Walking On Stars with Boundary Conditions

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Contents

1 Preliminaries

To get familiar with the context, first present the definition of the various boundary conditions:

Definition 1.1. Given an (ordinary/partial) differential equation with domain Ω,

- A **Dirichlet boundary condition** fixes the value of solution on the boundary of domain.
- A **Neumann boundary condition** fixes the derivative (normal) applied at the boundary of the domain.
- A **Robin boundary condition** is a weighed combination of the previous two. Explicitly, for given *functions* a, b, g defined on $\partial\Omega$ the associated Robin boundary condition is (for target function f)

$$a\, f + b\, \partial_n f = g \qquad \text{on } \partial\Omega$$

where $\partial_n(\cdot)$ denotes the normal derivative.

The followings are a collection some purely Mathematical definitions for formal description of objects introduced. They are not necessarily essential for understanding the objects, and serves as a reminder merely. You are welcome (and suggested) to skip this part, and refer back to these definitions when encountering anything unfamiliar.

Definition 1.2 (σ-algebra)**.** Given a set X with $\mathcal{P}(X)$ its power set, a subset $\Sigma \subseteq P(X)$ is a **σ-algebra** if it satisfies

1) $X \in \Sigma$.
2) Σ is closed under complement.
3) Σ is closed under countable unions.

Remark 1.3. By applying De Morgan's Law directly, σ-algebras are also closed under countable intersections.

Definition 1.4 (Borel (Measurable) Space)**.** A **Borel space**, (or **measurable space**), is a tuple $(X, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra on X.

Remark 1.5. This needs to be distinguished from the *measure space*: no measure is required for a measurable space. The "measurable" here refers to the sets in $\mathcal{F}$ are "measured", or considered, in $\mathcal{P}(X)$.

Definition 1.6 (Stochastic Process)**.** A **stochastic process** on a probability space $(\Omega, \mathcal{F}, \Pr)$ with a measurable space (S, Σ) and index set T (often time, subset of $\mathbb{R}$) is a collection of S-valued random variables with evaluations $\{X(t) \mid t \in T\}$.

2 Solving Dirichlet Problems by "Walk On Spheres" [Mul56]

Due to the complexity of algebraic manipulation/approximation, some advanced results will be used as black box, and will be marked with purple. Please refer to the original paper for details. Also, results with detailed proofs in the original paper will be reasoned with the "intuitions"; and important results presented without proof in the paper will be explained in detail.

Definition 2.1 (Harmonic Function)**.** A twice continuously differentiable $\mathbb{R}$-valued function $f : U \to \mathbb{R}$ is **harmonic** if $\nabla^2 f \equiv 0$ on U.

Definition 2.2 (Dirichlet Problem)**.** Given a domain (for simplicity assume that this is simply-connected) D with boundary ∂D, a **Dirichlet problem** associated to a function f continuously defined on ∂D is to find the function u defined on R s.t. u and f coincide on ∂D.

2.1 Discretization of Brownian Motion

Definition 2.3 (Brownian Motion)**.** A $\mathbb{R}^d$**-valued Brownian motion** starting at $x \in \mathbb{R}^d$ is a stochastic process $\{B(t) \mid t \in T := \mathbb{R}_{\geq 0}\}$ satisfying the following properties:

1) *Anchor:* $B(0) = x$.
2) *Independent incrementals:* for any increasing sequence $(t_n)_{n \in \mathbb{Z}_{\geq 0}}$ on T, $\{B(t_{i+1}) - B(t_i) \mid i \in \mathbb{Z}_{\geq 0}\}$ are independent random variables.
3) *Normality in each step:* For all $t \geq 0, h > 0$, the incremental $B(t+h) - B(t)$ follows a normal distribution $N(0, h)$.
4) *Continuity:* The function $t \mapsto B(t)$ is almost surely (i.e., has probability 1 of being) continuous.

Remark 2.4. Property 4) in the definition actually loosens the definition; but the discontinuity does not interfere with any numerical treatment, as it happens with probability 0.

Notation. For simplicity, the evaluation of Brownian motion B will be denoted as $X(t, \omega)$ with $\omega \in \mathbb{R}$ a random variable specifying the exact value of $B(t)$. That is, for $X(t, \omega) \in \mathbb{R}^N$.

In a domain D, given an arbitrary Brownian motion B, it can be discretized into an infinite sequence of points (associated to the corresponding time index t), where the distribution (w.r.t. t) is preserved:

Definition 2.5 (Maximum Sphere)**.** Let D be an arbitrary domain, and $x \in D$ be an arbitrary point. Then the **maximum sphere** centered at x in D, denoted $\boldsymbol{K}_D(x)$, is the sphere

$$S(x, \inf_{x_0 \in \partial D} \{\|x - x_0\|\})$$

Its boundary (surface of the sphere) is denoted as $\overline{\boldsymbol{K}}_D(x)$.

Definition 2.6 (Successive Intersections)**.** Given a Brownian motion B with arbitrary ambient domain D, its **successive intersections** w.r.t. D is a (possibly, and often, infinite) sequence $P_n(B(0), \boldsymbol{K}_D(P_{n-1}), \omega)$ defined as follows:

1. $P_0(B(0), -, \omega) := B(0)$.
2. $P_n(B(0), \overline{\boldsymbol{K}}(P_{n-1}), \omega)$ is the point where $X(t, \omega)$ for the first time intersects with $\overline{\boldsymbol{K}}_D(P_{n-1})$.

Notice that by the randomness defined in the Brownian motion, the distribution of P_n on $\overline{\boldsymbol{K}}_D(P_{n-1})$ is uniform; this is also a Markov chain since the local direction $P_n - P_{n-1}$ is uniformly random, and is independent over n (by requirement 2) in the definition of Brownian motion). Therefore this can be "perfectly simulated" with the following process:

Definition 2.7 (Spherical Process). Given a domain D with boundary ∂D and an arbitrary point $x \in D$, the **spherical process** defines a sequence of points $P_n(x, \phi)$ where ϕ is the random variable, as follows:

1. $P_0(x, \phi) := x$.
2. For all $i \in \mathbb{Z}_{\geq 1}$, designate P_i uniformly randomly from $\overline{\boldsymbol{K}}_D(P_{i-1})$.

It is then clear, that given a Brownian motion with random variable $\omega = \omega_0$ it corresponds uniquely to a spherical process with random variable $\phi = \phi(\omega_0)$; and vice versa. This allows application of the results proved for Brownian motion to obtain the solution of the Dirichlet problem; and use Monte Carlo methods on the spherical process for numerical quadrature.

2.2 Obtaining Solution from Spherical Process

The solution of the Dirichlet problem associated to function f is computed directly using the following result. A complete context of the result can be found in [Kak44]. Notice that the "measure" in the integration involves the aforementioned Brownian motion. This blog provides a good excursion into the topic.

Proposition 2.8. Given a Dirichlet problem defined on a domain D with boundary values $f(\zeta)$ for $\zeta \in \partial D$, for any $x_0 \in D$, the solution u of the Dirichlet problem evaluates to

$$u(x_0) = \int_{\partial D} \Pr(x_0, \zeta, D) f(\zeta) \mathrm{d}\zeta \qquad \forall x_0 \in D$$

where $\Pr(x_0, \zeta, D)$ is the probability density function of a Brownian motion starting at x_0 intersects for the first time with ∂D at ζ.

Proof. □

Remark 2.9. The probability function $\Pr(x_0, \zeta, D)\mathrm{d}\zeta =: \mathrm{d}\omega$ defines a measure, and is called the harmonic measure.

Numerically, the quadrature can be approximated via applying Monte Carlo analysis on the spherical process.

2.3 Convergence Analysis

First we need to show that the spherical process indeed (almost always, i.e. with probability 1) converges.

Proposition 2.10.

Intuition. □

Now we show the order of convergence:

Theorem 2.11. Let N be the dimension s.t. the problem domain $\Omega \subseteq \mathbb{R}^N$. Then the steps required for convergence of the method is linear in N.

Proof. □

3 Boundary Value Caching for *WoS* [Mil+23]

4 Walking on Stars [Saw+23]

5 Extending *WoSt* to Robin Boundary Conditions[Mil+24]

References

[Kak44] Shizuo Kakutani. "Two-dimensional Brownian motion and harmonic functions". In: *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 20.10 (Jan. 1944). ISSN: 0386-2194. DOI: 10.3792/pia/1195572706. URL: http://dx.doi.org/10.3792/pia/1195572706.

[Mil+23] Bailey Miller et al. "Boundary Value Caching for Walk on Spheres". In: *ACM Transactions on Graphics* 42.4 (July 2023), pp. 1–11. ISSN: 1557-7368. DOI: 10.1145/3592400. URL: http://dx.doi.org/10.1145/3592400.

[Mil+24] Bailey Miller et al. "Walkin' Robin: Walk on Stars with Robin Boundary Conditions". In: *ACM Transactions on Graphics* 4 (July 2024), pp. 1–18. URL: https://imaging.cs.cmu.edu/walk_on_stars_robin/index.html.

[Mul56] Mervin E. Muller. "Some continuous Monte Carlo methods for the Dirichlet problem". In: *The Annals of Mathematical Statistics* 27.3 (Sept. 1956), pp. 569–589. DOI: 10.1214/aoms/1177728169.

[Saw+23] Rohan Sawhney et al. "Walk on Stars: A Grid-Free Monte Carlo Method for PDEs with Neumann Boundary Conditions". In: *ACM Transactions on Graphics* 42.4 (July 2023), pp. 1–20. ISSN: 1557-7368. DOI: 10.1145/3592398. URL: http://dx.doi.org/10.1145/3592398.