MATH 635 - Basic Riemannian Constructions

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Contents

1 Riemannian Structure

Definition 1.1 (Riemannian Structure). Let M be a smooth n-manifold. Then a **Riemannian Structure** on it is an assignment $M \ni p \mapsto g_p$, where $g_p : T_pM \times T_pM \to \mathbb{R}$ is a bilinear, positive-definite, symmetric form, that depends smoothly on p. Such g is a **Riemannian metric**.

Specifically, if $(x^1, \dots, x^n)$ is a coordinate system on $U \subseteq M$, then for $i, j \in [\![1, n]\!]$, $p \in U$, define

$$g_{ij}(p) = g_p\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right)$$

where $\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$ for all i. Then g_{ij} is C^∞; and $g(p) = (g_{ij}(p))$ is a symmetric matrix that depends on p. The matrix is often referred to as the **metric tensor**. Evaluation on the metric can be done via $g_p(v_1, v_2) = v_1^T g(p) v_2$.

Example 1.2. Let $M \subseteq \mathbb{R}^N$ be a smooth manifold. Then for all $p \in M$, via embedding the tangent space into $\mathbb{R}^N$, $T_pM \subseteq \mathbb{R}^N$. The inner product in the usual sense (dot product in $\mathbb{R}^n$) gives M a Riemannian structure. This implies that Euclidean Space obtains a Riemannian structure.

Example 1.3. Take $M = S^2 \subset \mathbb{R}^3$, and let $U = S^2 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$. Specify the Riemannian structure as the inner product in $\mathbb{R}^3$, with the tangent space regarded as planes in $\mathbb{R}^3$, taking the local coordinate system as (x_1, x_2), then at (x_1, x_2), the metric tensor for the tangent space is given by

$$g = \frac{1}{1 - (x_1^2 + x_2^2)} \begin{pmatrix} 1 - x_2^2 & x_1 x_2 \\ x_1 x_2 & 1 - x_1^2 \end{pmatrix}$$

To see that this is indeed the metric, notice that at $T_{(x_1,x_2)}M$, the normal vector is given by $(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$. Therefore $\boldsymbol{\alpha} \in T_{(x_1,x_2)}M$ must be in the form of $(a, b, -\frac{ax_1 + bx_2}{\sqrt{1 - x_1^2 - x_2^2}})$ for $a, b \in \mathbb{R}$. Let $\boldsymbol{\beta} := (c, d, -\frac{cx_1 + dx_2}{\sqrt{1 - x_1^2 - x_2^2}}) \in T_{(x_1,x_2)}M$. Then

$$\begin{aligned} \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle &= ac + bd + \frac{acx_1^2 + bdx^2 + (ad + bc)(x_1 x_2)}{1 - x_1^2 - x_2^2} \\ &= \frac{1}{1 - x_1^2 - x_2^2}\left(ac(1 - x_2^2) + bd(1 - x_1^2) + ad(x_1 x_2) + bc(x_2 x_1)\right) \end{aligned}$$

where the entries of the metric tensor can be read off.

Observation 1.4. The length can thus be defined given the generalization of inner product on the structure. For $\gamma : [0, 1] \to M$, the length of γ

$$\|\gamma\| = \int_0^1 \sqrt{g_{\gamma(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t), \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right)}\mathrm{d}t =: \int_0^1 \sqrt{g_{\gamma(t)}(\dot\gamma, \dot\gamma)}\mathrm{d}t$$

If M is connected, Then the distance between $a, b \in M$ is $\inf_{\gamma(0)=a, \gamma(1)=b} \|\gamma\|$.

2 Vector Bundle

Definition 2.1 (Vector Bundle). A **(real) vector bundle of rank k over M** is a surjection $\pi : \mathcal{E} \to M$ satisfying the following properties:

1. For all $p \in M$, $\mathcal{E}_p := \pi^{-1}(p)$ has a real vector space structure of dimension k.
2. For all $p \in M$, there exists an open neighborhood $U \subset M$ containing p, and a diffeomorphism $\chi : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ together with the projection $p : U \times \mathbb{R}^n \to U$ s.t.
   - $\pi|_U = p \circ \chi$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\chi"] \arrow[dr, "\pi|_{\pi^{-1}(U)}"'] & U \times \mathbb{R}^n \arrow[d, "p"] \\
& U
\end{tikzcd}$$

   - For all $p \in U$, $\chi|_p : \mathcal{E}_p \to \{p\} \times \mathbb{R}^n$ is a linear isomorphism.

E is the total space, and M is the base. The map χ is called the local trivialization of E at p.

Example 2.2. The following gives some examples of vector bundle:

1. The vector bundle which associates every point in M the vector space $\mathbb{R}^n$, given by $\pi : M \times \mathbb{R}^n \to M$ is the trivial bundle.
2. Consider $\mathcal{E} = TM$ which is the tangent bundle (or isomorphically, the cotangent bundle T^*M), defined as $TM = \coprod_{p \in M} T_pM$. Then if $\phi = (x^1, \ldots, x^n)$ is a coordinate system on $U \subset M$, we have a basis $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ a basis of T_pM for each $p \in M$. The map χ is then given by

$$\chi : \pi^{-1}(U) = TU \to U \times \mathbb{R}^n, \quad (p, T_pM \ni v) \mapsto (p, \langle v^1, \ldots, v^n \rangle) \text{ s.t. } v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$$

   χ is a diffeomorphism as the tangent space TM has a smooth structure. Similarly one can consider the dual T^*M with basis $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^n\}$. Then the corresponding map becomes

$$\chi : \pi^{-1}(U) = T^*U \to U \times \mathbb{R}^n, \quad (p, T_p^*M \ni \alpha) \mapsto (p, \langle a_1, \ldots, a_n \rangle) \text{ s.t. } \alpha = \sum_{i=1}^n a_i \mathrm{d}x^i$$

   where $\alpha : T_pM \to \mathbb{R}$ is a linear map.
3. Let $M \subset \mathbb{R}^N$ be a n-manifold. Define $\mathcal{E} = \left\{(p, v) \mid p \in \mathcal{E}, v \in (T_pM)^\perp\right\}$, The map can be defined as

$$\chi : \pi^{-1}(U) \to U \times \mathbb{R}^{N-n}, \quad (p, v) \mapsto (p, v)$$

   which is the identity if one identifies T_pM with a subspace of $\mathbb{R}^N$.

Definition 2.3 (Section). Let $\pi : \mathcal{E} \to M$ be a vector bundle. Then a **(smooth) section** is a (smooth) map $s : M \to \mathcal{E}$ s.t. $\pi \circ s = \mathrm{Id}_M$. The set of all smooth sections is denoted as $\Gamma(\mathcal{E}, M)$, or simply $\Gamma(\mathcal{E})$.

Remark 2.4. Explained in plain words, a section selects an element $s(p) \in \mathcal{E}_p$ for each $p \in M$. Let $\mathcal{E} = TM$, and identifying the tangent space with $\mathbb{R}^N$ for some N, then a section s gives a vector field on M.

Remark 2.5. $\Gamma(\mathcal{E})$ defines a module structure over $C^\infty(M)$, with the scalar multiplication defined as $\chi(f \cdot s(x)) = (x, f(x) \cdot (p_r \circ \chi \circ s)(x))$, where $p_r : (U, \mathbb{R}^n) \to \mathbb{R}^n$ is the projection that takes the second field. This map is smooth, as both f, p_r, χ and s are smooth.

Definition 2.6 (Moving Frame). Let $\mathcal{E} \to M$ be a vector bundle, and $U \subset M$ an open set. Then a **moving frame** of $\mathcal{E}$ on U is an r-tuple $(E_1, \ldots, E_r)$ s.t. for all j, E_j is a section of $\mathcal{E} \supset \pi^{-1}(U) \to U$; and for all $p \in U$, $(E_1(p), \ldots, E_r(p))$ gives a basis of $\mathcal{E}_p$.

Remark 2.7. There is a bijection between moving frames of $\mathcal{E}$ on U and trivializations $\chi : \pi^{-1}(U) \to U \times \mathbb{R}^n$:

- Given a moving frame $(E_1, \ldots, E_r)$ of $\mathcal{E}$ on U, define

$$\chi : \pi^{-1}(U) \to U \times \mathbb{R}^n, \quad v \mapsto (\pi(v), \langle v^1, \ldots, v^n \rangle) \text{ s.t. } v = \sum_{i=1}^{n} v^i E_i(p)$$

- Conversely, given a trivialization $\chi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, define

$$E_i : M \supset U \to \pi^{-1}U \subset \mathcal{E}, \quad p \mapsto \chi^{-1}(p, \langle 0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{-th position}}}{1}, 0, \ldots, n \rangle)$$

The vector space structure is induced by the vector space structure of $\mathbb{R}^n$.

Definition 2.8 (Transition). Let $\pi : \mathcal{E} \to M$ be a vector bundle, and $U_\alpha, U_\beta \subset M$ two open subsets of M, and χ_α, χ_β be the corresponding local trivializations. Define $\phi_{\alpha,p} := p_r \circ \chi_\alpha$ where $p_r : (U, \mathbb{R}^n) \to \mathbb{R}^n$ is the projection. Then the **transition** between χ_α and χ_β is a map $\tau_{\alpha\beta}$ s.t. for $f_{\alpha\beta} : U \times \mathbb{R}^n \to U \times \mathbb{R}^n, (p, v) \mapsto (p, \tau_{\alpha\beta}(v))$, the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\chi_\alpha"] \arrow[dr, "\chi_\beta"'] & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \arrow[d, "f_{\alpha\beta}"] \\
& (U_\alpha \cap U_\beta) \times \mathbb{R}^r
\end{tikzcd}$$

Observation 2.9. $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{R})$. It is smooth as χ is a diffeomorphism.

Remark 2.10. By definition it is clear that $\tau_{\alpha\beta} = \tau_{\beta\alpha}^{-1}$; and $\tau_{\alpha\beta} \circ \tau_{\beta\gamma} = \tau_{\alpha\gamma}$.

Observation 2.11. By the fact that any object defined on the vector space can be transferred to the vector bundle via the local trivialization since it is a diffeomorphism, constructions on vector spaces can be identified with constructions on vector bundles.

Lemma 2.12 (Bundle Chart Lemma). Given M a smooth manifold, and for all $p \in M$ associate an r-dimensional vector space $\mathcal{E}_p$ to it. Define $\mathcal{E} := \coprod_{p\in M} \mathcal{E}_p$, and $\pi : \mathcal{E} \to M, (p, v) \mapsto p$. Further the followings are satisfied:

- There exists an open cover $\{U_\alpha\}$ of M.
- For all α, there exists a bijective map $\chi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ s.t. $\pi = p \circ \chi_\alpha$ where p is the projection $U_\alpha \times \mathbb{R}^n \to U_\alpha$.

Then there exists a unique C^∞ manifold structure on $\mathcal{E}$ s.t. χ_α is a diffeomorphism; and $\mathcal{E}$ is a C^∞ bundle over M.

The proof of the lemma is via verifying that $\mathcal{E}$ is indeed a manifold, and is omitted here. The more interesting aspect is that this provides a lot of operations on bundles, for example combination of two bundles:

Example 2.13. Let (U_α) be an open cover of M, and $\pi' : \mathcal{E}' \to M$, $\pi'' : \mathcal{E}'' \to M$ be two vector bundles over M. Define $\mathcal{E}' \oplus \mathcal{E}'' \to M$, where $\mathcal{E}' \oplus \mathcal{E}'' = \coprod_{p\in M}(\mathcal{E}'_p \oplus \mathcal{E}''_p)$, Let (U_α) be a cover of M s.t. both $\mathcal{E}'$ and $\mathcal{E}''$ trivialize over each U_α. Then it is valid to define the trivialization $\chi : \pi^{-1}(U_\alpha) \to U_\alpha \times (\mathbb{R}^{r'} \oplus \mathbb{R}^{r''})$ s.t. $\phi_{\alpha,p} = \phi'_{\alpha,p} \oplus \phi''_{\alpha,p}$. In matrices this gives a block diagonal matrix.

3 Tensors

Definition 3.1 (Free Vector Space). Let S be a (possibly infinite) set, and K a field. The **free vector space** generated by S over K is defined as the set

$$FS := \left\{ \sum_{s\in S} \alpha_s s \;\middle|\; \text{finite nonzero } \alpha_s \right\}$$

satisfying the axioms of vector spaces, i.e. linear in s with addition and scalar multiplication.

Definition 3.2 (Tensor Product). Let V, W be finite dimensional vector space. The **tensor product of V and W**, denoted $V \otimes W$ is the free vector space generated by $(V \times W)/\sim$, where $\sim$ follows the rules:

- $(v + v', w) \sim (v, w) + (v', w)$ for $v, v' \in V, w \in W$.
- $(v, w + w') \sim (v, w) + (v, w')$ for $v \in V, w, w' \in W$.
- $(cv, w) \sim c(v, w) \sim (v, cw)$ for $c \in K, v \in V, w \in W$.

This gives a map $\otimes : V \times W \to V \otimes W$, $(v, w) \mapsto [(v, w)]$ which is an equivalence class in $V \otimes W$.

Proposition 3.3 (Universal Property of Tensor Product). Let Z be a real vector space, and the map $b : V \times W \to Z$ bilinear. Then there exists a unique linear map $f : V \otimes W \to Z$ s.t. the following diagram commute:

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "b"'] & V \otimes W \arrow[d, "f"] \\ & Z \end{tikzcd}$$

Corollary 3.4. Let (v_i) be a basis of V, and (w_i) a basis of W. Then $(v_i \otimes w_j)$ gives a basis of $V \otimes W$, with $\dim V \otimes W = \dim V \cdot \dim W$.

Remark 3.5. Not every element in $V \otimes W$ can be expressed as $v \otimes w$ for some $v \in V, w \in W$. In particular, one can have $v' = \sum_{i \in I} v_i \otimes w_i \in V \otimes W$, whose representative cannot be further simplified.

Proposition 3.6. Let V and W be finite dimensional real vector space. Then there exists a natural isomorphism $V \otimes W \xrightarrow{\sim} \operatorname{Hom}(V^*, W)$. In particular, this gives $V^* \otimes V^* \simeq \operatorname{Hom}(V, V^*) \simeq \{\text{bilinear forms } V \times V \to \mathbb{R}\}$.

Proof. To construct the map it suffices to give a bilinear map $V \times W \to \operatorname{Hom}(V^*, W)$. Define it as

$$(v, w) \mapsto (\alpha \mapsto \alpha(v)w) \in \operatorname{Hom}(V^*, W)$$

Applying the universal property gives the desired result. □

Remark 3.7. This can be extended to multiple tensor products:

$$\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \quad \simeq \quad (\underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{\ell} \to \mathbb{R})$$

where RHS is a multilinear map.

Definition 3.8 (Tensor)**.** Denote

$$G = \underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell}$$

Then an element in G is called a (k, ℓ)-**tensor** on V. k is the contravariant degree, and ℓ is the covariant degree.

Remark 3.9. It is possible to define the tensor product of maps. Namely, for $f : V \to V'$ and $g : W \to W'$, there exists a unique linear map $f \otimes g : V \otimes W \to V' \otimes W'$ s.t. $(f \otimes g)(v \otimes w) = (f(v)) \otimes (g(w))$. This can be seen via consider first the $\mathbb{R}$-balanced linear map $V \times W \to V \otimes W$; and apply the universal property of tensor product gives the desired result.

Observation 3.10. Similar to Example 2.13 where we considered the direct sum of vector bundles as a new bundle, we can consider the tensor product of vector bundles. For $\mathcal{E}'$ and $\mathcal{E}''$ two vector bundles, $\coprod_{p \in M} \mathcal{E}'_p \otimes \mathcal{E}''_p$ gives a vector bundle structure over M. In particular, this can be extended to the tangent and cotangent spaces of a manifold.

Notation. Given a smooth manifold M, for $k, \ell \in \mathbb{Z}_{\geq 0}$, denote

$$\mathscr{T}^{(k,\ell)} M = \Pi^{(k,\ell)} M = \underbrace{TM \otimes \cdots \otimes TM}_{k} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{\ell}$$

In particular, this is TM for $k = 1, \ell = 0$, and is T^*M for $k = 0, \ell = 1$. By identification in Remark 3.7 maps $\Pi^{(k,\ell)} M \to M$ are always multilinear (with the identification with the Euclidean Space).

Definition 3.11 (Tensor Field). Let M be a smooth manifold. A (k,ℓ)-**tensor (field) on** M is a smooth section of $\Pi^{(k,\ell)}M$.

In local coordinates, if we have local coordinates $(x^1,\ldots,x^n)$ on $U \subseteq M$, then any element in $\Pi^{(k,\ell)}M$ is in the form of

$$f^{i_1,\ldots,i_k}_{j_1,\ldots,j_\ell} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j\ell}$$

where i_ks and j_ℓs run over $\{1,\ldots,n\}$ arbitrarily. A section on $\Pi^{(k,\ell)}M$ is then $\sum_{i,j} f^{i_1,\ldots,i_k}_{j_1,\ldots,j_\ell}$, with it being smooth implies that all such fs are smooth (as functions).

4 Riemannian Metric

Definition 4.1 (Riemannian Metric). Given a Riemannian manifold M, a **Riemannian Metric** on M is a (0, 2)-tensor g that is symmetric and positive definite. In local coordinates,

$$g = \sum_{i,j=1}^{n} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

That is, for all $p \in M$, $g_p : T_pM \times T_pM \to \mathbb{R}$ is bilinaer and symmetric; and for all $v \in T_pM$, $g_p(v,v) \geq 0$, with equality reached if and only if $v = 0$.

Observation 4.2. Notice that in local coordinates we have $v = v^i \frac{\partial}{\partial x^i}$, $w = w^j \frac{\partial}{\partial x^j}$. Evaluating the metric gives

$$(\mathrm{d}x^i \otimes \mathrm{d}x^j)(v,w) = v^i w^j = v^j w^i = (\mathrm{d}x^i \otimes \mathrm{d}x^j)(w,v) \implies g_{ij} = g_{ji}$$

Therefore, we can write

$$\begin{aligned} g &= \frac{1}{2}\left(g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{ji}\mathrm{d}x^j \otimes \mathrm{d}x^i\right) = \frac{1}{2} g_{ij}\left(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i\right) \\ &= g_{ij}\left(\frac{1}{2}\left(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i\right)\right) \end{aligned}$$

where $\left(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i\right) \in \mathrm{Sym}(T_pM)$ which is in the symmetric tensor, i.e. invariant w.r.t permutation of the entries.

Definition 4.3 (Riemannian Manifold). A **Riemannian Manifold** is a pair (M,g) where g is a Riemannian metric on M.

Definition 4.4 (Isometry). A C^∞-map $F : (M_1, g_1) \to (M_2, g_2)$ is an **isometry** if and only if

1) F is a diffeomorphism.

2) For all $p \in M_1$, $u, v \in T_pM_1$, $g_2(\mathrm{d}_pF(u), \mathrm{d}_pF(v)) = g_1(u,v)$.

F is a **local isometry** if only 2) holds.

Remark 4.5. Since g is nondegenerate, $\mathrm{d}_p F$ is bijective as otherwise LHS of the equality in 2) will vanish. This then implies that $\dim M_1 = \dim M_2$, otherwise $\mathrm{d}_p F$ will either be not injective ($\dim M_1 > \dim M_2$) or not surjective ($\dim M_1 < \dim M_2$).

Theorem 4.6. Any C^∞-manifold M has a Riemannian metric.

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas of M. By definition of the chart we have the embedding $\varphi_\alpha(U_\alpha) \longrightarrow \mathbb{R}^n$. Define in U_α the metric s.t. φ_α is an isometry. In local coordinates, the metric can be expressed as $g_\alpha = \sum_i \mathrm{d}x^i_\alpha \otimes \mathrm{d}x^j_\alpha$.

Let (χ_α) be a partition of unit subordinated to (U_α) and $g = \sum_\alpha \chi_\alpha g_\alpha$. Nondegeneracy of the metric g follows from the positivity of metric g_α and χ_α. $\square$

Example 4.7. The following gives two constructions of Riemannian Metrics:

1) Let (M, g) be a Riemannian manifold, with $N \subset M$ a submanifold. Then N inherits a Riemannian metric from the Riemannian structure on M, as for all $p \in N$, $T_p N \subset T_p M$.

2) Consider Cartesian product of Riemannian manifolds. Given two Riemannian manifolds (M_1, g_1), (M_2, g_2), consider $M_1 \times M_2$. Then for all $(p_1, p_2) \in M_1 \times M_2$ we have the identification between the tangent spaces

$$T_{(p_1,p_2)}(M_1 \times M_2) \simeq T_{p_1} M_1 \oplus T_{p_2} M_2$$

Then in coordinates, the combined metric is block diagonal:

$$g = \left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2 \end{array}\right)$$

5 Coverings

6 Metric on Lee Group

7 Common Objects on Riemannian Manifolds