MAA202: Analysis
Homework I

Alexandre Hirsch, André Renom, Malys Solal

4^{th} November 2019

Contents

1 Exercise 1

1.1

Let $B = \bar{B}(0,1) = \{x \in E \mid ||x|| \leq 1\}$ be the closed unit ball for the norm $|| \cdot ||$. Let us show that B is cool.

Let us prove that B is convex.
Take $x, y \in B$ and $t \in [0,1]$. We want to prove that $tx + (1-t)y \in B$. To do so, we prove that $||tx + (1-t)y|| \leq 1$.

$$\begin{aligned}
||tx + (1-t)y|| &\leq ||tx|| + ||(1-t)y|| \quad \text{by the triangle inequality} \\
&= |t| \cdot ||x|| + |1-t| \cdot ||y|| \quad \text{by homogeneity} \\
&\leq t + (1-t) \quad \text{as } ||x|| \leq 1 \text{ and } ||y|| \leq 1 \\
&= 1
\end{aligned}$$

Which finally proves that B is convex.

We now prove that B is bounded.
Take $x \in B$ then $||x|| \leq 1$ by definition of B. This proves that B is bounded by 1.

We now prove that B is symmetric with respect to 0.
Take $x \in B$ then $||x|| \leq 1$ by definition of B and thus $|-1| \cdot ||x|| \leq 1$ and by homogeneity $||-x|| \leq 1$ which proves $-x \in B$.
We have therefore proved that B is symmetric with respect to 0.

We finally prove that $0 \in \mathring{B}$
Let $\widetilde{B} = B(0,1) = \{x \mid ||x|| < 1\}$ be the open unit ball. We know $\widetilde{B} \subset B$ and $\widetilde{B}$ is open. As $\mathring{B}$ is the union of all open sets of B, we get $\widetilde{B} \subset \mathring{B}$.
By definition, $0 \in \widetilde{B}$ and hence $0 \in \mathring{B}$.

Finally, we have proved that B is cool.

1.2

We want to show that for a cool set X, and $\alpha, \beta \geq 0$, $\alpha X + \beta X = (\alpha + \beta)X$. We will proceed by double inclusion.

Let us first show that $\alpha X + \beta X \subseteq (\alpha + \beta)X$.
Take $a, b \in X$. Then $a\alpha + b\beta \in \alpha X + \beta X$. We then write:

$$a\alpha + b\beta = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta}a + \frac{\beta}{\alpha + \beta}b\right)$$

We define $t := \frac{\alpha}{\alpha+\beta} \in [0,1]$, and thus $\frac{\beta}{\alpha+\beta} = 1 - t$. We therefore have:

$$a\alpha + b\beta = (\alpha + \beta)\big(at + b(1-t)\big)$$

As X is a cool set, it is convex. Hence for $a, b \in X$, we have $c := ta + (1-t)b \in X$. Therefore:

$$a\alpha + b\beta = (\alpha + \beta)c \in (\alpha + \beta)X$$

We have therefore proved that $\alpha X + \beta X \subseteq (\alpha + \beta)X$.

We now want to show that $\alpha X + \beta X \supseteq (\alpha + \beta)X$.
Take $x \in (\alpha + \beta)X$. Then there exists $y \in X$ such that:

$$\begin{aligned} x &= (\alpha + \beta)y \\ &= \alpha y + \beta y \in \alpha X + \beta X \end{aligned}$$

We therefore have proved that $\alpha X + \beta X \supseteq (\alpha + \beta)X$.

Finally, by double inclusion, we have proved that $\alpha X + \beta X = (\alpha + \beta)X$.

1.3

We now define a function on E by setting for ever $x \in E$,

$$N_X(x) = \inf\{|\alpha| \mid x \in \alpha X\}$$

We now show that N_X is well-defined and that it defines a norm on E.

1.3.a

We want to show that for each $x \in E$, the set $N := \{\alpha \,|\, x \in \alpha X\}$ is not empty.

We start from the fact that $0 \in \mathring{X}$.
Then, there exists an $r > 0$ such that $B(0, r) \subset X$.
We define $\alpha := \frac{2||x||}{r}$. We know that $B(0, \alpha r) \subset \alpha X$. Since $B(0, \alpha r) = B(0, 2||x||)$, we have that $x \in \alpha X$, and hence $\alpha \in N$.

We have hence proved that N isn't empty.

1.3.b

We want to show N is homogeneous, that is $N_X(\lambda x) = |\lambda| N_X(x)$.

We start from:

$$N_X(\lambda x) = \inf\{|\alpha| \,|\lambda x \in \alpha X\}$$

But we know that $\lambda x \in \alpha X \Leftrightarrow \lambda x \in -\alpha X$, hence we can write:

$$\begin{aligned} N_X(\lambda x) &= \inf\{|\alpha| \mid |\lambda| x \in \alpha X\} \\ &= \inf\{|\lambda \alpha| \mid x \in \alpha X\} \\ &= |\lambda| \inf\{|\alpha| \mid x \in \alpha X\} \\ &= |\lambda| N_X(x) \end{aligned}$$

We have hence proved N is homogeneous.

1.3.c

We want to show that N is definite, that is $N_X(x) = 0 \Rightarrow x = 0$.

We start from:

$$\begin{aligned} & N_X(0) = \inf\{|\alpha| \,|\, x \in \alpha X\} = 0 \\ \Rightarrow & \, x \in 0 \times X \\ \Rightarrow & \, ||x|| \leq 0 \times M_X \\ \Rightarrow & \, x = 0 \end{aligned}$$

This concludes the proof.

1.3.d

We want to show that the triangular inequality is true for N_X.
We therefore take $x, y \in E$.

$$\begin{aligned} N_X(x+y) &= \inf\{|\alpha| \mid (x+y) \in \alpha X\} \\ &= \inf\left\{|\alpha| \mid (x+y) \in \frac{\alpha}{2}X + \frac{\alpha}{2}X\right\} \\ &= \inf\left\{|\alpha| \mid x \in \frac{\alpha}{2}X\right\} + \inf\left\{|\alpha| \mid y \in \frac{\alpha}{2}X\right\} \\ &\leq \inf\{|\alpha| \mid x \in \alpha X\} + \inf\{|\alpha| \mid y \in \alpha X\} \\ &= N_X(x) + N_X(y) \end{aligned}$$

This concludes the proof that the triangular inequality holds for N_X, and by extension, that N_X is a norm.

1.4

1.4.a

We want to show that $||x|| \leq MN_X(x) \quad \forall x \in E$. In order to do this, we start from the fact that X is bounded by M and that therefore $X \subset \bar{B}(0, M)$. We have from this that $x \in N_X(x)\bar{X} \subset N_X\bar{B}(0, M)$. Since $x \in N_X\bar{B}(0, M)$ then $||x|| \leq MN_X$. This concludes the proof that $|| \cdot ||$ is weaker than N_x.

1.4.b

We want to show that there exists an α such that for all x, $N_X(x) \leq \alpha||x||$. Since $0 \in \mathring{X}$, we can consider $r := sup\{r \,|\, B(0, r) \subset X\}$ with $r \neq 0$. We then have that for all x, $N_X B(0, r) \subset B(0, x)$. We therefore have that:

$$\begin{aligned} N_X r &\leq ||x|| \\ N_X &\leq \frac{1}{r}||x|| \end{aligned}$$

This concludes the proof that N_X is weaker than $|| \cdot ||$, and therefore also that they are equivalent.

1.5

We know that $x \in \bar{X}$ for the norm N_X is equivalent to

$$\exists (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}, (x_n) \text{ converges for } N_X \text{ to } x$$

But wee know that the norms $||\cdot||$ and N_X are equivalent, therefore if a sequence converrges under N_X, then it must also converge under $||\cdot||$. Therefore:

$$\begin{aligned} x \in \bar{X}_{N_X} &\Leftrightarrow \exists (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}, (x_n) \text{ converges for } ||\cdot|| \text{ to } x \\ &\Leftrightarrow x \in \bar{X}_{||\cdot||} \end{aligned}$$

We have therefore shown equivalence between the closure of X for the two norms. This concludes the proof.

2 Exercise 2

2.1

Let us prove that $\bar{B}(0,1)$ is compact if and only if $S(0,1) = \{x \in E \mid N(x) = 1\}$ is compact. We will prove this by double implication.

Assume $\bar{B}(0,1) = \{x \in E \mid N(x) \leq 1\}$ is compact.
We know $S(0,1) \subset \bar{B}(0,1)$ then $S(0,1)$ is compact provided it is closed as any closed subset of a compact set is compact. Let us now prove $S(0,1)$ is closed by proving its complement is open. We have:

$$\begin{aligned} (S(0,1))^c &= \{x \in E \mid N(x) \neq 1\} \\ &= \{x \in E \mid N(x) < 1 \lor N(x) > 1\} \\ &= \{x \in E \mid N(x) < 1\} \cup \{x \in E \mid N(x) > 1\} \\ &= B(0,1) \cup (\bar{B}(0,1))^c \end{aligned}$$

We know $B(0,1)$ is open and as $\bar{B}(0,1)$ is closed, $(\bar{B}(0,1))^c$ is open and hence $(S(0,1))^c$ is open as the union of open sets. Thus $S(0,1)$ is closed and therefore compact.

Let us now prove the converse statement. Assume $S(0,1)$ is compact.

Consider the function

$$\begin{aligned} f: \quad S(0,1) \times [0,1] &\to E \\ (x,t) &\mapsto tx \end{aligned}$$

Let us prove that f is Lipschitz continuous.
Let $x, y \in S(0,1)$ and let $t_x, t_y \in [0,1]$. Then $N(t_x x - t_y y) \leq N(x-y)$ and thus f is Lipschitz continuous.
Hence $f(S(0,1),[0,1])$ is compact.

Let us now prove that $f(S(0,1),[0,1]) = \bar{B}(0,1)$ by double inclusion.

- "$\subseteq$" Take $x \in S(0,1)$ and $t \in [0,1]$.
  Then $N(f(x,t)) = N(tx) = |t|\, N(x) \leq N(x) = 1$ hence $f(x,t) \in \bar{B}(0,1)$.
  Therefore we have proved $f(S(0,1),[0,1]) \subseteq \bar{B}(0,1)$
- "$\supseteq$" Take $x \in \bar{B}(0,1)$. We must prove that there exist $x' \in S(0,1)$ and $t' \in [0,1]$ such that $f(x',t') = x't' = x$.
  - If $x = 0$: then we can set $t' := 0$ and let x' be any element of $S(0,1)$. We then get $f(x',t') = x't' = 0 = x$ which proves $x \in f(S(0,1),[0,1])$

– If $x \neq 0$: Let $t := \frac{1}{N(x)}$, then $N(tx) = |t|\, N(x) = 1$ thus we can set $x' := tx \in S(0,1)$. We then set $t' := N(x) \in [0,1]$ so that $x = x't'$. We hence get $f(x', t') = x$ which proves $x \in f(S(0,1), [0,1])$

We therefore have proved that $\bar{B}(0,1) \subseteq S(0,1)$.

Finally, $f(S(0,1), [0,1]) = \bar{B}(0,1)$ and hence $\bar{B}(0,1)$ is compact which finally concludes the proof.

2.2

TO DO

2.3

Let's first prove that $d(x,F) \leq N(x)$.
From the previous question, we have that $x \mapsto d(x,F)$ is Lipschitz-continuous so $\forall x, y \in F$ $|d(x,F) - d(y,F)| \leq N(x-y)$. Now let $y = 0$ which is possible as $0 \in F$ because F is a vector subspace. We then get $|d(x,F) - d(0,F)| = |d(x,F)| = d(x,F) \leq N(x-0) = N(x)$ from which we get our result.

Let us now prove that $d(x,F) = \inf\{N(x-u) \,|\, u \in F,\, N(u) \leq 2N(x)\}$
TO DO