MATH 538 - Lie Algebra

ARessegetes Stery

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Contents

1 Overview of Lie Algebras

Definition 1.1 (Lie Algebra)**.** Let $\mathbb{F}$ be a field. A **Lie Algebra** is a vector space L over $\mathbb{F}$ equipped with a bilinear map $[\cdot,\cdot] : L \times L \to L$ (the **Lie bracket**) satisfying the following properties:

- *Alternating Property*: $[x, x] = 0$ for all $x \in L$. (For $\operatorname{char} \mathbb{F} \neq 2$, this is equivalent to antisymmetry: $[x, y] = -[y, x]$ for all $x, y \in L$.)
- *Jacobi Identity*: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

The **dimension** of a Lie algebra is the dimension of the underlying vector space.

Remark 1.2. Throughout this note, for simplicity we will assume that $\operatorname{char} \mathbb{F} = 0$. Nevertheless, the limitations on field characteristics will be pointed out when it is clear that the result will fall in certain cases.

Example 1.3. Consider $V = \mathbb{F}^n$. Notice that $\dim_{\mathbb{F}}(\operatorname{End}(V)) = \dim_{\mathbb{F}}(\operatorname{Mat}_n(\mathbb{F})) = n^2$ as vector spaces over $\mathbb{F}$; and they are further isomorphic. We can further show that they are isomorphic as Lie algebras.

Proposition 1.4. Define the Lie bracket on $\operatorname{End}(V)$ by $[f, g] = fg - gf$ (with the product the composition of functions). Then $\operatorname{End}(V)$ is a Lie Algebra.

Proof. It suffices to verify that Lie bracket satisfies the alternating property and the Jacobi identity.

- *Alternating Property*: $[f, f] = ff - ff = 0$.
- *Jacobi Identity*:

$$\begin{aligned}
[f, [g, h]] + [g, [h, f]] + [h, [f, g]] &= [f, gh - hg] + [g, hf - fh] + [h, fg - gf] \\
&= f(gh - hg) - (gh - hg)f + g(hf - fh) - (hf - fh)g + h(fg - gf) - (fg - gf)h \\
&= fgh - fgh - ghf + hgf + ghf - hfg - hfg + fhg + fhg - fgh - gfh + gfh \\
&= 0.
\end{aligned}$$

Bi-linearity results directly from the linearity of functions. □

Notation. The Lie algebra $(\operatorname{End}(V), [\cdot,\cdot])$ is denoted by $\mathfrak{gl}(V)$.

Example 1.5. Let $V = \mathbb{R}^n$. Then as vector spaces $\operatorname{End}(V) \simeq \operatorname{Mat}_n(\mathbb{R}) \simeq \mathfrak{gl}(V)$ where $[A, B] = AB - BA$ for $A, B \in \operatorname{Mat}_n(\mathbb{R})$.

The Lie Algebra $\operatorname{End}(V) \simeq \operatorname{Mat}_n(V) \simeq \mathfrak{gl}(V)$ has a basis $\{e_{ij}\}_{i,j=1}^n$, where $\{e_{ij}\}$ represents the matrix with all zero entries except for 1 in the (i, j)-th entry.

Definition 1.6 (Lie Subalgebra)**.** A **Lie Subalgebra** $K \subseteq L$ is a subspace of s.t. for all $x, y \in K$, $[x, y] \in K$.

Definition 1.7 (Linear Lie Algebra)**.** Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear Lie algebra**.

Theorem 1.8 (Ado-Iwasawa). Every finite dimensional Lie algebra is isomorphic to a linear Lie algebra.

The proof of this statement requires more structure and will be deferred.

TODO: search for proof

Example 1.9 (Classical Lie Algebra). The following examples of Lie algebra constitute the **classical Lie algebra** which are the bulk of existing Lie algebras. As expected, they are closely related to matrices.

1. Let $V = \mathbb{F}^{n+1}$. The **special linear algebra** A_n is

$$A_n = \mathfrak{sl}_{n+1}(V) = \{g \in \mathfrak{gl}_{n+1}(V) \mid \operatorname{Tr} g = 0\}$$

   As a vector space over $\mathbb{F}$ it has dimension $(n^2 + 2n)$.

2. Let $V = \mathbb{F}^{2n}$. The **symplectic algebra** C_n is

$$C_n = \mathfrak{sp}_{2n}(V) = \{g \in \mathfrak{gl}_{2n}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

   Considering its dimension, writing also $g \in \mathfrak{gl}_{2n}(V)$ as n-by-n blocks, we have

$$g = \left(\begin{array}{c|c} g_1 & g_2 \\ \hline g_3 & g_4 \end{array}\right) \quad \Longrightarrow \quad sg + g^T s = \left(\begin{array}{c|c} g_3 - g_3^T & g_4 + g_1^T \\ \hline g_1 + g_4^T & g_2 - g_2^T \end{array}\right) = 0$$

   Further notice that $(g_4 + g_1^T) = (g_1 + g_4^T)^T$. Then the condition of matrix vanishing becomes equivalent to $g_3 = g_3^T$, $g_2 = -g_2^T$ and $g_1 = -g_4^T$. For a symmetric matrix its dimension is $\frac{1}{2}n(n+1)$, and arbitrary g_1 fixes g_4, giving dimension n. Summing together gives the $\dim_{\mathbb{F}} C_n = 2n^2 + n$.

3. Let $V = \mathbb{F}^{2n+1}$. The **(odd) orthogonal algebra** B_n is

$$B_n = \mathfrak{o}_{2n+1}(V) = \{g \in \mathfrak{gl}_{2n+1}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_n \\ 0 & \mathrm{Id}_n & 0 \end{pmatrix}$$

   Considering its dimension, write also $g \in \mathfrak{gl}_{2n+1}(V)$ in block form:

$$g = \begin{pmatrix} x & a_{1\times n} & b_{1\times n} \\ c_{n\times 1} & m_{n\times n} & n_{n\times n} \\ d_{n\times 1} & p_{n\times n} & q_{n\times n} \end{pmatrix} \quad \Longrightarrow \quad sg + g^T s = \begin{pmatrix} 2x & a + d^T & b + c^T \\ a^T + d & p + p^T & q + m^T \\ b^T + c & m + q^T & n + n^T \end{pmatrix} = 0$$

   This translates to equalities

$$2x = 0, \quad a + d^T = (a^T + d)^T = 0, \quad b + c^T = (b^T + c)^T = 0, \quad q + m^T = (m + q^T)^T = 0, \quad p + p^T = 0, \quad n + n^T = 0$$

   This gives $x = 0$. Fixing a and b fixes d and c, of which there are $2n$ choices. Fixing m fixes q, of which there are n^2 choices. It is also required that both n and p are anti-symmetric matrices, i.e. they have zeros on the diagonal, and fixing upper triangle elements fixes the whole matrix, of which there are $\frac{1}{2}n(n-1)$ choices. Then

$$\dim_{\mathbb{F}} B_n = 2n + n^2 + 2 \cdot \frac{1}{2}n(n-1) = 2n^2 + n$$

4. Let $V = \mathbb{F}^{2n}$. The **(even) orthogonal algebra** D_n is

$$D_n = \mathfrak{o}_{2n}(V) = \{g \in \mathfrak{gl}_{2n}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$$

Considering its dimension, use the similar strategy as above. Write $g \in \mathfrak{gl}_{2n}(V)$ in block form:

$$g = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \quad \Longrightarrow \quad sg + g^T s = \begin{pmatrix} p + p^T & q + m^T \\ m + q^T & n + n^T \end{pmatrix} = 0$$

The equality translates to the following conditions

$$p + p^T = 0, \quad q + m^T = (m + q^T)^T = 0, \quad n + n^T = 0$$

Fixing p fixes m, and both p and n are anti-symmetric matrices, which have dimension $\frac{1}{2}n(n-1)$ over $\mathbb{F}$. This gives

$$\dim_{\mathbb{F}} D_n = n^2 + 2 \cdot \frac{1}{2}n(n-1) = 2n^2 - n$$

2 Analogy with other Algebraic Structures

Now we turn to discuss the algebraic structures of Lie algebra. Being an algebra itself, we are interested in some special properties, analogous to the study of groups or rings. In general, some of the definitions and theorems can be rephrased in a categorical sense, and thereby applies to all such similar structures.

Definition 2.1 ($\mathbb{F}$-Algebra)**.** Let $\mathbb{F}$ be a field. An $\mathbb{F}$-algebra is a vector space U over $\mathbb{F}$ equipped with a bilinear map $(\cdot) : U \times U \to U$, denoted $(u_1, u_2) \mapsto u_1 u_2$.

Remark 2.2. In this sense, the Lie algebra L (by definition is an $\mathbb{F}$-vector space) is indeed an $\mathbb{F}$-algebra (compatible with its name), with the bilinear map $(\cdot)$ being the Lie bracket.

Definition 2.3 (Derivation)**.** Let U be a vector space over field $\mathbb{F}$. A **derivation** of U is $S \in \mathfrak{gl}(U)$ s.t. for all $a, b \in U$, $S(ab) = aS(b) + S(a)b$ (i.e. satisfies the Leibniz Rule). The set of derivations on U is denoted by $\mathrm{Der}(U)$.

Remark 2.4. $\mathrm{Der}(U)$ is a subalgebra of $\mathfrak{gl}(U)$. By definition, it suffices to verify that for all $S, T \in \mathrm{Der}(U)$, $[S, T] \in \mathrm{Der}(U)$.

Check that the Leibniz rule is satisfied:

$$\begin{aligned}[S,T](ab) &= S(T(ab)) - T(S(ab)) \\ &= S(aT(b) + T(a)b) - T(aS(b) + S(a)b) \\ &= aS(T(b)) + bS(T(a)) - aT(S(b)) - bT(S(a)) \\ &= a[S,T](b) + b[S,T](a)\end{aligned}$$

Definition 2.5 (Adjoint Representation). The map $\mathrm{ad} : L \to \mathrm{Der}(L)$ is the **adjoint representation**, defined by $\mathrm{ad}(x)(y) = [x, y]$

An immediate thought is to verify that indeed $\mathrm{ad}(x)$ gives a derivation. Recall that the product defined in Lie algebra is the Lie bracket. Therefore, to verify the derivation property it suffices to check whether we have the equality

$$\mathrm{ad}(x)([y,z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]$$

Applying the definition and manipulating the terms, we get the Jacobi Identity, which verifies the equality:

$$\begin{aligned}\mathrm{ad}(x)([y,z]) = [x,[y,z]] = -[y,[z,x]] - [z,[x,y]] = -[y, \mathrm{ad}(x)(z)] - [z, \mathrm{ad}(x)(y)] \\ = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]\end{aligned}$$

Definition 2.6 (Structural Constants). Let $\{x_1, \dots, x_n\}$ be a basis of L as an $\mathbb{F}$-vector space. The **structural constants** of L are the coefficients c_{ij}^k s.t. $[x_i, x_j] = \sum_{k=1}^n c_{ij}^k x_k$.

It is clear that the structural constants are specified solely by the definition of Lie bracket, which is the sole extra structure given to any Lie algebra apart from the underlying vector space structure.

Remark 2.7. Using the structural constants we can rewrite the Jacobi Identity. Given a basis of L over $\mathbb{F}$ and its corresponding structural constants $a_{ij}^k \in \mathbb{F}$, the Jacobi Identity can be written as

$$\sum_{k=1}^n \left(a_{ij}^k a_{k\ell}^m + a_{j\ell}^k a_{ki}^m + a_{\ell i}^k a_{kj}^m\right) = 0, \quad \forall i, j, \ell, m \in \{1, \dots, n\}$$

Definition 2.8 (Abelian). A Lie algebra L is **abelian** if $[x, y] = 0$ for all $x, y \in L$.

Definition 2.9 (Ideal). Given a Lie algebra L, a subspace $I \subseteq L$ is an **ideal** if for all $x \in I$ and $y \in L$, $[x, y] \in I$.

Remark 2.10. This kind of "absorbing" property is analogous to the normal subgroup in group theory. Considering the Lie bracket as a multiplication on a ring, this is compatible with the definition of an ideal in a ring.

Remark 2.11. Given a Lie algebra L, if both I and J are ideals in L, then $I + J$, $I \cap J$ and $[I, J]$ are also ideals.

With the similar formulation of structure preserving sub-objects, we have similar structures as in group or ring theory.

Definition 2.12 (Quotient). Given a Lie algebra L and an ideal $I \subseteq L$, the **quotient** L/I is the vector space $L/I = \{x + I \mid x \in L\}$ with the Lie bracket $[x + I, y + I] = [x, y] + I$.

Definition 2.13 (Center). Given a Lie algebra L, the **center** of L is defined as $Z(L) = \{z \in L \mid [z, x] = 0 \text{ for all } x \in L\}$.

Definition 2.14 (Derived Algebra). Given a Lie algebra L, the **derived algebra** of L is $[L, L] = L' = L^{(1)}$, where $[L, L]$ can be interpreted as the set $[L, L] := \{[x, y] \mid x, y \in L\}$ By definition this is an ideal of L.

Definition 2.15 (Simple). A Lie algebra L is **simple** if $[L, L] \neq 0$ (i.e. it is non-trivial), and the only ideals of L are trivial (0 and L).

Example 2.16. Let $L = \mathfrak{gl}_n(\mathbb{F})$. Then $[e_{ij}, e_{k\ell}] = \delta_{jk} e_{i\ell} - \delta_{\ell i} e_{kj}$. Setting $k = j$ and $i = \ell$ gives $\mathrm{Tr}([e_{ij}, e_{k\ell}]) = 0$. By definitions in Example 1.9, $L^{(1)} = [L, L] \simeq \mathfrak{sl}_n(F)$. Since specifically $\mathfrak{sl}_n(\mathbb{F}) \subseteq \mathfrak{gl}_n(\mathbb{F})$ which gives an ideal, $\mathfrak{gl}_n(\mathbb{F})$ is not simple. In fact $\mathfrak{sl}_n(\mathbb{F})$ is simple, but proving this result requires more constructions.

Definition 2.17 (Normalizer). Let L be a Lie algebra and K a subalgebra of L. The **normalizer** of K is defined by $N_L(K) := \{x \in L \mid [x, K] \subseteq K\}$, the largest subalgebra of L in which K is an ideal.

Definition 2.18 (Centralizer). Let L be a Lie algebra and X an arbitrary set. The **centralizer** of X is defined by $C_L(X) := \{x \in L \mid [x, X] = 0\}$.

Definition 2.19 (Morphism). Let L and L' be two Lie algebras over $\mathbb{F}$. A **homomorphism** is an $\mathbb{F}$-linear transformation $\phi : L \to L'$ s.t. $\phi([x, y]) = [\phi(x), \phi(y)]$, i.e. it commutes with the multiplication, which is the Lie bracket here.

Adopting general categorical nomenclature, a homomorphism is a **monomorphism** if it is injective, an **epimorphism** if it is surjective, an **isomorphism** if it is bijective, and an **automorphism** if it is an isomorphism from L to itself.

Remark 2.20. With the definition of morphisms, we can easily translate common results from group or ring theory to Lie algebras. For all homomorphism $\phi : L \to L'$:

1. Both $\ker \phi$ and $\operatorname{im} \phi$ are ideals of L.

2. (First Isomorphism Theorem) $\operatorname{im} \phi \simeq L/\ker \phi$.

3. (Second Isomorphism Theorem) Given two ideals $I, J \subseteq L$, by Remark 2.11 $I + J$ is also an ideal of L. We have the isomorphism $(I + J)/J \simeq I/I \cap J$.

4. (Third Isomorphism Theorem) Let $I \subseteq J \subseteq L$ where both I and J are ideals of L. Then $L/J \simeq (L/I)/(J/I)$.

5. (Fourth Isomorphism Theorem, Correspondence) Let $I \subseteq \ker \phi$ be an ideal in $\ker \phi$. Then there exists a unique homomorphism $\psi : L/I \to L'$, i.e. making the following diagram commute:

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[dr] & L' \\ & L/I \arrow[u, dashed, "\psi"'] \end{tikzcd}$$

Definition 2.21 (Representation). Let L and L' be Lie algebras over $\mathbb{F}$, and V a vector space over $\mathbb{F}$. A **representation** of L on V is a homomorphism $\rho : L \to \mathfrak{gl}(V)$.

Example 2.22 (Adjoint Representation). For a Lie algebra L, recall that the adjoint map is $\mathrm{ad} : L \to \mathfrak{gl}(L)$, $\mathrm{ad}(x)(y) = [x, y]$. Since L is itself a vector space over $\mathbb{F}$, letting $V = L$ we have a representation. This is indeed a homomorphism as

$$\begin{aligned}[\mathrm{ad}(x), \mathrm{ad}(y)](z) &= \mathrm{ad}(x)\,\mathrm{ad}(y)(z) - \mathrm{ad}(y)\,\mathrm{ad}(x)(z) = [x, [y, z]] - [y, [x, z]] \\ &= [x, [y, z]] + [y, [z, x]] = -[z, [x, y]] = [[x, y], z] \\ &= \mathrm{ad}([x, y])(z)\end{aligned}$$

Proposition 2.23. Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof. Consider using the adjoint representation. The kernel of ad

$$\ker(\mathrm{ad}) = \{x \in L \mid \mathrm{ad}(x) = 0\} = \{x \in L \mid [x, z] = 0, \forall z \in L\} = Z(L)$$

by Remark 2.20 is an ideal in L. Since L is simple, $\ker(\mathrm{ad}) = 0$. Using First Isomorphism Theorem, $L/\ker(\mathrm{ad}) \simeq L \simeq \mathrm{im}\,(\mathrm{ad})$. Since $\mathrm{ad} : L \to \mathfrak{gl}(V)$, $\mathrm{im}\,(\mathrm{ad}) \subseteq \mathfrak{gl}(L)$. By Definition 1.7 L is a linear Lie algebra. □

Notation. It is often denoted $\mathrm{GL}(V) = \{g \in \mathfrak{gl}(V) \mid g \text{ invertible}\}$. This coincides with the general linear group.

Example 2.24 (Adjoint Representation of $\mathrm{GL}(V)$). Let $g \in \mathrm{GL}(V)$. Define $\mathrm{Ad}(g) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ by $\mathrm{Ad}(g)(x) = gxg^{-1}$, which is the adjoint representation of $\mathrm{GL}(V)$.

Definition 2.25 (Nilpotent (Element)). Assume that $\mathrm{char}\,\mathbb{F} = 0$. Let L be a Lie algebra on $\mathbb{F}$. Then $x \in L$ is **nilpotent** if $(\mathrm{ad}(x))^k = 0$ for some $k \in \mathbb{Z}_{\geq 0}$.

Definition 2.26 (Exponential Map). Let L be a Lie algebra, and $x \in L$ nilpotent of order k. Then the **exponential map** is defined by

$$\exp(\mathrm{ad}(x)) = \sum_{n=0}^{k-1} \frac{(\mathrm{ad}(x))^n}{n!}$$

which coincides with the Taylor expansion. ALl terms of order k vanishes due to x being k-nilpotent.

Lemma 2.27. $\exp(\mathrm{ad}(x)) \in \mathrm{Aut}(L)$. More generally, if $\delta \in \mathrm{Der}(L)$ is nilpotent, then $exp(\delta) \in \mathrm{Aut}(L)$.

Proof. We seek to prove only the general version of the statement. First verify that ad is a derivation, i.e. we have the equality $\mathrm{ad}([x, y])(z) = ([x, \mathrm{ad}(y)] + [\mathrm{ad}(x), y])(z)$ for all $x, y, z \in L$. This is a direct consequence of the Jacobi Identity:

$$\mathrm{ad}([x, y])(z) = [[x, y], z] = [x, [y, z]] - [y, [z, x]] = [x, \mathrm{ad}(y)(z)] - [y, \mathrm{ad}(x)(z)] = ([x, \mathrm{ad}(y)] + [\mathrm{ad}(x), y])(z)$$

Now for the general statement on derivations, first verify that δ is a homomorphism. Write out the expressions explicitly:

$$\begin{aligned}
\exp(\delta(x))\exp(\delta(y)) &= \left(\sum_{n=0}^{k-1}\frac{\delta^n(x)}{n!}\right)\left(\sum_{m=0}^{k-1}\frac{\delta^m(y)}{m!}\right) \\
&= \sum_{n=0}^{2k-2}\left(\sum_{i=0}^{n}\frac{\delta^i(x)}{i!}\cdot\frac{\delta^{n-i}(y)}{(n-i)!}\right) \\
&= \sum_{n=0}^{2k-2}\frac{\delta^n(xy)}{n!} && \text{(Leibniz Rule)} \\
&= \sum_{n=0}^{k-1}\frac{\delta^n(xy)}{n!} = \exp(\delta(xy)) && (\delta^k = 0)
\end{aligned}$$

The fact that $\exp(\delta)$ is a bijection can be verified via writing out its inverse: let $\exp(\eta) = \mathrm{Id} - \eta$. Then $\exp^{-1} = \sum_{n=0}^{k-1}\eta^n$, where we designate $\eta^0 = \mathrm{Id}$. □

Example 2.28. Let $\mathbb{F}$ be a field of characteristic 0, and $L = \mathbb{F}^2$ with basis $\{v_1, v_2\}$. Endow it with a Lie algebra structure by setting $[v_1, v_2] = v_1$. Then $\mathrm{ad} \in \mathfrak{gl}_2(L)$, with the matrix representation

$$\mathrm{ad}(v_1) = \left(\begin{array}{c|c} \mathrm{ad}(v_1)(v_1) & \mathrm{ad}(v_1)(v_2) \end{array}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathrm{ad}(v_1)^2 = 0$$

Since ad is 2-nilpotent,

$$\exp(\mathrm{ad}(v_1)) = \mathrm{Id}_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Using the same notation as in the proof above,

$$\eta = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \exp^{-1}(\mathrm{ad}(v_1)) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

3 Solvable and Nilpotent Lie Algebras

Definition 3.1 (Derived Series). Given a Lie algebra L, the **derived series** of L is the sequence of ideals

$$L^{(0)} = L, \quad L^{(1)} = [L, L], \quad \cdots \quad L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$$

Remark 3.2. Similar to the cases in group or ring theory, the relation of ideals is in general not transitive. That is, directly from the expression we know that for all n, $L^{(i)}$ is an ideal in $L^{(i-1)}$; but in general $L^{(n)}$ is not an ideal in L for $n \in \mathbb{Z}_{\geq 2}$.

Definition 3.3 (Solvable). A Lie algebra L is **solvable** if $L^{(n)} = 0$ for some n.

Example 3.4. Fix $\mathbb{F}$ to be a field. Let $\mathfrak{t}_n(\mathbb{F})$ be upper triangular n-by-n matrices over $\mathbb{F}$, $\mathfrak{d}_n(\mathbb{F})$ be diagonal n-by-n matrices and $\mathfrak{n}_n(\mathbb{F})$ strictly upper triangular n-by-n matrices. Then $\mathfrak{d}_n(\mathbb{F})$ is solvable since in general diagonal matrices commutes with all matrices in terms of multiplication (i.e. lies in the center of the Lie algebra). $\mathfrak{n}_n(\mathbb{F})$ is also solvable since strictly upper triangular matrices are nilpotent. Furthermore, as vector spaces $\mathfrak{t}_n(\mathbb{F}) = \mathfrak{d}_n(\mathbb{F}) \oplus \mathfrak{n}_n(\mathbb{F})$, giving $\mathfrak{t}_n(\mathbb{F})$ solvable.

In general we can express the k-th derived algebra of $\mathfrak{t}_n(\mathbb{F})$ as

$$\mathfrak{t}_n(\mathbb{F})^{(k)} = \{(g_{ij}) \mid g_{ij} = 0 \forall i + k - 1 \geq j\}$$

which is nilpotent of order $n - 1$.

Example 3.5. All simple Lie algebras are non-solvable. By definition $[L, L]$ is an ideal in L; and L being simple implies that $[L, L]$ must be either 0 or L. However being simple also requires that L is not abelian. Therefore $L^{(1)} = [L, L] = L$, giving $L^{(k)} = L$ for all k and therefore is non-solvable. One of such examples is $\mathfrak{sl}_n(\mathbb{F})$ (also mentioned but not proved in 2.16).

Remark 3.6. Similar to solvability in group theory, we have the following results:

1. If L is solvable, then so are its subalgebras and $\operatorname{im} \phi$ for all $\phi : L \to L'$ homomorphisms.
2. For $I \subseteq L$ an ideal, if both I and L/I are solvable, then L is solvable.
3. If I and J are solvable ideals in L, then $I + J$ is also solvable.

Corollary 3.7. If L is solvable, then L has a unique solvable ideal, called the **radical ideal**, denoted $\operatorname{Rad} L$.

Proof. Let S be a maximal solvable ideal of L, and I any solvable ideal of L. There fore $S \subseteq I + S$, and by Remark 3.6 $I + S$ is solvable. Maximality of S implies $I + S = S$, and therefore $I \subseteq S$ which verifies the uniqueness. Existence follows from the fact that the zero ideal is solvable. □

Definition 3.8 (Semisimple (Algebra))**.** A non-zero Lie algebra L is **solvable** if $\operatorname{Rad}(L) = 0$.

Remark 3.9. We have the following properties directly resulting from the definition:

1. A simple Lie algebra is semisimple (as expected), as the radical ideal is an ideal.
2. If a Lie algebra L is not solvable, then $L/\operatorname{Rad} L$ is semisimple, as by Correspondence Theorem a solvable ideal $J \subseteq L/\operatorname{Rad} L$ corresponds to a solvable ideal $I \subseteq L$ containing $\operatorname{Rad} L$. From definition of radical ideals $I = J$, and therefore $L/\operatorname{Rad} L$ is semisimple.

Definition 3.10 (Descending (Lower) Central Series)**.** Given a Lie algebra L, its **descending central series** (or **lower central series**) is the sequence of ideals

$$L^0 = L, \quad L^1 = [L, L], \quad \cdots \quad L^n = [L, L^{n-1}]$$

The notation is consistent with the treatment of Lie brackets in Lie algebras as multiplication.

Remark 3.11. Different from the every term in the descending central series is an ideal in L.

With the notion of "raising a power" for Lie algebras, we can extend the definition of nilpotency:

Definition 3.12 (Nilpotent (Lie Algebra))**.** A Lie algebra L is **nilpotent** if $L^n = 0$ for some n.

Example 3.13. Revisiting the example for derived series (Example 3.4) with the same notation, although similarly both $\mathfrak{n}_n(\mathbb{F})$ and $\mathfrak{d}_n(\mathbb{F})$ are nilpotent, $\mathfrak{t}_n(\mathbb{F})$ is not nilpotent, as one can always fill in a diagonal matrix for L. The explicit expression for $\mathfrak{n}_n(\mathbb{F})$ agrees the nilpotency of upper triangular matrices:

$$\mathfrak{n}_n(\mathbb{F})^k = \{(g_{ij}) \mid g_{ij} = 0 \forall i + k \geq j\}$$

Remark 3.14. Similarly the properties of nilpotent groups extend to nilpotent Lie algebras:

1. If a Lie algebra L is nilpotent, and all its subalgebras and $\operatorname{im}\phi$ for all $\phi : L \to L'$ homomorphism are nilpotent.
2. If $L/Z(L)$ is nilpotent, then L is nilpotent.
3. If $L \neq 0$ is nilpotent, then $Z(L) \neq 0$.

Definition 3.15 (Ad-nilpotent)**.** Given a Lie algebra L, an element $x \in L$ is **ad-nilpotent** if $\operatorname{ad}(x)$ is nilpotent.

Lemma 3.16. If L is nilpotent, then for all $x \in L$, x is ad-nilpotent.

Proof. Since L is nilpotent, there exists n s.t. $L^n = 0$. Since $\operatorname{ad}(x)^n(y) \in L^n$, $\operatorname{ad}(x)^n = 0$. □

We now seek prove the converse. From now on, assume that $\mathbb{F}$ is a field of characteristic and algebraically closed. Our main focus will be Lie algebras over such fields. For simplicity, when referring to the dimension of a Lie algebra, we will simply use $\dim(-)$ instead of $\dim_{\mathbb{F}}(-)$.

Theorem 3.17 (Engel)**.** Let L be a Lie algebra. If for all $x \in L$, x is ad-nilpotent, then L is nilpotent.

Lemma 3.18. If $x \in \mathfrak{gl}(V)$ is a nilpotent endomorphism, then $\operatorname{ad}(x)$ is nilpotent.

Proof. This can be verified via writing out the expression explicitly. Using induction, we have

$$\operatorname{ad}^n(x)(y) = \sum_{i=0}^{n} \binom{n}{i} x^i y x^{n-i}$$

Suppose that x is k-nilpotent. Choosing $n = 2k$ makes $\operatorname{ad}^n(x)$ vanish. □

To prove Engel's theorem, we first prove the following theorem:

Theorem 3.19. Let L be a subalgebra of $\mathfrak{gl}(V)$, for any vector space V s.t. $\dim V$ is finite. If L consists of nilpotent endomorphisms and $V \neq 0$, then there exists a nonzero $v \in V$ s.t. $L \cdot v = 0$, i.e. 0 is always an eigenvalue of L.

Remark 3.20. This is the analogy of a similar (yet under that context much easier to prove) result in linear algebra: given a homomorphism that is not surjective, the kernel is non-trivial; and therefore the dimension of the image is strictly less than the dimension of the domain.

Proof. Apply induction on the dimension of L:

- $\dim L = 0$. Then any nonzero $v \in V$ works (the result is vacuous).

- $\dim L = 1$. Then L can be written as the linear span of a single element $L = \text{span}\{\ell\}$. ℓ being nilpotent implies that its only eigenvalue is 0. Therefore for any v being an eigenvector of L, $\ell v = 0 \implies L \cdot v = 0$.

- *Inductive Step.* Let $K \subsetneq L$ be a maximal subalgebra. (This can be 0, which degenerates to the case of $\dim L = 0$) $K \neq L$ implies that $\dim K < \dim L$, giving $\dim(L/K) < \dim L$. Now take $V = L/K$, and apply the inductive hypothesis: there exists a nonzero $v + K \in L/K$ s.t. $(L/K) \cdot (v + K) = 0$, i.e. $L \cdot v \in K$. Recall that the action of Lie algebra on itself is via adjoint representation, i.e. we have $[y, x] \in K$ for all $y \in K$ and $x \in L$. Since K is maximal in L, $\dim K = \dim L - 1$. Therefore, we can write $L = \text{span}\{K, x\}$. By the inductive hypothesis, there exists a nonzero $v \in V$ s.t. $K \cdot v = 0$. Denote all such elements by $W := \{v \in V \mid K \cdot v = 0\}$. We now seek to prove that this set is also annihilated by L.

  First observe that for all $k \in K, w \in W$, $kxw = x(kw) - [x,k]w$. Inductive hypothesis on K implies that $kw = 0$ and therefore $x(kw) = 0$; and inductive hypothesis on L/K implies that $[x,k]w = 0$ for all $x \in L$. By definition of W, this gives $x \cdot w \in W$, i.e. $\text{span}\{x\}$ is a subalgebra of $\mathfrak{gl}(W)$. Since $\text{span}\{x\}$ is 1-dimensional, IH implies that there exists $w_0 \in W$ s.t. $x \cdot w_0 = 0$. Then decomposing application of L on w_0 gives $L \cdot w_0 = K \cdot w_0 + \mathbb{F}x \cdot w_0 = 0$.

□

Using this theorem it is then straightforward to prove Engel's theorem:

Proof of Theorem 3.17. Similarly apply induction on the dimension of L:

- $\dim L = 1$. In this case, L is abelian, i.e. $L^2 = [L, L] = 0$. By definition L is nilpotent.

- *Inductive step.* Suppose that the result holds for V s.t. $\dim V \leq n$. By hypothesis, for all $x \in L$, $\text{ad}(x)$ is nilpotent. Use the notation of Theorem 3.19, letting $V = L$ gives that $\text{ad}(L)$ is a subalgebra of $\mathfrak{gl}(L) = \mathfrak{gl}(V)$. The result of Theorem 3.19 gives that there exists $x \in L$ s.t. $\text{ad}(L)(x) = [L, x] = 0$. This implies that $Z(L) \ni x$ is nonzero. Since $Z(L)$ is a vector subspace, $\dim(L/Z(L)) < \dim L$. Inductive hypothesis gives that under the hypothesis, $\dim(L/Z(L))$ is nilpotent. Furthermore, $Z(L)$ is nilpotent by definition (as $(Z(L))^2 = 0$). By Remark 3.6, L is nilpotent.

□

Definition 3.21 (Flag). Assume that $\dim V$ is finite. A **flag** in V is a sequence of subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V \quad \text{s.t.} \quad \dim V_i = i$$

Definition 3.22 (Stabilizer on Flags). An element $x \in \text{End}(V)$ **stabilizes** a flag if $x(V_i) \subseteq V_i$ for all i.

Using "flags" we can describe the theorem above in a language more resembling that in vector spaces:

Corollary 3.23. If $L \in \mathfrak{gl}(V)$ consists of nilpotent endomorphisms, then there exists a flag (V_i) stable under L, i.e. in such basis $L \in \mathfrak{n}_n(\mathbb{F})$, which can be expressed by a strictly upper triangular matrix.

Proof. It suffices to give a construction of such flag. Theorem 3.19 implies that there exists nonzero $v \in V$ s.t. $L \cdot v = 0$. Set $V_1 = \mathbb{F}v$ and $W = V/V_1$. Repeat the process until $W = 0$; and "flatten" the quotient spaces by taking the preimage of the quotient map, and consider the elements as vector subspaces of V. □

Lemma 3.24. Let $L \neq 0$ be a nilpotent Lie algebra, and $I \subseteq L$ a nonzero ideal. Then $I \cap Z(L) \neq 0$.

Proof. L being nilpotent implies that the derived algebra $L' = [L, L] \subsetneq L$, i.e. $Z(L) \neq 0$. Consider again the action of L on itself via adjoint. Apply again Theorem 3.19 with $V = I$, there exists $x \in I$ s.t. $\operatorname{ad}(L)(x) = 0$, i.e. $x \in Z(L)$. This gives a nonzero element in $I \cap Z(L)$. □

Theorem 3.25. Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$, and assume that $\dim V$ is finite. If $V \neq 0$, then V contains a common eigenvector (but the eigenvalue may change), i.e. there exists $v \in V$ s.t. for all $\ell \in L$, $\ell v = \lambda(\ell)v$ where v is an eigenvector of ℓ.

Proof. Similar to the proof of the previous result, proceed via applying induction on $\dim L$. The case of $\dim L = 1$ is given by any scalar.

First sketch the framework of the proof:

- First define an ideal of codimension 1 in L. Using the inductive hypothesis to get a common eigenvector.
- Verify that L stabilizes a certain space containing that eigenvector.
- Find an eigenvector z in that space s.t. $L + I + \mathbb{F}z$.

Now proceed the proof:

- **Step 1**. Since $\dim L > 0$ and L is solvable, $[L, L] \subsetneq L$ (otherwise the derived series will have identical terms and thus do not terminate). By definition $L/[L, L]$ is abelian.[1] Then any subspace $I \subseteq L/[L, L]$ will be an ideal. Take $I' \subseteq L/[L, L]$ be an ideal of codimension 1. Take its preimage $I \subsetneq L$ of the quotient map, which is an ideal of codimension 1 in L. Inductive hypothesis implies that there exists nonzero $v \in V$ s.t. for all $x \in I$, $x \cdot v = \lambda(x)v$. Fix λ and define $W := \{w \in V \mid x \cdot w = \lambda w, \forall x \in I\}$. Then $v \in V$ implies that $W \neq 0$, which is a nontrivial space fixed by I.

- **Step 2**. We now want to show that the space is actually fixed by $L \supset I$. Use the same trick as the one in the previous proof: Let $w \in W, x \in L$; by definition, for all $y \in I$ we have the equality

  $$(yx)w = xyw - [x, y]w = \lambda(y)xw - \lambda([x, y])w \tag{$*$}$$

  To show that $xw \in W$, it suffices to show that $yxw = \lambda(y)xw$, i.e. $\lambda([x, y]) = 0$. Let $n > 0$ be the smallest integer s.t. $\{w, xw, \cdots, x^{n-1}w\}$ are linearly independent. Define $W_i = \operatorname{span}\left\{w, xw, \cdots, x^{i-1}w\right\}$. It is clear that there are finitely many distinct W_is, as $W_{n+j} = W_n$ for all $j \in \mathbb{Z}_{\geq 0}$; and $xW_n \subseteq W_n$. Eq. ($*$) implies that $yx^i w = \lambda(y)x^i w$ for all i, i.e. $IW_n \subseteq W_n$. That is, y is an upper triangular matrix w.r.t. the basis $\{w, xw \cdots, x^{n-1}w\}$; and diagonal entries are all $\lambda(y)$, i.e. for all $y \in I, \operatorname{Tr}_{W_n}(y) = n\lambda(y)$. Since I is an ideal, $y \in I \implies [x, y] \in I$, i.e. $\operatorname{Tr}_{W_n}([x, y]) = \lambda([x, y])$. However, $\operatorname{Tr}_{W_n}([x, y]) = 0$ as $\operatorname{Tr}_{W_n}(xy) = \operatorname{Tr}_{W_n}(yx)$ for al $x, y \in L$. This thus gives the desired result $\lambda([x, y]) = 0$.

- **Step 3**. Since I is codimension-1, we can write $L = I + \mathbb{F}z$ for some $z \in L$. Since $\mathbb{F}$ is assumed to be algebraically closed, there exists eigenvector $v_0 \in W$ s.t. $zv_0 = cv_0$ for all $z \in L$ with the corresponding $c \in \mathbb{F}$. Furthermore by definition of W, for all $y \in I$, $y \cdot v_0 = \lambda(y)v_0$, i.e. v_0 is a common eigenvector of L.

[1] This quotient is to ensure that there exists such a codimension-1 subspace.

□

With the above result, we can state the following theorem (as a corollary):

Theorem 3.26 (Lie). Let L be a solvable Lie subalgebra of $\mathfrak{gl}(V)$ with $\dim V$ finite. Then L stabilizes a flag of V, i.e. there exists a basis of V for which L can be represented as an upper triangular matrix.

Proof. Construct the basis by repeatedly including the common eigenvectors of L. □

Corollary 3.27. Let L be an n-dimensional Lie algebra. Then there exists ideals $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = L$ s.t. $\dim I_i = i$.

Proof. Consider $\mathrm{ad} : L \to \mathfrak{gl}(L)$. By Theorem 3.26, $\mathrm{ad}(L)$ stabilizes a flag of L. Definition of ad ensures that they are ideals. □

4 Semisimple Lie Algebras

Definition 4.1 (Semisimple (Endomorphism)). Let V be a finite dimensional vector space over $\mathbb{F}$. $x \in \mathrm{End}(V)$ is **semisimple** if the roots of its minimal polynomial over $\mathbb{F}$ are distinct. Assuming $\mathbb{F}$ is algebraically closed, this is equivalent to x being diagonalizable.

Proposition 4.2 (Jordan Decomposition). For every $x \in \mathrm{End}(V)$, there exists unique $x_s, x_n \in \mathrm{End}(V)$ s.t. $x = x_s + x_n$ where x_s is semisimple, x_n is nilpotent, and $x_s x_n = x_n x_s$. Furthermore, we have the following properties:

1) There exists $p, q \in \mathbb{F}[T]$ s.t. $p(0) = q(0) = 0$; and $x_s = p(x)$, $x_n = q(x)$ for that specific $x \in \mathrm{End}(V)$ above.

2) If we have $A \subseteq B \subseteq V$ subspaces, and $xB \subseteq A$, then both $x_s B \subseteq A$, and $x_n B \subseteq A$.

Proof. Let $a_1, \ldots, a_k$ be distinct eigenvalues of x, with multiplicities $m_1, \ldots, m_k$. By definition the characteristic polynomial is given by $\mathrm{char}(x) = \prod_{i=1}^k (T - a_i)^{m_i}$. This allows decomposing V into eigenspaces: denoting $V_i = \ker(T - a_i)^{m_i}$, we have $V = \oplus_{i=1}^k V_i$; and x preserves V_i. Furthermore, the characteristic polynomial of the endomorphism of x on V_i is given by the corresponding part of the characteristic polynomial $\mathrm{char}_{\ V_i}(x) = (T - a_i)^{m_i}$.

By Chinese Remainder Theorem gives the isomorphism

$$\mathbb{F}[T]/\,\mathrm{char}(x) \simeq \prod_{i=1}^k \mathbb{F}[T]/\,\mathrm{char}_{V_i}(x)$$

This is applicable as the eigenvalues a_i are distinct, indicating that the ideals generated by $\mathrm{char}_{V_i}(x)$ respectively are coprime. Furthermore there exists $p \in \mathbb{F}[T]$ s.t. for all i, $p \equiv a_i \pmod{\mathrm{char}_{V_i}(x)}$; and $p \equiv 0 \pmod{T}$. The second condition can be achieved via noticing that the ideals generated by T and $(T - a_i)^{m_i}$ are coprime if $a_i \neq 0$; otherwise the second condition is implied by the first one.

Set $q := \mathrm{Id}_{\mathbb{F}[x]} - p$ (where we view $x \in \mathbb{F}[x]$), and let $x_s = p(x)$, $x_n = q(x)$. Since we require $p \equiv 0 \pmod{T}$, $p(0) = 0$. Then $q(0) = \mathrm{Id}_{\mathbb{F}[x]}(0) - p(0) = 0$. This also implies 2). Since p and q commute (as elements of $\mathbb{F}[T]$ commute), x_s and x_n commute.

By definition of eigenspaces, V_i is stabilized by x for all i. This in particular implies that both $p(x)$ and $q(x)$ stabilize V_i for all i. As previously we have shown that $p \equiv a_i \pmod{\mathrm{char}_{V_i}(x)}$, $p(x)|_{V_i} = a_i \mathrm{Id}_{V_i}$, i.e. the matrix representation of p consists of only diagonal entries, which implies that p is semisimple. By definition, $x_n = x - x_s$, giving $x_n^m = (x - x_s)^m \equiv (x - a_i)^m \pmod{V_i}$, giving that x_n is m_i-nilpotent when restricted to V_i. Setting $\widetilde{m} = \max_i\{m_i\}$, we have x_n is $\widetilde{m}$-nilpotent.

It then remains to show that the decomposition $x = x_s + x_n$ is unique. Suppose that we have another decomposition $x = x'_s + x'_n$, then $x_s - x'_s = x'_n - x_n$, which is both semisimple and nilpotent. This can only be represented by the zero matrix, i.e. $x_s = x'_s$, and $x_n = x'_n$. □

Lemma 4.3. Similar to Lemma 3.18, if $x \in \mathfrak{gl}(V)$ is semisimple for some vector space V, then so is $\mathrm{ad}(x)$.

Proof. Use the fact that semisimple endomorphisms are diagonalizable. Fix a basis $\{v_1, \dots, v_n\}$ of V s.t. the matrix representation of x is $\mathrm{diag}(a_1, \dots, a_n)$. Denote the basis of $\mathfrak{gl}(V)$ by $\{e_{ij}\}$. Notice $\mathrm{ad}(x)(e_{ij}) - (a_i - a_j)e_{ij}$, which implies that $\mathrm{ad}(x)$ is diagonalizable. □

Lemma 4.4. Let $x \in \mathrm{End}(V)$ with its Jordan Decomposition given by $x = x_s + x_n$. Then the Jordan Decomposition of $\mathrm{ad}(x)$ is given by $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$.

Proof. By Lemma 3.18 amd 4.3, we know $\mathrm{ad}(x_s)$ is semisimple, and $\mathrm{ad}(x_n)$ is nilpotent. From Proposition 4.2 we know that such $\mathrm{ad}(x_s)$ and $\mathrm{ad}(x_n)$ are unique as long as we can show that they commute. This is indeed the case as $\mathrm{ad}(\cdot)$ gives a representation, i.e. the commutator is given by the Lie bracket

$$[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = \mathrm{ad}([x_s, x_n]) = \mathrm{ad}(0) = 0$$

as x_s and x_n commute. □

Lemma 4.5. Let $A \subseteq B \subseteq \mathfrak{gl}(V)$ be subspaces, and define $M := \{x \in \mathfrak{gl}(V) \mid [x, B] \subseteq A\}$. For $x \in M$, if for any $y \in M$ we have $\mathrm{Tr}(xy) = 0$, then x is nilpotent.

Proof. Try to use Jordan Decomposition, by showing that the semisimple component is zero. Write $x = x_s + x_n$ as Jordan Decomposition. Fix a basis $\{v_1, \dots, v_n\}$ of V s.t. x_s can be represented as a diagonal matrix $\mathrm{diag}(a_1, \dots, a_n)$. SInce we work under the setting of $\mathrm{char}\,\mathbb{F} = 0$, $\mathbb{Q} \subseteq \mathbb{F}$. Let E be the subspace spanned by the eigenvalues $a_1, \dots, a_n$ over $\mathbb{Q}$, in $\mathbb{F}$. We need to show $E = 0$. Since V is finite dimensional, it suffices to show that its dual $E^* := \mathrm{Hom}(E, \mathbb{Q}) = 0$ as we have the isomorphism $E \simeq E^*$.

Let $f \in E^*$, and $y \in \mathfrak{gl}(V)$ s.t. $y = \mathrm{diag}(f(a_1), \dots, f(a_n))$. Let $\{e_{ij}\}$ be the standard basis of $\mathfrak{gl}(V)$. In the proof of Lemma 4.3, we have shown that $\mathrm{ad}(x_s)(e_{ij}) = (a_i - a_j)e_{ij}$, giving $\mathrm{ad}(y)(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$. By Lagrange interpolation, there exists $r \in \mathbb{F}[T]$ s.t. $r(a_i - a_j) = f(a_i - a_j)$. By Lemma 4.3, $\mathrm{ad}(x_s)$ gives the semisimple component of $\mathrm{ad}(x)$, i.e. there exists $p \in \mathbb{F}[T]$ s.t. $p(\mathrm{ad}(x)) = \mathrm{ad}(x_s)$, and $p(0) = 0$ (by Proposition 4.2). Applying r on both side gives

$$r(p(0)) = 0, \quad r(p(\mathrm{ad}(x)))(e_{ij}) = r(\mathrm{ad}(x_s))(e_{ij}) = (f(a_i) - f(a_j))(e_{ij}) = \mathrm{ad}(y)e_{ij} \implies r(p(\mathrm{ad}(x))) = \mathrm{ad}(y)$$

Now use the hypothesis: suppose that $x \in M$, then we have $[x, B] \subseteq A \implies \mathrm{ad}(x)(B) \subseteq A$. Furthermore, since $r(p(0)) = 0$, i.e. $r \circ p$ has no constant term, $\mathrm{ad}(y) = (r \circ p)(\mathrm{ad}(x))$ applies at least once $\mathrm{ad}(x)$, giving $\mathrm{ad}(y)(B) = (u(\mathrm{ad}(x)))\,\mathrm{ad}(x)(B) \subseteq$

$(u(\mathrm{ad}(x)))(A) \subseteq A$ for some $u \in \mathbb{F}[T]$, as $A \subseteq B$ implies that $\mathrm{ad}^n(x)(B) \subseteq \mathrm{ad}^{n-1}(x)(A) \subseteq \mathrm{ad}^{n-1}(x)(B)$. This then gives $y \in M$, which by hypothesis implies that $\mathrm{Tr}(xy) = 0$.

Recall we have chosen the basis s.t. in matrix representation, $x = \mathrm{diag}(a_1, \dots, a_n)$ and $y = \mathrm{diag}(f(a_1), \dots, f(a_n))$. The condition $\mathrm{Tr}(xy) = 0$ then translates to

$$0 = \sum_{i=1}^{n} a_i \cdot f(a_i) = f\left(\sum_{i=1}^{n} a_i \cdot f(a_i)\right) = \sum_{i=1}^{n} f(a_i) \cdot f^2(a_i) = \sum_{i=1}^{n} (f(a_i))^2 \quad \Longrightarrow \quad f(a_i) = 0, \forall i$$

This then gives $f = 0$. □

> **Theorem 4.6** (Cartan's First Criterion). Let $L \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra. If for all $x \in [L, L]$, $y \in L$, $\mathrm{Tr}(xy) = 0$, then L is solvable.

Proof. By definition of solvability, it suffices to prove that $[L, L]$ is nilpotent. Engel's Theorem implies that it is sufficient to prove that every $x \in [L, L]$ is nilpotent (as elements in a Lie algebra acts on each other via Lie bracket).

Now use the result of Lemma 4.5, with $B = L$, $A = [L, L]$, with $M = \{x \in \mathfrak{gl}(V) \mid [x, L] \subseteq [L, L]\}$. Clearly $L \subseteq M$. To apply the result for lemma, we need to show that for all $x \in [L, L], z \in M$, $\mathrm{Tr}(xz) = 0$. That is, for all $a, b \in L$, we need to show $\mathrm{Tr}([a, b]z) = 0$. Notice

$$\mathrm{Tr}([a, b]z) = \mathrm{Tr}(abz - baz) = \mathrm{Tr}(abz) - \mathrm{Tr}(b(az)) = \mathrm{Tr}(abz) - \mathrm{Tr}((az)b) = \mathrm{Tr}(a[b, z]) = \mathrm{Tr}([b, z]a) \tag{1}$$

by using the property that for all A, B, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Hypothesis gives $\mathrm{Tr}([z, b]a) = 0$ by definition of W. Since $[z, b] = -[b, z]$, from Eq. (1) we have $\mathrm{Tr}([a, b]z) = 0$. Lemma 4.5 gives the desired result. □

> **Corollary 4.7.** Let L be a Lie algebra s.t. $Tr(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ for all $x \in [L, L]$ and $y \in L$. Then L is solvable.

Proof. Let $L' = \mathrm{ad}(L)$. This is solvable by applying Cartan's First Criterion. Further $\ker \mathrm{ad}(L) = Z(L)$ which is also solvable as $[Z(L), Z(L)] = 0$. Remark 3.6 implies that L is solvable. □