Ray-theoretic analysis of a mathematical model for SOFAR propagation

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The problem of acoustic wave propagation in an underwater sound channel has been discussed using the ray theory for a model consisting of two semi-infinite liquid media in which the propagation velocity increases exponentially in both directions from the interface. The results predicted by this model are in good agreement with the results of experimental work.

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INTRODUCTION

Several mathematical models for speed–depth profile have been used to examine the characteristics of SOFAR propagation analytically. Blum and Cohen[1] considered the sound speed $C(y)$ varying as $C(y)=C_\infty[1+M/\cosh^2 \times(\frac{1}{2}my)]^{-1/2}$, $M>0$, C_∞, m, and M being constants; Kornhauser and Yaghjian[2] assumed that the wavenumber $K(y)$ varies as $K(y)=K_0[\operatorname{sech}^2(\alpha y)-\beta^2]^{1/2}$, where K_0, α, β are constants. For the profile $C^2(y)=C_0^2(1-|\alpha y|^\beta)^{-1}$, where C_0, α, β are constants, Hirsch and Carter[3] have proved that when $\beta<2$ the off-axis arrivals are earlier. Recently, Munk[4] has computed the ray arrivals for an axial source and receiver for asymmetric profile $C(y)=C_1[1+\epsilon(\eta+e^{-\eta}-1)]$, where C_1 is the velocity at channel axis whose depth is y_0, ϵ is a perturbation coefficient and $\eta=(y-y_0)/\frac{1}{2}B$, B being a scale depth. In this profile the exponential term dominates above the channel axis while the linear term dominates below the axis. In this note we consider a model consisting of two semi-infinite liquid media in contact such that the velocity increases exponentially in both directions from the interface. The model allows for both symmetric and asymmetric velocity variations. Ray-theoretic treatment is presented here, while wave-theoretic solutions have been discussed elsewhere (Goda and Rao[5]).

I. THE MATHEMATICAL MODEL AND THE RAY SOLUTION

We consider two semi-infinite media in contact at the plane $y=0$, the y axis being positive downwards. Throughout this work the subscripts 1 and 2 shall be used for upper and lower media, respectively. We assume that the velocities $C_1(y)$ and $C_2(y)$ in the two media are given by

$$\begin{aligned} C_1^2(y)&=C_{-\infty}^2+(C_0^2-C_{-\infty}^2)e^{\epsilon_1 y}, \quad y\in[0,-\infty], \\ C_2^2(y)&=C_\infty^2+(C_0^2-C_\infty^2)e^{-\epsilon_2 y}, \quad y\in[0,\infty], \end{aligned} \tag{1}$$

where ϵ_1 and ϵ_2 are positive constants, $C_0=C(0)$, $C_{-\infty}=C(-\infty)$, $C_\infty=C(\infty)$, such that C_0 is less than both C_∞ and $C_{-\infty}$.

Consider a ray emitted from a source situated at $y=0$, $r=0$ at an angle θ_0 with the channel axis which is received at a point $P(r,y)$. Using the well-known techniques of ray theory (see Brekhovskikh[6]), the range $\Delta_2(\theta_0)$ and the travel time $\tau_2(\theta_0)$ as functions of the take-off angle θ_0 for a half-cycle in the lower medium are

$$\Delta_2(\theta_0)=2/\epsilon_2(W_2+2\theta_0), \tag{2}$$

$$\tau_2(\theta_0)=2/\epsilon_2[C_0/C_\infty^2\cos\theta_0)]W_2, \tag{3}$$

where

$$W_2=[1-C_0^2/(C_\infty^2\cos^2\theta_0)]^{-1/2}\{\pi/2+\sin^{-1}[(C_0^2/C_\infty^2-\cos 2\theta_0)/(1-C_0^2/C_\infty^2)]\} \tag{4}$$

The critical angle in the lower medium is given by $\cos^{-1}(C_0/C_\infty)$, substituting this value for θ_0 in Eq. 2, $\Delta_2\to\infty$ meaning that there is no shadow zone. If N_2 is the number of half-cycles in the lower medium, $T_2=N_2\tau_2(\theta_0)$ and $R_2=N_2\Delta_2(\theta_0)$ are the total travel time and ranges for these N_2 cycles, respectively. From Eqs. 2 and 3 we get

$$T_2=R_2[C_0/(C_\infty^2\cos\theta_0)][W_2/(W_2+2\theta_0)]. \tag{5}$$

Expanding the right-hand side of Eq. 5 for fixed R_2 and small θ_0, we get

$$T_2=(R_2/C_0)(1-\theta_0^2/6), \tag{6}$$

with a similar equation for the upper medium. Finally, we find

$$T=(R/C_0)(1-\theta_0^2/6), \tag{7}$$

where T, R are the total travel time and range, respectively, for a ray received at a point on the channel axis. When $\theta_0\to 0$, the travel time will increase with the axial ray being the last to arrive.

From Eqs. 2 and 3,

$$T_2=(C_0/[C_\infty^2\cos\theta_0])(R_2+4N_2\theta_0/\epsilon_2). \tag{8}$$

For very large R_2, the travel time of the first arrival $T_2^{(1)}$ is given by, $N_2=1$ and θ_0 should be in the neighborhood of $\cos^{-1}(C_0/C_\infty)$. The travel time for the axial ray is R_2/C_0 and hence the duration T_{2d} is given by

$$T_{2d}\approx R_2/C_0-R_2/C_\infty-4\cos^{-1}(C_0/C_\infty)/(\epsilon_2 C_\infty)=\alpha R_2+\beta, \tag{9}$$

where $\alpha=1/C_0-1/C_\infty$, $\beta=-4\cos^{-1}(C_0/C_\infty)/(\epsilon_2 C_\infty)$. The relation between the duration and the range, therefore, is asymptotically a straight line.

II. COMPARISON WITH OBSERVATIONS

We shall consider two finite layers instead of the semi-infinite layers. We assume the channel axis at a

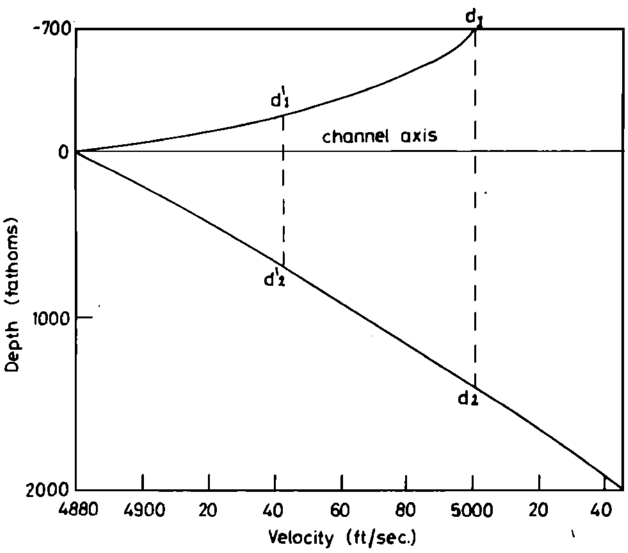

FIG. 1. Sound-speed profile given by Eq. 10.

depth of 700 fathoms from the surface, and that the velocities $C_1(y)$, $C_2(y)$ are given by

$$C_1^2(y) = (5025)^2 + [(4880)^2 - (5025)^2]\, e^{(2.6)10^{-3} y}, \quad y \in [0, -700], \qquad (10)$$

$$C_2^2(y) = (5800)^2 + [(4800)^2 - (5025)^2]\, e^{-(9.2)10^{-5} y}, \quad y \in [0, \bar{y}],$$

where $C_1(y)$, $C_2(y)$ are the velocities in feet/sec, y is the depth in fathoms, $\bar{y}$ is the depth of the lower layer. The velocity profile given by Eq. 10 is shown in Fig. 1. The velocity distribution in Fig. 1 agrees with the actual measurements in the mid-Atlantic given by Ewing, Pekeris, and Worzel[7] to within ± 0.5% up to a depth of 2700 fathoms. The ray diagram for our profile is shown in Fig. 2. The upper and lower limits of the sound channel are defined by the two depths of equal maximum velocity in the profile (see Urick[8]), and these in turn will define the maximum allowable θ_0 without reflections. The ray which arrives first for a certain large distance will have its initial angle in the neighborhood of this

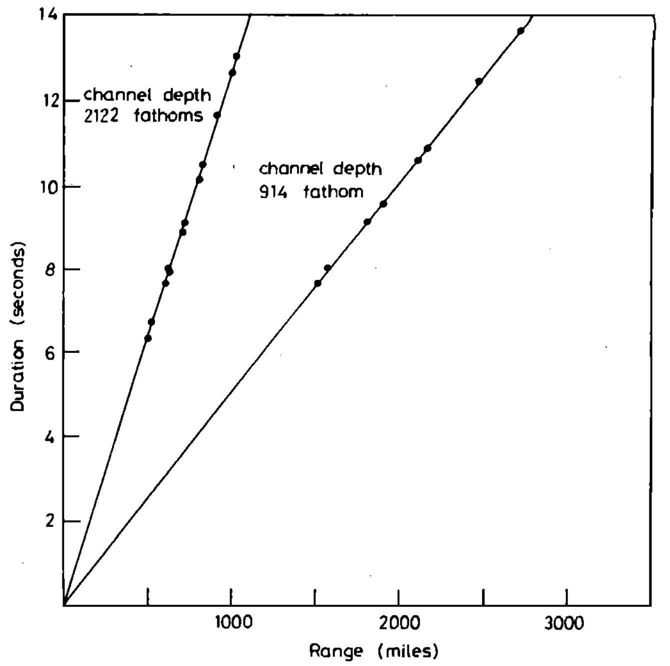

FIG. 3. Range-duration curves for two different sound channels.

maximum θ_0. We shall assume that the bottom is about 2000 fathoms below the channel axis, i.e., 2700 fathoms from the surface. From Fig. 1, the channel height $d_1 d_2 \simeq 700 + 1422 = 2122$ fathoms, while from Fig. 2 the maximum allowable value for θ_0 is about 12.67°. In order to draw the duration–range graph, we assume some large values of R and find those rays which arrive first with the least possible values of N_1 and N_2 (N_1 and N_2 being the number of half-cycles in the upper and lower medium, respectively) and then calculate the corresponding duration. The duration-range graph is shown in Fig. 3. Again if we assume that the depth of the lower layer from the surface is 1400 fathoms, which is the depth of the mid-Atlantic ridge (Bryan, Truchan, and Ewing[9]), then the height of the sound channel is $d_1' d_2'$

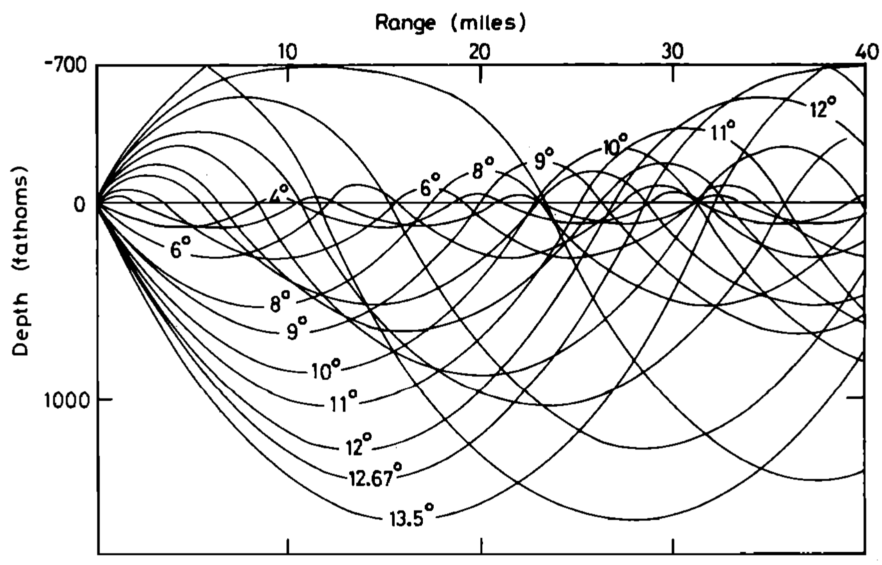

FIG. 2. Ray diagram for the profile given by Eq. 10.

$\approx 214 + 700 = 914$ fathoms and the maximum allowable $\theta_0 \approx 9.11°$. The duration-range curve is shown in Fig. 3, the straight line with less inclination. For ranges which include the ridge, the effective height of the SOFAR channel is restricted by the ridge. Our results, as seen from Fig. 3, indicate almost perfect accordance with the experimental results obtained by Bryan, Truchan, and Ewing.[9]

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