Effective range derivative for acoustic propagation loss in a horizontally stratified ocean

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(Received 3 February 1981; accepted for publication 24 August 1981)

A method for predicting acoustic propagation loss in a horizontally stratified ocean had been developed. We begin by comparing two widely used models, and demonstrate that unresolved discrepancies exist. Then classical ray tracing equations are developed by evaluating a multipath integral by stationary phase. The inaccuracy of ray theory is attributed to deficiencies in the stationary phase approach. In order to correct these shortcomings, we express multipaths in terms of Fresnel integrals and effective range derivatives. The approach is generalized to shadow zone propagation. Finally, we introduce a computer model that is based on the effective range derivative evaluation of the multipath expansion. *The main emphasis of this paper, however, is not model comparison, but the effective range derivative technique, itself.*

PACS numbers: 43.30.Bp, 92.10.Vz, 43.20.Dk

INTRODUCTION

This paper introduces a model for computing propagation loss in a horizontally stratified ocean. That is, the ocean's surface and bottom are flat, while the sound speed is a function of depth but not range. The problem has been solved in a theoretical sense for some time, but our ability to make practical predictions remains unsatisfactory.

For the majority of users, a propagation loss model must (1) be accurate, (2) provide physical insight, and (3) be computationally efficient. Although classical ray tracing models provide physical insight and are computationally efficient, they can be inaccurate near caustics. Normal mode models are theoretically accurate, but are inefficient and do not provide physical insight when the number of propagating modes is large.

The reader is probably aware of various asymptotic corrections and hybrid theories that have been incorporated into the more sophisticated computer models. Unfortunately, these models sometimes give different predictions for the same enviromental inputs. Section I compares two of the best propagation loss models in order to give the reader an indication of the magnitude of the errors that can be expected.

Since none of the existing propagation loss models satisfies conditions (1), (2), and (3) simultaneously, the ideal model remains to be written. Hopefully, the technique described here is a good compromise between accuracy, physical insight, and computational efficiency. Our basic result is similar to the propagation loss equation from classical ray theory. The trick is to replace the usual range derivative with an "effective range derivative."

Section II reviews the multipath expansion for acoustic propagation in a horizontally stratified ocean. Although this method is formally exact, the numerical evaluation of various integrals is complicated and time consuming. The first step taken to improve efficiency removes "slowly varying" terms from beneath the integral sign, leaving a phase integral for the spreading loss factor.

We show in Sec. III how classical ray tracing equations can be derived by applying the method of stationary phase to a multipath integral. The inaccuracy of ray theory at low frequencies and near caustics is attributed to deficiencies in the stationary phase approach.

The effective range derivative technique, introduced in Sec. IV, is similar to the method of stationary phase, but more accurate. It is recognized that the limits of integration for a particular multipath integral may be finite, and that the range derivative at the point of stationary phase may not be representative over the dominant interval of integration. To correct these shortcomings, we express the multipath in terms of Fresnel integrals and an "effective range derivative."

The main topic of Sec. V is propagation into a shadow zone that is bounded by a smooth caustic. Here the usual range derivative is zero. We first express the multipath integral in terms of two Airy functions and an "effective second range derivative." However, this leads to a numerically ill-conditioned result. The problem is resolved, and we obtain a more typical Airy function representation for the shadow.

The method of stationary phase predicts infinite intensities as one approaches a smooth caustic from the illuminated region. In contrast, Sec. VI demonstrates that the basic effective range derivative method of Sec. IV can be used at a smooth caustic without introducing large anomalies. It is shown that the effective range derivative gives good predictions for the amplitude, but introduces a phase error of $\pm\pi/4$.

Although many computer programs model smooth caustics, they omit the equally important caustic formed at an ocean boundary. We discuss boundary produced caustics in Sec. VII. The result involves a slight modification of the effective range derivative technique of Sec. IV. The reader is cautioned that this formulation may be inaccurate deep in the shadow. Hopefully, other multipaths will mask the error.

Finally, in Sec. VIII we introduce a computer program which uses the effective range derivative to predict propagation loss. This model compares favorably with data, other models, and closed form solutions.

I. FACT-RAYMODE COMPARISON

The 70's were very productive years for propagation loss modelers. International workshops were held where participants could compare their latest model with experimental data and other models. By the end of the decade, however, only a few models retained national prominence. Two widely used are FACT[1] and RAYMODE.[2]

FACT is a ray tracing program that uses asymptotic corrections to improve accuracy in the vicinity of caustics. A lesser known aspect of the model involves the "semicoherent" addition of eigenrays from a near surface source to a near surface receiver. It is assumed that four paths having the same number of bottom bounces (or lower vertexes), but a different number of surface reflections (see Fig. 1) will have similar propagation losses. The relative phase of these rays may differ in range producing an interference pattern. By finding only one of the four paths, and estimating relative phase differences for the others, FACT can reduce computation time by 75%.

RAYMODE is a hybrid model that is based on three methods. The intensities of nonvertexing paths are computed by classical ray tracing, while normal mode theory is used for low-frequency-vertexing waves. The remaining energy, consisting of high-frequency-vertexing waves, is given by the RAYMODE algorithm. Here normal mode summations are evaluated by an equivalent ray analogy. The transition from high to low frequency occurs when the number of modes in a particular wave bundle falls below a specified value.

Since FACT and RAYMODE are both designed to predict propagation loss versus range at a frequency in a horizontally stratified ocean, one is bound to ask "Which program is better?." Various comparisons have been made,[3] but the results are inconclusive. The comparison presented here was obtained using a "Generic Sonar Model," which contains modularized versions of FACT and RAYMODE. This allows us to use common environmental submodels (such as bottom loss) in contrast to the original versions.

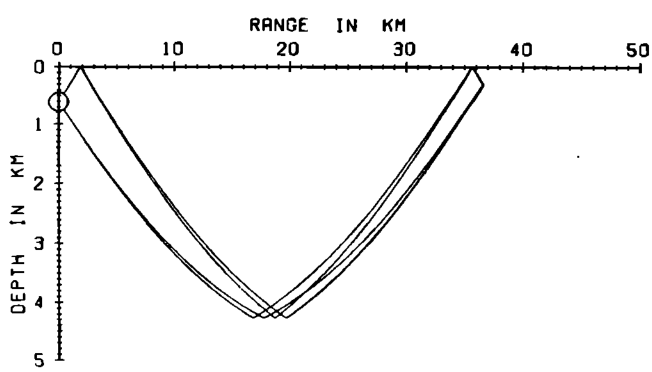

FIG. 1. Four bottom bounce rays connecting a source to a receiver.

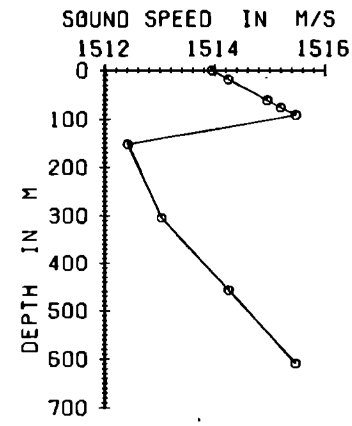

FIG. 2. Sound-speed profile used in FACT-RAYMODE comparison.

Figure 2 illustrates the sound-speed profile under investigation. Note the channel axis at 153 m. The most significant rays originating at a 195-m source depth are traced to a range of 50 km in Fig. 3. Figures 4 and 5 contain RAYMODE and FACT propagation loss predictions at 50 Hz and 9 kHz for a 205-m receiver. At 50 Hz and 200 kyd, FACT shows 83 dB loss as compared to 98 dB for RAYMODE. At 9 kHz and 1 kyd, FACT predicts 58 dB loss as compared to 66 dB for RAYMODE.

We are confident that FACT is more nearly correct in this example. Conversely, we have seen cases where FACT predictions are questionable. We are not trying to imply which program is better, but to demonstrate that since double digit discrepancies exist in our best propagation loss models, more work needs to be done.

II. MULTIPATH EXPANSION FOR HORIZONTALLY STRATIFIED OCEANS

The mathematical foundation for the model proposed here is based on the multipath expansion for acoustic propagation in a horizontally stratified ocean. Assume that a point harmonic source having radian frequency $\omega = 2\pi f$ is emitting acoustic energy at range $r = 0$, depth $z = z_s$. It has been shown[4] that the pressure field can then be written as a sum of integrals of the form

$$P(r,z,z_s;f) = \int_{\lambda_{\min}}^{\lambda_{\max}} \left(\frac{f}{r\lambda}\right)^{1/2} A(z;\lambda,f)A(z_s;\lambda,f) \times L_{\mathrm{srf}}^{n_{\mathrm{srf}}}(\lambda,f) L_{\mathrm{btm}}^{n_{\mathrm{btm}}}(\lambda,f) \times \exp[i\omega Q(z,z_s;\lambda) + i\omega r\lambda + i\pi/4]d\lambda. \quad (1)$$

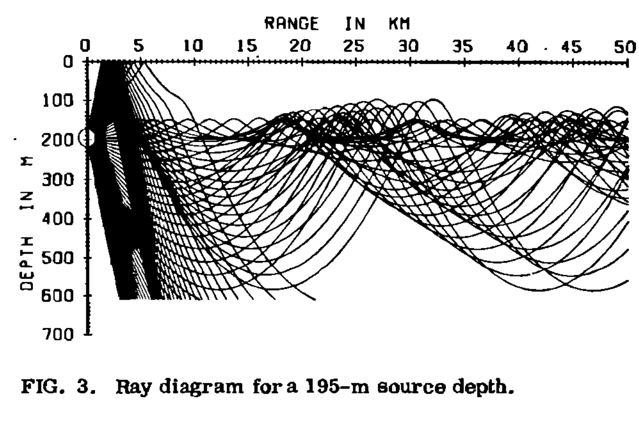

FIG. 3. Ray diagram for a 195-m source depth.

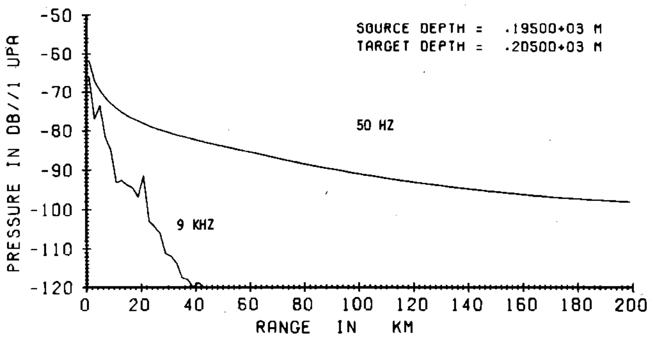

FIG. 4. Propagation loss for GENERIC RAYMODE.

Here, L_{srf} and L_{btm} denote complex valued surface and bottom reflection coefficients, respectively, n_{srf} and n_{btm} are the corresponding number of reflections (or vertexes),

$$Q(z,z_s;\lambda)=\int_{z_s}^{z}[c^{-2}(z')-\lambda^2]^{1/2}dz', \tag{2}$$

and $c(z)$ is the ocean sound speed. The depth integration in Eq. (2) must be performed over the entire depth traveled by the ray which originates at z_s with an inclination angle (with respect to the horizontal) of

$$\theta_s=\arccos[\lambda c(z_s)], \tag{3}$$

undergoes n_{srf} and n_{btm} surface and bottom reflections (or vertexes), and terminates at the depth z.

If one sets

$$A(z;\lambda,f)=\{\lambda^2/[c^{-2}(z)-\lambda^2]\}^{1/4}, \tag{4}$$

as in classical ray theory, the integral representation, Eq. (1), becomes a poor approximation in the vicinity of turning points, that is, depths z_t for which $c(z_t)=1/\lambda$. The Airy representation

$$A(z,\lambda,f)=\{\pi\lambda[\mathrm{Ai}(p)^2+\mathrm{Bi}(p)^2]/d\}^{1/2}, \tag{5}$$

where Ai and Bi are Airy functions of the first and second type, respectively,

$$p=-(3q/2)^{2/3}, \tag{6}$$

$$q=-\omega|Q(z,z_t;\lambda)|, \tag{7}$$

and

$$d=[c^{-2}(z)-\lambda^2]^{1/2}/|p|^{1/2}, \tag{8}$$

is better near a simple turning point.

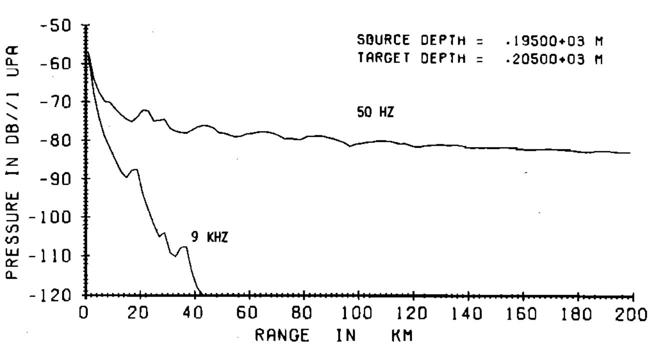

FIG. 5. Propagation loss for GENERIC FACT.

Although the accuracy of the integrand in Eq. (1) can be improved further, and in theory made exact, the effort at present is probably not justified. We shall concentrate instead on performing the integration. Our first step assumes the existence of a single $\underline{\lambda}$ between λ_{min} and λ_{max} such that

$$P(r,z,z_s;f)=N(r,z,z_s;\underline{\lambda},f)\exp\{[i\omega-a(\omega)]T(r,z,z_s;\underline{\lambda})\}, \tag{9}$$

where

$$T(r,z,z_s;\underline{\lambda})=Q(z,z_s;\underline{\lambda})+r\underline{\lambda}, \tag{10}$$

$$N(r,z,z_s;\underline{\lambda},f)=K(z,z_s;\underline{\lambda},f)I(r,z,z_s;\underline{\lambda},f)/(r\underline{\lambda})^{1/2}, \tag{11}$$

$$K(z,z_s;\underline{\lambda},f)=[A(z;\underline{\lambda},f)A(z_s;\underline{\lambda},f)L_{srf}^{n_{srf}}(\underline{\lambda},f)L_{btm}^{n_{btm}}(\underline{\lambda},f)]_{eff}, \tag{12}$$

and

$$I(r,z,z_s;\underline{\lambda},f)=f^{1/2}\int_{\lambda_{min}}^{\lambda_{max}}\exp\{i\omega[Q(z,z_s;\lambda)-Q(z,z_s;\underline{\lambda})+r(\lambda-\underline{\lambda})]+i\pi/4\}d\lambda. \tag{13}$$

The quantity T denotes travel time along the ray having vertex velocity $1/\lambda$, while N corresponds to the geometrical spreading loss factor from classical ray theory plus boundary effects. It is convenient to assume that the volume attentuation coefficient, $a(\omega)$, multiplies travel time instead of the usual ray arc length. We have also removed the "slowly varying" terms A and L from beneath the integral sign, and denoted their effective product by K. The expression $[\ \ ]_{eff}$ will be defined later. Our goal is to determine K and evaluate the phase integral in Eq. (13) as accurately and efficiently as possible.

III. EVALUATION OF MULTIPATH INTEGRALS BY STATIONARY PHASE

Classical ray tracing equations can be derived by applying the method of stationary phase to Eq. (13). It is assumed that, for large frequencies, the dominant contribution to the integral is due to an infinitesimal region about λ for which

$$Q'(z,z_s;\underline{\lambda})=\left.\frac{dQ(z,z_s;\lambda)}{d\lambda}\right|_{\lambda=\underline{\lambda}}=-r. \tag{14}$$

Outside this region, contributions cancel due to the rapidly oscillating integrand. Within the region, the "slowly varying" terms A and L are assumed constant, and Eq. (12) reduces to

$$K(z,z_s;\underline{\lambda},f)=A(z;\underline{\lambda},f)A(z_s;\underline{\lambda},f)L_{srf}^{n_{srf}}(\underline{\lambda},f)L_{btm}^{n_{btm}}(\underline{\lambda},f). \tag{15}$$

Next, the quantity

$$\psi(r,z,z_s;\lambda,\underline{\lambda})=Q(z,z_s;\lambda)-Q(z,z_s;\underline{\lambda})+r(\lambda-\underline{\lambda}) \tag{16}$$

is approximated by the leading term of its series expansion about $\lambda=\underline{\lambda}$. Since

$$\psi(r,z,z_s;\underline{\lambda},\underline{\lambda})=\psi'(r,z,z_s;\underline{\lambda},\underline{\lambda})=0, \tag{17}$$

one obtains

$$\psi(r,z,z_s;\lambda,\underline{\lambda}) = Q''(z,z_s;\underline{\lambda})(\lambda-\underline{\lambda})^2/2. \quad (18)$$

It is also argued that the limit of integration, λ_{min} and λ_{max}, can be extended to $-\infty$ and $+\infty$, respectively, without significantly affecting the value of I. This produces:

$$\begin{aligned} I(r,z,z_s;\underline{\lambda},f) &= f^{1/2}\int_{-\infty}^{+\infty}\exp[i\omega Q''(z,z_s;\underline{\lambda}) \\ &\quad\times(\lambda-\underline{\lambda})^2/2 + i\pi/4]\,d\lambda \\ &= \exp(i\pi/4 \pm i\pi/4)/\left|Q''(z,z_s;\underline{\lambda})\right|^{1/2}. \end{aligned} \quad (19)$$

The sign of $\pm i\pi/4$ must agree with that of Q''. Hence, if Q'' is negative, as it will in the case of the direct path, the phase of I will vanish.

There are numerous conditions under which the range derivative, $-Q''$, cannot be assumed constant near the point of stationary phase, $\underline{\lambda}$, or when both limits of integration cannot be extended to infinity without significantly affecting the result. The best known cases occur at a smooth caustic where $Q''=0$, and along boundary grazing rays where $Q''=-\infty$. The first condition produces infinite intensity predictions, while the second gives rise to an infinite propagation loss. Classical ray theory also tends to give misleading predictions in ducts, and in regions where A and L are not "slowly varying."

IV. EFFECTIVE RANGE DERIVATIVE FOR A REAL POINT OF STATIONARY PHASE

In order to arrive at more accurate predictions let us return our attention to the phase integral, Eq. (13). We now segment the interval of integration into two subintervals at the point of stationary phase. Thus,

$$I(r,z,z_s;\underline{\lambda},f) = I_-(r,z,z_s;\underline{\lambda},f) + I_+(r,z,z_s;\underline{\lambda},f), \quad (20)$$

where, by Eq. (16),

$$\begin{aligned} &I_-(r,z,z_s;\underline{\lambda},f) \\ &\quad= f^{1/2}\int_{\lambda_{min}}^{\underline{\lambda}}\exp[i\omega\psi(r,z,z_s;\lambda,\underline{\lambda}) + i\pi/4]\,d\lambda. \end{aligned} \quad (21)$$

and

$$\begin{aligned} &I_+(r,z,z_s;\underline{\lambda},f) \\ &\quad= f^{1/2}\int_{\underline{\lambda}}^{\lambda_{max}}\exp[i\omega\psi(r,z,z_s;\lambda,\underline{\lambda}) + i\pi/4]\,d\lambda. \end{aligned} \quad (22)$$

For sake of specificness, only I_+ will be investigated, and Q'' will be assumed nonzero in the open interval $(\underline{\lambda},\lambda_{max})$.

Using the method of stationary phase as a guide, we further assume the existence of an effective range derivative for $(\underline{\lambda},\lambda_{max})$ denoted by

$$\left[\frac{dr}{d\lambda}\right]_+ = -[Q'']_+ \quad (23)$$

such that

$$\begin{aligned} &f^{1/2}\int_{\underline{\lambda}}^{\lambda_{max}}\exp[i\omega\psi(r,z,z_s;\lambda,\underline{\lambda}) + i\pi/4]\,d\lambda \\ &\quad= f^{1/2}\int_{\underline{\lambda}}^{\lambda_{max}}\exp\{i\omega[Q'']_+(\lambda-\underline{\lambda})^2/2 + i\pi/4\}\,d\lambda. \end{aligned} \quad (24)$$

The left-hand side is, by definition, I_+, while the right-hand side can be expressed in terms of the Fresnel integrals

$$C(x) = \int_0^x \cos(\pi t/2)^2\,dt \quad (25)$$

and

$$S(x) = \int_0^x \sin(\pi t/2)^2\,dt. \quad (26)$$

The desired result is

$$\begin{aligned} &I_+(r,z,z_s;\underline{\lambda},f) \\ &\quad= [C(x) \pm iS(x)]\left|2[Q'']_+\right|^{-1/2}\exp(i\pi/4) \end{aligned} \quad (27)$$

where

$$x = \left|2f[Q'']_+\right|^{1/2}(\lambda_{max}-\underline{\lambda}). \quad (28)$$

The expression $C(x)\pm iS(x)$ can be computed quite easily using rational approximations from Ref. 5. In terms of auxiliary functions

$$f(x) = \frac{1+0.926x}{2+1.792x+3.104x^2}, \quad (29)$$

and

$$g(x) = (2+4.142x+3.492x^2+6.670x^3)^{-1}, \quad (30)$$

one obtains

$$C(x)\pm iS(x) = (1\pm i)/2 - [g(x)\pm if(x)]\exp(\pm i\pi x^2/2). \quad (31)$$

It follows that

$$2^*\text{Eq. (27)} \to \text{Eq. (19) as } \lambda_{max}\to\infty, \quad (32)$$

when the effective and usual range derivatives are equal. The factor 2 is present since only one side of the integration has been performed.

We have come to the crucial point of this paper, finding $[Q'']_+$ so that Eq. (24) is a good approximation. How accurate does $[Q'']_+$ have to be? According to the remarks in Sec. I regarding the present state of the art, not very. For example, a factor of 2 error in $[Q'']_+$ will by Eq. (27) produce at most a 3-dB error in propagation loss. Bearing this in mind, let us set

$$[Q'']_+ = 2\psi(r,z,z_s;\lambda_+,\underline{\lambda})/(\lambda_+-\underline{\lambda})^2, \quad (33)$$

where λ_+ is the smaller of the values for which

$$\lambda_+ = \lambda_{max} \quad (34)$$

or

$$\psi(r,z,z_s;\lambda_+,\underline{\lambda}) = 1/4f. \quad (35)$$

Roughly speaking, we have modified the method of stationary phase to consider a small region, $(\underline{\lambda},\lambda_+)$ in contrast to an infinitesimal one about $\underline{\lambda}$. In $(\underline{\lambda},\lambda_+)$,

$$0 < \omega\psi(r,z,z_s;\lambda,\underline{\lambda}) < \pi/2, \quad (36)$$

and we can almost justify the removal of A and L from beneath the integral sign in Eq. (1) by appealing to the mean value theorem for integrals. However, rather than evaluate K by selecting a particular λ in $(\underline{\lambda},\lambda_+)$ for the product in Eq. (12), we define $[\ \ ]_{eff}$ by

$$[\ \ ]_{eff} = \int_{\underline{\lambda}}^{\lambda_+}[\ \ ]\,d\lambda/(\lambda_+-\underline{\lambda}). \quad (37)$$

At low frequencies, Eq. (34) will hold. However, we see from Eq. (35) that

$$\psi(r,z,z_s;\lambda_+,\underline{\lambda}) \to 0 \text{ as } f\to\infty. \tag{38}$$

For this situation, $\lambda_+ \to \underline{\lambda}$, and

$$[Q'']_+ \to Q''(z,z_s;\underline{\lambda}). \tag{39}$$

Thus the effective range derivative technique approaches the method of stationary phase in the high-frequency limit, but is significantly more accurate at the lower frequencies [where $\omega\psi(r,z,z_s;\lambda_{max},\underline{\lambda}) < \pi/2$].

V. EFFECTIVE SECOND RANGE DERIVATIVE FOR A SMOOTH CAUSTIC

The effective range derivative technique will now be applied to the field near a smooth caustic, as in Fig. 6, and in particular, to the shadow zone. Numerous users of propagation loss models require pressure levels along individual multipaths as opposed to coherent or incoherent sums. We are therefore motivated by an attempt to maintain continuity of individual multipath levels as one approaches the caustic and enters the shadow, and initially follow the same procedure that was developed for a real point of stationary phase.

Let λ_c denote the value of λ at the caustic, and let r_c be the corresponding range. Equations (9)–(16) hold (with λ_c replacing $\underline{\lambda}$), but now

$$\psi(r,z,z_s;\lambda_c,\lambda_c)=0, \tag{40}$$

$$\psi'(r,z,z_s;\lambda_c,\lambda_c)=r-r_c, \tag{41}$$

and

$$\psi''(r,z,z_s;\lambda_c,\lambda_c)=0. \tag{42}$$

We assume the existence of an effective (second) range derivative,

$$\left[\frac{d^2r}{d\lambda^2}\right]_+ = -[Q''']_+, \tag{43}$$

such that the cubic approximation

$$\psi(r,z,z_s;\lambda,\lambda_c) = (r-r_c)(\lambda-\lambda_c)+[Q''']_+(\lambda-\lambda_c)^3/6, \tag{44}$$

is sufficiently accurate in the dominant region of integration. Hence,

$$I_+(r,z,z_s;\lambda_c,f) = f^{1/2}\int_{\lambda_c}^{\lambda_{max}} \exp\{i\omega(r-r_c)(\lambda-\lambda_c) + i\omega[Q''']_+(\lambda-\lambda_c)^3/6+i\pi/4\}\,d\lambda. \tag{45}$$

When λ_{max} can be extended to infinity without significantly affecting the value of the integral, Eq. (45) can be expressed in terms of the Airy functions

$$\mathrm{Ai}(p)=\pi^{-1}\int_0^\infty \cos(t^3/3+pt)\,dt \tag{46}$$

and

$$\mathrm{Gi}(p)=\pi^{-1}\int_0^\infty \sin(t^3/3+pt)\,dt. \tag{47}$$

Our initial result is

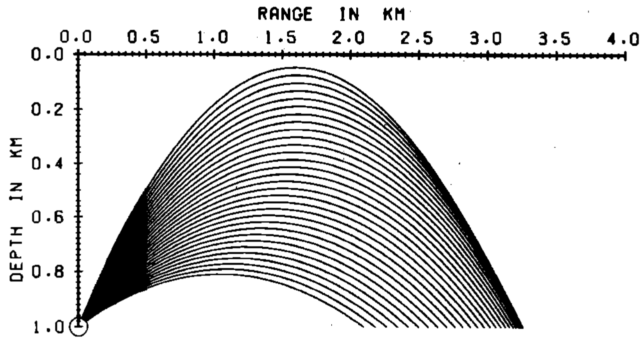

FIG. 6. Ray diagram for a smooth caustic.

$$I_+(r,z,z_s;\lambda_c,f) = f^{1/2}\pi(\omega[Q''']_+/2)^{-1/3}[\mathrm{Ai}(p)+i\,\mathrm{Gi}(p)]\exp(i\pi/4), \tag{48}$$

where

$$p=\omega(r-r_c)(\omega[Q''']_+/2)^{-1/3}. \tag{49}$$

In order to find the effective range derivative, we use equations that are similar to Eqs. (33)–(35) from the previous section. For a smooth caustic

$$[Q''']_+ = 6\psi(r_c,z,z_s;\lambda_+,\lambda_c)/(\lambda_+-\lambda_c)^3, \tag{50}$$

where λ_+ is the smaller of the values for which

$$\lambda_+=\lambda_{max} \tag{51}$$

or

$$\psi(r_c,z,z_s;\lambda_+,\lambda_c)=1/4f. \tag{52}$$

It will now be demonstrated that deep in the shadow zone, Eq. (48) is inappropriate, and how the problem can be remedied. Let us recall the asymptotic expansions

$$\mathrm{Ai}(p)\sim\pi^{-1/2}p^{-1/4}\exp(-q)/2 \tag{53}$$

and

$$\mathrm{Gi}(p)\sim\pi^{-1}p^{-1} \tag{54}$$

for the Airy functions, Ai and Gi, where $p>0$ and

$$q=2p^{3/2}/3. \tag{55}$$

As $p\to\infty$, Gi dominates Ai. At first glance, this seems to contradict the generally accepted result that the field decays exponentially with range in the shadow. However, at a smooth caustic, $[Q''']_+$ and $[Q''']_-$ should be nearly equal. The undesirable Gi term would cancel if I_+ were added coherently to I_-. It therefore appears reasonable to omit the Gi terms from Eq. (48) to obtain the desired expression

$$I_+(r,z,z_s;\lambda_c,f) = f^{1/2}\pi(\omega[Q''']_+/2)^{-1/3}\mathrm{Ai}(p)\exp(i\pi/4). \tag{56}$$

An alternative derivation of Eq. (56) is to initially assume a single $[Q''']$ over the entire interval $(-\infty,+\infty)$ so that Gi does not appear in the first place. The interval may then be segmented as before at λ_c without introducing numerically ill-conditioned expressions. Finally, compute $[Q''']$ by taking the average value of $[Q''']_+$ and $[Q''']_-$.

VI. BASIC EFFECTIVE RANGE DERIVATIVE TECHNIQUE USED AT A SMOOTH CAUSTIC

As stated earlier, classical ray theory predicts infinite intensity at a smooth caustic because the range derivative vanishes. This does not occur using the approach of Sec. IV since the effective range derivative will (neglecting possible computer errors) never vanish. We want to demonstrate now that the effective range derivative prediction is, in fact, accurate at the caustic.

Let us assume that ψ is given by Eq. (44) so that Eq. (56) is exact. At the caustic, $p=0$. Then, by virtue of Eqs. (50) and (52), Eq. (56) reduces to

$$I_+(r,z,z_s;\lambda_c,f) = f^{1/2}\pi(3\pi/2)^{-1/3}(\lambda_+-\lambda_c)\,\mathrm{Ai}(0)\exp(i\pi/4). \qquad (57)$$

If instead, we use Eq. (27) to compute I_+ with $\lambda_{max}=\infty$,

$$I_+(r,z,z_s;\underline{\lambda},f) = (1/2\pm i/2)\,|2[Q'']_+|^{-1/2}\exp(i\pi/4). \qquad (58)$$

By virtue of Eqs. (33) and (35), Eq. (58) reduces to

$$I_+(r,z,z_s;\underline{\lambda},f) = (1/2\pm i/2)f^{1/2}(\lambda_+-\underline{\lambda})\exp(i\pi/4). \qquad (59)$$

The relative error incurred by using the method of Sec. IV at the caustic is obtained by dividing Eq. (59) by Eq. (57) and setting $\underline{\lambda}=\lambda_c$:

$$\frac{I_+(r,z,z_s;\underline{\lambda},f)}{I_+(r,z,z_s;\lambda_c,f)} = \frac{(1\pm i)(3\pi/2)^{1/3}}{2\pi\,\mathrm{Ai}(0)} = 1.0628847\exp(\pm i\pi/4). \qquad (60)$$

According to the remarks made in Sec. I regarding the present state of the art, this error is small when making practical predictions.

VII. EFFECTIVE RANGE DERIVATIVE FOR A BOUNDARY PRODUCED CAUSTIC

In this section, we shall investigate the field near a boundary produced caustic such as the surface-grazing ray depicted in Fig. 7. According to classical ray theory, the range derivative of the reflected ray approaches infinity as the ray grazes the ocean boundary, while the vertexing ray is well behaved. If the intensity ratio of these arrivals is used in conjunction with measurements to estimate a bottom reflection coefficient, one would obviously obtain erroneous results. An advantage of the effective range derivative method over stationary phase is that the integration process implicit in the former method is not sensitive to range derivative discontinuities.

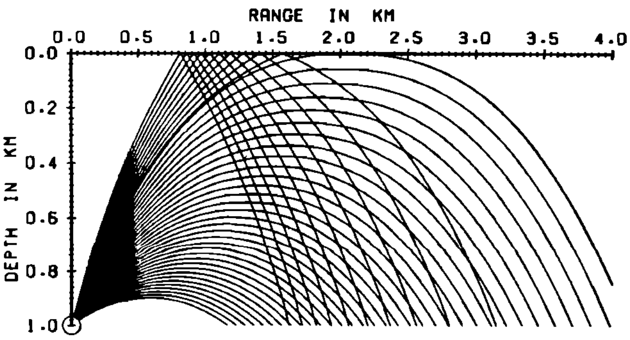

FIG. 7. Ray diagram for a boundary produced caustic.

Let λ_c denote the value of λ for the boundary-grazing ray, and let us assume that

$$\psi(r,z,z_s;\lambda,\lambda_c) = (r-r_c)(\lambda-\lambda_c)+[Q'']_+(\lambda-\lambda_c)^2/2 \qquad (61)$$

is an accurate representation for ψ in the vicinity of λ_c. We compute

$$\underline{\lambda} = \lambda_c-(r-r_c)/[Q'']_+, \qquad (62)$$

and view $\underline{\lambda}$ as being a point of stationary phase which lies outside the interval of integration $(\lambda_c,\lambda_{max})$. In Eq. (22), Sec. IV, $\underline{\lambda}$ was the lower limit of integration. In the present case, λ_c is the lower limit, and

$$I_+(r,z,z_s;\lambda_c,f) = f^{1/2}\int_{\lambda_c}^{\lambda_{max}}\exp[i\omega\psi(r,z,z_s;\lambda,\lambda_c)+i\pi/4]\,d\lambda \qquad (63)$$

is the generalization of Eq. (22) to model this effect.

We next complete the square in Eq. (61).

$$\psi(r,z,z_s;\lambda,\lambda_c) = -(r-r_c)^2/2[Q'']_+ + [Q'']_+(\lambda-\underline{\lambda})^2/2, \qquad (64)$$

and replace ψ in Eq. (63). Thus,

$$I_+(r,z,z_s;\lambda_c,f) = f^{1/2}\int_{\lambda_c}^{\lambda_{max}}\exp\{-i\omega(r-r_c)^2/2[Q'']_+ + i\omega[Q'']_+(\lambda-\underline{\lambda})^2/2+i\pi/4\}\,d\lambda, \qquad (65)$$

which can be integrated using Fresnel integrals. The desired result is

$$I_+(r,z,z_s;\lambda_c,f) = \{[C(x_2)\pm iS(x_2)] - [C(x_1)\pm iS(x_1)]\}\,|2[Q'']_+|^{-1/2} \times\exp\{-i\omega(r-r_c)^2/2[Q'']_+ + i\pi/4\}, \qquad (66)$$

where

$$x_1 = |2f[Q'']_+|^{1/2}(\lambda_c-\underline{\lambda}) \qquad (67)$$

and

$$x_2 = |2f[Q'']_+|^{1/2}(\lambda_{max}-\underline{\lambda}). \qquad (68)$$

Equation (68) would reduce to Eq. (27) if λ_c were set to $\underline{\lambda}$.

The acoustic field predicted by Eq. (68) has two desirable features. There will be a smooth transition from the illuminated region into the shadow, and the required terms are readily available from ray-tracing programs. We would like to show, as in Sec. VI, that the method is accurate at the caustic, but there are complications. Contrary to an initial assumption, the boundary reflection coefficient is not slowly varying in the vicinity of the grazing ray. In particular, the phase of the surface reflected ray changes from $-\pi$ to $-\pi/2$ in a small region. Reference 6 gives a more accurate description of propagation at boundary produced caustics. We maintain, however, that the method presented here is an improvement in our practical capability.

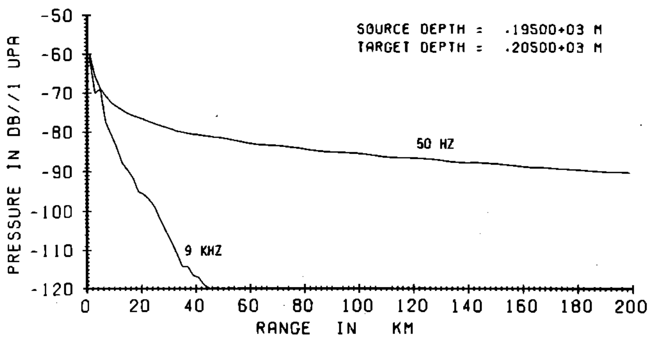

FIG. 8. Propagation loss for FAME.

Furthermore, errors in a shadow zone will normally be masked by other multipath arrivals.

VIII. FAST MULTIPATH EXPANSION

The effective range derivative technique was added to the Generic Sonar Model, as were FACT and RAYMODE. We call this addition the Fast Multipath Expansion (FAME). FAME is faster than the original multipath expansion model discussed in Ref. 4, but usually slower than FACT or RAYMODE. It is felt, however, that FAME is a better compromise between accuracy, physical insight, and computational efficiency.

A numerical problem encountered in FAME dealt with the evaluation of ψ using Eq. (16). Occasionally, one would have to compute Q to numerous significant digits in order that ψ be meaningful. Rather than use time consuming double precision arithmetic, we found that integrating

$$\psi(r,z,z_s;\lambda,\underline{\lambda}) = \int_{\underline{\lambda}}^{\lambda} [Q'(z,z_s;\lambda')+r]\,d\lambda' \qquad (69)$$

was sufficient for our needs. Here λ' is the integration variable, and $-Q'$ is the range of the ray corresponding to λ'.

After making this modification to the computer program, we used FAME to predict propagation loss for the example given in Sec. II. Results are shown in Fig. 8. Extensive testing will now be performed to see if FAME is, in fact, an improvement.

[1]Acoustical Environmental Support Detachment, "Fast Asymptotic Coherent Transmission (FACT) Model," Office of Naval Research Computer Listing, version PL 9D (April 1973).

[2]D. A. DiNapoli and R. L. Deavenport, "Numerical Models of Underwater Acoustic Propagation," *Topics in Current Physics* (Springer-Verlag, New York, 1979), pp. 107–121.

[3]Ref. 2, pp. 121–131.

[4]H. Weinberg, "Application of Ray Theory to Acoustic Propagation in Horizontally Stratified Oceans," J. Acoust. Soc. Am. 58, 97–109 (1975).

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[6]D. S. Jones, "High-Frequency Refraction and Diffraction in General Media," Philos. Trans. R. Soc. London 255, Ser. A. 1058, 431 (February 1963).