Walking On Stars with Boundary Conditions

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1 Prelimiaries

To get familiar with the context, first present the definition of the various boundary conditions:

Definition 1.1. Given an (ordinary/partial) differential equation with domain Ω,

- A **Dirichlet boundary condition** fixes the value of solution on the boundary of domain.
- A **Neumann boundary condition** fixes the derivative (normal) applied at the boundary of the domain.
- A **Robin boundary condition** is a weighed combination of the previous two. Explicitly, for given *functions* a, b, g defined on $\partial\Omega$ the associated Robin boundary condition is (for target function f)

  $$a\,f + b\,\partial_n f = g \qquad \text{on } \partial\Omega$$

  where $\partial_n(\cdot)$ denotes the normal derivative.

The followings are a collection some purely Mathematical definitions for formal description of objects introduced. They are not necessarily essential for understanding the objects, and serves as a reminder merely.

Definition 1.2 (σ-algebra)**.** Given a set X with $\mathcal{P}(X)$ its power set, a subset $\Sigma \subseteq P(X)$ is a σ**-algebra** if it satisfies

1) $X \in \Sigma$.
2) Σ is closed under complementation.
3) Σ is closed under countable unions.

Remark 1.3. By applying De Morgan's Law directly, σ-algebras are also closed under countable intersections.

Definition 1.4 (Borel (Measurable) Space)**.** A **Borel Space**, (or Measurable Space), is a tuple $(X, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra on X.

Remark 1.5. This needs to be distinguished from the *measure space*: no measure is required for a measurable space. The "measurable" here refers to the sets in $\mathcal{F}$ are "measured", or considered, in $\mathcal{P}(X)$.

Definition 1.6 (Stochastic Process)**.** A **stocahstic process** on a probability space $(\Omega, \mathcal{F}, \Pr)$ with a measureable space (S, Σ) and index set T (often time, subset of $\mathbb{R}$) is a collection of S-valued random variables with evaluations $\{X(t) \mid t \in T\}$.

2 Walk On Spheres [Mul56]

Definition 2.1 (Brownian Motion). A $\mathbb{R}^d$**-valued Brownian motion** starting at $x \in \mathbb{R}^d$ is a stochastic process $\{B(t) \mid t \in T := \mathbb{R}_{\geq 0}\}$ satisfying the following properties:

1) *Anchor:* $B(0) = x$.
2) *Independent incrementals:* for any increasing sequence $(t_n)_{n \in \mathbb{Z}_{\geq 0}}$ on T, $\{B(t_{i+1}) - B(t_i) \mid i \in \mathbb{Z}_{\geq 0}\}$ are independent random variables.
3) *Normality in each step:* For all $t \geq 0, h > 0$, the incremental $B(t+h) - B(t)$ follows a normal distribution $N(0, h)$.
4) *Continuity:* The function $t \mapsto B(t)$ is almost surely (i.e., has probability 1 of being) continuous.

Remark 2.2. Property 4) in the definition actually loosens the definition; but the discontinuity does not interfere with any numerical treatment, as it happens with probability 0.

3 Boundary Value Caching for *WoS* [Mil+23]

4 Walking on Stars [Saw+23]

5 Extending *WoSt* to Robin Boundary Conditions[Mil+24]

References

[Mil+23] Bailey Miller et al. “Boundary Value Caching for Walk on Spheres”. In: *ACM Transactions on Graphics* 42.4 (July 2023), pp. 1–11. ISSN: 1557-7368. DOI: 10.1145/3592400. URL: http://dx.doi.org/10.1145/3592400.

[Mil+24] Bailey Miller et al. “Walkin’ Robin: Walk on Stars with Robin Boundary Conditions”. In: *ACM Transactions on Graphics* 4 (July 2024), pp. 1–18. URL: https://imaging.cs.cmu.edu/walk_on_stars_robin/index.html.

[Mul56] Mervin E. Muller. “Some continuous Monte Carlo methods for the Dirichlet problem”. In: *The Annals of Mathematical Statistics* 27.3 (Sept. 1956), pp. 569–589. DOI: 10.1214/aoms/1177728169.

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