ECOS3021

Business Cycles and Asset Markets

Notes

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Contents

Introduction

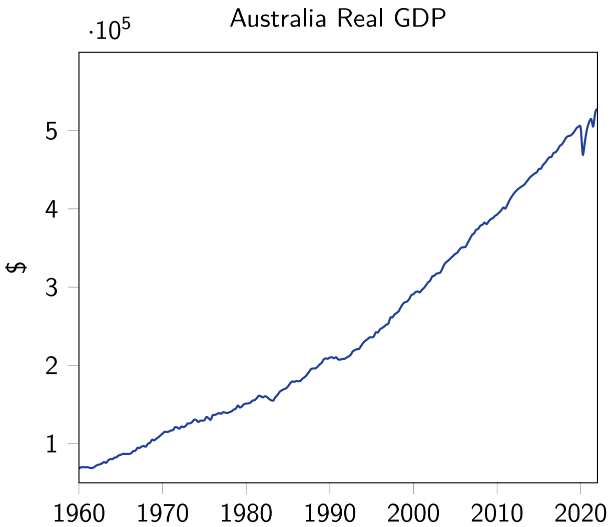
What are Business Cycles

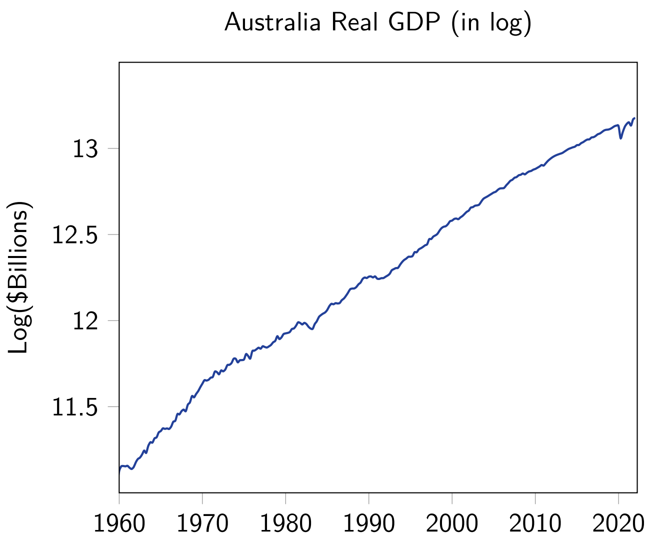
What is the business cycle?

- Distinguish long-run macroeconomic growth from short-run macroeconomic fluctuations
- Business cycles are fluctuations in aggregate- or macro-economic activity
- These fluctuations occur over the short to medium term

Stylised Features

- Trend: Long-run increase in economic activity
- Peak: Short-run/cyclical high in economic activity
- Though: Short-run/cyclical low in economic activity
- Boom/Expansion: Period of increasing economic activity following a recession
- Slump/Recession/Contraction: Period of decreasing economic activity following a boom
- Recovery: Post-recession period of growth that brings economic activity back up to its long-run trend
- Consider real Gross Domestic Product (GDP) for Australia, observed at a quarterly frequency





- Let y_t be real GDP at time t
- Let Δy_t be the growth rate of y (in percent) between dates $t-1$ and t

$$\begin{aligned}
\Delta y_t &= \frac{y_t - y_{t-1}}{y - t - 1} \\
\Delta y_t &= \frac{y_t}{y_{t-1}} - 1 \\
\log(1 + \Delta y_t) &= \log(y_t) - \log(y - t - 1) \\
\Delta y_t &\approx \log(y_t) - \log(y - t - 1)
\end{aligned}$$

- Plotting the log of GDP makes it easier to see growth rates
- In a log GDP plot, the slope characterizes the growth rate

Classical Business Cycles

According to Burns and Mitchell (1946):

- Business cycles are not defined as fluctuations in real GDP but as fluctuations in an undefined measure of "aggregate economic activity". (Why not GDP alone?)
- Dating of business cycle turning points is based on a mixture of mechanically applied rules and ad hoc judgments (e.g. NBER's Business Cycle Dating Committee). Requires careful interpretation of data!
- Harding and Pagan (2002) presented a now well-known algorithm for identifying turning points in classical business cycles using quarterly data (known as the BBQ procedure).
- A peak at time t occurs if:

$$[(y_t - y_{t-2}) > 0, (y_t - y_{t-1}) > 0], \text{ and}$$
$$[(y_{t+2} - y_t) < 0, (y_{t+1} - y_t) < 0],$$

- A trough at time t occurs if:

$$[(y_t - y_{t-2}) < 0, (y_t - y_{t-1}) < 0], \text{ and}$$
$$[(y_{t+2} - y_t) > 0, (y_{t+1} - y_t) > 0],$$

Growth Business Cycles

- According to Robert Lucas (1977), "aggregate fluctuations around the trend or growth path"
- "Refers to the same thing (as Classical cycles) in some detrended series"
- A growth recession requires a relative decline (i.e. growth can still be positive) in real GDP, but below the long-term growth trend
- A complete growth cycle in industralized countries typically takes between 18 months and 8 years, depending on how the trend is defined
- No clear asymmetry in growth cycles. (Why might this be?)
- Think of a time series yt with secular (i.e. uncorrelated) components decomposed as

$$\log y_t = g + c_t$$

- g is the long-run growth or trend component
- c_t is the cyclical (business cycle) component

How do we detrend a time series with growth components?

1. Difference the series. Let y_t be a quarterly time series

$$\begin{aligned} \text{Quarterly difference:} \quad & \log y_t - \log y_{t-1} = g + c_t - (g + c_{t-1}) = c_t - c_{t-1} \\ \text{Year-on-year difference:} \quad & \log y_t - \log y_{t-4} = g + c_t - (g + c_{t-4}) = c_t - c_{t-4} \end{aligned}$$

   - Differencing removes the growth component, leaving only fluctuations due to the cyclical components
   - However, differencing tends to remove too much information and displays short-term volatility. So not suitable to obtain medium-term movements.
   - But, easy and useful to interpret and assess economic conditions.

2. Assume the trend is a deterministic function of time
   - $y_t = g_t + c_t$
   - where the growth component is given by: $g_t = g + \alpha \cdot t + \beta \cdot t^2$
   - t is just time (e.g. the year 1990, 1991, 1992, etc)
   - g is a constant, α and β are coefficients on the linear and quadratic terms

3. Assume a stochastic trend (i.e. a random trend). Find via a filtering algorithm

- Many filters are borrowed from engineering applications, e.g., filtering noise from a signal
- Examples of filters in macroeconomics:
  - Hodrick-Prescott (1997) filter
  - Band-pass filter
  - Forecasting filters (e.g. Hamilton, 2017)
- These filters are used to find a smooth trend in the data visually similar to the trend that one can obtain with a free-hand drawing
- The cycle component is then consistent with the growth cycle definition of Lucas (1977)

How do we Understand the Price of an Asset?

- Several methods for "valuing" or "pricing" an asset:
  - **Discounted Cashflow Valuation**: present value of the expected cash flows of an asset
  - **Relative Valuation**: estimate value from price/value of similar or comparable assets
  - **Contingent Claim (Option) Valuation**: positive payoff if underlying value is higher than some "strike price" (e.g. a startup either starts to make money or its fixed assets are liquidated)

Asset Prices as Discounted Cash Flows

- The price of an asset is equal to its stream of cash flows, discounted by the interest rate

$$\text{Price}_t = \text{Cash}_t + \frac{\text{Cash}_{t+1}}{(1+r)} + \frac{\text{Cash}_{t+2}}{(1+r)^2} + \cdots + \frac{\text{Cash}_T}{(1+r)^T}$$

- Subscript t denotes the time (e.g. weeks, months, years)
- $Cash_t$ is the cash flow received from the asset at time t
- r is the interest rate
- Subcript T is the final period in which cash flows are received from the asset
- Why do we divide future cash flows by the (gross) interest rate, $1+r$?
  - Rather than buy the asset, could put money into bank account and wait for interest to accrue
  - These forgone interest earnings are the opportunity cost of investing in the asset
  - So we "discount" the value of future cash flows by the interest we could have earned
- How might the price of assets be affected by the business cycle?
  - Cash flows fluctuate over the business cycle
  - Interest rates fluctuate over the business cycle

Real Business Cycles and the RBC Model

Stylised Facts About Business Cycles

- We want to gather some "stylised" facts about business cycles
- Looking for statistics that explain what typically takes place during a business cycle
- But we should be aware that "every recession" is special in its own way
- And the existence of stylised facts does not mean that business cycles are predictable

Cyclical Relations: Definitions

A macroeconomic variable is:

- Pro-cyclical: if deviations from trend are positively correlated with real GDP deviations from its own trend
- Counter-cyclical: if deviations from trend are negatively correlated with real GDP deviations from its own trend
- Acyclical: if deviations from trend for each variable are not correlated

Time Series (Cyclical) Relations

1. Correlation (or, co-movement)
   - Measure the degree ofcontemporaneous synchronisation between any two variables
2. Leads and Lags
   - Measure the degree of synchronisation between any two variables across time
   - We measure these relationships via cross-time correlations:
     - $Corr(x_{t+j}, y_t)$ with $j < 0$ indicates x is a leading variable (e.g. x_{t-1} increases before y_t)
     - $Corr(x_{t+j}, y_t)$ with $j > 0$ indicates x is a leading variable (e.g. x_{t+1} increases before y_t)

Time Series Properties

3. Variability (a.k.a. volatility)
   - Measures the amplitude of deviations from a trend or mean
   - Measure variability via the standard deviation of a variable
4. Persistence
   - Measures the time dependence of a variable (i.e. high today $\Rightarrow$ high tomorrow)
   - Measure persistence via the autocorrelation function
   - $Corr(y_t, y_{t-j})$ with $j > 0$ (lags of y or $j < 0$ leads of y)

Documenting Business Cycle Facts

- Employment, consumption, investment are allpro-cyclical to GDP
- Employment and total hours worked fluctuate almost as much as GDP
- Consumption (of non-durables and services) is smooth and fluctuates less than GDP
- Investment fluctuates much more than GDP
- Productivity is slightly pro-cyclical to GDP
- Government expenditure is uncorrelated with GDP (acyclical)
- Net exports are pro-cyclical to GDP

How do we document and report these 'stylised facts'?

- Detrend the (log) time series data, removing growth components
- Compute summary statistics from detrended data (i.e. cyclical components)

Summary Statistics

1. For individual variables/time series, compute:
   - Mean (e.g. mean growth rate)
   - Standard deviation (i.e. volatility)
   - Autocorrelation (i.e. persistence)

2. For pairs of variables (e.g. consumption and GDP)
   - Relative standard deviation: $S.D.(x_t)/S.D.(y_t)$
   - Cross correlation (co-variance) at various leads/lags - $Corr(x_{t+j}, y_t)$ for j negative (x leads) or positive (x lags)

Brief History of Business Cycle Theories

- Business cycle theories of early 20th century quantitatively analysed economic fluctuations using mathematical and statistical approaches
- This research agenda was led by Ragnar Frisch and Jan Tinbergen, the first winners of the Nobel Prize in Economics
- This work on business cycles begun before John Maynard Keynes became one of the most well-known names in the study of macroeconomic fluctuations (i.e. the father of "Keynesian" economics)
- From the 1970s, Real Business Cycle (RBC) theory attempted to quantitatively explain macroeconomic fluctuations via shocks to aggregate production technology (i.e. productivity)
- This followed the tradition of Classical and Neo-Classical Economics
  - Households and firms behave as if they make rational choices subject to constraints
  - Macroeconomic outcomes are determined byequilibrium and market clearing
  - The "classical dichotomy": nominal variables do not affect real variables
- RBC model structure follows from Optimal Growth Theory (e.g. the Solow-Swan model)
- RBC models incorporate Neo-Classical growth with stochastic shifts or shocks as the driving force behind cyclical macroeconomic fluctuations
- The RBC research agenda uses the stochastic growth model to try to explain fluctuations that can be quantitatively assessed
- Another aim of RBC economists was to build small laboratories in which government policies could be tested
- Modern macroeconomic models used at central banks are rooted in the RBC framework
  - Federal Reserve Board ("FRB/US"); Norges Bank ("NEMO"); Swedish Riksbank ("RAMSES II"); Bank of Canada ("TOTEM"); Reserve Bank of Australia ("MARTIN"); Reserve Bank of New Zealand ("NZSIM")
- RBC models have evolved into Dynamic Stochastic General Equilibrium (DSGE) models
- Inspired by Robert Lucas (1977), Kydland and Prescott (1982) aimed to study growth and fluctuations in a single model framework asking the following question:
- "Can business cycle fluctuations occur as a natural consequence of the competitive economy where agents make optimal inter-temporal resource allocation decisions in response tostochastic shifts in technology and preferences?"
  - If the answer is **No** (as most economists at the time believed):
    * Market co-ordination failure
    * Large welfare losses from market outcomes
    * Role for active macroeconomic stabilization policy (e.g. Keynesian stimulus)
  - If answer is **Yes** (as RBC economists believed):
    * Business cycles are "efficient"
    * Negligible welfare costs from market outcomes
    * Active stabilisation policies can be disruptive/destabilising
- Why real business cycles?

  - 'Real' as opposed to 'nominal' or monetary forces
- Why are real business cycles efficient?
  - No economic frictions to distort optimal decisions
- Modern DSGE models incorporate many nominal/monetary features:
  - Price rigidity, nominal shocks, monetary and fiscal policies
- Modern DSGE models incorporate many economic rigidities/frictions:
  - Imperfect competition, search frictions, credit market frictions

Intra-Temporal Households in the RBC Model

Choice between Work and Leisure

- Households must decide how much to work (in order to earn income) and how much leisure to enjoy.
- The more a household works, the more income they have to spend, but the less leisure time they can enjoy (there are only so many hours in a day!)
- Leisure is a normal good
- The static optimisation problem is to maximise utility subject to a static budget constraint and a time endowment constraint
- A household's problem is to choose consumption C and leisure L

$$\begin{aligned} \max_{C,L} \quad & U(C) + V(L) \\ \text{s.t.} \quad & C = w \cdot N^S + \Pi \quad \text{Budget constraint} \\ & L + N^S = 1 \quad \text{Time endowment} \end{aligned}$$

- Where
  - N^S is hours worked, or the amount of labour supplied by the household
  - $L + N^S = 1$ refers to the total time available in a day
  - Π are the dividends paid out by the firms owned by households
  - $U'(C) > 0, U''(C) < 0$ implies diminishing marginal utility of consumption
  - $V'(N) > 0, V''(N) < 0$ implies diminishing marginal utility of leisure
- For tractability, let's simplify functional forms:

$$U(C) = \log(C), \quad V(L) = b\log(L)$$

- So the household problem becomes:

$$\begin{aligned} \max_{C,L} \quad & \log(C) + b\log(L) \\ \text{s.t.} \quad & C = w \cdot N^S + \Pi \quad \text{Budget constraint} \\ & L + N^S = 1 \quad \text{Time endowment} \end{aligned}$$

- Substitute the time endowment and the budget constraint into the utility function:

$$\max_{N^S} \log(wN^S + \Pi) + b\log(1 - N^S)$$

- Now take the derivative of the objective function with respect to N^S:

$$
\begin{aligned}
&= \frac{\partial \log(wN^S + \Pi)}{\partial N^S} + \frac{\partial b \log(1 - N^S)}{\partial N^S} \\
&= \frac{\partial \log(wN^S + \Pi)}{\partial (wN^S + \Pi)} \times \frac{\partial (wN^S + \Pi)}{\partial N^S} + \frac{\partial b \log(1 - N^S)}{\partial (1 - N^S)} \times \frac{\partial (1 - N^S)}{\partial N^S} \\
&= \frac{1}{(wN^S + \Pi)} \times w + \frac{b}{1 - N^S} \times (-1) \\
&= \frac{w}{c} - \frac{b}{1 - N^S}
\end{aligned}
$$

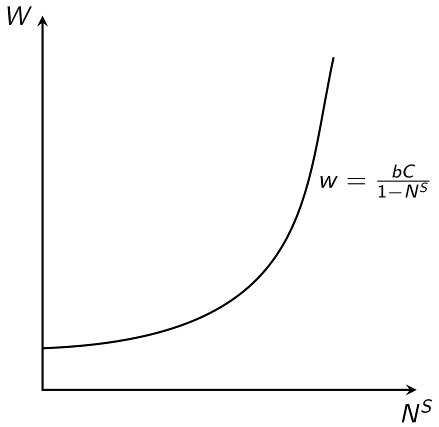
- Setting the derivative equal to zero yields the First Order Condition:

$$\underbrace{\frac{w}{c}}_{\substack{\text{Marginal Benefit of} \\ \text{Labour Supplied}}} - \underbrace{\frac{b}{1 - N^S}}_{\substack{\text{Marginal Cost of} \\ \text{Labour Supplied}}} = 0$$

- We can rewrite this as the Labour Supply curve of the household:

$$\underbrace{w}_{\substack{\text{Marginal Benefit} \\ \text{of Labour Supplied} \\ \text{in Consumption Units}}} = \underbrace{\frac{bC}{1 - N^S}}_{\substack{\text{Marginal Cost} \\ \text{of Labour Supplied} \\ \text{in Consumption Units}}}$$

- The labour supply curve slopes up
- So households supply more labour as wages rise

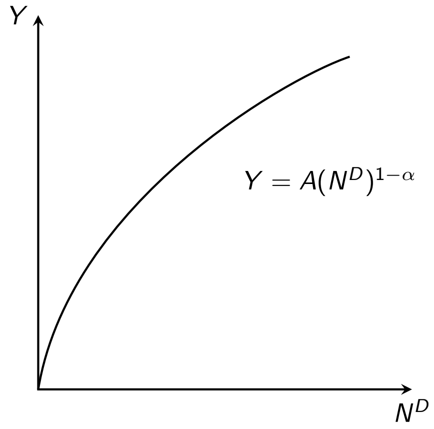


Firms in the Simple RBC Model

- Firms produce output using a production technology:

$$Y = A \times (N^D)^{1-\alpha}$$

- Where A is the exogenous level of technology;
- And where N^D is labour inputs demanded by firms



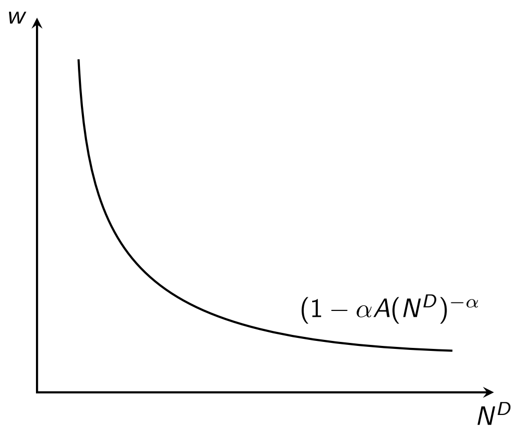
- A competitive firm chooses labour N^D to maximize profit Π (returned to households)

$$\begin{aligned}\Pi &= \max_{N^D} Y - wN^D \\ &= \max_{N^D} A(N^D)^{1-\alpha} - wN^D\end{aligned}$$

- where w is the wage or cost of hiring labour (and is taken as given)
- The first order condition yields:

$$\underbrace{(1-\alpha)A(N^D)^{-\alpha}}_{\substack{\text{Marginal Product}\\\text{of Labour}}} - \underbrace{w}_{\substack{\text{Marginal Cost}\\\text{of Labour}}}$$

- Marginal Product of Labour (MPN) = extra output generated by one additional labour input
- Firms demand less labour as wages increase



Equilibrium in the Simple RBC Model

- The Labour Market clearing condition holds:

$$\frac{bC}{1-N} = w = (1-\alpha)AN^{-\alpha} \tag{2.1}$$

- Aggregate production is determined by technology:

$$Y = AN^{1-\alpha} \tag{2.2}$$

- Firm output (i.e. goods supply) is equal to household consumption (i.e. goods demand):

$$Y = C \tag{2.3}$$

- First substitute equation (2.3) into (2.2), and substitute this into equation (2.1):

$$b\frac{AN^{1-\alpha}}{1-N} = (1-\alpha)AN^{-\alpha}$$

- Second, rearrange and solve for N:

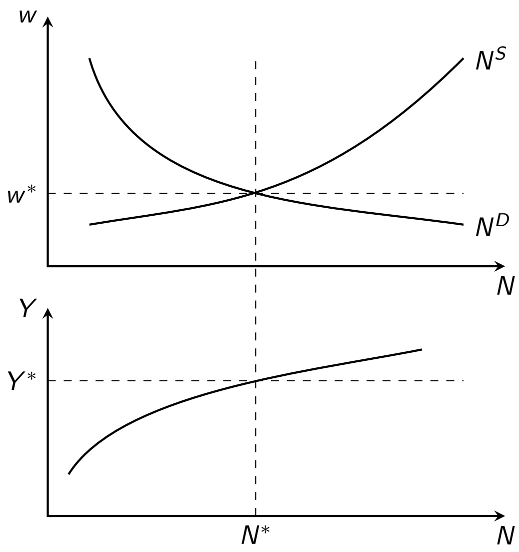
$$N = \frac{(1-\alpha)}{b+(1-\alpha)} \tag{2.4}$$

- Third, substitute into either the labour supply or demand curve to find w:

$$w = (1-\alpha)A\left(\frac{(1-\alpha)}{b+(1-\alpha)}\right)^{-\alpha} \tag{2.5}$$

- Finally use equation (2.4) and (2.2) and (2.3) to solve for Y and C:

$$Y = C = A\left(\frac{(1-\alpha)}{b+(1-\alpha)}\right)^{1-\alpha} \tag{2.6}$$



Business Cycle Fluctuations in the RBC Model

- Macroeconomic fluctuations in early RBC models were driven entirely by changes in aggregate productivity
- In our simple RBC model, changes in productivity A can drive fluctuations in each of the aggregate variables: C, Y, w, N

Limitations of the Simple (Intra-Temporal) RBC Model

- In our simple RBC model, households and firms only make static or intra-temporal decisions
- But these households do not care about the future!
  - inter-temporal decisions
  - Household savings
  - Productive capital
  - Financial assets (or asset prices!)
  - A relationship between the past and the future
  - A serious characterization of aggregate dynamics

Inter-Temporal Choice and the Business Cycle

Simple Inter-Temporal Households in the RBC Model

Choice between Consumption and Saving

- Households must decide how much to consume today, how much to save, and how much to consume tomorrow
- Because savings earn interest (returns), the more resources that are saved today, the more resources are available for consumption in the future
- But households are impatient as they discount the value of future consumption more than the value of current consumption
- The optimisation problem is to maximise life-time utility subject to an inter-temporal budget constraint

A Model of Consumption and Saving

- Assumptions:
  - Earn (net) real interest rate r on savings S
  - Future utility is discounted at the rate β Exogenous income in each period Y_1, Y_2
- Households use their savings to smooth consumption across time
- For now we ignore:
  - Risk
  - Inflation
  - Different types of assets
  - Other asset market participants
- A household chooses current consumption C_1, future consumption C_2, and savings S:

$$\begin{aligned} \max_{C_1, C_2, s} \quad & \log(C_1) + \beta \log(C_2) & \\ \text{s.t.} \quad & C_1 + S = Y_1 & \text{First period budget constraint} \\ & C_2 = Y_2 + S(1+r) & \text{Second period budget constraint} \end{aligned}$$

- Combine the within-period budget constraints:

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

- This is the inter-temporal budget constraint (or, life-time budget constraint)

Household Choice for Consumption and Saving

- The simplified household problem is:

$$\begin{aligned} \max_{C_1, C_2, S} \quad & \log(C_1) + \beta \log(C_2) \\ \text{s.t.} \quad & C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \end{aligned}$$

- The first order condition yield:

$$\underbrace{\frac{1}{C_1}}_{\substack{\text{Marginal Utility of} \\ \text{Consumption in Period 1}}} = \underbrace{(1+r)}_{\text{Return of Savings}} \times \underbrace{\beta \frac{1}{C_2}}_{\substack{\text{Marginal Utility of} \\ \text{Consumption in Period 2}}} \tag{3.1}$$

- This is called the Consumption Euler Equation, which describes efficient inter-temporal consumption choices
- Later, we will see that this equation is fundamental for understanding the price of assets!
- Rearrange equation (3.1) for C_2, then substitute into the inter-temporal budget constraint to find C_1 and C_2:

$$C_1 = \frac{1}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right), \quad c_2 = \frac{\beta(1+r)}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right)$$

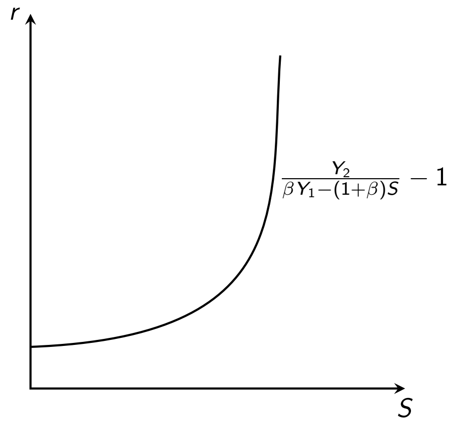
- To find S substitute either C_1 into the first period budget constraint or C_2 into the second period budget constraint:

$$S = \frac{\beta}{1+\beta}Y_1 - \frac{1}{(1+r)(1+\beta)}Y_2$$

- Rewrite the savings function for r:

$$r = \frac{Y_2}{\beta Y_1 - (1+\beta)S} - 1$$

- This represents the household's supply of savings



- When does the household choose to save (i.e. S > 0)?
  - Save when income in period 1 (Y_1) is larger than income in period 2 (Y_2)
  - Save more when the interest rate r is high
- What makes saving valuable?
- Savings transfers resources from periods of high income (when the MU of consumption is low) to periods of low income (when the MU of consumption is high)

Inter-Temporal Households and Capital Accumulation in the RBC Model

- Inter-temporal households generate a supply of savings (or a demand for loans!)
- In the canonical RBC model, households hold physical capital that is then used in production
- Here, we want to think of capital as a productive asset, but one whose return may fluctuate with the business cycle

Household Consumption Choice and Capital Accumulation

- A household chooses current consumptions C_1, C_2, and investment in capital I_1:

$$\begin{aligned}
\max_{C_1,C_2,I_1} \quad & \log(C_1) + \beta \log(C_2) \\
\text{s.t.} \quad & C_1 + I_1 = \Pi_1 + r_1 K_1 && \text{First period budget constraint} \\
& C_2 = \Pi_2 + (1 + r_2 - \delta) K_2 && \text{Second period budget constraint} \\
& K_2 = I_1 + K_1(1-\delta) && \text{Capital accumulation equation}
\end{aligned}$$

- Households are endowed with capital K_1 (cannot be adjusted)
- Households earn (net) real interest r_1, r_2 on their capital holdings
- Capital in period 2 is investment in new capital + undepreciated capital from period 1
- In period 2, after production takes place, households consume remaining capital: $(1-\delta)K_2$
- For simplicity, assume households don't supply labour, but they own firms and receive dividends Π_1, Π_2
- Substitute capital accumulation equation into first period budget constraint:

$$C_1 + K_2 = \Pi_1 + K_1(1 + r_1 - \delta)$$

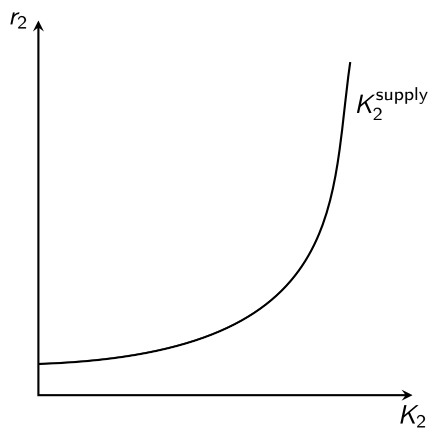
- Now, substitute the budget constraints into the utility function:

$$\max_{K_2} \log(\Pi_1 + K_1(1 + r_1 - \delta) - K_2) + \beta \log(\Pi_2 + K_2(1 + r_2 - \delta))$$

- Taking the FOC with respect to K_2:

$$\frac{1}{c} = \beta(1 + r_2 - \delta)\frac{1}{c_2}$$

- Which is also a Consumption Euler Equation
- Here, the return on savings/capital holding is $(1 + r_2 - \delta)$, but households value capital in the same way they valued savings in section 3.1

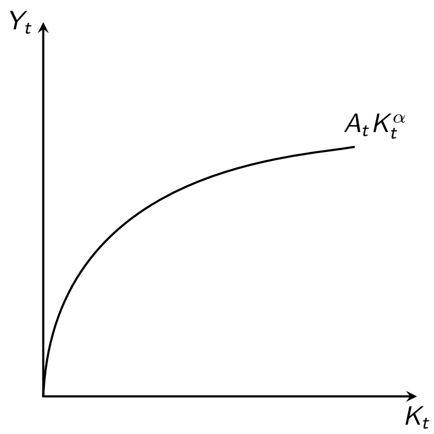


Firms in the Inter-Temporal RBC Model

- Firms produce output using the production technology:

$$Y_t = A_t K_t^{\alpha}, \text{where } t = 1, 2$$

- Where A_t is technology/productivity; and K_t is the capital inputs of firms



Firm's Choice of Capital Inputs

- A competitive firm chooses capital K_2^D to maximise profit Π_2:

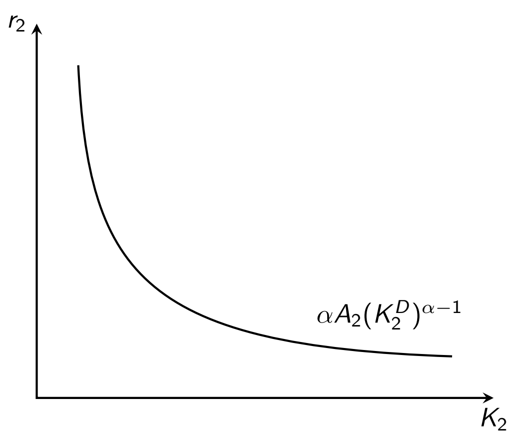
$$\Pi_2 = \max_{K_2^D} A_2(K_2^D)^\alpha - r_2 K_2^D$$

- where r_2 is the interest rate or rental rate of capital (taken as given)
- The first order condition yields:

$$\underbrace{\alpha A_2(K_2^D)^{\alpha-1}}_{\substack{\text{Marginal Utility}\\ \text{of Capital}}} - \underbrace{r_2}_{\substack{\text{Marginal Cost}\\ \text{of Capital}}} = 0$$

- Marginal Product of Capital (MPK) = extra output generated by additional capital input
- The FOC yields the capital demand curve:

$$r_2 = \alpha A_2(K_2^D)^{\alpha-1}$$



Firm's Profits

- Recall that households own the firms and receive the profits the firms generate:

$$\begin{aligned} C_1 + I_1 &= \Pi_1 + r_1 K_1 && \text{First period budget constraint} \\ C_2 &= \Pi_2 + (1 + r_2 - \delta)K_2 && \text{Second period budget constraint} \end{aligned}$$

- The firms' first order condition gives us $r_t = \alpha A_t (K_t^D)^{\alpha-1}$, so profits are:

$$\begin{aligned}\Pi_t &= A_t(K_t^D)\alpha - r_t K_t^D \\ &= A_t(K_t^D)\alpha - \alpha A_t(K_t^D)^{\alpha-1}K_t^D \\ &= A_t(K_t^D)\alpha - \alpha A_t(K_t^D)^{\alpha} \\ &= (1-\alpha)A_t(K_t^D)^{\alpha} > 0\end{aligned}$$

- Which means households are sensitive to changes in productivity *through* their ownership of firms

Equilibrium in the Inter-Temporal RBC Model

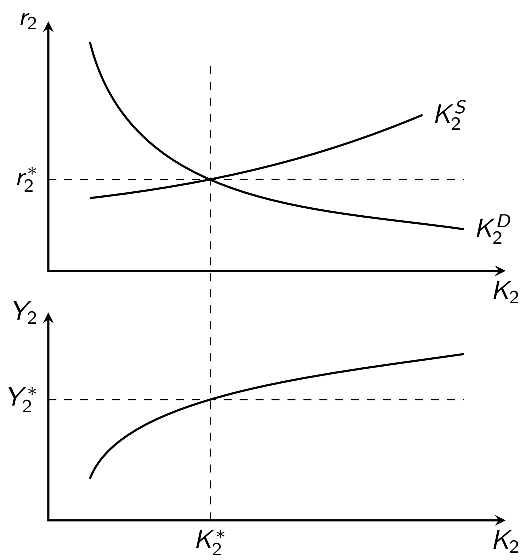
- The Capital Market clearing condition holds:
  - The real interest rate r_t ensures that the capital market clears
  - Capital supply (by households) is equal to capital demand (by firms) $K_t^S = K_t^D$
- Aggregate production is determined by technology:

$$Y_t = A_t K_t^{\alpha}, t = 1, 2 \tag{3.2}$$

- The aggregate resource constraint holds each period:

$$Y_1 = C_1 + I_1 \tag{3.3}$$
$$Y_2 + (1-\delta)K_2 = C_2 \tag{3.4}$$

- (Where total resources in period 2 include remaining undepreciated capital)
- Notice the real economy is tightly linked to the asset market (i.e. capital market)
- Equilibrium return on assets (i.e. interest rate), pins down amount of capital supplied
- Capital supply determines production/output in the economy
- So the macroeconomy and asset markets are very closely related!



<u>RBC Model Implications</u>

- Business cycles are due to "real" shocks (e.g. TFP or technology shocks)
- Productivity, real wages, employment, consumption, and investment are all pro-cyclical
- Markets are always in equilibrium.

- Prices and wages always adjust (flexibly) to ensure this equilibrium is efficient
- No involuntary unemployment in the model
- Money neutrality holds: changes in money supply do not affect real variables
- Government stabilization policies tend to be counter-productive

Limitations of RBC Models

- How do we measure TFP shocks? Solow Residuals?
- Do we really have frequent regressions in technological progress that cause recessions?
- What is the role of fiscal and monetary policy in the evolution of the macroeconomy?
- Most macroeconomists now convinced that money neutrality only holds in the long run
- Real wages are not pro-cyclical in the data. What does this imply?
  - "Real Wages and the Business Cycle", Abraham and Haltiwanger (JEL, 1995)
  - "Short-Run Equilibrium Dynamics of Unemployment, Vacancies, and Real Wages", Pissarides, (AER, 1985)
- To answer these questions, will typically need a DSGE model that incorporates price stickiness, wage stickiness, and policy shocks
- Most models in the RBC literature are solved using linear approximations to the model
- These linear approximations study deviations of the model from a well-defined steady state of the model economy
- But linear approximation means agents solve their problems under certainty equivalence:
- Certainty equivalence $\iff$ agents behave as if there is no risk!
- But risk is one of the primary reasons for holding financial assets:
  - We often want to insure against risks by holding financial assets that pay out if certain undesirable states of the world eventuate (e.g. unemployment, fire, theft, death)
  - In equilibrium, agents often want to share or smooth risks e.g. you payout when I am doing poorly, and I payout when you are doing poorly

Money and Savings in the New Keynesian Model

An Introduction to the New Keynesian Model

- The ideas of John Maynard Keynes dominated macroeconomics in the early 20th century
- Keynesian macroeconomics (e.g. the IS-LM-AS model) studied government policies that might stabilize output in response to shocks
- The RBC model, with its lack of government stabilisation policy, dominated macroeconomics from the 1970s
- But continuing to believe in the importance of government policy, macroeconomists then developed what is now called the New Keynesian Model
- Like the RBC model, the New Keynesian Model:
  - Has micro foundations of economic behaviour
  - Has agents with rational expectations about the future
  - Can be calibrated to match various business cycle statistics about the macroeconomy
- Unlike the RBC model, the New Keynesian Model:

- Features price and/or wages that are sticky (i.e. do not update in response to economic shocks)
- Describes a macroeconomy that does not respond efficiently to shocks
- May lead to output and employment being far from their socially optimal levels
- Allows a role for macroeconomic stabilisation via monetary policy and/or fiscal policy

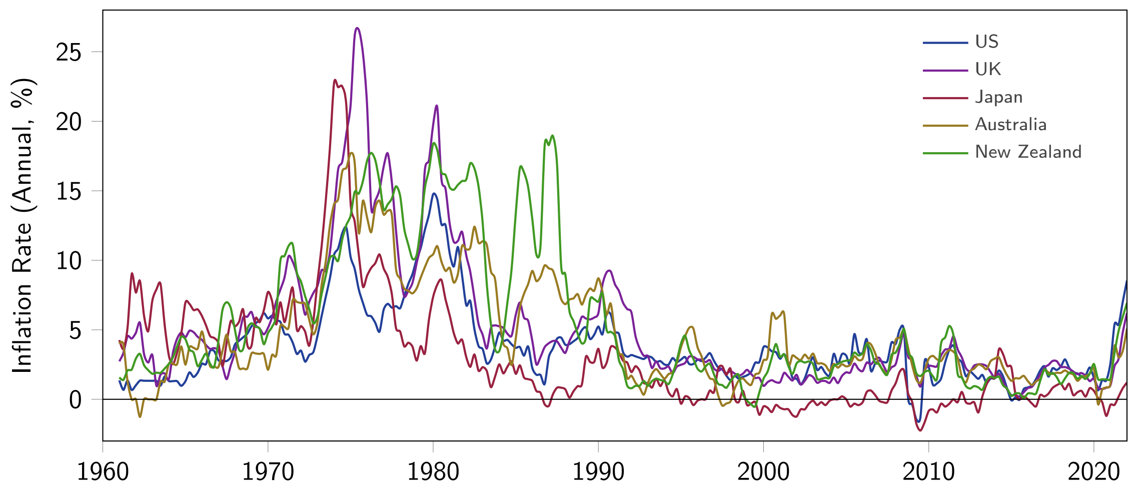
Inflation, and Nominal and Real Interest Rates

Inflation

- Define the general price level in an economy: $P_t \equiv$ price index
  - i.e the dollar cost of a representative basket of consumer goods
- Inflation: $\pi \equiv$ percent change in the price index:

$$\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

International Inflation Rates



Definitions of Nominal Variables

- Nominal interest rate: $r_t^n \equiv$ rate of return on an asset, in period t dollars
- Asset price: $S_t \equiv$ dollar price of a discount bond that pays one dollar next period
  - Discount bond: a bond that is issued or traded at less than its face–value
  - Face-value: amount the bond issuer pays to the bondholder once maturity is reached
  - Maturity: length of time a bond is held e.g. one month, one quarter, a year
- If r_n^t is the rate of return on the discount bond, then we we compute this as:

$$r_t^n = \frac{\text{Payoff}_{t+1} - \text{Bond Price}_t}{\text{Bond Price}_t} = \frac{1 - S_t}{S_t} = \frac{1}{S_t} - 1$$

$$\Rightarrow \; S_t = \frac{1}{r_t^n}$$

Real vs. Nominal Interest Rates and the Fisher Equation

- Purchasing power of one dollar $\equiv \frac{1}{P_t}$

  Purchasing power represents the number of consumption goods one dollar can buy

- The "ex-post" real interest rate $r_t \equiv$ realised return on the bond in units of consumption:

$$r_t = \frac{\frac{1}{P_{t+1}} - \frac{S_t}{P_t}}{\frac{S_t}{P_t}} = \frac{1}{S_t}\frac{P_t}{P_{t+1}} - 1$$
$$\Rightarrow\ 1 + r_t = \frac{1 + r_t^n}{1 + \pi_{t+1}}$$

- Rearranging

$$\begin{aligned} 1 + r_t^n &= (1 + r_t)(1 + \pi_{t+1}) \\ &= 1 + r_t + \pi_{t+1} + r_t\pi_{t+1} \end{aligned}$$

- Since $r_t\pi_{t+1} \approx 0$ for small values of r_t and π_{t+1}:

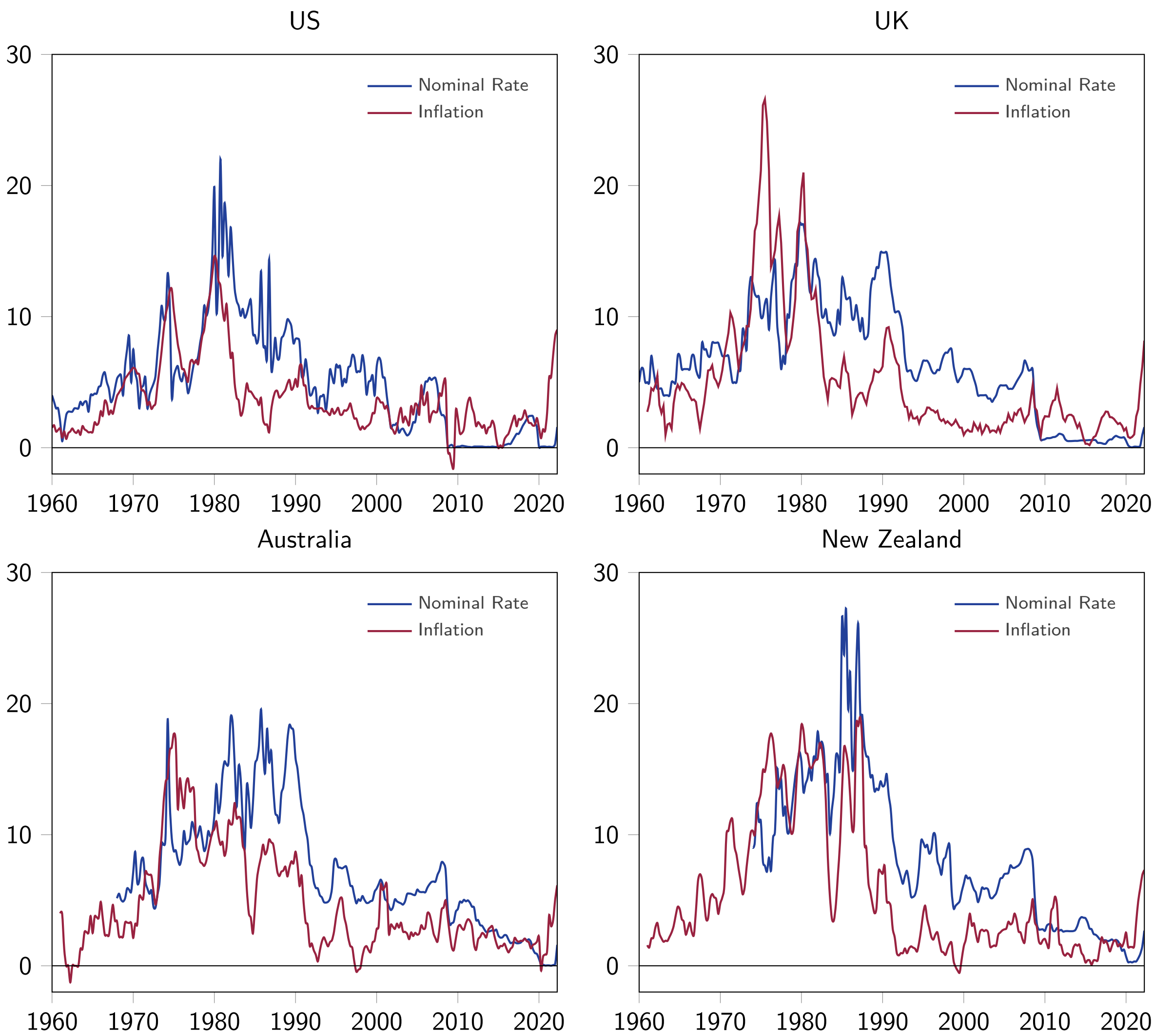
$$r_t \approx r_t^n - \pi_{t+1}$$

- Which is known as the Fisher Equation

Expected vs. Ex-Post Real Interest Rate

- The expected real rate is $E_t(r_t)$:

$$E_t(r_t) \approx E_t(r_t^n) - E_t(\pi_{t+1}) = r_t^n - E_t(\pi_{t+1})$$



Money and Inflation

The Rate of Return on Money

- We can also think of money as a type of asset.
- But what is the rate of return on money?
  - Since the nominal rate of return on money is $r^n_{m,t} = 0$, the real return is:

$$r_{m,t} - r^n_{m,t} - E_t(\pi_{t+1})$$

- **The return on money falls as expected inflation rises**
- So why do people hold money when its return is much lower than other assets?
  - Convenience: money has a role as a medium of exchange (i.e. used for trading goods and services)
  - Risk: fear of bank failures/financial market collapse (e.g. "money under the mattress")

Returns on Money vs. Bonds

- Recall:
  - Real rate of return on money:

$$r_{m,t} = -E_t(\pi_{t+1})$$

- (Expected) real rate of return on bonds:

$$E_t(r_t) \approx r_t^n - E_t(\pi_{t+1})$$

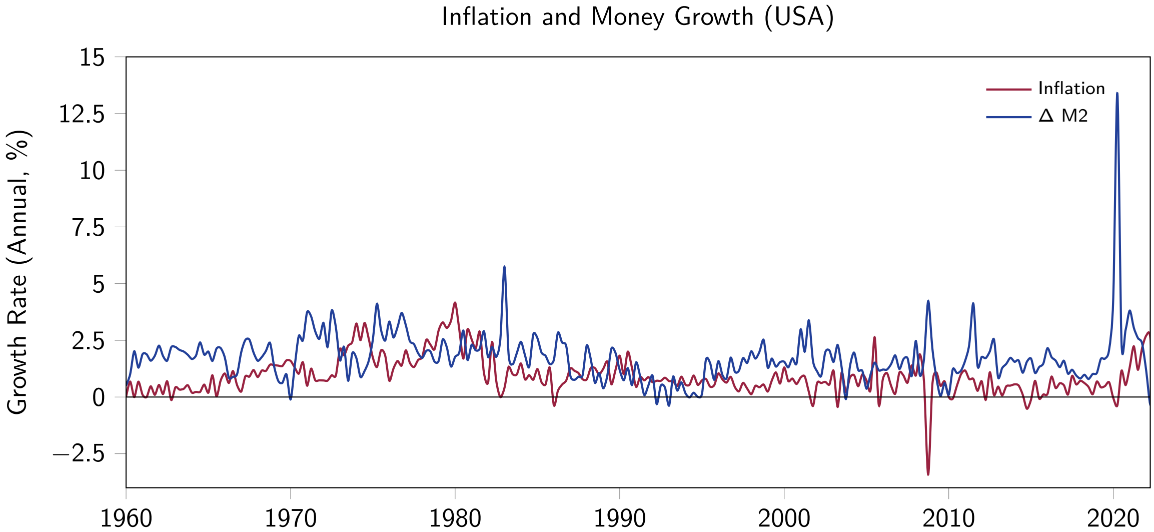
- Assuming the Fisher Hypothesis (i.e. that nominal rates move with inflation)
- Then fluctuations in inflation change return on money relative to the return on bonds
- Therefore, when monetary policy influences inflation, also affects the incentive to hold different kinds of assets

Quantity Theory of Money

- Consider again the Fisher Hypothesis:

$$r_t^n \approx E_t(r_t) + E_t(\pi_{t+1})$$

- If nominal interest rates move with inflation, what drives inflation?
- Much empirical evidence suggests a link between money growth and inflation
  - Evidence across time within a given country (mainly evidence over the long-run)
  - Evidence across countries



- Irving Fisher developed the Quantity Theory of Money (QTM):
  - A theory of the price level that explains what determines the value of a unity of money
- Begin with an accounting identity:

$$expenditures \equiv receipts$$

- Let $M \equiv$ stock of money; $V \equiv$ velocity of money (i.e. number of times a unit of money changes hands per period); $Y \equiv$ real output
- Then:

$$M \times V = expenditures$$
$$P \times Y = receipts$$
$$\Rightarrow MV = PY$$

- tart with the Quantity theory identity:

$$M_t V_t = P_t Y_t$$
$$\Rightarrow \Delta \ln M_t + \Delta \ln V_t = \Delta \ln P_t + \Delta \ln Y_t$$

- Rearranging:

$$\Delta \ln P_t = \Delta \ln M_t + \Delta \ln V_t - \Delta \ln Y_t$$

- The Quantity Theory then states:
  - Assumption (1) $\Delta \ln Y_t$ is independent of $\Delta \ln P_t, \Delta \ln M_t, \Delta \ln V_t$ (i.e. neo-classical assumption of **monetary neutrality**)
  - Assumption (2) $\Delta \ln V_t = 0$

$$\Rightarrow \Delta \ln P_t = \Delta \ln M_t - \Delta \ln Y_t$$

1. Why assume Y is independent of M, P, V?

- Neo-Classical theory argues only real factors matter for Y (e.g. technology)

2. Why assume stable velocity of money?

- Fisher assumed money demand was proportional to nominal income:

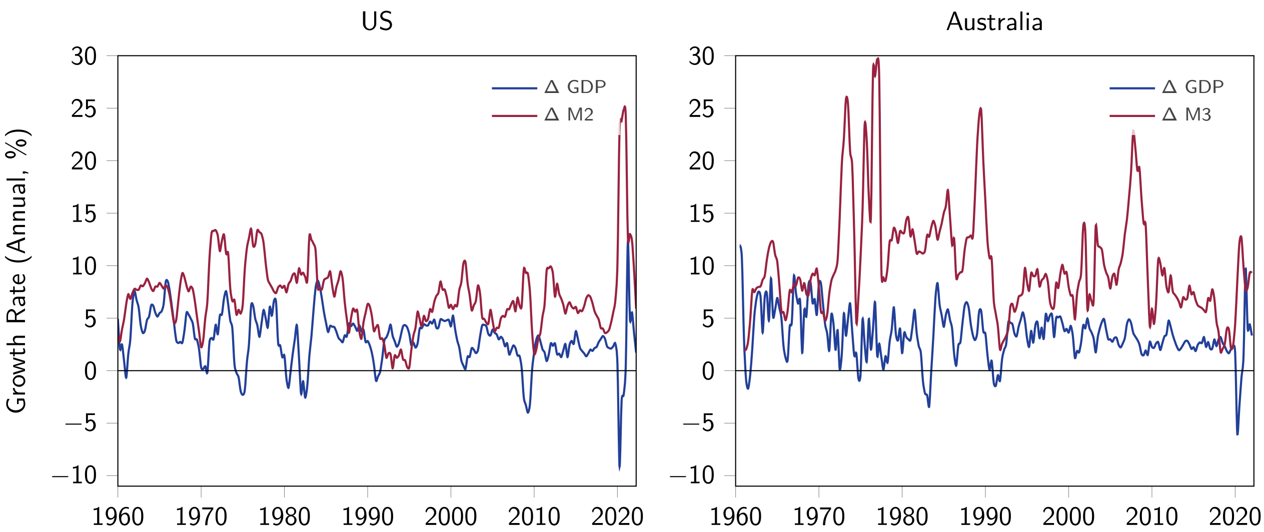
$$M = \kappa PY$$
$$\Rightarrow M\frac{1}{\kappa} = PY$$
$$\Rightarrow V = \frac{1}{\kappa}, \text{so V is constant}$$

- This might be true if financial institutions and technologies change slowly over time

1. Y is independent of M, P, V? NO!

- Much evidence shows that Y is clearly not independent of M
- Periods when central banks have sharply contracted the money supply have been followed by large real output declines
  - E.g.: Great Depression of the 1930s; large disinflations of the 1980s/1990s
- Why? Temporary **nominal price rigidities** mean M affects Y in short run
  - If P is sticky in the short run, then variation in M will affect Y

$$Y = V \times \underbrace{\frac{M}{P}}_{\text{Real Money Supply}}$$



2. Stable velocity of money? NO!

- Velocity is not constant and appears to be strongly pro-cyclical

- Problem:
  - Changes in financial technology provide easier to access money/substitutes (e.g. on-call savings accounts, EFTPOS, Pay-Wave), which changes velocity
  - The opportunity cost of holding money - i.e. the nominal interest rate on other assets r_t^n - also matters
  - Empirically, money demand does not have a simple proportional relationship to output

A Simple New Keynseian Model

- Household chooses consumption, nominal bonds, and money
- Simplified demand for money due to utility of holding real money balances
  - Represents the "convenience yield" of money holdings
  - But is something of a short-cut to characterize various desires for holding money

Household Choice Problem

- Household choice problem is:

$$\begin{aligned}\max_{C_1,C_2,M_1,B_2} \quad & \log(C_1) + \omega \log \frac{M_1}{P_1} + \beta \log(C_2) \\ \text{s.t.} \quad & P_1 C_1 + M_1 + B_2 = P_1 Y_1 \\ & P_2 C_2 = P_2 Y_2 + M_1 + B_2(1+r^n)\end{aligned}$$

- Where M_1/P_1 are real money balances
- The inter-temporal real budget constraint is:

$$C_1 + \frac{M_1}{P_1} + \frac{C_2}{1+r^n}\frac{P_2}{P_1} = Y_1 + \frac{Y_2}{1+r^n}\frac{P_2}{P_1} + \frac{M_1/P_1}{1+r^n}$$

- The Lagrangian Problem is:

$$\mathcal{L} = \log(C_1) + \omega \log \frac{M_1}{P_1} + \beta \log(C_2) + \lambda \left(Y_1 + \frac{Y_2}{1+r^n}\frac{P_2}{P_1} + \frac{M_1/P_1}{1+r^n} - C_1 - \frac{M_1}{P_1} - \frac{C_2}{1+r^n}\frac{P_2}{P_1} \right)$$

- The first order conditions for the problem are:

$$\begin{aligned} C_1: \quad & \frac{1}{C_1} - \lambda = 0 \\ C_2: \quad & \beta \frac{1}{C_2} - \lambda \frac{1}{1+r^n}\frac{P_2}{P_1} = 0 \\ M_1: \quad & \omega \frac{1}{P_1}\frac{1}{M_1/P_1} + \lambda \frac{1}{1+r^n}\frac{1}{P_1} - \lambda \frac{1}{P_1} = 0 \end{aligned}$$

- Combining the first two yields the consumption Euler equation:

$$\frac{1}{C_1} = \beta(1+r^n)\frac{P_1}{P_2}\frac{1}{C_1}$$

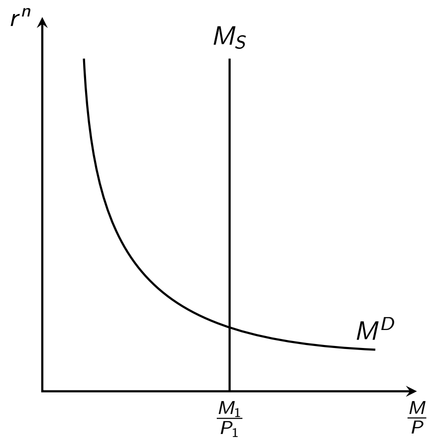
- Combining the first and third yields the consumption-money optimality condition:

$$\omega \frac{C_1}{M_1/P_1} = \left(\frac{r^n}{1+r^n} \right)$$

- which states that the marginal rate of substitution between consumption and money balances is governed by the nominal interest rate on bonds

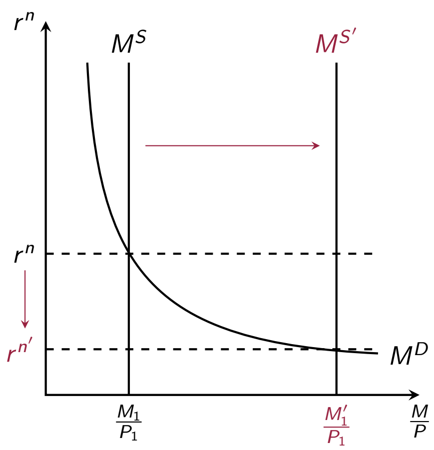
Demand and Supply for Money

- We can represent the consumption-money optimality condition as a money demand equation in $(M_1/P_1, r^n)$-space
- Suppose the central bank supplies money **inelastically** with respect to the interest rate



Simple Monetary Policy

- The New Keynesian model suggestions that money affects the real economy
- Simple example:
  - Assume that nominal price rigidities mean that prices are constant: $P_1 = P_2 = P$
  - Now consider an unexpected increase in money supply $\uparrow M_1^S$
  - What happens to consumption (C_1, C_2)?
- Note: These assumptions only hold in the short run!
- With sticky prices (i.e. P constant), an increase in the money supply **decreases** the nominal interest rate



- To solve for changes in consumption take the inter-temporal budget constraint, money demand, and

Euler equations (assuming that $P_1 = P_2$):

$$C_1 + \frac{M_1}{P_1} + \frac{C_2}{1+r^n} = Y_1 + \frac{Y_2}{1+r^n} + \frac{M_1/P_1}{1+r^n}$$
$$\omega\frac{C_1}{M_1/P_1} = \left(\frac{r^n}{1+r^n}\right)$$
$$\frac{1}{C_1} = \beta(1+r^n)\frac{1}{C_2}$$

- Substituting the money demand and Euler equations into the budget constraint, we get the consumption functions:

$$C_1 = \frac{1}{1+\omega+\beta}\left(Y_1 + \frac{Y_2}{1+r^n}\right), \quad C_2 = \frac{\beta(1+r^n)}{1+\omega+\beta}\left(Y_1 + \frac{Y_2}{1+r^n}\right)$$

- Remember the increase in money supply leads to a decrease in r^n
- Thus, consumption in period 1 rises:

$$\uparrow C_1 = \frac{1}{1+\omega+\beta}\left(Y_1 + \frac{Y_2}{\underbrace{(1+r^n)}_{\downarrow}}\right),$$

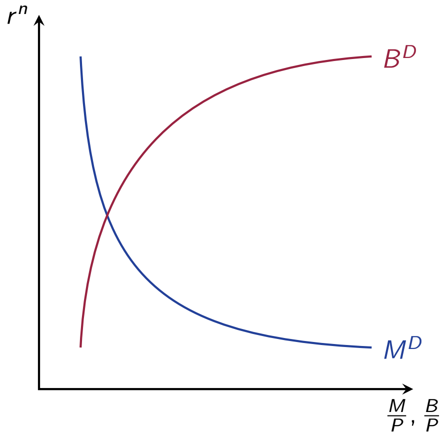
- And consumption in period 2 falls:

$$\downarrow C_2 = \frac{\beta(1+r^n)}{1+\omega+\beta}\left(Y_1 + \frac{Y_2}{1+r^n}\right) = \frac{\overbrace{\beta(1+r^n)}^{\downarrow}}{1+\omega+\beta}Y_1 + \frac{\beta}{1+\omega+\beta}Y_2$$

- So sticky prices mean that monetary policy is non-neutral in the short run
  - That is, monetary policy can have real effects on the macroeconomy!
- Changes in monetary policy also affect demand for assets!
- Derive the bond demand equation using the period 1 budget constraint and the money demand equation:

$$\frac{B_2}{P_1} = Y_1 - C_1 - \frac{M_1}{P_1}$$
$$= Y_1 - C_1 - \omega C_1\left(1 + \frac{1}{r^n}\right)$$

- Which shows that real bond demand is **increasing** in the nominal interest rate rn
- So households adjust their asset portfolio according to the return on bonds
- Changes in monetary policy affect real asset portfolio allocation decisions
- Household composition of assets varies with the relative return on the assets available
- So a decrease in money supply raises the nominal interest rate, which increases bond holdings
- Since the nominal return on money is zero, an increase in the nominal return on bonds leads to a shift away from money and towards bonds



Limitations of the New Keynesian Model

- The source of price rigidities is often not well-microfounded
  - Typically introduce ad-hoc "price stickiness" to models
- New Keynesian models often do not account for macroeconomic data much better than RBC models
- Despite their basis in monetary economics, New Keynesian models often do a poor job of explaining fluctuations in inflation
- As was the case for the RBC model, most New Keynesian models are not solved with economic risk in mind
- So, again, these models are not ideal for studying some of the main motives for asset holdings
- Both RBC and New Keynesian models contain a single, representative household
- This household has no one else to trade with, so the notion of a financial market is limited

Expectations, Uncertainty, and Asset Holdings

Risk Aversion and the Precautionary Savings Motive

- Risk aversion: a tendency to prefer economic outcomes with low uncertainty to those with more uncertainty
- Precautionary Savings: an increase in income uncertainty that leaves expected income unchanged reduces current consumption. But savings increase as a form of self **insurance** against low income states of the world.
- Risk aversion is a consequence of diminishing marginal utility
  - For utility function $u(\cdot)$, then $u' > 0$ and $u'' < 0$
  - Implies a loss of x matters more than a gain x
  - A risk averse agent would turn down a fair bet with even odds of an increase of x or a decrease in x
  - But risk aversion does not tell us how an agent *responds* to uncertainty or risk
- Precautionary Savings is a result of marginal utility declining at a decreasing rate:
  - For utility function $u(\cdot)$, then $u''' > 0$
  - This feature of utility functions/preferences is sometimes called **prudence**
- In this case, an increase in income uncertainty (holding expected income constant) raises expected **marginal utility**

- This means that the value of **additional** consumption is higher, which means that households save more in order to consume more in the periods of heightened uncertainty
- These additional savings in the face of greater uncertainty are called precautionary savings

The Precautionary Savings Motive: An Illustrative Model

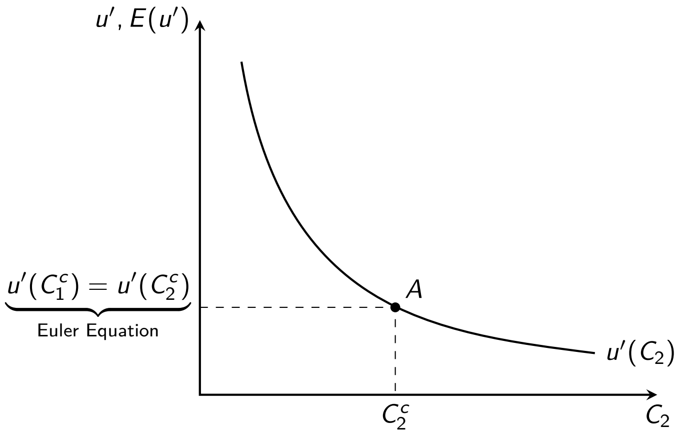
- A household makes consumption and savings decisions, subject to known and constant incomes
- Assume $\beta = 1$ return on savings is zero ($r = 0$), income in each period is $\bar{Y}$, utility function $u' > 0$, $u'' < 0$, $u''' > 0$

$$\begin{aligned} \max_{C_1, C_2} \quad & u(C_1) + u(C_2) \\ \text{s.t.} \quad & C_1 + C_2 = \bar{Y} + \bar{Y} \end{aligned}$$

- The first order condition yields the optimality condition (Euler Equation):

$$\begin{aligned} u'(C_1) &= u'(C_2) \\ \Rightarrow C_1 &= C_2 = \bar{Y} \end{aligned}$$

- Label these consumption choices C_1^c and C_2^c for the choices under **certainty**
- It will be helpful to plot marginal utility as a function of consumption in period 2
- First, plot marginal utility at our consumption choice under certainty C_2^c
- Note that because $u'' < 0$ and $u''' > 0$, marginal utility is decreasing and convex (i.e. curved out from the origin)



- Now suppose there are different **states** of the world
- These states affect income in period 2, with a chance of a good outcome and a chance of a bad outcome

$$Y_2 = \begin{cases} \bar{Y} + x & \text{with probability } 0.5 \quad \text{(Good Outcome)} \\ \bar{Y} - x & \text{with probability } 0.5 \quad \text{(Bad Outcome)} \end{cases}$$

- The first order condition in this case yields the Expected Euler Equation:

$$u'(C_1) = E(u'(C_2))$$

- Where $E(u'(C_2))$ is the expectation over **marginal utility** of consumption in period 2
- Note that we can compute this as:

$$\begin{aligned} E(u'(C_2)) &= 0.5 \times u'(C_2(good)) + 0.5 \times u'(C_2(bad)) \\ &= 0.5 \times u'(C_2(\bar{Y} + x + S)) + 0.5 \times u'(C_2(\bar{Y} - x + S)) \end{aligned}$$

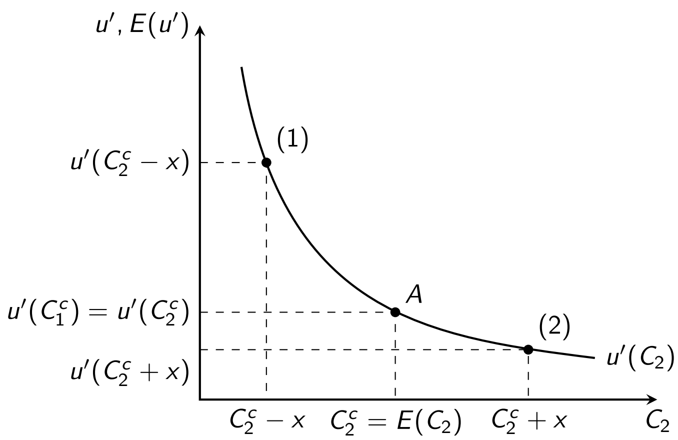
- Suppose the household were to choose period 1 consumption the same as under the certainty case: $C_1 = C_1^c = \bar{Y}$
- Then from the period 1 budget constraint, savings are: $S = \bar{Y} - C_1^c$
- And we can write consumption in period two as:

$$
\begin{aligned}
C_2 &= Y_2 + S \\
&= Y_2 + \bar{Y} - C_1^c \\
&= \begin{cases} \bar{Y} + x + \bar{Y} - C_1^c & \text{with probability 0.5} \\ \bar{Y} - x + \bar{Y} - C_1^c & \text{with probability 0.5} \end{cases} \\
&= \begin{cases} C_2^c + x & \text{with probability 0.5} \\ C_2^c + x & \text{with probability 0.5} \end{cases}
\end{aligned}
$$

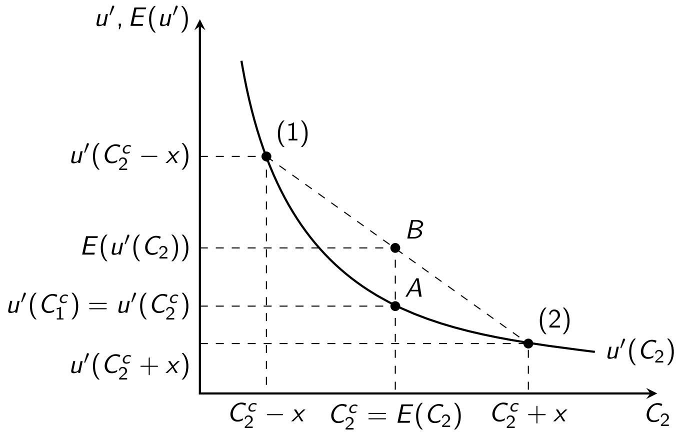
- If choosing the certainty consumption in period 1, period 2 consumption is equal to the certainty consumption (C_2^c) plus or minus the uncertain component of income x
- Now, write the expected marginal utility of consumption in period 2 as:

$$
\begin{aligned}
E(u'(C_2)) &= 0.5 \times u'(C_2^c + x) + 0.5 \times u'(C_2^c - x) \\
&\geq u'(C_2^c)
\end{aligned}
$$

- Because $u''' > 0$ the expected marginal utility of consumption in the uncertain case is greater than marginal utility in the certain case
- This means that the value of the certain consumption choice is greater than the value of the uncertain consumption outcomes
- Another way: households prefer certainty to uncertainty, even when the expected value of outcomes is the same in both cases
- Again consider plot of marginal utility as function of consumption in period 2
- Consumption is low/marginal utility is high in the bad state (1)
- Consumption is high/marginal utility is low in the good state (2)



- Notice that: $E(u'(C_2)) > \underbrace{u'(C_2^c) = u'(C_1^c)}_{\text{Euler Equation}}$
- Therefore:
  - Households want to increase C_2, and decrease C_1,
  - They accomplish this with higher (i.e. precautionary) savings S



- Point (A) corresponds to marginal utility of the certain consumption C_2^c
- This is the optimal consumption choice for the certainty case: $u'(C_1^c) = u'(C_2^c)$
- Point (B) is the **expected marginal utility** over consumption in the uncertain case: $E(u'(C_2))$
- Note that $E(u'(C_2)) > u'(C_2^c) = u'(C_1^c)$
- This means that consumption is **too low** in period 2 (i.e. marginal utility is too high)
- Therefore, the household should consume less in period 1: $C_1^u < C_1^c$
- This allows household to save more and so consume more in period 2: $C_2^u(s) > C_2^c(s)$

Precautionary Savings and Asset Prices

- Consider our two-period model:

$$\begin{aligned} \max_{C_1, C_2} \quad & u(C_1) + u(C_2) \\ \text{s.t.} \quad & C_1 + P_b B = \bar{Y} \\ & \quad C_2 = Y_2 + B \end{aligned}$$

- Where a one period bond B can be purchased at price P_b
- Income is again uncertain:

$$Y_2 = \begin{cases} \bar{Y} + x & \text{with probability } 0.5 \quad \text{(Good Outcome)} \\ \bar{Y} - x & \text{with probability } 0.5 \quad \text{(Bad Outcome)} \end{cases}$$

- The first order condition yields the Expected Euler Equation:

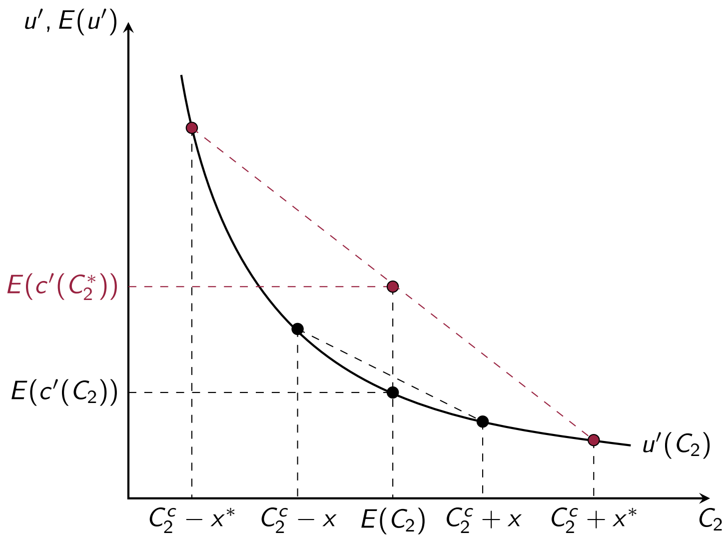
$$P_b u'(C_1) = \beta E(u'(C_2))$$

- And rearranging we have:

$$P_b = \beta \frac{E(u'(C_2))}{u'(C_1)}$$

- This is referred to as an Asset Pricing Equation
- Asset prices determined by the ratio of marginal utilities of consumption in each period
- Or, another way: asset prices are given by the marginal rate of substitution between consumption across periods. How does uncertainty affect prices?
- So now consider an increase from x to x^*:
  - Now $Y_2 = \bar{Y} + x^* > \bar{Y} + x$ with probability 0.5, and $Y_2 = \bar{Y} - x^* < \bar{Y} - x$ with probability 0.5

- But it is still the case that $E(Y_2) = \bar{Y}$
- This is called a mean-preserving spread in Y_2
- Uncertainty only affects period 2, so the effect on asset prices comes through $E(u'(C_2))$



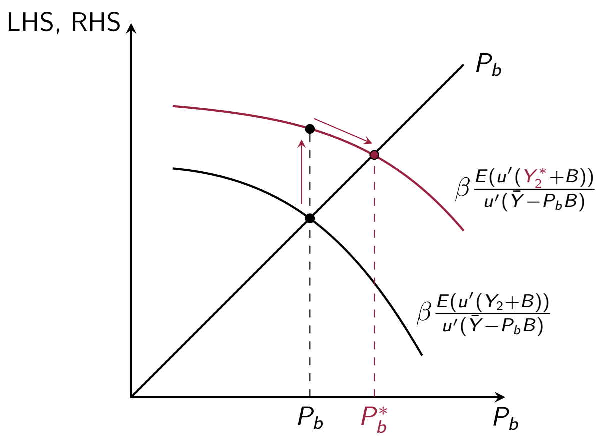
- Since $E(c'(C_2))$ increases, P_b increases also:

$$\uparrow P_b = \beta \frac{E(u'(C_2))\uparrow}{u'(C_1)}$$

- But C_1 also decreases in response to greater uncertainty, which increases $u'(C_1)$
- So what is the overall effect?
- Substitute in the budget constraints:

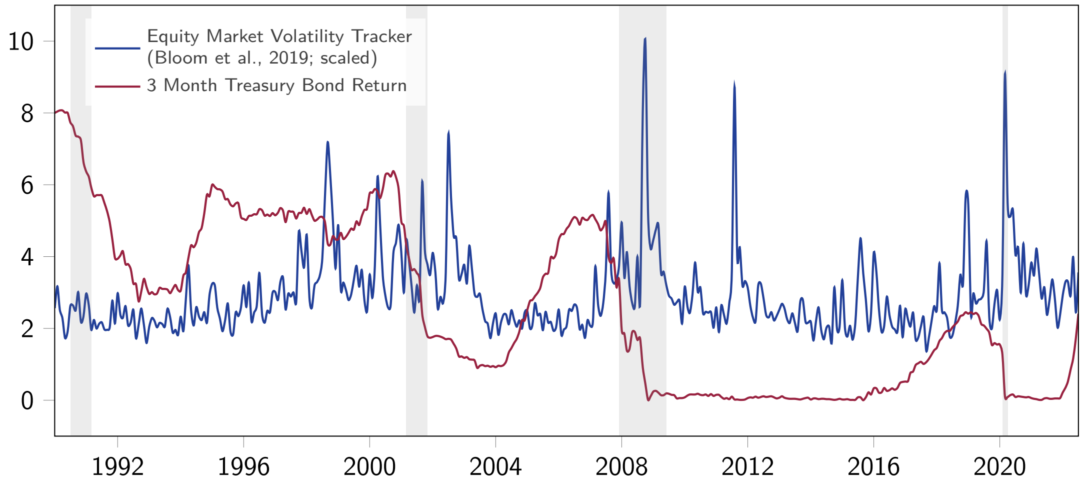
$$P_b = \beta \frac{E(u'(Y_2 + B))}{u'(\bar{Y} - P_b B)}$$

- Illustrate optimal choices graphically by plotting the left-hand-side and right-hand-side of the asset pricing equation

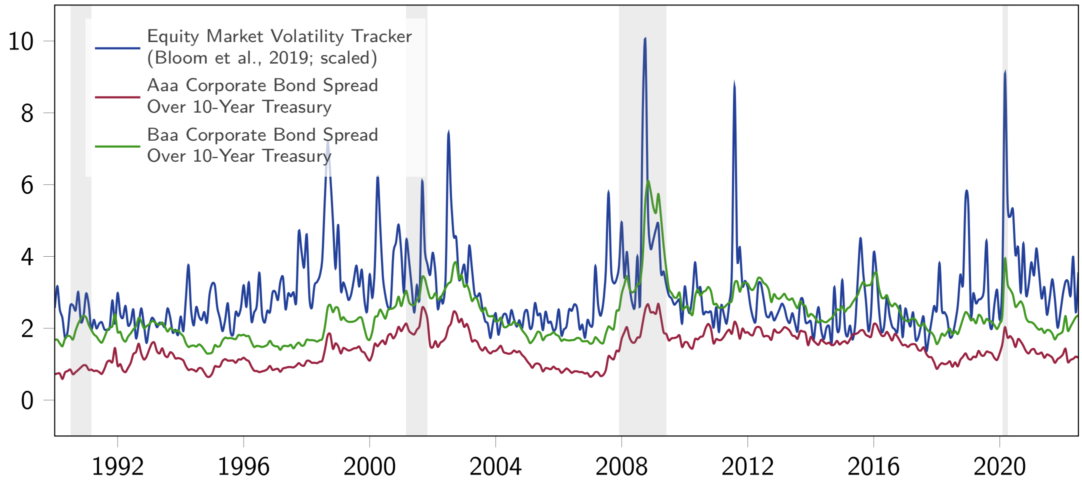


- An increase in uncertainty **increases** the price of assets
- This is intuitive:
  - Greater uncertainty induces precautionary savings which increases demand for assets

  - Higher asset demand is associated with higher asset
- Recall that the asset return: $R = \frac{1}{P_b}$
- So higher asset prices are associated with **lower** asset returns
- Do we observe this empirically?



- Note, we need to compare risk-free bonds
- For risky assets, demand and prices may increase or decrease depending on the nature of the asset risk



How Much Does the Precautionary Savings Motive Matter?

- The two motives for asset holding that we have studied so far:
  - Life-cycle motive: consumption smoothing across time
  - Precautionary savings motive: consumption smoothing across outcomes/states of the world
- Finds that precautionary savings matter much more for young households' asset decisions
- Finds that life-cycle motives matter much more for older households' asset decisions
- Young households start out with low wealth, need to save to build a **precautionary savings buffer**

Introduction to Asset Pricing: Concepts, Measurement, and a Simple Model

Definitions and Measurement

- Consider simple asset that was bought last period at P_{t-1} and sold this period at P_t
- The simple net return r_t on this asset between dates $t-1$ and t is:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

- The simple gross return is: $R_t = 1 + r_t$
- The gross return on the asset over k periods starting at date $t-k$ is:

$$\begin{aligned} R_t(k) &= R_t \cdot R_{t-1} \cdots R_{t-k+1} \\ &= \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} \cdots \frac{P_{t-k+1}}{P_{t-k}} \\ &= \frac{P_t}{P_{t-k}} \end{aligned}$$

- These multi-period returns are referred to as **compounded returns**
- Multi-year returns are often annualised in order to easily compare investments in assets over different horizons
- An annualised gross return $R_t^{ann}(k)$ is computed via:

$$R_t^{ann}(k) = [R_t \cdot R_{t-1} \cdots R_{t-k+1}]^{\frac{1}{k}} = \left[\prod_{j=0}^{k-1} R_{t-j}\right]^{\frac{1}{k}}$$

- This formula is known as a **geometric mean**
- For quick comparisons – that are less accurate! – we sometimes use an arithmetic mean as an approximation to the annualised return:

$$R_t^{ann}(k) \approx \frac{1}{k} \sum_{k-1}^{j=0} R_{t-j}$$

Continuous Compounding

- The continuously compounded return or log-return of an asset is defined as:

$$\tilde{r}_t = \log(R_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1}) = p_t - p_{t-1}$$

- where lower-case letters represent the log of a variable
- So the continuously compounded multi-period return over k periods is:

$$\begin{aligned} \tilde{r} &= \log(R_t(k)) \\ &= \log(R_t) + \log(R_{t-1}) + \cdots + \log(R_{t-k+1}) \\ &= \tilde{r} + \tilde{r}_{t-1} + \cdots + \tilde{r}_{t-k+1} \end{aligned}$$

- Compounding – a multiplicative operation – is converted to an additive operation by taking logarithms!

Dividend payments

- Some assets (e.g. stocks) pay out dividends, which make up part of the return on the asset
- For these assets, define returns as:

$$R_t = \frac{P_t + D_t}{P_{t-1}}$$
$$\Rightarrow r_t = \frac{P_t + D_t}{P_{t-1}} - 1$$

- To compute the log-return:

$$\tilde{r}_t = \log(R_t) = \log\left(\frac{P_t + D_t}{P_{t-1}}\right) = \log(P_t + D_t) - \log(P_{t-1})$$

Excess Returns

- We will often want to compare returns across different assets
- Consider the gross return on a benchmark asset R_t
- And consider the gross return R_t^i on a comparison asset i
- The **excess return** of asset i over the benchmark is:

$$R_t^i - R_t = r_t^i - r_t$$

- In many cases, the benchmark asset will be something approximating a riskless/risk-free asset such as a government bond

Equity Premium

- The equity premium is the expected excess return of an asset over the risk-free rate:

$$E_t(R_t^i - R_t)$$

- The equity premium tells us the excess return on asset i required to compensate investors for the additional risk of holding i over the risk-free asset

Present Value with Constant Discount Rates

- Suppose an asset paying a regular dividend has a constant expected return $R_t = R$:

$$R = E_t\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right)$$

- Rearrange for P_t:

$$P_t = E_t\left(\frac{P_{t+1} + D_{t+1}}{R}\right) \tag{6.1}$$

- Note that this is the same as the Discounted Cash Flow valuation model from section 1
- Now step the price forward one period to Pt+1:

$$P_{t+1} = E_t\left(\frac{P_{t+2} + D_{t+2}}{R}\right) \tag{6.2}$$

- Substitute back into equation (6.1):

$$P_t = E_t\left(E_{t+1}\left[\frac{P_{t+2} + D_{t+2}}{R^2}\right] + \frac{D_{t+1}}{R}\right)$$

- We can simplify this as:

$$P_t = E_t\left(\frac{P_{t+2}+D_{t+2}}{R^2}+\frac{D_{t+1}}{R}\right)$$

- Where the second equality follows from the Law of Iterated Expectations e.g.: $E_t(E_{t+h}[x_{t+h+1}]) = E_t(x_{t+h+1})$
- Iterating this process K times yields:

$$P_t = E_t\left(\sum_{k=1}^{K}\frac{D_{t+k}}{R^k}\right)+E_t\left(\frac{P_{t+k}}{R^k}\right)$$

- We typically assume that:

$$\lim_{K\to\infty} E_t\left(\frac{P_{t+k}}{R^k}\right)=0$$

- Which is referred to as the No Bubble Condition
- Then we have a simple asset valuation formula:

$$P_t = E_t\left(\sum_{k=1}^{K}\frac{D_{t+k}}{R^k}\right)$$

- Where the RHS is the discounted present value of future dividends (i.e. cashflows)

Introduction to Macroeconomic Models of Asset Pricing

- How do asset prices relate to the macroeconomic model we have been studying so far?
- We will show how the standard model of household behaviour can lead us to a theory of asset pricing known as the Consumption Capital Asset Pricing Model (C-CAPM)
- We will then put C-CAPM into the context of the broader study of finance and macro-finance
- Consider a household problem at some generic date "t"
- Two assets:
  - A risk-free bond B_t. Price $P_{B,t}$. At $t+1$, pays a face value of 1.
  - A risky asset A_t. Price $P_{A,t}$ At $t+1$, pays uncertain dividend D_{t+1}, and can be resold at price $P_{A,t+1}$
- The household problem is:

$$\begin{aligned}\max_{C_t,C_{t+1},B_t,A_t} \quad & u(C_t)+\beta E_t[u(C_{t+1})]\\ \text{s.t.} \quad & C_t + P_{B,t}B_t + P_{A,t}A_t = Y_t\\ & C_{t+1} = Y_{t+1} + B_t + D_{t+1}A_t + P_{A,t+1}A_t\end{aligned}$$

- Household chooses C_t, B_t, A_t at time t, but chooses C_{t+1} at time $t+1$
- But choice of B_t, A_t affects the budget constraint at time $t+1$ where outcomes are uncertain
- This uncertainty means the household must form expectations (E_t) about $t+1$ sing information available at time t
- Important:
  - Expectations over $t+1$ matter for decisions at t if those decisions affect outcomes at $t+1$!

- The Lagrangian Problem is:

$$
\begin{aligned}
\mathcal{L} = & \, u(C_t) + \beta E_t[u(C_{t+1})] \\
& + \lambda_t(Y_t - C_t + P_{B,t}B_t + P_{A,t}A_t) \\
& + E_t[\lambda_{t+1}(Y_{t+1} + B_t + D_{t+1}A_t + P_{A,t+1}A_t - C_{t+1})]
\end{aligned}
$$

- The Lagrange Multipliers λ_t, λ_{t+1} measure the shadow value of the budget constraints:
  - λ_t = the marginal value of an extra dollar allocated to the budget constraint in period t
  - λ_{t+1} = the marginal value of an extra dollar allocated to the budget constraint in period $t+1$
- The first order conditions with respect to C_t, B_t, A_t are:

$$
\begin{aligned}
C_t: & \quad u'(C_t) - \lambda_t = 0 \\
B_t: & \quad -\lambda_t P_{B,t} + E_t(\lambda_{t+1}) = 0 \\
A_t: & \quad -\lambda_{t+1} P_{A,t} + E_t(\lambda_{t+1}[D_{t+1} + P_{A,t+1}]) = 0
\end{aligned}
$$

- Where expectations enter the FOCs for B_t and A_t because those decisions affect outcomes during the uncertain period $t+1$
- The first order condition with respect to C_{t+1} is:

$$
C_{t+1}: \quad \beta u'(C_{t+1}) - \lambda_{t+1} = 0
$$

- Where there are **no expectations** because the decision C_{t+1} is made **after** the uncertainty in period $t+1$ has been resolved
- Tidying up the first order conditions:

$$
\begin{aligned}
C_t: & \quad \lambda_t = u'(C_t) && (6.3) \\
C_t + 1: & \quad \lambda_{t+1} = \beta u'(C_{t+1}) && (6.4) \\
B_t: & \quad \lambda_t P_{B,t} = E_t(\lambda_{t+1}) && (6.5) \\
A_t: & \quad \lambda_{t+1} P_{A,t} = E_t(\lambda_{t+1}[D_{t+1} + P_{A,t+1}]) && (6.6)
\end{aligned}
$$

The Macroeconomic Perspective on Asset Prices

- Combining equations (6.3), (6.4) and (6.5):

$$
\begin{aligned}
& u'(C_t)P_{B,t} = \beta E_t(u'(C_{t+1})) \\
\Rightarrow & \frac{u'(C_t)}{E_t(u'(C_{t+1}))} = \frac{1}{P_{B,t}} = R_{B,t}
\end{aligned}
$$

- Where the LHS represents the inter-temporal marginal rate of substitution
- And $R_{B,t}$ on the RHS is the (certain) return on the bond
- This says that:
  - The relative value of consumption across periods is tied to the interest rate (return) on bonds
  - Or, consumption growth across periods tied to price of transferring resources across periods
- Macroeconomic Perspective: inter-temporal consumption is all about interest rates

The Finance Perspective on Asset Prices

- Combining equations (6.5) and (6.6):

$$P_{B,t} = E_t(\frac{\lambda_{t+1}}{\lambda_t}), \qquad \text{With t+1 payoff} = 1$$
$$P_{A,t} = E_t\left(\frac{\lambda_{t+1}}{\lambda_t}[D_{t+1} + P_{A,t+1}]\right), \qquad \text{With t+1 payoff} = D_{t+1} + P_{A,t+1}$$

- These are known as asset pricing equations
- They state that the price of an asset is determined by the valuation of that asset's payoffs
- And those valuations are given by the Lagrange Multipliers:
  - $\lambda_t =$ the marginal value of an extra dollar allocated to the budget constraint in period t
  - $\lambda_{t+1} =$ the marginal value of an extra dollar allocated to the budget constraint in period $t+1$
- Looking closely, we can see that both assets' payoffs are valued at the same rate
- This valuation is given by the stochastic discount factor (SDF):

$$\frac{\lambda_{t+1}}{\lambda_t}$$

- For our particular model, we know that the SDF is given by:

$$\frac{\lambda_{t+1}}{\lambda_t} = \beta\frac{u'(C_{t+1})}{u'(C_t)}$$

- That is, assets are valued by variations in household consumption across time
- Since consumption is determined by developments in the macroeconomy: asset prices must be linked to business cycle fluctuations!
- Our asset pricing equations are:

$$P_{B,t} = E_t\left(\beta\frac{u'(C_{t+1})}{u'(C_t)}\right), \quad P_{A,t} = E_t\left(\beta\frac{u'(C_{t+1})}{u'(C_t)}[D_{t+1} + P_{A,t+1}]\right)$$

- Holding all else equal, asset prices are higher when:
  - The marginal utility of current consumption C_t is low (i.e. C_t is high)
  - The marginal utility of current consumption C_{t+1} is high (i.e. C_{t+1} is low)
- However, it will rarely be the case that C_t or C_{t+1} move independent of everything else
- Remember, macroeconomic and financial variables move together in equilibrium
- Example:
  - Consider shares in a firm A that trade at price $P_{A,t}$ and pay dividends D_{t+1}
  - Dividends are paid out of firm profits
  - Future profits and dividends will be low during recessions
  - But recessions are times when future consumption is also likely to be low

$$E_t\left(\beta\overbrace{\frac{u'(C_{t+1})}{u'(C_t)}}^{\uparrow}[\overbrace{D_{t+1}}^{\downarrow}+P_{A,t+1}]\right) = P_{A,t}(?)$$

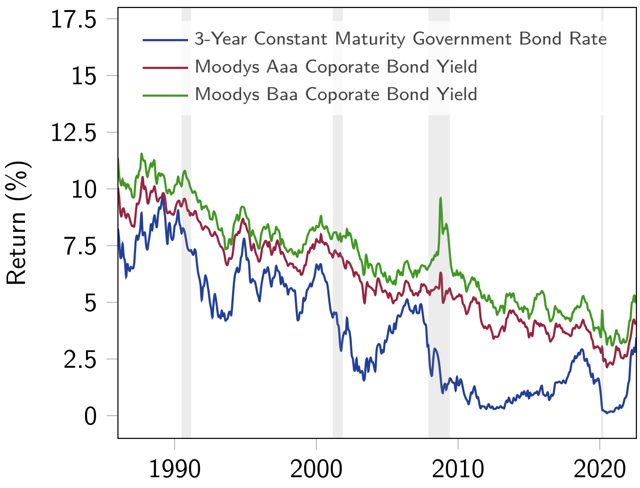
- So risky asset prices depend on co-movement between future consumption and uncertain payoffs

- However, the price of risk-free assets (i.e. bonds) only depends on the SDF
- Previous example: future recession decreases future consumption and dividends

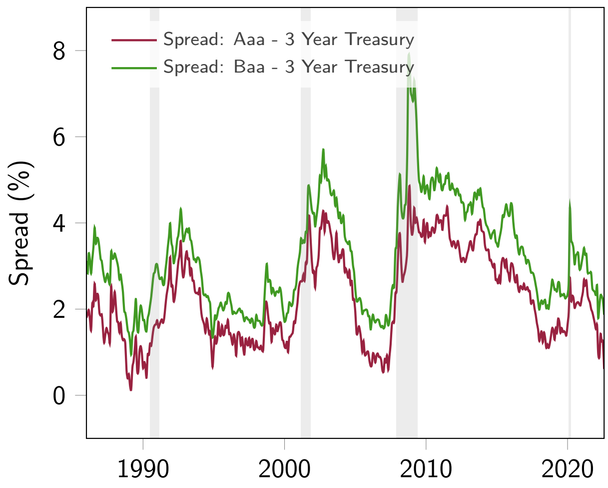
$$P_{B,t}\uparrow = E_t\left(\beta \overbrace{\frac{u'(C_{t+1})}{u'(C_t)}}^{\uparrow}\right), \quad P_{A,t}\downarrow = E_t\left(\beta \overbrace{\frac{u'(C_{t+1})}{u'(C_t)}}^{\uparrow}[\overbrace{D_{t+1}}^{\downarrow\downarrow}+P_{A,t+1}]\right)$$

- Which means that the price (return) of risk-free assets can move in opposite directions to the prices (returns) of risky assets

Various Asset Returns



Various Asset Spreads



Asset Prices, Consumption, and the Business Cycle

Historical Overview of Finance and Asset Pricing

1. **Market Efficiency View:**
   - Asks “Are market asset prices set conditional on all available information?”
   - Or “Can you beat the market return without taking on more than market risk?”
   - The
     textcolormyblueEfficient Markets Hypothesis:
     - [An] old economist and younger economist [are] walking down the street, and the younger economist says, ‘Look, there’s a hundred-dollar bill,’ and the older one says, ‘Nonsense, if it was there somebody would have picked it up already.’
2. **Portfolio Theory:**
   - How should we form asset portfolios? (Markowitz, 1952)
   - The variance of the return on an asset portfolio is much smaller than the average of the variances of the returns on the individual assets in the portfolio
   - So what is the optimal variance of an asset portfolio?
   - Something called the Mean-Variance Efficient Frontier can be constructed for a portfolio
3. **Capital Asset Pricing Model (CAPM):**
   - How much do individual asset prices move with the market? (Sharpe 1964; Litner 1965)
   - Want a model of the cross-sectional behaviour of stock returns

- Let R_m be the return on the "market", R_f is the risk free return, and R_i is the return on asset i
- Then the excess return on asset i is:

$$R_i - R_f = \beta_i(R_m - R_f), \text{ where } \beta_i = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}$$

- And β_i can be estimated with regression models (e.g. OLS)
- β_i measures an asset's systematic risk/exposure to market fluctuations
- Assets with high 'betas' are more sensitive to the market
- Since investors seek excess returns, this systematic risk is rewarded with higher prices
- However, the model does not account for idiosyncratic risk
- And while the model does well with cross-sectional data, it performs poorly with time-series data!

4. **No-Arbitrage Multi-Risk Theory:**
   - No-arbitrage relationships are the key intuition behind modern asset pricing developments
   - Arbitrage Pricing Theory (APT) is due to Ross, Sharpe, and Merton
   - Presents a multi-factor approach to asset pricing
   - This generalizes to multiple sources of risk including: inflation risk, business cycle risk, interest rate risk, exchange rate risk, and default risk.
   - Multiple 'betas' and multiple regression models required.
   - But the model assumes that risk and risk premiums are constant
5. **Market Microstructure:**
   - Studies how the market itself works
   - E.g. the role of information asymmetry, information trading, market networks, liquidity, trading volume, who is a buyer vs. who is a seller, bid-ask spreads, etc.
6. **Macro-Finance Models:**
   - C-CAPM (Consumption based CAPM) emerged in the 1980s
   - Shows how individual attitudes to risk and uncertainty are related to variations in asset prices
   - The inter-temporal macroeconomic model based on consumer utility functions is central.
   - The key concept is the Stochastic Discount Factor, otherwise called the Pricing Kernel
   - In macroeconomic models the SDF is tied to the marginal rate of substitution between consumption across periods
   - Consumption, which depends on income and wealth, provides the link between business cycles and asset prices in this model

Finance of CAPM

- The CAPM can be expressed as:

$$R_i - R_f = \beta_i(R_m - R_f)$$

- where r_i is return on asset i, r_f is the risk-free rate, and r_m is the market portfolio return
- The LHS is the excess return on asset i over the risk-free rate
- And the RHS is the excess return on the market portfolio over the risk free rate
- The parameter $\beta_{i,m}$ captures the covariance between the risky asset and the market portfolio (scaled by the variance of the market):

$$\beta_{i,m} = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}$$

- Notice that beta is the same as the OLS regression slope coefficient between returns for asset i and the market m
- The risk of an asset i is characterised by its covariance with the market portfolio
- This particular risk is called systematic risk, and cannot be diversified away
  - Why not? Market risk is embedded in all assets, so not possible for any investors to "take the other side" of the market
- For this reason, systematic risk needs to be rewarded with higher returns

Understanding the CAPM

- If the CAPM is thought of like a regression equation, it can be written as:

$$r_i - r_f = \alpha_i + \beta_{i,m}(r_m - r_f) + \varepsilon_i$$

- where α_i is the regression intercept/mean excess return
- And ε_i is the error term/idiosyncratic asset risk
- Our standard regression assumptions require:

$$E(\varepsilon_i) = 0$$
$$\text{Cov}(r_m, \varepsilon_i) = 0$$

1. **'Alpha'**
   - CAPM (but not necessarily the regression formula) predicts that 'alpha' should be zero for all assets
   - This is because the CAPM states that market risk is the only factor driving excess returns
   - In a regression specification, alpha measures an asset's excess return over and above its risk-adjusted reward
   - From outside the CAPM perspective, alpha may be picking up other (i.e. non-market) risks that are not captured by the model
2. **'Beta'**
   - Beta measures an asset's systematic risk
   - Assets with high betas are more sensitive to the market
3. **'Sigma'**
   - Sigma measures the (variance) of non-systematic risk
   - Non-systematic risk is uncorrelated with systematic risk
   - Often refer to this as idiosyncratic risk

- Total risk of an asset is decomposed as follows:

$$r_i - r_f = \overbrace{\beta_{i,m}(r_m - r_f)}^{\text{Systematic Component}} + \overbrace{\varepsilon_i}^{\text{Idiosyncratic Component}}$$

- Taking the variance of both sides of the equation:

$$\overbrace{\text{Var}(r_i - r_f)}^{\text{Total Risk}} = \overbrace{\beta_{i,m}\text{Var}(r_m - r_f)}^{\text{Systematic Risk}} + \overbrace{\text{Var}(\varepsilon_i)}^{\text{Idiosyncratic Risk}}$$

- Note: $\text{Var}(r_i - r_f) = \text{Var}(r_i)$ and $\text{Var}(r_m - r_f) = \text{Var}(r_m)$
- CAPM is attractive because:

- It is east to understand and sensible as it is built on modern portfolio theory
- It distinguishes between systematic and non-systematic risk
- It is very east to implement empirically

Limitations of CAPM

- Empirical evidence on the performance of CAPM is mixed
- The model does not work well with time series data
- This is because in CAPM investors follow myopic strategies as the investment horizon is short and investment opportunities are assumed to be constant over time
- But in general there are **two** types of systematic risks:
  - Static (temporal) - Market Risk
  - Dynamic (inter-temporal) - Changes in investment opportunities

Multi-Factor CAPM

- Many papers attempted to build on/improve the simple CAPM model by adding more "factors"
- This literature pioneered by Eugene Fama (Nobel Prize winner) and Kenneth French
- Try to identify other explanations for variation in excess returns on a given asset
- Find portfolios of traded securities that are highly correlated with different "factors"
- Hypothesize that the risk premium is linearly related to the risk premium on these portfolios:

$$r_i - r_f = \alpha_i + \beta_{i,1}(r_{F1} - r_f) + \beta_{i,2}(r_{F2} - r_f) + \cdots + \beta_{i,K}(r_{FK} - r_f)$$

- Where r_{FK} is the return on a portfolio correlated with the k-th factor only
- Factors might include: firm size, firm leverage, recent firm performance, etc
- Multiple regression used to estimate the factor betas

Limitations of Multi-Factor CAPM

- May not identify macroeconomic variables that constitute inter-temporal risks
- May not specify the relative importance of these inter-temporal risks
- Need to identify different sources of inter-temporal risks in asset returns and specifiy relative importance to investors
- But the CAPM theory itself gives little guidance on what these factors should be
- There are now hundreds (thousands?!) of proposed factors!
- This leads to the need for 'cleaning up' papers like *Taming the Factor Zoo: A Test of New Factors* by Feng, Giglio, Xiu (2020)

Macroeconomics of C-CAPM

C-CAPM Model Enviroment

- We now derive the macro-finance model known as the Consumption-CAPM
- This model states that asset prices must be closely related to fluctuations in consumption
- The reason being that fluctuations in consumption across time determine willingness to save and take risks
- The setup of the model closely follows the model of precautionary savings we studied in section 5
- x_{t+1} is a random variable

- Uncertainty in x_{t+1} is due to randomness of the **state** that occurs tomorrow
- How do we price or value an asset sold today with this payoff structure?
- To know how to value the asset, we need to know an investor's preferences
- Consider an investor who maximizes expected inter-temporal utility defined over two periods:

$$u(c_t) + \beta E_t[u(c_{t+1})]$$

- Where β is the rate at which future utility is discounted (**not** the CAPM beta!)
- $u(\cdot)$ is a general utility function defined over consumption
- And E_t is the expectations function taken with respect to information available at time t

Utility Functions

- The utility function captures the investor's attitude towards risk
- Note that the **level** of $u(C)$ does not matter
- Instead, it is **marginal utility** that matters
- Marginal utility measures 'hunger' rather than 'happiness', as it describes how much of an **improvement** an additional unit of consumption would make

Risk Aversion and Expected Utlity

- Consider a fair bet that would see you gain \$$x$ or lose \$$x$ with a 50-50 chance
- Starting from $\bar{c}$:

$$E[u(C)] = 0.5 \times u(\bar{C} + x) + 0.5 \times u(\bar{C} - x)$$

- For a risk-averse investor, the utility of expected consumption is greater than the expected utility of consumption: $u[E(C)] > E[u(C)]$
- Investors prefer a sure-thing to a risky bet

Measuring Risk Aversion

- How much do investors dislike risks?
- This can be measured with the Coefficient of Relative Risk Aversion (RRA)

$$RRA = -\frac{c \times u''(c)}{u'(c)}$$

- This measures how much **curvature** there is in the utility function
- Which in turn measures how much an investor is willing to take risks
- We will often work with a Constant Relative Risk Aversion (CRRA) utility function:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

- where γ is a parameter in the utility function
- Marginal utility for this function is:

$$u'(c) = c^{-\gamma}$$

- and the RRA for this utility function is:

$$RRA = -\frac{c \times u''(c)}{u'(c)} = \gamma$$

- and so γ is the risk aversion coefficient

The Asset Pricing Function

- What is the value of the payoff x_{t+1} to an investor with a utility function $u(c)$?
- The asset pricing formula is:

$$p_t = E_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right)$$

- When $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ we have:

$$p_t = E_t\left(\beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} x_{t+1}\right)$$

- The asset pricing equation provides the theoretical basis for understanding macro-asset pricing relationships
- We have seen before that the key element is called the Stochastic Discount Factor:

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

- So that we can rewrite out asset pricing equations as a function of the SDF:

$$p_t = E_t(m_{t+1} x_{t+1})$$

- The fundamental principle of modern asset pricing is that the price of an asset is equal to the expected discounted value of the asset's payoff
- And in microfinance, discounting depends on inter-temporal optimisation through the SDF
- The asset pricing equation can be written as:

$$\underbrace{p_t u'(c_t)}_{\text{Marginal Cost}} = \underbrace{\beta E_t[u'(c_{t+1}) x_{t+1}]}_{\text{Marginal Benefit}}$$

- Marginal cost is the opportunity cost of buying the asset in period 1
- Marginal benefit is the discounted/marginal-utility weighted payoff of the asset in period 2
- Consider an investor facing the following choices:
  1. Consume an extra \$1 today $\Rightarrow u'(c_0)$ or
  2. Invest \$1 in the asset
     - Receive $\$x_{t+1}$ tomorrow
     - Consume the payoff tomorrow $\Rightarrow \beta u'(c_1) x_{t+1}$
- When making optimal decisions, the investor is indifferent between the two choices

Interpretation

i) Before buying, an investor will consider the asset under-priced if:

$$p_t < \beta E_t[u'(c_{t+1})/u'(c_t) x_{t+1}]$$

- Investor reduces consumption today and reallocates resources towards the asset
- Investor keeps buying the asset until consumption in each period equalizes the pricing equation

ii) From investor's perspective, prices are fixed and the formula explains how to adjust consumption

- However, in the macroeconomy investors all together will affect prices
- If aggregate consumption is fixed by total output in the economy, then this pins down asset prices

Asset Pricing Examples

Example 1

- An investor's utility function is: $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$
- An asset purchased today has a payoff x_{t+1} with certainty
- And suppose the investor's income is constant so: $c_t = c_{t+1} = 1$
- Assume $\beta = 1$
- The asset pricing formula yields:

$$\begin{aligned} p_t &= \beta E_t \left(\frac{(u'(c_{t+1})}{u'(c_t)} \times x_{t+1} \right) \\ &= \beta \left(\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \times x_{t+1} \right) \\ &= 1 \times \left(\frac{1^{-\gamma}}{1^{-\gamma}} \times 1 \right) \\ &= 1 \end{aligned}$$

Example 2

- An investor's utility function is: $u(c_t) = \log(c_t)$. Assume $\beta = 1$
- There are two periods. Period 1 is certain, and the investor consumes $c_1 = 1$
- In period 2, $c_2 = Y_2 + x_2$, where $Y = 1$
- There are two states of the world in period 2 describing payoffs:

$$x_2 = \begin{cases} 1 & \text{with probability } 1/4 \\ 2 & \text{with probability } 3/4 \end{cases}$$

$$\begin{aligned} p_1 &= \beta E_1 \left(\frac{(c_2)^{-1}}{(c_1)^{-1} x_2} \right) \\ &= \frac{1}{4} \left(\frac{(1+1)^{-1}}{(c_1)^{-1} x_2} \right) + \frac{3}{4} \left(\frac{(1+2)^{-1}}{(1)^{-1} \times 2} \right) \\ &= \frac{1}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{2}{3} \\ &= \frac{5}{8} \end{aligned}$$

Risk-Free Rate and Consumption Growth

- Let's explore further with the risk-free rate and a specific utility function
- This will help to understand the relationship between asset returns and consumption growth and hence the business cycle
- The Constant Relative Risk Aversion (CRRA) utility function is:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \Rightarrow u'(c) = c^{-\gamma}$$

- Then the asset pricing formula is:

$$p_t = E_t[m_{t+1} x_{t+1}]$$

$$\text{where:} \quad m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t) = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}}$$

- Using some tricks, we can write the SDF with a linear approximation:

$$\begin{aligned} m_{t+1} &= \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \\ &= e^{\ln\beta}e^{\ln\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}} \\ &= e^{\ln\beta}e^{-\gamma\Delta C_{t+1}} \\ &\approx 1+\ln\beta-\gamma\Delta\ln c_{t+1} \end{aligned}$$

- A risk-free asset has a price $p_t = 1$ and payoff/return R_f

$$\begin{aligned} 1 &= E(m_{t+1}R_f) = (m_{t+1})R_f \\ \Rightarrow R_f &= \frac{1}{E(m_{t+1})} \end{aligned}$$

- Now use our linear approximation trick to work out the linear relationship between the risk-free rate and consumption growth:

$$\begin{aligned} R_f &\approx \frac{1}{1+\ln\beta-\gamma E_t\Delta\ln c_{t+1}} \\ &\approx 1-\ln\beta+\gamma E_t\Delta\ln c_{t+1} \end{aligned}$$

1. The risk-free rate is higher, all else equal, when:
   - People are more impatient (i.e. low β)
   - Expected consumption growth is high
2. The risk-free rate is more sensitive to consumption growth when γ is high
   - Risk aversion is governed by γ
   - The risk free rate is more sensitive to consumption when investors are more risk averse

C-CAPM and the Valuation of Risk

- How can we use C-CAPM to price/value risk?
  - That is, what is the value of an asset with a particular risk profile?
- Use the definition of covariance:

$$\begin{aligned} p &= E[mx] \\ &= E[m]E[x] + Cov[m,x] \\ &= \frac{E[x]}{R_f} + Cov[m,x] \end{aligned}$$

- Where we substitute $E[m] = \frac{1}{R_f}$ from the asset pricing formula for the risk-free asset
- Intuition for the value of an asset:

$$p = \underbrace{\frac{E[x]}{r_f}}_{\substack{\text{Present Value} \\ \text{of Payoff x}}} + \underbrace{Cov[m,x]}_{\text{Risk Correction}}$$

$$P_t = \frac{E_t[x_{t+1}]}{R_f} + Cov[m_{t+1}, x_{t+1}]$$

- Using our linear approximation: $m_{t+1} \approx 1 + \ln\beta - \gamma\ln c_{t+1}$

$$\begin{aligned} p_t &\approx \frac{E_t[x_{t+1}]}{R_f} + Cov[1+\ln\beta-\gamma\ln c_{t+1}] \\ &= \frac{E_t[x_{t+1}]}{R_f} - \gamma Cov[\Delta\ln C_{t+1}, x_{t+1}] \end{aligned}$$

- When $Cov[\Delta \ln C_{t+1}, x_{t+1}] > 0$:
  - Asset payoffs are high when future consumption is high
  - This increases consumption risk so has a lower price
- When $Cov[\Delta \ln C_{t+1}, x_{t+1}] < 0$:
  - Asset payoffs are high when future consumption is low
  - This decreases consumption risk so has higher price
- Why is m_{t+1} called the Stochastic Discount Factor?
- Consider the pricing equation for an asset i:

$$P_t = E_t[m_{t+1}x^j_{t+1}]$$

- Notice that m_{t+1} is unknown at time t (and sits with an investor's expectations)
- But the SDF m_{t+1} is the **same** for all assets
- What differs is the covariance between the SDF and the asset payoff x^i_{t+1}
- This asset-specific covariance gives different risk adjustments for each asset
- What matters for asset pricing is co-movement between the random (i.e stochastic) nature of the SDF and individual asser payoffs

How does C-CAPM Related to CAPM

- We can derive a 'beta' similar to the 'beta' in the CAPM
- Define the excess return on asset i as: $R^e_i = R_i - R_f$
  - We can always earn the excess return R^e_i by ***borrowing*** at rate R_f and ***investing*** in asset i with return R_i
  - Note that he price of the asset paying R^e_i is zero, since we borrow \$$x$ and invest \$$x$ at the same time
- The asset pricing formula for an asset with return R_f is: $1 = E_t[m_{t+1}R_f]$
- The asset pricing formula for an asset with return R_i is: $1 = E_t[m_{t+1}R_i]$
- So the asset pricing formula applied to the excess return is:

$$0 = E_t[m_{t+1}(R_i - R_f)] = E_t[m_t{+}1R^e_i]$$

- Again using the definition of covariance:

$$\begin{aligned}
0 &= E_t[m_{t+1}R^e_i] \\
&= E_t[m_{t+1}]E_t[R^e_i] + Cov[m_{t+1}, R^e_i] \\
\Rightarrow E_t[R^e_i] &= -\frac{Cov[m_{t+1}, R^e_i]}{E_t[m_{t+1}]} \\
&= -\frac{Cpv[m_{t=1}, R^e_j]}{Var[m_[t+1]]}\frac{Var[m_{t+1}]}{E_t[m_{t+1}]} \\
&= \beta_{i,m} \times \delta_m
\end{aligned}$$

- Where $\beta_{i,m}$ is like 'beta'/market loading for asset i, as in CAPM
- And λ_m is the 'market' risk
- For C-CAPM, 'market risk' is the risk associated with fluctuations in consumption

- From our earlier approximation to the SDF:

$$Cov(m_{t+1}, R_i^e) \approx Cov(1 + \ln\beta - \gamma\Delta \ln E_t C_{t+1}, R_i^e) = -\gamma Cov(E_t \Delta \ln E_t C_{t+1}, R_i^e)$$
$$Var(m_{t+1}) \approx Var(1 + \ln\beta - \gamma\Delta \ln E_t C_{t+1}) = \gamma^2 Var(E_t \Delta ln C_{t+1})$$

- So our expression for excess returns becomes:

$$\begin{aligned} E_t[R_i^e] &= -\frac{Cov[m_{t+1}, R_i^e]}{Var[m_{t+1}]}\frac{Var[m_{t+1}]}{E_t[m_{t+1}]} \\ &= \frac{Cov(E_t\Delta \ln c_{t+1}, R_i^e)}{Var[E_t\Delta \ln c_[t+1]]} \times \gamma \frac{Var[E_t\Delta \ln c_[t+1]]}{E_t[1 + \ln\beta - \gamma E_t \Delta \ln C_{t+1}]} \\ &= \beta_{i,\Delta C} \times \lambda_{\Delta C} \end{aligned}$$

Interpretation

1. When $Cov[\Delta \ln C_{t+1}, R_i^e] > 0$
   - Excess returns are high when consumption growth is high
   - This means payoffs are high when consumption growth is high
   - This increases consumption risk so has a lower price
   - High excess returns $E_t[R_i^e] \Leftrightarrow$ low price
2. When $Cov[\Delta \ln C_{t+1}, R_i^e] < 0$
   - Excess returns are high when consumption growth is low
   - This means payoffs are high when consumption growth is low
   - This decreases consumption risk so has a higher price
   - Low excess returns $E_t[R_i^e] \Leftrightarrow$ high price
3. Higher $\gamma \Rightarrow$ higher risk aversion $\Rightarrow$ larger effects on prices and returns
4. Only systematic risk matters for prices/returns
   - Systematic risk comes through co-variation with investors consumption growth

C-CAPM: Empirical Issues

There are (at least!) three major 'puzzles' about relationship between C-CAPM and data

1. **The equity premium puzzle**
   - Due to Mehra and Prescott (1985)
   - The equity premium in the data over the last century $\approx 6\%$
   - But C-CAPM calibrated to US business cycle statistics yields an equity premium $\approx 1\%$
   - For reasonable levels of risk aversion, the equity premium observed in the data is far **too high**, it over-compensates for risk
2. **The risk-free rate puzzle**
   - Suppose we can match the equity premium with an (implausibly) high level of risk aversion
   - Then the implied risk-free rate is also very high
   - So why is the risk-free rate observed in the data so **low**?
3. **The volatility puzzle**
   - Due to Shiller (1981)
   - Far too much volatility in stock prices given the (relatively low) volatility in future payoffs

Extensions of the C-CAPM

- The benchmark C-CAPM cannot solve the **equity premium puzzle** and the **risk-free rate puzzle** simultaneously
- This is largely due to the way that risk aversion and inter-temporal substitution are characterised in the model
- There have been many extensions of the C-CAPM model to try and solve these puzzles
- We briefly summarize three of them here:
    1. Habit formation in consumption
    2. Long-run risk model
    3. Heterogeneous agents with incomplete markets

1. Habit Formation in Consumption

- The idea behind habit formation is to generate persistent movements in utility over time
- This generates time-varying risk aversion
- If utility was high in the past, then it should also be high in the future and this reduces current risk aversion
- Consider the habit formation utility function:

$$U = \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma}$$

- Where X_t is the (external) habit stock of consumption
- This is sometimes referred to as 'Keeping Up with the Joneses': if you observe that the consumption of your neighbours is high, then you would like to increase your own consumption
- Habit formation and risk aversion.

$$RRA = -\frac{U'' \times C_t}{U'} = \frac{\gamma}{S_t}$$

- where $S_t \equiv \frac{C_t - X_t}{C_t}$ which is the surplus consumption ratio
- When consumption is high relative to the habit stock, risk aversion falls
- When consumption is low and gets close to the habit stock, risk aversion rises
- Marginal Utility is:

$$U' = (C_t - X_t)^{-\gamma} = S_t^{-\gamma} C_t^{-\gamma}$$

- So the Stochastic Discount Factor becomes:

$$M_{t+1} = \nu \left(u'\left(C_{t+1}\right) / u'\left(C_t\right)\right) = \beta \left(\frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

- The SDF becomes more volatile as the volatility of the surplus consumption ratio S_t rises
- Thus, risk aversion increases without having to increase the risk aversion coefficient γ
- When consumption falls relative to habit, the increase in risk aversion drives up the equity risk premium $\Rightarrow$ time variation in risk aversion (see Cochrane, 2011)

2. Long-Run Risk

- The long-run risks model also tries to generate persistent fluctuations in utility over time
- As with habit formation, this generates time-varying risk aversion
- The main mechanism for doing so is with **recursive** preferences
- These preferences allows for separation between risk aversion and inter-temporal substitution
- Additionally, these preferences imply that investors may prefer **early** or **late** resolution of uncertainty over future consumption
  - When investors prefer early resolution of uncertainty, they must be compensated for long-run risks over consumption
  - And so changes in views of long-run risks affect compensation for holding different assets today
  - And this necessarily affects the equity risk premium
- The long-run risk model uses more complicated Epstein-Zin-Weil preferences:

$$U_t = \left((1-\beta) C_t^{1-\rho} + \beta \left[E_t \left(U_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$

- where γ again governs risk aversion, but the ρ separately governs inter-temporal substitution
- Notice that **expected future utility**, $E_t(U_{t+1}^{1-\gamma})$, affects the value of consumption today
- The SDF for this model is:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{U_{t+1}}{E_t \left(U_{t+1}^{1-\gamma} \right)^{1/(1-\gamma)}} \right)^{\rho-\gamma}$$

- When γ is high, and future utility is risky (i.e. U_{t+1} dispersed), the SDF is higher:
  - ⇒ Higher excess returns (i.e. higher equity risk premium)
  - ⇒ Higher average SDF and so lower risk-free rate

3. Heterogeneous Agents with Incomplete Markets

- So far, everything we have done involves a representative agent: a single entity representing the entire economy
- In this world, individual consumption **is** aggregate consumption
- And so consumption only fluctuates with market-level risk
- But this has two problems empirically:
  1. Aggregate consumption is far too smooth over time, hence low risk premium in C-CAPM
  2. Non-aggregate/idiosyncratic risks seem to be much more important to individual income/consumption
- But the C-CAPM does not account for idiosyncratic risk, and **only prices market risk**
- Instead of a representative agent, assume there are multiple/many agents (i.e. heterogenity)
- These agents cannot diversify away all of their idiosyncratic risks (i.e. incomplete markets)
- Then asset pricing depends on **individual asset pricing equations**
- This opens many avenues for changing the pricing of risk
  - Income equality: if the distribution of income risk is correlated with market risk, this will increase the compensation required to hold risks (i.e. the risk premium)

- Limited market participation: only some people are active investors in a particular asset e.g. stocks, bonds, houses, currencies
- Thus, each asset class may be priced by a different type of investor with their own idiosyncratic risks, changing the risk premium on those assets

Why the C-CAPM is Important Despite its Limitations

- Why is the C-CAPM still of interest, despite worse empirical performance than 'reduced form' models like the CAPM or multi-factor models?
- *Macroeconomics and Finance*
  - Asset markets are the mechanism that helps us to understand the equation of savings to investment, and the allocation of consumption and investment across time and states of nature
  - The relationship between asset prices and the macroeconomy helps us understand important topics like: monetary policy, fiscal policy (i.e. government debt), mortgages, houses, investment, exchange rates, credit markets, etc
  - We need a theoretically consistent way of linking asset prices back to the macroeconomy
  - C-CAPM is the foundational model (however imperfect) to help us do this

Housing and the Business Cycle I

Simple Models of Housing Purchase Decisions

- When modeling housing decisions in the macroeconomy, we need to consider **three** primary assets associated with the housing market
  - Owner-occupied housing
  - Residential investment property
  - Mortgages to finance house purchase
- Developments in any one of these asset markets can influence each of the other markets, as well as the macroeconomy as a whole
- We will study simple decision models for each asset, starting with investment property

A Model of Housing Investment Decisions

- Investors choose the size of the investment property they want to hold, H_t
- Houses can be bought and sold at price P_t^h
- Investment property earns a rental return R_t^h in the period in which it is purchased
- Houses depreciates at rate δ, proportional to the value of the investment property
- Resale of houses is subject to a simple sales tax τ, proportional to the value of the investment property at date of sale (i.e. similar to a capital gains tax)
- Investors also have access to a bond B_t that pays return R_{t+1} next period
- An investor's infinite-horizon decision problem is:

$$\max_{C_t,B_t,H_t} \quad \sum_{t=0}^{\infty} \beta^t \log C_t$$

$$\text{s.t.} \quad C_t + \underbrace{P_t^h H_t}_{\substack{\text{Cost of}\\ \text{new}\\ \text{housing}}} + B_t = Y_t + \underbrace{R_t^h H_t}_{\substack{\text{rental}\\ \text{yield}\\ \text{on housing}}} + B_{t-1} R_t + \underbrace{P_t^h H_{t-1}}_{\substack{\text{Resale value}\\ \text{of previous}\\ \text{housing}}} - \underbrace{\delta P_t^h H_{t-1}}_{\substack{\text{Depreciation}\\ \text{cost on previous}\\ \text{housing}}} - \underbrace{\tau P_t^h H_{t-1}}_{\substack{\text{Sales tax}\\ \text{on previous}\\ \text{housing value}}}$$

$$H_t \geq 0$$

- Note that the investor cannot “borrow” in (or “short”) housing
- However, the investor may save or borrow in the risk free bond
- The Lagrangian is

$$\mathcal{L} = \beta^t \log C_t - \lambda(Y_t + R^h_t H_t + B_{t-1}R_t + P^h_t H_{t-1} - \delta P^h_t H_{t-1} - \tau P^h_t H_{t-1} - C_t - P^h_t H_t - B_t)$$

- The first order conditions for the investor are:

$$\begin{aligned} C_t: &\quad \lambda_t = \beta^t \frac{1}{C_t} \\ B_t: &\quad \lambda_t = \lambda_{t+1} R_{t+1} \\ H_t: &\quad 0 = \lambda_t (R^h_t - P^h_t)\lambda_{t+1}(1-\delta-\tau)P^h_{t+1} \end{aligned}$$

- Together, these form the two Euler equations:

$$\begin{aligned} \frac{1}{C_t} &= \beta\left[R_{t+1}\frac{1}{C_{t+1}}\right] && \text{Bond Euler Equation} \\ P^h_t \frac{1}{C_t} &= R^h_t \frac{1}{C_t} + \beta\left[\frac{1}{C_{t+1}}(1-\delta-\tau)P^h_{t+1}\right] && \text{Housing Euler Equation} \end{aligned}$$

- Substitute the bond Euler equation into the housing Euler equation:

$$P^h_t = R^h_t + (1-\delta-\tau)\frac{P_{t+1}}{R_{t+1}}$$

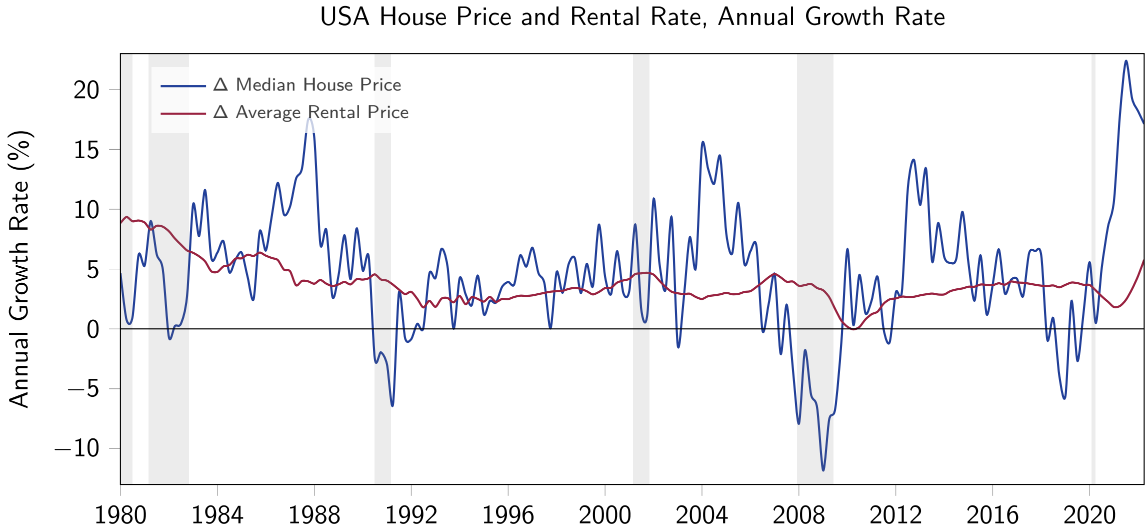
- Assume that $R_t = R$ for all t
- Step forward one period, and substitute P^h_{t+1} into the right hand side repeatly:

$$P^h_t = \sum_{s=0}^{\infty}\left(\frac{1-\delta-\tau}{R}\right)^s R^h_{t+s}$$

- So current house prices reflect the discounted stream of future rental flows

1. Higher interest rates
  - Can reflect higher opportunity cost of housing investment (i.e. alternative is to investment)
  - Can also reflect higher cost of borrowing using the bond
  - Higher opportunity/borrowing costs reduce future rental payoffs

2. Higher depreciation and taxes
  - Higher depreciation implies a higher carrying cost of holding as an investment
  - Higher taxes reduce the return due to capital gains
  - Both depreciation and taxes reduce the return to housing investment



A Model of Homeownership Decisions

Housing as an owner-occupied asset/durable good:

- Households may be renters or homeowners
  - Assume that households are indifferent between renting and owning
- For renters:
  - Choose size of the house to be rented, H_t^R
  - Enjoy utility from housing services rented
  - Rental cost of R_t^h per unit of housing rented
- For homeowners:
  - Choose size of the house to be purchased/owned, h_t^O
  - Enjoy utility from housing services owned
  - Houses can be bought and sold at price P_t^h
  - Houses depreciate at a rate δ, and houses sales are subject to tax τ
  - Owners also have access to a bond B_t, that pays return R_{t+1} next period
- Both renters and homeowners enjoy utility $u(C_t, H_t)$ over the houses they rent/own
- Common functional forms:

$$\begin{aligned} u(C_t, H_t) &= \alpha \log C_t + (1-\alpha) \log H_t && \text{Separable Utility} \\ u(C_t, H_t) &= C_t^{\alpha} H_t^{1-\alpha} && \text{Cobb-Douglas Utility} \\ u(C_t, H_t) &= \left[\alpha C_t^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha) H_t^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon-1}{\epsilon}} && \text{Constant Elasticity of Substitution Utility} \end{aligned}$$

- Where α is the share of total expenditure on consumption
- For CES utility, ϵ is the elasticity of substitution between consumption and housing services

A Model of Household Rental Decisions

- Renters have a simple "static" decision each period:

$$\begin{aligned} \max_{C_t^R, H_t^R} \quad & \alpha \log(C_t^R) + (1-\alpha) \log(H_t^R) \\ \text{s.t.} \quad & C_t^R + R_t^h H_t^R = Y_t \end{aligned}$$

- The first order conditions for the renting household yield:

$$\frac{(1-\alpha)}{\alpha}\frac{1/H_t^R}{1/C_t^R} = R_t^h$$

- Which says that the marginal rate of substitution between housing and consumption is equal to the rental cost of housing

A Model of Household Homeownership Decisions

- Homeowners solve an infinite-horizon problem:

$$\begin{aligned} \max_{C_t^0, B_t^0, H_t^0} \quad & \sum_{t=0}^{\infty} \beta^t (\alpha \log C_t + (1-\alpha) \log H_t^O) \\ \text{s.t.} \quad & C_t^O + P_t^h H_t^O + B_t = Y_t + B_{t-1}^O R_t + (1-\delta-\tau) P_t^h H_{t-1}^O \\ & H_t \geq 0 \end{aligned}$$

- As with the investor, the homeowner may save or borrow in the risk free bond
- Unlike the investor, a homeowner does not receive the rental yield from houses they occupy
- The Lagrangian is

$$\mathcal{L} = \beta^t(\alpha \log C_t + (1-\alpha)\log H_t^O) + \lambda(Y_t + B_{t-1}^O R_t + (1-\delta-\tau)P_t^h H_{t-1}^O - C_t^O - P_t^h H_t^O - B_t)$$

- The first order conditions for the homeowner are:

$$\begin{aligned} C_t^O: &\quad \lambda_t = \alpha \frac{1}{C_t^O} \\ B_t^O: &\quad \lambda_t = \beta \lambda_{t+1} R_{t+1} \\ H_t^O: &\quad 0 = (1-\alpha)\frac{1}{H_t^O} - \beta\lambda_{t+1}(1-\delta-\tau)P_{t+1}^h \end{aligned}$$

- Together, these form the two Euler equations:

$$\begin{aligned} \frac{1}{C_t^O} &= \beta R_{t+1} \frac{1}{C_{t+1}} && \text{Bond Euler Equation} \\ P_t^h \alpha \frac{1}{C_t} &= (1-\alpha)\frac{1}{C_t} + \beta\alpha \frac{1}{C_{t+1}^O}(1-\delta-\tau)P_{t+1}^h && \text{Housing Euler Equation} \end{aligned}$$

- Rewrite the housing Euler equation as:

$$\beta \frac{C_t^O}{C_{t+1}^O}(1-\delta-\tau)P_{t+1}^h P_t^h = \underbrace{\frac{(1-\alpha)}{\alpha}\frac{1/H_t^O}{1/C_t^O}}_{\substack{\text{Marginal rate of} \\ \text{substitution between} \\ \text{housing and consumption}}} \underbrace{+\beta \frac{C_t^O}{C_{t+1}^O}(1-\delta-\tau)P_{t+1}^h}_{\substack{\text{Present discounted value} \\ \text{of capital gain on housing}}}$$

- Where the MRS between housing and consumption is the flow value of housing services derived from homeownership
- Since households are indifferent between renting and owning a home, utility must be the same in every period:

$$U^R \equiv \alpha \log C_t^R + (1-\alpha)\log H_t^R \quad = \quad \alpha \log C_t^O + (1-\alpha)\log H_t^O \equiv U^O$$

- Which means that consumption and housing choices are the same in every period:

$$\begin{aligned} C_t^R &= C_t^O = C_t \\ H_t^R &= H_t^O = H_t \end{aligned}$$

- The renter and homeowner optimality conditions are:

$$R^h_t = \frac{(1-\alpha)}{\alpha}\frac{1/H_t}{1/C_t} \qquad \text{Renter Optimality Condition}$$
$$P^h_t = \frac{(1-\alpha)}{\alpha}\frac{1/H_t}{1/C_t} + \beta\frac{C_t}{C_{t+1}}(1-\delta-\tau)P^h_{t+1} \qquad \text{Homeowner Euler Equation}$$

- Combining:

$$P^h_t = R^h_t + \beta\frac{C_t}{C_{t+1}}(1-\delta-\tau)P^h_{t+1}$$

- So the flow value of housing services is equivalent to the rental rate on housing
- In fact, this is exactly how national statistical agencies aim to measure rents in macroeconomic data e.g. GDP, the CPI
- Finally, note the equivalence between the price of houses from the homeowner's perspective and from the investor's perspective

$$P^h_t = \frac{(1-\alpha)}{\alpha}\frac{1/H_t}{1/C_t} + \beta\frac{C_t}{C_{t+1}}(1-\delta-\tau)P^h_{t+1} \qquad \text{Homeowner's Asset Price Equation}$$
$$P^h_t = R^h_t + \beta\frac{C_t}{C_{t+1}}(1-\delta-\tau)P^h_{t+1} \qquad \text{Investors's Asset Price Equation}$$

- These asset pricing equations are the same, despite the fact that investors earn rents while homeowners enjoy the service flow of housing

Limitations of the Simple Housing Model

- There are several issues that complicate housing purchase/investment decisions relative to the simple models
- These complications are important for properly understanding:
  - Homeownership rates
  - Mortgage borrowing and indebtedness
  - The consumption decisions of homeowners
  - House prices
- Borrowing costs:
  - Mortgage interest rates are higher than risk-free asset/deposit rates
  - Borrowing costs also vary with risk e.g. high debt loads, less ability to repay
  - Implies different borrowers face different mortgage financing costs
  - Borrowing costs also vary over the business cycle and the credit cycle
- Borrowing constraints:
  - Both banks and governments restrict the ability to borrow against housing
  - Restrictions on: loan-to-value ratios, debt-to-income ratios, and payment-to-income ratios
  - These restrictions are in place to prevent risky borrowing by homeowners
- Housing adjustment costs:
  - Sales costs: e.g. real estate agent fees, stamp duty, moving costs
  - Home owners adjust infrequently, may stay in far-from-optimal housing for long periods
  - Following a shock, homeowners may be forced to dramatically cut consumption, rather than adjust housing

Housing and the Business Cycle II

A Simple Model with Mortgage Finance Decisions

- Consider a two-period homeowner decision problem:

$$\begin{aligned}\max_{C_t,B_t,H_t} \quad & \log C_1 + \alpha \log H + \beta \log C_2 \\ \text{s.t.} \quad & C_1 + P_1^h H + B = Y_1 \\ & C_2 = Y_2 + B\tilde{R}(B) + (1-\delta)P^H H \\ & H \geq 0\end{aligned}$$

- Where the interest rate depends on the savings/borrowing decision:

$$\tilde{R} = \begin{cases} R & \text{if } B \geq 0 \\ R^m & \text{if } B < 0 \end{cases}$$

- and $R^m > 0$ means that borrowing is more costly than the return of savings
- The Lagrangian equation is:

$$\mathcal{L} = \log C_1 + \alpha \log H + \beta \log C_2 + \lambda_1 (Y_1 - C_1 - P_1^h H - B) + \lambda_2 (Y_2 + B\tilde{R}(B) + (1-\delta)p^H H - C_2)$$

- And the first order conditions are

$$\begin{aligned} C_1: & \quad \frac{1}{C_1} = \lambda_1 \\ C_2: & \quad \beta \frac{1}{C_2} = \lambda_2 \\ B: & \quad \lambda_1 = \lambda_2 \tilde{R}(B) \\ H: & \quad \alpha \frac{1}{H} + \lambda_2 (1-\delta) P^H = \lambda_1 P_1^h \end{aligned}$$

- Combining the first order conditions
- The Euler equations for a homeowner with savings ($B \leq 0$) are:

$$\begin{aligned} \frac{1}{C_1^S} &= \beta R \frac{1}{C_2^S} && \text{Bond Euler Equation} \\ P_1^h \frac{1}{C_1^S} &= \alpha \frac{1}{H^S} + \beta \frac{1}{C_2^S}(1-\delta)p^H && \text{Housing Euler Equation} \end{aligned}$$

- The Euler equations for a homeowner that borrows ($B < 0$) are:

$$\begin{aligned} \frac{1}{C_1^B} &= \beta R^m \frac{1}{C_2^B} && \text{Bond Euler Equation} \\ P_1^h \frac{1}{C_1^B} &= \alpha \frac{1}{H^B} + \beta \frac{1}{C_2^B}(1-\delta)p^H && \text{Housing Euler Equation} \end{aligned}$$

- Substituting in for the bond Euler equation, the house price for a homeowner with savings:

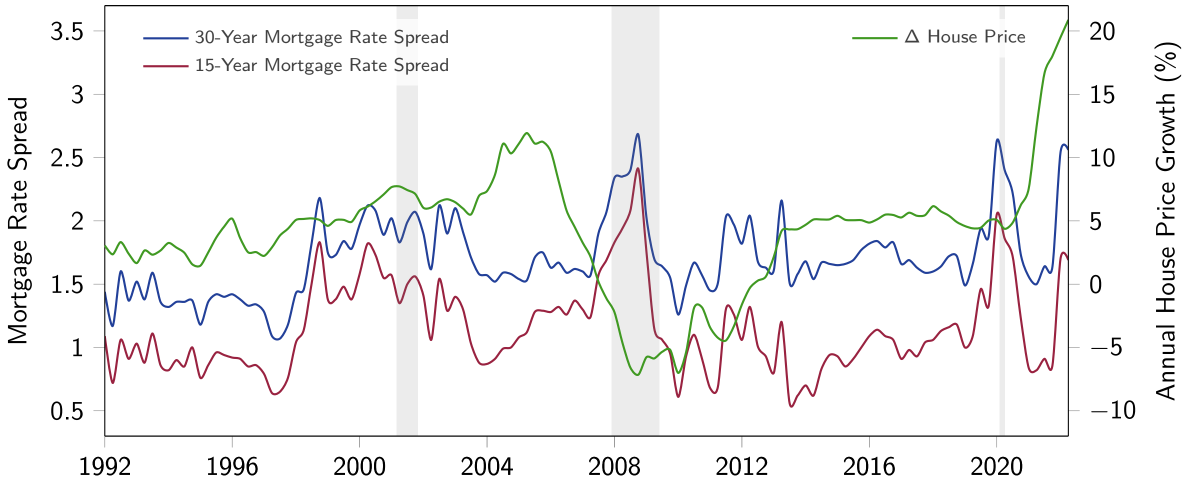
$$P_1^h = \alpha \frac{\frac{1}{H^S}}{\frac{1}{C_1^S}} + \frac{(1-\delta)P_2^h}{R}$$

- Substituting in for the bond Euler equation, the house price for a homeowner that borrows:

$$P_1^h = \alpha \frac{\frac{1}{H^B}}{\frac{1}{C_1^B}} + \frac{(1-\delta)P_2^h}{R^m}$$

- Borrowing at a higher interest rate increases the cost of house purchase
- This decreases demand for housing, and reduces willingness to pay for housing by borrowers
- Periods with a low mortgage interest rate spread are associated with higher house prices
- Spread = mortgage rate - 10 year bond

Mortgage Rate Spreads vs. House Price Growth



A Model of Mortgage Finance and Consumption Decisions

- We now want to study the joint mortgage-consumption decisions
- To simplify,
  - Suppose homeowners has already choosen the size of house H
  - Suppose house prices are constant: $P_1^h = P_2^h = P^h$
- Again, choose consumption and savings/debt subject to costly mortgage finance:

$$
\begin{aligned}
\max_{C_1, C_2, B} \quad & \log C_1 + \beta \log C_2 \\
\text{s.t.} \quad & C_1 + P^h H + B = Y_1 \\
& C_2 = Y_2 + B\tilde{R}(B) + (1-\delta)P^h H \\
& \tilde{R} = \begin{cases} R & \text{if } B \geq 0 \\ R^m & \text{if } B < 0 \end{cases}
\end{aligned}
$$

- And $R^m > R$
- The inter-temporal budget constraint is:

$$
C_1 + P^h H + \frac{C_2}{\tilde{R}(B)} = Y_1 + \frac{Y_2 + (1-\delta)P^h H}{\tilde{R}(B)}
$$

- When borrowing, $\tilde{R}(B) = R^m > R$ reduces total resources available to consume
- Additionally, the household saves/borrows wherever:

$$
\begin{aligned}
B > 0, \quad & \text{if } C_1 < Y_1 P^h H \\
B > 0, \quad & \text{if } C_1 \geq Y_1 P^h H
\end{aligned}
$$

- Borrow whenever consumption is greater than what is leftover after purchasing a house

Graphical Illustration

- We can plot the household's constraints and consumption decisions in (C_1, C_2)-space
- To plot the inter-temporal budget constraint, note that:
  - The most a household can consume in period 1 occurs when $C_2 = 0$

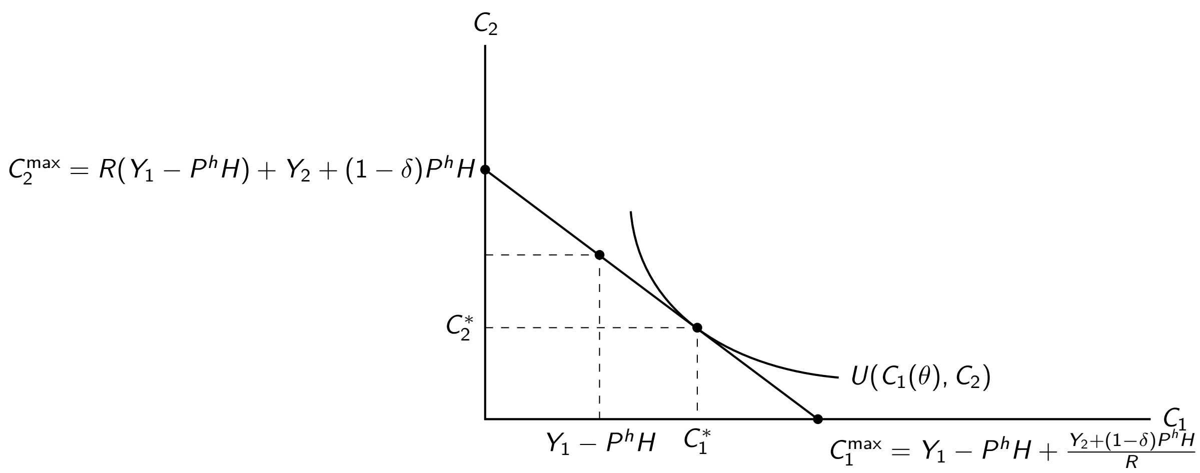
$$C_1^{\max} = Y_1 - P^h H + \frac{Y_2 + (1-\delta)P^h H}{\tilde{R}(B)}$$

  - The most a household can consume in period 2 occurs when $C_1 = 0$

$$C_2^{\max} = \tilde{R}(B)(Y_1 - P^h H) + Y_2 + (1-\delta)P^h H$$

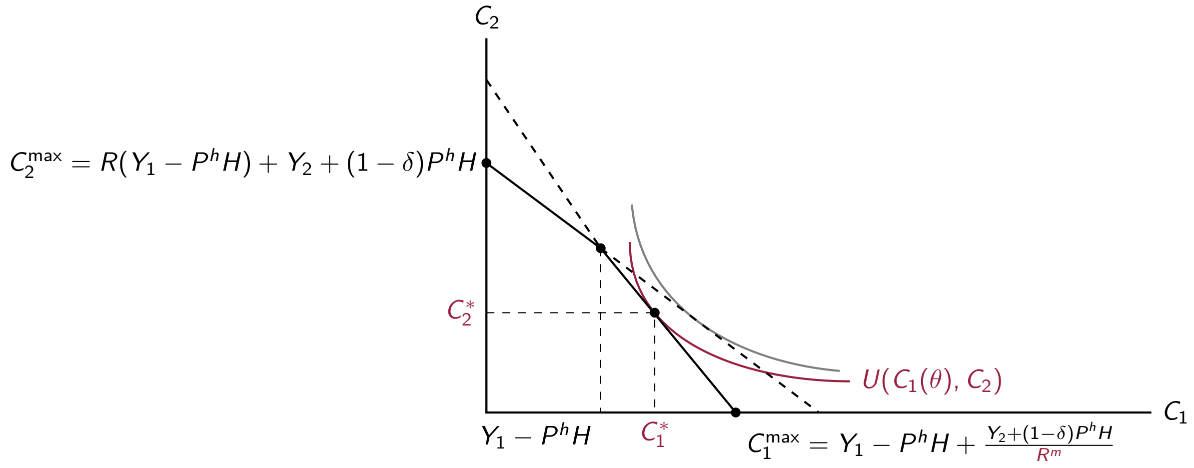
Graphical Illustration: No Additional Borrowing Costs

- First, let's suppose that $R^m = R$, so there is no additional cost for borrowing
- Given the cost of housing $P^h H$, the optimal consumption choices are c_1^*, c_2^*
- Because $C_1^* > Y_1 - P^h H$, the household is currently borrowing



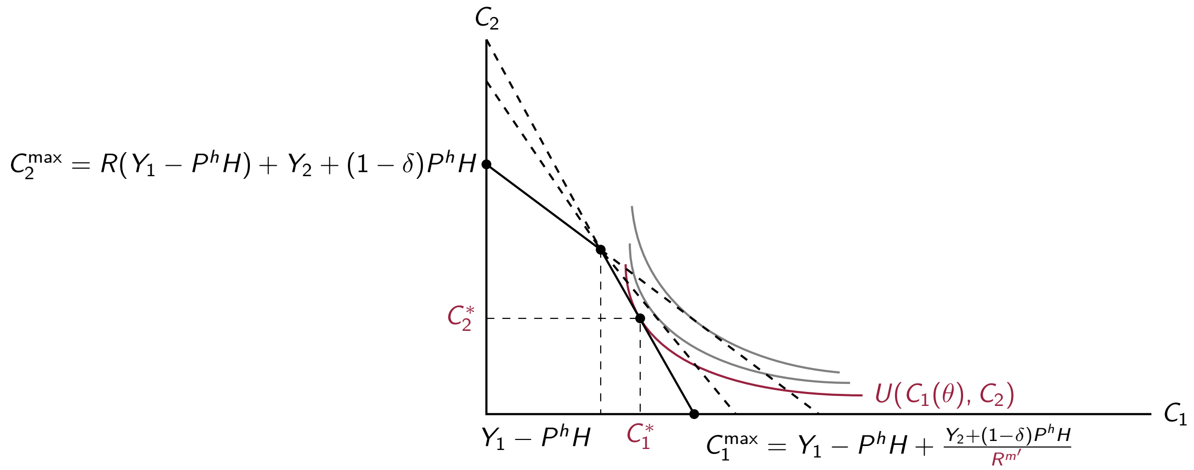
Graphical Illustration: Costly Mortgage Borrowing

- Now suppose $R^m > R$, so that borrowing to finance housing is expensive
- The budget constraint under R^m has a steeper slope (and higher y-intercept)
- Consumption below $C_1 = Y_1 - P^h H$ is unaffected by the borrowing cost since saving at rate R
- Consumption above $C_1 = Y_1 - P^h H$ implies lower C_2 due to higher borrowing cost R^m
- When borrowing, both C_1 and C_2 are lower as cost of borrowing is spread across time



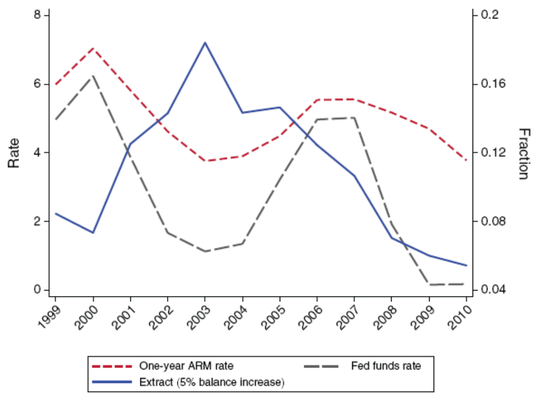
Graphical Illustration: An Increase in the Cost of Mortgage Borrowing

- Finally, suppose that the mortgage interest rate increases from R^m to $R^{m'}$
- Budget constraint rotates, with a fall in the maximum consumption possible in period 1
- More costly borrowing reduces resources available for consumption after repaying debt
- Again, spread the cost of borrowing across time, so both C_1 and C_2 decrease

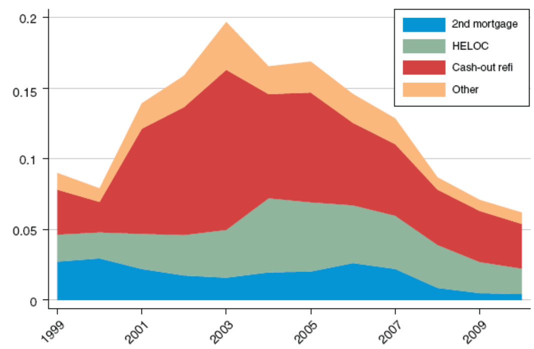


Mortgage Finance Costs, Borrowing, and Consumption: Emperical Evidence

- How do borrowing and consumption respond to changes in mortgage interest rates?
- Bhutta and Keys (2016) show that declining mortgage interest rates resulted in significant "housing equity extraction" in the form of "cash out refinancing"
- They show that very little of the cash extracted was used to repay other debts
- Instead, the cash was used to finance consumption expenditures (e.g. cars, home renovation, holidays, etc)



Probability of Extracting Equity in a Given Year versus Interest Rate



Method of Equity Extraction, by Year

Source: Bhutta and Keys (2016) Interest Rates and Equity Extraction During the Housing Boom

A Model of Constrained Mortgage Finance Decisions

- Again consider a two period model where the homeowner has already chosen a house, H
- The house is purchased in period 1 at P_1^h, and is sold in period 2 at price P_2^h
- Here, the homeowner is restricted in the amount that can be borrowed, B

$$\begin{aligned} \max_{C_1, C_2, B} \quad & \log C_1 + \beta \log C_2 \\ \text{s.t.} \quad & C_1 + P_1^h H = Y_1 + B \\ & C_2 + RB = Y_2 + (1-\delta) P_2^h H \\ & B \leq \bar{\theta} P_1^h H \end{aligned}$$

- The final inequality is a loan to value constraint
- The amount borrowed cannot exceed a fraction $\bar{\theta}$ of the value of the house
- We can rewrite the budget constraints in terms of the LTV borrowing ratio $\bar{\theta} = \frac{B}{P_1^h H}$
- For period 1:

$$\begin{aligned} C_1 + P_1^h H &= Y_1 + B \\ C_1 + P_1^h H &= Y_1 + \frac{B}{P_1^h H} P_1^h H \\ C_1 + P_1^h H &= Y_1 + \bar{\theta} P_1^h H \end{aligned}$$

- For period 2:

$$\begin{aligned} C_2 + RB &= Y_2 + (1-\delta) P_2^h H \\ C_2 + R \frac{B}{P_1^h H} P_1^h H &= Y_2 + (1-\delta) P_2^h \\ C_2 + R\bar{\theta} P_1^h H &= Y_2 + (1-\delta) P_2^h H \end{aligned}$$

- And where the LTV choice must be less than the maximum LTV: $\theta \leq \bar{\theta}$
- To illustrate the importance of the LTV constraint, we will make a figure in (θ, C_2)-space
  - This is similar to our previous figures in (C_1, C_2)-space
  - θ governs the amount borrowed, which has a direct effect on C_1

- Take the period 2 budget constraint:

$$C_2 + R\bar{\theta}P_1^h H = Y_2 + (1-\delta)P_2^h H$$

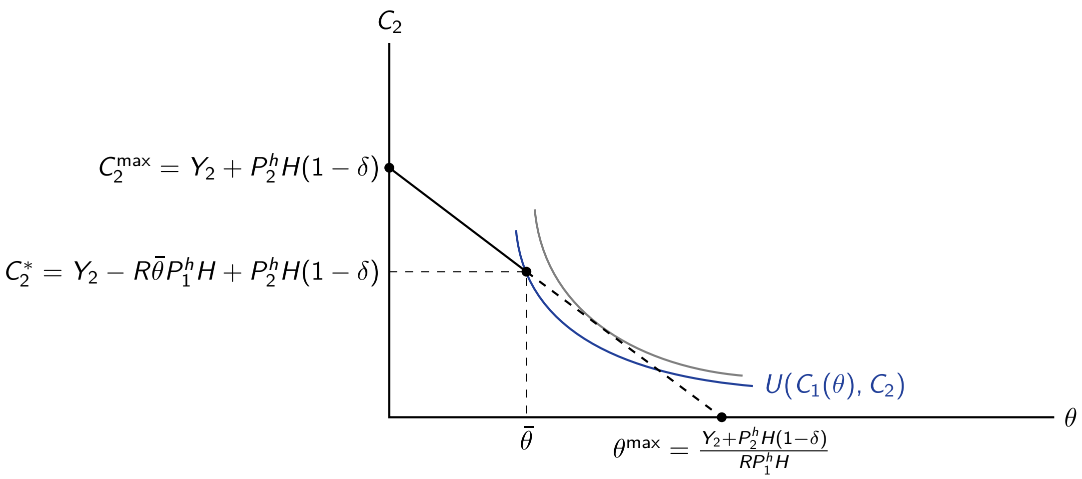
- When the household borrows nothing, $\theta = 0$ and maximum consumption is:

$$C_2^{\max} = Y_2 + (1-\delta)P_2^h H$$

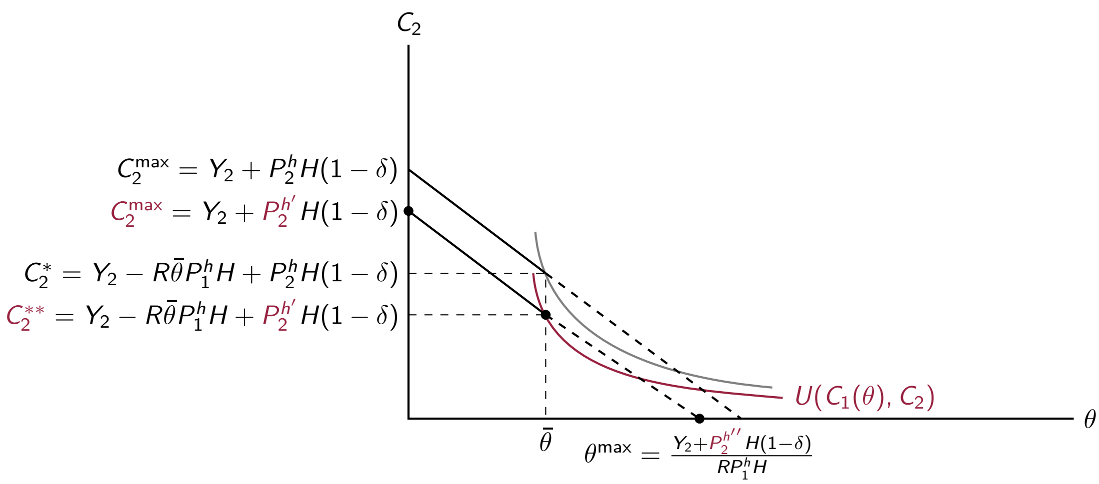
- If there were no constraint on the LTV choice and $C_2 = 0$, then the maximum LTV would be:

$$\theta^{\max} = \frac{Y_2 + (1-\delta)P_2^h H}{RP_1^h H}$$

- Because of the LTV constraint ($\theta \leq \bar{\theta}$), the household is restricted in its borrowing
- This yields a truncated budget constrain in $(\theta - C_2)$-space
- Note: Higher θ implies higher C_1, so the indifference curve is convex in $(\theta - C_2)$-space
- When constrained, consume more in period 2 and less in period 1 than if unconstrained



- Now consider a fall in the period 2 house price: $P_2^h \rightarrow P_2^{h'}$



- What is the difference between constrained vs. unconstrained households?
- The optimal consumption choices for each type of household are:

$$C_2^{\text{con}} = Y_2 - R\bar{\theta}P_1^h H + (1-\delta)P_2^h H, \qquad C_2^{\text{unc}} = \frac{\beta R}{1+\beta}\left(Y_1 - P_1^h H + \frac{Y_2 + (1-\delta P_2^h H)}{R}\right)$$

- And the consumption responses to P_2^h are:

$$\frac{\partial C_2^{\text{con}}}{\partial P_2^h} = (1-\delta)H, \qquad \frac{\partial C_2^{\text{unc}}}{\partial P_2^h} = \frac{\beta}{1+\beta}(1-\delta)H$$

- Constrained consumption much more sensitive than unconstrained consumption!

Housing Booms and Busts

A Simple Model of Mortgage Credit and Housing Booms and Busts

- How do credit conditions affect the housing marker?
- Consider a simple model of a household that purchases housing using a mortgage
- The size of the mortgage is determined by credit conditions:
  - The cost of borrowing, i.e. the interest rate
  - The maximum Loan-to-Value constraint on mortgage borrowing
- Housing market equilibrium:
  - House prices adjust to ensure that housing demand equals housing supply

  The household problem is:

$$\begin{aligned}
\max_{C_t, C_{t+1}, B_{t+1}, H_{t+1}} \quad & u(C_t) + \beta[u(C_{t+1}) + v(H_{t+1})] \\
\text{s.t.} \quad & C_t + P_t H_{t+1} = Y_t + B_{t+1} \\
& C_{t+1} + (1 + r_{t+1})B_{t+1} = Y_{t+1} + (1-\delta)P_{t+1}H_{t+1} \\
& B_{t+1} \leq \theta_t P_t H_{t+1}
\end{aligned}$$

- Choose housing at time t to be enjoyed at time $t+1$
- Sell housing at time $t+1$
- Borrow B_{t+1} to finance housing, subject to a maximum LTV constraint
- Make three very useful simplifying assumptions:
  - Linear utility in consumption: $u(C) = C$
  - Log utlity in housing: $v(H) = \log H$
  - Household is always constrained, i.e. always borrowing as much as allowed by the LTV constraint

$$\begin{aligned}
\max_{C_t, C_{t+1}, B_{t+1}, H_{t+1}} \quad & C_t + \beta[C_{t+1} + \log H_{t+1}] \\
\text{s.t.} \quad & C_t + P_t H_{t+1} = Y_t + B_{t+1} \\
& C_{t+1} + (1 + r_{t+1})B_{t+1} = Y_{t+1} + (1-\delta)P_{t+1}H_{t+1} \\
& B_{t+1} = \theta_t P_t H_{t+1}
\end{aligned}$$

- The Lagrangian function:

$$\begin{aligned}
\mathcal{L} = C_t + \beta[C_{t+1} + \log H_{t+1}] + \lambda_t(Y_t + B_{t+1} - C_t - P_t H_{t+1}) \\
+ \lambda_{t+1}(Y_{t+1} + (1-\delta)P_{t+1}H_{t+1} - C_{t+1} - (1 + r_{t+1})B_{t+1}) + \mu_t(\theta_t P_t H_{t+1} - B_{t+1})
\end{aligned}$$

- λ_t, λ_{t+1} are Lagrange multipliers on the budget constraints
- μ_t is the Lagrange multiplier on the LTV constraint

- The first order conditions are:

$$\begin{aligned}
C_t: &\quad 1 = \lambda_t \\
C_{t+1}: &\quad \beta = \lambda_{t+1} \\
B_{t+1}: &\quad \lambda_t = \lambda_{t+1}(1 + r_{t+1}) + \mu_t \\
H_{t+1}: &\quad \lambda_t P_t = \beta \frac{1}{H_{t+1}} + \lambda_{t+1}(1-\delta)P_{t+1} + \mu_t \theta_t P_{t+1}
\end{aligned}$$

- Combining the first order conditions, and repeating the LTV constraint, we have:

$$\begin{aligned}
1 - \mu_t &= \beta(1 + r_{t+1}) && \text{Consumption Euler Equation} \\
P_t &= \frac{\beta}{1 - \mu_t \theta_t}\left(\frac{1}{H_{t+1}} + (1-\delta)P_{t+1}\right) && \text{Housing Euler Equation} \\
B_{t+1} &= \theta_t P_t H_{t+1} && \text{LTV Constraint}
\end{aligned}$$

- Note μ_t is the Lagrange multiplier on the borrowing constraint
- It tells us the marginal value of an extra dollar borrowed to finance housing

Housing Market Equilibrium

- Housing market equilibrium:

$$\underbrace{H_{t+1}}_{\text{Housing Demand}} = \underbrace{\bar{H}}_{\text{Housing Supply}}$$

- The house price p_t adjusts to ensure housing market clears in each period t

Model Experiment: Expansion of Mortgage Credit

- First, find the steady state of the model
- Assume that all variables are the same forever eg. $r_t = t_{t+1} = r$
- Our model equations in the steady state are:

$$1 - \mu = \beta(1 + r) \tag{10.1}$$

$$P = \frac{\beta}{1 - \mu\theta}\left(\frac{1}{\bar{H}} + (1-\delta)P\right) \tag{10.2}$$

$$B = \theta P \bar{H} \tag{10.3}$$

- Solve for the house price P (using equation (10.1) and equation (10.2)):

$$P = \frac{\beta}{\bar{H}((1-\theta)(1-\beta) + \beta(\theta r + \delta))}$$

- Solve for mortgage debt (using equation (10.3))

$$B = \frac{\beta\theta}{((1-\theta)(1-\beta) + \beta(\theta r + \delta))}$$

- First, consider the effect of a change in the interest rate r

$$\frac{\partial P}{\partial r} = -\bar{H}\beta\theta \times \frac{\beta}{(\bar{H}((1-\theta)(1-\beta) + \beta(\theta r + \delta)))^2} < 0$$

$$\frac{\partial B}{\partial r} = -\beta\theta \times \frac{\beta\theta}{((1-\theta)(1-\beta) + \beta(\theta r + \delta))^2} < 0$$

- Decrease in interest rates leads to: 1. increase in house prices; 2. increase in mortgage debt

  - Lower mortgage finance costs increase housing demand
  - With fixed housing supply $\bar{H}$, prices must increase
  - To finance higher-priced houses, households must increase borrowing

- Second, consider the effect of a change in the maximum LTV ratio θ

$$
\begin{aligned}
\frac{\partial P}{\partial \theta} &= -\bar{H}(1-\beta(1+r)) \times \frac{\beta}{(\bar{H}((1-\theta)(1-\beta)+\beta(\theta r+\delta)))^2} \\
\frac{\partial B}{\partial \theta} &= \frac{\beta\theta}{((1-\theta)(1-\beta)+\beta(\theta r+\delta))} + (1-\beta(1+r)) \times \frac{\beta\theta}{((1-\theta)(1-\beta)+\beta(\theta r+\delta))^2}
\end{aligned}
$$

- If $\beta(1+r) > 1$, households are patient and/or have high costs of borrowing
  - Demand for housing does not rise with increased borrowing opportunities

$$\frac{\partial P}{\partial \theta} < 0, \qquad \frac{\partial B}{\partial \theta} < 0$$

- If $\beta(1+r) < 1$, households are impatient and/or have with costs of borrowing
  - Demand for housing rises with increased borrowing opportunities
  - Seems most likely case since we observe households borrow a lot to finance housing

$$\frac{\partial P}{\partial \theta} > 0, \qquad \frac{\partial B}{\partial \theta} > 0$$

Dynamics of Mortgage and Housing Markets

Dynamic Model Experiment: Expansion of Mortgage Credit

- Now study the dynamics of the model in response to an expansion of mortgage credit
- We will consider effect of our two shocks:
  1. A decrease in the mortgage interest rate
  2. An increase in the maximum LTV ratio on mortgage borrowing
- Recall the FOCs/optimal decisions of the household:

$$
\begin{aligned}
1-\mu_t &= \beta(1+r_{t+1}) && \text{Consumption Euler Equation} \\
P_t &= \frac{\beta}{1-\mu_t\theta_t}\left(\frac{1}{H_{t+1}} + (1-\delta)P_{t+1}\right) && \text{Housing Euler Equation} \\
B_{t+1} &= \theta_t P_t \bar{H} && \text{LTV Constraint}
\end{aligned}
$$

- To begin, suppose "beliefs" about future house prices are held fixed: $P_{t+1} = P$
- Trace out effect of a decline in r_{t+1}

$$
\begin{aligned}
1-\underbrace{\mu_t}_{\uparrow} &= \beta(1+\underbrace{r_{t+1}}_{\downarrow}) && \text{Consumption Euler Equation} \\
\underbrace{P_t}_{\uparrow} &= \frac{\beta}{1-\underbrace{\mu_t}_{\uparrow}\theta_t}\left(\frac{1}{H_{t+1}} + (1-\delta)P_{t+1}\right) && \text{Housing Euler Equation} \\
\underbrace{B_{t+1}}_{\uparrow} 1 &= \theta_t \underbrace{P_t}_{\uparrow} H_{t+1} && \text{LTV Constraint}
\end{aligned}
$$

  - Lower interest rates $\Rightarrow$ increase marginal utility of extra dollar borrowed
  - Increased demand for housing $\Rightarrow$ with fixed supply, current prices must rise

- Borrowing increases to pay for higher price of houses

- Trace out effect of a rise in θ_t:

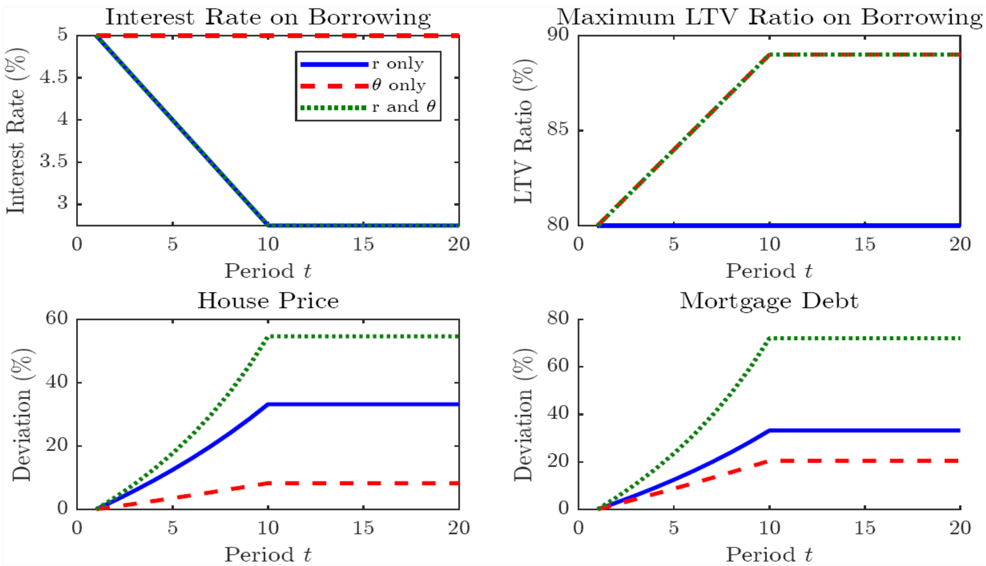
$$\begin{aligned} 1 - \mu_t &= \beta(1 + r_{t+1}) && \text{Consumption Euler Equation} \\ \underbrace{P_t}_{\uparrow} &= \frac{\beta}{1 - \mu_t \underbrace{\theta_t}_{\uparrow}} \left(\frac{1}{H_{t+1}} + (1 - \delta) P_{t+1} \right) && \text{Housing Euler Equation} \\ \underbrace{B_{t+1}}_{\uparrow} &= \underbrace{\theta_t}_{\uparrow} \underbrace{P_t}_{\uparrow} \bar{H} && \text{LTV Constraint} \end{aligned}$$

  - Higher LTV borrowing limits $\Rightarrow$ increase amount that can be borrowed
  - Since households always borrow as much as they can, borrowing increases
  - Demand for housing increases in line with borrowing $\Rightarrow$ with fixed supply, prices rise

- Now to solve the model in practice (e.g. on a computer!)
- Create **exogenous** paths (i.e shocks) for r_{t+1} and θ_t
- Assume model is in new steady state at some point in the future (e.g. some period T)
- Iterating backwards from T, take P_{t+1} as given, then solve for P_t, B_{t+1}:

$$\begin{aligned} 1 - \mu_t &= \beta(1 + r_{t+1}) \\ P_t &= \frac{\beta}{1 - \mu_t \theta_t} \left(\frac{1}{H} + (1 - \delta) P_{t+1} \right) \\ B_{t+1} &= \theta_t P_t \bar{H} \end{aligned}$$

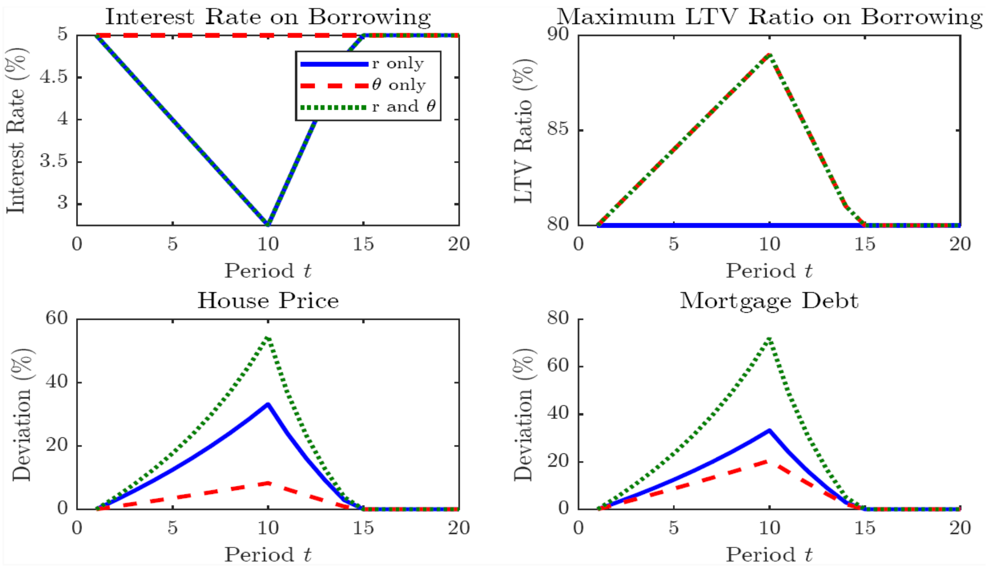
- For our dynamic experiments:
  - The shocks last for 8 quarters (i.e. two years)
  - The shocks are unanticipated each period (i.e. a complete surprise)
  - We run experiments separately for: (1) interest rate shocks, (2) LTV ratio shocks, (3) both interest rates and LTV ratio shocks



- The boom is the same as our previous experiment
  - Each model period represents 1 quarter
  - The credit expansion shock lasts for 8 quarters (i.e. two years)

- The shocks are unanticipated each period (i.e. a complete surprise)

- But now for the credit bust shock:
  - In the 9th quarter, interest rates and the LTV ratio revert to their initial steady state values in just 4 quarters (i.e. one year)



Welfare and Redistribution Through Asset Price Movements

A Simple Model of Welfare Gains and Losses From Asset Price Movements

- Simple model of asset choice and asset price movements
- Hold initial asset stock, rebalance asset portfolio, earn cash flow next period
- A household's problem is:

$$\underbrace{V}_{\text{Value Function}} = \max_{C_1, C_2, A_2} U(C_1) + \beta U(C_2)$$

$$\text{s.t} \quad C_1 + (A_2 - A_1)P_1 = Y_1$$

$$C_2 = Y_2 + A_2 D_2$$

- where:
  - P_1 = price of asset when buying/selling at time 1
  - D_2 = cash flow/dividends from asset at time 2
  - $(A_2 - A_1)$ = net transactions of the asset in period 1
  - V = Value Function, the total utility of the consumption and asset choices for the household

- The Lagrangian equation is:

$$\mathcal{L} = U(C_1) + \beta U(C_2) + \lambda_1(Y_1 - C_1 - (A_2 - A_1)P_1) + \lambda_2(Y_2 + A_2 D_2 - C_2)$$

- The first order conditions are:

$$\begin{aligned} C_1 &: \quad Y'(C_1) = \lambda_1 \\ C_2 &: \quad \beta U'(C_2) = \lambda_2 \\ A_2 &: \quad \lambda_1 P_1 = \lambda_2 D_2 \end{aligned}$$

- And, combining the FOCs, we find the Euler equation:

$$\underbrace{U'(C_1)}_{\text{Marginal Utility of } C_1} = \underbrace{\beta U'(C_2)}_{\text{Marginal Utility of } C_2} \times \underbrace{\frac{D_2}{P_1}}_{\text{Return on asset}}$$

- Recall, Euler equation describes optimal inter-temporal decisions of the household
- Characterises trade-off between consumption today and investment for consumption tomorrow
- Note that the price of the asset P_1 directly affects asset returns $\frac{D_2}{P_1}$
- All else equal, higher prices reduce returns which discourages further investment in the asset
- But asset price P_1 has **indirect** effects through valuation of household wealth
- Recall the period 1 budget constraint is:

$$C_1 = Y_1 + \underbrace{P_1 A_1 - P_1 A_2}_{\text{Net change in asset position}}$$

- E.g. an increase in the price P_1 increases the value of the households initial portfolio: $P_1 A_1$
- This change in portfolio values is called the wealth effect
- We want to understand the welfare gains/losses from a change in asset prices
- Overall, are households better off or worse off when asset prices rise?
- Depends on size of effects on returns and wealth
  - Higher asset prices reduce asset returns, making households worse off
  - Higher asset prices increase value of initial wealth, making households better off
- What is the effect of an increase in asset prices P_1?

$$\begin{aligned}\frac{\partial V}{\partial P_1} &= \frac{\partial U(C_1)}{\partial C_1} \times \frac{\partial C_1}{\partial P_1} + \frac{\partial U(C_2)}{\partial C_2} \times \frac{\partial C_2}{\partial P_1} \\ &= U'(c_1) \times (A_1 - A_2)\end{aligned}$$

- Where $(A_1 - A_2)$ is net asset portfolio transactions
- Another way to understand this: rewrite the budget constraint as:
- And the budget constraints:

$$\begin{aligned}C_1 &= Y_1 + (A_1 - A_2)P_1 \\ &= Y_1 + \frac{P_1}{P_0} P_0 A_1 - \frac{P_1}{D_2} D_2 A_2 \\ &= Y_1 + R_1 P_0 A_1 - \frac{1}{R_2} D_2 A_2\end{aligned}$$

- Where P_0 is the initial price assets were purchased at, R_1 is the return on assets bought prior to time 1, and R_2 is the return on assets purchased at time 1
- Now what is the effect of an increase in asset prices P_1?

$$\begin{aligned}\frac{\partial V}{\partial P_1} &= \frac{\partial U(C_1)}{\partial C_1} \times \left(\underbrace{\frac{\partial C_1}{\partial R_1} \times \frac{\partial R_1}{\partial P_1}}_{\text{Wealth effect}} + \underbrace{\frac{\partial C_1}{\partial R_2} \times \frac{\partial R_2}{\partial P_1}}_{\text{Investment returns effect}} \right) \\ &= \underbrace{U'(C_1) \times A_1 P_0 \times \frac{\partial R_1}{\partial P_1}}_{\text{Wealth effects}} + \underbrace{U'(C_1) \times A_2 D_2 \times A_1^{-2} \times \frac{\partial R_1}{\partial P_1}}_{\text{Investment returns effect}}\end{aligned}$$

- Changes in P_1 may have different implications for welfare via returns:
  - Higher P_1 increases returns in period 1 (i.e. a wealth effect)
  - Higher P_1 increases returns in period 2 (holding D_2 constant)
- Important! Overall effect on welfare is not the same as wealth effect
  - Welfare effect $= \frac{\partial V}{\partial P_1}$
  - Wealth effect $= U'(C_1) \times A_1 P_0 \times \frac{\partial R_1}{\partial P_1}$
- The welfare effect of asset price movements is summarised by:

$$U'(c_1) \times (A_1 - A_2)$$

- First, consider a **net seller** of assets: $A_1 > A_2$

$$\frac{\partial V^{seller}}{\partial P_1} = U'(C_1) \times (A_1 - A_2) > 0$$

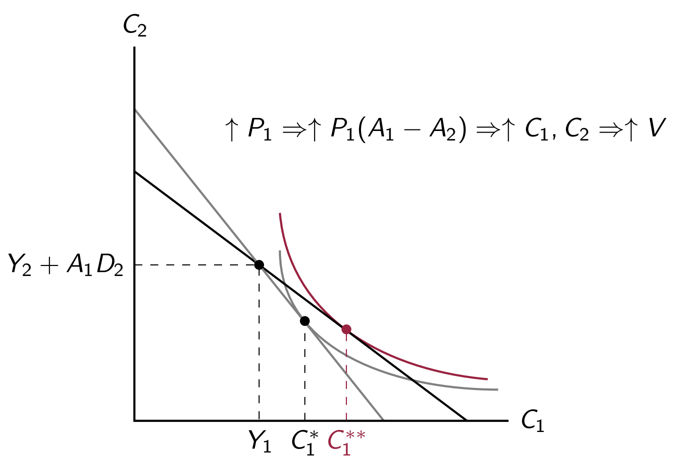
- Earn higher returns on assets sold, net gain
- Second, consider a **net buyer** of assets: $A_1 < A_2$

$$\frac{\partial V^{buyer}}{\partial P_1} = U'(C_1) \times (A_1 - A_2) < 0$$

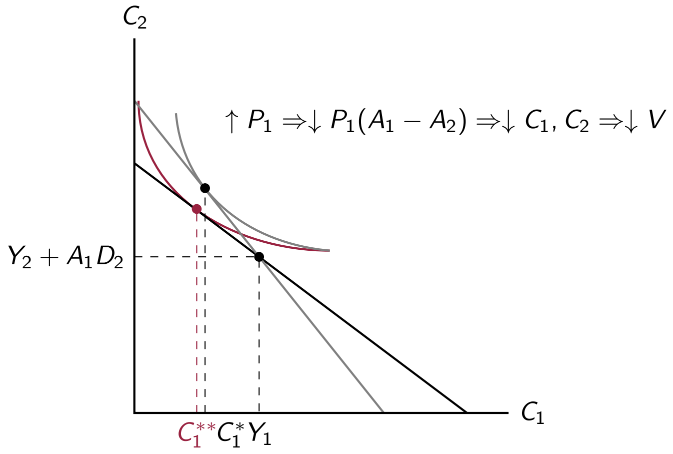
- Earn lower returns on asset investments, net loss

Graphical Illustration

- Increase in P_1 tilts budget constraint out
- Can spend down more net asset wealth
- For **net seller** of assets, consume more in both periods
- Higher indifference curve ⇔ higher utility ⇔ higher welfare



- Increase in P_1 tilts budget constraint out
- Investment in net assets is more expensive
- For **net buyer** of assets, consume less in both periods
- Lower indifference curve ⇔ lower utility ⇔ lower welfare


C_2
$\uparrow P_1 \Rightarrow \downarrow P_1(A_1 - A_2) \Rightarrow \downarrow C_1, C_2 \Rightarrow \downarrow V$
$Y_2 + A_1 D_2$
C_1
C_1^{**}
C_1^{*}
Y_1