MATH 538 - Lie Algebra

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1 Lie Algebra

Definition 1.1 (Lie Algebra). Let $\mathbb{F}$ be a field. A **Lie Algebra** is a vector space L over $\mathbb{F}$ equipped with a bilinear map $[\cdot,\cdot] : L \times L \to L$ (the **Lie Bracket**) satisfying the following properties:

- *Alternating Property*: $[x, x] = 0$ for all $x \in L$. (For char $\mathbb{F} \neq 2$, this is equivalent to antisymmetry: $[x, y] = -[y, x]$ for all $x, y \in L$.)
- *Jacobi Identity*: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Example 1.2. Consider $V = \mathbb{F}^n$. Notice that $\dim(\mathrm{End}(V)) = \dim(\mathrm{Mat}_n(\mathbb{F})) = n^2$ as vector spaces over $\mathbb{F}$; and they are further isomorphic. We can further show that they are isomorphic as Lie algebras.

Proposition 1.3. Define the Lie bracket on $\mathrm{End}(V)$ by $[f, g] = fg - gf$ (with the product the composition of functions). Then $\mathrm{End}(V)$ is a Lie Algebra.

Proof. It suffices to verify that Lie bracket satisfies the alternating property and the Jacobi identity.

- *Alternating Property*: $[f, f] = ff - ff = 0$.
- *Jacobi Identity*:

$$\begin{aligned}
[f, [g, h]] + [g, [h, f]] + [h, [f, g]] &= [f, gh - hg] + [g, hf - fh] + [h, fg - gf] \\
&= f(gh - hg) - (gh - hg)f + g(hf - fh) - (hf - fh)g + h(fg - gf) - (fg - gf)h \\
&= fgh - fgh - ghf + hgf + ghf - hfg - hfg + fhg + fhg - fgh - gfh + gfh \\
&= 0.
\end{aligned}$$

Bi-linearity results directly from the linearity of functions. □

Notation. The Lie algebra $(\mathrm{End}(V), [\cdot,\cdot])$ is denoted by $\mathfrak{gl}(V)$.

Example 1.4. Let $V = \mathbb{R}^n$. Then as vector spaces $\mathrm{End}(V) \simeq \mathrm{Mat}_n(\mathbb{R}) \simeq \mathfrak{gl}(V)$ where $[A, B] = AB - BA$ for $A, B \in \mathrm{Mat}_n(\mathbb{R})$.

Definition 1.5 (Lie Subalgebra). A **Lie Subalgebra** $K \subseteq L$ is a subspace of s.t. for all $x, y \in K$, $[x, y] \in K$.

Definition 1.6 (Linear Algebra). Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear algebra**.

2 An Algebraic Perspective on Lie Algebra