}_A^2) \subseteq \mathfrak{m}_\epsilon^2 = 0$. Thus, we deduce that for any such φ, $\mathrm{Ker}\,(\varphi) \supseteq \mathfrak{m}_A^2$. It follows from universal property of quotients that any such φ is in one-to-one correspondence with k-algebra homomorphisms

$$\tilde{\varphi} : A/\mathfrak{m}_A^2 \cong k \oplus (\mathfrak{m}_A/\mathfrak{m}_A^2) \longrightarrow k[\epsilon].$$

As $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_\epsilon$, therefore $\tilde{\varphi}(\mathfrak{m}_A/\mathfrak{m}_A^2) \subseteq \mathfrak{m}_\epsilon$. Thus, we obtain a k-linear map of k-modules

$$\theta : \mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow k \cong \mathfrak{m}_\epsilon$$

where $\mathfrak{m}_\epsilon \cong k$ as k-modules. It suffices to now show that from any such θ, one can obtain a unique k-algebra map $\tilde{\varphi} : k \oplus (\mathfrak{m}_A/\mathfrak{m}_A^2) \to k[\epsilon]$, which furthermore sets up a bijection between all such $\tilde{\varphi}$ and θ.

Indeed, from k-linear map θ, we may construct the following k-algebra map

$$\begin{aligned}\tilde{\varphi} : k \oplus (\mathfrak{m}_A/\mathfrak{m}_A^2) &\longrightarrow k[\epsilon]\\ (k + \bar{m}) &\longmapsto k + \theta(\bar{m})\epsilon.\end{aligned}$$

Then we observe that $\tilde{\varphi}$ is a k-algebra homomorphism as

$$\begin{aligned}\tilde{\varphi}((k_1 + \bar{m}_1)(k_2 + \bar{m}_2)) &= \tilde{\varphi}(k_1k_2 + k_1\bar{m}_2 + k_2\bar{m}_1 + \bar{m}_1\bar{m}_2)\\ &= k_1k_2 + k_1\theta(\bar{m}_2)\epsilon + k_2\theta(\bar{m}_1)\epsilon + \theta(\bar{m}_1\bar{m}_2)\epsilon\\ &= k_1k_2 + k_1\theta(\bar{m}_2)\epsilon + k_2\theta(\bar{m}_1)\epsilon\\ &= (k_1 + \theta(\bar{m}_1)\epsilon) \cdot (k_2 + \theta(\bar{m}_2)\epsilon)\\ &= \tilde{\varphi}(k_1 + \bar{m}_1) \cdot \tilde{\varphi}(k_2 + \bar{m}_2).\end{aligned}$$

Hence, from θ one obtain $\tilde{\varphi}$ back, thus setting up a bijection and completing the proof. $\square$

23.1.3 Structure theorem

Let M be a finitely generated R-module. We can understand the structure of such modules completely in terms of the ring R, when R is a PID (so that it's UFD). This is the content of the structure theorem. We first give the following few propositions which is used in the proof of the structure theorem but is of independent interest as well, in order to derive a usable variant of structure theorem. The following theorem tells us a direct sum decomposition exists for any finitely free torsion module over a PID.

Proposition 23.1.3.1. *Let M be a finitely generated torsion module over a PID R. If $\mathrm{Ann}(M) = \langle c\rangle$ where $c = p_1^{k_1} \dots p_r^{k_r}$ and $p_i \in R$ are prime elements, then*

$$M \cong M_1 \oplus \cdots \oplus M_r$$

where $M_i = \{x \in M \mid p_i^{r_i}x = 0\} \leq M$ for all $i = 1, \dots, r$, that is, where $\mathrm{Ann}(M_i) = \langle p_i^{r_i}\rangle$ for all $i = 1, \dots, r$.

The next result tells us that we can further write each of the above M_is as a direct sum decomposition of a special kind.

Proposition 23.1.3.2. *Let M be a finitely generated torsion module over a PID R. If $\mathrm{Ann}M = \langle p^r\rangle$ where $p \in R$ is a prime element, then there exists $r_1 \geq r_2 \geq \cdots \geq r_k \geq 1$ such that*

$$M \cong R/\langle p^{r_1}\rangle \oplus \cdots \oplus R/\langle p^{r_k}\rangle.$$

The structure theorem is as follows.

Theorem 23.1.3.3. *(Structure theorem) Let R be a PID and M be a finitely generated R-module. Then there exists an unique $n \in \mathbb{N} \cup \{0\}$ and $q_1, \dots q_r \in R$ unique upto units such that $q_{i-1}|q_i$ for all $i = 2, \dots, r$ and*

$$M \cong R^n \oplus R/\langle q_1\rangle \oplus \cdots \oplus R/\langle q_r\rangle.$$

The most useful version of this is the following:

Corollary 23.1.3.4. *Let M be a finitely generated torsion module over a PID R. Then, there exists k-many prime elements $p_1, \dots, p_k \in R$, $n_j \in \mathbb{N}$ for each $j = 1, \dots, k$ and $1 \leq r_{1j} \leq \cdots \leq r_{n_j j} \in \mathbb{N}$ for each $j = 1, \dots, k$ such that*

$$M \cong \bigoplus_{j=1}^{k} \left(R/\langle p_j^{r_{1j}} \rangle \oplus \cdots \oplus R/\langle p_j^{n_j j} \rangle \right).$$

Proof. This is a consequence of Propositions 23.1.3.1 and 23.1.3.2. $\square$

This is the famous structure theorem for finitely generated modules over a PID. Note that the ring $\mathbb{Z}$ is PID and any abelian group is a $\mathbb{Z}$-module. Thus, we can classify finitely generated abelian groups using the structure theorem.

Example 23.1.3.5. An example of a module which is not finitely generated is the polynomial module $R[x]$ over a ring R. Indeed, the collection $\{1, x, x^2, \dots\}$ will make it free but not finitely generated.

Example 23.1.3.6. Classification of all abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$, for example, can be achieved via structure theorem. Indeed using Corollary 23.1.3.4, we will get that there are 6 total such abelian groups given by

- $\left(\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)$
- $\left(\frac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)$
- $\left(\frac{\mathbb{Z}}{2^3\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)$
- $\left(\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{3^2\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)$
- $\left(\frac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{3^2\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)$
- $\left(\frac{\mathbb{Z}}{2^3\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{3^2\mathbb{Z}}\right) \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right)$

23.1.4 UFDs

23.1.5 Gauss' lemma

Add results surrounding primitive polynomials and Gauss' lemma here from notebook.

Spectra of polynomial rings over UFDs

We now calculate the prime spectra of polynomial rings over UFDs. For that, we need the following two lemmas.

Lemma 23.1.5.1. *Let R be a UFD and $I \leq R[x]$ be an ideal containing two elements with no common factors. Then I contains a non-zero constant from R.*

Proof. Indeed, let $f, g \in R[x]$ be two elements with no common factors. Let Q denote the fraction field of R. We first claim that $f, g \in Q[x]$ have no common factor as well. Indeed, suppose $h(x) \in Q[x]$ is a common factor of $f(x)$ and $g(x)$. It follows from the result on primitive polynomials that we can write $h(x) = ch_0(x)$ where $c \in Q$ and $h_0(x) \in R[x]$ is primitive. Hence, we see that $h_0(x) \in R[x]$ is a polynomial such that $h_0|f$ and $h_0|g$ in $Q[x]$.

Again, by general results in UFD, we then conclude that $h_0|f$ and $h_0|g$ in $R[x]$. As f and g have no common factor, therefore $h_0(x) \in R[x]$ is a unit. Hence $h(x) \in Q[x]$ is a unit. Thus, there is no common factor of $f(x)$ and $g(x)$ in $Q[x]$ if there is none in $R[x]$.

Hence, $f(x), g(x)$ in $Q[x]$ have gcd 1, where $Q[x]$ is a PID. Consequently, $f(x)$ and $g(x)$ generates the unit ideal in $Q[x]$. It follows that there exists $p(x), q(x) \in Q[x]$ such that

$$1 = p(x)f(x) + q(x)g(x).$$

By theorem on primitive polynomials, we may write $p(x) = \frac{a}{b}p_0(x)$ and $q(x) = \frac{c}{d}q_0(x)$ where $a/b, c/d \in Q$ and $p_0(x), q_0(x) \in R[x]$ are primitive. The above equation hence becomes

$$\begin{aligned} 1 &= \frac{a}{b}p_0(x)f(x) + \frac{c}{d}q_0(x)g(x) \\ &= \frac{adp_0(x)f(x) + bcq_0(x)g(x)}{bd}, \end{aligned}$$

which thus yields

$$bd = adp_0(x)f(x) + bcq_0(x)g(x)$$

where RHS is in $I \leq R[x]$ because $ad, p_0, bc, q_0 \in R[x]$ and $f, g \in I$ and LHS is in R. Hence $I \cap R$ is not zero. $\square$

Lemma 23.1.5.2. *Let R be a PID and $f, g \in R[x]$ be non-zero polynomials such that f and g have no common factors. Then,*

1. *any prime ideal $\mathfrak{p} \lneq R[x]$ containing f and g is maximal,*
2. *any maximal ideal $\mathfrak{m} \lneq R[x]$ containing f and g is of the form $\langle p, h(x)\rangle$ where $p \in R$ is prime and $h(x)$ is prime modulo p,*
3. *there are only finitely many maximal ideals of $R[x]$ containing f and g.*

Proof. 1. : Let $\mathfrak{p} \lneq R[x]$ be a prime ideal containing f and g. Observe by Lemma 23.1.5.1 that there exists $b \in R \setminus 0$ such that $b \in \mathfrak{p} \cap R$, that is, $\mathfrak{p} \cap R \neq 0$. As R is a PID and $\mathfrak{p} \cap R$ is a prime ideal of R, therefore $\mathfrak{p} \cap R = pR$ for some prime element $p \in \mathfrak{p} \cap R$. We wish to show that $R[x]/\mathfrak{p}$ is a field. Indeed, we see that (note $\langle p, \mathfrak{p}\rangle = \mathfrak{p}$ as $p \in \mathfrak{p}$)

$$\begin{aligned} \frac{R[x]}{\mathfrak{p}} &\cong \frac{\frac{R[x]}{pR[x]}}{\frac{\langle p, \mathfrak{p}\rangle}{pR[x]}} = \frac{\frac{R[x]}{pR[x]}}{\frac{\mathfrak{p}}{pR[x]}} \\ &\cong \frac{\frac{R}{pR}[x]}{\overline{\mathfrak{p}}} \end{aligned}$$

where $\overline{\mathfrak{p}} = \pi(\mathfrak{p})$ where $\pi : R[x] \twoheadrightarrow \frac{R}{pR}[x]$ is the quotient map. As R is a PID and pR is a non-zero prime ideal, therefore it is maximal. Consequently, R/pR is a field and hence $\frac{R}{pR}[x]$ is a PID. Suppose $\overline{\mathfrak{p}} = 0$, then f and g have a common factor given by $p \in R$, which is not possible. Consequently, $\overline{\mathfrak{p}}$ is a proper prime ideal of $\frac{R}{pR}[x]$ by correspondence theorem. But in PIDs, non-zero prime ideals are maximal ideals, hence we obtain that $\frac{R}{pR}[x]/\overline{\mathfrak{p}}$ is a field, as required.

2. : Let $\mathfrak{m} \lneq R[x]$ be a maximal ideal of $R[x]$ containing f and g. Hence, from Lemma

23.1.5.1 and R being a PID, there exists $p \in R$ a prime such that $\mathfrak{m} \cap R = pR$. Hence R/pR is a field as R is a PID and pR a non-zero prime ideal (so maximal). Consequently, we have a quotient map

$$\pi : R[x] \twoheadrightarrow \frac{R[x]}{pR[x]} \cong \frac{R}{pR}[x]$$

As $p \in \mathfrak{m}$, therefore by correspondence thereom $\pi(\mathfrak{m}) = \overline{\mathfrak{m}}$ is a maximal ideal of $\frac{R}{pR}[x]$. As R/pR is a field, therefore $\frac{R}{pR}[x]$ is a PID. Hence, $\overline{\mathfrak{m}} = \langle \overline{h(x)} \rangle$ for some $h(x) \in R[x]$ such that $\overline{h(x)}$ is irreducible (so it generates a maximal ideal). Again, by correspondence theorem we have $\pi^{-1}(\overline{\mathfrak{m}}) = \mathfrak{m} = h(x)R[x] + pR[x] = \langle p, h(x) \rangle$, as required.

3. : We will use notations of proof of 2. above. Take any maximal ideal $\mathfrak{m} = \langle p, h(x) \rangle \lneq R[x]$ which contains $f(x)$ and $g(x)$, $p \in R$ is prime and $h(x)$ is irreducible modulo p. As R is a PID, so it is a UFD, hence $R[x]$ is a UFD by Gauss' lemma. Hence, writing $f(x)$ and $g(x)$ as product of prime factors in $R[x]$, we observe that there exists distinct primes $p(x), q(x) \in R[x]$ such that $p(x), q(x) \in \mathfrak{m}$. Replacing f by p and g by q, we may assume f and g are irreducible (or prime) in $R[x]$.

By Lemma 23.1.5.1, there exists $b \in R \setminus 0$ such that $b \in \mathfrak{m} \cap R$. As the proof of 2. above shows, $p|b$ in R. As R is a PID, so it is a UFD, hence there are only finitely many choices for p.

Now, going modulo prime p, we see that $\overline{f(x)}, \overline{g(x)} \in \overline{\mathfrak{m}} \lneq \frac{R}{pR}[x]$ has a common factor in $\frac{R}{pR}[x]$, given by $\overline{h(x)}$ as $\overline{\mathfrak{m}} = \langle \overline{h(x)} \rangle$ (by proof of 2.). As $\overline{h(x)}$ generates a maximal ideal in $\frac{R}{pR}[x]$, therefore $\overline{h(x)}$ is a prime element of $\frac{R}{pR}[x]$, which has to divide $\overline{f(x)}$ and $\overline{g(x)}$. As $\frac{R}{pR}[x]$ is a PID, therefore there are only finitely many choices for $\overline{h(x)}$, and since $\mathfrak{m} = \pi^{-1}(\langle \overline{h(x)} \rangle)$, therefore every choice of p as above, yields finitely many choices for $\mathfrak{m}$.

Consequently, there are finitely many choices for p and once p is fixed, there are only finitely many choices for the ideal $\overline{\mathfrak{m}}$. As $\mathfrak{m} = \pi^{-1}(\overline{m})$, therefore there are finitely many maximal ideals containing f and g. $\square$

We now classify Spec $(R[x])$ for a UFD R.

Theorem 23.1.5.3. *Let R be a PID. Any prime ideal $\mathfrak{p} \lneq R[x]$ is of one of the following forms*

1. $\mathfrak{p} = \mathfrak{o}$,
2. $\mathfrak{p} = \langle f(x) \rangle$ *for some irreducible* $f(x) \in R[x]$,
3. $\mathfrak{p} = \langle p, h(x) \rangle$ *for some prime* $p \in R$ *and* $h(x) \in R[x]$ *irreducible modulo* p *and this is also a maximal ideal.*

Proof. Indeed, pick any prime ideal $\mathfrak{p} \lneq R[x]$. If $\mathfrak{p}$ is 0, then it is prime as $R[x]$ is a domain. We now have two cases. If $\mathfrak{p}$ is principal, then $\mathfrak{p} = \langle f(x) \rangle$ for some $f(x) \in R[x]$. As $\langle f(x) \rangle$ is prime therefore $f(x)$ is a prime element. As $R[x]$ is a UFD by Gauss' lemma, therefore $f(x)$ is also irreducible. Consequently, $\mathfrak{p} = \langle f(x) \rangle$ where $f(x)$ is irreducible.

On the other hand if $\mathfrak{p}$ is not principal, there exists $f(x), g(x) \in \mathfrak{p}$ such that $f(x) \not| g(x)$ and $g(x) \not| f(x)$. As $R[x]$ is a UFD and $\mathfrak{p}$ is prime, therefore there exists prime factors of f and g which are in $\mathfrak{p}$. Replacing f and g by these prime factors, we may assume f and g are distinct irreducibles in $\mathfrak{p}$. Consequently, by Lemma 23.1.5.2, we see that $\mathfrak{p} = \langle p, h(x)\rangle$ for some prime $p \in R$ and $h(x)$ irreducible modulo p. Moreover by Lemma 23.1.5.2 we know that $\mathfrak{p}$ in this case is maximal. $\square$

We now portray their use in the following.

Lemma 23.1.5.4. *Let F be an algebraically closed field. Then,*

1. *every non-constant polynomial $f(x, y) \in F[x, y]$ has at least one zero in F^2,*
2. *every maximal ideal of $F[x, y]$ is of the form $\mathfrak{m} = \langle x - a, y - b\rangle$ for some $a, b \in F$.*

Proof. 1. : Take any polynomial $f(x, y) \in F[x, y]$. Going modulo y, we see that $\overline{f(x, y)} \in F[x, y]/\langle y\rangle = F[x]$. If $\overline{f(x, y)} = 0$, then $(a, 0)$ is a root of $f(x, y)$ for any $a \in F$. if $\overline{f(x, y)} \neq 0$, then since F is algebraically closed, therefore we may write $\overline{f(x, y)} = (x - a_1) \dots (x - a_n)$. Consequently, any $(a_i, 0)$ is a zero of $f(x, y)$. Hence, in any case, $f(x, y)$ has a root in F^2.
2. : Let $R = F[x]$. We know that R is a PID. Take any maximal ideal $\mathfrak{m} \subsetneq R[y] = F[x, y]$. Then by Theorem 23.1.5.3, we have that either $\mathfrak{m} = \langle f(x, y)\rangle$ where $f(x, y)$ is irreducible or $\mathfrak{m} = \langle p(x), h(x, y)\rangle$ where $p(x) \in R$ is prime and $h(x, y)$ is irreducible modulo $p(x)$.

In the former, we claim that $\langle f(x, y)\rangle$ is not maximal. Indeed, by item 1, we have that $f(x, y)$ has a zero in F^2, say (a, b). Dividing $f(x, y)$ by $y - b$ in $R[y]$, we obtain $f(x, y) = h(x, y)(y - b) + k(x)$, where $k(x) \in R$. Consequently, $k(a) = 0$. Hence, $k(x) = (x - a)l(x)$. Thus, we have $f(x, y) = h(x, y)(y - b) + (x - a)l(x)$, showing $f(x, y) \in \langle x - a, y - b\rangle$. By Theorem 23.1.5.3 above, we know that $\langle x - a, y - b\rangle \in R[y]$ is a maximal ideal and we also know that it contains $f(x, y)$. We hence need only show that $\langle f(x, y)\rangle \subsetneq \langle x - a, y - b\rangle$. Indeed, observe that $x - a \notin \langle f(x, y)\rangle$ as if it is, then $f(x, y)|x - a$. But then $f(x, y)$ is in R, hence $y - b \notin \langle f(x, y)\rangle$. So in either case, $\langle f(x, y)\rangle$ is properly contained in $\langle x - a, y - b\rangle$, showing that $\langle f(x, y)\rangle$ cannot be maximal. Thus, no maximal ideal of $R[y]$ can be of the form $\langle f(x, y)\rangle$.

In the latter, where $\mathfrak{m} = \langle p(x), h(x, y)\rangle$ where $p(x) \in R$ is prime and $h(x, y)$ is irreducible modulo $p(x)$, we first see that $p(x) = x - a$ for some $a \in F$ as $R = F[x]$ and only primes of $F[x]$ are of this type. Let $\pi : R \twoheadrightarrow \frac{R}{p(x)R}[y] \cong \frac{R}{\langle x-a\rangle}[y] \cong F[y]$ be the quotient map by the ideal $p(x)R[y]$. Then we see that by correspondence theorem, $\pi(\mathfrak{m}) = \overline{\mathfrak{m}} = \langle\overline{h(x, y)}\rangle$ is a prime ideal of $F[y]$. Hence, $\overline{\mathfrak{m}} = \langle k(y)\rangle$ for some $k(y) \in F[y]$. Further, since $\overline{\mathfrak{m}}$ is prime and F algebraically closed, therefore $k(y) = y - b$. Thus, we see that modulo $p(x)$ we have $h(x, y) = k(y) = y - b$. We then see that $\mathfrak{m} = \pi^{-1}(\overline{\mathfrak{m}}) = \pi^{-1}(\langle\overline{y - b}\rangle) = \langle p(x), y - b\rangle = \langle x - a, y - b\rangle$, as required. $\square$

Another example gives us finiteness of intersection of two algebraic curves over an algebraically closed field.

Proposition 23.1.5.5. *Let F be an algebraically closed field and $f, g \in F[x, y]$ be two polynomials with no common factors. Then, $Z(f) \cap Z(g)$ is a finite set, that is, f and g intersects at finitely many points in $\mathbb{A}^2_F$.*

Proof. We first show that for any $h(x, y) \in F[x, y]$, $h(a, b) = 0$ for some $(a, b) \in F^2$ if and only if $h \in \langle x - a, y - b\rangle$. Clearly, $(\Leftarrow)$ is immediate. For $(\Rightarrow)$, we proceed as follows. Going modulo $y - b$ in $F[x, y]$, we obtain $\overline{h(x, y)} \in F[x, y]/\langle y - b\rangle \cong F[x]$. Observe that $\langle y - b\rangle$

is the kernel of the map $F[x,y] \to F[x]$ taking $y \mapsto b$, hence $\overline{h(x,y)} = \overline{h(x,b)}$. As F is algebraically closed, therefore we may write

$$\overline{h(x,b)} = \overline{h(x,y)} = (x-c_1)\dots(x-c_n)$$

for $c_i \in F$. As, $h(a,b) = 0$, therefore $(x-a)|\overline{h(x,b)}$. Hence, for some i, we must have $c_i = a$. This allows us to write

$$h(x,y) - (x-a)k(x) \in \langle y-b \rangle$$

for some $k(x) \in F[x]$. It follows that for some $q(x,y) \in F[x,y]$ we have

$$h(x,y) - (x-a)k(x) = (y-b)q(x,y)$$

Thus, $h(x,y) \in \langle x-a, y-b \rangle$. This completes the proof of the claim above.

Now, using above claim $f(a,b) = 0 = g(a,b)$ if and only if $f, g \in \langle x-a, y-b \rangle$. By Lemma 23.1.5.2, as f and g have no common factors, therefore there are finitely many maximal ideals containing f and g. Further, by Lemma 23.1.5.4, we know that each such maximal ideal is of the form $\langle x-a, y-b \rangle$. Hence, there are only finitely many maximal ideals containing f and g, each of which looks like $\langle x-a, y-b \rangle$. Hence, by above claim, there are finitely many points $(a,b) \in F^2$ such that $f(a,b) = 0 = g(a,b)$. $\square$

23.1.6 Finite type k-algebras

We discuss basic theory of finite type k-algebras, that is, algebras of form $k[x_1, \dots, x_n]/I$.

Recall that for a field k, we denote by $k[x]$ the polynomial ring in one variable and we denote the rational function field $k(x)$ to be the field obtained by localizing at prime $\mathfrak{o}$. Further if K/k is a field extension and $\alpha \in K$, then $k[\alpha]$ is a subring of K generated by $\alpha \in K$ and it contains k. Whereas, $k(\alpha)$ is a field extension $k \hookrightarrow k(\alpha) \hookrightarrow K$. The following lemma shows that if K is algebraic, then $k(\alpha) = k[\alpha]$.

Lemma 23.1.6.1. *Let k be a field and K/k be an algebraic extension. If $\alpha_1, \dots, \alpha_n \in K$, then $k[\alpha_1, \dots, \alpha_n] = k(\alpha_1, \dots, \alpha_n)$.*

Proof. The proof uses a standard observation in field theory. First, let $f_1(x) \in k[x]$ be the minimal polynomial of α_1. Consequently, by a standard result in field theory, $k[\alpha_1] = k[x]/f_1(x)$ is a field. Thus $k[\alpha_1] = k(\alpha_1)$. Now observe that $K/k(\alpha_1)$ is an algebraic extension. Consequently, the same argument will yield $k(\alpha_1)[\alpha_2]$ to be a field. By above, we thus obtain $k(\alpha_1)[\alpha_2] = k[\alpha_1][\alpha_2] = k[\alpha_1, \alpha_2]$ to be a field. Consequently, $k[\alpha_1, \alpha_2] = k(\alpha_1, \alpha_2)$. One completes the proof now by induction. $\square$

Lemma 23.1.6.2. *Let k be a field and K/k be an algebraic extension. Then the homomorphism*

$$\begin{aligned} k[x_1, \dots, x_n] &\longrightarrow k(\alpha_1, \dots, \alpha_n) \\ x_i &\longmapsto \alpha_i \end{aligned}$$

has kernel which is a maximal ideal generated by n elements.

Proof. (*Sketch*) Use the proof of Lemma 23.1.6.1 to obtain that for each $1 \leq i \leq n$, we have that $k(\alpha_1, \dots, \alpha_{i-1})[\alpha_i] \cong k(\alpha_1, \dots, \alpha_{i-1})[x_i]/p_i(\alpha_1, \dots, \alpha_{i-1}, x_i)$ and divide an element $p \in k[x_1, \dots, x_n]$ in the kernel inductively by p_i and replacing p_i by remainder, starting at $i = n$. $\square$

Let us now observe a basic fact. Next, we observe that residue fields of any point in an affine n-space over k is an algebraic extension of k. **TODO : Till Jacobson rings from Matsumura.**

23.2 Graded rings & modules

We now study a very important class of rings, which have an extra structure of having their additive abelian group being graded by $\mathbb{Z}$[1]. These include polynomial algebras and quotient of polynomial algebras by homogeneous ideals. In particular, they are the algebraic counterparts of projective varieties. These will also be essential while discussing dimension theory.

Definition 23.2.0.1 (**Graded rings & homogeneous ideals**). A ring S is said to be graded if the additive subgroup of S has a decomposition

$$S = \bigoplus_{d \geq 0} S_d$$

where $S_d \subseteq S$ is a subgroup which is called the subgroup of degree d homogeneous elements, such that for all $d, e \geq 0$, we have

$$S_d \cdot S_e \subseteq S_{d+e}.$$

An ideal $\mathfrak{a} \leq S$ is said to be homogeneous if the additive subgroup of $\mathfrak{a}$ has a decomposition

$$\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a} \cap S_d.$$

Polynomial rings $S = k[x_0, \ldots, x_n]$ are graded rings where S_d is the abelian subgroup of all degree d homogeneous monomials. We will see more examples once we show how to construct quotients and localizations of graded rings. But first we see some important properties of homogeneous ideals.

Proposition 23.2.0.2. *Let S be a graded ring and $\mathfrak{a} \leq S$ be any ideal. Then,*

1. *$\mathfrak{a}$ is homogeneous if and only if there exists $G \subseteq S$ a subset of homogeneous elements such that G generates $\mathfrak{a}$.*
2. *Let $\mathfrak{a}, \mathfrak{b}$ be two homogeneous ideals of S. Then $\mathfrak{a}+\mathfrak{b}, \mathfrak{a}\cdot\mathfrak{b}$ and $\sqrt{\mathfrak{a}}$ are again homogeneous ideals.*
3. *The homogeneous ideal $\mathfrak{a}$ is prime if and only if for any two homogeneous $f, g \in \mathfrak{a}$ it follows that $fg \in \mathfrak{a}$ implies either $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.*

Proof. content... □

We now define the notion of graded map of graded rings.

Definition 23.2.0.3 (**Map of graded rings**). Let S, T be graded rings. A ring homomorphism $\varphi : S \to T$ is said to be a graded map if for all $d \geq 0$ we get $\varphi|_{S_d} : S_d \to T_d$. That is, φ preserves degree.

23.2.1 Constructions on graded rings

We now do familiar constructions on graded rings, like quotients, fraction fields and localizations.

[1] we choose to not work in excessive generality; $\mathbb{Z}$-grading is sufficient for us.

Quotients

Fraction field of a graded domain

Homogeneous localization

The following is a discussion on localization of a graded ring S at a homogeneous prime ideal $\mathfrak{p}$. Let T denote the multiplicative subset of S consisting of all homogeneous elements not contained in $\mathfrak{p}$. Then $T^{-1}S$ is a graded ring whose degree d-elements are a/f where $a \in S_{d+e}$ and $f \in T$ of degree e. These form an additive abelian group where $a/f + b/g = ag + bf/fg$ where $a \in S_{d+k}, b \in S_{d+l}$ and $f, g \in T$ are of degree k and l respectively. Indeed, then $ag + bf \in S_{d+k+l}$ and $fg \in T$ of degree $k + l$. Consequently, we define

$$S_{(\mathfrak{p})} := (T^{-1}S)_0$$

where $(T^{-1}S)_0$ is the degree 0 elements in the localization $T^{-1}S$. We call this the *homogeneous localization* of the graded ring S at the homogeneous prime ideal $\mathfrak{p}$. Thus $S_{(\mathfrak{p})} = (S_{\mathfrak{p}})_0$, i.e. homogeneous localization just picks out degree 0 elements from the usual localization. Note that the usual localization $T^{-1}S$ is a graded ring where grading is given by subtracting the degree of numerator by degree of denominator.

Lemma 23.2.1.1. *Let S be a graded ring and $\mathfrak{p}$ be a homogeneous prime ideal of S. Then, the homogeneous localization $S_{\mathfrak{p}}$ is a local ring.*

Proof. Consider the set $\mathfrak{m} := (\mathfrak{p} \cdot T^{-1}S) \cap S_{\mathfrak{p}}$. Then, $\mathfrak{m}$ is a maximal ideal of $S_{\mathfrak{p}}$ as any element not in $\mathfrak{m}$ in $S_{\mathfrak{p}}$ is a fraction f/g where $\deg f = \deg g$ and $f \notin \mathfrak{p}$ and thus it is invertible. Consequently, $S_{\mathfrak{p}}$ is local. $\square$

Remark 23.2.1.2. Note that if S is a graded domain, then $S_{(\langle 0 \rangle)}$ yields a field whose elements are of the form f/g where $\deg f = \deg g$ and f, g g is a non-zero homogeneous element of S. This field is called the *homogeneous fraction field* of graded domain S. This is a subfield of usual fraction field $S_{\langle 0 \rangle}$.

Let S be a graded ring and $g \in S$ be a homogeneous element. The *homogeneous localization of S at g* is defined to be the following subring of S_g:

$$S_{(g)} := \{f/g^n \in S_g \mid f \text{ is homogeneous with } \deg f = n \deg g,\ n \in \mathbb{N}\} \leq S_g.$$

Let S be a graded ring. Then an S-module M is said to be *graded S-module* if $M = \bigoplus_{d \in \mathbb{Z}} M_d$ where $M_d \leq M$ is a subgroup of M such that $S_d \cdot M_e \subseteq M_{d+e}$. Then, for a homogeneous element $g \in S$, we denote by $M_{(g)}$ the following submodule of M_g:

$$M_{(g)} := \{m/g^n \mid m \text{ is homogeneous with } \deg m = n \deg g,\ n \in \mathbb{N}\} \leq M_g.$$

For each graded S-module M, one can attach a sequence of graded modules.

Definition 23.2.1.3. (**Twisted modules**) Let S be a graded ring and M a graded S-module. Then, define

$$M(l) := \bigoplus_{d \in \mathbb{Z}} M_{d+l}$$

to be the *l-twisted graded module of M*.

An important lemma with regards to localization of a graded ring at a positive degree element is as follows, it will prove its worth in showing that projective spectrum of a graded ring is a scheme (see Lemma **??**).

Lemma 23.2.1.4. *Let S be a graded ring and $f \in S_d$, $d > 0$. Then we have a bijection*

$$D_+(f) \cong \operatorname{Spec}\left(S_{(f)}\right)$$

where $D_+(f) \subseteq \operatorname{Spec}(S)$ is the set of all homogeneous prime ideals of S which does not contain f and does not contain S_+.

Proof. Consider the following map

$$\begin{aligned}\varphi : D_+(f) &\longrightarrow \operatorname{Spec}\left(S_{(f)}\right)\\ \mathfrak{p} &\longmapsto (\mathfrak{p} \cdot S_f)_0,\end{aligned}$$

that is, the degree zero elements of the prime ideal $\mathfrak{p} \cdot S_f$ of S_f. Indeed, $\varphi(\mathfrak{p})$ is a prime ideal of $S_{(f)}$. Further, if $(\mathfrak{p} \cdot S_f)_0 = (\mathfrak{q} \cdot S_f)_0$ for $\mathfrak{p}, \mathfrak{q} \in D_+(f)$, then for any $g \in \mathfrak{p}$, one observes via above equality that $g \in \mathfrak{q}$. Consequently, $\mathfrak{p} = \mathfrak{q}$. Thus φ is injective. For surjectivity, pick any prime ideal $\mathfrak{p} \in \operatorname{Spec}\left(S_{(f)}\right)$. We will construct a prime ideal $\mathfrak{q} \in D_+(f)$ such that $\varphi(\mathfrak{q}) = \mathfrak{p}$. Indeed, let $K = \{g \in S \mid g \text{ is homogeneous } \& \ \exists n > 0 \text{ s.t. } g/f^n \in \mathfrak{p}\}$ and consider the ideal

$$\mathfrak{q} = \langle K \rangle.$$

We thus need to check the following statements to complete the bijection:

1. $\mathfrak{q}$ is not the unit ideal of S,
2. $\mathfrak{q}$ is homogeneous in S,
3. $\mathfrak{q}$ is prime in S,
4. $\mathfrak{q}$ doesn't contain f,
5. $(\mathfrak{q} \cdot S_f)_0 = \mathfrak{p}$.

Statement 4 tells us that $\mathfrak{q}$ doesn't contain S_+. Statement 1 follows from a degree argument; if $1 \in \mathfrak{q}$, then $1 = a_1 g_1 + \cdots + a_m g_m$ for $g_i \in K$ and $a_i \in S$, but 1 is a degree 0 element whereas the minimum degree of the right is atleast > 0. Statement 2 is immediate as $\mathfrak{q}$ is generated by homogeneous elements. For statement 3, it is enough to check for homogeneous elements $h, k \in S$ that $hk \in \mathfrak{q} \implies h \in \mathfrak{q}$ or $k \in \mathfrak{q}$. This is immediate, after observing that any homogeneous element of $\mathfrak{q}$ is in K because K is the set of all homogeneous elements of S of positive degree which is not a power of f. Statements 4 and 5 are immediate checks. $\square$

23.3 Noetherian modules and rings

Let R be a ring. An R-module M is said to be *noetherian* if it satisfies either of the following equivalent properties:

1. Every increasing chain of submodules of M eventually stabilizes.
2. Every non-empty family of submodules of M has a maximal element.
3. Every submodule is finitely generated.

We prove the equivalence of 1 and 3 as in Proposition 23.3.0.3. But before, let us see that noetherian hypothesis descends to submodules and to quotients:

Lemma 23.3.0.1. *Let R be a ring and M be a noetherian R-module.*

1. *If N is a submodule of M, then N is noetherian.*
2. *If M/N is a quotient of M, then M/N is noetherian.*

Proof. 1. Take any submodule of M which is in N, then it is a submodule of N which is finitely generated.
2. Take any submodule of M/N, which is of the form K/N where $K \subseteq M$ is a submodule of M containing N. Hence K is finitely generated and so is N. Thus K/N is finitely generated. □

We also have that a finitely generated module over noetherian ring necessarily has to be noetherian, so every submodule is also finitely generated, which is not usually the case. This is another hint why having noetherian hypothesis can greatly ease calculations.

Lemma 23.3.0.2. *Let R be a noetherian ring and let M be an R-module. Then M is a noetherian module if and only if M is finitely generated.*

Proof. The only non-trivial side is R $\Rightarrow$ L. Since M is finitely generated, therefore there is a surjection $f : R^n \twoheadrightarrow M$ where R^n is noetherian as R is noetherian (you may like to see it as a consequence of Corollary 23.3.0.5). Now take an increasing chain of submodules $N_0 \subseteq N_1 \subseteq \dots$ of M. This yields an increasing chain of ideals $f^{-1}(N_0) \subseteq f^{-1}(N_1) \subseteq \dots$, which stabilizes as R is noetherian. Applying f to the chain again we get that $N_0 \subseteq N_1 \subseteq \dots$ stabilizes. □

Here's the proof of equivalence as promised.

Proposition 23.3.0.3. *Let R be a ring. An R-module M is noetherian if and only if every submodule of M is finitely generated.*

Proof. (L $\implies$ R) Suppose R-module M is noetherian and let $S \subseteq M$ be a submodule of M. Note S is also noetherian. This means that any subcollection of submodules of S has a maximal element. Let such a subcollection be the collection of all finitely generated submodules of S, which clearly isn't empty as $\{0\}$ is there. This would have a maximal element, say N. If $N = S$, we are done. If not, then take $x \in S \setminus N$ and look at $N + Rx \subset S$. Clearly this is a submodule of S strictly containing N and is also finitely generated as N is too. This contradicts the maximality of N. Hence every submodule of M is finitely generated.
(R $\implies$ L) Let every submodule of M be finitely generated. We wish to show that this

makes M into a noetherian module. So take any ascending chain of submodules $S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$. Consider the union $S = \cup_{i=0}^{\infty} S_i$. S is also a submodule because for any $x, y \in S$, since $\{S_i\}$ is an ascending chain, there exists S_i such that $x, y \in S_i$, and so $x + y \in S_i \subseteq S$. By hypothesis, $S = \langle x_1, \dots, x_k \rangle$. Let S_{n_i} be the smallest submodule containing x_i. Then $S_{\max n_i}$ is a member of the chain which contains each of the x_is, which thus means that the $S_{\max n_i}$ is generated by x_is because if it didn't then S would have either a smaller or a larger generating set, contradicting the generation by $x_1, \dots, x_k$. Hence the chain stabilizes after $S_{\max n_i}$. □

The reason one dwells with the noetherian hypothesis is reflected in the following properties enjoyed by it. Given a short exact sequence of modules, it is possible to figure out whether the middle module is noetherian or not by checking the same for the other two:

Proposition 23.3.0.4. *Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence of R-modules. Then, the module M is noetherian if and only if M' and M'' are noetherian.*

Proof. (L $\implies$ R) Let M be noetherian. Then if we consider any ascending chain in M' or M'', then we get an ascending chain in M because of the maps f and g. Remember inverse image of an injective and direct image of a surjective module homomorphism of a submodule is also a submodule.
(R $\implies$ L) Consider an ascending chain of submodules $S_0 \subseteq S_1 \subseteq \dots$ in M. We then have two more ascending chains $\{(f)^{-1}(S_i)\}$ and $\{g(S_i)\}$ in M' and M'' respectively. Since these are noetherian, therefore for both of them $\exists k \in \mathbb{N}$ such that these two chains stabilizes after k. Now, we wish to show that $\{S_i\}$ also stabilizes after k. For this, we just need to show that $S_{k+1} \subseteq S_k$. Hence take any $m \in S_{k+1}$. We have $g(m) \in g(S_k)$, therefore $\exists s \in S_k$ such that $g(m) = g(s) \implies g(m-s) = 0$ in M''. Since the sequence is exact, therefore $\exists m' \in M'$ such that $f(m') = m - s$, or, $m - s \in \operatorname{im}(f)$. Since $m \in S_{k+1}$ and $s \in S_k \subseteq S_{k+1}$, therefore $m - s \in S_{k+1}$. Hence $m - s \in \operatorname{im}(f) \cap S_{k+1}$ and since $\operatorname{im}(f) \cap S_{k+1} = \operatorname{im}(f) \cap S_k$, therefore $m - s \in S_k$ and thus $m \in S_k$. This proves $S_{k+1} \subseteq S_k$, proving $S_k = S_{k+1} = \dots$. □

An easy consequence of the above is that direct sum of finitely many noetherian modules is again noetherian:

Corollary 23.3.0.5. *Suppose $\{M_i\}_{i=1}^n$ be a collection of noetherian R-modules. Then $\bigoplus_{i=1}^n M_i$ is also a noetherian R-module.*

Proof. Since the sum $\bigoplus_{i=1}^n M_i$ sits at the middle of the following short exact sequence:

$$0 \longrightarrow M_1 \xrightarrow{f} \bigoplus_{i=1}^{n} M_i \xrightarrow{g} \bigoplus_{i=2}^{n} M_i \longrightarrow 0$$

where f is given by $m \longmapsto (m, 0, \dots, 0)$ and g is given by $(m_1, \dots, m_n) \longmapsto (m_2, \dots, m_n)$. The fact that this is indeed exact is simple to see. One can next use induction to complete the proof. □

An important result in the theory of noetherian rings is the following, which gives us few more (but highly important) examples of noetherian rings in nature. In particular it tells us that the one of the major class of rings which are studied in algebraic geometry, polynomial rings over algebraically closed fields, are noetherian.

Theorem 23.3.0.6. *(Hilbert basis theorem) Let R be a ring. If R is noetherian, then*
1. *$R[x_1, \dots, x_n]$ is noetherian,*
2. *$R[[x_1, \dots, x_n]]$ is noetherian.*

Proof. 1. We need only show that if R is noetherian then so is $R[x]$. Pick any ideal $I \leq R[x]$. We wish to show it is finitely generated. We go by contradiction, let I not be finitely generated.

Let $f_1 \in I$ be the smallest degree non-constant polynomial[2] and denote $I_1 = \langle f_1 \rangle$. Let $f_2 \in I \setminus I_1$ be the smallest degree non-constant polynomial and denote $I_2 = \langle f_1, f_2 \rangle$. Inductively, we define $I_n = \langle f_1, \dots, f_n \rangle$ where $f_n \in I \setminus I_{n-1}$ is of least degree non-constant. As I is not finitely generated, therefore for all $n \in \mathbb{N}$, $I_n \lneq I$. Let $f_n(x) = a_n x^m +$ other terms for each $n \in \mathbb{N}$ so that $a_n \in R$ represents the coefficient of the leading term of $f_n(x)$. Consequently, we obtain a sequence $\{a_n\} \subseteq R$. Let $J = \langle a_1, \dots, a_n, \dots \rangle$. As R is noetherian, therefore there exists $n \in \mathbb{N}$ such that $J = \langle a_1, \dots, a_n \rangle$. It follows that for some $r_1, \dots, r_n \in R$ we have

$$a_{n+1} = r_1 a_1 + \cdots + r_n a_n.$$

We claim that $I = \langle f_1, \dots, f_n \rangle =: I_n$.

If not then $f_{n+1} \in I \setminus I_n$ is of least degree non-constant. We will now show that $f_{n+1} \in I_n$, thus obtaining a contradiction. Indeed, we have by the way of choice of f_{n+1} that $\deg f_{n+1} \geq \deg f_i$ for each $i = 1, \dots, n$. Consequently the polynomial

$$g = \sum_{i=1}^{n} r_i f_i \cdot x^{\deg f_{n+1} - \deg f_i}$$

has the property that its degree is equal to $\deg f_{n+1}$ and the coefficient of its leading term is equal to f_{n+1}. It follows that the polynomial $g - f_{n+1} \in I$ has degree strictly less than that of f_{n+1}. By minimality of f_{n+1}, it follows that $g - f_{n+1} \in I_n$. Note that by construction $g \in I_n$. Hence $f_{n+1} \in I_n$, as required.

2. **TODO** : *Write it from your exercise notebook.* □

Any localization of noetherian ring is again noetherian.

Proposition 23.3.0.7. *Let R be a noetherian ring and $S \subset R$ be a multiplicative set. Then $S^{-1}R$ is a noetherian ring.*

Proof. Any ideal of R is $S^{-1}I$ where $I \subseteq R$ is an ideal by exactness of localization (Lemma 23.1.2.2). As I is finitely generated as an R-module, therefore $S^{-1}I$ is finitely generated as an $S^{-1}R$-module, as needed. □

Lemma 23.3.0.8. *Let R be a ring with $\langle f_1, \dots, f_n \rangle = R$. If each R_{f_i} is noetherian, then R is noetherian.*

Proof. Pick any ideal $I \subseteq R$. We wish to show it is finitely generated. By exactness of localization (Lemma 23.1.2.2), we get $I_{f_i} \subseteq R_{f_i}$ is an ideal, thus finitely generated as R_{f_i}-module. By Lemma 23.1.2.10, I is finitely generated as an R-module. □

[2] this exists by well-ordering by degree.

Corollary 23.3.0.9. *Let R be a ring. Then, R is noetherian if and only if R_f is noetherian for all $f \in R$.* □

23.4 Supp (M), Ass (M) and primary decomposition

Let R be a ring and M be a finitely generated R-module. In the classical case when R is a field and M is then a finite dimensional R-vector space, if $x \in M$ then if even a single element of R annihilate x, then all elements of R annihilate x. This luxury is not enjoyed when R is a ring because not all elements of R may be invertible. What one does then is to study the associated annihilating ideals corresponding to each element of M. The global version of this idea is exactly the concept of *annihilator ideal of* M, i.e. $\mathfrak{a}_M := \{r \in R \mid rM = 0\}$. A module M is then called *faithful* if $\mathfrak{a}_M = 0$.

Now, if we have an R-module M, then we get an ideal of R. This gives us a closed subset of Spec (R) (see Section 1.2). A basic question that then arises is what is the relationship between the module M and the closed set $V(\mathfrak{a}_M) \hookrightarrow \operatorname{Spec}(R)$. The following answers that.

Lemma 23.4.0.1. *Let R be a ring and M be a finitely generated R-module. If $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathfrak{a}_M = \operatorname{Ann}(M)$ be the annihilator ideal, then the following are equivalent:*

1. *$M_\mathfrak{p} \neq 0$.*
2. *$\mathfrak{p} \in V(\mathfrak{a}_M)$.*

Proof. If we can show that $\operatorname{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) = (\mathfrak{a}_M)_\mathfrak{p}$, then we have the following equivalence

$$M_\mathfrak{p} \neq 0 \iff \operatorname{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq R_\mathfrak{p} \iff (\mathfrak{a}_M)_\mathfrak{p} \lneq R_\mathfrak{p} \iff \mathfrak{a}_M \subseteq \mathfrak{p}$$

where last equivalence follows from a modified version of Lemma 23.1.2.3. Hence we reduce to showing that $\operatorname{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) = (\mathfrak{a}_M)_\mathfrak{p}$. It is easy to see that $\operatorname{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p}) \supseteq (\mathfrak{a}_M)_\mathfrak{p}$. Let $r/s \in \operatorname{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p})$. We wish to show that $r/s \in (\mathfrak{a}_M)_\mathfrak{p}$. Since M is finitely generated, therefore let $\{m_1, \dots, m_n\}$ be a generating set of M. We thus reduce to showing that $r/s \cdot m_i/1 = 0$ for each $i = 1, \dots, n$. This is exactly the data provided by the fact that $r/s \in \operatorname{Ann}_{R_\mathfrak{p}}(M_\mathfrak{p})$. □

The above lemma hence gives us a closed subset of Spec (R) attached to each finitely generated R-module M. This has a name.

Definition 23.4.0.2. (**Support of a module**) Let R be a ring and M be a finitely generated R-module. Let $\mathfrak{a}_M$ be the annihilator ideal of M. Then, the support of the module M is defined to be the closed set $\operatorname{Supp}(M) := V(\mathfrak{a}_M) \hookrightarrow \operatorname{Spec}(R)$. By Lemma 23.4.0.1, it is equivalently given by the set of all those points $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $M_\mathfrak{p} \neq 0$.

We then define prime ideals associated to an R-module.

Definition 23.4.0.3. (**Associated prime ideals**) Let R be a noetherian ring and M be an R-module. A prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is said to be associated to M if there exits $m \in M$ such that

$$\mathfrak{p} = \{r \in R \mid rm = 0\}.$$

The subspace of Spec (R) of all prime ideals associated to M is denoted $\operatorname{Ass}(M) \hookrightarrow \operatorname{Spec}(R)$.

One can have the following alternate definition of an associated prime ideal.

Lemma 23.4.0.4. *Let R be a noetherian ring and M be an R-module. Then,*

$$\mathfrak{p} \in \operatorname{Ass}(M) \iff \exists N \leq M \text{ such that } N \cong R/\mathfrak{p}.$$

Proof. L $\Rightarrow$ R is easy, just consider the map $R \to M$ given by $r \mapsto rm$ where $m \in M$ corresponds to $\mathfrak{p}$. Conversely, take any $0 \neq n \in N$. Then $\mathfrak{p} = \{r \in R \mid rn = 0\}$ as if $r \in R$ is such that $rn = 0$ and $n = s + \mathfrak{p}$, then $rn = rs + \mathfrak{p} = \mathfrak{p}$, that is $rs \in \mathfrak{p}$ and since $s \notin \mathfrak{p}$, therefore $r \in \mathfrak{p}$. Conversely, if $r \in \mathfrak{p}$ then for all $n \in N$, $rn = 0$. $\square$

So, for an R-module M, we get two subspaces of $\operatorname{Spec}(R)$, one is the closed subspace called support $\operatorname{Supp}(M)$ and the other is $\operatorname{Ass}(M)$. Support will be used later, but the concept of associated prime ideals of M have a deeper connection with the ring R. They are not unrelated.

Lemma 23.4.0.5. *Let M be an R-module. Then $\operatorname{Ass}(M) \hookrightarrow \operatorname{Supp}(M) \hookrightarrow \operatorname{Spec}(R)$.*

Proof. For $\mathfrak{p} \in \operatorname{Ass}(M)$ let $m \in M$ such that its annihilator is $\mathfrak{p}$. Then, for any $r \in \mathfrak{a}_M$, $rm = 0$ and hence $r \in \mathfrak{p}$. Thus $\mathfrak{p} \in V(\mathfrak{a}_M) = \operatorname{Supp}(M)$. $\square$

We wish to show the following result from which primary decomposition follows.

Theorem 23.4.0.6. *Let R be a noetherian ring and M be a finitely generated R-module. Then there exists an injective map*

$$M \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Ass}(M)} E_{\mathfrak{p}}$$

where for each $\mathfrak{p} \in \operatorname{Ass}(M)$, $E_{\mathfrak{p}}$ is an R-module where $\operatorname{Ass}(E_{\mathfrak{p}}) \hookrightarrow \operatorname{Spec}(R)$ is a singleton given by $\{\mathfrak{p}\}$.

This result clearly tells us that points of $\operatorname{Ass}(M)$ are somewhat special. Let us investigate[3].

Lemma 23.4.0.7. *Let R be a noetherian ring and M be a finite R-module[4]. Then,*

1. *If $N \subseteq M$ is a submodule, then $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M)$.*
2. *If $N \subseteq M$ is a submodule, then $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$.*
3. *If $N \subseteq M$ is a submodule, then $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(N) \cup \operatorname{Ass}(M/N)$.*
4. *For any point $\mathfrak{p} \in \operatorname{Spec}(R)$, we have $\mathfrak{a}_{R/\mathfrak{p}} := \operatorname{Ann}(R/\mathfrak{p}) = \mathfrak{p}$. Thus, $\operatorname{Supp}(R/\mathfrak{p}) = V(\mathfrak{p})$ is an irreducible closed subset of $\operatorname{Spec}(R)$.*
5. *For any point $\mathfrak{p} \in \operatorname{Spec}(R)$, we have $\operatorname{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$. Thus, $\operatorname{Ass}(R/\mathfrak{p})$ is exactly the generic point of $\operatorname{Supp}(R/\mathfrak{p})$.*
6. *For all $\mathfrak{p} \in \operatorname{Spec}(R)$, there exists a maximal submodule $N \subseteq M$ such that $\mathfrak{p} \notin \operatorname{Ass}(N)$.*
7. *For all $\mathfrak{p} \in \operatorname{Ass}(M)$, there exists a maximal submodule $N \subsetneq M$ such that $\mathfrak{p} \notin \operatorname{Ass}(N)$ and none of these maximal submodules are isomorphic to $R/\mathfrak{p}$.*

[3]The following was a personal investigation of the author, who, in the process of overcoming his inexperience as a true researcher, would like to apologize for the mess that is the Lemma 23.4.0.7.

[4]this is just another name for finitely generated R-modules.

Proof. Note that by Lemma 23.3.0.2, M is a Noetherian module.

1. If $\mathfrak{p} \in \operatorname{Ass}(N)$, then for some $n \in N$, $\mathfrak{p} = \{r \in R \mid rn = 0\}$. Result follows as $n \in M$.
2. If $\mathfrak{p} \in \operatorname{Supp}(N)$, then $\mathfrak{p} \supseteq \mathfrak{a}_N$. Result follows as $\mathfrak{a}_N \supseteq \mathfrak{a}_M$.
3. By 1, we need only show $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(N) \cup \operatorname{Ass}(M/N)$. Pick $\mathfrak{p} \in \operatorname{Ass}(M)$. By the Lemma 23.4.0.4 and it's proof, the submodule E of M containing of all elements of M who have annihilator as $\mathfrak{p}$ is isomorphic to $R/\mathfrak{p}$. If $E \cap N =$, then M/N has a submodule isomorphic to $R/\mathfrak{p}$ and hence $\mathfrak{p} \in \operatorname{Ass}(M/N)$. Otherwise if $E \cap N \neq \emptyset$, then pick $x \in E \cap N$. Since $x \in E$, so annihilator of x is $\mathfrak{p}$ and thus $\mathfrak{p} \in \operatorname{Ass}(E \cap N)$. By another use of Lemma 23.4.0.4, there is a submodule $F \subseteq E \cap N$ which is isomorphic to $R/\mathfrak{p}$. It follows that N has a submodule isomorphic to $R/\mathfrak{p}$. By a final use of Lemma 23.4.0.4, we conclude that $\mathfrak{p} \in \operatorname{Ass}(N)$.
4. $\operatorname{Ann}(R/\mathfrak{p}) = \{r \in R \mid r(R/\mathfrak{p}) = \mathfrak{p}\}$. It follows from primality of $\mathfrak{p}$ that $\operatorname{Ann}(R/\mathfrak{p}) = \mathfrak{p}$.
5. As above, this reduces to primality of $\mathfrak{p}$.
6. The set of all submodules N of M satisfying $\operatorname{Ass}(N) \notin p$ has a maximal element as M is a noetherian module.
7. If $\mathfrak{p} \in \operatorname{Ass}(M)$, then the maximal N obtained from 5 cannot be M. The other fact follows from 4.

□

With the above investigation, we are now ready to prove Theorem 23.4.0.6.

Proof of Theorem 23.4.0.6. **TODO**. □

The primary decomposition now is a corollary.

Corollary 23.4.0.8. *(Primary decomposition theorem[5])*

Complete this fr
Local Algebra, C
ter 23.

[5] for finitely generated modules over a Noetherian ring.

23.5 Tensor, symmetric & exterior algebras

23.5.1 Results on tensor products

We collect some important results on tensor products in this section which are used all over the text. The following results are immediate corollaries of definition of tensor product, but are of immense use in general.

Proposition 23.5.1.1. *Following are some basic properties of tensor products.*

1. *Tensor product is associative and commutative upto isomorphism.*
2. *If $\{M_\lambda\}$ is a family of R-modules and N is an R-module, then*
$$\left(\bigoplus_\lambda M_\lambda\right) \otimes_R N \cong \bigoplus_\lambda M_\lambda \otimes_R N.$$
3. *Let $\varphi : R \to S$ be a ring homomorphism and M, N be two R-modules. Then the scalar extended modules $M \otimes_R S$ and $N \otimes_R S$ satisfy the following*
$$(M \otimes_R S) \otimes_S (N \otimes_R S) \cong (M \otimes_R N) \otimes_R S.$$
4. *Let R be a ring and M be an R-module. If $I, J \leq R$ are two ideals, then*
$$R/I \otimes_R R/J \cong R/I + J$$
as rings.
5. *If R, S are two rings, then*
$$R \otimes_S S[x] \cong R[x]$$
as rings.

Proof. **TODO.** □

The following is a helpful lemma showing that tensor product commutes with direct limits in all positions.

Lemma 23.5.1.2. *Let M_i, N_i bet R_i-modules where I is directed set and $\{M_i\}, \{N_i\}$ and $\{R_i\}$ are directed systems of modules and rings. Let $M := \varinjlim_{i\in I} M_i$, $N := \varinjlim_{i\in I} N_i$ and $R := \varinjlim_{i\in I} R_i$. Then,*
$$\varinjlim_{i\in I}(M_i \otimes_{R_i} N_i) \cong M \otimes_R N$$
as R-modules.

Proof. We will construct R-linear maps $f : \varinjlim_{i\in I}(M_i \otimes_{R_i} N_i) \longleftrightarrow M \otimes_R N : g$ which will be inverses to each other. We first construct f as follows. For each $i \in I$, we have
$$f_i : M_i \otimes_{R_i} N_i \to M \otimes_{R_i} N \to M \otimes_R N$$

given by $(m_i \otimes n_i) \mapsto ((m_i) \otimes (n_i)) \mapsto ((m_i) \otimes (n_i))$. Note that M, N are R_i-modules canonically. By universal property of $\varinjlim_{i \in I}$, we obtain f as above. To construct g, we need only construct an R-bilinear map

$$\begin{aligned} M \times N &\longrightarrow \varinjlim_{i \in I}(M_i \otimes_{R_i} N_i) \\ ((m_i)_{i\in I}, (n_i)_{i \in I}) &\longmapsto ((m_i \otimes n_i)_{i\in I}). \end{aligned}$$

This can be said to be R-bilinear, thus yielding a map g as required. It is straightforward to see they are inverses to each other. $\square$

The following says that localization commutes with tensor products.

Lemma 23.5.1.3. *Let M, N be two R-modules and $S \subseteq R$ be a multiplicative set. Then,*

$$S^{-1}(M \otimes_R N) \cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N.$$

Proof. We may write by Lemma 23.1.2.1 the following

$$\begin{aligned} S^{-1}M \otimes_{S^{-1}R} S^{-1}N &\cong (M \otimes_R S^{-1}R) \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \\ &\cong M \otimes_R (S^{-1}R \otimes_{S^{-1}R} (S^{-1}R \otimes_R N)) \\ &\cong M \otimes_R (N \otimes_R S^{-1}R) \\ &\cong (M \otimes_R N) \otimes_R S^{-1}R \\ &\cong S^{-1}(M \otimes_R N). \end{aligned}$$

This completes the proof. $\square$

Next, we discuss the notion of fiber of a map of rings. This is easily understood in the scheme language.

Definition 23.5.1.4 (Fiber at a prime ideal). Let $\varphi : R \to S$ be a ring homomorphism and let $\mathfrak{p} \lneq R$ be a prime ideal. Then the fiber of φ at $\mathfrak{p}$ is defined to be $S \otimes_R \kappa(\mathfrak{p})$.

One of the fundamental observation about fiber at a prime ideal is that it is indeed the fiber of the corresponding map of schemes (see Proposition 1.6.5.1), so that the notation makes sense.

23.5.2 Determinants

Fix a commutative ring R with unity for the remainder of this section. We shall show in this section that there exists a unique determinant map over $M_n(R)$. This will motivate further notions discussed in later sections.

We begin by defining a multilinear map over $M_n(R)$.

Definition 23.5.2.1. (**Multilinear map over** $M_n(R)$) Let $n \in \mathbb{N}$ and consider $M_n(R)$. An n-linear map over $M_n(R)$ is a function

$$D : M_n(R) \longrightarrow R$$

which is linear in each row. That is, if A_i denotes the i^{th}-row of matrix A and $c \in R$, then for each $i = 1, \dots, n$, we have

$$\begin{aligned} D(A_1, \dots, A_{i-1}, cA_i + B_i, A_{i+1}, \dots, A_n) = cD(A_1, \dots, A_{i-1}, A_i, A_{i+1}, \dots, A_n) \\ + D(A_1, \dots, A_{i-1}, B_i, A_{i+1}, \dots, A_n). \end{aligned}$$

We may abbreviate the above by simply writing $D(cA_i + B_i) = cD(A_i) = D(B_i)$.

Example 23.5.2.2. The map

$$\begin{aligned} D : M_n(R) &\longrightarrow R \\ A &\longmapsto cA_{1k_1}A_{2k_2} \dots A_{nk_n} \end{aligned}$$

is an n-linear map where $c \in R$ is a constant and $1 \leq k_i \leq n$ are n integers.

We first see that linear combination of n-linear maps is again n-linear.

Lemma 23.5.2.3. *Let $D_1, \dots, D_r$ be n-linear maps and $c_1, \dots, c_r \in R$. Then $c_1D_1 + \dots + c_rD_r$ is an n-linear map.*

Proof. By induction, we may assume $r = 2$. Now this is a straightforward check. □

We now come more closer to determinants by defining the following type of n-linear maps.

Definition 23.5.2.4. (**Alternating & determinant maps**) An n-linear map $D : M_n(R) \to R$ is said to be alternating if
1. $D(A) = 0$ if $A_i = A_j$ for any $i \neq j$,
2. $D(\sigma_{ij}(A)) = -D(A)$ where σ_{ij} swaps rows A_i and A_j.

An alternating n-linear map $D : M_n(R) \to R$ is said to be determinant if $D(I_n) = 1$.

Proposition 23.5.2.5. *If $D : M_n(R) \to R$ is an n-linear map such that $D(A) = 0$ whenever $A_i = A_{i+1}$ for some $1 \leq i \leq n$, then D is alternating.*

Proof. Let $A \in M_n(R)$ and $1 \leq i \neq j \leq n$ be such that $A_i = A_j$. We first wish to show that $D(\sigma_{ij}(A)) = -D(A)$. We may assume $j > i$. We go by strong induction over $j - i$. We first show this for $j = i + 1$. Indeed, we then have $D(\sigma_{i,i+1}(A)) = D(A_{i+1}, A_i)$. Writing $0 = D(A_{i+1} + A_i, A_i + A_{i+1}) = D(A_{i+1}, A_i) + D(A_i, A_{i+1})$. Thus we get $D(A_{i+1}, A_i) = -D(A_i, A_{i+1})$.

In the inductive case, suppose $D(\sigma_{ij}(A)) = -D(A)$ for all $j - i \leq k$. We wish to show that if $j - i = k + 1$, then the same holds. As $\sigma_{i,i+k+1}(A) = \sigma_{i+k,i+k+1} \circ \sigma_{i,i+k} \circ \sigma_{i+k,i+k+1}(A)$, therefore we are done.

To get that $D(A) = 0$ for A such that $A_i = A_j$ for some $j > i$, we may simply swap rows till they are adjacent, which will be zero by our hypothesis. □

We now define the main candidate for the determinant function over $M_n(R)$.

Definition 23.5.2.6. (E_j) Let $D : M_{n-1}(R) \to R$ be an $n-1$-linear map. For each $1 \le j \le n$, define the following map

$$\begin{aligned} E_j : M_n(R) &\longrightarrow R \\ A &\longmapsto \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D(A[i|j]). \end{aligned}$$

Further denote $D_{ij}(A) := D(A[i|j])$.

Theorem 23.5.2.7. *Let $n \in \mathbb{N}$ and $D : M_{n-1}(R) \to R$ be an alternating $n-1$-linear map. For each $1 \le j \le n$, the map $E_j : M_n(R) \to R$ defined as above is an alternating n-linear map. If moreover D is a determinant map, then so is each E_j.*

Proof. Fix $1 \le j \le n$. We first wish to show that E_j is n-linear. As $D_{ij} : M_n(R) \to R$ is linear in every row except i. Thus $A \mapsto A_{ij} D_i j(A)$ is n-linear. It follows from Lemma 23.5.2.3 that E_j is n-linear.

To show that E_j is alternating, it would suffice from Proposition 23.5.2.5 to show that $E_j(A) = 0$ if A has any two adjacent rows equal, say $A_k = A_{k+1}$. This one checks directly by the definition of E_j.

To see that E_j is determinant if D is determinant is also easy to see. □

We now show the uniqueness of determinants and alternating n-linear maps (upto the value on I_n).

Theorem 23.5.2.8. *Let $D : M_n(R) \to R$ be an alternating n-linear map over $M_n(R)$. Then,*

1. *D is given explicitly on $A \in M_n(R)$ by*

$$D(A) = \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \dots A_{n\sigma(n)} \right) D(I),$$

   hence D is unique upto its value over I,
2. *if D is determinant map, then it is uniquely given by*

$$D(A) = \det A := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \dots A_{n\sigma(n)},$$

3. *any alternating map D on $M_n(R)$ is thus uniquely determined by its value on I as*

$$D(A) = (\det A) \cdot D(I).$$

Proof. The proof is straightforward but tedious. **TODO**. □

Corollary 23.5.2.9. *Let $n \in \mathbb{N}$.*

1. *If $A, B \in M_n(R)$, then $\det(AB) = \det(A) \cdot \det(B)$.*
2. *If $B \in M_n(R)$ is obtained by $B_j = A_j + cA_i$ for some fixed $1 \le i, j \le n$ and rest of the rows of B are identical to A, then $\det(B) = \det(A)$.*

3. *If* $M \in M_{r+s}(R)$ *is given by*

$$M = \begin{bmatrix} A_{r\times r} & B_{r\times s} \\ 0 & C_{s\times s} \end{bmatrix}$$

then $\det(M) = \det(A) \cdot \det(C)$.

4. *For each* $1 \leq j \leq n$*, we have*

$$\det(A) = E_j(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(A[i|j]).$$

Proof. (*Sketch*) For 1. we can contemplate

$$\begin{aligned} D : M_n(R) &\longrightarrow R \\ A &\mapsto \det(AB). \end{aligned}$$

One claims that D is an n-linear alternating map. Then apply Theorem 23.5.2.8, 3.

2. Follows by multilinearity of det.

3. As elementary row operations only change determinant upto sign and restricting an $r+s$-linear alternating map to first r or last s entries keeps it r-linear and s-linear alternating respectively, therefore the result follows.

4. Follows from Theorem 23.5.2.7 and Theorem 23.5.2.8. □

Construction 23.5.2.10. (*Adjoint of a matrix*) Let $A \in M_n(R)$ be a square matrix. By Corollary 23.5.2.9, the sum $E_j(A) = \det(A)$ for each $1 \leq j \leq n$

$$\det(A) = \sum_{i=1}^{n} A_{ij} (-1)^{i+j} \det(A[i|j]).$$

Hence, let us define $C_{ij} := (-1)^{i+j} \det(A[i|j])$ as the ij^{th}*-cofactor of* A. Consequently, we get a matrix $(\mathrm{Adj}A)_{ij} = C_{ji}$, called the *adjoint matrix*. Hence, we may rewrite the determinant as

$$\begin{aligned} \det(A) &= \sum_{i=1}^{n} A_{ij} C_{ij} \\ &= \sum_{i=1}^{n} (\mathrm{Adj}A)_{ji} A_{ij}. \end{aligned}$$

Thus,

$$\det(A) I = \mathrm{Adj}(A) \cdot A.$$

This also allows us to write that in the case when A is invertible, we have

$$A^{-1} = \frac{1}{\det A} \mathrm{Adj}(A).$$

As similar matrices have same determinant, therefore each linear operator on a finite dimensional vector space has a unique determinant. Thus determinants are invariants of linear operators upto similarity.

23.5.3 Multilinear maps

We now put the previous discussion in a more abstract framework where we work with modules over a commutative ring with 1. We first recall that the rank of a finitely generated module is the size of the smallest generating set. Further recall that a finitely generated free R-module V has a well-defined rank and the smallest generating set is moreover a basis of V (i.e. linearly independent set of generators).

For this section, we would hence fix a commutative ring R with 1.

Definition 23.5.3.1. (**r-linear forms over a module**) Let V be an R-module. An r-linear form L over V is a function

$$L : V^r = V \times \cdots \times V \longrightarrow R$$

such that for any $c \in R$, $\beta_i \in V$ and $(\alpha_1, \ldots, \alpha_r) \in V^r$, we have

$$L(\alpha_1, \ldots, c\alpha_i + \beta_i, \ldots, \alpha_n) = cL(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) + L(\alpha_1, \ldots, \beta_i, \ldots, \alpha_n)$$

for any $1 \leq i \leq r$. An r-linear form is usually also called an r-tensor. A 2-linear form/tensor is also usually called a bilinear form. Note that an r-linear form may not be linear. Denote the R-module of all r-linear forms by $M^r(V)$.

Remark 23.5.3.2. Let $f_1, \ldots, f_r \in V^* = \operatorname{Hom}_R(V, R) = M^1(V)$ be a collection of linear functionals. We then obtain $L \in M^r(V)$ given by

$$L(\alpha_1, \ldots, \alpha_r) = f_1(\alpha_1) \cdot \cdots \cdot f_r(\alpha_r).$$

Example 23.5.3.3. We give some examples.

1. Let $V = R^n$ be a free R-module of rank n. Then for a fixed matrix $A \in M_n(R)$, the map

$$\begin{aligned} V \times V &\longrightarrow R \\ (x, y) &\longmapsto x^t A y \end{aligned}$$

is a bilinear form over V.

2. Let $V = R^n$ be a free R-module of rank n. Then we obtain the following n-linear form

$$\begin{aligned} \det : V^n &\longrightarrow R \\ (\alpha_1, \ldots, \alpha_n) &\longmapsto \det(A) \end{aligned}$$

where $A \in M_n(R)$ whose i^{th}-row is α_i. Hence, determinant is an n-tensor/n-linear form over V.

Remark 23.5.3.4. (General expression of an r-linear form) Let $L \in M^r(V)$ be an r-form over an R-module V where V is a free module of rank n. Further denote $e_1, \ldots, e_n$ be a basis of V. For any $(\alpha_1, \ldots, \alpha_r) \in V^r$, we may write $\alpha_i = \sum_{j=1}^n A_{ij} e_j$. Hence we have $A \in M_{r\times n}(R)$. This yields by n-linearity of L that

$$\begin{aligned} L(\alpha_1, \ldots, \alpha_r) &= \sum_{j_r=1}^n \cdots \sum_{j_1=1}^n A_{1j_1} \ldots A_{rj_r} L(e_{j_1}, \ldots, e_{j_r}) \\ &= \sum_{J=\{j_1,\ldots,j_r\}} A_J L(e_J) \end{aligned}$$

where $J \in X$ where X is the set of all r-tuples with entries in $\{1, \ldots, n\}$. There are therefore n^r terms in the above sum.

Definition 23.5.3.5. (**Tensor product of linear forms**) Let M be an R-module. We then define

$$\begin{aligned} - \otimes - : M^r(V) \times M^s(V) &\longrightarrow M^{r+s}(V) \\ (L, M) &\longmapsto L \otimes M \end{aligned}$$

where $L\otimes M : V^{r+s} \to R$ is given by $(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) \mapsto L(\alpha_1, \ldots, \alpha_r)M(\beta_1, \ldots, \beta_s)$.

Remark 23.5.3.6. We have following observations about tensor of forms:
1. $L \otimes (T + S) = L \otimes T + L \otimes S$,
2. $(L \otimes T) \otimes N = L \otimes (T \otimes N)$,
3. $c(L + T) \otimes S = cL \otimes S + cT \otimes S$,
4. $L \otimes T \neq T \otimes L$.

We now come to an important theorem about $M^r(V)$

Theorem 23.5.3.7. *Let V be a free R-module of rank n and $B = \{e_1, \ldots, e_n\} \subseteq V$ be a basis of V. Let X denote the set of all r-tuples with entries in $\{1, \ldots, n\}$. Then,*
1. *the R-module $M^r(V)$ is free of rank n^r,*
2. *a basis of $M^r(V)$ is given by $f_J = f_{j_1} \otimes \ldots \otimes f_{j_r}$ where $B^* = \{f_1, \ldots, f_n\} \subseteq V^* = M^1(V)$ is the dual basis of B, where $J = \{j_1, \ldots, j_r\}$ varies over all elements of X.*

Proof. (*Sketch*) We claim that $\{f_J\}_{J\subseteq X}$ forms a basis of $M^r(V)$. Pick any $(\alpha_1, \ldots, \alpha_r) \in V^r$, then by Remark 23.5.3.4, we first have $\alpha_i = \sum_{j=1}^n f_j(\alpha_i)e_j$. Consequently,

$$\begin{aligned} L(\alpha_1, \ldots, \alpha_r) &= \sum_{J=\{j_1,\ldots,j_r\}} L(e_{j_1}, \ldots, e_{j_r}) \cdot f_{j_1} \otimes \ldots \otimes f_{j_r}(\alpha_1, \ldots, \alpha_r) \\ &= \sum_{J=\{j_1,\ldots,j_r\}} L(e_J) f_{j_1} \otimes \ldots \otimes f_{j_r}(\alpha_1, \ldots, \alpha_r). \end{aligned}$$

Thus, $\{f_J\}_{J\subseteq X}$ spans $M^r(V)$. For linear independence, take any combination

$$\sum_{J\subseteq X} c_J f_J = 0.$$

On the LHS, apply e_I to get $c_I = 0$ for each $I \subseteq X$. $\square$

Definition 23.5.3.8. (**Alternating r-linear forms**) Let V be an R-module. An r-linear form $L \in M^r(V)$ is said to be alternating if

1. $L(\alpha_1, \ldots, \alpha_r) = 0$ if $\alpha_i = \alpha_j$ for $i \neq j$,
2. $L(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)}) = \operatorname{sgn}(\sigma) L(\alpha_1, \ldots, \alpha_r)$ for all $\sigma \in S_r$.

The collection of all alternating r-linear forms is denoted by $\Lambda^r(V)$ and its a submodule of $M^r(V)$. Note that the second axiom follows from 1, but is important to keep it in mind.

Observe that $\Lambda^1(V) = M^1(V) = V^*$.

Remark 23.5.3.9. Consider $V = R^n$, a free R-module of rank n. We saw earlier that $\det \in M^n(V)$ is an n-linear form.. Theorem 23.5.2.8 shows that det is moreover an unique alternating form with $\det(e_1, \ldots, e_n) = 1$. Thus, $\det \in \Lambda^n(V) \subsetneq M^n(V)$ is the unique alternating n-linear form over V such that $\det(e_1, \ldots, e_n) = 1$, i.e. $\Lambda^n(V)$ is a free R-module of rank 1.

Construction 23.5.3.10. Let V be an R-module. We now construct an R-linear map $\pi_r : M^r(V) \to \Lambda^r(V)$. For each $L \in M^r(V)$, define $L_\sigma \in M^r(V)$ given by $L_\sigma(\alpha_1, \ldots, \alpha_r) = L(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)})$ for $(\alpha_1, \ldots, \alpha_r) \in V^r$. Consequently, we claim that the following map is well-defined:

$$\begin{aligned} \pi_r : M^r(V) &\longrightarrow \Lambda^r(V) \\ L &\longmapsto \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) L_\sigma. \end{aligned}$$

Indeed, we have to show that $\pi_r L$ is an alternating form. Let $(\alpha_1, \ldots, \alpha_r) \in V^r$ be such that $\alpha_i = \alpha_j$ for $i \neq j$. We wish to show that $\pi_r L(\alpha_1, \ldots, \alpha_r) = 0$. Let $\tau = (ij)$ be the transposition swapping i and j. First observe that the map $S_r \to S_r$ given by $\sigma \mapsto \tau\sigma$ is a bijection. Consequently, if we let $\sigma_1, \ldots, \sigma_{\frac{n!}{2}}$ to be any $\frac{n!}{2}$ elements of S_r, then the rest $\frac{n!}{2}$ are given by $\tau\sigma_i$, $i = 1, \ldots, n!/2$. Consequently,

$$\begin{aligned} \pi_r L(\alpha_1, \ldots, \alpha_r) &= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) L(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)}) \\ &= \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) L(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)}) \\ &= \sum_{i=1}^{\frac{n!}{2}} \operatorname{sgn}(\sigma_i) L(\alpha_{\sigma_i(1)}, \ldots, \alpha_{\sigma_i(r)}) + \sum_{i=1}^{\frac{n!}{2}} \operatorname{sgn}(\tau\sigma_i) L(\alpha_{\tau\sigma_i(1)}, \ldots, \alpha_{\tau\sigma_i(r)}) \\ &= \sum_{i=1}^{\frac{n!}{2}} \operatorname{sgn}(\sigma_i) L(\alpha_{\sigma_i(1)}, \ldots, \alpha_{\sigma_i(r)}) + \sum_{i=1}^{\frac{n!}{2}} -\operatorname{sgn}(\sigma_i) L(\alpha_{\sigma_i(1)}, \ldots, \alpha_{\sigma_i(r)}) \\ &= 0. \end{aligned}$$

Hence, π_r is indeed an R-linear map from $M^r(V)$ into $\Lambda^r(V)$.

Finally note that if $L \in \Lambda^r(V)$, then $\pi_r L = r!L$ as $L_\sigma = \operatorname{sgn}(\sigma) L$ for any $\sigma \in S_r$.

Example 23.5.3.11. Let $V = R^n$ be the free R-module of rank n. Let $e_1, \ldots, e_n \in V$ be the standard R-basis of V. Further, let $f_1, \ldots, f_n \in M^1(V)$ be the associated dual basis. Note that for any $\alpha \in V$, we have $\alpha = f_1(\alpha)e_1 + \ldots f_n(\alpha)e_n$. Then, we get an n-form

$$L = f_1 \otimes \ldots \otimes f_n \in M^n(V).$$

Consequently we obtain an alternating n-form given by

$$\pi_r L = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}).$$

Observe that for any $(\alpha_1, \ldots, \alpha_n) \in V^n$, we obtain

$$\begin{aligned}\pi_r L(\alpha_1, \ldots, \alpha_n) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}\right)(\alpha_1, \ldots, \alpha_n) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(f_{\sigma(1)}(\alpha_1) \cdot \cdots \cdot f_{\sigma(n)}(\alpha_n)\right).\end{aligned}$$

This is exactly the determinant of the $n \times n$ matrix over R given by $A = (f_j(\alpha_i))$. That is, $\pi_r L = \det$.

The following properties of π_r will become important later on.

Proposition 23.5.3.12. *Let V be an R-module and $L \in M^r(V)$ and $M \in M^s(V)$ be r and s-forms over V respectively. Then,*

$$\pi_{r+s}(\pi_r(L) \otimes \pi_s(M)) = r!s!\pi_{r+s}(L \otimes M).$$

Proof. ***TODO*** *: Magnum tedium.* □

The above has a very nice and useful corollary.

Corollary 23.5.3.13. *Let V be a free R-module of rank n with $f_1, \ldots, f_n \in V^*$ be a dual basis of V^*. Let $I \in X_r$ and $J \in X_s$ where X_r and X_s are the sets of r and s combinations of $\{1, \ldots, n\}$, respectively, such that I and J are disjoint ($i_k \neq j_l$ for any $1 \leq k \leq r$, $1 \leq l \leq s$). Denote $D_I = \pi_r(f_I)$ and $D_J = \pi_s(f_J)$ where $f_I = f_{i_1} \otimes \ldots \otimes f_{i_r} \in M^r(V)$ and $f_J = f_{j_1} \otimes \ldots \otimes f_{j_s} \in M^s(V)$. Then,*

$$\pi_{r+s}(D_I \otimes D_J) = r!s!D_{I \amalg J}.$$

Proof. Follows immediately from Proposition 23.5.3.12 □

We now come to the main result about alternating forms.

Theorem 23.5.3.14. *Let V be a free module of rank n over R.*

1. *If $r > n$, then $\Lambda^r(V) = 0$.*
2. *If $0 \leq r \leq n$, then rank of $\Lambda^r(V)$ is ${}^nC_r$.*

Proof. (*Sketch*) Using Remark 23.5.3.4, statement 1 is straightforward. For 2, observe that we can write for $(\alpha_1, \dots, \alpha_r) \in V^r$, $r \leq n$ as follows

$$L(\alpha_1, \dots, \alpha_r) = \sum_{J=\{j_1,\dots,j_r\}\in X} L(e_J)(f_{j_1} \otimes \dots \otimes f_{j_r})(\alpha_1, \dots, \alpha_r)$$

where X is the set of all r-permutations of $\{1, \dots, n\}$ (as for any repeatitions, the corresponding term is 0). Now, partitioning the set X into classes in which permutations represent the same combination, we obtain an indexing set $\hat{X}$ of size ${}^nC_r$. Again, by the fact that L is alternating, we observe $\text{sgn}(\sigma)L(e_{j_1}, \dots, e_{j_r}) = L(e_{j_{\sigma(1)}}, \dots, e_{j_{\sigma(r)}})$. Consequntly we may write the above sum as

$$L(\alpha_1, \dots, \alpha_r) = \sum_{J=\{j_1,\dots,j_r\}\in \hat{X}} L(e_{j_1}, \dots, e_{j_r}) \sum_{\sigma\in S_r} \text{sgn}(\sigma) \left(f_{j_{\sigma(1)}} \otimes \dots \otimes f_{j_{\sigma(r)}}\right)(\alpha_1, \dots, \alpha_r).$$

Therefore denote for each $J \in \hat{X}$ the following

$$D_J = \sum_{\sigma\in S_r} \text{sgn}(\sigma) \left(f_{j_{\sigma(1)}} \otimes \dots \otimes f_{j_{\sigma(r)}}\right).$$

One can observe that the D_J for each $J \in \hat{X}$ can alternatively be written as

$$D_J = \pi_r(f_{j_1} \otimes \dots \otimes f_{j_r}).$$

The above shows that D_J is in $\Lambda^r(V)$ and that it spans $\Lambda_r(V)$. The claim now is that these are also linearly independent. Indeed, that follows immediately by using the fact that f_js are dual basis of e_js. $\square$

We can now abstractly obtain the determinant of a linear operator $T : V \to V$ on a free R-module V of rank n.

Corollary 23.5.3.15. *Let V be a free R-module of rank n and $T : V \to V$ be an R-linear operator. Then,*

1. *rank of $\Lambda^n(V) = 1$,*
2. *there exists a unique $c_T \in R$ such that for all $L \in \Lambda^n(V)$,*

$$L \circ T = c_T L.$$

This c_T is defined to be the determinant of the operator T.

Proof. Statement 1 follows from Theorem 23.5.3.14. For statement 2, one need only observe that $L \circ T$ is again an alternating n-tensor and then use statement 1. $\square$

23.5.4 Exterior algebra over characteristic 0 fields

Let us first make the exterior algebra over characteristic 0 fields, before moving to arbitrary ring.

Definition 23.5.4.1. (**Wedge product**) Let K be a field of characteristic 0 and V be an R-vector space. For any $r, s \in \mathbb{N}$, define

$$\begin{aligned} \Lambda^r(V) \times \Lambda^s(V) &\longrightarrow \Lambda^{r+s}(V) \\ (L, M) &\longmapsto L \wedge M := \frac{1}{r!s!}\pi_{r+s}(L \otimes M). \end{aligned}$$

Observe that $D_I \wedge D_J = \frac{1}{r!s!}\pi_{r+s}(\pi_r(f_I) \otimes \pi_s(f_J)) = \frac{r!s!}{r!s!}\pi_{r+s}(f_I \otimes f_J)$ and the latter is either 0 if I and J have a common index or $D_{I \amalg J}$ if they are distinct. This follows from Proposition 23.5.3.12.

In the following result, we see that wedge product is a anti-commutative, distributive and associative operation.

Proposition 23.5.4.2. *Let V be a K-vector space over a field K of characteristic 0.*

1. *Let $\omega, \eta \in \Lambda^k(V), \phi \in \Lambda^l(V)$. Then, wedge product is distributive as*

$$(\omega + \eta) \wedge \phi = \omega \wedge \phi + \eta \wedge \phi,$$

2. *Let $\omega \in \Lambda^k(V), \eta \in \Lambda^l(V)$. Then, wedge product is anti-commutative as*

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega,$$

3. *Let $\omega \in \Lambda^k(V), \eta \in \Lambda^l(V), \phi \in \Lambda^m(V)$. Then, wedge product is associative as*

$$(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi).$$

Proof. We need only check these identities on the basis elements $\{D_I\}$ of each $\Lambda^r(V)$.

1. Let $\omega = D_I, \eta = D_J$ and $\varphi = D_M$. Then,

$$\begin{aligned} (D_I + D_J) \wedge D_M &= \pi_{k+l}((D_I + D_J) \otimes D_M) = \pi_{k+l}(D_I \otimes D_M + D_J \otimes D_M) \\ &= \pi_{k+l}(D_I \otimes D_M) + \pi_{k+l}(D_J \otimes D_M) = D_I \wedge D_M + D_J \wedge D_M \end{aligned}$$

as required.

2. **TODO**. □

Using above, we come to the following definition.

Definition 23.5.4.3. (**Exterior algebra**) Let V be a K-vector space where K is a field of characteristic 0. Then the exterior algebra over V is

$$\begin{aligned} \Lambda(V) &= K \oplus \Lambda^1(V) \oplus \Lambda^2(V) \dots \\ &= K \oplus \bigoplus_{k \geq 1} \Lambda^k(V) \end{aligned}$$

where the product is given by wedge product which by Proposition 23.5.4.2 is associative, unital, distributive but non-commutative. This is also sometimes called the Grassmann algebra over V.

Remark 23.5.4.4. Observe that if V is of dimension n, then

$$\Lambda(V) = K \oplus \bigoplus_{k=1}^{n} \Lambda^k(V)$$

as all the higher forms are automatically 0. Consequently, the dimension of $\Lambda(V)$ by Theorem 23.5.3.14 is seen to be

$$\begin{aligned}\dim_K \Lambda(V) &= 1 + \sum_{k=1}^{n} {}^nC_k \\ &= \sum_{k=0}^{n} {}^nC_k \\ &= 2^n.\end{aligned}$$

Remark 23.5.4.5. Let V be a K-vector space of dimension n, where K is of characteristic 0. The exterior algebra $\Lambda(V)$ is a graded K-algebra of dimension 2^n over K. Indeed, the grading is correct as if $\omega \in \Lambda^k(V), \eta \in \Lambda^l(V)$, then $\omega \wedge \eta \in \Lambda^{k+l}(V)$.

23.5.5 Tensor, symmetric & exterior algebras

We now define the three algebras TM, SM and $\wedge M$ associated to a module M over R

Definition 23.5.5.1 (TM, SM **and** $\wedge M$)**.** Let R be a ring and M be an R-module.

1. The tensor algebra over M is defined to be
$$TM = \bigoplus_{n\geq 0} T^n M$$
where $T^nM = M\otimes \ldots \otimes M$ n-times and $T^0M = R$. This is a non-commutative graded R-algebra where the multiplication is given by tensor product.
2. The symmetric algebra over M is defined to be the quotient
$$SM = TM/I = \bigoplus_{n\geq 0} S^n M$$
where I is the two-sided graded ideal of TM given by
$$I = \langle x \otimes y - y \otimes x | x, y \in M\rangle.$$
This is a commutative graded R-algebra where S^nM denotes $T^nM/I \cap T^nM$.
3. The exterior algebra over M is defined to be the quotient
$$\wedge M = TM/J = \bigoplus_{n\geq 0} \wedge^n M$$
where J is the two-sided graded ideal of TM given by
$$J = \langle x \otimes x \mid x \in M\rangle.$$
This is a skew-commutative[6] graded R-algebra where $\wedge^n M$ denotes $T^nM/J \cap T^nM$.

We now give a canonical basis for each of them in the case when M is a free R-module of rank n. **TODO.**

[6] as J contains $x \otimes y + y \otimes x$ by opening $(x + y) \otimes (x + y) \in J$.

23.6 Field theory

23.6.1 Finite and algebraic extensions and compositum

Recall that a field extension K/F is said to be *finite* if K/F is a finite dimensional F-vector space and then we denote $[K : F] := \dim_F K$. It is said to be *algebraic* if for every $\alpha \in K$, there exists $p(x) \in F[x]$ such that $p(\alpha) = 0$, that is, the inclusion $F \hookrightarrow K$ is integral. Let $I = \{p(x) \in F[x] \mid p(\alpha) = 0\} \leq F[x]$ be an ideal. The generating element $m_{\alpha,F}(x)$ of I is called the *minimal polynomial* of $\alpha \in K$. Note that this is irreducible as I is a prime ideal as it is kernel of a map.

The main basic result connecting algebraic and finite extensions is that finitely generated algebraic extensions are equivalent to finite extensions. This is immediate from Proposition 23.7.1.8, but we give an elementary proof. We first begin by elementary observations.

Theorem 23.6.1.1. *Let K/F be a field extension and $\alpha \in K$.*

1. *If K/F is finite, then it is algebraic.*
2. *If $K/L/F$ are extensions, then*
$$[K : F] = [K : L] \cdot [L : F]$$
where $[K : L]$ or $[L : F]$ is infinity if and only if $[K : F]$ is infinity.
3. *If $\alpha_1, \dots, \alpha_n$ are algebraic over F, then $F(\alpha_1, \dots, \alpha_n) = F[\alpha_1, \dots, \alpha_n]$.*
4. *We have $[F(\alpha) : F] = \deg m_{\alpha,F}$.*
5. *The extension $F(\alpha_1, \dots, \alpha_n)/F$ is algebraic if and only if $\alpha_1, \dots, \alpha_n$ are algebraic over F.*
6. *K/F is a finite-type algebraic extension if and only if K/F is finite.*
7. *If K/L and L/F are both algebraic, then K/F is algebraic.*
8. *The set of all algebraic elements in K over F forms a subfield of K containing F denoted $K^{\mathrm{alg}/F}$.*

Proof. 1. Pick any element $x \in K$ and consider $\{1, x, x^2, \dots\}$. Finiteness of K/F makes sure that there is a finite subset of above which is linearly depenedent.

2. Take bases of K/L and L/F and consider their pairwise product. One sees that this new collection is linearly independent and its F-span is K.

3. As $F[\alpha]$ is a field as it is isomorphic to $F[x]/\langle m_{\alpha,F}(x)\rangle$ and $m_{\alpha,F}(x)$ is irreducible. By universal property of quotients, we get $F[\alpha] = F(\alpha)$. By induction, we wish to show that $F(\alpha_1, \dots, \alpha_{n-1})[\alpha_n] = F(\alpha_1, \dots, \alpha_{n-1})(\alpha_n) = F(\alpha_1, \dots, \alpha_{n-1}, \alpha_n)$, which completes the proof.

4. We have $F(\alpha) = F[\alpha] = \frac{F[x]}{m_{\alpha,F}(x)}$ and this is of dimension $\deg m_{\alpha,F}(x)$ over F.

5. Forward is immediate. For converse, proceed by induction. Clearly, $F(\alpha_1)/F$ is algebraic as it is finite. Composition of finite is finite, so $F(\alpha_1, \dots, \alpha_n)/F$ is finite, thus algebraic.

6. Forward is the only non-trivial side. Let $K = F(\alpha_1, \dots, \alpha_n)$ and by algebraicity, α_i are algebraic. Now $F(\alpha_1)/F$ is finite as algebraic. By induction, we get the result.

7. Pick $\alpha \in K$ and consider $m_{\alpha,L}(x) \in L[x]$ as $m_{\alpha,L}(x) = x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0$, $c_i \in L$. Then, consider $F(c_0, \dots, c_{n-1}) \subseteq L$. As L/F is algebraic, thus $c_i \in L$ are algebraic and thus by previous, we get $F(c_0, \dots, c_{n-1})/F$ is algebraic and finite. As

$F(c_0, \ldots, c_{n-1})(\alpha)/F(c_0, \ldots, c_{n-1})$ is algebraic as it is finite, thus $F(c_0, \ldots, c_{n-1}, \alpha)/F$ is algebraic as it is composite of two finite extensions.

8. Indeed, pick any two algebraic elements $\alpha, \beta \in K$ over F. Then $F(\alpha, \beta)$ is an algebraic extension over F and thus $F(\alpha, \beta) \subseteq K^{\mathrm{alg}/F}$. □

Next, we define compositum, the smallest field containing two subfields.

Definition 23.6.1.2 (**Compositum of fields**)**.** Let F, K be two fields in a field L. Then compositum of F and K in L is the smallest field in L containing both F and K. This is denoted by $F \cdot K$.

The following are the main results for compositum. We will see more later when needed.

Theorem 23.6.1.3. *Let K/F be a field extension and $K_1, K_2 \subseteq K$ be two subfields containing F. Then,*

1. *If $K_1 = F(\alpha_1, \ldots, \alpha_n)$ and $K_2 = F(\beta_1, \ldots, \beta_m)$, then $K_1 \cdot K_2 = F(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$.*
2. *If K_1/F and K_2/F are algebraic, then $K_1 \cdot K_2/F$ is algebraic.*
3. *If K_1/F and K_2/F are finite, then $K_1 \cdot K_2/F$ is finite.*
4. *If $[K_1 : F]$ and $[K_2 : F]$ are coprime, then $[K_1 \cdot K_2 : F] = [K_1 : F] \cdot [K_2 : F]$.*
5. *We have $[K_1 \cdot K_2 : F] \leq [K_1 : F] \cdot [K_2 : F]$.*

Proof. 1. It is clear that $K_1 \cdot K_2 \supseteq F(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ since $K_1 \cdot K_2$ contains both K_1, K_2 and F. For the converse, as $K_1 \cdot K_2$ is the smallest field containing both K_1 and K_2 therefore $K_1 \cdot K_2 \subseteq F(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$.

2. Let L be the algebraic closure of F in $K_1 \cdot K_2$. By hypothesis, $L \supseteq K_1, K_2$. Thus $L \supseteq K_1 \cdot K_2$.

3. By Theorem 23.6.1.1, 6, $K_1 = F(\alpha_1, \ldots, \alpha_n)$ and $K_2 = F(\beta_1, \ldots, \beta_m)$ where α_i, β_j are algebraic elements over F. By item 1, $K_1 \cdot K_2 = F(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ is a finitely generated algebraic extension, thus finite, as required.

4. Since we have

$$\begin{aligned}
[K_1 \cdot K_2 : F] &= [K_1 \cdot K_2 : K_1][K_1 : F] \\
&= [K_1 \cdot K_2 : K_2][K_2 : F].
\end{aligned}$$

By hypothesis, $[K_1 \cdot K_2 : F]$ is a multiple of $[K_1 : F] \cdot [K_2 : F]$. Thus we redude to showing $[K_1 \cdot K_2 : F] \leq [K_1 : F] \cdot [K_2 : F]$. Note by above equations, it suffices to show that

$$[K_1 \cdot K_2 : K_1] \leq [K_2 : F].$$

To this end, let $\alpha_1, \ldots, \alpha_n \in K_2$ be an F-basis of K_2. It thus suffices to show that K_1-span of $\alpha_1, \ldots, \alpha_n$ is whole of $K_1 \cdot K_2$, that is, we wish to show

$$L := K_1 \cdot \alpha_1 + \cdots + K_1 \cdot \alpha_n = K_1 \cdot K_2.$$

Note that it suffices to show that L is a field containing both K_1 and K_2. Indeed, the fact that L contains K_2 is immediate as L contains F and $\alpha_1, \ldots, \alpha_n$. Further L contains K_1 as L contains 1 since L contains K_2 and that it is a K_1-vector space. Thus, $L \supseteq K_1, K_2$. We thus reduce to showing that L is a field.

To this end, observe that if $l \in L$, then $l = c_1\alpha_1 + \cdots + c_n\alpha_n$ for $c_i \in K_1$. Now, $l \in K_2(c_1, \ldots, c_n)$. Thus $l^{-1} \in K_2(c_1, \ldots, c_n) = K_2[c_1, \ldots, c_n]$, that is, l^{-1} is a polynomial in c_i with coefficients in K_2. But any element of K_2 is an F-linear combination of $\alpha_1, \ldots, \alpha_n$. As $K_1 \supseteq F$, therefore l^{-1} is a linear combination of $\alpha_1, \ldots, \alpha_n$ with coefficients in K_1 (powers of c_i multiplied by elements of F, so in K_1). Thus, $l^{-1} \in L$, as needed. The fact that L is multiplicatively closed is immediate. This completes the proof.

5. Follows from proof of item 4 above. $\square$

We now see that a finite algebra over a domain which is a domain induces a finite extension of fraction fields.

Lemma 23.6.1.4. *Let $B \hookrightarrow A$ be a finite B-algebra where both A, B are domains. Then $Q(A)$ is a finite extension of $Q(B)$.*

Proof. Let $\alpha_1, \ldots, \alpha_n \in A$ be a generating set of A as a B-module and let $\varphi : B \hookrightarrow A$ be the structure map of the finite B-algebra structure on A. Now let $S = B - \{0\}$. Now we get a map $S^{-1}\varphi : Q(B) \hookrightarrow S^{-1}A$. This is a finite map since $S^{-1}A$ as the $Q(B)$ span of $\alpha_1, \ldots, \alpha_n$ in $S^{-1}A$ is $S^{-1}A$. To complete the proof, we need only show that the natural inclusion $S^{-1}A \hookrightarrow Q(A)$ given by $\frac{a}{b} \mapsto \frac{a}{b}$ is a finite map. We see something stronger: $Q(A) = S^{-1}A$. Indeed, this is true because $S^{-1}A$ is a field containing A as $S^{-1}A$ is a domain which is finite over the field $Q(B)$, so that by Lemma 23.7.1.13, we get that $S^{-1}(A)$ is a field. As it contains A, so it also contains $Q(A)$. This completes the proof. $\square$

23.6.2 Maps of field extensions

There are some important results which allow us to extend a field homomorphism from a smaller field to a bigger field. These come in handy while discussing splitting fields and algebraic closures.

Proposition 23.6.2.1 (Extension-I)**.** *Let $\varphi : F \to F'$ be a field isomorphism. Let $p(x) \in F[x]$ be an irreducible polynomial and let $\varphi(p(x)) \in F'[x]$ be the irreducible polynomial in the image. If α is a root of $p(x)$ and β is a root of $\varphi(p(x))$, then there exists a field isomorphism $\tilde{\varphi} : F(\alpha) \to F'(\beta)$ mapping $\alpha \mapsto \beta$ and extending φ:*

$$\begin{tikzcd} F(\alpha) \arrow[r, "\tilde{\varphi}", "\cong"'] & F'(\beta) \\ F \arrow[u, hook] \arrow[r, "\cong", "\varphi"'] & F' \arrow[u, hook] \end{tikzcd} .$$

Proof. Since $F(\alpha) = F[x]/p(x)$ and $F'(\beta) = F'[x]/\varphi(p(x))$, therefore we need only construct an isomorphism between them via φ which takes $\bar{x}$ to $\bar{x}$ (as $\bar{x}$ in $F(\alpha)$ is the root of $p(x)$ in $F(\alpha)$ and similarly for $F(\beta)$).

Indeed, consider the map

$$\begin{aligned} \varphi : F[x] &\to F'[x] \\ x &\mapsto x. \end{aligned}$$

Then, we get $\tilde{\varphi} : \frac{F[x]}{\varphi^{-1}(\varphi(p(x)))} \xrightarrow{\cong} \frac{F'[x]}{\varphi(p(x))}$. This completes the proof. $\square$

Corollary 23.6.2.2. *If $p(x) \in F[x]$ is irreducible and $\alpha \neq \beta$ are two roots, then there is an isomorphism*

$$\begin{aligned} F(\alpha) &\longrightarrow F(\beta) \\ \alpha &\longmapsto \beta \end{aligned}$$

which is id *on F.*

Proof. Use $\varphi = \mathrm{id}_F$ with $F' = F$ on Proposition 23.6.2.1 to get the result. □

We next show that transcendental elements are mapped to transcendental elements under a field homomorphism.

Proposition 23.6.2.3. *Let $\varphi : F \to F'$ be a morphism of fields. If K/F is a field extension, $\psi : K \to F'$ is a morphism extending φ, then the following are equivalent:*
1. *$\alpha \in K$ is transcendental over F,*
2. *$\psi(\alpha) \in F'$ is transcendental over $\varphi(F) \subseteq F'$.*

Proof. The main observation is that for transcendental element $\alpha \in K$ over F, we have that $F[\alpha]$ is isomorphic to polynomial ring $F[x]$. Using this, we consider the restriction $\psi : F(\alpha) \to F'$. Note that $\alpha \in F(\alpha)$ is transcendental over F if and only if $\mathrm{Ker}\,(\psi) = 0$. Further $\psi(\alpha)$ is transcendental over $\psi(F)$ if and only if $\mathrm{Ker}\,(\psi) = 0$. We win. □

23.6.3 Splitting fields & algebraic closure

Given a polynomial, we will now construct the smallest field where that polynomial splits into linear factors. We will then see that splitting fields are exactly what are called normal extensions.

Definition 23.6.3.1 (**Splitting field**)**.** Let $f(x) \in F$ be a field and $f(x) \in F[x]$ be a polynomial. The splitting field of $f(x)$ over F is the smallest field extension K/F such that $f(x) \in K[x]$ is product of linear factors, that is, K is the smallest field containing all roots of $f(x)$.

Theorem 23.6.3.2. *Splitting field exists.*

Proof. Let $f(x) \in F$ be a field and $f(x) \in F[x]$ be a polynomial. We wish to construct the smallest field K/F containing all roots of F. We induct over $\deg f(x) = n$. If $n = 1$, then $K = F$ will do. Suppose for every polynomial $g(x)$ of degree $n-1$ or lower has a splitting field, which we denote by K_g/F. Pick $f(x) \in F[x]$ be of degree n. We wish to construct the splitting field of $f(x)$. We have two cases. If $f(x)$ is reducible, then $f(x) = g(x)h(x)$ where $\deg g, \deg h < n$. We thus have splitting fields K_g and K_h for g and h respectively. We claim that $K_g \cdot K_h$ is a splitting field of $f(x)$. Indeed, $K_g \cdot K_h$ contains all roots of $f(x)$ so splitting field is a subfield of $K_g \cdot K_h$. But since splitting field of $f(x)$ also contains roots of $g(x)$ and $h(x)$, it follows that it must contain K_g and K_h and thus $K_g \cdot K_h$ as well. Hence splitting field is exactly $K_g \cdot K_h$.

On the other hand if $f(x)$ is irreducible, then let $K = \frac{F[x]}{\langle f(x) \rangle}$ which is a finite extension of F. Now, K has atleast one root of $f(x)$, namely $\bar{x}$, which we label as $\alpha \in K$. Thus, we have that $f(x) = (x - \alpha)g(x)$ in $K[x]$. Thus $g(x) \in K[x]$ is of degree $n - 1$. Hence by inductive hypothesis, there exists a field $L_g/K/F$ such that $g(x)$ splits into linear factor/L_g contains all roots of $g(x)$. Thus $L_g(\alpha)$ contains all roots of $f(x)$. We claim that $L_g(\alpha)$ is contains a splitting field of $f(x)$. Indeed, we may take intersection of all sub-fields of $L_g(\alpha)$ which contains all roots of $f(x)$. Such a collection is non-empty as $L_g(\alpha)$ contains all roots of $f(x)$. As intersection of subfields is a subfield, we win the induction step. □

We now show that splitting fields are unique upto isomorphism.

Proposition 23.6.3.3 (Extension-II)**.** *Let $\varphi : F \to F'$ be a field isomorphism and $f(x) \in F[x]$ be a polynomial. Let $\varphi(f(x)) \in F'[x]$ be the image of $f(x)$ under φ. Then, φ lifts to an isomorphism $\tilde{\varphi} : K \to K'$ where K/F is the splitting field of $f(x)$ and K'/F' is the splitting field of $\varphi(f(x))$:*

$$\begin{tikzcd} K \arrow[r, "\tilde{\varphi}", "\cong"'] & K' \\ F \arrow[u, hook] \arrow[r, "\cong", "\varphi"'] & F' \arrow[u, hook] \end{tikzcd}.$$

Proof. We will induct on degree of $f(x)$. If $\deg f(x) = 1$, then F has the root of f and thus we may take $\tilde{\varphi}$ to be φ itself. Let $\deg f = n$ and suppose that for any polynomial of degree $n - 1$ or lower over any extension of F, we have the required map. Let $f(x) = p(x)g(x)$ where $p(x) \in F[x]$ is an irreducible factor of $f(x)$. Thus $\deg p(x) \leq n - 1$. Now, let α be a root of $p(x)$ and α' be a root of $\varphi(p(x))$. Thus by Extension-I (Proposition 23.6.2.1), it follows that we have an extension $\chi : F(\alpha) \to F'(\alpha')$ which extends φ. Now consider $h(x) = f(x)/x - \alpha$ in $F(\alpha)[x]$. Then, $h(x)$ has degree $n - 1$ over $F(\alpha)$, so by inductive hypothesis, we get an extension $\tilde{\varphi} : K_h \to K'_h$ where $K_h/F(\alpha)$ and $K'_h/F'(\alpha')$ are splitting fields of $h(x)$ and $\chi(h(x))$ respectively. We claim that K_h is the splitting field of $f(x)$. Indeed, K_h has all roots of $f(x)$, so it contains the splitting field. But roots of $h(x)$ are just those of $f(x)$ except α, so K_h is the splitting field of $f(x)$. This completes the proof. □

Algebraic closure

We now discuss some basic properties of algebraic closure. Note that there is a subtlety to the definition of an extension being algebraically closed.

Definition 23.6.3.4 (**Algebraically closed fields & extensions**)**.** A field K is algebraically closed if every polynomial in $K[x]$ has a root. An extension K/F is called an algebraically closed extension if K/F is algebraic and K is algebraically closed. In this case, K is called the algebraic closure of F.

Remark 23.6.3.5. The linguistic subtlety here is that $\mathbb{C}/\mathbb{Q}$ is not algebraically closed extension as it is not algebraic. But $\bar{\mathbb{Q}}/\mathbb{Q}$ is an algebraically closed extension.

We will omit the statement that an algebraic closed extension of any field exists as it can be found in any standard book. We however state the following important results about equivalence conditions for a field to be algebraically closed.

Theorem 23.6.3.6. *Let F be a field. Then the following are equivalent:*

1. *F is algebraically closed.*
2. *Only irreducible polynomial in $F[x]$ are linear.*
3. *If K/F is algebraic, then $K = F$.*

Proof. The only non-trivial part is that of 3. $\Rightarrow$ 1. Indeed, pick any $f(x) \in F[x]$. Then, consider the splitting field K/F of $f(x)$. As K/F is finite, therefore K/F is algebraic and thus by hypothesis we have $K = F$. It follows that F has all roots of F, as required. $\square$

23.6.4 Separable, normal extensions & perfect fields

Let us begin with definitions.

Definition 23.6.4.1 (**Separable polynomials & extensions**). A polynomial $f(x) \in F[x]$ is said to be separable if $f(x)$ has no repeated roots. That is, there doesn't exists $\alpha \in \bar{F}$ such that $(x-\alpha)^2 | f(x)$. An extension K/F is said to be separable if it is algebraic and for all $\alpha \in K$, the minimal polynomial $m_{\alpha,F}(x) \in F[x]$ is separable.

Definition 23.6.4.2 (**Normal extensions**). An extension K/F is said to be normal if it is algebraic and for all $\alpha \in K$, the minimal polynomial $m_{\alpha,F}(x) \in F[x]$ has all roots in K and is thus a product of linear factors in $K[x]$.

Remark 23.6.4.3. Note that if K/F is normal, then K contains the splitting field of all $f(x) \in F[x]$. Thus every splitting field of some $f(x) \in F[x]$ is an intermediate extension of K/F.

Definition 23.6.4.4 (**Frobenius & perfect fields**). Let K be a field of characteristic $p > 0$. Then the Frobenius is the field map $\mathrm{Fr} : K \to K$ mapping $x \mapsto x^p$. A field K is perfect if either $\mathrm{char}(K) = 0$ or the Frobenius $\mathrm{Fr} : K \to K$ is an isomorphism.

Basic properties

For finite normal extensions, we essentially the following as the most important observation.

Theorem 23.6.4.5. *Let K/F be a finite normal extension. If $\alpha \in K$ and $Z(m_{\alpha,F}(x)) \subseteq K$ is the set of all F-conjugates of α, then $\mathrm{Aut}\,(K/F)$ acts on $Z(m_{\alpha,F}(x))$ transitively.*

We prove this using the following statements.

Proposition 23.6.4.6. *Let K/F be an algebraic extension and $\alpha \in K$. Then,*
1. *For any $\sigma \in \mathrm{Aut}\,(K/F)$, $\sigma(\alpha) \in K$ is an F-conjugate of α.*
2. *If $\beta \in \bar{K}$ is an F-conjugate of α, then there exists a map*
$$\sigma : K \longrightarrow \bar{K}$$
such that $\sigma(\alpha) = \beta$, $\sigma|_F = \mathrm{id}$ and $\sigma(\alpha) = \beta$.
3. *If K/F is a finite normal extension and $\sigma : K \to \bar{K}$ is a field homomorphism such that $\sigma|_F = \mathrm{id}_F$, then $\sigma(K) = K$. That is, if σ is an F-homomorphism, then $\sigma \in \mathrm{Aut}\,(K/F)$.*

Proof. 1. Apply σ on $m_{\alpha,F}(\alpha) = 0$ to get the desired result.

2. By Extension-I (Proposition 23.6.2.1), we have an extension of id : $F \to F$ denoted $\chi : F(\alpha) \to F(\beta)$. By a generalization of Extension-II (Proposition 23.6.3.3) which gives us the same result but for splitting fields of arbitrary collection, we get an extension of χ to $\tilde{\sigma} : \bar{K} \to \bar{K}$ extending χ. Defining $\sigma = \tilde{\sigma}|_K : K \to \bar{K}$, we get that σ extends id_F and $\sigma(\alpha) = \beta$, as required.

3. Pick any $\alpha \in K$. We first wish to show that $\sigma(\alpha) \in K$. By item 1, $\sigma(\alpha) \in \bar{K}$ is an F-conjugate of α. As the minimal polynomial $m_{\alpha,F}(x) \in F[x]$ splits linearly in K, this shows that $\sigma(\alpha) \in K$, hence showing that $\sigma(K) \subseteq K$. To show equality, we need only show that $[K : \sigma(K)] = 1$. Indeed, since

$$[K : F] = [K : \sigma(K)] \cdot [\sigma(K) : F] < \infty$$

and since $\sigma : K \to \sigma(K)$ is an F-isomorphism, therefore $[K : F] = [\sigma(K) : F]$. It follows that $[K : \sigma(K)] = 1$, as required. □

Theorem 23.6.4.7 is now immediate.

Proof of Theorem 23.6.4.7. Pick any two root $\beta \in Z(m_{\alpha,F}(x))$. It suffices to show that there exists $\sigma \in \mathrm{Aut}\,(K/F)$ which maps $\alpha \mapsto \beta$. Indeed, by Proposition 23.6.4.6, 2 & 3, we have such an F-automorphism. □

Characterization of normality and separability

Our goal is to study two questions. First is to understand the relationship between splitting fields and normal extensions (we will see that they are equivalent). Second is to understand the relationship between separability and the automorphisms of the extension.

Understanding these two problems will give us the tool which will allow us to show when a field extension is separable or normal, which will come in handy while doing Galois theory.

Let us begin by the first question.

Theorem 23.6.4.7. *Let K/F be an extension. The following are equivalent:*

1. *K/F is a splitting field of some $S \subseteq F[x]$.*
2. *K/F is a normal extension.*

We now build towards answering the second question.

Definition 23.6.4.8 (**Separable degree**)**.** Let K/F be a finite extension. Then the separable degree of K/F is defined to be

$$[K : F]_s = \left|\mathrm{Hom}_F\left(K, \bar{F}\right)\right|$$

where $\mathrm{Hom}_F\left(K, \bar{F}\right)$ is finite in size since K/F is finite.

There's a tower law for separable degree as well.

Proposition 23.6.4.9. *Let $L/K/F$ be field extensions and L/F be finite. Then,*

$$[L : F]_s = [L : K]_s \cdot [K : F]_s.$$

The following is an easy lemma.

Lemma 23.6.4.10. *Let K/F be a finite extension. Then*

$$[K : F]_s \leq [K : F].$$

Proof. For $K = F(\alpha)$, this is immediate as any $\sigma \in \mathrm{Hom}_F\left(K, \bar{F}\right)$ takes α to some F-conjugate of α. Thus, $[K : F]_s = \#$ conjugates of α in $\bar{F} \leq \deg m_{\alpha,F}(x) = [K : F]$. Now proceed by induction via tower law (Proposition 23.6.4.9). □

Theorem 23.6.4.11. *Let K/F be a field extension. Then the following are equivalent:*

1. $[K : F]_s = [K : F]$.
2. *K/F is a separable extension.*

Another important criterion for separability of a polynomial is to check its derivatives. This is useful in positive characteristic settings.

Lemma 23.6.4.12. *Let $f(x) \in F[x]$ be a polynomial where F is a field. If $f(x)$ is irreducible, then the following are equivalent.*

1. *$f(x)$ is separable.*
2. $f'(x) \neq 0$.

Proof. (1. $\Rightarrow$ 2.) If $f'(x)$ is zero, then $f(x)$ and $f'(x)$ will have a common root, which implies that $f(x)$ has a repeated root, a contradiction.

(2. $\Rightarrow$ 1.) Suppose $f(x)$ is inseparable, that is, it has a repeated root. This is equivalent to stating that there is a non-trivial common factor of $f'(x)$ and $f(x)$, say $p(x)$, which we may assume to be the gcd of $f(x)$ and $f'(x)$. As $f(x)$ is irreducible and $p(x)|f(x)$, therefore $p(x) = f(x)$. But $p(x)|f'(x)$, so $f(x)|f'(x)$. This is not possible as $\deg f' \leq \deg f - 1$. □

Using the above theorems, we obtain the following useful criterion usually used in induction steps and allows us to reduce to checking the separability and normality for a single element.

Proposition 23.6.4.13. *Let K/F be a field extension and $\alpha \in K$ be an algebraic element. If the minimal polynomial $m_{\alpha,F}(x) \in F[x]$*

1. *is a separable polynomial, then $F(\alpha)/F$ is a separable extension,*
2. *has all roots in $F(\alpha)$, then $F(\alpha)/F$ is a normal extension.*

Proof. 1. Note that since $m_{\alpha,F}(x)$ is separable, we get

$$[F(\alpha) : F]_s = |S(\mathrm{id}, F(\alpha)/F)| = \#\text{conjugates of } \alpha = \deg m_{\alpha,F}(x) = [F(\alpha) : F].$$

By Theorem 23.6.4.11, we win.

2. We claim that $F(\alpha)/F$ is the splitting field of $m_{\alpha,F}(x)$ in this case. Indeed, $F(\alpha)/F$ is the smallest field containing F and α. By hypothesis, it contains all the roots of $m_{\alpha,F}(x)$, of which α is one. It follows that $F(\alpha)/F$ is the smallest field containing all roots of $m_{\alpha,F}(x)$, as required. □

23.6.5 Galois extensions

For simplicity, let us only work with finite Galois extensions.

Definition 23.6.5.1 (Galois extensions & Galois group). An extension K/F is Galois if it is finite, separable and normal. That is, for all $\alpha \in K$, the minimal polynomial $m_{\alpha,F}(x) \in F[x]$ has all roots in K and each of them is distinct. The Galois group of a Galois extension K/F, denoted $\mathrm{Gal}\,(K/F)$, is defined to be the automorphism group $\mathrm{Aut}\,(K/F)$.

Let us first see that every splitting field of a separable polynomial is a Galois extension over the base.

Proposition 23.6.5.2. *Let F be a field and $f(x) \in F[x]$ be a separable polynomial. Let K/F be the splitting field of $f(x)$ over F. Then K/F is a Galois extension and* $\mathrm{Gal}\,(K/F)$ *is called the Galois group of the polynomial $f(x)$.*

Proof. We first establish that K/F is Galois. Indeed K/F is finite as it is a splitting field of a polynomial. As it is a splitting field, so it is normal (Theorem 23.6.4.7). To show separability, it suffices to show that the separable degree $[K : F]_s = [K : F]$ (Theorem 23.6.4.11). To this end, we first have $K = F(\alpha_1, \dots, \alpha_n)$ for $\alpha_i \in K$ elements algebraic over F. Consequently, by the tower law for separable degree (Proposition 23.6.4.9), we obtain

$$[K : F]_s = [K : F(\alpha_1, \dots, \alpha_{n-1})]_s \cdot \dots \cdot [F(\alpha_1, \alpha_2) : F(\alpha_1)]_s \cdot [F(\alpha_1) : F]_s.$$

By Proposition 23.6.4.13, it suffices to show that $m_{\alpha_i,F(\alpha_1,\dots,\alpha_{i-1})}(x) \in F(\alpha_1, \dots, \alpha_{i-1})[x]$ is a separable polynomial for each i. Indeed, since $f(\alpha_i) = 0$, thus $m_{\alpha_i,F(\alpha_1,\dots,\alpha_{i-1})}(x)|f(x)$ in $F(\alpha_1, \dots, \alpha_{i-1})[x]$. As $f(x)$ is separable, and $\overline{F(\alpha_1, \dots, \alpha_{i-1})} = \overline{F}$, it follows that $m_{\alpha_i,F(\alpha_1,\dots,\alpha_{i-1})}(x)$ is separable, as required. □

There's a converse to the above result as well.

Proposition 23.6.5.3. *Let K/F be a Galois extension. Then there exists $f(x) \in F[x]$ a separable polynomial whose splitting field is K.*

Proof. As K/F is Galois, therefore finite and hence we may write $K = F(\alpha_1, \dots, \alpha_n)$ for $\alpha_i \in K$ such that no α_i and α_j are conjugate for $i \neq j$ (by normality of K/F, this is possible). As K/F is separable, therefore each $m_{\alpha_i,F}(x) \in F[x]$ is a distinct separable polynomial. Let $f(x) = \prod_{i=1}^n m_{\alpha_i,F}(x)$. This is a separable polynomial as no α_i are conjugates. Moreover, $f(x)$ splits into linear factors over K. It follows that the splitting field of $f(x)$, denoted L, is contained in K. As L contains each of the α_i and F, it follows that $L = K$, as required. □

Thus, for the purposes of clarity, we summarize the above two results in the following corollary.

Corollary 23.6.5.4. *Let K/F be a field extension. Then the following are equivalent.*

1. *K/F is a Galois extension.*
2. *There is a separable polynomial $f(x) \in F[x]$ whose splitting field is K.*

Proof. Follows from Proposition 23.6.5.2 and 23.6.5.3. □

We have the following equivalent criterion to be Galois.

Theorem 23.6.5.5. *Let K/F be a finite extension. Then the following are equivalent:*
1. *K/F is a Galois extension.*
2. *$|\mathrm{Aut}(K/F)| = [K : F]$.*

An extremely important result to keep in mind is the following, telling us that a fixed field by a finite subgroup of the automorphism group always gives a Galois extension(!)

Theorem 23.6.5.6. *Let K be a field and $G \leq \mathrm{Aut}(K)$ be a finite subgroup. Then,*
1. *The extension K/K^G is a Galois extension.*
2. *The Galois group of K/K^G is equal to G;*

$$\mathrm{Gal}\left(K/K^G\right) = G.$$

Théorème fondamental de la théorie de Galois

Theorem 23.6.5.7 (Fundamental theorem)**.** *Let K/F be a Galois extension with Galois group $G = \mathrm{Gal}(K/F)$. Then the maps*

$$\begin{tikzcd} \{L \mid K/L/F \text{ is an intermediate extension}\} \arrow[d, shift left, "\mathrm{Gal}(K/-)"] \\ \{H \mid H \leq G \text{ is a subgroup}\} \arrow[u, shift left, "K^{(-)}"] \end{tikzcd}$$

establish a bijection. Moreover, we have the following:
1. *For any intermediate $K/L/F$, the extension K/L is a Galois extension.*
2. *Both the maps above are antitone, i.e. they reverse the order.*
3. *For any intermediate extension $K/L/F$, the following are equivalent:*
   (a) L/F is a Galois extension.
   (b) $\mathrm{Gal}(K/L)$ is a normal subgroup of G and in this case,

$$\mathrm{Gal}(L/F) \cong \frac{G}{\mathrm{Gal}(K/L)}.$$

4. *For any intermediate extension $K/L/F$[7] we have a bijection (where $\bar{F}$ is an algebraic closure of F containing K)*

$$[L : F]_s = \mathrm{Hom}_F\left(L, \bar{F}\right) = \{\sigma : L \to \bar{F} \mid \sigma|_F = \mathrm{id}_F\} \cong \frac{G}{\mathrm{Gal}(K/L)}$$

   where the RHS is the set of cosets of $\mathrm{Gal}(K/L) \leq G$.
5. *For any two intermediate extensions $K/L_1, L_2/F$ with $H_i = \mathrm{Gal}(K/L_i)$, we have*
   (a) $\mathrm{Gal}(K/L_1 \cdot L_2) = H_1 \cap H_2$ in G,
   (b) $\mathrm{Gal}(K/L_1 \cap L_2) = \langle H_1, H_2 \rangle$ in G.

[7] even if L/F is not Galois, i.e. $\mathrm{Gal}(K/L)$ is not normal.

23.6.6 Consequences of Galois theory

We now portray several consequences of Galois theory (not just fundamental theorem, but field theory in general as well). We begin from observing that finite fields are Galois theoretically quite simple.

Galois groups of finite fields

We begin by showing that any finite extension of a finite field (so itself is finite) is Galois with cyclic Galois group.

Proposition 23.6.6.1. *Let $\mathbb{F}_{p^m}$ be a characteristic p finite field. If $K/\mathbb{F}_{p^m}$ is a finite extension of degree n, then $K/\mathbb{F}_{p^m}$ is a Galois extension with Galois group*

$$\mathrm{Gal}\,(K/\mathbb{F}_{p^m}) \cong \mathbb{Z}/n\mathbb{Z}.$$

Proof. content... □

Primitive element theorem

An important theorem in Galois theory is the observation that a finite separable extension is always simple. In particular, every Galois extension is a singly generated field extension.

Theorem 23.6.6.2 (Primitive element theorem)**.** *Let K/F be a finite separable extension. Then there exists $\alpha \in K$ such that $K = F(\alpha)$.*

Proof. content... □

Compositum & Galois closure

We now study how Galois extensions behave with compositums. One calls it the *sliding lemma* as it says that Galois extensions slides through arbitrary extensions.

Proposition 23.6.6.3 (Sliding lemma)**.** *Let K/F be a Galois extension and F'/F be an arbitrary extension such that $K, F' \subseteq \Omega$ where Ω is some large field. Then,*

1. *The extension $K \cdot F'/F'$ is a Galois extension.*
2. *There is an injective group homomorphism*

$$\mathrm{Gal}\left(K \cdot F'/F'\right) \hookrightarrow \mathrm{Gal}\,(K/F)$$

whose image is $\mathrm{Gal}\,(K/F' \cap K)$. That is,

$$\mathrm{Gal}\left(K \cdot F'/F'\right) \cong \mathrm{Gal}\left(K/F' \cap K\right).$$

Proof. □

The following tells us that compositum and intersections of Galois is Galois.

Proposition 23.6.6.4 (Compositum & intersection of Galois)**.** *Let K_1/F and K_2/F be Galois extensions where $K_1, K_2 \subseteq \Omega$ for some large field Ω. Then,*

1. *Extension $K_1 \cdot K_2/F$ is Galois.*
2. *Extension $K_1 \cap K_2/F$ is Galois.*
3. *There is an injective group homomorphism*

$$\begin{aligned}\varphi : \mathrm{Gal}\,(K_1 \cdot K_2/F) &\hookrightarrow \mathrm{Gal}\,(K_1/F) \times \mathrm{Gal}\,(K_2/F) \\ \sigma &\mapsto (\sigma|_{K_1}, \sigma|_{K_2})\end{aligned}$$

whose image is

$$\begin{aligned}\mathrm{Im}\,(\varphi) &= \{(\sigma,\tau) \mid \ \sigma|_{K_1\cap K_2} = \tau|_{K_1\cap K_2}\} \\ &= \mathrm{Gal}\,(K_1/F) \times_{\mathrm{Gal}(K_1\cap K_2/F)} \mathrm{Gal}\,(K_2/F)\,.\end{aligned}$$

Hence, in particular, if $K_1 \cap K_2 = F$, then

$$\mathrm{Gal}\,(K_1 \cdot K_2/F) \cong \mathrm{Gal}\,(K_1/F) \times \mathrm{Gal}\,(K_2/F)\,.$$

Proof. content... □

We now show that any finite separable extension admits a Galois closure.

Lemma 23.6.6.5. *Let K/F be a finite separable extension. Then there exists a Galois extension L/F such that $L \supseteq K$ which is smallest with respect to containing K.*

Proof. We first show that there exists a Galois extension of F containing K. Indeed, consider $K = F\alpha_1 + \cdots + F\alpha_n$ and let $m_{\alpha_i,F}(x) \in F[x]$ be minimal polynomial of α_i. As K is separable, each of $m_{\alpha_i,F}(x)$ is a separable polynomial in $F[x]$. Thus let K_i/F be the splitting field of $m_{\alpha_i,F}(x)$. By Proposition 23.6.5.2, it follows that K_i/F are all Galois. By compositum of Galois (Proposition 23.6.6.4), we deduce that $L = K_1 \cdot \cdots \cdot K_n$ is a Galois extension of F which contains K as it contains $\alpha_1, \dots, \alpha_n$. Thus we have found a Galois extension of F containing K, as required.

We now wish to show that there is a smallest Galois extension of F containing K. Indeed, consider $E = \bigcap_{L/A/K/F} A$ where A/F is a Galois extension containing K. By fundamental theorem (Theorem 23.6.5.7), it follows that there are only finitely many intermediate extensions of L/F, thus finitely many such A. Thus E is Galois by intersection of Galois (Proposition 23.6.6.4). Clearly, by construction E is the smallest field extension of F containing K and is Galois. This completes the proof. □

Definition 23.6.6.6 (Galois closure of a finite separable extension). Let K/F be a finite separable extension. Then the smallest extension L/F containing K such that L/F is Galois is called the Galois closure of K/F. Lemma 23.6.6.5 shows that it always exists.

Norm & trace of a finite extension

Let K/F be an extension. A main technique in field theory is to construct non-trivial elements in K not in F. To this end one of the important set of tools available are those provided by norm & trace of a finite separable extension.

Definition 23.6.6.7 (Norm & Trace). Let K/F be a finite separable extension and $L/K/F$ be the Galois closure. Let $H = \mathrm{Gal}\,(L/K) \leq \mathrm{Gal}\,(L/F)$. Let $\alpha \in K$ and consider a fixed algebraic closure $\bar{F}$ of F. Define

$$N_{K/F}(\alpha) = \prod_{\sigma \in \mathrm{Hom}_F\left(K, \bar{F}\right)} \sigma(\alpha)$$

and

$$\mathrm{Tr}_{K/F}(\alpha) = \sum_{\sigma \in \mathrm{Hom}_F\left(K, \bar{F}\right)} \sigma(\alpha)$$

which we respectively call the norm and trace of α in K/F. Note that $\mathrm{Hom}_F\left(K, \bar{F}\right)$ is finite by Lemma 23.6.4.10.

We can give an alternate definition norm and trace.

Lemma 23.6.6.8. *Let K/F be a finite separable extension. Let $L/K/F$ be the Galois closure of K/F and let $\{\sigma_1, \ldots, \sigma_k\} \in \mathrm{Gal}\,(L/F)$ be distinct coset representatives of $\mathrm{Gal}\,(L/K)$ in $\mathrm{Gal}\,(L/F)$. Then*

$$N_{K/F}(\alpha) = \prod_{i=1}^{k} \sigma_i(\alpha)$$

and

$$\mathrm{Tr}_{K/F}(\alpha) = \sum_{i=1}^{k} \sigma_i(\alpha).$$

If K/F is Galois, then $N_{K/F}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma(\alpha)$ and $\mathrm{Tr}_{K/F}(\alpha) = \sum_{\sigma \in \mathrm{Gal}(K/F)} \sigma(\alpha)$.

Proof. By fundamental theorem 23.6.5.7, 4, we have a bijection of sets (which is an isomorphism of groups if K/F is Galois by fundamental theorem):

$$\mathrm{Hom}_F\left(K, \bar{F}\right) \cong \frac{\mathrm{Gal}\,(L/F)}{\mathrm{Gal}\,(L/K)}.$$

The result now follows from definition of norm and trace. $\square$

We now state some basic properties of these two functions.

Proposition 23.6.6.9. *Let K/F be a finite separable extension. Let $L/K/F$ be the Galois closure of K/F.*

1. *For any $\alpha \in K$, $N_{K/F}(\alpha) \in F$ and $\mathrm{Tr}_{K/F}(\alpha) \in F$.*
2. *For any $\alpha, \beta \in K$, we have*

$$N_{K/F}(\alpha\beta) = N_{K/F}(\alpha) N_{K/F}(\beta)$$

and

$$\mathrm{Tr}_{K/F}(\alpha + \beta) = \mathrm{Tr}_{K/F}(\alpha) + \mathrm{Tr}_{K/F}(\beta).$$

3. *If $K = F(\sqrt{D})$ for some $D \in F$, then for $a, b \in F$ we have*
$$N_{K/F}(a + b\sqrt{D}) = a^2 - b^2 D$$
and
$$\operatorname{Tr}_{K/F}(a + b\sqrt{D}) = 2a.$$

Proof. For item 1, since these are coefficients of $m_{\alpha,F}(x)$, so they are in F. Item 2 follows immediately from Lemma 23.6.6.8. For item 3, observe that there is only one other conjugate of $\alpha = a + b\sqrt{D}$ (as minimal polynomial is quadratic) given by $\bar{\alpha} = a - b\sqrt{D}$. Now use Lemma 23.6.6.8. □

Lemma 23.6.6.10. *Let K/F be a finite extension of degree n and $\alpha \in K$. Then*
1. *Element α acting by left multiplication on K is an F-linear transformation, which we denote by $T_\alpha : K \to K$.*
2. *The minimal polynomial of element $\alpha \in K$, denoted $m_{\alpha,F}(x)$ is same as the minimal polynomial of the F-linear map $T_\alpha : K \to K$, denoted $p(x) \in F[x]$.*
3. *The norm $N_{K/F}(\alpha)$ and trace $\operatorname{Tr}_{K/F}(\alpha)$ are respectively the determinanat and trace of the F-linear map T_α.*

Proof. 1. Indeed, $T_\alpha : K \to K$ is given by $x \mapsto \alpha x$ which F-linear as $T_\alpha(x + cy) = \alpha(x + cy) = \alpha x + c\alpha y = T_\alpha(x) + cT_\alpha(y)$ where $c \in F$.

2. As $m_{\alpha,F}(x)$ is irreducible, we need only show that $p(x)|m_{\alpha,F}(x)$. Note that $m_{\alpha,F}(T_\alpha) = 0$ since for any $z \in K$, we have
$$m_{\alpha,F}(T_\alpha)(z) = m_{\alpha,F}(\alpha)z = 0.$$
Hence $p(x)|m_{\alpha,F}(x)$, as required.

3. Let $m_{\alpha,F}(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ in $F[x]$ and $[K : F] = n$. By item 2, the minimal polynomial $p(x)$ of T_α is also $m_{\alpha,F}(x)$. Determinant of T_α is the product of all eigenvalues (with repetitions) and trace of T_α is the sum of all eigenvalues. One can then deduce[8] that
$$N_{K/F}(\alpha) = (-1)^n a_0^{n/d}$$
and
$$\operatorname{Tr}_{K/F}(\alpha) = \frac{-n}{d} a_{d-1}.$$
As K/F is separable, therefore we may write $p(x) = m_{\alpha,F}(x) = (x - \lambda_1) \cdot \cdots \cdot (x - \lambda_d)$ where λ_i are distinct eigenvalues of T_α or equivalently, F-conjugates of α. It is now sufficient to show that each eigenvalue λ_i has algebraic multiplicity n/d.

Let $\Phi(x) \in F[x]$ be the characteristic polynomial of T_α. Since $p(x)$ and $\Phi(x)$ have same irreducible factors and $p(x)$ is irreducible, it follows that $\Phi(x) = p(x)^k$ for some $k \geq 1$. As $\Phi(x)$ has degree n and $p(x)$ has degree d, therefore we conclude that $k = n/d$, as required. □

[8]by Questions 17 and 18 of Section 14.2 of DF, cite[DummitFoote]

23.6.7 Cyclotomic extensions

We discuss the extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ where ζ_n is an n^{th}-root of unity, that is, a solution of $x^n - 1$ in $\mathbb{C}$. We will see that n^{th}-roots of unity form a cyclic group $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$, therefore we define a *primitive* n^{th} root of unity to be a generator of $\mathbb{Z}/n\mathbb{Z}$. Thus, there are $\varphi(n)$ many primitive n-th roots of unity, where φ is the Euler totient function.

We denote the group of n-th roots of unity as μ_n. Some basic facts about μ_n are as follows.

Lemma 23.6.7.1. *Let $n \in \mathbb{N}$. Then,*

1. *μ_n is a finite cyclic group isomorphic to $\mathbb{Z}/n\mathbb{Z}$.*
2. *If $d|n$, then $\mu_d \hookrightarrow \mu_n$.*

Proof. 1. μ_n is finite of size n since its the set of roots of $x^n - 1$ in $\mathbb{C}$. This is a group since product of any two n-th roots of unity is an n-th root of unity. Thus μ_n is a finite subgroup of the multiplicative group $\mathbb{C}^\times$. It follows that μ_n is cyclic.

2. Consider the map

$$\begin{aligned} \varphi : \mu_d &\longrightarrow \mu_n \\ \zeta &\longmapsto \zeta. \end{aligned}$$

This is well-defined since a d-th root of unity is also an n-th root of unity if $d|n$. Further, this is clearly a group homomorphism. $\square$

Thus $\mu_d \leq \mu_n$ is precisely the subgroup of order d-elements of μ_n.

Definition 23.6.7.2 (n^{th}**-cyclotomic polynomial**)**.** Let $n \in \mathbb{N}$. The n^{th}-cyclotomic polynomial is defined to be the polynomial $\Phi_n(x) = \prod_{\zeta \in \mu_n^\times}(x - \zeta)$, that is, the polynomial whose all roots are the primitive n^{th}-roots of unity.

We immediately have the following observations.

Lemma 23.6.7.3. *Let $\Phi_n(x)$ be the n^{th}-cyclotomic polynomial. Then,*

1. *$\Phi_n(x)|x^n - 1$.*
2. *$x^n - 1 = \prod_{d|n} \Phi_d(x)$.*

Proof. Follows from the observation that $x^n - 1 = \prod_{\zeta^n=1}(x - \zeta)$. $\square$

Remark 23.6.7.4. Using Lemma 23.6.7.3, we see that we can calculate $\Phi_n(x)$ recursively by finding Φ_d for all $d|n$ and $d \neq n$. In particular,

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n, d\neq n} \Phi_d(x)}.$$

We now state and prove the following theorem, which in particular tells us that cyclotomic polynomial $\Phi_n(x)$ is monic irreducible of degree $\varphi(n)$. Once shown, we would be able to conclude that the the minimal polynomial of a primite n^{th}-root of unity is $\Phi_n(x)$.

Theorem 23.6.7.5. *Let $n \in \mathbb{N}$. Then,*

1. *$\Phi_n(x)$ is a monic polynomial of degree $\varphi(n)$ in $\mathbb{Z}[x]$.*

2. $\Phi_n(x)$ is an irreducible polynomial in $\mathbb{Z}[x]$.
3. $\Phi_n(x)$ is the minimal polynomial of any primitive n^{th}-root of unity $\zeta_n \in \mathbb{C}$.
4. If ζ_n is a primitive n^{th}-root of unity, then $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is a degree $\varphi(n)$ extension.

Proof. 1. The fact that degree of $\Phi_n)(x)$ is $\varphi(n)$ follows from the fact that in $\mathbb{C}$ it is a product of $\varphi(n)$ many linear factors. This also shows that $\Phi_n(x)$ is a monic polynomial. We need only show that coefficients lie in $\mathbb{Z}$. To this end, we proceed by induction. For $n = 1$, $\Phi_n(x) = x - 1 \in \mathbb{Z}[x]$. For $n = 2$, $\Phi_2(x) = x + 1 \in \mathbb{Z}[x]$. Now suppose that for all $d < n$ $\Phi_d(x) \in \mathbb{Z}[x]$. Then we have

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n,d\neq n} \Phi_d(x)},$$

thus $f(x) := \prod_{d|n,d\neq n} \Phi_d(x) \in \mathbb{Z}[x]$ by inductive hypothesis. As $f(x)|x^n - 1$ in $\mathbb{Q}[x]$ and $f(x) \in \mathbb{Z}[x]$, therefore by results surrounding Gauss' lemma, we get $f(x)|x^n - 1$ in $\mathbb{Z}[x]$, that is, $\Phi_n(x) \in \mathbb{Z}[x]$.

2. Let $\Phi_n(x) = f(x)g(x)$ in $\mathbb{Z}[x]$ where we assume that $f(x)$ is an irreducible factor of $\Phi_n(x)$ (by $\mathbb{Z}[x]$ being an UFD). We claim that $f(x)$ has all primitive n^{th}-roots of unity as a root over $\mathbb{C}$, so that $f(x) = \Phi_n(x)$ over $\mathbb{Z}$. Indeed, let $\zeta^a \in \mu_n$ be any other primitive root, then $(a, n) = 1$ and so we may write $a = p_1 \dots p_k$ where p_i are primes not dividing n. We wish to show that ζ^a is a root of $f(x)$. It suffices to show that if ζ is a root of $f(x)$, then ζ^p is a root of $f(x)$ as well for any prime p not dividing n. This is what we will show now.

Indeed, let $\zeta \in \mu_n$ a primitive n^{th}-root of unity which is a root of $f(x)$. As $f(x)$ is irreducible over $\mathbb{Z}[x]$, therefore irreducible over $\mathbb{Q}[x]$ as well, hence $f(x)$ is the minimal polynomial of ζ over $\mathbb{Q}$. Consider p a prime not dividing n. We wish to show that ζ^p is also a root of $f(x)$. Indeed, as $\Phi_n(x)$ has ζ^p as a root, therefore either $f(\zeta^p) = 0$ or $g(\zeta^p) = 0$ over $\mathbb{C}$. Suppose the latter is true. Thus $g(x^p)$ has ζ as a root. As $g(x^p) \in \mathbb{Q}[x]$, therefore $f(x)|g(x^p)$ in $\mathbb{Q}[x]$. As $f(x), g(x^p) \in \mathbb{Z}[x]$, therefore by results surrounding Gauss' lemma, we conclude that $f(x)|g(x^p)$ in $\mathbb{Z}[x]$. Let $g(x^p) = f(x) \cdot h(x)$ where $h(x) \in \mathbb{Z}[x]$. Going modulo p, we get that $\bar{g}(x^p) = (\bar{g}(x))^p$. Thus, $(\bar{g}(x))^p = \bar{f}(x)\bar{h}(x)$ in $\mathbb{F}_p[x]$. Thus, $\bar{g}$ and $\bar{f}$ have a common factor in $\mathbb{F}_p[x]$ as both have ζ as a root. Thus, $\bar{\Phi}_n(x) = \bar{f}(x)\bar{g}(x)$ has a repeated factor, thus, $\Phi_n(x)$ is not separable over over $\mathbb{F}_p$. But since $\Phi_n'(x) = nx^{n-1} \neq 0$ has only $x = 0$ as a root, therefore $\Phi_n(x)$ is separable. It follows that we have a contradiction to the separability of $x^n - 1$ as $\Phi_n(x)$ is a factor of $x^n - 1$, thus ζ^p cannot be a root of $g(x)$, as required.

3. As $\Phi_n(\zeta_n) = 0$ for any primitive n^{th}-root of unity, therefore we get that $m_{\zeta_n,\mathbb{Q}}|\Phi_n(x)$. As $m_{\zeta_n,\mathbb{Q}}$ is irreducible and so is $\Phi_n(x)$, thus $m_{\zeta_n,\mathbb{Q}} = \Phi_n$, as required.

4. As $\Phi_n(x)$ is the minimal polynomial of ζ_n which has degree $\varphi(n)$, the result follows. □

We now wish to study the Galois group of a cyclotomic extension.

Definition 23.6.7.6 (Cyclotomic extension). Let $\zeta_n \in \mathbb{C}$ be a primitive n^{th}-root of unity. The extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is called a cyclotomic extension.

It is easy to see that every cyclotomic extension is Galois.

Lemma 23.6.7.7. *Let $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ be a cyclotomic extension. Then $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is a Galois extension.*

Proof. By Theorem 23.6.7.5, 4, it follows that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is finite. Observe that $m_{\zeta_n,\mathbb{Q}}(x) \in \mathbb{Q}[x]$ is $\Phi_n(x)$ by Theorem 23.6.7.5, 3 which is separable. As ζ_n is the primitive n^{th}-root of unity, therefore it generates all other roots of unity. Consequently, $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is normal as well, as required. □

Calculation of Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is quite easy.

Theorem 23.6.7.8. *Let $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ be a cyclotomic extension where ζ_n is a primitive n^{th}-root. Then, the map*

$$\begin{aligned} (\mathbb{Z}/n\mathbb{Z})^\times &\longrightarrow \operatorname{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \\ a &\longmapsto \sigma_a : \zeta_n \mapsto \zeta_n^a \end{aligned}$$

is an isomorphism.

Proof. Immediate. □

Cyclotomic extensions are a particular example of an abelian extension.

Definition 23.6.7.9 (Abelian extension). Let K/F be a field extension. If K/F is Galois and $\operatorname{Gal}(K/F)$ is an abelian group, then K/F is called an abelian extension.

Remark 23.6.7.10. If $K_1, K_2/F$ are abelian extensions, then any subfield $K_1/L/F$ is an abelian extension by fundamental theorem (Theorem 23.6.5.7) and compositum $K_1 \cdot K_2/F$ is also abelian by Proposition 23.6.6.3.

An important result in the theory of finite abelian extensions is the fact that any extension of $\mathbb{Q}$ is abelian if and only if it is contained in a cyclotomic extension. Using this result, one can heuristically say that finite abelian groups are to groups what are cyclotomic extensions are to field extensions(!)

Theorem 23.6.7.11 (Kronecker-Weber)**.** *Let $K/\mathbb{Q}$ be an extension. Then the following are equivalent:*

1. *$K/\mathbb{Q}$ is a finite abelian.*
2. *$K \subseteq \mathbb{Q}(\zeta_n)$ for some $n \in \mathbb{N}$.*

Moreover, if G is any finite abelian group, then there exists $K/\mathbb{Q}$ finite abelian such that $\operatorname{Gal}(K/\mathbb{Q}) \cong G$.

Another important line of thought around cyclotomic extensions is the situation when Galois group is cyclic. We have seen that Galois groups of finite fields are cyclic (Proposition 23.6.6.1). Moreover, if p is a prime, then by Theorem 23.6.7.8, the cyclotomic extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ also has cyclic Galois group for ζ_p a primitive p^{th}-root of unity. We now see that such Galois extensions are of a very simple type.

Definition 23.6.7.12 (Cyclic extensions). An extension K/F is said to be cyclic if it is Galois and $\operatorname{Gal}(K/F)$ is cyclic.

Theorem 23.6.7.13 (Kummer)**.** *Let F be a characteristic $p > 0$ field and $\zeta_n \in F$ where ζ_n is a primitive n^{th}-root of unity for $\gcd(n, p) = 1$. Then, the following are equivalent.*

1. *$K = F(a^{1/n})$ for some non-zero $a \in F$.*
2. *K/F is a cyclic extension.*

Proof. (1. $\Rightarrow$ 2.) We first show that K/F is Galois. Let $\alpha = a^{1/n}$. Finiteness is clear as $m_{\alpha,F}(x)|x^n - a$. We wish to show that $m_{\alpha,F}(x)$ is separable. Indeed, since $x^n - a$ has derivative nx^{n-1} which is a non-zero polynomial (as $\gcd(n, p) = 1$) whose only root is 0, therefore $x^n - a$ is separable and thus so is $m_{\alpha,F}(x)$. Finally, as all roots of $x^n - a$ are $\{\zeta_n^k \alpha\}_{k=0,\dots,n-1}$, which are in K as $\zeta_n \in F$, therefore $x^n - a$ splits in K into linear factors, and hence so does $m_{\alpha,F}(x)$. Indeed, K is the splitting field of $x^n - a$ over F.

Next, we show that K/F is cyclic. Indeed, consider the map

$$\begin{aligned} \varphi : \operatorname{Gal}(K/F) &\longrightarrow \mu_n \\ \sigma &\longmapsto \frac{\sigma(\alpha)}{\alpha}. \end{aligned}$$

This is well defined as $\sigma(\alpha) = \zeta_n^{k_\sigma} \alpha$, some conjugate of α. Thus, $\varphi(\sigma) = \zeta_n^{k_\sigma}$. We claim that this is an injective group homomorphism, and thus $\operatorname{Gal}(K/F)$ is cyclic.

Indeed, this is a group homomorphism as $\varphi(\sigma \circ \tau) = \sigma(\tau(\alpha)) = \sigma(\zeta_n^{k_\tau} \alpha)/\alpha = \zeta_n^{k_\sigma} \zeta_n^{k_\tau}$. Hence, it is a group homomorphism. It is moreover injective as if $\zeta_n^{k_\sigma} = \zeta_n^{k_\tau}$, then $\sigma(\alpha) = \tau(\alpha)$. As σ, τ are F-automorphisms of $K = F(\alpha)$ mapping α to the same element, therefore $\sigma = \tau$, as needed.

(2. $\Rightarrow$ 1.) We wish to find an n^{th}-root of some a in K and show that it generates K. As $\operatorname{Gal}(K/F) = \langle \sigma \rangle$ is cyclic, therefore consider the following element of K constructed out of any $\alpha \in K$:

$$\beta = \alpha + \zeta_n \sigma(\alpha) + \zeta_n^2 \sigma^2(\alpha) + \cdots + \zeta_n^{n-1} \sigma^{n-1}(\alpha).$$

Observe that

$$\begin{aligned} \sigma(\beta) &= \sigma(\alpha) + \zeta_n \sigma^2(\alpha) + \zeta_n^2 \sigma^3(\alpha) + \cdots + \zeta_n^{n-1} \alpha \\ &= \zeta_n^{n-1} \beta. \end{aligned}$$

Similarly, we get for each $0 \le k \le n-1$ the following relation:

$$\sigma^k(\beta) = \zeta_n^{n-k} \beta.$$

Hence, we see that $p(x) = x^n - \beta^n$ has all roots in K given by $\{\zeta_n^{n-k}\beta\}_{0 \le k \le n-1}$.

We claim that $\beta^n \in F$. Indeed, we show that for $G = \operatorname{Gal}(K/F) = \langle \sigma \rangle$, the element β^n is in K^G and since $K^G = F$ by fundamental theorem (Theorem 23.6.5.7), hence we will be done. As $\sigma(\beta^n) = (\zeta_n^{n-1}\beta)^n = \beta^n$, therefore $\beta^n \in K^G = F$, as required. Hence, $\beta = a^{1/n}$ for $a = \beta^n \in F$.

We finally claim that $F(\beta) = K$. Indeed, as $K/F(\beta)/F$ is an intermediate extension and $\operatorname{Gal}(K/F)$ is cyclic hence abelian, therefore $F(\beta)/F$ is Galois by fundamental theorem (Theorem 23.6.5.7). As $\sigma \in \operatorname{Gal}(F(\beta)/F)$, therefore $|\operatorname{Gal}(F(\beta)/F)| \ge n$. But by fundamental theorem, $\operatorname{Gal}(F(\beta)/F) = \frac{\operatorname{Gal}(K/F)}{\operatorname{Gal}(K/F(\beta))}$, thus, $|\operatorname{Gal}(K/F(\beta))| = [K : F(\beta)] = 1$, thus, $[K : F] = [K : F(\beta)][F(\beta) : F] = [F(\beta) : F]$, thus showing that $F(\beta) = K$, as required. $\square$

23.6.8 Transcendence degree

Definition 23.6.8.1. (**Transcendence**) Let K/k be a field extension.

1. A collection of elements $\{\alpha_i\}_{i\in I}$ of K is said to be *algebraically independent* if the map
$$\begin{aligned} k[x_i \mid i \in I] &\longrightarrow K \\ x_i &\longmapsto \alpha_i \end{aligned}$$
is injective.
2. A *transcendence basis* of K/k is defined to be an algebraically independent set $\{\alpha_i \mid i \in I\}$ of K/k such that $K/k(\alpha_i \mid i \in I)$ is an algebraic extension.
3. The extension K/k is said to be *purely transcendental* if $K \cong k(x_i \mid i \in I)$ for some indexing set I.

Lemma 23.6.8.2. *Let K/k be a field extension. Then, $\{\alpha_i\}_{i\in I}$ is a transcendence basis of K/k if and only if $\{\alpha_i\}_{i\in I}$ is a maximal algebraically independent set of K/k.*

Proof. (L $\Rightarrow$ R) If $\{\alpha_i\}_{i\in I}$ is not maximal, then there exists $S \subset K$ containing $\{\alpha_i\}_{i\in I}$ such that S is algebraically independent. Let $\beta \in S \setminus \{\alpha_i\}_{i\in I}$. But since $K/k(\{\alpha_i\}_{i\in I})$ is an algebraic extension and $\beta \notin k(\{\alpha_i\}_{i\in I})$ by algebraic independence of S, therefore we have a contradiction to algebraic nature of the extension $K/k(\{\alpha_i\}_{i\in I})$.
(R $\Rightarrow$ L) Suppose $K/k(\{\alpha_i\}_{i\in I})$ is not algebraic. Then there exists $\beta \in K$ which is transcendental over $k(\{\alpha_i\}_{i\in I})$. Thus the set $\{\alpha_i\}_{i\in I} \cup \{\beta\}$ is a larger algebraically independent set, contradicting the maximality. □

Lemma 23.6.8.3. *Let K/k be a field extension. Then any two transcendence basis have the same cardinality.*

Proof. See Tag 030F of cite[Stacksproject]. □

Definition 23.6.8.4. (**Transcendence degree**) Let K/k be a field extension. The cardinality of any transcendence basis is said to be the transcendence degree, denoted trdeg K/k. Furthermore, if A is a domain containing k, then we define trdeg A/k to be the transcendence degree of A_0, the field of fractions of A, over k.

Remark 23.6.8.5. Let K/k be a field extension. If trdeg $K/k = 1$, then there exists $\alpha \in K$ such that α is not an algebraic element over k but $K/k(\alpha)$ is algebraic. In particular, for any transcendental element $\alpha \in K$ over k, the set $\{\alpha\}$ is algebraically independent over k. Precisely, there is a one-to-one bijection between the set of all singletons which are algebraically independent and all transcendental elements of K/k.

Example 23.6.8.6. There are some basic examples which reader might have encountered. For example, one knows that $\mathbb{Q}(\pi)/\mathbb{Q}$ is transcendental as $\pi \in \mathbb{Q}(\pi)$ is not algebraic over $\mathbb{Q}$. Consequently, trdeg $\mathbb{Q}(\pi)/\mathbb{Q}$ is 1, as $\mathbb{Q}(\pi)/\mathbb{Q}(\pi)$ is algebraic.

For another example, consider the next obvious situation of $\mathbb{Q}(e,\pi)/\mathbb{Q}$. Since $\{e\}$ and $\{\pi\}$ are algebraically independent sets over $\mathbb{Q}$, therefore trdeg in this case is ≥ 1. But it is an unknown problem whether $\{e, \pi\}$ forms an algebraically independent set over $\mathbb{Q}$(!) Consequently, if they do, then trdeg $\mathbb{Q}(e,\pi)/\mathbb{Q} = 2$ and if they don't, then the best we can say is trdeg $\mathbb{Q}(e,\pi)/\mathbb{Q} \geq 1$.

Example 23.6.8.7. We have trdeg $k(x_1, \ldots, x_n)/k = n$ as $\{x_1, \ldots, x_n\}$ forms a maximal algebraically independent set.

We observe some basic first properties of transcendence degree.

Lemma 23.6.8.8. *Let $A = k[\alpha_1, \ldots, \alpha_n]$ be an integral domain where $\alpha_i \in K$ for some field extension K/k. If* trdeg $A/k = r > 0$*, then there exists $\alpha_{i_1}, \ldots, \alpha_{i_r}$ which are transcendental over k.*

23.7 Integral dependence and normal domains

The main topic of interest of study in this section is the following question: "*let R be a ring and S be an R-algebra. How do all those elements of S behave like which satisfy a polynomial with coefficients in R?*".

23.7.1 Definitions and basic theory

In order to investigate this further, let us bring some definitions.

Definition 23.7.1.1. (**Integral elements and integral algebra**) Let R be a ring and S be an R-algebra. An element $s \in S$ for which there exists $p(x) \in R[x]$ such that $p(s) = 0$ in S is said to be an *integral element* over R. Further, S is said to be *integral over* R if every element of S is integral over R.

To begin deriving properties, we would need a fundamental result about endomorphisms of finitely generated modules.

Theorem 23.7.1.2. *(Cayley-Hamilton theorem) Let R be a ring, M be a finitely generated R-module generated by n elements and $I \leq R$ be an ideal. If $\varphi : M \to M$ is an R-linear map such that*

$$\varphi(M) \subseteq IM,$$

then there exists a monic polynomial

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

in $R[x]$ such that $p(\varphi) = 0$ in $\operatorname{Hom}_R(M, M)$ and $a_k \in I^k$ for $k = 1, \ldots, n$.

Proof. See Theorem 4.3, pp 120, [cite Eisenbud]. □

There are two immediate corollaries of Cayley-Hamilton which will remind the reader of finite-dimensional vector space case.

Corollary 23.7.1.3. *Let R be a ring and M be a finitely generated R-module. If $\phi : M \to M$ is a surjective R-module homomorphism, then ϕ is an isomorphism.*

Proof. Using ϕ, we may regard M as an $R[z]$-module. Note that M is a finitely generated $R[z]$-module. Let $I = \langle z \rangle \leq R[z]$. Since the action of z on M is by ϕ and ϕ is surjective, therefore $IM = M$. We may use Cayley-Hamilton with $\varphi = \mathrm{id}$ to deduce that there is a polynomial $p(x, z) \in R[x, z]$ such that $p(z, \mathrm{id}) = 0$ and $p(x, z)$ is a monic polynomial in $R[z][x]$. Consequently, we can write $0 = p(z, \mathrm{id}) = 1 + q(z)z$ for some $q(z) \in R[z]$. It follows that $-q(z)$ is the inverse of z in $R[z]$. Since $z \in R[z]$ denotes the endomorphism ϕ, so we have found an R-linear inverse of ϕ, namely the one corresponding to $-q(z)$, as required. □

Corollary 23.7.1.4. *Let R be a ring and M be a finitely generated R-module. If $M \cong R^n$, then any generating set of n elements of M is linearly independent. In particular, any generating set of n elements of M is a basis.*

Proof. Denote $f : M \to R^n$ to be the given isomorphism. Pick $S = \{s_1, \dots, s_n\}$ to be a generating set of M. This yields a surjection $g : R^n \to M$. We wish to show that g is an isomorphim. Observe that $gf : M \to M$ is surjective. It follows from Corollary 23.7.1.3 that gf is an isomorphism. Since f is an isomorphism, hence it follows that g is an isomorphism, as required. □

The fundamental result which drives the basic results about integral algebras is the following equivalence.

Proposition 23.7.1.5. *Let $R \to S$ be an R-algebra and $s \in S$. Then the following are equivalent.*

1. *$s \in S$ is integral over R.*
2. *$R[s] \subseteq S$ is a finite R-algebra.*
3. *$R[s] \subseteq S$ is contained in a finite R-algebra.*
4. *There is a faithful $R[s]$-module M which when restricted to R is finitely generated as an R-module.*

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ follows at once. We do $4 \Rightarrow 1$. Indeed, let $I = \langle s \rangle \leq R[s]$ be the ideal generated by $s \in R[s]$. Consequently, s induces an endomorphism $m_s : M \to M$ by scalar multiplication. Observe that $m_s(M) = IM$. It follows by Cayley-Hamilton (Theorem 23.7.1.2) that there exists a monic $p(x) \in R[s][x]$ such that $p(m_s) = 0$ as an $R[s]$-linear map $M \to M$. Consequently, for any $a \in M$, we have $p(m_s)(a) = 0$, where upon expanding one sees that $p(m_s) = m_{q(s)}$ for some $q(s) \in R$, $q(x) \in R[x]$. But since M is faithful, therefore $q(s) = 0$, as required. □

Lemma 23.7.1.6. *Let $R \to S$ be an R-algebra and $s_1, \dots, s_n \in S$ be integral over R. Then $R[s_1, \dots, s_n]$ is a finite R-algebra.*

Proof. We proceed by induction over n. Base case follows from Proposition 23.7.1.5. Assume that $R_k = R[s_1, \dots, s_k]$ is a finite R-algebra. Since $s_{k+1} \in S$ is integral over R, therefore it is integral over R_k. It follows from Proposition 23.7.1.5 that $R_k[s_{k+1}]$ is a finite R_k-algebra. Since R_k is a finite R-algebra, therefore $R_k[s_{k+1}] = R[s_1, \dots, s_{k+1}]$ is a finite R-algebra, as required. □

One then obtains that finite generation of an algebra by integral elements as an algebra is equivalent to finite generation as an R-module.

Lemma 23.7.1.7. *Any finite R-algebra is integral over R.*

Proof. Let S be a finite R-algebra and let $s \in S$ be an element. Let $m_s : S \to S$ be the R-linear given by multiplication by s. As S is a finitely generated R-module, then by Cayley-Hamilton (Theorem 23.7.1.2), it follows that there is a monic $p(x) \in R[x]$ such that $p(m_s) = 0$ as an R-linear map. Applying $p(m_s)$ to $1 \in S$ yields $p(s) = 0$, as required. □

Proposition 23.7.1.8. *Let R be a ring and S be an R-algebra. Then the following are equivalent.*

1. *S is a finite R-algebra.*
2. *$S = R[s_1, \dots, s_n]$ where $s_1, \dots, s_n \in S$ are integral over R. In particular, S is integral over R.*

That is, an R-algebra is finite if and only if it is a finite type and integral R-algebra.

Proof. Observe that 2. $\Rightarrow$ 1. is just Lemma 23.7.1.6. For 1. $\Rightarrow$ 2. proceed as follows. By Lemma 23.7.1.7, it follows that S is integral over R. Let $s_1, \dots, s_n \in S$ be a generating set of S as an R-module. It is now clear that $R[s_1, \dots, s_n] = S$ as S is finitely generated. $\square$

The following result show that all integral elements form a subring of S.

Proposition 23.7.1.9. *Let R be a ring and S be an R-algebra. The set of all elements of S integral over R forms a subalgebra of S, called the integral closure of R in S.*

Proof. Let $s, t \in S$ be integral over R. Then $R[s,t]$ is a subalgebra of S. It suffices to show that every element of $R[s,t]$ is integral over R. By Proposition 23.7.1.8, the algebra $R[s,t]$ is integral over R as it is finite by Lemma 23.7.1.6. $\square$

With this, a natural situation is when every element of S is integral over R.

Definition 23.7.1.10. (**Normalization & integral extension**) Let R be a ring and S be an R-algebra. The subalgebra A of all integral elements of S over R is said to be the *integral closure of S over R*. One also calls A the *normalization of R in S*. If S is fraction field of R, then A is also denoted by $\tilde{R}$. Further, if $R \hookrightarrow S$ is a ring extension and every element of S is integral over R, then S is said to be an integral extension of R. If $f : R \to S$ is an integral R-algebra, then the map f is said to be *integral*.

Composition of integral maps is integral.

Lemma 23.7.1.11. *Let $R \to S$ and $S \to T$ be integral maps. Then the composite $R \to S \to T$ is integral.*

Proof. Pick any element $t \in T$. We wish to show that $R[t]$ is contained in a finite R-algebra by Proposition 23.7.1.5. As $S \to T$ is integral, there exists $p(x) \in S[x]$ monic such that $p(t) = 0$. So we have

$$t^n + s_{n-1}t^{n-1} + \cdots + s_1 t + s_0 = 0$$

in T where $s_i \in S$. Let $S' = R[s_0, \dots, s_{n-1}]$. As $R \to S$ is integral, therefore S' is a finite R-algebra by Lemma 23.7.1.6. Note that $R \subseteq S'$. By the above equation, it then follows that $S'[t]$ is a finite S'-algebra. As composition of finite maps is finite, therefore $S'[t]$ is a finite R-algebra containing $R[t]$, as required. $\square$

Another trivial observation is that a map which factors an integral map becomes integral.

Lemma 23.7.1.12. *Let $A \to C$ be an integral map. If there is a map $A \to B$ such that*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \\ B \arrow[ur] & \end{tikzcd}$$

commutes, then $B \to C$ is an integral map.

Proof. Pick any element $c \in C$. There exists non-zero monic $p(x) \in A[x]$ such that $p(x)$ is non-zero in $C[x]$ and $p(c) = 0$ in C. Observe that $p(x) \in B[x]$ is also a non-zero monic as if not then $p(x)$ would be zero in $C[x]$ because the above triangle commutes. The result then follows. $\square$

The following observation is simple to see, but comes in very handy while handling intermediate rings that pop-up while subsequent localizations.

Lemma 23.7.1.13. *Let k be a field and A be an integral k-algebra. Then A is a field.*

Proof. Pick any element $a \in A$. By integrality, there exists $c_i \in k$ such that

$$a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = 0$$

in A. Consider this equation in the fraction field $Q(A)$ to multiply by a^{-1}, so that we may get

$$a^{n-1} + c_{n-1}a^{n-2} + \cdots + c_2 a + c_1 + c_0 a^{-1} = 0$$

in $Q(A)$. It thus follows that a^{-1} is a polynomial in A with coefficients in k, that is, $a^{-1} \in A$, as required. $\square$

23.7.2 Normalization & normal domains

A special situation in Definition 23.7.1.10 is when R is a domain and S is its fraction field. These domains will play a crucial role later on, especially in arithmetic.

Definition 23.7.2.1. (**Normal domain**) Let R be a domain and S be its fraction field. If the normalization of R in S is R itself, then R is said to be a normal domain.

Example 23.7.2.2. Let R be a domain, K its fraction field and $\tilde{R}$ be the normalization of R in K. It follows that $\tilde{R} \hookrightarrow K$ is a normal domain. Indeed, let $\hat{R}$ be normalization of $\tilde{R}$ in K. Then, we have maps

$$R \hookrightarrow \tilde{R} \hookrightarrow \hat{R}$$

where both inclusions are integral maps by construction. It follows from Lemma 23.7.1.11 that the inclusion $R \hookrightarrow \hat{R}$ is integral, forcing $\hat{R} \subseteq \tilde{R}$ which further implies $\tilde{R} = \hat{R}$.

Further investigation into normal domains lets us identify all UFDs as normal domains.

Proposition 23.7.2.3. *All unique factorization domains are normal domains.*

Proof. Let R be a UFD and K be its fraction field. Let $\frac{a}{b} \in K$ with $\gcd(a, b) = 1$. Suppose $\frac{a}{b}$ is integral over R so that there exists $p(x) = x^n + c_{n-1}x^{n-1} + \ldots c_1 x + c_0 \in R[x]$ such that $p(a/b) = 0$. It follows by rearrangement that

$$a^n + c_{n-1}ba^{n-1} + \ldots c_1 b^{n-1} + c_0 b^n = 0.$$

Hence, $b|a^n$. As $\gcd(a, b) = 1$, hence we deduce thay $b|a$, a contradiction. $\square$

Example 23.7.2.4. Consequently, $\mathbb{Z}$ and $\mathbb{Z}[x_1, \dots, x_n]$ are normal as well. Moreover, as Gauss' lemma states that R is UFD if and only if $R[x]$ is UFD, therefore we deduce that $R[x_1, \dots, x_n]$ is a normal domain if R is UFD.

We have something similar to Gauss' lemma for normal domains.

Proposition 23.7.2.5. *A ring R is normal if and only if $R[x]$ is normal.*

Proof. **TODO.** □

Further, we can obtain a generalization of the fact that a monic irreducible in $\mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$.

Proposition 23.7.2.6. *Let $R \hookrightarrow S$ be a ring extension and let $f \in R[x]$ be a monic polynomial. If $f = gh$ in $S[x]$ where g and h are monic, then the coefficients of g and h are integral over R.*

We also obtain that any monic irreducible in the polynomial ring in one variable over a normal domain is prime.

Lemma 23.7.2.7. *Let R be a ring and $f(x) \in R[x]$ be a monic irreducible. If R is a normal domain, then $f(x)$ is a prime element.*

Thus, for normal domains R, monic irreducible and monic prime polynomials are equivalent concepts.

We now show that normalization is a very hereditary process as it preserves many properties of the original ring. Indeed, we first show that normalization and localization commutes.

Proposition 23.7.2.8. *Let $f : R \to S$ be an R-algebra and $M \subseteq R$ be a multiplicative set. If $A \subseteq S$ is the integral closure of R in S, then $M^{-1}A$ is the integral closure of $M^{-1}R$ in $M^{-1}S$.*

Proof. We may assume that f is inclusion of a subring of S by replacing R by $f(R)$ and M by $f(M)$. Consequently, we have inclusions $R \hookrightarrow A \hookrightarrow S$ which induces inclusions $M^{-1}R \hookrightarrow M^{-1}A \hookrightarrow M^{-1}S$. We wish to show that $M^{-1}A$ is the integral closure of $M^{-1}R$ in $M^{-1}S$. Pick an element $s/m \in M^{-1}S$ where $m \in M$ which is integral over $M^{-1}R$. Consequently, there exists $r_i/m_i \in M^{-1}R$ for $0 \le i \le k-1$ such that

$$\left(\frac{s}{m}\right)^k + \frac{r_{k-1}}{m_{k-1}}\left(\frac{s}{m}\right)^{k-1} + \dots \frac{r_1}{s_1}\left(\frac{s}{m}\right) + \frac{r_0}{m_0} = 0$$

in $M^{-1}S$. Multiplying by product of denominators and absorbing coefficients into r_i, we get

$$m's^k + r_{k-1}s^{k-1} + \dots + r_1 s + r_0 = 0$$

which we may multiply by $(m')^{k-1}$ to get

$$(m's)^k + r_{k-1}(m's)^{k-1} + \dots + r_1(m')^{k-2}(m's) + r_0(m')^{k-1} = 0.$$

It follows that $m's \in A$, thus $s/1 \in M^{-1}A$ and thus $s/m \in M^{-1}A$.

Conversely, pick an element $a/m \in M^{-1}A$. We wish to show that it is integral over $M^{-1}R$. As $a \in A$, therefore we have

$$a^n + r_{n-1}a^{n-1} + \dots a_1 r + a_0 = 0$$

for $r_i \in R$. This equation in $M^{-1}S$ can be divided by m^n to obtain

$$\left(\frac{a}{m}\right)^n + \frac{r_{n-1}}{m}\left(\frac{a}{m}\right)^{n-1} + \dots + \frac{r_1}{m^{n-1}}\left(\frac{a}{m}\right) + \frac{r_0}{m^n} = 0.$$

It follows that a/m is integral over $M^{-1}R$, as required. □

An immediate, but important corollary of the above is the following.

Corollary 23.7.2.9. *Let A be a domain, K be its fraction field and $\tilde{A}$ be its normalization. Then, for all $g \in A$, we have $\tilde{A}_g = \widetilde{A_g}$ in K.* □

Another important corollary is that being a normal domain is alocal property.

Proposition 23.7.2.10. *Let R be a domain. Then the following are equivalent:*

1. *R is a normal domain.*
2. *$R_{\mathfrak{p}}$ is a normal domain for each prime $\mathfrak{p} \in \operatorname{Spec}(R)$.*
3. *$R_{\mathfrak{m}}$ is a normal domain for each maximal $\mathfrak{m} \in \operatorname{Spec}(R)$.*

Proof. By Proposition 23.7.2.8, we immediately have that (1. $\Rightarrow$ 2.) and (1. $\Rightarrow$ 3.). The (2. $\Rightarrow$ 3.) is immediate. We thus show (3. $\Rightarrow$ 1.). Let K be the fraction field of R. Observe that each $R_{\mathfrak{m}}$ is a domain and have fraction field K again, where $\mathfrak{m} \in \operatorname{Spec}(R)$ is a maximal ideal. Thus we have $R \hookrightarrow R_{\mathfrak{m}} \hookrightarrow K$. Pick $x \in K$ which satisfies a monic polynomial over R. It follows that x satisifes a monic polynomial over $R_{\mathfrak{m}}$ for each maximal $\mathfrak{m} \in \operatorname{Spec}(R)$. Thus $x \in R_{\mathfrak{m}}$ for each $\mathfrak{m}$ as $R_{\mathfrak{m}}$ is a normal domain. We thus deduce from Lemma 23.1.2.12 that $x \in \bigcap_{\mathfrak{m} \neq R} R_{\mathfrak{m}} = R$, as required. □

Remark 23.7.2.11 (*Normalization is a strongly local construction*)**.** Let A be an arbitrary domain. Then we get an inclusion $\varphi_A : A \hookrightarrow \tilde{A}$ where $\tilde{A}$ is the normalization of A in its fraction field. We claim that the collection of maps $\{\varphi_A : A \hookrightarrow \tilde{A}\}$ one for each domain is a construction which is strongly local on domains (see Definitions 1.6.2.3 & 1.6.2.4).

Indeed, first $\{\varphi_A : A \hookrightarrow \tilde{A}\}$ is a construction on domains as if $\eta : A \to B$ is an isomorphism, then we have an isomorphism $\tilde{\eta} : \tilde{A} \to \tilde{B}$ given as follows: we have an isomorphism $\bar{\eta} : K_A \to K_B$ between their fraction fields, given by $a/a' \mapsto \eta(a)/\eta(b)$. Now $a/a' \in K_A$ is integral over A if and only if $\eta(a)/\eta(a') \in K_B$ is integral over B. This shows that $\bar{\eta} : K_A \to K_B$ restricts to an isomorphism $\tilde{\eta} : \tilde{A} \to \tilde{B}$. Moreover, if $\eta : A \to A$ is id, then so is $\tilde{\eta}$ and it satisfies the square and cocycle condition as well of Definition 1.6.2.3. We now claim that normalization is strongly local.

Indeed, pick $g \in A$ non-zero. Then, the localization of the inclusion $\varphi_A : A \hookrightarrow \tilde{A}$ at element g yields $(\varphi_A)_g : A_g \hookrightarrow \tilde{A}_g = \widetilde{A_g}$ which is equal to the normalization of the domain $\varphi_{A_g} : A_g \hookrightarrow \widetilde{A_g}$. It follows that any integral scheme X admits a *normalization* in light of Theorem 1.6.2.10. Indeed, this is what is the content of Theorem 1.6.6.3.

We have a universal property for normalization of domains.

Proposition 23.7.2.12. *Let A be a domain and $\tilde{A}$ be the normalization of A in its fraction field. Then for any normal domain B and an injective map $A \hookrightarrow B$, there exists a unique map $\tilde{A} \to B$ such that following commutes:*

$$\begin{tikzcd} \tilde{A} \arrow[r, dashed] & B \\ A \arrow[u, hook] \arrow[ur, hook] & \end{tikzcd}.$$

Proof. Let $f : A \hookrightarrow B$. This, by universal property of fraction fields, induces a unique injective map $\varphi : K \hookrightarrow L$ from fraction field of A to that of B such that $\varphi|_A = f$. Let $x \in \tilde{A}$. Then

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

holds in K where $a_i \in A$. Applying φ on the above equation yields

$$\varphi(x)^n + f(a_{n-1})\varphi(x)^{n-1} + \cdots + f(a_1)\varphi(x) + f(a_0) = 0$$

in L. It follows that $\varphi(x)$ is an integral element of L over B. As B is normal it follows that $\varphi(x) \in B$. Consequently, we have a unique map

$$\varphi|_{\tilde{A}} : \tilde{A} \to B$$

such that the triangle commutes, as required. □

In certain situations (especially those arising in geometry and arithmetic), normalization preserves noetherian property. **TODO.**

23.7.3 Noether normalization lemma

Finally, as a big use of normalization in geometry, we obtain the following famous result.

Theorem 23.7.3.1. [9] *Let k be a field and A be a finite type k-algebra. Then, there exists elements $y_1, \dots, y_r \in A$ algebraically independent over k such that the inclusion $k[y_1, \dots, y_r] \hookrightarrow A$ is an integral map.*

Proof. Let us assume that k is infinite. Let $x_1, \dots, x_n \in A$ be generators of A as a k-algebra. Suppose there is no algebraically independent subset of $\{x_1, \dots, x_n\}$. Thus, each $x_1, \dots, x_n$ is integral over k. As $A = k[x_1, \dots, x_n]$, therefore by Proposition 23.7.1.8 it follows that A is integral over k, so there is nothing to show here.

Consequently, we may assume that there is a largest algebraically independent subset of $\{x_1, \dots, x_n\}$, denoted $\{x_1, \dots, x_r\}$. It follows that each $x_{r+1}, \dots x_n$ is integral/algebraic over k. If $r = n$, then A is the affine n-ring over k, so there is nothing to show. Consequently, we may assume that $n > r$. We now proceed by induction over n.

In the base case, we have $n = 1$, and thus $r < 1$. It follows that $A = k[x]$ where $x \in A$

[9] Exercise 5.16 of AMD.

is algebraically dependent over k, that is, x is integral over k. Consequently, A is integral over k by Lemma 23.7.1.6 and there is nothing to show. We now do the inductive case.

Assume that every finite type k-algebra $B \subseteq A$ with $n-1$ generators have elements $\{y_1, \dots, y_m\} \subseteq B$ algebraically independent over k such that B is integral over $k[y_1, \dots, y_m]$. Denote $A_{n-1} = k[x_1, \dots, x_{n-1}] \subseteq A$. It now suffices to find a finite type k-algebra $B \subseteq A$ generated by $n-1$ elements not containing x_n such that the following two statements hold about B:

1. $x_n \in A$ is integral over B,
2. $B[x_n] = A$.

For if such a B exists, then we have integral maps $k[y_1, \dots, y_m] \hookrightarrow B$ and $B \hookrightarrow B[x_n] = A$ (Proposition 23.7.1.5). Then, by Lemma 23.7.1.11, it follows that $k[y_1, \dots, y_m] \hookrightarrow A$ is integral, as needed.

Indeed, first observe that since x_n is algebraic over k and $k \subseteq A_{n-1}$, therefore x_n is algebraic over A_{n-1}. Consequently, there is a polynomial $f(z_1, \dots, z_{n-1}, z_n) \in k[z_1, \dots, z_n]$ of total degree N such that $f(x_1, \dots, x_{n-1}, x_n) = 0$. Using this, we now construct the required algebra B as follows. Let F be the highest degree homogeneous part of f and denote it by

$$F(z_1, \dots, z_n) = \sum_{i_1 + \dots + i_n = N} c_{i_1 \dots i_n} z_1^{i_1} \dots z_n^{i_n}$$

where $c_{i_1 \dots i_n}$ and is 0 for those indices which are not present in F and is 1 for those which are present. Let $(\lambda_1, \dots, \lambda_{n-1}) \in k^{n-1}$ be a tuple such that $F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$. Such a tuple exists because the field is infinite (n might be arbitrarily large). Consequently, for each $0 \leq i \leq n-1$, consider the following elements of A:

$$x_i' = x_i - \lambda_i x_n.$$

Let $B = k[x_1', \dots, x_{n-1}'] \subseteq A$. We now show that above two hypotheses are satisfied by B. This will conclude the proof. First, we immediately have the second hypothesis as $B[x_n] = k[x_1', \dots, x_{n-1}', x_n] = k[x_1, \dots, x_n] = A$. We thus need only show that x_n is integral over B. This also follows by the way of construction of B; consider the polynomial

$$g(z_1, \dots, z_{n-1}, z_n) := f(z_1 + \lambda_1 z_n, \dots, z_{n-1} + \lambda_{n-1} z_n, z_n)$$

in $k[z_1, \dots, z_{n-1}, z_n]$. We wish to show the following two items

1. $g(z_1, \dots, z_{n-1}, z_n)$ is monic in z_n,
2. $g(x_1', \dots, x_{n-1}', x_n) = 0$.

This would suffice as a polynomial in $B[z_n]$ is just a polynomial in $k[x_1', \dots, x_{n-1}', z_n]$.

Indeed, we see that

$$\begin{aligned}
g(z_1, \dots, z_{n-1}, z_n) &= f(z_1 + \lambda_1 z_n, \dots, z_{n-1} + \lambda_{n-1} z_n, z_n) \\
&= F(z_1 + \lambda_1 z_n, \dots, z_{n-1} + \lambda_{n-1} z_n, z_n) + \cdots \\
&= \sum_{i_1 + \cdots + i_n = N} c_{i_1 \dots i_n} (z_1 + \lambda_1 z_n)^{i_1} \dots (z_{n-1} + \lambda_{n-1} z_n)^{i_{n-1}} z_n^{i_n} + \cdots \\
&= \left(\sum_{i_1 + \cdots + i_n = N} c_{i_1 \dots i_n} \lambda_1^{i_1} z_n^{i_1} \dots \lambda_{n-1}^{i_{n-1}} z_n^{i_{n-1}} z_n^{i_n} \right) + \dots \\
&= z_n^N \left(\sum_{i_1 + \cdots + i_n = N} c_{i_1 \dots i_n} \lambda_1^{i_1} \dots \lambda_{n-1}^{i_{n-1}} \right) + \cdots \\
&= z_n^N F(\lambda_1, \dots, \lambda_{n-1}, 1) + \cdots .
\end{aligned}$$

It follows that g is monic in z_n and $g(x'_1, \dots, x'_{n-1}, x_n) = f(x_1, \dots, x_{n-1}, x_n) = 0$. This completes the proof. $\square$

23.7.4 Dimension of integral algebras

We will cover Cohen-Seidenberg theorems about primes in an integral extension. The main theorem will allow us to deduce that, apart from other things, dimension of an integral R-algebra is equal to that of R.

23.8 Dimension theory

We will discuss the notion of dimension of rings and how that notion corresponds to dimension of the corresponding affine scheme. Further, the notion of dimension applied to algebraic geometry will garnish us with a concrete geometric intuition to situations which otherwise may feel completely sterile.

23.8.1 Dimension, height & coheight

As usual, all rings are commutative with 1.

Definition 23.8.1.1. (**Dimension of a ring**) Let R be a ring. Then $\dim R$ is defined as follows

$$\dim R := \sup_r \{\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r \mid \mathfrak{p}_i \text{ are prime ideals of } R\}.$$

Definition 23.8.1.2. (**Height/coheight of a prime ideal**) Let R be a ring and $\mathfrak{p} \lneq R$ be a prime ideal. Then height of $\mathfrak{p}$ is defined as follows:

$$\text{ht } \mathfrak{p} := \sup_r \{\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r \mid \mathfrak{p}_i \text{ are prime ideals of } R\}.$$

Similarly, the coheight of $\mathfrak{p}$ is defined by

$$\text{coht } \mathfrak{p} := \sup_r \{\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \mid \mathfrak{p}_i \text{ are prime ideals of } R\}$$

Remark 23.8.1.3. Note that the dimension of a prime ideal $\mathfrak{p}$ as a ring may not be same as its height in R, as there might be many more primes in $\mathfrak{p}$ which may fail to be primes in the ring R. But clearly, $\dim \mathfrak{p} \geq \text{ht } \mathfrak{p}$.

Recall that the *dimension of a topological space* X is defined as

$$\dim X = \sup_r \{Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r \mid Z_i \text{ are irreducible closed subsets of } X\}.$$

We now have some immediate observations about height, coheight and dimension.

Lemma 23.8.1.4. *Let R be a ring. Then,*

1. $\text{ht } \mathfrak{p} = \dim R_{\mathfrak{p}}$,
2. $\text{coht } \mathfrak{p} = \dim R/\mathfrak{p}$,
3. $\text{ht } \mathfrak{p} + \text{coht } \mathfrak{p} \leq \dim R$.

Proof. Prime ideals of $R/\mathfrak{p}$ are in one-to-one order preserving bijection with prime ideals of R containing $\mathfrak{p}$. Prime ideals of $R_{\mathfrak{p}}$ are in one-to-one order preserving bijection with prime ideals of R contained in $\mathfrak{p}$. Let Y denote the length of all chains of prime ideals of R passing through $\mathfrak{p}$. Consequently, $\sup Y \leq \dim X$. But $\sup Y = \text{ht } \mathfrak{p} + \text{coht } \mathfrak{p}$. □

Lemma 23.8.1.5. *Let R be a PID. Then, $\dim R = 1$. Consequently, $\mathbb{Z}$ and $k[x]$ are one dimensional rings for any field k*[10].

[10] as the intuition agrees!

Proof. Any chain is either of the form $\langle 0\rangle$ or $\langle x\rangle \supsetneq \langle 0\rangle$. □

Further, by Theorem 23.1.5.3 we see the following.

Lemma 23.8.1.6. *If R is a PID which is not a field, then* $\dim R[x] = 2$.

Proof. Indeed, by Theorem 23.1.5.3, the longest chain of prime ideals of the form $\mathfrak{o} \lneq \langle f(x)\rangle \lneq \langle p, h(x)\rangle$ where $f(x)$ is irreducible and $h(x)$ is irreducible modulo prime $p \in R$, as one can see immediately. □

The following is also a simple assertion, which basically is why one introduces dimension of a ring.

Lemma 23.8.1.7. *Let R be a ring. Then,*

$$\dim \operatorname{Spec}(A) = \dim A.$$

Proof. Immediate from definitions and Lemma 1.2.1.1. □

Let us now give some more helpful notions, especially the dimension of an R-module.

Definition 23.8.1.8. (**Dimension of a module and height of ideals**) Let M be an R-module. Then the dimension of M is defined as

$$\dim M := \dim R/\operatorname{Ann}(M).$$

Further, for an ideal $I \leq R$, we define the height of I as the infimum of heights of all prime ideals above I:

$$\text{ht } I := \inf\{\text{ht } \mathfrak{p} \mid \mathfrak{p} \supseteq I,\ \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

We have the corresponding topological result.

Lemma 23.8.1.9. *Let R be a ring and M be a finitely generated R-module. Then,*

$$\dim M = \dim \operatorname{Supp}(M)$$

where $\operatorname{Supp}(M) \subseteq \operatorname{Spec}(R)$ *is the support of the module M.*

Proof. The result follows as $\operatorname{Supp}(M)$ is the closed subset $V(\operatorname{Ann}M)$ so that any irreducible closed set in $\operatorname{Supp}(M)$ will be irreducible closed in $\operatorname{Spec}(R)$ and then we can use Lemma 1.2.1.1. □

23.8.2 Dimension of finite type k-algebras

In algebraic geometry, one is principally interested in finite type algebras over a field. Thus it is natural to engage in the study of their dimensions. We discuss some elementary results in this direction in this section. See Section 23.1.6 for basics of finite type k-algebras.

The main results are as follows.

Theorem 23.8.2.1. *Let k be a field and A be a finite type k-algebra which is a domain*[11]*. Then,*

$$\dim A = \operatorname{trdeg} A/k.$$

Theorem 23.8.2.2. *Let k be a field and A be a finite type k-algebra which is a domain and let $\mathfrak{p} \lneq A$ be a prime ideal. Then,*

$$\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} = \dim A.$$

We now prove these results.

[11] note that such algebras are exactly the ones which correspond to affine algebraic varieties.

23.9 Completions

Do from Chapter 7 of Eisenbud

23.10 Valuation rings

We begin with the basic theory of valuation rings.

23.10.1 General theory

Definition 23.10.1.1. (**Valuation on a field**) Let K be a field and G be an abelian group. A function $v : K \to G \cup \{\infty\}$ is said to be a valuation of K with values in G if v satisfies

1. $v(xy) = v(x) + v(y)$,
2. $v(x+y) \geq \min\{v(x), v(y)\}$,
3. $v(x) = \infty$ if and only if $x = 0$.

Let $\mathrm{Val}(K, G)$ denote the set of all valuations over K with values in G.

Few immediate observations are in order.

Lemma 23.10.1.2. *Let K be a field, G be an abelian group and $v \in \mathrm{Val}(K, G)$ be a valuation. Then,*

1. $R = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$ *is a subring of K,*
2. $\mathfrak{m} = \{x \in K \mid v(x) > 0\} \cup \{0\}$ *is a maximal ideal of R,*
3. $(R, \mathfrak{m})$ *is a local ring,*
4. *R is an integral domain,*
5. $R_{\langle 0 \rangle} = K$,
6. $\forall x \in K$, $x \in R$ *or* $x^{-1} \in R$.

Proof. Items 1 and 4 are immediate from the axioms of valuations. Items 2 and 3 are immediate from the observation that $\{x \in K \mid v(x) = 0\} \cup \{0\}$ is a field in R. For items 5 and 6, we need to observe that $v(1) = 0$ and for any $x \in K^\times$, $v(x^{-1}) = -v(x)$. $\square$

Remark 23.10.1.3. We call the subring $R \subset K$ above corresponding to a valuation v over K to be the *value ring of v*.

Definition 23.10.1.4. (**Valuation rings**) Let R be an integral domain. Then R is said to be a valuation ring if it is the value ring of some valuation over $K = R_{\langle 0 \rangle}$.

Definition 23.10.1.5. (**Domination**) Let K be a field and $A, B \subset K$ be two local rings in K. Then B is said to dominate A if $B \supseteq A$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$.

There is an important characterization of valuation rings inside a field K with respect to all local rings in K.

Theorem 23.10.1.6. *Let K be a field and $R \subset K$ be a local ring. Denote $\mathrm{Loc}(K)$ to be the set of all local rings in K together with the partial order of domination. Then, the following are equivalent,*

1. *R is a valuation ring.*
2. *R is a maximal element of the poset $\mathrm{Loc}(K)$.*

Furthermore, for every local ring $S \in \mathrm{Loc}(K)$, there exists a valuation ring $R \in \mathrm{Loc}(K)$ which dominates S.

Proof. See Tag 00I8 of cite[Stacksproject]. □

An important type of valuation rings are where the value group is the integers.

Definition 23.10.1.7. (**Discrete valuation rings**) Let R be a domain. Then R is said to be a discrete valuation ring (DVR) if the value group of R is the integers $\mathbb{Z}$.

It turns out that noetherian local domains of dimension 1 have some important characterizations.

Theorem 23.10.1.8. *Let A be a noetherian local domain of dimension 1. Then the following are equivalent*

1. *A is a DVR,*
2. *A is a normal domain,*
3. *A is a regular local ring,*
4. *the maximal ideal of A is principal.*

Proof. Do it from Atiyah-Macdonald page 94. □

23.10.2 Absolute values

We discuss the basics of absolute values and places, which will be used to state Ostrowski's theorem which classifies the places of $\mathbb{Q}$.

23.11 Dedekind domains

We will now discuss a class of rings which forms the right context for number theory. We give here the barebones, rest will be developed as needed elsewhere.

Definition 23.11.0.1 (**Dedekind domain**)**.** A noetherian normal domain of dimension 1 is defined to be a Dedekind domain.

Theorem 23.11.0.2. *Let R be a noetherian domain of dimension 1. Then the following are equivalent:*

1. *R is normal (equivalently, Dedekind).*
2. *Every primary ideal $\mathfrak{q}$ of R is of the form $\mathfrak{q} = \mathfrak{p}^n$ for some prime ideal $\mathfrak{p}$ and $n \geq 0$.*
3. *$R_{\mathfrak{p}}$ is a DVR for each non-zero prime $\mathfrak{p}$.*

Proof. **TODO**. □

23.12 Tor and Ext functors

Start doing from Appendix 3 of Eisenbud.

23.12.1 Some computations

Do exercises from Bruzzo.

23.13 Projective and injective modules

In this section we define an important object in the study of algebraic K-theory, projective modules. These generalize finitely generated free R-modules. This notion is further used in a very important geometric concept called depth and Cohen-Macaulay condition. In order to reach there, we would need a concept called projective dimension, which we cover here.

23.13.1 Divisible modules and Baer's criterion

Baer's criterion gives a characterization of injective R-modules. It consequently helps to show that divisible modules are injective in $\mathbf{Mod}(R)$ and thus that $\mathbf{Mod}(R)$ has enough injectives.

Definition 23.13.1.1 (**Divisible modules**). An R-module M is said to be divisible if for every $r \in R$, the multiplication by r map $\mu_r : M \to M$ is surjective.

Theorem 23.13.1.2. *(Baer's criterion) Let R be a ring and M be an R-module. The following are equivalent:*

1. *M is an injective R-module.*
2. *For any ideal $\mathfrak{a} \leq R$ and any map $f : \mathfrak{a} \to M$, there exists an extension $g : R \to M$ such that the following commutes:*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dashed, "g"] \\ & M & \end{tikzcd}.$$

That is, one needs to check injectivity condition along inclusions of submodules of R.

Proof. 1. $\Rightarrow$ 2. is immediate from definition. For 2. $\Rightarrow$ 1. we proceed as follows. Pick $i : A \to B$ an injection of submodule $A \leq B$ and a map $f : A \to M$. We wish to extend this to $g : B \to M$. Indeed, consider the poset $\mathcal{P}$ of tuples (A', f'), $f' : A' \to M$ an extension of f with $(A', f') \leq (A'', f'')$ such that $A' \subseteq A''$ and f'' extends f'. By Zorn's lemma, we have a maximal extension $\bar{f} : \bar{A} \to M$. We reduce to showing that $\bar{A} = B$. If not, then there is $b \in B \setminus \bar{A}$. Consider $\tilde{A} = Rb + \bar{A}$. We claim that there is a map $\tilde{f} : \tilde{A} \to M$ extending f. Indeed, consider the ideal $\mathfrak{a} = \{r \in R \mid rb \in \bar{A}\}$. The map $\bar{f}$ defines a map $\mathfrak{a} \to M$ given by $r \mapsto \bar{f}(rm)$. By hypothesis, this has an extension, say $\kappa : R \to M$. Thus, we may define $g : \tilde{A} \to M$ as $rb + \bar{a} \mapsto \kappa(r) + \bar{f}(\bar{a})$. This extends f as if $rb + \bar{a} \in A$, then $rb \in \bar{A}$. Consequently, $\kappa(r) + \bar{f}(\bar{a}) = \bar{f}(rb) + \bar{f}(\bar{a}) = \bar{f}(rb + \bar{a}) = f(rb + \bar{a})$, as needed. □

As a corollary, we see that injective R-modules are divisible.

Corollary 23.13.1.3. *Let R be a ring and M be an R-module. If M is injective, then M is divisible.*

Proof. Pick any $m \in M$ and $r \in R$. Then, we have an R-linear map $\mu_r : \langle r \rangle \to M$ given by $r \mapsto m$. By Theorem 23.13.1.2, 2, this extends to an R-linear homomorphism $g : R \to M$ where $\mu_r(r) = g(r) = rg(1) = m$, Thus $g(1) \in M$ is such that $rg(1) = m$, as needed. □

23.14 Multiplicities

23.15 Kähler differentials

23.16 Depth and Cohen-Macaulay

23.17 Filtrations

Do from Chapter 5 of Eisenbud

23.18 Flatness

Complete this from Eisenbud Chapter 7 and appendix of Sernesi on Flatness, especially Proposition A.2.

This is one of the important parts of commutative algebra, as this notion corresponds to the idea of a continuous family of schemes, in some sense, as is discussed in the respective part above.

Definition 23.18.0.1. (**Flat modules and flat map of rings**) Let R be a ring. An R-module M is said to be flat if for any short exact sequence of R-modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

the following sequence is exact

$$0 \longrightarrow M \otimes_R N_1 \longrightarrow M \otimes_R N_2 \longrightarrow M \otimes_R N_3 \longrightarrow 0 \ .$$

A map $\varphi : A \to B$ is a flat map if B is a flat A-module. In this case one also calls B to be a flat A-algebra.

Remark 23.18.0.2.
1. By right exactness of tensor products, it is sufficient to check that the s.e.s. $0 \to N_1 \to N_2$ is taken to s.e.s $0 \to M \otimes_R N_1 \to M \otimes_R N_2$.
2. Since localisation is an exact functor (Lemma 23.1.2.2), thus the natural map $A \to S^{-1}A$ is a flat map for any multiplicative set $S \subseteq A$.

23.19 Lifting properties : Étale maps

23.20 Lifting properties : Unramified maps

23.21 Lifting properties : Smooth maps

23.22 Simple, semisimple and separable algebras

These algebras are at the heart of the Galois phenomenology, i.e. all things related to polynomials splitting in a bigger field or not. Our study of these objects will thus motivate the study of the corresponding geometrical picture.

Find and write r
about these alge
Chapter 23.

23.22.1 Semisimple algebras

Definition 23.22.1.1. (**Semisimple algebras over a field** k) Let A be a k-algebra. Then A is a semisimple k-algebra if the Jacobson radical of A is 0.

23.22.2 Separable algebras

We will first study a rather special type of separable algebras, which are finitely generated and free as modules. Let us first give an example of such an algebra which is motivating our definition given later.

Example 23.22.2.1. Consider a ring A and the A-algebra A^n. There is something special about A^n; it is "separated" into finitely pieces which looks like A. This can be formalized. Indeed, we have the most obvious fact about such algebras that the obvious map

$$\begin{aligned} \varphi : A^n &\longrightarrow \operatorname{Hom}_A(A^n, A) \\ (a_1, \dots, a_n) &\longmapsto e_i \mapsto a_i \end{aligned}$$

is an isomorphism of A-algebras. More specifically, the map φ takes $(a_i) = (a_1, \dots, a_n)$ to the following mapping

$$\begin{aligned} \varphi((a_i)) : A^n &\longrightarrow A \\ (b_1, \dots, b_n) &\longmapsto a_1 b_1 + \dots + a_n b_n. \end{aligned}$$

We now wish to generalize this. That is to say, taking above phenomenon as a definition we want to generalize when an A-algebra B "separates" into simple pieces. For this to work, we need to find an alternate characterization of the above phenomenon. For this, a little bit of thought shows that the above map is obtained as the dual map of the $\phi \in \operatorname{Hom}_A(A^n, \operatorname{Hom}_A(A^n, A))$ under the $\otimes$-Hom adjunction

$$\operatorname{Hom}_A(A^n \times A^n, A) \cong \operatorname{Hom}_A(A^n, \operatorname{Hom}_A(A^n, A))$$

where the isomorphism is given by

$$\left(A^n \times A^n \xrightarrow{f} A\right) \longmapsto \left((a_i) \mapsto \left((b_i) \mapsto f((a_i), (b_i))\right)\right).$$

Now, consider the map

$$\begin{aligned} \tilde{\phi} : A^n \times A^n &\longrightarrow A \\ ((a_i), (b_i)) &\longmapsto \sum_{i=1}^{n} a_i b_i. \end{aligned}$$

The tensor-hom isomorphism tells us that $\tilde{\phi}$ is the dual map of ϕ above. Now notice that this dual map $\tilde{\phi}$ has a very simple description; it is given by the following commutative diagram:

$$\begin{tikzcd} A^n \times A^n \arrow[r, "\tilde{\phi}"] \arrow[d] & A \\ \mathrm{Hom}_A(A^n, A^n) \arrow[ur, "\mathrm{Tr}"'] & \end{tikzcd}.$$

It is this dual map that we shall generalize to the setting of arbitrary A-algebra B which is finitely generated and free of rank n. Indeed, for any A-algebra B and chose any generating set of B as an A-module, so that for any element $b \in B$, we can write $b = (b_1, \ldots, b_n) \in A^n$. We thus get a natural map $\tilde{\phi}$ as in the diagram below

$$\begin{tikzcd} (b, c) \arrow[d] & B \times B \arrow[r, "\tilde{\phi}"] \arrow[d, "\kappa"'] & A \\ (b_i c_j)_{1 \le i,j \le n} & \mathrm{Hom}_A(B, B) \arrow[ur, "\mathrm{Tr}"'] & \end{tikzcd}.$$

Now, consider the tensor-hom dual of $\tilde{\phi}$ to obtain

$$\begin{aligned} \phi : B &\longrightarrow \mathrm{Hom}_A(B, A) \\ b &\longmapsto \left(c \mapsto \tilde{\phi}(b, c)\right). \end{aligned}$$

In order to mimic the case of A^n, we would require the map ϕ to be an isomorphism. Indeed, this is what we do in the definition given below.

Before defining a nice class of separable algebras, let us define an A-algebra B to be *finitely free* if B is finitely generated and free as an A-module.

Definition 23.22.2.2. (**Free separable algebras**) Let A be a ring and B be a finitely free A-algebra of rank n and chose a generating set of B, so for $b \in B$, we can write $b = (b_1, \ldots, b_n)$ for $b_i \in A$. Define $\tilde{\varphi}$ to be the following map

$$\begin{tikzcd} (b, c) \arrow[d] & B \times B \arrow[r, "\tilde{\varphi}"] \arrow[d, "\kappa"'] & A \\ (b_i c_j)_{1 \le i,j \le n} & \mathrm{Hom}_A(B, B) \arrow[ur, "\mathrm{Tr}"'] & \end{tikzcd}.$$

Then B is said to be a separable A-algebra if the tensor-hom dual map $\varphi : B \to \mathrm{Hom}_A(B, A)$ is an isomorphism of A-algebras.

We would now like to show how separable algebras become familiar in the case of algebras over a field.

Proposition 23.22.2.3. *Let k be a field and A be an k-algebra. Then, the following are equivalent*

1. *A is a free separable k-algebra.*

2. $A = \prod_{i=1}^{n} K_i$ *where* K_i *are finite separable extensions of field* k.

tant exercise, er 23.

Proof. ☐

Another characterization of separable algebras is as follows.

Lemma 23.22.2.4. *Let* A *be a ring and* B *be a finitely free* A*-algebra. Then the following are equivalent.*

1. B *is a separable* A*-algebra.*
2. *For all* $\{w_1, \ldots, w_n\}$ *in* B *which is a generating set of free* A*-module* B*, we have*

$$\det\left(\mathrm{Tr}(w_i w_j)_{1\leq i,j\leq n}\right) \in A^{\times}.$$

Proof. ☐

Important exerc Chapter 23.

23.23 Miscellaneous

We collect in this section results which so far doesn't fit in any other prior section. Perhaps this means our arrangement of material is not optimal.

The following result is a generalization of Lagrange interpolation formula.

Lemma 23.23.0.1. *Let* K/F *be an algebraic field extension. Then for any* $\alpha_1, \ldots, \alpha_n \in K$*, such that* α_i *is not equal to any* α_j *nor any of its conjugate, and for any choice* $\beta_1, \ldots, \beta_n \in K$*, there exists a polynomial* $f(x) \in F[x]$ *such that* $f(\alpha_i) = \beta_i$ *for all* $i = 1, \ldots, n$.

Proof. Let $\alpha_1, \ldots, \alpha_n \in K$ be such that α_j is not equal to α_i nor any of its conjugates for any $j \neq i$. Let $\beta_1, \ldots, \beta_n \in K[\alpha_i]$. We wish to find a polynomial $f(x) \in F[x]$ such that $f(\alpha_i) = \beta_i$ for each $i = 1, \ldots, n$.

We first observe that as K is an algebraic extension of F, therefore there exists $p_i(x) \in F[x]$ which is the minimal polynomial of $\alpha_i \in K$. This polynomial is obtained by looking at the kernel of evaluation at α_i, $\varphi_i : F[x] \to K$ where $x \mapsto \alpha_i$. Consequently, $p_i(x)$ is a monic irreducible polynomial of least degree in $F[x]$ such that $p_i(\alpha_i) = 0$, for each $i = 1, \ldots, n$.

As $\mathfrak{m}_i := \langle p_i(x)\rangle \leq F[x]$ are maximal ideals and $p_i(x) \neq p_j(x)$ because $\alpha_i \neq \alpha_j, \overline{\alpha_j}$[12], therefore $\mathfrak{m}_i + \mathfrak{m}_j = F[x]$ for all $i \neq j$. Hence $\mathfrak{m}_i$ are comaximal. Consequently, we obtain by Chinese remainder theorem that

$$\begin{array}{ccccccc}
F[x] & \twoheadrightarrow & \frac{F[x]}{\mathfrak{m}_1\ldots\mathfrak{m}_n} & \xrightarrow{\cong} & \frac{F[x]}{\mathfrak{m}_1}\times\cdots\times\frac{F[x]}{\mathfrak{m}_n} & \xrightarrow{\cong} & F[\alpha_1]\times\cdots\times F[\alpha_n] \\
f(x) & \longmapsto & f(x)+\mathfrak{m}_1\ldots\mathfrak{m}_n & \longmapsto & (f(x)+\mathfrak{m}_i)_i & \longmapsto & (f(\alpha_1),\ldots,f(\alpha_n))
\end{array}.$$

Consequently, by above diagram, for the elements $(\beta,1\ldots,\beta_n) \in F[\alpha_1]\times\cdots\times F[\alpha_n]$, there exists a polynomial $f(x) \in F[x]$ such that $(f(\alpha_1),\ldots,f(\alpha_n)) = (\beta_1,\ldots,\beta_n)$. Hence $f(\alpha_i) = \beta_i$ for each $i = 1, \ldots, n$. This completes the proof. ☐

The following is a general exercise in basic ideal theory.

[12] because conjugates have same minimal polynomials.

Lemma 23.23.0.2. *Let R be a commutative ring with unity. Let $\mathfrak{p} \lneq R$ be a prime ideal and $I, J \leq R$ be ideals. Then,*

1. *$I^k \subseteq \mathfrak{p}$ for some $k \geq 0$ implies $I \subseteq \mathfrak{p}$,*
2. *the following are equivalent:*
   (a) *$\sqrt{I} + \sqrt{J} = R$,*
   (b) *$I + J = R$,*
   (c) *$I^k + J^l = R$ for all $k, l > 0$.*

Proof. 1. Let $I \leq R$ be an ideal and $\mathfrak{p} \lneq \mathbb{R}$ be a prime ideal. Then, we wish to show that $I^k \subseteq \mathfrak{p} \implies I \subseteq \mathfrak{p}$ for any $k \in \mathbb{N}$.

Indeed, pick any $x \in I$. As $x^k \in I$, therefore $x^k \in \mathfrak{p}$. As $x^k = x \cdot x^{k-1} \in \mathfrak{p}$, therefore either $x \in \mathfrak{p}$ of $x^{k-1} \in \mathfrak{p}$. If the former, then we are done. If the latter, then we have $x^{k-1} = x \cdot x^{k-2} \in \mathfrak{p}$. Continuing in this manner, we eventually reach to the conclusion that $x \in \mathfrak{p}$.

2. ((a) $\Rightarrow$ (b)) : As we have $x \in \sqrt{I}$ and $y \in \sqrt{J}$ such that $x + y = 1$, therefore for some $n, m \in \mathbb{N}$ we have $x^n \in I$ and $y^m \in J$. Now, observe that

$$\begin{aligned} 1 = 1^{n+m} = (x+y)^{n+m} &= \sum_{r=0}^{n+m} {}^{n+m}C_r x^r y^{n+m-r} \\ &= \sum_{r=0}^{n} {}^{n+m}C_r x^r y^{n+m-r} + \sum_{r=n+1}^{n+m} {}^{n+m}C_r x^r y^{n+m-r}. \end{aligned}$$

If $0 \leq r \leq n$, then $y^{n+m-r} \in J$ and if $n+1 \leq r \leq n+m$, then $x^r \in I$. Hence $\sum_{r=0}^{n} {}^{n+m}C_r x^r y^{n+m-r} \in J$ and $\sum_{r=n+1}^{n+m} {}^{n+m}C_r x^r y^{n+m-r} \in I$. This shows that there exists $a \in I$ and $b \in J$ then $a + b = 1$.

((b) $\Rightarrow$ (c)) : As we have $x \in I$ and $y \in J$ such that $x + y = 1$, thus writing $1 = 1^{k+l}$ again, we see

$$\begin{aligned} 1 &= 1^{k+l} = (x+y)^{k+l} \\ &= \sum_{r=0}^{k+l} {}^{k+l}C_r x^r y^{k+l-r} \\ &= \sum_{r=0}^{k} {}^{k+l}C_r x^r y^{k+l-r} + \sum_{r=k+1}^{k+l} {}^{k+l}C_r x^r y^{k+l-r}. \end{aligned}$$

If $0 \leq r \leq k$, then $y^{k+l-r} \in J^l$ and if $k+1 \leq r \leq k+l$, then $x^r \in I^k$. Consequently, we have $\sum_{r=0}^{k} {}^{k+l}C_r x^r y^{k+l-r} \in J^l$ and $\sum_{r=k+1}^{k+l} {}^{k+l}C_r x^r y^{k+l-r} \in I^k$. Hence there exists $a \in I^k$ and $b \in J^l$ such that $a + b = 1$.

((c) $\Rightarrow$ (a)) : Setting $k = l = 1$, we have that there exists $x \in I$ and $y \in J$ such that $x + y = 1$. As $\sqrt{I} \supseteq I$ and $\sqrt{J} \supseteq J$, therefore $x \in \sqrt{I}$ and $y \in \sqrt{J}$ such that $x + y = 1$. Hence $\sqrt{I} + \sqrt{J} = R$. This completes the proof. $\square$

The following is a counterexample to the claim that a sub-algebra of a finite type algebra is a finite type algebra.

Lemma 23.23.0.3. *Let R be a ring. The ring $R[t, tx, tx^2, \dots, tx^i, \dots]$ is neither a finite type R-algebra nor a finite type $R[t]$-algebra.*

Proof. Let $S = R[t, tx, tx^2, tx^3, \dots]$. We wish to show that S is not a finitely generated R or $R[t]$ algebra.

a) We first show that S is not finitely generated R-algebra. Indeed, let $p_1, \dots, p_n \in S$ be generators of S as an R-algebra. Then, we have that $p_i \in R[t, tx, \dots, tx^{m_i}]$ as a polynomial can atmost be in finitely many indeterminates. Hence, letting $M = \max_i m_i$, we obtain that $p_1, \dots, p_n \in R[t, tx, \dots, tx^M]$. It then follows that the R-algebra generated by $p_1, \dots, p_n$ will only be inside $R[t, tx, \dots, tx^M]$. We consequently reduce to showing that $R[t, tx, \dots, tx^M] \neq S$.

Let $tx^{M+1} \in S$. We claim that $tx^{M+1} \notin R[t, tx, \dots, tx^M]$. Assuming to the contrary, we have that for some $a_{k_0,\dots,k_M} \in R$

$$\begin{aligned} tx^{M+1} &= \sum_{k_0,\dots,k_M} a_{k_0,\dots,k_M} t^{k_0} \dots (tx^M)^{k_M} \\ &= \sum_{k_0,\dots,k_M} a_{k_0,\dots,k_M} t^{k_0+\dots+k_M} \cdot x^{k_1+2k_2+\dots+Mk_M}. \end{aligned}$$

We thus deduce that $a_{k_0,\dots,k_M} \neq 0$ if and only if $k_0 + \dots + k_M = 1$. As $k_i \in \mathbb{Z}_{\geq 0}$, we further deduce that the only non-zero coefficients are $a_{1,0,\dots,0}, a_{0,1,\dots,0}, \dots, a_{0,0,\dots,1}$. Hence, the above equation reduces to

$$tx^{M+1} = a_{1,0,\dots,0} t + a_{0,1,\dots,0} tx + \dots + a_{0,0,\dots,1} tx^M.$$

Clearly, for no choice of coefficients $a_{1,0,\dots,0}, a_{0,1,\dots,0}, \dots, a_{0,0,\dots,1}$ in R can we make both sides equal in $R[t, x]$. This is a contradiction.

b) We now wish to show that S is not finitely generated as an $R[t]$-algebra. Assuming to the contrary, there exists $p_1, \dots, p_n \in S$ such that S is generated by them as an $R[t]$-algebra. Again for the same reason as in a), we see that $p_1, \dots, p_n \in R[t, tx, \dots, tx^M]$ for some $M \in \mathbb{Z}_{>0}$. Now, as $R[t, tx, \dots, tx^M] = R[t][tx, tx^2, \dots, tx^M]$, therefore the $R[t]$-algebra generated by $p_1, \dots, p_n$ will only be inside $R[t][tx, tx^2, \dots, tx^M]$. Hence, we reduce to showing that $R[t][tx, tx^2, \dots, tx^M] \neq S$. To this end, the exact same technique as in part a) works verbatim, as we need only show that $tx^{M+1} \notin R[t][tx, tx^2, \dots, tx^M] = R[t, tx, \dots, tx^M]$.

This completes the proof. $\square$

The following result characterizes all ideals of $F[[x]]$, yielding that $F[[x]]$ is a local PID, i.e. a DVR, and tells us that localization of $F[[x]]$ at the local parameter x yields the Laurent series ring, i.e. the fraction field of $F[[x]]$.

Proposition 23.23.0.4. *Let F be a field and $R = F[[x]]$.*

1. *An element in $a = a_0 + a_1 x + \dots \in R$ is a unit if and only if $a_0 \neq 0$.*
2. *Every non-zero ideal of R is of the form $x^k R$.*
3. *$R[x^{-1}] = Q(R) = F((x))$.*

Proof. 1. ($\Rightarrow$) Since $\sum_{i\geq 0} a_i x^i$ is a unit in $F[[x]]$, therefore there exists $\sum_{i\geq 0} b_i x^i$ which is an inverse of $\sum_{i\geq 0} a_i x^i$. Consequently, we have

$$\begin{aligned} (a_0 + a_1 x + \dots) \cdot (b_0 + b_1 x + \dots) &= 1 \\ (a_0 b_0 + (a_1 b_0 + a_0 b_1) x + \dots) &= 1. \end{aligned}$$

Comparing the degree 0 term both sides, we obtain $a_0b_0 = 1$. Therefore, if $a_0 = 0$, then $a_0b_0 = 0$ and we would thus obtain a contradiction.
($\Leftarrow$) Suppose $a_0 \neq 0$. We wish to find $\sum_{i\geq 0} b_ix^i$ such that $\left(\sum_{i\geq 0} a_ix^i\right) \cdot \left(\sum_{j\geq 0} b_jx^j\right) = 1$. We can calculate what b_is should be by observing the following:

$$\left(\sum_{i\geq 0} a_ix^i\right) \cdot \left(\sum_{j\geq 0} b_jx^j\right) = \sum_{k\geq 0} c_kx^k$$

where $c_k = \sum_{i+j=k} a_ib_j$. We now claim that there exists a unique solution for each b_i in the equations given by setting $c_0 = 1$ and $c_k = 0$ for all $k \geq 1$. We show this by strong induction. Indeed, for $c_0 = a_0b_0 = 1$ yields that $b_0 = a_0^{-1}$. For $k = 1$, we have $c_1 = a_1b_0 + a_0b_1 = 0$ which thus yields $b_1 = -a_0^{-1}a_1b_0$. We now wish to show that if b_l has a unique solution for all $l = 0, \dots k-1$, then b_k has a unique solution as well. Indeed, b_k satisfies the following equation coming from $c_k = 0$:

$$\begin{aligned} 0 &= \sum_{i+j=k} a_ib_j \\ &= a_0b_k + \sum_{i+j=k, j<k} a_ib_j. \end{aligned}$$

By inductive hypothesis, for all $0 \leq j < k$, b_j has a unique solution. Consequently by the above, b_k has a unique solution as well. This completes the induction which yields the required formal power series.

$\sum_{j\geq 0} b_jx^j$ which acts as the inverse of $\sum_{i\geq 0} a_ix^i$. 2. We wish to show that any non-zero ideal $I \leq R$ is of the form $I = x^kR$ where $k \in \mathbb{N}$. Pick any ideal $I \leq R$. For any power series $p(x) = c_nx^n + c_{n+1}x^{n+1} + \dots$ where $c_n \neq 0$, we define n to be the ***co-degree*** of $p(x)$. Then, let $p(x) = c_kx^k + c_{k+1}x^{k+1} + \dots$ be the element of I with least co-degree (such an element exists by virtue of well-ordering of $\mathbb{N}$). Consequently, we obtain $p(x) = x^k(c_k + c_{k+1}x + \dots)$.

We thus claim that $I = x^kR$. Indeed, pick any $f(x) \in I$. Then, $f(x) = d_nx^n + d_{n+1}x^{n+1} + \dots$ where $d_n \neq 0$. Hence, we may write $f(x) = x^n(d_n + d_{n+1}x + \dots)$. By item 1, we know that $d_n + d_{n+1}x + \dots$ is a unit in R, so that we may write $f(x) = x^nu$, $u \in R$ is a unit. Now, as $f(x) \in I$, thus co-degree of f is atleast k as $p(x) \in I$ with co-degree k is the least co-degree element. Consequently, we may write $f(x) = x^kx^{n-k}u$. Hence $f(x) \in x^kR$. Conversely, pick any $x^kg(x) \in x^kR$. Since $p(x) = x^k(c_k + c_{k+1}x + \dots)$ where $c_k \neq 0$, therefore $c_k + c_{k+1}x + \dots$ is a unit, hence $p(x) = x^kv$ for some unit $v \in R$. Thus, $x^k \in I$ and hence $x^kg(x) \in I$. This completes the proof.

3. We wish to show that $R[\frac{1}{x}] = Q(R)$, the fraction field of R, i.e. $F((x))$. Indeed, as $x \in R$ is a non-zero element, therefore $1/x \in Q(R)$ and consequently, $R[\frac{1}{x}] \subseteq Q(R)$. We now wish to show that converse also holds.

Pick any $\frac{f(x)}{g(x)} \in Q(R)$ where $f(x), g(x) \in R$ are power series. Let $f(x)$ have co-degree n and $g(x)$ have co-degree m. We may then write

$$\frac{f(x)}{g(x)} = \frac{c_nx^n + c_{n+1}x^{n+1} + \dots}{d_mx^m + d_{m+1}x^{m+1} + \dots}$$

where $c_n, d_m \neq 0$. We may further write above as

$$\frac{f(x)}{g(x)} = \frac{x^n u}{x^m v}$$

for units $u = c_n + c_{n+1}x + \ldots, v = d_m + d_{m+1}x + \cdots \in R$ (by item 1).

If $n > m$, then $\frac{f(x)}{g(x)} = \frac{x^{n-m}w}{1}$ for some unit $w \in R$ and we know that $\frac{x^{n-m}}{1} \in R[\frac{1}{x}]$. If $n < m$, then $\frac{f(x)}{g(x)} = \frac{w}{x^{m-n}}$ for some unit $w \in R$ and we know that $\frac{1}{x^{m-n}} \in R[\frac{1}{x}]$. Finally if $n = m$, then $\frac{f(x)}{g(x)}$ is a unit of R and hence of $R[\frac{1}{x}]$.

Hence in all cases, $\frac{f(x)}{g(x)} \in R[\frac{1}{x}]$. We thus conclude $Q(R) \subseteq R[\frac{1}{x}]$, completing the proof. □

In the following theorem, we show some important properties of the ring $\mathbb{Z}[\omega]$, where ω is a third root of unity.

Theorem 23.23.0.5. *Let $R = \mathbb{Z}[\omega]$ where $\omega = e^{\frac{2\pi i}{3}}$ is a cube root of unity.*

1. *R is a Euclidean domain.*
2. *The function given by*

$$f : \operatorname{Spec}(\mathbb{Z}[\omega]) \longrightarrow \operatorname{Spec}(\mathbb{Z})$$
$$\pi \longmapsto \begin{cases} p & \text{if } \pi = p \text{ upto associates,} \\ \pi\bar{\pi} & \text{else.} \end{cases}$$

is surjective such that $f^{-1}(p)$ is either $\{\pi, \bar{\pi}\}$ or $\{p\}$ (upto associates) for any prime $p \in \operatorname{Spec}(\mathbb{Z})$.

3. *Let $p \in \mathbb{Z}$ be a prime. The following are equivalent:*
   (i) p splits in $\mathbb{Z}[\omega]$, that is $p = \alpha\bar{\alpha}$ for some $\alpha \in \mathbb{Z}[\omega]$,
   (ii) $x^2 \pm x + 1$ has a root in $\mathbb{F}_p$, that is, $\exists a \in \mathbb{F}_p$ such that $a \neq 1$ and $a^3 = \pm 1$,
   (iii) either $p = 3$ or $p = 1 \mod 3$.
4. *Take any $n \in \mathbb{Z}$. The following are equivalent:*
   (i) $n = a^2 \pm ab + b^2$ for some $a, b \in \mathbb{Z}$,
   (ii) primes $2 \mod 3$ occurs evenly many times in the prime factorization of n.

Proof. 1. We first wish to show that R is a Euclidean domain. We claim that the following function

$$d : R \setminus \{0\} \longrightarrow \mathbb{N} \cup \{0\}$$
$$\alpha = a + b\omega \longmapsto \alpha\bar{\alpha} = a^2 + b^2 - ab$$

satisfies the axiom of size function for R. Indeed, pick any $\alpha, \beta \in R$ where $\beta \neq 0$. We may then write

$$\frac{\alpha}{\beta} = \frac{\alpha\bar{\beta}}{\beta\bar{\beta}} = \frac{\alpha\bar{\beta}}{c} = a + ib$$

where $a, b \in \mathbb{Q}$. As any rational $x \in \mathbb{Q}$ can be written as $x = n + q$ where $n \in \mathbb{Z}$ and $0 \leq q \leq 1/2$, therefore we may write

$$\frac{\alpha}{\beta} = a + ib = (n_1 + r_1) + \omega(n_2 + r_2)$$

where $n_1, n_2 \in \mathbb{Z}$ and $0 \leq r_1, r_2 \leq 1/2$. Thus,

$$\alpha = \beta(n_1 + \omega n_2) + \beta(r_1 + \omega r_2) \tag{1.1}$$

As $\alpha, \beta(n_1 + \omega n_2) \in R$, therefore by (1.1) we deduce that $\beta(r_1 + \omega r_2) \in R$.
Note that since the size function d is the norm map, which is actually a multiplicative map defined on whole of $\mathbb{C}$ as

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{R} \\ z &\longmapsto z\bar{z}, \end{aligned}$$

hence, we see that

$$\begin{aligned} d(\beta(r_1 + \omega r_2)) &= \beta\bar{\beta}(r_1^2 + r_2^2 - r_1 r_2) \\ &\leq \beta\bar{\beta}\left(\frac{1}{2^2} + \frac{1}{2^2}\right) \\ &= \frac{\beta\bar{\beta}}{2} \\ &< \beta\bar{\beta} \\ &= d(\beta). \end{aligned}$$

Thus, Eq. (1.1) is the required division of α by β. This proves that R is a Euclidean domain.

2. Let R be an arbitrary Euclidean domain and let $\operatorname{Spec}(R)$ denote the set of all prime ideals of R. As R is a Euclidean domain, therefore it is a PID. Consequently, $\operatorname{Spec}(R)$ is in one-to-one bijection with prime/irreducible elements of R together with 0. Hence, we write $p \in \operatorname{Spec}(R)$ to mean a prime element of R. We know that $\mathbb{Z}[\omega]$ and $\mathbb{Z}$ are Euclidean domains.
We wish to show that there is a surjective map

$$\begin{aligned} f : \operatorname{Spec}(\mathbb{Z}[\omega]) &\longrightarrow \operatorname{Spec}(\mathbb{Z}) \\ \pi &\longmapsto \begin{cases} p & \text{if } \pi = p \text{ upto associates,} \\ \pi\bar{\pi} & \text{else.} \end{cases} \end{aligned}$$

such that $f^{-1}(p)$ is either $\{\pi, \bar{\pi}\}$ or $\{p\}$ (upto associates) for any prime $p \in \operatorname{Spec}(\mathbb{Z})$ where $\pi \in \operatorname{Spec}(\mathbb{Z}[\omega])$ is a prime element.
We first observe that $\mathbb{Z}[\omega]$ has a non-trivial automorphism given by $\alpha = a + b\omega \mapsto \bar{\alpha} = a + b\omega^2$. Pick $\pi \in \operatorname{Spec}(\mathbb{Z}[\omega])$ a non-zero prime element. Observe that automorphisms takes a prime element to a prime element. As $\mathbb{Z}$ is a UFD, therefore for $p_1, \dots, p_l \in$

Spec $(\mathbb{Z})$ non-zero primes, and $\pi_1, \dots, \pi_k \in$ Spec $(\mathbb{Z}[\omega])$ non-zero primes, we may write

$$\begin{aligned} \pi\bar{\pi} &= a^2 + b^2 - ab \\ &= p_1 \dots p_l \\ &= \pi_1 \dots \pi_k \end{aligned}$$

where the last equality comes from writing prime factorization of each p_i in $\mathbb{Z}[\omega]$. Now, as $\mathbb{Z}[\omega]$ is a UFD, therefore $k = 2$ and hence $l \leq 2$. We now have two cases

(i) If $l = 2$, then $\pi\bar{\pi} = p_1 p_2$. Expanding each p_i into product of primes in $\mathbb{Z}[\omega]$, we immediately deduce by unique factorization in $\mathbb{Z}[\omega]$ that $p_1 = \pi$ and $p_2 = \bar{\pi}$ upto associates (wlog). Hence, $\bar{\pi} = p_2 = p_1$. That is,

$$\pi\bar{\pi} = p^2.$$

(ii) If $l = 1$, then

$$\pi\bar{\pi} = p$$

for some non-zero prime $p \in$ Spec $(\mathbb{Z})$.

This defines the function $f : \text{Spec}\,(\mathbb{Z}[\omega]) \to \text{Spec}\,(\mathbb{Z})$. Next, we wish to show that this is surjective. Indeed, pick any non-zero $p \in$ Spec $(\mathbb{Z})$. Using prime factorization in $\mathbb{Z}[\omega]$, we obtain primes $\pi_1, \dots, \pi_k$ in $\mathbb{Z}[\omega]$ such that

$$p = \pi_1 \dots \pi_k.$$

Again using the conjugation automorphism yields us

$$p^2 = (\pi_1\bar{\pi}_1) \dots (\pi_k\bar{\pi}_k).$$

Note $\pi_i\bar{\pi}_i \in \mathbb{Z}$. Hence, by unique factorization of $\mathbb{Z}$, we obtain $k \leq 2$. We now have two cases

(i) If $k = 2$, then $p^2 = (\pi_1\bar{\pi}_1)(\pi_2\bar{\pi}_2)$. As π_i are not units, we deduce that $p = \pi_1\bar{\pi}_1$ and $p = \pi_2\bar{\pi}_2$. Consequently, we have $\pi_1\bar{\pi}_1 = \pi_2\bar{\pi}_2$. Thus, by unique factorization of $\mathbb{Z}[\omega]$, we further deduce that $\pi_1 = \pi_2$ or $\bar{\pi}_2$. Hence, $p = \pi\bar{\pi}$ for a unique $\pi \in$ Spec $(\mathbb{Z}[\omega])$.

(ii) If $k = 1$, then

$$p^2 = \pi\bar{\pi}$$

for some $\pi \in$ Spec $(\mathbb{Z}[\omega])$. Writing p as a product of primes in $\mathbb{Z}[\omega]$, we immediately deduce of unique factorization of $\mathbb{Z}[\omega]$ that $p = \pi'$ upto units for some non-zero prime $\pi' \in$ Spec $(\mathbb{Z}[\omega])$. Consequently, $p^2 = \pi'\bar{\pi}' = \pi\bar{\pi}$. Again by unique factorization of $\mathbb{Z}[\omega]$, we immediately deduce that $\pi = \pi'$ upto units.

This shows the surjectivity of the map f.

3. (i) $\iff$ (ii) : By part b), p splits in $\mathbb{Z}[\omega]$ iff p is not prime in $\mathbb{Z}[\omega]$. This happens iff $\mathbb{Z}[\omega]/p$ is not a domain. We now observe
$$\begin{aligned}\frac{\mathbb{Z}[\omega]}{p\mathbb{Z}[\omega]} &\cong \frac{\frac{\mathbb{Z}[x]}{\langle x^2+x+1\rangle}}{\frac{\langle p,x^2+x+1\rangle}{\langle x^2+x+1\rangle}}\\ &\cong \frac{\mathbb{Z}[x]}{\langle p, x^2+x+1\rangle}\\ &\cong \frac{\frac{\mathbb{Z}[x]}{p\mathbb{Z}[x]}}{\frac{\langle p,x^2+x+1\rangle}{p\mathbb{Z}[x]}}\\ &\cong \frac{\mathbb{F}_p[x]}{\langle x^2+x+1\rangle}.\end{aligned}$$
Hence, p is not prime in $\mathbb{Z}[\omega]$ iff x^2+x+1 is reducible in $\mathbb{F}_p[x]$. As a polynomial of degree 2 or 3 over a field is reducible iff it has a root in the field, therefore p is not prime in $\mathbb{Z}[\omega]$ iff x^2+x+1 has a root in $\mathbb{F}_p$. Similarly, since ω^2 has minimal polynomial x^2-x+1 and $\mathbb{Z}[\omega]=\mathbb{Z}[\omega^2]$, hence repeating the above yields p is not prime in $\mathbb{Z}[\omega]$ iff x^2-x+1 has a root in $\mathbb{F}_p[x]$.

(ii) $\Rightarrow$ (iii) : If $p=2$, then $x^2\pm x+1$ has no roots in $\mathbb{F}_2$. Consequently, let $p\neq 2,3$. We then wish to show that $p=1 \mod 3$. Let $a\in\mathbb{F}_p$ be the root of $f(x)=x^2\pm x+1$. Thus, $a^3=\pm 1$. Observe that $a\neq\pm 1$ as if $a=1$, then $f(1)$ and $f(-1)$ are either 1 or 3 and since $p\neq 3$, therefore $f(1), f(-1)\neq 0$, a contradiction.
As $a^3=\pm 1$ and $a\neq\pm 1$, therefore the order of $a\in\mathbb{F}_p^*$ is either 3 or 6. In either case, as $\left|\mathbb{F}_p^*\right|=p-1$, therefore by Lagrange's theorem, $3|p-1$ or $6|p-1$. But in both cases, we have $p=1 \mod 3$.

(iii) $\Rightarrow$ (ii) : If $p=3$, then $1\in\mathbb{F}_3$ is root of x^2+x+1 and 2 is the root of x^2-x+1. If $p=1 \mod 3$, then we proceed as follows. As $\mathbb{F}_p^*$ is a cyclic group of order $p-1$ and since $p-1=3k$ for some $k\in\mathbb{Z}$, hence there exists an element $a\in\mathbb{F}_p$ of order 3. Consequently, we have $a^3=1$ and thus x^3-1 in $\mathbb{F}_p[x]$ has a root. As $x^3-1=(x-1)(x^2+x+1)$ and $a\neq 1$, hence a is a root of x^2+x+1.
Now since
$$\frac{\mathbb{F}_p[x]}{\langle x^2+x+1\rangle}\cong\frac{\mathbb{F}_p[x-1]}{\langle (x-1)^2+(x-1)+1\rangle}=\frac{\mathbb{F}_p[x]}{\langle x^2-x+1\rangle}$$
therefore if x^2+x+1 has a root in $\mathbb{F}_p$, then so does x^2-x+1.

4. (i) $\Rightarrow$ (ii) : Write the prime factorization of n in $\mathbb{Z}[\omega]$ as follows
$$\begin{aligned}n &= (a+b\omega)(a+b\omega^2)\\ &= (\pi_1\dots\pi_k)(\bar{\pi_1}\dots\bar{\pi_k})\\ &= (\pi_1\bar{\pi_1})\dots(\pi_k\bar{\pi_k}).\end{aligned}$$
From parts b) and c), we know that for any prime element $\pi\in\mathbb{Z}[\omega]$, we have $\pi\bar{\pi}=p$ iff $p=3$ or $1 \mod 3$ and $\pi\bar{\pi}=p^2$ iff $p=2 \mod 3$. Consequently, we have
$$n=(p_1\dots p_m)(p_{m+1}^2\dots p_k^2)$$

where we call primes $p_1, \dots, p_m$ which are either 3 or 1 mod 3 of ***split type***. Similarly, we call the primes $p_{m+1}, \dots, p_k$ which are 2 mod 3 of ***unsplit type***. From above it is clear that unsplit type primes appear evenly many times (they appear in squares) in the prime factorization of n.

(ii) $\Rightarrow$ (i) : Let $n \in \mathbb{Z}$ be such that its prime factorization in $\mathbb{Z}$ is as follows

$$n = (p_1 \dots p_m)(q_1^{2k_1} \dots q_n^{2k_n})$$

where q_i are primes of unsplit type, that is, $q_i = 2 \mod 3$ and p_i are of split type, that is, 3 or 1 mod 3. Now, by part b), we may write $p_i = \pi_i \bar{\pi_i}$ as they split in $\mathbb{Z}[\omega]$ and $q_i = \xi_i$, where ξ_i, π_i are primes in $\mathbb{Z}[\omega]$.
It follows that we may write

$$\begin{aligned} n &= (\pi_1 \bar{\pi_1} \dots \pi_m \bar{\pi_m}) \left(\xi_1^{2k_1} \dots \xi_n^{2k_n}\right) \\ &= (\xi_1^{k_1} \dots \xi_n^{k_n})(\pi_1 \dots \pi_m) \cdot (\xi_1^{k_1} \dots \xi_n^{k_n})(\bar{\pi_1} \dots \bar{\pi_m}) \\ &= \alpha \bar{\alpha} \end{aligned}$$

where $\alpha = (\xi_1^{k_1} \dots \xi_n^{k_n})(\pi_1 \dots \pi_m) = a + b\omega$, as required.
This completes the proof. $\square$

Example 23.23.0.6. As an example use of above we may now find all ordered tuples $(a, b) \in \mathbb{Z}^2$ such that $2100 = a^2 - ab + b^2$.

Observe that

$$\begin{aligned} 2100 &= 2^2 \cdot 3 \cdot 5^2 \cdot 7 \\ &= 2^2 \cdot 5^2 \cdot (2 + \omega)(2 + \omega^2)(3 + \omega)(3 + \omega^2). \end{aligned}$$

We now wish to find the distinct $\alpha \in \mathbb{Z}[\omega]$ such that $2100 = \alpha\bar{\alpha}$. For this, we first need to find all units of $\mathbb{Z}[\omega]$.

Indeed, we claim that the units of $\mathbb{Z}[\omega]$ are $1, -1, \omega, -\omega, 1 + \omega, -1 - \omega$. We give a terse proof of this fact as follows. Let $a + b\omega \in \mathbb{Z}[\omega]$ be a unit, so that there exists $c + d\omega$ such that $(a + b\omega)(c + d\omega) = 1$. Then, the multiplicative map

$$\begin{aligned} \mathbb{Z}[\omega] &\to \mathbb{Z} \\ \alpha &\mapsto \alpha\bar{\alpha} \end{aligned}$$

yields in $\mathbb{Z}$ that $(a^2 + b^2 - ab)(c^2 + d^2 - cd) = 1$. This forces $a^2 + b^2 - ab = 1 = c^2 + d^2 - cd$. From these equations one can deduce that $c + d\omega = (a - b) - b\omega$. Hence, $a + b\omega$ is a unit iff $a^2 + b^2 - ab = 1$. It follows by AM-GM inequality on a^2 and b^2 that $ab \leq 1$. Hence, we deduce that $a = 1, b = 1$ or $a = -1, b = -1$ or $a = 0$ or $b = 0$. Correspondingly, we get the six units of $\mathbb{Z}[\omega]$ as mentioned above.

In order to count the number of distinct pairs $(a, b) \in \mathbb{Z}^2$ such that $n = a^2 + b^2 - ab = (a + b\omega)(a + b\omega^2)$ properly, let us bring some notations. Let $X_n = \{(a + b\omega) \mid (a + b\omega)(a + b\omega^2) = n\} \subseteq \mathbb{Z}[\omega]$. Denote $f : \mathbb{Z}[\omega] \to \mathbb{Z}$ to be the multiplicative map $\alpha \mapsto \alpha\bar{\alpha}$. We thus have $X_n = f^{-1}(n)$. Now observe that

1. for each $a + b\omega \in X_n$, we have $b + a\omega \in X_n$,
2. for each $a + b\omega \in X_n$, we have $a + b\omega^2 \in X_n$,
3. for each $a + b\omega \in X_n$ and $u \in \mathbb{Z}[\omega]$ a unit, we have $u(a + b\omega) \in X_n$. This is because in $\mathbb{Z}[\omega]$, inverse of a unit is its conjugate.

Our goal is to count ordered tuples $(a, b) \in \mathbb{Z}^2$ such that $n = a^2 + b^2 - ab$. Immediately, we see that such ordered tuples are in bijection with X_n. Hence, we reduce to counting X_n.

From the above discussion, we see the elements in X_n obtained by multiplying by units are

- $2 \cdot 5 \cdot 1 \cdot (2 + \omega)(3 + \omega) = 50 + 40\omega$,
- $2 \cdot 5 \cdot -1 \cdot (2 + \omega)(3 + \omega) = -50 - 40\omega$,
- $2 \cdot 5 \cdot \omega \cdot (2 + \omega)(3 + \omega) = -40 + 10\omega$,
- $2 \cdot 5 \cdot (-\omega) \cdot (2 + \omega)(3 + \omega) = 40 - 10\omega$,
- $2 \cdot 5 \cdot (1 + \omega) \cdot (2 + \omega)(3 + \omega) = 10 + 50\omega$,
- $2 \cdot 5 \cdot (-1 - \omega) \cdot (2 + \omega)(3 + \omega) = -10 - 50\omega$,
- $2 \cdot 5 \cdot 1 \cdot (2 + \omega^2)(3 + \omega) = 40 - 10\omega$.
- $2 \cdot 5 \cdot -1 \cdot (2 + \omega^2)(3 + \omega) = -40 + 10\omega$,
- $2 \cdot 5 \cdot \omega \cdot (2 + \omega^2)(3 + \omega) = 10 + 50\omega$,
- $2 \cdot 5 \cdot (-\omega) \cdot (2 + \omega^2)(3 + \omega) = -10 - 50\omega$,
- $2 \cdot 5 \cdot (1 + \omega) \cdot (2 + \omega^2)(3 + \omega) = 50 + 40\omega$,
- $2 \cdot 5 \cdot (-1 - \omega) \cdot (2 + \omega^2(3 + \omega) = -50 - 40\omega$.

Similarly, those obtained by swapping are

- $40 + 50\omega$,
- $-40 - 50\omega$,
- $10 - 40\omega$,
- $50 + 10\omega$,
- $-50 - 10\omega$.

Hence, there are 12 such ordered tuples $(a, b) \in \mathbb{Z}^2$ given by $(40, 50), (-40, -50), (10, -40)$, $(50, 10), (-50, 10), (50, 40), (-50, -40), (-40, 10), (10, 50), (-10, -50)$.

The following is a simple but powerful lemma about certain type of k-algebras.

Lemma 23.23.0.7. *Let k be a field and A be a k-algebra such that there is a maximal ideal $\mathfrak{m} \lneq A$ for which $A/\mathfrak{m} \cong k$. Then,*

$$A \cong k \oplus \mathfrak{m}$$

where $k \oplus \mathfrak{m}$ obtains the k-algebra structure from A.

Proof. Consider the triangle

$$\begin{tikzcd} A \arrow[d, two heads, "\pi"'] & k \arrow[l, hook'] \arrow[dl, "\cong"] \\ A/\mathfrak{m} & \end{tikzcd}.$$

Pick any $a \in A$. We have $\pi(a) \in A/\mathfrak{m} \cong k$, so let $\pi(a) \in k$ by identifying under that isomorphism. Consequently, we may write $a = \pi(a) + (a - \pi(a))$. Note since $\pi(a - \pi(a)) = \pi(a) - \pi(\pi(a)) = \pi(a) - \pi(a) = 0$ by the commutativity of the above, therefore $a \in \mathfrak{m}$.

Furthermore $\mathfrak{m} \cap k = 0$ is immediate as $\mathfrak{m}$ is a proper ideal. It follows that $A = k \oplus \mathfrak{m}$ as k-linear subspaces, and thus $k \oplus \mathfrak{m}$ is a k-algebra as well, isomorphic to A, where, since $(k_1 + m_1) \cdot (k_2 + m_2) = k_1k_2 + k_1m_2 + k_2m_1 + m_1m_2$ inside of A, hence we may define the k-algebra structure on $k \oplus \mathfrak{m}$ as

$$(k_1, m_1) \cdot (k_2, m_2) = (k_1k_2, k_1m_2 + k_2m_1 + m_1m_2)$$

for $(k_i, m_i) \in k \oplus \mathfrak{m}$. $\square$

The following proposition shows that any submodule of a free module over a PID is free (which is not true in general).

Proposition 23.23.0.8. *Let R be a PID and X an indexing set. Then any submodule of $R^{\oplus X}$ is free.*

Proof. Let $M \leq R^{\oplus X}$ be a submodule. For each $Y \subseteq X$, consider the submodule

$$M_Y := M \cap R^{\oplus Y}.$$

Denote by $\mathbb{T}$ the following partially ordered set

$$\mathbb{T} = \left\{ (B, Y) \mid Y \subseteq X,\ B \subseteq M \text{ s.t. } M_Y = \bigoplus_{b \in B} Rb \right\}$$

where $(B_1, Y_1) \leq (B_2, Y_2)$ if and only if $B_1 \subseteq B_2$ and $Y_1 \subseteq Y_2$.

We first claim that $\mathbb{T}$ is non-empty. Indeed, consider any finite subset $Y \subseteq X$. We claim that $M \cap R^{\oplus Y}$ is free. To this end, first observe that $M \cap R^{\oplus Y} \leq R^{\oplus Y}$. As finite direct sum of noetherian modules is noetherian, therefore $R^{\oplus Y}$ is noetherian. As a module is noetherian if and only if every submodule is finitely generated, therefore $M \cap R^{\oplus Y}$ is finitely generated.

By structure theorem of finitely generated modules over a PID, we deduce that

$$M \cap R^{\oplus Y} \cong \frac{R}{d_1R} \oplus \cdots \oplus \frac{R}{d_kR} \oplus R^n. \tag{5.1}$$

As R is a PID, so in particular a domain, therefore $R^{\oplus Y}$ has no R-torsion element. Consequently, in Eq. (5.1), we conclude that $d_i = 1$ for each $i = 1, \ldots, k$, that is, $M \cap R^{\oplus Y} \cong R^n$. Hence, $M \cap R^{\oplus Y}$ is free, as required. More generally this argument shows that any submodule of R^X where X is finite is free. This shows that $\mathbb{T}$ is non-empty.

We next wish to show that $\mathbb{T}$ has a maximal element. We will use Zorn's lemma on $\mathbb{T}$ for this. Pick any totally ordered subset $\mathcal{T} \subseteq \mathbb{T}$. We wish to show that $\mathcal{T}$ has an upper bound. Indeed, denote

$$C = \bigcup_{(B,Y) \in \mathcal{T}} B \ \&\ Z = \bigcup_{(B,Y) \in \mathcal{T}} Y.$$

We claim that

$$M_Z := M \cap R^{\oplus Z} = \bigoplus_{c \in C} Rc.$$

For $(\subseteq)$, pick an element $m \in M_Z$. We may write

$$m = (m_\alpha)_{\alpha \in Z}$$

where $m_\alpha \in R$ for each $\alpha \in Z$ and $m_{\alpha_i} \neq 0$ only for $i = 1, \dots, k$. As $\alpha_i \in Z$ and $\mathcal{T}$ is totally ordered, therefore for some $(B, Y) \in \mathcal{T}$, we have $\alpha_i \in Y$ for each $i = 1, \dots, k$. Thus, $m \in M \cap R^Y = \bigoplus_{b \in B} Rb$. In particular, $m \in \bigoplus_{b \in B} Rb \subseteq \bigoplus_{c \in C} Rc$ as $B \subseteq C$. This shows $(\subseteq)$. For $(\supseteq)$, pick any $(m_c)_{c \in C} \in \bigoplus_{c \in C} Rc$. Then $m_c = 0$ for all but finitely many $c_1, \dots, c_k$. As $\mathcal{T}$ is totally ordered and $m_{c_i} \in Rc_i$, therefore there exists $(B, Y) \in \mathcal{T}$ such that all $c_i \in B$ for $i = 1, \dots, k$. We then conclude that $m \in \bigoplus_{b \in B} Rb = M \cap R^{\oplus Y} \subseteq M \cap R^{\oplus Z}$, as needed. This shows that $(C, Z) \in \mathbb{T}$.

It is clear that for any $(B, Y) \in \mathcal{T}$, we have $(B, Y) \leq (C, Z)$ by construction. Hence we have produced an upper bound for any toset of $\mathbb{T}$. It follows by Zorn's lemma that $\mathbb{T}$ has a maximal element. Let it be denoted by $(\tilde{B}, \tilde{Y})$.

It now suffices to show that $\tilde{Y} = X$ as it would imply $M = M \cap R^{\oplus X} \in \mathbb{T}$, and hence is free. To this end, suppose $\tilde{Y} \subsetneq X$. Then there exists $\tilde{Y} \subsetneq Y'$ such that $Y' \setminus \tilde{Y}$ is finite. We shall now construct an element $(B', Y') \in \mathbb{T}$ such that $(\tilde{B}, \tilde{Y}) \leq (B', Y')$ and $(\tilde{B}, \tilde{Y}) \neq (B', Y')$, thus contradicting the maximality of $(\tilde{B}, \tilde{Y})$.

We first have the following exact sequence

$$0 \longrightarrow M \cap R^{\oplus \tilde{Y}} \xhookrightarrow{i} M \cap R^{\oplus Y'} \xrightarrow{\pi} \operatorname{CoKer}(()i) \longrightarrow 0 \tag{5.2}$$

We claim that $\operatorname{CoKer}(()i)$ is a free module. To this end, we first claim that

$$\operatorname{CoKer}(i) = \frac{M \cap R^{\oplus Y'}}{M \cap R^{\oplus \tilde{Y}}} \cong K$$

where $K \leq R^{\oplus Y' \setminus \tilde{Y}}$ is a submodule. Indeed, consider the map $\tilde{\varphi}$ obtained by the universal property of quotients

$$\begin{tikzcd}
M \cap R^{\oplus Y'} \arrow[r, "\varphi"] \arrow[d, two heads] & R^{\oplus Y' \setminus \tilde{Y}} \\
\frac{M \cap R^{\oplus Y'}}{M \cap R^{\oplus \tilde{Y}}} \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

where φ is the R-linear map which takes $(m_\alpha)_{\alpha \in Y'} \mapsto (m_\alpha)_{\alpha \in Y' \setminus \tilde{Y}}$. It is clear that $\operatorname{Ker}(\varphi) = M \cap R^{\oplus \tilde{Y}}$. Consequently, $\tilde{\varphi}$ is an inclusion and let $K \leq R^{\oplus Y' \setminus \tilde{Y}}$ be its image.

As $Y' \setminus \tilde{Y}$ is finite and we showed above that every submodule of a finitely generated free module is free, therefore

$$K = \bigoplus_{z \in Z} Rz \cong R^{\oplus Z}.$$

where $Z \subseteq R^{\oplus Y' \setminus Y}$. This shows that $\operatorname{CoKer}(()i) \cong R^{\oplus Z}$ is a free R-module. In particular, it is projective. Consequently, the exact sequence of (5.2) is split exact so that there exists $j : \operatorname{CoKer}(()i) \hookrightarrow M \cap R^{\oplus Y'}$ such that $\pi j = \mathrm{id}_{\operatorname{CoKer}(()i)}$. It now follows immediately that

$$\begin{aligned}
M \cap R^{\oplus Y'} &= \operatorname{Ker}(\pi) \oplus j(\operatorname{CoKer}(()i)) \\
&= \left(M \cap R^{\oplus \tilde{Y}}\right) \oplus j(\operatorname{CoKer}(()i))
\end{aligned}$$

where $j\left(\operatorname{CoKer}\left(()i\right)\right) \cong R^{\oplus Z}$ so it is free. Hence, we see that $B' \supseteq \tilde{B}$. This shows that $(B', Y') \geq (\tilde{B}, \tilde{Y})$, completing the proof. $\square$

Chapter 24

K-Theory of Rings

Contents

Author's note : This chapter will be added around March, during the talks on K-theory by GST IISER Mohali. The goal is to understand K_0, K_1, K_2 of rings concretely and then define higher K-groups categorically.
A lot of the subsections made below are not permanent as they will be absorbed into the sections in Special Topics, like projective and free modules should go into Algebra (only the commutative case, the non-commutative case shall be given in a small remark). A lot others will be given a theorem or a lemma.

24.1 Lower K-theory of rings

Complete this se tion, Chapter 24

We shall study the basics of K-theory here, atleast the algebraic picture which was classically the first one to be developed. Algebraic K-theory, or atleast the idea thereof, was first developed by Grothendieck in his far reaching generalization of Riemann-Roch theorem while studying intersection theory of varieties.

The lower K-theory of rings was the first view of algebraic K-theory that people found. Indeed, $K_0(R)$ for a ring R with unity was defined and used by Grothendieck whereas later $K_1(R), K_2(R)$ was defined later by Hymann Bass and John Milnor. The existence of a nine term long-exact sequence then raised a question whether one can define higher K-groups. Indeed, this was done by Daniel Gray Quillen who won Fields medal for this work. The general procedure in which he defined $K_n(R)$ was to first obtain a topological space from a ring R in such a manner so to make its first 3 homotopy groups identical to K_0, K_1, K_2 and then defined higher K-groups by the higher homotopy groups of that space.

Let us now see the various places that we will find ourself in during the process of finding the right notions of K-groups.

24.1.1 The grand schema

The following is the picture which we shall traverse througout these notes.

$$\begin{tikzcd}
\mathbf{Ring} \arrow[rr, "K_n"] \arrow[dd, "\mathbf{Proj}\text{-}(-)"'] & & \mathbf{AbGrp} \arrow[dl, hook, shift left] \arrow[d, hook, shift right] & \\
& \mathbf{ComMon} \arrow[ur, "G(-)", shift left] & \mathbf{Grp} \arrow[u, "-/[-:-]"', shift right] \arrow[d, hook] & \mathbf{Htpy} \arrow[l, "\pi_n"'] \arrow[ul, "H_n"'] \\
\mathbf{SymMonCat} \arrow[ur, "(-)^{\cong}"] \arrow[rr, hook] & & \mathbf{Cat} \arrow[ur, "B"'] &
\end{tikzcd} \cdot$$

We shall see.

24.1.2 Group completion

For a commutative semigroup $(S, \star)$, we can construct an equivalence relation on it which mimics the way on obtains the group of integers out of the semigroup of naturals. Indeed, define the following on $S \times S$:

$$(a, b) \sim (c, d) \iff \exists t \in S \text{ s.t. } a \star d \star t = b \star c \star t.$$

Define $G(S) = (S \times S)/\sim$ to be the *Grothendieck group* of S, also called group completion of S. The operation on the elements is given as follows

$$[(a, b)] + [(c, d)] := [(a \star c, b \star d)].$$

Indeed since $\star$ is commutative, therefore $+$ is commutative. Moreover, identity is given by $[(a, a)]$ for any $a \in S$. The inverse of $[(a, b)]$ is obviously given by $[(b, a)]$. Examples are as follows:

1. $G(\mathbb{N}, +) = (\mathbb{Z}, +)$,
2. $G(\mathbb{N}, \cdot) = 0$ as there is an absorbing element in $(\mathbb{N}, \cdot)$,

3. for a linear order $(L, \leq)$, the operation of taking minimums is associative, hence $(L, \min)$ is a semigroup. Its Grothendieck group is 0 as for $\min(a,d)$ and $\min(b,c)$, because of linearity, there is an element t which is below both of them.

We now see that group completion has a right adjoint.

Theorem 24.1.2.1. *The following is an adjunction*

$$\begin{tikzcd} \mathbf{ComSGrp} \arrow[r, shift left, "G"] \arrow[r, phantom, "\perp"] & \mathbf{AbGrp} \arrow[l, hook, shift left] \end{tikzcd} .$$

Proof. The unit of this adjunction is given as follows. For any semigroup S, define the following map:

$$\begin{aligned} i_S : S &\longrightarrow G(S) \\ s &\longmapsto [(s \star s, s)]. \end{aligned}$$

Indeed, this makes the following triangle commutative for any abelian group A and a map of semigroups $f : S \to A$:

$$\begin{tikzcd} S \arrow[r, "i_S"] \arrow[dr, "f"'] & G(S) \arrow[d, dashed, "\tilde{f}"] \\ & A \end{tikzcd}$$

where $\tilde{f}([(a,b)]) = f(a) - f(b)$. This is clearly a map of semigroups. □

Call the element $[(s \star s, s)] \in G(S)$ for an $s \in S$ to be the *klasse* of $s \in S$ and is denoted by $[s] \in G(S)$. This is where the subject K-theory got its name.

24.1.3 Cofinality

An important aspect of the Grothendieck construction is that of cofinality. This in form of a prose means *"Going to end together"*. Indeed, for a semigroup S, a sub-semigroup $T \leq S$ is said to be *cofinal* in S if for all $s \in S$, there exists $s' \in S$ such that $s \star s' \in T$. Indeed, for example, the subsemigroup of all multiples of 3 in $(\mathbb{N}, +)$ forms a subsemigroup which is cofinal in $\mathbb{N}$.

The main thrust behind this notion is because of the following theorem, which in particular tells us that cofinal sub-semigroups becomes subgroups of the Grothendieck group.

Theorem 24.1.3.1. *(Cofinality theorem) Let $(S, \star)$ be a semigroup and T be a cofinal sub-semigroup of S. Then,*

1. *$G(T) \leq G(S)$,*
2. *if $m, m' \in S$ are such that $[m] = [m'] \in G(S)$, then there exists $n \in T$ such that $m \star n = m' \star n$,*
3. *for all $x \in G(S)$, there exists $m \in S$ and $n \in T$ such that $x = [m] - [n]$.*

Proof. **TODO**. □

24.1.4 Source of examples : SymMonCat

A good source of examples of interesting semigroups (and thus their group completions) is provided by the categorification of the notion of semigroups in the name of *symmetric monoidal categories*. We define symmetric monoidal categories as follows.

Definition 24.1.4.1. (**Symmetric monoidal categories**) A category $\mathbf{C}$ is said to be symmetric monoidal if there exists following additional structures on top of $\mathbf{C}$

1. $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, the tensor bifunctor,
2. $e \in \mathbf{C}$, the unit object,
3. **TODO.**

SymMonCat → ComMon

We can obtain a commutative monoid by identifying all isomorphic objects in a symmetric monoidal category.

Construction 24.1.4.2. (*Commutative monoid from symmetric monoidal categories*) **TODO**.

24.1.5 $K_0(R)$

We now construct K_0 of a ring with unity.

Construction 24.1.5.1. Let R be a ring with 1. Consider the category of all right projective R-modules, denoted $\mathbf{Proj}$-R. Note that it forms a symmetric monoidal category, where the tensor is the direct sum and the unit object is the zero module:

$$(\mathbf{Proj}\text{-}R, \oplus, 0)\,.$$

By Construction 24.1.4.2, we obtain a commutative monoid

$$(\mathbf{Proj}\text{-}R, \oplus, 0)^{\cong}\,.$$

Now, commutative monoids are commutative semi-groups, therefore we have

$$\mathbf{ComMon} \hookrightarrow \mathbf{ComSGrp}.$$

Using the group completion functor of Theorem 24.1.2.1, we obtain an abelian group, which is defined to be the K_0 *of the ring* R:

$$K_0(R) := G\,(\mathbf{Proj}\text{-}R, \oplus, 0)^{\cong}\,.$$

Thus K_0 can be treated as a composition of functors:

$$\begin{tikzcd}
\mathbf{SymMonCat} \arrow[r, "(-)^{\cong}"] & \mathbf{ComSGrp} \arrow[d, "G(-)"] \\
\mathbf{Ring} \arrow[u, "\mathbf{Proj}\text{-}(-)"] \arrow[r, dashed, "K_0(-)"'] & \mathbf{AbGrp}
\end{tikzcd}\,.$$

Examples

Discussion 24.1.5.2. Let us discuss what we are about to do. Continuing from what we have done, we will now look at some more exotic examples of Grothendieck groups, notably the connection between class groups in algebraic number theory and reduced Grothendieck group for Dedekind domains. The functor K_0 behaves well with respect to finite direct sums and direct limits. Reinterpreting finitely generated projective modules as infinite idempotent matrices, one gets an alternate presentation of K_0, which directly shows that it is a Morita invariant.

K-theory behaves a lot like a homology theory (Definition 10.3.1.2), thus facilitating computations. In fact, given a two-sided ideal I of a ring R, there is an exact sequence relating the K-groups of the rings R, R/I and a relative K-group of the pair (R, I). Moreover, the relative K-group $K_0(R, I)$ only depends on the difference between R and R/I, which is the ideal I thought of as a ring without unity—an analogue of the excision theorem in homology.

24.1.6 K_0 of Dedekind domains

24.1.7 Idempotent matrices and K_0

24.1.8 Reduced and relative K_0

24.1.9 Excision and relative K_0-sequence

Discussion 24.1.9.1. Let us discuss what we are next going to do. It is said that the Grothendieck group $K_0(R)$ is orthogonal to all higher K-groups in the sense that the former looks at the spread of projective modules while the rest only look at the eventual behaviour as the size of those modules grows.

Hyman Bass's group $K_1(R)$ is the intended "value group for the determinant" of an invertible matrix over R and can be defined as the abelianization of the direct limit of automorphism groups of finitely generated projective modules. We will look at two different, yet equivalent, approaches to this group. Thanks to Whitehead, we will be dealing with invertible matrices without worrying about their sizes. Gaussian elimination and Dieudonne's determinant map will help us to compute $K_1(R)$ in familiar examples. Finally we will extend the exact sequence from last few sections to a six term long exact sequence.

24.1.10 $K_1(R)$

24.1.11 Elementary matrices and K_1

24.1.12 Examples

24.1.13 Relative K_1 and relative K_1-sequence

24.1.14 Clutching and K_0-K_1-sequence

Discussion 24.1.14.1. Let us discuss what we are about to do. Motivated by topological K-theory, the ring $R[t]$ of polynomials and the ring $R[t, t^{-1}]$ of Laurent polynomials can be viewed as "algebraic version" of cylinder and desuspension for the ring R respectively. The study of K-groups for these rings by Grothendieck and Bass-Heller-Swan resulted in the definition of negative K-groups for R, which extend the 6-term long exact sequence on the right.
Positive K-theory of a ring R is supposed to measure abelian invariants of the highly non-abelian group $GL(R)$. At the end we will briefly study universal central extension of the group $E(R)$ of elementary matrices to define $K_2(R)$ and see how to extend the 6-term exact sequence on the left.

$\vdots$

24.2 Higher K-theory of rings

Our goal in this section is to construct the K-theory space. For that we would need ideas coming from the world of simplicial sets, which is present in Chapter 19, Language of ∞-Categories.

Chapter 25

Abstract Analysis

Contents

We will discuss some topics from integration theory and functional analysis (topological vector spaces). Please mind that this chapter is *very* incomplete.

25.1 Integration theory

We would like to state and portray the uses of some of the important and highly usable results of integration theory, elucidating in the process the analytical thought which is of paramount importance in any route of exploration in this field[1]. We give bare-bone proofs as all this is standard, but we will highlight the main part of the proof by $\heartsuit$ or if there are many main parts, then by $\heartsuit\heartsuit\ldots(!)$ Let us first begin with some motivation behind modern measure theory.

We know that the class of all Riemann integrable functions on $[a, b]$, denoted $R([a, b])$, is not complete under pointwise limit (a sequential approximation of Dirichlet's function shows that). Further, motivated by Weierstrass approximation, one would like to have commutability results between lim and $\int$, which again $R([a, b])$ lacks. Consequently, one is motivated to find a larger class of "integrable" functions for which these defects would be rectified.

The idea that H. Lebesgue had was quite simple. He continued the idea of Riemann (that is, of partitions) but made sure that the function under investigation is much more intertwined in with it. Indeed, for a bounded function $f : [a, b] \to \mathbb{R}$, we contain the image $\mathrm{Im}\,(f) \subseteq [\alpha, \beta]$ and then consider a partition $\mathcal{P} = \{I_i\}_{i=1}^n$ where I_i is an interval. Now

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[1]One may argue, instead, in whole of mathematics.

choose $\xi_i \in f^{-1}(I_i) =: J_i$ for each i. Consequently, we may naturally define *Lebesgue sum of f w.r.t. $\mathcal{P}$* as follows

$$L(f, \mathcal{P}) := \sum_{i=1}^{n} f(\xi_i) m(J_i),$$

where $m(J_i)$ is supposed to be some sort of measure of J_i. Note that J_i in general might be very bad (may not even be an interval!). To complete this idea of "integration", we are naturally led to considering more general notions of measures. Indeed, this is what we will pursue in this course.

Remark 25.1.0.1. (***Pseudo-definition of measure***) First, what do we expect from a notion of measure on $\mathbb{R}$? Perhaps the following is the minimum conditions we would require to call a function "measure": A function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ is said to be a *pseudo-measure* if it satisfies the following

1. (*measure of intervals*) for any interval I, the measure $\mu(I) = l(I)$ where l is the length function,
2. (*measure of disjoint unions*) for any disjoint sequence of subsets $\{A_n\}$, $\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$,
3. (*translation invariance*) for any subset A and $x \in \mathbb{R}$, we have $\mu(A + x) = \mu(A)$.

We will call such a function a *pseudo-measure on* $\mathbb{R}$. Observe that for $A \subseteq B$, we obtain $\mu(A) \leq \mu(B)$ by breaking $B = A \cup B \setminus A$. We call μ a pseudo-measure because it does not exists!

Theorem 25.1.0.2. *(Vitali set) There exists no pseudo-measure on $\mathbb{R}$. In paritcular, there exists a set $V \subseteq \mathbb{R}$ such that for a pseudo-measure μ, $\mu(V) \notin [0, \infty]$.*

Proof. We will construct such a set V. Begin with the closed interval $J = [0, 1]$. Define an equivalence relation $\sim$ on J given as follows:

$$x \sim y \iff x - y \in \mathbb{Q}.$$

This can easily be seen to be an equivalence relation on J. We have first some observations to make about this equivalence relation and the consequent partition of J that it entails.

1. Observe that the class of any rational r in J under $\sim$ is simply $[0]$, as $r - 0 \in \mathbb{Q}$.
2. Every equivalence class is countable in size. Indeed, for any $x \in J$, the class $[x]$ is just translate of x by rationals, which is countable.
3. There are uncountably many equivalence classes. Indeed, if there were atmost countably many equivalence classes, then by statement 2 above, it would follow there are atmost countably many elements in J, which is a contradiction.

Consequently, this equivalence relation partitions J into following classes:

$$J = \bigcup_{\alpha \in \mathcal{I}} [\alpha]$$

where $\mathcal{I}$ is an uncountable set.

We would now construct the set V as follows. First, let us assume axiom of choice, so that for each class $[\alpha]$, we may pick an element $r_\alpha \in [\alpha]$ and would thus obtain a subset of J, denoted $V = \{r_\alpha \mid \alpha \in I\}$. We call this the Vitali set.

Consider the set $Q = [-1, 1] \cap \mathbb{Q}$. Since it is countable so consider an enumeration $Q = \{q_n\}$. Now consider the translates $V + q_n$ for all $n \in \mathbb{N}$ and their union $X = \bigcup_n V + q_n$. We now observe the following two facts about X.

1. If $n \neq m$, then $(V + q_n) \cap (V + q_m) = \emptyset$. Indeed, if $x \in (V + q_n) \cap (V + q_m)$, then $x = r_a + q_n = r_b + q_m$. Consequently, $r_a - r_b \in \mathbb{Q}$ and hence $[a] = [b]$. But by single choice of r_c for each $c \in \mathcal{I}$, we get $r_a = r_b$ and thus $q_n = q_m$ from above, which is a contradiction.
2. $J = [0, 1] \subseteq X$. Indeed, for any $x \in [0, 1]$, consider the class $[a]$ in which x is present. Consequently we have a unique $r_a \in V$ corresponding to x which satisfies $x \in [r_a]$. Thus, $x = r_a + t$ where $t \in \mathbb{Q}$. We may write $t = q_n$ to obtain that $x \in V + q_n$, as desired.
3. $X \subseteq [-1, 2]$. Indeed, this follows immediately since $X = \bigcup_n V + q_n$ where q_ns are rationals in $[-1, 1]$ and $V \subseteq [0, 1]$.

With the above three observations, we obtain the following inclusions:

$$[0, 1] \subseteq \bigcup_n V + q_n \subseteq [-1, 2].$$

Now, if we apply the pseudo-measure μ on the above inclusions, we will obtain the following:

$$1 \leq \sum_n \mu(V) \leq 3.$$

If $\mu(V) = 0, \infty$, then we have an immediate contradiction. Else if $0 < \mu(V) < \infty$, then $\sum_n \mu(V) = \infty$ and we again have a contradiction. Thus, $\mu(V) \notin [0, \infty]$, a contradiction. $\square$

Remark 25.1.0.3. The main issue in pseudo-measures is that we trying to get a measure on *all* of the subsets of $\mathbb{R}$. By Theorem 25.1.0.2, this is hopeless. What we shall now do instead is to obtain a measure not on all of the subsets of $\mathbb{R}$, but rather on only a subcollection of subsets of $\mathbb{R}$, and we shall choose this subcollection in a manner so that we don't allow sets like Vitali sets. Indeed, this becomes our point of departure for the abstract definition of σ-algebras and measure/measurable spaces, the need for the right domain of a measure function.

25.1.1 σ-algebras and measures

We first define an algebra and a σ-algebra over a set X.

Definition 25.1.1.1. (**Algebra and σ-algebra**) Let X be a set. An algebra is a subset of the power set $\mathcal{A} \subseteq P(X)$ such that

1. $X \in \mathcal{A}$,
2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,
3. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^n A_i \in \mathcal{A}$.

If $\mathcal{A}$ further satisfies that

4. $\{A_n\}$ is a sequence in $\mathcal{A}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$,

then $\mathcal{A}$ is said to be a σ-algebra.

Remark 25.1.1.2. The name "algebra" is indeed coming from algebra(!) Consider the tuple $(\mathcal{P}(X), \Delta, \cap)$ where $A\Delta B$ is the symmetric difference. One observe that then forms a commutative ring with unity with addition being symmetric difference and multiplication being intersection. One now sees that an algebra in the sense of Definition 25.1.1.1 indeed is closed under symmetric differences, has X and is closed under intersections. Consequently, an algebra $\mathcal{A}$ (in the sense of Definition 25.1.1.1) is a subalgebra of the algebra $\mathcal{P}(X)$ (in the usual sense). This therefore motivates the name.

For fun, we also define the following notion.

Definition 25.1.1.3. (**Constructible algebra and Borel σ-algebra**) Let (X, τ) be a topological space. Then a constructible algebra $\mathcal{C}$ is defined to be the smallest algebra containing the topology τ and the Borel σ-algebra is defined to be the smallest σ-algebra containing τ.

We may later dwell on the former for Zariski spaces (also known as spectral spaces) and their importance in algebraic geometry.

Definition 25.1.1.4. (**σ-algebra generated by a class**) Let X be a set and $S \subseteq P(X)$ be a class of sets of X. Then the σ-algebra generated by S is the smallest σ-algebra containing S, denoted $\sigma(S)$.

Consequently, one defines the Borel σ-algebra to be the σ-algebra generated by τ.

Lemma 25.1.1.5. *Let X be a set and $S \subseteq P(X)$ a class of subsets of X. Let $A \subseteq X$ be a subset. Denote by $S \cap A = \{B \cap A \mid B \in S\}$. Then,*

$$\sigma_A(S \cap A) = \sigma(S) \cap A.$$

where $\sigma_A(S\cap A)$ denotes the smallest σ-algebra over A generated by the class $S\cap A \subseteq P(A)$.

Proof. It is easy to see that $\sigma_A(S \cap A) \hookrightarrow \sigma(S) \cap A$ by considering that $S \cap A \subseteq \sigma(S) \cap A$. Conversely, we use the generating set principle. That is, since we wish to show that for any $B \in \sigma(S)$, we have $B \cap A \in \sigma_A(S \cap A)$, therefore we define

$$\mathcal{S} := \{B \in \sigma(S) \mid B \cap A \in \sigma_A(S \cap A)\}$$

and then observe quite easily that $\mathcal{S}$ is a σ-algebra over X inside $\sigma(S)$ containing S. Thus $\mathcal{S} = \sigma(S)$, as needed. $\square$

We now define a measure.

Definition 25.1.1.6. (**Measure and measure spaces**) Let X be a non-empty set together with a σ-algebra $\mathcal{A}$ of X. The tuple $(X, \mathcal{A})$ is defined to be a measurable space. A measure on $(X, \mathcal{A})$ is a set function

$$\mu : \mathcal{A} \longrightarrow [0, \infty]$$

such that $\mu(\emptyset) = 0$ and for a disjoint sequence of sets $\{A_n\} \subseteq \mathcal{A}$, we have $\mu(\cup_n A_n) = \sum_n \mu(A_n)$. The triple $(X, \mathcal{A}, \mu)$ is said to be a measure space.

The following are some finiteness conditions we would like to have on measure spaces.

Definition 25.1.1.7. (**Finiteness conditions**) Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,

1. X is said to be *finite* if $\mu(X) < \infty$,
2. X is said to be σ*-finite* if there exists $\{A_n\} \subseteq \mathcal{A}$ such that $\bigcup_n A_n = X$ and $\mu(A_n) < \infty$,
3. X is said to be *semi-finite* if for all $A \in \mathcal{A}$ such that $\mu(A) = \infty$, there exists $B \subseteq A$ such that $B \in \mathcal{A}$ and $\mu(B) < \infty$.

X-indexed $\mathbb{R}$-series

We would now like to make sense of the sum $\sum_{x \in X} f(x)$ where $f : X \to [0, \infty]$ is an arbitrary function.

Definition 25.1.1.8. (**X-indexed $\mathbb{R}$-series**) Let $f : X \to [0, \infty]$ be a function where X is a set. We define the series $\sum_{x \in X} f(x)$ as follows:

$$\sum_{x \in X} f(x) = \sup \left\{ \sum_{x \in F} f(x) \mid F \subseteq X \text{ is finite} \right\}.$$

The following are some basic properties of X-indexed $\mathbb{R}$-series.

Proposition 25.1.1.9. *Let X be a set and $f : X \to [0, \infty]$ be a function. Denote $S = \{x \in X \mid f(x) > 0\}$.*

1. *If S is uncountable then $\sum_{x \in X} f(x) = \infty$.*
2. *If S is countably finite then for any bijection $\varphi : \mathbb{N} \to S$, we have*

$$\sum_{x \in X} f(x) = \sum_{n \in \mathbb{N}} f(\varphi(n)).$$

Proof. 1. Write $S = \bigcup_n S_n$ where $S_n = \{f(x) > 1/n\}$. Note that S_n forms an increasing sequence of sets. As S is uncountable, there exists $N \in \mathbb{N}$ such that S_N is uncountable. Consequently, for any finite set $F \subseteq S_N$, we have $\sum_{x \in F} f(x) \geq \frac{|F|}{N}$. As $\sum_{x \in F} f(x) \leq \sum_{x \in X} f(x)$, therefore

$$\frac{|F|}{N} \leq \sum_{x \in X} f(x). \tag{$\heartsuit$}$$

As $F \subseteq S_N$ is arbitrary finite set and S_N is uncountable, therefore we get the desired result.

2. Pick any bijection $\varphi : \mathbb{N} \to S$ and pick a finite set $F \subseteq X$. We have $\sum_{x \in F} f(x) = \sum_{x \in F \cap S} f(x)$, so replace $F \subseteq X$ by a finite set $F \subseteq S$. Let $n \in \mathbb{N}$ be large enough so that $\varphi(\{1, \dots, n\}) \supseteq F$. Consequently, we have

$$\sum_{x \in F} f(x) \leq \sum_{k=1}^{n} f(\varphi(k)) \leq \sum_{x \in X} f(x). \tag{$\heartsuit$}$$

Take $n \to \infty$ in the above inequality to obtain

$$\sum_{x \in F} f(x) \leq \sum_{k=1}^{\infty} f(\varphi(k)) \leq \sum_{x \in X} f(x).$$

Take sup over all finite subsets F of X in the above inequality to obtain

$$\sum_{x \in X} f(x) \leq \sum_{n \in \mathbb{N}} f(\varphi(n)) \leq \sum_{x \in X} f(x),$$

which yields the desired result. $\square$

25.1.2 Basic results on measure spaces

We have the following first result.

Proposition 25.1.2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

1. *If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*
2. *If $A, B \in \mathcal{A}$ and $A \subseteq B$ where $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.*
3. *For any sequence $\{A_n\} \subseteq \mathcal{A}$, we have*

$$\mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n).$$

4. *If $\{A_n\} \subseteq \mathcal{A}$ is an increasing sequence of measurable sets, then*

$$\mu\left(\bigcup_n A_n\right) = \lim_n \mu(A_n).$$

5. *If $\{A_n\} \subseteq \mathcal{A}$ is a decreasing sequence of measurable sets where $\mu(A_1) < \infty$, then*

$$\mu\left(\bigcap_n A_n\right) = \lim_n \mu(A_n).$$

6. *If X is σ-finite, then X is semi-finite.*

Proof. Statements 1. and 2. are immediate from the disjoint decomposition $B = A \amalg (B \setminus A)$. For 3. note that for any $\{A_n\} \subseteq \mathcal{A}$, we can form a disjoint sequence $\{B_n\} \subseteq \mathcal{A}$ such that $\bigcup_n A_n = \coprod_n B_n$ and $\mu(B_n) \leq \mu(A_n)$. Statement 4. also follows from similar reasons, where we can now let $B_n = A_n \setminus A_{n-1}$. Let us do statement 5. in some detail.

Observe that the sequence $C_1 = \emptyset$ and $C_n = A_1 \setminus A_n$ is an increasing sequence of sets. Thus, we have by statement 4. that

$$\mu\left(\bigcup_n C_n\right) = \lim_n \mu\left(C_n\right). \qquad (\heartsuit)$$

We can write $A_1 = (A_1 \setminus A_n) \amalg A_n$. Using statement 2. we obtain that

$$\begin{aligned} \mu(A_1) &= \mu(C_n) + \mu(A_n) \\ \mu(A_1) - \mu(A_n) &= \mu(C_n). \end{aligned} \tag{$\heartsuit\heartsuit$}$$

We now claim that $\bigcap_n A_n = A_1 \setminus \bigcup_n C_n$. Indeed, for $x \in \bigcap_n A_n$, $x \in A_n \subseteq A_1$ for all n and thus $x \in A_1$. But if $x \in C_n$ for some n, then $x \notin A_n$, consequently a contradiction. Hence $x \in A_1 \setminus \bigcup_n C_n$. Conversely, for $x \in A_1 \setminus \bigcup_n C_n$ and any $n \in \mathbb{N}$, we have that if $x \notin A_n$, then $x \in A_1 \setminus A_n = C_n$, a contradiction. Hence the claim is proved.

As each $C_n \subseteq A_1$, thus $\bigcup_n C_n \subseteq A_1$. Consequently, by statement 2. and above claim we obtain that

$$\begin{aligned} \mu\left(\bigcap A_n\right) &= \mu(A_1) - \mu\left(\bigcup_n C_n\right) \\ &= \mu(A_1) - \lim_n \mu(C_n) \\ &= \mu(A_1) - \lim_n \left(\mu(A_1) - \mu(A_n)\right) \\ &= \lim_n \mu(A_n). \end{aligned}$$

This proves statement 5.

For statement 6. pick any $A \in \mathcal{A}$ with $\mu(A) = \infty$. We wish to construct a subset $B \subseteq A$ with $B \in \mathcal{A}$ and $0 < \mu(B) < \infty$. Let $\{D_n\} \subseteq \mathcal{A}$ be a collection of finite measure sets such that $\bigcup_n D_n = X$. Note that we can assume D_n are disjoint by suitably replacing D_n by $D_n \setminus D_1 \cup \cdots \cup D_{n-1}$. Assume to the contrary, so that for each $B \subseteq A$ with $B \in \mathcal{A}$, either $\mu(B) = 0$ or $\mu(B) = \infty$. Let $D_n \cap A$ be such that $D_n \cap A \neq \emptyset$. Consequently, $\mu(D_n \cap A) = 0$ or ∞. The latter isn't possible, therefore $\mu(D_n \cap A) = 0$ for all $n \in \mathbb{N}$.

Since we have $A = \coprod_n D_n \cap A$, therefore $\mu(A) = \sum_n \mu(D_n \cap A) = 0$, a contradiction to the fact that $\mu(A) = \infty$. $\square$

We now cover an important example of a measure.

Construction 25.1.2.2. (*Measures from a positive function*) Let $(X, \mathcal{A})$ be a measurable space and $f : X \to [0, \infty]$ be a function. We construct the following map

$$\{\text{All functions } X \to [0,\infty]\} \longrightarrow \{\text{Measures on } (X, \mathcal{A})\}.$$

Indeed, define

$$\begin{aligned} \mu_f : \mathcal{A} &\longrightarrow [0, \infty] \\ A &\longmapsto \sum_{x \in A} f(x). \end{aligned}$$

We claim that μ_f forms a measure.

It is clear that $\mu_f(\emptyset) = 0$. Consequently we need to show that for a disjoint collection $\{A_n\} \subseteq \mathcal{A}$, we have

$$\mu_f\left(\coprod_n A_n\right) = \sum_n \mu_f(A_n).$$

We first have that

$$\mu_f\left(\coprod_n A_n\right) = \sup\left\{\sum_{x\in F} f(x) \mid F \subseteq \coprod_n A_n \text{ is finite}\right\} \tag{1}$$

and

$$\sum_n \mu_f(A_n) = \sum_n \sup\left\{\sum_{x\in G} f(x) \mid G \subseteq A_n \text{ is finite}\right\}. \tag{2}$$

We first show that (1) $\leq$ (2). We need only show that for a finite set $F \subseteq \amalg_n A_n$, we have $\sum_{x\in F} f(x) \leq (2)$. Indeed, as $F_n := F \cap A_n$ is a collection of disjoint finite set where $F_n \subseteq A_n$ and only for finitely many n is F_n non-empty, therefore $\sum_{x\in F} f(x) = \sum_n \sum_{x\in F_n} f(x) \leq (2)$.

Conversely, we now wish to show that (2) $\leq$ (1). We use a standard technique for this. Pick any $\epsilon > 0$. For each $n \in \mathbb{N}$, we obtain a finite set $G_n \subseteq A_n$ such that

$$\mu_f(A_n) - \frac{\epsilon}{2^n} \leq \sum_{x\in G_n} f(x). \tag{$\heartsuit$}$$

Summing this till $N \in \mathbb{N}$, we obtain

$$\sum_{n=1}^{N}\left(\mu_f(A_n) - \frac{\epsilon}{2^n}\right) \leq \sum_{n=1}^{N}\sum_{x\in G_n} f(x) = \sum_{x\in \amalg_{n=1}^{N} G_n} f(x) \leq (1).$$

Now take $N \to \infty$ and $\epsilon \to 0$ to obtain the result[2].

Observe that the map defined above in Construction 25.1.2.2 is neither injective nor surjective, and that's good, otherwise measure theory would have been redundant. We now study completions of a measure space.

Remark 25.1.2.3. The goal of next few sections is to establish a good measure on $\mathbb{R}^n$ through which we can proceed to a theory of integration of measurable functions. Indeed, this goal was achieved by Lebesgue and he constructed what will be called the Lebesgue measure on $\mathbb{R}^n$. Hence, one should view the goal of the next few sections as to construct this measure space $(\mathbb{R}^n, \mathcal{M}, m)$, which is highly usable (as we will see in the integration theory) and is the gold standard of modern analysis.

25.1.3 Completion of a measure space

Definition 25.1.3.1. (**Null sets and complete measure spaces**) Let $(X, \mathcal{A}, \mu)$ be a measure space. A null set is an element $A \in \mathcal{A}$ such that $\mu(A) = 0$. The collection of all null sets is written as $\text{Null}(\mathcal{A}) \subseteq \mathcal{A}$. A measure space $(X, \mathcal{A}, \mu)$ is said to be complete if for all $A \in \text{Null}(\mathcal{A})$, $\mathcal{P}(A) \subseteq \mathcal{A}$.

Remark 25.1.3.2. Note that for a measure space $(X, \mathcal{A}, \mu)$, the collection of all null sets $\text{Null}(\mathcal{A})$ contains $\emptyset$ and is closed under countable union. Indeed, for $\{A_n\} \subseteq \text{Null}(\mathcal{A})$, we have $\mu(\cup_n A_n) \leq \sum_n \mu(A_n) = 0$ by Proposition 25.1.2.1, 3.

[2] We call this the *ϵ-wiggle around* inf *and* sup *technique.*

Definition 25.1.3.3. (**Extension of measure spaces**) Let $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}', \mu')$ be two measure spaces. Then we say that $(X, \mathcal{A}', \mu')$ is an extension of $(X, \mathcal{A}, \mu)$ if $\mathcal{A}' \supseteq \mathcal{A}$ and $\mu'|_{\mathcal{A}} = \mu$.

We will now for each measure space $(X, \mathcal{A}, \mu)$ will construct an extension of it which will be complete.

Construction 25.1.3.4. Let $(X, \mathcal{A}, \mu)$ be a measure space. Consider the following collection

$$\hat{\mathcal{A}} := \{A \cup B \mid A \in \mathcal{A}, B \subseteq N, N \in \mathrm{Null}(\mathcal{A})\}.$$

Define $\hat{\mu} : \hat{\mathcal{A}} \to [0, \infty]$ as $A \cup B \mapsto \mu(A)$.

Theorem 25.1.3.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then, $(X, \hat{\mathcal{A}}, \hat{\mu})$ is a complete measure space extending $(X, \mathcal{A}, \mu)$. We call it the completion of $(X, \mathcal{A}, \mu)$.*

Proof. We need to show the following things.
1. $\hat{\mathcal{A}}$ is a σ-algebra,
2. $\hat{\mu}$ is a measure,
3. $\hat{\mu}|_{\mathcal{A}} = \mu$,
4. $(X, \hat{\mathcal{A}}, \hat{\mu})$ is complete.

The first three are straightforward. We show 4. in some detail.

Pick $A \cup B \in \hat{\mathcal{A}}$ such that $\hat{\mu}(A \cup B) = \mu(A) = 0$. Then $A \in \mathrm{Null}(\mathcal{A})$. Further, $B \subseteq N$ where $N \in \mathrm{Null}(\mathcal{A})$. Let $C \subseteq A \cup B$. Then $C = (C \cap A) \cup (C \cap B)$. Since $C \cap A \subseteq A$ and $C \cap B \subseteq N$, therefore $C \subseteq A \cup N$ where $A \cup N \in \mathrm{Null}(\mathcal{A})$. Consequently, we may write $C = \emptyset \cup C$ where C is a subset of a null set. Hence $C \in \hat{\mathcal{A}}$. $\square$

Example 25.1.3.6. Let $X = \{1, 2, 3\}$ and $\mathcal{A} = \{\emptyset, X, \{1\}, \{2, 3\}\}$. Define $\mu : \mathcal{A} \to [0, \infty]$ by $\mu(\emptyset) = 0 = \mu(\{2, 3\})$ and $\mu(\{1\}) = \mu(X)$. Clearly, $(X, \mathcal{A}, \mu)$ is a measure space which is not complete. We calculate its completion $(X, \hat{\mathcal{A}}, \hat{\mu})$. By Construction 25.1.3.4, as the only null set is $\{2, 3\}$, we have

$$\hat{\mathcal{A}} = \{\emptyset, X, \{1\}, \{2, 3\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}.$$

Hence $\hat{\mathcal{A}} = \mathcal{P}(X)$. Similarly, $\hat{\mu}$ is easy to find by the definition in Construction 25.1.3.4.

25.1.4 Outer and Lebesgue measures

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We wish to understand the type of sets of $\mathbb{R}$ we obtain by looking at the *area below* f. That is the set $A = \{(x, y) \in [a, b] \times \mathbb{R} \mid f(x) < y\}$. The high-school Newtonian integration proceeds by approximating the set A from the *outside* in the sense that we take bigger rectangles E containing A and then try to make it small so that in the limit we obtain A. We would now like to formalize this idea and this is the main purpose of outer measures. Indeed, note that in the above example, we have that the target set A has the following property with respect to all $E \subseteq \mathbb{R}^2$: the area of E is recovered by taking area of $E \cap A$ and adding it to area of $E \cap A^c$. This will be the main force behind our definition of outer measure.

Definition 25.1.4.1. (**Outer measures and measurable sets**) Let X be a non-empty set. A function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is said to be an outer measure if it satisfies the following properties:

1. $\mu^*(\emptyset) = 0$,
2. $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$,
3. $\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n)$.

A subset $A \subseteq X$ is said to be μ^*-measurable if for all $E \subseteq X$ we have that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We will denote the collection of all μ^*-measurable sets by $\mathcal{M}$.

Remark 25.1.4.2. Let X be a set with an outer measure μ^*. By subadditivity (the 3rd axiom of outer measure), we obtain that $A \subseteq X$ is μ^*-measurable if and only if for all $E \subseteq X$ we have the inequality

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

We now construct an important outer measure on $\mathbb{R}^n$. This is the precursor to the measure that we finally aim for (Remark 25.1.2.3). This also portrays the use of the idea we discussed in the beginning.

Construction 25.1.4.3. (*Lebesgue outer measure on* $\mathbb{R}^n$) Consider $\mathbb{R}^m$ and for an box $I \subseteq \mathbb{R}^n$, by which we mean a product of interval $I = I_1 \times \cdots \times I_m$ for $I_i \subseteq \mathbb{R}$, denote $v(I)$ to be its volume; $v(I) = \prod_{i=1}^m l(I_i)$. For any $A \subseteq \mathbb{R}^n$, we define

$$\mu^*(A) = \inf \left\{ \sum_n v(I_n) \mid \bigcup_n I_n \supseteq A,\ I_n \text{ are boxes} \right\}.$$

We claim that μ^* forms an outer measure on $\mathbb{R}^n$.

Indeed, $\mu^*(\emptyset) = 0$ as $\emptyset \subseteq (-1/k, 1/k)^m$ for all $n \in \mathbb{N}$ so we have $\mu^*(A) \leq 2^m/n^m$. Taking $n \to \infty$ does the job.

Let $A \subseteq B$ in $\mathbb{R}^m$. Observe that to show $\mu^*(A) \leq \mu^*(B)$ we need only show that $\{\sum_n v(I_n) \mid \bigcup_n I_n \supseteq A,\ I_n \text{ are boxes}\} \supseteq \{\sum_n v(I_n) \mid \bigcup_n I_n \supseteq B,\ I_n \text{ are boxes}\}$. But this is trivial as and sequence of boxes $\{I_n\}$ covering B also covers A.

Finally we wish to show countable subadditivity. Pick $\{A_n\} \subseteq \mathcal{P}(\mathbb{R}^m)$. We wish to show that

$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n).$$

We use the ϵ-wiggle around sup and inf technique to show this, as discussed earlier in Construction 25.1.2.2. Pick any $\epsilon > 0$ and observe that we have a sequence of boxes $\{I_{n,k}\}_k$ for each $n \in \mathbb{N}$ such that $\bigcup_k I_{n,k} \supseteq A_n$ and

$$\mu^*(A_n) + \frac{\epsilon}{2^n} \geq \sum_k v(I_{n,k}). \tag{$\heartsuit$}$$

Observe further that $\bigcup_n \bigcup_k I_{n,k} \supseteq \bigcup_n A_n$. Consequently, we have $\sum_n \sum_k v(I_{n,k}) \geq \mu^*(\bigcup_n A_n)$. Hence,

$$\sum_n \left(\mu^*(A_n) + \frac{\epsilon}{2^n}\right) \geq \sum_n \sum_k v(I_{n,k}) \geq \mu^*\left(\bigcup_n A_n\right).$$

Hence μ^* is an outer measure on $\mathbb{R}^n$.

Note that the only place we required knowledge about boxes explicitly was only to show that $\mu^*(\emptyset) = 0$. This motivates the following simple result

Theorem 25.1.4.4. *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ be a collection of sets containing $\emptyset$ and X. Let $l : \mathcal{S} \to [0, \infty]$ be a function such that $l(\emptyset) = 0$. Then μ^* defined by*

$$\begin{aligned}\mu^* : \mathcal{P}(X) &\longrightarrow [0, \infty]\\ A &\longmapsto \inf\left\{\sum_n l(I_n) \mid \bigcup_n I_n \supseteq A,\ I_n \in \mathcal{S}\right\}\end{aligned}$$

is an outer measure on X.

Proof. Verbatim to Construction 25.1.4.3, except that $\mu^*(\emptyset) = 0$ follows now by the assumption that $l(\emptyset) = 0$ and $\emptyset \in \mathcal{P}(X)$ so that $\emptyset$ forms its own covering. □

25.1.5 Extension theorems-I : Carathéodory's theorem

Let X be a set and μ^* be an outer measure on X. We now show that one can construct a complete measure space on X out of this data. This means that we get a measure on $\mathbb{R}$ corresponding to Lebesguw outer measure on $\mathbb{R}$. However there will still be some issue which we will fix in the next section. This is attributed to Constantin Carathéodory.

Theorem 25.1.5.1. *Let X be a set and μ^* be an outer measure on X. Denote the collection of all μ^*-measurable sets by $\mathcal{M}$. Then $(X, \mathcal{M}, \mu)$ is a complete measure space where $\mu = \mu^*|_{\mathcal{M}}$.*

Proof. We begin by showing that the collection $\mathcal{M}$ is a σ-algebra. Note that $A \in \mathcal{M}$ if and only if for each $E \subseteq X$, we have $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Clearly $A = \emptyset$ trivially satisfies this. Similarly it is clear that if $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$. So we need only show that $\mathcal{M}$ is closed under countable union. To this end, we first show that $\mathcal{M}$ is closed under finite union.

Consider $A_1, A_2 \in \mathcal{M}$. We wish to show that $A_1 \cup A_2 \in \mathcal{M}$. Indeed, we have the following by the fact that $A_i \in \mathcal{M}$

$$\begin{aligned}\mu^*(E) &\geq \mu^*(E \cap A_1) + \mu^*(E \cap A_1^c)\\ &\geq \mu^*(E \cap A_1 \cap A_2) + \mu^*(E \cap A_1 \cap A_2^c) + \mu^*(E \cap A_1^c)\\ &\geq \mu^*(E \cap A_1 \cap A_2) + \mu^*(E \cap A_1 \cap A_2^c) + \mu^*(E \cap A_1^c \cap A_2) + \mu^*(E \cap A_1^c \cap A_2^c).\end{aligned}$$

By countable subadditivity of outer measures and the fact that $E \cap (A_1 \cup A_2) = (E \cap (A_1 \cap A_2^c)) \cup (E \cap (A_2 \cap A_1^c)) \cup (E \cap A_1 \cap A_2)$, we further obtain that

$$\mu^*(E) \geq \mu^*(E \cap (A_1 \cup A_2)) + \mu^*(E \cap (A_1 \cup A_2)^c).$$

We will now show that $\mathcal{M}$ is closed under countable union by using above. Let $\{A_n\} \subseteq \mathcal{A}$. We can always assume that $\{A_n\}$ are disjoint by replacing A_n by $A_n \setminus A_1 \cup \cdots \cup A_{n-1}$ as this does not changes the union. Let $A = \coprod_n A_n$. We wish to show that $A \in \mathcal{M}$, that is, for any $E \subseteq X$ we wish to show that

$$\mu^*(E) \geq m^*(E \cap A) + \mu^*(E \cap A^c).$$

Denote $B_n = \coprod_{k=1}^n A_k$. By the above, we have that $B_n \in \mathcal{M}$. Consequently, we have

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c). \tag{$*$}$$

We first observe that by the fact that $A_n \in \mathcal{M}$, we obtain the following recursion relation for $\mu^*(E \cap B_n)$

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \qquad (\heartsuit) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &\vdots \\ &= \sum_{k=1}^n \mu^*(E \cap A_k). \end{aligned}$$

Using the above on $(*)$ and the fact that $B_n^c \supseteq A^c$ together with continuity of measure, we obtain

$$\begin{aligned} \mu^*(E) &= \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \cap B_n^c) \\ &\geq \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \cap A^c). \end{aligned}$$

Taking $n \to \infty$, we obtain

$$\mu^*(E) \geq \sum_n \mu^*(E \cap A_n) + \mu^*(E \cap A^c). \tag{$**$}$$

By countable subadditivity of μ^*, we further obtain

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

This shows that $A \in \mathcal{M}$ and hence this completes the proof that $\mathcal{M}$ is a σ-algebra.

We next wish to show that μ is a measure on $(X, \mathcal{M})$. As $\mu^*(\emptyset) = 0$, therefore $\mu(\emptyset) = 0$. Now pick a disjoint collection $\{A_n\} \subseteq \mathcal{M}$. We wish to show that $\mu^*\left(\coprod_n A_n\right) = \sum_n \mu^*(A_n)$.

Indeed, by the countable subadditivity of outer measure, we already have that $\mu^*\left(\coprod_n A_n\right) \le \sum_n \mu^*(A_n)$. Hence we wish to show that $\mu^*\left(\coprod_n A_n\right) \ge \sum_n \mu^*(A_n)$. This follows at once by letting $E = \coprod_n A_n$ in $(**)$ above.

We finally wish to show that μ is a complete measure. Pick any $N \in \mathcal{M}$ such that $\mu(N) = 0$ and then pick any $A \subseteq N$. We wish to show that $A \in \mathcal{M}$. Pick any $E \subseteq X$. We wish to show that

$$\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Since $\mu^*(E \cap A) \le \mu^*(A)\mu^*(N) = 0$, so we reduce to showing that

$$\mu^*(E) \ge \mu^*(E \cap A^c).$$

As $N \in \mathcal{M}$ and $\mu^*(E \cap N) = 0$ for similar reasons as above, therefore we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap N^c)\\ &\ge \mu^*(E \cap A^c)\end{aligned}$$

as $A \subseteq N$. Hence $A \in \mathcal{M}$ and this completes the proof. $\square$

25.1.6 Extension theorems-II : Premeasures and Carathéodory-Hahn theorem

In the previous section, we constructed a complete measure by the data of an outer measure. By Theorem 25.1.4.4, we also saw that one can obtain an outer measure on any set through a "length function" defined on a collection of "good sets". We then in Theorem 25.1.5.1 constructed a complete measure space out of it. Unfortunately, the notion of outer measures as constructed in Theorem 25.1.4.4 is to broad for applications. Moreover, the aforementioned "length function" and "good sets" are not actually good enough to see some light of applications; these are too structureless. For this reason, we would jazz up our notion of "length function" by considering a *premeasure* and "good sets" by considering a collection of sets which should form an *algebra*. We would then prove an analogue of Carathéodory's theorem in this setup and as an application, we would see in the next section how to make sense of the traditionally difficult to precisely understand notion of a distribution.

25.1.7 Extension theorems-III : Distribution functions

25.1.8 Lebesgue-Stieltjes measure

25.1.9 Applications-I : Measure spaces and measurable functions

We now present applications of the above theory. This is, in particular, to showcase the true strength of abstract analysis. This can also be used to strengthen one's intuition about the topic.

σ-algebras and measure spaces

Lemma 25.1.9.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that μ is σ-finite if and only if there exists a countable disjoint family of measurable sets $\{A_n\}$ such that $X = \coprod_n A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.*

Proof. Note that R $\implies$ L is immediate from definition. Let μ be σ-finite. Then there exists $\{B_n\} \subseteq \mathcal{A}$ such that $\mu(B_n) < \infty$ and $\bigcup_n B_n = X$. Define $A_1 = B_1$ and $A_n = B_n \setminus B_1 \cup \cdots \cup B_{n-1}$. As $\mathcal{A}$ is a σ-algebra, so $\{A_n\} \subseteq \mathcal{A}$. Moreover, $A_n \cap A_m = \emptyset$ for all $n \neq m$ because if $m > n$[3] and $x \in A_m \cap A_n$, then $x \in B_m \setminus B_1 \cup \cdots \cup B_n \cup \ldots B_{m-1}$ and $x \in B_n \setminus B_1 \cup \ldots B_{n-1}$, a contradiction. As $A_n \subseteq B_n$, therefore $\mu(A_n) \leq \mu(B_n) < \infty$. To complete the proof, we need only show that $\bigcup_n A_n = \bigcup_n B_n$.

Pick any $x \in \bigcup_n A_n$. Then $x \in B_n \setminus B_1 \cup \cdots \cup B_{n-1}$ for some $n \in \mathbb{N}$. Thus, $x \in B_n$ and hence $x \in \bigcup_n B_n$. Conversely, pick $x \in \bigcup_n B_n$. Then $x \in B_n$ for some $n \in \mathbb{N}$. Now, either $x \in B_n \setminus B_1 \cup \cdots \cup B_{n-1}$ or $x \in B_n \cap (B_1 \cup \cdots \cup B_{n-1})$. If the former is true, then $x \in A_n$ and we are done. If the latter is true, then we may assume $x \in B_{n-1} \cap B_n$. Now again either $x \in B_{n-1} \setminus B_1 \cup \cdots \cup B_{n-2}$ or $x \in B_n \cap B_{n-1} \cap (B_1 \cup \cdots \cup B_{n-2})$. Repeating this process inductively, we will end up in either of the following cases:

1. $x \in A_k$ for some $1 \leq k \leq n$,
2. $x \in B_1 \cap \cdots \cap B_n$.

As $B_1 = A_1$ by construction, therefore in either case we are done. □

Lemma 25.1.9.2. *Given $\mathcal{S} \subseteq \mathcal{P}(X)$, denote by $\mathcal{A}(\mathcal{S})$ the σ-algebra generated by $\mathcal{S}$. Then,*

$$\mathcal{A}(\mathcal{S}) = \mathcal{A}(\mathcal{A}(\mathcal{S})).$$

Proof. Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$ be an arbitrary collection of subsets of X. If X is empty then the statement is vacuously true, so let X be non-empty. Since the σ-algebra generated by $\mathcal{A}(\mathcal{S})$ is the intersection of all σ-algebras containing $\mathcal{A}(\mathcal{S})$, therefore we have that $\mathcal{A}(\mathcal{A}(\mathcal{S})) = \bigcap_{\mathcal{C} \supseteq \mathcal{A}(\mathcal{S})} \mathcal{C}$. Since $\mathcal{A}(S)$ is a σ-algebra containing $\mathcal{A}(\mathcal{S})$, therefore $\mathcal{A}(\mathcal{A}(\mathcal{S})) \subseteq \mathcal{A}(\mathcal{S})$. Since $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{C}$ for all σ-algebras $\mathcal{C}$ containing $\mathcal{A}(\mathcal{S})$, therefore $\mathcal{A}(\mathcal{A}(\mathcal{S})) \supseteq \mathcal{A}(\mathcal{S})$. □

Lemma 25.1.9.3. *Let $\mathcal{A}(\mathcal{S})$ be the σ-algebra generated by a set $\mathcal{S} \subseteq \mathcal{P}(X)$. Then, $\mathcal{A}(\mathcal{S})$ is the union of the σ-algebras generated by Y as Y ranges over all countable subsets of $\mathcal{S}$.*

Proof. Let X be a non-empty set and $\mathcal{S} \subseteq \mathcal{P}(X)$. We wish to show that

$$\mathcal{A}(\mathcal{S}) = \bigcup_{\mathcal{Y} \subseteq \mathcal{S},\ \text{countable}} \mathcal{A}(\mathcal{Y}).$$

[3] which we may assume wlog.

Let $\mathcal{Y} \subseteq \mathcal{S}$ be a countable subcollection. Then, $\mathcal{A}(\mathcal{Y}) \subseteq \mathcal{A}(\mathcal{S})$. Consequently, $\bigcup_{\mathcal{Y}\subseteq\mathcal{S},\text{ countable}} \mathcal{A}(\mathcal{Y}) \subseteq \mathcal{A}(\mathcal{S})$. Conversely, we need to show that

$$\mathcal{A}(\mathcal{S}) \subseteq \bigcup_{\mathcal{Y}\subseteq\mathcal{S},\text{ countable}} \mathcal{A}(\mathcal{Y}).$$

We claim that $\bigcup_{\mathcal{Y}\subseteq\mathcal{S},\text{ countable}} \mathcal{A}(\mathcal{Y})$ is a σ-algebra containing $\mathcal{S}$. This would complete the proof as $\mathcal{A}(\mathcal{S})$ is the smallest σ-algebra containing $\mathcal{S}$.

Denote $\mathcal{Z} = \bigcup_{\mathcal{Y}\subseteq\mathcal{S},\text{ countable}} \mathcal{A}(\mathcal{Y})$. As $\mathcal{A}(\mathcal{Y})$s are σ-algebras, therefore $\mathcal{Z}$ contains X and $\emptyset$. Let $A \in \mathcal{Z}$. Then $A \in \mathcal{A}(\mathcal{Y})$ for some $\mathcal{Y} \subseteq \mathcal{S}$ countable. Consequently, $A^c \in \mathcal{A}(\mathcal{Y})$ and thus $A^c \in \mathcal{Z}$. Let $\{A_n\} \subseteq \mathcal{Z}$ be a countable collection of sets. Then $A_n \in \mathcal{A}(\mathcal{Y}_n)$ for all $n \in \mathbb{N}$. Further, we have that $\mathcal{Y}_k \subseteq \mathcal{A}\left(\bigcup_n \mathcal{Y}_n\right)$ for all $k \in \mathbb{N}$ as $\mathcal{Y}_k \subseteq \bigcup_n \mathcal{Y}_n$. As $\mathcal{Y}_k$ are countable and countable union of countable sets is countable, therefore $\bigcup_n \mathcal{Y}_n$ is countable. Thus, we have

$$A_k \in \mathcal{A}(\mathcal{Y}_k) \subseteq \mathcal{A}\left(\bigcup_n \mathcal{Y}_n\right) \subseteq \mathcal{Z}\ \forall k \in \mathbb{N}.$$

Thus from above, we obtain that

$$\bigcup_k A_k \in \left(\bigcup_n \mathcal{Y}_n\right) \subseteq \mathcal{Z}.$$

Hence, $\mathcal{Z}$ is a σ-algebra. To complete the proof, we need only show that $\mathcal{Z}$ contains $\mathcal{S}$.

Let $A \in \mathcal{S}$. Then since $\{A\}$ is a countable subset of $\mathcal{S}$, therefore $\mathcal{A}(\{A\})$ is contained in $\mathcal{Z}$ and thus $A \in \mathcal{Z}$. □

Lemma 25.1.9.4. *The σ-algebra generated by*

1. $\mathcal{S} = \{(a, b] \mid a < b \in \mathbb{Q}\}$,
2. $\mathcal{S} = \{(a, n] \mid a \in \mathbb{Q}, n \in \mathbb{Z}\}$,

is the Borel σ-algebra on $\mathbb{R}$.

Proof. 1. Let $\mathcal{S} = \{(a, b] \mid a, b \in \mathbb{Q}\}$. We wish to show that $\mathcal{A}(\mathcal{S}) = \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$. Since $(a, b]$ for $a, b \in \mathbb{Q}$ is contained in $\mathcal{B}$ as $(a, b] = (a, b) \cup \bigcap_{n\in\mathbb{N}}(b - 1/n, b + 1/n)$, therefore $\mathcal{S} \subseteq \mathcal{B}$. Consequently, $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{B}$ as $\mathcal{B}$ is the smallest σ-algebra containing open intervals.

Since we also know that $\mathcal{B}$ is generated by the collection of all closed intervals $[a, b]$ in $\mathbb{R}$, therefore to show that $\mathcal{B} \subseteq \mathcal{A}(\mathcal{S})$, it would suffice to show $[a, b] \in \mathcal{A}(\mathcal{S})$ where $a < b$ in $\mathbb{R}$. Pick $a < b$ in $\mathbb{R}$. By density of $\mathbb{Q}$, we may pick $\{a_n\}$ to be an increasing sequence such that $a_n \in \mathbb{Q}$, $a_n < a$ and $\lim_{n\to\infty} a_n = a$. Similarly, we may pick a decreasing sequence $\{b_n\}$ such that $b_n \in \mathbb{Q}$, $b_n > b$ and $\lim_{n\to\infty} b_n = b$. Consequently, we claim that

$$[a, b] = \bigcap_n (a_n, b_n]$$

where $(a_n, b_n] \in \mathcal{S}$. Indeed, $(\subseteq)$ is clear. For $(\supseteq)$, take $x \in \bigcap_n (a_n, b_n]$. Hence $a_n < x \leq b_n$. Taking $n \to \infty$, we get $a \leq x \leq b$ as desired. Thus, $[a, b] \in \mathcal{A}(\mathcal{S})$.

2. Let $\mathcal{S} = \{(a,n] \ |a \in \mathbb{Q}, \ n \in \mathbb{N}\}$. We wish to show that $\mathcal{A}(\mathcal{S}) = \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$. Since $(a,n]$ for $a \in \mathbb{Q}$ and $n \in \mathbb{N}$ is contained in $\mathcal{B}$ as $(a,n] = (a,n) \cup \bigcap_{k\in\mathbb{N}}(n-1/k, n+1/k)$, therefore $\mathcal{S} \subseteq \mathcal{B}$. Consequently, $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{B}$.

Since we also know that $\mathcal{B}$ is generated by the collection of all open intervals of the form (a,∞), $a \in \mathbb{R}$, therefore to show that $\mathcal{B} \subseteq \mathcal{A}(\mathcal{S})$, it would suffice to show $(a,\infty) \in \mathcal{A}(\mathcal{S})$ for all $a \in \mathbb{R}$. Pick (a,∞) for some $a \in \mathbb{R}$. By density of $\mathbb{Q}$, there exists a decreasing sequence $\{a_n\}$ in $\mathbb{R}$ such that $a_n \in \mathbb{Q}$, $a_n > a$ and $\lim_{n\to\infty} a_n = a$. Consequently, we claim that

$$(a,\infty) = \bigcup_n (a_n, n]$$

where $(a_n, n] \in \mathcal{S}$. Indeed, for $(\subseteq)$, take $x \in (a,\infty)$. We therefore have $a < x < \infty$. As $\lim_{n\to\infty} a_n = a$ and $a_n > a$ for all $n \in \mathbb{N}$, therefore there exists $N \in \mathbb{N}$ such that $a < a_n \le a_N < x$ for all $n \ge N$. Consequently, for some large $n \in \mathbb{N}$ greater than N such that $x \le n$, we obtain $a_n < x \le n$ and hence $x \in (a_n, n]$. For $(\supseteq)$, take $x \in \bigcup_n (a_n, n]$ and thus we get $a < a_n < x \le n < \infty$. Thus, $(a,\infty) \in \mathcal{A}(\mathcal{S})$. $\square$

Lemma 25.1.9.5. *The Borel σ-algebra on $\mathbb{R}^2$ is generated by*

$$\{(I \times \mathbb{R}) \cup (\mathbb{R} \times J) \mid I, J \subseteq \mathbb{R}, \ \textit{open intervals}\}.$$

Proof. Let $\mathcal{S} = \{(I \times \mathbb{R}) \cup (\mathbb{R} \times J) \mid I, J \subseteq \mathbb{R} \text{ is open}\}$. We wish to show that $\mathcal{A}(\mathcal{S}) = \mathcal{B}$ where $\mathcal{B}$ is the σ-algebra of $\mathbb{R}^2$.

As $\mathcal{S}$ is a collection of open sets of $\mathbb{R}^2$ and $\mathcal{B}$ is generated by all open sets of $\mathbb{R}^2$, therefore $\mathcal{S} \subseteq \mathcal{B}$ and thus $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{B}$.

We now wish to show that $\mathcal{B} \subseteq \mathcal{A}(\mathcal{S})$. It would suffice to show that any open set $U \subseteq \mathbb{R}^2$ is in $\mathcal{A}(\mathcal{S})$. Note that $\mathcal{A}(\mathcal{S})$ consists of all open rectanlges $I \times J = (I \times \mathbb{R}) \cap (\mathbb{R} \times J)$. Thus, it would suffice to show that U can be written as countable union of open rectangles. Recall that open rectangles forms a basis for the usual topology on $\mathbb{R}^2$. Consider the collection of all open rectangles K inside U whose vertices have both rational coordinates. We claim that the union of such open rectangles is equal to U. Indeed, their union is inside U and for any $x \in U$, there exists an open ball $x \in B \subseteq U$, so there exists an open rectangle K inside B which contains x and has vertices which have both rational coordinates. Thus U is equal to the union of all such rectangles. Since there are only countably many such open rectangles as they are parameterized by choice of 4 points in $\mathbb{Q}^2 \cap U$ which is atmost countably many, therefore we have obtained a countable cover of U by open rectangles. This completes the proof. $\square$

Lemma 25.1.9.6. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $A, B \in \mathcal{A}$. Then,*

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

Proof. Observe that we can write

$$A \cup B = (A \setminus (A \cap B)) \cup B$$

where the right side is a disjoint union. Consequently, we have

$$\mu(A \cup B) = \mu(A \setminus A \cap B) + \mu(B). \tag{6.1}$$

We now have two cases. If $\mu(A \cap B) = \infty$, then since $\mu(A \cap B) \leq \mu(A), \mu(B)$ and $\mu(A) \leq \mu(A \cup B)$, therefore we get $\mu(A \cup B) = \mu(A \cap B) = \mu(A) = \mu(B) = \infty$, so that the statement to be proven is a tautology. Else if $\mu(A \cap B) < \infty$, then we can write

$$\mu(A \setminus A \cap B) = \mu(A) - \mu(A \cap B).$$

Consequently, by Eq. (6.1) and the fact that $\mu(A \cap B) < \infty$, we have

$$\begin{aligned} \mu(A \cup B) &= \mu(A) - \mu(A \cap B) + \mu(B) \\ \mu(A \cup B) + \mu(A \cap B) &= \mu(A) + \mu(B). \end{aligned}$$

This completes the proof. □

Lemma 25.1.9.7. *Let $x \in \mathbb{R}$ and let B be a Borel subset of $\mathbb{R}$. Then, $x + B$ and xB are Borel subsets of $\mathbb{R}$ (that is, Borel subsets of $\mathbb{R}$ are translation and dilation invariant).*

Proof. 1. Let $x \in \mathbb{R}$ and $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. We wish to show that for all $B \in \mathcal{B}$, the translate $x + B \in \mathcal{B}$. Consequently, we wish to show

$$x + \mathcal{B} \subseteq \mathcal{B}$$

where $x + \mathcal{B} = \{x + B \mid B \in \mathcal{B}\}$. We use the following standard technique to show this.

Consider the following collection

$$\mathcal{C} = \{B \in \mathcal{B} \mid x + B \in \mathcal{B}\}.$$

Our goal is to show that $\mathcal{C} = \mathcal{B}$. Note that $\mathcal{C} \subseteq \mathcal{B}$. Conversely, we wish to show that $\mathcal{B} \subseteq \mathcal{C}$. This would follow immediately if we show that $\mathcal{C}$ is a σ-algebra containing all open intervals, as $\mathcal{B}$ is the σ-algebra generated by all open intervals.

We now establish that $\mathcal{C}$ is a σ-algebra. Since $x + \mathbb{R} = \mathbb{R}$ and $x + \emptyset = \emptyset$, therefore $\mathbb{R}, \emptyset \in \mathcal{C}$. Let $A \in \mathcal{C}$. We wish to show that $A^c \in \mathcal{C}$. Since $x + A \in \mathcal{B}$, therefore $(x + A)^c \in \mathcal{B}$. Thus it suffices to show that $(x + A)^c = x + A^c$. Indeed, we have the following equalities

$$\begin{aligned} (x + A)^c &= \{y \in \mathbb{R} \mid y \notin x + A\} \\ &= \{y \in \mathbb{R} \mid y - x \notin A\} \\ &= \{y \in \mathbb{R} \mid y - x \in A^c\} \\ &= \{y \in \mathbb{R} \mid y \in x + A^c\} \\ &= x + A^c. \end{aligned}$$

Let $\{A_n\} \subseteq \mathcal{C}$. We wish to show that $\bigcup_n A_n \in \mathcal{C}$. We have that for each $n \in \mathbb{N}$, $x + A_n \in \mathcal{B}$. It would thus suffice to show that

$$x + \bigcup_n A_n = \bigcup_n (x + A_n).$$

Indeed, take $x+a \in x+\bigcup_n A_n$. Hence $a \in A_n$ for some $n \in \mathbb{N}$. Consequently, $x+a \in x+A_n$. Thus $x+a \in \bigcup_n (x+A_n)$. Conversely, let $z \in \bigcup_n (x+A_n)$. Then $z = x+y_n$ for $y_n \in A_n$. Consequently, $z \in x+\bigcup_n A_n$. This show that $\mathcal{C}$ is a σ-algebra.

To complete the proof, we now need only show that $\mathcal{C}$ has all open intervals. This is immediate, as we show now. Take any $(a,b) \subseteq \mathbb{R}$. Since $x+(a,b) = (x+a, x+b) \in \mathcal{B}$, therefore $(a,b) \in \mathcal{C}$.

2. Let $x \in \mathbb{R}$ and $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. We wish to show that for all $B \in \mathcal{B}$, the dilate $x \cdot B \in \mathcal{B}$. Note that $x \cdot B = \{xb \mid b \in B\}$. Consequently, we wish to show

$$x \cdot \mathcal{B} \subseteq \mathcal{B}$$

where $x \cdot \mathcal{B} = \{x \cdot B \mid B \in \mathcal{B}\}$. If $x = 0$, then $x \cdot \mathcal{B} = \{0\}$ and that is trivially inside $\mathcal{B}$ as $\{0\} = \bigcap_n (-1/n, 1/n)$. Thus we now assume that $x \neq 0$. We use the following standard technique to show the above inclusion.

Consider the following collection

$$\mathcal{C} = \{B \in \mathcal{B} \mid x \cdot B \in \mathcal{B}\}.$$

Our goal is to show that $\mathcal{C} = \mathcal{B}$. Note that $\mathcal{C} \subseteq \mathcal{B}$. Conversely, we wish to show that $\mathcal{B} \subseteq \mathcal{C}$. This would follow immediately if we show that $\mathcal{C}$ is a σ-algebra containing all open intervals, as $\mathcal{B}$ is the σ-algebra generated by all open intervals.

We now establish that $\mathcal{C}$ is a σ-algebra. Observe that $x \cdot \mathbb{R} = \mathbb{R}$. Indeed, as $x \cdot \mathbb{R} \subseteq \mathbb{R}$ is clear, we can also write any $a \in \mathbb{R}$ as $x \cdot x^{-1}a$. We also have $x \cdot \emptyset = \emptyset$. Therefore $\mathbb{R}, \emptyset \in \mathcal{C}$. Let $A \in \mathcal{C}$. We wish to show that $A^c \in \mathcal{C}$. Since $x \cdot A \in \mathcal{B}$, therefore $(x \cdot A)^c \in \mathcal{B}$. Thus it suffices to show that $(x \cdot A)^c = x \cdot A^c$. Indeed, we have the following equalities

$$\begin{aligned}
(x \cdot A)^c &= \{y \in \mathbb{R} \mid y \notin x \cdot A\} \\
&= \{y \in \mathbb{R} \mid x^{-1}y \notin A\} \\
&= \{y \in \mathbb{R} \mid x^{-1}y \in A^c\} \\
&= \{y \in \mathbb{R} \mid y \in x \cdot A^c\} \\
&= x \cdot A^c.
\end{aligned}$$

Let $\{A_n\} \subseteq \mathcal{C}$. We wish to show that $\bigcup_n A_n \in \mathcal{C}$. We have that for each $n \in \mathbb{N}$, $x \cdot A_n \in \mathcal{B}$. It would thus suffice to show that

$$x \cdot \bigcup_n A_n = \bigcup_n (x \cdot A_n).$$

Indeed, take $x \cdot a \in x \cdot \bigcup_n A_n$. Hence $a \in A_n$ for some $n \in \mathbb{N}$. Consequently, $x \cdot a \in x \cdot A_n$. Thus $x \cdot a \in \bigcup_n (x \cdot A_n)$. Conversely, let $z \in \bigcup_n (x \cdot A_n)$. Then $z = x \cdot y_n$ for $y_n \in A_n$. Consequently, $z \in x \cdot \bigcup_n A_n$. This show that $\mathcal{C}$ is a σ-algebra.

To complete the proof, we now need only show that $\mathcal{C}$ has all open intervals. This is immediate, as we show now. Take any $(a,b) \subseteq \mathbb{R}$. If $x > 0$, then we have $x \cdot (a,b) = (x \cdot a, x \cdot b) \in \mathcal{B}$, therefore $(a,b) \in \mathcal{C}$. If $x < 0$, then we have $x \cdot (a,b) = (x \cdot b, x \cdot a) \in \mathcal{B}$, therefore $(a,b) \in \mathcal{C}$. $\square$

Lemma 25.1.9.8. *Let $(X, \mathcal{A})$ be a measurable space and let $\{\mu_i\}_{i=1}^n$ be a finite collection of measures on $(X, \mathcal{A})$. If $r_1, \dots, r_n \in \mathbb{R}_{\geq 0}$, then $\sum_i r_i\mu_i$ is a measure on $(X, \mathcal{A})$ (that is, positive linear combination of measures is a measure).*

Proof. Let $(X, \mathcal{A})$ be a measurable space and $\{\mu_i\}_{i=1}^n$ be a collection of measures on it. Let $\{r_i\}_{i=1}^n \subseteq \mathbb{R}_{\geq 0}$. We wish to show that $\mu = \sum_{i=1}^n r_i\mu_i$ is a measure on $(X, \mathcal{A})$. First we may assume that each $r_i > 0$ as if any $r_j = 0$, then $\mu(A) = \sum_{i=1}^n r_i\mu_i(A) = \sum_{i\neq j} r_i\mu_i(A) + r_j\mu_j(A)$, therefore if $\mu_j(A) < \infty$, then $r_j\mu_j(A) = 0$ and if $\mu_j(A) = \infty$, then since $0 \cdot \infty = 0$, therefore still $r_j\mu_j(A) = 0$. Further, if all $r_i = 0$, then $\mu = 0$, which is the trivial measure. Consequently, we assume that $r_i > 0$ for all $i = 1, \dots, n$.

We now show that μ is a measure on $(X, \mathcal{A})$. We have $\mu(\emptyset) = \sum_{i=1}^n r_i\mu_i(\emptyset) = \sum_{i=1}^n r_i \cdot 0 = 0$. Let $\{A_n\} \subseteq \mathcal{A}$ be a collection of disjoint measurable sets. We wish to show that

$$\mu\left(\coprod_k A_k\right) = \sum_k \mu(A_k).$$

We have

$$\begin{aligned}\mu\left(\coprod_k A_k\right) &= \sum_{i=1}^n r_i\mu_i\left(\coprod_k A_k\right)\\ &= \sum_{i=1}^n r_i \sum_{k=1}^\infty \mu_i(A_k).\end{aligned}$$

We now claim that

$$\sum_{i=1}^n r_i \sum_{k=1}^\infty \mu_i(A_k) = \sum_{k=1}^\infty \sum_{i=1}^n r_i\mu_i(A_k) \tag{8.1}$$

and showing this will complete the proof as

$$\sum_{k=1}^\infty \sum_{i=1}^n r_i\mu_i(A_k) = \sum_{k=1}^\infty \mu(A_k).$$

We have few cases for establishing Eq. (8.1).

1. *If for all $i = 1, \dots, n$, the series $\sum_{k=1}^\infty \mu_i(A_k)$ is finite.* Then, $\sum_{i=1}^n r_i \sum_{k=1}^\infty \mu_i(A_k) = \sum_{i=1}^n \sum_{k=1}^\infty r_i\mu_i(A_k)$. Now, if $\sum_n x_n, \sum_n y_n$ are two convergent positive series, then their linear combination $c\sum_n x_n + d\sum_n y_n$ is equal to $\sum_n cx_n + dy_n$, where $c, d \in \mathbb{R}_{\geq 0}$. Indeed, this follows at once from the equality $c\lim_{n\to\infty} \sum_{k=1}^n x_k + d\lim_{n\to\infty} \sum_{k=1}^n y_k = \lim_{n\to\infty} \sum_{k=1}^n cx_k + dy_k$, which follows from the fact that both the limit exists and $c, d \in \mathbb{R}$. Consequently, we have

   $$\sum_{i=1}^n \sum_{k=1}^\infty r_i\mu_i(A_k) = \sum_{k=1}^\infty \sum_{i=1}^n r_i\mu_i(A_k),$$

   which is what we needed.

2. *If there exists $i_0 = 1, \ldots, n$ such that the series $\sum_{k=1}^{\infty} \mu_{i_0}(A_k) = \infty$.* In this case, in the Eq. (8.1), the left side is ∞. The right side is also infinity as shown below:

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{i=1}^{n} r_i\mu_i(A_k) &\geq \sum_{k=1}^{\infty} r_{i_0}\mu_{i_0}(A_k)\\ &= \infty\end{aligned}$$

   where the first inequality follows from $r_i > 0$ for all $i = 1, \ldots, n$ and measure being positive by definition. Consequently, Eq. (8.1) follows in this case as well.

This completes the proof. $\square$

Lemma 25.1.9.9. *For any set X and a subset $S \subseteq X$, the collection*

$$\mathcal{A}_S = \{A \subseteq X \mid A \subseteq S \text{ or } A^c \subseteq S\}$$

is a σ-algebra on X.

Proof. Let X be a non-empty set, $S \subseteq X$ and define

$$\mathcal{A}_S := \{A \subseteq X \mid A \subseteq S \text{ or } A^c \subseteq S\}.$$

We claim that this forms a σ-algebra on X. As $X^c = \emptyset \subseteq S$, therefore $X \in \mathcal{A}_S$ and $\emptyset \in \mathcal{A}_S$. Let $A \in \mathcal{A}_S$. If $A \subseteq S$, then A^c is such that $(A^c)^c = A \subseteq S$, so $A^c \in \mathcal{A}_S$. If $A^c \subseteq S$, then A^c is such that $A^c \subseteq S$, so $A^c \in \mathcal{A}_S$. So in both cases $\mathcal{A}_S$ is closed uncer complements.

Let $\{A_n\} \subseteq \mathcal{A}_S$ be a collection of subsets. We wish to show that $\bigcup_n A_n \in \mathcal{A}_S$. We have three cases.

C1. *$A_n \subseteq S$ for all $n \in \mathbb{N}$.* Then $\bigcup_n A_n \subseteq S$ and thus $\bigcup_n A_n \in \mathcal{A}_S$.

C2. *$\exists A_m$ such that $A_m \not\subseteq S$.* Then $A_m^c \subseteq S$. We then observe by De-Morgan's law that

$$\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c \subseteq A_m^c \subseteq S.$$

Consequently, $\bigcup_n A_n \in \mathcal{A}_S$.

C3. *$A_n \not\subseteq S$ for all $n \in \mathbb{N}$.* Then $A_n^c \subseteq S$ for all $n \in \mathbb{N}$. We again observe by De-Morgan's law that

$$\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c \subseteq A_m^c \subseteq S \; \forall m \in \mathbb{N}.$$

Consequently, $\bigcup_n A_n \in \mathcal{A}_S$.

In all three cases, $\bigcup_n A_n \in \mathcal{A}_S$. Hence $\mathcal{A}_S$ is a σ-algebra. $\square$

Lemma 25.1.9.10. *Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space, and let $\mu(A) = \infty$ for some $A \in \mathcal{A}$. If $M > 0$, then there exists $B \subseteq A$ such that $M < \mu(B) < \infty$.*

Proof. Let $(X, \mathcal{A}, \mu)$ be a semi-finite measure space and $A \in \mathcal{A}$ such that $\mu(A) = \infty$. We wish to show that for all $M > 0$, there exists a subset $B \subseteq A$ such that $B \in \mathcal{A}$ and $M < \mu(B) < \infty$.

We wish to show that there exists measurable subsets of A of arbitrarily large size. Therefore, consider the collection

$$S = \{\mu(B) \mid B \subseteq A, B \in \mathcal{A}, \mu(B) < \infty\}.$$

Denote $l = \sup S$. We wish to show that $l = \infty$. Pick a sequence $\{B_n\} \subseteq S$ such that $\lim_{n\to\infty}\mu(B_n) = l$. We first claim that

$$\mu\left(\bigcup_n B_n\right) = l \tag{10.1}$$

Clearly, $\bigcup_n B_n \in \mathcal{A}$. Observe that since

$$\mu(B_k) \leq \mu\left(\bigcup_n B_n\right)$$

for all $k \in \mathbb{N}$, therefore taking $k \to \infty$, we easily obtain

$$l \leq \mu\left(\bigcup_n B_n\right).$$

Conversely, we wish to show that

$$\mu\left(\bigcup_n B_n\right) \leq l.$$

Let $D_1 = B_1$, $D_2 = B_1 \cup B_2$ and in general $D_n = B_1 \cup \cdots \cup B_n$. Then we observe that $\{D_n\} \subseteq \mathcal{A}$ forms an increasing sequence of sets with $\bigcup_n D_n = \bigcup_n B_n$. Consequently,

$$\mu\left(\bigcup_n B_n\right) = \mu\left(\bigcup_k D_k\right) = \lim_{k\to\infty}\mu(D_k).$$

Since $D_k \subseteq A$ is such that $\mu(D_k) \leq \sum_{i=1}^k \mu(B_i) < \infty$ (by subadditivity), therefore $\mu(D_k) \in S$ for all $k \in \mathbb{N}$. Consequently,

$$\lim_{k\to\infty}\mu(D_k) \leq l.$$

Therefore we obtain $\mu\left(\bigcup_n B_n\right) \leq l$. Hence this completes the proof of Eq. (10.1).

Since we wish to show that $l = \infty$, so assume to the contrary that $l < \infty$. It follows from Eq. (10.1) that $\mu\left(\bigcup_n B_n\right) < \infty$ and therefore $\bigcup_n B_n \in S$. Let $C = \bigcup_n B_n$. Then consider $A_1 = A \setminus C$. Since $\mu(A_1) = \mu(A) - \mu(C)$ as $\mu(C) < \infty$, therefore we have $\mu(A_1) = \infty - \mu(C) = \infty$. It follows from semifiniteness that there exists $C_1 \subseteq A_1$ such that $C_1 \in \mathcal{A}$ and $0 < \mu(C_1) < \infty$. Note that C_1 and C are disjoint. It follows that the disjoint union $C_1 \cup C \subseteq A$ is such that $\mu(C \cup C_1) \in S$. But since $\mu(C_1 \cup C) = \mu(C_1) + \mu(C) > \mu(C) = l$, therefore S contains an element which is strictly larger than its supremum, a contradiction. Hence $l = \infty$ and this completes the proof. $\square$

Lebesgue measure on $\mathbb{R}$

In this section $(\mathbb{R}, \mathcal{M}, m)$ denotes the Lebesgue measure space on $\mathbb{R}$ and m^* denotes the Lebesgue outer measure on $\mathbb{R}$.

Lemma 25.1.9.11. *Every Borel subset of $\mathbb{R}$ is Lebesgue measurable.*

Proof. Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space over $\mathbb{R}$. We wish to show that the σ-algebra of Borel sets denoted $\mathcal{B}$ is in $\mathcal{M}$. Denote by $\mathcal{A}$ the following:

$$\mathcal{A} = \{\text{disjoint finite union of intervals of form } (-\infty, a], (b, \infty), (a, b] \text{ for } a < b \in \mathbb{R}\}. \quad (1.1)$$

By construction of Lebesgue measure, we know that $\mathcal{A} \subseteq \mathcal{M}$. We thus claim that the σ-algebra generated by $\mathcal{A}$ contains $\mathcal{B}$, that is, $\langle \mathcal{A} \rangle \supseteq \mathcal{B}$. This will conclude the proof.

Indeed, as we know that $\mathcal{B}$ is generated by all closed intervals of the form $(-\infty, a]$ for all $a \in \mathbb{R}$, therefore it suffices to show that $(-\infty, a] \in \langle \mathcal{A} \rangle$, but that is a tautology as $(-\infty, a]$ is in $\mathcal{A}$. Hence $\mathcal{B} \subseteq \langle A \rangle$. $\square$

Lemma 25.1.9.12. *Let A be a subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Then,*

1. *$m^*(A + c) = m^*(A)$,*
2. *$A \in \mathcal{M}$ if and only if $A + c \in \mathcal{M}$,*
3. *if $A \in \mathcal{M}$, then $m(A + c) = m(A)$.*

Proof. Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$. Take $A \subseteq \mathbb{R}$ and for $c \in \mathbb{R}$ define $A + c = \{a + c \in \mathbb{R} \mid a \in A\}$. Let us set up some notation. For any $E \subseteq \mathbb{R}$, we denote

$$C(E) = \left\{ \{I_n\} \mid \bigcup_n I_n \supseteq A,\ I_n = (a_n, b_n] \in \mathcal{A} \right\} \quad (*)$$

where $\mathcal{A}$ is the algebra defined in Eq. (1.1). Further, let us denote

$$\Sigma C(E) = \left\{ \sum_n l(I_n) \in [0, \infty] \mid \{I_n\} \in C(E) \right\} \quad (**)$$

where $l((a, b]) = b - a$ is the length function. By definition, we have $m^*(E) := \inf \Sigma C(E)$.

(i) : We first wish to show that the Lebesgue outer measure m^* is translation invariant. That is, $m^*(A + c) = m^*(A)$. We first show $m^*(A + c) \geq m^*(A)$. Pick any $\{I_n\} \in C(A)$. Then we claim that $\{I_n + c\}$ is an element of $C(A + c)$. Indeed, denoting $I_n = (a_n, b_n]$, we immediately get $I_n + c = (a_n + c, b_n + c]$. Now to see that $\bigcup_n (I_n + c) \supseteq A + c$, pick any $a + c \in A + c$ where $a \in A$. Then, as $\bigcup_n I_n \supseteq A$, therefore $a \in I_n$ for some n and thus $a + c \in I_n + c$. It follows that $\{I_n + c\} \in C(A + c)$. Further note that $l(I_n) = l(I_n + c)$ by definition. Consequently, we have

$$\Sigma C(A) \subseteq \Sigma C(A + c).$$

Taking infima, we yield $m^*(A) = \inf \Sigma C(A) \leq \inf \Sigma C(A + c) = m^*(A + c)$, that is $m^*(A) \leq m^*(A + c)$.

Conversely, we wish to show that $m^*(A) \geq m^*(A+c)$. For this, we use the standard technique of *ϵ-wiggle around* inf. Fix $\epsilon > 0$. By definition of $m^*(A)$, there exists $\{I_n\} \in C(A)$ where $I_n = (a_n, b_n]$ such that

$$m^*(A) + \epsilon > \sum_n b_n - a_n. \tag{2.1}$$

Note that we can write the above as

$$\begin{aligned} m^*(A) + \epsilon &> \sum_n (b_n + c) - (a_n + c) \\ &= \sum_n l((a_n + c, b_n + c]) \\ &= \sum_n l(I_n + c). \end{aligned}$$

We have $\{I_n + c\} \in C(A+c)$ as shown previously, therefore we obtain

$$m^*(A) + \epsilon > \sum_n l(I_n + c) \geq \inf \Sigma C(A+c) = m^*(A+c).$$

Hence we have $m^*(A) + \epsilon > m^*(A+c)$. Taking $\epsilon \to 0$, we obtain $m^*(A) \geq m^*(A+c)$. This completes the proof.

(ii) : We next wish to show that $A + c \in \mathcal{M}$ if and only if $A \in \mathcal{M}$. Observe that it suffices to show that $A \in \mathcal{M} \implies A + c \in \mathcal{M}$. Indeed, for the converse, take $B = A + c \in \mathcal{M}$. To show that $A \in \mathcal{M}$, it would suffice to show that $B - c \in \mathcal{M}$, which would follow at once by previous. Hence, we may only show that $A \in \mathcal{M} \implies A + c \in \mathcal{M}$.

Pick $A \in \mathcal{M}$. Fix $\epsilon > 0$. By regularity theorems, there exists open $U \supseteq A$ such that $m^*(U \setminus A) < \epsilon$. We now claim the following three statements:

1. *$U + c$ is open* : Indeed, pick any $x + c \in U + c$ where $x \in U$. As U is open, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq U$. Consequently, $(x - \delta + c, x + \delta + c) \subseteq U + c$, hence $U + c$ is open.
2. *$U + c$ contains $A + c$* : Pick any $a + c \in A + c$ where $a \in A$. As $U \supseteq A$, therefore $a + c \in U + c$.
3. *$(U + c) \setminus (A + c)$ equals $(U \setminus A) + c$* : We first show $(U + c) \setminus (A + c) \subseteq (U \setminus A) + c$. Pick any $x + c \in (U + c) \setminus (A + c)$. Then $x + c \in U + c$ and $x + c \notin A + c$. Thus, $x \in U$ and $x \notin A$. Hence $x \in U \setminus A$ and thus $x + c \in U \setminus A + c$.
   Conversely, pick $x + c \in (U \setminus A) + c$. Then $x \in U \setminus A$ and thus $x + c \in U + c$ and $x + c \notin A + c$. Thus $x + c \in (U + c) \setminus (A + c)$. This completes the proof of this claim.

By above three claims, we conclude that $U + c$ is an open set containing $A + c$ such that

$$m^*(U + c \setminus A + c) = m^*((U \setminus A) + c) \overset{(ii)}{=} m^*(U \setminus A) < \epsilon.$$

By regularity theorems, we conclude the proof.

(iii) : We wish to show that if $A \in \mathcal{M}$, then $m(A + c) = m(A)$. This is immediate from *(i)* as $m = m^*|_{\mathcal{M}}$. $\square$

Lemma 25.1.9.13. *Let A be a subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Then,*

1. $m^*(cA) = |c|\, m^*(A)$,
2. *for $c \neq 0$, $A \in \mathcal{M}$ if and only if $cA \in \mathcal{M}$,*
3. *if $A \in \mathcal{M}$, then $m(cA) = |c|\, m(A)$.*

Proof. Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space. Take any $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$.

(i) : We first wish to show that $m^*(cA) = |c|\, m^*(A)$. If $c = 0$, then the equality is immediate as $cA = \{0\}$ and we know that $m^*(\{0\}) = 0$ as $0 \in (-1/n, 1/n]$ for all $n \in \mathbb{N}$ so that $m^*(\{0\}) \leq 2/n$. Taking $n \to \infty$, we get that $m^*(\{0\}) = 0$. So we assume from now on that $c \neq 0$. We first immediately reduce to showing either one of

$$m^*(cA) \geq |c|\, m^*(A) \text{ or } m^*(cA) \leq |c|\, m^*(A)$$

Indeed, the other side follows by replacing A by cA and replacing c by $1/c$ in either of the above. We now have two cases based on c being positive or negative.

If $c > 0$, then we proceed as follows. We follow the convention of Eqns $(*)$ and $(**)$ as set up in Q2. We use the standard technique of ϵ-wiggle around inf. Fix $\epsilon > 0$. By definition of outer measure, there exists $\{I_n\} \in C(cA)$ where $I_n = (a_n, b_n]$ such that

$$m^*(cA) + \epsilon > \sum_n l(I_n). \tag{3.1}$$

As $\bigcup_n I_n \supseteq cA$ and $c > 0$, therefore we claim that $\bigcup_n (\frac{1}{c} I_n) \supseteq A$. Indeed, for any $a \in A$, $cA \in I_n$. Thus $ca \in (a_n, b_n]$. Consequently, $a \in (a_n/c, b_n/c] = (\frac{1}{c} I_n)$. Thus, $\{\frac{1}{c} I_n\} \in C(A)$. Consequently, we have

$$\sum_n l\left(\frac{1}{c} I_n\right) = \sum_n \frac{1}{c} l(I_n) \geq m^*(A).$$

Consequently, $\sum_n l(I_n) \geq c m^*(A)$. Using this in Eq. (3.1), we thus obtain

$$m^*(cA) + \epsilon > \sum_n l(I_n) \geq c m^*(A).$$

Taking $\epsilon \to 0$, we obtain $m^*(cA) \geq c m^*(A)$, as required.

If $c < 0$, then we begin similarly to the previous case. Fix $\epsilon > 0$. There exists $\{I_n\} \in C(A)$ where $I_n = (a_n, b_n]$ such that

$$m^*(A) + \epsilon > \sum_n l(I_n). \tag{3.2}$$

Note that $cI_n = c(a_n, b_n] = [cb_n, ca_n)$ as $c < 0$ and this type of set is not half-open and is thus not in $\mathcal{A}$, the algebra of half-opens of Eq. (1.1). Consequently, we have to use ϵ-wiggle to find a new collection of intervals obtained via cI_n which are half open but their sum of lengths in only in ϵ-neighborhood of those $\{cI_n\}$. Indeed, for each $n \in \mathbb{N}$, we may construct

$$J_n = \left(cb_n - \frac{\epsilon}{2^{n+1}}, ca_n + \frac{\epsilon}{2^{n+1}}\right].$$

Note that $J_n \supseteq cI_n$. As $\bigcup_n cI_n \supseteq cA$, therefore $\bigcup_n J_n \supseteq cA$. Thus $\{J_n\} \in C(cA)$. Consequently,

$$\begin{aligned} m^*(cA) &\leq \sum_n l(J_n) \\ &= \sum_n c(a_n - b_n) + \frac{2\epsilon}{2^{n+1}} \\ &= \sum_n -c(b_n - a_n) + \sum_n \frac{\epsilon}{2^n} \\ &= -c\sum_n (b_n - a_n) + \epsilon \\ &= -c\sum_n l(I_n) + \epsilon \end{aligned}$$

where the third line follows from the series being positive and thus we can rearrange such a series. It thus follows by Eq. (3.2) and above that

$$\begin{aligned} m^*(cA) &< -c(m^*(A) + \epsilon) + \epsilon \\ &= -cm^*(A) + \epsilon(1 - c). \end{aligned}$$

Taking $\epsilon \to 0$, we obtain ($-c = |c|$ as $c < 0$)

$$m^*(cA) \leq |c|\, m^*(A)$$

as required. This completes the proof.

(ii) : We now wish to show that for $c \neq 0$, $A \in \mathcal{M}$ if and only if $cA \in \mathcal{M}$. Note that this is not true for $c = 0$ as if we take a non-measurable set $V \subseteq \mathbb{R}$, then $cV = \{0\}$ is measurable but V is not.

Pick $c \neq 0$. We first note that showing only $A \in \mathcal{M} \implies cA \in \mathcal{M}$ is sufficient. Indeed, the other side follows by replacing c by $1/c$ in the above. So we reduce to showing $A \in \mathcal{M} \implies cA \in \mathcal{M}$.

Pick $A \in \mathcal{M}$ and $c \neq 0$ in $\mathbb{R}$. Fix $\epsilon > 0$. By regularity theorems, there exists open $U \setminus A$ such that $m^*(U \setminus A) < \frac{\epsilon}{|c|}$. We now claim the following statements:

1. *cU is open* : Pick $cx \in cU$ where $x \in U$. As U is open therefore there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq U$. Consequently, $c(x - \delta, x + \delta) = (c(x + \delta), c(x - \delta)) \subseteq cU$ and contain cx. Hence cU is open.
2. *cU contains cA* : Pick any cx in cA. Then $x \in A$. As $U \subseteq A$, therefore $x \in U$ and hence $cx \in cU$.
3. *$cU \setminus cA$ equals $c(U \setminus A)$* : For ($\subseteq$), pick any $cx \in cU \setminus cA$. Then $cx \in cU$ and $cx \notin cA$. Thus, $x \in U$ and $x \notin A$, that is $\in U \setminus A$ and thus $cx \in c(U \setminus A)$. Conversely to show ($\supseteq$), pick any $cx \in c(U \setminus A)$ where $x \in U \setminus A$. Thus, $x \in U$ and $x \notin A$. Thus $cx \in cU$ and $cx \notin cA$. Thus $cx \in cU \setminus cA$.

Following the above three lemmas, we conclude that cU is an open set containing cA such that

$$m^*(cU \setminus cA) = m^*(c(U \setminus A)) \overset{(i)}{=} |c|\, m^*(U \setminus A) < |c|\,\frac{\epsilon}{|c|} = \epsilon.$$

Thus by regularity theorems, $cA \in \mathcal{M}$ as well.

(iii) : We wish to show that if $A \in \mathcal{M}$, then $m(cA) = |c|\, m(A)$. But this is immediate from *(i)* as $m = m^*|_{\mathcal{M}}$. This completes the whole proof. $\square$

Lemma 25.1.9.14. *For each subseteq $A \subseteq \mathbb{R}$, there exists a Borel subset $B \supseteq A$ such that*

$$m^*(A) = m(B).$$

Proof. We wish to show that for each $A \subseteq \mathbb{R}$, there exists a Borel set $B \supseteq A$ such that $m(B) = m^*(A)$. We divide into two cases based on outer measure of A. We will follow the notations of Eq. $(*)$ and $(**)$.

If $m^*(A) = \infty$. In this case, we claim that $B = \mathbb{R}$ will work. Indeed $\mathbb{R}$ is open and thus Borel. We thus claim that $m(\mathbb{R}) = \infty$. Indeed, for $I_n = (n, n+1]$, $n \in \mathbb{Z}$, we have that $\{I_n\}$ are disjoint and $\coprod_n I_n = \mathbb{R}$. As m is a measure and I_n are measurable, therefore

$$m(\mathbb{R}) = \sum_n m(I_n) = \sum_n 1 = \infty.$$

Hence $B = \mathbb{R}$ will work.

If $m^*(A) < \infty$, then we proceed as follows. For each $N \in \mathbb{N}$, there exists $\{I_n^N\} \in C(A)$ such that

$$m^*(A) + \frac{1}{N} > \sum_n l(I_n^N).$$

Define $U_N = \bigcup_n I_n^N$. As each half open interval $(a, b] = \bigcap_{n\in\mathbb{N}} (a, b + 1/n)$ is a Borel set, therefore U_N is a Borel set. Observe that

$$m(U_N) \leq \sum_n m^*(I_n^N) = \sum_n l(I_n^N) < m^*(A) + \frac{1}{N}.$$

Note that in the above we have used the fact that Lebesgue measure restricted to half opens is exactly the length function. We thus have for each $N \in \mathbb{N}$ a Borel set U_N containing A such that

$$m(U_N) < m^*(A) + \frac{1}{N}. \tag{4.1}$$

Denote $B_K = \bigcap_{N=1}^K U_N$. Then each B_K is Borel and $\{B_K\}$ is a decreasing sequence of sets. Furthermore, $\bigcap_{K=1}^\infty B_K = \bigcap_{N=1}^\infty U_N$. Denote $B = \bigcap_{K=1}^\infty B_K$. Observe that $B \supseteq A$ as $B_K \supseteq A$ for each $K \in \mathbb{N}$. Consequently, by continuity of m^* we have

$$m(B) \geq m^*(A).$$

For the converse, first note that by Eq. (4.1), $m(U_1) < \infty$. Thus by monotone convergence property of measures, we obtain that $\lim_{K\to\infty} m(B_K) = m\left(\bigcap_{K=1}^\infty B_K\right)$. It follows from above, $B_K \subseteq U_K$ and Eq. (4.1) that

$$\begin{aligned}
m(B) &= m\left(\bigcap_{K=1}^{\infty} B_K\right) \\
&= \lim_{K\to\infty} m(B_K) \\
&\leq \lim_{K\to\infty} m(U_K) \\
&\overset{(4.1)}{<} \lim_{K\to\infty}\left(m^*(A) + \frac{1}{K}\right) \\
&\leq m^*(A).
\end{aligned}$$

Thus $m(B) \leq m^*(A)$ and we are done. □

Lemma 25.1.9.15. *TODO : Type up Q5-Q10 of HW#2.*

Lemma 25.1.9.16. *A bounded set $E \subseteq \mathbb{R}$ is measurable if and only if $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ for all bounded subsets $A \subseteq \mathbb{R}$.*

Proof. Let E be a bounded set of $\mathbb{R}$. We wish to show that E is measurable if and only if for all bounded sets $A \subseteq \mathbb{R}$, we get $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$.

The ($\Rightarrow$) is immediate from definitions. For ($\Leftarrow$), we proceed as follows. We wish to show that for any $F \subseteq \mathbb{R}$, we have

$$m^*(F) \geq m^*(F \cap E) + m^*(F \cap E^c).$$

Indeed, if $m^*(F) = \infty$, then there is nothing to show. So we assume $m^*(F) < \infty$. Observe then that $m^*(F \cap E), m^*(F \cap E^c) \leq m^*(F) < \infty$. Fix $\epsilon > 0$. There exists a sequence $\{I_n\}$ of half-opens such that $\bigcup_n I_n \supseteq F$ and

$$m^*(F) + \epsilon > \sum_n m^*(I_n)$$

where we are using the fact that measure of a half-open interval is its length. Observe that for each $n \in \mathbb{N}$, we have $m^*(F) + \epsilon > m^*(I_n)$, thus each I_n is a half-open interval with bounded length, hence I_n is bounded as a set. Consequently, we have

$$\begin{aligned}
m^*(F) + \epsilon &> \sum_n m^*(I_n) \\
\text{(by hypothesis)} \qquad &\geq \sum_n m^*(I_n \cap E) + m^*(I_n \cap E^c) \\
\text{(by rearrangement of +ve series)} \qquad &= \sum_n m^*(I_n \cap E) + \sum_n m^*(I_n \cap E^c) \\
\text{(by subadditivity)} \qquad &\geq m^*\left(\bigcup_n I_n \cap E\right) + m^*\left(\bigcup_n I_n \cap E^c\right) \\
\text{(by } \cup_n I_n \supseteq F\text{)} \qquad &\geq m^*(F \cap E) + m^*(F \cap E^c).
\end{aligned}$$

This completes the proof. □

Measurable functions

Notation 25.1.9.17. At times, we will write a subset of X as follows:

$$\{x \in X \mid \mathcal{P}_x \text{ is true}\} = \{\mathcal{P}_x \text{ is true}\}.$$

This makes some constructions much more clearer to see and interpret.

Lemma 25.1.9.18. *Let $f : X \to Y$ be a function and $\mathcal{A}$ be an algebra on Y. Then,*

$$\langle f^{-1}(\mathcal{A})\rangle = f^{-1}(\langle \mathcal{A}\rangle).$$

Proof. Let $f : X \to Y$ be a function and $\mathcal{A}$ be an algebra over Y. We wish to show that

$$\langle f^{-1}(\mathcal{A})\rangle = f^{-1}(\langle \mathcal{A}\rangle). \tag{2.1}$$

We first claim that $f^{-1}(\langle \mathcal{A}\rangle)$ is a σ-algebra over X. Indeed, as $Y, \emptyset \in \langle \mathcal{A}\rangle$, we have $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$. Further, if $B \in f^{-1}(\langle \mathcal{A}\rangle)$, then $B = f^{-1}(A)$ for some $A \in \langle \mathcal{A}\rangle$. Hence $B^c = f^{-1}(A)^c = f^{-1}(A^c)$ and $A^c \in \langle A\rangle$ as $\langle A\rangle$ is a σ-algebra. Finally, pick $\{B_n\} \subseteq f^{-1}(\langle A\rangle)$. Then $B_n = f^{-1}(A_n)$ for $A_n \in \langle \mathcal{A}\rangle$. Consequently, $\bigcup_n B_n = \bigcup_n f^{-1}(A_n) = f^{-1}\left(\bigcup_n A_n\right)$ and since $\bigcup_n A_n \in \langle A\rangle$, hence this proves that $f^{-1}(\langle \mathcal{A}\rangle)$ is a σ-algebra.

We now show ($\subseteq$) part of Eq. (2.1). Indeed, by above, it would suffice to show that $f^{-1}(\mathcal{A})$ is contained in the σ-algebra $f^{-1}(\langle \mathcal{A}\rangle)$. Pick any $B \in f^{-1}(\mathcal{A})$, so that $B = f^{-1}(A)$ where $A \in \mathcal{A}$. As $\mathcal{A} \subseteq \langle \mathcal{A}\rangle$, therefore $A \in \langle \mathcal{A}\rangle$. It follows that $B = f^{-1}(A) \in f^{-1}(\langle \mathcal{A}\rangle)$. This shows that $\langle f^{-1}(\mathcal{A})\rangle \subseteq f^{-1}(\langle \mathcal{A}\rangle)$.

We now show ($\supseteq$) part of Eq. (2.1). We will use the standard technique of *good sets* for this. Consider

$$\mathcal{C} := \{A \in \langle \mathcal{A}\rangle \mid f^{-1}(A) \in \langle f^{-1}(\mathcal{A})\rangle\} \subseteq \langle \mathcal{A}\rangle$$

We now claim the following two statements:

1. *$\mathcal{C}$ is a σ-algebra on Y* : Indeed, $Y = f^{-1}(X)$ and $\emptyset = f^{-1}(\emptyset)$ where $X, \emptyset \in \langle \mathcal{A}\rangle$ and $X, \emptyset \in \langle f^{-1}(\mathcal{A})\rangle$. Further, for $A \in \mathcal{C}$, we have $f^{-1}(A) \in \langle f^{-1}(\mathcal{A})\rangle$ and thus $(f^{-1}(A))^c = f^{-1}(A^c) \in \langle f^{-1}(\mathcal{A})\rangle$. Thus $A^c \in \mathcal{C}$. Finally, pick $\{A_n\} \subseteq \mathcal{C}$. Then $f^{-1}(A_n) \in \langle f^{-1}(\mathcal{A})\rangle$ for each $n \in \mathbb{N}$. Thus, $\bigcup_n f^{-1}(A_n) = f^{-1}\left(\bigcup_n A_n\right) \in \langle f^{-1}(\mathcal{A})\rangle$. It then follows that $\bigcup_n A_n \in \mathcal{C}$. This shows that $\mathcal{C}$ is a σ-algebra.
2. $\mathcal{C} \supseteq \mathcal{A}$: Pick any $A \in \mathcal{A}$. As $\langle f^{-1}(\mathcal{A})\rangle$ contains $f^{-1}(\mathcal{A})$, so $f^{-1}(A) \in \langle f^{-1}(\mathcal{A})\rangle$.

We now conclude the proof. As $\mathcal{C}$ is a σ-algebra containing $\mathcal{A}$ and inside $\langle \mathcal{A}\rangle$, therefore $\mathcal{C} = \langle \mathcal{A}\rangle$. It follows that for each $A \in \langle \mathcal{A}\rangle$, we have $f^{-1}(A) \in \langle f^{-1}(\mathcal{A})\rangle$, that is $f^{-1}(\langle \mathcal{A}\rangle) \subseteq \langle f^{-1}(\mathcal{A})\rangle$, as required. This completes the proof. $\square$

Lemma 25.1.9.19. *Let $(X, \mathcal{M}, m)$ be the Lebesgue measure space. Let $A \in \mathcal{M}$ be a bounded set such that $0 < m(A) < \infty$. For each $0 < M < m(A)$, there exists a $B \subsetneq A$ such that $B \in \mathcal{M}$ and $m(B) = M$.*

Proof. There are two proofs that we wish to present, one uses Lemma 25.1.9.20 and other is independent. The latter uses a nice technique which we would like to write down concretely.

Method 1 : *(Using Lemma 25.1.9.20)* Consider the map

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto m(A \cap (-\infty, x]). \end{aligned}$$

As A is a bounded set, therefore $m(A) < \infty$ as there exists a bounded interval $I \supseteq A$ where $I = [c, d]$. By Lemma 25.1.9.20, the map f is a continuous map. Let $a \in \mathbb{R}$ be such that $a < c$. Then $f(a) = m(A \cap (-\infty, a]) = m(\emptyset) = 0$. Let $b \in \mathbb{R}$ such that $b > d$. Then, $f(b) = m(A \cap (-\infty, b]) = m(A)$. On the interval $J = [a, b]$ we have $f(a) = 0$ and $f(b) = m(A)$. By intermediate value property of f, there exists $c \in J$ such that $f(c) = M$. Consequently, $A \cap (-\infty, c]$ is a measurable subset of A whose measure is M.

Method 2 : *(Exponential subdivision technique)* We shall explicitly construct $B \subsetneq A$ such that $m(B) = M$. First, we observe that the question is invariant under translation and dilation. Hence we may, after suitable dilation and translation, assume that $A \subseteq [0, 1)$. For each $n \in \mathbb{N}$, consider the following partition of $[0, 1)$

$$P_n : 0 < x_1 = \frac{1}{2^n} < x_2 = 2 \cdot \frac{1}{2^n} < \cdots < x_{2^n - 1} = (2^n - 1) \cdot \frac{1}{2^n} < 1.$$

Denote $I_{n,j} = [\frac{j-1}{2^n}, \frac{j}{2^n})$ for each $j = 1, \ldots, 2^n$. Observe that $I_{n,j}$ are disjoint and, denoting $A_{n,j} = A \cap I_{n,j}$, we further have a disjoint collection $\{A_{n,j}\}$ of measurable subsets[4] of A such that

$$\coprod_{j=1}^{2^n} A_{n,j} = A.$$

Further, we have that

$$\begin{aligned} \sum_{j=1}^{2^n} m(A_{n,j}) &= m\left(\coprod_{j=1}^{2^n} A_{n,j}\right) \\ &= m(A) \end{aligned}$$

and that

$$m(A_{n,j}) \leq m(I_{n,j}) = \frac{1}{2^n}.$$

Now, for each $n \in \mathbb{N}$, let N_n be the largest index such that

$$\sum_{j=1}^{N_n} m(A_{n,j}) \leq M.$$

[4] measurable because A and $I_{n,j}$ are measurable

By the choice of index N_n, we observe that

$$\begin{aligned}M &< \sum_{j=1}^{N_n+1} m(A_{n,j}) \\ &= \sum_{j=1}^{N_n} m(A_{n,j}) + m(A_{n,N_n+1}) \\ &\le \sum_{j=1}^{N_n} m(A_{n,j}) + \frac{1}{2^n}.\end{aligned}$$

Denoting $C_n = \amalg_{j=1}^{N_n} A_{n,j}$, we obtain,

$$M - \frac{1}{2^n} < \sum_{j=1}^{N_n} m(A_{n,j}) = m\left(C_n\right) \le M. \tag{3.1}$$

We now claim that $\{C_n\}$ is an increasing sequence of measurable subsets of A. First observe that for each $n \in \mathbb{N}$, we have that N_{n+1} is either $2N_n - 1$ or $2N_n$. Indeed, pick any $x \in C_n$. Then $x \in A_{n,j}$ where $j = 1, \dots, N_n$. Expanding this, we have

$$\begin{aligned}x &\in A_{n,j} \\ &= A \cap \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right) \\ &= A \cap \left(\left[\frac{2(j-1)}{2^{n+1}}, \frac{2j-1}{2^{n+1}}\right) \amalg \left[\frac{2j-1}{2^{n+1}}, \frac{2j}{2^{n+1}}\right)\right) \\ &= A_{n+1,2j-1} \amalg A_{n+1,2j}. \end{aligned} \tag{3.2}$$

As $N_{n+1} = 2N_n - 1$ or $2N_n$, therefore for $j = 1, \dots, N_n$, $2j = 2, \dots, 2N_n$, hence in Eq. (3.2), we obtain that $x \in A_{n+1,2j-1}$ or $x \in A_{n+1,2j}$ and as $2j \le 2N_n$, hence $x \in C_{n+1}$. This shows that $C_n \subseteq C_{n+1}$.

Applying $\lim_{n\to\infty}$ on Eq. (3.1), we thus obtain

$$M \le \lim_{n\to\infty} m(C_n) \le M.$$

Thus, by monotone convergence of measures, we conclude

$$\begin{aligned}M &= \lim_{n\to\infty} m(C_n) \\ &= m\left(\bigcup_n C_n\right).\end{aligned}$$

As $C_n \subseteq A$ for each $n \in \mathbb{N}$, therefore $\bigcup_n C_n \subseteq A$. Consequently we have obtained a subset of A whose measure is M. □

Lemma 25.1.9.20. *Let $(X, \mathcal{M}, \mu)$ be the Lebesgue measure space and $A \in \mathcal{M}$ be a bounded set. Then the function*

$$\begin{aligned}f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto m(A \cap (-\infty, x])\end{aligned}$$

is continuous.

Proof. Let $A \in \mathcal{M}$ which has finite measure. We wish to show that

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto m(A \cap (-\infty, x]) \end{aligned}$$

is continuous. Pick any $a \in \mathbb{R}$ and any $\epsilon > 0$. We wish to find a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. We now have two cases with respect to the position of x and a in $\mathbb{R}$.

1. *If* $a \leq x$: then $f(x) - f(a)$ can be rewritten as follows:

$$\begin{aligned} |f(x) - f(a)| &= m(A \cap (-\infty, x]) - m(A \cap (-\infty, a]) \\ &= m(A \cap (-\infty, x] \setminus A \cap (-\infty, a]) \\ &= m(A \cap (a, x]) \\ &\leq m((a, x]) \\ &= x - a. \end{aligned}$$

   Therefore taking $\delta = \epsilon$, we would be done.
2. *If* $a > x$: then $|f(x) - f(a)|$ can be written as

$$\begin{aligned} |f(x) - f(a)| &= |f(a) - f(x)| \\ &= m(A \cap (-\infty, a]) - m(A \cap (-\infty, x]) \\ &= m(A \cap (x, a]) \\ &\leq m((x, a]) \\ &= a - x. \end{aligned}$$

   Thus, again, taking $\delta = \epsilon$ would do the job.

This completes the proof. $\square$

Lemma 25.1.9.21. *Let X be a measurable space and let $f : X \to \mathbb{R}$ be a function. Suppose $\{x \in X \mid a \leq f(x) < b\}$ is measurable for all $a < b$. Then f is a measurable function.*

Proof. As the Borel σ-algebra on $\mathbb{R}$ is generated by sets of the form $[a, \infty)$ for $a \in \mathbb{R}$, therefore for a fixed $a \in \mathbb{R}$ we need only show that $f^{-1}([a, \infty))$ is measurable in X.

We can write

$$\begin{aligned} f^{-1}([a, \infty)) &= \{a \leq f(x)\} \\ &= \bigcup_{n > a \text{ in } \mathbb{N}} \{a \leq f(x) < n\}. \end{aligned}$$

As we are given that $\{a \leq f(x) < b\}$ are measurable for all $a < b \in \mathbb{R}$ and countable union of measurable sets is measurable, therefore $f^{-1}([a, \infty))$ is measurable. $\square$

Lemma 25.1.9.22. *All monotone functions $f : \mathbb{R} \to \mathbb{R}$ are measurable.*

Proof. We wish to show that all monotone functions $f : \mathbb{R} \to \mathbb{R}$ are measurable. Note that we may first reduce to assuming that f is non-decreasing as if f is non-increasing, then $-f$

will be non-decreasing.

Hence let $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing. As Borel σ-algebra of $\mathbb{R}$ is generated by intervals of the form $[a, \infty)$, $a \in \mathbb{R}$, therefore it suffices to check that $f^{-1}([a, \infty))$ is measurable in $\mathbb{R}$. Observe

$$f^{-1}([a, \infty)) = \{a \le f(x)\}.$$

We now have two cases to handle.

1. *If $a \in f(\mathbb{R})$:* Then there exists $b \in \mathbb{R}$ such that $f(b) = a$. We may write

$$\{a \le f(x)\} = \{a < f(x)\} \amalg \{a = f(x)\}.$$

Now since f is non-decreasing, therefore $f(x) > f(y)$ implies $x > y$. Further, we have that $f^{-1}(a) = \{a = f(x)\}$ is measurable as singletons are Borel. Consequently, we have

$$\begin{aligned}\{a \le f(x)\} &= \{f(b) < f(x)\} \amalg \{a = f(x)\} \\ &= (b, \infty) \amalg f^{-1}(a).\end{aligned}$$

Hence $f^{-1}([a, \infty))$ is measurable.

2. *If $a \notin f(\mathbb{R})$:* We further have two cases.
   (a) *If there exists $b \in \mathbb{R}$ such that $b \notin \{a \le f(x)\}$:* Observe first that in this case $f(b) < a$. We claim that in this case $\{a \le f(x)\}$ is lower bounded by b. Indeed, suppose not. Then there exists $y < b$ such that $y \in \{a \le f(x)\}$. Then $a \le f(y) \le f(b) < a$, a contradiction. Hence $\{a \le f(x)\}$ is bounded below.
   Let $c = \inf\{a \le f(x)\}$, which now exists. Consequently, we have two more cases:
   - *If $f(c) \ge a$:* That is, if $c \in \{a \le f(x)\}$. Then we claim

   $$\{a \le f(x)\} = [c, \infty).$$

   which is clearly a measurable. Indeed, for some $x \in \mathbb{R}$ such that $f(x) \ge a$, then $x \ge c$. Conversely, if $b \ge c$ in $\mathbb{R}$, then $f(b) \ge f(c) \ge a$, so $b \in \{a \le f(x)\}$. This proves the claim.
   - *If $f(c) < a$:* That is, if $c \notin \{a \le f(x)\}$. Then we claim

   $$\{a \le f(x)\} = (c, \infty)$$

   which is clearly a measurbale set. Indeed, for $x \in \mathbb{R}$ such that $f(x) \ge a$, $x > c$. Further $x \ne c$ as otherwise $f(x) < a$. Conversely, if $b > c$, then there exists $d \in \{a \le f(x)\}$ such that $c < d < b$ as c is the infimum. Consequently, $a \le f(d) \le f(b)$. Hence $b \in \{a \le f(x)\}$. This proves the claim.

   (b) *If there doesn't exists any $b \in \mathbb{R}$ such that $f(b) < a$:* Then for all $b \in \mathbb{R}$ we have $f(b) \ge a$. Consequently, $f^{-1}([a, \infty)) = \{a \le f(x)\} = \mathbb{R}$, which is measurable.

Hence in all cases $f^{-1}([a, \infty))$ is a measurable set. This completes the proof. □

Lemma 25.1.9.23. *Let $f : X \to \mathbb{C}$ be a complex measurable function on a measurable space X. Then, there exists a complex measurable function $g : X \to \mathbb{C}$ such that $|g| = 1$ and $f = g\,|f|$.*

Proof. Let $f : X \to \mathbb{C}$ be a measurable function. We wish to find a measurable function $g : X \to \mathbb{C}$ such that $|g| = 1$ and $f = g\,|f|$.

As $|f| = f\chi_{\{f(x)\geq 0\}} - f\chi_{\{f(x)<0\}}$, therefore $|f|$ is a measurable function. Denote $E = \{|f(x)| = 0\}$. Consequently, we define g as follows:

$$g(x) = \begin{cases} \frac{f(x)}{|f(x)|} & \text{if } x \in E^c \\ 1 & \text{if } x \in E. \end{cases}$$

We first wish to show that g is measurable. For this, we shall use the fact that measurability of g can be checked on a cover $\{E_\alpha\}$ of X such that $g|_{E_\alpha}$ is measurable. Thus in our case, we need only show that $g|_E$ and $g|_{E^c}$ are measurable. On E, g is a constant, hence measurable. On E^c, g is $f/|f|$. As $|f|$ is not zero on E^c, therefore by Lemma 25.1.9.25, $f/|f|$ is measurable. Hence, g is measurable.

We now see that $|g|(x) = \left|\frac{f(x)}{|f(x)|}\right| = 1$ on E^c and $|g(x)| = 1$ on E. Thus $|g| = 1$ on X. Further, if $x \in E$, then $f(x) = 0 = g(x)\,|f|(x)$. If $x \in E^c$, then $g(x) = \frac{f(x)}{|f(x)|}$ which implies $|f(x)|\,g(x) = f(x)$. This shows that in all cases, $f = g\,|f|$. □

Example 25.1.9.24. It is not true that if $f : [0, 1] \to \mathbb{R}$ is a function whose each fibre is measurable, then f is measurable.

Consider the following function

$$\begin{aligned} f : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} x & \text{if } x \in V^c \\ x + N & \text{if } x \in V \end{cases} \end{aligned}$$

where $V \subseteq [0, 1]$ denotes the Vitali set and $N = 3$. Then, for each $y \in \mathbb{R}$, we have that $f^{-1}(y)$ is atmost a singleton, which is measurable in $[0, 1]$. However, for any $1 < b < N$, we see that $f^{-1}((b,\infty)) = V$, which is not measurable. Hence f is a non-measurable function whose fibres are measurable.

Lemma 25.1.9.25. *Let $f, g : X \to \mathbb{C}$ be a measurable function such that $\{g(x) \neq 0\} = X$. Then f/g is measurable.*

Proof. Let $f, g : X \to \mathbb{C}$ be a measurable function such that $\{g(x) \neq 0\} = X$. Then we wish to show that f/g is measurable.

We first have that $(f, g) : X \to \mathbb{R}^2$ given by $x \mapsto (f(x), g(x))$ is measurable. Consequently, we consider the composite

$$X \xrightarrow{(f,g)} \mathbb{R}^2 \setminus \{y = 0\} \xrightarrow{\Phi} \mathbb{R}$$

where $\Phi(x, y) = \frac{x}{y}$. As Φ is continuous, therefore the composite $\Phi \circ (f, g)$ is measurable. Consequently, we obtain that the map $x \mapsto \frac{f(x)}{g(x)}$ is measurable, but this is exactly f/g over X. This completes the proof. □

Lemma 25.1.9.26. *Let $f, g : X \to \overline{\mathbb{R}}$ be measurable functions and pick any $r_0 \in \overline{\mathbb{R}}$. Then the map*

$$\begin{aligned} h : X &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto \begin{cases} r_0 & \textit{if } f(x) = -g(x) = \pm\infty \\ f(x) + g(x) & \textit{else} \end{cases} \end{aligned}$$

is measurable[5].

Proof. Let $f, g : X \to \overline{\mathbb{R}}$ be measurable functions and pick any $r_0 \in \overline{\mathbb{R}}$. Then we wish to show that the map

$$\begin{aligned} h : X &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto \begin{cases} r_0 & \text{if } f(x) = -g(x) = \pm\infty \\ f(x) + g(x) & \text{else} \end{cases} \end{aligned}$$

is measurable.

Define the following sets

$$\begin{aligned} E &= \{f(x) = \infty = -g(x)\} \\ F &= \{f(x) = -\infty = -g(x)\}. \end{aligned}$$

As $E = f^{-1}(\infty) \cap g^{-1}(-\infty)$ and $F = f^{-1}(-\infty) \cap g^{-1}(\infty)$, therefore they are measurable. Observe that E and F are disjoint. We thus need only show that h restricted to E, F and $X \setminus (E \amalg F)$ is measurable.

1. On E : As $h|_E$ is constant r_0, therefore $h|_E$ is measurable.
2. On F : As $h|_F$ is again constant r_0, therefore $h|_F$ is measurable.
3. On $X \setminus (E \amalg F)$: We first deduce that

$$\begin{aligned} X \setminus (E \amalg F) &= X \cap E^c \cap F^c \\ &= E^c \cap F^c \\ &= (\{f(x) \neq \infty\} \cup \{g(x) \neq -\infty\}) \bigcap (\{f(x) \neq -\infty\} \cup \{g(x) \neq \infty\}) \end{aligned}$$

Let $G = \{f(x) \in \mathbb{R}\}$ and $H = \{g(x) \in \mathbb{R}\}$. Then we may write $X \setminus (E \amalg F)$ as

$$\begin{aligned} X \setminus (E \amalg F) &= (G \cup H \cup \{f(x) = -\infty\} \cup \{g(x) = \infty\}) \bigcap (G \cup H \cup \{f(x) = \infty\} \cup \{g(x) = -\infty\}) \\ &= (G \cup H) \cup \Big((\{f(x) = -\infty\} \cup \{g(x) = \infty\}) \bigcap (\{f(x) = \infty\} \cup \{g(x) = -\infty\})\Big) \\ &= (G \cup H) \cup \underbrace{\{f(x) = -\infty = g(x)\}}_{=:A} \cup \underbrace{\{f(x) = \infty = g(x)\}}_{=:B}. \end{aligned}$$

As $h|_{G \cup H}$ is $(f+g)|_{G \cup H}$ and on $G \cup H$, $f + g : G \cup H \to \mathbb{R}$, therefore h is measurable. We thus reduce to checking that $h|_A$ and $h|_B$ are measurable. On both of them, one immediately observes that h is constant $-\infty$ and ∞ respectively. Hence, $h|_A$ and $h|_B$ are measurable. As h restricted to $G \cup H$, A and B is measurable therefore h restricted to $X \setminus (E \amalg F)$ is measurable.

[5] This question in particular shows that modifying a measurable function at a single point doesn't affect measurability at all.

This completes the proof. □

Example 25.1.9.27. It is not true in general that if for a function $f : X \to \mathbb{R}$, the $|f| : X \to [0, \infty]$ is measurable then f is measurable.

Indeed, consider the following function where $V \subseteq [0, 1]$ denotes the Vitali set:

$$\begin{aligned} f : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} -x & \text{if } x \in V \\ x & \text{if } x \in V^c. \end{cases} \end{aligned}$$

Then, $|f| = \mathrm{id}_{[0,1]}$ which is measurable whereas f is not measurable as $f^{-1}((-\infty, 0)) = V$, which is not a measurable set.

Lemma 25.1.9.28. *Let $(X_1, \mathcal{A}_{X_1}, \mu_1)$ be a measure space, $(X_2, \mathcal{A}_{X_2})$ be a measurable space and $f : X_1 \to X_2$ be a measurable function. Then*

$$\begin{aligned} \mu_2 : \mathcal{A}_{X_2} &\longrightarrow [0, \infty] \\ B &\longmapsto \mu_1(f^{-1}(B)) \end{aligned}$$

is a measure on $(X_2, \mathcal{A}_{X_2})$.

Proof. We first immediately observe that $\mu_2(\emptyset) = \mu_1(f^{-1}(\emptyset)) = \mu_1(\emptyset) = 0$. We thus reduce to showing that for any disjoint collection $\{B_n\} \subseteq \mathcal{A}_{X_2}$, we have $\mu_2(\coprod_n B_n) = \sum_n \mu_2(B_n)$. To this end, observe that

$$\begin{aligned} \mu_2\left(\coprod_n B_n\right) &= \mu_1\left(f^{-1}\left(\coprod_n B_n\right)\right) \\ &= \mu_1\left(\coprod_n f^{-1}(B_n)\right) \\ &= \sum_n \mu_1(f^{-1}(B_n)) \\ &= \sum_n \mu_2(B_n). \end{aligned}$$

This completes the proof. □

Lemma 25.1.9.29. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a measurable function such that $\mu(\{|f(x)| \geq \epsilon\}) = 0$ for all $\epsilon > 0$. Then $f = 0$ almost everywhere.*

Proof. We first claim that it suffices to show that $\{|f(x)| > 0\}$ is a null set. Indeed, this is because $\{f(x) \neq 0\} = \{|f(x)| > 0\}$. Hence it suffices to show that $|f| = 0$ a.e.

Define for each $n \in \mathbb{N}$ the following subset of X

$$E_n = \{|f(x)| > 1/n\}.$$

We claim that

$$\{|f(x)| > 0\} = \bigcup_{n\in\mathbb{N}} E_n.$$

Indeed, for ($\subseteq$), pick $x \in X$ such that $|f(x)| > 0$. Then there exists $n \in \mathbb{N}$ such that $|f(x)| > 1/n$. Hence $x \in E_n$. Conversely pick $x \in E_n$, then by way of construction of E_n, we have $|f(x)| > 1/n > 0$.

Observe that $\{E_n\}$ is an increasing sequence of sets as if $x \in E_n$ then $|f(x)| > \frac{1}{n} > \frac{1}{n+1}$, so $x \in E_{n+1}$. It then follows by monotone convergence property of measures that

$$\mu(\{|f(x) > 0|\}) = \mu\left(\bigcup_n E_n\right) = \lim_{n\to\infty}\mu(E_n) = \lim_{n\to\infty}0 = 0.$$

This completes the proof. □

Example 25.1.9.30. The statement of Egoroff's theorem depends crucially on the fact that each function in the sequence $\{f_n\}$ is measurable. Indeed, we show by the way of an example that the conclusion of Egoroff's theorem is not true when f_n's are not measurable.

We wish to show that the statement of Egoroff's theorem fails if we drop the condition that functions be measurable.

Consider the measure space $(\mathbb{Z}, \mathcal{A}, \mu)$ where $\mathcal{A} = \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, \mathbb{Z}\backslash 2\mathbb{Z}\}$ and $\mu(\emptyset) = 0 = \mu(2\mathbb{Z})$, $\mu(\mathbb{Z}) = 1 = \mu(\mathbb{Z} \setminus 2\mathbb{Z})$. Consider the functions $f_n : (\mathbb{Z}, \mathcal{A}, \mu) \to \mathbb{R}$ where $\mathbb{R}$ has the Borel measure, given by

$$f_n(k) = \frac{k}{n}$$

for all $k \in \mathbb{Z}$. Observe that $\{f_n\}$ pointwise converges to the constant 0 function at all points of $\mathbb{Z}$. Further note that f_n is not measurable as $f_n^{-1}(\{k/n\}) = \{k\}$ is not a measurable set in $\mathcal{A}$ but $\{k/n\}$ is Borel measurable.

To show that this is a counterexample, it would suffice to show that there exists an $\epsilon_0 > 0$ such that for all measurable sets $E \in \mathcal{A}$, either $\mu(E^c) \geq \epsilon_0$ or f_n does not converges uniformly to 0 on E. We claim that in our situation, $\epsilon_0 = 1/2$ works. Indeed, for $E = \emptyset, 2\mathbb{Z}$, we have $\mu(E^c) = 1 > 1/2$. Thus we reduce to showing that f_n does not converges uniformly on $\mathbb{Z}$ and $\mathbb{Z} \setminus 2\mathbb{Z}$. Indeed, observe that $\sup_{k\in\mathbb{Z}} |f_n(k)| = \sup_{k\in\mathbb{Z}} k/n = \infty$ for each $n \in \mathbb{N}$. As f_n converges uniformly if and only if $\sup_{k\in\mathbb{Z}} |f_n(k)| \to 0$ as $n \to \infty$, therefore we deduce that f_n does not converge uniformly over $\mathbb{Z}$. Similarly, it doesn't converge uniformly over $\mathbb{Z} \setminus 2\mathbb{Z}$.

Lemma 25.1.9.31. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. If $f = g$ almost everywhere, then $f = g$.*

Proof. Indeed, consider $h = f - g$. Suppose $h \neq 0$, therefore there exists $x_0 \in \mathbb{R}$ such that $h(x_0) \neq 0$. By continuity of h, there exists $\epsilon > 0$ such that $(x_0 - \epsilon, x_0 + \epsilon) \subseteq \{h(x) \neq 0\}$. Hence, $2\epsilon < m(\{h(x) \neq 0\}) = 0$, which yields $0 < 2\epsilon \leq 0$, a contradiction. □

Lemma 25.1.9.32. *Let $(X, \mathcal{S}, \mu)$ be a measure space and $f_n, f : X \to \overline{\mathbb{R}}$ be measurable functions such that $f_n \to f$ pointwise almost everywhere. Then, there exists measurable functions $g_n : X \to \overline{\mathbb{R}}$ such that $f_n = g_n$ almost everywhere and $g_n \to f$ pointwise.*

Proof. Indeed, as f_n converges pointwise to f almost everywhere, therefore the set $E = \{\lim_{n\to\infty} f_n(x) \neq f(x)\}$ is a zero measure set. Consequently, we may define

$$\begin{aligned} g_n : X &\longrightarrow \overline{\mathbb{R}} \\ x &\longmapsto \begin{cases} f_n(x) & \text{if } x \notin E \\ f(x) & \text{if } x \in E. \end{cases} \end{aligned}$$

We then observe that $\{g_n(x) \neq f_n(x)\} = E$, which is of measure zero, hence $g_n = f_n$ almost everywhere. Furthermore, we see that for any $x \in X$,

$$\lim_{n\to\infty} g_n(x) = \begin{cases} \lim_{n\to\infty} f_n(x) = f(x) & \text{if } x \notin E \\ \lim_{n\to\infty} f(x) = f(x) & \text{if } x \in E. \end{cases}$$

Thus, $\lim_{n\to\infty} g_n = f$ pointwise. This completes the proof. □

Example 25.1.9.33. We wish to show that there exists continuous function $f : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f : \mathbb{R} \to \mathbb{R}$ is not Lebesgue measurable.

While learning about the existence of a non-Borel measurable set, one learns about the existence of a homeomorphism $\varphi : [0,1] \to [0,2]$ such that $m(\varphi(C)) = 1 > 0$ where $C \subseteq [0,1]$ is the Cantor set. Indeed, if $\mathcal{C} : [0,1] \to [0,1]$ denotes the Cantor function, then φ is constructed by defining $\varphi(x) = \mathcal{C}(x) + x$. As, $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = 1$, therefore $\varphi(0) = 0$ and $\varphi(1) = 2$. Consequently, we may define a continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x - 1 & \text{if } x > 2 \\ \varphi^{-1}(x) & \text{if } x \in [0,2] \\ x & \text{if } x < 0. \end{cases}$$

Observe that f is continuous as f is obtained by gluing three continuous functions at points where they agree.

As $m(\varphi(C)) = 1 > 0$ for Cantor set C, therefore there exists a non-measurable set $V \subseteq \varphi(C) \subseteq [0,2]$. But since $f(V) = \varphi^{-1}(V) \subseteq \varphi^{-1}(\varphi(C)) = C$ and C is a null set, therefore by completeness of Lebesgue measure, it follows that $f(V)$ is a Lebesgue measurable set. Consequently, we may define $g = \chi_{f(V)} : \mathbb{R} \to \mathbb{R}$, which is Lebesgue measurable as $f(V)$ is Lebesgue measurable. We thus have

$$\mathbb{R} \xrightarrow[\text{cont.}]{f} \mathbb{R} \xrightarrow[\text{Leb. msble}]{g} \mathbb{R} .$$

We claim that $h := g \circ f$ is not Lebesgue measurable. Indeed, observe that $h^{-1}(\{1\}) = (g \circ f)^{-1}(\{1\}) = f^{-1}(g^{-1}(\{1\})) = f^{-1}(f(V))$. But as f restricted to $[0,2]$ is a homeomorphism from $[0,2]$ to $[0,1]$ because on $[0,2]$, f is equal to φ^{-1}, hence $f^{-1}(f(V)) = V$. Hence $h^{-1}(\{1\}) = V$, where $\{1\}$ is measurable but $V \subseteq [0,2]$ is non-measurable. This shows that h is not measurable. This completes the proof.

25.1.10 Integration of $\mathbb{C}$-valued measurable functions

We now discuss integration theory. The plan is as follows. We first define integrals of $[0, \infty]$-valued measurable functions. Using this we will define the integral of $\overline{\mathbb{R}}$-valued measurable functions which doesn't achieve both $\pm\infty$. Finally, using this we will define integral of $\mathbb{C}$-valued measurable functions.

Let $(X, \mathcal{M}, \mu)$ be a fixed measure space. A *simple function* $\varphi \in \text{Simp}(X)$ is a $\overline{\mathbb{R}}$-valued measurable function whose range is a finite set. Consequently, one can write

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

where $E_i \in \mathcal{M}$ are disjoint and $a_i \in \overline{\mathbb{R}}$ are disjoint. One calls this the *standard form* of φ. The class of all non-negative simple functions on X will be denoted by $\text{Simp}_{\geq 0}(X)$. Define the integral of a non-negative simple function $\varphi \in \text{Simp}_{\geq 0}(X)$ as

$$\int_X \varphi d\mu := \sum_{i=1}^{n} a_i \mu(E_i)$$

where $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ is in the standard form, but one checks that a presentation of φ in non-standard form will also yield the same integration value. One further checks that integral of simple functions is additive and scalar invariant.

We begin with the simple approximation theorem of positive measurable functions.

Theorem 25.1.10.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a positive measurable map. Then, there exists a sequence of simple functions $\{s_n\}_{n\in\mathbb{N}}$ such that*

1. $0 \leq s_n(x) \leq s_{n+1}(x)$ *for all* $n \in \mathbb{N}$ *and* $x \in X$,
2. $\lim_{n\to\infty} s_n(x) = f(x)$ *for all* $x \in X$.

By simple approximation theorem (Theorem 25.1.10.1), we know that for any measurable function $f : X \to [0, \infty]$, there exists a sequence of simple functions $\{\varphi_n\} \subseteq \text{Simp}_{\geq 0}(X)$ such that $0 \leq \varphi_n \leq \varphi_{n+1} \leq f$ and $\lim_{n\to\infty} \varphi_n = f$. Using this we define the integral of measurable function $f : X \to [0, \infty]$ as

$$\int_X f d\mu := \sup \left\{ \int_X \varphi \mid \varphi \in \text{Simp}_{\geq 0}(X) \;\&\; \varphi \leq f. \right\}$$

We denote the class of non-negative measurable functions on X as $L^+(X)$ or simply L^+ if the measure space is clear from context. If $f : \mathbb{R} \to [0, \infty]$ is a measurable function where $(\mathbb{R}, \mathcal{M}, m)$ is the Lebesgue measure space, then $\int_{\mathbb{R}} f dm$ is defined to be the *Lebesgue integral* of f.

One then checks that the by above definition, the integral of a simple function is really the same as the one defined prior. We now have some important results, beginning with the famous monotone convergence theorem.

Theorem 25.1.10.2. *(MCT) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n : X \to [0, \infty]$ be a sequence of positive measurable maps. Suppose*

1. $\lim_n f_n(x)$ *exists and is equal to* $f(x)$ *for some measurable* $f : X \to \mathbb{R}$,
2. $f_n(x) \le f_{n+1}(x)$ *for all* $x \in X$ *and* $n \in \mathbb{N}$.

Then,

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu.$$

An important corollary of MCT is that the integral itself is a measure.

Proposition 25.1.10.3. *Let* $f \in L^+(X)$. *Then the function*

$$\begin{aligned} \nu_f : \mathcal{M} &\longrightarrow [0, \infty] \\ A &\longmapsto \int_A f d\mu \end{aligned}$$

is a measure on $(X, \mathcal{M})$. *We call the* ν_f *the measure associated to the non-negative measurable function* f.

This is an important result, as it implies the following.

Lemma 25.1.10.4. *Let* $f \in L^+(X)$ *and* $\{A_n\} \subseteq \mathcal{M}$ *be an increasing sequence of measurable sets such that* $X = \bigcup_n A_n$. *Then,*

$$\lim_n \int_{A_n} f d\mu = \int_X f d\mu.$$

Next we see the usual properties of integration reflected in this abstract setting.

Lemma 25.1.10.5. *Let* $f, g \in L^+(X)$ *and* $c \in \geq 0$. *Then*
1. $\int_X cf d\mu = c \int_X f d\mu$,
2. $f \le g \implies \int_X f d\mu \le \int_X g d\mu$,
3. $\int_X f + g d\mu = \int_X f d\mu + \int_X g d\mu$.

Proof. Items 1 and 2 are immediate from definition of integral of non-negative measurable functions. For item 3, we will make use of MCT, which is done below (Theorem 25.1.10.2). By simple approximation (Theorem 25.1.10.1), there exists increasing sequences $\{\varphi_n\}$ bounded by f and $\{\psi_n\}$ in $\mathrm{Simp}_{\geq 0}(X)$ bounded by g such that $\lim_n \varphi_n = f$ and $\lim_n \psi_n = g$. Hence, $\lim_n(\varphi_n + \psi_n) = f + g$ and $\varphi_n + \psi_n$ is an increasing sequence as well. By MCT, we obtain $\int_X f + g d\mu = \int_X \lim_n(\varphi_n + \psi_n) d\mu = \lim_n \int_X \varphi_n + \psi_n d\mu = \lim_n \int \varphi_n d\mu + \lim_n \int_X \psi_n d\mu$ which again by MCT yields $\int_X f d\mu + \int_X g d\mu$. □

As a nice application, we see the following.

Proposition 25.1.10.6. *Let* $f, g \in L^+(X)$. *Then*

$$\int_X g d\nu_f = \int_X f g d\mu.$$

Proof. The proof is straightforward. First prove this for $g = \chi_E$, a characteristic function. Then prove this for $g = \sum_{i=1}^n a_i \chi_{E_i}$ a non-negative simple function and then using simple approximation prove this for the general L^+ case. This portrays the fundamental reduction technique that one can employ to prove results in measure theory. □

TODO!

25.1.11 Main results : Commutation of $\int$ and $\lim$

We have the Fatou's lemma, which is the best thing we got without any hypotheses on the sequence of functions under consideration.

Theorem 25.1.11.1. *(Fatou's lemma) Let (X, M, μ) be a measure space and let $f_n : X \to [0, \infty]$ be a sequence of positive measurable maps. Then,*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.$$

Similarly, we have the famous dominated convergence theorem.

Theorem 25.1.11.2. *(DCT) Let (X, M, μ) be a measure space and let $f_n : X \to \mathbb{C}$ be a sequence of measurable maps. Suppose*

1. *$\lim_n f_n(x)$ exists and is equal to $f(x)$ for some measurable $f : X \to \mathbb{C}$,*
2. *there exists a measurable $g : X \to \mathbb{R}$ of class L^1 such that $|f_n(x)| \leq g(x)$ for all $x \in X$ and $n \in \mathbb{N}$.*

Then,

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu.$$

A simple corollary of the MCT tells us an equivalent story for decreasing sequence of maps where first term is L^1, as compared to the statement of MCT.

Corollary 25.1.11.3. *Let (X, M, μ) be a measure space and let $f_n : X \to \mathbb{R}$ be a sequence of positive measurable maps. Suppose*

1. *$\lim_n f_n(x)$ exists and is equal to $f(x)$ for some measurable $f : X \to \mathbb{R}$,*
2. *$f_n(x) \geq f_{n+1}(x)$ for all $x \in X$ and $n \in \mathbb{N}$,*
3. *$f_1(x) \in L^1$.*

Then,

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu.$$

Proof. Since $f \leq f_n \leq f_1$, therefore $f \in L^1$. Now, consider the (not necessarily positive!) measurable sequence $g_n = f - f_n$. Since f_n decreases, therefore g_n increases. Now, $\lim_n g_n = 0$ as $\lim_n f_n = f$. Since $0 \in L^1$, therefore Hence, by MCT Theorem 25.1.10.2, we get that $\lim_n \int_X g_n dm = \int_X \lim_n g_n dm$. Expanding it and using the fact that f is in L^1 (so you can cancel $\int_X f dm$ both sides!) gives the desired result. $\square$

Another important result which is of tremendous usability is the fact that Riemann and Lebesgue agree on compact domains(!)

Theorem 25.1.11.4. *(Riemann = Lebesgue on $[a,b]$) Let $[a,b] \subseteq \mathbb{R}$ be a closed bounded interval and $f : [a,b] \to \mathbb{R}$ be a Riemann integrable map. Then, the Riemann integral and Lebesgue integral of f agrees on $[a,b]$. That is,*

$$\int_a^b f(x)dx = \int_{[a,b]} f dm$$

where m is the Lebesgue measure of $\mathbb{R}$.

25.1.12 Properties of L^1 maps

We would in this section quickly portray some of the easy properties of L^1-maps which are good to keep in mind. The first tells us that a high schooler's dream of claiming a map to be zero if integral is zero is *almost* true for L^1 maps.

Lemma 25.1.12.1. *Let $f : X \to \mathbb{C}$ be a measurable map where (X, M, m) is a measure space. Suppose $f \in L^1$. Then, $\int_F f dm = 0$ for all $F \in M$ if and only if $f = 0$ almost everywhere.*

Proof. One side is trivial. For the other, we may reduce to the case when f is real valued. Let $A = \{x \in X \mid f^-(x) > 0\}$. As f^- is measurable, therefore $A \in M$. Since $\int_A f dm = 0$, therefore $\int_A f^+ - f^- dm = \int_A f^+ = \int_A f^- dm$. If $x \in A$, then $f^-(x) > 0$, and hence $f^+(x) = 0$. Hence $\int_A f^+ dm = 0$ and hence $\int_A f^- dm = 0$. Since $f^- \geq 0$, therefore $f^- = 0$ almost everywhere. We thus have $\int_X f dm = \int_X f^+ dm = 0$ as $X \in M$. Since $f^+ \geq 0$, therefore $f^+ = 0$ almost everywhere. $\square$

Lemma 25.1.12.2. *Let (X, M, m) be a measure space and $f : X \to \mathbb{R}$ be a measurable map with $f \geq 0$. Then,*

$$m(\{x \in X \mid f(x) = \infty\}) = 0.$$

Proof. This again uses the standard idea of breaking the set which we wish to measure into sets whose bounds on measure is known. Indeed, observe that

$$E := \{f(x) = \infty\} = \bigcap_{n \in \mathbb{N}} \{f(x) > n\} =: \bigcap_{n \in \mathbb{N}} E_n.$$

Moreover, $\{E_n\}$ is decreasing. Thus,

$$m(E) = \lim_{n\to\infty} m(E_n).$$

Now we obtain bound on $m(E_n)$. Indeed,

$$nm(E_n) = \int_{E_n} n dm \leq \int_{E_n} f(x) dm \leq \int_X f(x) dm =: I < \infty.$$

Thus $m(E_n) \leq I/n$. Hence $\lim_{n\to\infty} m(E_n) = 0$. $\square$

25.1.13 Applications-II : Integration

TODO : Add HW# 5 material here.

We present important applications of the above results, showcasing the power of their usage. At parts here, we are proving results from Folland's exercises ([**??**]).

Lemma 25.1.13.1. *The Lebesgue integral*

$$\int_0^1 \frac{x^p - 1}{\log x} dx$$

exists for $p > -1$.

Proof. The first idea is to break p into cases. In some cases, it is obvious why the above integral exists, in others, we have to work. Denote $f_p(x) = \frac{x^p-1}{\log x}$.

Act 1 : $p > 0$

In this regime, we can bound the $\int_0^1 f_p(x)dx$ by a fixed quantity. Indeed, since $f_p(x)$ is positive, it will suffice. Observe that

$$\frac{x^p - 1}{\log x} = \frac{1 - x^p}{-\log x} \leq \frac{1}{-\log x}.$$

Now, $-\log x$ can be lower bounded by $1 - ax$ for some $0 < a < 1$ by an easy graphical observation. Hence, continuing above, we get

$$\frac{x^p - 1}{\log x} \leq \frac{1}{1 - ax}.$$

The integral then translates to

$$\int_0^1 \frac{x^p - 1}{\log x} dx \leq -\int_0^1 \frac{1}{1 - ax} dx = -\frac{\log(1 - a)}{a} < \infty.$$

Act 2 : $-1 < p < 0$

This is the regime in which we got to work a bit. First, from some graphical observations about $x^p - 1$ and $\log x$, we conclude the following:

1. $x^p - 1$ is positive and $\log x$ is negative, so that $\frac{x^p-1}{\log x}$ is negative.
2. Viewing $1/\log x$ as an **attenuating factor**[6], we see that $0 < 1/\log x < -1$ for $0 < x < 1/e$ and $1/\log x \leq -1$ for $1/e \leq x < 1$.
3. On $1/e < x < 1$, $\log x > 1 - x$. Hence $1/\log x < 1/1 - x$.

With this, we write our integral as

$$\begin{aligned}\int_0^1 \frac{x^p - 1}{\log x} dx &= \int_0^{1/e} \frac{x^p - 1}{\log x} dx + \int_{1/e}^1 \frac{x^p - 1}{\log x} dx \\ &< \int_0^{1/e} (1 - x^p) dx + \int_0^1 \frac{x^p - 1}{x - 1} dx\end{aligned}$$

[6]we view $1/\log x$ as an attenuating factor instead of $x^p - 1$ as if we remove $1/\log x$, then we would be left with $x^p - 1$, whose integral is easy to find.

Now the first integral is bounded while the second is bounded as the derivative of x^p exists at $x = 1$.

$\square$

Lemma 25.1.13.2. *Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty, -\infty\}$ be a measurable map with $(\mathbb{R}, M, m)$ be a measure structure on $\mathbb{R}$. If there exists $M > 0$ such that for all $E \in M$ such that $0 < m(E) < \infty$ we have that*

$$\left| \frac{1}{m(E)} \int_E f dm \right| < M,$$

then

$$|f(x)| \leq M \ \textit{a.e.}.$$

Proof. Let $A = \{x \in \mathbb{R} \mid |f(x)| > M\}$. We can write it as $A = A_+ \cup A_-$ where $A_+ = \{x \in \mathbb{R} \mid f(x) > M\}$ and $A_- = \{x \in \mathbb{R} \mid f(x) < -M\}$. Clearly these are disjoint and covers A. Hence, we wish to show

$$m(A) = m(A_+) + m(A_-) = 0$$

which is equivalent to showing that $m(A_+) = m(A_-) = 0$ as measures are always positive.

$$\textbf{Act 1} : m(A_+) = 0.$$

The way A_+ and A_- are defined, it is natural for the next step to be a consideration of integral of f over these. Indeed, we observe that, due to the fact that $f \in L^1$ and $A_+ \subseteq \mathbb{R}$

$$Mm(A_+) = \int_{A_+} M \leq \int_{A_+} |f| \leq \int_{\mathbb{R}} |f| \, dm < \infty.$$

Thus, $\infty > \int_{A_+} fdm \geq Mm(A_+)$. Note we dropped the absolute sign as f is positive on A_+. Hence $m(A_+) \neq \infty$.

Now suppose $0 < m(A_+) < \infty$. Then by hypothesis, we can write

$$\int_{A_+} fdm < Mm(A_+),$$

which is a contradiction. Hence $m(A_+) = 0$.

$$\textbf{Act 2} : m(A_-) = 0.$$

Again using $f \in L^1$ and $A_- \subseteq \mathbb{R}$, we get

$$\int_{A_-} |f| \, dm \leq \int_{\mathbb{R}} |f| \, dm < \infty.$$

Since $\left|\int_{A_-} f dm\right| \leq \int_{A_-} |f|\, dm$ and since $\int_{A_-} f dm < \int_{A_-} -M dm = -Mm(A_-)$ so that $\left|\int_{A_-} f dm\right| > Mm(A_-)$, therefore we get

$$Mm(A_-) < \left|\int_{A_-} f dm\right| \leq \int_{A_-} |f|\, dm < \infty.$$

Hence $m(A_-) \neq \infty$. Now with this, if we assume $\infty > m(A_-) > 0$, then by hypothesis, we obtain

$$\left|\int_{A_-} f dm\right| \leq m(A_-)M,$$

which contradicts the above inequality. Hence $m(A_-) = 0$.

$\square$

Lemma 25.1.13.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable map where the domain $\mathbb{R}$ has a measure structure $(\mathbb{R}, M, m)$. If $f \in L^1$ and $f \geq 0$, then for all $E \in M$*

$$\lim_{n\to\infty} \int_E f^{\frac{1}{n}} dm = m(E).$$

Proof. The fundamental observation that one has to make here is that if $y \in [0, \infty)$, then $y^{1/n}$ increases to 1 on $(0, 1]$ and $y^{1/n}$ decreases to 1 on $(1, \infty)$. Indeed, pick any $E \in M$ and define

$$E_{\leq} := E \cap \{x \in \mathbb{R} \mid f(x) \leq 1\}$$
$$E_{>} := E \cap \{x \in \mathbb{R} \mid f(x) > 1\}.$$

We thus have a disjoint measurable cover of E and hence $m(E) = m(E_{\leq}) + m(E_{>})$. Hence we get that

$$\lim_{n\to\infty} \int_E f^{\frac{1}{n}} dm = \lim_{n\to\infty} \int_{E_{\leq}} f^{\frac{1}{n}} dm + \lim_{n\to\infty} \int_{E_{>}} f^{\frac{1}{n}} dm.$$

Now, we have two integrals to consider.

$$\textbf{Act 1}: \lim_{n\to\infty} \textstyle\int_{E_{\leq}} f^{\frac{1}{n}} dm = m(E_{\leq}).$$

Since $f^{\frac{1}{n}}$ is a sequence of positive measurable maps increasing to 1, therefore by MCT (Theorem 25.1.10.2), we get that

$$\begin{aligned}\lim_{n\to\infty} \int_{E_{\leq}} f^{\frac{1}{n}} dm &= \int_{E_{\leq}} \lim_{n\to\infty} f^{\frac{1}{n}} dm \\ &= \int_{E_{\leq}} 1 dm \\ &= m(E_{\leq}).\end{aligned}$$

$$\textbf{Act 2}: \lim_{n\to\infty} \textstyle\int_{E_{>}} f^{\frac{1}{n}} dm = m(E_{>}).$$

It is this place where we will have to use the fact that $f \in L^1$. Since $f^{\frac{1}{n}}$ is a sequence of positive measurable maps decreasing to 1 where f is L^1. Hence, by Corollary 25.1.11.3 of MCT, we get that

$$\begin{aligned}\lim_{n\to\infty} \int_{E_>} f^{\frac{1}{n}} dm &= \int_{E_>} \lim_{n\to\infty} f^{\frac{1}{n}} dm \\ &= \int_{E_>} 1 dm \\ &= m(E_>).\end{aligned}$$

This completes the proof, as we have showed $\lim_{n\to\infty} \int_E f^{\frac{1}{n}} dm = \lim_{n\to\infty} \int_{E_\leq} f^{\frac{1}{n}} dm + \lim_{n\to\infty} \int_{E_>} f^{\frac{1}{n}} dm = m(E_\leq) + m(E_>) = m(E)$. □

Lemma 25.1.13.4. *Let $(\mathbb{R}, M, m)$ be a measure space with $E \in M$ and $f_n : E \to \mathbb{R} \cup \{\pm\infty\}$ be a sequence of measurable maps. If there exists a measurable map $f : E \to \mathbb{R} \cup \{\pm\infty\}$ such that*

$$|f_n(x)| \leq |f(x)| \;\; a.e. \; \forall n \in \mathbb{N},$$

then

$$\int_E \limsup_n f_n dm \leq \limsup_n \int_E f_n dm \leq \liminf_n \int_E f_n dm \leq \int_E \liminf f_n dm.$$

Proof. □

Complete the pr of Lemma 25.1.1 Chapter 25.

Lemma 25.1.13.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be an L^1 map where domain has a measure structure given by $(\mathbb{R}, M, m)$. Pick $E \in M$ and let $E_n := \{x \in E \mid f(x) \geq n\}$. Then,*

1. $m(E_n) \leq \frac{1}{n} \int_{E_n} |f| \, dm$,
2. $\lim_{n\to\infty} \int_{E_n} |f| \, dm$,
3. $\lim_{n\to\infty} n m(E_n) = 0$.

Proof. □

Complete the pr of Lemma 25.1.1 Chapter 25.

Lemma 25.1.13.6. *Let E be a non-measurable set and $f : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$f(x) = \begin{cases} e^x & if \; x \in E \\ -e^x & if \; x \notin E \end{cases}.$$

Then

1. $\{x \in \mathbb{R} \mid f(x) = c\}$ *is measurable for all* $x \in \mathbb{R}$.
2. f *is not a measurable map.*

Proof. □

Complete the pr of Lemma 25.1.1 Chapter 25.

Lemma 25.1.13.7. *Let (X, M, m) be a measure space and $f : X \times [a, b] \to \mathbb{C}$ be a function such that $f(x, t) : X \to \mathbb{C}$ is measurable for all $t \in [a, b]$. Let $F(t) := \int_X f(x, t)dm$. Suppose there exists $g \in L^1$ such that*

$$|f(x, t)| \leq |g(x)| \, \forall x \in X$$

for every $t \in [a, b]$. If $\lim_{t\to t_0} f(x, t) = f(x, t_0)$ for every $x \in X$, then

$$\lim_{t\to t_0} F(t) = F(t_0).$$

Proof. Clearly we should use DCT. However, we first need to get a sequence of functions for it. Indeed, since we know that $\lim_{t_n\to t_0} f(x, t) = f(x, t_0)$, thus for any sequence $t_n \to t_0$, we have $\lim_{n\to\infty} f(x, t_n) = f(x, t_0)$. Hence we may define $f_n(x) = f(x, t_n)$ which are by definition measurable. Moreover, we have $|f_n(x)| \leq |g(x)|$ for all $x \in X$ where $g \in L^1$. Hence, by DCT, we obtain

$$\begin{aligned}
\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \int_X f_n(x)dm &= \int_X \lim_{n\to\infty} f_n(x)dm \\
&= \int_X f(x, t_0)dm \\
&= F(t_0).
\end{aligned}$$

Since $t_n \to t_0$ is arbitrary, therefore $\lim_{t\to t_0} F(t) = F(t_0)$. $\square$

25.1.14 Product measures

25.1.15 Fubini-Tonelli theorem

25.1.16 Applications-III : Product and Fubini-Tonelli

Lemma 25.1.16.1. *Let (X, Σ_1, μ) and (Y, Σ_2, ν) be two σ-finite measure space with $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^1(\nu)$. Then the function $h(x, y) = f(x)g(y)$ is in $\mathcal{L}^1(\mu \times \nu)$ and that*

$$\int_{X\times Y} h d\mu \times \nu = \left(\int_X f d\mu\right)\left(\int_Y g d\nu\right). \tag{1.1}$$

Proof. We first show that h is measurable. Indeed, as $X\times Y \to \mathbb{C}$ given by $(x, y) \mapsto f(x)$ and $X \times Y \to \mathbb{C}$ given by $(x, y) \mapsto g(y)$ are measurable as they are composites $X \times Y \overset{\pi_1}{\to} X \overset{f}{\to} \mathbb{C}$ and $X \times Y \overset{\pi_2}{\to} X \overset{g}{\to} \mathbb{C}$ respectively, where we know that the projection π_i are measurable, therefore their pointwise product $h(x, y) = f(x)g(y)$ is measurable as well. This shows that h is measurable.

Now note that we have $\int_X |f|\, d\mu = M < \infty$ and $\int_Y |g|\, d\nu = N < \infty$. Furthermore, we have $|h|_x = (|f|\,|g|)_x = |f(x)|\,|g|$ and similarly $|h|^y = (|f|\,|g|)^y = |g(y)|\,|f|$. Consequently by Fubini-Tonelli for $L^+(\mu \times \nu)$, we obtain

$$\begin{aligned}
\int_{X\times Y} |h|\, d\mu \times \nu &= \int_X \int_Y |h|\, d\nu d\mu \\
&= \int_X \int_Y |f|\,|g|\, d\nu d\mu \\
&= \int_X |f| \left(\int_Y |g|\, d\nu\right) d\mu \\
&= \int_X N\,|f|\, d\mu \\
&= NM < \infty.
\end{aligned}$$

Hence, $h \in \mathcal{L}^1(\mu \times \nu)$.

We now wish to show Eq. (1.1). Indeed, as $h \in \mathcal{L}^1(\mu \times \nu)$, therefore by Fubini-Tonelli for $\mathcal{L}^1(\mu \times \nu)$, we obtain

$$\begin{aligned}
\int_{X\times Y} h d\mu \times \nu &= \int_X \int_Y h_x d\nu d\mu \\
&= \int_X \int_Y f(x) g d\nu d\mu \\
&= \int_X f(x) \left(\int_Y g d\nu\right) d\mu \\
&= \left(\int_X f d\mu\right)\left(\int_Y g d\nu\right)
\end{aligned}$$

as needed. □

Example 25.1.16.2. For $X = Y = \mathbb{N}$, $\Sigma_1 = \Sigma_2 = \mathcal{P}(\mathbb{N})$ and $\mu = \nu = \#$ the counting measure, we wish to restate the Fubini-Tonelli theorem in this setting.

First of all, we observe that both the spaces (X, Σ_1, μ) and (Y, Σ_2, ν) are σ-finite as $\mathbb{N}$ can be covered by $\{E_n\}$ where $E_n = \{n\}$ is a finite measure subset. Hence the Fubini-Tonelli applies.

For any measurable $h : X \to \mathbb{C}$, we first claim that the integral $\int_X h d\mu = \sum_n h(n)$. Indeed, we first have by definition

$$\int_X h d\mu = \int_X \Re(h)^+ d\mu - \int_X \Re(h)^- d\mu + i\left(\int_X \Im(h)^+ - \int_X \Im(h)^- d\mu\right)$$

where each $\Re(h)^\pm, \Im(h)^\pm$ are measurable functions $X \to [0, \infty)$. Hence we reduce to assuming h is a non-negative measurable function. In this case, we observe the following. Consider $g_n = \sum_{k=1}^n h(k)\chi_{\{k\}}$. Observe that g_n are increasing and converges to f pointwise. Then by MCT, we have

$$\begin{aligned}
\int_X h d\mu &= \lim_{n\to\infty} \int_X g_n d\mu \\
&= \lim_{n\to\infty} \int_X \sum_{k=1}^n h(k)\chi_{\{k\}} d\mu \\
&= \lim_{n\to\infty} \sum_{k=1}^n \int_X h(k)\chi_{\{k\}} d\mu \\
&= \lim_{n\to\infty} \sum_{k=1}^n h(k) \\
&= \sum_{k=1}^\infty h(k)
\end{aligned}$$

as needed.

Now pick any $h \in L^+(\mu \times \nu)$. We first claim that $\int_{X\times Y} h d\mu \times \nu = \sum_{n,m} h(n,m)$. Indeed, we claim that $\int_{X\times Y} h d\mu \times \nu = \sup\{\int_{X\times Y} \varphi d\mu \times \nu \mid 0 \le \varphi \le h,\ \varphi \text{ is simple}\} = \sup\{\sum_{(n,m)\in F} h(n,m) \mid F \subseteq \mathbb{N}\times\mathbb{N} \text{ is finite}\} = \sum_{n,m} h(n,m)$, as needed. Let $A = \{\int_{X\times Y} \varphi d\mu \times \nu \mid 0 \le \varphi \le h,\ \varphi \text{ is simple}\}$ and $B = \{\sum_{(n,m)\in F} h(n,m) \mid F \subseteq \mathbb{N}\times\mathbb{N} \text{ is finite}\}$. To show the above claim, we need only show that

$$\sup A = \sup B.$$

First suppose that B is not bounded. Then there exists a sequence $b_k \in B$ such that $b_k \to \infty$ as $k \to \infty$. Let $b_k = \sum_{(n,m)\in F_k} h(n,m) \to \infty$ as $k \to \infty$, where F_k are finite sets. Hence, construct $\varphi_k = \sum_{(n,m)\in F_k} h(n,m)\chi_{\{(n,m)\}}$. Clearly, $\varphi_k \in A$ is a simple function below h. As $\int_{X\times Y} \varphi_k d\mu \times \nu = \sum_{(n,m)\in F_k} h(n,m) = b_k$, therefore we get that A is unbounded as well.

Now suppose B is bounded. Then, A is bounded as well because for any simple function $0 \le \varphi \le h$, φ cannot be supported on an infinite cardinality set as otherwise B will be unbounded. Hence both $\sup A$ and $\sup B$ exists and we wish to show that they are equal. Note that the above argument shows that for any simple function $0 \le \varphi \le h$ given by $\varphi = \sum_{k=1}^n a_k \chi_{E_k}$, the integral $\int_{X\times Y} \varphi d\mu \times \nu = \sum_{k=1}^n a_k \#(E_k)$ is finite. Hence for any $\varphi \in A$, there exists a finite set F such that $\int_{X\times Y} \varphi d\mu \times \nu \le \sum_{(n,m)\in F} h(n,m)$. Thus, $\sup A \le \sup B$.

Conversely, pick any $\sum_{(n,m)\in F} h(n,m) \in B$ for some finite F. Then, the simple function $\varphi = \sum_{(n,m)\in F} h(n,m)\chi_{\{(n,m)\}} \in A$ is such that $\int_{X\times Y} \varphi d\mu \times \nu = \sum_{(n,m)\in F} h(n,m)$. Hence $\sup B \le \sup A$. This completes the proof that integral of h over $X \times Y$ is just the double sum.

Now by Fubini-Tonelli for L^+, we obtain that

$$\begin{aligned}
\int_{X\times Y} hd(\mu\times\nu) &= \sum_{n,m} h(n,m) \\
&= \int_X \int_Y h_n d\nu d\mu \\
&= \int_X \sum_m h(n,m) d\mu \\
\text{(by MCT)} &= \sum_m \int_X h(n,m) d\mu \\
&= \sum_m \sum_n h(n,m).
\end{aligned}$$

Similarly, we also yield by an application of MCT that

$$\begin{aligned}
\int_{X\times Y} hd(\mu\times\nu) &= \sum_{n,m} h(n,m) \\
&= \int_Y \int_X h^m d\mu d\nu \\
&= \sum_n \sum_m h(n,m).
\end{aligned}$$

Now suppose $h \in \mathcal{L}^1(\mu\times\nu)$. Then by Fubini-Tonelli, we yield that

$$\begin{aligned}
\int_{X\times Y} hd\mu\times\nu &= \sum_{n,m} h(n,m) \\
&= \int_X \int_Y h_n d\nu d\mu \\
&= \int_X \sum_m h(n,m) d\mu \\
\text{(by DCT as each } h^m \in \mathcal{L}^1(\mu) \text{ by Fubini)} &= \sum_m \int_X h(n,m) d\mu \\
&= \sum_m \sum_n h(n,m).
\end{aligned}$$

Similarly, we yield

$$\int_{X\times Y} hd\mu\times\nu = \sum_n \sum_m h(n,m).$$

Hence, we yield the following two statements from this discussion:

1. *Let* $\sum_{n,m} a_{n,m}$ *be a double series of non-negative real numbers. Then,*

$$\sum_{n,m} a_{n,m} = \sum_n \sum_m a_{n,m} = \sum_m \sum_n a_{n,m}.$$

2. *Let* $\sum_{n,m} a_{n,m}$ *be a double series of complex numbers such that*

$$\sum_{n,m} |a_{n,m}| < \infty.$$

Then,

$$\sum_{n,m} a_{n,m} = \sum_n \sum_m a_{n,m} = \sum_m \sum_n a_{n,m}.$$

This completes the analysis.

Example 25.1.16.3. Let $c \in \mathbb{R}$ and define $f : [0, \infty) \to \mathbb{R}$ a map given by

$$f(x) = \frac{\sin x^2}{x} + \frac{cx}{1+x}.$$

Let $a > 0$. Then we wish to show that

$$\mathrm{lim}_{n\to\infty} \int_0^a f(nx)dx = ac.$$

We claim that $\frac{\sin x}{x}$ is a bounded function over $[0, \infty)$. Indeed, fix $\epsilon > 0$. As $\lim_{x\to 0} \frac{\sin x^2}{x} = 0$, therefore there exists $\delta > 0$ such that for $x \in (0, \delta)$, we have $\left|\frac{\sin x^2}{x}\right| < \epsilon$. Furthermore, for $x \geq \delta$ we have $\left|\frac{\sin x^2}{x}\right| \leq \frac{1}{|x|} \leq \frac{1}{\delta}$. Hence taking $M = \max\{\epsilon, 1/\delta\}$, we see that $\left|\frac{\sin x^2}{x}\right| \leq M$ over $[0, \infty)$. Consequently, over $[0, \infty)$, we have

$$\begin{aligned} |f(x)| &= \left|\frac{\sin x^2}{x} + \frac{cx}{1+x}\right| \\ &\leq |M| + \left|\frac{cx}{1+x}\right| \\ &\leq M + |c|. \end{aligned}$$

Thus, the sequence of measurable functions $|f(nx)|$ is upper bounded by $|g(x)| = M + |c|$ over $[0, a]$, which is $\mathcal{L}^1$ over $[0, a]$. Furthermore, we see that $f(nx) \to c$ over $(0, a]$ pointwise as $n \to \infty$. Hence, by DCT, we obtain

$$\begin{aligned} \mathrm{lim}_{n\to\infty} \int_0^a f(nx)dx &= \int_0^a \mathrm{lim}_{n\to\infty} f(nx)dx \\ &= \int_0^a cdx \\ &= ca \end{aligned}$$

as needed.

Example 25.1.16.4. Let $X = Y = [0,1]$, $\Sigma_1 = \Sigma_2 = \mathcal{B}_{[0,1]}$ the Borel σ-algebra on $[0,1]$ and μ = Lebesgue measure over $[0,1]$ and ν = counting measure over $[0,1]$. We wish to show that Fubini-Tonelli doesn't holds here for the function $\chi_D : X \times Y \to \mathbb{R}$ where $D = \{(x,x) \mid x \in X\}$.

Let us first calculate $\int_{X\times Y} \chi_D d\mu \times \nu$. As χ_D is just a characteristic function, therefore we simply have

$$\int_{X\times Y} \chi_D d\mu \times \nu = \mu \times \nu(D).$$

1. We claim that $\mu \times \nu(D) = \infty$. Indeed, by definition, we have

$$\mu \times \nu(D) = \inf \left\{ \sum_n \mu(I_n)\nu(J_n) \mid \bigcup_n I_n \times J_n \supseteq D, I_n \times J_n \in \mathcal{R} \right\}$$

where $\mathcal{R}$ is the elementary family of all rectangles. We claim that for any such cover $D \subseteq \bigcup_n I_n \times J_n$, we have $\sum_n \mu(I_n)\nu(J_n) = \infty$. Indeed, it suffices to show that there is an $n \in \mathbb{N}$ such that $\mu(I_n) \neq 0$ and J_n is infinite. Suppose there is no such n. It then follows that if $\mu(I_n) \neq 0$, then J_n is finite. Further, if $\mu(I_n) = 0$, then J_n can be finite or infinite. Let

$$K := \{n \in \mathbb{N} \mid \mu(I_n) \neq 0\}$$

and

$$L := \{n \in \mathbb{N} \mid \mu(I_n) = 0\}.$$

Consequently, $K \cup L = \mathbb{N}$.

Pick $n \in K$. Then, $\mu(I_n) \neq 0$ and J_n is finite. It follows that $(I_n \times J_n) \cap D$ is atmost a finite set. Thus, $\bigcup_{n\in K} I_n \times J_n$ covers atmost a countable subset of D. Hence, it follows that $\bigcup_{n\in L} I_n \times J_n$ covers an uncountable subset of D. Furthermore,

$$\begin{aligned} V &:= D \setminus \left(\bigcup_{n\in L} (I_n \times J_n) \cap D \right) \\ &= \bigcup_{n\in K} (I_n \times J_n) \cap D \ \text{ is countable.} \end{aligned} \tag{4.1}$$

For any $n \in \mathbb{N}$, observe that

$$(I_n \times J_n) \cap D = \{(x,x) \in D \mid x \in I_n \cap J_n\}. \tag{4.2}$$

From the preceding remark, it is thus clear that the set $\bigcup_{n\in L}(I_n \times J_n) \cap D = \{(x,x) \in D \mid x \in I_n \cap J_n \text{ for some } n \in L\}$ is uncountable, which further makes $A := \bigcup_{n\in L} I_n \cap J_n \subseteq [0,1]$ uncountable. We claim that $[0,1] \setminus A$ is countable. Indeed, by (4.1), we first see that

$$\begin{aligned} V &= \{(x,x) \mid x \in I_n \cap J_n \text{ for some } n \in K\} \\ &\cong \bigcup_{n\in K} I_n \cap J_n. \end{aligned}$$

Thus, $\bigcup_{n\in K} I_n \cap J_n$ is countable.

Observe that

$$[0,1] = \left(\bigcup_{n\in K} I_n \cap J_n\right) \cup \left(\bigcup_{n\in L} I_n \cap J_n\right)$$

because $\{I_n \times J_n\}_{n\in\mathbb{N}}$ covers D. Consequently, as A is uncountable, therefore

$$[0,1] \setminus A \subseteq \bigcup_{n\in K} I_n \cap J_n$$

is countable by Eq. (4.3), as required.

As $A \subseteq [0,1]$ is such that $[0,1] \setminus A$ is countable therefore $\mu(A) = 1$. But, $A \subseteq \bigcup_{n\in L} I_n$, therefore $1 = \mu(A) = \sum_{n\in L} m(I_n) = \sum_n 0 = 0$ as I_n for $n \in L$ is of measure 0. Hence we have $1 = \mu(A) \leq 0$, a contradiction. This shows that $\sum_n \mu(I_n)\nu(J_n) = \infty$ for each $\{I_n \times J_n\} \subseteq \mathcal{R}$ such that $\bigcup_n I_n \times J_n \supseteq D$. Thus,

$$\mu \times \nu(D) = \infty.$$

2. We claim that $\int_Y \int_X \chi_D d\mu d\nu = 0$. Indeed, we have

$$\begin{aligned}\int_Y \int_X (\chi_D)^y \, d\mu d\nu &= \int_Y \int_X \chi_{D^y} d\mu d\nu \\ &= \int_Y \mu(\{(y,y)\}) d\nu \\ &= \int_Y 0 d\nu \\ &= 0,\end{aligned}$$

as required.

3. We claim that $\int_X \int_Y \chi_D d\nu d\mu = 1$. Indeed, we have

$$\begin{aligned}\int_X \int_Y (\chi_D)_x d\nu d\mu &= \int_X \int_Y \chi_{D_x} d\nu d\mu \\ &= \int_X \nu(\{(x,x)\}) d\mu \\ &= \int_X 1 d\mu \\ &= \mu(X) \\ &= 1,\end{aligned}$$

as needed.

Hence, we have shown that for Fubini-Tonelli to work, we require both spaces to be σ-finite (which is not the case here as Y is not σ-finite).

Example 25.1.16.5. We wish to construct an example of a monotone class of subsets of a non-empty set X such that it is not a σ-algebra. Indeed, consider $X = \{1., 2, 3\}$. Define $\mathcal{C} := \{\emptyset, \{1\}, X\}$. Then $\mathcal{C}$ is a monotone class as the only non-trivial increasing sequence of sets is $\emptyset \subseteq \{1\}$ and their union is clearly $\{1\}$ which is in $\mathcal{C}$. Furthermore the only non-trivial decreasing sequence is $X \supseteq \{1\}$, whose intersection is $\{1\}$, which is in $\mathcal{C}$. However, $\mathcal{C}$ is not a σ-algebra as $\{1\}^c = \{2, 3\} \notin \mathcal{C}$.

Lemma 25.1.16.6. *Let (X, Σ, μ) be a measure space and $f : X \to \mathbb{C}$ be an $\mathcal{L}^1(\mu)$ map. For each $E \in \Sigma$, define*

$$\nu(E) = \int_E f d\mu.$$

1. *If $\mu(E) = 0$, then $\nu(E) = 0$.*
2. *If $\{E_n\} \subseteq \Sigma$ is a disjoint collection, then*

$$\nu\left(\coprod_n E_n\right) = \sum_n \nu(E_n).$$

3. *For all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\mu(E) < \delta \implies |\nu(E)| < \epsilon.$$

Proof. 1. Note that $\nu(E) = 0$ iff $|\nu(E)| = 0$. Consequently, we see that

$$\begin{aligned} |\nu(E)| = \left|\int_E f d\mu\right| &\leq \int_E |f| \, d\mu \\ &\leq \infty \cdot \int_E d\mu \\ &= \infty \mu(E) \\ &= \infty \cdot 0 = 0, \end{aligned}$$

as needed.

2. Pick $\{E_n\} \subseteq \Sigma$ to be a disjoint collection. Consider the sequence of measurable functions $g_n = f\chi_{\coprod_{k=1}^n E_k}$. Observe that $g_n \to f\chi_{\coprod_{k=1}^\infty E_k}$ pointwise as $n \to \infty$. Furthermore, observe that $|g_n| \leq |f|$ and as $f \in \mathcal{L}^1(\mu)$, therefore we may apply DCT on $\{g_n\}$.

Applying DCT, we yield

$$\begin{aligned}\int_{\coprod_{k=1}^{\infty} E_k} f d\mu = \int_X f\chi_{\coprod_{k=1}^{\infty} E_k} d\mu &= \lim_{n\to\infty} \int_X f\chi_{\coprod_{k=1}^{n} E_k} d\mu \\ &= \lim_{n\to\infty} \int_{\coprod_{k=1}^{n} E_k} f d\mu \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{E_k} f d\mu \\ &= \sum_{k=1}^{\infty} \int_{E_k} f d\mu \\ &= \sum_{k=1}^{\infty} \nu(E_k),\end{aligned}$$

as needed.

3. As $f \in \mathcal{L}^1(\mu)$, therefore there exists a sequence of bounded functions $g_n \in \mathcal{L}^1(\mu)$ such that $g_n \to f$ pointwise as $n \to \infty$ and $|g_n| \leq |f|$ over X. Fix $E \in \Sigma$ of finite measure. It follows from DCT applied on g_n over X that

$$\lim_{n\to\infty} \int_E |f - g_n|\, d\mu \leq \lim_{n\to\infty} \int_X |f - g_n|\, d\mu = 0.$$

Fix $\epsilon > 0$. The convergence of above limit yields that there exists $N \in \mathbb{N}$ such that

$$\int_E |f - g_n|\, d\mu < \epsilon/2$$

for all $n \geq N$. Thus, in particular,

$$\int_E |f| - |g_N|\, d\mu \leq \int_E |f - g_N|\, d\mu < \epsilon/2.$$

Now, from above, we yield that

$$\begin{aligned}|\nu(E)| = \left| \int_E f d\mu \right| &\leq \int_E |f|\, d\mu \\ &\leq \epsilon/2 + \int_E |g_N|\, d\mu.\end{aligned}$$

As $|g_n|$ is bounded, therefore let $|g_N| \leq M_n$ for some $M_N \in [0, \infty)$. Consequently,

$$\begin{aligned}|\nu(E)| &\leq \epsilon/2 + \int_E |g_N|\, d\mu \\ &\leq \epsilon/2 + \int_E M_N d\mu \\ &\leq \epsilon/2 + M_N \mu(E).\end{aligned}$$

Hence, letting $\delta = \epsilon/2M_N$, we yield that for any $E \in \Sigma$ such that $\mu(E) < \delta$ we have

$$\begin{aligned}|\nu(E)| &< \epsilon/2 + \epsilon/2 \\ &= \epsilon.\end{aligned}$$

This completes the proof. $\square$

Example 25.1.16.7. Let $X = Y = \mathbb{N}$, $\Sigma_1 = \Sigma_2 = \mathcal{P}(\mathbb{N})$ and $\mu = \nu =$ counting measure. Further, define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ given by

$$f(m,n) = \begin{cases} 1 & \text{if } m = n, \\ -1 & \text{if } m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

We wish to show that

1. $\int_{X\times Y} |f|\, d(\mu \times \nu) = \infty$,
2. $\int_X \int_Y f d\nu d\mu = 1$,
3. $\int_Y \int_X f d\mu d\nu = 0$.

Before proving, we would first like to show that f is indeed measurable. Indeed, we may write $f = \chi_D - \chi_S$ where $D = \{(m,m) \mid m \in \mathbb{N}\}$ is the diagonal and $S = \{(n+1,n) \mid n \in \mathbb{N}\}$. Both are measurable subsets of $\Sigma_1 \otimes \Sigma_2$ as $D = \bigcup_m \{(m,m)\}$ and $S = \bigcup_n \{(n+1,n)\}$. Note that singletons of $X \times Y$ are measurable as singletons in X and Y are measurable.

1. Observe that $|f| = \chi_D + \chi_S = \chi_{D \amalg S}$ where D and S are two disjoint subsets as defined above. Consequently

$$\begin{aligned}\int_{X\times Y} |f|\, d\mu \times \nu &= \mu \times \nu(D \amalg S) \\ &= \mu \times \nu(D) + \mu \times \nu(S).\end{aligned}$$

We claim that both $\mu \times \nu(D)$ and $\mu \times \nu(S)$ are ∞.

Indeed, for any $\{I_n \times J_n\}$ for $I_n \times J_n \in \mathcal{R}$ rectangles such that $\bigcup_n I_n \times J_n \supseteq D$, we see that if $(I_n \times J_n) \cap D \neq \emptyset$, then $\mu(I_n)\nu(J_n) \geq 1$ as in this case $I_n \cap J_n \neq \emptyset$. As D is an infinite set and $\bigcup_n (I_n \times J_n) \cap D = D$, therefore $\sum_n \mu(I_n)\nu(J_n) \geq \mu \times \nu\left(\bigcup_n I_n \times J_n\right) \geq \mu \times \nu(D) = \infty$. This shows $\mu \times \nu(D) = \infty$.

Similarly, if $\{I_n \times J_n\}$ for $I_n \times J_n \in \mathcal{R}$ is a collection of rectangles such that $\bigcup_n I_n \times J_n \supseteq S$, then for each n for which $(I_n \times J_n) \cap D \neq \emptyset$ we deduce that $\mu(I_n)\nu(J_n) \geq 1$. Hence, as above, we again get that $\sum_n \mu(I_n)\nu(J_n) = \infty$. This proves that $\int_{X\times Y} |f|\, d\mu \times \nu = \infty$.

2. We simply observe that by definition we have $D_m = \{m\}$ and $S_m = \{m-1\}$. Conse-

quently,

$$\begin{aligned}
\int_X \int_Y f_m d\nu d\mu &= \int_X \int_Y \chi_{D_m} - \chi_{S_m} d\nu d\mu \\
&= \int_X \nu(D_m) - \nu(S_m) d\mu \\
&= \int_{X \setminus \{1\}} (1-1) d\mu + \int_{\{1\}} (1-0) d\mu \\
&= 1.
\end{aligned}$$

3. We simply observe that by definition $D^n = \{n\}$ and $S^n = \{n+1\}$. Consequently,

$$\begin{aligned}
\int_Y \int_X f^n d\mu d\nu &= \int_Y \int_X (\chi_{D^n} - \chi_{S^n}) d\mu d\nu \\
&= \int_Y \mu(D^n) - \mu(S^n) d\nu \\
&= \int_Y 1 - 1 d\nu \\
&= 0.
\end{aligned}$$

This completes the proof.

25.1.17 Signed measures

25.1.18 Applications-IV : Signed spaces

Lemma 25.1.18.1. *Let $(X, \mathcal{A})$ be a measurable space and μ, ν be two signed measures on it. Then $\nu \ll \mu$ and $\mu \perp \nu$ if and only if $\nu = 0$.*

Proof. ($\Rightarrow$) As $\mu \perp \nu$, therefore there exists a μ-null set A and a ν-null set B such that $A \amalg B = X$. For any measurable set $E \subseteq X$, we have $E = (E \cap A) \amalg (E \cap B)$. As $E \cap A \subseteq A$, therefore $\mu(E \cap A) = 0$. As $\nu \ll \mu$, therefore $\nu(E \cap A) = 0$. Furthermore, since $E \cap B \subseteq B$, therefore $\nu(E \cap B) = 0$. Hence,

$$\begin{aligned}\nu(E) &= \nu(E \cap A) + \nu(E \cap B) \\ &= 0,\end{aligned}$$

as needed.

($\Leftarrow$) As for any measurable set $E \subseteq X$, we have $\nu(E) = 0$, hence $\nu \ll \mu$. Further, as X is now ν-null and $\emptyset$ is μ-null, therefore $X = X \amalg \emptyset$ gives us the required decomposition to claim that $\mu \perp \nu$. $\square$

Lemma 25.1.18.2. *Let $(X, \mathcal{A})$ be a measurable space and μ, ν be two positive measures on it. The following are equivalent.*

1. *$\nu \perp \mu$,*
2. *there exists a sequence $\{E_n\} \subseteq \mathcal{A}$ such that $\mu(E_n) \to 0$ and $\nu(X \setminus E_n) \to 0$ as $n \to \infty$.*

Proof. (1. $\Rightarrow$ 2.) As $\nu \perp \mu$, therefore there exists a ν-null set A and a μ-null set B such that $X = A \amalg B$. Hence, we may take $E_n = A$ and $X \setminus E_n = B$ for each $n \in \mathbb{N}$. This provides the required sequence.

(2. $\Rightarrow$ 1.) We wish to construct $A, B \subseteq X$ such that $A \amalg B = X$ and A is ν-null and B is μ-null. To construct A and B, we proceed as follows.

We first observe that since $\mu(E_n) \to 0$, therefore there exists a subsequence of $\mu(E_n)$ say $\mu(E_{n_k})$ such that $\sum_k \mu(E_{n_k}) < \infty$. Indeed, this is a consequence of a general result : for any positive sequence $\{a_n\}$ such that $\lim_n a_n = 0$, we have that there exists a subsequence $\{a_{n_k}\}$ such that $\sum_k a_{n_k} < \infty$. Indeed, for each $k \in \mathbb{N}$ there exists an $n_k \in \mathbb{N}$ such that $a_n \leq 1/2^k$ for all $n \geq n_k$. Consequently, we see that $\sum_{k=1}^{\infty} a_{n_k} \leq \sum_{k=1}^{\infty} 1/2^k < \infty$, as required.

We apply the above result to $\{\mu(E_n)\}$ to obtain a subsequence $\{E_{n_k}\}$. We now replace $\{E_n\}$ by $\{E_{n_k}\}$ so that we may assume $\sum_n \mu(E_n) < \infty$.

Consider the sequence

$$\begin{aligned}F_n &= X \setminus \bigcup_{k=n}^{\infty} E_k \\ &= \bigcap_{k=n}^{\infty} X \setminus E_k.\end{aligned}$$

Observe that F_n is an increasing sequence and that each $F_n \subseteq X \setminus E_n$. Hence,

$$\nu(F_n) \leq \nu(X \setminus E_n). \tag{2.1}$$

Moreover, observe that since $\lim_{n\to\infty}\nu(X \setminus E_n) = 0$, therefore $\lim_{n\to\infty}\nu(F_n) = 0$. Hence, we deduce by monotone property of measures that

$$\lim_{n\to\infty}\nu(F_n) = \nu\left(\bigcup_n F_n\right).$$

Hence, by previous discussion, we further deduce that

$$\lim_{n\to\infty}\nu(F_n) = 0 = \nu\left(\bigcup_n F_n\right).$$

Thus $A := \bigcup_n F_n$ is a ν-null set. It now suffices to show that $X \setminus A$ is a μ-null set.

Observe that $X \setminus A$ can be written as

$$\begin{aligned} X \setminus A &= \bigcap_{n=1}^{\infty} X \setminus F_n \\ &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k. \end{aligned}$$

We claim that $X \setminus A$ is a μ-null set. Indeed, denote

$$S_n = \bigcup_{k=n}^{\infty} E_k.$$

We wish to show that

$$\mu\left(\bigcap_{n=1}^{\infty} S_n\right) = \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 0.$$

Observe that S_n is a decreasing sequence. Furthermore, as $\mu(S_n) \le \sum_{k=n}^{\infty} \mu(E_k) < \infty$, therefore we may apply the monotone property of measures. Consequently, we yield the following

$$\lim_{n\to\infty}\mu(S_n) = \mu\left(\bigcap_{n=1}^{\infty} S_n\right). \tag{2.2}$$

We now show that $\lim_{n\to\infty}\mu(S_n) = 0$. Indeed, denoting $l = \sum_k \mu(E_k)$ and $l_n = \sum_{k=1}^{n-1} \mu(E_k)$, we first see that $\lim_{n\to\infty} l_n = l$. Now observe that

$$\mu(S_n) \le \sum_{k=n}^{\infty} \mu(E_k) = l - l_{n-1}$$

where the last equality follows from rearrangement of a positive convergent series. Hence, taking $\lim_{n\to\infty}$, we obtain that

$$\lim_{n\to\infty}\mu(S_n) \le l - l = 0,$$

that is, $\lim_{n\to\infty}\mu(S_n) = 0$. By Eq. (2.2), we deduce that $X \setminus A = \bigcap_n S_n$ is a μ-null set, as needed. This completes the proof. $\square$

Example 25.1.18.3. We wish to find Lebesgue decomposition of $\nu = m + \delta_0$ where m is the Lebesgue measure and δ_0 is the Dirac delta measure at $0 \in \mathbb{R}$.

Indeed, as $(\mathbb{R}, \mathcal{M})$ is a σ-algebra, therefore the Lebesgue decomposition theorem holds. We see an immediate candidate for Lebesgue decomposition of ν with respect to m as follows:

$$\nu = \nu_a + \nu_s$$

where we set $\nu_a = m$ and $\nu_s = \delta_0$. Indeed, this works as $m \ll m$ holds trivially and $\delta_0 \perp m$ because of the decomposition $\mathbb{R} = \{0\} \amalg (\mathbb{R}^2 \setminus \{0\})$ where we see immediately that $\{0\}$ is m-null and $\mathbb{R}^2 \setminus \{0\}$ is δ_0-null.

Example 25.1.18.4. Let $p(x) = x^2 - 6x + 1$ be a function $\mathbb{R} \to \mathbb{R}$. Consider the signed measure

$$\nu(E) = \int_E pdm$$

on $(\mathbb{R}, \mathcal{M})$.

1. We first wish to show that $(\mathbb{R}, \mathcal{M}, \nu)$ is σ-finite. Indeed, let $X_n = [n, n+1]$. We claim that $-\infty < \nu(X_n) < \infty$ for each $n \in \mathbb{N}$. Now, observe that over X_n, the polynomial is a continuous function supported on a compact interval, hence it achieves a maxima and a minima, say M_n and m_n respectively. Consequently, we have $m_n \leq p \leq M_n$ over X_n.

$$\int_{X_n} m_n dm \leq \int_{X_n} pdm \leq \int_{X_n} M_n dm$$

and thus $-\infty < m_n \leq \nu(X_n) \leq M_n < \infty$ for each n. Hence ν is σ-finite.

2. We wish to find the Hahn-decomposition of $\mathbb{R}$ w.r.t. ν. That is, we wish to find a decomposition $\mathbb{R} = P \amalg N$ such that P is a ν-positive set and N is a ν-negative set.

Observe that $p(x)$ has two real roots $c_1, c_2 \in \mathbb{R}$. Consequently, we see that over $N = [c_1, c_2]$ the polynomial $p(x)$ is negative and hence $\nu(E) = \int_E pdm \leq 0$ for any measurable $E \subseteq N$. Thus N is a negative set. Similarly, define $P = (-\infty, c_1) \cup (c_2, \infty)$. Then observe that $p(x)$ is positive over p, thus $\nu(E) \geq 0$ for any measurable $E \subseteq P$.

3. We now wish to find the Jordan decomposition of ν. Indeed, define $\nu^+(E) := \nu(E \cap P)$ and $\nu^-(E) := -\nu(E \cap N)$ where $X = P \amalg N$ is the Hahn decomposition. These are positive measures such that $\nu = \nu^+ - \nu^-$. Furthermore, $\nu^+ \perp \nu^-$ as P is ν^--null and N is ν^+-null by construction.

4. We wish to find the Lebesgue decomposition of ν with respect to the Lebesgue measure m. Indeed, we claim that $\nu \ll m$, which will immediately show that the Lebesgue decomposition of ν with respect to m is simply $\nu = \nu + 0$ where $\nu \ll m$ and $0 \perp m$. Indeed, take any measurable set $E \subseteq X$ such that $m(E) = 0$. As p is measurable therefore

$$\nu(E) = \int_E pdm = 0.$$

Hence $\nu \ll m$, completing the proof.

Lemma 25.1.18.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{E_n\}_{n=1}^N \subseteq \mathcal{A}$ and $\{c_n\}_{n=1}^N \subseteq \mathbb{R}_{\geq 0}$. Consider the positive measure*

$$\nu(E) = \sum_{n=1}^{N} c_n \mu(E \cap E_n)$$

for some fixed $E_n \in \mathcal{A}$. Then,

1. *$\nu \ll \mu$,*
2. *$d\nu/d\mu = \sum_{n=1}^N c_n \chi_{E_n}$.*

Proof. 1. We wish to show that $\nu \ll \mu$. Indeed, pick any $E \in \mathcal{A}$ such that $\mu(E) = 0$. As μ is positive, consequently $\mu(E \cap E_n) = 0$ for each $n = 1, \ldots, N$ as $E \cap E_n \subseteq E$. Hence, we deduce that $\nu(E) = 0$. Thus $\nu \ll \mu$.

2. We now wish to find the Radon-Nikodym derivative $d\nu/d\mu$, which exists as $\nu \ll \mu$. Indeed, this means we need to find a measurable function $f : X \to [0, \infty]$ such that

$$\nu(E) = \int_E f d\mu$$

for each $E \in \mathcal{A}$. We claim that the following simple function

$$f = \sum_{n=1}^{N} c_n \chi_{E_n}$$

is the required derivative. Indeed, observe that

$$\begin{aligned}
\int_E f d\mu &= \int_E \sum_{n=1}^{N} c_n \chi_{E_n} d\mu \\
&= \sum_{n=1}^{N} c_n \int_E \chi_{E_n} d\mu \\
&= \sum_{n=1}^{N} c_n \int_X \chi_{E_n \cap E} d\mu \\
&= \sum_{n=1}^{N} c_n \mu(E \cap E_n) \\
&= \nu(E),
\end{aligned}$$

as required. $\square$

Lemma 25.1.18.6. *Let $(X, \mathcal{A})$ be a measurable space and μ, ν be two positive measures. Suppose $\nu \ll \mu$. Then,*

1. *if the derivative $d\nu/d\mu > 0$ μ-almost everywhere, then $\mu \ll \nu$,*
2. *Assuming both μ and ν are σ-finite, if the derivative $d\nu/d\mu > 0$ μ-almost everywhere, then $\mu \ll \nu$ and*

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1} \mu\text{-almost everywhere.}$$

Proof. 1. We first wish to show that if the derivative $d\nu/d\mu > 0$ μ-almost everywhere, then $\mu \ll \nu$.

Denote $f = d\nu/d\mu$. Suppose $E \in \mathcal{A}$ is such that $\nu(E) = 0$. Thus $\nu(E) = \int_E f d\mu = 0$. As $f > 0$ μ-almost everywhere, therefore consider the sequence $E_n = \{f(x) \geq 1/n\}$. Clearly, $\bigcup_n E_n = X \setminus N$ as $f > 0$ over X, where $N = \{f(x) = 0\}$ is a μ-null set. Hence, $\bigcup_n E \cap E_n = E \setminus N$. Thus, $\mu(E \setminus N) \leq \sum_n \mu(E \cap E_n)$. Now,

$$\frac{1}{n}\mu(E \cap E_n) \leq \int_{E \cap E_n} f d\mu \leq \int_E f d\mu = 0.$$

Thus, $\mu(E \cap E_n) = 0$ for each $n \in \mathbb{N}$. Hence,

$$\mu(E \setminus N) \leq \sum_n \nu(E \cap E_n) = 0$$

and thus $\mu(E) = \mu(E \cap N) + \mu(E \setminus N) = 0 + 0 = 0$.

2. Assuming both μ and ν are σ-finite, we now wish to show that if the derivative $d\nu/d\mu > 0$ μ-almost everywhere, then $\mu \ll \nu$ and

$$\frac{d\mu}{d\nu} = \left(\frac{d\nu}{d\mu}\right)^{-1} \mu\text{-almost everywhere.}$$

We have shown that $\mu \ll \nu$ in the item 1 above. By Radon-Nikodym theorem, we have the derivative $g = d\mu/d\nu$ which is a measurable function $g : X \to [0, \infty]$ such that

$$\mu(E) = \int_E g d\nu.$$

Denote $f = d\nu/d\mu : X \to [0, \infty]$ which is such that

$$\nu(E) = \int_E f d\mu.$$

We are given that $f > 0$ μ-almost everywhere. We wish to show that $g = 1/f$ μ-almost everywhere.

As we have seen that for an L^+ function h, we obtain a positive measure given by $\mu_h = \int_E h d\mu$, therefore we deduce that $\mu = \nu_g$ and $\nu = \mu_f$. Consequently, denoting $N = \{f(x) = 0\}$ to be the μ-null set, we obtain

$$\begin{aligned}\int_E \frac{1}{f} d\nu &= \int_{E \setminus N} \frac{1}{f} f d\mu + \int_{E \cap N} \frac{1}{f} d\nu \\ &= \int_{E \setminus N} d\mu + 0 \\ &= \mu(E \setminus N).\end{aligned}$$

As $\mu(E \cap N) = 0$, therefore adding this to above we add

$$\int_E \frac{1}{f} d\nu = \mu(E \setminus N) + \mu(E \cap N) = \mu(E).$$

Thus by almost everywhere uniqueness of Radon-Nikodym derivative of μ w.r.t. ν, we see that $1/f = g$ μ-almost everywhere. $\square$

Lemma 25.1.18.7. *Let $(X, \mathcal{A})$ be a measurable space with μ and ν be two finite positive measures. Suppose*

$$f = \frac{d\nu}{d(\mu + \nu)}.$$

Assume that $1 - f > 0$. Then,

$$\nu(E) = \int_E \frac{f}{1-f} d\mu,$$

equivalently, that

$$\frac{d\nu}{d\mu} = \frac{f}{1-f}.$$

Proof. We first show that $g := 1 - f$ is equal to the derivative $d\mu/d(\mu + \nu)$. Observe that $g > 0$. Indeed, for this, we need to show that for any $E \in \mathcal{A}$, we have

$$\mu(E) = \int_E g d(\mu + \nu).$$

To this end, we see that by definition of f and finiteness of μ, ν and thus $\mu + \nu$ as measures, we may deduce

$$\begin{aligned}
\int_E g d(\mu + \nu) &= \int_E (1 - f) d(\mu + \nu) \\
&= \int_E d(\mu + \nu) - \int_E f d(\mu + \nu) \\
&= \mu(E) + \nu(E) - \nu(E) \\
&= \mu(E),
\end{aligned}$$

as required. We may therefore write $\mu = (\mu + \nu)_g$ as the notation introduced in the class for positive measures defined by positive measurable functions.

Next, we claim that the function f/g is the derivative $d\nu/d\mu$. For this, we wish to show that for any measurable $E \in \mathcal{A}$, we have that

$$\nu(E) = \int_E \frac{f}{g} d\mu.$$

As $\mu = (\mu + \nu)_g$, hence we see that

$$\begin{aligned}
\int_E \frac{f}{g} d\mu &= \int_E \frac{f}{g} g d(\mu + \nu) \\
&= \int_E f d(\mu + \nu) \\
&= \nu(E),
\end{aligned}$$

as required. $\square$

Example 25.1.18.8. Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ be a measurable space and ν be a σ-finite signed measure. Further, let μ be the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

1. We wish to show that $\nu \ll \mu$. This is immediate, as $\mu(E) = 0$ if and only if $E = \emptyset$, and hence $\nu(E) = 0$ by definition.

2. We now wish to compute the derivative $d\nu/d\mu$. This is straightforward, for we first observe that the following function

$$\begin{aligned} f : \mathbb{N} &\longrightarrow [0, \infty] \\ n &\longmapsto \nu(\{n\}) \end{aligned}$$

is measurable. Indeed, this is because the σ-algebra on $\mathbb{N}$ is the power set $\mathcal{P}(\mathbb{N})$. We thus claim that

$$f = \frac{d\nu}{d\mu}.$$

Indeed, pick any measurable set $E \subseteq \mathbb{N}$. Note that it is countable in size. Observe that

$$\begin{aligned} \int_E f d\mu &= \sum_{n \in E} f(n) \\ &= \sum_{n \in E} \nu(\{n\}) \\ &= \nu\left(\coprod_{n \in E} \{n\}\right) \\ &= \nu(E) \end{aligned}$$

where the second-to-last equality is obtained from the fact ν is a measure. This completes the proof.

Lemma 25.1.18.9. *Let $(X, \mathcal{A})$ be a measurable space and μ, ν be two σ-finite positive measures on $(X, \mathcal{A})$. Let $\lambda = \mu + \nu$. Then the following are equivalent*

1. $\mu \perp \nu$,
2. *if $f = d\mu/d\lambda$ and $g = d\nu/d\lambda$, then*

$$fg = 0 \ \lambda\text{-almost everywhere.}$$

Proof. (1. $\Rightarrow$ 2.) As $\mu \perp \nu$, therefore there exists a ν-null set A and a μ-null set B such that

$$A \amalg B = X. \tag{9.1}$$

For any measurable $E \subseteq X$, we have

$$\begin{aligned} \mu(E) &= \int_E f d\lambda \\ \nu(E) &= \int_E g d\lambda. \end{aligned}$$

We first observe that if any of the μ or ν is the zero measure, then we are done. Indeed, for if $\mu = 0$, then we deduce that $\mu(X) = 0$ and hence $f = 0$ λ-a.e. Consequently, $fg = 0$ λ-a.e. Hence, we may now assume that none of the μ and ν are 0 measures.

Observe that since $\mu(B) = 0$, therefore $\int_B f d\lambda = 0$. As $\lambda(B) = \mu(B) + \nu(B) = \nu(B)$, therefore we deduce from the fact that $\nu \neq 0$ and $\nu(A) = \nu(X \setminus B) = 0$ that $\nu(B) \neq 0$. Hence,

$$\lambda(B) \neq 0. \tag{9.2}$$

For exactly the same reasoning applied on $\nu(A) = 0$, we deduce that

$$\lambda(A) \neq 0. \tag{9.3}$$

Hence, we have that $\int_B f d\lambda = 0 = \int_A g d\lambda$. By Eqns (9.2) and (9.3), we conclude that $f = 0$ λ-a.e. over B and $g = 0$ λ-a.e. over A.

Consider the set $N = \{f(x) \neq 0\} \cap \{g(x) \neq 0\}$. Writing

$$N = (N \cap A) \amalg (N \cap B),$$

we observe that

1. $N \cap A$ is ν-null as A is ν-null,
2. $N \cap A$ is μ-null as $\{g(x) \neq 0\} \cap A$ is λ-null and over A, we have $\lambda = \mu$,
3. $N \cap B$ is μ-null as B is μ-null,
4. $N \cap B$ is ν-null as $\{f(x) \neq 0\} \cap B$ is λ-null and over B, $\lambda = \nu$.

Hence, we see that $N \cap A$ and $N \cap B$ both are λ-null. Consequently, N is λ-null.

(2. $\Rightarrow$ 1.) For any measurable $E \subseteq X$, we have

$$\begin{aligned} \mu(E) &= \int_E f d\lambda \\ \nu(E) &= \int_E g d\lambda. \end{aligned}$$

Consider the following measurable sets

$$\begin{aligned} A &= \{g(x) = 0\} \\ B &= \{g(x) \neq 0\} \cap \{f(x) = 0\} \\ N &= \{g(x) \neq 0\} \cap \{f(x) \neq 0\}. \end{aligned}$$

Clearly, $X = A \amalg B \amalg N$. Furthermore, as $fg = 0$ λ-a.e, therefore N is λ-null. Over A we see that ν is 0 and over B we see that μ is 0. As N is λ-null, therefore it is both μ and ν-null as well. Consequently, we have

$$X = A \amalg (B \amalg N)$$

where A is ν-null and $B \amalg N$ is μ-null, as required. $\square$

Lemma 25.1.18.10. *Let $(X, \mathcal{A}, \nu)$ be a signed space. Then,*

1. $\frac{d\nu^+}{d|\nu|} = \chi_P$,
2. $\frac{d\nu^-}{d|\nu|} = \chi_N$.

Proof. First, observe that these derivatives exists because $\nu^+ \ll |\nu|$ and $\nu^- \ll |\nu|$. By Jordan decomposition of ν, we have

$$\nu = \nu^+ - \nu^-$$

where $\nu^+(E) = \nu(P \cap E)$ and $\nu^-(E) = -\nu(N \cap E)$, where $X = P \amalg N$ is the Hahn-decomposition of X into a positive set P and a negative set N obtained by ν and $E \in \mathcal{A}$.

1. We claim that $\frac{d\nu^+}{d|\nu|}$ is given by χ_P. To this end, we need only show that

$$\nu^+(E) = \int_E \chi_P d|\nu|$$

as by Radon-Nikodym theorem, we know that the derivatives are unique $|\nu|$-almost everywhere, and therefore ν-almost everywhere.

Now, we see that

$$\begin{aligned}\int_E \chi_P d|\nu| &= |\nu|(E \cap P)\\ &= \nu^+(E \cap P) + \nu^-(E \cap P)\\ &= \nu(E \cap P \cap P) - \nu(E \cap P \cap N)\\ &= \nu(E \cap P) - \nu(\emptyset)\\ &= \nu^+(E),\end{aligned}$$

as needed.

2. We proceed similarly as above and claim that χ_N is the derivative $\frac{d\nu^-}{d|\nu|}$. Indeed, we see that

$$\begin{aligned}\int_E \chi_N d|\nu| &= |\nu|(E \cap N)\\ &= \nu^+(E \cap N) + \nu^-(E \cap N)\\ &= \nu(E \cap N \cap P) - \nu(E \cap N \cap N)\\ &= \nu(\emptyset) - \nu(E \cap N)\\ &= \nu^-(E),\end{aligned}$$

as required. □

Lemma 25.1.18.11. *Let $(X, \mathcal{A}, \nu)$ be a signed space and let $f : X \to \mathbb{C}$ be a measurable function. Define*

$$\int_X f d\nu = \int_X f d\nu^+ - \int_X f d\nu^-$$

where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν. Then,

1. *we have*

$$\left|\int_X f d\nu\right| \leq \int_X |f|\, d\,|\nu|\,,$$

2. *for any $E \in \mathcal{A}$, we have*

$$|\nu|\,(E) = \sup\left\{\left|\int_E f d\nu\right| \;\mid\; |f| \leq 1\right\}.$$

Proof. 1. We may write

$$\begin{aligned}\left|\int_X f d\nu\right| &= \left|\int_X f d\nu^+ - \int_X f d\nu^-\right| \\ &\leq \left|\int_X f d\nu^+\right| + \left|\int_X f d\nu^-\right| \\ &\leq \int_X |f|\, d\nu^+ + \int_X |f|\, d\nu^-. \end{aligned} \tag{11.1}$$

We now claim that $\int_X |f|\, d\nu^+ + \int_X |f|\, d\nu^- = \int_X |f|\, d\,|\nu|$. Indeed, we first observe that for any $E \in \mathcal{A}$, we have $\nu^+(E) = \int_E \chi_P d\,|\nu|$ and $\nu^-(E) = \int_E \chi_N d\,|\nu|$. Consequently, we get

$$\begin{aligned}\int_X |f|\, d\nu^+ + \int_X |f|\, d\nu^- &= \int_X |f|\, \chi_P d\,|\nu| + \int_X |f|\, \chi_N d\,|\nu| \\ &= \int_X |f|\, (\chi_P + \chi_N) d\,|\nu| \\ &= \int_X |f| \cdot 1 d\,|\nu| \\ &= \int_X |f|\, d\,|\nu|\,, \end{aligned} \tag{11.2}$$

as required. Hence we conclude by Eqns (11.1) and (11.2).

2. Let $\mathcal{Z} := \{\left|\int_E f d\nu\right| \mid |f| \leq 1\}$. We first see that for any measurable $f : X \to \mathbb{C}$ with $|f| \leq 1$, we have the following by item 1 above

$$\begin{aligned}\left|\int_E f d\nu\right| &\leq \int_E |f|\, d\,|\nu| \\ &\leq \int_E d\,|\nu| \\ &\leq |\nu|\,(E). \end{aligned}$$

Hence, $\sup \mathcal{Z} \leq |\nu|\,(E)$.

For the converse, we wish to show that $|\nu|\,(E) \leq \sup \mathcal{Z}$. If $\sup \mathcal{Z} = \infty$, then there is nothing to be shown. So we may assume $\sup \mathcal{Z} < \infty$. As the constant function 1 is in the collection, therefore

$$|\nu(E)| \leq \sup \mathcal{Z} < \infty. \tag{11.3}$$

In order to show $|\nu|(E) \leq \sup \mathcal{Z}$, it suffices to find a measurable function $f : X \to \mathbb{C}$ such that $|f| \leq 1$ and $|\nu|(E) \leq \left|\int_E f d\nu\right|$. Indeed, denoting by $X = P \amalg N$ to be the Hahn-decomposition of X obtained by ν, we consider $f = \chi_P - \chi_N$. Clearly, image of f is $\{-1, 0, 1\}$ as $A \cap B = \emptyset$, hence $|f| \leq 1$. Moreover, we observe that

$$\begin{aligned}\left|\int_E f d\nu\right| &= \left|\int_E f d\nu^+ - \int_E f d\nu^-\right| \\ &= \left|\int_E (\chi_P - \chi_N) d\nu^+ - \int_E (\chi_P - \chi_N) d\nu^-\right|. \end{aligned} \tag{11.4}$$

By Eq. (11.3), we deduce that $\nu^+(E)$ and $\nu^-(E)$ are finite. Furthermore, over E we have that χ_P and χ_N are both in $\mathcal{L}^1(\nu^+)$ and $\mathcal{L}^1(\nu^-)$. With this, we may continue Eq. (11.4) as follows

$$\begin{aligned}\left|\int_E f d\nu\right| &= \left|\int_E \chi_P d\nu^+ - \int_E \chi_N d\nu^+ - \int_E \chi_P d\nu^- + \int_E \chi_N d\nu^-\right| \\ &= \left|\int_E \chi_P d\nu^+ - 0 - 0 + \int_E \chi_N d\nu^-\right| \\ &= \nu^+(E \cap P) + \nu^-(E \cap N) \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E). \end{aligned}$$

where in the second equality we have used the fact the fact that $\nu^+(E) := \nu(E \cap P)$ and $\nu^-(E) := \nu(E \cap N)$. This shows that for some $f : X \to \mathbb{C}$ measurable with $|f| \leq 1$ we have $\left|\int_E f d\nu\right| = |\nu|(E)$, which consequently shows that $|\nu|(E) \leq \sup \mathcal{Z}$. This completes the proof. □

Example 25.1.18.12. We wish to find those signed spaces $(X, \mathcal{A}, \nu)$ which satisfies property 1 below. Further, we also wish to find those which satisfies 2 as below:

1. For c the counting measure on $(X, \mathcal{A})$, we have $c \ll \nu$.
2. For $x_0 \in X$ and the Dirac measure δ_{x_0}, we have $\delta_{x_0} \ll \nu$.

1. Let $E \in \mathcal{A}$. We know that $c(E) = 0$ iff $E = \emptyset$. Consequently, if $\nu(E) = 0$, then $c(E) = 0$ iff $E = \emptyset$. That is, $\nu(E) = 0$ iff $E = \emptyset$. Hence all those signed spaces $(X, \mathcal{A}, \nu)$ whose only null set is $\emptyset$ can only be such that $c \ll \nu$.

2. Let $E \in \mathcal{A}$. We know that $\delta_{x_0}(E) = 0$ iff $x_0 \notin E$. Thus if $\nu(E) = 0$ and $\delta_{x_0} \ll \nu$, then $x_0 \notin E$. Hence, $(X, \mathcal{A}, \nu)$ is a signed space such that all its null sets does not contain x_0. This completes the characterizations.

Lemma 25.1.18.13. *Let $(X, \mathcal{A}, \nu)$ be a signed space. Then,*

1. *If $\{E_n\} \subseteq \mathcal{A}$ be an increasing collection of measurable sets, then*

$$\nu\left(\bigcup_n E_n\right) = \lim_{n\to\infty} \nu(E_n).$$

2. *If $\{E_n\} \subseteq \mathcal{A}$ be a decreasing collection of measurable sets such that $\nu(A_1)$ is finite, then*

$$\nu\left(\bigcap_n E_n\right) = \lim_{n\to\infty}\nu(E_n).$$

Proof. 1. Denote $F_1 = E_1$ and $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Observe that $\{F_n\}$ are disjoint, but

$$\bigcup_n E_n = \coprod_n F_n. \tag{9.1}$$

Now observe that $E_n = F_n \amalg E_{n-1}$. This is recursive relation, which when unravelled, yields

$$E_n = F_n \amalg F_{n-1} \amalg \cdots \amalg F_1.$$

Applying ν yields

$$\nu(E_n) = \sum_{k=1}^{n} \nu(F_k). \tag{9.2}$$

It follows from Eqns (9.1) and (9.2) that

$$\begin{aligned}
\nu\left(\bigcup_n E_n\right) &= \nu\left(\coprod_n F_n\right) \\
&= \sum_{k=1}^{\infty} \nu(F_k) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \nu(F_k) \\
&= \lim_{n\to\infty}\nu(E_n),
\end{aligned}$$

as needed.

2. Consider the sequence $F_1 = E_1$ and $F_n = E_1 \setminus E_n$ for $n \geq 2$. Note that $\{F_n\}$ is increasing. Hence by item 1, we have

$$\nu\left(\bigcup_n F_n\right) = \lim_{n\to\infty}\nu(F_n). \tag{9.3}$$

Now observe that

$$E_1 = F_n \amalg E_n$$

for each $n \in \mathbb{N}$. Hence, applying ν we yield

$$\nu(E_1) = \nu(F_n) + \nu(E_n).$$

As $\nu(E_1)$ is finite, therefore the RHS in above equation is finite. Consequently, each term in the above equation is finite. Hence we may write it as

$$\nu(E_1) - \nu(F_n) = \nu(E_n).$$

Taking $n \to \infty$ yields

$$\nu(E_1) - \mathrm{lim}_{n\to\infty}\nu(F_n) = \mathrm{lim}_{n\to\infty}\nu(E_n)$$

which by Eq. (9.3), yields

$$\nu(E_1) - \nu\left(\bigcup_n F_n\right) = \mathrm{lim}_{n\to\infty}\nu(E_n). \tag{9.4}$$

We now claim that if $A \in \mathcal{A}$ and $B \subseteq A$ in $\mathcal{A}$ is such that $\nu(B)$ is finite, then $\nu(A \setminus B) = \nu(A) - \nu(B)$. Indeed, we may write $A = (A \setminus B) \amalg B$ where $A \setminus B$ is measurable as well. Applying ν, we yield $\nu(A) = \nu(A \setminus B) + \nu(B)$. As $\nu(B)$ is finite, therefore we may subtract both sides by $\nu(B)$ to yield $\nu(A \setminus B) = \nu(A) - \nu(B)$, as desired.

Using the above proved statement on Eq. (9.4), we obtain

$$\begin{aligned}
\mathrm{lim}_{n\to\infty} &= \nu\left(E_1 \setminus \bigcup_n F_n\right) \\
&= \nu\left(E_1 \cap \bigcap_n F_n^c\right) \\
&= \nu\left(\bigcap_n E_1 \cap F_n^c\right) \\
&= \nu\left(\bigcap_n E_n\right),
\end{aligned}$$

as desired. □

Lemma 25.1.18.14. *Let (X, Σ, μ) be a measure space and $f, g : X \to [0, \infty)$ be two non-negative measurable functions such that $f(x)g(x) = 0$ for almost all $x \in X$. Suppose for each $E \in \Sigma$ we have*

$$\mu(E) = \int_E f d\mu.$$

Define for each $E \in \Sigma$

$$\nu(E) = \int_E g d\mu.$$

Then $\mu \perp \nu$.

Proof. We know that ν as defined is a positive measure. Let $N = \{f(x)g(x) \neq 0\}$. This is a null-set. Consequently, we wish to find A and B measurable subsets such that $X = A \amalg B$ with A being μ-null and B being ν-null.

Define $A = \{f(x) = 0\}$ and $B = \{g(x) = 0 \ \& \ f(x) \neq 0\}$. Observe that $X = A \amalg B \amalg N$. Let $X_1 = A \amalg N$ and $X_2 = B$. Consequently $X = X_1 \amalg X_2$. Now, for any measurable $A' \subseteq X_1$, we may write $A' = (A' \cap A) \amalg (A' \cap N)$

$$\mu(A') = \int_{A' \cap A} f d\mu + \int_{A' \cap N} f d\mu = \int_{A' \cap A} 0 d\mu + \int_{A' \cap N} f d\mu = 0 + 0 = 0$$

where the latter term is zero because it is an integral over a measure 0 subset. Similarly, for any measurable $B' \subseteq X_2$, we see that

$$\nu(B') = \int_{B'} g d\mu = \int_{B'} 0 d\mu = 0.$$

Hence we have shown that X_1 is μ-null and X_2 is ν-null, as required. □

Lemma 25.1.18.15. *Let $(X, \mathcal{A}, \nu)$ be a signed space. Then,*

1. *If $A \in \mathcal{A}$ is a positive set, then $B \subseteq A$ such that $B \in \mathcal{A}$ is also a positive set.*
2. *If $\{A_n\} \subseteq \mathcal{A}$ is a sequence of positive sets, then $\bigcup_n A_n$ is a positive set.*

Proof. 1. Pick any $C \subseteq B$. As $B \subseteq A$, therefore $C \subseteq A$. As A is positive, thus $\nu(C) \geq 0$, as needed.

2. Let $B_1 = A_1$ and $B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$. Then observe that $\{B_n\}$ is a disjoint sequence of sets in $\mathcal{A}$, each positive as well by item 1. Furthermore, observe that

$$\coprod_n B_n = \bigcup_n A_n.$$

Now pick any $E \subseteq \bigcup_n A_n$ and denote $E_n = E \cap B_n$. Then, since B_n are disjoint, thus so is $\{E \cap B_n\}$. Furthermore $E = \coprod_n E_n$. Hence, we obtain

$$\mu(E) = \mu\left(\coprod_n E_n\right) = \sum_n \mu(E_n).$$

As each B_n is a positive set, so $E_n = E \cap B_n$ is a positive set as well by item 1. Consequently, $\mu(E_n) \geq 0$ for all $n \in \mathbb{N}$. Hence, from above, we deduce that

$$\mu(E) = \sum_n \mu(E_n) \geq 0,$$

as needed. □

25.1.19 Elements of L^p spaces

Let (X, M, m) be a measure space. The $L^p(X)$ is the set of all measurable maps $f : X \to \mathbb{C}$ such that $(\int_X |f|^p \, dm)^{1/p} < \infty$. This is a complete normed linear space for $1 \leq p \leq \infty$, where the triangle inequality is also sometimes called Minkowski's inequality, which follows from Hölder's inequality, which say that if $f \in L^p, g \in L^{p^*}$, then $fg \in L^1$ and $\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p^*}}$ where $1/p + 1/p^* = 1$. Egoroff's theorem tells us that a pointwise convergence of a sequence of measurable functions in a finite measure space is almost uniformly convergent, except on a set of finite measure and this set's measure can be chosen as small as we want. Further, convergence in L^p norm implies that there is a subsequence which converges pointwise. The L^∞ norm is also defined. It is given by the essential supremum of the measurable function f:

$$\|f\|_{L^\infty} := \inf\{M > 0 \mid m(\{|f(x)| \geq M\}) = 0\}.$$

On any finite measure space X, we further have that $L^q(X) \hookrightarrow L^p(X)$ for all $\infty \geq q \geq p \geq 1$.

25.2 Banach spaces

We now discuss some techniques of normed linear and Banach spaces.

25.2.1 Definitions and examples

Remark 25.2.1.1. a) We claim that any linear space could be normed. Let X be a linear space and $\{b_j\}$ be a Hamel basis. Then for each $x \in X$ there are unique finitely many non-zero elements $c_{x_1}, \dots, c_{x_k} \in \mathbb{K}$ such that $x = c_{x_1} b_{j_1} + \dots c_{x_k} b_{j_k}$. Define the following map

$$\begin{aligned} \| - \| : X &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto \max\{|c_{x_1}|, \dots, |c_{x_k}|\}. \end{aligned}$$

We claim that $\| - \|$ is a norm. Indeed, if $\|x\| = 0$, then $c_{x_i} = 0$ for all $i = 1, \dots, k$. Consequently, $x = 0$. If $x = 0$, then it is clear by uniqueness of c_{x_i} that all $c_{x_i} = 0$.

Consider $c \in \mathbb{K}$ and $x \in X$. Then $\|cx\| = \max |cc_{x_1}|, \dots, |cc_{x_k}| = |c| \max\{|c_{x_1}|, \dots, |c_{x_k}|\} = |c| \, \|x\|$.

We finally wish to show triangle inequality. Pick $x, y \in X$. Then, (we allow c_{x_i} and c_{y_i} to be zero)

$$\begin{aligned} \|x + y\| &= \max\{|c_{x_1} + c_{y_1}|, \dots, |c_{x_k} + c_{y_k}|\} \\ &\leq \max\{|c_{x_1}| + |c_{y_1}|, \dots, |c_{x_k}| + |c_{y_k}|\} \\ &\leq \max\{|c_{x_1}|, \dots, |c_{x_k}|\} + \max\{|c_{y_1}|, \dots, |c_{y_k}|\} \\ &= \|x\| + \|y\|. \end{aligned}$$

Hence every linear space is normable.

b) We claim that not all metric on a linear space X comes from a norm on X. Indeed, consider the following metric:

$$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R}_{\geq 0} \\ (x,y) &\longmapsto \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases} \end{aligned}$$

Indeed it is clear that $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$. For triangle inequality, we need only consider the case when $x \neq y$ and to show that for any $z \in X$ we have

$$1 = d(x,y) \leq d(x,z) + d(y,z).$$

It is clear that we need only show that $d(x,z)$ and $d(y,z)$ are both not simultaneously 0. Indeed, if both are simultaneously 0, then $x = z = y$, a contradiction. Hence d is indeed a metric.

We claim that d is not induced by any norm. Indeed, assume to the contrary it is induced by a norm $\| - \|$. It follows that

$$\|x\| = d(x,0) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Since $\| - \|$ is a norm, it follows that for any $c \neq 1$ in $\mathbb{K}$ and $x \neq 0$ in X, we must have $\|cx\| = 1$ as $cx \neq 0$. We now have the following contradiction

$$1 = \|cx\| = |c| \, \|x\| = |c| \neq 1.$$

This completes the proof.

Remark 25.2.1.2. We wish to show that the following are equivalent for a linear space X with a function $\| : \|X \to \mathbb{R}_{\geq 0}$ satisfying $\|x\| = 0$ iff $x = 0$ and $\|cx\| = |c| \, \|x\|$ for all $c \in \mathbb{K}$ and $x \in X$:

1. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.
2. The closed unit ball $B_1[0] = \{x \in X \mid \|x\| \leq 1\}$ is convex.

(1. $\Rightarrow$ 2.) Pick $x, y \in B_1[0]$ and $c \in [0,1]$. We wish to show that $cx + (1-c)y \in B_1[0]$. Indeed, since $\|x\|, \|y\| \leq 1$, therefore we have

$$\begin{aligned} |cx + (1-c)y| &\leq |c| \, \|x\| + |1-c| \, \|y\| \\ &\leq c + (1-c) \\ &= 1. \end{aligned}$$

(2. $\Rightarrow$ 1.) Pick $x, y \in X$. If any of the x or y is 0, then triangle inequality is immediate. Hence we may assume x and y are both not 0. Then $\frac{x}{\|x\|}, \frac{y}{\|y\|} \in B_1[0]$. Let $c = \frac{\|x\|}{\|x\|+\|y\|}$ so that $1 - c = \frac{\|y\|}{\|x\|+\|y\|}$. It is clear that $c \in [0,1]$. By convexity of $B_1[0]$, it follows that

$$c\frac{x}{\|x\|} + (1-c)\frac{y}{\|y\|} \in B_1[0].$$

But we have

$$c\frac{x}{\|x\|} + (1-c)\frac{y}{\|y\|} = \frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|}$$

hence the RHS above is in $B_1[0]$. Taking norm, we see

$$\|\frac{x}{\|x\| + \|y\|} + \frac{y}{\|x\| + \|y\|}\| = \frac{\|x+y\|}{\|x\| + \|y\|} \leq 1$$

from which we get

$$\|x+y\| \leq \|x\| + \|y\|,$$

as required.

Example 25.2.1.3. Consider $C[a,b]$ be the $\mathbb{R}$-vector space of all continuous functions on $[a,b]$. Define for any $1 \leq p < \infty$

$$\|f\|_p := \left(\int_a^b |f(t)|^p \, dt\right)^{\frac{1}{p}}.$$

a) We wish to show that $\|-\|_p$ is a norm on $C[a,b]$. Indeed, if $f \in C[a,b]$ such that $\|f\|_p = 0$, then we have

$$\int_a^b |f(t)^p| \, dt = 0.$$

We wish to show that $f = 0$. Suppose not, so that $f(t_0) \neq 0$ at a point $t_0 \in [a,b]$. If $t_0 = a$ or b, then by continuity there is a point in (a,b) where f is non-zero. Replace t by that point in (a,b). It follows by continuity that there exists $\delta > 0$ such that f is non-zero on $I = [t_0 - \delta, t_0 + \delta] \subseteq (a,b)$. Let $m = \min_{t \in I} |f(t)|^p > 0$ which exists as f is continuous on compact I and $f \neq 0$ on I. Then

$$0 = \int_a^b |f(t)|^p \, dt \geq \int_{t_0-\delta}^{t_0+\delta} m dt = m \cdot (2\delta) > 0,$$

a contradiction. It follows that $f = 0$ on $[a,b]$.

We now wish to show triangle inequality. For this, we invoke the fact that $C[a,b]$ is contained inside the linear space $L^p[a,b]$ of $\mathbb{R}$-valued Lebesgue measurable functions on $[a,b]$. Moreover, the function

$$\|f\|_p := \left(\int_{[a,b]} |f|^p \, dm\right)^{1/p}$$

for $f \in L^p[a,b]$ defines a norm. Moreover if f is continuous, then the above Lebesgue integral on $[a,b]$ agrees with the usual Riemann integral. So we may conclude that there is an inclusion of linear spaces

$$(C[a,b], \|-\|_p) \subseteq (L^p[a,b], \|-\|_p).$$

We know that $(L^p[a,b], \|-\|_p)$ forms a normed linear space, where triangle inequality is established by Minkowski's inequality. Using the same theorem on the subspace $(C[a,b], \|-\|_p)$, we get the desired result.

b) We claim that $(C[0,2], \|-\|_1)$ is not complete. It suffices to show a Cauchy sequence which is not convergent. Indeed consider $f_n(x)$ as follows:

$$f_n(x) = \begin{cases} x^n & \text{if } x \in [0,1] \\ 1 & \text{if } x \in (1,2]. \end{cases}$$

We first claim that (f_n) is Cauchy in $C[0,2]$. Indeed, for $n \geq m$, we have

$$\begin{aligned} \|f_n - f_m\|_1 &= \int_0^2 |f_n(x) - f_m(x)|\,dx \\ &= \int_0^1 |x^n - x^m|\,dx \\ &= \int_0^1 x^m - x^n dx \\ &= \int_0^1 x^m dx - \int_0^1 x^n dx \\ &= \frac{1}{m+1} - \frac{1}{n+1} \\ &\leq \frac{1}{m+1}. \end{aligned}$$

So for a fixed $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $1/N < \epsilon$. Then for all $n, m \geq N$, we have

$$\|f_n - f_m\|_1 \leq \frac{1}{m+1} \leq \frac{1}{N+1} < \epsilon,$$

as needed. Next, we claim that (f_n) doesn't converge in $C[0,2]$. Indeed, it would suffice to show that it converges in $L^1[0,2]$ to a non-continuous function. Indeed, consider the following simple function

$$f = \chi_{[1,2]}.$$

This is not continuous in $[0,2]$. We claim that $f_n \to f$ in $L^1[0,2]$. Indeed, we have

$$\begin{aligned} \|f_n - f\|_1 = \int_{[0,2]} |f_n - f|\,dm &= \int_{[0,1]} |f_n - f|\,dm + \int_{[1,2]} |f_n - f|\,dm \\ = \int_{[0,1]} |f_n - f|\,dm &= \int_0^1 |x^n|\,dx, \end{aligned}$$

where the last equality comes from Riemann and Lebesgue integrals being equal on compact intervals for Riemann integrable functions. Consequently, we have

$$\|f_n - f\|_1 = \frac{1}{n+1}$$

which converges to 0 as $n \to \infty$. Hence in $L^1[0,2]$, $f_n \to f$. As $C[0,2] \subseteq L^1[0,2]$ with the given norm, it follows that $(f_n) \subseteq C[0,2]$ does not converge in $C[0,2]$.

Example 25.2.1.4. Let $X = (C[0,1], \|\cdot\|_\infty)$. We wish to calculate the following:

1. $d(f_1, C)$ where $f_1(t) = t$ and C is the linear subspace of all constant functions,
2. $d(f_2, P)$ where $f_2(t) = t^2$ and P is the linear subspace of polynomials of degree at most 1.

1. We claim that $d(f_1, C) = 1/2$. Indeed, we have

$$\begin{aligned} d(f_1, C) &= \inf_{c \in C} \|f_1 - c\| = \inf_{c \in C} \sup_{t \in [0,1]} |t - c| \\ &= \inf_{c \in C} \begin{cases} c & \text{if } \frac{1}{2} \leq c < \infty \\ 1 - c & \text{if } -\infty < c < \frac{1}{2}. \end{cases} \\ &= \frac{1}{2}, \end{aligned}$$

as needed.

2. We claim that $d(f_2, P) = 1/8$. Pick any $at + b \in P$ for $a, b \in \mathbb{R}$. We first show that

$$\sup_{t \in [0,1]} \left| t^2 - at - b \right| = \max \left\{ -b, 1 - a - b, \frac{a^2}{4} + b \right\}. \tag{$*$}$$

Indeed, consider the discriminant $a^2 + 4b$ of $f(t) = t^2 - at - b$. There are two cases to be had here:

1. If $a^2 + 4b \leq 0$: Then the maximum of $|f(t)|$ is equal to that of $f(t)$ and is achieved only on the boundary at $t = 0$ or 1 because $f(t) \geq 0$ for all $t \in [0,1]$. Consequently, $\sup_{t \in [0,1]} |f(t)| = -b$ or $1 - a - b$.
2. If $a^2 + 4b > 0$: Then the maximum of $|f(t)|$ is either on boundary at $t = 0, 1$ or at the point of minima of $f(t)$ at $t = a/2$, which thus becomes a point of maxima for $|f(t)|$. It follows that $\sup_{t \in [0,1]} |f(t)| = -b, 1 - a - b$ or $\frac{a^2}{4} + b$.

These two cases shows the claim in Eqn $(*)$.

Consider now $f(a, b) = \max \left\{ -b, 1 - a - b, \frac{a^2}{4} + b \right\}$ as a function $f : \mathbb{R}^2 \to \mathbb{R}$. We wish to find $\inf_{(a,b) \in \mathbb{R}^2} f(a, b)$. First we observe the following three regions:

1. The region R_1 : This is
$$R_1 = \{(a, b) \in \mathbb{R}^2 \mid f(a, b) = -b\}.$$
2. The region R_2 : This is
$$R_2 = \{(a, b) \in \mathbb{R}^2 \mid f(a, b) = 1 - a - b\}.$$
3. The region R_3 : This is
$$R_3 = \left\{ (a, b) \in \mathbb{R}^2 \mid f(a, b) = \frac{a^2}{4} + b \right\}.$$

We now analyze bounds on a point $(a, b) \in R_i$ as follows.

1. If $(a,b) \in R_1$: Then we have

$$\begin{aligned} -b &> 1-a-b \\ -b &> a^2/4+b \end{aligned}$$

solving which, we get bounds

$$\begin{aligned} a &< 1 \\ b &< -\frac{a^2}{8}. \end{aligned}$$

Hence, to minimize b, we need to maximize a, thus to get that $b < -1/8$. So we have $(a,b) = (1,-1/8)$ as a point of minima for $-b$.

2. If $(a,b) \in R_2$: Then we have

$$\begin{aligned} 1-a-b &> -b \\ 1-a-b &> \frac{a^2}{4}+b \end{aligned}$$

solving which we get bounds

$$\begin{aligned} a &< 1 \\ b &< \frac{1}{2}-\frac{a}{2}-\frac{a^2}{8}. \end{aligned}$$

Hence to minimize $1-a-b$, we have to maximize a and b. Doing so yields $a = 1$ and $b = -1/8$. Hence $(a,b) = (1,-1/8)$ is a point of minima for $1-a-b$.

3. If $(a,b) \in R_3$: Then we have

$$\begin{aligned} \frac{a^2}{4}+b &< -b \\ \frac{a^2}{4}+b &< 1-a-b \end{aligned}$$

solving which, we get bounds

$$\begin{aligned} b &> -\frac{a^2}{8} \\ b &> -\frac{a^2}{8}-\frac{a}{2}+\frac{1}{2}. \end{aligned}$$

Hence to minimize $\frac{a^2}{4}+b$, we have to minimize b and a. Doing so, we obtain $b = -a^2/8$ which thus yields

$$a > 1.$$

Hence to minimize a, we have to take $a = 1$. Consequently, $(a,b) = (1,-1/8)$ is a point of minima for $a^2/4+b$.

From all the three cases above, we see that f minimizes at the point $(a,b) = (1,-1/8)$. Indeed, we see that $(1,-1/8) \in R_1 \cap R_2 \cap R_3$ as all three functions $-b, 1-a-b$ and $a^2/4+b$ are equal at it. Consequently, the $\inf_{(a,b)\in\mathbb{R}^2} f(a,b) = 1/8$, as required.

25.2.2 Properties

Proposition 25.2.2.1. *Let X be a normed linear space. The following are equivalent:*

1. *X is a Banach space.*
2. *$S^1(X) = \{x \in X \mid \|x\| = 1\}$ is a complete subset of X.*

Proof. content... □

Proposition 25.2.2.2. *Let X be a normed linear space. Then the following are equivalent:*

1. *X is a Banach space.*
2. *Absolutely convergent series in X are convergent in X.*

Proof. 1. $\Rightarrow$ 2. Pick an absolutely convergent series $\sum_n x_n$ in X so that

$$\sum_n \|x_n\| < \infty.$$

It follows that $T_n = \sum_{k=1}^n \|x_k\|$ is a Cauchy sequence in $\mathbb{R}$. We wish to show that $\sum_n x_n$ converges in X. It suffices to show that the sequence $S_n = \sum_{k=1}^n x_k$ converges in X. We reduce to showing that (S_n) is Cauchy. Fix $\epsilon > 0$. For any $n \geq m$, we have

$$\begin{aligned}
\|S_n - S_m\| &= \|x_{m+1} + \cdots + x_n\| \\
&\leq \|x_{m+1}\| + \cdots + \|x_n\| \\
&= \left| \left(\sum_{k=1}^{n} \|x_k\| \right) - \left(\sum_{k=1}^{m} \|x_k\| \right) \right| \\
&= |T_n - T_m| < \epsilon
\end{aligned}$$

some $N \in \mathbb{N}$ and $n, m \geq N$ since (T_n) is Cauchy in $\mathbb{R}$. This shows that (S_n) is Cauchy, as required.

2. $\Rightarrow$ 1. Pick a Cauchy sequence $(x_n) \subseteq X$. We wish to show that there is a convergent subsequence of (x_n). We first find a subsequence of (x_n) which is better behaved. Indeed, by Cauchy condition, we find for each $k \geq 0$ a positive integer $N_k \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \frac{1}{2^k}$$

for all $n, m \geq N_k$. We may assume N_k to be the least such possible by well-ordering on $\mathbb{N}$. Then we see that $N_{k+1} \geq N_k$ by minimality hypothesis. Thus consider the subsequence (x_{N_k}) of (x_n). Observe that

$$\|x_{N_{k+1}} - x_{N_k}\| < \frac{1}{2^k}$$

as $N_{k+1}, N_k \geq N_k$. We replace (x_n) by the subsequence (x_{N_k}) so that we may assume

$$\|x_{n+1} - x_n\| < \frac{1}{2^n} \; \forall n \in \mathbb{N}. \tag{3}$$

We now find the limit to which (x_n) converges. Indeed, define the following sequence in X:

$$\begin{aligned} y_{n-1} &= \sum_{k=1}^{n-1} x_{k+1} - x_k \\ &= x_n - x_1. \end{aligned}$$

We claim that $\sum_n x_{n+1} - x_n$ is an absolutely convergent series. Indeed, denote

$$\begin{aligned} S_{n-1} &:= \sum_{k=1}^{n-1} \|x_{k+1} - x_k\| \\ &\leq \sum_{k=1}^{n-1} \frac{1}{2^k} \end{aligned}$$

where the latter bound follows from Eqn. (3). Then, we see that for any $n \in \mathbb{N}$

$$S_n \leq \sum_{k=1}^{n} \frac{1}{2^k} < \sum_{k=1}^{\infty} \frac{1}{2^k} = M < \infty. \tag{4}$$

That is, (S_n) is a monotonically increasing positive bounded sequence in $\mathbb{R}$, therefore (S_n) is convergent. This shows that the series $\sum_n x_{n+1} - x_n$ is absolutely convergent. By our hypothesis, it follows that $\sum_n x_{n+1} - x_n$ is convergent in X. That is, the sequence

$$y_{n-1} = \sum_{k=1}^{n-1} x_k$$

of partial sums is convergent in X. But since $y_{n-1} = x_n - x_1$, it follows that (x_n) is a convergent sequence in X, as required. $\square$

25.2.3 Bases & quotients

Lemma 25.2.3.1. *If X is a normed linear space with a Schauder basis, then X is separable.*

Proof. Let $(b_n) \subseteq X$ be a Schauder basis. Consider the following subset

$$D = \left\{ \sum_{k=1}^{n} q_k b_k \mid q_k \in E,\ n \in \mathbb{N} \right\}$$

where $E \subseteq \mathbb{K}$ is a countable dense subset. It is clear that D is countable. We claim that D is dense in X.

Pick any point $x \in X$. Since (b_n) is a Schauder basis, there exists $(c_k) \subseteq \mathbb{K}$ such that

$$x = \sum_{k=1}^{\infty} c_k b_k$$

where the series converges in X. Pick a ball $B_\epsilon(x)$ around x. We wish to show that $B_\epsilon(x) \cap D \neq \emptyset$. Indeed, consider $N \in \mathbb{N}$ large enough such that

$$\|x - \sum_{k=1}^{N} c_k b_k\| < \frac{\epsilon}{2}. \tag{9}$$

Moreover, for each $k = 1, \dots, N$, consider $q_k \in E$ such that

$$|c_k - q_k| < \frac{\epsilon}{2 \cdot 2^k \|b_k\|} \tag{10}$$

which exists by density of E in $\mathbb{K}$. Hence, we have by Eqns (9) and (10) the following inequalities:

$$\begin{aligned}
\|x - \sum_{k=1}^{N} q_k b_k\| &\leq \|x - \sum_{k=1}^{N} c_k b_k\| + \|\sum_{k=1}^{N} (c_k - q_k) b_k\| \\
&< \frac{\epsilon}{2} + \sum_{k=1}^{N} |c_k - q_k| \, \|b_k\| \\
&< \frac{\epsilon}{2} + \frac{1}{2} \sum_{k=1}^{N} \frac{\epsilon}{2^k} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \left(1 - \frac{1}{2^N}\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

as needed. This shows that $\sum_{k=1}^{N} q_k b_k \in B_\epsilon(x)$, that is D is dense in X. □

Proposition 25.2.3.2 (2 out of 3 property)**.** *Let X be a normed linear space and $Y \subseteq X$ be a closed linear subspace. Then,*

1. *X, Y Banach implies X/Y Banach.*
2. *$X, X/Y$ Banach implies Y Banach.*
3. *$Y, X/Y$ Banach implies X Banach.*

Proof. 1. We have done in class that if X is Banach then for any closed linear subspace Y, X/Y is Banach.

2. We need the following lemma here:

Lemma 25.2.3.3. *Let X be a Banach space and $Y \subseteq X$ be a linear subspace. Then the following are equivalent:*

1. *Y is complete.*
2. *Y is closed.*

Proof of Lemma 25.2.3.3. 1. $\Rightarrow$ 2. Take $(y_n) \subseteq Y$ be a convergent sequence in X such that it converges to $x \in X$. We wish to show that $x \in Y$. Indeed, as $(y_n) \subseteq Y$ is convergent, so it is Cauchy. Since Y is complete, it follows that (y_n) converges to a point in Y. By uniqueness of point of convergence in a Hausdorff space, $x \in Y$.

2. $\Rightarrow$ 1. Pick a Cauchy sequence $(y_n) \subseteq Y$. We wish to show that it converges in Y. Indeed, (y_n) as a sequence in X is Cauchy and thus by completeness of X, we deduce that $y_n \to x$ in X. But since Y is closed, therefore by uniqueness of point of convergence, we must have $x \in Y$, as required. □

Since X is Banach and Y is closed, it follows from Lemma 25.2.3.3 that Y is complete.

3. Pick a Cauchy sequence $(x_n) \subseteq X$. We wish to show that that it converges. We have a sequence $(x_n + Y) \subseteq X/Y$. We first claim that $(x_n + Y)$ is Cauchy. Indeed, we have

$$\begin{aligned}\|x_n - x_m + Y\| &= \inf_{y \in Y} \|x_n - x_m + y\| \\ &\leq \|x_n - x_m\| < \epsilon\end{aligned}$$

for all $n, m \geq N$ for some $N \in \mathbb{N}$ as $(x_n) \subseteq X$ is Cauchy. As X/Y is Banach, it follows that $(x_n + Y) \to (x + Y)$ in X/Y. Consequently, for a fixed $\epsilon > 0$, we get

$$\|x_n - x + Y\| = \inf_{y \in Y} \|x_n - x + y\| < \epsilon/2 < \epsilon$$

for all $n \geq N$ for some $N \in \mathbb{N}$. It follows from above that there is a sequence $(y_n) \subseteq Y$ such that

$$\|x_n - x + y_n\| \leq \epsilon/2 < \epsilon. \tag{1}$$

We claim that $(y_n) \subseteq Y$ is Cauchy. Indeed, we first see from Eqn. (1) that

$$\|x_n + y_n - x\| < \epsilon$$

for all $n \geq N$. Consequently, the sequence $(x_n + y_n) \subseteq X$ converges to $x \in X$. Hence, $(x_n + y_n) \subseteq X$ is Cauchy, from which we get $N \in \mathbb{N}$ such that

$$\|x_n + y_n - x_m - y_m\| = \|x_n - x_m - (y_m - y_n)\| < \epsilon$$

for each $n, m \geq N$. We may write by triangle inequality the following:

$$|\|x_n - x_m\| - \|y_n - y_m\|| \leq \|x_n - x_m - (y_m - y_n)\| < \epsilon$$

so that

$$\|y_n - y_m\| < \|x_n - x_m\| + \epsilon \tag{2}$$

for all $n, m \geq N$. As $(x_n) \subseteq X$ is Cauchy, so for some $N' \in \mathbb{N}$ we have $\|x_n - x_m\| < \epsilon$ for all $n, m \geq N'$. Replacing N by maximum of N' and N, we obtain from Eqn. (2) the following:

$$\|y_n - y_m\| < 2\epsilon \; \forall n, m \geq N.$$

This shows that $(y_n) \subseteq Y$ is Cauchy. As Y is complete, therefore $y_n \to y \in Y$. As $x_n + y_n \to x$ in X, therefore $x_n \to x - y$ in X, thus showing that X is complete. □

Proposition 25.2.3.4. *The Banach space ℓ^p is separable for all $1 \leq p < \infty$.*

Proof. Recall that

$$\ell^p = \left\{ (x_n) \mid x_n \in \mathbb{K} \ \& \ \sum_n |x_n|^p < \infty \right\}$$

with the norm being $\|(x_n)\|_p = \left(\sum_n |x_n|^p\right)^{1/p}$. Let $D \subseteq \mathbb{K}$ be a countable dense subset of $\mathbb{K}$ (which exists as $\mathbb{R}$ and $\mathbb{C}$ are separable in their usual topology). Using D we will construct a countable dense subset $F \subseteq \ell^p$. Indeed, consider the following subset of ℓ^p:

$$F = \bigcup_{N \geq 0} F_N$$

where

$$F_N = \{(x_n) \in \ell^p \mid x_n \in D,\ x_n = 0\ \forall n \geq N\}.$$

We see that F_N is countable as finite product of countable sets is countable and thus F is a countable union of countable sets, showing that F is countable. We next claim that F is dense in ℓ^p.

Pick any open set $B_r(y) \subseteq \ell^p$. Note that

$$B_r(y) = \left\{ (x_n) \in \ell^p \mid \sum_n |x_n - y_n|^p < r^p \right\}.$$

As $y = (y_n) \in \ell^p$, therefore $\sum_n |y_n|^p = M < \infty$. Now observe that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=n}^{\infty} |y_n|^p < \epsilon \tag{5}$$

for all $n \geq N$. As $D \subseteq \mathbb{K}$ is dense and each $y_n \in \mathbb{K}$, therefore choose

$$x_n \in B_{r_n}(y_n) \cap D \subseteq \mathbb{K}$$

where $r_n = \frac{r}{2^{\frac{n+1}{p}}}$ for all $n \in \mathbb{N}$. Hence,

$$|x_n - y_n|^p < \frac{r^p}{2 \cdot 2^n}$$

for all $n \in \mathbb{N}$. As $\sum_{n=1}^{\infty} r^p/2^{n+1} = r^p/2$, therefore

$$\sum_{n \geq 1} |x_n - y_n|^p < \frac{r^p}{2}. \tag{6}$$

This shows that the element $(x_n) \in B_r(y) \subseteq \ell^p$.

Now, fix $\epsilon > 0$ so that there exists $K \in \mathbb{N}$ large enough using Eqn. (5) such that

$$\sum_{n=K}^{\infty} |y_n|^p < \epsilon. \tag{7}$$

Using Eqn. (6) and (7), we can write

$$\begin{aligned}\sum_{n=1}^{K-1}|x_n-y_n|^p+\sum_{n=K}^{\infty}|y_n|^p &< \sum_{n=1}^{K-1}\frac{r^p}{2\cdot 2^n}+\sum_{n=K}^{\infty}|y_n|^p\\ &< \frac{r^p}{2}\left(1-\frac{1}{2^K}\right)+\sum_{n=K}^{\infty}|y_n|^p\\ &< \frac{r^p}{2}\left(1-\frac{1}{2^K}\right)+\epsilon\\ &< \frac{r^p}{2}\left(1-\frac{1}{2^N}\right)+\epsilon\end{aligned}$$

for all $N \geq K$. So let $N \to \infty$ so that we obtain

$$\sum_{n=1}^{K-1}|x_n-y_n|^p+\sum_{n=K}^{\infty}|y_n|^p \leq \frac{r^p}{2}+\epsilon.$$

Thus taking $\epsilon = \frac{r^p}{4}$, we get $\tilde{K} \in \mathbb{N}$ such that

$$\sum_{n=1}^{\tilde{K}-1}|x_n-y_n|^p+\sum_{n=\tilde{K}}^{\infty}|y_n|^p \leq \frac{3r^p}{4} < r^p. \tag{8}$$

Define $\tilde{x} \in \ell^p$ as follows:

$$\tilde{x}_n = \begin{cases} x_n & \text{if } n \leq \tilde{K}-1 \\ 0 & \text{if } n \geq \tilde{K}. \end{cases}$$

Then $\tilde{x} \in F_{\tilde{K}}$ and by Eqn. (8) it follows that

$$\sum_{n=1}^{\infty}|\tilde{x}_n-y_n|^p < r^p.$$

Consequently, $\tilde{x} \in F \cap B_r(y)$, as needed. □

Example 25.2.3.5 (ℓ^∞ is not separable)**.** We wish to show that ℓ^∞ does not have a Schauder basis. By Lemma 25.2.3.1, it suffices to show that ℓ^∞ is not separable. Suppose to the contrary that $D \subseteq \ell^\infty$ is a countable dense set. We will derive a contradiction to countability of D. Indeed, consider $\kappa = \{0,1\}$ and the subset $\kappa^\infty \subseteq \ell^\infty$ of all sequences formed by 1 and 0. Observe that κ^∞ is uncountable.

Pick any $x \in \kappa^\infty$. We first claim that $B_{1/2}(x) \cap \kappa^\infty = \{x\}$. Indeed, if $y \in B_{1/2}(x)$, then $\sup_n |x_n - y_n| < 1/2$. It follows that there exists $0 < \epsilon < 1/2$ such that

$$|x_n - y_n| < \epsilon \ \forall n \in \mathbb{N}.$$

As $x_n = 0$ or 1, therefore

$$\begin{cases} -\epsilon < y_n < \epsilon & \text{if } x_n = 0 \\ 1-\epsilon < y_n < 1+\epsilon & \text{if } x_n = 1. \end{cases} \tag{9}$$

Hence, if $y \in \kappa^\infty$, then by Eqn. (9) it follows that $y_n = x_n$ for all $n \in \mathbb{N}$ and thus $x = y$.

We next show that for $x \neq x' \in \kappa^\infty$, the open balls $B_{1/2}(x) \cap B_{1/2}(x') = \emptyset$. Since $x \neq x'$, we may assume WLOG that there exists $m \in \mathbb{N}$ such that $x_m = 0$ and $x'_m = 1$. Thus, if $y \in B_{1/2}(x) \cap B_{1/2}(x')$, then by Eqn. (9), it follows that

$$\begin{aligned} -\epsilon < y_m < \epsilon \\ 1-\epsilon < y_m < 1+\epsilon. \end{aligned}$$

Since $\epsilon = 1/2$, therefore the above inequalities give a contradiction. Hence $B_{1/2}(x) \cap B_{1/2}(x') = \emptyset$.

We now complete the proof. As $D \subseteq \ell^\infty$ is dense, therefore $D \cap B_{1/2}(x) \neq \emptyset$ for all $x \in \kappa^\infty$. Pick one $d_x \in D \cap B_{1/2}(x)$ for each $x \in \kappa^\infty$. By above two claims, it follows that we have an injective map

$$f : \kappa^\infty \hookrightarrow D,$$

but κ^∞ is uncountable and D is countable, a contradiction. This completes the proof.

25.2.4 Continuous linear transformations

Example 25.2.4.1. We wish to show that the inverse of a bounded linear operator may not be bounded. Indeed consider $X = (P[0,1]_1, \|\cdot\|_{\text{sup}})$ to be the normed linear space of all polynomials whose least degree term is of degree 1. Similarly, consider $Y = (P[0,1]_2, \|\cdot\|_{\text{sup}})$ to be the normed linear space of all polynomials whose least degree term is of degree 2. We consider the following linear map

$$\begin{aligned} T : X &\longrightarrow Y \\ p &\longmapsto \int p dx \end{aligned}$$

so that if $p(x) = a_n x^n + \dots a_1 x$, then $T(p) = \frac{a_n}{n+1}x^{n+1} + \dots + \frac{a_1}{2}x^2$. We claim that T is bounded. Indeed,

$$\begin{aligned} \|T(p)\| &= \|\frac{a_n}{n+1}x^{n+1} + \dots + \frac{a_1}{2}x^2\| \\ &= \sup_{x\in[0,1]} \left|\frac{a_n}{n+1}x^{n+1} + \dots + \frac{a_1}{2}x^2\right| \\ &= \sup_{x\in[0,1]} \left|x \cdot \left(\frac{a_n}{n+1}x^n + \dots + \frac{a_1}{2}x\right)\right| \\ &\leq \sup_{x\in[0,1]} |x| \sup_{x\in[0,1]} \left|\left(\frac{a_n}{n+1}x^n + \dots + \frac{a_1}{2}x\right)\right| \\ &\leq 1 \cdot \sup_{x\in[0,1]} |a_n x^n + \dots a_1 x| \\ &= \sup_{x\in[0,1]} |p(x)| \\ &= \|p\|. \end{aligned}$$

Thus indeed, T is a bounded linear transformation. We next claim that the following linear transform is an inverse of T:

$$\begin{aligned} U : Y &\longrightarrow X \\ q &\longmapsto q' \end{aligned}$$

so that if $q(x) = a_n x^n + \dots a_2 x^2$, then $U(q) = n a_n x^{n-1} + \dots + 2a_2 x$. Indeed, we see that

$$\begin{aligned} T \circ U(q) &= T\left(n a_n x^{n-1} + \dots + 2a_2 x\right) \\ &= n a_n \frac{x^n}{n} + \dots 2a_2 \frac{x^2}{2} \\ &= q. \end{aligned}$$

Similarly, for $p(x) = a_n x^n + \dots a_1 x$, we see that

$$\begin{aligned} U \circ T(p) &= U\left(\frac{a_n}{n+1} x^{n+1} + \dots + \frac{a_1}{2} x^2\right) \\ &= \frac{a_n}{n+1}(n+1)x^n + \dots + \frac{a_1}{2}(2)x \\ &= p. \end{aligned}$$

This shows that U is inverse of T. We now show that U is unbounded. Indeed,

$$\begin{aligned} \|U(x^n)\| &= \|n x^{n-1}\| \\ &= \sup_{x \in [0,1]} \left|n x^{n-1}\right| \\ &= n \cdot 1 \\ &= n \cdot \|x^n\|. \end{aligned}$$

This shows that for all $n \geq 2$, there exists $q_n(x) \in Y$ given by $q_n(x) = x^{n+1}$ such that

$$\|U(q_n)\| = n + 1 > n = n\|q_n\|,$$

making U unbounded. This completes the proof.

25.2.5 Isometries & norms

25.2.6 Finite dimensional Banach spaces

Compactness of the unit sphere determines the finite dimensionality of a Banach space.

Theorem 25.2.6.1. *Let X be a normed linear space. Then the following are equivalent:*

1. *X is a finite dimensional linear space.*
2. *$S^1(X) = \{x \in X \mid \|x\| = 1\}$ is a compact subspace.*
3. *$D(X) = \{x \in X \mid \|x\| \leq 1\}$ is a compact subspace.*

Proof. content... $\square$

25.2.7 Functionals and evaluations

Let X be a normed linear space. The evaluation map $X \to X^{**}$ is injective. Later, by an application of Hahn-Banach theorem, we would be able to deduce that this map is further norm preserving.

25.2.8 Miscellaneous applications

Example 25.2.8.1. We wish to construct an additive function $f : \mathbb{R} \to \mathbb{R}$ which is not continuous. Indeed, consider the Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and denote it by $\mathcal{B}$. We know that $\mathcal{B}$ is not finite. Observe that any additive map $f : \mathbb{R} \to \mathbb{R}$ is $\mathbb{Q}$-linear as

$$f\left(\frac{p}{q}x\right) = pf\left(\frac{1}{q}x\right)$$

and since $qf\left(\frac{1}{q}x\right) = f(x)$, thus,

$$f\left(\frac{p}{q}x\right) = \frac{p}{q}f(x).$$

Since any function $\mathcal{B} \to \mathbb{R}$ can be extended $\mathbb{Q}$-linearly to $\mathbb{R} \to \mathbb{R}$, therefore we now construct a function $f : \mathcal{B} \to \mathbb{R}$ and show that its $\mathbb{Q}$-linear extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ cannot be continuous at 0.

Pick any sequence $(b_n) \subseteq \mathcal{B}$ and consider the following sequence in $\mathbb{R}$

$$x_n = \frac{b_n}{n^{\lceil |b_n| \rceil + n}}$$

where $\lceil z \rceil$ is the ceiling function (smallest integer larger than z). Note that the denominator of x_n is a positive integer. Observe that $x_n \to 0$ as $n \to \infty$.

Define the following function $f : \mathcal{B} \to \mathbb{R}$:

$$f(b) = \begin{cases} n^{\lceil |b_n| \rceil + n} & \text{if } b = b_n \\ 1 & \text{else.} \end{cases}$$

Extend this function to a $\mathbb{Q}$-linear map $\tilde{f} : \mathbb{R} \to \mathbb{R}$, so that it is additive. We claim that $\tilde{f}$ is not continuous at 0. Indeed, we have $x_n \to 0$ as $n \to \infty$, but

$$\tilde{f}(x_n) = \tilde{f}\left(\frac{b_n}{n^{\lceil |b_n| \rceil + n}}\right) = \frac{1}{n^{\lceil |b_n| \rceil + n}}\tilde{f}(b_n) = \frac{1}{n^{\lceil |b_n| \rceil + n}}n^{\lceil |b_n| \rceil + n} = 1$$

and thus $\tilde{f}(x_n) = 1 \not\to \tilde{f}(0) = 0$ as $n \to \infty$, making $\tilde{f}$ discontinuous at 0, as needed.

Proposition 25.2.8.2. *Let X be a normed linear space over field $\mathbb{K}$ and $T : X \to \mathbb{K}$ be a linear functional. If T is unbounded, then* $\operatorname{Ker}(T) \subseteq X$ *is dense.*

Proof. Since T is unbounded, therefore we first claim that T is unbounded on each $B_{1/n}[0]$. Indeed, if there exists $n_0 \in \mathbb{N}$ such that T is bounded on $B_{1/n_0}[0]$, then for any $x \in X$, we have $\frac{x}{n_0\|x\|} \in B_{1/n_0}[0]$. Thus, by boundedness of T on $B_{1/n_0}[0]$, there exists $K \in \mathbb{R}_{>0}$ such that

$$\left|T\left(\frac{x}{n_0\|x\|}\right)\right| \leq K.$$

By linearity it follows from above that

$$|Tx| \leq Kn_0\|x\|$$

for all $x \in X$. This makes T bounded, a contradiction. Hence T is unbounded on each $B_{1/n}[0]$.

Consequently, for each $n \in \mathbb{N}$, there exists $y_n \in B_{1/n}[0]$ such that $\|Ty_n\| \geq n$. It follows that $y_n \to 0$ as $n \to \infty$ since $y_n \in B_{1/n}[0]$. Further, observe that

$$z_n = \frac{y_n}{Ty_n} - \frac{x}{Tx} \in \operatorname{Ker}(T).$$

Now we claim that $z_n \to \frac{x}{Tx}$ as $n \to \infty$. Indeed, since

$$\|\frac{y_n}{Ty_n}\| = \frac{1}{|Ty_n|}\|y_n\| \leq \frac{1}{n}\|y_n\| < \|y_n\|$$

and since $\|y_n\| \to 0$ as $n \to \infty$, therefore this shows that $\frac{y_n}{Ty_n} \to 0$ as $n \to \infty$. It follows that $z_n \to \frac{x}{Tx}$ as $n \to \infty$, as required.

As $z_n \in \operatorname{Ker}(T)$, therefore $T(x)z_n \in \operatorname{Ker}(T)$ by linearity. Thus $T(x)z_n \to x$ as $n \to \infty$. This shows the density of $\operatorname{Ker}(T)$, thus completing the proof. □

The following is a generalization of Riesz lemma to $r = 1$.

Proposition 25.2.8.3. *Let X be a normed linear space and $Y \subseteq X$ be a finite dimensional proper linear subspace. Then there exists $x_1 \in S^1(X) = \{x \in X \mid \|x\| = 1\}$ such that*

$$d(x_1, Y) = 1.$$

Proof. Pick $x \in X \setminus Y$. As Y is finite-dimensional in X, therefore it is closed in X. Hence, $d(x, Y) > 0$. We first claim that there exists $\tilde{y} \in Y$ such that

$$d(x, Y) = d(x, \tilde{y}). \tag{10}$$

Indeed, since $d(x, Y) = \inf_{y \in Y} d(x, y) = M$, therefore there exists a sequence $(y_n) \subseteq Y$ such that $d(x, y_n) \to M$ as $n \to \infty$. Fix $\epsilon > 0$. Thus, there exists $N \in \mathbb{N}$ such that $|d(x, y_n) - M| < \epsilon$ for all $n \geq N$. That is, $0 < d(x, y_n) < M + \epsilon$ for all $n \geq N$. Since $g(y) := d(x, y)$ is a continuous map on Y, therefore we have that

$$(y_n)_{n \geq N} \subseteq K = g^{-1}([0, M + \epsilon])$$

where $K \subseteq Y$ is a closed subset of Y. We now claim that K is bounded. Pick $y \in K$. Then

$$\begin{aligned}\|y\| = d(y, 0) &\leq d(y, x) + d(x, 0)\\ &< M + \epsilon + d(x, 0).\end{aligned}$$

This shows that K is bounded. As Y is finite-dimensional normed linear space, therefore generalized Heine-Borel holds and we deduce that K is a compact subset of Y. Since in a metric space compactness is equivalent to sequentially compact, therefore K is sequentially compact. It follows that $(y_n)_{n\geq N} \subseteq K$ has a subsequence which converges, say to $\tilde{y} \in K \subseteq Y$. Replace (y_n) by that subsequence so that we may write $y_n \to \tilde{y}$ and $d(x, y_n) \to M$. By continuity of g, it follows that $g(y_n) = d(x, y_n) \to g(\tilde{y}) = d(x, \tilde{y})$, but $d(x, y_n) \to M$, thus by uniqueness of limits in a Hausdorff space, it follows that $d(x, \tilde{y}) = M$, as needed. This completes the proof of claim in Eqn. (10).

We now complete the proof. Consider the vector

$$x_1 = \frac{x - \tilde{y}}{\|x - \tilde{y}\|} \in X.$$

We claim that $d(x_1, Y) = 1$. Indeed,

$$\begin{aligned} d(x_1, Y) &= \inf_{y \in Y} \|\frac{x - \tilde{y}}{\|x - \tilde{y}\|} - y\| \\ &= \frac{1}{\|x - \tilde{y}\|} \inf_{y \in Y} \|x - (\tilde{y} + \|x - \tilde{y}\| y)\| \\ &= \frac{1}{\|x - \tilde{y}\|} \inf_{y \in Y} \|x - y\| \end{aligned}$$

where the last equality follows from the bijection provided by affine transformations $Y \to Y$ mapping as $y \mapsto ay + x$ for $a \in \mathbb{K}$ and $x \in Y$, using the linearity of Y. From above equalities, it follows from Eqn. (10) that

$$\begin{aligned} d(x_1, Y) &= \frac{1}{\|x - \tilde{y}\|} \inf_{y \in Y} \|x - y\| \\ &= \frac{1}{\|x - \tilde{y}\|} d(x, Y) \\ &= \frac{1}{d(x, \tilde{y})} d(x, Y) = 1, \end{aligned}$$

as required to complete the proof. $\square$

25.3 Hilbert spaces

A general exposition on Hilbert spaces.

25.3.1 $\mathbb{C}$-inner product & Hilbert spaces

After setting the definitions, we will show Cauchy-Schwartz inequality which is of fundamental utility. Two important identities stasfied by an inner product and the associated norm are the parallelogram identity and polarization identity. The former in particular can be used to show when a norm does not comes from an inner product. A famous example of a space which is not an IPS is ℓ^p for $p \neq 2$, which will be established by the above technique.

25.3.2 Banach-Hilbert theorem

The only requirement for a Banach space to be a Hilbert space is that of the parallelogram identity.

25.3.3 Orthogonality, norm minimization

After defining orthogonal vectors in an IPS, we deduce the Pythagoras theorem. We'll then show that in a convex closed subspaces of a Hilbert space, there is a unique element of minimum norm. An application of this result would yield that for any point outside of a convex closed subspace, there is a point in it which is ay minimum distance from it.

In a $\mathbb{C}$-IPS, one can define all those vectors which are orthogonal to a fixed set, called the annihilaor. This turns out to form a $\mathbb{C}$-vector space. An important theorem on annihilators is the projection theorem, which tells us that for any closed linear subspace of a Hilbert space together with its annihilator gives a direct sum decomposition of the space. We further show that a subset of a Hilbert space is linear and closed iff the double annihilator is the same set.

25.3.4 Orthonormal sets and Hilbert dimension

An orthonormal set in a Hilbert space is a collections of vectors mutually orthogonal with norm 1. Two fundamental tools in their analysis are produced by Bessel's inequality and Parseval's theorem. Proving this finite orthonormal set is straightforward, but for arbitrary orthonormal sets require work. To this end we will prove a result which in some sense tells us about uniqueness properties of series defined by a countable orthonormal sets. We then define orthonormal basis and after showing that they exists for Hilber spaces, we will prove equivalent criterion for an orthonormal set to be a basis, one of which equality in the Bessel's inequality.

25.3.5 Duals and Riesz representation theorem

25.3.6 Sesquilinear forms and adjoints

25.3.7 Self-adjoint, unitary and normal operators

25.4 Extension problems-I : Hahn-Banach theorem

25.5 Major theorems : UBP, OMT, BIT, CGT

There are four major theorems in basic functional analysis, which we discuss now.

Theorem 25.5.0.1 (Uniform boundedness principle). *Let X be a Banach space and Y be a normed linear space. Consider a collection of bounded linear transformations $(T_i)_{i\in I} \subseteq B(X,Y)$ such that for each $x \in X$, the subset $(T_i x)_{i\in I} \subseteq Y$ is bounded. Then, $(\|T_i\|)_{i\in I}$ is bounded in $\mathbb{R}$, that is, $(T_i)_{i\in I} \subseteq B(X,Y)$ is a bounded set.*

Theorem 25.5.0.2 (Open mapping & bounded inverse theorem). *Let X and Y be Banach spaces and $T : X \to Y$ be a surjective bounded linear map. Then,*

1. *T is an open map.*
2. *If T is a bijection, then T is a homeomorphism.*

Theorem 25.5.0.3 (Closed graph theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be a linear transformation. Then the following are equivalent:*
1. *T is continuous/bounded.*
2. *The graph $\Gamma(T) = \{(x, Tx) \in X \times Y \mid x \in X\}$ is closed in $X \times Y$.*

We now show how all three are equivalent.

Theorem 25.5.0.4. *Let X and Y be Banach spaces. Then the following implications are true:*
1. *CGT $\implies$ UBP.*
2. *BIT $\implies$ OMP.*
3. *CGT $\implies$ OMP.*

Proof. 1. Closed graph theorem (CGT) states that a linear map $T : X \to Y$ is bounded if and only if $\Gamma(T) = \{(x, Tx) \in X \times Y \mid x \in X\}$ is a closed set in $X \times Y$. We wish to show that uniform boundedness principle (UBP) holds, that is, if $(T_i)_{i\in I}$ is a non-empty collection of bounded linear maps from X to Y such that for each $x \in X$, the set $(T_i(x))_{i\in I} \subseteq Y$ is bounded, then the set $(\|T_i\|)_{i\in I} \subseteq \mathbb{R}$ is a bounded set.

Pick any collection $(T_i)_{i\in I} \subseteq B(X, Y)$ such that for all $x \in X$, there exists $M_x \in \mathbb{R}_+$ such that $\sup_{i\in I} \|T_i x\| \leq M_x$. We wish to show that $(\|T_i\|)_{i\in I}$ is bounded. Indeed, to this end, we contstruct a new norm on X, using which, we will show the above.

Define the following for each $x \in X$:

$$\|x\|_1 := \|x\| + \sup_{i\in I} \|T_i x\|.$$

This is well-defined, as $(T_i x)$ is a bounded set in Y. We now make the following claims:
C1. $(X, \|\cdot\|_1)$ is a normed linear space.
C2. $(X, \|\cdot\|_1)$ is a Banach space.
Assuming the above two claims to be true, let us first show how this will complete the proof. We consider the map

$$\mathrm{id} : (X, \|\cdot\|) \to (X, \|\cdot\|_1).$$

We claim that this is a continuous linear transformation. Indeed, by CGT, we need only show that $\Gamma(\mathrm{id})$ is closed. That is, (denote $X_1 = (X, \|\cdot\|_1)$)

$$\Gamma(\mathrm{id}) = \{(x, x) \in X \times X_1 \mid x \in X\} \subseteq X \times X_1$$

is closed. Indeed, consider any sequence $(x_n, x_n) \subseteq \Gamma(\mathrm{id})$ which is convergent in $X \times X_1$. Then suppose $x_n \to x$ in X and $x_n \to x'$ in X_1. We claim that $x = x'$, so that $(x_n, x_n) \to (x, x)$ and since $(x, x) \in \Gamma(\mathrm{id})$, so this will show that $\Gamma(\mathrm{id})$ is closed.

Indeed, we have $x_n \to x$ in X, so $\|x_n - x\| \to 0$ as $n \to \infty$. Similarly, $\|x_n - x'\|_1 \to 0$ as $n \to \infty$. Since

$$\|x_n - x'\|_1 = \|x_n - x'\| + \sup_{i\in I} \|T_i x_n - T_i x'\| \to 0$$

as $n \to \infty$, therefore $\sup_{i\in I} \|T_i x_n - T_i x'\| \to 0$ and $\|x_n - x'\| \to 0$ as well. The latter says that $x_n \to x$ in X. By uniqueness of limits, we conclude that $x = x'$, as required. This shows that $\mathrm{id} : X \to X_1$ is continuous linear transform by CGT, hence bounded.

We wish to bound $\sup_{\|x\|=1} \|T_i x\|$. Pick any $x \in X$ with $\|x\| = 1$. Then we have for each $i \in I$ that

$$\begin{aligned} \|x\|_1 &= \|x\| + \sup_{i\in I} \|T_i x\| \\ &\geq 1 + \|T_i x\|. \end{aligned}$$

Thus, for each $i \in I$, we have

$$\|T_i x\| \leq \|x\|_1 - 1 \leq \|x\|_1.$$

It follows that

$$\sup_{\|x\|=1} \|T_i x\| \leq \sup_{\|x\|=1} \|x\|_1 = \|\mathrm{id}\| < \infty,$$

as required. Hence we now need only prove the claims C1 and C2.

To see claim C1, proceed as follows. Observe that if $\|x\|_1 = 0$, then $\|x\| = 0$, so $x = 0$. Further we have for any $c \in \mathbb{K}$ that $\|cx\|_1 = \|cx\| + \sup_{i\in I} \|T_i(cx)\| = |c|\,\|x\| + |c| \sup_{i\in I} \|T_i x\| = |c|\,\|x\|_1$. Finally, to see triangle inequality, we see that

$$\begin{aligned} \|x+y\|_1 &= \|x+y\| + \sup_{i\in I} \|T_i x + T_i y\| \\ &\leq \|x\| + \|y\| + \sup_{i\in I} (\|T_i x\| + \|T_i y\|) \\ &\leq \|x\| + \|y\| + \sup_{i\in I} \|T_i x\| + \sup_{i\in I} \|T_i y\| \\ &= \|x\|_1 + \|y\|_1, \end{aligned}$$

as required. This shows claim C1.

To see claim C2, proceed as follows. Take any Cauchy sequence $(x_n) \subseteq X_1$. We wish to show that it converges. We claim that (x_n) is Cauchy in X. Indeed, for any $\epsilon > 0$, we have $N \in \mathbb{N}$ such that for any $n, m \geq N$ we have

$$\|x_n - x_m\| \leq \|x_n - x_m\|_1 < \epsilon$$

and for each $j \in I$, we have

$$\|T_j x_n - T_j x_m\| \leq \sup_{i\in I} \|T_i x_n - T_i x_m\| \leq \|x_n - x_m\|_1 < \epsilon/2.$$

Thus, we get by former that (x_n) is Cauchy, so convergent to say $x \in X$. We claim that (x_n) converges to x in X_1. In the latter, by letting $m \to \infty$, we obtain that for each $j \in I$ and each $n \geq N$, we have

$$\|T_j x_n - T_j x\| \leq \epsilon/2 < \epsilon.$$

Thus, taking $\sup_{i\in I}$, we further obtain that for each $n \geq N$ we have

$$\sup_{i\in I} \|T_i x_n - T_i x\| \leq \epsilon/2 < \epsilon.$$

Now, we may write

$$\begin{aligned}\|x_n - x\|_1 &= \|x_n - x\| + \sup_{i\in I} \|T_i x_n - T_i x\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

for $n \geq N$, as requird. This completes the proof.

2. Consider any bounded linear map $T : X \to Y$ which is surjective. We wish to show that T is an open mapping using bounded inverse theorem. Indeed, as T is bounded, therefore $\mathrm{Ker}\,(T)$ is a closed linear subspace. Going modulo $\mathrm{Ker}\,(T)$, we get a linear transformation $\tilde{T} : X/\mathrm{Ker}\,(T) \to Y$ such that the following commutes:

$$\begin{tikzcd} X \arrow[r, two heads, "T"] \arrow[d, two heads, "\pi"'] & Y \\ X/\mathrm{Ker}\,(T) \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

We first claim that $\tilde{T}$ is bounded. Indeed, as for any $x + \mathrm{Ker}\,(T) \in X/\mathrm{Ker}\,(T)$ we have $\tilde{T}(x + \mathrm{Ker}\,(T)) = Tx$, therefore

$$\|\tilde{T}(x + \mathrm{Ker}\,(T))\| = \inf_{z\in \mathrm{Ker}(T)} \|T(x+z)\| = \inf_{z\in \mathrm{Ker}(T)} \|Tx\| = \|Tx\|.$$

This shows that $\tilde{T}$ is a bounded linear map which is injective and surjective. Thus, $\tilde{T}$ is a bijection and thus by BIT, we get that $\tilde{T}$ is a homeomorphism. In particular, we see that $\tilde{T}$ is an open map. Now consider the map $\pi : X \to X/\mathrm{Ker}\,(T)$. We wish to show that π is an open map. Let $U \subseteq X$ be an open set and pick any point $x + \mathrm{Ker}\,(T) \in \pi(U) \subseteq X/\mathrm{Ker}\,(T)$ where $x \in U$. As there exists $B_\epsilon(x) \subseteq U$, thus we claim that $B_\epsilon(x + \mathrm{Ker}\,(T)) \subseteq \pi(U)$. Indeed, if $y + \mathrm{Ker}\,(T) \in B_\epsilon(x + \mathrm{Ker}\,(T))$, then $\|x - y + \mathrm{Ker}\,(T)\| < \epsilon$. As

$$\|x - y + \mathrm{Ker}\,(T)\| = \inf_{z\in \mathrm{Ker}(T)} \|x - y + z\| < \epsilon,$$

thus there exists $z \in Z$ such that $\|x - y + z\| < \epsilon$. Thus, $y - z \in B_\epsilon(x) \subseteq U$. Hence, $y - z + \mathrm{Ker}\,(T) = y + \mathrm{Ker}\,(T) \subseteq \pi(U)$, as needed.

3. We first show that closed graph theorem (CGT) implies bounded inverse theorem (BIT). Indeed, this combined with item 2 above will show that CGT $\implies$ OMP. Let $T : X \twoheadrightarrow Y$ be a surjective bounded linear transformation which is a bijection. We then wish to show that the inverse linear transformation of T, $T^{-1} : Y \to X$, is also bounded. By CGT, it is equivalent to showing that the graph $\Gamma(T^{-1}) \subseteq Y \times X$ is a closed set. Since T is a bijection, we get

$$\begin{aligned}\Gamma(T^{-1}) &= \{(y, T^{-1}y) \in Y \times X \mid y \in Y\} \\ &= \{(Tx, x) \in Y \times X \mid x \in X\} \\ &\cong \{(x, Tx) \in X \times Y \mid x \in X\}\end{aligned}$$

where the last homeomorphism is induced by restricting the natural homeomorphism $Y \times X \to X \times Y$. It follows that $\Gamma(T^{-1})$ is closed in $Y \times X$ since $\Gamma(T)$ is closed in $X \times Y$ by CGT (as it is continuous), as required. $\square$

We next see that it is important in closed graph theorem for X and Y to be Banach.

Example 25.5.0.5. We wish to show that there exists a linear map $T : X \to Y$ where X and Y are normed linear spaces such that T is unbounded and the graph $\Gamma(T) \subseteq X \times Y$ is closed.

Indeed, consider $X = C^1[0,1]^*$ to be the subspace of $C^1[0,1]$ of those functions f such that $f(a) = 0$ and $Y = C[0,1]^*$ both with sup norm. Define

$$\begin{aligned} T : X &\longrightarrow Y \\ f(x) &\longmapsto f'(x) \end{aligned}$$

to be the derivative map. We know that T is unbounded as $f_n(x) = x^n \in C^1[0,1]$ has norm 1 but its derivative has unbounded norm. We wish to show that $\Gamma(T)$ is closed in $X \times Y$. Indeed, consider any sequence $(f_n) \subseteq X$ such that $(f_n, Tf_n) \subseteq \Gamma(T)$ is convergent in $X \times Y$. As projection map are continuous, it follows that $(f_n) \subseteq X$ and $(Tf_n) = (f'_n) \subseteq Y$ are convergent. Let $f_n \to f$ in X and $f'_n \to g$ in Y. As X and Y are in sup norm, it follows that $f_n \to f$ and $f'_n \to g$ uniformly. As $f_n(0) = 0$, it follows by the theorem on uniform convergence and derivatives that f_n converges uniformly to a differentiable function which we know is f and $f' = g$. That is $Tf = g$. This shows that $(f_n, Tf_n) \to (f, Tf)$ in $X \times Y$, that is, (f_n, Tf_n) converges in $\Gamma(T)$. This shows that $\Gamma(T)$ is closed. Yet, T is unbounded, as required.

Similarly, the hypothesis of completeness is essential in uniform boudnedness principle.

Example 25.5.0.6. We wish to show that the hypothesis of completeness of the domain in uniform boundedness principle is essential.

Indeed, let $X = \mathbb{R}^\infty \subseteq (\ell^2, \|\cdot\|_2)$ of all eventually zero sequences in ℓ^2 with the induced norm. Then X is not Banach as $(x_k^{(n)}) = (1, 1/2, \dots, 1/n, 0, \dots)$ is a sequence in X which is Cauchy but it is not convergent. We now construct a sequence of functionals $f_n : X \to \mathbb{K}$ such that for all $(x_k) \in X$, the sequence $(f_n((x_k)))_n$ is bounded in $\mathbb{K}$ but still $(\|f_n\|)_n \subseteq \mathbb{R}$ is unbounded.

Consider

$$\begin{aligned} f_n : X &\longrightarrow \mathbb{K} \\ (x_k) &\longmapsto \sum_{k=1}^{n} x_k. \end{aligned}$$

Pick any $(x_k) \in X$. Then,

$$|f_n((x_k))| = \left| \sum_{k=1}^{n} x_k \right| \leq \left| \sum_{k=1}^{\infty} x_k \right| < \infty$$

as there are only finitely many non-zero elements, thus for each $(x_k) \in X$, $(f_n((x_k)))_n$ is bounded. Moreover,

$$\|f_n\| = \sup_{(x_k)\in X} \frac{|f_n((x_k))|}{\|(x_k)\|} \geq \frac{|\sum_{k=1}^{n} x_k|}{\left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2}}$$

for any (x_k) in X. We claim that $\|f_n\| \to \infty$ as $n \to \infty$. Indeed, consider $(x_k^{(n)}) = (1, 1/2, \dots, 1/n, 0, \dots)$. Then, $\|(x_k^{(n)})\| = 1 + 1/2^2 + \dots 1/n^2 < M$ for a fixed $M > 0$ and for all n. Further, by above we have

$$\begin{aligned}\|f_n\| &\geq \frac{|\sum_{k=1}^{n} 1/k|}{\|(x_k^{(n)})\|} \\ &> \frac{1}{M}\sum_{k=1}^{n}\frac{1}{k} \to \infty\end{aligned}$$

as $n \to \infty$, as required.

We wish to next prove the main theorems using an important technical lemma.

Theorem 25.5.0.7 (Zabreiko's lemma)**.** *Let X be a Banach space and $p : X \to \mathbb{R}_{\geq 0}$ be a seminorm. If p is countably subadditive, then p is continuous.*

Proof. Let us first define a seminorm on a Banach space.

Definition 25.5.0.8 (**Seminorm and countably subadditive functions**)**.** Let X be a normed linear space. A function $p : X \to \mathbb{R}_{\geq 0}$ is said to be a seminorm if $p(\alpha x) = |\alpha|\, p(x)$ for all $\alpha \in \mathbb{K}$ and $x \in X$ and $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

The function p is said to be countably subadditive if for every convergent series $\sum_n x_n$ in X, we have

$$p\left(\sum_{n=1}^{\infty} x_n\right) \leq \sum_{n=1}^{\infty} p(x_n)$$

In proving Zabreiko's lemma, we would need a notion of absorbing sets.

Definition 25.5.0.9 (**Absorbing set**)**.** Let X be a normed linear space. A subset $A \subseteq X$ is said to be absorbing if for all $x \in X$, there exists $s_x \in \mathbb{R}_{>0}$ such that $x \in tA$ for all $t \geq s_x$.

Note that if A is absorbing, then $-A$ is also absorbing. We now state the following proposition, which will be used in proving Zabreiko's lemma.

Proposition 25.5.0.10. *Let X be a normed linear space, $p : X \to \mathbb{R}_{\geq 0}$ be a function and $A \subseteq X$.*

1. *If A is absorbing, then $0 \in A$.*
2. *If X is Banach and A is closed convex and absorbing, then A contains a neighborhood of 0.*
3. *If p is a seminorm, then if p is continuous at 0, then p is continuous on X.*

Proof. 1. As A is absorbing, therefore for all $x \in X$, there exists $s_x \in \mathbb{R}_{>0}$ such that $x \in tA$ for all $t \geq s_x$. Let $x = 0$. Then, there exists $s_0 \in \mathbb{R}_{>0}$ such that $x \in tA$ for all $t \geq s_0$. Pick any $t \geq s_0$, we get $0 = ta$ for some $a \in A$. As $t \neq 0$, it follows that $a = 0$, as required.

2. Let $A \subseteq X$ be closed convex and absorbing. Then first observe that

$$D = A \cap (-A) \subseteq A$$

is non-empty as $0 \in A$ (and thus so is in $-A$). We claim that for any $S \subseteq D$, we have

$$\frac{1}{2}S + \frac{1}{2}(-S) \subseteq D.$$

Indeed, pick any $\frac{s_1 - s_2}{2} \in \frac{1}{2}S + \frac{1}{2}(-S)$. We wish to show that $\frac{s_1-s_2}{2} \in A$ and $\frac{s_1-s_2}{2} \in -A$. Thus, we reduce to showing that $\frac{s_1-s_2}{2}, \frac{s_2-s_1}{2} \in A$. It is easy to see that $A \cap -A$ is convex as A and $-A$ are convex. As $s_1, s_2 \in S \subseteq A \cap -A$ thus $-s_1, -s_2 \in S \subseteq A \cap -A$ as well. Now, by convexity of $A \cap -A$, we get

$$\frac{s_1 - s_2}{2}, \frac{s_2 - s_1}{2} \in A \cap -A,$$

as required.

We claim that D° is non-empty. This will complete the proof as by above we will have that $\frac{1}{2}D^\circ + \frac{1}{2}(-D^\circ) \subseteq D$ is open in D and since it contains 0, we would have shown that A contains an open set containing 0.

Suppose to the contrary that $D^\circ = \emptyset$. We wish to derive a contradiction to the fact that A is an absorbing set. Indeed, first observe that for all $n \in \mathbb{N}$, we have $(nD)^\circ = \emptyset$ and nD is closed. This gives us that for each $n \in \mathbb{N}$, the set $Y_n = X - (nD)$ is an open dense subset of X. Pick any $x \in X - D$. As $X - D$ is open, there exists $B_1 = \overline{B_{r_1}(x)} \subseteq X - D$ where $r_1 < 1$. As $X - 2D$ is dense, therefore $(X - 2D) \cap (B_1)^\circ$ is non-empty and thus we get a closed ball B_2 of radius less than $1/2$ in B_1. Continuing this, we have a sequence of closed balls $B_1 \supseteq B_2 \supseteq \dots \supseteq B_n \supseteq \dots$ with radius of B_n less than $1/n$ and $B_n \cap nD = \emptyset$. Let x_n be the center of B_n. We claim that (x_n) is a Cauchy sequence. Indeed, for any $1/k$ we have

$$\|x_n - x_m\| < 2/k$$

for all $n, m \geq k$. As X is complete therefore there exists $x \in X$ such that $x_n \to x$. Thus $x \in B_n$ for all $n \in \mathbb{N}$, that is, $x \notin nD$ for all $n \in \mathbb{N}$. As A is absorbing, therefore there exists $s_x \in \mathbb{R}_{>0}$ such that $x \in tA$ for all $t \geq s_x$. As $-A$ is also absorbing, thus we get $s'_x \in \mathbb{R}_{>0}$ such that $x \in -tA$ for all $t \geq s'_x$. Let $n \in \mathbb{N}$ be larger than both s_x, s'_x. Then we have that $x \in nA$ and $x \in -nA$. It follows that $x \in nA \cap (-nA) = nD$, a contradiction to the fact that $x \notin D$. This completes the proof of item 2.

3. Let $x_n \to x$ in X where $x \neq 0$. We wish to show that $p(x_n) \to p(x)$. Indeed, since $x_n - x \to 0$ and p is continuous at 0, we get $p(x_n - x) \to p(0) = 0$. Thus for any $\epsilon > 0$, we have $p(x_n - x) = |p(x_n - x)| < \epsilon$ for all $n \geq N$. As $p(x_n) - p(x) \leq p(x_n - x)$ by seminorm crieterion, we get $|p(x_n) - p(x)| < \epsilon$ for all $n \geq N$. It follows that $p(x_n) \to p(x)$, as required. $\square$

Using the above proposition, we now prove Zabreiko's lemma.

Proof of Theorem 25.5.0.7. By Proposition 25.5.0.10, 3, we reduce to proving that p is continuous at 0. We claim that it is sufficient to show that there is an open ball $B_r(0)$ of radius $r > 0$ at 0 such that $p(B_r(0))$ is a bounded set in $\mathbb{R}_{\geq 0}$. Indeed, for any sequence (x_n) in X converging to 0, which we may assume to be contained in $B_r(0)$, we get that $p(x_n) \in p(B_r(0))$ for all $n \in \mathbb{N}$. We wish to show that $p(x_n) \to p(0) = 0$. Indeed, if $p(B_r(0))$ is upper bounded by $M > 0$, we thus get for any $x \in B_r(0)$ the following bound:

$$p(x) = \|x\| p\left(\frac{x}{\|x\|}\right) \leq M\|x\|.$$

Consequently, we have

$$p(x_n) \leq M\|x_n\|.$$

As $\|x_n\| \to 0$ as $n \to \infty$, it follows by above that $p(x_n) \to 0$ as $n \to \infty$, as required.

So we reduce to showing that there exists an open $B_r(0)$ of 0 in X such that $p(B_r(0))$ is a bounded set. Consider $A_! = \{x \in X \mid p(x) < 1\}$. We claim that A is an absorbing set. Indeed, for any $x \in X$, we have $\|x\|$ such that for all $t \geq \|x\|$ we have $x \in tA_!$ since $p(x/t) = p(x)/t < 1/t$, so $p(t \cdot x/t) < 1$, as required. This shows that $A_!$ is absorbing. We claim that $A = \overline{A_!}$ is absorbing as well. Indeed, observe that since A contains an absorbing set, namely $A_!$, then A is absorbing as well.

We next show that A is convex. Note that since closure of convex set is convex and $A_!$ is convex since if $x, y \in A_!$, then $p((1-t)x+ty) \leq (1-t)p(x)+tp(y) < (1-t)+t = 1$, therefore A is convex. Thus, A is closed convex absorbing set in a Banach space. By Proposition 25.5.0.10, 2, it follows that A has a neighborhood of 0.

We now find the required ball $B_r(0)$ so that $p(B_r(0))$ is bounded. Indeed consider $r > 0$ such that $\overline{B_r(0)} \subseteq \bar{A}$ and fix a point $x \in B_r(0)$. Pick a point $x_1 \in A$ such that $\|x - x_1\| < r/2$, that is, $x_1 \in B_{r/2}(x) \cap B_r(0) \subseteq \frac{1}{2}A$. Thus $x - x_1 \in \frac{1}{2}B_r(0) \subseteq \frac{1}{2}A \subseteq \frac{1}{2}\bar{A}$. Now there exists $x_2 \in \frac{1}{2}A$ such that $\|x - x_1 - x_2\| \leq r/2^2$, that is, $x_2 \in B_{r/2^2}(x - x_1) \cap B_{r/2}(0) \subseteq \frac{1}{2^2}A$. Continuing this, we get a sequence (x_n) in A such that $x_n \in \frac{1}{2^{n-1}}A$ and $\|x - \sum_{k=1}^n x_k\| < \frac{r}{2^n}$. It follows that $\sum_{k=1}^n x_k \to x$ as $n \to \infty$.

By countable sub-additivity of p, it follows that

$$p(x) = p\left(\sum_{k=1}^{\infty} x_k\right) \leq \sum_{k=1}^{\infty} p\left(x_k\right).$$

As $x_k \in \frac{1}{2^k}A$, therefore $p(x_k) < \frac{1}{2^k}$ by definition of A. Thus, $\sum_{k=1}^{\infty} p(x_k) \leq 1$, and thus $p(x) \leq 1$. As $x \in B_r(0)$ was arbitrary, we have thus shown that $p(B_r(0)) \leq 2$, as required. $\square$

Theorem 25.5.0.11. *One can derive OMT, UBP, CGT from Zabreiko's lemma (Theorem 25.5.0.7).*

Proof. (Zabreiko $\Rightarrow$ OMT) Let $T : X \twoheadrightarrow Y$ be a surjective linear transformation between Banach spaces. By translation and scaling homeomorphism, we reduce to showing that

$T(B_1(0))$ is open. Define

$$\begin{aligned} p : Y &\longrightarrow \mathbb{R}_{\geq 0} \\ y &\longmapsto \inf\{\|x\| \mid Tx = y\}. \end{aligned}$$

We claim that p is a countably subadditive semi-norm, so that by Theorem 25.5.0.7, we will get p is continuous. This is sufficient as

$$T(B_1(0)) = p^{-1}([0,1))$$

which is easy to see. So we reduce to showing that p is a countably subadditive seminorm.

1. p is countably subadditive : Let $\sum_n y_n$ be a covergent series in Y. We wish to show that $p\left(\sum_n y_n\right) \leq \sum_n p(y_n)$. Indeed, fix $\epsilon > 0$. We get the following

   $$p(y_n) + \frac{\epsilon}{2^n} \geq \|x_n\|$$

   for each $n \in \mathbb{N}$ where $x_n \in X$ is such that $Tx_n = y_n$. Summing till N we get

   $$\sum_{n=1}^{N} p(y_n) + \sum_{n=1}^{N} \frac{\epsilon}{2^n} \geq \sum_{n=1}^{N} \|x_n\| \geq \|\sum_{n=1}^{N} x_n\|$$

   and since $T(x_1 + \cdots + x_n) = \sum_{n=1}^{N} y_n$, we get that $\|x_1 + \cdots + x_n\| \geq p(\sum_{n=1}^{N} y_n)$. This yields that

   $$\sum_{n=1}^{N} p(y_n) + \sum_{n=1}^{N} \frac{\epsilon}{2^n} \geq p\left(\sum_{n=1}^{N} y_n\right).$$

   Taking $N \to \infty$ and then $\epsilon \to 0$, the result follows.
2. p is a seminorm : Fact that $p(cy) = |c|\, y$ is immediate from definition. Subadditivity follows from item 1.

(Zabreiko $\Rightarrow$ UBP) Let X, Y be Banach and $(T_i)_{i\in I} \subseteq B(X,Y)$ be a family of bounded linear transformations such that for all $x \in X$, the set $(T_i(x))_{i\in I} \subseteq Y$ is bounded. We wish to show that $(\|T_i\|)_{i\in I}$ is bounded in $\mathbb{R}$.

Consider

$$\begin{aligned} p : X &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto \sup_{i\in I} \|T_i(x)\|. \end{aligned}$$

We claim that p is a countably subadditive seminorm. Indeed, then it would follow by Theorem 25.5.0.7 that p is continuous. Then there exists $\delta > 0$ such that $\|x\| < \delta$ implies $|p(x)| \leq 1$. As p is a seminorm, therefore we would obtain

$$\|x\| < 1 \implies p(x) < 1/\delta.$$

As $\|T_i\| = \sup_{\|x\|<1} \|T_i x\|$ and $p(x) < 1/\delta$ for $\|x\| < 1$ where

$$p(x) = \sup_{i\in I} \|T_i x\| < 1/\delta$$

therefore $\|T_i x\| < 1/\delta$ for all $\|x\| < 1$, which would thus tield $\|T_i\| \leq 1/\delta$, as required. So we reduce to showing that p is a countably subadditive seminorm.

1. p is countably subadditive : Let $\sum_n x_n$ be a convergent series in X. We wish to show that $p\left(\sum_n x_n\right) \leq \sum_n p(x_n)$. Indeed, we have
$$p\left(\sum_n x_n\right) = \sup_{i\in I} \|T_i\left(\sum_n x_n\right)\| = \sup_{i\in I} \|\sum_n T_i x_n\| \leq \sup_{i\in I}\sum_n \|T_i x_n\| \leq \sum_n \sup_{i\in I} \|T_i x_n\| = \sum_n p(x_n)$$
where $\sup_{i\in I} \|T_i x_n\|$ exists and is bounded as by hypothesis, the set $(T_i x)_{i\in I}$ is bounded for any $x \in X$. This shows that p is countably subadditive.
2. p is a seminorm : Fact that $p(cy) = |c|\, y$ is immediate from definition. Subadditivity follows from item 1.

(Zabreiko $\Rightarrow$ CGT) Let $T : X \to Y$ be a linear transformation between Banach spaces. We wish to show that T is bounded if and only if $\Gamma(T) \subseteq X \times Y$ is closed.

($\Rightarrow$) is immediate by considering the inverse image at 0 of $X\times Y \to Y$ of $(x, y) \mapsto Tx-y$.

($\Leftarrow$) Consider the following function
$$\begin{aligned} p : X &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto \|Tx\|. \end{aligned}$$
We claim that p is a countably subadditive seminorm. Indeed, this would imply that p is continuous by Theorem 25.5.0.7. Note that it is sufficient to show that $\{\|Tx\| \mid \|x\| < 1\}$ is bounded. But this set is same as $p(B_1(0))$. Thus, we reduce to showing that $p(B_1(0))$ is bounded. Indeed, this follows as there exists $\delta > 0$ such that
$$\|x\| < \delta \implies p(x) < 1$$
which by seminorm property is equivalent to
$$\|x\| < 1 \implies p(x) < 1/\delta.$$
This shows that $p(B_1(0)) < 1/\delta$, as needed. We thus reduce to showing that p is a countably subadditive seminorm.

1. p is countably subadditive : Let $\sum_n x_n$ be a convergent series in X. We wish to show that $p\left(\sum_n x_n\right) \leq \sum_n p(x_n)$. Indeed, we have
$$p\left(\sum_n x_n\right) = \|T\left(\sum_n x_n\right)\|$$
where since $\left(\sum_{k=1}^n x_k, \sum_{k=1}^n Tx_k\right)$ is in the graph and is convergent where graph is closed, therefore $T(\sum_n x_n) = \sum_n Tx_n$. Thus,
$$\|T\left(\sum_n x_n\right)\| = \|\sum_n Tx_n\| \leq \sum_n \|Tx_n\| = \sum_n p(x_n).$$
This shows that p is countably subadditive.
2. p is a seminorm : Fact that $p(cy) = |c|\, y$ is immediate from definition. Subadditivity follows from item 1.

This completes the proof of Theorem 25.5.0.7. □

This completes the proof. □

25.6 Strong & weak convergence

These are important definitions as these protray that how fundamental importance this topic gives to functionals, anyways, its *functional* analysis so we must be very comfortable with constructing and manipulating functionals on a normed linear space.

Definition 25.6.0.1 (**Strong & weak convergence**). Let X be a normed linear space and $(x_n) \subseteq X$ be a sequence in X. Then, (x_n) is said to be strongly convergent if there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Further (x_n) is said to be weakly convergent if there exists $x \in X$ such that for all functionals $f \in X^*$, the sequence $(f(x_n)) \to f(x)$ in $\mathbb{K}$. In the former case x is said to be the strong limit and in the latter case x is said to be the weak limit.

We now state some basic properties of the above definition. In particular we see that strong convergence implies weak convergence and if domain is finite dimensional then both the notions are equivalent. **TODO**.

The following showcases a nice property of weak convergence.

Proposition 25.6.0.2. *Let X be a normed linear space and $x_n \to x$ weakly in X. Then*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Proof. As $f(x_n) \to f(x)$ for all $f \in X^*$, therefore we will construct a functional using Hahn-Banach through which the desrired inequality is straightforward. Indeed, by separation theorem applied on point x, we get that there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Consequntly, we get by weak convergence that

$$f(x_n) \to f(x) = \|x\|.$$

Now, for each $n \in \mathbb{N}$ we have

$$|f(x_n)| \leq \|f\|\|x_n\| = \|x_n\|.$$

Taking liminf both sides, we obtain

$$\liminf_{n\to\infty} |f(x_n)| \leq \liminf_{n\to\infty} \|x_n\|.$$

As $f(x_n) \to f(x)$, therefore $\liminf_{n\to\infty} |f(x_n)| = |f(x)| = \|x\|$. Thus we get

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|,$$

as required. □

Definition 25.6.0.3 (**Weakly Cauchy and complete**). A normed linear space X is *weakly complete* if every weakly Cauchy sequence is weakly convergent, where a sequence (x_n) in X is weakly Cauchy if for all $f \in X^*$, the sequence $(f(x_n))$ is Cauchy. Thus, unravelling this, we have that X is weakly complete if for any sequence (x_n) in X such that $(f(x_n))$ is Cauchy in $\mathbb{K}$ for each $f \in X^*$, there exists $x \in X$ such that $f(x_n) \to f(x)$ for each $f \in X^*$.

Proposition 25.6.0.4. *Any reflexive normed linear space X is weakly complete.*

Proof. Recall X is reflexive if the James map $\mathrm{ev} : X \to X^{**}$ is surjective. Since we have seen that ev is an isometric embedding, therefore reflexivity tells us ev is an isometric isomorphism.

To show that X is weakly complete, pick any weakly Cauchy sequence (x_n) in X. Then, for each $f \in X^*$, the sequence $f(x_n)$ is Cauchy in $\mathbb{K}$. As $\mathbb{K}$ is complete, it follows that $f(x_n)$ converges and let $f(x_n) \to c_f$ where $c_f \in \mathbb{K}$. We claim that the mapping

$$\begin{aligned} \varphi : X^* &\longrightarrow \mathbb{K} \\ f &\longmapsto c_f \end{aligned}$$

is a bounded linear map. This will complete the proof as by reflexivity we will have a unique $x \in X$ such that $\mathrm{ev}_x = \varphi$ and thus $\mathrm{ev}_x(f) = f(x) = c_f = \varphi(f)$, that is,

$$f(x) = \varinjlim_{n\to\infty} f(x_n)$$

for all $f \in X^*$, which shows that (x_n) weakly convergent, as required. We thus reduce to proving that φ is a bounded linear map.

To see linearity, pick any $f, g \in X^*$ and $\alpha \in \mathbb{K}$ to observe that

$$\varphi(f + \alpha g) = c_{f+\alpha g} = \varinjlim_n (f + \alpha g)(x_n) = \varinjlim_n f(x_n) + \alpha \varinjlim_n g(x_n) = c_f + \alpha c_g$$

since each $f(x_n)$ and $g(x_n)$ converges because they are Cauchy. To see boundedness, we first show that the set $\{x_n\} \subseteq X$ is a bounded set. Indeed, by a corollary of uniform boundedness principle we have that a set $Y \subseteq X$ is bounded if and only if $f(Y) \subseteq \mathbb{K}$ is bounded for each $f \in X^*$. For $Y = \{x_n\}$ and any $f \in X^*$, we see that $f(Y) = (f(x_n))$ is bounded as $f(x_n) \to c_f$. It follows that $\{x_n\}$ is a bounded set, as required. Consequently, let $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. We thus have

$$|\varphi(f)| = |c_f| = \varinjlim_n |f(x_n)| \leq \limsup_n \|f\|\|x_n\| \leq \|f\| \cdot M.$$

Hence, φ is a bounded linear map, as required. □

25.6.1 Operator convergence

We now define strong and weak operator convergence which mimics the definitions made above.

Definition 25.6.1.1 (Uniform, strong & weak operator covergence). Let X and Y be normed linear spaces and $(T_n) \subseteq B(X, Y)$ be a sequence. We say that

1. (T_n) is uniformly operator convergent if there exists $T \in B(X, Y)$ such that $\|T_n - T\| \to 0$ as $n \to \infty$,
2. (T_n) is strongly operator convergent if there exists $T \in B(X, Y)$ such that for all $x \in X$, we have $\|T_n x - Tx\| \to 0$ as $n \to \infty$, that is, if $(T_n x)$ is strongly convergent in Y,
3. (T_n) is weakly operator convergent if there exists $T \in B(X, Y)$ such that for all $x \in X$ and all $f \in Y^*$, we have $|f(T_n x) - f(Tx)| \to 0$ as $n \to \infty$, that is, if $(T_n x)$ is weakly convergent in Y.

25.7 Spectral theory

25.8 Compact operators

Chapter 26

Homological Methods

Contents

Methods employed in homological algebra comes in handy to attack certain type of local-global problems in geometry. We would like to discuss some foundational homological algebra in this chapter in the setting of additive and abelian categories. The main goal is not to illuminate foundations but to quickly get to the working theory which can allow us to develop deeper results elsewhere in this notebook. Using the Freyd-Mitchell embedding theorem, we can always assume that any (small) abelian category $\mathbf{A}$ is a full subcategory of $\mathbf{Mod}(R)$ over some ring R. Thus we will freely do the technique of diagram chasing in the following, implicitly assuming $\mathbf{A}$ to be embedded in a module category. Consequently, the main example to keep in mind throughout this chapter is of-course the category of R-modules, $\mathbf{Mod}(R)$.

26.1 The setup : abelian categories

Let us begin with the basic definitions. Let $\mathbf{A}$ be a category. Then $\mathbf{A}$ is said to be *preadditive* if for any $x, y \in \mathbf{A}$, the homset $\mathrm{Hom}\,(x, y)$ is an abelian group and the composition $\mathrm{Hom}\,(x, y) \times \mathrm{Hom}\,(y, z) \to \mathrm{Hom}\,(x, z)$ is a group homomorphism. For two preadditive categories $\mathbf{A}, \mathbf{B}$ a functor $F : \mathbf{A} \to \mathbf{B}$ is called *additive* if for all $x, y \in \mathbf{A}$, the function $\mathrm{Hom}_{\mathbf{A}}\,(x, y) \to \mathrm{Hom}_{\mathbf{B}}\,(Fx, Fy)$ is a group homomorphism.

Let $\mathbf{A}$ be a preadditive category and $f : x \to y$ be an arrow. This mean for any two object $w, z \in \mathbf{A}$, there is a zero arrow $0 \in \mathrm{Hom}\,(w, z)$. Then, we can define the usual notions of algebra as follows.

1. $i : \mathrm{Ker}\,(f) \to x$ is defined by the following universal property w.r.t. $fi = 0$:

$$\begin{tikzcd} x' \arrow[d, dashed, "\exists!"'] \arrow[dr, "\forall g \text{ s.t. } fg=0"] & & \\ \mathrm{Ker}\,(f) \arrow[r, "i"] & x \arrow[r, "f"] & y \end{tikzcd}$$

2. $j : y \to \mathrm{CoKer}\,(f)$ defined by the following universal property w.r.t. $jf = 0$:

$$\begin{tikzcd} & & y' \\ x \arrow[r, "f"] & y \arrow[ur, "\forall g \text{ s.t. } gf=0"] \arrow[r, "j"] & \mathrm{CoKer}\,(f) \arrow[u, dashed, "\exists!"'] \end{tikzcd}.$$

3. $k : x \to \mathrm{CoIm}\,(f)$ is defined to be the cokernel of the kernel map $i : \mathrm{Ker}\,(f) \to x$.
4. $l : \mathrm{Im}\,(f) \to y$ is defined to be the kernel of the cokernel map $j : y \to \mathrm{CoKer}\,(f)$.

Hence, for each $f : x \to y$ in a preadditive category $\mathbf{A}$, we can contemplate the following four type of maps:

$$\begin{tikzcd} \mathrm{Ker}\,(f) \arrow[d] & \mathrm{CoKer}\,(f) \\ x \arrow[r, "f"] \arrow[d] & y \arrow[u] \\ \mathrm{CoIm}\,(f) & \mathrm{Im}\,(f) \arrow[u] \end{tikzcd}$$

Lemma 26.1.0.1. *In a preadditive category, if a coproduct $x \oplus y$ exists, then so does the product $x \times y$ and vice versa. In such a case, $x \oplus y \cong x \times y$.* □

A preadditive category $\mathbf{A}$ is said to be *additive* if it contains all finite products, including the empty ones. By the above lemma, we require zero objects and sums of objects to exist.

An additive category $\mathbf{A}$ is said to be *abelian* if all kernels and cokernels exist and the natural map for each $f : x \to y$ in $\mathbf{A}$

$$\mathrm{CoIm}\,(f) \to \mathrm{Im}\,(f)$$

is an isomorphism. This intuitively means that the first isomorphism theorem holds in abelian categories by definition.

26.2 (co)Homology, resolutions and derived functors

In this section, we shall discuss basic topics of homological algebra in abelian categories, which we shall need to setup the sheaf cohomology in geometry and Lie group cohomology in algebra and etcetera, etcetera.

26.2.1 (co)Homology

We define cochain complexes and maps, cohomology and homotopy of such. Since this section is mostly filled with *trivial matters*, therefore we shall allow ourselves to be a bit sketchy with proofs.

Definition 26.2.1.1. (**Cochain complexes, maps and cohomology**) A cochain complex $A^\bullet$ is a sequence of object $\{A^i\}_{i\in\mathbb{Z}}$ with a map $d^i : A^i \to A^{i+1}$ called the coboundary maps which satisfies $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. A map $f : A^\bullet \to B^\bullet$ of cochain complexes is defined as a collection of maps $f^i : A^i \to B^i$ such that the following commutes

$$\begin{tikzcd} A^i \arrow[r, "d^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d^i"'] & B^{i+1} \end{tikzcd} .$$

That is, $d^i f^i = f^{i+1} d^i$ for each $i \in \mathbb{Z}$. For a cochain complex $A^\bullet$, we define the i^{th} cohomology object as the quotient

$$h^i(A^\bullet) := \operatorname{Ker}\left(d^i\right)/\operatorname{Im}\left(d^{i-1}\right).$$

With the obvious notion of composition, we thus obtain a category of cochain complexes $\operatorname{coCh}(\mathbf{A})$ over the abelian category $\mathbf{A}$.

We now show that h^i forms a functor over $\operatorname{coCh}(\mathbf{A}\ )$.

Lemma 26.2.1.2. *Let* $\mathbf{A}$ *be an abelian category. The* i^{th}*-cohomology assignment is a functor*

$$\begin{aligned} h^i : \operatorname{coCh}(\mathbf{A}\ ) &\longrightarrow \mathbf{A} \\ A^\bullet &\longmapsto h^i(A^\bullet). \end{aligned}$$

Proof. For a map of complexes $f : A^\bullet \to B^\bullet$, we first define the map $h^i(f)$

$$\begin{aligned} h^i(f) : h^i(A^\bullet) &\longrightarrow h^i(B^\bullet) \\ a + \operatorname{Im}\left(d^{i-1}\right) &\longmapsto f^i(a) + \operatorname{Im}\left(d^{i-1}\right). \end{aligned}$$

This is well defined group homomorphism. Further, it is clear that this is functorial. □

With this, we obtain the cohomology long-exact sequence.

Lemma 26.2.1.3. *Let* $\mathbf{A}$ *be an abelian category and* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ *a short-exact sequence in* $\operatorname{coCh}(\mathbf{A})$*. Then there is a map* $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ *for each* $i \in \mathbb{Z}$ *such that the following is a long exact sequence*

$$
\begin{tikzcd}
 & & & {} \arrow[llld, dotted] \\
h^i(A^\bullet) \arrow[r] & h^i(B^\bullet) \arrow[r] & h^i(C^\bullet) \arrow[lld, "\delta^i"'] & \\
h^{i+1}(A^\bullet) \arrow[r] & h^{i+1}(B^\bullet) \arrow[r] & h^{i+1}(C^\bullet) \arrow[lld, dotted] & \\
{} & & &
\end{tikzcd}
.
$$

Proof. (*Sketch*) The proof relies on chasing an element $c \in \operatorname{Ker}(d)$ of C^i till we obtain an element $a \in A^{i+1}$ in $\operatorname{Ker}(d)$, in the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & A^{i-1} \arrow[r] \arrow[d, "d"'] & B^{i-1} \arrow[r] \arrow[d, "d"'] & C^{i-1} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{i} \arrow[r] \arrow[d, "d"'] & B^{i} \arrow[r] \arrow[d, "d"'] & C^{i} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r] & B^{i+1} \arrow[r] & C^{i+1} \arrow[r] & 0
\end{tikzcd}
$$

The chase is straightforward and is thus omitted. The resultant map is indeed a well-defined group homomorphism. □

We now define homotopy of maps of complexes

Definition 26.2.1.4. (**Homotopy between maps**) Let $\mathbf{A}$ be an abelian category and $f, g : A^\bullet \to B^\bullet$ be two maps of cochain complexes. Then a homotopy between f and g is defined to be a collection of maps $k := \{k^i : A^i \to B^{i-1}\}_{i \in \mathbb{Z}}$ such that $f^i - g^i = dk^i + k^{i+1}d$ for each $i \in \mathbb{Z}$:

$$
\begin{tikzcd}
 & A^i \arrow[r, "d"'] \arrow[dl, "k^i"'] \arrow[d, shift right, "g^i"'] \arrow[d, shift left, "f^i"] & A^{i+1} \arrow[dl, "k^{i+1}"] \\
B^{i-1} \arrow[r, "d"'] & B^i &
\end{tikzcd}
.
$$

As one might expect, homotopic maps induces *same* (not isomorphic, but actually same) maps on cohomology.

Lemma 26.2.1.5. *Let* $\mathbf{A}$ *be an abelian category and* $f, g : A^\bullet \to B^\bullet$ *be two maps of cochain complexes. If* $k : f \sim g$ *is a homotopy between* f *and* g*, then* $h^i(f) = h^i(g)$ *as maps* $h^i(A^\bullet) \to h^i(B^\bullet)$ *for all* $i \in \mathbb{Z}$*.*

Proof. (*Sketch*) Pick any $a \in \operatorname{Ker}(d)$ in A^i. We wish to show that $f^i(a) - g^i(a) \in \operatorname{Im}(d)$. This follows from unravelling the definition of homotopy $k : f \sim g$. □

We now define the notion of exact functors between two abelian categories.

Definition 26.2.1.6. (**Exactness of functors**) Let $\mathbf{A}$ and $\mathbf{B}$ be abelian categories. A functor $F : \mathbf{A} \to \mathbf{B}$ is said to be

1. *additive* if the map $\mathrm{Hom}_{\mathbf{A}}(A, B) \to \mathrm{Hom}_{\mathbf{B}}(FA, FB)$ is a group homomorphism,
2. *left exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $0 \to FA' \to FA \to FA''$ is exact,
3. *right exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $FA' \to FA \to FA'' \to 0$ is exact,
4. *exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $0 \to FA' \to FA \to FA'' \to 0$ is exact,
5. *exact at middle* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $FA' \to FA \to FA''$ is exact.

Remark 26.2.1.7. It is important to keep in mind that all the above definitions are made for *short* exact sequences; a left exact A functor may not map a long exact sequence $0 \to A_1 \to \dots$ to a long exact sequence $0 \to FA_1 \to \dots$.

There are two prototypical examples of such functors in the category of R-modules.

Example 26.2.1.8. ($- \otimes_R M$ and $\mathrm{Hom}_R(M, -)$) Let R be a commutative ring and M be an R-module. It is a trivial matter to see that the functor $- \otimes_R M : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ is right exact but not left exact as applying $- \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ on the following shows where $\gcd(n, m) = 1$:

$$0 \to n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Indeed, $n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is not injective as the former is an infinite ring whereas the latter is finite.

Consider the covariant hom-functor $\mathrm{Hom}_R(M, -) : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$. This can easily be seen to be left exact. This is not right exact as applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ to the above exact sequence would yield (note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$).

We next dualize the above theory study the dual notion of homology, without much change. **TODO.**

26.2.2 Resolutions

We begin with injective objects, resolutions and having enough injectives.

Definition 26.2.2.1. (**Injective objects and resolutions**) Let $\mathbf{A}$ be an abelian category. An object $I \in \mathbf{A}$ is said to be injective if the functor thus represented, $\mathrm{Hom}_{\mathbf{A}}(-, I) : \mathbf{A}^{\mathrm{op}} \to \mathbf{AbGrp}$ is exact. An injective resolution of an object $A \in \mathbf{A}$ is an exact cochain complex

$$A \xrightarrow{\epsilon} I^0 \to I^1 \to \dots$$

where each I^i is an injective object. We denote an injective resolution of A by $\epsilon : A \to I^\bullet$.

The following are equivalent characterizations of injective objects.

Proposition 26.2.2.2. *Let* $\mathbf{A}$ *be an abelian category and* $I \in \mathbf{A}$*. Then the following are equivalent*

1. *The functor* $\mathrm{Hom}_{\mathbf{A}}(-, I)$ *is exact.*
2. *For any monomorphism* $i : A \to B$ *and any map* $f : A \to I$*, there is an extension* $\tilde{f} : B \to I$ *to make following commute*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\tilde{f}"] \\ & I & \end{tikzcd}.$$

3. *Any exact sequence*

$$0 \to I \to A \to B \to 0$$

splits.

Proof. 1. $\Rightarrow$ 2. is immediate from definition. 2. $\Rightarrow$ 3. follows from using the universal property of item 2 on $\mathrm{id} : I \to I$ and monomorphism $0 \to I \to A$. For 3. $\Rightarrow$ 1., we need only check right exactness of $\mathrm{Hom}_{\mathbf{A}}(-, I)$, which follows immediately from item 3. □

The following are some properties of injective objects.

Proposition 26.2.2.3. *Let* $\mathbf{A}$ *be an abelian category. If* $\{I_i\}_i$ *is a collection of injective objects of* $\mathbf{A}$ *and* $\prod_i I_i$ *exists, then it is injective.*

Proof. As $\mathrm{Hom}_{\mathbf{A}}(-, \prod_i I_i) \cong \prod_i \mathrm{Hom}_{\mathbf{A}}(-, I_i)$ and arbitrary product of surjective maps is surjective, therefore the claim follows. □

We see that any two injective resolutions of an object are homotopy equivalent.

Lemma 26.2.2.4. *Let* $\mathbf{A}$ *be an abelian category and* $A \in \mathbf{A}$ *be an object with two injective resolutions* $\epsilon : A \to I^\bullet$ *and* $\eta : A \to J^\bullet$*. Then there exists a homotopy* $k : \epsilon \sim \eta$*.*

Proof. Comparison Theorem 2.3.7, pp 40, [cite Weibel Homological Algebra]. □

We then define when an abelian category has enough injectives.

Definition 26.2.2.5. (**Enough injectives**) An abelian category $\mathbf{A}$ is said to have enough injectives if for each object $A \in \mathbf{A}$, there is an injective object $I \in \mathbf{A}$ such that A is a subobject of I, $A \leq I$.

In such abelian categories, all objects have injective resolutions.

Lemma 26.2.2.6. *Let* $\mathbf{A}$ *be an abelian category with enough injectives. Then all objects* $A \in \mathbf{A}$ *admit injective resolutions* $\epsilon : A \to I^\bullet$*.*

Proof. Pick any object $A \in \mathbf{A}$. As $\mathbf{A}$ has enough injectives, therefore we have $0 \to A \xrightarrow{\epsilon} I^0$. Consider $\mathrm{CoKer}(\epsilon)$ and let it be embedded in some injective object I^1, which yields the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook, "\epsilon"] & I^0 \arrow[r, "d"] \arrow[d, two heads] & I^1 \\
& & \mathrm{CoKer}(\epsilon) \arrow[ur, hook] &
\end{tikzcd}.$$

Continue this diagram by considering $\mathrm{CoKer}(d)$ which embeds in some other injective I^2 to further yield the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook, "\epsilon"] & I^0 \arrow[rr, "d"] \arrow[d, two heads] & & I^1 \arrow[r, "d"] \arrow[d, two heads] & I^2 \\
& & \mathrm{CoKer}(\epsilon) \arrow[urr, hook] & & \mathrm{CoKer}(d) \arrow[ur, hook] &
\end{tikzcd}.$$

This builds the required injective resolution. $\square$

We now give examples of abelian categories with enough injectives. Recall that a *divisible group* G is an abelian group such that for any $g \in G$ and ay $n \in \mathbb{Z}$ there exists $h \in G$ such that $g = nh$ (see Definition 23.13.1.1).

Theorem 26.2.2.7. *Let R be a commutative ring with 1. Then,*

1. *Any divisible group in* **AbGrp** *is an injective object.*
2. *If G is an injective abelian group, then* $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ *is an injective R-module.*
3. **AbGrp** *is an abelian category which has enough injectives.*
4. $\mathbf{Mod}(R)$ *is an abelian category which has enough injectives.*

Proof. The main idea of the proofs of the later parts is to use injective objects constructed in a bigger category and an adjunction to a lower category to construct injectives in the smaller subcategory. Further, embedding each object in a large enough product of injectives (which would remain injective by Proposition 26.2.2.3) would show enough injectivity.

1. By Corollary 23.13.1.3, the statement follows.

2. Recall that $F(-) : \mathbf{Mod}(R) \rightleftarrows \mathbf{AbGrp} : \mathrm{Hom}_{\mathbb{Z}}(R, -)$ is an adjunction, where F is the forgetful functor. Consequently $\mathrm{Hom}_{\mathbb{Z}}(F(M), G) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, G))$. It then follows that $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ is injective.

3. Observe that $\mathbb{Q}/\mathbb{Z}$ is a divisible, thus injective abelian group by item 1. Let G be an abelian group. Consider the abelian group

$$I = \prod_{\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}.$$

By Proposition 26.2.2.3, I is an injective abelian group. We now construct an injection $\varphi : G \to I$, which would complete the proof. We have the canonical map

$$\begin{aligned} \theta : G &\longrightarrow I \\ g &\longmapsto (\varphi(g))_{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})}. \end{aligned}$$

For this to be well-defined, we need to show that $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ is non-zero. Indeed, we claim that for any element $g \in G$, there is a $\mathbb{Z}$-linear map $\varphi_g : G \to \mathbb{Q}/\mathbb{Z}$ such that $\varphi_g(g) \neq 0$. This would suffice as if $\theta(g) = 0$ for some $g \in G$, then $\varphi(g) = 0$ for all $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. Consequently, $\varphi_g(g) = 0$, which cannot happen, hence θ is injective. So we need only show the existence of φ_g. Indeed, if $|g| = \infty$, then we have an injection $\mathbb{Z} \hookrightarrow G$ taking $1 \mapsto g$. Pick any non-zero map $f : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. By injectivity of $\mathbb{Q}/\mathbb{Z}$, f extends to $\varphi_g : G \to \mathbb{Q}/\mathbb{Z}$ which is non-zero at g. On the other hand, if $|g| = k < \infty$, then consider the inclusion $\mathbb{Z}/k\mathbb{Z} \hookrightarrow G$ taking $\bar{1} \mapsto g$. Then, consider the map $f : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ taking $\bar{1} \mapsto \frac{1}{k}$. Then, by injectivity of $\mathbb{Q}/\mathbb{Z}$, it extends to $\varphi_g : G \to \mathbb{Q}/\mathbb{Z}$ which is non-zero at g.

4. Pick any R-module M. We wish to find an injective R-module I such that $M \leq I$. By items 1 and 2, we know that $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is an injective R-module. By the proof of item 2, we also know that

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})).$$

Conseuqently, by Proposition 26.2.2.3, we have an injective module

$$I = \prod_{\mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))} \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}),$$

We claim that the following map

$$\begin{aligned} \theta : M &\longrightarrow I \\ m &\longmapsto (\varphi(m))_{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))} \end{aligned}$$

is injective. Indeed, we claim that for each $m \in M$, there exists $\varphi_m \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$ such that $\varphi_m(m) \neq 0$. By the above isomorphism, we equivalently wish to show the existence of $g_m \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ such that $g_m(m) \neq 0$. This is immediate from the proof of item 3. $\square$

We next dualize the above theory and study projective objects, projective resolutions and having enough projectives. **TODO**.

26.2.3 Derived functors and general properties

For each covariant left exact functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories, we will produce a sequence of functors R^iF for each $i \geq 0$.

Definition 26.2.3.1. (**Right derived functors of a left-exact functor**) Let $F : \mathbf{A} \to \mathbf{B}$ be a left exact functor of abelian categories where $\mathbf{A}$ has enough injectives. Then, define for each $i \geq 0$ the following

$$\begin{aligned} R^iF : \mathbf{A} &\longrightarrow \mathbf{B} \\ A &\longmapsto h^i(F(I^\bullet)) \end{aligned}$$

where $\epsilon : A \to I^\bullet$ is any injective resolution of A. We call R^iF the i^{th} right derived functor of the left exact functor F.

Remark 26.2.3.2. Indeed the above definition is well-defined, by Lemmas 26.2.1.5 and 26.2.2.4. Further, keep in mind the Remark 26.2.1.7.

Some of the basic properties of right derived functors are as follows. First, the 0^{th}-right derived functor of F is canonically isomorphic to F.

Lemma 26.2.3.3. *Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ has enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor. Then, there is a natural isomorphism*

$$R^0 F \cong F.$$

Proof. Pick any object $A \in \mathbf{A}$ with an injective resolution $0 \to A \stackrel{\epsilon}{\to} I^\bullet$. Consequently, $R^0 F(A)$ is the cohomology of

$$0 \to F(I^0) \stackrel{Fd^0}{\to} F(I^1),$$

that is, $R^0 F(A) = \operatorname{Ker}\left(Fd^0\right)$. But since F is left-exact and we have the following exact sequence

$$0 \to A \stackrel{\epsilon}{\to} I^0 \stackrel{d^0}{\to} \operatorname{Im}\left(d^0\right),$$

therefore we get that $\operatorname{Ker}\left(Fd^0\right) = \operatorname{Im}(F\epsilon)$. This also needs the observation that if F is left-exact, then for any map $f \in \mathbf{A}$, we have $F(\operatorname{Im}(f)) \cong \operatorname{Im}(Ff)$. Since ϵ is injective, then so is $F\epsilon$ and thus $\operatorname{Im}(F\epsilon) \cong FA$. $\square$

Remark 26.2.3.4. Let $I \in \mathbf{A}$ be an injective object. Then we claim that $R^i F(I) = 0$ for all $i \geq 1$. Indeed, this follows immediately because we have $0 \to I \stackrel{\text{id}}{\to} I \to 0$ as a trivial injective resolution of I.

The following is an important property of right derived functors which makes them ideal for defining the general notion of cohomology, because they always have long exact sequene in cohomology.

Theorem 26.2.3.5. *Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ has enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor. If*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathbf{A}$, then we have a long exact sequence in right derived functors of F as in

$$\begin{tikzcd}
0 \arrow[r] & R^0FA \arrow[r] & R^0FB \arrow[r] & R^0FC \arrow[dll, "\delta_0"'] \\
 & R^1FA \arrow[r] & R^1FB \arrow[r] & R^1FC \arrow[dll, dotted, "\delta_1"'] \\
 & {} & &
\end{tikzcd}.$$

It follows from above theorem that if F is exact, then $R^i F$ are trivial for $i \geq 1$.

Corollary 26.2.3.6. *Let* $\mathbf{A}, \mathbf{B}$ *be two abelian categories where* $\mathbf{A}$ *has enough injectives. Let* $F : \mathbf{A} \to \mathbf{B}$ *be an exact functor. Then,*

$$R^i F = 0$$

for all $i \geq 1$.

Proof. Pick any object $A \in \mathbf{A}$ and let $I \in A$ be an an injective object such that $0 \to A \to I$ is an injective map. Then we have a short-exact sequence

$$0 \to A \to I \to B \to 0$$

where $B = I/A$. By Theorem 26.2.3.5, Lemma 26.2.3.3 and Remark 26.2.3.4, it follows that we have a long exact sequence in right derived functors of F as in

$$\begin{tikzcd}
R^iFA \arrow[r] & 0 \arrow[r] & R^iFB \arrow[dll, "\delta_i"'] \\
R^{i+1}FA \arrow[r] & 0 \arrow[r] & R^{i+1}FB
\end{tikzcd}.$$

It follows from exactness of the above sequence that $R^iFB \cong R^{i+1}FA$ for all $i \geq 1$. Repeating the same process for B (embedding B into an injective object and observing the resultant long exact sequence), we obtain that

$$R^{i+1}FA \cong R^1FC$$

for some object $C \in \mathbf{A}$. Replacing A by C, it thus suffices to show that $R^1FA = 0$.

In the beginning of the above long exact sequence we have

$$\begin{tikzcd}
0 \arrow[r] & FA \arrow[r] & FI \arrow[r, two heads] & FB \arrow[dll, "\delta_0"'] \\
& R^1FA \arrow[r] & 0 \arrow[r] & R^1FB
\end{tikzcd}$$

from which it follows via exactness that δ_0 is surjective and $\operatorname{Ker}(\delta_0) = FB$. We then deduce that $R^1FA = 0$, as required. □

Injective resolutions might be hard to find in general, but given a left exact functor F, it would be somewhat easier to find objects J such that $R^iF(J) = 0$ for all $i \geq 1$. The remarkable property of such objects is that it can help to calculate the value of right derived functors of F for objects admitting resolutions by them.

Definition 26.2.3.7 (**Acyclic resolution**)**.** Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ have enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor. An object $J \in \mathbf{A}$ is said to be acyclic if $R^iF(J) = 0$ for all $i \geq 1$. An acyclic resolution of $A \in \mathbf{A}$ is an exact sequence of the form

$$0 \to A \xrightarrow{\epsilon} J^0 \to J^1 \dots$$

where each J^i is acyclic.

The name "acyclic" is justified since they have zero cohomology, so all cocycles are coboundaries, so there are no cycles for that object.

Remark 26.2.3.8. Note that for an acyclic resolution $0 \to A \xrightarrow{\epsilon} J^\bullet$, we have $h^0(F(J^\bullet)) \cong FA$ by following the steps as in the proof of Lemma 26.2.3.3.

We then have the following useful theorem.

Proposition 26.2.3.9. *Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ have enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor. For $A \in \mathbf{A}$, let $0 \to A \xrightarrow{\epsilon} J^\bullet$ be an acyclic resolution. Then for all $i \geq 0$, there is a natural isomorphism*

$$R^iF(A) \cong h^i(F(J^\bullet)).$$

Derived functors are equivalent to datum of what is defined to be a universal δ-functor. In the rest of this section we setup the definitions and only state the result.

TODO : Universal δ-functors.

26.3 Filtrations

26.4 Gradings

26.5 Spectral sequences

Do from Weibel Chapter 5, Eisenbud Appendix 3 and Stacksproject.

Chapter 27

Foundational Sheaf Theory

Contents

The notion of sheaves plays perhaps the most important role in modern viewpoint of geometry. It is thus important to understand the various constructions that one can make on them. We assume the reader knows the definition of a sheaf on a space X and morphism of sheaves. We begin with some recollections.

27.1 Recollections

Remark 27.1.0.1 (*Map on stalks*)**.** Recall that a map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X defines for each point $x \in X$ a map of stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ given by $s_x \mapsto \varphi_U(s)_x$ where s is a section of $\mathcal{F}$ over $U \subseteq X$. One can check quite easily that this is well-defined and that this map φ_x is in-fact the unique map given by the universal property of the colimit in the diagram below:

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r] & \varinjlim_{V \ni x} \mathcal{G}(V) \\ \mathcal{F}(U) \arrow[r] \arrow[u, "\varphi_U"] & \varinjlim_{V \ni x} \mathcal{F}(V) \arrow[u, dashed, "\varphi_x"'] \end{tikzcd}.$$

Hence φ_x is the unique map which makes the above diagram commute.

Remark 27.1.0.2 (*Subsheaves*)**.** Recall that $\mathcal{F} \hookrightarrow \mathcal{G}$ is a subsheaf if $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ such that for $U \hookrightarrow V$, the restriction map $\rho_{V,U} : \mathcal{G}(V) \to \mathcal{G}(U)$ restricts to $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$.

Remark 27.1.0.3 (*Constant sheaves*)**.** For an abelian group A and a space X, one defines the constant sheaf $\mathcal{A}$ as the sheaf which for each open set $U \subseteq X$ assigns $\mathcal{A}(U) = \{s : U \to A \mid s \text{ is continuous}\}$, where A is given the discrete topology. One sees instantly that this is a sheaf. Further one observes that if $U = U_1 \amalg \cdots \amalg U_k$ where U_i are components of open set U and U_i are open, then $\mathcal{A}(U) \cong A \oplus \cdots \oplus A$ k-times. In particular, for any open connected subset U, we get $\mathcal{A}(U) = A$.

We now begin by showing how to construct a sheaf out of a presheaf over X.

27.2 The sheafification functor

Let X be a topological space, denote the category of presheaves on X by $\mathbf{PSh}(X)$ and denote the category of sheaves over X by $\mathbf{Sh}(X)$. We have a canonical inclusion functor $i : \mathbf{PSh}(X) \hookrightarrow \mathbf{Sh}(X)$. We construct it's left adjoint commonly known as the process of sheafifying a presheaf.

Theorem 27.2.0.1. *(Sheafification) Let X be a topological space and let F be a presheaf over X. Then there exists a pair $(\mathcal{F}, i_F)$ of a sheaf $\mathcal{F}$ and a map $i_F : F \to \mathcal{F}$ such that for any sheaf $\mathcal{G}$ and a morphism of presheaves $\varphi : F \to \mathcal{G}$, there exists a unique morphism of sheaves $\tilde{\varphi} : \mathcal{F} \to \mathcal{G}$ such that the following commutes*

$$\begin{tikzcd} \mathcal{F} \arrow[r, dashed, "\tilde{\varphi}"] & \mathcal{G} \\ F \arrow[u, "i_F"] \arrow[ur, "\varphi"'] & \end{tikzcd},$$

that is, we have a natural bijection

$$\mathrm{Hom}_{\mathbf{PSh}(X)}(F, \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, \mathcal{G}).$$

Moreover:

1. *$(\mathcal{F}, i_F)$ is unique upto unique isomorphism.*
2. *For every $x \in X$, the map on stalks $i_{F,x} : F_x \to \mathcal{F}_x$ is bijective.*
3. *For any map of presheaves $\varphi : F \to G$, we get a map of sheaves $\tilde{\varphi} : \mathcal{F} \to \mathcal{G}$ such that the natural square commutes:*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\tilde{\varphi}"] & \mathcal{G} \\ F \arrow[u, "i_F"] \arrow[r, "\varphi"'] & G \arrow[u, "i_G"'] \end{tikzcd}.$$

Hence we have a functor

$$\begin{aligned} (-)^{++} : \mathbf{PSh}(X) &\longrightarrow \mathbf{Sh}(X) \\ F &\longmapsto F^{++} := \mathcal{F}. \end{aligned}$$

Proof. We explicitly construct the sheaf $\mathcal{F}$ out of F. We define the local sections of $\mathcal{F}$ by using germs and turning the gluing condition of sheaf definition onto itself. In particular, define

$$\mathcal{F}(U) := \left\{ ((s_{i_x})_x) \in \prod_{x \in U} F_x \mid \forall x \in U,\ \exists \text{ open } W \ni x \;\&\; t \in F(W) \text{ s.t. } \forall p \in W,\ t_p = (s_{i_p})_p \right\}.$$

The restriction map for $U \hookrightarrow V$ of $\mathcal{F}$ is given by $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, $((s_{i_x})_x) \mapsto ((s_{i_x})_x)_{x \in U}$, that is, $\rho_{V,U}$ is just the projection map. Next, we show that $\mathcal{F}$ satisfies the gluing criterion and that is where we will see how the above definition of sections of $\mathcal{F}$ came about. Take an open set $U \subseteq X$ and an open cover $U = \bigcup_{i \in I} U_i$. Let $s_i \in \mathcal{F}(U_i)$ be a corresponding collection of sections such that for all $i, j \in I$, we have $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$. We wish to thus construct a section $t \in \mathcal{F}(U)$ such that $\rho_{U,U_i}(t) = s_i$ for all $i \in I$. Indeed let $((t_{i_x})_x) \in \prod_{x \in U} F_x$ where $t := (t_{i_x})_x = (s_i)_x$ if $x \in U_i$. Then since for any $x \in U$, there exists $U \supseteq U_i \ni x$ and $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U,U_i}(t) = ((t_{i_x})_x)_{x \in U_i} = ((s_i)_x)_{x \in U_i}$, we thus conclude that $t \in \mathcal{F}(U)$. So $\mathcal{F}$ satisfies the gluing condition. The locality is quite simple. Next the map i_F is given as follows on sections:

$$\begin{aligned} i_{F,U} : F(U) &\longrightarrow \mathcal{F}(U) \\ s &\longmapsto (s_x). \end{aligned}$$

Now, it can be seen by definition of colimits that $\mathcal{F}_x = F_x$. Finally, let $\mathcal{G}$ be a sheaf and let $\varphi : F \to \mathcal{G}$ be a morphism of presheaves, then we can define $\tilde{\varphi}$ by gluing the germs as follows:

$$\begin{aligned} \tilde{\varphi}_U : \mathcal{F}(U) &\longrightarrow \mathcal{G}(U) \\ ((s_{i_x})_x) &\longmapsto [\varphi_{W_x}(s_{i_x})] \end{aligned}$$

where $[\varphi_{W_x}(s_{i_x})]$ denotes the unique section in $\mathcal{G}(U)$ that one gets by considering the open cover $\bigcup_{x \in U} W_x$ where $s_{i_x} \in \mathcal{F}(W_x)$ and considering the gluing of corresponding sections $\varphi_{W_x}(s_{i_x}) \in \mathcal{G}(W_x)$. These sections agree on intersections because φ is a natural transformation and (s_{i_x}) agree on intersections as sections of $\mathcal{F}(U)$. Hence we have the unique map $\tilde{\varphi}$. Moreover, it is clear that $\tilde{\varphi} \circ i_F = \varphi$. □

Corollary 27.2.0.2. *Let F be a presheaf over a topological space X, then for all $x \in X$, $F_x = (F^{++})_x$.*

Proof. By construction of F^{++}. □

Corollary 27.2.0.3. *If $\mathcal{F}$ is a sheaf over a topological space X, then $\mathcal{F}^{++} = \mathcal{F}$.*

Proof. Follows immediately from the universal property of the sheafification, Theorem 27.2.0.1. □

Remark 27.2.0.4. The sections of sheaf $\mathcal{F}$ in an open set U containing x is defined in such a manner so that $f \in \mathcal{F}(U)$ can be constructed locally out of sections of F. In particular, we can write $\mathcal{F}(U)$ more clearly as follows

$$\mathcal{F}(U) = \left\{ s : U \to \coprod_{x \in U} F_x \mid \forall x \in U,\ s(x) \in F_x \;\&\; \exists \text{ open } x \in V \subseteq U \;\&\; \exists t \in F(V) \text{ s.t. } s(y) = t_y\ \forall y \in V \right\}.$$

Note that this is exactly the realization that $\mathcal{F}(U)$ is the set of section of the étale space of the sheaf $\mathcal{F}$ (see Section 27.4). Most of the time in practice, we would work with the universal property of $\mathcal{F}$ in Theorem 27.2.0.1 as it is much more amenable, but the above must be kept in mind as it is used, for example, to make sure that certain algebraic constructions of $\mathcal{O}_X$-modules remains $\mathcal{O}_X$-modules (no matter how trivial they may sound).

We note that sheafification and restrictions to open sets commute.

Lemma 27.2.0.5. *Let X be a space, $U \subseteq X$ be an open subset and F be a presheaf over X. Then,*

$$(F|_U)^{++} \cong (F^{++})\big|_U .$$

Proof. Immediate from universal property of sheafification (Theorem 27.2.0.1). □

27.3 Morphisms of sheaves

All sheaves are abelian sheaves in this section. One of the most important aspects of using sheaves is that the injectivity and bijectivity of φ_x can be checked on sections. We first show that taking stalks is functorial

Lemma 27.3.0.1. *Let X be a topological space, $\mathcal{F}, \mathcal{G}$ be two sheaves over X and $x \in X$ be a point. Then the following mapping is functorial:*

$$\begin{aligned} \mathbf{Sh}(X) &\longrightarrow \mathbf{AbGrp} \\ \mathcal{F} &\longmapsto \mathcal{F}_x \\ \mathcal{F} \stackrel{f}{\to} \mathcal{G} &\longmapsto \mathcal{F}_x \stackrel{f_x}{\to} \mathcal{G}_x. \end{aligned}$$

Proof. Immediate, just remember how composition of two natural transforms is defined. □

Another simple lemma about sheaves and stalks is that equality of two sections can be checked at the stalk level.

Lemma 27.3.0.2. *Let X be a topological space and $\mathcal{F}$ be a sheaf over X. If $s, t \in \mathcal{F}(U)$ for some open $U \subseteq X$ such that $(U, s)_x = (U, t)_x \; \forall x \in U$, then $s = t$ in $\mathcal{F}(U)$.*

Proof. By equality on stalks, it follows that we have an open set $W_x \ni x$ in U for all $x \in U$ such that $\rho_{U,W_x}(s) = \rho_{U,W_x}(t)$. The result follows from the unique gluing property of sheaf $\mathcal{F}$. □

The above result therefore show why almost all the time it is enough to work with stalks in geometry. Let us now define an injective and surjective map of sheaves.

Definition 27.3.0.3. (**Injective & surjective maps**) Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves on X. A map of sheaves $f : \mathcal{F} \to \mathcal{G}$ is said to be

1. *injective* if for all opens $U \subseteq X$, the local homomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective,

2. *surjective* if for all opens $U \subseteq X$ and all $s \in \mathcal{G}(U)$, there exists an open covering $\{U_i\}_{i\in I}$ such that $\rho_{U,U_i}(s) \in \mathrm{Im}\,(f_{U_i})$,
3. *bijective* if f is injective and surjective.

Heuristically, one may understand the notion of f being surjection by saying that every local section of $\mathcal{G}$ is locally constructible by the image of $\mathcal{F}$ under the map f.

For each map of sheaves, we can also define two corresponding sheaves which are global algebraic analogues of the local algebraic constructions.

Definition 27.3.0.4. (**Quotient sheaf**) Let X be a topological space and $\mathcal{F}$ be a sheaf on X. For a subsheaf $\mathcal{S} \subseteq \mathcal{F}$, one defines the quotient sheaf $\mathcal{F}/\mathcal{S}$ as the sheafification of the presheaf F/S defined on open sets $U \subseteq X$ by

$$F/S(U) := \mathcal{F}(U)/\mathcal{S}(U).$$

Definition 27.3.0.5. (**Image & kernel sheaves**) Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X and $f : \mathcal{F} \to \mathcal{G}$ be a morphism. Then,

1. *image sheaf* is the sheafification of the presheaf $\mathrm{Im}\,(f)$ defined on open sets $U \subseteq X$ by

$$(\mathrm{Im}\,(f))(U) := \mathrm{Im}\,(f_U),$$

   and we denote it by the same symbol, $\mathrm{Im}\,(f)$,
2. *kernel sheaf* is the sheafification of the presheaf $\mathrm{Ker}\,(f)$ defined on open sets $U \subseteq X$ by

$$(\mathrm{Ker}\,(f))(U) := \mathrm{Ker}\,(f_U)$$

   and we denote it by the same symbol, $\mathrm{Ker}\,(f)$.

In both the above definitions, the important aspect is the sheafification of the canonical presheaves.

The main point is that one can check all the three notions introduced in Definition 27.3.0.3 for $f : \mathcal{F} \to \mathcal{G}$ by checking on stalks $f_x : \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$.

Theorem 27.3.0.6. [1] *Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X. Then, a map $f : \mathcal{F} \to \mathcal{G}$ is*

1. *injective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$,*
2. *surjective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$,*
3. *bijective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective for all $x \in X$,*
4. *an isomorphism if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective for all $x \in X$*[2].
5. *an isomorphism if and only if $f : \mathcal{F} \to \mathcal{G}$ is bijective.*

[1]Exercise II.1.2, II.1.3 and II.1.5 of Hartshorne.

[2]In general, we should write "*... if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism*", but since we are in the setting of abelian sheaves and bijective homomorphism of abelian groups is an isomorphism, so we can get away with this.

Proof. The proof is more of an exercise to get a familiarity with the flexibility of sheaf language. The main idea almost everywhere is to do some local calculations and use sheaf axioms to construct a unique section out of local sections.

1. (L $\Rightarrow$ R) We wish to show that f_x is injective. Suppose for two $(U,s)_x, (V,t)_y \in \mathcal{F}_x$ we have $f_x((U,s)_x) = f_x((V,t)_x) \in \mathcal{G}_x$, which translates to $(U, f_U(s))_x = (V, f_U(t))_x$. We wish to show that $(U,s)_x = (V,t)_y$. By definition of equality on stalks, we obtain open $W \subseteq U \cap V$ containing x such that
$$\rho_{U,W}(f_U(s)) = \rho_{V,W}(f_V(t)).$$
By the fact that f is a natural transformation, we further translate the above equality to
$$f_W(\rho_{U,W}(s)) = f_W(\rho_{V,W}(t)).$$
By injectivity of homomorphism f_W, we obtain
$$\rho_{U,W}(s) = \rho_{V,W}(t)$$
in $\mathcal{F}(W)$. Hence by the definition of equality on stalks, we obtain $(U,s)_x = (V,t)_x$.

   (R $\Rightarrow$ L) Pick any open $U \subseteq X$. We wish to show that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. Let $s \in \mathcal{F}(U)$ be such that $f_U(s) = 0$. Thus for all $x \in U$, we have $(U, f_U(s))_x = 0$. Further, by definition of the map f_x, we obtain $f_x((U,s))_x = (U, f_U(s))_x = 0$. By injectivity of f_x, we obtain $(U,s)_x = 0$ for all $x \in U$[3]. By definition of equality on stalks, we obtain an open cover $\{W_x\}_{x\in U}$ such that $x \in W_x$ and $s|_{W_x} := \rho_{U,W_x}(s) = 0$. Since f is a natural transformation, we therefore obtain that $\{s|_{W_x}\}_{x\in U}$ is a matching family, i.e. on intersections of W_x, W_y, the corresponding sections agree. Hence, there is a unique glue of $\{s|_{W_x}\}_{x\in U}$ denote $t \in \mathcal{F}(U)$. Since each $s|_{W_x} = 0$, therefore we have two glues of the family over U, one is 0 and the other is s. By uniqueness of the glue, it follows that $s = 0$.
2. (L $\Rightarrow$ R) Pick any $x \in X$. We wish to show that $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective. Pick any $(V,t)_x \in \mathcal{G}_x$. We wish to show that for some open $U \ni x$, we have $(U,s)_x \in \mathcal{F}_x$ such that
$$(V,t)_x = (U, f_U(s))_x.$$
Since $t \in \mathcal{G}(V)$, therefore by surjectivity of f that there exists an open cover $\{V_i\}_{i\in I}$ of V such that
$$\rho_{V,V_i}(t) \in \mathrm{Im}\,(f_{V_i}).$$
Therefore we may pick $s_i \in \mathcal{F}(V_i)$ such that
$$\begin{aligned}\rho_{V,V_i}(t) &= f_{V_i}(s_i)\\ &= f_{V_i}(\rho_{V_i,V_i}(s_i))\\ &= \rho_{V_i,V_i}(f_{V_i}(s_i)).\end{aligned}$$

[3] We could be done right here by Lemma 27.3.0.2.

Thus, $(V,t)_x$ and $(V_i, f_{V_i}(s_i))_x$ are same.

(R $\Rightarrow$ L) We wish to show that $f : \mathcal{F} \to \mathcal{G}$ is surjective. Pick any open set $V \subseteq X$ and $t \in \mathcal{G}(V)$. We wish to find an open cover $\{W_i\}$ of V such that $s_i \in \mathcal{F}(V_i)$ and $f_{V_i}(s_i) = \rho_{V,V_i}(t)$. Since we have $(V,t)_x \in \mathcal{G}_x$ for all $x \in V$, therefore by surjectivity of each $f_x : \mathcal{F}_x \to \mathcal{G}_x$, we obtain germs $(W_x, s_x)_x \in \mathcal{F}_x$ such that $(W_x, f_{W_x}(s_x))_x = (V,t)_x$ for all $x \in V$. By shrinking W_x and restricting s_x, we may assume $\{W_x\}$ covers V. Thus we have an open cover of V such that for all $s_x \in \mathcal{F}(W_x)$, we have $f_{W_x}(s_x) = \rho_{V,W_x}(t)$.

3. Trivially follows from 1. and 2.
4. (L $\Rightarrow$ R) Use the fact that taking stalks is a functor (Lemma 27.3.0.1).
   (R $\Rightarrow$ L) Let $g_x : \mathcal{G}_x \to \mathcal{F}_x$ be the inverse homomorphism of f_x for each $x \in X$. Using g_x, we can easily construct a sheaf homorphism $g : \mathcal{G} \to \mathcal{F}$ which will be the inverse of f. Indeed, consider the following map for any open $U \subseteq X$

$$\begin{aligned} g_U : \mathcal{G}(U) &\longrightarrow \mathcal{F}(U) \\ t &\longmapsto s \end{aligned}$$

   where $s \in \mathcal{F}(U)$ is formed as the unique glue of the matching family

$$\{s_x \in \mathcal{F}(U_x)\}_{x \in U}$$

   where $(U,t)_x = (U_x, f_{U_x}(s_x))_x$ for each $x \in U$ and $U_x \subseteq U$. In particular, $s_x = g_x((U_x, \rho_{U,U_x}(t))_x)$. This is obtained via the bijectivity of f_x. Consequently, g is a sheaf homomorphism, which is naturally the inverse of f.
5. Follows from 3. and 4.

$\square$

The following theorem further tells us that our intuition about algebra can be globalized, and equality of sheaf morphisms can be checked on each stalk.

Theorem 27.3.0.7. *Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X. Then, a map $f : \mathcal{F} \to \mathcal{G}$*

1. *is injective if and only if the kernel sheaf* $\mathrm{Ker}\,(f)$ *is the zero sheaf,*
2. *is surjective if and only if the image sheaf* $\mathrm{Im}\,(f)$ *is* $\mathcal{G}$*,*
3. *is equal to another map $g : \mathcal{F} \to \mathcal{G}$ if and only if $f_x = g_x$ for all $x \in X$.*

Proof. The main idea in most of the proofs below is to either use the definition or the universal property of sheafification.

1. (L $\Rightarrow$ R) Let $f : \mathcal{F} \to \mathcal{G}$ be injective. We wish to show that $\mathrm{Ker}\,(f) = 0$. Since the kernel presheaf $\ker f = 0$, therefore its sheafification $\mathrm{Ker}\,(f) = 0$.
   (R $\Rightarrow$ L) Let $\mathrm{Ker}\,(f) = 0$. We wish to show that f is injective. Suppose to the contrary that f is not injective. We have that $(\mathrm{Ker}\,(f))_x = 0$ for all $x \in X$. Thus there exists an open set $U \subseteq X$ such that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is not injective. Hence, there exists, $0 \neq s \in \mathcal{F}(U)$ such that $f_U(s) = 0$. Thus, we have an element $(U,s)_x \in (\ker f)_x = (\mathrm{Ker}\,(f))_x = 0$ for all $x \in U$. Hence $s = 0$ by Lemma 27.3.0.2, which is a contradiction.

2. (L $\Rightarrow$ R) Let $f : \mathcal{F} \to \mathcal{G}$ be a surjective map. In order to show that $\mathrm{Im}\,(f) = \mathcal{G}$, we will show that $\mathcal{G}$ satisfies the universal property of sheafification (Theorem 27.2.0.1). For this, consider a sheaf $\mathcal{H}$ and a presheaf map $h : \mathrm{im}\,(f) \to \mathcal{H}$. Consider the inclusion map $\iota : \mathrm{im}\,(f) \hookrightarrow \mathcal{G}$. We will construct a unique sheaf map $\tilde{h} : \mathcal{G} \to \mathcal{H}$ which will be natural such that $\tilde{h} \circ \iota = h$. Pick any open set $U \subseteq X$. We wish to define the map
$$\tilde{h}_U : \mathcal{G}(U) \longrightarrow \mathcal{H}(U).$$
Take $t \in \mathcal{G}(U)$. By surjectivity of f, there exists a covering $\{U_i\}$ of U and matching family $s_i \in \mathcal{F}(U_i)$ for all i such that
$$f_{U_i}(s_i) = \rho_{U,U_i}(t) =: t_i.$$
We shall construct an element $\tilde{h}_U(t) \in \mathcal{H}(U)$. Indeed, we first claim that
$$\{h_{U_i}(t_i) \in \mathcal{H}(U_i)\}_i$$
is a matching family. This can be shown by keeping the following diagram in mind and the fact that $\{s_i\}$ is a matching family:
$$\begin{tikzcd}
\mathcal{H}(U_i) \arrow[d, "\rho_{U_i,U_i\cap U_j}"'] & \mathrm{im}\,(f_{U_i}) \arrow[l, "h_{U_i}"'] \arrow[d, "\rho_{U_i,U_i\cap U_j}"] & \mathcal{F}(U_i) \arrow[l, "f_{U_i}"'] \arrow[d, "\rho_{U_i,U_i\cap U_j}"] \\
\mathcal{H}(U_i \cap U_j) & \mathrm{im}\left(f_{U_i\cap U_j}\right) \arrow[l, "h_{U_i\cap U_j}"] & \mathcal{F}(U_i \cap U_j) \arrow[l, "f_{U_i\cap U_j}"]
\end{tikzcd}.$$
Thus we get a unique glue which we define to be the image of $\tilde{h}_U$ for the section $t \in \mathcal{G}(U)$, denoted $\tilde{h}_U(t) \in \mathcal{H}(U)$. Uniqueness and naturality follows from construction.
(R $\Rightarrow$ L) We have that $(\mathrm{im}\,(f))^{++} = \mathcal{G}$. Pick any open set $U \subseteq X$ and a section $t \in \mathcal{G}$. We wish to find an open cover $\{U_i\}_{i\in I}$ of U and $s_i \in \mathcal{F}(U_i)$ such that $f_{U_i}(s_i) = \rho_{U,U_i}(t)$ for all $i \in I$. Indeed, by Corollary 27.2.0.2, we obtain that $\mathcal{G}_x = \mathrm{im}\,(f)_x$ for all $x \in X$. Hence for the chosen (U, t), we obtain for each $x \in U$, by appropriately shrinking and restricting, an open set $W_x \subseteq U$ containing x and a section $s_x \in \mathcal{F}(W_x)$ satisfying $\rho_{U,W_x}(t) = f_{W_x}(s_x)$.
3. (L $\Rightarrow$ R) Trivial.
(R $\Rightarrow$ L) Suppose for all $x \in X$ we have $f_x = g_x : \mathcal{F}_x \to \mathcal{G}_x$. We wish to show that $f = g$. Pick an open set $U \subseteq X$ and consider $s \in \mathcal{F}(U)$. We wish to show that $f_U(s) = g_U(s)$. For each $x \in U$, we have $(U, s)_x \in \mathcal{F}_x$ and by the fact that $f_x = g_x$, we further have
$$(U, f_U(s))_x = (U, g_U(s))_x.$$
Hence for all $x \in U$, there exists open $x \in W_x \subseteq U$ such that
$$\rho_{U,W_x}(f_U(s)) = \rho_{U,W_x}(g_U(s)).$$
It is then an easy observation that both $\{\rho_{U,W_x}(g_U(s))\}_{x\in U}$ and $\{\rho_{U,W_x}(f_U(s))\}_{x\in U}$ forms the same matching family. Hence we have a unique glue by sheaf axiom of $\mathcal{G}$ to obtain $f_U(s) = g_U(s)$ in $\mathcal{G}(U)$.

□

Lemma 27.3.0.8. *Let X be a topological space. Then, the following are equivalent:*

1. *The following is an exact sequence of sheaves over X*
$$\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'',$$
that is, $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$.
2. *The following is an exact sequence of stalks for each $x \in X$*
$$\mathcal{F}'_x \xrightarrow{f_x} \mathcal{F}_x \xrightarrow{g_x} \mathcal{F}''_x.$$

Proof. (1. $\Rightarrow$ 2.) Pick any $(U, s)_x \in \mathcal{F}_x$ which is in $\ker g_x$. Thus, there exists $V \subseteq U$ open such that $\rho_{U,V}(g_U(s)) = g_V(\rho_{U,V}(s)) = 0$. Thus $\rho_{U,V}(s) \in \mathcal{F}(V)$ is in $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$ and thus $(V, \rho_{U,V}(s))_x = (U, s)_x \in \mathcal{F}_x$ is in $\mathrm{im}\,(f_x)$. Conversely, for $(U, f_x(t))_x \in \mathrm{im}\,(f_x)$, we see that since $g \circ f = 0$, then $(U, g_x(f_x(t)))_x = 0$.

(2. $\Rightarrow$ 1.) This is immediate, by looking at a section of $\mathcal{F}$ at any open set (use Remark 27.2.0.4). □

Given an open subset U of X and a sheaf over U, we can extend it to a sheaf over X by zeros. This in particular means extending a sheaf from a subspace in such a way so that stalks outside of the subspace are always zero. This operation would be fundamental in cohomology and other places as it yields a nice exact sequence corresponding to any closed or open subset of X.

Definition 27.3.0.9 (Extending a sheaf by zeros). Let X be a space and $i : Z \hookrightarrow X$ be an inclusion of a closed set and $j : U \hookrightarrow X$ be an inclusion of an open set.

1. If $\mathcal{F}$ is a sheaf over Z, then $i_*\mathcal{F}$ is a sheaf over X called the extension of $\mathcal{F}$ to X by zeros.
2. If $\mathcal{F}$ is a sheaf over U, then the extension of $\mathcal{F}$ to X by zeroes, denoted $j_!\mathcal{F}$ is the sheafification of the presheaf over X given by
$$V \longmapsto \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U \\ 0 & \text{else.} \end{cases}$$

The main result is as follows.

Proposition 27.3.0.10. [4] *Let X be a space, $i : Z \hookrightarrow X$ be closed and $j : U \hookrightarrow X$ be open. Then,*

1. *If $\mathcal{F}$ is a sheaf over Z, then for any $p \in X$, we have*
$$(i_*\mathcal{F})_p = \begin{cases} \mathcal{F}_p & \text{if } p \in Z \\ 0 & \text{if } p \notin Z. \end{cases}$$

[4] Exercise II.1.19 of Hartshorne.

2. *If $\mathcal{F}$ is a sheaf over U, then for any $p \in X$, we have*

$$(j_!\mathcal{F})_p = \begin{cases} \mathcal{F}_p & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$$

Moreover, $(j_!\mathcal{F})_{|U} = \mathcal{F}$ and $j_!\mathcal{F}$ is unique w.r.t these two properties.

Proof. The first item follows immediately from the fact that $Z \subseteq X$ is a closed subset. In particular, if $p \notin Z$, then there is a cofinal collection of open sets containing p on which $i_*\mathcal{F}$ is 0.

For the second item, we proceed as follows. Let G be the presheaf as in Definition 27.3.0.9, 2. Note that

$$G_p = \begin{cases} \mathcal{F}_p & \text{if } p \in V \subseteq U \text{ for some open } V \subset X, \\ 0 & \text{else.} \end{cases}$$

In particular, if $p \in U$, then $G_p = \mathcal{F}_p$ and if $p \notin U$, then $G_p = 0$. Since stalks before and after sheafification are same, therefore we have our result for stalks. Next, $(j_!\mathcal{F})_{|U} = \mathcal{F}$ because over U, the presheaf $G_{|U}$ itself is a sheaf, so sheafification of G will yield a sheaf equal to $\mathcal{F}$ over U. Further $j_!\mathcal{F}$ is unique with the two properties as if for any other sheaf $\mathcal{G}$ which satisfies that $\mathcal{G}_{|U} = \mathcal{F}$, then we get an map of presheaves $G \to \mathcal{G}$ which induces an isomorphism on stalks. By universal property of sheafification (Theorem 27.2.0.1), we deduce that $j_!\mathcal{F} \cong \mathcal{G}$. □

With the above result, we have a useful short exact sequence.

Corollary 27.3.0.11. *Let X be a space and $\mathcal{F}$ be a sheaf over X. Let $i : Z \hookrightarrow X$ be a closed subspace and $j : U = X \setminus Z \hookrightarrow X$ be the corresponding open subspace. Then there is a short exact sequence*

$$0 \longrightarrow j_!\mathcal{F}_{|U} \longrightarrow \mathcal{F} \longrightarrow i_*\mathcal{F}_{|Z} \longrightarrow 0$$

where $\mathcal{F}_{|Z} = i^{-1}\mathcal{F}$. We call this the extension by zero short exact sequence.

Proof. Following the notation of proof of Proposition 27.3.0.10, we see that we have an injective map $G \to \mathcal{F}$, which then by universal property and local nature of injectivity gives an injective map $j_!\mathcal{F}_{|U} \to \mathcal{F}$. The map $\mathcal{F} \to i_*\mathcal{F}_{|Z}$ is obtained by considering the unit map of the adjunction $i_* \vdash i^{-1}$. This is surjective because on the stalks, we obtain $(i_*\mathcal{F}_{|Z})_p = \mathcal{F}_p$ if $p \in Z$ or 0 otherwise by above result. To show exactness at middle, we again go to stalks (Lemma 27.3.0.8) and observe that if $p \in U$, then we get exact sequence $0 \to \mathcal{F}_p \xrightarrow{\text{id}} \mathcal{F}_p \to 0 \to 0$ and if $p \in Z$, then we get the exact sequence $0 \to 0 \to \mathcal{F}_p \xrightarrow{\text{id}} \mathcal{F}_p \to 0$. □

27.4 Sheaves are étale spaces

Another important and in some sense dual viewpoint of sheaves over X is that they can be equivalently defined as a certain type of bundle over X and all such bundles arises only from a sheaf. This is important because this viewpoint naturally extends the usual concepts of covering spaces, bundles and vector bundles to that of sheaves. In particular, a lot of classical constructs in algebraic topology can be equivalently be seen as specific instantiates of the notion of étale space of the sheaf.

Definition 27.4.0.1. (**Étale space**) Let X be a topological space and let $\pi : E \to X$ be a bundle over X. Then (E, π, X) is said to be étale over X or just étale if for all $e \in E$, there exists an open set $V \ni e$ of E such that $p(V)$ is open and $p|_V : V \to p(V)$ is a homeomorphism, that is, if p is a local homeomorphism. A morphism of étale spaces $(E_1, \pi_1, X), (E_2, \pi_2, X)$ over X is given by a continuous map $f : E_1 \to E_2$ such that $\pi_2 \circ f = \pi_1$. Denote the category of étale spaces over X by $\mathbf{Et}/X$.

Clearly, covering spaces over X are étale spaces over X, but not all étale spaces over X are covering, of-course. We now wish to show that the sheafification functor factors through a functor mapping a presheaf to an étale space. In particular, we want to show the existence of functor F, G so that the following commutes

$$\begin{tikzcd}
& \mathbf{Et}/X \arrow[dr, "F"] & \\
\mathbf{PSh}(X) \arrow[ur, "G"] \arrow[rr, shift left, "(-)^{++}"] & & \mathbf{Sh}(X) \arrow[ll, hook', shift left]
\end{tikzcd}.$$

Construction 27.4.0.2. (*Étale space of a sheaf*) Let us now show the construction of the above functors:

1. (*The functor G*) Let P be a presheaf over X. The étale space $E := G(P)$ is given by the disjoint union of all stalks:

$$E := \coprod_{x \in X} P_x.$$

The topology on E is given by the initial topology of the map

$$\begin{aligned} \pi : E &\longrightarrow X \\ s_x &\longmapsto x. \end{aligned}$$

In particular, E has a basis given by sets of the form $B_{U,s} \subseteq E$ where $B_{U,s} = \{s_x \in E \mid x \in U\}$ and $s \in P(U)$. Next, we wish to establish that π is a local homeomorphism. So take any $s_x \in E$ and consider the basic open set $B_{U,s} \ni s_x$. The map $\pi|_{B_{U,s}} : B_{U,s} \to \pi(B_{U,s})$ takes $s_x \mapsto x$. This is a homeomorphism because we can construct an inverse given by $x \mapsto s_x$. A simple calculation checks that this is continuous. Hence indeed, (E, π, X) is an étale space over X.

Next consider a map of presheaves $\varphi : F \to G$. We can construct a map of corresponding étale spaces as

$$\begin{aligned} \hat{\varphi} : (E_F, \pi_F, X) &\longrightarrow (E_G, \pi_G, X) \\ s_x &\longmapsto \varphi_x(s_x). \end{aligned}$$

This map is continuous and a valid bundle map over X. This defines the functor G.

2. (*The functor F*) Let $\pi : E \to X$ be an étale space over X. Then, we can construct a sheaf $\mathcal{E}$ over X out of it. This is done in a very natural way by considering the set of sections over U of $\mathcal{E}$ to be quite literally the set of *cross-sections*[5] of map π on U. That is, define:

$$\mathcal{E}(U) := \{s : U \to E \mid \pi \circ s = \mathrm{id}_U\}.$$

The fact that this is indeed a sheaf can be seen by a general phenomenon that for any continuous map $f : X \to Y$, the set of all cross-sections of f over open subsets of Y assembles itself into a sheaf. Hence, we have constructed a sheaf $\mathcal{E}$ out of an étale space E over X.

Next consider a map of étale spaces $\xi : (E_1, \pi_1, X) \to (E_2, \pi_2, X)$. we can construct a map of corresponding sheaves $\tilde{\xi} : \mathcal{E}_1 \to \mathcal{E}_2$ by defining the following for open $U \subseteq X$:

$$\begin{aligned} \tilde{\xi}_U : \mathcal{E}_1(U) &\longrightarrow \mathcal{E}_2(U) \\ s &\longmapsto \xi \circ s. \end{aligned}$$

One can check that this is indeed a valid sheaf morphism. This defines the functor F.

We then see that the categories $\mathbf{Et}/X$ and $\mathbf{Sh}(X)$ are equivalent.

Theorem 27.4.0.3. [6] *(The étale viewpoint of sheaves) Let X be a topological space. The functors F and G as defined in Construction 27.4.0.2 defines an equivalence of categories*

$$\mathbf{Sh}(X) \equiv \mathbf{Et}/X.$$

We will prove this result in many small lemmas below. We would first like to observe that for any étalé bundle E over X yields a sheaf by $F(E)$ whose stalks are bijective to fibres of E.

Lemma 27.4.0.4. *Let (E, π, X) be an étalé bundle over X and let $\mathcal{E}$ be the sheaf obtained by $F((E, \pi, X))$. Then, for any $x \in X$, the following is a bijection*

$$\begin{aligned} \tau_x : \mathcal{E}_x &\longrightarrow E_x := \pi^{-1}(x) \\ (U, s)_x &\longmapsto s(x). \end{aligned}$$

Proof. We first show that τ_x is injective. Let $(U, s)_x, (V, t)_x$ be two germs such that $p = s(x) = t(x)$. We wish to show that s and t are equal on an open subset in $U \cap V$. As E is étaleé, therefore we have an open $A \subseteq E$ with $p \in A$ such that $\pi|_A : A \to \pi(A)$ is a homeomorphism. Consequently, we see that the open set $W = \pi(A) \cap U \cap V$ would do just fine.

We now show surjectivity. Pick $e \in E_x$. As E is étalé, we thus get an open set $A \ni e$ in E such that $\pi|_A : A \to \pi(A)$ is a homeomorphism. Denote the inverse of this homeomorphism by $g : \pi(A) \to A$. This is therefore a section of E over $\pi(A)$ where $x \in \pi(A)$. Consequently, $(\pi(A), g)_x \in \mathcal{E}_x$ is such that τ_x maps it to e. □

[5] In-fact, historically the notion of sheaf was really that of this étale space, and that is why to this day, we still use the terminology of "sections" of a sheaf over an open subset.

[6] Exercise II.1.13 of Hartshorne.

Proof of Theorem 27.4.0.3. We first show that $F \circ G$ is naturally isomorphic to sheafification functor. Let $\mathcal{E}$ be a presheaf, $(E, \pi) = G(\mathcal{E})$ and $F(E, \pi) = \mathcal{E}'$. We wish to show that there is a natural isomorphism $\mathcal{E}^{++} \to \mathcal{E}'$. By Theorem 27.2.0.1 and 27.3.0.6, 3, it suffices to show that there is a map of presheaves $\mathcal{E} \to \mathcal{E}'$ which is isomorphism on stalks.

Consider the map $\varphi : \mathcal{E} \to \mathcal{E}'$ which on an open set $U \subseteq X$ gives the following map

$$\begin{aligned} \varphi_U : \mathcal{E}(U) &\longrightarrow \mathcal{E}'(U) \\ s &\longmapsto U \overset{f_{U,s}}{\to} E \end{aligned}$$

where $f_{U,s} : U \to E$ maps as $x \mapsto (U, s)_x$. Since $f_{U,s}$ is just the stalk map, then as in Construction 27.4.0.2, $f_{U,s}$ is continuous. Now on the stalks, we get the following commutative diagram by Lemma 27.4.0.4:

$$\begin{tikzcd} \mathcal{E}_x \arrow[r, "\varphi_x"] \arrow[d] & \mathcal{E}'_x \arrow[dl, dash, "\cong"] \\ E_x & \end{tikzcd},$$

where the vertical map takes a germ $(U, s)_x$ and maps it to the element represented in $E_x = \mathcal{E}_x$, as $E_x = \pi^{-1}(x) = \{x \in E \mid \pi(e) = x\} = \{(U, s)_y \in E \mid \pi((U, s)_y) = y = x\}$. Consequently, the vertical map is a bijection and thus φ_x is a bijection. The naturality of this isomorphism can be checked trivially.

We now wish to show that $G \circ F$ is naturally isomorphic to the identity functor on $\mathbf{Et}/X$. Pick an étalé bundle (E, π) over X, denote $F(E, \pi) = \mathcal{E}$ and $G(\mathcal{E}) = (E', \pi')$. We wish to find a homeomorphism φ so that the following commutes:

$$\begin{tikzcd} E' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & E \arrow[dl, "\pi"] \\ X & \end{tikzcd}.$$

Consider the following map

$$\begin{aligned} \varphi : E' &\longrightarrow E \\ (U, s)_x &\longmapsto s(x). \end{aligned}$$

By Lemma 27.4.0.4, φ is a bijective map. We thus reduce to showing that φ is a continuous open map.

To show continuity, consider an open set $A \subseteq E$ and then observe that

$$\begin{aligned} \varphi^{-1}(A) &= \{(U, s)_x \in E' \mid s(x) \in A\} \\ &= \{(U, s)_x \in E' \mid x \in s^{-1}(A)\} \\ &= \bigcup_{U \ni x, s : U \to E} B_{U,s} \end{aligned}$$

and since $B_{U,s} \subseteq E'$ is a basic open, therefore φ is continuous.

Finally, to show that φ is open, one reduces to showing that if $s : U \to E$ is a continuous section of bundle (E, π) and $U \subseteq X$ is an open set, then $s(U)$ is an open set in E (by working with a basic open $B_{U,s} \subseteq E'$). This follows from the fact that since π is a local homeomorphism, therefore for each $e \in s(U)$, there exists an open set $A \ni e$ in E such that $s(U) \cap A \ni e$ and since $\pi : s(U) \cap A \to \pi(s(U) \cap A) = U \cap \pi(A)$ is a homeomorphism, we further get that $s(U) \cap A$ is open (as $U \cap \pi(A)$ is open). Consequently, $s(U)$ is open. $\square$

Remark 27.4.0.5. (*The sheaf associated to a covering space*) By the above equivalence, each covering space space over X, which is an étale map, determines a unique sheaf (upto isomorphism). We analyze this sheaf. **TODO**.

27.5 Direct and inverse image

Let $f : X \to Y$ be a continuous map of topological spaces. Then one can derive two functors $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ which are adjoint of each other, called direct and inverse image functors respectively. While f_* is easy to define, it is usually the inverse image of a sheaf that causes trouble for its obscurity if one works with the definition that inverse image functor is left-adjoint to direct image functor. This is resolved by working with the corresponding étale spaces (Theorem 27.4.0.3). In this section we will show how to construct them.

Let us first define the direct image functor.

Definition 27.5.0.1. (**Direct image**) Let $f : X \to Y$ be a continuous map. Then, for any sheaf $\mathcal{F}$ on X, we can define its direct image under f as $f_*\mathcal{F}$ whose sections on open $V \subseteq Y$ are given by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)).$$

This can easily be seen to be a sheaf. For any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X, we can define the map of direct image sheaves as

$$\begin{aligned} (f_*\varphi)_V : f_*\mathcal{F}(V) &\longrightarrow f_*\mathcal{G}(V) \\ s &\longmapsto \varphi_{f^{-1}(V)}(s). \end{aligned}$$

This defines a functor

$$f_* : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y).$$

One defines the inverse image of a sheaf as follows:

Definition 27.5.0.2. (**Inverse image**) Let $f : X \to Y$ be a continuous map and let $\mathcal{G}$ be a sheaf over Y. Consider a presheaf F over X constructed by the data of $\mathcal{G}$ as follows. Let $U \subseteq X$ be open, then define

$$f^+\mathcal{G}(U) := \varinjlim_{\text{open } V \supseteq f(U)} \mathcal{G}(V),$$

where restriction maps of $f^+\mathcal{G}$ is given by the unique map obtained by universality of colimits. Then, $f^+\mathcal{G}$ is a presheaf over X and this construction is functorial again by universal property of colimits:

$$f^+ : \mathbf{PSh}(Y) \longrightarrow \mathbf{PSh}(X).$$

Let $f^{-1}\mathcal{G} = (f^+\mathcal{G})^{++}$ denote the sheafification of $f^+\mathcal{G}$. This sheaf is called the inverse sheaf of $\mathcal{G}$ under f. Now for any map of sheaves $\varphi : \mathcal{G} \to \mathcal{H}$ over Y, we get a corresponding map of inverse image sheaves $f^{-1}\varphi : f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{H}$ by composition of two functors. This yields a functor

$$f^{-1} : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X).$$

As is visible, this definition is quite obscure if one likes elemental definitions. We thus give some general properties enjoyed by inverse sheaf.

Lemma 27.5.0.3. *Let $f : X \to Y$ be a continuous map and $\mathcal{G}$ be a sheaf over Y.*

1. *If f is open, then $f^{-1}\mathcal{G} = \mathcal{G}(f(-))$.*
2. *If f is constant to $y \in Y$, then $f^{-1}\mathcal{G}$ is the constant sheaf on X with sections $\mathcal{G}_y$.*
3. *If $X = \{x\}$ is a singleton space, then $f^{-1}\mathcal{G}$ is the constant sheaf on X with sections $\mathcal{G}_{f(x)}$.*
4. *If $x \in X$, then*

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}.$$

Proof. 1. One notes that $f^+\mathcal{G}(U) := \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \mathcal{G}(f(U))$. The mapping $\mathcal{G}(f(-))$ is a sheaf, hence sheafifying it will yield the same sheaf.
2. We see that $f^+\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \varinjlim_{V \ni y} \mathcal{G}(V) = \mathcal{G}_y$ and presheaves with constant values are sheaves, as restrictions are identity.
3. We see that $f^+\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \varinjlim_{V \ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}$ and presheaves with constant values are sheaves, as restrictions are identity.
4. By passing to the right adjoint, one observes that for $f : X \to Y$ and $g : Y \to Z$ continuous maps, one can obtain the following natural isomorphism of functors

$$(g \circ f)^{-1} \cong f^{-1} \circ g^{-1}.$$

Consider the composite $f \circ \iota$ where $\iota : \{x\} \hookrightarrow X$ is the inclusion map. Consequently, by 3. above, we obtain the following

$$\begin{aligned}
\mathcal{G}_{f(x)} &\cong (f \circ \iota)^{-1}(\mathcal{G})(\{x\}) \\
&\cong (\iota^{-1} \circ f^{-1})(\mathcal{G})(\{x\}) \\
&\cong \iota^{-1}(f^{-1}\mathcal{G})(\{x\}) \\
&\cong (f^{-1}\mathcal{G})_{f(x)}.
\end{aligned}$$

∎

The following is a fundamental duality between inverse and direct image functors.

Theorem 27.5.0.4. [7] *(Direct and inverse image adjunction) Let $f : X \to Y$ be a continuous map. Then the the inverse image functor is the left adjoint of direct image functor* [8]

$$\begin{tikzcd} \mathbf{Sh}(Y) \arrow[r, shift left, "f^{-1}"] & \mathbf{Sh}(X) \arrow[l, shift left, "f_*"] \end{tikzcd}.$$

In particular, we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}\left(f^{-1}\mathcal{F}, \mathcal{G}\right) \cong \mathrm{Hom}_{\mathbf{Sh}(Y)}\left(\mathcal{F}, f_*\mathcal{G}\right).$$

Proof. We will show the construction of unit and counit and the duality between $\mathcal{F} \to f_*\mathcal{G}$ and $f^{-1}\mathcal{F} \to \mathcal{G}$. □

lete the proof
eorem 27.5.0.4,
er 27.

One situation that we will find ourselves a lot in algebraic geometry is when $f : X \to Y$ will be a closed immersion of topological spaces ($f : X \to f(X)$ is homeomorphism and $f(X) \subseteq Y$ is closed) and for a sheaf $\mathcal{F}$ over X, we would like to find $(f_*\mathcal{F})_{f(x)}$ for each point $x \in X$. This is a situation where the stalk of direct image can be calculated quite easily.

Lemma 27.5.0.5. *Let $f : X \to Y$ will be a closed immersion of topological spaces and $\mathcal{F}$ a sheaf over X. Then, there is a natural isomorphism*

$$(f_*\mathcal{F})_{f(x)} \cong \mathcal{F}_x.$$

Proof. From a straightforward unravelling of definitions of the two stalks, the result follows from the observation that each open set $U \ni x$ in X is in one-to-one correspondence with open set $f(U) \ni f(x)$ in Y. □

Remark 27.5.0.6. We wish to know how the inverse image of sheaves changes the stalk. Let $f : X \to Y$ be a continuous map and let $\mathcal{F}$ be a sheaf on Y. Consider the inverse sheaf $f^{-1}\mathcal{F}$ on X. Let $x \in X$. Then we have that (Lemma 27.5.0.3, 4)

$$(f^{-1}\mathcal{F})_x \cong \mathcal{F}_{f(x)}.$$

The importance of this is that, suppose $f : X \to Y$ is given together with $\mathcal{F}$ and $\mathcal{G}$ are sheaves over X and Y respectively and a map $\varphi^\flat : \mathcal{G} \to f_*\mathcal{F}$ over Y, which is equivalent to $\varphi^\sharp : f^{-1}\mathcal{G} \to \mathcal{F}$ over X. Now, most of the time, our interest in a sheaf is only limited to stalks (functions defined in *some* open subset around a point), therefore we are mostly interested in considering only the map induced at the level of stalks at a point $f(x) \in Y$:

$$\varphi^\flat_{f(x)} : \mathcal{G}_{f(x)} \longrightarrow (f_*\mathcal{F})_{f(x)}.$$

But the description of the stalk $(f_*\mathcal{F})_{f(x)}$ is usually not simple to derive. But dually, we may ask the map of stalks of the other map at $x \in X$, and we directly land into the stalks

$$\varphi^\sharp_x : \mathcal{G}_{f(x)} \cong (f^{-1}\mathcal{G})_x \longrightarrow \mathcal{F}_x.$$

[7] Exercise II.1.18 of Hartshorne.

[8] admirers of topoi may see this as a quintessential example of geometric map of topoi.

However, this is a strange map as the stalks are of sheaves which are not on same space. In particular, this map is given as follows. For any open $V \ni f(x)$ in Y, we have the following maps:

$$\mathcal{G}(V) \xrightarrow{\varphi_V^\flat} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}_x \ .$$

Passing to colimits ($\varphi_V^\flat$ commutes with restrictions), one can see that we get the map $\varphi_x^\sharp : \mathcal{G}_{f(x)} \to \mathcal{F}_x$ back.

It is a good principle to keep in mind that if we wish to work with explicit local sections, then we should look for the "flat" map and if it is enough to work with germs, then we should look for the "sharp" map, even though the above remark telling us how to construct the map of stalks from the "flat" maps on each open set.

This map $\varphi_x^\sharp$ can be heuristically be defined as the map which on sections which makes sure that a non-invertible section remains non-invertible after going through the map. Hence we mostly work only with maps $f^{-1}\mathcal{G} \to \mathcal{F}$ if we are interested only at the stalk level (which is more than enough for us).

27.6 Category of abelian sheaves

We will discuss some basic properties of the category of sheaves of abelian groups over X, denoted $\mathbf{Sh}(X)$. This is important as we wish to calculate cohomology of its objects, hence we would require the notion of injective and projective resolutions of sheaves. We covered the homological methods necessary for this section in the Homological Methods, Chapter 26.

Write the prope of $\mathbf{Sh}(X)$. Chap 27.

The main theorem is that the category of abelian sheaves on a topological space is a Grothendieck-abelian category (Theorem **??**) and thus it has enough injectives.

27.6.1 Direct and inverse limits in $\mathbf{Sh}(X)$

Since Grothendieck-abelian categories have all colimits, therefore it also has direct limits. We now show that the direct limits in $\mathbf{Sh}(X)$ are obtained by sheafifying the corresponding direct limit in $\mathbf{PSh}(X)$.

Lemma 27.6.1.1. [9] *Let X be a topological space and $\{\mathcal{F}_i\}$ be a direct system of sheaves over X. Then, the direct limit $\varinjlim_i \mathcal{F}_i$ in $\mathbf{Sh}(X)$ is formed by sheafification of the presheaf $U \mapsto \varinjlim_i \mathcal{F}_i(U)$.*

Proof. Let F denote the presheaf obtained by $U \mapsto \varinjlim_i \mathcal{F}_i(U)$ and further denote $\mathcal{F} = F^{++}$, the sheafification of F. Note that we have $\mathcal{F}_i \xrightarrow{j_i} F \to \mathcal{F}$. We wish to show that $\mathcal{F}$ satisfies the universal property of direct limits in $\mathbf{Sh}(X)$. Indeed, take any other sheaf $\mathcal{G}$ for which there are maps $f_i : \mathcal{F}_i \to \mathcal{G}$ which further satisfies that for any $j \geq i$ in the direct set

[9] Exercise II.1.10 of Hartshorne.

indexing the system, we have that the following triangle commutes:

$$\begin{tikzcd} & \mathcal{G} & \\ \mathcal{F}_i \arrow[ur, "f_i"] \arrow[rr, "\varphi_{ij}"'] & & \mathcal{F}_j \arrow[ul, "f_j"'] \end{tikzcd}.$$

We wish to show that there exists a unique map $\tilde{f} : \mathcal{F} \to \mathcal{G}$ such that for all i, the following commutes:

$$\begin{tikzcd} \mathcal{G} & \mathcal{F} \arrow[l, dashed, "\tilde{f}"'] \\ \mathcal{F}_i \arrow[u, "f_i"] \arrow[r, "j_i"'] & F \arrow[u] \end{tikzcd}.$$

But this is straightforward, as by the universal property of direct limits in $\mathbf{PSh}(X)$, we first have a map $f : F \to \mathcal{G}$ which makes the bottom left triangle in the above commute. Then, by the universal property of sheafification (Theorem 27.2.0.1), we get a corresponding $\tilde{f} : \mathcal{F} \to \mathcal{G}$ which makes the top right triangle in the above commute. Consequently, we have obtained $\tilde{f}$ which makes the square commute. $\square$

27.7 Classical Čech cohomology

Sheaf cohomology becomes an important tool to any attempt at understanding any sophisticated geometric situation in topology. It is a tool which measures the obstructions faced in extending a local construction (which are usually not too difficult to make) to a global one (which are most of the time very difficult to make). To get a feel of why such questions and tools developed to solve them are important, one may look no further than basic analysis; say in case of $\mathbb{R}^n$, we wish to extend a local isometry from an open set of $\mathbb{R}^n$ to $\mathbb{R}^m$, into a global one between $\mathbb{R}^n$ and $\mathbb{R}^m$. Clearly the former is much, much easier than the latter. In the same vein, we wish to understand obstructions faced in making local-to-global leaps in the context of schemes, which covers almost all range of algebro-geometric situations.

Construction 27.7.0.1 (*Čech cochain complex and Čech cohomology of an abelian presheaf.*)**.** Let X be a topological space and F be an abelian presheaf over X. We will construct and discuss the Čech cohomology groups $\check{H}^q(X;F)$. After giving the basic constructions, we will specialize to the case of schemes in Chapter 1, to prove the Serre's theorem on invariance of affine refinements of cohomology of coherent sheaves.

We first construct the Čech cochain complex of F w.r.t. to an open cover $\mathcal{U}$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be a fixed open cover of X. We can then define for each $i = 0, 1, 2, \dots,$ a group called the group of *i-cochains of F w.r.t. $\mathcal{U}$*:

$$C^i(\mathcal{U}, F) := \prod_{(\alpha_0,\dots,\alpha_i)\in I^{i+1}} F(U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_i}).$$

where the product runs over all increasing $i+1$-tuples with entries in I[10]. A typical element $s \in C^i(\mathcal{U}, F)$ is called an i-cochain, whose part corresponding to $(\beta_0, \dots, \beta_i) \in I^{i+1}$ is denoted by $s(\beta_0, \dots, \beta_i) \in F(U_{\beta_0} \cap \dots \cap U_{\beta_i})$. For example, the set of all 0-cochains is $\prod_{\alpha_0\in I} F(U_{\alpha_0})$, which is equivalent to choosing a section for each element of the cover. Similarly, choosing an element from $C^1(\mathcal{U}, F)$ can be thought of as choosing a section for each intersection of two open sets from $\mathcal{U}$. Similarly one can interpret the higher cochains.

Next, we give the sequence of groups $\{C^i(\mathcal{U}, F)\}_{i\in\mathbb{N}\cup 0}$ the structure of a cochain complex. Indeed, one defines the required *differential* in quite an obvious manner, if one knows the construction of singular homology. Define a map

$$\begin{aligned} d : C^i(\mathcal{U}, F) &\longrightarrow C^{i+1}(\mathcal{U}, F) \\ s = (s(\alpha_0, \dots, \alpha_i)) &\longmapsto ds \end{aligned}$$

where the components of ds are given as follows for $\beta_0, \dots, \beta_{i+1} \in I$:

$$\begin{aligned} (ds)(\beta_0, \dots, \beta_{i+1}) &:= \sum_{j=0}^{i+1} (-1)^j \rho_j(s(\beta_0, \dots, \beta_{j-1}, \beta_{j+1}, \dots, \beta_{i+1})) \\ &= \sum_{j=0}^{i+1} (-1)^j \rho_j(s(\widehat{\beta_j})) \end{aligned}$$

[10] we choose increasing tuples only to make sure we don't repeat an open set in the product.

where ρ_j is the following restriction map of the presheaf F:

$$\rho_j : F(U_{\beta_0} \cap \cdots \cap U_{\beta_{j-1}} \cap U_{\beta_{j+1}} \cap U_{\beta_{i+1}}) \longrightarrow F(U_{\beta_0} \cap \cdots \cap U_{\beta_{j-1}} \cap U_{\beta_j} \cap U_{\beta_{j+1}} \cap U_{\beta_{i+1}}),$$

that is, the one where the open set U_{β_j} is dropped from intersection.

This differential can be understood in the simple case of $i = 0$ as follows. Take $s = (s(\alpha_0)) \in C^0(\mathcal{U}, F)$. Then $ds \in C^1(\mathcal{U}, F)$ and it corresponds to a choice of a section in the intersection on each pair of open sets in $\mathcal{U}$. For $\beta_0, \beta_1 \in I$, this choice is given by

$$(ds)(\beta_0, \beta_1) = \rho_0(s(\beta_1)) - \rho_1(s(\beta_0)).$$

This is interpreted as "how much far away $s(\beta_1) \in F(U_{\beta_1})$ and $s(\beta_0) \in F(U_{\beta_0})$ are in the intersection $U_{\beta_1} \cap U_{\beta_0}$". If $d(s) = 0$, then $s \in C^0(\mathcal{U}, F)$ corresponds to a matching family.

Similarly, for a $s \in C^1(\mathcal{U}, F)$, we can think of it as a choice of a section on each intersecting pair of open sets of $\mathcal{U}$. Then, the differential $ds \in C^2(\mathcal{U}, F)$ for any $(\beta_0, \beta_1, \beta_2) \in I^3$ has the component

$$(ds)(\beta_0, \beta_1, \beta_2) = \rho_0(s(\beta_1, \beta_2)) - \rho_1(s(\beta_0, \beta_2)) + \rho_2(s(\beta_0, \beta_1)).$$

If this quantity is non zero, then it measures "how much the three elements $s(\beta_1, \beta_2) \in F(U_{\beta_1} \cap U_{\beta_2})$, $s(\beta_0, \beta_2) \in F(U_{\beta_0} \cap U_{\beta_2})$ and $s(\beta_0, \beta_1) \in F(U_{\beta_0} \cap U_{\beta_1})$ differs in the combined intersection $U_{\beta_0} \cap U_{\beta_1} \cap U_{\beta_2}$". Indeed, suppose the three agree on $F(U_{\beta_0} \cap U_{\beta_1} \cap U_{\beta_2})$. Then, we have $\rho_0(s(\beta_1, \beta_2)) = \rho_1(s(\beta_0, \beta_2)) = \rho_2(s(\beta_0, \beta_1))$. Consequently, $ds(\beta_0, \beta_1, \beta_2) = \rho_2(s(\beta_0, \beta_1))$.

Now it is quite obvious that in order to measure the failure of an element of $C^i(\mathcal{U}, F)$ to "match up in one level above" will be measured by the homology of the cochain complex. Indeed that is what we do now.

For any $s \in C^i(\mathcal{U}, F)$, it is observed by doing the summation and using the fact that the restriction maps ρ are group homomorphisms that

$$d^2 = 0.$$

Hence, we have a cochain complex, called the **Čech cochain complex w.r.t. $\mathcal{U}$**:

$$C^0(\mathcal{U}, F) \xrightarrow{d} C^1(\mathcal{U}, F) \xrightarrow{d} C^2(\mathcal{U}, F) \xrightarrow{d} \cdots$$

The cohomology of this complex is denoted by

$$H^q(\mathcal{U}; F) := \frac{\mathrm{Im}\,(d)}{\mathrm{Ker}\,(d)}$$

for $C^{q+1}(\mathcal{U}, F) \leftarrow C^q(\mathcal{U}, F) \leftarrow C^{q-1}(\mathcal{U}, F)$. The subgroup $\mathrm{Im}\,(d) \subseteq C^q(\mathcal{U}, F)$ is called the *group of q-coboundaries*, whereas the group $\mathrm{Ker}\,(d) \subseteq C^q(\mathcal{U}, F)$ is called the *group of q-cocycles.*

To define the general Čech cohomology groups, we need to take limit of cohomology groups with respect to finer and finer open covers. To this end, we first define the following.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ and $\mathcal{V} = \{V_\beta\}_{\beta\in J}$ be two open covers. Then, $\mathcal{V}$ is said to be *finer* than $\mathcal{U}$ if for all $j \in J$, there is an $i \in I$ such that $V_j \subseteq U_i$. We therefore obtain a function $\sigma : J \to I$ such that $V_j \subseteq U_{\sigma(j)}$.

For two open covers $\mathcal{U}, \mathcal{V}$ where $\mathcal{V}$ is finer than $\mathcal{U}$ as above, we first get a map of cochain complexes given by

$$\begin{aligned} r_{\mathcal{U},\mathcal{V}} : C^q(\mathcal{U}, F) &\longrightarrow C^q(\mathcal{V}, F) \\ s &\longmapsto r_{\mathcal{U},\mathcal{V}}(s) \end{aligned}$$

where for any $(\beta_0, \dots, \beta_q) \in J^{q+1}$, we define

$$r_{\mathcal{U},\mathcal{V}}(s)(\beta_0, \dots, \beta_q) = \rho\left(s(\sigma\beta_0, \dots, \sigma\beta_q)\right)$$

for $\rho : F(U_{\sigma\beta_0} \cap \dots \cap U_{\sigma\beta_q}) \longrightarrow F(V_{\beta_0} \cap \dots \cap V_{\beta_q})$ is the restriction map of F. As restriction homomorphisms commute with themselves, therefore we have that the following square commutes

$$\begin{array}{ccc} C^q(\mathcal{U}, F) & \xrightarrow{d} & C^{q+1}(\mathcal{U}, F) \\ {\scriptstyle r_{\mathcal{U},\mathcal{V}}}\big\downarrow & & \big\downarrow{\scriptstyle r_{\mathcal{U},\mathcal{V}}} \\ C^q(\mathcal{V}, F) & \xrightarrow[d]{} & C^{q+1}(\mathcal{V}, F) \end{array},$$

showing that $r_{\mathcal{U},\mathcal{V}} : C^\bullet(\mathcal{U}, F) \to C^\bullet(\mathcal{V}, F)$ is a map of cochain complexes. Consequently, we get a map at the level of cohomology also denoted by

$$r_{\mathcal{U},\mathcal{V}} : H^q(\mathcal{U}, F) \longrightarrow H^q(\mathcal{V}, F).$$

We call the above the *refinement homomorphism*.

We now wish to show that if $\mathcal{V}$ is a refinement of $\mathcal{U}$ via $\sigma : J \to I$, then the refinement homomorphism $r_{\mathcal{U},\mathcal{V}}$ on cohomology doesn't depend on σ; there might be many such σ making $\mathcal{V}$ finer than $\mathcal{U}$, but all give same refinement homomorphism on cohomology.

Lemma 27.7.0.2. *The refinement homomorphism $r_{\mathcal{U},\mathcal{V}}$ is independent of σ.*

Proof. Let $r, r' : C^q(\mathcal{U}, F) \to C^q(\mathcal{V}, F)$ be the refinement homomorphisms on cochain level for $\sigma, \tau : J \to I$ respectively. Pick any q-cocycle $s \in C^q(\mathcal{U}, F)$. We wish to show that $r(s) - r'(s)$ is a q-coboundary w.r.t. $\mathcal{V}$. The following $t \in C^{q-1}(\mathcal{V}, F)$

$$t(\alpha_0, \dots, \alpha_{q-1}) := \sum_{j=0}^{q-1} (-1)^j \rho\left(s\left(\sigma\alpha_0, \dots, \sigma\alpha_j, \tau\alpha_j, \tau\alpha_{j+1}, \dots, \tau\alpha_{i-1}\right)\right)$$

where $\rho : F(U_{\sigma\alpha_0} \cap \dots \cap U_{\sigma\alpha_j} \cap U_{\tau\alpha_j} \cap \dots \cap U_{\tau\alpha_{i-1}}) \longrightarrow F(V_{\alpha_0} \cap \dots \cap V_{\alpha_j} \cap \dots \cap V_{\alpha_{i-1}})$ is such that

$$r(s) - r'(s) = dt$$

in $C^q(\mathcal{V}, F)$. This can be checked by expanding dt and using the fact that $ds = 0$. This calculation is omitted for being too cumbersome to write. $\square$

This finally allows us to define Čech cohomology of a presheaf over a topological space as follows. Let $\mathcal{O}$ be the poset of all open covers of X ordered by refinement. The **Čech cohomology groups of presheaf** F are then defined to be

$$\check{H}^q(X,F) := \varinjlim_{\mathcal{U}\in\mathcal{O}} H^q(\mathcal{U},F).$$

Diagrammatically, we have for any two open covers $\mathcal{U}$ and $\mathcal{V}$ where $\mathcal{V}$ is a refinement of $\mathcal{U}$ the following

$$\begin{tikzcd} & \check{H}^q(X,F) \\ H^q(\mathcal{U},F) \arrow[ur] \arrow[r, "r_{\mathcal{U},\mathcal{V}}"'] & H^q(\mathcal{V},F) \arrow[u] \end{tikzcd}.$$

This completes the construction of Čech cohomology groups.

Let us first see something that we hinted during the construction.

Lemma 27.7.0.3. *Let X be a space and $\mathcal{F}$ be a sheaf over X. Then, for any open cover $\mathcal{U}$ of X, we have*

$$H^0(\mathcal{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F}).$$

Consequently, we have $\check{H}^0(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F})$.

Proof. We first have $H^1(X,F) = \mathrm{Ker}\,(d)$ where $d : C^0(\mathcal{U},F) \to C^1(\mathcal{U},F)$. But any $s \in \mathrm{Ker}\,(d)$ is equivalent to the data of a matching family over $\mathcal{U}$. As $\mathcal{F}$ is a sheaf, therefore this gives rise to a unique element in $\Gamma(X,\mathcal{F})$. Conversely, by restriction, we get an element of $\mathrm{Ker}\,(d)$ via a global section. $\square$

Let us first see an example computation of $\check{H}^1(X,F)$.

Example 27.7.0.4. Let $X = S^1$ and $F = \mathcal{K}$ be the constant sheaf of a field K. Further, let $\mathcal{U}$ be the open cover obtained by dividing S^1 into n-open intervals $U_1,\dots,U_n$ where $U_i \cap U_{i+1}$ and $U_i \cap U_{i-1}$ are non-empty and $U_i \cap U_j$ is empty for all $j \neq i, i+1, i-1$. We wish to calculate $H^1(\mathcal{U},\mathcal{K})$. To this end, we first see that

$$C^0(\mathcal{U},\mathcal{K}) = \prod_{i=1}^n \mathcal{K}(U_i) = K^{\oplus n}$$

and

$$C^1(\mathcal{U},\mathcal{K}) = \prod_{i=1}^n \mathcal{K}(U_i \cap U_{i+1}) = K^{\oplus n}.$$

For $q \geq 2$, we clearly have $C^q(\mathcal{U},\mathcal{K}) = 0$ as there are no higher intersections. The differential $d : C^0(\mathcal{U},\mathcal{K}) \to C^1(\mathcal{U},\mathcal{K})$ maps as

$$d(x_1,\dots x_n) = (x_2 - x_1, x_3 - x_2, \dots, x_1 - x_n).$$

Consequently,

$$H^0(\mathcal{U},\mathcal{K}) = \mathrm{Ker}\,(d) = \{(x_1,\dots,x_n) \in C^0(\mathcal{U},\mathcal{K}) \mid x_1 = x_2 = \cdots = x_n\} \cong K$$

and

$$H^1(\mathcal{U},\mathcal{K}) = \frac{C^1(\mathcal{U},\mathcal{K})}{\mathrm{Im}\,(d)} \cong K$$

as $C^1(\mathcal{U},\mathcal{K})$ is an n-dimensional K-vector space and $\mathrm{Im}\,(d)$ is of dimension $n-1$ because its defined by one equation deeming the sum of all entries to be 0.

The main tool for calculations with cohomology theories is the cohomology long exact sequence. We put below, without proof, the main theorem of Čech cohomology which gives a condition for an exact sequence of sheaves to induce this long exact sequence in cohomology. Recall X is paracompact if it is Hausdorff and every open cover has a locally finite refinement. Such spaces are always normal.

Theorem 27.7.0.5. *Let X be a paracompact space and the following be an exact sequence of sheaves over X*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0.$$

Then, there is a long exact sequence in cohomology

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(X,\mathcal{F}_1) \arrow[r] & \check{H}^0(X,\mathcal{F}_2) \arrow[r] & \check{H}^0(X,\mathcal{F}_3) \arrow[dll] \\
& \check{H}^1(X,\mathcal{F}_1) \arrow[r] & \check{H}^1(X,\mathcal{F}_2) \arrow[r] & \check{H}^1(X,\mathcal{F}_3) \arrow[dl, dashed, no head] \\
& & {} &
\end{tikzcd}.$$

27.8 Derived functor cohomology

We will here define the cohomology of abelian sheaves over a topological space as right derived functors of the left exact global-sections functor (see Section 26.2 for preliminaries on derived functors).

Let X be a topological space. In Section 27.6, we showed that the category of abelian sheaves $\mathbf{Sh}(X)$ has enough injectives. We now use it to define cohomology of $\mathcal{F} \in \mathbf{Sh}(X)$.

Definition 27.8.0.1. (**Sheaf cohomology functors**) Let X be a topological space and $\mathbf{Sh}(X)$ be the category of abelian sheaves over X. The i^{th}-cohomology functor $H^i(X,-) : \mathbf{Sh}(X) \to \mathbf{AbGrp}$ is defined to be the i^{th}-right derived functor of the global sections functor $\Gamma(-,X) : \mathbf{Sh}(X) \to \mathbf{AbGrp}$. In other words, $H^i(X,\mathcal{F})$ for $\mathcal{F} \in \mathbf{Sh}(X)$ is defined by choosing an injective resolution $0 \to \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^\bullet$ in $\mathbf{Sh}(X)$ and then

$$H^i(X,\mathcal{F}) := h^i(\Gamma(X,\mathcal{I}^\bullet)).$$

As sheaf cohomology functors are in particular derived functors, so they satisfy results from Section 26.2.3. The main point in particular being that sheaf cohomology induces a long exact sequence in cohomology from a short exact sequence of sheaves. This will be our primary source of computations.

27.8.1 Flasque sheaves & cohomology of $\mathcal{O}_X$-modules

We would like to see the following theorem.

Theorem 27.8.1.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the right derived functors of $\Gamma(-, X) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{AbGrp}$ is equal to the restriction of the cohomology functors $H^i(X, -) : \mathbf{Sh}(X) \to \mathbf{AbGrp}$.*

Remember that $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives (Theorem 8.5.2.2) but, apriori, the above two functors might be different because an injective object in $\mathbf{Mod}(\mathcal{O}_X)$ may not be injective in $\mathbf{Sh}(X)$. Consequently, the above result is important because its relevance in rectifying the cohomology of $\mathcal{O}_X$-modules (which are of the only utmost interest in algebraic geometry) to that of the usual sheaf cohomology functors. Hence, we may completely work inside the module category $\mathbf{Mod}(\mathcal{O}_X)$. Clearly to prove such a result, we need a bridge between injective modules in $\mathbf{Mod}(\mathcal{O}_X)$ and either injective or acyclic objects in $\mathbf{Sh}(X)$. Indeed, we will see that this bridge is provided by the realization that injective modules in $\mathbf{Mod}(\mathcal{O}_X)$ are acyclic because they are *flasque*.

Definition 27.8.1.2 (Flasque sheaves). A sheaf $\mathcal{F}$ on X is said to be flasque if all restriction maps of $\mathcal{F}$ are surjective.

The following is a simple, yet important class of examples of flasque sheaves.

Example 27.8.1.3. Let X be an irreducible topological space and $\mathcal{A}$ be the constant sheaf over X for an abelian group A. We claim that $\mathcal{A}$ is flasque. Indeed, first recall that any open subspace $U \subseteq X$ is irreducible, therefore connected. Conseuqently, all restrictions are $\rho : \mathcal{A}(V) \to \mathcal{A}(U)$ are identity maps $\mathrm{id} : A \to A$ (see Remark 27.1.0.3). In-fact this shows that on an irreducible space, any constant sheaf $\mathcal{A}$ of abelian group A has section over any open set U as $\mathcal{A}(U) = A$ and all restrictions are identities.

An important property of flasque sheaves is that they have no obstruction to lifting of sections, a hint to their triviality in cohomology. However, the proof of this is quite non-constructive and thus a bit enlightening.

Theorem 27.8.1.4. *Let X be a space. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of sheaves and $\mathcal{F}_1$ is flasque, then we have an exact sequence of sections over any open $U \subseteq X$*

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U) \to 0.$$

Proof. By left-exactness of global sections functor, we need only show the surjectivity of $\Gamma(\mathcal{F}_2, X) \to \Gamma(\mathcal{F}_3, X)$. To this end, pick any $s \in \Gamma(\mathcal{F}_3, X)$. We wish to lift this to an element of $\Gamma(\mathcal{F}_2, X)$. Consider the following poset

$$\mathcal{P} = \{(U, t) \mid U \subseteq X \text{ open } \& \ t \in \mathcal{F}_2(U) \text{ is a lift of } \ s|_U\}$$

where $(U, t) \le (U', t')$ iff $U' \supseteq U$ and $t'|_U = t$. We reduce to showing that $\mathcal{P}$ has a maximal element and it is of the form (X, t). This will conclude the proof.

To show the existence of a maximal element, we will use Zorn's lemma. Pick any toset of $\mathcal{P}$ denoted $\mathcal{T}$. We wish to show that it is upper bounded. Indeed, let $V = \bigcup_{(U,t) \in \mathcal{T}} U$

and $\tilde{t} \in \mathcal{F}_2(V)$ be the section obtained by gluing $t \in \mathcal{F}_2(U)$ for each $(U,t) \in \mathcal{T}$ (they form a matching family because $\mathcal{T}$ is totally ordered). We thus have $(V, \tilde{t})$ which we wish to show is in $\mathcal{P}$. Indeed, as $\tilde{t}$ is obtained by lifts of restrictions of s, therefore $\tilde{t}$ is a lift of $s|_V$ by locality of sheaf $\mathcal{F}_3$. This shows that $\mathcal{P}$ has a maximal element, denote it by $(V, \tilde{t})$.

We finally wish to show that $V = X$. Indeed, if not, then $V \subsetneq X$. Pick any point $x \in X \setminus V$. Since we have a surjective map on stalks $\mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$, hence the germ $(X,s)_x \in \mathcal{F}_{3,x}$ can be lifted to $(U,a)_x$ for some open $U \ni x$ and $a \in \mathcal{F}_2(U)$. We now have two cases. If $U \cap V = \emptyset$, then $(V \cup U, \tilde{t} \amalg a)$ is a lift of $s|_{V\cup U}$, contradicting the maximality of $(V, \tilde{t})$. On the other hand, suppose we have $U \cap V \neq \emptyset$. Let $W = U \cap V$. Since $W \subseteq V$, therefore we have $t_W \in \mathcal{F}_2(W)$ a lift of $s|_W$. Moreover, by restriction, we have $a \in \mathcal{F}_2(W)$ also a lift of $s|_W$. It follows that $a - t_W \in \mathcal{F}_1(W)$. As $\mathcal{F}_1$ is flasque, therefore there exists $b \in \Gamma(\mathcal{F}_1, X)$ which extends $a - t_W$. Consequently, we have $a - b = t_W \in \mathcal{F}_2(W)$. Observe that $a - b \in \mathcal{F}_2(U)$ is also a lift of $s|_U$ because $b = 0$ in $\Gamma(\mathcal{F}_3, X)$ by the left-exactness of global sections functor. It follows that $(U, a-b)$ and $(V, \tilde{t})$ is a matching family, which glues to $(U \cup V, c)$ where c is a lift of $s|_{U\cup V}$ as well, contradicting the maximality of $(V, \tilde{t})$. □

Corollary 27.8.1.5. *Let X be a space. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of sheaves where $\mathcal{F}_2$ is flasque, then $\mathcal{F}_3$ is flasque.*

Proof. This is immediate from Theorem 27.8.1.4 and the following diagram where $U \supseteq V$ an inclusion of open subsets of X:

$$\begin{tikzcd}
\mathcal{F}_2(U) \arrow[r] \arrow[d, two heads, "\rho"'] & \mathcal{F}_3(U) \arrow[r] \arrow[d, "\rho"] & 0 \\
\mathcal{F}_2(V) \arrow[r] & \mathcal{F}_3(V) \arrow[r] & 0
\end{tikzcd}.$$

□

Lemma 27.8.1.6. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{O}_U = i_!\mathcal{O}_{X|U}$ to be the extension by zeros of $\mathcal{O}_{X|U}$ for any open set $i : U \hookrightarrow X$. Then,*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{F}) \cong \mathcal{F}(U).$$

Proof. Indeed, we have the following isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{U|U}, \mathcal{F}_{|U}\right) \cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right) \cong \mathcal{F}(U).$$

The first isomorphism follows from the universal property of sheafification. The second isomorphism follows from the observation that $\mathcal{O}_{U|U} = \mathcal{O}_{X|U}$ as is clear from Definition 27.3.0.9 and the fact that sheafification of a sheaf is that sheaf back. The last isomorphism follows from Lemma 8.5.1.19, 2. □

Proposition 27.8.1.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{I}$ is flasque.*

Proof. Let $i : U \hookrightarrow X$ be an open set. Denote $\mathcal{O}_U = i_!\mathcal{O}_{X|U}$ (see Definition 27.3.0.9). We know from Lemma 27.8.1.6 that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \cong \mathcal{I}(U)$ for any open $U \subseteq X$. Now, let $U \subseteq V$ be an inclusion of open sets. To this, we get $\rho : \mathcal{I}(V) \to \mathcal{I}(U)$ the restriction map. Restricting to open set V, we get the following injective map by Corollary 27.3.0.11

$$0 \to \mathcal{O}_U \to \mathcal{O}_V.$$

Using injectivity of $\mathcal{I}$, we obtain a surjection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \to 0.$$

Consequently, we have

$$\mathcal{I}(V) \to \mathcal{I}(U) \to 0$$

where the map is the restriction map of sheaf $\mathcal{I}$. Indeed, this follows from the explicit isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \cong \mathcal{I}(X)$ constructed in the proof of Lemma 8.5.1.19, 2. $\square$

Finally, we see that flasque sheaves have trivial cohomology.

Proposition 27.8.1.8. *Let X be a space and $\mathcal{F}$ be a flasque sheaf over X. Then*

$$H^i(X, \mathcal{F}) = 0$$

for all $i \geq 1$. That is, flasque sheaves are acyclic for the global sections functor.

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}$ be an injective map where $\mathcal{I}$ is an injective sheaf. Consequently, we have an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$$

where $\mathcal{G} = \mathcal{I}/\mathcal{F}$. It follows from Proposition 27.8.1.7 that $\mathcal{I}$ is flasque. By Corollary 27.8.1.5 it follows that $\mathcal{G}$ is flasque. By Theorem 26.2.3.5, we have a long exact sequence in cohomology

$$\begin{tikzcd}
H^i(X,\mathcal{F}) \arrow[r] & H^i(X,\mathcal{I}) \arrow[r] & H^i(X,\mathcal{G}) \arrow[dll, "\delta_i"] \\
H^{i+1}(X,\mathcal{F}) \arrow[r] & H^{i+1}(X,\mathcal{I}) \arrow[r] & H^{i+1}(X,\mathcal{G}) \arrow[dll, dotted, "\delta_{i+1}"] \\
{} & {} & {}
\end{tikzcd}.$$

Since $\mathcal{I}$ is injective, therefore by Remark 26.2.3.4, we have $H^i(X, \mathcal{I}) = 0$ for all $i \geq 1$. It follows from exactness of the above diagram that δ_i are isomorphisms for each $i \geq 1$, that is,

$$H^i(X, \mathcal{G}) \cong H^{i+1}(X, \mathcal{F}).$$

But since $\mathcal{G}$ is also flasque, therefore by repeating the above process, we deduce that $H^{i+1}(X,\mathcal{F}) \cong H^1(X,\mathcal{H})$ where $\mathcal{H}$ is some other flasque sheaf. It thus suffices to show that $H^1(X,\mathcal{F}) = 0$. This follows immediately as we have an exact sequence

$$0 \to \Gamma(\mathcal{F},X) \to \Gamma(\mathcal{I},X) \to \Gamma(\mathcal{G},X) \to H^1(X,\mathcal{F}) \to 0$$

where by Theorem 27.8.1.4, the map $\Gamma(\mathcal{I},X) \to \Gamma(\mathcal{G},X)$ is surjective and since $\Gamma(\mathcal{G},X) \to H^1(X,\mathcal{F})$ is surjective by exactness, it follows that the map $\Gamma(\mathcal{G},X) \to H^1(X,\mathcal{F})$ is the zero map and $H^1(X,\mathcal{F}) = 0$, as required. □

An immediate corollary is the proof of Theorem 27.8.1.1.

Proof of Theorem 27.8.1.1. Pick any $\mathcal{F} \in \mathbf{Mod}(\mathcal{O}_X)$ and pick an injective resolution of $\mathcal{F}$ in $\mathbf{Mod}(\mathcal{O}_X)$

$$0 \to \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^\bullet.$$

By Proposition 27.8.1.7, it follows that each $\mathcal{I}^i$ is flasque. By Proposition 27.8.1.8, it follows that the above is an acyclic resolution for the sheaf $\mathcal{F}$ in $\mathbf{Sh}(X)$. Denote by $\bar{\Gamma} : \mathbf{Mod}(\mathcal{O}_X) \to$ **AbGrp** the restriction of the global sections functor. We wish to show that $R^i\bar{\Gamma}(\mathcal{F}) \cong H^i(X,\mathcal{F})$. By Proposition 26.2.3.9, we have the following isomorphism

$$R^i\bar{\Gamma}(\mathcal{F}) \cong h^i(\bar{\Gamma}(\mathcal{I}^\bullet)) = h^i(\Gamma(\mathcal{I}^\bullet)) \cong H^i(X,\mathcal{F}),$$

as needed. □

An important property of flasque sheaves over noetherian spaces is that it is closed under direct limits.

Proposition 27.8.1.9. *Let X be a noetherian space and $\{\mathcal{F}_\alpha\}$ be a directed system of flasque sheaves. Then $\varinjlim \mathcal{F}_\alpha$ is a flasque sheaf as well.*

Proof. **TODO.** □

Examples

We now present some computations.

Example 27.8.1.10. [11] Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k and $\mathbb{Z}$ be the constant sheaf over X. Let $P, Q \in X$ be two distinct closed points and let $U = X \setminus C$ where $C = \{P, Q\}$ be an open set. Denote $\mathbb{Z}_U$ to be the extension by zero sheaf of $\mathbb{Z}_{|U}$ over X. We claim that

$$H^1(X,\mathbb{Z}_U) \neq 0.$$

We will use the extension by zero short exact sequence of Corollary 27.3.0.11. Denote $i : C \hookrightarrow X$ to be the inclusion. Then, we have

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to i_*\mathbb{Z}_{|C} \to 0.$$

[11]Exercise III.2.1, a) of Hartshorne.

By Theorem 26.2.3.5 and Example 27.8.1.3, it follows that the following sequence is exact

$$0 \to \Gamma(\mathbb{Z}_U, X) \to \Gamma(\mathbb{Z}, X) \to \Gamma(i_*\mathbb{Z}_{|C}, X) \to H^1(X, \mathbb{Z}_U) \to 0.$$

Now suppose that $H^1(X, \mathbb{Z}_U) = 0$. It follows that the map $\Gamma(\mathbb{Z}, X) \to \Gamma(i_*\mathbb{Z}_{|C}, X)$ is surjective. Since X is irreducible and hence connected, we yield $\Gamma(\mathbb{Z}, X) = \mathbb{Z}$. Consequently, we have a surjective map $\mathbb{Z} \to \Gamma(i_*\mathbb{Z}_{|C}, X)$. It follows that $\Gamma(i_*\mathbb{Z}_{|C}, X) = \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. We claim that this is not possible by showing that $\Gamma(i_*\mathbb{Z}_{|C}, X)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, which will yield a contradiction.

We first observe that $\Gamma(i_*\mathbb{Z}_{|C}, X) = \Gamma(\mathbb{Z}_{|C}, C)$. Recall that $\mathbb{Z}_{|C} = i^{-1}\mathbb{Z}$. Note that $(\mathbb{Z}_{|C})_P = \mathbb{Z}_p = \mathbb{Z} = (\mathbb{Z}_{|C})_Q$ by Lemma 27.5.0.3. Hence, by Definition 27.5.0.2 and Remark 27.2.0.4, we deduce that $(i^+\mathbb{Z})_P = (i^+\mathbb{Z})(\{P\}) = \mathbb{Z}_P = \mathbb{Z} = (i^+\mathbb{Z})_Q$ and

$$\Gamma(\mathbb{Z}_{|C}, C) = \left\{ \begin{matrix} (s,t) \in \mathbb{Z} \oplus \mathbb{Z} \mid \exists \text{ opens } U_P \ni P, U_Q \ni \\ Q \text{ in } C \ \& \ s' \in i^+\mathbb{Z}(U_P) \ \& \ t' \in \\ i^+\mathbb{Z}(U_Q) \text{ s.t. } s = s'_P,\ t = t'_Q,\ s = t'_P \text{ if } P \in \\ U_Q \ \& \ t = s'_Q \text{ if } Q \in U_P. \end{matrix} \right\}$$

With this, we observe that for each $(s,t) \in \mathbb{Z} \oplus \mathbb{Z}$, if we keep $U_P = \{P\}$ and $U_Q = \{Q\}$ (which is possible since $P \neq Q$ are closed points in X), we obtain $i^+\mathbb{Z}(U_P) = \mathbb{Z} = i^+\mathbb{Z}(U_Q)$. Then, we may take $s' = s$ and $t' = t$ to obtain that $\Gamma(\mathbb{Z}_{|C}, C) \cong \mathbb{Z} \oplus \mathbb{Z}$. This completes the proof.

Moreover, one can see that the only properties of $\mathbb{A}^1_k$ that we needed was that it is irreducible and $P, Q \in \mathbb{A}^1_k$ are distinct closed points. Consequently, the above result holds true for X an arbitrary irreducible space and $U = X \setminus \{P, Q\}$ where P, Q are two distinct closed points.

Example 27.8.1.11. Consider the notations of Example 27.8.1.10. We give another simple calculation of

$$\Gamma(i_*\mathbb{Z}_{|C}, X) = \mathbb{Z} \oplus \mathbb{Z}.$$

TODO.

Example 27.8.1.12. Consider the notations of Example 27.8.1.10. As an exercise in working with sheaves and sheafification, we also show that

$$\Gamma(\mathbb{Z}_U, X) = 0.$$

TODO.

27.8.2 Čech-to-derived functor spectral sequence

We wish to now observe how the Čech cohomology and derived functor cohomology are related. This is done by a spectral sequence.

Chapter 28

Quadratic Forms

We review here the language and results of the theory of quadratic forms.

Chapter 29

Curiosities

In this chapter we discuss some line of thoughts that we tend to get while discussing problems with others. Some of these might be useless, but nonetheless, fun.

29.1 Table of topological properties

The following table (Table 29.1) consists of some basic topological spaces and some basic topological properties that they do and do not satisfy. These properties cover almost all of those which one encounters in a basic course in topology. Note that $\mathbb{R}_K$ stands for K-topology on $\mathbb{R}$ defined by declaring $(a, b) \setminus K$ to be open as well in $\mathbb{R}$ apart from the (a, b). Further, $\mathbb{R}_l$ is the lower-limit topology on $\mathbb{R}$ generated by basic opens of the form $[a, b)$. The space I^2 in dictionary order is the subspace of $\mathbb{R}^2$ in dictionary order.

S.No.		D	P	U	B	J	K	L
		I^2, dictionary	$(\mathbb{R}^\omega$, product$)$	$(\mathbb{R}^\omega$, uniform$)$	$(\mathbb{R}^\omega$, box$)$	$\mathbb{R}^J$, J uncountable	$\mathbb{R}_K$	$\mathbb{R}_l$
1.	T_0							
2.	T_1							
3.	T_2 (Hausdorff)							
4.	T_3 (Regular)							
5.	$T_{3\frac{1}{2}}$ (Completely regular)							
6.	T_4 (Normal)							
7.	Connected							
8.	Locally connected							
9.	Path-connected							
10.	Locally path-connected							
11.	Compact							
12.	Locally compact							
13.	1^{st}-countable							
14.	2^{nd}-countable							
15.	Separable							
16.	Lindelöf							
17.	Metrizable							
18.	Compactly generated							

Figure 29.1: Table of topological notions compared to some basic examples.

We present a brief proof or counterexample for each of the boxes as below.

Proof of entries in Table 29.1. We label each block in the table by its row and column labels.

`1.E` : □

Statistics:-

- # Parts = 6
- # Chapters = 29
- # Definitions = ??
- # Remarks = ??
- # Lemmas = ??
- # Propositions = ??
- # Theorems = ??

Milestones:-

- Started : Somewhere in middle of December 2022.
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- Total 400 pages : 8^{th} October, 2023.
- The Arithmetic Viewpoint initiated on : 24^{th} December, 2023.
- First 500 pages : 30^{th} December, 2023.