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Energy losses due to elastic wave propagation during an elastic impact

J Reed
Central Electricity Generating Board, Berkeley Nuclear Laboratories, Berkeley, Gloucestershire GL13 9PB, UK

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Abstract. During the impact of an elastic sphere on an elastic, massive, substrate some of the kinetic energy is radiated into the massive substrate as elastic waves and is not available for subsequent recovery as kinetic energy. This loss of recoverable energy then determines the maximum possible value for the coefficient of restitution for any impact.

In evaluating the radiated energy loss Hunter used an approximation to the displacement–time relationship in a Hertzian impact so as to obtain an analytic solution to the problem. However, in doing so he derived a force–time relationship which had a different power dependence from that of the Hertz equations. In this paper the energy loss is re-evaluated using Hunter's approximation for the displacement–time relationship but applying it in such a way as to maintain similarity with the Hertz equations. This re-evaluation predicts the energy loss as being some 4.5 times greater than predicted by Hunter's analysis. It is shown that the present analysis results in a generally better fit to experimental data.

1. Introduction

The adhesion of particles upon impact with a surface is of interest in many fields of technology, e.g. powder transport, surface contamination, etc. An energy balance criterion for the adhesion of an impacting particle may be readily obtained by noting that the kinetic energy of a rebounding particle is derived from the release of elastic energy stored in the bodies as a result of the impact. Thus if the maximum elastic energy stored in the bodies during the impact is less than the adhesive energy between the bodies then the impacting particle will adhere to the surface. Conversely, if the stored elastic energy is greater than the adhesive energy then the particle will probably rebound.

In any impact between two bodies there are several mechanisms by which the initial kinetic energy may be dissipated. These include, for example, the propagation of elastic and flexural waves, plastic deformation, and internal friction. The quantities of energy dissipated by these different mechanisms depend upon the details of the impact. However, the propagation of elastic waves occurs in any impact and thus the energy dissipated by this mechanism represents the absolute minimum energy loss in any impact regardless of the contribution of any other mechanisms which may be operative. This energy loss then determines the maximum possible value of the coefficient of restitution for any impact. (In particular, we note that if it can be shown that a particle will adhere upon impact when this energy-loss mechanism alone is considered then there is little need to consider any other mechanisms of energy loss.)

There will, in general, be no energy loss in an impact between two perfectly elastic bodies where the time of the impact is very long in comparison with the time taken for an elastic wave to traverse either body (Love 1944). However, in an impact between a sphere and a massive plane body where the time taken for an elastic wave to traverse the massive body is very long in comparison with the duration of the impact the energy contained in these elastic waves is lost as regards its recovery in the rebound stage of the impact.

This paper presents an improvement to the Hunter (1957) theoretical evaluation of the energy dissipated due to elastic wave propagation in the elastic impact of a sphere with a massive, plane body. This is shown to result in better agreement with experimental data for the coefficient of restitution.

2. Theory

The impact is considered to be both normal and elastic in nature, i.e. neither body suffers permanent deformation as a result of the impact, and to be describable by the equations of Hertz (1881). (Love's (1944) criterion for application of the Hertz equations to dynamic situations would imply that they are not applicable to the present case of impact between a sphere and a massive, plane body. However, there is considerable experimental evidence that they are applicable in such cases, e.g. Tait (1890, 1892), Vincent *et al* (1957), and Hunter (1957) provides some theoretical justification for their use in such cases.) The derivation of the Hertz equation is readily accessible (e.g. Love 1944) and so only the salient features will be presented here.

The force acting during the contact of a sphere, radius R, with a plane surface is given from the Hertz equations as

$$F = -R^{1/2} K \alpha^{3/2}$$

where α is the distance of approach of the centres of the two bodies when in contact,

$$K = \frac{4}{3}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)^{-1}$$

where for body i

$$E_i = \text{Young's modulus}$$

and

$$\nu_i = \text{Poisson's ratio.}$$

Subscript one refers to the particle and subscript two refers to the massive body.

Assuming the Hertz equations to apply during an elastic impact then

$$F(t) = -R^{1/2} K \alpha(t)^{3/2}. \tag{1}$$

From Newtons laws

$$F(t) = m\,\mathrm{d}^2\alpha/\mathrm{d}t^2 \tag{2}$$

where m is the mass of the particle. Combining equations (1) and (2) gives

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \frac{-R^{1/2}K}{m}\,\alpha(t)^{3/2}. \tag{3}$$

This has a solution given by

$$\alpha(t) = \alpha_0 \varphi(vt/\alpha_0)$$

where

$$\alpha_0 = \text{maximum compression} = (5mv^2/4R^{1/2}K)^{2/5}$$

$$v = \text{impact velocity.}$$

$\varphi(vt/\alpha_0)$ is given by the inverse relationship

$$\frac{vt}{\alpha_0} = \int_0^{\varphi} \frac{\mathrm{d}x}{(1 - x^{5/2})^{1/2}}. \tag{4}$$

Hunter (1957) has shown

$$\varphi(vt/\alpha_0) \simeq \sin\left(\frac{\pi}{2.94}\frac{vt}{\alpha_0}\right) \tag{5}$$

and hence

$$\alpha(t) \simeq \alpha_0 \sin\left(\frac{\pi}{2.94}\frac{vt}{\alpha_0}\right) \qquad 0 < t < 2.94\alpha_0/v. \tag{6}$$

Figure 1 shows the comparison between equations (4) and (5) and illustrates that there is little error in this approximation. It is at this point that the present analysis differs from that of Hunter. In his analysis Hunter substituted the approximation for $\alpha(t)$, given by equation (6), into equation (2) to obtain (his equation (38))

$$\left.\begin{aligned} F(t) &= -m\alpha_0 \left(\frac{\pi v}{2.94\alpha_0}\right)^2 \sin\left(\frac{\pi vt}{2.94\alpha_0}\right) \\ &= -m\left(\frac{\pi v}{2.94\alpha_0}\right)^2 \alpha(t) \qquad 0 \leqslant t \leqslant 2.94\alpha_0/v \\ F(t) &= 0 \qquad\qquad t < 0;\, t > 2.94\alpha_0/v. \end{aligned}\right\} \tag{7}$$

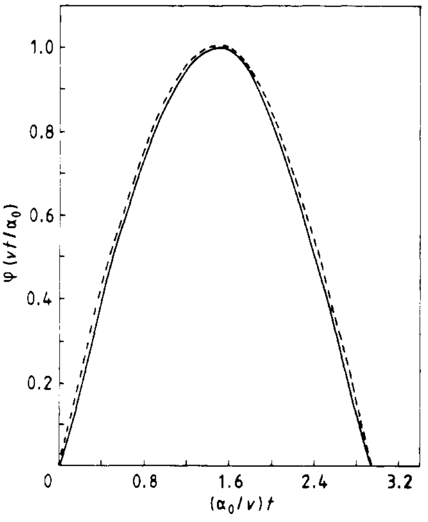

Figure 1. The comparison between: equation (4), full curve, the inverse relationship of φ; and equation (5), broken curve, $\varphi \simeq \sin \pi vt/2.94\alpha_0$.

This is clearly at variance with equation (1) where $F(t)$ is stated to be proportional to $\alpha(t)^{3/2}$ rather than $\alpha(t)$. In the present analysis this difficulty is avoided by substituting the approximation for $\alpha(t)$ directly into equation (1) to give

$$\left.\begin{aligned} &F(t) = -R^{1/2}K\alpha(t)^{3/2} = -R^{1/2}K\alpha_0^{3/2}\sin^{3/2}\left(\frac{\pi v t}{2.94\alpha_0}\right) && 0 \leqslant t \leqslant 2.94\alpha_0/v \\ &F(t) = 0 && t < 0;\, t > 2.94\alpha_0/v. \end{aligned}\right\} \tag{8}$$

Figure 2 shows $F(t)$ given by equation (7) and equation (8). The difference between the two force-time curves can readily be seen.

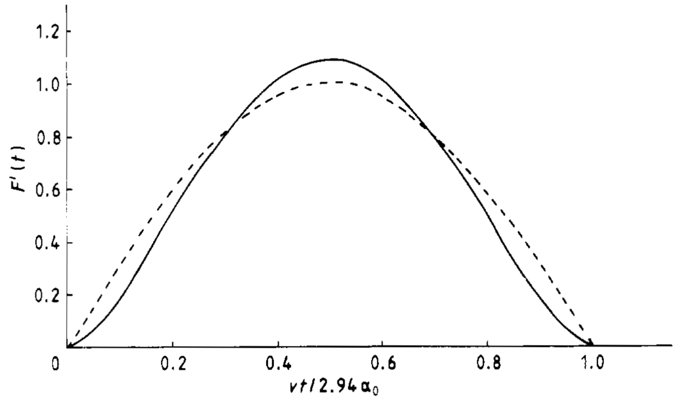

Figure 2. The comparison between the force–time curve used by Hunter (1957), equation (7), broken curve; and the force–time curve in this paper, equation (8), full curve. Here $F'(t) = F(t)/1.044\, m^{3/5} v^{6/5} K^{2/5} R^{1/5}$.

The present analysis now proceeds in the same manner as Hunter's except that the force–time relationship of equation (8) is used rather than that of equation (7). The Fourier components of the total force are given by

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\exp(-\mathrm{i}\omega t)\,\mathrm{d}t$$

therefore, using equation (8),

$$f(\omega) = -\frac{1}{2\pi}R^{1/2}K\frac{2.94\alpha_0^{5/2}}{\pi v}\int_0^{\pi}\sin^{3/2}(x)\exp(-\mathrm{i}\beta x)\,\mathrm{d}x$$

where

$$\beta = \frac{2.94\alpha_0\omega}{\pi v}.$$

Integration leads to

$$f(\omega) = \frac{-R^{1/2}K}{\pi^{1/2}}\frac{\alpha_0^{5/2}}{v}0.7796\frac{\exp(-\mathrm{i}\pi\beta/2)}{\Gamma(\frac{7}{4} + 2.94\alpha_0\omega/2\pi v)\Gamma(\frac{7}{4} - 2.94\alpha_0\omega/2\pi v)}. \tag{9}$$

The total energy in the form of elastic vibrations, W, is given by (Hunter (1957), equation (20))

$$W = \frac{8\delta(1+\nu_2)}{\rho_2 C_0^3}\left(\frac{1-\nu_2^2}{1-2\nu_2}\right)^{1/2}\int_0^\infty \omega^2|f(\omega)|^2\,\mathrm{d}\omega \tag{10}$$

where δ is the imaginary part of

$$\int_0^\infty \frac{\xi(\xi^2-1)^{1/2}\,\mathrm{d}\xi}{F_0(\xi)}$$

where

$$F_0(\xi) = (2\xi^2-\gamma^2)^2 - 4\xi^2[(\xi^2-1)(\xi^2-\gamma^2)]^{1/2}$$

$$\gamma = [2(1-\nu_2)/(1-2\nu_2)]^{1/2}$$

$$C_0 = (E_2/\rho_2)^{1/2}$$

ρ_2 = density of the substrate.

Putting equation (9) into equation (10) and numerically integrating, the fractional kinetic energy lost, λ, is obtained as

$$\lambda \equiv \frac{W}{1/2mv^2} = 7.267\frac{\delta}{\rho_2 C_0^3}(1+\nu_2)\left(\frac{1-\nu_2^2}{1-2\nu_2}\right)^{1/2}\rho_1^{-1/5}K^{6/5}v^{3/5} \tag{11}$$

where ρ_1 is the density of the particle. Rewriting Hunter's equation (45) then

$$\lambda_{\text{Hunter}} = 1.5965\frac{\delta}{\rho_2 C_0^3}(1+\nu_2)\left(\frac{1-\nu_2^2}{1-2\nu_2}\right)^{1/2}\rho_1^{-1/5}K^{6/5}v^{3/5}. \tag{12}$$

It is of interest, here, to note that despite the use of different force–time relationships the only difference between equations (11) and (12) is the numerical constant, that is, the parameteric dependence is the same in both cases. It should also be noted that the fractional energy loss is independent of particle size.

Finally, the coefficient of restitution, e, of the impacting bodies may be derived from the relationship

$$e = (1-\lambda)^{1/2}. \tag{13}$$

3. Measurement of the coefficient of restitution

Measurements have been made of the coefficient of restitution for the impact of steel ball bearings onto various flat surfaces. The impact and rebound velocities were measured in some cases using a stroboscope and camera. In the other cases the time between successive bounces was used to calculate the required velocities. The spheres were released from various heights (50–500 mm) above the flat surface and allowed to accelerate under the influence of gravity alone. (For further details of the experimental technique see Rogers and Reed (1984).) The size of the ball bearings ranged from 0.5 to 1 mm in diameter. The flat bodies used were a circular glass block (approximately 70 mm diameter × 35 mm thick and 350 gm weight) and a tungsten block (approximately 40 mm × 40 mm × 20 mm thick and 400 gm weight). The glass block was used in the as

Table 1. Elastic constants for materials used in the impact experiments, taken from Kaye and Laby (1973).

	Steel	Glass	Tungsten
Young's modulus (GPa)	211.9	80.1	411.0
Poisson's ratio	0.291	0.27	0.28
Density† ($Mg\,m^{-3}$)	7.83	2.47	12.5
Elastic yield limits (GPa)	9–15	>4	15–35

† The densities were measured for the specimens used.

received condition with one smooth surface. The tungsten block had one face highly polished (i.e. mirror finish).

Values for the elastic constants, etc, for these materials are given in table 1. The variation in the coefficient of restitution for different impact velocities is shown in figures 3 and 4 for the impact on glass and tungsten surfaces respectively.

It should be pointed out that all the data points are averages of several values. The line drawn at each data point gives the actual spread of the data.

4. Discussion

It may be readily shown (e.g. Rogers and Reed (1984)) that the impact velocities above which plastic deformation occurs in one of the bodies are 4.5 and 6.0 m s^{-1} for the glass and tungsten surfaces, respectively. Since the actual impact velocities are below these values, it is safe to disregard plastic deformation as an energy-loss mechanism. It may, furthermore, be shown from Zener (1941) that less than 0.005% of the impact energy would have been dissipated by flexural waves. It is, therefore, reasonable to consider that elastic wave propagation was the major cause of energy loss in the impacts.

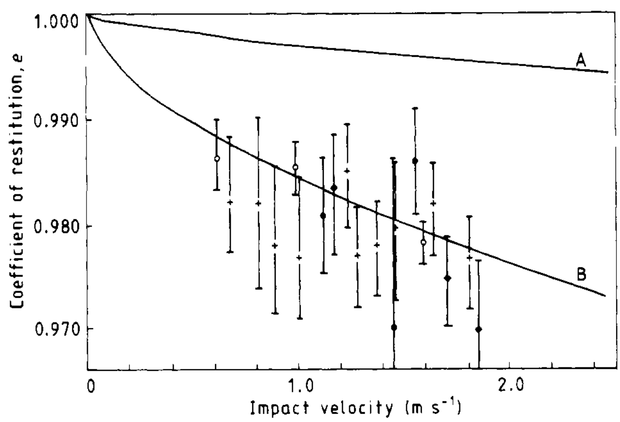

Figure 3. The variation of the coefficient of restitution with impact velocity for steel spheres colliding with glass. Theory: A, Hunter; B, this paper. Data taken with stroboscope and with spheres of diameter: +, 1 mm; ◆, 0.75 mm; ●, 0.50 mm. Data taken from time between bounces: ○, 0.75 mm.

In figures 3 and 4 it can be seen that, generally, the coefficient of restitution decreases with increasing impact velocity, although there is a large spread in the results. It also appears that the coefficient is independent of size for the combinations used here. Both of these trends were expected upon examination of equations (11) or (12). Figures 3 and 4 show the theoretical curves for the variation of the coefficient of restitution given by equations (11), (12) and (13). In deriving these curves the values of $\delta(\nu)$ for $\nu = 0.27$ and 0.28 have not been obtained from the definition below equation (10) but rather by assuming a linear relationship for $\delta(\nu)$ over the range of $\nu = \frac{1}{4}$ to $\frac{1}{3}$. Therefore, for glass–steel impacts $\delta(\nu) = 0.50$ and for tungsten–steel impacts $\delta(\nu) = 0.49$.

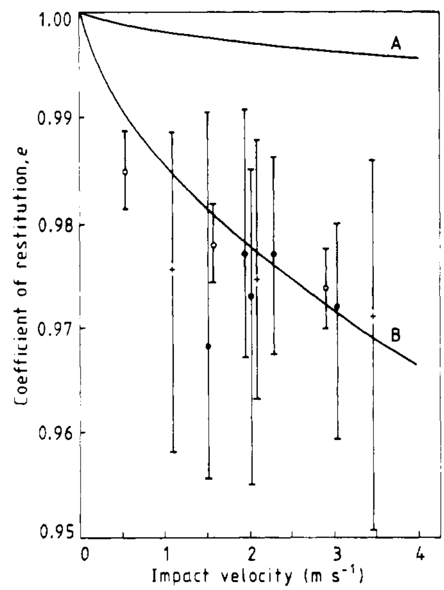

Figure 4. Same as figure 3 but for steel spheres colliding with tungsten.

Before discussing the fit between the data and theory shown in figures 3 and 4, it is worthwhile to outline some of the assumptions built into the theoretical equations.

Both the present analysis and that of Hunter use the response of an isotropic, elastic, semi-infinite medium to a time-dependent force evaluated by Miller and Pursey (1954, 1955). Miller and Pursey calculated their results for a uniform pressure over the contact circle while in the present case they are applied to a Hertzian contact where the pressure distribution is not uniform and nor is the area of contact constant. It is here assumed that the history of the force in time, rather than the local pressure distribution, determines the wave energy radiated out into the body. Additionally the Hertz equations apply for perfectly elastic impacts (i.e. $e = 1$) while in the present case, the energy losses are considered to necessarily occur.

It is believed that these assumptions will introduce negligible error although this cannot be proved.

Considering figures 3 and 4 it can then be seen that in both cases the theoretical curves from the analysis presented here fit the data far better than do Hunter's.

Hunter compared his equation with values for the coefficient of restitution taken by Tillet (1954). For the impact at 0.9 m s^{-1} of a steel sphere onto glass Tillet gave $e = 0.985$ (± 0.006).

Hunter evaluated the coefficient of restitution to be $e = 0.996$ at this impact velocity. This shows some agreement but not within the limits of experimental error. However, the present analysis (using the same constants as Hunter) leads to $e = 0.984$ at 0.9 m s^{-1}. This is in excellent agreement with Tillet's result. It should be noted, however, that Tillet quotes a $\frac{1}{2}\%$ variation in the value of e over an impact velocity range from 0.1 to 3 m s^{-1}. Hunter's analysis leads to a predicted variation of e over this range of 0.6% (0.999 to 0.993) while the analysis presented here results in a variation of 3.0% (0.996 to 0.966). This lends support to Hunter's analysis but Tillet gives insufficient information for further discussion of this point.

Tillet also gives data for the impact of steel spheres onto steel surfaces. For an impact velocity of 0.01 m s^{-1} Tillet found $e = 0.95$. Hunter calculates $e = 0.9998$ while this analysis predicts $e = 0.9989$ at this velocity (see Hunter (1957) for elastic constants, etc). In this case neither of the two theoretical values agree with the experimentally measured value. Since the present analysis gives a good fit to the results reported here it seems likely that other energy-loss mechanisms were present in this case. Hunter suggests strain rate effects as one possibility. Plastic deformation of surface asperities is another possible cause.

5. Conclusions

The energy loss due to the propagation of elastic waves generated in an elastic impact between a sphere and a massive plane body has been evaluated by means of an approach similar to that used by Hunter (1957). The difference between this work and Hunter's is that a better approximation to the force–time relationship for an elastic impact is used. This results in a factor of about 4.5 times more energy being predicted as lost during the impact. The present analysis gives generally better agreement with experimental measurement than does that of Hunter.

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