

# MAA202: Analysis

## Homework I

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4<sup>th</sup> November 2019

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# 1 Exercise 1

## 1.1

Let  $B = \bar{B}(0, 1) = \{x \in E \mid \|x\| \leq 1\}$  be the closed unit ball for the norm  $\|\cdot\|$ . Let us show that  $B$  is cool.

Let us prove that  $B$  is convex.

Take  $x, y \in B$  and  $t \in [0, 1]$ . We want to prove that  $tx + (1 - t)y \in B$ . To do so, we prove that  $\|tx + (1 - t)y\| \leq 1$ .

$$\begin{aligned}\|tx + (1 - t)y\| &\leq \|tx\| + \|(1 - t)y\| \quad \text{by the triangle inequality} \\ &= |t| \cdot \|x\| + |1 - t| \cdot \|y\| \quad \text{by homogeneity} \\ &\leq t + (1 - t) \quad \text{as } \|x\| \leq 1 \text{ and } \|y\| \leq 1 \\ &= 1\end{aligned}$$

Which finally proves that  $B$  is convex.

We now prove that  $B$  is bounded.

Take  $x \in B$  then  $\|x\| \leq 1$  by definition of  $B$ . This proves that  $B$  is bounded by 1.

We now prove that  $B$  is symmetric with respect to 0.

Take  $x \in B$  then  $\|x\| \leq 1$  by definition of  $B$  and thus  $|-1| \cdot \|x\| \leq 1$  and by homogeneity  $\|-x\| \leq 1$  which proves  $-x \in B$ .

We have therefore proved that  $B$  is symmetric with respect to 0.

We finally prove that  $0 \in \mathring{B}$

Let  $\tilde{B} = B(0, 1) = \{x \mid \|x\| < 1\}$  be the open unit ball. We know  $\tilde{B} \subset B$  and  $\tilde{B}$  is open. As  $\mathring{B}$  is the union of all open sets of  $B$ , we get  $\tilde{B} \subset \mathring{B}$ .

By definition,  $0 \in \tilde{B}$  and hence  $0 \in \mathring{B}$ .

Finally, we have proved that  $B$  is cool.

## 1.2

We want to show that for a cool set  $X$ , and  $\alpha, \beta \geq 0$ ,  $\alpha X + \beta X = (\alpha + \beta)X$ . We will proceed by double inclusion.

Let us first show that  $\alpha X + \beta X \subseteq (\alpha + \beta)X$ .

Take  $a, b \in X$ . Then  $a\alpha + b\beta \in \alpha X + \beta X$ . We then write:

$$a\alpha + b\beta = (\alpha + \beta) \left( \frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b \right)$$

We define  $t := \frac{\alpha}{\alpha + \beta} \in [0, 1]$ , and thus  $\frac{\beta}{\alpha + \beta} = 1 - t$ . We therefore have:

$$a\alpha + b\beta = (\alpha + \beta)(at + b(1 - t))$$

As  $X$  is a cool set, it is convex. Hence for  $a, b \in X$ , we have  $c := ta + (1 - t)b \in X$ . Therefore:

$$a\alpha + b\beta = (\alpha + \beta)c \in (\alpha + \beta)X$$

We have therefore proved that  $\alpha X + \beta X \subseteq (\alpha + \beta)X$ .

We now want to show that  $\alpha X + \beta X \supseteq (\alpha + \beta)X$ .

Take  $x \in (\alpha + \beta)X$ . Then there exists  $y \in X$  such that:

$$\begin{aligned} x &= (\alpha + \beta)y \\ &= \alpha y + \beta y \in \alpha X + \beta X \end{aligned}$$

We therefore have proved that  $\alpha X + \beta X \supseteq (\alpha + \beta)X$ .

Finally, by double inclusion, we have proved that  $\alpha X + \beta X = (\alpha + \beta)X$ .

### 1.3

We now define a function on  $E$  by setting for ever  $x \in E$ ,

$$N_X(x) = \inf\{|\alpha| \mid x \in \alpha X\}$$

We now show that  $N_X$  is well-defined and that it defines a norm on  $E$ .

#### 1.3.a

We want to show that for each  $x \in E$ , the set  $N := \{\alpha \mid x \in \alpha X\}$  is not empty.

We start from the fact that  $0 \in \overset{\circ}{X}$ .

Then, there exists an  $r > 0$  such that  $B(0, r) \subset X$ .

We define  $\alpha := \frac{2\|x\|}{r}$ . We know that  $B(0, \alpha r) \subset \alpha X$ . Since  $B(0, \alpha r) = B(0, 2\|x\|)$ , we have that  $x \in \alpha X$ , and hence  $\alpha \in N$ .

We have hence proved that  $N$  isn't empty.

#### 1.3.b

We want to show  $N$  is homogeneous, that is  $N_X(\lambda x) = |\lambda|N_X(x)$ .

We start from:

$$N_X(\lambda x) = \inf\{|\alpha| \mid \lambda x \in \alpha X\}$$

But we know that  $\lambda x \in \alpha X \Leftrightarrow \lambda x \in -\alpha X$ , hence we can write:

$$\begin{aligned} N_X(\lambda x) &= \inf\{|\alpha| \mid |\lambda|x \in \alpha X\} \\ &= \inf\{|\lambda\alpha| \mid x \in \alpha X\} \\ &= |\lambda| \inf\{|\alpha| \mid x \in \alpha X\} \\ &= |\lambda|N_X(x) \end{aligned}$$

We have hence proved  $N$  is homogeneous.

### 1.3.c

We want to show that  $N$  is definite, that is  $N_X(x) = 0 \Rightarrow x = 0$ .

We start from:

$$\begin{aligned} N_X(0) &= \inf\{|\alpha| \mid x \in \alpha X\} = 0 \\ \Rightarrow x &\in 0 \times X \\ \Rightarrow \|x\| &\leq 0 \times M_X \\ \Rightarrow x &= 0 \end{aligned}$$

This concludes the proof.

### 1.3.d

We want to show that the triangular inequality is true for  $N_X$ .

We therefore take  $x, y \in E$ .

$$\begin{aligned} N_X(x+y) &= \inf\{|\alpha| \mid (x+y) \in \alpha X\} \\ &= \inf\left\{|\alpha| \mid (x+y) \in \frac{\alpha}{2}X + \frac{\alpha}{2}X\right\} \\ &= \inf\left\{|\alpha| \mid x \in \frac{\alpha}{2}X\right\} + \inf\left\{|\alpha| \mid y \in \frac{\alpha}{2}X\right\} \\ &\leq \inf\{|\alpha| \mid x \in \alpha X\} + \inf\{|\alpha| \mid y \in \alpha X\} \\ &= N_X(x) + N_X(y) \end{aligned}$$

This concludes the proof that the triangular inequality holds for  $N_X$ , and by extension, that  $N_X$  is a norm.

## 1.4

### 1.4.a

We want to show that  $\|x\| \leq MN_X(x) \quad \forall x \in E$ . In order to do this, we start from the fact that  $X$  is bounded by  $M$  and that therefore  $X \subset \bar{B}(0, M)$ . We have from this that  $x \in N_X(x)\bar{X} \subset N_X\bar{B}(0, M)$ . Since  $x \in N_X\bar{B}(0, M)$  then  $\|x\| \leq MN_X$ . This concludes the proof that  $\|\cdot\|$  is weaker than  $N_x$ .

### 1.4.b

We want to show that there exists an  $\alpha$  such that for all  $x$ ,  $N_X(x) \leq \alpha\|x\|$ . Since  $0 \in \overset{\circ}{X}$ , we can consider  $r := \sup\{r \mid B(0, r) \subset X\}$  with  $r \neq 0$ . We then have that for all  $x$ ,  $N_X B(0, r) \subset B(0, x)$ . We therefore have that:

$$\begin{aligned} N_X r &\leq \|x\| \\ N_X &\leq \frac{1}{r}\|x\| \end{aligned}$$

This concludes the proof that  $N_X$  is weaker than  $\|\cdot\|$ , and therefore also that they are equivalent.

## 1.5

We know that  $x \in \bar{X}$  is equivalent to

## 2 Exercise 2

### 2.1

Let us prove that  $\bar{B}(0, 1)$  is compact if and only if  $S(0, 1) = \{x \in E \mid N(x) = 1\}$  is compact. We will prove this by double implication.

Assume  $\bar{B}(0, 1) = \{x \in E \mid N(x) \leq 1\}$  is compact.

We know  $S(0, 1) \subset \bar{B}(0, 1)$  then  $S(0, 1)$  is compact provided it is closed as any closed subset of a compact set is compact. Let us now prove  $S(0, 1)$  is closed by proving its complement is open. We have:

$$\begin{aligned} (S(0, 1))^c &= \{x \in E \mid N(x) \neq 1\} \\ &= \{x \in E \mid N(x) < 1 \vee N(x) > 1\} \\ &= \{x \in E \mid N(x) < 1\} \cup \{x \in E \mid N(x) > 1\} \\ &= B(0, 1) \cup (\bar{B}(0, 1))^c \end{aligned}$$

We know  $B(0, 1)$  is open and as  $\bar{B}(0, 1)$  is closed,  $(\bar{B}(0, 1))^c$  is open and hence  $(S(0, 1))^c$  is open as the union of open sets. Thus  $S(0, 1)$  is closed and therefore compact.

Let us now prove the converse statement. Assume  $S(0, 1)$  is compact.

Consider the function

$$\begin{array}{ccc} f : & S(0, 1) \times [0, 1] \rightarrow & E \\ & (x, t) \mapsto & tx \end{array}$$

Let us prove that  $f$  is Lipschitz continuous.

Let  $x, y \in S(0, 1)$  and let  $t_x, t_y \in [0, 1]$ . Then  $N(t_x x - t_y y) \leq N(x - y)$  and thus  $f$  is Lipschitz continuous.

Hence  $f(S(0, 1), [0, 1])$  is compact.

Let us now prove that  $f(S(0, 1), [0, 1]) = \bar{B}(0, 1)$  by double inclusion.

- " $\subseteq$ " Take  $x \in S(0, 1)$  and  $t \in [0, 1]$ .  
Then  $N(f(x, t)) = N(tx) = |t|N(x) \leq N(x) = 1$  hence  $f(x, t) \in \bar{B}(0, 1)$ .  
Therefore we have proved  $f(S(0, 1), [0, 1]) \subseteq \bar{B}(0, 1)$
- " $\supseteq$ " Take  $x \in \bar{B}(0, 1)$ . We must prove that there exist  $x' \in S(0, 1)$  and  $t' \in [0, 1]$  such that  $f(x', t') = x't' = x$ .
  - If  $x = 0$ : then we can set  $t' := 0$  and let  $x'$  be any element of  $S(0, 1)$ . We then get  $f(x', t') = x't' = 0 = x$  which proves  $x \in f(S(0, 1), [0, 1])$
  - If  $x \neq 0$ : Let  $t := \frac{1}{N(x)}$ , then  $N(tx) = |t|N(x) = 1$  thus we can set  $x' := tx \in S(0, 1)$ . We then set  $t' := N(x) \in [0, 1]$  so that  $x = x't'$ . We hence get  $f(x', t') = x$  which proves  $x \in f(S(0, 1), [0, 1])$

We therefore have proved that  $\bar{B}(0, 1) \subseteq S(0, 1)$ .

Finally,  $f(S(0, 1), [0, 1]) = \bar{B}(0, 1)$  and hence  $\bar{B}(0, 1)$  is compact.