MAA202: Analysis
Homework I

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1 Exercise 1

1.1

Let $B = \bar{B}(0,1) = \{x \in E \mid ||x|| \leq 1\}$ be the closed unit ball for the norm $||\cdot||$. Let us show that B is cool.

Let us prove that B is convex.
Take $x, y \in B$ and $t \in [0,1]$. We want to prove that $tx + (1-t)y \in B$. To do so, we prove that $||tx + (1-t)y|| \leq 1$.

$$\begin{aligned} ||tx + (1-t)y|| &\leq ||tx|| + ||(1-t)y|| \quad \text{by the triangle inequality} \\ &= |t| \cdot ||x|| + |1-t| \cdot ||y|| \quad \text{by homogeneity} \\ &\leq t + (1-t) \quad \text{as } ||x|| \leq 1 \text{ and } ||y|| \leq 1 \\ &= 1 \end{aligned}$$

Which finally proves that B is convex.

We now prove that B is bounded.
Take $x \in B$ then $||x|| \leq 1$ by definition of B. This proves that B is bounded by 1.

We now prove that B is symmetric with respect to 0.
Take $x \in B$ then $||x|| \leq 1$ by definition of B and thus $|-1| \cdot ||x|| \leq 1$ and by homogeneity $||-x|| \leq 1$ which proves $-x \in B$.
We have therefore proved that B is symmetric with respect to 0.

We finally prove that $0 \in \mathring{B}$
Let $\widetilde{B} = B(0,1) = \{x \mid ||x|| < 1\}$ be the open unit ball. We know $\widetilde{B} \subset B$ and $\widetilde{B}$ is open. As $\mathring{B}$ is the union of all open sets contained in B, we get $\widetilde{B} \subset \mathring{B}$.
By definition, $0 \in \widetilde{B}$ and hence $0 \in \mathring{B}$.

Finally, we have proved that B is cool.

1.2

We want to show that for a cool set X, and $\alpha, \beta \geq 0$, $\alpha X + \beta X = (\alpha + \beta)X$. We will proceed by double inclusion.

Let us first show that $\alpha X + \beta X \subseteq (\alpha + \beta)X$.
Take $a, b \in X$. Then $a\alpha + b\beta \in \alpha X + \beta X$. We then write:

$$a\alpha + b\beta = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta}a + \frac{\beta}{\alpha + \beta}b\right)$$

We define $t := \frac{\alpha}{\alpha+\beta} \in [0,1]$, and thus $\frac{\beta}{\alpha+\beta} = 1 - t$. We therefore have:

$$a\alpha + b\beta = (\alpha + \beta)\big(at + b(1-t)\big)$$

As X is a cool set, it is convex. Hence for $a, b \in X$, we have $c := ta + (1-t)b \in X$. Therefore:

$$a\alpha + b\beta = (\alpha + \beta)c \in (\alpha + \beta)X$$

We have therefore proved that $\alpha X + \beta X \subseteq (\alpha + \beta)X$.

We now want to show that $\alpha X + \beta X \supseteq (\alpha + \beta)X$.
Take $x \in (\alpha + \beta)X$. Then there exists $y \in X$ such that:

$$\begin{aligned} x &= (\alpha + \beta)y \\ &= \alpha y + \beta y \in \alpha X + \beta X \end{aligned}$$

We therefore have proved that $\alpha X + \beta X \supseteq (\alpha + \beta)X$.

Finally, by double inclusion, we have proved that $\alpha X + \beta X = (\alpha + \beta)X$.

1.3

We now define a function on E by setting for every $x \in E$,

$$N_X(x) = \inf\{|\alpha| \mid x \in \alpha X\}$$

We now show that N_X is well-defined and that it defines a norm on E.

1.3.a

We want to show that for each $x \in E$, the set $N := \{\alpha \,|\, x \in \alpha X\}$ is not empty.

We start from the fact that $0 \in \overset{\circ}{X}$.
Then, there exists an $r > 0$ such that $B(0, r) \subset X$.
We define $\alpha := \frac{2||x||}{r}$. We know that $B(0, \alpha r) \subset \alpha X$. Since $B(0, \alpha r) = B(0, 2||x||)$, we have that $x \in \alpha X$, and hence $\alpha \in N$.

We have hence proved that N isn't empty.

1.3.b

We want to show N is homogeneous, that is $N_X(\lambda x) = |\lambda| N_X(x)$.

We start from:

$$N_X(\lambda x) = \inf\{|\alpha| \,|\lambda x \in \alpha X\}$$

But we know that $\lambda x \in \alpha X \Leftrightarrow \lambda x \in -\alpha X$, hence we can write:

$$\begin{aligned} N_X(\lambda x) &= \inf\{|\alpha| \mid |\lambda| x \in \alpha X\} \\ &= \inf\{|\lambda \alpha| \mid x \in \alpha X\} \\ &= |\lambda| \inf\{|\alpha| \mid x \in \alpha X\} \\ &= |\lambda| N_X(x) \end{aligned}$$

We have hence proved N is homogeneous.

1.3.c

We want to show that N is definite, that is $N_X(x) = 0 \Rightarrow x = 0$.

We start from:

$$\begin{aligned}
&N_X(0) = \inf\{|\alpha| \,|\, x \in \alpha X\} = 0 \\
&\Rightarrow x \in 0 \times X \\
&\Rightarrow ||x|| \leq 0 \times M_X \\
&\Rightarrow x = 0
\end{aligned}$$

This concludes the proof.

1.3.d

We want to show that the triangular inequality is true for N_X.
We therefore take $x, y \in E$.

$$\begin{aligned}
N_X(x+y) &= \inf\{|\alpha| \mid (x+y) \in \alpha X\} \\
&= \inf\left\{|\alpha| \mid (x+y) \in \frac{\alpha}{2}X + \frac{\alpha}{2}X\right\} \\
&= \inf\left\{|\alpha| \mid x \in \frac{\alpha}{2}X\right\} + \inf\left\{|\alpha| \mid y \in \frac{\alpha}{2}X\right\} \\
&\leq \inf\{|\alpha| \mid x \in \alpha X\} + \inf\{|\alpha| \mid y \in \alpha X\} \\
&= N_X(x) + N_X(y)
\end{aligned}$$

This concludes the proof that the triangular inequality holds for N_X, and by extension, that N_X is a norm.

1.4

1.4.a

We want to show that $||x|| \leq MN_X(x) \quad \forall x \in E$. In order to do this, we start from the fact that X is bounded by M and that therefore $X \subset \bar{B}(0, M)$. We have from this that $x \in N_X(x)\bar{X} \subset N_X\bar{B}(0, M)$. Since $x \in N_X\bar{B}(0, M)$ then $||x|| \leq MN_X$. This concludes the proof that $|| \cdot ||$ is weaker than N_x.

1.4.b

We want to show that there exists an α such that for all x, $N_X(x) \leq \alpha||x||$. Since $0 \in \mathring{X}$, we can consider $r := sup\{r \,|\, B(0, r) \subset X\}$ with $r \neq 0$. We then have that for all x, $N_X B(0, r) \subset B(0, x)$. We therefore have that:

$$\begin{aligned}
N_X r &\leq ||x|| \\
N_X &\leq \frac{1}{r}||x||
\end{aligned}$$

This concludes the proof that N_X is weaker than $|| \cdot ||$, and therefore also that they are equivalent.

1.5

We know that $x \in \bar{X}$ for the norm N_X is equivalent to

$$\exists (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}, (x_n) \text{ converges for } N_X \text{ to } x$$

But wee know that the norms $||\cdot||$ and N_X are equivalent, therefore if a sequence converrges under N_X, then it must also converge under $||\cdot||$. Therefore:

$$\begin{aligned} x \in \bar{X}_{N_X} &\Leftrightarrow \exists (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}, (x_n) \text{ converges for } ||\cdot|| \text{ to } x \\ &\Leftrightarrow x \in \bar{X}_{||\cdot||} \end{aligned}$$

We have therefore shown equivalence between the closure of X for the two norms. This concludes the proof.

2 Exercise 2

2.1

Let us prove that $\bar{B}(0,1)$ is compact if and only if $S(0,1) = \{x \in E \mid N(x) = 1\}$ is compact. We will prove this by double implication.

Assume $\bar{B}(0,1) = \{x \in E \mid N(x) \leq 1\}$ is compact.
We know $S(0,1) \subset \bar{B}(0,1)$ then $S(0,1)$ is compact provided it is closed as any closed subset of a compact set is compact. Let us now prove $S(0,1)$ is closed by proving its complement is open. We have:

$$\begin{aligned} (S(0,1))^c &= \{x \in E \mid N(x) \neq 1\} \\ &= \{x \in E \mid N(x) < 1 \vee N(x) > 1\} \\ &= \{x \in E \mid N(x) < 1\} \cup \{x \in E \mid N(x) > 1\} \\ &= B(0,1) \cup (\bar{B}(0,1))^c \end{aligned}$$

We know $B(0,1)$ is open and as $\bar{B}(0,1)$ is closed, $(\bar{B}(0,1))^c$ is open and hence $(S(0,1))^c$ is open as the union of open sets. Thus $S(0,1)$ is closed and therefore compact.

Let us now prove the converse statement. Assume $S(0,1)$ is compact.

Consider the function

$$\begin{aligned} f : S(0,1) \times [0,1] &\to E \\ (x,t) &\mapsto tx \end{aligned}$$

Let us prove that f is Lipschitz continuous.
Let $x, y \in S(0,1)$ and let $t_x, t_y \in [0,1]$. Then $N(t_x x - t_y y) \leq N(x - y)$ and thus f is Lipschitz continuous.
Hence $f(S(0,1),[0,1])$ is compact.

Let us now prove that $f(S(0,1),[0,1]) = \bar{B}(0,1)$ by double inclusion.

- "$\subseteq$" Take $x \in S(0,1)$ and $t \in [0,1]$.
  Then $N(f(x,t)) = N(tx) = |t|\, N(x) \leq N(x) = 1$ hence $f(x,t) \in \bar{B}(0,1)$.
  Therefore we have proved $f(S(0,1),[0,1]) \subseteq \bar{B}(0,1)$
- "$\supseteq$" Take $x \in \bar{B}(0,1)$. We must prove that there exist $x' \in S(0,1)$ and $t' \in [0,1]$ such that $f(x',t') = x't' = x$.
  - If $x = 0$: then we can set $t' := 0$ and let x' be any element of $S(0,1)$. We then get $f(x',t') = x't' = 0 = x$ which proves $x \in f(S(0,1),[0,1])$

– If $x \neq 0$: Let $t := \frac{1}{N(x)}$, then $N(tx) = |t|\, N(x) = 1$ thus we can set $x' := tx \in S(0,1)$. We then set $t' := N(x) \in [0,1]$ so that $x = x't'$. We hence get $f(x', t') = x$ which proves $x \in f(S(0,1), [0,1])$

We therefore have proved that $\bar{B}(0,1) \subseteq S(0,1)$.

Finally, $f(S(0,1), [0,1]) = \bar{B}(0,1)$ and hence $\bar{B}(0,1)$ is compact which finally concludes the proof.

2.2

We first recall the definition of Lischitz continuity for two normed vector spaces each endowed with their own norm, $(E, N_E), (F, N_F)$. The map f is 1-Lipschitz continuous if:

$$\forall x, y \in E,\ N_F\big(f(x) - f(y)\big) \leq N_E(x - y)$$

In this case, we are considering the function $f : E \to \mathbb{R},\, x \mapsto d(x, A) = inf\{N(x-u) \,|\, u \in A\}$ with $A \subset E$. We want to show that the map f is 1-Lipschitz continuous. That is to say that:

$$\big|f(x) - f(y)\big| \leq N(x - y)$$

First we consider an element $z_1 \in \bar{A}$ such that $d(y, z_1) = d(y, A)$. By the triangular inequality for distances, we have:

$$d(x, z_1) \leq d(x, y) + d(y, z_1)$$

However, we know that by definition $d(x, A) \leq d(x, z_1)$, and $d(y, z_1) = d(y, A)$, hence:

$$\begin{aligned} d(x, A) &\leq d(x, y) + d(y, A) \\ d(x, A) - d(y, A) &\leq d(x, y) \end{aligned}$$

We will now consider an element $z_2 \in \bar{A}$ such that $d(x, z_1) = d(x, A)$. By the same logic as above:

$$d(y, z_2) \leq d(y, x) + d(x, z_2)$$

However, we know that by definition $d(y, A) \leq d(y, z_2)$, and $d(x, z_2) = d(x, A)$. Also, we have that $d(x, y) = d(y, x)$. Hence:

$$\begin{aligned} d(y, A) &\leq d(y, x) + d(x, A) \\ -\big(d(x, A) - d(y, A)\big) &\leq d(x, y) \end{aligned}$$

By combining these two results, we hae that:

$$\begin{aligned} \big|d(x, A) - d(y, A)\big| &\leq d(x, y) \\ \big|f(x) - f(y)\big| &\leq N(x - y) \end{aligned}$$

This concludes the proof of 1-Lipschitz continuity.

2.3

Let us first prove that $d(x, F) \leq N(x)$.
From the previous question, we have that $x \mapsto d(x, F)$ is Lipschitz-continuous so $\forall x, y \in F\ |d(x, F) - d(y, F)| \leq N(x - y)$. Now let $y = 0$ which is possible as $0 \in F$ because F is a vector subspace of E. We

then get $|d(x,F) - d(0,F)| = |d(x,F)| = d(x,F) \leq N(x-0) = N(x)$ from which we get our result.

Let us now prove that $d(x,F) = \inf\{N(x-u) \mid u \in F,\, N(u) \leq 2N(x)\}$. We have from the previous question that since F is a subset of E, $d(x,F) = \inf\{N(x-u) \mid u \in F\}$. We also know that $0 \in F$. Assume therefore that $N(u) > 2N(x)$, then $d(u,x) > N(x) = d(x,0)$. In other words, since zero is in F, we need not consider elements of F whose distance from x will be greater than the distance between x and 0.

2.4

We want to show that $d(x,F)$ is attained at some $u_x \in F$. That is to say $\exists\, u_x \in F, d(x,u_x) = d(x,F)$. We therefore define a function

$$\begin{aligned} f : F &\to \mathbb{R}_+ \\ u &\mapsto N(x-u) = d(x,u) \end{aligned}$$

We will also define a set $U := \{u \in F \mid N(u) \leq 2N(x)\}$.
We have from previous questions that $d(x,F) = \inf\{N(x-u) | u \in U\} = \inf\{f(u) | u \in U\}$. The necessary conditons for $d(x,F)$ to be attained are that U is compact and f is continuous.
We have by definition that U is bounded, with $M = 2N(x)$. Snce F is finite dimensional, it now suffices to prove that U is closed to satisfy compactness. We will argue by contradiction, and assume that there exists a sequence $(x_n)_{n\in\mathbb{N}} \in U^{\mathbb{N}}$ converging to $l \notin U$. Then for some $\epsilon > 0$, $\exists n \in \mathbb{N}$, $N(l - x_n) < \epsilon$.

$$\begin{aligned} N(l) &= N(l - x_n + x_n) \\ &\leq N(x_n - l) + N(x_n) \\ &< \epsilon + N(x_n) \\ &\leq \epsilon + 2N(x) \end{aligned}$$

As $\epsilon \to 0$, we find that l is in fact in U, showing that U is therefore closed.
We will now show that f is continuous. We will do this using the triangular inequality, taking $y, z \in F$.

$$\begin{aligned} N(y-x) = N(y-z+z-x) \leq N(y-z) + N(z-x) &\Leftrightarrow \quad f(y) - f(z) \ \leq N(y-z) \\ N(z-x) = N(z-y+y-x) \leq N(y-z) + N(y-x) &\Leftrightarrow -(f(y) - f(z)) \leq N(y-z) \end{aligned}$$

We therefore have that $|f(y) - f(z)| \leq N(y-z)|$ showing that f is a continuous function.
Combining the compactness of U and the continuity of F concludes the proof that finally gives the result that $d(x,F)$ is attained at some $u_x \in F$.

2.5

We want to show that for a function $y : x - u_x$,

$$\sup_{N(y)=1} d(y,F) \in \{0,1\}$$

We will first consider the case where $F = E$, where $d(y,F)$ can take no other value than 0. In that case we trivially have

$$\sup_{N(y)=1} d(y,F) = 0 \in \{0,1\}$$

We now consider the case where $F \subsetneq E$. We have from a previous question that $d(y,F) \leq N(y) = 1$. We take $u_y \in F$ to be the element of F for which $d(y,F)$ is attained. We therefore have that $\forall u \in F$, $N(y - u_y) \leq N(y-u)$. Along the same principle, we have that $\forall u \in F$, $N(x - u_x) \leq N(x-u)$.

Since $u_x + u_y \in F$, $N(x - u_x) \leq N\big(xi(u_x + u_y)\big)$.
We now use the initial statement that $\forall x \in E$, $d(x, f) \leq N(x)$, to say that $N(y - u_y) \leq N(y)$, using the definition of y to rewrite this as $N\big(x - (u_x + u_y)\big) \leq N(x - u_x)$.
The two sides of the inequality conclude that $N\big(x - (u_x + u_y)\big) = N(x - u_x)$, that is to say $N(y - u_y) = N(y) = 1$. This concludes the proof that $\sup_{N(y)=1} d(y, F) \in \{0, 1\}$.

2.6