MATH 635 - Basic Riemannian Constructions

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Contents

Notation (Eisntein Summation Notation)**.** Throughout this course, we adopt the **Einstein summation notation**, which omits the summation symbol, and any free indices appeared indicates that we are summing over them, with the bounds referred from the context. Typical such free indices start with letter i, and goes onwards as needed.

1 Riemannian Structure

Definition 1.1 (Riemannian Structure)**.** Let M be a smooth n-manifold. Then a **Riemannian Structure** on it is an assignment $M \ni p \mapsto g_p$, where $g_p : T_pM \times T_pM \to \mathbb{R}$ is a bilinear, positive-definite, symmetric form, that depends smoothly on p. Such g is a **Riemannian metric**.

Specifically, if $(x^1, \dots, x^n)$ is a coordinate system on $U \subseteq M$, then for $i, j \in [\![1, n]\!]$, $p \in U$, define

$$g_{ij}(p) = g_p\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right)$$

where $\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$ for all i. Then g_{ij} is C^∞; and $g(p) = (g_{ij}(p))$ is a symmetric matrix that depends on p. The matrix is often referred to as the **metric tensor**. Evaluation on the metric can be done via $g_p(v_1, v_2) = v_1^T g(p) v_2$.

Example 1.2. Let $M \subseteq \mathbb{R}^N$ be a smooth manifold. Then for all $p \in M$, via embedding the tangent space into $\mathbb{R}^N$, $T_pM \subseteq \mathbb{R}^N$. The inner product in the usual sense (dot product in $\mathbb{R}^n$) gives M a Riemannian structure. This implies that Euclidean Space obtains a Riemannian structure.

Example 1.3. Take $M = S^2 \subset \mathbb{R}^3$, and let $U = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$. Specify the Riemannian structure as the inner product in $\mathbb{R}^3$, with the tangent space regarded as planes in $\mathbb{R}^3$, taking the local coordinate system as (x_1, x_2), then at (x_1, x_2), the metric tensor for the tangent space is given by

$$g = \frac{1}{1 - (x_1^2 + x_2^2)} \begin{pmatrix} 1 - x_2^2 & x_1 x_2 \\ x_1 x_2 & 1 - x_1^2 \end{pmatrix}$$

To see that this is indeed the metric, notice that at $T_{(x_1,x_2)}M$, the normal vector is given by $(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$. Therefore $\boldsymbol{\alpha} \in T_{(x_1,x_2)}M$ must be in the form of $(a, b, -\frac{ax_1+bx_2}{\sqrt{1-x_1^2-x_2^2}})$ for $a, b \in \mathbb{R}$. Let $\boldsymbol{\beta} := (c, d, -\frac{cx_1+dx_2}{\sqrt{1-x_1^2-x_2^2}}) \in T_{(x_1,x_2)}M$. Then

$$\begin{aligned} \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle &= ac + bd + \frac{acx_1^2 + bdx^2 + (ad + bc)(x_1 x_2)}{1 - x_1^2 - x_2^2} \\ &= \frac{1}{1 - x_1^2 - x_2^2}\left(ac(1 - x_2^2) + bd(1 - x_1^2) + ad(x_1 x_2) + bc(x_2 x_1)\right) \end{aligned}$$

where the entries of the metric tensor can be read off.

Observation 1.4. The length can thus be defined given the generalization of inner product on the structure. For $\gamma : [0, 1] \to M$, the length of γ

$$\|\gamma\| = \int_0^1 \sqrt{g_{\gamma(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t), \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right)}\mathrm{d}t =: \int_0^1 \sqrt{g_{\gamma(t)}\left(\dot{\gamma}, \dot{\gamma}\right)}\mathrm{d}t$$

If M is connected, Then the distance between $a, b \in M$ is $\inf\limits_{\gamma(0)=a,\gamma(1)=b} \|\gamma\|$.

2 Vector Bundle

Definition 2.1 (Vector Bundle). A **(real) vector bundle of rank k over M** is a surjection $\pi : \mathcal{E} \to M$ satisfying the following properties:

1. For all $p \in M$, $\mathcal{E}_p := \pi^{-1}(p)$ has a real vector space structure of dimension k.
2. For all $p \in M$, there exists an open neighborhood $U \subset M$ containing p, and a diffeomorphism $\chi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ together with the projection $p : U \times \mathbb{R}^n \to U$ s.t.
   - $\pi|_U = p \circ \chi$, i.e. the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\chi"] \arrow[dr, "\pi|_{\pi^{-1}(U)}"'] & U \times \mathbb{R}^n \arrow[d, "p"] \\ & U \end{tikzcd}$$

   - For all $p \in U$, $\chi|_p : \mathcal{E}_p \to \{p\} \times \mathbb{R}^n$ is a linear isomorphism.

E is the total space, and M is the base. The map χ is called the local trivialization of E at p.

Example 2.2. The following gives some examples of vector bundle:

1. The vector bundle which associates every point in M the vector space $\mathbb{R}^n$, given by $\pi : M \times \mathbb{R}^n \to M$ is the trivial bundle.
2. Consider $\mathcal{E} = TM$ which is the tangent bundle (or isomorphically, the cotangent bundle T^*M), defined as $TM = \coprod_{p \in M} T_pM$. Then if $\phi = (x^1, \dots, x^n)$ is a coordinate system on $U \subset M$, we have a basis $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ a basis of T_pM for each $p \in M$. The map χ is then given by

$$\chi : \pi^{-1}(U) = TU \to U \times \mathbb{R}^n, \quad (p, T_pM \ni v) \mapsto (p, \langle v^1, \dots, v^n \rangle) \text{ s.t. } v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$$

χ is a diffeomorphism as the tangent space TM has a smooth structure. Similarly one can consider the dual T^*M with basis $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$. Then the corresponding map becomes

$$\chi : \pi^{-1}(U) = T^*U \to U \times \mathbb{R}^n, \quad (p, T_p^*M \ni \alpha) \mapsto (p, \langle a_1, \dots, a_n \rangle) \text{ s.t. } \alpha = \sum_{i=1}^n a_i \mathrm{d}x^i$$

where $\alpha : T_pM \to \mathbb{R}$ is a linear map.

3. Let $M \subset \mathbb{R}^N$ be a n-manifold. Define $\mathcal{E} = \left\{(p, v) \mid p \in \mathcal{E}, v \in (T_pM)^\perp\right\}$, The map can be defined as

$$\chi : \pi^{-1}(U) \to U \times \mathbb{R}^{N-n}, \quad (p, v) \mapsto (p, v)$$

which is the identity if one identifies T_pM with a subspace of $\mathbb{R}^N$.

Definition 2.3 (Section). Let $\pi : \mathcal{E} \to M$ be a vector bundle. Then a **(smooth) section** is a (smooth) map $s : M \to \mathcal{E}$ s.t. $\pi \circ s = \mathrm{Id}_M$. The set of all smooth sections is denoted as $\Gamma(\mathcal{E}, M)$, or simply $\Gamma(\mathcal{E})$.

Remark 2.4. Explained in plain words, a section selects an element $s(p) \in \mathcal{E}_p$ for each $p \in M$. If $\mathcal{E} = TM$, and identifying the tangent space with $\mathbb{R}^N$ for some N, a section s gives a vector field on M.

Remark 2.5. $\Gamma(\mathcal{E})$ defines a module structure over $C^\infty(M)$, with the scalar multiplication defined as $\chi(f \cdot s(x)) = (x, f(x) \cdot (p_r \circ \chi \circ s)(x))$, where $p_r : (U, \mathbb{R}^n) \to \mathbb{R}^n$ is the projection that takes the second field. This map is smooth, as both f, p_r, χ and s are smooth.

Definition 2.6 (Moving Frame). Let $\mathcal{E} \to M$ be a vector bundle, and $U \subset M$ an open set. Then a **moving frame** of $\mathcal{E}$ on U is an r-tuple $(E_1, \dots, E_r)$ s.t. for all j, E_j is a section of $\mathcal{E} \supset \pi^{-1}(U) \to U$; and for all $p \in U$, $(E_1(p), \dots, E_r(p))$ gives a basis of $\mathcal{E}_p$.

Remark 2.7. There is a bijection between moving frames of $\mathcal{E}$ on U and trivializations $\chi : \pi^{-1}(U) \to U \times \mathbb{R}^n$:

- Given a moving frame $(E_1, \dots, E_r)$ of $\mathcal{E}$ on U, define

$$\chi : \pi^{-1}(U) \to U \times \mathbb{R}^n, \quad v \mapsto (\pi(v), \langle v^1, \dots, v^n \rangle) \text{ s.t. } v = \sum_{i=1}^{n} v^i E_i(p)$$

- Conversely, given a trivialization $\chi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, define

$$E_i : M \supset U \to \pi^{-1}U \subset \mathcal{E}, \quad p \mapsto \chi^{-1}(p, \langle 0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th position}}}{1}, 0, \dots, n \rangle)$$

The vector space structure is induced by the vector space structure of $\mathbb{R}^n$.

Definition 2.8 (Transition). Let $\pi : \mathcal{E} \to M$ be a vector bundle, and $U_\alpha, U_\beta \subset M$ two open subsets of M, and χ_α, χ_β be the corresponding local trivializations. Define $\phi_{\alpha,p} := p_r \circ \chi_\alpha$ where $p_r : (U, \mathbb{R}^n) \to \mathbb{R}^n$ is the projection. Then the **transition** between χ_α and χ_β is a map $\tau_{\alpha\beta}$ s.t. for $f_{\alpha\beta} : U \times \mathbb{R}^n \to U \times \mathbb{R}^n, (p, v) \mapsto (p, \tau_{\alpha\beta}(v))$, the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\chi_\alpha"] \arrow[dr, "\chi_\beta"'] & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \arrow[d, "f_{\alpha\beta}"] \\
& (U_\alpha \cap U_\beta) \times \mathbb{R}^r
\end{tikzcd}$$

Observation 2.9. $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{R})$. It is smooth as χ is a diffeomorphism.

Remark 2.10. By definition it is clear that $\tau_{\alpha\beta} = \tau_{\beta\alpha}^{-1}$; and $\tau_{\alpha\beta} \circ \tau_{\beta\gamma} = \tau_{\alpha\gamma}$.

Observation 2.11. By the fact that any object defined on the vector space can be transferred to the vector bundle via the local trivialization since it is a diffeomorphism, constructions on vector spaces can be identified with constructions on vector bundles.

Lemma 2.12 (Bundle Chart Lemma). Given M a smooth manifold, and for all $p \in M$ associate an r-dimensional vector space $\mathcal{E}_p$ to it. Define $\mathcal{E} := \coprod_{p\in M} \mathcal{E}_p$, and $\pi : \mathcal{E} \to M, (p, v) \mapsto p$. Further the followings are satisfied:

- There exists an open cover $\{U_\alpha\}$ of M.
- For all α, there exists a bijective map $\chi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ s.t. $\pi = p \circ \chi_\alpha$ where p is the projection $U_\alpha \times \mathbb{R}^n \to U_\alpha$.

Then there exists a unique C^∞ manifold structure on $\mathcal{E}$ s.t. χ_α is a diffeomorphism; and $\mathcal{E}$ is a C^∞ bundle over M.

The proof of the lemma is via verifying that $\mathcal{E}$ is indeed a manifold, and is omitted here. The more interesting aspect is that this provides a lot of operations on bundles, for example combination of two bundles:

Example 2.13. Let (U_α) be an open cover of M, and $\pi' : \mathcal{E}' \to M, \pi'' : \mathcal{E}'' \to M$ be two vector bundles over M. Define $\mathcal{E}' \oplus \mathcal{E}'' \to M$, where $\mathcal{E}' \oplus \mathcal{E}'' = \coprod_{p\in M}(\mathcal{E}'_p \oplus \mathcal{E}''_p)$, Let (U_α) be a cover of M s.t. both $\mathcal{E}'$ and $\mathcal{E}''$ trivialize over each U_α. Then it is valid to define the trivialization $\chi : \pi^{-1}(U_\alpha) \to U_\alpha \times (\mathbb{R}^{r'} \oplus \mathbb{R}^{r''})$ s.t. $\phi_{\alpha,p} = \phi'_{\alpha,p} \oplus \phi''_{\alpha,p}$. In matrices this gives a block diagonal matrix.

3 Tensors

Definition 3.1 (Free Vector Space). Let S be a (possibly infinite) set, and K a field. The **free vector space** generated by S over K is defined as the set

$$FS := \left\{ \sum_{s\in S} \alpha_s s \;\middle|\; \text{finite nonzero } \alpha_s \right\}$$

satisfying the axioms of vector spaces, i.e. linear in s with addition and scalar multiplication.

Definition 3.2 (Tensor Product). Let V, W be finite dimensional vector space. The **tensor product of V and W**, denoted $V \otimes W$ is the free vector space generated by $(V \times W)/\sim$, where $\sim$ follows the rules:

- $(v + v', w) \sim (v, w) + (v', w)$ for $v, v' \in V, w \in W$.
- $(v, w + w') \sim (v, w) + (v, w')$ for $v \in V, w, w' \in W$.
- $(cv, w) \sim c(v, w) \sim (v, cw)$ for $c \in K, v \in V, w \in W$.

This gives a map $\otimes : V \times W \to V \otimes W$, $(v, w) \mapsto [(v, w)]$ which is an equivalence class in $V \otimes W$.

Proposition 3.3 (Universal Property of Tensor Product). Let Z be a real vector space, and the map $b : V \times W \to Z$ bilinear. Then there exists a unique linear map $f : V \otimes W \to Z$ s.t. the following diagram commute:

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "b"'] & V \otimes W \arrow[d, "f"] \\ & Z \end{tikzcd}$$

Corollary 3.4. Let (v_i) be a basis of V, and (w_i) a basis of W. Then $(v_i \otimes w_j)$ gives a basis of $V \otimes W$, with $\dim V \otimes W = \dim V \cdot \dim W$.

Remark 3.5. Not every element in $V \otimes W$ can be expressed as $v \otimes w$ for some $v \in V, w \in W$. In particular, one can have $v' = \sum_{i \in I} v_i \otimes w_i \in V \otimes W$, whose representative cannot be further simplified.

Proposition 3.6. Let V and W be finite dimensional real vector space. Then there exists a natural isomorphism $V \otimes W \xrightarrow{\sim} \mathrm{Hom}(V^*, W)$. In particular, this gives $V^* \otimes V^* \simeq \mathrm{Hom}(V, V^*) \simeq \{\text{bilinear forms } V \times V \to \mathbb{R}\}$.

Proof. To construct the map it suffices to give a bilinear map $V \times W \to \mathrm{Hom}(V^*, W)$. Define it as

$$(v, w) \mapsto (\alpha \mapsto \alpha(v)w) \in \mathrm{Hom}(V^*, W)$$

Applying the universal property gives the desired result. □

Remark 3.7. This can be extended to multiple tensor products:

$$\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \quad \simeq \quad (\underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{\ell} \to \mathbb{R})$$

where RHS is a multilinear map.

Definition 3.8 (Tensor). Denote

$$G = \underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell}$$

Then an element in G is called a (k, ℓ)-**tensor** on V. k is the contravariant degree, and ℓ is the covariant degree.

Remark 3.9. It is possible to define the tensor product of maps. Namely, for $f : V \to V'$ and $g : W \to W'$, there exists a unique linear map $f \otimes g : V \otimes W \to V' \otimes W'$ s.t. $(f \otimes g)(v \otimes w) = (f(v)) \otimes (g(w))$. This can be seen via consider first the $\mathbb{R}$-balanced linear map $V \times W \to V \otimes W$; and apply the universal property of tensor product gives the desired result.

Observation 3.10. Similar to Example 2.13 where we considered the direct sum of vector bundles as a new bundle, we can consider the tensor product of vector bundles. For $\mathcal{E}'$ and $\mathcal{E}''$ two vector bundles, $\coprod_{p \in M} \mathcal{E}'_p \otimes \mathcal{E}''_p$ gives a vector bundle structure over M. In particular, this can be extended to the tangent and cotangent spaces of a manifold.

Notation. Given a smooth manifold M, for $k, \ell \in \mathbb{Z}_{\geq 0}$, denote

$$\mathscr{T}^{(k,\ell)}M = \Pi^{(k,\ell)}M = \underbrace{TM \otimes \cdots \otimes TM}_{k} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{\ell}$$

In particular, this is TM for $k = 1, \ell = 0$, and is T^*M for $k = 0, \ell = 1$. By identification in Remark 3.7 maps $\Pi^{(k,\ell)}M \to M$ are always multilinear (with the identification with the Euclidean Space).

Definition 3.11 (Tensor Field)**.** Let M be a smooth manifold. A (k, ℓ)**-tensor (field) on** M is a smooth section of $\Pi^{(k,\ell)}M$.

In local coordinates, if we have local coordinates $(x^1, \ldots, x^n)$ on $U \subseteq M$, then any element in $\Pi^{(k,\ell)}M$ is in the form of

$$f^{i_1,\ldots,i_k}_{j_1,\ldots,j_\ell} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_\ell}$$

where i_ks and j_ℓs run over $\{1, \ldots, n\}$ arbitrarily. A section on $\Pi^{(k,\ell)}M$ is then $\sum_{i,j} f^{i_1,\ldots,i_k}_{j_1,\ldots,j_\ell}$, with it being smooth implies that all such fs are smooth (as functions).

4 Riemannian Metric

Definition 4.1 (Riemannian Metric)**.** Given a Riemannian manifold M, a **Riemannian Metric** on M is a (0, 2)-tensor g that is symmetric and positive definite. In local coordinates,

$$g = \sum_{i,j=1}^{n} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

That is, for all $p \in M$, $g_p : T_pM \times T_pM \to \mathbb{R}$ is bilinaer and symmetric; and for all $v \in T_pM$, $g_p(v, v) \geq 0$, with equality reached if and only if $v = 0$.

Observation 4.2. Notice that in local coordinates we have $v = v^i \frac{\partial}{\partial x^i}$, $w = w^j \frac{\partial}{\partial x^j}$. Evaluating the metric gives

$$(\mathrm{d}x^i \otimes \mathrm{d}x^j)(v, w) = v^i w^j = v^j w^i = (\mathrm{d}x^i \otimes \mathrm{d}x^j)(w, v) \implies g_{ij} = g_{ji}$$

Therefore, we can write

$$\begin{aligned} g &= \frac{1}{2}\left(g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ji}\mathrm{d}x^j \otimes \mathrm{d}x^i\right) = \frac{1}{2} g_{ij}\left(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i\right) \\ &= g_{ij}\left(\frac{1}{2}\left(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i\right)\right) \end{aligned}$$

where $\left(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i\right) \in \mathrm{Sym}(T_pM)$ which is in the symmetric tensor, i.e. invariant w.r.t permutation of the entries.

Definition 4.3 (Riemannian Manifold)**.** A **Riemannian Manifold** is a pair (M, g) where g is a Riemannian metric on M.

Definition 4.4 (Isometry). A C^∞-map $F : (M_1, g_1) \to (M_2, g_2)$ is an **isometry** if and only if

1) F is a diffeomorphism.

2) For all $p \in M_1$, $u, v \in T_p M_1$, $g_2(\mathrm{d}_p F(u), \mathrm{d}_p F(v)) = g_1(u, v)$.

F is a **local isometry** if only 2) holds.

Remark 4.5. Since g is nondegenerate, $\mathrm{d}_p F$ is bijective as otherwise LHS of the equality in 2) will vanish. This then implies that $\dim M_1 = \dim M_2$, otherwise $\mathrm{d}_p F$ will either be not injective ($\dim M_1 > \dim M_2$) or not surjective ($\dim M_1 < \dim M_2$).

Theorem 4.6. Any C^∞-manifold M has a Riemannian metric.

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas of M. By definition of the chart we have the embedding $\varphi_\alpha(U_\alpha) \hookrightarrow \mathbb{R}^n$. Define in U_α the metric s.t. φ_α is an isometry. In local coordinates, the metric can be expressed as $g_\alpha = \sum_i \mathrm{d}x^i_\alpha \otimes \mathrm{d}x^j_\alpha$.

Let (χ_α) be a partition of unit subordinated to (U_α) and $g = \sum_\alpha \chi_\alpha g_\alpha$. Nondegeneracy of the metric g follows from the positivity of metric g_α and χ_α. □

Example 4.7. The following gives two constructions of Riemannian Metrics:

1) Let (M, g) be a Riemannian manifold, with $N \subset M$ a submanifold. Then N inherits a Riemannian metric from the Riemannian structure on M, as for all $p \in N$, $T_p N \subset T_p M$.

2) Consider Cartesian product of Riemannian manifolds. Given two Riemannian manifolds (M_1, g_1), (M_2, g_2), consider $M_1 \times M_2$. Then for all $(p_1, p_2) \in M_1 \times M_2$ we have the identification between the tangent spaces

$$T_{(p_1,p_2)}(M_1 \times M_2) \simeq T_{p_1} M_1 \oplus T_{p_2} M_2$$

Then in coordinates, the combined metric is block diagonal:

$$g = \left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2 \end{array}\right)$$

5 Coverings

Definition 5.1 (Covering). A C^∞ surjective map $\pi : M \to N$ is a C^∞-**covering** if for all $q \in N$, there exists U, a connected neighborhood of q s.t. if $\pi^{-1}(U) = \amalg_j V_j$, where V_j are connected components of $\pi^{-1}(U)$; and for all j, $\pi|_{V_j} : V_j \to U$ is a diffeomorphism.

Definition 5.2 (Automorphism of Covering). The automorphism on the covering is defined as

$$\mathrm{Aut}(\pi) := \{F : M \to M \text{ diffeomorphisms} \mid \pi \circ F = \pi\}$$

For any $f \in \mathrm{Aut}(\pi)$, it shuffles the fibers $\{\pi^{-1}(q) \mid q \in N\}$.

Definition 5.3 (Normal Covering). A covering π is **normal** if $\mathrm{Aut}(\pi)$ acts transitively on fibers.

Proposition 5.4. Suppose that we have a manifold M has a Riemannian metric, $\pi : M \to N$ is a covering. If for all $f \in \mathrm{Aut}(\pi)$ it is an isometry, then there exists a unique Riemannian metric on N s.t. π is a local isometry.

Proof. Fix $q \in N$. Choose $p \in \pi^{-1}(q)$. Since π restricted to a neighborhood of p is a diffeomorphism, the differential $\mathrm{d}\pi_p : T_pM \to T_qN$ is bijective. Define the metric on T_qN s.t. $d\pi_p$ is a linear isometry.

Check that this is compatible with the covering: if we have another choice $p' \in \pi^{-1}(q)$, by assumption there exists $F \in \mathrm{Aut}(\pi)$ an isometry s.t. $F(p) = F(p')$; and $\mathrm{d}F_p : T_pM \to T_{p'}M$ is a linear isometry. Then the composition is also a linear isometry:

$$\begin{tikzcd}
T_pM \arrow[rr, "\mathrm{d}F_p"] \arrow[dr, "\mathrm{d}\pi_p"'] & & T_{p'}M \arrow[dl, "\mathrm{d}\pi_{p'}"] \\
& T_pN &
\end{tikzcd}$$

□

Example 5.5. The following gives some simple examples of coverings:

- The covering $\pi : S^n \to \mathbb{RP}^n$, where the antipodal points are identified. $\mathrm{Aut}(\pi)$ is the antipodal map.
- Let $\Lambda \subset \mathbb{R}^n$ be a lattice (e.g. $\Lambda = \mathbb{Z}^n$). Considering $\mathbb{R}^n \to \mathbb{R}^n/\Lambda$ gives a covering map. $\mathrm{Aut}(\pi)$ are translations by Λ.
- The hyperbolic space. Let M be the upper half plane $\mathbb{H} = \{(x, y) \mid y > 0\}$ with metric $\mathrm{d}s^2 = \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. It turns out that any compact surface of genus > 1 is covered by $\mathbb{H}$. The covering $\pi : \mathbb{H} \to S$ is such that $\mathrm{Aut}(\pi)$ acts by isometries; and the metrics on S is locally isometric to the hyperbolic metric.

6 Metric on Lie Group

Definition 6.1 (Invariance). A Riemannian metric on a Lie Group is **left-invariant** if and only if for all $g \in G$, the map

$$\ell_g : G \to G, \quad k \mapsto gk$$

is an isometry. The metric being right-invariant is defined similarly.

Proposition 6.2. Given a Lie group G, there is a one-to-one correspondence between left-invariant metrics on G, and positive definite inner products on T_eG.

Proof. Identify G with T_eG as e acting on G is the identity. Start with $\langle -, - \rangle_e : T_eG \times T_eG \to R$ positive-definite, to define the metric $\langle -, - \rangle_g : T_gG \times T_gG \to \mathbb{R}$. Impose that $\mathrm{d}(\ell_g)_e : T_eG \to T_gG$ is a linear isometry. To show that the resulting metric is left-invariant we need to show that for all $g, k \in G$, $\mathrm{d}(\ell_g)_k : T_kG \to T_{gk}G$ is a linear isometry. Consider the transition

$$\begin{tikzcd} T_kG \arrow[rr, "\mathrm{d}(\ell_g)_k"] & & T_{gk}G \\ & T_eG \arrow[ul, "\mathrm{d}(\ell_g)_e"] \arrow[ur, "\mathrm{d}(\ell_{gk})_e"'] & \end{tikzcd}$$

$\mathrm{d}(\ell_g)_k$ is then a linear isometry as both $\mathrm{d}(\ell_g)_e$ and $\mathrm{d}(\ell_{gk})_e$ are linear isometries. Smoothness follows similarly from composition of diffeomorphisms. The diagram commutes from by definition $\ell_g \circ \ell_k = \ell_{gk}$. ☐

Remark 6.3. There exists metrics that are bi-invariant (both left- and right-invariant).

Example 6.4. Take $S^{2n+1} \subset \mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$. The group $S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$ acts on S^{2n+1} via

$$e^{i\theta}(z_0, \ldots, z_n) = (e^{i\theta} z_0, \ldots, e^{i\theta} z_n) \quad z_0, \ldots, z_n \text{ not all zero}$$

A side remark is that via identifying elements by the action of S^1 we have a map $S^{2n+1} \to S^{2n+1}/S^1 \simeq \mathbb{CP}^n$. The case for $n = 1$: $S^3 \to S^2 \simeq \mathbb{CP}^1$ is the Hopf map.

Now take $w \in S^{2n+1}$, Decompose $T_wS^{2n+1} = \mathbb{R}\partial_\theta \oplus H_W$, which are the vertical and horizontal subspaces. Explicitly, ∂_θ is the vector field given by $\frac{\mathrm{d}}{\mathrm{d}\theta}(e^{i\theta} w)\big|_{\theta=0}$; and H_W is the orthogonal complement of $\mathbb{R}\partial_\theta$.

The map

$$\rho(e^{i\theta}) : S^{2n+1} \to S^{2n+1}, \quad z \mapsto e^{i\theta} z$$

maps $H_w to H_{e^{i\theta}w}$ isometrically, i.e. with the previous decomposition we have the isometry

$$\mathrm{d}\ \pi_w\big|_{H_w} : H_w \simeq T_{\pi(w)}\mathbb{CP}^n$$

where π_w is the projection onto the horizontal subspace. This is the Fubini-Study Metric. It will appear again in the last chapter when we briefly discuss complex manifolds.

7 Common Objects on Riemannian Manifolds

The common objects defined on Riemannian manifolds are the **Length of Curves** extending lengths on the Euclidean space; and **Volume Density/Form** extending definition of volumes. Since a Riemannian manifold may not be orientable (the transition functions of charts on the atlas may have negative Jacobian determinant), we have forms or densities separating these cases.

Definition 7.1 (Length of Curve). Let $\gamma : \mathbb{R} \supset [a, b] \to M$ be a curve. Then the **length** of γ is defined as

$$\ell(\gamma) = \int_a^b (\|\gamma'(t)\|)|_{\gamma(t)}\, \mathrm{d}t$$

By chain rule this is independent of the parameterization.

Definition 7.2 (Distance). Given a manifold M and $p, q \in M$, the **distance** between p and q is defined as

$$d(p,q) = \inf_{\gamma:\gamma(a)=p,\gamma(b)=q} \ell(\gamma)$$

Remark 7.3. The distance indeed gives a metric. For $d \geq 0$, the triangle inequality results from the triangle inequality of $\|-\|$; and we have $d(p,q) = d(q,p)$ via reversing the parameterization. It is much more delicate to show that $d(p,q) = 0$, which we will prove in a later section. Furthermore, the infimum above does not have to be realized; but this will be the case if the space is compact.

Now we seek to generalize the notion of volume. First consider the oriented case:

Definition 7.4 (Volume Form). Given (M, g) an oriented Riemannian manifold. Let $n = \dim M$. For all $p \in M$, let $\{e_1, \ldots, e_n\}$ be a positive orthonormal basis of T_pM. The **volume form** Ω_p is the differential n-form satisfying $\Omega_p(e_1, \ldots, e_n) = 1$.

In coordinates, for $(x^1, \ldots, x^n)$ a positive chart, $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ a positive coordinate system for the tangent space at each point. Applying Gram-Schmidt orthonormalization at each point gives an orthonormal moving frame $\{E_1, \ldots, E_n\}$. The moving frame is smooth as Gram-Schmidt only applies linear transformations. Express the coordinate in the moving frame:

$$\frac{\partial}{\partial x^i} = \sum_\ell a_i^\ell E_\ell \qquad A = \left(a_i^\ell\right)_{-\text{column index}}^{-\text{row index}}$$

Then compute the metric:

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = a_i^j a_j^m \underbrace{\langle E_\ell, E_m \rangle}_{=\delta_{\ell m}} = \sum_\ell a_i^\ell a_j^\ell = \left(A^T A\right)_j^i$$

as from the Einstein summation notation $\left(a_i^\ell b_\ell^j\right) = (BA)_i^j$, with the same notation for indices in matrices as above. The written in matrix the metric tensor is given by $A^T A$. On the other hand, apply linearity for all n-forms we have

$$\Omega\left(\frac{\partial}{\partial x^1}, \cdots \frac{\partial}{\partial x^n}\right) = \det A \underbrace{\Omega(E_1, \ldots, E_n)}_{=1 \text{ by orthonormality}} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

as we need top-degree differential form. Then in coordinates this gives

$$\Omega_i^j = \sqrt{\det(g_{ij})} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

Now consider the case where M is not necessarily orientable. For proper definition we would like the domain where the function is supported is compact:

Definition 7.5 (Compact Support). A function has **compact support** I, denoted $C_c(I)$, if I is compact, and f is zero outside I.

Definition 7.6 (Volume Density). If U is the domain of a chart $U \subset M$, and $f \in C_c^\infty(M)$. Then the **volume density** is

> defined as
> $$\underbrace{\int_M f(x^1, \ldots, x^n)\sqrt{\det g}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n}_{\text{Riemann Integral}}$$

This is independent of the choice of coordinates, as change of coordinate involves $\left|\det\left(\frac{\partial y^i}{\partial x^k}\right)\right|$, which does not change the Riemann integral, as the magnitude is taken care of by the determinant of the metric; and the orientation (sign) of the transition does not show up.