# MATH 437 - Manifolds

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## Contents

# 1 Preliminary Results from Calculus

This section is dedicated to proving the following two theorems, which has various implications in differential geometry:

**Theorem 1.1** (Inverse Function Theorem). *Let $X, Y$ be finite-dimensional normed vector space, $\Omega \subseteq X$ an open set, and $f : \Omega \to Y$ a continuously differentiable function. Let $x_0 \in \Omega$. If $Df|_{x_0}$ is invertible, then $f$ is locally $C^1$-invertible at $x_0$ (i.e. $f^{-1}$ is $C^1$-continuous), whose derivative satisfies*

$$Df^{-1}\big|_{f(x)} = (Df|_x)^{-1}$$

**Theorem 1.2** (Implicit Function Theorem). *Let $\Omega \subseteq \mathbb{R}^n \times R^m$ be open, with $F \in C^1\Omega, \mathbb{R}^m$ s.t. $F(x_0, y_0) = 0$ for $(x_0, y_0) \in \Omega$. If*

$$\det DF(x_0, -)|_{y_0} \neq 0$$

*then there exists $\varepsilon \in \mathbb{R}$, and $g \in C^1(B_\varepsilon(x_0), \mathbb{R}^m)$ s.t.*

$$F(x, g(x)) = 0 \qquad \forall\, x \in B_\varepsilon(x_0)$$

The general setup is to first prove the inverse function theorem, and use it to vastly simplify the proof for the implicit function theorem. An alternative approach is to first prove the implicit function theorem, and then express the derivative of the inverse function in an implicit manner.

First setup some lemmas for further use in the proof:

**Lemma 1.3** (Contraction Mapping Principle). *Let $X$ be a complete normed vector space, and $M$ be a closed subset of $X$. If $f : M \to M$ is Lipschitz continuous with Lipschitz constant less than 1 (a contraction), then there exists a unique $x_0 \in M$ s.t. $f(x_0) = x_0$.*

*Proof.* Since $X$ is complete and $M$ is closed, it suffices to prove that the sequence given by $(a_n) := f^n(x)$ is Cauchy. The limit must lie in $M$ as otherwise $M$ is not equal to its closure. By hypothesis there exists $C \in [0, 1)$ s.t.

$$\left|f^2(x) - f(x)\right| < C \cdot |f(x) - x| \qquad \forall\, x \in M$$

This gives the approximation

$$|f^m(x) - x| < \sum_{k=0}^{m-1} C^k\,|f(x) - x| < \left(\sum_{k=0}^{\infty} C^k\right) \cdot |f(x) - x| = \frac{1}{1-C} \cdot |f(x) - x|$$

which implies that $f^m(x)$ is bounded for all $m$. Further for $m < n$ we have

$$|f^n(x) - f^m(x)| \leq \sum_{k=m}^{n-1} \left|f^{k+1}(x) - f^k(x)\right| < \sum_{k=m}^{n-1} C^k\,|f(x) - x| < C^m \left(\sum_{k=0}^{\infty} C^k\,|f(x) - x|\right) = \frac{C^m}{1-C}\,|f(x) - x|$$

which can be arbitrarily small via specifying $m$ to be arbitrarily large. Therefore $(a_n)$ admits a limit $a$. Then

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(a)$$

since $f$ is continuous (as it is Lipschitz continuous), it commutes with taking the limit.

Uniqueness follows from the fact that if there exists $x_0 \neq x_1$ s.t. $f(x_0) = x_0, f(x_1) = x_1$, then $|x_0 - x_1| = |f(x_0) - f(x_1)|$ which contradicts with the hypothesis. □

**Lemma 1.4.** *Let $X$ be a normed vector space, with $L \in \{L \in \mathrm{GL}(X) \mid \|L\| < 1\}$ then $(\mathrm{Id} - L) \in \mathrm{GL}(X)$.*

*Proof.* Consider the series $S = \sum_{i=k}^{\infty} L^k$. This converges as

$$\|S\| \leq \sum_{i=k}^{\infty} \|L\|^k = \frac{1}{1 - \|L\|}$$

which is finite and well-defined as $\|L\| \in [0,1)$. Then consider

$$\lim_{n\to\infty} (\mathrm{Id} - L) \cdot S = \mathrm{Id} - \lim_{n\to\infty} L^n = \mathrm{Id}$$

as $\|L\| < 1$. □

*Proof of Theorem 1.1.* Without loss of generality, assume $Df|_{x_0} = \mathrm{Id}$; and $x_0 = 0, f(x_0) = 0$. These are done via applying an affine transformation, which preserves the property of being $C^1$-invertible.

Consider the approximation of $f$ near $x_0 = 0$: let $g(x) = f(x) - x$. Since $g(0) = 0$ and both $f$ and $\mathrm{Id}$ are continuous, there exists $r$ s.t. $\overline{B_r(x)} \subseteq \Omega$, and $\| Dg|_x \| \leq \frac{1}{2}$ for all $x \in \Omega$. This implies that $g$ is a contraction in $\overline{B_r(0)}$, as

$$g(x) - g(0) = \int_0^x Dg|_t \,\mathrm{d}t = \int_0^1 Dg|_{xt}\, x\mathrm{d}t = x \cdot \int_0^1 Dg|_{xt} \,\mathrm{d}t \implies \|g(x)\| \leq \sup_{t\in[0,1]} \| Dg|_{xt} \| \cdot \|x\| = \frac{1}{2}\|x\|$$

This implies that $f$ maps surjectively onto $\overline{B_{r/2}(0)}$: consider $h := y + g(x)$, which is a map from $\overline{B_r(0)}$ to itself. $h$ is also a contraction as

$$\|h(x_1) - h(x_2)\| = \|g(x_1) - g(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|$$

Further the space on which $h$ acts is closed.[1] This implies that there exists a fixed point $x_0$ for $h$; but this implies a preimage:

$$x_0 = y + x_0 - f(x_0) \implies f(x_0) = y$$

Then $f$ is a bijection on $\Omega_1 \to f(\Omega_1)$, where $\Omega_1 = f^{-1}B_{r/2}(0) \cap B_r(0)$. The continuity of $f$, and the fact that $g$ is a contraction is preserved through restriction of domain. Therefore we have

$$\begin{aligned} \|x_1 - x_2\| &= \|(f(x_1) - f(x_2)) - (g(x_1) - g(x_2))\| \\ &\leq \|f(x_1) - f(x_2)\| + \|g(x_1) - g(x_2)\| \\ &\leq \|f(x_1) - f(x_2)\| + \frac{1}{2}\|x_1 - x_2\| \end{aligned}$$

which implies that

$$\frac{1}{2}\|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\| \implies \|x_1 - x_2\| = \left\|f^{-1}(y_1) - f^{-1}(y_2)\right\| \leq 2\|y_1 - y_2\| \tag{$*$}$$

[1]The point of taking the closure of $B_r(0)$ in the construction is to apply the contraction mapping principle here, as it requires the space to be closed.

i.e. $y$ is continuous. It is safe to take the inverse as $f$ is a bijection in the domain specified.

Now prove the theorem. By the setup we have $\left\|\mathrm{Id} - Df|_{x_1}\right\| = \left\|Dg|_{x_1}\right\| \leq \frac{1}{2}$ for all $x_1 \in \Omega$, which by Lemma 1.4 implies that $Df|_x$ is invertible for all such $x$. Since $f$ is differentiable, there exists $Df|_{x_1}$ s.t.

$$f(x) - f(x_1) = Df|_{x_1}(x - x_1) + o(\|x - x_1\|)$$

for $x \in B_\eta(x_1)$ for some $\eta$. This implies that

$$(Df|_{x_1})^{-1}(f(x) - f(x_1)) = (x - x_1) + o(\|x - x_1\|)$$

Further restrict $x_1 \in \Omega_1$ where we have the linear bound $(*)$. After substituting the variables we have

$$(Df|_{f^{-1}(y_1)})^{-1}(y - y_1) = (f^{-1}(y) - f^{-1}(y_1)) + o(\|y - y_1\|)$$

which gives by the definition of derivative

$$(Df|_x)^{-1} = Df^{-1}\big|_{f(x)}$$

Now consider the chain of functions

$$y \xrightarrow{f^{-1}} f^{-1}y \xrightarrow{Df} Df|_{f^{-1}y} \xrightarrow{(-)^{-1}} (Df|_{f^{-1}y})^{-1}$$

where $Df$ is by hypothesis continuous; and $f^{-1}$ is continuous according to the proof above. This implies that the composition, which is the derivative of the inverse function, is continuous, i.e. $f$ is $C^1$-invertible. □

This greatly simplifies the proof of the Implicit Function Theorem:

*Proof of Theorem 1.2.* Consider the map $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, $(x, y) \mapsto (x, F(x, y))$. Since $\det DF = \det D\Phi \neq 0$ by Laplace expansion, which implies that $D\Phi$ is invertible. This allows application of Inverse Function Theorem, which implies that there exists $\Psi = \Phi^{-1}$ taking the form of $(x, y) \mapsto (x, G(x, y))$. Now consider specifically

$$(x, 0) = \Phi \circ \Psi(x, 0) = \Phi(x, G(x, 0)) = (x, F(x, G(x, 0)))$$

which implies that there exists $g := G(-, 0)$ satisfying the requirement. Since the inverse function is uniquely determined, such $g$ is also uniquely defined. □

## 2 Preliminary Topology

**Definition 2.1** (Topological Space)**.** *A **topological space** is a pair (X, $\mathcal{O}$), where $X$ is an arbitrary set, and $\mathcal{O}$ a class of subsets of $X$, which satisfies the follows:*

- *$\emptyset \in \mathcal{O}$, and $X \in \mathcal{O}$.*
- *Let $(\Omega_i)_{i \in I}$ be a (possibly infinite) class of subsets of $X$, then $\bigcup_{i \in I} \Omega_i \in \mathcal{O}$.*
- *Let $(\omega_i)_{i \in I}$ be a finite class of subsets of $X$, then $\bigcap_{i \in I} \omega_i \in \mathcal{O}$.*

*Elements of $\mathcal{O}$ are called the **open sets**. $\mathcal{O}$ (as a class) gives the topology of $X$.*

**Remark 2.2.** If the topology of $X$ is clear, one often denotes the space by simply $X$.

**Definition 2.3** (Hausdorff Space)**.** *A topological space $(X, \mathcal{O})$ is a **Hausdorff space** if for all $x, y \in X$ there exists $\Omega_x, \Omega_y \in \mathcal{O}$ s.t. $x \in \Omega_x, y \in \Omega_y$; and $\Omega_x \cap \Omega_y = \emptyset$.*

**Example 2.4.** Choice of topology is very important. Consider the following examples:

- Consider $(\mathbb{R}, \mathcal{O})$ where $\Omega \in \mathcal{O}$ if and only if for all $x \in \Omega$, there exists $\varepsilon \in \mathbb{R}$ s.t. $(x - \varepsilon, x + \varepsilon) \subseteq \Omega$. This space is Hausdorff as one could choose $\varepsilon_x = \varepsilon_y = |x - y| / 4$. This also illustrates why the intersection must be finite, as otherwise one could construct a converging sequence of $\varepsilon$, whose corresponding class of subsets intersecting to a closed interval.
- Consider $(\mathbb{R}, \mathcal{O})$ to be the trivial topology, where $\mathcal{O} := \{\mathbb{R}, \emptyset\}$. Then the only subset in $\mathcal{O}$ containing $x$ and $y$ is $\mathbb{R}$, which implies that the space is not Hausdorff.

**Definition 2.5** (Basis of a Topology)**.** *Let $(X, \mathcal{O})$ be a topology space. Then a subset $\mathcal{B} \subseteq \mathcal{O}$ is a **basis of topology** $\mathcal{O}$ if for any $\Omega \in \mathcal{O}$ it can be expressed as union of elements in $\mathcal{B}$. If there exists such $\mathcal{B}$ s.t. it is countably infinite, then $X$ admits a **countable basis of topology**.*

**Proposition 2.6.** *$(\mathbb{R}, \mathcal{O})$ with topology defined as in the first case in the example above admits a countable basis of topology.*

*Proof.* First consider $\mathcal{B} := \{(a, b) \mid a < b, a, b \in \mathbb{R}\}$. This gives a basis for the topology of $(\mathbb{R}, \mathcal{O})$ in the sense of the first case above:

- *Any union of elements in $\mathcal{B}$ is open.* Denote such union to be $\Omega$. By construction, for $x \in \mathbb{R}$ s.t. $x \in \Omega$, there exists $a_x < b_x$ s.t. $x \in (a_x, b_x)$. Then there exists $\varepsilon = \min\{x - a_x, b_x - x\}/2$ that satisfies the definition.
- *Any open subset of $\mathbb{R}$ can be expressed as a union of open intervals.* Recall that the union can be infinite. Now consider for $\Omega$ an open interval
$$\Omega = \bigcup_{x \in \Omega} (x - \varepsilon_x, x + \varepsilon_x)$$
where for each $x$, $\varepsilon_x$ is the corresponding radius s.t. the definition is satisfied; and for all $x \in \Omega$, $(\mathbb{R} \smallsetminus \Omega) \cap (x - 2\varepsilon_x, x + 2\varepsilon_x) \neq \emptyset$. By definition for all $x \in \Omega$ there exists such an interval that $x$ is in it.

Then consider $\mathcal{B}' := \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for any $x' \in \mathbb{R} \cap \Omega$ there exists a sequence $(x_n)$ s.t. $\lim_{n \to \infty} x_n = x'$. Choose the sequence s.t. $|x_n - x'| > 2\,|x_{n+1} - x'|$. Suppose that for all $n$, $x' \notin (x_n - \varepsilon_{x_n}, x_n + \varepsilon_{x_n})$. Then there exists some $n_0$ s.t. $(x_{n+1} - \varepsilon_{x_{n+1}}, x_{n+1} + \varepsilon_{x_{n+1}}) \subsetneq (x_n, x')$ assuming $x_n < x'$ without loss of generality, which is a contradiction.

Since $\mathbb{Q} \simeq \mathbb{N} \times \mathbb{N}$ which is countable, $\mathbb{Q} \times \mathbb{Q}$ is also countable, indicating that $\beta'$ gives a countable basis of topology on $\mathbb{R}$. ☐

**Remark 2.7.** For $\mathbb{R}^n$, the standard topology is defined as where a set $\Omega$ is open if and only if for every $x \in \Omega$ there exists $\varepsilon > 0$ s.t. $B^n_\varepsilon(x) \subseteq \Omega$.

**Corollary 2.8.** By the same proof, and considering open intervals as 1-balls on $\mathbb{R}^1$, this gives that $\mathbb{R}^n$ with the standard topology admits a countable basis of topology.

**Definition 2.9** (Induced Topology)**.** *Let $(X, \mathcal{O}_X)$ be a topological space, and $Y \subseteq X$. Then there exists a definition for $\mathcal{O}_Y$ where $\Omega' \in \mathcal{O}_Y$ if and only if there exists $\Omega \in \mathcal{O}_X$ s.t. $\Omega' = \Omega \cap Y$. This is the **induced topology** on $Y$.*

**Definition 2.10.** *Let $(X_1, \mathcal{O}_1)$, $(X_2, \mathcal{O}_2)$ be topological spaces. Then*

- *A map $f : X_1 \to X_2$ is **continuous** if for all $\Omega \in \mathcal{O}_2$, $f^{-1}(\Omega) \in \mathcal{O}_1$.*
- *A map $f : X_1 \to X_2$ is **homeomorphic** if it is invertible, and both $f$ and $f^{-1}$ are continuous.*

**Example 2.11.** The map $f : [0, 2\pi) \to S^1, x \mapsto (\cos x, \sin x)$ is not homeomorphic, as for the arcs wrapped around the origin (e.g. $f([0, \pi/6) \cup (11\pi/6, 2\pi))$) the image is open, but the interval itself is not open.

**Definition 2.12** (Diffeomorphism)**.** *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be open sets. Then a homeomorphism $f : \Omega_1 \to \Omega_2$ is a **diffeomorphism** if both $f$ and $f^{-1}$ are differentiable. Similarly one can consider $C^k$-diffeomorphism for $f$ and $f^{-1}$ being $k$-times differentiable.*

**Proposition 2.13.** *If $f$ is a diffeomorphism which is $n$-times differentiable, then $f^{-1}$ is also $n$-times differentiable.*

*Proof.* Proceed to prove this via induction on $k$:

- *Base case.* For $k = 1$, this is by the definition of diffeomorphisms.
- *Inductive step.* Suppose that this is proven for $k = m$. For $k = m + 1 \leq n$, since $f$ is $C^n$-differentiable and locally injective everywhere, by inverse function theorem this gives

  $$g'(x) = (f'(g(x)))^{-1}$$

  where by hypothesis $f'$ and $g$ are both $C^m$-differentiable, and therefore so is their composition and $g'$.

This gives $g'$ is $C^{n-1}$-differentiable, which gives $g$ being $C^n$-differentiable. □

**Remark 2.14.** A differentiable homeomorphism is not necessarily a diffeomorphism, as $f$ is not necessarily locally injective everywhere. Consider $f : x \mapsto x^3$ whose inverse is not differentiable at $0$. $f$ is not locally injective at 0.

# 3 Differentiable Manifolds