

MATH 635 - Differential Topology Preliminaries

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March 17, 2024

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1 Riemannian Structure

Definition 1.1 (Riemannian Structure). Let M be a smooth n -manifold. Then a **Riemannian Structure** on it is an assignment $M \ni p \mapsto g_p$, where $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is a bilinear positive definite symmetric form, that depends smoothly on p . Such g is a **Riemannian metric**.

Specifically, if (x^1, \dots, x^n) is a coordinate system on $U \subseteq M$, then for $i, j \in \llbracket 1, n \rrbracket$, $p \in U$, define

$$g_{ij}(p) = g_p \left(\left. \frac{\partial}{\partial x^i} \right|_p, \left. \frac{\partial}{\partial x^j} \right|_p \right)$$

where $\left. \frac{\partial}{\partial x^i} \right|_p \in T_p M$ for all i . Then g_{ij} is C^∞ ; and $g(p) = (g_{ij}(p))$ is a symmetric matrix that depends on p . The matrix is often referred to as the **metric tensor**. Evaluation on the metric can be done via $g_p(v_1, v_2) = v_1^T g(p) v_2$.

Definition 1.2 (Riemannian Manifold). A **Riemannian Manifold** is a smooth manifold M endowed with a Riemannian Metric g , often denoted as pair (M, g) .

Example 1.3. Let $M \subseteq \mathbb{R}^N$ be a smooth manifold. Then for all $p \in M$, via embedding the tangent space into \mathbb{R}^N , $T_p M \subseteq \mathbb{R}^N$. The inner product in the usual sense (dot product in \mathbb{R}^n) gives M a Riemannian structure. This implies that Euclidean Space obtains a Riemannian structure.

Example 1.4. Take $M = S^2 \subset \mathbb{R}^3$, and let $U = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$. Specify the Riemannian structure as the inner product in \mathbb{R}^3 , with the tangent space regarded as planes in \mathbb{R}^3 , taking the local coordinate system as (x_1, x_2) , then at (x_1, x_2) , the metric tensor for the tangent space is given by

$$g = \frac{1}{1 - (x_1^2 + x_2^2)} \begin{pmatrix} 1 - x_2^2 & x_1 x_2 \\ x_1 x_2 & 1 - x_1^2 \end{pmatrix}$$

To see that this is indeed the metric, notice that at $T_{(x_1, x_2)} M$, the normal vector is given by $(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$. Therefore $\alpha \in T_{(x_1, x_2)} M$ must be in the form of $(a, b, -\frac{ax_1 + bx_2}{\sqrt{1 - x_1^2 - x_2^2}})$ for $a, b \in \mathbb{R}$. Let $\beta := (c, d, -\frac{cx_1 + dx_2}{\sqrt{1 - x_1^2 - x_2^2}}) \in T_{(x_1, x_2)} M$. Then

$$\begin{aligned} \langle \alpha, \beta \rangle &= ac + bd + \frac{acx_1^2 + bdx_2^2 + (ad + bc)(x_1 x_2)}{1 - x_1^2 - x_2^2} \\ &= \frac{1}{1 - x_1^2 - x_2^2} (ac(1 - x_2^2) + bd(1 - x_1^2) + ad(x_1 x_2) + bc(x_2 x_1)) \end{aligned}$$

where the entries of the metric tensor can be read off.

Observation 1.5. The length can thus be defined given the generalization of inner product on the structure. For $\gamma : [0, 1] \rightarrow M$, the length of γ

$$\|\gamma\| = \int_0^1 \sqrt{g_{\gamma(t)} \left(\frac{d}{dt} \gamma(t), \frac{d}{dt} \gamma(t) \right)} dt =: \int_0^1 \sqrt{g_{\gamma(t)} (\dot{\gamma}, \dot{\gamma})} dt$$

If M is connected, Then the distance between $a, b \in M$ is $\inf_{\gamma(0)=a, \gamma(1)=b} \|\gamma\|$.

2 Vector Bundle

Definition 2.1 (Vector Bundle). A **(real) vector bundle of rank k over M** is a surjection $\pi : \mathcal{E} \rightarrow M$ satisfying the following properties:

1. For all $p \in M$, $\mathcal{E}_p := \pi^{-1}(p)$ has a real vector space structure of dimension k .
2. For all $p \in M$, there exists an open neighborhood $U \subset M$ containing p , and a diffeomorphism $\chi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ together with the projection $p : U \times \mathbb{R}^n \rightarrow U$ s.t.
 - $\pi|_U = p \circ \chi$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\chi} & U \times \mathbb{R}^n \\ & \searrow \pi|_{\pi^{-1}(U)} & \downarrow p \\ & & U \end{array}$$

- For all $p \in U$, $\chi|_p : \mathcal{E}_p \rightarrow \{p\} \times \mathbb{R}^n$ is a linear isomorphism.

E is the total space, and M is the base. The map χ is called the local trivialization of E at p .

Example 2.2. The following gives some examples of vector bundle:

1. The vector bundle which associates every point in M the vector space \mathbb{R}^n , given by $\pi : M \times \mathbb{R}^n \rightarrow M$ is the trivial bundle.
2. Consider $\mathcal{E} = TM$ which is the tangent bundle (or isomorphically, the cotangent bundle T^*M), defined as $TM = \coprod_{p \in M} T_p M$. Then if $\phi = (x^1, \dots, x^n)$ is a coordinate system on $U \subset M$, we have a basis $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$ a basis of $T_p M$ for each $p \in M$. The map χ is then given by

$$\chi : \pi^{-1}(U) = TU \rightarrow U \times \mathbb{R}^n, \quad (p, T_p M \ni v) \mapsto (p, \langle v^1, \dots, v^n \rangle) \text{ s.t. } v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$$

χ is a diffeomorphism as the tangent space TM has a smooth structure. Similarly one can consider the dual T^*M with basis $\{dx^1, \dots, dx^n\}$. Then the corresponding map becomes

$$\chi : \pi^{-1}(U) = T^*U \rightarrow U \times \mathbb{R}^n, \quad (p, T_p^* M \ni \alpha) \mapsto (p, \langle a_1, \dots, a_n \rangle) \text{ s.t. } \alpha = \sum_{i=1}^n a_i dx^i$$

where $\alpha : T_p M \rightarrow \mathbb{R}$ is a linear map.

3. Let $M \subset \mathbb{R}^N$ be a n -manifold. Define $\mathcal{E} = \{(p, v) \mid p \in \mathcal{E}, v \in (T_p M)^\perp\}$, The map can be defined as

$$\chi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{N-n}, \quad (p, v) \mapsto (p, v)$$

which is the identity if one identifies $T_p M$ with a subspace of \mathbb{R}^N .

Definition 2.3 (Section). Let $\pi : \mathcal{E} \rightarrow M$ be a vector bundle. Then a **(smooth) section** is a (smooth) map $s : M \rightarrow \mathcal{E}$ s.t. $\pi \circ s = \text{Id}_M$. The set of all smooth sections is denoted as $\Gamma(\mathcal{E}, M)$, or simply $\Gamma(\mathcal{E})$.

Remark 2.4. Explained in plain words, a section selects an element $s(p) \in \mathcal{E}_p$ for each $p \in M$. Let $\mathcal{E} = TM$, and identifying the tangent space with \mathbb{R}^N for some N , then a section s gives a vector field on M .

Remark 2.5. $\Gamma(\mathcal{E})$ defines a module structure over $C^\infty(M)$, with the scalar multiplication defined as $\chi(f \cdot s(x)) = (x, f(x) \cdot (p_r \circ \chi \circ s)(x))$, where $p_r : (U, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is the projection that takes the second field. This map is smooth, as both f, p_r, χ and s are smooth.

Definition 2.6 (Moving Frame). Let $\mathcal{E} \rightarrow M$ be a vector bundle, and $U \subset M$ an open set. Then a **moving frame** of \mathcal{E} on U is an r -tuple (E_1, \dots, E_r) s.t. for all j , E_j is a section of $\mathcal{E} \supset \pi^{-1}(U) \rightarrow U$; and for all $p \in U$, $(E_1(p), \dots, E_r(p))$ gives a basis of \mathcal{E}_p .

Remark 2.7. There is a bijection between moving frames of \mathcal{E} on U and trivializations $\chi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$:

- Given a moving frame (E_1, \dots, E_r) of \mathcal{E} on U , define

$$\chi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n, \quad v \mapsto (\pi(v), \langle v^1, \dots, v^n \rangle) \text{ s.t. } v = \sum_{i=1}^n v^i E_i(p)$$

- Conversely, given a trivialization $\chi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$, define

$$E_i : M \supset U \rightarrow \pi^{-1}U \subset \mathcal{E}, \quad p \mapsto \chi^{-1}(p, \underbrace{\langle 0, \dots, 0, 1, 0, \dots, n \rangle}_{i\text{-th position}})$$

The vector space structure is induced by the vector space structure of \mathbb{R}^n .

Definition 2.8 (Transition). Let $\pi : \mathcal{E} \rightarrow M$ be a vector bundle, and $U_\alpha, U_\beta \subset M$ two open subsets of M , and χ_α, χ_β be the corresponding local trivializations. Define $\phi_{\alpha,p} := p_r \circ \chi_\alpha$ where $p_r : (U, \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is the projection. Then the **transition** between χ_α and χ_β is a map $\tau_{\alpha\beta}$ s.t. for $f_{\alpha\beta} : U \times \mathbb{R}^n \rightarrow U \times \mathbb{R}^n$, $(p, v) \mapsto (p, \tau_{\alpha\beta}(v))$, the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{\chi_\alpha} & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \\ & \searrow \chi_\beta & \downarrow f_{\alpha\beta} \\ & & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \end{array}$$

Observation 2.9. $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}_r(\mathbb{R})$. It is smooth as χ is a diffeomorphism.

Remark 2.10. By definition it is clear that $\tau_{\alpha\beta} = \tau_{\beta\alpha}^{-1}$; and $\tau_{\alpha\beta} \circ \tau_{\beta\gamma} = \tau_{\alpha\gamma}$.

Observation 2.11. By the fact that any object defined on the vector space can be transferred to the vector bundle via the local trivialization since it is a diffeomorphism, constructions on vector spaces can be identified with constructions on vector bundles.

Lemma 2.12 (Bundle Chart Lemma). Given M a smooth manifold, and for all $p \in M$ associate an r -dimensional vector space \mathcal{E}_p to it. Define $\mathcal{E} := \coprod_{p \in M} \mathcal{E}_p$, and $\pi : \mathcal{E} \rightarrow M, (p, v) \mapsto p$. Further the followings are satisfied:

- There exists an open cover $\{U_\alpha\}$ of M .
- For all α , there exists a bijective map $\chi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ s.t. $\pi = p \circ \chi_\alpha$ where p is the projection $U_\alpha \times \mathbb{R}^n \rightarrow U_\alpha$.

Then there exists a unique C^∞ manifold structure on \mathcal{E} s.t. χ_α is a diffeomorphism; and \mathcal{E} is a C^∞ bundle over M .

The proof of the lemma is via verifying that \mathcal{E} is indeed a manifold, and is omitted here. The more interesting aspect is that this provides a lot of operations on bundles, for example combination of two bundles:

Example 2.13. Let (U_α) be an open cover of M , and $\pi' : \mathcal{E}' \rightarrow M, \pi'' : \mathcal{E}'' \rightarrow M$ be two vector bundles over M . Define $\mathcal{E}' \oplus \mathcal{E}'' \rightarrow M$, where $\mathcal{E}' \oplus \mathcal{E}'' = \coprod_{p \in M} (\mathcal{E}'_p \oplus \mathcal{E}''_p)$. Let (U_α) be a cover of M s.t. both \mathcal{E}' and \mathcal{E}'' trivialize over each U_α . Then it is valid to define the trivialization $\chi : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times (\mathbb{R}^{r'} \oplus \mathbb{R}^{r''})$ s.t. $\phi_{\alpha,p} = \phi'_{\alpha,p} \oplus \phi''_{\alpha,p}$. In matrices this gives a block diagonal matrix.

3 Tensors

Definition 3.1 (Free Vector Space). Let S be a (possibly infinite) set, and K a field. The **free vector space** generated by S over K is defined as the set

$$FS := \left\{ \sum_{s \in S} \alpha_s s \mid \text{finite nonzero } \alpha_s \right\}$$

satisfying the axioms of vector spaces, i.e. linear in s with addition and scalar multiplication.

Definition 3.2 (Tensor Product). Let V, W be finite dimensional vector space. The **tensor product of V and W** , denoted $V \otimes W$ is the free vector space generated by $(V \times W) / \sim$, where \sim follows the rules:

- $(v + v', w) \sim (v, w) + (v', w)$ for $v, v' \in V, w \in W$.
- $(v, w + w') \sim (v, w) + (v, w')$ for $v \in V, w, w' \in W$.
- $(cv, w) \sim c(v, w) \sim (v, cw)$ for $c \in K, v \in V, w \in W$.

This gives a map $\otimes : V \times W \rightarrow V \otimes W, (v, w) \mapsto [(v, w)]$ which is an equivalence class in $V \otimes W$.

Proposition 3.3 (Universal Property of Tensor Product). Let Z be a real vector space, and the map $b : V \times W \rightarrow Z$ bilinear. Then there exists a unique linear map $f : V \otimes W \rightarrow Z$ s.t. the following diagram commute:

$$\begin{array}{ccc}
 V \times W & \xrightarrow{\otimes} & V \otimes W \\
 & \searrow b & \downarrow f \\
 & & Z
 \end{array}$$

Corollary 3.4. Let (v_i) be a basis of V , and (w_i) a basis of W . Then $(v_i \otimes w_j)$ gives a basis of $V \otimes W$, with $\dim V \otimes W = \dim V \cdot \dim W$.

Remark 3.5. Not every element in $V \otimes W$ can be expressed as $v \otimes w$ for some $v \in V, w \in W$. In particular, one can have $v' = \sum_{i \in I} v_i \otimes w_i \in V \otimes W$, whose representative cannot be further simplified.

Proposition 3.6. Let V and W be finite dimensional real vector space. Then there exists a natural isomorphism $V \otimes W \xrightarrow{\sim} \text{Hom}(V^*, W)$. In particular, this gives $V^* \otimes V^* \simeq \text{Hom}(V, V^*) \simeq \{\text{bilinear forms } V \times V \rightarrow \mathbb{R}\}$.

Proof. To construct the map it suffices to give a bilinear map $V \times W \rightarrow \text{Hom}(V^*, W)$. Define it as

$$(v, w) \mapsto (\alpha \mapsto \alpha(v)w) \in \text{Hom}(V^*, W)$$

Applying the universal property gives the desired result. □

Remark 3.7. This can be extended to multiple tensor products:

$$\underbrace{V \otimes \cdots \otimes V}_k \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_\ell \simeq (\underbrace{V^* \times \cdots \times V^*}_k \times \underbrace{V \times \cdots \times V}_\ell \rightarrow \mathbb{R})$$

where RHS is a multilinear map.

Definition 3.8 (Tensor). Denote

$$G = \underbrace{V \otimes \cdots \otimes V}_k \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_\ell$$

Then an element in G is called a (k, ℓ) -**tensor** on V . k is the contravariant degree, and ℓ is the covariant degree.

Remark 3.9. It is possible to define the tensor product of maps. Namely, for $f : V \rightarrow V'$ and $g : W \rightarrow W'$, there exists a unique linear map $f \otimes g : V \otimes W \rightarrow V' \otimes W'$ s.t. $(f \otimes g)(v \otimes w) = (f(v)) \otimes (g(w))$. This can be seen via consider first the \mathbb{R} -balanced linear map $V \times W \rightarrow V \otimes W$; and apply the universal property of tensor product gives the desired result.

Observation 3.10. Similar to Example 2.13 where we considered the direct sum of vector bundles as a new bundle, we can consider the tensor product of vector bundles. For \mathcal{E}' and \mathcal{E}'' two vector bundles, $\coprod_{p \in M} \mathcal{E}'_p \otimes \mathcal{E}''_p$ gives a vector bundle structure over M . In particular, this can be extended to the tangent and cotangent spaces of a manifold.

Notation. Given a smooth manifold M , for $k, \ell \in \mathbb{Z}_{\geq 0}$, denote

$$\Pi^{(k,\ell)} M = \underbrace{TM \otimes \cdots \otimes TM}_k \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_\ell$$

In particular, this is TM for $k = 1, \ell = 0$, and is T^*M for $k = 0, \ell = 1$. By identification in Remark 3.7 maps $\Pi^{(k,\ell)} M \rightarrow M$ are always multilinear (with the identification with the Euclidean Space).

Definition 3.11 (Tensor Field). Let M be a smooth manifold. A (k, ℓ) -**tensor (field) on M** is a smooth section of $\Pi^{(k,\ell)} M$.

In local coordinates, if we have local coordinates (x^1, \dots, x^n) on $U \subseteq M$, then any element in $\Pi^{(k,\ell)} M$ is in the form of

$$f_{j_1, \dots, j_\ell}^{i_1, \dots, i_k} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}$$

where i_k s and j_ℓ s run over $\{1, \dots, n\}$ arbitrarily. A section on $\Pi^{(k,\ell)} M$ is then $\sum_{i,j} f_{j_1, \dots, j_\ell}^{i_1, \dots, i_k}$, with it being smooth implies that all such f s are smooth (as functions).

4 Riemannian Metric

5 Coverings

6 Metric on Lee Group

7 Common Objects on Riemannian Manifolds