

# MAA202: Analysis

## Homework I

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# 1 Exercise 1

## 1.1

Let  $B = \bar{B}(0, 1) = \{x \in E \mid \|x\| \leq 1\}$  be the closed unit ball for the norm  $\|\cdot\|$ . Let us show that  $B$  is cool.

Let us prove that  $B$  is convex.

Take  $x, y \in B$  and  $t \in [0, 1]$ . We want to prove that  $tx + (1 - t)y \in B$ . To do so, we prove that  $\|tx + (1 - t)y\| \leq 1$ .

$$\begin{aligned}\|tx + (1 - t)y\| &\leq \|tx\| + \|(1 - t)y\| \quad \text{by the triangle inequality} \\ &= |t| \cdot \|x\| + |1 - t| \cdot \|y\| \quad \text{by homogeneity} \\ &\leq t + (1 - t) \quad \text{as } \|x\| \leq 1 \text{ and } \|y\| \leq 1 \\ &= 1\end{aligned}$$

Which finally proves that  $B$  is convex.

We now prove that  $B$  is bounded.

Take  $x \in B$  then  $\|x\| \leq 1$  by definition of  $B$ . This proves that  $B$  is bounded by 1.

We now prove that  $B$  is symmetric with respect to 0.

Take  $x \in B$  then  $\|x\| \leq 1$  by definition of  $B$  and thus  $|-1| \cdot \|x\| \leq 1$  and by homogeneity  $\|-x\| \leq 1$  which proves  $-x \in B$ .

We have therefore proved that  $B$  is symmetric with respect to 0.

We finally prove that  $0 \in \mathring{B}$

Let  $\tilde{B} = B(0, 1) = \{x \mid \|x\| < 1\}$  be the open unit ball. We know  $\tilde{B} \subset B$  and  $\tilde{B}$  is open. As  $\mathring{B}$  is the union of all open sets of  $B$ , we get  $\tilde{B} \subset \mathring{B}$ .

By definition,  $0 \in \tilde{B}$  and hence  $0 \in \mathring{B}$ .

Finally, we have proved that  $B$  is cool.

## 1.2

We want to show that for a cool set  $X$ , and  $\alpha, \beta \geq 0$ ,  $\alpha X + \beta X = (\alpha + \beta)X$ . We will proceed by double inclusion.

Let us first show that  $\alpha X + \beta X \subseteq (\alpha + \beta)X$ .

Take  $a, b \in X$ . Then  $a\alpha + b\beta \in \alpha X + \beta X$ . We then write:

$$a\alpha + b\beta = (\alpha + \beta) \left( \frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b \right)$$

We define  $t := \frac{\alpha}{\alpha + \beta} \in [0, 1]$ , and thus  $\frac{\beta}{\alpha + \beta} = 1 - t$ . We therefore have:

$$a\alpha + b\beta = (\alpha + \beta)(at + b(1 - t))$$

As  $X$  is a cool set, it is convex. Hence for  $a, b \in X$ , we have  $c := ta + (1 - t)b \in X$ . Therefore:

$$a\alpha + b\beta = (\alpha + \beta)c \in (\alpha + \beta)X$$

We have therefore proved that  $\alpha X + \beta X \subseteq (\alpha + \beta)X$ .

We now want to show that  $\alpha X + \beta X \supseteq (\alpha + \beta)X$ .

Take  $x \in (\alpha + \beta)X$ . Then there exists  $y \in X$  such that:

$$\begin{aligned} x &= (\alpha + \beta)y \\ &= \alpha y + \beta y \in \alpha X + \beta X \end{aligned}$$

We therefore have proved that  $\alpha X + \beta X \supseteq (\alpha + \beta)X$ .

Finally, by double inclusion, we have proved that  $\alpha X + \beta X = (\alpha + \beta)X$ .

### 1.3

We now define a function on  $E$  by setting for ever  $x \in E$ ,

$$N_X(x) = \inf\{|\alpha| \mid x \in \alpha X\}$$

We now show that  $N_X$  is well-defined and that it defines a norm on  $E$ .

#### 1.3.a

We want to show that for each  $x \in E$ , the set  $N := \{\alpha \mid x \in \alpha X\}$  is not empty.

We start from the fact that  $0 \in \overset{\circ}{X}$ .

Then, there exists an  $r > 0$  such that  $B(0, r) \subset X$ .

We define  $\alpha := \frac{2\|x\|}{r}$ . We know that  $B(0, \alpha r) \subset \alpha X$ . Since  $B(0, \alpha r) = B(0, 2\|x\|)$ , we have that  $x \in \alpha X$ , and hence  $\alpha \in N$ .

We have hence proved that  $N$  isn't empty.

#### 1.3.b

We want to show  $N$  is homogeneous, that is  $N_X(\lambda x) = |\lambda|N_X(x)$ .

We start from:

$$N_X(\lambda x) = \inf\{|\alpha| \mid \lambda x \in \alpha X\}$$

But we know that  $\lambda x \in \alpha X \Leftrightarrow \lambda x \in -\alpha X$ , hence we can write:

$$\begin{aligned} N_X(\lambda x) &= \inf\{|\alpha| \mid |\lambda|x \in \alpha X\} \\ &= \inf\{|\lambda\alpha| \mid x \in \alpha X\} \\ &= |\lambda| \inf\{|\alpha| \mid x \in \alpha X\} \\ &= |\lambda|N_X(x) \end{aligned}$$

We have hence proved  $N$  is homogeneous.

### 1.3.c

We want to show that  $N$  is definite, that is  $N_X(x) = 0 \Rightarrow x = 0$ .

We start from:

$$\begin{aligned} N_X(0) &= \inf\{|\alpha| \mid x \in \alpha X\} = 0 \\ \Rightarrow x &\in 0 \times X \\ \Rightarrow \|x\| &\leq 0 \times M_X \\ \Rightarrow x &= 0 \end{aligned}$$

This concludes the proof.

### 1.3.d

We want to show that the triangular inequality is true for  $N_X$ .

We therefore take  $x, y \in E$ .

$$\begin{aligned} N_X(x+y) &= \inf\{|\alpha| \mid (x+y) \in \alpha X\} \\ &= \inf\left\{|\alpha| \mid (x+y) \in \frac{\alpha}{2}X + \frac{\alpha}{2}X\right\} \\ &= \inf\left\{|\alpha| \mid x \in \frac{\alpha}{2}X\right\} + \inf\left\{|\alpha| \mid y \in \frac{\alpha}{2}X\right\} \\ &\leq \inf\{|\alpha| \mid x \in \alpha X\} + \inf\{|\alpha| \mid y \in \alpha X\} \\ &= N_X(x) + N_X(y) \end{aligned}$$

This concludes the proof that the triangular inequality holds for  $N_X$ , and by extension, that  $N_X$  is a norm.

## 1.4

### 1.4.a

We want to show that  $\|x\| \leq MN_X(x) \quad \forall x \in E$ . In order to do this, we start from the fact that  $X$  is bounded by  $M$  and that therefore  $X \subset \bar{B}(0, M)$ . We have from this that  $x \in N_X(x)\bar{X} \subset N_X\bar{B}(0, M)$ . Since  $x \in N_X\bar{B}(0, M)$  then  $\|x\| \leq MN_X$ . This concludes the proof that  $\|\cdot\|$  is weaker than  $N_X$ .

### 1.4.b

We want to show that there exists an  $\alpha$  such that for all  $x$ ,  $N_X(x) \leq \alpha\|x\|$ . Since  $0 \in \overset{\circ}{X}$ , we can consider  $r := \sup\{r \mid B(0, r) \subset X\}$  with  $r \neq 0$ . We then have that for all  $x$ ,  $N_X B(0, r) \subset B(0, x)$ . We therefore have that:

$$\begin{aligned} N_X r &\leq \|x\| \\ N_X &\leq \frac{1}{r}\|x\| \end{aligned}$$

This concludes the proof that  $N_X$  is weaker than  $\|\cdot\|$ , and therefore also that they are equivalent.

## 1.5

We know that  $x \in \bar{X}$  for the norm  $N_X$  is equivalent to

$$\exists (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}, (x_n) \text{ converges for } N_X \text{ to } x$$

But we know that the norms  $\|\cdot\|$  and  $N_X$  are equivalent, therefore if a sequence converges under  $N_X$ , then it must also converge under  $\|\cdot\|$ . Therefore:

$$\begin{aligned} x \in \bar{X}_{N_X} &\Leftrightarrow \exists (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}, (x_n) \text{ converges for } \|\cdot\| \text{ to } x \\ &\Leftrightarrow x \in \bar{X}_{\|\cdot\|} \end{aligned}$$

We have therefore shown equivalence between the closure of  $X$  for the two norms. This concludes the proof.

## 2 Exercise 2

### 2.1

Let us prove that  $\bar{B}(0, 1)$  is compact if and only if  $S(0, 1) = \{x \in E \mid N(x) = 1\}$  is compact. We will prove this by double implication.

Assume  $\bar{B}(0, 1) = \{x \in E \mid N(x) \leq 1\}$  is compact.

We know  $S(0, 1) \subset \bar{B}(0, 1)$  then  $S(0, 1)$  is compact provided it is closed as any closed subset of a compact set is compact. Let us now prove  $S(0, 1)$  is closed by proving its complement is open. We have:

$$\begin{aligned} (S(0, 1))^c &= \{x \in E \mid N(x) \neq 1\} \\ &= \{x \in E \mid N(x) < 1 \vee N(x) > 1\} \\ &= \{x \in E \mid N(x) < 1\} \cup \{x \in E \mid N(x) > 1\} \\ &= B(0, 1) \cup (\bar{B}(0, 1))^c \end{aligned}$$

We know  $B(0, 1)$  is open and as  $\bar{B}(0, 1)$  is closed,  $(\bar{B}(0, 1))^c$  is open and hence  $(S(0, 1))^c$  is open as the union of open sets. Thus  $S(0, 1)$  is closed and therefore compact.

Let us now prove the converse statement. Assume  $S(0, 1)$  is compact.

Consider the function

$$\begin{aligned} f : S(0, 1) \times [0, 1] &\rightarrow E \\ (x, t) &\mapsto tx \end{aligned}$$

Let us prove that  $f$  is Lipschitz continuous.

Let  $x, y \in S(0, 1)$  and let  $t_x, t_y \in [0, 1]$ . Then  $N(t_x x - t_y y) \leq N(x - y)$  and thus  $f$  is Lipschitz continuous.

Hence  $f(S(0, 1), [0, 1])$  is compact.

Let us now prove that  $f(S(0, 1), [0, 1]) = \bar{B}(0, 1)$  by double inclusion.

- " $\subseteq$ " Take  $x \in S(0, 1)$  and  $t \in [0, 1]$ .  
Then  $N(f(x, t)) = N(tx) = |t|N(x) \leq N(x) = 1$  hence  $f(x, t) \in \bar{B}(0, 1)$ .  
Therefore we have proved  $f(S(0, 1), [0, 1]) \subseteq \bar{B}(0, 1)$
- " $\supseteq$ " Take  $x \in \bar{B}(0, 1)$ . We must prove that there exist  $x' \in S(0, 1)$  and  $t' \in [0, 1]$  such that  $f(x', t') = x't' = x$ .
  - If  $x = 0$ : then we can set  $t' := 0$  and let  $x'$  be any element of  $S(0, 1)$ . We then get  $f(x', t') = x't' = 0 = x$  which proves  $x \in f(S(0, 1), [0, 1])$

- If  $x \neq 0$ : Let  $t := \frac{1}{N(x)}$ , then  $N(tx) = |t| N(x) = 1$  thus we can set  $x' := tx \in S(0, 1)$ . We then set  $t' := N(x) \in [0, 1]$  so that  $x = x't'$ . We hence get  $f(x', t') = x$  which proves  $x \in f(S(0, 1), [0, 1])$

We therefore have proved that  $\bar{B}(0, 1) \subseteq S(0, 1)$ .

Finally,  $f(S(0, 1), [0, 1]) = \bar{B}(0, 1)$  and hence  $\bar{B}(0, 1)$  is compact which finally concludes the proof.

## 2.2

We first recall the definition of Lischitz continuity for two normed vector spaces each endowed with their own norm,  $(E, N_E), (F, N_F)$ . The map  $f$  is 1-Lipschitz continuous if:

$$\forall x, y \in E, N_F(f(x) - f(y)) \leq N_E(x - y)$$

In this case, we are considering the function  $f : E \rightarrow \mathbb{R}, x \mapsto d(x, A) = \inf\{N(x - u) \mid u \in A\}$  with  $A \subset E$ . We want to show that the map  $f$  is 1-Lipschitz continuous. That is to say that:

$$|f(x) - f(y)| \leq N(x - y)$$

First we consider an element  $z_1 \in \bar{A}$  such that  $d(y, z_1) = d(y, A)$ . By the triangular inequality for distances, we have:

$$d(x, z_1) \leq d(x, y) + d(y, z_1)$$

However, we know that by definition  $d(x, A) \leq d(x, z_1)$ , and  $d(y, z_1) = d(y, A)$ , hence:

$$\begin{aligned} d(x, A) &\leq d(x, y) + d(y, A) \\ d(x, A) - d(y, A) &\leq d(x, y) \end{aligned}$$

We will now consider an element  $z_2 \in \bar{A}$  such that  $d(x, z_2) = d(x, A)$ . By the same logic as above:

$$d(y, z_2) \leq d(y, x) + d(x, z_2)$$

However, we know that by definition  $d(y, A) \leq d(y, z_2)$ , and  $d(x, z_2) = d(x, A)$ . Also, we have that  $d(x, y) = d(y, x)$ . Hence:

$$\begin{aligned} d(y, A) &\leq d(y, x) + d(x, A) \\ -(d(x, A) - d(y, A)) &\leq d(x, y) \end{aligned}$$

By combining these two results, we have that:

$$\begin{aligned} |d(x, A) - d(y, A)| &\leq d(x, y) \\ |f(x) - f(y)| &\leq N(x - y) \end{aligned}$$

This concludes the proof of 1-Lipschitz continuity.

## 2.3

Let us first prove that  $d(x, F) \leq N(x)$ .

From the previous question, we have that  $x \mapsto d(x, F)$  is Lipschitz-continuous so  $\forall x, y \in F |d(x, F) - d(y, F)| \leq N(x - y)$ . Now let  $y = 0$  which is possible as  $0 \in F$  because  $F$  is a vector subspace of  $E$ . We

then get  $|d(x, F) - d(0, F)| = |d(x, F)| = d(x, F) \leq N(x - 0) = N(x)$  from which we get our result.

Let us now prove that  $d(x, F) = \inf\{N(x - u) \mid u \in F, N(u) \leq 2N(x)\}$ . We have from the previous question that since  $F$  is a subset of  $E$ ,  $d(x, F) = \inf\{N(x - u) \mid u \in F\}$ . We also know that  $0 \in F$ . Assume therefore that  $N(u) > 2N(x)$ , then  $d(u, x) > N(x) = d(x, 0)$ . In other words, since zero is in  $F$ , we need not consider elements of  $F$  whose distance from  $x$  will be greater than the distance between  $x$  and 0.

## 2.4

We want to show that  $d(x, F)$  is attained at some  $u_x \in F$ . That is to say  $\exists u_x \in F, d(x, u_x) = d(x, F)$ . We therefore define a function

$$\begin{aligned} f : F &\rightarrow \mathbb{R}_+ \\ u &\mapsto N(x - u) = d(x, u) \end{aligned}$$

We will also define a set  $U := \{u \in F \mid N(u) \leq 2N(x)\}$ .

We have from previous questions that  $d(x, F) = \inf\{N(x - u) \mid u \in U\} = \inf\{f(u) \mid u \in U\}$ . The necessary conditions for  $d(x, F)$  to be attained are that  $U$  is compact and  $f$  is continuous.

We have by definition that  $U$  is bounded, with  $M = 2N(x)$ . Since  $F$  is finite dimensional, it now suffices to prove that  $U$  is closed to satisfy compactness. We will argue by contradiction, and assume that there exists a sequence  $(x_n)_{n \in \mathbb{N}} \in U^{\mathbb{N}}$  converging to  $l \notin U$ . Then for some  $\epsilon > 0$ ,  $\exists n \in \mathbb{N}, N(l - x_n) < \epsilon$ .

$$\begin{aligned} N(l) &= N(l - x_n + x_n) \\ &\leq N(x_n - l) + N(x_n) \\ &< \epsilon + N(x_n) \\ &\leq \epsilon + 2N(x) \end{aligned}$$

As  $\epsilon \rightarrow 0$ , we find that  $l$  is in fact in  $U$ , showing that  $U$  is therefore closed.

We will now show that  $f$  is continuous. We will do this using the triangular inequality, taking  $y, z \in F$ .

$$\begin{aligned} N(y - x) &= N(y - z + z - x) \leq N(y - z) + N(z - x) \Leftrightarrow f(y) - f(z) \leq N(y - z) \\ N(z - x) &= N(z - y + y - x) \leq N(y - z) + N(y - x) \Leftrightarrow -(f(y) - f(z)) \leq N(y - z) \end{aligned}$$

We therefore have that  $|f(y) - f(z)| \leq N(y - z)$  showing that  $f$  is a continuous function.

Combining the compactness of  $U$  and the continuity of  $F$  concludes the proof that finally gives the result that  $d(x, F)$  is attained at some  $u_x \in F$ .

## 2.5

We want to show that for a function  $y : x - u_x$ ,

$$\sup_{N(y)=1} d(y, F) \in \{0, 1\}$$

We will first consider the case where  $F = E$ , where  $d(y, F)$  can take no other value than 0. In that case we trivially have

$$\sup_{N(y)=1} d(y, F) = 0 \in \{0, 1\}$$

We now consider the case where  $F \subsetneq E$ . We have from a previous question that  $d(y, F) \leq N(y) = 1$ . We take  $u_y \in F$  to be the element of  $F$  for which  $d(y, F)$  is attained. We therefore have that  $\forall u \in F, N(y - u_y) \leq N(y - u)$ . Along the same principle, we have that  $\forall u \in F, N(x - u_x) \leq N(x - u)$ .

Since  $u_x + u_y \in F$ ,  $N(x - u_x) \leq N(xi(u_x + u_y))$ .

We now use the initial statement that  $\forall x \in E$ ,  $d(x, f) \leq N(x)$ , to say that  $N(y - u_y) \leq N(y)$ , using the definition of  $y$  to rewrite this as  $N(x - (u_x + u_y)) \leq N(x - u_x)$ .

The two sides of the inequality conclude that  $N(x - (u_x + u_y)) = N(x - u_x)$ , that is to say  $N(y - u_y) = N(y) = 1$ . This concludes the proof that  $\sup_{N(y)=1} d(y, F) \in \{0, 1\}$ .