MATH 635 - Differential Topology Preliminaries

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1 Riemannian Structure

Definition 1.1 (Riemannian Structure). *Let M be a smooth n-manifold. Then a* ***Riemannian Structure*** *on it is an assignment $M \ni p \mapsto g_p$, where $g_p : T_pM \times T_pM \to \mathbb{R}$ is a bilinear positive definite symmetric form, that depends smoothly on p. Such g is a* ***Riemannian metric****.*

Specifically, if $(x^1, \dots, x^n)$ is a coordinate system on $U \subseteq M$, then for $i, j \in [\![1, n]\!]$, $p \in U$, define

$$g_{ij}(p) = g_p \left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right)$$

where $\left.\frac{\partial}{\partial x^i}\right|_p \in M$ for all i. Then g_{ij} is C^∞; and $g(p) = (g_{ij}(p))$ is a symmetric matrix that depends on p. The matrix is often referred to as the ***metric tensor****. The calculation can be defined via specifying $g_p(v_1, v_2) = v_1^T g(p) v_2$.*

Definition 1.2 (Riemannian Manifold). *A* ***Riemannian Manifold*** *is a smooth manifold M endowed with a Riemannian Metric g, often denoted as pair (M, g).*

Example 1.3. Let $M \subseteq \mathbb{R}^N$ be a smooth manifold. Then for all $p \in M$, via embedding the tangent space into $\mathbb{R}^N$, $T_pM \subseteq \mathbb{R}^N$. The inner product in the usual sense (dot product in $\mathbb{R}^n$) gives M a Riemannian structure.

Example 1.4. Take $M = S^2 \subset \mathbb{R}^3$, and let $U = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$. Specify the Riemannian structure as the inner product in $\mathbb{R}^3$, with the tangent space regarded as planes in $\mathbb{R}^3$, taking the local coordinate system as (x_1, x_2), then at (x_1, x_2), the metric tensor for the tangent space is given by

$$g = \frac{1}{1 - (x_1^2 + x_2^2)} \begin{pmatrix} 1 - x_2^2 & x_1 x_2 \\ x_1 x_2 & 1 - x_1^2 \end{pmatrix}$$

To see that this is indeed the metric, notice that at $T_{(x_1,x_2)}M$ for $(x_1, x_2) \in U$, the normal vector is given by $(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$. Therefore $\boldsymbol{\alpha} \in T_{(x_1,x_2)}M$ must be in the form of $(a, b, -\frac{ax_1 + bx_2}{\sqrt{1 - x_1^2 - x_2^2}})$ for $a, b \in \mathbb{R}$. Let $\boldsymbol{\beta} := (c, d, -\frac{cx_1 + dx_2}{\sqrt{1 - x_1^2 - x_2^2}}) \in T_{(x_1,x_2)}M$. Then

$$\begin{aligned}
\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle &= ac + bd + \frac{acx_1^2 + bdx^2 + (ad + bc)(x_1 x_2)}{1 - x_1^2 - x_2^2} \\
&= \frac{1}{1 - x_1^2 - x_2^2} \left(ac(1 - x_2^2) + bd(1 - x_1^2) + ad(x_1 x_2) + bc(x_1 x_2) \right)
\end{aligned}$$

where the entries of the metric tensor can be read off.

Observation 1.5. The length can thus be defined given the generalization of inner product on the structure. For $\gamma : [0, 1] \to M$, the length of γ

$$\|\gamma\| = \int_0^1 \sqrt{g_{\gamma(t)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t), \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \right)} =: \int_0^1 \sqrt{g_{\gamma(t)} \left(\dot{\gamma}, \dot{\gamma} \right)}$$

If M is connected, Then the distance between $a, b \in M$ is $\inf\limits_{\gamma(0)=a, \gamma(1)=b} \|\gamma\|$.

2 Vector Bundle

3 Tensor Product

4 Riemannian Metric

5 Coverings

6 Metric on Lee Group

7 Common Objects on Riemannian Manifolds