

# PEAT8002 - SEISMOLOGY

## Lecture 7: Rays in spherically symmetric media

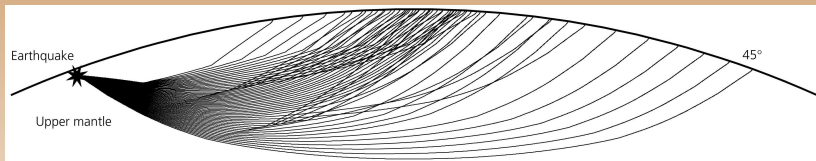
Nick Rawlinson

Research School of Earth Sciences  
Australian National University

# Rays in spherically symmetric media

## Introduction

- On a global scale, the structure of the Earth is, to a good approximation, spherically symmetric. In this case, slowness is a function of radius only:  $U(r)$ .



- Seismic rays generally begin at a source on or near the surface, and are recorded at a seismograph elsewhere on the surface. The ray path connecting the source and receiver is refracted and/or reflected as it traverses the Earth.

# Rays in spherically symmetric media

## Ray parameter

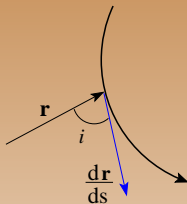
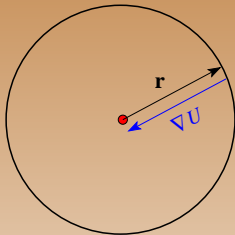
- To examine the behaviour of rays in spherically symmetric media, we begin by investigating how the quantity  $\mathbf{r} \times (U \mathbf{dr}/ds)$ , where  $U \mathbf{dr}/ds$  is the slowness vector, varies along the raypath (i.e. with pathlength  $s$ )

$$\frac{d}{ds} \left[ \mathbf{r} \times \left( U \frac{d\mathbf{r}}{ds} \right) \right] = \frac{d\mathbf{r}}{ds} \times \left( U \frac{d\mathbf{r}}{ds} \right) + \mathbf{r} \times \frac{d}{ds} \left( U \frac{d\mathbf{r}}{ds} \right) = 0$$

- The first term on the RHS is zero because the cross product of any vector with itself is zero.
- The second term equals  $\mathbf{r} \times \nabla U$  from the ray equation, since  $\mathbf{r}$  and  $\nabla U$  are parallel for a spherically symmetric medium,  $\mathbf{r} \times \nabla U = 0$ .

# Rays in spherically symmetric media

Ray parameter



- As a result, we reach the conclusion that

$$\mathbf{r} \times \left( U \frac{d\mathbf{r}}{ds} \right) = \text{constant}$$

which means that it is preserved along the ray, and its direction is normal to the plane that contains  $d\mathbf{r}/ds$  and  $\mathbf{r}$ .

# Rays in spherically symmetric media

## Ray parameter

- The raypath therefore lies in a plane that contains the origin of the coordinate system.
- The magnitude of the preserved quantity  $p$  is referred to as the **ray parameter**:

$$p = \left| \mathbf{r} \times U \frac{d\mathbf{r}}{ds} \right| = rU \sin i$$

since for any vector  $\mathbf{a}$  and  $\mathbf{b}$ ,  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin i$ .

- $p$  is constant along the path for a particular ray.
- $p = rU \sin i$  is a general form of Snell's law for spherically symmetric media.
- The radius at which the ray bottoms out is given by  $r = p/U$  since this occurs when  $i = 90^\circ$ .

# Rays in spherically symmetric media

## Traveltime determination

- The goal of this section is to determine the traveltime of a raypath in a spherically symmetric medium characterised by a continuous variation of slowness with radius  
 $U = U(r)$ .

- The gradient operator in spherical coordinates is given by

$$\nabla = \left[ \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right]$$

- The eikonal equation in spherical coordinates can then be written:

$$\nabla T \cdot \nabla T = \left( \frac{\partial T}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial T}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial T}{\partial \phi} \right)^2 = [U(\mathbf{r})]^2$$

# Rays in spherically symmetric media

## Travelttime determination

- In spherically symmetric media, we can set the last term of the central equality in the above equation to zero since variations in  $\phi$  do not effect  $T$  (ray travels in plane of great circle).
- In order to solve for  $T(r, \theta)$ , assume a separation of variables:

$$T(r, \theta) = f(\theta) + g(r)$$

- Substitution into the eikonal equation gives:

$$r^2 \left[ \frac{dg(r)}{dr} \right]^2 - r^2 [U(r)]^2 = - \left[ \frac{df(\theta)}{d\theta} \right]^2 = -k^2 \quad (1)$$

- Both sides must be constant, because varying the independent variable on one side will leave the other side unaltered. Here,  $k^2$  is a separation constant.

# Rays in spherically symmetric media

## Traveltime determination

- Taking the RHS of Equation (1),

$$\frac{df(\theta)}{d\theta} = k \quad \text{so} \quad f(\theta) = \int_0^\theta k d\theta = k\theta$$

- From the LHS of Equation (1),

$$\left[ \frac{dg(r)}{dr} \right]^2 = U^2 - \frac{k^2}{r^2}$$

so that

$$\frac{dg(r)}{dr} = \pm \sqrt{U^2 - \left( \frac{k}{r} \right)^2}$$



# Rays in spherically symmetric media

## Traveltime determination

- These terms can now be combined to give:

$$T(r, \theta) = k\theta \pm \int_0^r \frac{\sqrt{r'^2 U^2 - k^2}}{r'} dr'$$

- From earlier,

$$\frac{dr}{ds} = \frac{\nabla T}{U}$$

Therefore

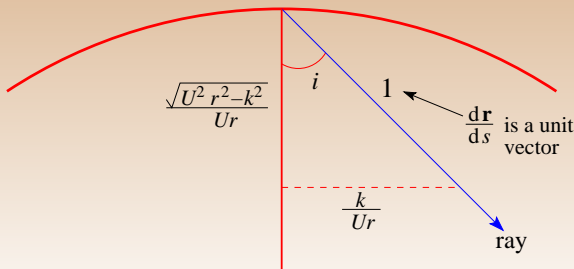
$$\begin{aligned} U \frac{dr}{ds} &= \nabla T = \left[ \frac{\partial T}{\partial r}, \frac{1}{r} \frac{\partial T}{\partial \theta}, 0 \right] \\ &= \left[ \frac{\pm \sqrt{U^2 r^2 - k^2}}{r}, \frac{k}{r}, 0 \right] \end{aligned}$$

# Rays in spherically symmetric media

## Traveltime determination

- This expression can be rearranged to give a unit vector in the direction of the ray:

$$\frac{d\mathbf{r}}{ds} = \left[ \frac{\pm \sqrt{U^2 r^2 - k^2}}{Ur}, \frac{k}{Ur}, 0 \right]$$



# Rays in spherically symmetric media

## Traveltime determination

- From the diagram, it is evident that

$$\sin i = \frac{k}{Ur} \quad \cos i = \frac{\sqrt{U^2 r^2 - k^2}}{Ur} \quad \tan i = \frac{k}{\sqrt{U^2 r^2 - k^2}}$$

- We see that  $k = Ur \sin i = p$  so our separation constant is simply equal to the ray parameter. Therefore:

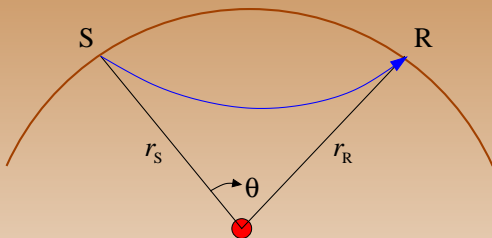
$$T(r, \theta) = p\theta \pm \int_{r_S}^r \frac{\sqrt{r^2 U^2 - p^2}}{r} dr$$

- Since  $T$  is traveltime, it must be positive and increase with distance along the ray.
  - Take +ve for ascending ray since  $dr > 0$ .
  - Take -ve for descending ray since  $dr < 0$ .

# Rays in spherically symmetric media

## Traveltime determination

- Therefore, the integral needs to be split up if the ray turns.



- If we differentiate the above expression with respect to  $\theta$ , we get:

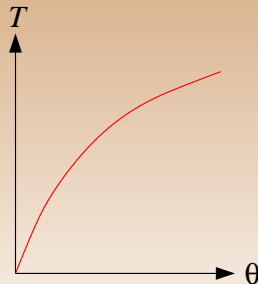
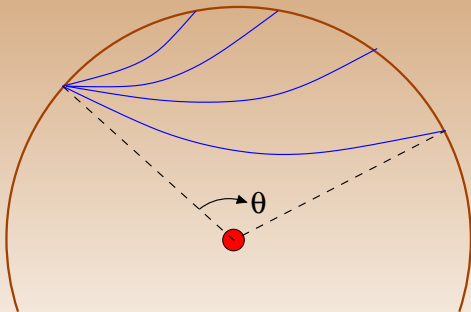
$$\left( \frac{\partial T}{\partial \theta} \right) = p$$

which tells us that the ray parameter can be determined from data if the source separation is known.

# Rays in spherically symmetric media

## Traveltime determination

- The ray parameter can be measured from the gradient of a traveltime curve.



# Rays in spherically symmetric media

## Traveltime determination

- From Fermat's principle,  $\partial T / \partial p = 0$  - traveltime is stationary with respect to variations in the path, so

$$\frac{\partial T}{\partial p} = \theta \pm \int_{r_s}^r \frac{-p}{r \sqrt{r^2 U^2 - p^2}} dr = 0$$

- Therefore:

$$\theta = \pm p \int_{r_s}^r \frac{dr}{r \sqrt{r^2 U^2 - p^2}}$$

- Substitution back into the expression for  $T$  now gives:

$$\begin{aligned} T &= \pm \int_{r_s}^r \left[ \frac{p^2}{r \sqrt{r^2 U^2 - p^2}} + \frac{\sqrt{r^2 U^2 - p^2}}{r} \right] dr \\ &= \pm \int_{r_s}^r \frac{p^2 + r^2 U^2 - p^2}{r \sqrt{r^2 U^2 - p^2}} dr \end{aligned}$$

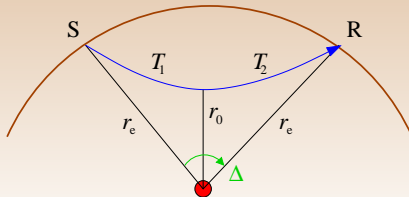
# Rays in spherically symmetric media

## Traveltime determination

- Thus, we end up with an expression that allows us to compute traveltime using only  $p$  and  $U(r)$ :

$$T = \pm \int_{r_S}^r \frac{rU^2}{\sqrt{r^2U^2 - p^2}} dr$$

- If the ray begins and ends at the surface, then the ascending and descending contributions are equal.



# Rays in spherically symmetric media

## Traveltime determination

- The total traveltime is thus given by ( $c$  = wavespeed):

$$T = 2 \int_{r_0}^{r_e} \frac{rU^2}{\sqrt{r^2 U^2 - p^2}} dr = 2 \int_{r_0}^{r_e} \frac{r/c}{\sqrt{r^2 - c^2 p^2}} dr$$

- If  $\Delta$  is the total angular distance covered by the ray, then:

$$\Delta = 2 \int_{r_0}^{r_e} \frac{pc/r}{\sqrt{r^2 - c^2 p^2}} dr$$

since from before:

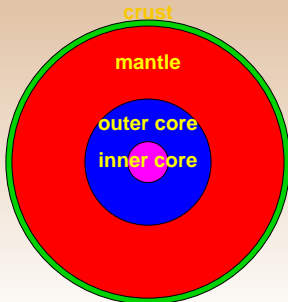
$$\theta = \pm p \int_{r_s}^r \frac{dr}{r \sqrt{r^2 U^2 - p^2}}$$



# Rays in spherically symmetric media

## Body waves in the Earth

- Beginning in the early 1900s, travel time tables for the Earth were compiled by combining data from many earthquakes observed at various epicentral distances.
- These seismological observations provide the basis for our first-order understanding of Earth structure i.e. A layered Earth composed of a thin crust, a mantle, a liquid outer core and a solid inner core.



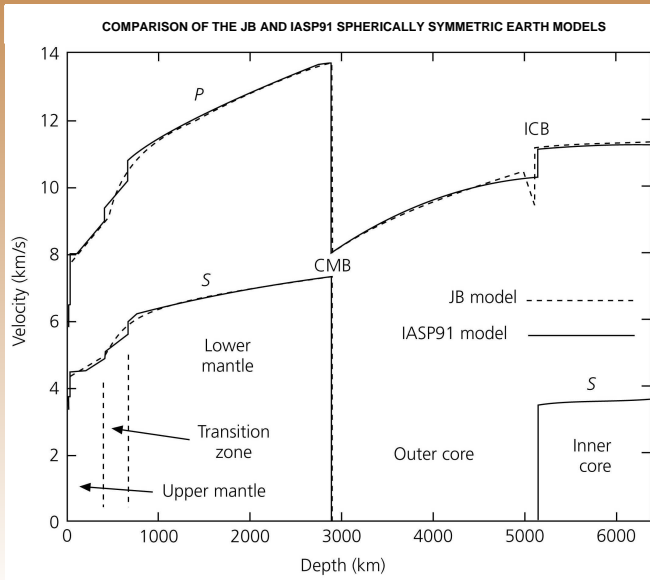
# Rays in spherically symmetric media

## Body waves in the Earth

- The classic Jeffreys-Bullen model of the internal seismic structure of the Earth, derived from earthquake traveltime observations, was first published in 1940.
- The JB model divided the mantle into an upper and lower mantle, each with smooth velocity gradients. The transition zone between these two regions features rapid velocity increase with depth.
- Below the core-mantle boundary (CMB), the core was divided into an outer core and inner core.
- Subsequent studies have also derived spherically symmetric models of seismic structure, including IASP91 and AK135. These later models have improved resolution in the mantle transition zone and core, but otherwise confirm the results obtained by Jeffreys and Bullen.

# Rays in spherically symmetric media

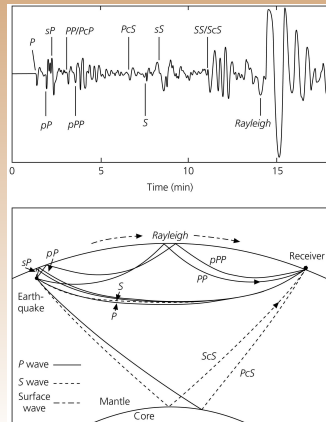
## Body waves in the Earth



# Rays in spherically symmetric media

## Body waves in the Earth

- Wavetrains from distant earthquakes (or teleseisms) are likely to contain a variety of arrivals that can be attributed to features present in spherically symmetric Earth models like IASP91.



# Rays in spherically symmetric media

## Body wave nomenclature

Name	Description
<i>P</i>	<i>Compressional wave</i>
<i>S</i>	<i>Shear wave</i>
<i>K</i>	<i>P-wave through outer core</i>
<i>I</i>	<i>P-wave through inner core</i>
<i>J</i>	<i>S-wave through inner core</i>
<i>PP</i>	<i>P-wave reflected at free surface</i>
<i>PPP</i>	<i>P-wave reflected at free surface twice</i>
<i>SS</i>	<i>S-wave reflected at free surface</i>
<i>SSS</i>	<i>S-wave reflected at free surface twice</i>
<i>SP</i>	<i>S-wave reflected at free surface as P-wave</i>
<i>PS</i>	<i>P-wave reflected at free surface as S-wave</i>
<i>pP</i>	<i>P-wave upgoing from source, reflected at free surface</i>
<i>sP</i>	<i>S-wave upgoing from source, converted to P at free surface</i>

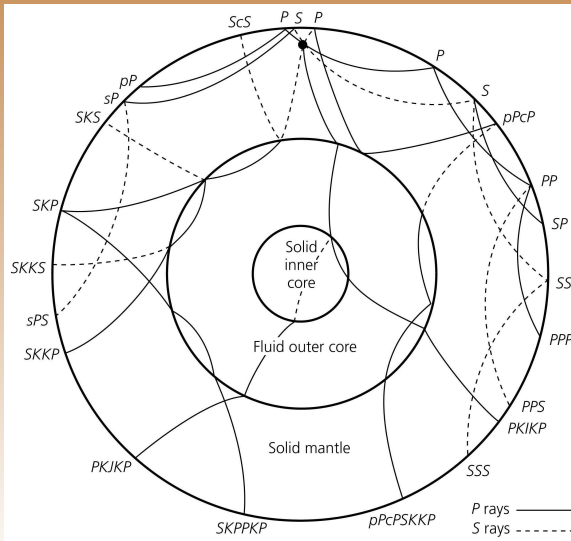
# Rays in spherically symmetric media

## Body wave nomenclature

Name	Description
$c$	Wave reflected at core–mantle boundary (e.g. ScS)
$i$	Wave reflected at inner core boundary (e.g. PKiKP)
$P'$	Abbreviation for PKP
$P_d$ or $P_{diff}$	P–wave diffracted along core–mantle boundary
$Pg$	P–wave refracted in crust
$Sg$	S–wave refracted in crust
$Pn$	P–wave refracted at Moho (head wave)
$Sn$	S–wave refracted at Moho (head wave)

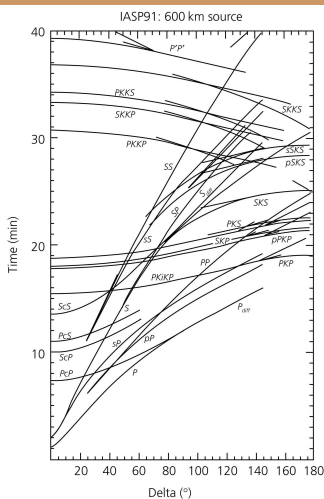
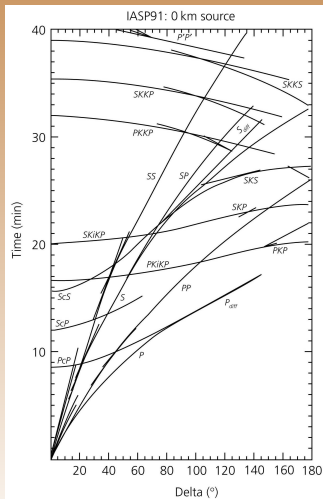
# Rays in spherically symmetric media

## Body wave nomenclature



# Rays in spherically symmetric media

## IASP91 traveltime curves

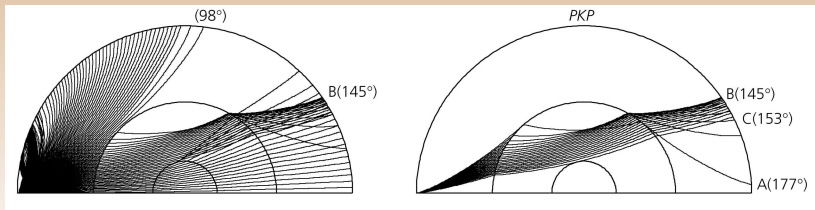




# Rays in spherically symmetric media

## Core shadow zone

- For purely transmissive phases, the core shadow zone occurs between about  $98^\circ$  and  $145^\circ$ .
- In reality, body waves are observed in the shadow zone e.g. PP, SS etc., although usually only for larger events. Their amplitudes may also vary significantly due to the presence of lateral structure.



# Rays in spherically symmetric media

## Core phases

- The contrast in properties between the solid mantle and the liquid core, which has lower velocity than the overriding mantle, makes the core well suited to seismological studies using refracted, transmitted, converted and diffracted arrivals.
- The inner core boundary also represents a strong contrast in physical properties, and along with the outer core boundary is the subject of much research.
- The strongly contrasting structure of the core means that a variety of subtly different phases can be identified, and used to study structure. These phases have another sub-class of nomenclature (e.g. PKP-AB, PKP-DF), which are based on the various branches of the traveltime curves they generate.

# Rays in spherically symmetric media

## Core phases

