

MAA202: Analysis

Homework I

Maëlys Solal
Alexandre Hirsch
André Renom

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1 Exercise 1

1.1

Let $B = \bar{B}(0, 1) = \{x \in E \mid \|x\| \leq 1\}$ be the closed unit ball for the norm $\|\cdot\|$. Let us show that B is cool.

Let us prove that B is convex.

Take $x, y \in B$ and $t \in [0, 1]$. We want to prove that $tx + (1 - t)y \in B$. To do so, we prove that $\|tx + (1 - t)y\| \leq 1$.

$$\begin{aligned}\|tx + (1 - t)y\| &\leq \|tx\| + \|(1 - t)y\| \quad \text{by the triangle inequality} \\ &= |t| \cdot \|x\| + |1 - t| \cdot \|y\| \quad \text{by homogeneity} \\ &\leq t + (1 - t) \quad \text{as } \|x\| \leq 1 \text{ and } \|y\| \leq 1 \\ &= 1\end{aligned}$$

Which finally proves that B is convex.

We now prove that B is bounded.

Take $x \in B$ then $\|x\| \leq 1$ by definition of B . This proves that B is bounded by 1.

We now prove that B is symmetric with respect to 0.

Take $x \in B$ then $\|x\| \leq 1$ by definition of B and thus $|-1| \cdot \|x\| \leq 1$ and by homogeneity $\|-x\| \leq 1$ which proves $-x \in B$.

We have therefore proved that B is symmetric with respect to 0.

We finally prove that $0 \in \mathring{B}$

Let $\tilde{B} = B(0, 1) = \{x \mid \|x\| < 1\}$ be the open unit ball. We know $\tilde{B} \subset B$ and \tilde{B} is open. As \mathring{B} is the union of all open sets of B , we get $\tilde{B} \subset \mathring{B}$.

By definition, $0 \in \tilde{B}$ and hence $0 \in \mathring{B}$.

Finally, we have proved that B is cool.

1.2

We want to show that for a cool set X , and $\alpha, \beta \geq 0$, $\alpha X + \beta X = (\alpha + \beta)X$.

We will proceed by double inclusion.

Let us first show that $\alpha X + \beta X \subseteq (\alpha + \beta)X$.

Take $a, b \in X$. Then $a\alpha + b\beta \in \alpha X + \beta X$. We then write:

$$a\alpha + b\beta = (\alpha + \beta) \left(\frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b \right)$$

We define $t := \frac{\alpha}{\alpha + \beta} \in [0, 1]$, and thus $\frac{\beta}{\alpha + \beta} = 1 - t$. We therefore have:

$$a\alpha + b\beta = (\alpha + \beta)(at + b(1 - t))$$

As X is a cool set, it is convex. Hence for $a, b \in X$, we have $c := ta + (t-1)b \in X$. Therefore:

$$a\alpha + b\beta = (\alpha + \beta)c \in (\alpha + \beta)X$$

We have therefore proved that $\alpha X + \beta X \subseteq (\alpha + \beta)X$.

We now want to show that $\alpha X + \beta X \supseteq (\alpha + \beta)X$. Take $x \in (\alpha + \beta)X$. Then there exists $y \in X$ such that:

$$\begin{aligned} x &= (\alpha + \beta)y \\ &= \alpha y + \beta y \in \alpha X + \beta X \end{aligned}$$

We therefore have proved that $\alpha X + \beta X \supseteq (\alpha + \beta)X$.

Finally, by double inclusion, we have proved that $\alpha X + \beta X = (\alpha + \beta)X$.

1.3

We now define a function on E by setting for ever $x \in E$,

$$N_X(x) = \inf\{|\alpha| \mid x \in \alpha X\}$$

We now show that N_X is well-defined and that it defines a norm on E .

1.3.a

We want to show that for each $x \in E$, the set $N := \{\alpha \mid x \in \alpha X\}$ is not empty.

We start from the fact that $0 \in \overset{\circ}{X}$.

Then, there exists an $r > 0$ such that $B(0, r) \subset X$.

We define $\alpha := \frac{2\|x\|}{r}$. We know that $B(0, \alpha r) \subset \alpha X$. Since $B(0, \alpha r) = B(0, 2\|x\|)$, we have that $x \in \alpha X$, and hence $\alpha \in N$.

We have hence proved that N isn't empty.

1.3.b

We want to show N is homogeneous, that is $N_X(\lambda x) = |\lambda|N_X(x)$.

We start from:

$$N_X(\lambda x) = \inf\{|\alpha| \mid \lambda x \in \alpha X\}$$

But we know that $\lambda x \in \alpha X \Leftrightarrow \lambda x \in -\alpha X$, hence we can write:

$$\begin{aligned} N_X(\lambda x) &= \inf\{|\alpha| \mid |\lambda|x \in \alpha X\} \\ &= \inf\{|\lambda\alpha| \mid x \in \alpha X\} \\ &= |\lambda| \inf\{|\alpha| \mid x \in \alpha X\} \\ &= |\lambda| N_X(x) \end{aligned}$$

We have hence proved N is homogeneous.

1.3.c

We want to show that N is definite, that is $N_X(x) = 0 \Rightarrow x = 0$.

We start from:

$$\begin{aligned} N_X(0) &= \inf\{|\alpha| \mid 0 \in \alpha X\} = 0 \\ \Rightarrow x &\in 0 \times X \\ \Rightarrow \|x\| &\leq 0 \times M_X \\ \Rightarrow x &= 0 \end{aligned}$$

This concludes the proof.

1.3.d

We want to show that the triangular inequality is true for N_X .

We therefore take $x, y \in E$.

$$\begin{aligned} N_X(x + y) &= \inf\{|\alpha| \mid (x + y) \in \alpha X\} \\ &= \inf\left\{|\alpha| \mid (x + y) \in \frac{\alpha}{2}X + \frac{\alpha}{2}X\right\} \\ &= \inf\left\{|\alpha| \mid x \in \frac{\alpha}{2}X\right\} + \inf\left\{|\alpha| \mid y \in \frac{\alpha}{2}X\right\} \\ &\leq \inf\{|\alpha| \mid x \in \alpha X\} + \inf\{|\alpha| \mid y \in \alpha X\} \\ &= N_X(x) + N_X(y) \end{aligned}$$

This concludes the proof that the triangular inequality holds for N_X , and by extension, that N_X is a norm.

1.4

1.4.a

We want to show that $\|x\| \leq MN_X(x) \quad \forall x \in E$. In order to do this, we start from the fact that X is bounded by M and that therefore $X \subset \bar{B}(0, M)$. We have from this that $x \in N_X(x)\bar{X} \subset N_X\bar{B}(0, M)$. Since $x \in N_X\bar{B}(0, M)$ then $\|x\| \leq MN_X$. This concludes the proof that $\|\cdot\|$ is weaker than N_x .

1.4.b

We want to show that there exists an α such that for all x , $N_X(x) \leq \alpha \|x\|$. Since $0 \in \overset{\circ}{X}$, we can consider $r := \sup\{r \mid B(0, r) \subset X\}$ with $r \neq 0$. We then have that for all x , $N_X B(0, r) \subset B(0, x)$. We therefore have that:

$$\begin{aligned} N_X r &\leq \|x\| \\ N_X &\leq \frac{1}{r} \|x\| \end{aligned}$$

This concludes the proof that N_X is weaker than $\|\cdot\|$, and therefore also that they are equivalent.

1.5

We know that $x \in \bar{X}$ is equivalent to