ECOS3021

Business Cycles and Asset Markets

Notes

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Contents

1 Introduction

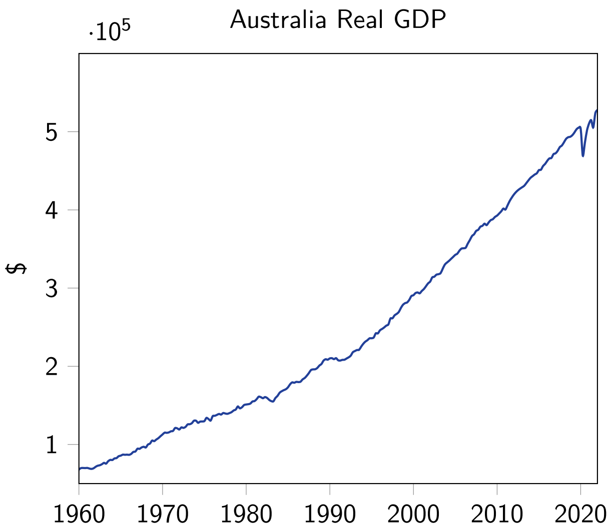
1.1 What are Business Cycles

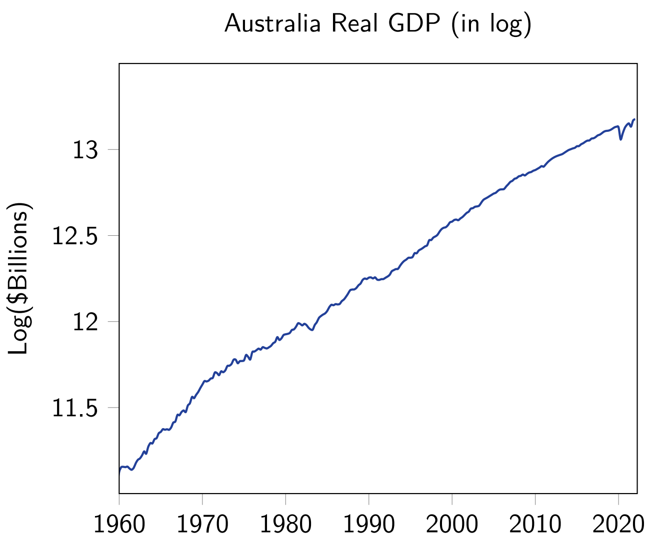
What is the business cycle?

- Distinguish long-run macroeconomic growth from short-run macroeconomic fluctuations
- Business cycles are fluctuations in aggregate- or macro-economic activity
- These fluctuations occur over the short to medium term

Stylised Features

- Trend: Long-run increase in economic activity
- Peak: Short-run/cyclical high in economic activity
- Though: Short-run/cyclical low in economic activity
- Boom/Expansion: Period of increasing economic activity following a recession
- Slump/Recession/Contraction: Period of decreasing economic activity following a boom
- Recovery: Post-recession period of growth that brings economic activity back up to its long-run trend
- Consider real Gross Domestic Product (GDP) for Australia, observed at a quarterly frequency





- Let y_t be real GDP at time t
- Let Δy_t be the growth rate of y (in percent) between dates $t-1$ and t

$$\Delta y_t = \frac{y_t - y_{t-1}}{y - t - 1}$$
$$\Delta y_t = \frac{y_t}{y_{t-1}} - 1$$
$$\log(1 + \Delta y_t) = \log(y_t) - \log(y - t - 1)$$
$$\Delta y_t \approx \log(y_t) - \log(y - t - 1)$$

- Plotting the log of GDP makes it easier to see growth rates
- In a log GDP plot, the slope characterizes the growth rate

Classical Business Cycles

According to Burns and Mitchell (1946):

- Business cycles are not defined as fluctuations in real GDP but as fluctuations in an undefined measure of "aggregate economic activity". (Why not GDP alone?)
- Dating of business cycle turning points is based on a mixture of mechanically applied rules and ad hoc judgments (e.g. NBER's Business Cycle Dating Committee). Requires careful interpretation of data!
- Harding and Pagan (2002) presented a now well-known algorithm for identifying turning points in classical business cycles using quarterly data (known as the BBQ procedure).
- A peak at time t occurs if:

$$[(y_t - y_{t-2}) > 0, (y_t - y_{t-1}) > 0], \text{ and}$$
$$[(y_{t+2} - y_t) < 0, (y_{t+1} - y_t) < 0],$$

- A trough at time t occurs if:

$$[(y_t - y_{t-2}) < 0, (y_t - y_{t-1}) < 0], \text{ and}$$
$$[(y_{t+2} - y_t) > 0, (y_{t+1} - y_t) > 0],$$

Growth Business Cycles

- According to Robert Lucas (1977), "aggregate fluctuations around the trend or growth path"
- "Refers to the same thing (as Classical cycles) in some detrended series"
- A growth recession requires a relative decline (i.e. growth can still be positive) in real GDP, but below the long-term growth trend
- A complete growth cycle in industralized countries typically takes between 18 months and 8 years, depending on how the trend is defined
- No clear asymmetry in growth cycles. (Why might this be?)
- Think of a time series yt with secular (i.e. uncorrelated) components decomposed as

$$\log y_t = g + c_t$$

- g is the long-run growth or trend component
- c_t is the cyclical (business cycle) component

How do we detrend a time series with growth components?

1. Difference the series. Let y_t be a quarterly time series

$$\text{Quarterly difference:} \quad \log y_t - \log y_{t-1} = g + c_t - (g + c_{t-1}) = c_t - c_{t-1}$$
$$\text{Year-on-year difference:} \quad \log y_t - \log y_{t-4} = g + c_t - (g + c_{t-4}) = c_t - c_{t-4}$$

   - Differencing removes the growth component, leaving only fluctuations due to the cyclical components
   - However, differencing tends to remove too much information and displays short-term volatility. So not suitable to obtain medium-term movements.
   - But, easy and useful to interpret and assess economic conditions.

2. Assume the trend is a deterministic function of time
   - $y_t = g_t + c_t$
   - where the growth component is given by: $g_t = g + \alpha \cdot t + \beta \cdot t^2$
   - t is just time (e.g. the year 1990, 1991, 1992, etc)

   - g is a constant, α and β are coefficients on the linear and quadratic terms
3. Assume a stochastic trend (i.e. a random trend). Find via a filtering algorithm
   - Many filters are borrowed from engineering applications, e.g., filtering noise from a signal
   - Examples of filters in macroeconomics:
     - Hodrick-Prescott (1997) filter
     - Band-pass filter
     - Forecasting filters (e.g. Hamilton, 2017)
   - These filters are used to find a smooth trend in the data visually similar to the trend that one can obtain with a free-hand drawing
   - The cycle component is then consistent with the growth cycle definition of Lucas (1977)

1.2 How do we Understand the Price of an Asset?

- Several methods for "valuing" or "pricing" an asset:
  - **Discounted Cashflow Valuation**: present value of the expected cash flows of an asset
  - **Relative Valuation**: estimate value from price/value of similar or comparable assets
  - **Contingent Claim (Option) Valuation**: positive payoff if underlying value is higher than some "strike price" (e.g. a startup either starts to make money or its fixed assets are liquidated)

Asset Prices as Discounted Cash Flows

- The price of an asset is equal to its stream of cash flows, discounted by the interest rate

$$\text{Price}_t = \text{Cash}_t + \frac{\text{Cash}_{t+1}}{(1+r)} + \frac{\text{Cash}_{t+2}}{(1+r)^2} + \cdots + \frac{\text{Cash}_T}{(1+r)^T}$$

- Subscript t denotes the time (e.g. weeks, months, years)
- $Cash_t$ is the cash flow received from the asset at time t
- r is the interest rate
- Subcript T is the final period in which cash flows are received from the asset
- Why do we divide future cash flows by the (gross) interest rate, $1+r$?
  - Rather than buy the asset, could put money into bank account and wait for interest to accrue
  - These forgone interest earnings are the opportunity cost of investing in the asset
  - So we "discount" the value of future cash flows by the interest we could have earned
- How might the price of assets be affected by the business cycle?
  - Cash flows fluctuate over the business cycle
  - Interest rates fluctuate over the business cycle

2 Real Business Cycles and the RBC Model

2.1 Stylised Facts About Business Cycles

- We want to gather some "stylised" facts about business cycles
- Looking for statistics that explain what typically takes place during a business cycle
- But we should be aware that "every recession" is special in its own way
- And the existence of stylised facts does not mean that business cycles are predictable

Cyclical Relations: Definitions

A macroeconomic variable is:

- Pro-cyclical: if deviations from trend are positively correlated with real GDP deviations from its own trend
- Counter-cyclical: if deviations from trend are negatively correlated with real GDP deviations from its own trend
- Acyclical: if deviations from trend for each variable are not correlated

Time Series (Cyclical) Relations

1. Correlation (or, co-movement)
   - Measure the degree ofcontemporaneous synchronisation between any two variables
2. Leads and Lags
   - Measure the degree of synchronisation between any two variables across time
   - We measure these relationships via cross-time correlations:
     - $Corr(x_{t+j}, y_t)$ with $j < 0$ indicates x is a leading variable (e.g. x_{t-1} increases before y_t)
     - $Corr(x_{t+j}, y_t)$ with $j > 0$ indicates x is a leading variable (e.g. x_{t+1} increases before y_t)

Time Series Properties

3. Variability (a.k.a. volatility)
   - Measures the amplitude of deviations from a trend or mean
   - Measure variability via the standard deviation of a variable
4. Persistence
   - Measures the time dependence of a variable (i.e. high today $\Rightarrow$ high tomorrow)
   - Measure persistence via the autocorrelation function
   - $Corr(y_t, y_{t-j})$ with $j > 0$ (lags of y or $j < 0$ leads of y)

Documenting Business Cycle Facts

- Employment, consumption, investment are allpro-cyclical to GDP
- Employment and total hours worked fluctuate almost as much as GDP
- Consumption (of non-durables and services) is smooth and fluctuates less than GDP
- Investment fluctuates much more than GDP
- Productivity is slightly pro-cyclical to GDP
- Government expenditure is uncorrelated with GDP (acyclical)
- Net exports are pro-cyclical to GDP

How do we document and report these 'stylised facts'?

- Detrend the (log) time series data, removing growth components
- Compute summary statistics from detrended data (i.e. cyclical components)

Summary Statistics

1. For individual variables/time series, compute:
   - Mean (e.g. mean growth rate)
   - Standard deviation (i.e. volatility)
   - Autocorrelation (i.e. persistence)
2. For pairs of variables (e.g. consumption and GDP)
   - Relative standard deviation: $S.D.(x_t)/S.D.(y_t)$
   - Cross correlation (co-variance) at various leads/lags - $Corr(x_{t+j}, y_t)$ for j negative (x leads) or positive (x lags)

2.2 Brief History of Business Cycle Theories

- Business cycle theories of early 20th century quantitatively analysed economic fluctuations using mathematical and statistical approaches
- This research agenda was led by Ragnar Frisch and Jan Tinbergen, the first winners of the Nobel Prize in Economics
- This work on business cycles begun before John Maynard Keynes became one of the most well-known names in the study of macroeconomic fluctuations (i.e. the father of "Keynesian" economics)
- From the 1970s, Real Business Cycle (RBC) theory attempted to quantitatively explain macroeconomic fluctuations via shocks to aggregate production technology (i.e. productivity)
- This followed the tradition of Classical and Neo-Classical Economics
  - Households and firms behave as if they make rational choices subject to constraints
  - Macroeconomic outcomes are determined byequilibrium and market clearing
  - The "classical dichotomy": nominal variables do not affect real variables
- RBC model structure follows from Optimal Growth Theory (e.g. the Solow-Swan model)
- RBC models incorporate Neo-Classical growth with stochastic shifts or shocks as the driving force behind cyclical macroeconomic fluctuations
- The RBC research agenda uses the stochastic growth model to try to explain fluctuations that can be quantitatively assessed
- Another aim of RBC economists was to build small laboratories in which government policies could be tested
- Modern macroeconomic models used at central banks are rooted in the RBC framework
  - Federal Reserve Board ("FRB/US"); Norges Bank ("NEMO"); Swedish Riksbank ("RAMSES II"); Bank of Canada ("TOTEM"); Reserve Bank of Australia ("MARTIN"); Reserve Bank of New Zealand ("NZSIM")
- RBC models have evolved into Dynamic Stochastic General Equilibrium (DSGE) models
- Inspired by Robert Lucas (1977), Kydland and Prescott (1982) aimed to study growth and fluctuations in a single model framework asking the following question:
- "Can business cycle fluctuations occur as a natural consequence of the competitive economy where agents make optimal inter-temporal resource allocation decisions in response tostochastic shifts in technology and preferences?"
  - If the answer is **No** (as most economists at the time believed):
    * Market co-ordination failure

    - Large welfare losses from market outcomes
    - Role for active macroeconomic stabilization policy (e.g. Keynesian stimulus)
  - If answer is **Yes** (as RBC economists believed):
    - Business cycles are "efficient"
    - Negligible welfare costs from market outcomes
    - Active stabilisation policies can be disruptive/destabilising
- Why real business cycles?
  - 'Real' as opposed to 'nominal' or monetary forces
- Why are real business cycles efficient?
  - No economic frictions to distort optimal decisions
- Modern DSGE models incorporate many nominal/monetary features:
  - Price rigidity, nominal shocks, monetary and fiscal policies
- Modern DSGE models incorporate many economic rigidities/frictions:
  - Imperfect competition, search frictions, credit market frictions

2.3 Intra-Temporal Households in the RBC Model

Choice between Work and Leisure

- Households must decide how much to work (in order to earn income) and how much leisure to enjoy.
- The more a household works, the more income they have to spend, but the less leisure time they can enjoy (there are only so many hours in a day!)
- Leisure is a normal good
- The static optimisation problem is to maximise utility subject to a static budget constraint and a time endowment constraint
- A household's problem is to choose consumption C and leisure L

$$\begin{aligned} \max_{C,L} \quad & U(C) + V(L) \\ \text{s.t.} \quad & C = w \cdot N^S + \Pi \quad \text{Budget constraint} \\ & L + N^S = 1 \quad \text{Time endowment} \end{aligned}$$

- Where
  - N^S is hours worked, or the amount of labour supplied by the household
  - $L + N^S = 1$ refers to the total time available in a day
  - Π are the dividends paid out by the firms owned by households
  - $U'(C) > 0, U''(C) < 0$ implies diminishing marginal utility of consumption
  - $V'(N) > 0, V''(N) < 0$ implies diminishing marginal utility of leisure
- For tractability, let's simplify functional forms:

$$U(C) = \log(C), \quad V(L) = b\log(L)$$

- So the household problem becomes:

$$\begin{aligned} \max_{C,L} \quad & \log(C) + b\log(L) \\ \text{s.t.} \quad & C = w \cdot N^S + \Pi \quad \text{Budget constraint} \\ & L + N^S = 1 \quad \text{Time endowment} \end{aligned}$$

- Substitute the time endowment and the budget constraint into the utility function:

$$\max_{N^S} \log(wN^S + \Pi) + b\log(1 - N^S)$$

- Now take the derivative of the objective function with respect to N^S:

$$\begin{aligned}
&= \frac{\partial \log(wN^S + \Pi)}{\partial N^S} + \frac{\partial b\log(1 - N^S)}{\partial N^S} \\
&= \frac{\partial \log(wN^S + \Pi)}{\partial (wN^S + \Pi)} \times \frac{\partial (wN^S + \Pi)}{\partial N^S} + \frac{\partial b\log(1 - N^S)}{\partial (1 - N^S)} \times \frac{\partial (1 - N^S)}{\partial N^S} \\
&= \frac{1}{(wN^S + \Pi)} \times w + \frac{b}{1 - N^S} \times (-1) \\
&= \frac{w}{c} - \frac{b}{1 - N^S}
\end{aligned}$$

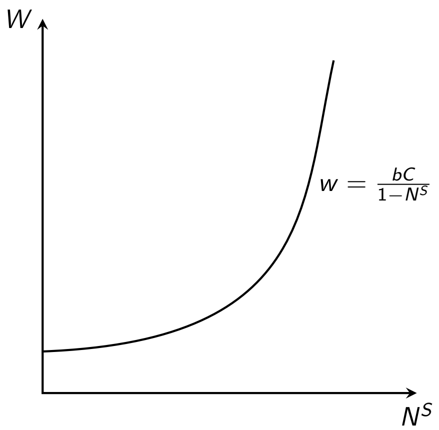
- Setting the derivative equal to zero yields the First Order Condition:

$$\underbrace{\frac{w}{c}}_{\substack{\text{Marginal Benefit of} \\ \text{Labour Supplied}}} - \underbrace{\frac{b}{1 - N^S}}_{\substack{\text{Marginal Cost of} \\ \text{Labour Supplied}}} = 0$$

- We can rewrite this as the Labour Supply curve of the household:

$$\underbrace{w}_{\substack{\text{Marginal Benefit} \\ \text{of Labour Supplied} \\ \text{in Consumption Units}}} = \underbrace{\frac{bC}{1 - N^S}}_{\substack{\text{Marginal Cost} \\ \text{of Labour Supplied} \\ \text{in Consumption Units}}}$$

- The labour supply curve slopes up
- So households supply more labour as wages rise

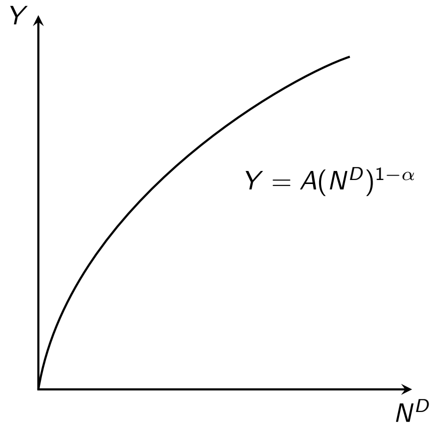


2.4 Firms in the Simple RBC Model

- Firms produce output using a production technology:

$$Y = A \times (N^D)^{1-\alpha}$$

- Where A is the exogenous level of technology;
- And where N^D is labour inputs demanded by firms



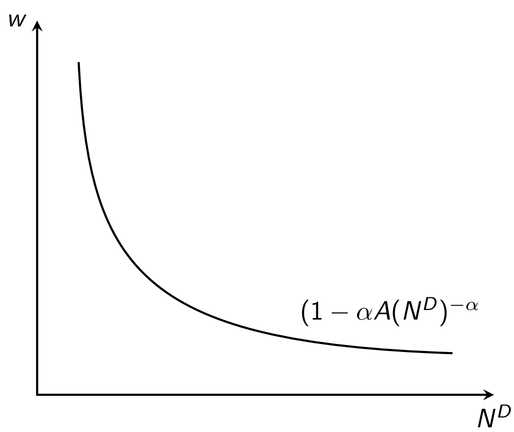
- A competitive firm chooses labour N^D to maximize profit Π (returned to households)

$$\begin{aligned}\Pi &= \max_{N^D} Y - wN^D \\ &= \max_{N^D} A(N^D)^{1-\alpha} - wN^D\end{aligned}$$

- where w is the wage or cost of hiring labour (and is taken as given)
- The first order condition yields:

$$\underbrace{(1-\alpha)A(N^D)^{-\alpha}}_{\substack{\text{Marginal Product}\\ \text{of Labour}}} - \underbrace{w}_{\substack{\text{Marginal Cost}\\ \text{of Labour}}}$$

- Marginal Product of Labour (MPN) = extra output generated by one additional labour input
- Firms demand less labour as wages increase



2.5 Equilibrium in the Simple RBC Model

- The Labour Market clearing condition holds:

$$\frac{bC}{1-N} = w = (1-\alpha)AN^{-\alpha} \tag{2.1}$$

- Aggregate production is determined by technology:

$$Y = AN^{1-\alpha} \tag{2.2}$$

- Firm output (i.e. goods supply) is equal to household consumption (i.e. goods demand):

$$Y = C \tag{2.3}$$

- First substitute equation (2.3) into (2.2), and substitute this into equation (2.1):

$$b\frac{AN^{1-\alpha}}{1-N} = (1-\alpha)AN^{-\alpha}$$

- Second, rearrange and solve for N:

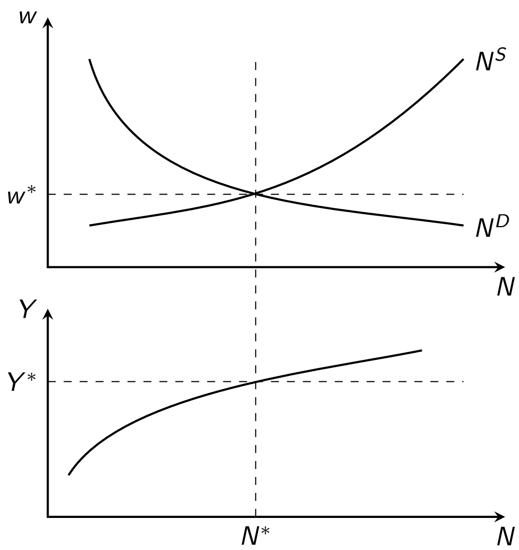
$$N = \frac{(1-\alpha)}{b+(1-\alpha)} \tag{2.4}$$

- Third, substitute into either the labour supply or demand curve to find w:

$$w = (1-\alpha)A\left(\frac{(1-\alpha)}{b+(1-\alpha)}\right)^{-\alpha} \tag{2.5}$$

- Finally use equation (2.4) and (2.2) and (2.3) to solve for Y and C:

$$Y = C = A\left(\frac{(1-\alpha)}{b+(1-\alpha)}\right)^{1-\alpha} \tag{2.6}$$



Business Cycle Fluctuations in the RBC Model

- Macroeconomic fluctuations in early RBC models were driven entirely by changes in aggregate productivity
- In our simple RBC model, changes in productivity A can drive fluctuations in each of the aggregate variables: C, Y, w, N

2.6 Limitations of the Simple (Intra-Temporal) RBC Model

- In our simple RBC model, households and firms only make static or intra-temporal decisions
- But these households do not care about the future!
  - inter-temporal decisions
  - Household savings
  - Productive capital
  - Financial assets (or asset prices!)
  - A relationship between the past and the future
  - A serious characterization of aggregate dynamics

3 Inter-Temporal Choice and the Business Cycle

3.1 Simple Inter-Temporal Households in the RBC Model

Choice between Consumption and Saving

- Households must decide how much to consume today, how much to save, and how much to consume tomorrow
- Because savings earn interest (returns), the more resources that are saved today, the more resources are available for consumption in the future
- But households are impatient as they discount the value of future consumption more than the value of current consumption
- The optimisation problem is to maximise life-time utility subject to an inter-temporal budget constraint

A Model of Consumption and Saving

- Assumptions:
  - Earn (net) real interest rate r on savings S
  - Future utility is discounted at the rate β Exogenous income in each period Y_1, Y_2
- Households use their savings to smooth consumption across time
- For now we ignore:
  - Risk
  - Inflation
  - Different types of assets
  - Other asset market participants
- A household chooses current consumption C_1, future consumption C_2, and savings S:

$$\begin{aligned} \max_{C_1, C_2, s} \quad & \log(C_1) + \beta \log(C_2) & \\ \text{s.t.} \quad & C_1 + S = Y_1 & \text{First period budget constraint} \\ & C_2 = Y_2 + S(1+r) & \text{Second period budget constraint} \end{aligned}$$

- Combine the within-period budget constraints:

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

- This is the inter-temporal budget constraint (or, life-time budget constraint)

Household Choice for Consumption and Saving

- The simplified household problem is:

$$\begin{aligned} \max_{C_1, C_2, S} \quad & \log(C_1) + \beta \log(C_2) \\ \text{s.t.} \quad & C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \end{aligned}$$

- The first order condition yield:

$$\underbrace{\frac{1}{C_1}}_{\substack{\text{Marginal Utility of} \\ \text{Consumption in Period 1}}} = \underbrace{(1+r)}_{\text{Return of Savings}} \times \underbrace{\beta \frac{1}{C_2}}_{\substack{\text{Marginal Utility of} \\ \text{Consumption in Period 2}}} \tag{3.1}$$

- This is called the Consumption Euler Equation, which describes efficient inter-temporal consumption choices
- Later, we will see that this equation is fundamental for understanding the price of assets!
- Rearrange equation (3.1) for C_2, then substitute into the inter-temporal budget constraint to find C_1 and C_2:

$$C_1 = \frac{1}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right), \quad c_2 = \frac{\beta(1+r)}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right)$$

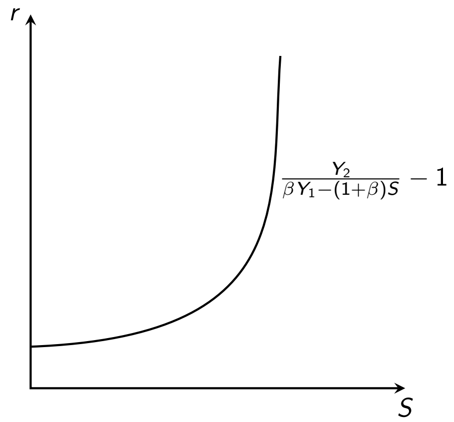
- To find S substitute either C_1 into the first period budget constraint or C_2 into the second period budget constraint:

$$S = \frac{\beta}{1+\beta}Y_1 - \frac{1}{(1+r)(1+\beta)}Y_2$$

- Rewrite the savings function for r:

$$r = \frac{Y_2}{\beta Y_1 - (1+\beta)S} - 1$$

- This represents the household's supply of savings



- When does the household choose to save (i.e. $S > 0$)?
  - Save when income in period 1 (Y_1) is larger than income in period 2 (Y_2)
  - Save more when the interest rate r is high
- What makes saving valuable?
- Savings transfers resources from periods of high income (when the MU of consumption is low) to periods of low income (when the MU of consumption is high)

3.2 Inter-Temporal Households and Capital Accumulation in the RBC Model

- Inter-temporal households generate a supply of savings (or a demand for loans!)
- In the canonical RBC model, households hold physical capital that is then used in production
- Here, we want to think of capital as a productive asset, but one whose return may fluctuate with the business cycle

Household Consumption Choice and Capital Accumulation

- A household chooses current consumptions C_1, C_2, and investment in capital I_1:

$$\begin{aligned} \max_{C_1,C_2,I_1} \quad & \log(C_1) + \beta \log(C_2) & \\ \text{s.t.} \quad & C_1 + I_1 = \Pi_1 + r_1 K_1 & \text{First period budget constraint} \\ & C_2 = \Pi_2 + (1 + r_2 - \delta) K_2 & \text{Second period budget constraint} \\ & K_2 = I_1 + K_1(1-\delta) & \text{Capital accumulation equation} \end{aligned}$$

- Households are endowed with capital K_1 (cannot be adjusted)
- Households earn (net) real interest r_1, r_2 on their capital holdings
- Capital in period 2 is investment in new capital + undepreciated capital from period 1
- In period 2, after production takes place, households consume remaining capital: $(1-\delta)K_2$
- For simplicity, assume households don't supply labour, but they own firms and receive dividends Π_1, Π_2
- Substitute capital accumulation equation into first period budget constraint:

$$C_1 + K_2 = \Pi_1 + K_1(1 + r_1 - \delta)$$

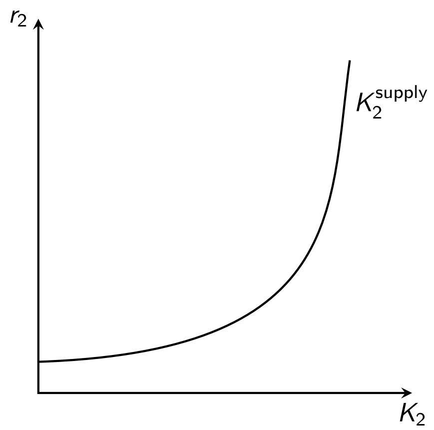
- Now, substitute the budget constraints into the utility function:

$$\max_{K_2} \log(\Pi_1 + K_1(1 + r_1 - \delta) - K_2) + \beta \log(\Pi_2 + K_2(1 + r_2 - \delta))$$

- Taking the FOC with respect to K_2:

$$\frac{1}{c} = \beta(1 + r_2 - \delta)\frac{1}{c_2}$$

- Which is also a Consumption Euler Equation
- Here, the return on savings/capital holding is $(1 + r_2 - \delta)$, but households value capital in the same way they valued savings in section 3.1

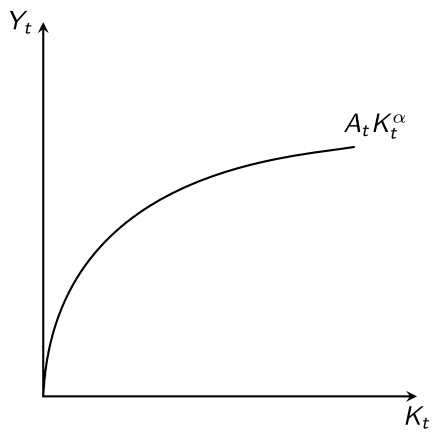


3.3 Firms in the Inter-Temporal RBC Model

- Firms produce output using the production technology:

$$Y_t = A_t K_t^{\alpha}, \text{where } t = 1, 2$$

- Where A_t is technology/productivity; and K_t is the capital inputs of firms



Firm's Choice of Capital Inputs

- A competitive firm chooses capital K_2^D to maximise profit Π_2:

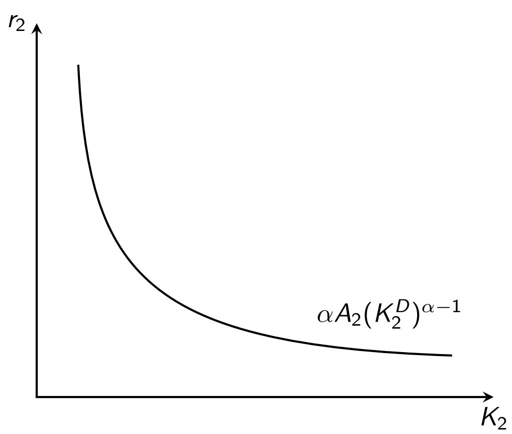
$$\Pi_2 = \max_{K_2^D} A_2 (K_2^D)^\alpha - r_2 K_2^D$$

- where r_2 is the interest rate or rental rate of capital (taken as given)
- The first order condition yields:

$$\underbrace{\alpha A_2 (K_2^D)^{\alpha-1}}_{\substack{\text{Marginal Utility} \\ \text{of Capital}}} - \underbrace{r_2}_{\substack{\text{Marginal Cost} \\ \text{of Capital}}} = 0$$

- Marginal Product of Capital (MPK) = extra output generated by additional capital input
- The FOC yields the capital demand curve:

$$r_2 = \alpha A_2 (K_2^D)^{\alpha-1}$$



Firm's Profits

- Recall that households own the firms and receive the profits the firms generate:

$$\begin{aligned} C_1 + I_1 &= \Pi_1 + r_1 K_1 && \text{First period budget constraint} \\ C_2 &= \Pi_2 + (1 + r_2 - \delta) K_2 && \text{Second period budget constraint} \end{aligned}$$

- The firms' first order condition gives us $r_t = \alpha A_t (K_t^D)^{\alpha-1}$, so profits are:

$$\begin{aligned}\Pi_t &= A_t(K_t^D)\alpha - r_t K_t^D \\ &= A_t(K_t^D)\alpha - \alpha A_t(K_t^D)^{\alpha-1}K_t^D \\ &= A_t(K_t^D)\alpha - \alpha A_t(K_t^D)^{\alpha} \\ &= (1-\alpha)A_t(K_t^D)^{\alpha} > 0\end{aligned}$$

- Which means households are sensitive to changes in productivity *through* their ownership of firms

3.4 Equilibrium in the Inter-Temporal RBC Model

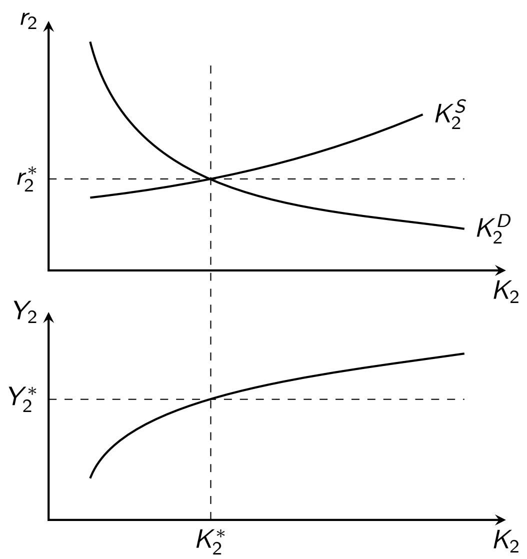
- The Capital Market clearing condition holds:
  - The real interest rate r_t ensures that the capital market clears
  - Capital supply (by households) is equal to capital demand (by firms) $K_t^S = K_t^D$
- Aggregate production is determined by technology:

$$Y_t = A_t K_t^{\alpha}, t = 1, 2 \tag{3.2}$$

- The aggregate resource constraint holds each period:

$$Y_1 = C_1 + I_1 \tag{3.3}$$
$$Y_2 + (1-\delta)K_2 = C_2 \tag{3.4}$$

- (Where total resources in period 2 include remaining undepreciated capital)
- Notice the real economy is tightly linked to the asset market (i.e. capital market)
- Equilibrium return on assets (i.e. interest rate), pins down amount of capital supplied
- Capital supply determines production/output in the economy
- So the macroeconomy and asset markets are very closely related!



RBC Model Implications

- Business cycles are due to "real" shocks (e.g. TFP or technology shocks)
- Productivity, real wages, employment, consumption, and investment are all pro-cyclical
- Markets are always in equilibrium.

- Prices and wages always adjust (flexibly) to ensure this equilibrium is efficient
- No involuntary unemployment in the model
- Money neutrality holds: changes in money supply do not affect real variables
- Government stabilization policies tend to be counter-productive

3.5 Limitations of RBC Models

- How do we measure TFP shocks? Solow Residuals?
- Do we really have frequent regressions in technological progress that cause recessions?
- What is the role of fiscal and monetary policy in the evolution of the macroeconomy?
- Most macroeconomists now convinced that money neutrality only holds in the long run
- Real wages are not pro-cyclical in the data. What does this imply?
  - "Real Wages and the Business Cycle", Abraham and Haltiwanger (JEL, 1995)
  - "Short-Run Equilibrium Dynamics of Unemployment, Vacancies, and Real Wages", Pissarides, (AER, 1985)
- To answer these questions, will typically need a DSGE model that incorporates price stickiness, wage stickiness, and policy shocks
- Most models in the RBC literature are solved using linear approximations to the model
- These linear approximations study deviations of the model from a well-defined steady state of the model economy
- But linear approximation means agents solve their problems under certainty equivalence:
- Certainty equivalence $\Longleftrightarrow$ agents behave as if there is no risk!
- But risk is one of the primary reasons for holding financial assets:
  - We often want to insure against risks by holding financial assets that pay out if certain undesirable states of the world eventuate (e.g. unemployment, fire, theft, death)
  - In equilibrium, agents often want to share or smooth risks e.g. you payout when I am doing poorly, and I payout when you are doing poorly

4 Money and Savings in the New Keynesian Model

4.1 An Introduction to the New Keynesian Model

- The ideas of John Maynard Keynes dominated macroeconomics in the early 20th century
- Keynesian macroeconomics (e.g. the IS-LM-AS model) studied government policies that might stabilize output in response to shocks
- The RBC model, with its lack of government stabilisation policy, dominated macroeconomics from the 1970s
- But continuing to believe in the importance of government policy, macroeconomists then developed what is now called the New Keynesian Model
- Like the RBC model, the New Keynesian Model:
  - Has micro foundations of economic behaviour
  - Has agents with rational expectations about the future
  - Can be calibrated to match various business cycle statistics about the macroeconomy
- Unlike the RBC model, the New Keynesian Model:
  - Features price and/or wages that are sticky (i.e. do not update in response to economic shocks)
  - Describes a macroeconomy that does not respond efficiently to shocks
  - May lead to output and employment being far from their socially optimal levels
  - Allows a role for macroeconomic stabilisation via monetary policy and/or fiscal policy

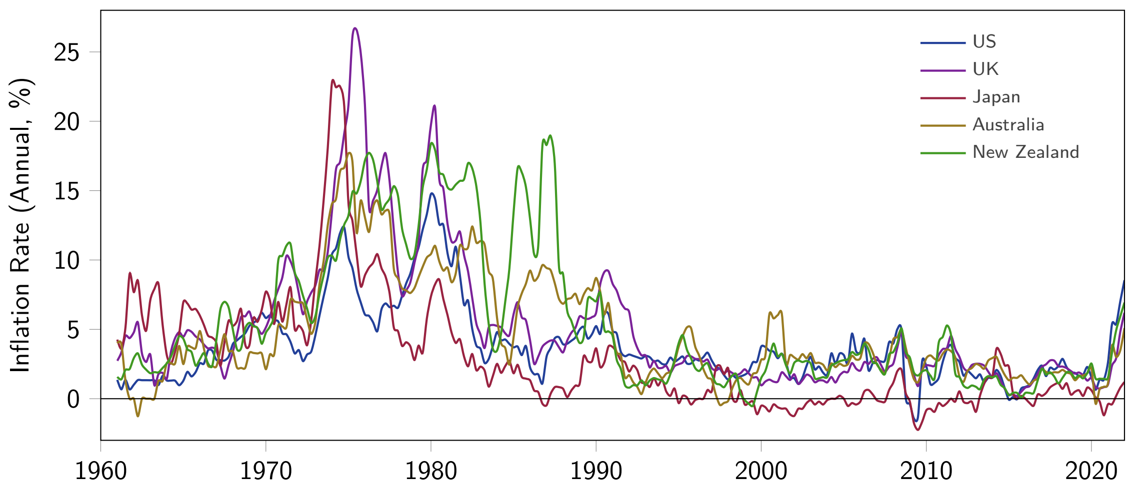
4.2 Inflation, and Nominal and Real Interest Rates

Inflation

- Define the general price level in an economy: $P_t \equiv$ price index
  - i.e the dollar cost of a representative basket of consumer goods
- Inflation: $\pi \equiv$ percent change in the price index:

$$\begin{aligned}\pi_t &= \frac{P_t - P_{t-1}}{P_{t-1}} \\ &= \frac{P_t}{P_{t-1}} - 1\end{aligned}$$

International Inflation Rates



Definitions of Nominal Variables

- Nominal interest rate: $r_t^n \equiv$ rate of return on an asset, in period t dollars
- Asset price: $S_t \equiv$ dollar price of a discount bond that pays one dollar next period
  - Discount bond: a bond that is issued or traded at less than its face–value
  - Face-value: amount the bond issuer pays to the bondholder once maturity is reached
  - Maturity: length of time a bond is held e.g. one month, one quarter, a year
- If r_n^t is the rate of return on the discount bond, then we we compute this as:

$$
\begin{aligned}
r_t^n &= \frac{\text{Payoff}_{t+1} - \text{Bond Price}_t}{\text{Bond Price}_t} \\
&= \frac{1 - S_t}{S_t} = \frac{1}{S_t} - 1 \\
\Rightarrow \; S_t &= \frac{1}{r_t^n}
\end{aligned}
$$

Real vs. Nominal Interest Rates and the Fisher Equation

- Purchasing power of one dollar $\equiv \frac{1}{P_t}$

  Purchasing power represents the number of consumption goods one dollar can buy

- The "ex-post" real interest rate $r_t \equiv$ realised return on the bond in units of consumption:

$$
\begin{aligned}
r_t &= \frac{\frac{1}{P_{t+1}} - \frac{S_t}{P_t}}{\frac{S_t}{P_t}} = \frac{1}{S_t}\frac{P_t}{P_{t+1}} - 1 \\
\Rightarrow \; 1 + r_t &= \frac{1 + r_t^n}{1 + \pi_{t+1}}
\end{aligned}
$$

- Rearranging

$$
\begin{aligned}
1 + r_t^n &= (1 + r_t)(1 + \pi_{t+1}) \\
&= 1 + r_t + \pi_{t+1} + r_t\pi_{t+1}
\end{aligned}
$$

- Since $r_t\pi_{t+1} \approx 0$ for small values of r_t and π_{t+1}:

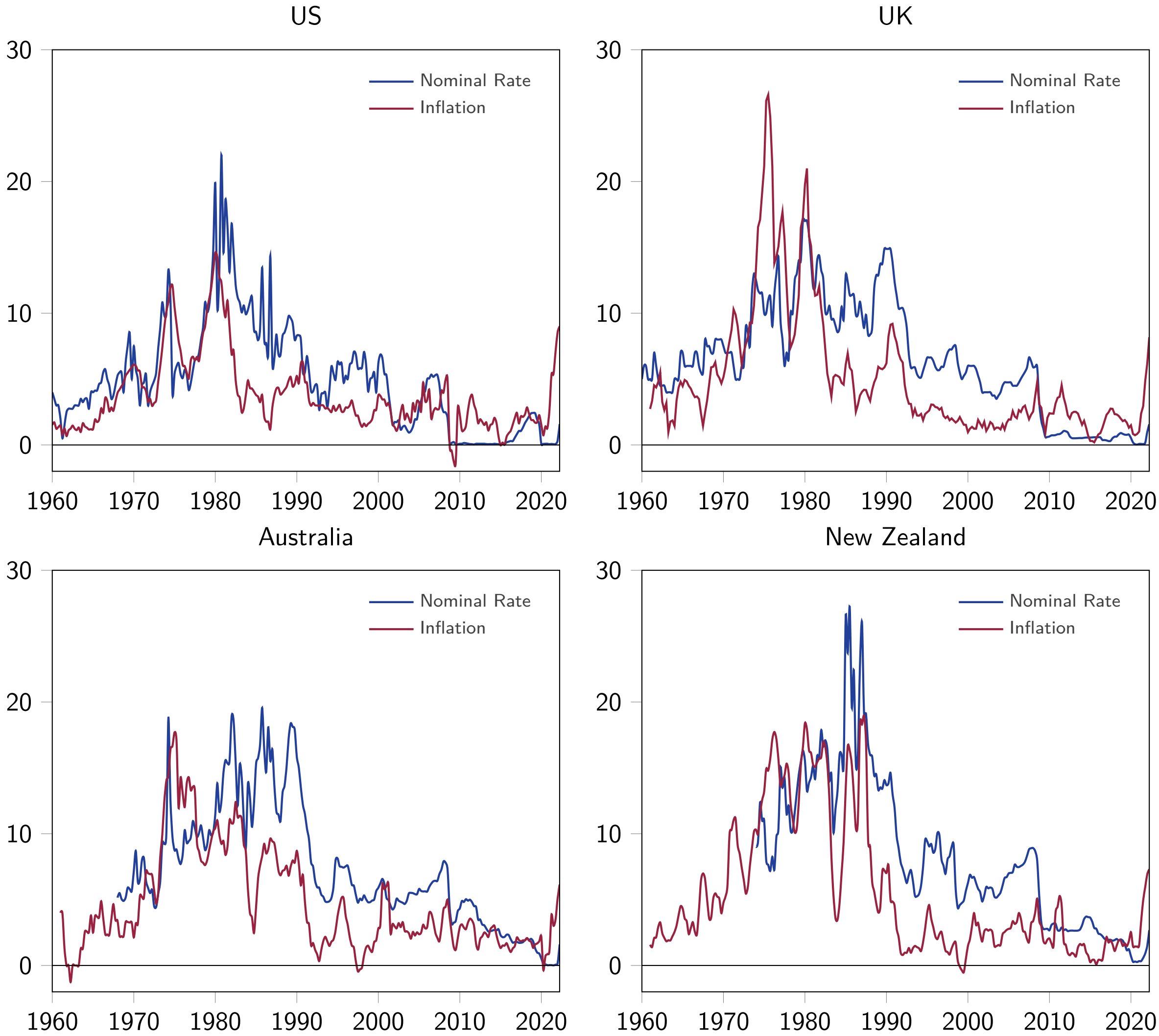
$$
r_t \approx r_t^n - \pi_{t+1}
$$

- Which is known as the Fisher Equation

Expected vs. Ex-Post Real Interest Rate

- The expected real rate is $E_t(r_t)$:

$$
E_t(r_t) \approx E_t(r_t^n) - E_t(\pi_{t+1}) = r_t^n - E_t(\pi_{t+1})
$$



4.3 Money and Inflation

The Rate of Return on Money

- We can also think of money as a type of asset.
- But what is the rate of return on money?
  - Since the nominal rate of return on money is $r^n_{m,t} = 0$, the real return is:

$$r_{m,t} - r^n_{m,t} - E_t(\pi_{t+1})$$

- **The return on money falls as expected inflation rises**
- So why do people hold money when its return is much lower than other assets?
  - Convenience: money has a role as a medium of exchange (i.e. used for trading goods and services)
  - Risk: fear of bank failures/financial market collapse (e.g. "money under the mattress")

Returns on Money vs. Bonds

- Recall:
  - Real rate of return on money:

$$r_{m,t} = -E_t(\pi_{t+1})$$

- (Expected) real rate of return on bonds:

$$E_t(r_t) \approx r_t^n - E_t(\pi_{t+1})$$

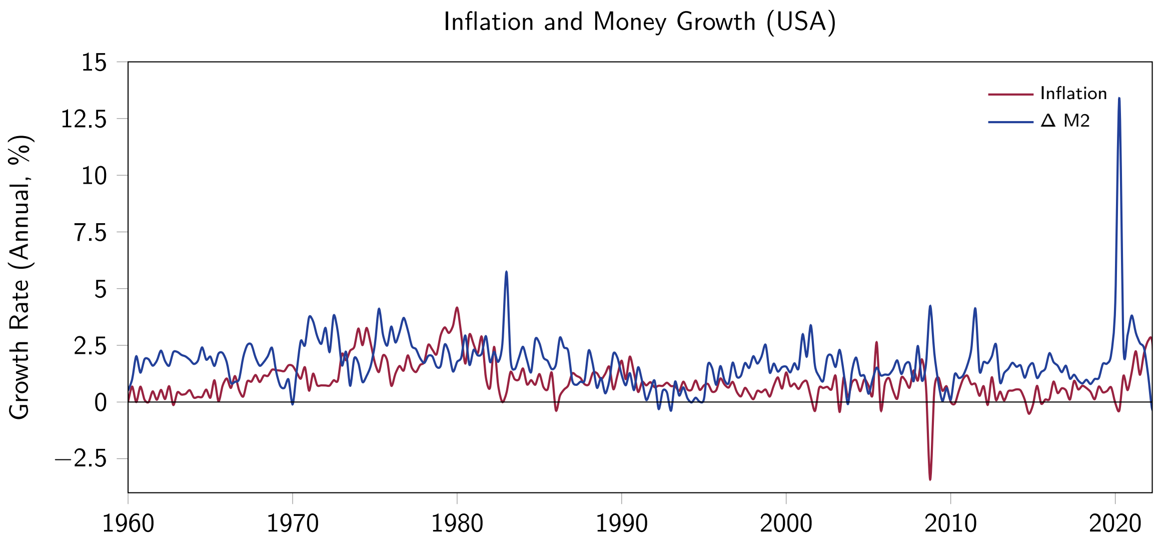
- Assuming the Fisher Hypothesis (i.e. that nominal rates move with inflation)
- Then fluctuations in inflation change return on money relative to the return on bonds
- Therefore, when monetary policy influences inflation, also affects the incentive to hold different kinds of assets

Quantity Theory of Money

- Consider again the Fisher Hypothesis:

$$r_t^n \approx E_t(r_t) + E_t(\pi_{t+1})$$

- If nominal interest rates move with inflation, what drives inflation?
- Much empirical evidence suggests a link between money growth and inflation
  - Evidence across time within a given country (mainly evidence over the long-run)
  - Evidence across countries



- Irving Fisher developed the Quantity Theory of Money (QTM):
  - A theory of the price level that explains what determines the value of a unity of money
- Begin with an accounting identity:

$$expenditures \equiv receipts$$

- Let $M \equiv$ stock of money; $V \equiv$ velocity of money (i.e. number of times a unit of money changes hands per period); $Y \equiv$ real output
- Then:

$$\begin{aligned} M \times V &= expenditures \\ P \times Y &= receipts \\ \Rightarrow MV &= PY \end{aligned}$$

- tart with the Quantity theory identity:

$$M_t V_t = P_t Y_t$$
$$\Rightarrow \Delta \ln M_t + \Delta \ln V_t = \Delta \ln P_t + \Delta \ln Y_t$$

- Rearranging:

$$\Delta \ln P_t = \Delta \ln M_t + \Delta \ln V_t - \Delta \ln Y_t$$

- The Quantity Theory then states:
  - Assumption (1) $\Delta \ln Y_t$ is independent of $\Delta \ln P_t, \Delta \ln M_t, \Delta \ln V_t$ (i.e. neo-classical assumption of **monetary neutrality**)
  - Assumption (2) $\Delta \ln V_t = 0$

$$\Rightarrow \Delta \ln P_t = \Delta \ln M_t - \Delta \ln Y_t$$

1. Why assume Y is independent of M, P, V?

- Neo-Classical theory argues only real factors matter for Y (e.g. technology)

2. Why assume stable velocity of money?

- Fisher assumed money demand was proportional to nominal income:

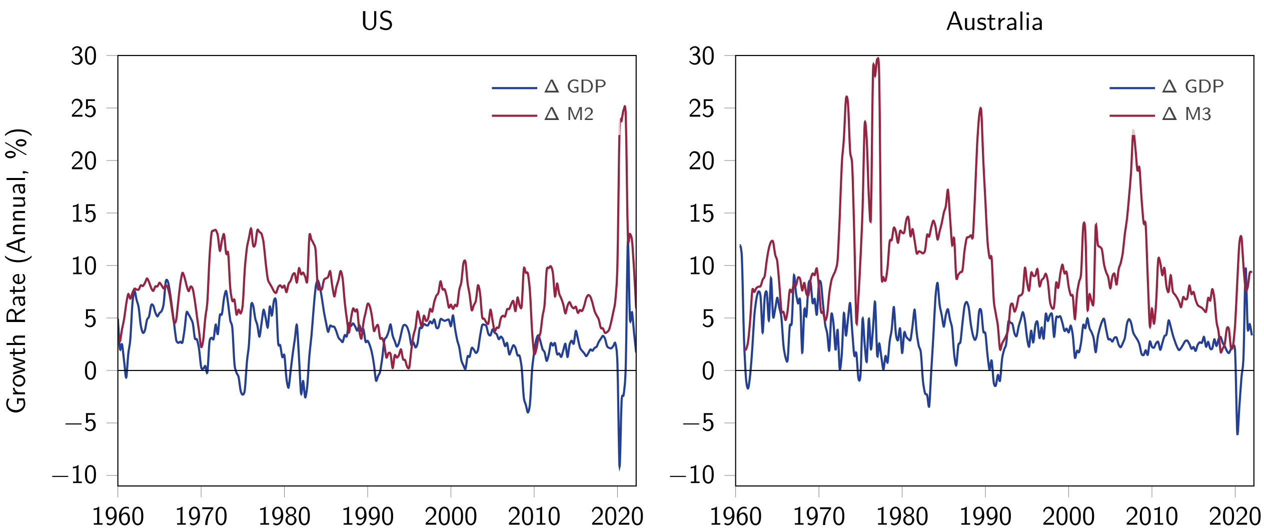
$$M = \kappa PY$$
$$\Rightarrow M\frac{1}{\kappa} = PY$$
$$\Rightarrow V = \frac{1}{\kappa}, \text{ so V is constant}$$

- This might be true if financial institutions and technologies change slowly over time

1. Y is independent of M, P, V? NO!

- Much evidence shows that Y is clearly not independent of M
- Periods when central banks have sharply contracted the money supply have been followed by large real output declines
  - E.g.: Great Depression of the 1930s; large disinflations of the 1980s/1990s
- Why? Temporary **nominal price rigidities** mean M affects Y in short run
  - If P is sticky in the short run, then variation in M will affect Y

$$Y = V \times \underbrace{\frac{M}{P}}_{\text{Real Money Supply}}$$



2. Stable velocity of money? NO!

- Velocity is not constant and appears to be strongly pro-cyclical

- Problem:
  - Changes in financial technology provide easier to access money/substitutes (e.g. on-call savings accounts, EFTPOS, Pay-Wave), which changes velocity
  - The opportunity cost of holding money - i.e. the nominal interest rate on other assets r_t^n - also matters
  - Empirically, money demand does not have a simple proportional relationship to output

4.4 A Simple New Keynseian Model

- Household chooses consumption, nominal bonds, and money
- Simplified demand for money due to utility of holding real money balances
  - Represents the "convenience yield" of money holdings
  - But is something of a short-cut to characterize various desires for holding money

Household Choice Problem

- Household choice problem is:

$$\begin{aligned}
\max_{C_1, C_2, M_1, B_2} \quad & \log(C_1) + \omega \log \frac{M_1}{P_1} + \beta \log(C_2) \\
\text{s.t.} \quad & P_1 C_1 + M_1 + B_2 = P_1 Y_1 \\
& P_2 C_2 = P_2 Y_2 + M_1 + B_2(1 + r^n)
\end{aligned}$$

- Where M_1/P_1 are real money balances
- The inter-temporal real budget constraint is:

$$C_1 + \frac{M_1}{P_1} + \frac{C_2}{1+r^n}\frac{P_2}{P_1} = Y_1 + \frac{Y_2}{1+r^n}\frac{P_2}{P_1} + \frac{M_1/P_1}{1+r^n}$$

- The Lagrangian Problem is:

$$\mathcal{L} = \log(C_1) + \omega \log \frac{M_1}{P_1} + \beta \log(C_2) + \lambda \left(Y_1 + \frac{Y_2}{1+r^n}\frac{P_2}{P_1} + \frac{M_1/P_1}{1+r^n} - C_1 - \frac{M_1}{P_1} - \frac{C_2}{1+r^n}\frac{P_2}{P_1} \right)$$

- The first order conditions for the problem are:

$$\begin{aligned}
C_1 : \quad & \frac{1}{C_1} - \lambda = 0 \\
C_2 : \quad & \beta \frac{1}{C_2} - \lambda \frac{1}{1+r^n}\frac{P_2}{P_1} = 0 \\
M_1 : \quad & \omega \frac{1}{P_1}\frac{1}{M_1/P_1} + \lambda \frac{1}{1+r^n}\frac{1}{P_1} - \lambda \frac{1}{P_1} = 0
\end{aligned}$$

- Combining the first two yields the consumption Euler equation:

$$\frac{1}{C_1} = \beta(1+r^n)\frac{P_1}{P_2}\frac{1}{C_1}$$

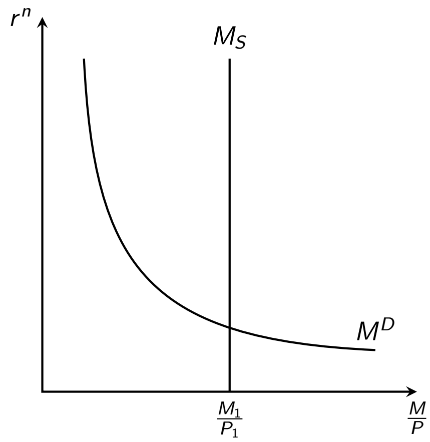
- Combining the first and third yields the consumption-money optimality condition:

$$\omega \frac{C_1}{M_1/P_1} = \left(\frac{r^n}{1+r^n} \right)$$

- which states that the marginal rate of substitution between consumption and money balances is governed by the nominal interest rate on bonds

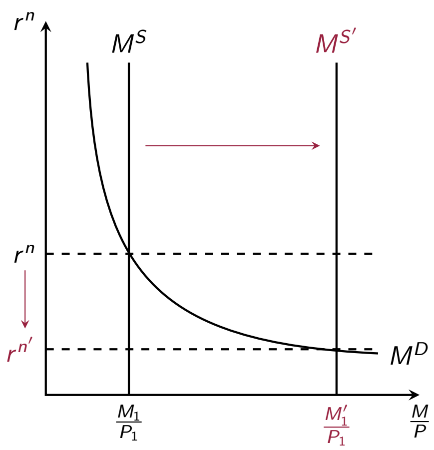
Demand and Supply for Money

- We can represent the consumption-money optimality condition as a money demand equation in $(M_1/P_1, r^n)$-space
- Suppose the central bank supplies money **inelastically** with respect to the interest rate



Simple Monetary Policy

- The New Keynesian model suggestions that money affects the real economy
- Simple example:
  - Assume that nominal price rigidities mean that prices are constant: $P_1 = P_2 = P$
  - Now consider an unexpected increase in money supply $\uparrow M_1^S$
  - What happens to consumption (C_1, C_2)?
- Note: These assumptions only hold in the short run!
- With sticky prices (i.e. P constant), an increase in the money supply **decreases** the nominal interest rate



- To solve for changes in consumption take the inter-temporal budget constraint, money demand, and

Euler equations (assuming that $P_1 = P_2$):

$$C_1 + \frac{M_1}{P_1} + \frac{C_2}{1+r^n} = Y_1 + \frac{Y_2}{1+r^n} + \frac{M_1/P_1}{1+r^n}$$
$$\omega\frac{C_1}{M_1/P_1} = \left(\frac{r^n}{1+r^n}\right)$$
$$\frac{1}{C_1} = \beta(1+r^n)\frac{1}{C_2}$$

- Substituting the money demand and Euler equations into the budget constraint, we get the consumption functions:

$$C_1 = \frac{1}{1+\omega+\beta}\left(Y_1 + \frac{Y_2}{1+r^n}\right), \quad C_2 = \frac{\beta(1+r^n)}{1+\omega+\beta}\left(Y_1 + \frac{Y_2}{1+r^n}\right)$$

- Remember the increase in money supply leads to a decrease in r^n
- Thus, consumption in period 1 rises:

$$\uparrow C_1 = \frac{1}{1+\omega+\beta}\left(Y_1 + \frac{Y_2}{\underbrace{(1+r^n)}_{\downarrow}}\right),$$

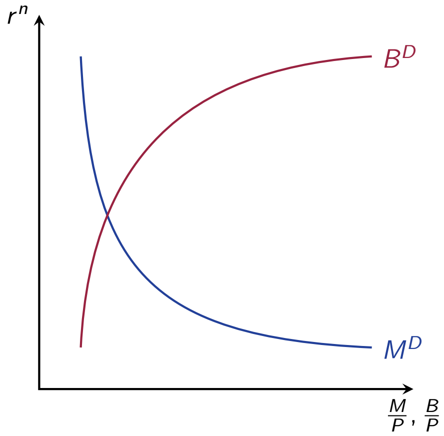
- And consumption in period 2 falls:

$$\downarrow C_2 = \frac{\beta(1+r^n)}{1+\omega+\beta}\left(Y_1 + \frac{Y_2}{1+r^n}\right) = \frac{\beta\overbrace{(1+r^n)}^{\downarrow}}{1+\omega+\beta}Y_1 + \frac{\beta}{1+\omega+\beta}Y_2$$

- So sticky prices mean that monetary policy is non-neutral in the short run
  - That is, monetary policy can have real effects on the macroeconomy!
- Changes in monetary policy also affect demand for assets!
- Derive the bond demand equation using the period 1 budget constraint and the money demand equation:

$$\frac{B_2}{P_1} = Y_1 - C_1 - \frac{M_1}{P_1}$$
$$= Y_1 - C_1 - \omega C_1\left(1 + \frac{1}{r^n}\right)$$

- Which shows that real bond demand is **increasing** in the nominal interest rate rn
- So households adjust their asset portfolio according to the return on bonds
- Changes in monetary policy affect real asset portfolio allocation decisions
- Household composition of assets varies with the relative return on the assets available
- So a decrease in money supply raises the nominal interest rate, which increases bond holdings
- Since the nominal return on money is zero, an increase in the nominal return on bonds leads to a shift away from money and towards bonds



4.5 Limitations of the New Keynesian Model

- The source of price rigidities is often not well-microfounded
  - Typically introduce ad-hoc "price stickiness" to models
- New Keynesian models often do not account for macroeconomic data much better than RBC models
- Despite their basis in monetary economics, New Keynesian models often do a poor job of explaining fluctuations in inflation
- As was the case for the RBC model, most New Keynesian models are not solved with economic risk in mind
- So, again, these models are not ideal for studying some of the main motives for asset holdings
- Both RBC and New Keynesian models contain a single, representative household
- This household has no one else to trade with, so the notion of a financial market is limited

5 Expectations, Uncertainty, and Asset Holdings

5.1 Risk Aversion and the Precautionary Savings Motive

- Risk aversion: a tendency to prefer economic outcomes with low uncertainty to those with more uncertainty
- Precautionary Savings: an increase in income uncertainty that leaves expected income unchanged reduces current consumption. But savings increase as a form of self **insurance** against low income states of the world.
- Risk aversion is a consequence of diminishing marginal utility
  - For utility function $u(\cdot)$, then $u' > 0$ and $u'' < 0$
  - Implies a loss of x matters more than a gain x
  - A risk averse agent would turn down a fair bet with even odds of an increase of x or a decrease in x
  - But risk aversion does not tell us how an agent *responds* to uncertainty or risk
- Precautionary Savings is a result of marginal utility declining at a decreasing rate:
  - For utility function $u(\cdot)$, then $u''' > 0$
  - This feature of utility functions/preferences is sometimes called **prudence**
- In this case, an increase in income uncertainty (holding expected income constant) raises expected **marginal utility**
- This means that the value of **additional** consumption is higher, which means that households save more in order to consume more in the periods of heightened uncertainty
- These additional savings in the face of greater uncertainty are called precautionary savings

5.2 The Precautionary Savings Motive: An Illustrative Model

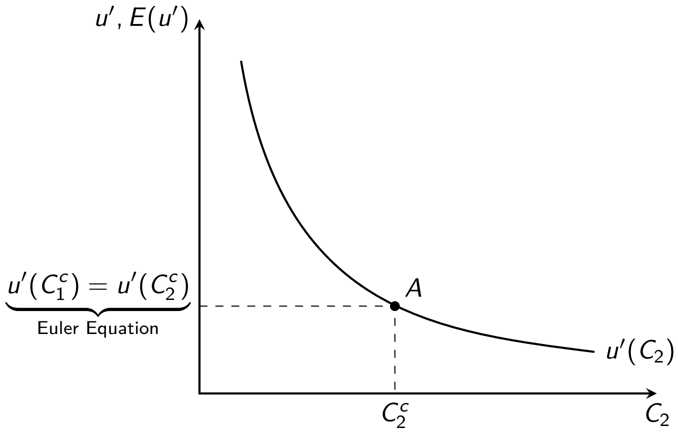
- A household makes consumption and savings decisions, subject to known and constant incomes
- Assume $\beta = 1$ return on savings is zero ($r = 0$), income in each period is $\bar{Y}$, utility function $u' > 0, u'' < 0, u''' > 0$

$$\max_{C_1, C_2} \quad u(C_1) + u(C_2)$$
$$\text{s.t.} \quad C_1 + C_2 = \bar{Y} + \bar{Y}$$

- The first order condition yields the optimality condition (Euler Equation):

$$u'(C_1) = u'(C_2)$$
$$\Rightarrow C_1 = C_2 = \bar{Y}$$

- Label these consumption choices C_1^c and C_2^c for the choices under **certainty**
- It will be helpful to plot marginal utility as a function of consumption in period 2
- First, plot marginal utility at our consumption choice under certainty C_2^c
- Note that because $u'' < 0$ and $u''' > 0$, marginal utility is decreasing and convex (i.e. curved out from the origin)



- Now suppose there are different **states** of the world
- These states affect income in period 2, with a chance of a good outcome and a chance of a bad outcome

$$Y_2 = \begin{cases} \bar{Y} + x & \text{with probability } 0.5 \quad \text{(Good Outcome)} \\ \bar{Y} - x & \text{with probability } 0.5 \quad \text{(Bad Outcome)} \end{cases}$$

- The first order condition in this case yields the Expected Euler Equation:

$$u'(C_1) = E(u'(C_2))$$

- Where $E(u'(C_2))$ is the expectation over **marginal utility** of consumption in period 2
- Note that we can compute this as:

$$\begin{aligned} E(u'(C_2)) &= 0.5 \times u'(C_2(good)) + 0.5 \times u'(C_2(bad)) \\ &= 0.5 \times u'(C_2(\bar{Y} + x + S)) + 0.5 \times u'(C_2(\bar{Y} - x + S)) \end{aligned}$$

- Suppose the household were to choose period 1 consumption the same as under the certainty case: $C_1 = C_1^c = \bar{Y}$
- Then from the period 1 budget constraint, savings are: $S = \bar{Y} - C_1^c$
- And we can write consumption in period two as:

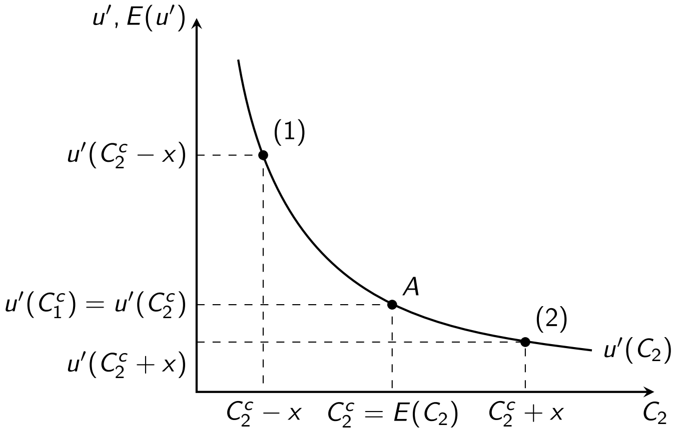
$$\begin{aligned} C_2 &= Y_2 + S \\ &= Y_2 + \bar{Y} - C_1^c \\ &= \begin{cases} \bar{Y} + x + \bar{Y} - C_1^c & \text{with probability } 0.5 \\ \bar{Y} - x + \bar{Y} - C_1^c & \text{with probability } 0.5 \end{cases} \\ &= \begin{cases} C_2^c + x & \text{with probability } 0.5 \\ C_2^c + x & \text{with probability } 0.5 \end{cases} \end{aligned}$$

- If choosing the certainty consumption in period 1, period 2 consumption is equal to the certainty consumption (C_2^c) plus or minus the uncertain component of income x
- Now, write the expected marginal utility of consumption in period 2 as:

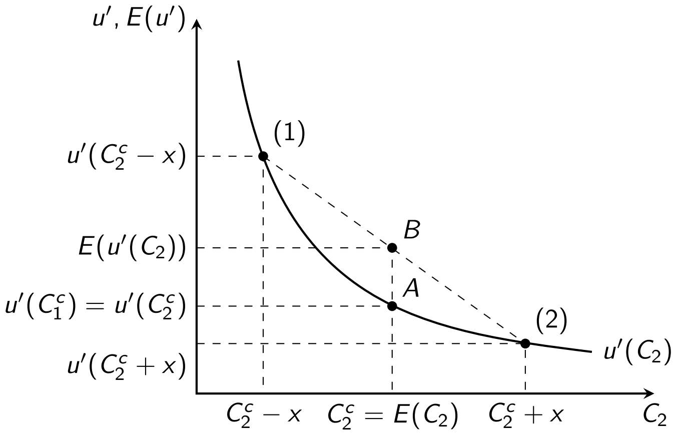
$$\begin{aligned} E(u'(C_2)) &= 0.5 \times u'(C_2^c + x) + 0.5 \times u'(C_2^c - x) \\ &\geq u'(C_2^c) \end{aligned}$$

- Because $u''' > 0$ the expected marginal utility of consumption in the uncertain case is greater than marginal utility in the certain case

- This means that the value of the certain consumption choice is greater than the value of the uncertain consumption outcomes
- Another way: households prefer certainty to uncertainty, even when the expected value of outcomes is the same in both cases
- Again consider plot of marginal utility as function of consumption in period 2
- Consumption is low/marginal utility is high in the bad state (1)
- Consumption is high/marginal utility is low in the good state (2)



- Notice that: $E(u'(C_2)) > \underbrace{u'(C_2^c) = u'(C_1^c)}_{\text{Euler Equation}}$
- Therefore:
  - Households want to increase C_2, and decrease C_1,
  - They accomplish this with higher (i.e. precautionary) savings S



- Point (A) corresponds to marginal utility of the certain consumption C_2^c
- This is the optimal consumption choice for the certainty case: $u'(C_1^c) = u'(C_2^c)$
- Point (B) is the **expected marginal utility** over consumption in the uncertain case: $E(u'(C_2))$
- Note that $E(u'(C_2)) > u'(C_2^c) = u'(C_1^c)$
- This means that consumption is **too low** in period 2 (i.e. marginal utility is too high)
- Therefore, the household should consume less in period 1: $C_1^u < C_1^c$
- This allows household to save more and so consume more in period 2: $C_2^u(s) > C_2^c(s)$

5.3 Precautionary Savings and Asset Prices

- Consider our two-period model:

$$\begin{aligned}\max_{C_1, C_2} \quad & u(C_1) + u(C_2) \\ \text{s.t.} \quad & C_1 + P_b B = \bar{Y} \\ & C_2 = Y_2 + B\end{aligned}$$

- Where a one period bond B can be purchased at price P_b
- Income is again uncertain:

$$Y_2 = \begin{cases} \bar{Y} + x & \text{with probability } 0.5 \quad \text{(Good Outcome)} \\ \bar{Y} - x & \text{with probability } 0.5 \quad \text{(Bad Outcome)} \end{cases}$$

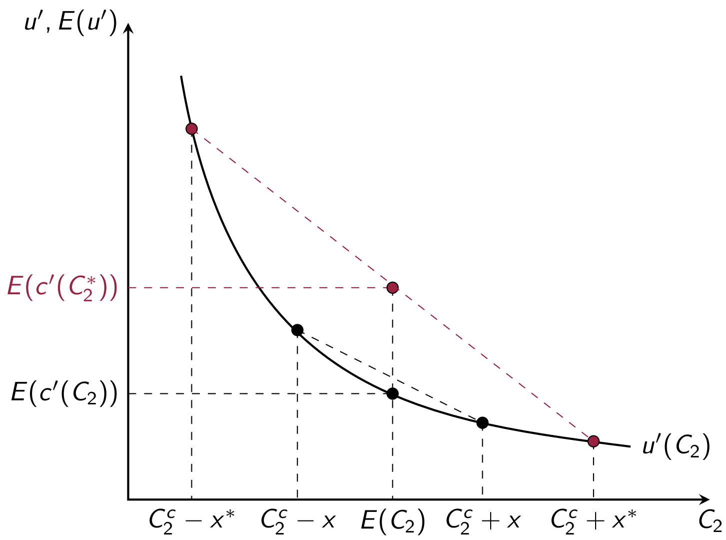
- The first order condition yields the Expected Euler Equation:

$$P_b u'(C_1) = \beta E(u'(C_2))$$

- And rearranging we have:

$$P_b = \beta \frac{E(u'(C_2))}{u'(C_1)}$$

- This is referred to as an Asset Pricing Equation
- Asset prices determined by the ratio of marginal utilities of consumption in each period
- Or, another way: asset prices are given by the marginal rate of substitution between consumption across periods. How does uncertainty affect prices?
- So now consider an increase from x to x^*:
  - Now $Y_2 = \bar{Y} + x^* > \bar{Y} + x$ with probability 0.5, and $Y_2 = \bar{Y} - x^* < \bar{Y} - x$ with probability 0.5
  - But it is still the case that $E(Y_2) = \bar{Y}$
  - This is called a mean-preserving spread in Y_2
  - Uncertainty only affects period 2, so the effect on asset prices comes through $E(u'(C_2))$



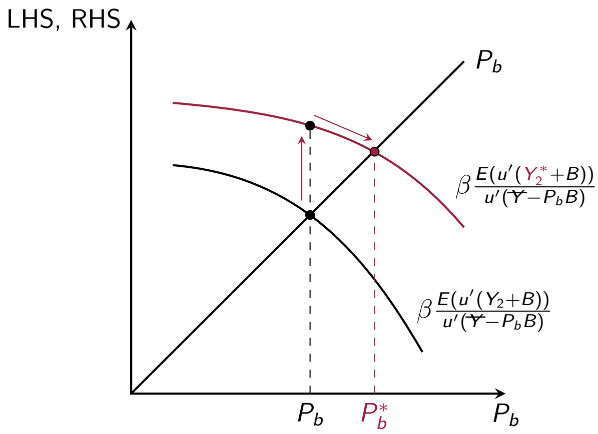
- Since $E(c'(C_2))$ increases, P_b increases also:

$$\uparrow P_b = \beta \frac{E(u'(C_2))\uparrow}{u'(C_1)}$$

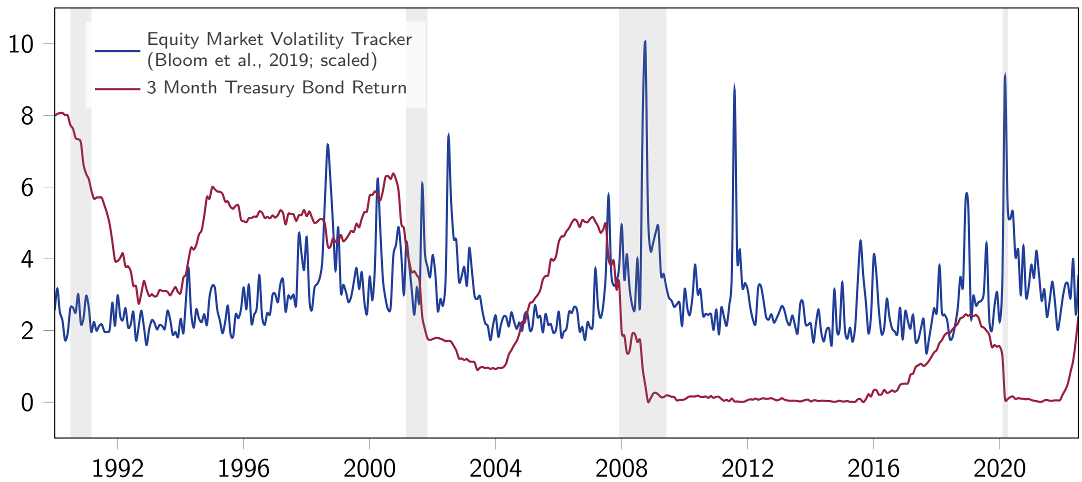
- But C_1 also decreases in response to greater uncertainty, which increases $u'(C_1)$
- So what is the overall effect?
- Substitute in the budget constraints:

$$P_b = \beta \frac{E(u'(Y_2 + B))}{u'(\cancel{Y} - P_b B)}$$

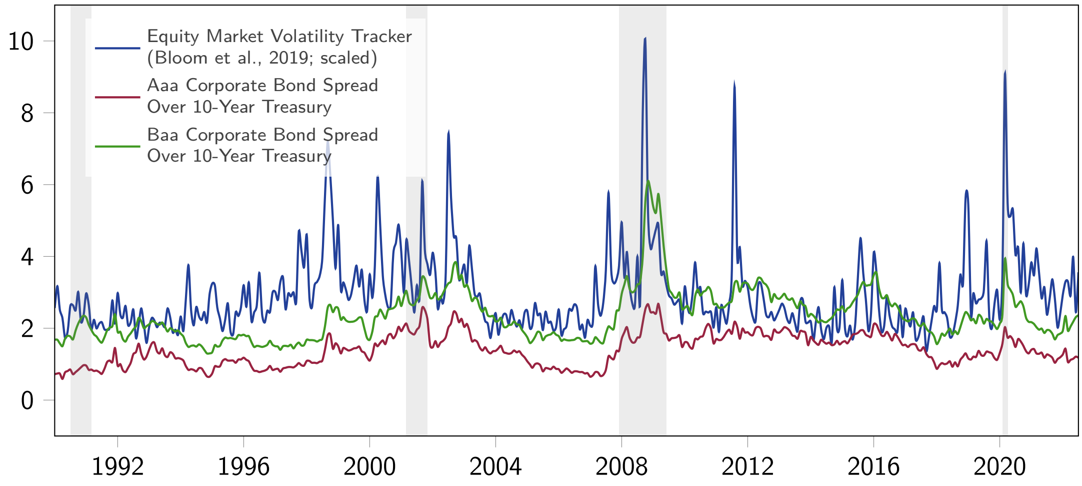
- Illustrate optimal choices graphically by plotting the left-hand-side and right-hand-side of the asset pricing equation



- An increase in uncertainty **increases** the price of assets
- This is intuitive:
  - Greater uncertainty induces precautionary savings which increases demand for assets
  - Higher asset demand is associated with higher asset
- Recall that the asset return: $R = \frac{1}{P_b}$
- So higher asset prices are associated with **lower** asset returns
- Do we observe this empirically?



- Note, we need to compare risk-free bonds
- For risky assets, demand and prices may increase or decrease depending on the nature of the asset risk



5.4 How Much Does the Precautionary Savings Motive Matter?

- The two motives for asset holding that we have studied so far:
  - Life-cycle motive: consumption smoothing across time
  - Precautionary savings motive: consumption smoothing across outcomes/states of the world
- Finds that precautionary savings matter much more for young households' asset decisions
- Finds that life-cycle motives matter much more for older households' asset decisions
- Young households start out with low wealth, need to save to build a **precautionary savings buffer**

6 Introduction to Asset Pricing: Concepts, Measurement, and a Simple Model

6.1 Definitions and Measurement

- Consider simple asset that was bought last period at P_{t-1} and sold this period at P_t
- The simple net return r_t on this asset between dates $t-1$ and t is:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

- The simple gross return is: $R_t = 1 + r_t$
- The gross return on the asset over k periods starting at date $t-k$ is:

$$\begin{aligned} R_t(k) &= R_t \cdot R_{t-1} \cdots R_{t-k+1} \\ &= \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} \cdots \frac{P_{t-k+1}}{P_{t-k}} \\ &= \frac{P_t}{P_{t-k}} \end{aligned}$$

- These multi-period returns are referred to as **compounded returns**
- Multi-year returns are often annualised in order to easily compare investments in assets over different horizons
- An annualised gross return $R_t^{ann}(k)$ is computed via:

$$R_t^{ann}(k) = [R_t \cdot R_{t-1} \cdots R_{t-k+1}]^{\frac{1}{k}} = \left[\prod_{j=0}^{k-1} R_{t-j}\right]^{\frac{1}{k}}$$

- This formula is known as a **geometric mean**
- For quick comparisons – that are less accurate! – we sometimes use an arithmetic mean as an approximation to the annualised return:

$$R_t^{ann}(k) \approx \frac{1}{k} \sum_{k-1}^{j=0} R_{t-j}$$

<u>**Continuous Compounding**</u>

- The continuously compounded return or log-return of an asset is defined as:

$$\not{r}_t = \log(R_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1}) = p_t - p_{t-1}$$

- where lower-case letters represent the log of a variable
- So the continuously compounded multi-period return over k periods is:

$$\begin{aligned} \not{r} &= \log(R_t(k)) \\ &= \log(R_t) + \log(R_{t-1}) + \cdots + \log(R_{t-k+1}) \\ &= \not{r} + \not{r}_{t-1} + \cdots + \not{r}_{t-k+1} \end{aligned}$$

- Compounding – a multiplicative operation – is converted to an additive operation by taking logarithms!

Dividend payments

- Some assets (e.g. stocks) pay out dividends, which make up part of the return on the asset
- For these assets, define returns as:

$$R_t = \frac{P_t + D_t}{P_{t-1}}$$
$$\Rightarrow r_t = \frac{P_t + D_t}{P_{t-1}} - 1$$

- To compute the log-return:

$$\cancel{r}_t = \log(R_t) = \log\left(\frac{P_t + D_t}{P_{t-1}}\right) = \log(P_t + D_t) - \log(P_{t-1})$$

Excess Returns

- We will often want to compare returns across different assets
- Consider the gross return on a benchmark asset R_t
- And consider the gross return R_t^i on a comparison asset i
- The **excess return** of asset i over the benchmark is:

$$R_t^i - R_t = r_t^i - r_t$$

- In many cases, the benchmark asset will be something approximating a riskless/risk-free asset such as a government bond

Equity Premium

- The equity premium is the expected excess return of an asset over the risk-free rate:

$$E_t(R_t^i - R_t)$$

- The equity premium tells us the excess return on asset i required to compensate investors for the additional risk of holding i over the risk-free asset

Present Value with Constant Discount Rates

- Suppose an asset paying a regular dividend has a constant expected return $R_t = R$:

$$R = E_t\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right)$$

- Rearrange for P_t:

$$P_t = E_t\left(\frac{P_{t+1} + D_{t+1}}{R}\right) \tag{6.1}$$

- Note that this is the same as the Discounted Cash Flow valuation model from section 1
- Now step the price forward one period to Pt+1:

$$P_{t+1} = E_t\left(\frac{P_{t+2} + D_{t+2}}{R}\right) \tag{6.2}$$

- Substitute back into equation (6.1):

$$P_t = E_t\left(E_{t+1}\left[\frac{P_{t+2} + D_{t+2}}{R^2}\right] + \frac{D_{t+1}}{R}\right)$$

- We can simplify this as:

$$P_t = E_t \left(\frac{P_{t+2} + D_{t+2}}{R^2} + \frac{D_{t+1}}{R} \right)$$

- Where the second equality follows from the Law of Iterated Expectations e.g.: $E_t(E_{t+h}[x_{t+h+1}]) = E_t(x_{t+h+1})$
- Iterating this process K times yields:

$$P_t = E_t \left(\sum_{k=1}^{K} \frac{D_{t+k}}{R^k} \right) + E_t \left(\frac{P_{t+k}}{R^k} \right)$$

- We typically assume that:

$$\lim_{K \to \infty} E_t \left(\frac{P_{t+k}}{R^k} \right) = 0$$

- Which is referred to as the No Bubble Condition
- Then we have a simple asset valuation formula:

$$P_t = E_t \left(\sum_{k=1}^{K} \frac{D_{t+k}}{R^k} \right)$$

- Where the RHS is the discounted present value of future dividends (i.e. cashflows)

6.2 Introduction to Macroeconomic Models of Asset Pricing

- How do asset prices relate to the macroeconomic model we have been studying so far?
- We will show how the standard model of household behaviour can lead us to a theory of asset pricing known as the Consumption Capital Asset Pricing Model (C-CAPM)
- We will then put C-CAPM into the context of the broader study of finance and macro-finance
- Consider a household problem at some generic date "t"
- Two assets:
  - A risk-free bond B_t. Price $P_{B,t}$. At $t+1$, pays a face value of 1.
  - A risky asset A_t. Price $P_{A,t}$ At $t+1$, pays uncertain dividend D_{t+1}, and can be resold at price $P_{A,t+1}$
- The household problem is:

$$\begin{aligned} \max_{C_t, C_{t+1}, B_t, A_t} \quad & u(C_t) + \beta E_t[u(C_{t+1})] \\ \text{s.t.} \quad & C_t + P_{B,t} B_t + P_{A,t} A_t = Y_t \\ & C_{t+1} = Y_{t+1} + B_t + D_{t+1} A_t + P_{A,t+1} A_t \end{aligned}$$

- Household chooses C_t, B_t, A_t at time t, but chooses C_{t+1} at time $t+1$
- But choice of B_t, A_t affects the budget constraint at time $t+1$ where outcomes are uncertain
- This uncertainty means the household must form expectations (E_t) about $t+1$ sing information available at time t
- Important:
  - Expectations over $t+1$ matter for decisions at t if those decisions affect outcomes at $t+1$!

- The Lagrangian Problem is:

$$\begin{aligned}\mathcal{L} = & u(C_t) + \beta E_t[u(C_{t+1})] \\ & + \lambda_t(Y_t - C_t + P_{B,t}B_t + P_{A,t}A_t) \\ & + E_t[\lambda_{t+1}(Y_{t+1} + B_t + D_{t+1}A_t + P_{A,t+1}A_t - C_{t+1})]\end{aligned}$$

- The Lagrange Multipliers λ_t, λ_{t+1} measure the shadow value of the budget constraints:
  - λ_t = the marginal value of an extra dollar allocated to the budget constraint in period t
  - λ_{t+1} = the marginal value of an extra dollar allocated to the budget constraint in period $t+1$
- The first order conditions with respect to C_t, B_t, A_t are:

$$\begin{aligned}C_t: &\quad u'(C_t) - \lambda_t = 0 \\ B_t: &\quad -\lambda_t P_{B,t} + E_t(\lambda_{t+1}) = 0 \\ A_t: &\quad -\lambda_{t+1} P_{A,t} + E_t(\lambda_{t+1}[D_{t+1} + P_{A,t+1}]) = 0\end{aligned}$$

- Where expectations enter the FOCs for B_t and A_t because those decisions affect outcomes during the uncertain period $t+1$
- The first order condition with respect to C_{t+1} is:

$$C_{t+1}: \quad \beta u'(C_{t+1}) - \lambda_{t+1} = 0$$

- Where there are **no expectations** because the decision C_{t+1} is made **after** the uncertainty in period $t+1$ has been resolved
- Tidying up the first order conditions:

$$\begin{aligned}C_t: &\quad \lambda_t = u'(C_t) && (6.3) \\ C_t + 1: &\quad \lambda_{t+1} = \beta u'(C_{t+1}) && (6.4) \\ B_t: &\quad \lambda_t P_{B,t} = E_t(\lambda_{t+1}) && (6.5) \\ A_t: &\quad \lambda_{t+1} P_{A,t} = E_t(\lambda_{t+1}[D_{t+1} + P_{A,t+1}]) && (6.6)\end{aligned}$$

The Macroeconomic Perspective on Asset Prices

- Combining equations (6.3), (6.4) and (6.5):

$$\begin{aligned}& u'(C_t)P_{B,t} = \beta E_t(u'(C_{t+1})) \\ \Rightarrow & \frac{u'(C_t)}{E_t(u'(C_{t+1}))} = \frac{1}{P_{B,t}} = R_{B,t}\end{aligned}$$

- Where the LHS represents the inter-temporal marginal rate of substitution
- And $R_{B,t}$ on the RHS is the (certain) return on the bond
- This says that:
  - The relative value of consumption across periods is tied to the interest rate (return) on bonds
  - Or, consumption growth across periods tied to price of transferring resources across periods
- Macroeconomic Perspective: inter-temporal consumption is all about interest rates

The Finance Perspective on Asset Prices

- Combining equations (6.5) and (6.6):

$$P_{B,t} = E_t(\frac{\lambda_{t+1}}{\lambda_t}), \qquad \text{With t+1 payoff} = 1$$
$$P_{A,t} = E_t\left(\frac{\lambda_{t+1}}{\lambda_t}[D_{t+1} + P_{A,t+1}]\right), \qquad \text{With t+1 payoff} = D_{t+1} + P_{A,t+1}$$

- These are known as asset pricing equations
- They state that the price of an asset is determined by the valuation of that asset's payoffs
- And those valuations are given by the Lagrange Multipliers:
  - $\lambda_t =$ the marginal value of an extra dollar allocated to the budget constraint in period t
  - $\lambda_{t+1} =$ the marginal value of an extra dollar allocated to the budget constraint in period $t+1$
- Looking closely, we can see that both assets' payoffs are valued at the same rate
- This valuation is given by the stochastic discount factor (SDF):

$$\frac{\lambda_{t+1}}{\lambda_t}$$

- For our particular model, we know that the SDF is given by:

$$\frac{\lambda_{t+1}}{\lambda_t} = \beta\frac{u'(C_{t+1})}{u'(C_t)}$$

- That is, assets are valued by variations in household consumption across time
- Since consumption is determined by developments in the macroeconomy: asset prices must be linked to business cycle fluctuations!
- Our asset pricing equations are:

$$P_{B,t} = E_t\left(\beta\frac{u'(C_{t+1})}{u'(C_t)}\right), \quad P_{A,t} = E_t\left(\beta\frac{u'(C_{t+1})}{u'(C_t)}[D_{t+1} + P_{A,t+1}]\right)$$

- Holding all else equal, asset prices are higher when:
  - The marginal utility of current consumption C_t is low (i.e. C_t is high)
  - The marginal utility of current consumption C_{t+1} is high (i.e. C_{t+1} is low)
- However, it will rarely be the case that C_t or C_{t+1} move independent of everything else
- Remember, macroeconomic and financial variables move together in equilibrium
- Example:
  - Consider shares in a firm A that trade at price $P_{A,t}$ and pay dividends D_{t+1}
  - Dividends are paid out of firm profits
  - Future profits and dividends will be low during recessions
  - But recessions are times when future consumption is also likely to be low

$$E_t\left(\beta\overbrace{\frac{u'(C_{t+1})}{u'(C_t)}}^{\uparrow}[\overbrace{D_{t+1}}^{\downarrow} + P_{A,t+1}]\right) = P_{A,t}(?)$$

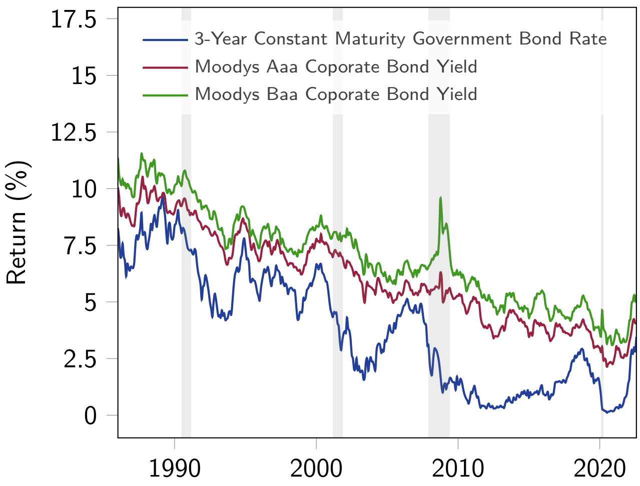
- So risky asset prices depend on co-movement between future consumption and uncertain payoffs

- However, the price of risk-free assets (i.e. bonds) only depends on the SDF
- Previous example: future recession decreases future consumption and dividends

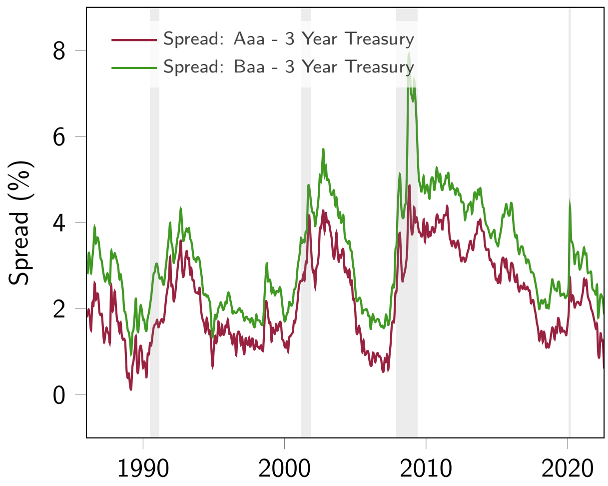
$$P_{B,t}\uparrow = E_t\left(\beta\frac{\overbrace{u'(C_{t+1})}^{\uparrow}}{u'(C_t)}\right), \quad P_{A,t}\downarrow = E_t\left(\beta\frac{\overbrace{u'(C_{t+1})}^{\uparrow}}{u'(C_t)}[\overbrace{D_{t+1}}^{\downarrow\downarrow}+P_{A,t+1}]\right)$$

- Which means that the price (return) of risk-free assets can move in opposite directions to the prices (returns) of risky assets

Various Asset Returns



Various Asset Spreads



7 Asset Prices, Consumption, and the Business Cycle

7.1 Historical Overview of Finance and Asset Pricing

1. **Market Efficiency View:**
   - Asks "Are market asset prices set conditional on all available information?"
   - Or "Can you beat the market return without taking on more than market risk?"
   - The
     textcolormyblueEfficient Markets Hypothesis:
     – [An] old economist and younger economist [are] walking down the street, and the younger economist says, 'Look, there's a hundred-dollar bill,' and the older one says, 'Nonsense, if it was there somebody would have picked it up already.'
2. **Portfolio Theory:**
   - How should we form asset portfolios? (Markowitz, 1952)
   - The variance of the return on an asset portfolio is much smaller than the average of the variances of the returns on the individual assets in the portfolio
   - So what is the optimal variance of an asset portfolio?
   - Something called the Mean-Variance Efficient Frontier can be constructed for a portfolio
3. **Capital Asset Pricing Model (CAPM):**
   - How much do individual asset prices move with the market? (Sharpe 1964; Litner 1965)
   - Want a model of the cross-sectional behaviour of stock returns
   - Let R_m be the return on the "market", R_f is the risk free return, and R_i is the return on asset i
   - Then the excess return on asset i is:

$$R_i - R_f = \beta_i(R_m - R_f), \text{ where } \beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}$$

   - And β_i can be estimated with regression models (e.g. OLS)
   - β_i measures an asset's systematic risk/exposure to market fluctuations
   - Assets with high 'betas' are more sensitive to the market
   - Since investors seek excess returns, this systematic risk is rewarded with higher prices
   - However, the model does not account for idiosyncratic risk
   - And while the model does well with cross-sectional data, it performs poorly with time-series data!
4. **No-Arbitrage Multi-Risk Theory:**
   - No-arbitrage relationships are the key intuition behind modern asset pricing developments
   - Arbitrage Pricing Theory (APT) is due to Ross, Sharpe, and Merton
   - Presents a multi-factor approach to asset pricing
   - This generalizes to multiple sources of risk including: inflation risk, business cycle risk, interest rate risk, exchange rate risk, and default risk.
   - Multiple 'betas' and multiple regression models required.
   - But the model assumes that risk and risk premiums are constant
5. **Market Microstructure:**
   - Studies how the market itself works

- E.g. the role of information asymmetry, information trading, market networks, liquidity, trading volume, who is a buyer vs. who is a seller, bid-ask spreads, etc.

6. **Macro-Finance Models:**
   - C-CAPM (Consumption based CAPM) emerged in the 1980s
   - Shows how individual attitudes to risk and uncertainty are related to variations in asset prices
   - The inter-temporal macroeconomic model based on consumer utility functions is central.
   - The key concept is the Stochastic Discount Factor, otherwise called the Pricing Kernel
   - In macroeconomic models the SDF is tied to the marginal rate of substitution between consumption across periods
   - Consumption, which depends on income and wealth, provides the link between business cycles and asset prices in this model

7.2 Finance of CAPM

- The CAPM can be expressed as:

$$R_i - R_f = \beta_i(R_m - R_f)$$

- where r_i is return on asset i, r_f is the risk-free rate, and r_m is the market portfolio return
- The LHS is the excess return on asset i over the risk-free rate
- And the RHS is the excess return on the market portfolio over the risk free rate
- The parameter $\beta_{i,m}$ captures the covariance between the risky asset and the market portfolio (scaled by the variance of the market):

$$\beta_{i,m} = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}$$

- Notice that beta is the same as the OLS regression slope coefficient between returns for asset i and the market m
- The risk of an asset i is characterised by its covariance with the market portfolio
- This particular risk is called systematic risk, and cannot be diversified away
  – Why not? Market risk is embedded in all assets, so not possible for any investors to "take the other side" of the market
- For this reason, systematic risk needs to be rewarded with higher returns

Understanding the CAPM

- If the CAPM is thought of like a regression equation, it can be written as:

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \varepsilon_i$$

- where α_i is the regression intercept/mean excess return
- And ε_i is the error term/idiosyncratic asset risk
- Our standard regression assumptions require:

$$E(\varepsilon_i) = 0$$
$$\text{Cov}(r_m, \varepsilon_i) = 0$$

1. **'Alpha'**
   - CAPM (but not necessarily the regression formula) predicts that 'alpha' should be zero for all assets
   - This is because the CAPM states that market risk is the only factor driving excess returns

- In a regression specification, alpha measures an asset's excess return over and above its risk-adjusted reward
- From outside the CAPM perspective, alpha may be picking up other (i.e. non-market) risks that are not captured by the model

2. 'Beta'

- Beta measures an asset's systematic risk
- Assets with high betas are more sensitive to the market

3. 'Sigma'

- Sigma measures the (variance) of non-systematic risk
- Non-systematic risk is uncorrelated with systematic risk
- Often refer to this as idiosyncratic risk

- Total risk of an asset is decomposed as follows:

$$r_i - r_f = \overbrace{\beta_{i,m}(r_m - r_f)}^{\text{Systematic Component}} + \overbrace{\varepsilon_i}^{\text{Idiosyncratic Component}}$$

- Taking the variance of both sides of the equation:

$$\overbrace{\text{Var}(r_i - r_f)}^{\text{Total Risk}} = \overbrace{\beta_{i,m}\text{Var}(r_m - r_f)}^{\text{Systematic Risk}} + \overbrace{\text{Var}(\varepsilon_i)}^{\text{Idiosyncratic Risk}}$$

- Note: $\text{Var}(r_i - r_f) = \text{Var}(r_i)$ and $\text{Var}(r_m - r_f) = \text{Var}(r_m)$
- CAPM is attractive because:
  - It is east to understand and sensible as it is built on modern portfolio theory
  - It distinguishes between systematic and non-systematic risk
  - It is very east to implement empirically

Limitations of CAPM

- Empirical evidence on the performance of CAPM is mixed
- The model does not work well with time series data
- This is because in CAPM investors follow myopic strategies as the investment horizon is short and investment opportunities are assumed to be constant over time
- But in general there are **two** types of systematic risks:
  - Static (temporal) - Market Risk
  - Dynamic (inter-temporal) - Changes in investment opportunities

Multi-Factor CAPM

- Many papers attempted to build on/improve the simple CAPM model by adding more "factors"
- This literature pioneered by Eugene Fama (Nobel Prize winner) and Kenneth French
- Try to identify other explanations for variation in excess returns on a given asset
- Find portfolios of traded securities that are highly correlated with different "factors"
- Hypothesize that the risk premium is linearly related to the risk premium on these portfolios:

$$r_i - r_f = \alpha_i + \beta_{i,1}(r_{F1} - r_f) + \beta_{i,2}(r_{F2} - r_f) + \cdots + \beta_{i,K}(r_{FK} - r_f)$$

- Where r_{FK} is the return on a portfolio correlated with the k-th factor only

- Factors might include: firm size, firm leverage, recent firm performance, etc
- Multiple regression used to estimate the factor betas

Limitations of Multi-Factor CAPM

- May not identify macroeconomic variables that constitute inter-temporal risks
- May not specify the relative importance of these inter-temporal risks
- Need to identify different sources of inter-temporal risks in asset returns and specifiy relative importance to investors
- But the CAPM theory itself gives little guidance on what these factors should be
- There are now hundreds (thousands?!) of proposed factors!
- This leads to the need for 'cleaning up' papers like *Taming the Factor Zoo: A Test of New Factors* by Feng, Giglio, Xiu (2020)

7.3 Macroeconomics of C-CAPM

C-CAPM Model Enviroment

- We now derive the macro-finance model known as the Consumption-CAPM
- This model states that asset prices must be closely related to fluctuations in consumption
- The reason being that fluctuations in consumption across time determine willingness to save and take risks
- The setup of the model closely follows the model of precautionary savings we studied in section 5
- x_{t+1} is a random variable
- Uncertainty in x_{t+1} is due to randomness of the **state** that occurs tomorrow
- How do we price or value an asset sold today with this payoff structure?
- To know how to value the asset, we need to know an investor's preferences
- Consider an investor who maximizes expected inter-temporal utility defined over two periods:

$$u(c_t) + \beta E_t[u(c_{t+1})]$$

- Where β is the rate at which future utility is discounted (**not** the CAPM beta!)
- $u(\cdot)$ is a general utility function defined over consumption
- And E_t is the expectations function taken with respect to information available at time t

Utility Functions

- The utility function captures the investor's attitude towards risk
- Note that the **level** of $u(C)$ does not matter
- Instead, it is **marginal utility** that matters
- Marginal utility measures 'hunger' rather than 'happiness', as it describes how much of an **improvement** an additional unit of consumption would make

Risk Aversion and Expected Utlity

- Consider a fair bet that would see you gain \$$x$ or lose \$$x$ with a 50-50 chance
- Starting from $\bar{c}$:

$$E[u(C)] = 0.5 \times u(\bar{C} + x) + 0.5 \times u(\bar{C} - x)$$

- For a risk-averse investor, the utility of expected consumption is greater than the expected utility of consumption: $u[E(C)] > E[u(C)]$

- Investors prefer a sure-thing to a risky bet

Measuring Risk Aversion

- How much do investors dislike risks?
- This can be measured with the Coefficient of Relative Risk Aversion (RRA)

$$RRA = -\frac{c \times u''(c)}{u'(c)}$$

- This measures how much **curvature** there is in the utility function
- Which in turn measures how much an investor is willing to take risks
- We will often work with a Constant Relative Risk Aversion (CRRA) utility function:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

- where γ is a parameter in the utility function
- Marginal utility for this function is:

$$u'(c) = c^{-\gamma}$$

- and the RRA for this utility function is:

$$RRA = -\frac{c \times u''(c)}{u'(c)} = \gamma$$

- and so γ is the risk aversion coefficient

The Asset Pricing Function

- What is the value of the payoff x_{t+1} to an investor with a utility function $u(c)$?
- The asset pricing formula is:

$$p_t = E_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right)$$

- When $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ we have:

$$p_t = E_t\left(\beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} x_{t+1}\right)$$

- The asset pricing equation provides the theoretical basis for understanding macro-asset pricing relationships
- We have seen before that the key element is called the Stochastic Discount Factor:

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

- So that we can rewrite out asset pricing equations as a function of the SDF:

$$p_t = E_t(m_{t+1} x_{t+1})$$

- The fundamental principle of modern asset pricing is that the price of an asset is equal to the expected discounted value of the asset's payoff
- And in microfinance, discounting depends on inter-temporal optimisation through the SDF

- The asset pricing equation can be written as:

$$\underbrace{p_t u'(c_t)}_{\text{Marginal Cost}} = \underbrace{\beta E_t[u'(c_{t+1})x_{t+1}]}_{\text{Marginal Benefit}}$$

- Marginal cost is the opportunity cost of buying the asset in period 1
- Marginal benefit is the discounted/marginal-utility weighted payoff of the asset in period 2
- Consider an investor facing the following choices:
  1. Consume an extra \$1 today $\Rightarrow u'(c_0)$ or
  2. Invest \$1 in the asset
     - Receive $\$x_{t+1}$ tomorrow
     - Consume the payoff tomorrow $\Rightarrow \beta u'(c_1)x_{t+1}$
- When making optimal decisions, the investor is indifferent between the two choices

Interpretation

i) Before buying, an investor will consider the asset under-priced if:

$$p_t < \beta E_t[u'(c_{t+1})/u'(c_t)x_{t+1}]$$

- Investor reduces consumption today and reallocates resources towards the asset
- Investor keeps buying the asset until consumption in each period equalizes the pricing equation

ii) From investor's perspective, prices are fixed and the formula explains how to adjust consumption

- However, in the macroeconomy investors all together will affect prices
- If aggregate consumption is fixed by total output in the economy, then this pins down asset prices

Asset Pricing Examples

Example 1

- An investor's utility function is: $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$
- An asset purchased today has a payoff x_{t+1} with certainty
- And suppose the investor's income is constant so: $c_t = c_{t+1} = 1$
- Assume $\beta = 1$
- The asset pricing formula yields:

$$\begin{aligned} p_t &= \beta E_t\left(\frac{(u'(c_{t+1})}{u'(c_t)} \times x_{t+1}\right) \\ &= \beta\left(\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \times x_{t+1}\right) \\ &= 1 \times \left(\frac{1^{-\gamma}}{1^{-\gamma}} \times 1\right) \\ &= 1 \end{aligned}$$

Example 2

- An investor's utility function is: $u(c_t) = \log(c_t)$. Assume $\beta = 1$
- There are two periods. Period 1 is certain, and the investor consumes $c_1 = 1$
- In period 2, $c_2 = Y_2 + x_2$, where $Y = 1$

- There are two states of the world in period 2 describing payoffs:

$$x_2 = \begin{cases} 1 & \text{with probability } 1/4 \\ 2 & \text{with probability } 3/4 \end{cases}$$

$$\begin{aligned} p_1 &= \beta E_1 \left(\frac{(c_2)^{-1}}{(c_1)^{-1} x_2} \right) \\ &= \frac{1}{4} \left(\frac{(1+1)^{-1}}{(c_1)^{-1} x_2} \right) + \frac{3}{4} \left(\frac{(1+2)^{-1}}{(1)^{-1} \times 2} \right) \\ &= \frac{1}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{2}{3} \\ &= \frac{5}{8} \end{aligned}$$

Risk-Free Rate and Consumption Growth

- Let's explore further with the risk-free rate and a specific utility function
- This will help to understand the relationship between asset returns and consumption growth and hence the business cycle
- The Constant Relative Risk Aversion (CRRA) utility function is:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \Rightarrow u'(c) = c^{-\gamma}$$

- Then the asset pricing formula is:

$$\begin{aligned} p_t &= E_t[m_{t+1} x_{t+1}] \\ \text{where:} \quad m_{t+1} &= \beta \frac{u'(c_{t+1})}{u'(c_t) = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}} \end{aligned}$$

- Using some tricks, we can write the SDF with a linear approximation:

$$\begin{aligned} m_{t+1} &= \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \\ &= e^{\ln \beta} e^{\ln \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}} \\ &= e^{\ln \beta} e^{-\gamma \Delta C_{t+1}} \\ &\approx 1 + \ln \beta - \gamma \Delta \ln c_{t+1} \end{aligned}$$

- A risk-free asset has a price $p_t = 1$ and payoff/return R_f

$$\begin{aligned} 1 &= E(m_{t+1} R_f) = (m_{t+1}) R_f \\ \Rightarrow R_f &= \frac{1}{E(m_{t+1})} \end{aligned}$$

- Now use our linear approximation trick to work out the linear relationship between the risk-free rate and consumption growth:

$$\begin{aligned} R_f &\approx \frac{1}{1 + \ln \beta - \gamma E_t \Delta \ln c_{t+1}} \\ &\approx 1 - \ln \beta + \gamma E_t \Delta \ln c_{t+1} \end{aligned}$$

1. The risk-free rate is higher, all else equal, when:
   - People are more impatient (i.e. low β)
   - Expected consumption growth is high

2. The risk-free rate is more sensitive to consumption growth when γ is high
    - Risk aversion is governed by γ
    - The risk free rate is more sensitive to consumption when investors are more risk averse

C-CAPM and the Valuation of Risk

- How can we use C-CAPM to price/value risk?
    - That is, what is the value of an asset with a particular risk profile?
- Use the definition of covariance:

$$\begin{aligned} p &= E[mx] \\ &= E[m]E[x] + Cov[m, x] \\ &= \frac{E[x]}{R_f} + Cov[m, x] \end{aligned}$$

- Where we substitute $E[m] = \frac{1}{R_f}$ from the asset pricing formula for the risk-free asset
- Intuition for the value of an asset:

$$p = \underbrace{\frac{E[x]}{r_f}}_{\substack{\text{Present Value} \\ \text{of Payoff x}}} + \underbrace{Cov[m, x]}_{\text{Risk Correction}}$$

$$P_t = \frac{E_t[x_{t+1}]}{R_f} + Cov[m_{t+1}, x_{t+1}]$$

- Using our linear approximation: $m_{t+1} \approx 1 + \ln \beta - \gamma \ln c_{t+1}$

$$\begin{aligned} p_t &\approx \frac{E_t[x_{t+1}]}{R_f} + Cov[1 + \ln \beta - \gamma \ln c_{t+1}] \\ &= \frac{E_t[x_{t+1}]}{R_f} - \gamma Cov[\Delta \ln C_{t+1}, x_{t+1}] \end{aligned}$$

- When $Cov[\Delta \ln C_{t+1}, x_{t+1}] > 0$:
    - Asset payoffs are high when future consumption is high
    - This increases consumption risk so has a lower price
- When $Cov[\Delta \ln C_{t+1}, x_{t+1}] < 0$:
    - Asset payoffs are high when future consumption is low
    - This decreases consumption risk so has higher price
- Why is m_{t+1} called the Stochastic Discount Factor?
- Consider the pricing equation for an asset i:

$$P_t = E_t[m_{t+1} x^j_{t+1}]$$

- Notice that m_{t+1} is unknown at time t (and sits with an investor's expectations)
- But the SDF m_{t+1} is the **same** for all assets
- What differs is the covariance between the SDF and the asset payoff x^i_{t+1}
- This asset-specific covariance gives different risk adjustments for each asset
- What matters for asset pricing is co-movement between the random (i.e stochastic) nature of the SDF and individual asser payoffs

How does C-CAPM Related to CAPM

- We can derive a 'beta' similar to the 'beta' in the CAPM
- Define the excess return on asset i as: $R_i^e = R_i - R_f$
  - We can always earn the excess return R_i^e by ***borrowing*** at rate R_f and ***investing*** in asset i with return R_i
  - Note that he price of the asset paying R_i^e is zero, since we borrow \$$x$ and invest \$$x$ at the same time
- The asset pricing formula for an asset with return R_f is: $1 = E_t[m_{t+1}R_f]$
- The asset pricing formula for an asset with return R_i is: $1 = E_t[m_{t+1}R_i]$
- So the asset pricing formula applied to the excess return is:

$$0 = E_t[m_{t+1}(R_i - R_f)] = E_t[m_t + 1R_i^e]$$

- Again using the definition of covariance:

$$\begin{aligned} 0 &= E_t[m_{t+1}R_i^e] \\ &= E_t[m_{t+1}]E_t[R_i^e] + Cov[m_{t+1}, R_i^e] \\ \Rightarrow E_t[R_i^e] &= -\frac{Cov[m_{t+1}, R_i^e]}{E_t[m_{t+1}]} \\ &= -\frac{Cpv[m_{t=1}, R_j^e]}{Var[m_[t+1]]}\frac{Var[m_{t+1}]}{E_t[m_{t+1}]} \\ &= \beta_{i,m} \times \delta_m \end{aligned}$$

- Where $\beta_{i,m}$ is like 'beta'/market loading for asset i, as in CAPM
- And λ_m is the 'market' risk
- For C-CAPM, 'market risk' is the risk associated with fluctuations in consumption
- From our earlier approximation to the SDF:

$$\begin{aligned} Cov(m_{t+1}, R_i^e) &\approx Cov(1 + \ln\beta - \gamma\Delta\ln E_t C_{t+1}, R_i^e) = -\gamma Cov(E_t\Delta\ln E_t C_{t+1}, R_i^e) \\ Var(m_{t+1}) &\approx Var(1 + \ln\beta - \gamma\Delta\ln E_t C_{t+1}) = \gamma^2 Var(E_t \Delta ln C_{t+1}) \end{aligned}$$

- So our expression for excess returns becomes:

$$\begin{aligned} E_t[R_i^e] &= -\frac{Cov[m_{t+1}, R_i^e]}{Var[m_{t+1}]}\frac{Var[m_{t+1}]}{E_t[m_{t+1}]} \\ &= \frac{Cov(E_t\Delta\ln c_{t+1}, R_i^e)}{Var[E_t\Delta\ln c_[t+1]]} \times \gamma\frac{Var[E_t\Delta\ln c_[t+1]]}{E_t[1 + \ln\beta - \gamma E_t\Delta\ln C_{t+1}]} \\ &= \beta_{i,\Delta C} \times \lambda_{\Delta C} \end{aligned}$$

Interpretation

1. When $Cov[\Delta\ln C_{t+1}, R_i^e] > 0$
   - Excess returns are high when consumption growth is high
   - This means payoffs are high when consumption growth is high
   - This increases consumption risk so has a lower price
   - High excess returns $E_t[R_i^e] \Leftrightarrow$ low price
2. When $Cov[\Delta\ln C_{t+1}, R_i^e] < 0$
   - Excess returns are high when consumption growth is low

- This means payoffs are high when consumption growth is low
- This decreases consumption risk so has a higher price
- Low excess returns $E_t[R_i^e] \Leftrightarrow$ high price

3. Higher $\gamma \Rightarrow$ higher risk aversion $\Rightarrow$ larger effects on prices and returns

4. Only systematic risk matters for prices/returns

- Systematic risk comes through co-variation with investors consumption growth

7.4 C-CAPM: Empirical Issues

There are (at least!) three major 'puzzles' about relationship between C-CAPM and data

1. **The equity premium puzzle**
   - Due to Mehra and Prescott (1985)
   - The equity premium in the data over the last century $\approx 6\%$
   - But C-CAPM calibrated to US business cycle statistics yields an equity premium $\approx 1\%$
   - For reasonable levels of risk aversion, the equity premium observed in the data is far **too high**, it over-compensates for risk
2. **The risk-free rate puzzle**
   - Suppose we can match the equity premium with an (implausibly) high level of risk aversion
   - Then the implied risk-free rate is also very high
   - So why is the risk-free rate observed in the data so **low**?
3. **The volatility puzzle**
   - Due to Shiller (1981)
   - Far too much volatility in stock prices given the (relatively low) volatility in future payoffs

7.5 Extensions of the C-CAPM

- The benchmark C-CAPM cannot solve the **equity premium puzzle** and the **risk-free rate puzzle** simultaneously
- This is largely due to the way that risk aversion and inter-temporal substitution are characterised in the model
- There have been many extensions of the C-CAPM model to try and solve these puzzles
- We briefly summarize three of them here:
  1. Habit formation in consumption
  2. Long-run risk model
  3. Heterogeneous agents with incomplete markets

1. Habit Formation in Consumption

- The idea behind habit formation is to generate persistent movements in utility over time
- This generates time-varying risk aversion
- If utility was high in the past, then it should also be high in the future and this reduces current risk aversion
- Consider the habit formation utility function:

$$U = \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma}$$

- Where X_t is the (external) habit stock of consumption
- This is sometimes referred to as 'Keeping Up with the Joneses': if you observe that the consumption of your neighbours is high, then you would like to increase your own consumption
- Habit formation and risk aversion.

$$RRA = -\frac{U'' \times C_t}{U'} = \frac{\gamma}{S_t}$$

- where $S_t \equiv \frac{C_t - X_t}{C_t}$ which is the surplus consumption ratio
- When consumption is high relative to the habit stock, risk aversion falls
- When consumption is low and gets close to the habit stock, risk aversion rises
- Marginal Utility is:

$$U' = (C_t - X_t)^{-\gamma} = S_t^{-\gamma} C_t^{-\gamma}$$

- So the Stochastic Discount Factor becomes:

$$M_{t+1} = \nu \left(u' \left(C_{t+1} \right) / u' \left(C_t \right) \right) = \beta \left(\frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

- The SDF becomes more volatile as the volatility of the surplus consumption ratio S_t rises
- Thus, risk aversion increases without having to increase the risk aversion coefficient γ
- When consumption falls relative to habit, the increase in risk aversion drives up the equity risk premium $\Rightarrow$ time variation in risk aversion (see Cochrane, 2011)

2. Long-Run Risk

- The long-run risks model also tries to generate persistent fluctuations in utility over time
- As with habit formation, this generates time-varying risk aversion
- The main mechanism for doing so is with **recursive** preferences
- These preferences allows for separation between risk aversion and inter-temporal substitution
- Additionally, these preferences imply that investors may prefer **early** or **late** resolution of uncertainty over future consumption
  - When investors prefer early resolution of uncertainty, they must be compensated for long-run risks over consumption
  - And so changes in views of long-run risks affect compensation for holding different assets today
  - And this necessarily affects the equity risk premium
- The long-run risk model uses more complicated Epstein-Zin-Weil preferences:

$$U_t = \left((1-\beta) C_t^{1-\rho} + \beta \left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$

- where γ again governs risk aversion, but the ρ separately governs inter-temporal substitution
- Notice that **expected future utility**, $E_t(U_{t+1}^{1-\gamma})$, affects the value of consumption today
- The SDF for this model is:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{U_{t+1}}{E_t \left(U_{t+1}^{1-\gamma} \right)^{1/(1-\gamma)}} \right)^{\rho-\gamma}$$

- When γ is high, and future utility is risky (i.e. U_{t+1} dispersed), the SDF is higher:

⇒ Higher excess returns (i.e. higher equity risk premium)

⇒ Higher average SDF and so lower risk-free rate

3. Heterogeneous Agents with Incomplete Markets

- So far, everything we have done involves a representative agent: a single entity representing the entire economy
- In this world, individual consumption **is** aggregate consumption
- And so consumption only fluctuates with market-level risk
- But this has two problems empirically:
  1. Aggregate consumption is far too smooth over time, hence low risk premium in C-CAPM
  2. Non-aggregate/idiosyncratic risks seem to be much more important to individual income/consumption
- But the C-CAPM does not account for idiosyncratic risk, and **only prices market risk**
- Instead of a representative agent, assume there are multiple/many agents (i.e. heterogenity)
- These agents cannot diversify away all of their idiosyncratic risks (i.e. incomplete markets)
- Then asset pricing depends on **individual asset pricing equations**
- This opens many avenues for changing the pricing of risk
  - Income equality: if the distribution of income risk is correlated with market risk, this will increase the compensation required to hold risks (i.e. the risk premium)
  - Limited market participation: only some people are active investors in a particular asset e.g. stocks, bonds, houses, currencies
  - Thus, each asset class may be priced by a different type of investor with their own idiosyncratic risks, changing the risk premium on those assets

7.6 Why the C-CAPM is Important Despite its Limitations

- Why is the C-CAPM still of interest, despite worse empirical performance than 'reduced form' models like the CAPM or multi-factor models?
- *Macroeconomics and Finance*
  - Asset markets are the mechanism that helps us to understand the equation of savings to investment, and the allocation of consumption and investment across time and states of nature
  - The relationship between asset prices and the macroeconomy helps us understand important topics like: monetary policy, fiscal policy (i.e. government debt), mortgages, houses, investment, exchange rates, credit markets, etc
  - We need a theoretically consistent way of linking asset prices back to the macroeconomy
  - C-CAPM is the foundational model (however imperfect) to help us do this

8 Housing and the Business Cycle I

8.1 Simple Models of Housing Purchase Decisions

- When modeling housing decisions in the macroeconomy, we need to consider **three** primary assets associated with the housing market
  - Owner-occupied housing
  - Residential investment property
  - Mortgages to finance house purchase
- Developments in any one of these asset markets can influence each of the other markets, as well as the macroeconomy as a whole
- We will study simple decision models for each asset, starting with investment property

8.2 A Model of Housing Investment Decisions

- Investors choose the size of the investment property they want to hold, H_t
- Houses can be bought and sold at price P_t^h
- Investment property earns a rental return R_t^h in the period in which it is purchased
- Houses depreciates at rate δ, proportional to the value of the investment property
- Resale of houses is subject to a simple sales tax τ, proportional to the value of the investment property at date of sale (i.e. similar to a capital gains tax)
- Investors also have access to a bond B_t that pays return R_{t+1} next period
- An investor's infinite-horizon decision problem is:

$$\max_{C_t, B_t, H_t} \quad \sum_{t=0}^{\infty} \beta^t \log C_t$$

$$\text{s.t.} \quad C_t + \underbrace{P_t^h H_t}_{\substack{\text{Cost of}\\ \text{new}\\ \text{housing}}} + B_t = Y_t + \underbrace{R_t^h H_t}_{\substack{\text{rental}\\ \text{yield}\\ \text{on housing}}} + B_{t-1} R_t + \underbrace{P_t^h H_{t-1}}_{\substack{\text{Resale value}\\ \text{of previous}\\ \text{housing}}} - \underbrace{\delta P_t^h H_{t-1}}_{\substack{\text{Depreciation}\\ \text{cost on previous}\\ \text{housing}}} - \underbrace{\tau P_t^h H_{t-1}}_{\substack{\text{Sales tax}\\ \text{on previous}\\ \text{housing value}}}$$

$$H_t \geq 0$$

- Note that the investor cannot "borrow" in (or "short") housing
- However, the investor may save or borrow in the risk free bond
- The Lagrangian is

$$\mathcal{L} = \beta^t \log C_t - \lambda(Y_t + R_t^h H_t + B_{t-1} R_t + P_t^h H_{t-1} - \delta P_t^h H_{t-1} - \tau P_t^h H_{t-1} - C_t - P_t^h H_t - B_t)$$

- The first order conditions for the investor are:

$$\begin{aligned} C_t: &\quad \lambda_t = \beta^t \frac{1}{C_t} \\ B_t: &\quad \lambda_t = \lambda_{t+1} R_{t+1} \\ H_t: &\quad 0 = \lambda_t (R_t^h - P_t^h) \lambda_{t+1} (1 - \delta - \tau) P_{t+1}^h \end{aligned}$$

- Together, these form the two Euler equations:

$$\frac{1}{C_t} = \beta \left[R_{t+1} \frac{1}{C_{t+1}} \right] \qquad \text{Bond Euler Equation}$$

$$P_t^h \frac{1}{C_t} = R_t^h \frac{1}{C_t} + \beta \left[\frac{1}{C_{t+1}} (1 - \delta - \tau) P_{t+1}^h \right] \qquad \text{Housing Euler Equation}$$

- Substitute the bond Euler equation into the housing Euler equation:

$$P_t^h = R_t^h + (1 - \delta - \tau)\frac{P_{t+1}}{R_{t+1}}$$

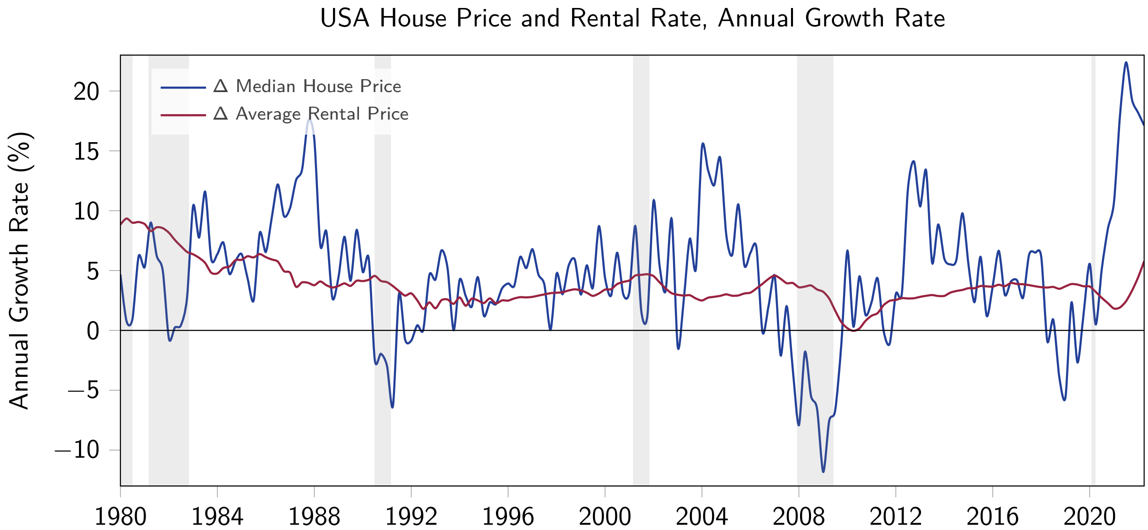
- Assume that $R_t = R$ for all t
- Step forward one period, and substitute P_{t+1}^h into the right hand side repeatly:

$$P_t^h = \sum_{s=0}^{\infty} \left(\frac{1 - \delta - \tau}{R}\right)^s R_{t+s}^h$$

- So current house prices reflect the discounted stream of future rental flows

1. Higher interest rates
  - Can reflect higher opportunity cost of housing investment (i.e. alternative is to investment)
  - Can also reflect higher cost of borrowing using the bond
  - Higher opportunity/borrowing costs reduce future rental payoffs

2. Higher depreciation and taxes
  - Higher depreciation implies a higher carrying cost of holding as an investment
  - Higher taxes reduce the return due to capital gains
  - Both depreciation and taxes reduce the return to housing investment



8.3 A Model of Homeownership Decisions

Housing as an owner-occupied asset/durable good:

- Households may be renters or homeowners
  - Assume that households are indifferent between renting and owning
- For renters:
  - Choose size of the house to be rented, H_t^R
  - Enjoy utility from housing services rented
  - Rental cost of R_t^h per unit of housing rented
- For homeowners:

  - Choose size of the house to be purchased/owned, h_t^O
  - Enjoy utility from housing services owned
  - Houses can be bought and sold at price P_t^h
  - Houses depreciate at a rate δ, and houses sales are subject to tax τ
  - Owners also have access to a bond B_t, that pays return R_{t+1} next period
- Both renters and homeowners enjoy utility $u(C_t, H_t)$ over the houses they rent/own
- Common functional forms:

$$
\begin{aligned}
u(C_t, H_t) &= \alpha \log C_t + (1-\alpha)\log H_t && \text{Separable Utility} \\
u(C_t, H_t) &= C_t^{\alpha} H_t^{1-\alpha} && \text{Cobb-Douglas Utility} \\
u(C_t, H_t) &= \left[\alpha C_t^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha) H_t^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon-1}{\epsilon}} && \text{Constant Elasticity of Substitution Utility}
\end{aligned}
$$

- Where α is the share of total expenditure on consumption
- For CES utility, ϵ is the elasticity of substitution between consumption and housing services

A Model of Household Rental Decisions

- Renters have a simple "static" decision each period:

$$
\begin{aligned}
\max_{C_t^R, H_t^R} \quad & \alpha \log(C_t^R) + (1-\alpha)\log(H_t^R) \\
\text{s.t.} \quad & C_t^R + R_t^h H_t^R = Y_t
\end{aligned}
$$

- The first order conditions for the renting household yield:

$$
\frac{(1-\alpha)}{\alpha} \frac{1/H_t^R}{1/C_t^R} = R_t^h
$$

- Which says that the marginal rate of substitution between housing and consumption is equal to the rental cost of housing

A Model of Household Homeownership Decisions

- Homeowners solve an infinite-horizon problem:

$$
\begin{aligned}
\max_{C_t^0, B_t^0, H_t^0} \quad & \sum_{t=0}^{\infty} \beta^t (\alpha \log C_t + (1-\alpha)\log H_t^O) \\
\text{s.t.} \quad & C_t^O + P_t^h H_t^O + B_t = Y_t + B_{t-1}^O R_t + (1-\delta-\tau) P_t^h H_{t-1}^O \\
& H_t \geq 0
\end{aligned}
$$

- As with the investor, the homeowner may save or borrow in the risk free bond
- Unlike the investor, a homeowner does not receive the rental yield from houses they occupy
- The Lagrangian is

$$
\mathcal{L} = \beta^t (\alpha \log C_t + (1-\alpha)\log H_t^O) + \lambda (Y_t + B_{t-1}^O R_t + (1-\delta-\tau) P_t^h H_{t-1}^O - C_t^O - P_t^h H_t^O - B_t)
$$

- The first order conditions for the homeowner are:

$$
\begin{aligned}
C_t^O: &\quad \lambda_t = \alpha \frac{1}{C_t^O} \\
B_t^O: &\quad \lambda_t = \beta \lambda_{t+1} R_{t+1} \\
H_t^O: &\quad 0 = (1-\alpha)\frac{1}{H_t^O} - \beta \lambda_{t+1} (1-\delta-\tau) P_{t+1}^h
\end{aligned}
$$

- Together, these form the two Euler equations:

$$\frac{1}{C_t^O} = \beta R_{t+1} \frac{1}{C_{t+1}} \qquad \text{Bond Euler Equation}$$
$$P_t^h \alpha \frac{1}{C_t} = (1-\alpha)\frac{1}{C_t} + \beta\alpha \frac{1}{C_{t+1}^O}(1-\delta-\tau)P_{t+1}^h \qquad \text{Housing Euler Equation}$$

- Rewrite the housing Euler equation as:

$$\beta \frac{C_t^O}{C_{t+1}^O}(1-\delta-\tau)P_{t+1}^h P_t^h = \underbrace{\frac{(1-\alpha)}{\alpha}\frac{1/H_t^O}{1/C_t^O}}_{\substack{\text{Marginal rate of}\\ \text{substitution between}\\ \text{housing and consumption}}} + \underbrace{\beta \frac{C_t^O}{C_{t+1}^O}(1-\delta-\tau)P_{t+1}^h}_{\substack{\text{Present discounted value}\\ \text{of capital gain on housing}}}$$

- Where the MRS between housing and consumption is the flow value of housing services derived from homeownership
- Since households are indifferent between renting and owning a home, utility must be the same in every period:
$$U^R \equiv \alpha \log C_t^R + (1-\alpha)\log H_t^R \quad = \quad \alpha \log C_t^O + (1-\alpha)\log H_t^O \equiv U^O$$
- Which means that consumption and housing choices are the same in every period:

$$C_t^R = C_t^O = C_t$$
$$H_t^R = H_t^O = H_t$$

- The renter and homeowner optimality conditions are:

$$R_t^h = \frac{(1-\alpha)}{\alpha}\frac{1/H_t}{1/C_t} \qquad \text{Renter Optimality Condition}$$
$$P_t^h = \frac{(1-\alpha)}{\alpha}\frac{1/H_t}{1/C_t} + \beta\frac{C_t}{C_{t+1}}(1-\delta-\tau)P_{t+1}^h \qquad \text{Homeowner Euler Equation}$$

- Combining:
$$P_t^h = R_t^h + \beta\frac{C_t}{C_{t+1}}(1-\delta-\tau)P_{t+1}^h$$
- So the flow value of housing services is equivalent to the rental rate on housing
- In fact, this is exactly how national statistical agencies aim to measure rents in macroeconomic data e.g. GDP, the CPI
- Finally, note the equivalence between the price of houses from the homeowner's perspective and from the investor's perspective

$$P_t^h = \frac{(1-\alpha)}{\alpha}\frac{1/H_t}{1/C_t} + \beta\frac{C_t}{C_{t+1}}(1-\delta-\tau)P_{t+1}^h \qquad \text{Homeowner's Asset Price Equation}$$
$$P_t^h = R_t^h + \beta\frac{C_t}{C_{t+1}}(1-\delta-\tau)P_{t+1}^h \qquad \text{Investors's Asset Price Equation}$$

- These asset pricing equations are the same, despite the fact that investors earn rents while homeowners enjoy the service flow of housing

8.4 Limitations of the Simple Housing Model

- There are several issues that complicate housing purchase/investment decisions relative to the simple models
- These complications are important for properly understanding:

  - Homeownership rates
  - Mortgage borrowing and indebtedness
  - The consumption decisions of homeowners
  - House prices
- Borrowing costs:
  - Mortgage interest rates are higher than risk-free asset/deposit rates
  - Borrowing costs also vary with risk e.g. high debt loads, less ability to repay
  - Implies different borrowers face different mortgage financing costs
  - Borrowing costs also vary over the business cycle and the credit cycle
- Borrowing constraints:
  - Both banks and governments restrict the ability to borrow against housing
  - Restrictions on: loan-to-value ratios, debt-to-income ratios, and payment-to-income ratios
  - These restrictions are in place to prevent risky borrowing by homeowners
- Housing adjustment costs:
  - Sales costs: e.g. real estate agent fees, stamp duty, moving costs
  - Home owners adjust infrequently, may stay in far-from-optimal housing for long periods
  - Following a shock, homeowners may be forced to dramatically cut consumption, rather than adjust housing

9 Housing and the Business Cycle II

9.1 A Simple Model with Mortgage Finance Decisions

- Consider a two-period homeowner decision problem:

$$\begin{aligned}
\max_{C_t, B_t, H_t} \quad & \log C_1 + \alpha \log H + \beta \log C_2 \\
\text{s.t.} \quad & C_1 + P_1^h H + B = Y_1 \\
& C_2 = Y_2 + B\bar{R}(B) + (1-\delta) P^H H \\
& H \geq 0
\end{aligned}$$

- Where the interest rate depends on the savings/borrowing decision:

$$\bar{R} = \begin{cases} R & \text{if } B \geq 0 \\ R^m & \text{if } B < 0 \end{cases}$$

- and $R^m > 0$ means that borrowing is more costly than the return of savings
- The Lagrangian equation is:

$$\mathcal{L} = \log C_1 + \alpha \log H + \beta \log C_2 + \lambda_1 (Y_1 - C_1 - P_1^h H - B) + \lambda_2 (Y_2 + B\bar{R}(B) + (1-\delta) p^H H - C_2)$$

- And the first order conditions are

$$\begin{aligned}
C_1: \quad & \frac{1}{C_1} = \lambda_1 \\
C_2: \quad & \beta \frac{1}{C_2} = \lambda_2 \\
B: \quad & \lambda_1 = \lambda_2 \bar{R}(B) \\
H: \quad & \alpha \frac{1}{H} + \lambda_2 (1-\delta) P^H = \lambda_1 P_1^h
\end{aligned}$$

- Combining the first order conditions
- The Euler equations for a homeowner with savings ($B \leq 0$) are:

$$\begin{aligned}
\frac{1}{C_1^S} &= \beta R \frac{1}{C_2^S} && \text{Bond Euler Equation} \\
P_1^h \frac{1}{C_1^S} &= \alpha \frac{1}{H^S} + \beta \frac{1}{C_2^S} (1-\delta) p^H && \text{Housing Euler Equation}
\end{aligned}$$

- The Euler equations for a homeowner that borrows ($B < 0$) are:

$$\begin{aligned}
\frac{1}{C_1^B} &= \beta R^m \frac{1}{C_2^B} && \text{Bond Euler Equation} \\
P_1^h \frac{1}{C_1^B} &= \alpha \frac{1}{H^B} + \beta \frac{1}{C_2^B} (1-\delta) p^H && \text{Housing Euler Equation}
\end{aligned}$$

- Substituting in for the bond Euler equation, the house price for a homeowner with savings:

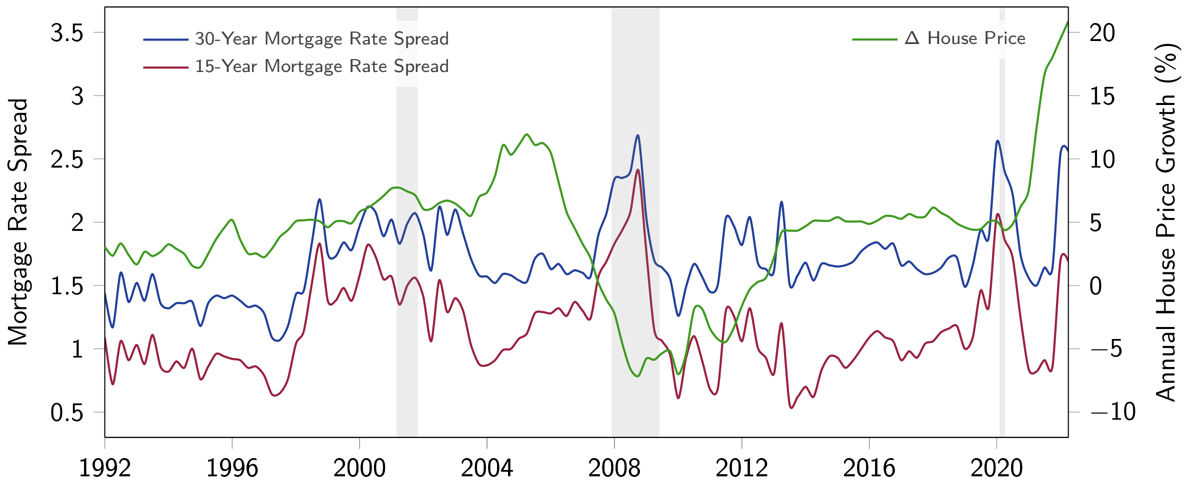
$$P_1^h = \alpha \frac{\frac{1}{H^S}}{\frac{1}{C_1^S}} + \frac{(1-\delta) P_2^h}{R}$$

- Substituting in for the bond Euler equation, the house price for a homeowner that borrows:

$$P_1^h = \alpha \frac{\frac{1}{H^B}}{\frac{1}{C_1^B}} + \frac{(1-\delta) P_2^h}{R^m}$$

- Borrowing at a higher interest rate increases the cost of house purchase
- This decreases demand for housing, and reduces willingness to pay for housing by borrowers
- Periods with a low mortgage interest rate spread are associated with higher house prices
- Spread = mortgage rate - 10 year bond

Mortgage Rate Spreads vs. House Price Growth



9.2 A Model of Mortgage Finance and Consumption Decisions

- We now want to study the joint mortgage-consumption decisions
- To simplify,
  - Suppose homeowners has already choosen the size of house H
  - Suppose house prices are constant: $P_1^h = P_2^h = P^h$
- Again, choose consumption and savings/debt subject to costly mortgage finance:

$$\begin{aligned} \max_{C_1, C_2, B} \quad & \log C_1 + \beta \log C_2 \\ \text{s.t.} \quad & C_1 + P^h H + B = Y_1 \\ & C_2 = Y_2 + B\bar{R}(B) + (1-\delta)P^h H \\ & \bar{R} = \begin{cases} R & \text{if } B \geq 0 \\ R^m & \text{if } B < 0 \end{cases} \end{aligned}$$

- And $R^m > R$
- The inter-temporal budget constraint is:

$$C_1 + P^h H + \frac{C_2}{\bar{R}(B)} = Y_1 + \frac{Y_2 + (1-\delta)P^h H}{\bar{R}(B)}$$

- When borrowing, $\bar{R}(B) = R^m > R$ reduces total resources available to consume
- Additionally, the household saves/borrows wherever:

$$\begin{aligned} B > 0, \quad & \text{if } C_1 < Y_1 P^h H \\ B > 0, \quad & \text{if } C_1 \geq Y_1 P^h H \end{aligned}$$

- Borrow whenever consumption is greater than what is leftover after purchasing a house

Graphical Illustration

- We can plot the household's constraints and consumption decisions in (C_1, C_2)-space
- To plot the inter-temporal budget constraint, note that:
  - The most a household can consume in period 1 occurs when $C_2 = 0$

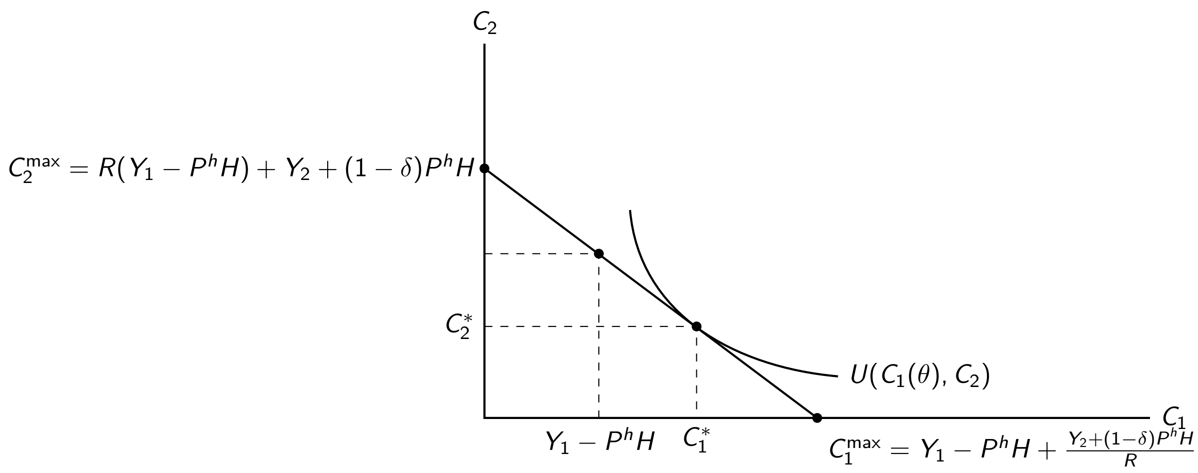
$$C_1^{\max} = Y_1 - P^h H + \frac{Y_2 + (1-\delta)P^h H}{\bar{R}(B)}$$

  - The most a household can consume in period 2 occurs when $C_1 = 0$

$$C_2^{\max} = \bar{R}(B)(Y_1 - P^h H) + Y_2 + (1-\delta)P^h H$$

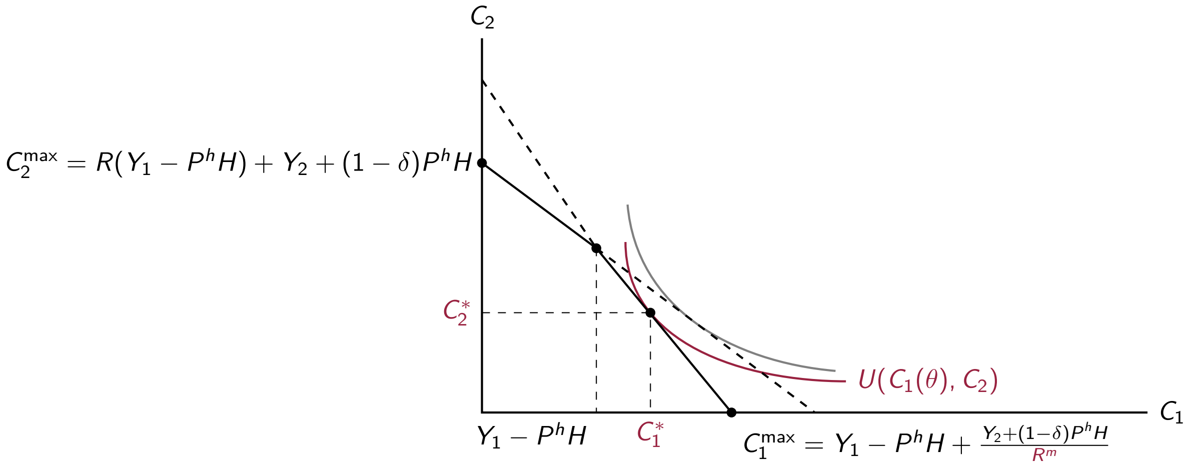
Graphical Illustration: No Additional Borrowing Costs

- First, let's suppose that $R^m = R$, so there is no additional cost for borrowing
- Given the cost of housing $P^h H$, the optimal consumption choices are c_1^*, c_2^*
- Because $C_1^* > Y_1 - P^h H$, the household is currently borrowing



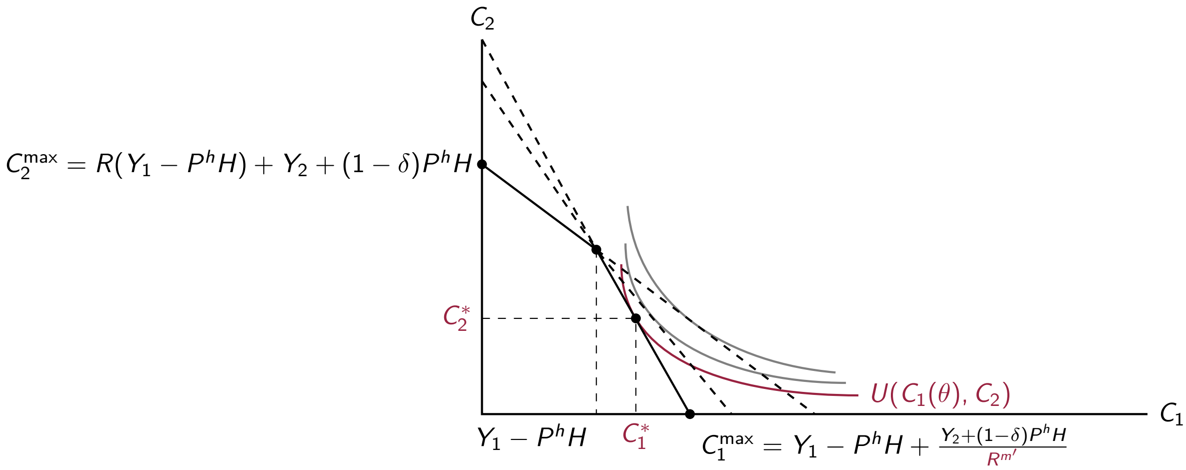
Graphical Illustration: Costly Mortgage Borrowing

- Now suppose $R^m > R$, so that borrowing to finance housing is expensive
- The budget constraint under R^m has a steeper slope (and higher y-intercept)
- Consumption below $C_1 = Y_1 - P^h H$ is unaffected by the borrowing cost since saving at rate R
- Consumption above $C_1 = Y_1 - P^h H$ implies lower C_2 due to higher borrowing cost R^m
- When borrowing, both C_1 and C_2 are lower as cost of borrowing is spread across time



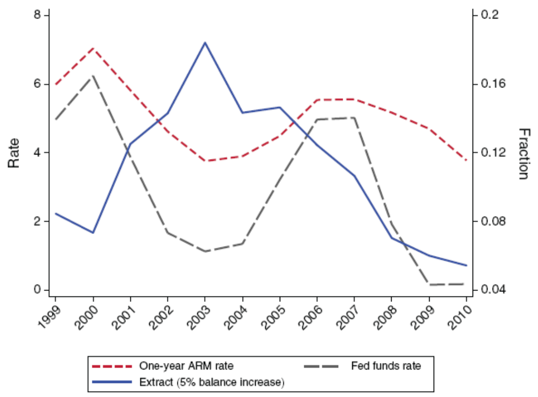
Graphical Illustration: An Increase in the Cost of Mortgage Borrowing

- Finally, suppose that the mortgage interest rate increases from R^m to $R^{m'}$
- Budget constraint rotates, with a fall in the maximum consumption possible in period 1
- More costly borrowing reduces resources available for consumption after repaying debt
- Again, spread the cost of borrowing across time, so both C_1 and C_2 decrease

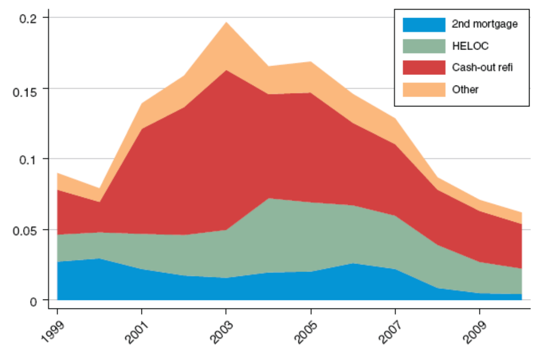


Mortgage Finance Costs, Borrowing, and Consumption: Emperical Evidence

- How do borrowing and consumption respond to changes in mortgage interest rates?
- Bhutta and Keys (2016) show that declining mortgage interest rates resulted in significant "housing equity extraction" in the form of "cash out refinancing"
- They show that very little of the cash extracted was used to repay other debts
- Instead, the cash was used to finance consumption expenditures (e.g. cars, home renovation, holidays, etc)



Probability of Extracting Equity in a Given Year versus Interest Rate



Method of Equity Extraction, by Year

Source: Bhutta and Keys (2016) Interest Rates and Equity Extraction During the Housing Boom

9.3 A Model of Constrained Mortgage Finance Decisions

- Again consider a two period model where the homeowner has already chosen a house, H
- The house is purchased in period 1 at P_1^h, and is sold in period 2 at price P_2^h
- Here, the homeowner is restricted in the amount that can be borrowed, B

$$\begin{aligned}
\max_{C_1, C_2, B} \quad & \log C_1 + \beta \log C_2 \\
\text{s.t.} \quad & C_1 + P_1^h H = Y_1 + B \\
& C_2 + RB = Y_2 + (1-\delta) P_2^h H \\
& B \leq \bar{\theta} P_1^h H
\end{aligned}$$

- The final inequality is a loan to value constraint
- The amount borrowed cannot exceed a fraction $\bar{\theta}$ of the value of the house
- We can rewrite the budget constraints in terms of the LTV borrowing ratio $\theta = \frac{B}{P_1^h H}$
- For period 1:

$$\begin{aligned}
C_1 + P_1^h H &= Y_1 + B \\
C_1 + P_1^h H &= Y_1 + \frac{B}{P_1^h H} P_1^h H \\
C_1 + P_1^h H &= Y_1 + \theta P_1^h H
\end{aligned}$$

- For period 2:

$$\begin{aligned}
C_2 + RB &= Y_2 + (1-\delta) P_2^h H \\
C_2 + R\frac{B}{P_1^h H} P_1^h H &= Y_2 + (1-\delta) P_2^h \\
C_2 + R\theta P_1^h H &= Y_2 + (1-\delta) P_2^h H
\end{aligned}$$

- And where the LTV choice must be less than the maximum LTV: $\theta \leq \bar{\theta}$
- To illustrate the importance of the LTV constraint, we will make a figure in (θ, C_2)-space
  - This is similar to our previous figures in (C_1, C_2)-space
  - θ governs the amount borrowed, which has a direct effect on C_1

- Take the period 2 budget constraint:

$$C_2 + R\bar{\theta}P_1^h H = Y_2 + (1-\delta)P_2^h H$$

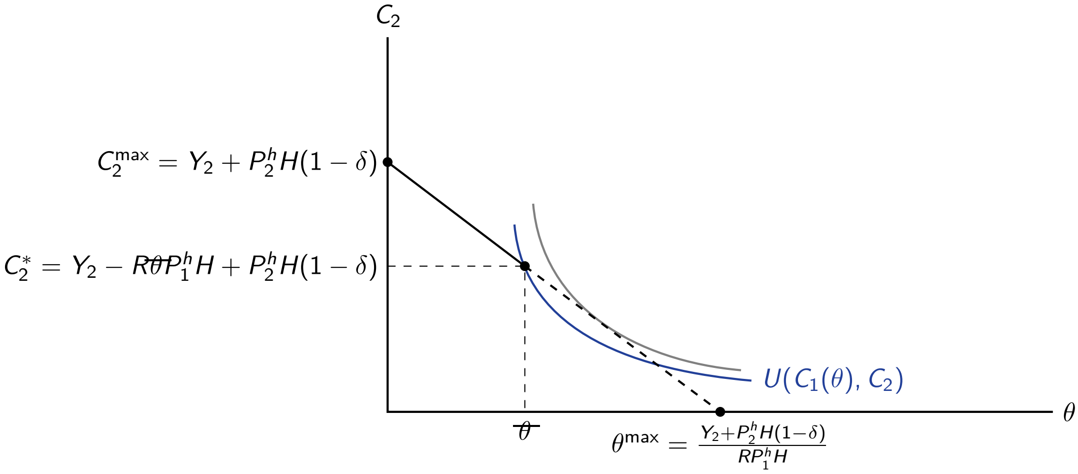
- When the household borrows nothing, $\theta = 0$ and maximum consumption is:

$$C_2^{\max} = Y_2 + (1-\delta)P_2^h H$$

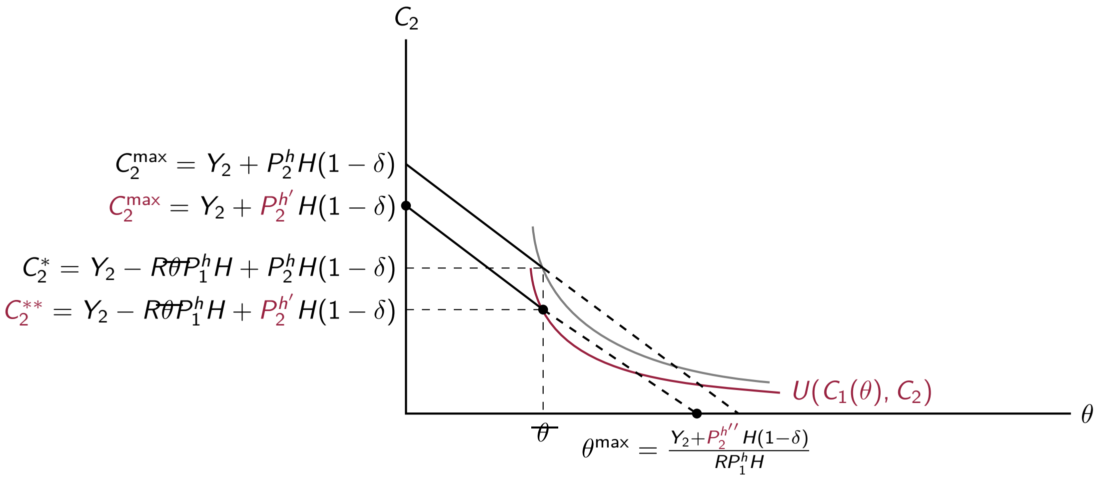
- If there were no constraint on the LTV choice and $C_2 = 0$, then the maximum LTV would be:

$$\theta^{\max} = \frac{Y_2 + (1-\delta)P_2^h H}{RP_1^h H}$$

- Because of the LTV constraint ($\theta \leq \bar{\theta}$), the household is restricted in its borrowing
- This yields a truncated budget constrain in $(\theta - C_2)$-space
- Note: Higher θ implies higher C_1, so the indifference curve is convex in $(\theta - C_2)$-space
- When constrained, consume more in period 2 and less in period 1 than if unconstrained



- Now consider a fall in the period 2 house price: $P_2^h \to P_2^{h'}$



- What is the difference between constrained vs. unconstrained households?
- The optimal consumption choices for each type of household are:

$$C_2^{\text{con}} = Y_2 - R\bar{\theta}P_1^h H + (1-\delta)P_2^h H, \qquad C_2^{\text{unc}} = \frac{\beta R}{1+\beta}\left(Y_1 - P_1^h H + \frac{Y_2 + (1-\delta P_2^h H)}{R}\right)$$

- And the consumption responses to P_2^h are:

$$\frac{\partial C_2^{\text{con}}}{\partial P_2^h} = (1-\delta)H, \qquad \frac{\partial C_2^{\text{unc}}}{\partial P_2^h} = \frac{\beta}{1+\beta}(1-\delta)H$$

- Constrained consumption much more sensitive than unconstrained consumption!

10 Housing Booms and Busts

10.1 A Simple Model of Mortgage Credit and Housing Booms and Busts

- How do credit conditions affect the housing marker?
- Consider a simple model of a household that purchases housing using a mortgage
- The size of the mortgage is determined by credit conditions:
  - The cost of borrowing, i.e. the interest rate
  - The maximum Loan-to-Value constraint on mortgage borrowing
- Housing market equilibrium:
  - House prices adjust to ensure that housing demand equals housing supply

The household problem is:

$$\begin{aligned}
\max_{C_t, C_{t+1}, B_{t+1}, H_{t+1}} \quad & u(C_t) + \beta[u(C_{t+1}) + v(H_{t+1})] \\
\text{s.t.} \quad & C_t + P_t H_{t+1} = Y_t + B_{t+1} \\
& C_{t+1} + (1 + r_{t+1})B_{t+1} = Y_{t+1} + (1-\delta)P_{t+1}H_{t+1} \\
& B_{t+1} \leq \theta_t P_t H_{t+1}
\end{aligned}$$

- Choose housing at time t to be enjoyed at time $t+1$
- Sell housing at time $t+1$
- Borrow B_{t+1} to finance housing, subject to a maximum LTV constraint
- Make three very useful simplifying assumptions:
  - Linear utility in consumption: $u(C) = C$
  - Log utlity in housing: $v(H) = \log H$
  - Household is always constrained, i.e. always borrowing as much as allowed by the LTV constraint

$$\begin{aligned}
\max_{C_t, C_{t+1}, B_{t+1}, H_{t+1}} \quad & C_t + \beta[C_{t+1} + \log H_{t+1}] \\
\text{s.t.} \quad & C_t + P_t H_{t+1} = Y_t + B_{t+1} \\
& C_{t+1} + (1 + r_{t+1})B_{t+1} = Y_{t+1} + (1-\delta)P_{t+1}H_{t+1} \\
& B_{t+1} = \theta_t P_t H_{t+1}
\end{aligned}$$

- The Lagrangian function:

$$\begin{aligned}
\mathcal{L} = {} & C_t + \beta[C_{t+1} + \log H_{t+1}] + \lambda_t(Y_t + B_{t+1} - C_t - P_t H_{t+1}) \\
& + \lambda_{t+1}(Y_{t+1} + (1-\delta)P_{t+1}H_{t+1} - C_{t+1} - (1+r_{t+1})B_{t+1}) + \mu_t(\theta_t P_t H_{t+1} - B_{t+1})
\end{aligned}$$

- λ_t, λ_{t+1} are Lagrange multipliers on the budget constraints
- μ_t is the Lagrange multiplier on the LTV constraint
- The first order conditions are:

$$\begin{aligned}
C_t : \quad & 1 = \lambda_t \\
C_{t+1} : \quad & \beta = \lambda_{t+1} \\
B_{t+1} : \quad & \lambda_t = \lambda_{t+1}(1 + r_{t+1}) + \mu_t \\
H_{t+1} : \quad & \lambda_t P_t = \beta \frac{1}{H_{t+1}} + \lambda_{t+1}(1-\delta)P_{t+1} + \mu_t \theta_t P_{t+1}
\end{aligned}$$

- Combining the first order conditions, and repeating the LTV constraint, we have:

$$
\begin{aligned}
1-\mu_t &= \beta(1+r_{t+1}) && \text{Consumption Euler Equation}\\
P_t &= \frac{\beta}{1-\mu_t\theta_t}\left(\frac{1}{H_{t+1}}+(1-\delta)P_{t+1}\right) && \text{Housing Euler Equation}\\
B_{t+1} &= \theta_t P_t H_{t+1} && \text{LTV Constraint}
\end{aligned}
$$

- Note μ_t is the Lagrange multiplier on the borrowing constraint
- It tells us the marginal value of an extra dollar borrowed to finance housing

Housing Market Equilibrium

- Housing market equilibrium:

$$\underbrace{H_{t+1}}_{\text{Housing Demand}} = \underbrace{\bar{H}}_{\text{Housing Supply}}$$

- The house price p_t adjusts to ensure housing market clears in each period t

Model Experiment: Expansion of Mortgage Credit

- First, find the steady state of the model
- Assume that all variables are the same forever eg. $r_t = t_{t+1} = r$
- Our model equations in the steady state are:

$$1-\mu = \beta(1+r) \tag{10.1}$$

$$P = \frac{\beta}{1-\mu\theta}\left(\frac{1}{\bar{H}}+(1-\delta)P\right) \tag{10.2}$$

$$B = \theta P\bar{H} \tag{10.3}$$

- Solve for the house price P (using equation (10.1) and equation (10.2)):

$$P = \frac{\beta}{\bar{H}((1-\theta)(1-\beta)+\beta(\theta r+\delta))}$$

- Solve for mortgage debt (using equation (10.3))

$$B = \frac{\beta\theta}{((1-\theta)(1-\beta)+\beta(\theta r+\delta))}$$

- First, consider the effect of a change in the interest rate r

$$
\begin{aligned}
\frac{\partial P}{\partial r} &= -\bar{H}\beta\theta \times \frac{\beta}{(\bar{H}((1-\theta)(1-\beta)+\beta(\theta r+\delta)))^2} < 0\\
\frac{\partial B}{\partial r} &= -\beta\theta \times \frac{\beta\theta}{((1-\theta)(1-\beta)+\beta(\theta r+\delta))^2} < 0
\end{aligned}
$$

- Decrease in interest rates leads to: 1. increase in house prices; 2. increase in mortgage debt
  - Lower mortgage finance costs increase housing demand
  - With fixed housing supply $\bar{H}$, prices must increase
  - To finance higher-priced houses, households must increase borrowing
- Second, consider the effect of a change in the maximum LTV ratio θ

$$
\begin{aligned}
\frac{\partial P}{\partial \theta} &= -\bar{H}(1-\beta(1+r)) \times \frac{\beta}{(\bar{H}((1-\theta)(1-\beta)+\beta(\theta r+\delta)))^2}\\
\frac{\partial B}{\partial \theta} &= \frac{\beta\theta}{((1-\theta)(1-\beta)+\beta(\theta r+\delta))} + (1-\beta(1+r)) \times \frac{\beta\theta}{((1-\theta)(1-\beta)+\beta(\theta r+\delta))^2}
\end{aligned}
$$

- If $\beta(1+r) > 1$, households are patient and/or have high costs of borrowing
  - Demand for housing does not rise with increased borrowing opportunities

$$\frac{\partial P}{\partial \theta} < 0, \qquad \frac{\partial B}{\partial \theta} < 0$$

- If $\beta(1+r) < 1$, households are impatient and/or have with costs of borrowing
  - Demand for housing rises with increased borrowing opportunities
  - Seems most likely case since we observe households borrow a lot to finance housing

$$\frac{\partial P}{\partial \theta} > 0, \qquad \frac{\partial B}{\partial \theta} > 0$$

10.2 Dynamics of Mortgage and Housing Markets

Dynamic Model Experiment: Expansion of Mortgage Credit

- Now study the dynamics of the model in response to an expansion of mortgage credit
- We will consider effect of our two shocks:
  1. A decrease in the mortgage interest rate
  2. An increase in the maximum LTV ratio on mortgage borrowing
- Recall the FOCs/optimal decisions of the household:

$$\begin{aligned} 1-\mu_t &= \beta(1+r_{t+1}) && \text{Consumption Euler Equation} \\ P_t &= \frac{\beta}{1-\mu_t\theta_t}\left(\frac{1}{H_{t+1}} + (1-\delta)P_{t+1}\right) && \text{Housing Euler Equation} \\ B_{t+1} &= \theta_t P_t \bar{H} && \text{LTV Constraint} \end{aligned}$$

- To begin, suppose "beliefs" about future house prices are held fixed: $P_{t+1} = P$
- Trace out effect of a decline in r_{t+1}

$$\begin{aligned} 1-\underbrace{\mu_t}_{\uparrow} &= \beta(1+\underbrace{r_{t+1}}_{\downarrow}) && \text{Consumption Euler Equation} \\ \underbrace{P_t}_{\uparrow} &= \frac{\beta}{1-\underbrace{\mu_t}_{\uparrow}\theta_t}\left(\frac{1}{H_{t+1}} + (1-\delta)P_{t+1}\right) && \text{Housing Euler Equation} \\ \underbrace{B_{t+1}}_{\uparrow} 1 &= \theta_t \underbrace{P_t}_{\uparrow} H_{t+1} && \text{LTV Constraint} \end{aligned}$$

  - Lower interest rates $\Rightarrow$ increase marginal utility of extra dollar borrowed
  - Increased demand for housing $\Rightarrow$ with fixed supply, current prices must rise
  - Borrowing increases to pay for higher price of houses
- Trace out effect of a rise in θ_t:

$$\begin{aligned} 1-\mu_t &= \beta(1+r_{t+1}) && \text{Consumption Euler Equation} \\ \underbrace{P_t}_{\uparrow} &= \frac{\beta}{1-\mu_t\underbrace{\theta_t}_{\uparrow}}\left(\frac{1}{H_{t+1}} + (1-\delta)P_{t+1}\right) && \text{Housing Euler Equation} \\ \underbrace{B_{t+1}}_{\uparrow} &= \underbrace{\theta_t}_{\uparrow}\underbrace{P_t}_{\uparrow}\bar{H} && \text{LTV Constraint} \end{aligned}$$

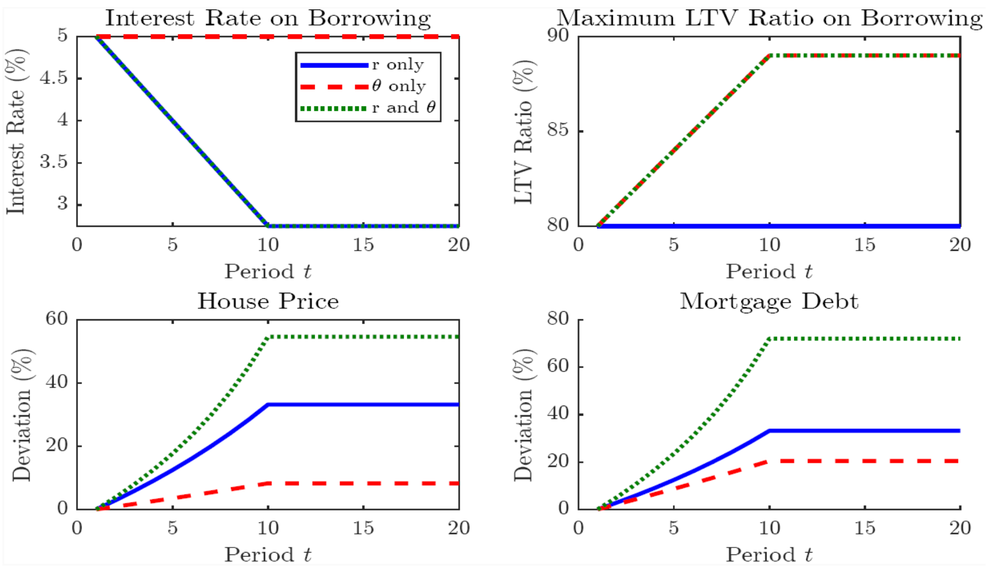
  - Higher LTV borrowing limits $\Rightarrow$ increase amount that can be borrowed

  - Since households always borrow as much as they can, borrowing increases
  - Demand for housing increases in line with borrowing $\Rightarrow$ with fixed supply, prices rise

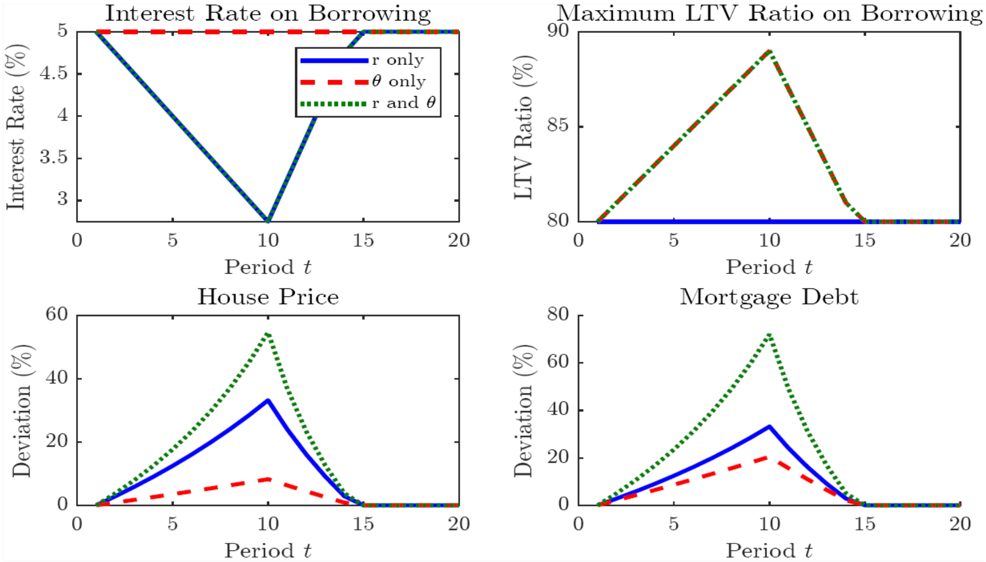
- Now to solve the model in practice (e.g. on a computer!)
- Create **exogenous** paths (i.e shocks) for r_{t+1} and θ_t
- Assume model is in new steady state at some point in the future (e.g. some period T)
- Iterating backwards from T, take P_{t+1} as given, then solve for P_t, B_{t+1}:

$$\begin{aligned} 1 - \mu_t &= \beta(1 + r_{t+1}) \\ P_t &= \frac{\beta}{1 - \mu_t \theta_t}\left(\frac{1}{H} + (1 - \delta)P_{t+1}\right) \\ B_{t+1} &= \theta_t P_t \bar{H} \end{aligned}$$

- For our dynamic experiments:
  - The shocks last for 8 quarters (i.e. two years)
  - The shocks are unanticipated each period (i.e. a complete surprise)
  - We run experiments separately for: (1) interest rate shocks, (2) LTV ratio shocks, (3) both interest rates and LTV ratio shocks



- The boom is the same as our previous experiment
  - Each model period represents 1 quarter
  - The credit expansion shock lasts for 8 quarters (i.e. two years)
  - The shocks are unanticipated each period (i.e. a complete surprise)
- But now for the credit bust shock:
  - In the 9th quarter, interest rates and the LTV ratio revert to their initial steady state values in just 4 quarters (i.e. one year)


Interest Rate on Borrowing
Interest Rate (%)
Period t
r only
θ only
r and θ
Maximum LTV Ratio on Borrowing
LTV Ratio (%)
Period t
House Price
Deviation (%)
Period t
Mortgage Debt
Deviation (%)
Period t

11 Welfare and Redistribution Through Asset Price Movements

11.1 A Simple Model of Welfare Gains and Losses From Asset Price Movements

- Simple model of asset choice and asset price movements
- Hold initial asset stock, rebalance asset portfolio, earn cash flow next period
- A household's problem is:

$$\begin{aligned}\underbrace{V}_{\text{Value Function}} &= \max_{C_1, C_2, A_2} U(C_1) + \beta U(C_2) \\ \text{s.t} \quad & C_1 + (A_2 - A_1)P_1 = Y_1 \\ & C_2 = Y_2 + A_2 D_2\end{aligned}$$

- where:
  - P_1 = price of asset when buying/selling at time 1
  - D_2 = cash flow/dividends from asset at time 2
  - $(A_2 - A_1)$ = net transactions of the asset in period 1
  - V = Value Function, the total utility of the consumption and asset choices for the household
- The Lagrangian equation is:

$$\mathcal{L} = U(C_1) + \beta U(C_2) + \lambda_1(Y_1 - C_1 - (A_2 - A_1)P_1) + \lambda_2(Y_2 + A_2 D_2 - C_2)$$

- The first order conditions are:

$$\begin{aligned} C_1 : &\quad Y'(C_1) = \lambda_1 \\ C_2 : &\quad \beta U'(C_2) = \lambda_2 \\ A_2 : &\quad \lambda_1 P_1 = \lambda_2 D_2 \end{aligned}$$

- And, combining the FOCs, we find the Euler equation:

$$\underbrace{U'(C_1)}_{\text{Marginal Utility of } C_1} = \underbrace{\beta U'(C_2)}_{\text{Marginal Utility of } C_2} \times \underbrace{\frac{D_2}{P_1}}_{\text{Return on asset}}$$

- Recall, Euler equation describes optimal inter-temporal decisions of the household
- Characterises trade-off between consumption today and investment for consumption tomorrow
- Note that the price of the asset P_1 directly affects asset returns $\frac{D_2}{P_1}$
- All else equal, higher prices reduce returns which discourages further investment in the asset
- But asset price P_1 has **indirect** effects through valuation of household wealth
- Recall the period 1 budget constraint is:

$$C_1 = Y_1 + \underbrace{P_1 A_1 - P_1 A_2}_{\text{Net change in asset position}}$$

- E.g. an increase in the price P_1 increases the value of the households initial portfolio: $P_1 A_1$
- This change in portfolio values is called the wealth effect
- We want to understand the welfare gains/losses from a change in asset prices
- Overall, are households better off or worse off when asset prices rise?
- Depends on size of effects on returns and wealth

  - Higher asset prices reduce asset returns, making households worse off
  - Higher asset prices increase value of initial wealth, making households better off

- What is the effect of an increase in asset prices P_1?

$$\begin{aligned}\frac{\partial V}{\partial P_1} &= \frac{\partial U(C_1)}{\partial C_1} \times \frac{\partial C_1}{\partial P_1} + \frac{\partial U(C_2)}{\partial C_2} \times \frac{\partial C_2}{\partial P_1} \\ &= U'(c_1) \times (A_1 - A_2)\end{aligned}$$

- Where $(A_1 - A_2)$ is net asset portfolio transactions
- Another way to understand this: rewrite the budget constraint as:
- And the budget constraints:

$$\begin{aligned}C_1 &= Y_1 + (A_1 - A_2)P_1 \\ &= Y_1 + \frac{P_1}{P_0}P_0A_1 - \frac{P_1}{D_2}D_2A_2 \\ &= Y_1 + R_1P_0A_1 - \frac{1}{R_2}D_2A_2\end{aligned}$$

- Where P_0 is the initial price assets were purchased at, R_1 is the return on assets bought prior to time 1, and R_2 is the return on assets purchased at time 1
- Now what is the effect of an increase in asset prices P_1?

$$\begin{aligned}\frac{\partial V}{\partial P_1} &= \frac{\partial U(C_1)}{\partial C_1} \times \left(\underbrace{\frac{\partial C_1}{\partial R_1} \times \frac{\partial R_1}{\partial P_1}}_{\text{Wealth effect}} + \underbrace{\frac{\partial C_1}{\partial R_2} \times \frac{\partial R_2}{\partial P_1}}_{\text{Investment returns effect}} \right) \\ &= \underbrace{U'(C_1) \times A_1P_0 \times \frac{\partial R_1}{\partial P_1}}_{\text{Wealth effects}} + \underbrace{U'(C_1) \times A_2D_2 \times A_1^{-2} \times \frac{\partial R_1}{\partial P_1}}_{\text{Investment returns effect}}\end{aligned}$$

- Changes in P_1 may have different implications for welfare via returns:
  - Higher P_1 increases returns in period 1 (i.e. a wealth effect)
  - Higher P_1 increases returns in period 2 (holding D_2 constant)
- Important! Overall effect on welfare is not the same as wealth effect
  - Welfare effect $= \frac{\partial V}{\partial P_1}$
  - Wealth effect $= U'(C_1) \times A_1P_0 \times \frac{\partial R_1}{\partial P_1}$
- The welfare effect of asset price movements is summarised by:

$$U'(c_1) \times (A_1 - A_2)$$

- First, consider a **net seller** of assets: $A_1 > A_2$

$$\frac{\partial V^{seller}}{\partial P_1} = U'(C_1) \times (A_1 - A_2) > 0$$

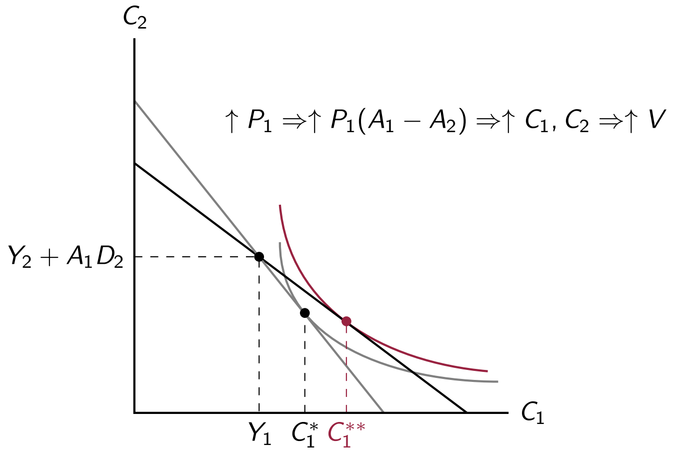
- Earn higher returns on assets sold, net gain
- Second, consider a **net buyer** of assets: $A_1 < A_2$

$$\frac{\partial V^{buyer}}{\partial P_1} = U'(C_1) \times (A_1 - A_2) < 0$$

- Earn lower returns on asset investments, net loss

Graphical Illustration

- Increase in P_1 tilts budget constraint out
- Can spend down more net asset wealth
- For **net seller** of assets, consume more in both periods
- Higher indifference curve $\Leftrightarrow$ higher utility $\Leftrightarrow$ higher welfare



- Increase in P_1 tilts budget constraint out
- Investment in net assets is more expensive
- For **net buyer** of assets, consume less in both periods
- Lower indifference curve $\Leftrightarrow$ lower utility $\Leftrightarrow$ lower welfare