

# MATH 635 - Differential Topology Preliminaries

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## Contents

1	Riemannian Structure	2
2	Vector Bundle	3
3	Tensors	5
4	Riemannian Metric	7
5	Coverings	7
6	Metric on Lee Group	7
7	Common Objects on Riemannian Manifolds	7

# 1 Riemannian Structure

**Definition 1.1 (Riemannian Structure).** Let  $M$  be a smooth  $n$ -manifold. Then a **Riemannian Structure** on it is an assignment  $M \ni p \mapsto g_p$ , where  $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$  is a bilinear positive definite symmetric form, that depends smoothly on  $p$ . Such  $g$  is a **Riemannian metric**.

Specifically, if  $(x^1, \dots, x^n)$  is a coordinate system on  $U \subseteq M$ , then for  $i, j \in \llbracket 1, n \rrbracket$ ,  $p \in U$ , define

$$g_{ij}(p) = g_p \left( \left. \frac{\partial}{\partial x^i} \right|_p, \left. \frac{\partial}{\partial x^j} \right|_p \right)$$

where  $\left. \frac{\partial}{\partial x^i} \right|_p \in T_p M$  for all  $i$ . Then  $g_{ij}$  is  $C^\infty$ ; and  $g(p) = (g_{ij}(p))$  is a symmetric matrix that depends on  $p$ . The matrix is often referred to as the **metric tensor**. Evaluation on the metric can be done via  $g_p(v_1, v_2) = v_1^T g(p) v_2$ .

**Definition 1.2 (Riemannian Manifold).** A **Riemannian Manifold** is a smooth manifold  $M$  endowed with a Riemannian Metric  $g$ , often denoted as pair  $(M, g)$ .

**Example 1.3.** Let  $M \subseteq \mathbb{R}^N$  be a smooth manifold. Then for all  $p \in M$ , via embedding the tangent space into  $\mathbb{R}^N$ ,  $T_p M \subseteq \mathbb{R}^N$ . The inner product in the usual sense (dot product in  $\mathbb{R}^n$ ) gives  $M$  a Riemannian structure. This implies that Euclidean Space obtains a Riemannian structure.

**Example 1.4.** Take  $M = S^2 \subset \mathbb{R}^3$ , and let  $U = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ . Specify the Riemannian structure as the inner product in  $\mathbb{R}^3$ , with the tangent space regarded as planes in  $\mathbb{R}^3$ , taking the local coordinate system as  $(x_1, x_2)$ , then at  $(x_1, x_2)$ , the metric tensor for the tangent space is given by

$$g = \frac{1}{1 - (x_1^2 + x_2^2)} \begin{pmatrix} 1 - x_2^2 & x_1 x_2 \\ x_1 x_2 & 1 - x_1^2 \end{pmatrix}$$

To see that this is indeed the metric, notice that at  $T_{(x_1, x_2)} M$ , the normal vector is given by  $(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$ . Therefore  $\alpha \in T_{(x_1, x_2)} M$  must be in the form of  $(a, b, -\frac{ax_1 + bx_2}{\sqrt{1 - x_1^2 - x_2^2}})$  for  $a, b \in \mathbb{R}$ . Let  $\beta := (c, d, -\frac{cx_1 + dx_2}{\sqrt{1 - x_1^2 - x_2^2}}) \in T_{(x_1, x_2)} M$ . Then

$$\begin{aligned} \langle \alpha, \beta \rangle &= ac + bd + \frac{acx_1^2 + bdx_2^2 + (ad + bc)(x_1 x_2)}{1 - x_1^2 - x_2^2} \\ &= \frac{1}{1 - x_1^2 - x_2^2} (ac(1 - x_2^2) + bd(1 - x_1^2) + ad(x_1 x_2) + bc(x_2 x_1)) \end{aligned}$$

where the entries of the metric tensor can be read off.

**Observation 1.5.** The length can thus be defined given the generalization of inner product on the structure. For  $\gamma : [0, 1] \rightarrow M$ , the length of  $\gamma$

$$\|\gamma\| = \int_0^1 \sqrt{g_{\gamma(t)} \left( \frac{d}{dt} \gamma(t), \frac{d}{dt} \gamma(t) \right)} dt =: \int_0^1 \sqrt{g_{\gamma(t)} (\dot{\gamma}, \dot{\gamma})} dt$$

If  $M$  is connected, Then the distance between  $a, b \in M$  is  $\inf_{\gamma(0)=a, \gamma(1)=b} \|\gamma\|$ .

## 2 Vector Bundle

**Definition 2.1 (Vector Bundle).** A **(real) vector bundle of rank  $k$  over  $M$**  is a surjection  $\pi : \mathcal{E} \rightarrow M$  satisfying the following properties:

1. For all  $p \in M$ ,  $\mathcal{E}_p := \pi^{-1}(p)$  has a real vector space structure of dimension  $k$ .
2. For all  $p \in M$ , there exists an open neighborhood  $U \subset M$  containing  $p$ , and a diffeomorphism  $\chi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$  together with the projection  $p : U \times \mathbb{R}^n \rightarrow U$  s.t.
  - $\pi|_U = p \circ \chi$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\chi} & U \times \mathbb{R}^n \\ & \searrow \pi|_{\pi^{-1}(U)} & \downarrow p \\ & & U \end{array}$$

- For all  $p \in U$ ,  $\chi|_p : \mathcal{E}_p \rightarrow \{p\} \times \mathbb{R}^n$  is a linear isomorphism.

$E$  is the total space, and  $M$  is the base. The map  $\chi$  is called the local trivialization of  $E$  at  $p$ .

**Example 2.2.** The following gives some examples of vector bundle:

1. The vector bundle which associates every point in  $M$  the vector space  $\mathbb{R}^n$ , given by  $\pi : M \times \mathbb{R}^n \rightarrow M$  is the trivial bundle.
2. Consider  $\mathcal{E} = TM$  which is the tangent bundle (or isomorphically, the cotangent bundle  $T^*M$ ), defined as  $TM = \coprod_{p \in M} T_p M$ . Then if  $\phi = (x^1, \dots, x^n)$  is a coordinate system on  $U \subset M$ , we have a basis  $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$  a basis of  $T_p M$  for each  $p \in M$ . The map  $\chi$  is then given by

$$\chi : \pi^{-1}(U) = TU \rightarrow U \times \mathbb{R}^n, \quad (p, T_p M \ni v) \mapsto (p, \langle v^1, \dots, v^n \rangle) \text{ s.t. } v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$$

$\chi$  is a diffeomorphism as the tangent space  $TM$  has a smooth structure. Similarly one can consider the dual  $T^*M$  with basis  $\{dx^1, \dots, dx^n\}$ . Then the corresponding map becomes

$$\chi : \pi^{-1}(U) = T^*U \rightarrow U \times \mathbb{R}^n, \quad (p, T_p^* M \ni \alpha) \mapsto (p, \langle a_1, \dots, a_n \rangle) \text{ s.t. } \alpha = \sum_{i=1}^n a_i dx^i$$

where  $\alpha : T_p M \rightarrow \mathbb{R}$  is a linear map.

3. Let  $M \subset \mathbb{R}^N$  be a  $n$ -manifold. Define  $\mathcal{E} = \{(p, v) \mid p \in \mathcal{E}, v \in (T_p M)^\perp\}$ , The map can be defined as

$$\chi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{N-n}, \quad (p, v) \mapsto (p, v)$$

which is the identity if one identifies  $T_p M$  with a subspace of  $\mathbb{R}^N$ .

**Definition 2.3 (Section).** Let  $\pi : \mathcal{E} \rightarrow M$  be a vector bundle. Then a **(smooth) section** is a (smooth) map  $s : M \rightarrow \mathcal{E}$  s.t.  $\pi \circ s = \text{Id}_M$ . The set of all smooth sections is denoted as  $\Gamma(\mathcal{E}, M)$ , or simply  $\Gamma(\mathcal{E})$ .

**Remark 2.4.** Explained in plain words, a section selects an element  $s(p) \in \mathcal{E}_p$  for each  $p \in M$ . Let  $\mathcal{E} = TM$ , and identifying the tangent space with  $\mathbb{R}^N$  for some  $N$ , then a section  $s$  gives a vector field on  $M$ .

**Remark 2.5.**  $\Gamma(\mathcal{E})$  defines a module structure over  $C^\infty(M)$ , with the scalar multiplication defined as  $\chi(f \cdot s(x)) = (x, f(x) \cdot (p_r \circ \chi \circ s)(x))$ , where  $p_r : (U, \mathbb{R}^n) \rightarrow \mathbb{R}^n$  is the projection that takes the second field. This map is smooth, as both  $f, p_r, \chi$  and  $s$  are smooth.

**Definition 2.6 (Moving Frame).** Let  $\mathcal{E} \rightarrow M$  be a vector bundle, and  $U \subset M$  an open set. Then a **moving frame** of  $\mathcal{E}$  on  $U$  is an  $r$ -tuple  $(E_1, \dots, E_r)$  s.t. for all  $j$ ,  $E_j$  is a section of  $\mathcal{E} \supset \pi^{-1}(U) \rightarrow U$ ; and for all  $p \in U$ ,  $(E_1(p), \dots, E_r(p))$  gives a basis of  $\mathcal{E}_p$ .

**Remark 2.7.** There is a bijection between moving frames of  $\mathcal{E}$  on  $U$  and trivializations  $\chi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ :

- Given a moving frame  $(E_1, \dots, E_r)$  of  $\mathcal{E}$  on  $U$ , define

$$\chi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n, \quad v \mapsto (\pi(v), \langle v^1, \dots, v^n \rangle) \text{ s.t. } v = \sum_{i=1}^n v^i E_i(p)$$

- Conversely, given a trivialization  $\chi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ , define

$$E_i : M \supset U \rightarrow \pi^{-1}U \subset \mathcal{E}, \quad p \mapsto \chi^{-1}(p, \langle 0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th position}}}{1}, 0, \dots, n \rangle)$$

The vector space structure is induced by the vector space structure of  $\mathbb{R}^n$ .

**Definition 2.8 (Transition).** Let  $\pi : \mathcal{E} \rightarrow M$  be a vector bundle, and  $U_\alpha, U_\beta \subset M$  two open subsets of  $M$ , and  $\chi_\alpha, \chi_\beta$  be the corresponding local trivializations. Define  $\phi_{\alpha,p} := p_r \circ \chi_\alpha$  where  $p_r : (U, \mathbb{R}^n) \rightarrow \mathbb{R}^n$  is the projection. Then the **transition** between  $\chi_\alpha$  and  $\chi_\beta$  is a map  $\tau_{\alpha\beta}$  s.t. for  $f_{\alpha\beta} : U \times \mathbb{R}^n \rightarrow U \times \mathbb{R}^n$ ,  $(p, v) \mapsto (p, \tau_{\alpha\beta}(v))$ , the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{\chi_\alpha} & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \\ & \searrow \chi_\beta & \downarrow f_{\alpha\beta} \\ & & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \end{array}$$

**Observation 2.9.**  $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}_r(\mathbb{R})$ . It is smooth as  $\chi$  is a diffeomorphism.

**Remark 2.10.** By definition it is clear that  $\tau_{\alpha\beta} = \tau_{\beta\alpha}^{-1}$ ; and  $\tau_{\alpha\beta} \circ \tau_{\beta\gamma} = \tau_{\alpha\gamma}$ .

**Observation 2.11.** By the fact that any object defined on the vector space can be transferred to the vector bundle via the local trivialization since it is a diffeomorphism, constructions on vector spaces can be identified with constructions on vector bundles.

**Lemma 2.12** (Bundle Chart Lemma). Given  $M$  a smooth manifold, and for all  $p \in M$  associate an  $r$ -dimensional vector space  $\mathcal{E}_p$  to it. Define  $\mathcal{E} := \coprod_{p \in M} \mathcal{E}_p$ , and  $\pi : \mathcal{E} \rightarrow M$ ,  $(p, v) \mapsto p$ . Further the followings are satisfied:

- There exists an open cover  $\{U_\alpha\}$  of  $M$ .
- For all  $\alpha$ , there exists a bijective map  $\chi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$  s.t.  $\pi = p \circ \chi_\alpha$  where  $p$  is the projection  $U_\alpha \times \mathbb{R}^n \rightarrow U_\alpha$ .

Then there exists a unique  $C^\infty$  manifold structure on  $\mathcal{E}$  s.t.  $\chi_\alpha$  is a diffeomorphism; and  $\mathcal{E}$  is a  $C^\infty$  bundle over  $M$ .

The proof of the lemma is via verifying that  $\mathcal{E}$  is indeed a manifold, and is omitted here. The more interesting aspect is that this provides a lot of operations on bundles, for example combination of two bundles:

**Example 2.13.** Let  $(U_\alpha)$  be an open cover of  $M$ , and  $\pi' : \mathcal{E}' \rightarrow M$ ,  $\pi'' : \mathcal{E}'' \rightarrow M$  be two vector bundles over  $M$ . Define  $\mathcal{E}' \oplus \mathcal{E}'' \rightarrow M$ , where  $\mathcal{E}' \oplus \mathcal{E}'' = \coprod_{p \in M} (\mathcal{E}'_p \oplus \mathcal{E}''_p)$ . Let  $(U_\alpha)$  be a cover of  $M$  s.t. both  $\mathcal{E}'$  and  $\mathcal{E}''$  trivialize over each  $U_\alpha$ . Then it is valid to define the trivialization  $\chi : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times (\mathbb{R}^{r'} \oplus \mathbb{R}^{r''})$  s.t.  $\phi_{\alpha,p} = \phi'_{\alpha,p} \oplus \phi''_{\alpha,p}$ . In matrices this gives a block diagonal matrix.

### 3 Tensors

**Definition 3.1** (Free Vector Space). Let  $S$  be a (possibly infinite) set, and  $K$  a field. The **free vector space** generated by  $S$  over  $K$  is defined as the set

$$FS := \left\{ \sum_{s \in S} \alpha_s s \mid \text{finite nonzero } \alpha_s \right\}$$

satisfying the axioms of vector spaces, i.e. linear in  $s$  with addition and scalar multiplication.

**Definition 3.2** (Tensor Product). Let  $V, W$  be finite dimensional vector space. The **tensor product of  $V$  and  $W$** , denoted  $V \otimes W$  is the free vector space generated by  $(V \times W) / \sim$ , where  $\sim$  follows the rules:

- $(v + v', w) \sim (v, w) + (v', w)$  for  $v, v' \in V, w \in W$ .
- $(v, w + w') \sim (v, w) + (v, w')$  for  $v \in V, w, w' \in W$ .
- $(cv, w) \sim c(v, w) \sim (v, cw)$  for  $c \in K, v \in V, w \in W$ .

This gives a map  $\otimes : V \times W \rightarrow V \otimes W$ ,  $(v, w) \mapsto [(v, w)]$  which is an equivalence class in  $V \otimes W$ .

**Proposition 3.3** (Universal Property of Tensor Product). Let  $Z$  be a real vector space, and the map  $b : V \times W \rightarrow Z$  bilinear. Then there exists a unique linear map  $f : V \otimes W \rightarrow Z$  s.t. the following diagram commute:

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes} & V \otimes W \\ & \searrow b & \downarrow f \\ & & Z \end{array}$$

**Corollary 3.4.** Let  $(v_i)$  be a basis of  $V$ , and  $(w_i)$  a basis of  $W$ . Then  $(v_i \otimes w_j)$  gives a basis of  $V \otimes W$ , with  $\dim V \otimes W = \dim V \cdot \dim W$ .

**Remark 3.5.** Not every element in  $V \otimes W$  can be expressed as  $v \otimes w$  for some  $v \in V, w \in W$ . In particular, one can have  $v' = \sum_{i \in I} v_i \otimes w_i \in V \otimes W$ , whose representative cannot be further simplified.

**Proposition 3.6.** Let  $V$  and  $W$  be finite dimensional real vector space. Then there exists a natural isomorphism  $V \otimes W \xrightarrow{\sim} \text{Hom}(V^*, W)$ . In particular, this gives  $V^* \otimes V^* \simeq \text{Hom}(V, V^*) \simeq \{\text{bilinear forms } V \times V \rightarrow \mathbb{R}\}$ .

*Proof.* To construct the map it suffices to give a bilinear map  $V \times W \rightarrow \text{Hom}(V^*, W)$ . Define it as

$$(v, w) \mapsto (\alpha \mapsto \alpha(v)w) \in \text{Hom}(V^*, W)$$

Applying the universal property gives the desired result. □

**Remark 3.7.** This can be extended to multiple tensor products:

$$\underbrace{V \otimes \cdots \otimes V}_k \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_\ell \simeq \underbrace{(V^* \times \cdots \times V^*)_k \times \underbrace{(V \times \cdots \times V)_\ell}_{\rightarrow \mathbb{R}}}$$

where RHS is a multilinear map.

**Definition 3.8 (Tensor).** Denote

$$G = \underbrace{V \otimes \cdots \otimes V}_k \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_\ell$$

Then an element in  $G$  is called a  $(k, \ell)$ -**tensor** on  $V$ .  $k$  is the contravariant degree, and  $\ell$  is the covariant degree.

**Remark 3.9.** It is possible to define the tensor product of maps. Namely, for  $f : V \rightarrow V'$  and  $g : W \rightarrow W'$ , there exists a unique linear map  $f \otimes g : V \otimes W \rightarrow V' \otimes W'$  s.t.  $(f \otimes g)(v \otimes w) = (f(v)) \otimes (g(w))$ . This can be seen via consider first the  $\mathbb{R}$ -balanced linear map  $V \times W \rightarrow V \otimes W$ ; and apply the universal property of tensor product gives the desired result.

**Observation 3.10.** Similar to Example 2.13 where we considered the direct sum of vector bundles as a new bundle, we can consider the tensor product of vector bundles. For  $\mathcal{E}'$  and  $\mathcal{E}''$  two vector bundles,  $\coprod_{p \in M} \mathcal{E}'_p \otimes \mathcal{E}''_p$  gives a vector bundle structure over  $M$ . In particular, this can be extended to the tangent and cotangent spaces of a manifold.

**Notation.** Given a smooth manifold  $M$ , for  $k, \ell \in \mathbb{Z}_{\geq 0}$ , denote

$$\Pi^{(k, \ell)} M = \underbrace{TM \otimes \cdots \otimes TM}_k \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_\ell$$

In particular, this is  $TM$  for  $k = 1, \ell = 0$ , and is  $T^*M$  for  $k = 0, \ell = 1$ . By identification in Remark 3.7 maps  $\Pi^{(k, \ell)} M \rightarrow M$  are always multilinear (with the identification with the Euclidean Space).

**Definition 3.11 (Tensor Field).** Let  $M$  be a smooth manifold. A  $(k, \ell)$ -**tensor (field) on  $M$**  is a smooth section of  $\Pi^{(k, \ell)} M$ .

In local coordinates, if we have local coordinates  $(x^1, \dots, x^n)$  on  $U \subseteq M$ , then any element in  $\Pi^{(k, \ell)} M$  is in the form of

$$f_{j_1, \dots, j_\ell}^{i_1, \dots, i_k} := \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_\ell}$$

where  $i_k$ s and  $j_\ell$ s run over  $\{1, \dots, n\}$  arbitrarily. A section on  $\Pi^{(k, \ell)} M$  is then  $\sum_{i, j} f_{j_1, \dots, j_\ell}^{i_1, \dots, i_k}$ , with it being smooth implies that all such  $f$ s are smooth (as functions).

## 4 Riemannian Metric

## 5 Coverings

## 6 Metric on Lee Group

## 7 Common Objects on Riemannian Manifolds