MAA202 - Homework 1

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Exercise 1: We let $(E, \|\cdot\|)$ a normed real vector space. We have that $X \subset E$ is cool if the following conditions hold:

- X is convex: $\forall x, y \in X, \forall t \in [0,1], tx + (1-t)y \in X$
- X is bounded: $\exists M > 0, \forall x \in X, \|x\| \leq M$
- X is symmetric with respect to 0: $-X = X$
- $0 \in \mathring{X}$

1. We want to prove that $B = \overline{B}(0,1)$ is a cool set.

   We then first wish to show B is convex.
   Let $x, y \in B$ and let $t \in [0,1]$.
   We thus have:

   $$\|tx + (1-t)y\| \leq \|tx\| + \|(1-t)y\| = t\|x\| + (1-t)\|y\| \leq tr + (1-t)r = r$$

   Thus we have that $tx + (1-t)y$ so B is convex.

   We now show B is bounded. Let $x \in B \Rightarrow \|x\| \leq 1$.
   Thus B is bounded.

   We show $-B = B$. We show let $x \in B \Leftrightarrow \|x\| \leq 1 \Leftrightarrow \|-x\| \leq 1 \Leftrightarrow x \in -B$.
   Thus $-B = B$.

   Finally we show $0 \in \mathring{X}$. We have that $0 \in X$ and $B(0, 1/2) \subset B \Leftrightarrow 0 \in \mathring{X}$.

   Thus we have that B is cool.

2. We let X a cool set and $\alpha, \beta \geq 0$. We show $\alpha X + \beta X = (\alpha + \beta)X$.

   "$\subset$" We let $x \in \alpha X + \beta X$. Then $\exists a, b \in X$

   $$\begin{aligned} x &= \alpha a + \beta b \\ &= (\alpha + \beta)\left(\frac{\alpha}{\alpha+\beta}a + \frac{\beta}{\alpha+\beta}b\right) \end{aligned}$$

We let $t = \frac{\alpha}{\alpha+\beta}$. Then let $c = ta + (1-t)b$. We have that $t \in [0,1]$ and since X is convex $c \in X$. Thus we have that $x \in (\alpha+\beta)X$.

"$\supset$" We let $x \in (\alpha+\beta)X \Rightarrow \exists y \in X, x = (\alpha+\beta)y \Rightarrow x = \alpha y + \beta y \Rightarrow x \in \alpha X + \beta X$.

Thus we indeed have that $\alpha X + \beta X = (\alpha+\beta)X$.

3. We now define a function on E, $N_X(x) = \inf\{|\alpha| \mid x \in \alpha X\}$. We show that N_X is well defined and that it defines a norm on E.

   (a) We show that the set $N = \{\alpha \mid x \in \alpha X\}$ for each $x \in E$ is non empty. Let $x \in E$. We have that since $0 \in \mathring{X}, \exists\ r > 0, B(0,r) \subset X$. Then we consider $\alpha = \|2x\|/r$. We have that $x \in B(0, \alpha * r) \subset \alpha X$. Thus N is not empty.

   (b) We show that N_X is homogenous. Let $x \in E$ and $\lambda \in \mathbb{R}$.

$$\begin{aligned} N_X(\lambda x) &= \inf\{|\alpha| \mid \lambda x \in \alpha X\} \\ &= \inf\{|\lambda\alpha| \mid x \in \lambda\alpha X\} \\ &= |\lambda| \inf\{|\alpha| \mid x \in \lambda\alpha X\} \\ &= |\lambda| N_X(x) \end{aligned}$$

   (c) We show that N_X is definite. Let $x \in E, N_X(x) = 0$

$$\begin{aligned} N_X(x) = 0 &\Leftrightarrow \inf\{|\alpha| \mid x \in \alpha X\} = 0 \\ &\Rightarrow x \in 0 * X \\ &\Rightarrow \exists M, \|x\| \leq 0 * M \\ &\Leftrightarrow \|x\| \leq 0 \\ &\Rightarrow x = 0 \end{aligned}$$

   Thus since X is bounded, we have that N_X is definite.

   (d) We show that the triangular inequality holds for N_X. Let $x, y \in E$.

$$\begin{aligned} N_X(x+y) &= \inf\{|\alpha| \mid (x+y) \in \alpha X\} \\ &= \inf\left\{|\alpha| \mid (x+y) \in \frac{\alpha}{2}X + \frac{\alpha}{2}X\right\} \\ &= \inf\left\{|\alpha| \mid x \in \frac{\alpha}{2}X\right\} + \inf\left\{|\alpha| \mid y \in \frac{\alpha}{2}X\right\} \\ &\leq \inf\{|\alpha| \mid x \in \alpha X\} + \inf\{|\alpha| \mid y \in \alpha X\} \\ &= N_X(x) + N_X(y) \end{aligned}$$

   Thus using question 2.2 we have that N_X obeys the triangular inequality and that it indeed defines a norm.

4. We now show that N_X and $\|\cdot\|$ are equivalent norms.

(a) We first show that $\|x\| \leq MN_X(x)$ for all $x \in E$.
Since X is bounded we have that $\exists\ M > 0, \|x\| \leq M$. Thus $X \subset \overline{B}(0, M)$.
We let $x \in E, x \in N_X(x)\overline{X} \subset \overline{B}(0, N_X(x) * M)$. Thus we have that $\|x\| \leq M * N_X(x)\ \forall x \in E$

(b) Next we show that N_X is weaker than $\|\cdot\|$.

5. We finally consider the closure of X in both norms and prove that $\overline{X}_{N_X} = \overline{X}_{\|\cdot\|}$.

$$\begin{aligned} x \in \overline{X}_{N_X} &\Leftrightarrow \exists\ (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}, (x_n) \text{ converges for } N_X \text{ to } x \\ &\Leftrightarrow (x_n) \text{ converges for } \|\cdot\| \text{ to } x \\ &\Leftrightarrow x \in \overline{X}_{\|\cdot\|} \end{aligned}$$

And so we have proven that $\overline{X}_{N_X} = \overline{X}_{\|\cdot\|}$.

Exercise 2: In this exercise we will show that the closed unit ball $\overline{B}(0, 1) \subset E$ a normed vector space is compact if and only if E is finite dimensional.

1. We consider $S(0, 1) = \{x \in E \mid N(x) = 1\}$ the unit sphere. We show that $\overline{B}(0, 1)$ is compact if and only if $S(0, 1)$ is compact.
"$\Rightarrow$" Assume $\overline{B}(0, 1)$ is compact. We have that $S \subset \overline{B}$ thus it is bounded. We show it is closed. We show that $S^c = \{x \in E \mid N(x) \neq 1\}$ is open.
Let $x \in S^c$. We let $r = |N(x) - 1|/2 > 0$. We show that $B(x, r) \subset S^c$. Let $y \in B(x, r)$.
$$N(y) = N(y - x + x) \leq N(y - x) + N(x) \leq r + N(x)$$
We have that $N(x) \neq 1$ and that $r < N(x) - 1$. Thus $N(y) \neq 1$ and S^c is open.
Thus S is compact.
"$\Leftarrow$" Assume $S(0, 1)$ is compact. Let $f : S(0, 1) \times [0, 1] \to E, f(x, t) = tx$. We show that f is Lipschhitz continuous. Let $x, y \in E, t_x, t_y \in [0, 1]$
$$N(t_x x - t_y y) \leq N(1 * x - 1 * y) = N(x - y)$$
Thus we have that since f is 1-Lipschitz continuous $f(S(0, 1), [0, 1])$ is compact.
We show that $f(S(0, 1), [0, 1]) = \overline{B}(0, 1)$.
Let $x \in S, t \in [0, 1]$. We have that $N(tx) = tN(x) \leq 1$, thus $tx \in \overline{B}(0, 1)$
Let $x \in \overline{B}(0, 1)$. If $x = 0$ then $t = 0$ and x' any element of S. We now assume x is non-zero.
Let $t = 1/N(x)$. Then $N(tx) = 1$. Thus $x' = tx \in S$. We let $t' = N(x) \in [0, 1]$.
Then we have that there exists $x' \in S$ and $t' \in [0, 1], x = x't'$
Thus we have that $\overline{B}(0, 1)$ is compact.

2. Let $A \subset E$ and $x \in E$. We define $d(x, A) = inf\{N(x - u) \mid u \in A\}$ and show that $x \mapsto d(x, A)$ is 1-Lipschitz continuous.
Let $x, y \in E$.
$$|d(x, A) - d(y, A)| = |\inf\{N(x - u) \mid u \in A\} - \inf\{N(y - u) \mid u \in A\}|$$

3. We now let F a finite dimensional vector subspace. We have that $d(x, F)$ is 1-Lipschitz continuous so we have that $\forall x, y \in E$, $|d(x, F) - d(y, F)| \leq N(x - y)$. We consider $d(0, F)$. Since F is a vector subspace we have that $0 \in F$ thus $\inf\{N(u) \mid u \in F\} = 0$. We let $y = 0$ and find:

   $$|d(x, F) - d(y, F)| = |d(x, F) - 0| = d(x, F) \leq N(x - 0) = N(x)$$

   We also have that:

   $$d(x, F) = \inf\{N(x - u) \mid u \in F\} \leq N(x - 0) = N(x)$$

   Let $u \in F$ such that $d(x, F) = N(x - u)$. By contradiction assume that $N(u) > 2N(x)$.

4. We show that the $d(x, F)$ is attained at some $u_x \in F$

Exercise 3:

Exercise 4:

Exercise 5: