ling. Thus all types of coupling necessary come in our scheme with equal importance.

Finally it should be noted that in our theory all quantities must be expressible in the tensor form in consistence with the assumption of the five-dimensional space, or, in other words, all quantities must be invariant under the five-dimensional rotation. As a consequence the coupling constants must be alike for all kinds of coupling.

IV. CONCLUSION

We have obtained the field equations and the interactions of the field and the particles with the assumption of the five-dimensional space-time. The electrodynamics in our theory is in agreement with the classical theory. In the case of the meson theory we obtained the vector, pseudoscalar and pseudovector couplings. But an important and also rather stringent consequence of our theory is that all coupling must appear in equal importance, i.e., the coupling constants must be alike for all kinds of coupling. Moreover, our theory corresponds to the weak coupling of the current theories.*

* Cf. W. Pauli and S. Kusaka, Phys. Rev. **63**, 400 (1943). In this paper they have shown that weak coupling is in better agreement with the experiment.

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Accuracy of the Earth-Flattening Approximation in the Theory of Microwave Propagation*

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A study was made of the maximum ranges and elevations for which the earth-flattening approximation in the theory of microwave propagation is valid. It is found that at a range equal to half the radius of the earth the error introduced by the earth-flattening approximation is only 2 percent, and this independently of the wave-length. The fractional error Δ in the height-gain functions is found to be proportional to the 5/2th power of the elevation, and to the inverse power of the wave-length. Values of Δ for various wave-lengths and elevations are shown in Table I. For wave-lengths of the order of several centimeters the earth-flattening approximation breaks down at elevations greater than a few thousand feet.

1. INTRODUCTION

THE central problem in the theory of microwave propagation is the determination of the electromagnetic field produced by a dipole antenna situated at some elevation above the ground. The electromagnetic field is affected primarily by the polarization of the antenna (vertical or horizontal dipole), the properties of the ground, the variation with elevation of the refractive index of the air, and by the spherical shape of the earth's surface. The last mentioned factor is, of course, very serious for propagation into regions below the horizon. It also introduces great complexity into the mathematical solution of the problem,[1] especially in the presence of a variable refractive index in the atmosphere.

Considerable simplification of the analysis has been achieved in recent years through a device due originally to Schelleng, Burrows, and Ferrell,[2] and later developed by M. H. L. Pryce,[3] whereby

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[1] See Arnold Sommerfeld's article in Frank-Mises, *Differentialgleichungen der Physik* Vol. II, p. 918.

[2] Schelling, Burrows, and Ferrell, Proc. I. R. E. **21**, 427 (1933).

[3] M. H. L. Pryce, unpublished report. See also J. E. Freehafer, Radiation Laboratory Report 447, 1943.

space is transformed so as to make the surface of the earth plane and the rays curved.[4] The trans- ormation is accomplished simply by the use in the wave equation of a modified index of refraction N, in place of the actual index of refraction μ, defined by

$$N(r) = r\mu(r)/a\mu(a), \tag{1}$$

where r denotes the distance from the center of the earth, and a the radius of the earth. It is known qualitatively that the flattening-of-the-earth-approximation is valid for determining the field up to moderate ranges and elevations.[5] The purpose of this investigation is to determine precisely the maximum ranges and elevations for which the flattening of the earth can be considered a good approximation.

2. THE EARTH-FLATTENING APPROXIMATION

When the fractional change of the index of refraction over a distance of a wave-length is small, the electromagnetic field is determined by a Hertzian potential Ψ which satisfies the wave equation

$$\nabla^2\Psi + k^2\mu^2\Psi = 0, \tag{2}$$

$$k = 2\pi/\lambda \text{ in air}, \tag{3}$$

$$k^2\mu^2 = (\epsilon\omega^2 - i\sigma\omega)/c^2 \text{ in the ground}, \tag{4}$$

where a time factor $e^{i\omega t}$ has been assumed. For vertical polarization Ψ represents the vertical component of the electric Hertzian potential, and the field intensity is proportional to $|\Psi|^2$. Similarly, in the case of horizontal polarization the field is still essentially determined by a single Hertzian potential Ψ.[6]

In the problem of radiation from a dipole antenna, the solution of Eq. (2) must satisfy the following boundary conditions:

(*a*) Ψ should reduce to the form e^{-ikR}/R for small distances R from the source,
(*b*) Ψ should represent an outgoing wave at great distances from the source, and
(*c*) The tangential components of the electric and magnetic fields should be continuous at the earth's surface.

We shall limit the discussion to the special, but important, case when the index of refraction μ is a function of elevation only, i.e., $\mu = \mu(r)$. Without loss of generality we can also confine the discussion to solutions of Ψ which are symmetrical with respect to the radius vector through the source. Using polar coordinates (r, θ), we seek to build up a solution of (2) satisfying the boundary conditions in terms of elementary solutions of the form

$$\psi = P(\theta) \cdot U(r). \tag{5}$$

Substituting in (2) we find that P and U must satisfy the equations

$$\frac{d^2P}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dP}{d\theta} + a^2k_m{}^2P = 0, \tag{6}$$

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} + \left[k^2\mu^2(r) - \frac{a^2k_m{}^2}{r^2}\right]U = 0. \tag{7}$$

Here we have written $a^2k_m{}^2$ (a denoting the radius of the earth, and k_m having the dimension of a reciprocal length) for the customary separation constant $n(n+1)$. The appropriate solutions of (6) are of course $P_n(\cos\theta)$, where n must be an integer in order to secure finiteness of P at $\theta = \pi$. We shall, however, relinquish the requirement of finiteness of P at $\theta = \pi$, as well as $\theta = 0$, by using what amounts to a linear combination of $P_n(\cos\theta)$ and $Q_n(\cos\theta)$. The resulting solution for Ψ, which can be made to satisfy the boundary conditions, will be useful everywhere except in the immediate

[4] In a memorandum submitted by the writer to Professor Carl Eckart on October 23, 1943, it was pointed out that the portion of the coverage diagram situated below the horizon shown in Fig. 5 of the paper by B. v. d. Pol and H. Bremmer, in Phil. Mag. **27**, 270 (1939), agrees with the solution for a flat earth when the index of refraction of the air μ varies with elevation h like $(1+h/a)$, where a denotes the radius of the earth. The solution of the latter problem had been obtained by the writer in May, 1943 (see a paper by the writer in J. Acous. Soc. Am. **18**, 295 (1946) In the field in which the writer was then active the flat earth with a variable index of refraction presented itself *ab initio*. It was suggested on many occasions that the treatment of that problem could perhaps be simplified by curving the earth and straightening the rays, a procedure the opposite of which was then being practiced by microwave propagation researchers.

[5] See Freehafer's discussion in Radiation Laboratory Report 447.

[6] See reference 1, page 948; M. C. Gray, Phil. Mag. **27**, 421 (1939); R. Burrows and M. C. Gray, Proc. I. R. E. **29**, 16 (1941).

neighborhood of the radius vector through the source and through the antipodes. The k_m are then found to be a set of complex characteristic values determined by the boundary conditions.

In the earth-flattening approximation one adopts, instead of the polar coordinates r and θ, the set of coordinates (ρ, h) defined by

$$\rho = a\theta, \tag{8}$$

$$h = r - a. \tag{9}$$

In terms of these coordinates, Eqs. (6) and (7) can be written as

$$\frac{d^2P}{d\rho^2}+\frac{1}{\rho}\frac{dP}{d\rho}+k_m^2P = -\frac{1}{\rho}\frac{dP}{d\rho}\left(\frac{1}{3}\frac{\rho^2}{a^2}+\frac{1}{45}\frac{\rho^4}{a^4}+\cdots\right), \tag{10}$$

$$\frac{d^2U}{dh^2}+\left[k^2\mu^2(h)-k_m^2+2k_m^2\frac{h}{a}\right]U = -\frac{2}{a}\frac{dU}{dh}+3k_m^2\frac{h^2}{a^2}U+\cdots, \tag{11}$$

upon expanding $\cos\theta/\sin\theta$ in powers of d/a, and $1/r$ in powers of h/a. Mathematically the earth-flattening approximation consists essentially in neglecting the right-hand sides of Eqs. (10) and (11), which are roughly of the order of ρ/a and h/a relative to the terms on the left-hand sides. One is then left with a system of equations which are appropriate for a flat earth, the only change being that the term $k^2\mu^2(h)$ is replaced by $[k^2\mu^2(h)+(2k_m^2h)/a]$. Since under actual conditions $\mu(h)\simeq 1$ and $k_m\simeq k$, the latter expression is equal to $k^2N^2(r)$ to within terms of the order of h^2/a^2, where $N(r)$ is defined in Eq. (1).

Barring certain pathological cases which we need not consider here[4] it is possible to obtain solutions of (2) satisfying the boundary conditions in terms of *normal modes* as follows

$$\Psi = -i\pi\sum_1^\infty H_0^{(2)}(k_m\rho)U_m(h_1)U_m(h_2), \tag{12}$$

where h_1 and h_2 denote the heights of transmitter and receiver respectively, and $U_m(h)$ satisfies the equation

$$\frac{d^2U_m}{dh^2}+(k^2N^2-k_m^2)U_m=0. \tag{13}$$

The solution (12) was arrived at in recent years by W. H. Furry, T. Pearcey, G. L. Roe, and the writer, the method of derivation having been used for the first time by H. Lamb in a classical paper written in 1904.[7]

3. THE ERROR INVOLVED IN THE EARTH-FLATTENING APPROXIMATION

We now turn to our principal task, which is an estimation of the error involved in the neglect of the terms appearing on the right-hand sides of Eqs. (10) and (11). Taking first Eq. (10), we seek a solution of the form

$$P(\rho)=P_0(\rho)+(1/a^2k_m^2)P_2(\rho)+(1/a^4k_m^4)P_4(\rho)+\cdots. \tag{14}$$

Substituting in (10) we find that

$$\frac{d^2P_0}{d\rho^2}+\frac{1}{\rho}\frac{dP_0}{d\rho}+k_m^2P_0=0,\quad P_0(\rho)=H_0^{(2)}(k_m\rho), \tag{15}$$

$$\frac{d^2P_2}{d\rho^2}+\frac{1}{\rho}\frac{dP_2}{d\rho}+k_m^2P_2=\frac{\rho}{3}\frac{dP_0}{d\rho},\qquad P_2(\rho)=-\frac{1}{12}(k_m\rho)^2H_2^{(2)}(k_m\rho). \tag{16}$$

Hence

$$P(\rho)=H_0^{(2)}(k_m\rho)-\frac{\rho^2}{12a^2}H_2^{(2)}(k_m\rho)+0(1/a^4k_m^4). \tag{17}$$

$P_2(\rho)$ is certainly small relative to $P_0(\rho)$ for small values of ρ, while for large values of $k_m\rho$, $H_2^{(0)}(k_m\rho)\rightarrow -H_0^{(0)}(k_m\rho)$, so that the correction term to $P_0(\rho)$ is of the order of $\rho^2/12a^2$, independently of frequency. For $\rho=a/3$ this correction term is less than one percent, and for $\rho=a/2$ it is around two percent. *We therefore conclude*

[7] H. Lamb, Phil. Trans. Roy. Soc. **A203**, 1 (1904).

*that the earth-flattening approximation is valid to within two percent up to ranges of about half the radius of the earth, and this independently of the frequency.**

Coming now to the height-gain functions $U(h)$, which in the earth-flattening approximation satisfy the homogeneous equation (13) rather than the exact inhomogeneous equation (11), we must consider separately the part of the solution in the interior of the earth and the part in the atmosphere. With regard to the solution in the ground it is clear qualitatively that the earth-flattening approximation will be valid only if the electromagnetic energy is confined to a thin skin layer of the ground. This condition will be realized the more closely the more conductive the ground is. Over the sea we may expect relatively little energy to be carried in the water, especially for the low order modes.

Our principal interest is to estimate the maximum heights for which the earth-flattening is a valid approximation. For this purpose we need to investigate the asymptotic behavior of the solutions of (11) for large heights. Now at great heights $\mu^2(h)$ decreases linearly with h under standard atmospheric conditions, and the expression in brackets in (11) becomes a linear function of h, which is characteristic for the so-called *standard atmosphere*. It follows that in our particular application we can limit ourselves to the case of a standard atmosphere where

$$\mu^2(h) = 1 - \frac{1}{2}\frac{h}{a}. \tag{18}$$

As far as the boundary condition at the surface is concerned we can assume for the sake of simplicity that $U_m(0) = 0$, because the mode of variation of $U_m(h)$ at great heights is not affected thereby.†

It is convenient to introduce non-dimensional coordinates as follows

$$N^2 = 1 + qh, \quad q = \frac{3}{2a}, \quad k_m{}^2 = k^2(1 - \Lambda_m), \tag{19}$$

$$H = (k^2 q)^{-\frac{1}{3}}, \quad z = h/H, \quad \Lambda_m = (k/q)^{-\frac{2}{3}} D_m. \tag{20}$$

Using these in (11) we obtain

$$\frac{d^2U}{dz^2} + (z + D_m)U = \frac{2H}{a}\left(-\frac{dU}{dz} + z^2 U\right). \tag{21}$$

As before, we seek a solution of (21) of the form

$$U(z) = U_0(z) + \frac{H}{a}U_1(z) + \cdots, \tag{22}$$

and find that

$$\frac{d^2U_0}{dz^2} + (z + D_m)U_0 = 0, \tag{23}$$

$$U_0 = Cu^{\frac{1}{3}} H_{1/3}{}^{(2)}(u), \quad u = \tfrac{2}{3}(z + D_m)^{\frac{3}{2}}, \tag{24}$$

$$\frac{d^2U_1}{dz^2} + (z + D_m)U_1 = -2\frac{dU_0}{dz} + 2z^2 U_0, \tag{25}$$

$$U_1 = \left(-\frac{2}{5}z^2 + \frac{8}{15}D_m z - \frac{16}{15}D_m{}^2\right)\dot{U}_0 - \frac{3}{5}zU_0. \tag{26}$$

Now

$$\dot{U}_0 = C(z + D_m)^{\frac{1}{2}} u^{\frac{1}{3}} H_{-2/3}{}^{(2)}(u),$$

so that

$$\frac{1}{U_0}\frac{dU_0}{dz} = \frac{(z + D_m)^{\frac{1}{2}} H^{(2)}{}_{-2/3}(u)}{H_{1/3}{}^{(2)}(u)} \rightarrow -i(z + D_m)^{\frac{1}{2}}, \tag{27}$$

for large z. It follows that for large z

$$U_1 \rightarrow i\tfrac{2}{5} z^{5/2} U_0.$$

The correction factor to be applied to the solution $U_0(z)$ obtained under the earth-flattening approximation is therefore

$$1 + i\frac{2H}{5a} z^{5/2} = 1 + i\frac{2h^{5/2}}{5a^{3/2}} \equiv 1 + i\Delta, \tag{28}$$

$$\Delta = \frac{2}{5}\frac{h^{5/2}}{aH^{3/2}} = \frac{\sqrt{6}}{5}\frac{kh^{5/2}}{a^{3/2}}. \tag{29}$$

* This investigation was prompted by alarming results which the writer obtained with the earth-flattening approximation when instead of the distance along the surface of the earth $\rho = a\theta$, the distance from the earth's axis $\rho = a \sin\theta$ was adopted for the horizontal coordinate. In this case the correction term to $P_0(\rho)$ is of the order of $\rho^3 k/6a^2$ instead of $\rho^2/12a^2$. The former varies with wave-length, and for a wave-length of 10 cm is greater than 1 already at a range of 10 miles!

† Essentially what is needed for the following argument is merely relation (27).

TABLE I. Fractional error $\Delta(h)$ of the height-gain function under the earth-flattening approximation for various elevations h and wave-lengths λ.

λ_{cm}	H	$\Delta(1000')$	$\Delta(2000')$	$\Delta(3000')$	$\Delta(4000')$	$\Delta(5000')$	$\Delta(6000')$
10	33.62′	0.0031	0.0176	0.0484	0.0993	0.1735	0.2737
3	15.07′	0.0103	0.0585	0.1613	0.3310	0.5783	0.9122
1	7.24′	0.0310	0.1756	0.4839	0.9930	1.735	2.737

It is seen that the correction term Δ increases with elevation like $h^{5/2}$, and is proportional to the frequency. In Table I are listed values of Δ for various elevations and wave-lengths.

It is clear from the table that for the wave-lengths listed the earth-flattening approximation breaks down at elevations of several thousand feet. In the case of a 10-cm wave, for example, the height-gain function is in error by 17 percent at an elevation of 5000 feet.

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Surface Layers on Quartz and Topaz*

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X-ray studies of single crystals of quartz and topaz show that there exist on the surfaces of these crystals disturbed layers which are not removed by etching, and that they exert an effect on etched surfaces to a depth of 5 or more microns. The disturbance is not caused by plastic flow. Good, single crystal quartz wafers, 35 microns thick, change when they are etched to 15 microns. The thin plates no longer appear to be single crystal. This "polycrystalline" state is stable at room temperature. Thin crystals undergo further changes when flexed, resulting in a two to threefold broadening of the Cu $K\alpha$ rocking curves. Similar results were observed with topaz. It is shown that this effect is different from, and may not be related to, the well-known broadening of the Bragg reflections that result from grinding crystal surfaces. As an explanation for the observed phenomena, it is suggested that at least part of the surface energy is in the form of structure irregularity and there results a very thin, glass-like layer that is unstable because it is effectively at a higher temperature than the rest of the crystal. Consequently, recrystallization to a more stable form occurs.

INTRODUCTION

MANY workers have reported that the surfaces of crystals, particularly if they have been ground or polished, are covered with misaligned material.[1-5] In 1943 C. J. Davisson[6] and E. H. Armstrong[7] investigated the surface of ground quartz crystals and found crystallites which were turned through more than 3°. Davisson measured the actual misalignment. A quartz crystal was adjusted in a Bragg spectrometer to give maximum reflection for the Cu $K\alpha$ line from the $(20\bar{2}3)$ planes. Photographs were taken, the first with the crystal as adjusted, the second with the crystal turned 15′ from the maximum reflecting position, the next with the crystal turned 30′, etc., until the quartz had been rotated through several degrees. The series showed two lines, one from the background radiation with position dependent on crystal orientation, and the second at the Bragg position for Cu $K\alpha$. The intensity of the undisplaced line fell off to zero with increasing angle. Since the crystallites showed a decided preference for the orientation of the main crystal, Davisson concluded that the line could not be caused by small particles of quartz broken off in grind-

* A report on part of this work was presented at the January, 1946, Meeting of the American Physical Society. Cf. D. D'Eustachio and S. B. Brody, Phys. Rev. **69**, 256 (1946).

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