MATH 538 - Root System

ARessegetes Stery

October 16, 2024

Contents

1 Root System

Definition 1.1 (Euclidean Space). A vector space E is an **Euclidean space** if it is finite dimensional over $\mathbb{R}$, with an inner product (bilinear, and $(v, v) \geq 0$ for all v) $(-, -)$.

Notation. Throughout this chapter, we will use parenthesis $(\cdot, \cdot)$ to denote the inner product.

Definition 1.2 (Root System; Roots). A **root system** $\Phi \subseteq E$ is a finite set in E s.t.

1. $0 \notin \Phi$ (in particular Φ cannot be a subspace of E), and $\operatorname{span}\{\Phi\} = E$.
2. If $\alpha \in \Phi$, then the only real multiple of α in Φ is $\pm\alpha$.
3. For all $\alpha, \beta \in \Phi$,
$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}.$$
4. If $\alpha, \beta \in \Phi$, then $\sigma_\alpha(\beta) := \beta - \langle \beta, \alpha \rangle \alpha \in \Phi$.

Elements in a root system are called **roots**.

Definition 1.3 (Orthogonal Hyperplane). The **orthogonal hyperplane** of $\alpha \in E$ is $P_\alpha := \{\beta \in E \mid (\beta, \alpha) = 0\}$.

Definition 1.4 (Reflection). The fourth condition, $\sigma_\alpha(\beta)$ is also called the **reflection**, which essentially reflects β w.r.t. the orthogonal hyperplane of α.

Remark 1.5. By the fourth axiom in the definition, for a root system Φ and any root α in it, $\sigma_\alpha(\Phi) = \Phi$.

Example 1.6 (Root System A_2). Consider the root system $A_2 \subseteq \mathbb{R}^2$ given by $A_2 = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$:

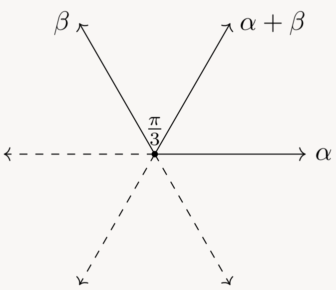

Figure 1: Visualization of the root system A_2 in $\mathbb{R}^2$

It suffices to verify that this root system is closed w.r.t. reflections. Check for example: $\sigma_\alpha(\beta) = (\alpha + \beta)$, and $\sigma_\beta(\alpha) = -(\alpha + \beta)$. Actually we can read off the reflections from the figure.

Definition 1.7 (Weyl Group). The **Weyl Group** W is the subgroup of $\operatorname{End}(E)$ generated by reflections $\{\sigma_\alpha \mid \alpha \in \Phi\}$.

Example 1.8 (Weyl Group of A_2). It is clear that $W(A_2) = \langle \mathrm{Id}, \sigma_\alpha, \sigma_\beta, \sigma_{\alpha+\beta} \rangle$. Further notice that

$$\sigma_\beta\sigma_\alpha(\alpha) = -(\alpha + \beta), \quad \sigma_\alpha(\alpha) = -\alpha \quad \sigma_\beta(\alpha) = \alpha + \beta \quad \sigma_{\alpha+\beta}(\alpha) = -\beta$$

which implies that $\sigma_\beta\sigma_\alpha$ is another distinct element in W. Further we could verify that $(\sigma_\alpha\sigma_\beta) = (\sigma_\beta\sigma_\alpha)^{-1}$. It can be verified that these are the only elements in $W(A_2)$, i.e. we have

$$W(A_2) = \{1, \sigma_\alpha, \sigma_\beta, \sigma_{\alpha+\beta}, \sigma_\alpha\sigma_\beta, \sigma_\beta\sigma_\alpha\} \simeq S_3$$

Remark 1.9. The example above can be extended to general cases: The root system

$$A_n := \{\pm(e_i - e_j) \mid 1 \leq i < j \leq (n+1)\}$$

is a root system for

$$E := \left\{ x = (x_i)_{i=1}^{n+1} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n+1} x_i = 0 \right\}$$

with $W(A_n) \simeq S_{n+1}$.

Example 1.10 (Root System D_n)**.** The root system D_n is given by $D_n := \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq n\}$. The Weyl group for D_n is complicated. The fact is $D_3 \simeq A_3$.

Example 1.11 (Root System B_n and C_n)**.**

Remark 1.12.

Lemma 1.13. Suppose that $\alpha, \beta \in \Phi$, and $\beta \neq \pm\alpha$. Then $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$.

Proof. By definition, we have

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{4(\alpha, \beta)(\beta, \alpha)}{(\alpha, \alpha)(\beta, \beta)} \leq 4.$$

by Cauchy-Schwarz inequality. Axiom 3) in the definition gives $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \mathbb{Z}$. Further since $\beta \neq \pm\alpha$ the equality cannot be reached. The result follows. □