MATH 538 - Lie Algebra

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1 Lie Algebra

Definition 1.1 (Lie Algebra). Let $\mathbb{F}$ be a field. A **Lie Algebra** is a vector space L over $\mathbb{F}$ equipped with a bilinear map $[\cdot,\cdot] : L \times L \to L$ (the **Lie Bracket**) satisfying the following properties:

- *Alternating Property*: $[x,x] = 0$ for all $x \in L$. (For $\operatorname{char} \mathbb{F} \neq 2$, this is equivalent to antisymmetry: $[x,y] = -[y,x]$ for all $x, y \in L$.)
- *Jacobi Identity*: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x, y, z \in L$.

Remark 1.2. Throughout this note, for simplicity we will assume that $\operatorname{char} \mathbb{F} = 0$. Nevertheless, the limitations on field characteristics will be pointed out when it is clear that the result will fall in certain cases.

Example 1.3. Consider $V = \mathbb{F}^n$. Notice that $\dim_{\mathbb{F}}(\operatorname{End}(V)) = \dim_{\mathbb{F}}(\operatorname{Mat}_n(\mathbb{F})) = n^2$ as vector spaces over $\mathbb{F}$; and they are further isomorphic. We can further show that they are isomorphic as Lie algebras.

Proposition 1.4. Define the Lie bracket on $\operatorname{End}(V)$ by $[f,g] = fg - gf$ (with the product the composition of functions). Then $\operatorname{End}(V)$ is a Lie Algebra.

Proof. It suffices to verify that Lie bracket satisfies the alternating property and the Jacobi identity.

- *Alternating Property*: $[f,f] = ff - ff = 0$.
- *Jacobi Identity*:

$$\begin{aligned}
[f,[g,h]] + [g,[h,f]] + [h,[f,g]] &= [f, gh - hg] + [g, hf - fh] + [h, fg - gf] \\
&= f(gh - hg) - (gh - hg)f + g(hf - fh) - (hf - fh)g + h(fg - gf) - (fg - gf)h \\
&= fgh - fgh - ghf + hgf + ghf - hfg - hfg + fhg + fhg - fgh - gfh + gfh \\
&= 0.
\end{aligned}$$

Bi-linearity results directly from the linearity of functions. ☐

Notation. The Lie algebra $(\operatorname{End}(V), [\cdot,\cdot])$ is denoted by $\mathfrak{gl}(V)$.

Example 1.5. Let $V = \mathbb{R}^n$. Then as vector spaces $\operatorname{End}(V) \simeq \operatorname{Mat}_n(\mathbb{R}) \simeq \mathfrak{gl}(V)$ where $[A,B] = AB - BA$ for $A, B \in \operatorname{Mat}_n(\mathbb{R})$.

The Lie Algebra $\operatorname{End}(V) \simeq \operatorname{Mat}_n(V) \simeq \mathfrak{gl}(V)$ has a basis $\{e_{ij}\}_{i,j=1}^n$, where $\{e_{ij}\}$ represents the matrix with all zero entries except for 1 in the (i,j)-th entry.

Definition 1.6 (Lie Subalgebra). A **Lie Subalgebra** $K \subseteq L$ is a subspace of s.t. for all $x, y \in K$, $[x,y] \in K$.

Definition 1.7 (Linear Lie Algebra). Any subalgebra of $\mathfrak{gl}(V)$ is called a **linear Lie algebra**.

Theorem 1.8 (Ado-Iwasawa). Every finite dimensional Lie algebra is isomorphic to a linear Lie algebra.

The proof of this statement requires more structure and will be deferred; for now we will use this results without proof as this theorem is really powerful.

TODO: search for proof

Example 1.9 (Classical Lie Algebra). The following examples of Lie algebra constitute the **classical Lie algebra** which are the bulk of existing Lie algebras. As expected, they are closely related to matrices.

1. Let $V = \mathbb{F}^{n+1}$. The **special linear algebra** A_n is

$$A_n = \mathfrak{sl}_{n+1}(V) = \{g \in \mathfrak{gl}_{n+1}(V) \mid \operatorname{Tr} g = 0\}$$

   As a vector space over $\mathbb{F}$ it has dimension $(n^2 + 2n)$.

2. Let $V = \mathbb{F}^{2n}$. The **symplectic algebra** C_n is

$$C_n = \mathfrak{sp}_{2n}(V) = \{g \in \mathfrak{gl}_{2n}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$$

   Considering its dimension, writing also $g \in \mathfrak{gl}_{2n}(V)$ as n-by-n blocks, we have

$$g = \left(\begin{array}{c|c} g_1 & g_2 \\ \hline g_3 & g_4 \end{array}\right) \quad \Longrightarrow \quad sg + g^T s = \left(\begin{array}{c|c} g_3 - g_3^T & g_4 + g_1^T \\ \hline g_1 + g_4^T & g_2 - g_2^T \end{array}\right) = 0$$

   Further notice that $(g_4 + g_1^T) = (g_1 + g_4^T)^T$. Then the condition of matrix vanishing becomes equivalent to $g_3 = g_3^T$, $g_2 = -g_2^T$ and $g_1 = -g_4^T$. For a symmetric matrix its dimension is $\frac{1}{2}n(n+1)$, and arbitrary g_1 fixes g_4, giving dimension n. Summing together gives the $\dim_{\mathbb{F}} C_n = 2n^2 + n$.

3. Let $V = \mathbb{F}^{2n+1}$. The **(odd) orthogonal algebra** B_n is

$$B_n = \mathfrak{o}_{2n+1}(V) = \{g \in \mathfrak{gl}_{2n+1}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_n \\ 0 & \mathrm{Id}_n & 0 \end{pmatrix}$$

   Considering its dimension, write also $g \in \mathfrak{gl}_{2n+1}(V)$ in block form:

$$g = \begin{pmatrix} x & a_{1\times n} & b_{1\times n} \\ c_{n\times 1} & m_{n\times n} & n_{n\times n} \\ d_{n\times 1} & p_{n\times n} & q_{n\times n} \end{pmatrix} \quad \Longrightarrow \quad sg + g^T s = \begin{pmatrix} 2x & a + d^T & b + c^T \\ a^T + d & p + p^T & q + m^T \\ b^T + c & m + q^T & n + n^T \end{pmatrix} = 0$$

   This translates to equalities

$$2x = 0, \quad a+d^T = (a^T+d)^T = 0, \quad b+c^T = (b^T+c)^T = 0, \quad q+m^T = (m+q^T)^T = 0, \quad p+p^T = 0, \quad n+n^T = 0$$

This gives $x = 0$. Fixing a and b fixes d and c, of which there are $2n$ choices. Fixing m fixes q, of which there are n^2 choices. It is also required that both n and p are anti-symmetric matrices, i.e. they have zeros on the diagonal, and fixing upper triangle elements fixes the whole matrix, of which there are $\frac{1}{2}n(n-1)$ choices. Then

$$\dim_{\mathbb{F}} B_n = 2n + n^2 + 2 \cdot \frac{1}{2}n(n-1) = 2n^2 + n$$

4. Let $V = \mathbb{F}^{2n}$. The **(even) orthogonal algebra** D_n is

$$D_n = \mathfrak{o}_{2n}(V) = \{g \in \mathfrak{gl}_{2n}(V) \mid sg + g^T s = 0\}, \quad \text{where } s = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$$

Considering its dimension, use the similar strategy as above. Write $g \in \mathfrak{gl}_{2n}(V)$ in block form:

$$g = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \quad \Longrightarrow \quad sg + g^T s = \begin{pmatrix} p + p^T & q + m^T \\ m + q^T & n + n^T \end{pmatrix} = 0$$

The equality translates to the following conditions

$$p + p^T = 0, \quad q + m^T = (m + q^T)^T = 0, \quad n + n^T = 0$$

Fixing p fixes m, and both p and n are anti-symmetric matrices, which have dimension $\frac{1}{2}n(n-1)$ over $\mathbb{F}$. This gives

$$\dim_{\mathbb{F}} D_n = n^2 + 2 \cdot \frac{1}{2}n(n-1) = 2n^2 - n$$

2 Solvable and Nilpotent Lie Algebra