MATH 437 - Manifolds

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Contents

1 Preliminary Topology

Definition 1.1 (Topological Space)**.** *A **topological space** is a pair (X, $\mathcal{O}$), where X is an arbitrary set, and $\mathcal{O}$ a class of subsets of X, which satisfies the follows:*

- *$\emptyset \in \mathcal{O}$, and $X \in \mathcal{O}$.*
- *Let $(\Omega_i)_{i\in I}$ be a (possibly infinite) class of subsets of X, then $\bigcup_{i\in I} \Omega_i \in \mathcal{O}$.*
- *Let $(\omega_i)_{i\in I}$ be a finite class of subsets of X, then $\bigcap_{i\in I} \omega_i \in \mathcal{O}$.*

*Elements of $\mathcal{O}$ are called the **open sets**. $\mathcal{O}$ (as a class) gives the topology of X.*

Remark 1.2. If the topology of X is clear, one often denotes the space by simply X.

Definition 1.3 (Hausdorff Space)**.** *A topological space $(X, \mathcal{O})$ is a **Hausdorff space** if for all $x, y \in X$ there exists $\Omega_x, \Omega_y \in \mathcal{O}$ s.t. $x \in \Omega_x, y \in \Omega_y$; and $\Omega_x \cap \Omega_y = \emptyset$.*

Example 1.4. Choice of topology is very important. Consider the following examples:

- Consider $(\mathbb{R}, \mathcal{O})$ where $\Omega \in \mathcal{O}$ if and only if for all $x \in \Omega$, there exists $\varepsilon \in \mathbb{R}$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq \Omega$. This space is Hausdorff as one could choose $\varepsilon_x = \varepsilon_y = |x-y|/4$. This also illustrates why the intersection must be finite, as otherwise one could construct a converging sequence of ε, whose corresponding class of subsets intersecting to a closed interval.
- Consider $(\mathbb{R}, \mathcal{O})$ to be the trivial topology, where $\mathcal{O} := \{\mathbb{R}, \emptyset\}$. Then the only subset in $\mathcal{O}$ containing x and y is $\mathbb{R}$, which implies that the space is not Hausdorff.

Definition 1.5 (Basis of a Topology)**.** *Let $(X, \mathcal{O})$ be a topology space. Then a subset $\mathcal{B} \subseteq \mathcal{O}$ is a **basis of topology** $\mathcal{O}$ if for any $\Omega \in \mathcal{O}$ it can be expressed as union of elements in $\mathcal{B}$. If there exists such $\mathcal{B}$ s.t. it is countably infinite, then X admits a **countable basis of topology**.*

Proposition 1.6. *$(\mathbb{R}, \mathcal{O})$ with topology defined as in the first case in the example above admits a countable basis of topology.*

Proof. First consider $\mathcal{B} := \{(a,b) \mid a < b, a, b \in \mathbb{R}\}$. This gives a basis for the topology of $(\mathbb{R}, \mathcal{O})$ in the sense of the first case above:

- *Any union of elements in $\mathcal{B}$ is open.* Denote such union to be Ω. By construction, for $x \in \mathbb{R}$ s.t. $x \in \Omega$, there exists $a_x < b_x$ s.t. $x \in (a_x, b_x)$. Then there exists $\varepsilon = \min\{x - a_x, b_x - x\}/2$ that satisfies the definition.
- *Any open subset of $\mathbb{R}$ can be expressed as a union of open intervals.* Recall that the union can be infinite. Now consider for Ω an open interval
$$\Omega = \bigcup_{x\in\Omega} (x - \varepsilon_x, x + \varepsilon_x)$$
where for each x, ε_x is the corresponding radius s.t. the definition is satisfied; and for all $x \in \Omega$, $(\mathbb{R}\backslash\Omega) \cap (x - 2\varepsilon_x, x + 2\varepsilon_x) \neq \emptyset$. By definition for all $x \in \Omega$ there exists such an interval that x is in it.

Then consider $\mathcal{B}' := \{(a,b) \mid a < b, a, b \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for any $x' \in \mathbb{R} \cap \Omega$ there exists a sequence (x_n) s.t. $\lim_{n\to\infty} x_n = x'$. Choose the sequence s.t. $|x_n - x'| > 2|x_{n+1} - x'|$. Suppose that for all n, $x' \notin (x_n - \varepsilon_{x_n}, x_n + \varepsilon_{x_n})$. Then there exists some n_0 s.t. $(x_{n+1} - \varepsilon_{x_{n+1}}, x_{n+1} + \varepsilon_{x_{n+1}}) \subsetneq (x_n, x')$ assuming $x_n < x'$ without loss of generality, which is a contradiction.

Since $\mathbb{Q} \simeq \mathbb{N} \times \mathbb{N}$ which is countable, $\mathbb{Q} \times \mathbb{Q}$ is also countable, indicating that β' gives a countable basis of topology on $\mathbb{R}$. □

Remark 1.7. For $\mathbb{R}^n$, the standard topology is defined as where a set Ω is open if and only if for every $x \in \Omega$ there exists $\varepsilon > 0$ s.t. $B^n_\varepsilon(x) \subseteq \Omega$.

Corollary 1.8. By the same proof, and considering open intervals as 1-balls on $\mathbb{R}^1$, this gives that $\mathbb{R}^n$ with the standard topology admits a countable basis of topology.

Definition 1.9 (Induced Topology)**.** *Let $(X, \mathcal{O}_X)$ be a topological space, and $Y \subseteq X$. Then there exists a definition for $\mathcal{O}_Y$ where $\Omega' \in \mathcal{O}_Y$ if and only if there exists $\Omega \in \mathcal{O}_X$ s.t. $\Omega' = \Omega \cap Y$. This is the **induced topology** on Y.*

Definition 1.10. *Let $(X_1, \mathcal{O}_1)$, $(X_2, \mathcal{O}_2)$ be topological spaces. Then*

- *A map $f : X_1 \to X_2$ is **continuous** if for all $\Omega \in \mathcal{O}_2$, $f^{-1}(\Omega) \in \mathcal{O}_1$.*
- *A map $f : X_1 \to X_2$ is **homeomorphic** if it is invertible, and both f and f^{-1} are continuous.*

Example 1.11. The map $f : [0, 2\pi) \to S^1, x \mapsto (\cos x, \sin x)$ is not homeomorphic, as for the arcs wrapped around the origin (e.g. $f([0, \pi/6) \cup (11\pi/6, 2\pi))$) the image is open, but the interval itself is not open.

Definition 1.12 (Diffeomorphism)**.** *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ be open sets. Then a homeomorphism $f : \Omega_1 \to \Omega_2$ is a **diffeomorphism** if both f and f^{-1} are differentiable. Similarly one can consider C^k-diffeomorphism for f and f^{-1} being k-times differentiable.*

Proposition 1.13. *If f is a diffeomorphism which is n-times differentiable, then f^{-1} is also n-times differentiable.*

Proof. Proceed to prove this via induction on k:

- *Base case.* For $k = 1$, this is by the definition of diffeomorphisms.
- *Inductive step.* Suppose that this is proven for $k = m$. For $k = m + 1 \leq n$, since f is C^n-differentiable and locally injective everywhere, by inverse function theorem this gives

  $$g'(x) = \frac{1}{f'(g(x))}$$

  where by hypothesis f' and g are both C^m-differentiable, and therefore so is their composition and g'.

This gives g' is C^{n-1}-differentiable, which gives g being C^n-differentiable. □

Remark 1.14. A differentiable homeomorphism is not necessarily a diffeomorphism, as f is not necessarily locally injective everywhere. Consider $f : x \mapsto x^3$ whose inverse is not differentiable at 0. f is not locally injective at 0.

2 Differentiable Manifolds