MATH 538 - Lie Algebra

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1 Lie Algebras

Definition 1.1 (Lie Algebra). Let $\mathbb{F}$ be a field. A **Lie Algebra** is a vector space L over $\mathbb{F}$ equipped with a bilinear map $[\cdot,\cdot] : L \times L \to L$ (the **Lie bracket**) satisfying the following properties:

- *Alternating Property*: $[x, x] = 0$ for all $x \in L$. (For $\operatorname{char} \mathbb{F} \neq 2$, this is equivalent to antisymmetry: $[x, y] = -[y, x]$ for all $x, y \in L$.)
- *Jacobi Identity*: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

The **dimension** of a Lie algebra is the dimension of the underlying vector space.

Remark 1.2. Throughout this note, for simplicity we will assume that $\operatorname{char} \mathbb{F} = 0$. Nevertheless, the limitations on field characteristics will be pointed out when it is clear that the result will fall in certain cases.

Example 1.3. Consider $V = \mathbb{F}^n$. Notice that $\dim_{\mathbb{F}}(\operatorname{End}(V)) = \dim_{\mathbb{F}}(\operatorname{Mat}_n(\mathbb{F})) = n^2$ as vector spaces over $\mathbb{F}$; and they are further isomorphic. We can further show that they are isomorphic as Lie algebras.

Proposition 1.4. Define the Lie bracket on $\operatorname{End}(V)$ by $[f, g] = fg - gf$ (with the product the composition of functions). Then $\operatorname{End}(V)$ is a Lie Algebra.

Proof. It suffices to verify that Lie bracket satisfies the alternating property and the Jacobi identity.

- *Alternating Property*: $[f, f] = ff - ff = 0$.
- *Jacobi Identity*:

$$\begin{aligned} [f, [g, h]] + [g, [h, f]] + [h, [f, g]] &= [f, gh - hg] + [g, hf - fh] + [h, fg - gf] \\ &= f(gh - hg) - (gh - hg)f + g(hf - fh) - (hf - fh)g + h(fg - gf) - (fg - gf)h \\ &= fgh - fgh - ghf + hgf + ghf - hfg - hfg + fhg + fhg - fgh - gfh + gfh \\ &= 0. \end{aligned}$$

Bi-linearity results directly from the linearity of functions. ☐

Notation. The Lie algebra $(\operatorname{End}(V), [\cdot,\cdot])$ is denoted by $\mathfrak{gl}(V)$.

Example 1.5. Let $V = \mathbb{R}^n$. Then as vector spaces $\operatorname{End}(V) \simeq \operatorname{Mat}_n(\mathbb{R}) \simeq \mathfrak{gl}(V)$ where $[A, B] = AB - BA$ for $A, B \in \operatorname{Mat}_n(\mathbb{R})$.

The Lie Algebra $\operatorname{End}(V) \simeq \operatorname{Mat}_n(V) \simeq \mathfrak{gl}(V)$ has a basis $\{e_{ij}\}_{i,j=1}^n$, where $\{e_{ij}\}$ represents the matrix with all zero entries except for 1 in the (i, j)-th entry.

Definition 1.6 (Lie Subalgebra). A **Lie Subalgebra** $K \subseteq L$ is a subspace of s.t. for all $x, y \in K$, $[x, y] \in K$.

Definition 1.7 (Linear Lie Algebra). Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear Lie algebra**.

Theorem 1.8 (Ado-Iwasawa). Every finite dimensional Lie algebra is isomorphic to a linear Lie algebra.

The proof of this statement requires more structure and will be deferred.

TODO: search for proof

Example 1.9 (Classical Lie Algebra). The following examples of Lie algebra constitute the **classical Lie algebra** which are the bulk of existing Lie algebras. As expected, they are closely related to matrices.

1. Let $V = \mathbb{F}^{n+1}$. The **special linear algebra** A_n is

$$A_n = \mathfrak{sl}_{n+1}(V) = \{g \in \mathfrak{gl}_{n+1}(V) \mid \operatorname{Tr} g = 0\}$$

As a vector space over $\mathbb{F}$ it has dimension $(n^2 + 2n)$.

2. Let $V = \mathbb{F}^{2n}$. The **symplectic algebra** C_n is

$$C_n = \mathfrak{sp}_{2n}(V) = \{g \in \mathfrak{gl}_{2n}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

Considering its dimension, writing also $g \in \mathfrak{gl}_{2n}(V)$ as n-by-n blocks, we have

$$g = \left(\begin{array}{c|c} g_1 & g_2 \\ \hline g_3 & g_4 \end{array}\right) \quad \Longrightarrow \quad sg + g^T s = \left(\begin{array}{c|c} g_3 - g_3^T & g_4 + g_1^T \\ \hline g_1 + g_4^T & g_2 - g_2^T \end{array}\right) = 0$$

Further notice that $(g_4 + g_1^T) = (g_1 + g_4^T)^T$. Then the condition of matrix vanishing becomes equivalent to $g_3 = g_3^T$, $g_2 = -g_2^T$ and $g_1 = -g_4^T$. For a symmetric matrix its dimension is $\frac{1}{2}n(n+1)$, and arbitrary g_1 fixes g_4, giving dimension n. Summing together gives the $\dim_{\mathbb{F}} C_n = 2n^2 + n$.

3. Let $V = \mathbb{F}^{2n+1}$. The **(odd) orthogonal algebra** B_n is

$$B_n = \mathfrak{o}_{2n+1}(V) = \{g \in \mathfrak{gl}_{2n+1}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_n \\ 0 & \mathrm{Id}_n & 0 \end{pmatrix}$$

Considering its dimension, write also $g \in \mathfrak{gl}_{2n+1}(V)$ in block form:

$$g = \begin{pmatrix} x & a_{1\times n} & b_{1\times n} \\ c_{n\times 1} & m_{n\times n} & n_{n\times n} \\ d_{n\times 1} & p_{n\times n} & q_{n\times n} \end{pmatrix} \quad \Longrightarrow \quad sg + g^T s = \begin{pmatrix} 2x & a + d^T & b + c^T \\ a^T + d & p + p^T & q + m^T \\ b^T + c & m + q^T & n + n^T \end{pmatrix} = 0$$

This translates to equalities

$$2x = 0, \quad a + d^T = (a^T + d)^T = 0, \quad b + c^T = (b^T + c)^T = 0, \quad q + m^T = (m + q^T)^T = 0, \quad p + p^T = 0, \quad n + n^T = 0$$

This gives $x = 0$. Fixing a and b fixes d and c, of which there are $2n$ choices. Fixing m fixes q, of which there are n^2 choices. It is also required that both n and p are anti-symmetric matrices, i.e. they have zeros on the diagonal, and fixing upper triangle elements fixes the whole matrix, of which there are $\frac{1}{2}n(n-1)$ choices. Then

$$\dim_{\mathbb{F}} B_n = 2n + n^2 + 2 \cdot \frac{1}{2}n(n-1) = 2n^2 + n$$

4. Let $V = \mathbb{F}^{2n}$. The **(even) orthogonal algebra** D_n is

$$D_n = \mathfrak{o}_{2n}(V) = \{g \in \mathfrak{gl}_{2n}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$$

Considering its dimension, use the similar strategy as above. Write $g \in \mathfrak{gl}_{2n}(V)$ in block form:

$$g = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \quad \Longrightarrow \quad sg + g^T s = \begin{pmatrix} p + p^T & q + m^T \\ m + q^T & n + n^T \end{pmatrix} = 0$$

The equality translates to the following conditions

$$p + p^T = 0, \quad q + m^T = (m + q^T)^T = 0, \quad n + n^T = 0$$

Fixing p fixes m, and both p and n are anti-symmetric matrices, which have dimension $\frac{1}{2}n(n-1)$ over $\mathbb{F}$. This gives

$$\dim_{\mathbb{F}} D_n = n^2 + 2 \cdot \frac{1}{2}n(n-1) = 2n^2 - n$$

2 Structure of Lie Algebras

Now we turn to discuss the algebraic structures of Lie algebra. Being an algebra itself, we are interested in some special properties, analogous to the study of groups or rings. In general, some of the definitions and theorems can be rephrased in a categorical sense, and thereby applies to all such similar structures.

Definition 2.1 ($\mathbb{F}$-Algebra)**.** Let $\mathbb{F}$ be a field. An $\mathbb{F}$-algebra is a vector space U over $\mathbb{F}$ equipped with a bilinear map $(\cdot) : U \times U \to U$, denoted $(u_1, u_2) \mapsto u_1 u_2$.

Remark 2.2. In this sense, the Lie algebra L (by definition is an $\mathbb{F}$-vector space) is indeed an $\mathbb{F}$-algebra (compatible with its name), with the bilinear map $(\cdot)$ being the Lie bracket.

Definition 2.3 (Derivation)**.** Let U be a vector space over field $\mathbb{F}$. A **derivation** of U is $S \in \mathfrak{gl}(U)$ s.t. for all $a, b \in U$, $S(ab) = aS(b) + S(a)b$ (i.e. satisfies the Leibniz Rule). The set of derivations on U is denoted by $\mathrm{Der}(U)$.

Remark 2.4. $\mathrm{Der}(U)$ is a subalgebra of $\mathfrak{gl}(U)$. By definition, it suffices to verify that for all $S, T \in \mathrm{Der}(U)$, $[S, T] \in \mathrm{Der}(U)$.

Check that the Leibniz rule is satisfied:

$$\begin{aligned}[S,T](ab) &= S(T(ab)) - T(S(ab)) \\ &= S(aT(b) + T(a)b) - T(aS(b) + S(a)b) \\ &= aS(T(b)) + bS(T(a)) - aT(S(b)) - bT(S(a)) \\ &= a[S,T](b) + b[S,T](a)\end{aligned}$$

Definition 2.5 (Adjoint Representation). The map $\mathrm{ad} : L \to \mathrm{Der}(L)$ is the **adjoint representation**, defined by $\mathrm{ad}(x)(y) = [x, y]$

An immediate thought is to verify that indeed $\mathrm{ad}(x)$ gives a derivation. Recall that the product defined in Lie algebra is the Lie bracket. Therefore, to verify the derivation property it suffices to check whether we have the equality

$$\mathrm{ad}(x)([y,z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]$$

Applying the definition and manipulating the terms, we get the Jacobi Identity, which verifies the equality:

$$\begin{aligned}\mathrm{ad}(x)([y,z]) = [x,[y,z]] = -[y,[z,x]] - [z,[x,y]] &= -[y, \mathrm{ad}(x)(z)] - [z, \mathrm{ad}(x)(y)] \\ &= [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]\end{aligned}$$

Definition 2.6 (Structure Constants). Let $\{x_1, \dots, x_n\}$ be a basis of L as an $\mathbb{F}$-vector space. The **structure constants** of L are the coefficients c_{ij}^k s.t. $[x_i, x_j] = \sum_{k=1}^n c_{ij}^k x_k$.

It is clear that the structure constants are specified solely by the definition of Lie bracket, which is the sole extra structure given to any Lie algebra apart from the underlying vector space structure.

Remark 2.7. Using the structure constants we can rewrite the Jacobi Identity. Given a basis of L over $\mathbb{F}$ and its corresponding structure constants $a_{ij}^k \in \mathbb{F}$, the Jacobi Identity can be written as

$$\sum_{k=1}^n \left(a_{ij}^k a_{k\ell}^m + a_{j\ell}^k a_{ki}^m + a_{\ell i}^k a_{kj}^m\right) = 0, \quad \forall i, j, \ell, m \in \{1, \dots, n\}$$

Definition 2.8 (Abelian). A Lie algebra L is **abelian** if $[x, y] = 0$ for all $x, y \in L$.

Definition 2.9 (Ideal). Given a Lie algebra L, a subspace $I \subseteq L$ is an **ideal** if for all $x \in I$ and $y \in L$, $[x, y] \in I$.

Remark 2.10. This kind of "absorbing" property is analogous to the normal subgroup in group theory. Considering the Lie bracket as a multiplication on a ring, this is compatible with the definition of an ideal in a ring.

Remark 2.11. Given a Lie algebra L, if both I and J are ideals in L, then $I + J$, $I \cap J$ and $[I, J]$ are also ideals.

With the similar formulation of structure preserving sub-objects, we have similar structures as in group or ring theory.

Definition 2.12 (Quotient). Given a Lie algebra L and an ideal $I \subseteq L$, the **quotient** L/I is the vector space $L/I = \{x + I \mid x \in L\}$ with the Lie bracket $[x + I, y + I] = [x, y] + I$.

Definition 2.13 (Center). Given a Lie algebra L, the **center** of L is defined as $Z(L) = \{z \in L \mid [z, x] = 0 \text{ for all } x \in L\}$.

Definition 2.14 (Derived Algebra). Given a Lie algebra L, the **derived algebra** of L is $[L, L] = L' = L^{(1)}$, where $[L, L]$ can be interpreted as the set $[L, L] := \{[x, y] \mid x, y \in L\}$ By definition this is an ideal of L.

Definition 2.15 (Simple). A Lie algebra L is **simple** if $[L, L] \neq 0$ (i.e. it is non-trivial), and the only ideals of L are trivial (0 and L).

Example 2.16. Let $L = \mathfrak{gl}_n(\mathbb{F})$. Then $[e_{ij}, e_{k\ell}] = \delta_{jk} e_{i\ell} - \delta_{\ell i} e_{kj}$. Setting $k = j$ and $i = \ell$ gives $\mathrm{Tr}([e_{ij}, e_{k\ell}]) = 0$. By definitions in Example 1.9, $L^{(1)} = [L, L] \simeq \mathfrak{sl}_n(F)$. Since specifically $\mathfrak{sl}_n(\mathbb{F}) \subseteq \mathfrak{gl}_n(\mathbb{F})$ which gives an ideal, $\mathfrak{gl}_n(\mathbb{F})$ is not simple. In fact $\mathfrak{sl}_n(\mathbb{F})$ is simple, but proving this result requires more constructions.

Definition 2.17 (Normalizer). Let L be a Lie algebra and K a subalgebra of L. The **normalizer** of K is defined by $N_L(K) := \{x \in L \mid [x, K] \subseteq K\}$, the largest subalgebra of L in which K is an ideal.

Definition 2.18 (Centralizer). Let L be a Lie algebra and X an arbitrary set. The **centralizer** of X is defined by $C_L(X) := \{x \in L \mid [x, X] = 0\}$.

Definition 2.19 (Morphism). Let L and L' be two Lie algebras over $\mathbb{F}$. A **homomorphism** is an $\mathbb{F}$-linear transformation $\phi : L \to L'$ s.t. $\phi([x, y]) = [\phi(x), \phi(y)]$, i.e. it commutes with the multiplication, which is the Lie bracket here.

Adopting general categorical nomenclature, a homomorphism is a **monomorphism** if it is injective, an **epimorphism** if it is surjective, an **isomorphism** if it is bijective, and an **automorphism** if it is an isomorphism from L to itself.

Remark 2.20. With the definition of morphisms, we can easily translate common results from group or ring theory to Lie algebras. For all homomorphism $\phi : L \to L'$:

1. Both $\ker \phi$ and $\operatorname{im} \phi$ are ideals of L.
2. (First Isomorphism Theorem) $\operatorname{im} \phi \simeq L/\ker \phi$.
3. (Second Isomorphism Theorem) Given two ideals $I, J \subseteq L$, by Remark 2.11 $I + J$ is also an ideal of L. We have the isomorphism $(I + J)/J \simeq I/I \cap J$.
4. (Third Isomorphism Theorem) Let $I \subseteq J \subseteq L$ where both I and J are ideals of L. Then $L/J \simeq (L/I)/(J/I)$.
5. (Fourth Isomorphism Theorem, Correspondence) Let $I \subseteq \ker \phi$ be an ideal in $\ker \phi$. Then there exists a unique homomorphism $\psi : L/I \to L'$, i.e. making the following diagram commute:

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[dr] & L' \\ & L/I \arrow[u, dashed, "\psi"'] \end{tikzcd}$$

Definition 2.21 (Representation). Let L and L' be Lie algebras over $\mathbb{F}$, and V a vector space over $\mathbb{F}$. A **representation** of L on V is a homomorphism $\rho : L \to \mathfrak{gl}(V)$.

Example 2.22 (Adjoint Representation). For a Lie algebra L, recall that the adjoint map is $\operatorname{ad} : L \to \mathfrak{gl}(L)$, $\operatorname{ad}(x)(y) = [x, y]$. Since L is itself a vector space over $\mathbb{F}$, letting $V = L$ we have a representation. This is indeed a homomorphism as

$$\begin{aligned}[\operatorname{ad}(x), \operatorname{ad}(y)](z) &= \operatorname{ad}(x)\operatorname{ad}(y)(z) - \operatorname{ad}(y)\operatorname{ad}(x)(z) = [x, [y, z]] - [y, [x, z]] \\ &= [x, [y, z]] + [y, [z, x]] = -[z, [x, y]] = [[x, y], z] \\ &= \operatorname{ad}([x, y])(z)\end{aligned}$$

Proposition 2.23. Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof. Consider using the adjoint representation. The kernel of ad

$$\ker(\operatorname{ad}) = \{x \in L \mid \operatorname{ad}(x) = 0\} = \{x \in L \mid [x, z] = 0, \forall z \in L\} = Z(L)$$

by Remark 2.20 is an ideal in L. Since L is simple, $\ker(\operatorname{ad}) = 0$. Using First Isomorphism Theorem, $L/\ker(\operatorname{ad}) \simeq L \simeq \operatorname{im}(\operatorname{ad})$. Since $\operatorname{ad} : L \to \mathfrak{gl}(V)$, $\operatorname{im}(\operatorname{ad}) \subseteq \mathfrak{gl}(L)$. By Definition 1.7 L is a linear Lie algebra. □

Notation. It is often denoted $\mathrm{GL}(V) = \{g \in \mathfrak{gl}(V) \mid g \text{ invertible}\}$. This coincides with the general linear group.

Example 2.24 (Adjoint Representation of $\mathrm{GL}(V)$). Let $g \in \mathrm{GL}(V)$. Define $\operatorname{Ad}(g) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ by $\operatorname{Ad}(g)(x) = gxg^{-1}$, which is the adjoint representation of $\mathrm{GL}(V)$.

Definition 2.25 (Nilpotent (Element)). Assume that $\operatorname{char} \mathbb{F} = 0$. Let L be a Lie algebra on $\mathbb{F}$. Then $x \in L$ is **nilpotent** if $(\operatorname{ad}(x))^k = 0$ for some $k \in \mathbb{Z}_{\geq 0}$.

Definition 2.26 (Exponential Map). Let L be a Lie algebra, and $x \in L$ nilpotent of order k. Then the **exponential map** is defined by

$$\exp(\operatorname{ad}(x)) = \sum_{n=0}^{k-1} \frac{(\operatorname{ad}(x))^n}{n!}$$

which coincides with the Taylor expansion. ALl terms of order k vanishes due to x being k-nilpotent.

Lemma 2.27. $\exp(\operatorname{ad}(x)) \in \operatorname{Aut}(L)$. More generally, if $\delta \in \operatorname{Der}(L)$ is nilpotent, then $exp(\delta) \in \operatorname{Aut}(L)$.

Proof. We seek to prove only the general version of the statement. First verify that ad is a derivation, i.e. we have the equality $\operatorname{ad}([x, y])(z) = ([x, \operatorname{ad}(y)] + [\operatorname{ad}(x), y])(z)$ for all $x, y, z \in L$. This is a direct consequence of the Jacobi Identity:

$$\operatorname{ad}([x, y])(z) = [[x, y], z] = [x, [y, z]] - [y, [z, x]] = [x, \operatorname{ad}(y)(z)] - [y, \operatorname{ad}(x)(z)] = ([x, \operatorname{ad}(y)] + [\operatorname{ad}(x), y])(z)$$

Now for the general statement on derivations, first verify that δ is a homomorphism. Write out the expressions explicitly:

$$\begin{aligned}
\exp(\delta(x))\exp(\delta(y)) &= \left(\sum_{n=0}^{k-1} \frac{\delta^n(x)}{n!}\right)\left(\sum_{m=0}^{k-1} \frac{\delta^m(y)}{m!}\right) \\
&= \sum_{n=0}^{2k-2}\left(\sum_{i=0}^{n} \frac{\delta^i(x)}{i!}\cdot\frac{\delta^{n-i}(y)}{(n-i)!}\right) \\
&= \sum_{n=0}^{2k-2} \frac{\delta^n(xy)}{n!} && \text{(Leibniz Rule)} \\
&= \sum_{n=0}^{k-1} \frac{\delta^n(xy)}{n!} = \exp(\delta(xy)) && (\delta^k = 0)
\end{aligned}$$

The fact that $\exp(\delta)$ is a bijection can be verified via writing out its inverse: let $\exp(\eta) = \mathrm{Id} - \eta$. Then $\exp^{-1} = \sum_{n=0}^{k-1} \eta^n$, where we designate $\eta^0 = \mathrm{Id}$. □

Example 2.28. Let $\mathbb{F}$ be a field of characteristic 0, and $L = \mathbb{F}^2$ with basis $\{v_1, v_2\}$. Endow it with a Lie algebra structure by setting $[v_1, v_2] = v_1$. Then $\mathrm{ad} \in \mathfrak{gl}_2(L)$, with the matrix representation

$$\mathrm{ad}(v_1) = \left(\begin{array}{c|c} \mathrm{ad}(v_1)(v_1) & \mathrm{ad}(v_1)(v_2) \end{array}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathrm{ad}(v_1)^2 = 0$$

Since ad is 2-nilpotent,

$$\exp(\mathrm{ad}(v_1)) = \mathrm{Id}_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Using the same notation as in the proof above,

$$\eta = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \exp^{-1}(\mathrm{ad}(v_1)) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

Definition 2.29 (Derived Series)**.** Given a Lie algebra L, the **derived series** of L is the sequence of ideals

$$L^{(0)} = L, \quad L^{(1)} = [L, L], \quad \cdots \quad L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$$

Remark 2.30. Similar to the cases in group or ring theory, the relation of ideals is in general not transitive. That is, directly from the expression we know that for all n, $L^{(i)}$ is an ideal in $L^{(i-1)}$; but in general $L^{(n)}$ is not an ideal in L for $n \in \mathbb{Z}_{\geq 2}$.

Definition 2.31 (Solvable)**.** A Lie algebra L is **sovlable** if $L^{(n)} = 0$ for some n.

Example 2.32. Fix $\mathbb{F}$ to be a field. Let $\mathfrak{t}_n(\mathbb{F})$ be upper triangular n-by-n matrices over $\mathbb{F}$, $\mathfrak{d}_n(\mathbb{F})$ be diagonal n-by-n matrices and $\mathfrak{n}_n(\mathbb{F})$ strictly upper triangular n-by-n matrices. Then $\mathfrak{d}_n(\mathbb{F})$ is solvable since in general diagonal matrices commutes with all matrices in terms of multiplication (i.e. lies in the center of the Lie algebra). $\mathfrak{n}_n(\mathbb{F})$ is also solvable since strictly upper triangular matrices are nilpotent. Furthermore, as vector spaces $\mathfrak{t}_n(\mathbb{F}) = \mathfrak{d}_n(\mathbb{F}) \oplus \mathfrak{n}_n(\mathbb{F})$, giving $\mathfrak{t}_n(\mathbb{F})$ solvable.

In general we can express the k-th derived algebra of $\mathfrak{t}_n(\mathbb{F})$ as

$$\mathfrak{t}_n(\mathbb{F})^{(k)} = \{(g_{ij}) \mid g_{ij} = 0 \forall i + k - 1 \geq j\}$$

which is nilpotent of order $n-1$.

Example 2.33. All simple Lie algebras are non-solvable. By definition $[L, L]$ is an ideal in L; and L being simple implies that $[L, L]$ must be either 0 or L. However being simple also requires that L is not abelian. Therefore $L^{(1)} = [L, L] = L$, giving $L^{(k)} = L$ for all k and therefore is non-solvable. One of such examples is $\mathfrak{sl}_n(\mathbb{F})$ (also mentioned but not proved in 2.16).

Remark 2.34. Similar to solvability in group theory, we have the following results:

1. If L is solvable, then so are its subalgebras and $\operatorname{im}\phi$ for all $\phi : L \to L'$ homomorphisms.
2. For $I \subseteq L$ an ideal, if both I and L/I are solvable, then L is solvable.
3. If I and J are solvable ideals in L, then $I + J$ is also solvable.

Corollary 2.35. If L is solvable, then L has a unique solvable ideal, called the **radical ideal**, denoted $\operatorname{Rad} L$.

Proof. Let S be a maximal solvable ideal of L, and I any solvable ideal of L. There fore $S \subseteq I + S$, and by Remark 2.34 $I + S$ is solvable. Maximality of S implies $I + S = S$, and therefore $I \subseteq S$ which verifies the uniqueness. Existence follows from the fact that the zero ideal is solvable. □

Definition 2.36 (Semisimple). A non-zero Lie algebra L is **solvable** if $\operatorname{Rad}(L) = 0$.

Remark 2.37. We have the following properties directly resulting from the definition:

1. A simple Lie algebra is semisimple (as expected), as the radical ideal is an ideal.
2. If a Lie algebra L is not solvable, then $L/\operatorname{Rad} L$ is semisimple, as by Correspondence Theorem a solvable ideal $J \subseteq L/\operatorname{Rad} L$ corresponds to a solvable ideal $I \subseteq L$ containing $\operatorname{Rad} L$. From definition of radical ideals $I = J$, and therefore $L/\operatorname{Rad} L$ is semisimple.

Definition 2.38 (Descending (Lower) Central Series). Given a Lie algebra L, its **descending central series** (or **lower central series**) is the sequence of ideals

$$L^0 = L, \quad L^1 = [L, L], \quad \cdots \quad L^n = [L, L^{n-1}]$$

The notation is consistent with the treatment of Lie brackets in Lie algebras as multiplication.

Remark 2.39. Different from the every term in the descending central series is an ideal in L.

With the notion of "raising a power" for Lie algebras, we can extend the definition of nilpotency:

Definition 2.40 (Nilpotent (Lie Algebra)). A Lie algebra L is **nilpotent** if $L^n = 0$ for some n.

Example 2.41. Revisiting the example for derived series (Example 2.32) with the same notation, although similarly both $\mathfrak{n}_n(\mathbb{F})$ and $\mathfrak{d}_n(\mathbb{F})$ are nilpotent, $\mathfrak{t}_n(\mathbb{F})$ is not nilpotent, as one can always fill in a diagonal matrix for L. The explicit expression for $\mathfrak{n}_n(\mathbb{F})$ agrees the nilpotency of upper triangular matrices:

$$\mathfrak{n}_n(\mathbb{F})^k = \{(g_{ij}) \mid g_{ij} = 0 \forall i + k \geq j\}$$

Remark 2.42. Similarly the properties of nilpotent groups extend to nilpotent Lie algebras:

1. If a Lie algebra L is nilpotent, and all its subalgebras and $\operatorname{im} \phi$ for all $\phi : L \to L'$ homomorphism are nilpotent.
2. If $L/Z(L)$ is nilpotent, then L is nilpotent.
3. If $L \neq 0$ is nilpotent, then $Z(L) \neq 0$.