Course

$\boxed{20/9/2022}$

$h = 6{,}6 \cdot 10^{-34}$ $J \cdot s$

energy × time is called "action" in engeneering

$\lambda \longrightarrow$ wave

Planck Hypotesis

$\circledast$ quanta $\longrightarrow$ $E_n = h \times \nu = h \frac{c}{\lambda} = 1$ single quanta

ν absolute freq

$\circledast$ 10 quanta $\longrightarrow$ $E_{beam} = 10 \times h \nu$

integer

energy is discrete
↳ Quatization

e.m. radiation = made by quanta = photons

Quatization values: | momentum | angular momentum
| energy | velocity ....

Planck's hypotesis leads to a 2 phenomena

① Photoelectric Effect

② Bohr Hydrogen Model

→ Black Body Radiation

From 1850 to 1900 there was no explanation for the shape of the plot

Planck's hypotesis gave the explanation for the full shape

This part could be explained

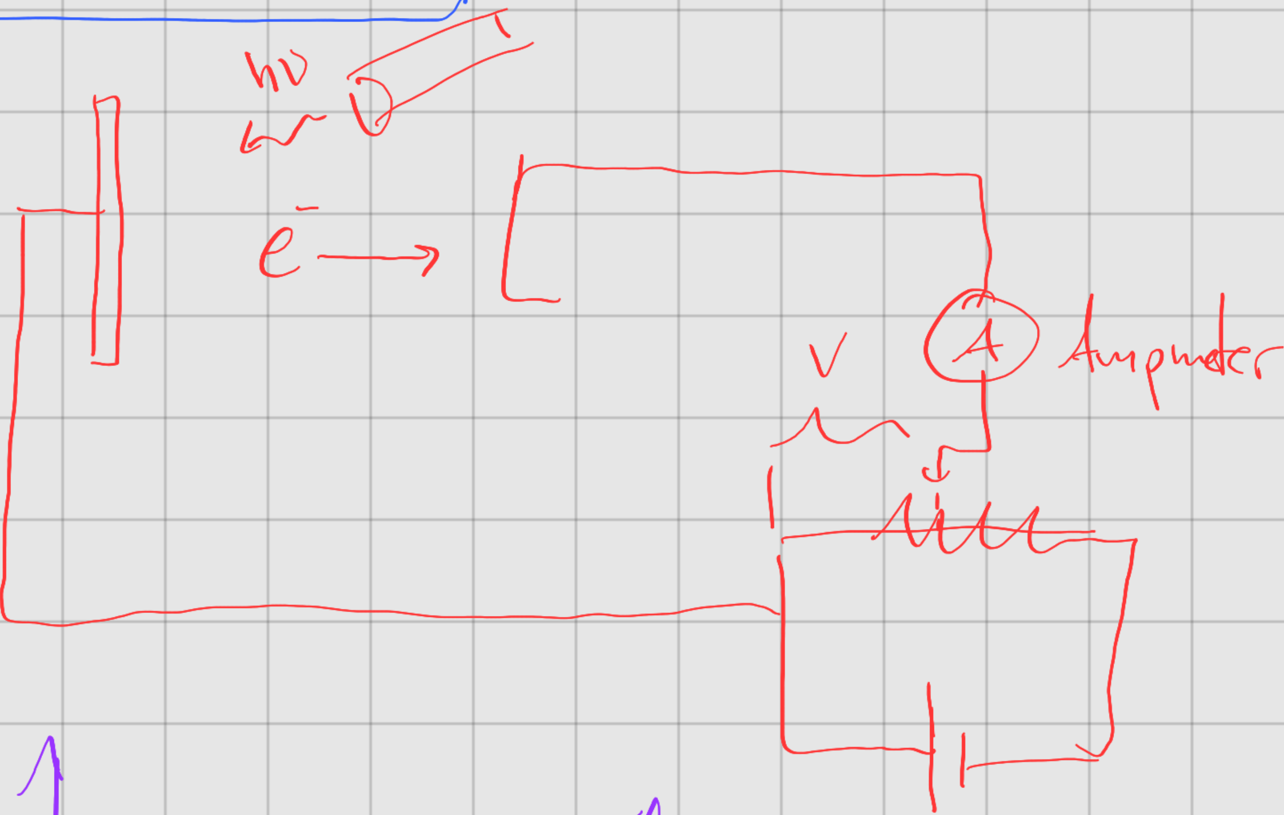
this also was explaines

but

→ Duality of light

Dual
- → Wave: Maxwell $\{\lambda, \nu\}$
- → Particle: Planck $\{photons\}$

$\boxed{\text{Photoelectric Effect}}$



1905 A. Einstein

1 quanta → 1 e^-

approximation

Before of the impact of the light on the metal we only have the light ($E = h\nu$)

Energy balance:

$$h\nu = E_k + \Phi$$

a single photon energy

kinetik energy of a single e^-

↳ the potential energy we need to extract the e^- from the metal (work function) → material dependent parameter.

$$E_k = h\nu - \Phi$$

Since kinetik energy can't be negative:

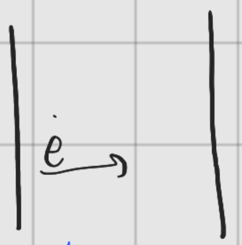
$$h\nu - \Phi \geq 0$$

$$h\nu \geq \Phi$$

$$\nu \geq \frac{\Phi}{h} = \nu_{th}$$

There is a threshold freq

To measure the kinetik energy



An electron is going from catod to anodo and it will create a current.

electrostatik potential energy $= e \times V$

If we charge negatively the right

it will make the e^- stop eventually.

When there is an energy that makes e^- stop there will be no more current. At that point $\boxed{eV_{stop} = E_k}$

That's how we calculate kinetik energy

→ Now:

We have $I = 2\ W/m^2 \rightarrow 2 \cdot \frac{J}{s} \cdot \frac{1}{m^2}$

$\lambda = 250\,nm$

If I want to know the flux

$$flux = \frac{n^o\ part}{A \cdot t} \Rightarrow \frac{I}{h\nu}$$

→ If $\lambda = 250\ nm$

$$E = h\nu = \frac{hc}{\lambda} = \frac{12{,}4 \cdot 10^3\ eV\ \mathring{A}}{250\ nm} \simeq 5eV$$

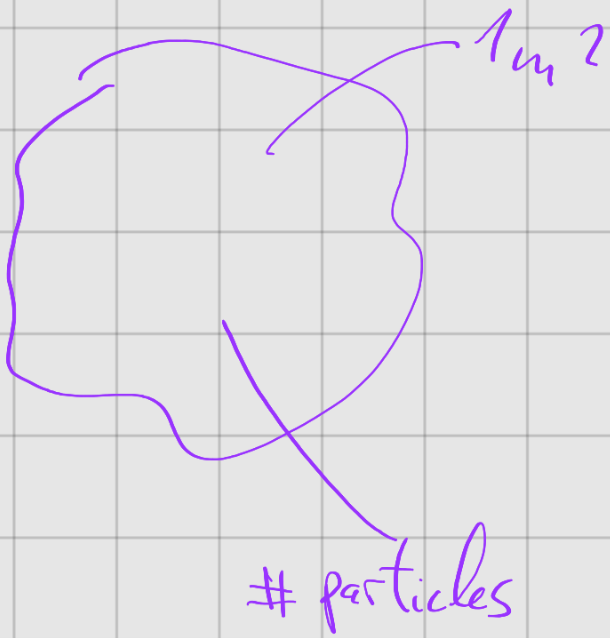
$$\frac{2J}{m^2 s} \cdot \frac{1}{E} = \frac{2J}{m^2 s} \cdot \frac{1}{h\nu} = \frac{\#\ part.}{m^2 s} =$$

$$= \frac{2J}{5eV} \frac{1}{m^2 s} = \frac{2J}{5 \times 1{,}6 \cdot 10^{-19} J\ m^2 s} \sim 10^{18}$$

How many photons?

$1\ m^2$ $1\ sec$

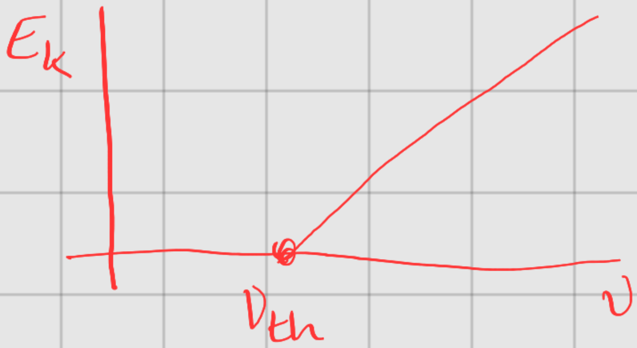
$$\frac{\text{total energy}}{\text{energy of a single quanta}}$$



Review

* Fotoelectric effect equation: $\boxed{E_k = h\nu - \Phi}$

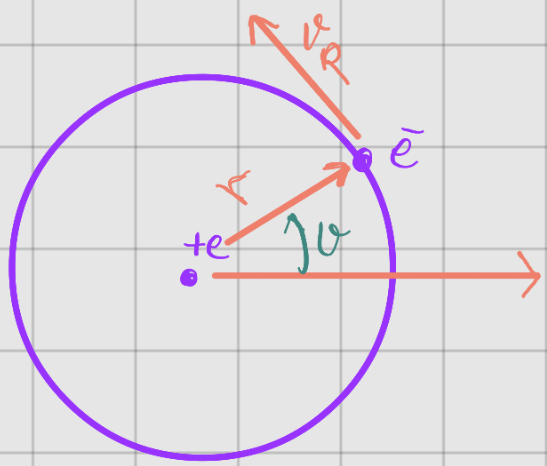
* Threshold frequency: $h\nu = \Phi \Rightarrow \boxed{\nu_{th} = \phi / h}$



* Measure the kinetic energy: $\boxed{E_k = e V_{stop}}$

$\boxed{21/9/2022}$

Bohr's Model H

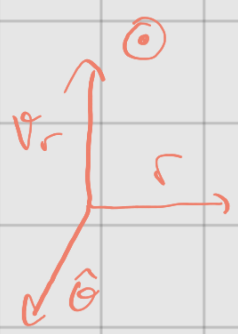


Angular momentum: $\vec{L} = \vec{r} \wedge \vec{p}$

In model H:

$\vec{L} = r m v_R \, \hat{\theta}$ → pointing out of circle: $\odot$

$\vec{L} = L_\theta \, \hat{\theta}$



Bohr's hypotesis

$$\int_G L_\theta \, d\theta = h n$$

line integral

$$\int_G L_\theta \, d\theta = L_\theta \int_G d\theta = L_\theta \, 2\pi$$

$$L_\theta = \frac{h n}{2\pi} = \hbar n$$

$\Rightarrow$ $L_\theta = n\hbar$ The angular momentum is quantazied

① Force Balance

$$\underbrace{m\frac{v_R^2}{r}}_{\text{centrifuge}} = \underbrace{\frac{e^2}{r^2}}_{\text{coulomb}} \Rightarrow \frac{L_0^2}{mr^3} = \frac{e^2}{r^2} \Rightarrow r = \frac{L_0^2}{me^2} = \boxed{\frac{\hbar^2 n^2}{me^2}}$$

r_0 fundamental constants

$r_0 \approx 0{,}5\,\text{Å}$

$$\Rightarrow r = r_0\, n^2$$

The radius of the orbits are quantazied. Discrete orbits

$r_1 = r_0$

$r_2 = 4r_0$

$r_3 = 9r_0$

② Velocity

$$v_R = \frac{L_0}{mr} = \frac{1}{\cancel{m}}\cdot\frac{\hbar n}{\frac{\hbar^2}{\cancel{m}e^2}n^2} = \frac{e^2}{\hbar}\cdot\frac{1}{n} \Rightarrow v_R = \alpha c\,\frac{1}{n}$$

velocity is quantazied

$$n=1 \longrightarrow v_{R,1} = \frac{e^2}{\hbar c}\cdot c = \alpha c$$

α: fine structure constant $= \frac{1}{137}$

③ Energy

$$E = \frac{1}{2} m v_R^2 - \frac{e^2}{r} = \frac{1}{2} m \left[\frac{e^2}{\hbar}\frac{1}{n}\right]^2 - e^2 \frac{me^2}{\hbar^2 n^2} =$$

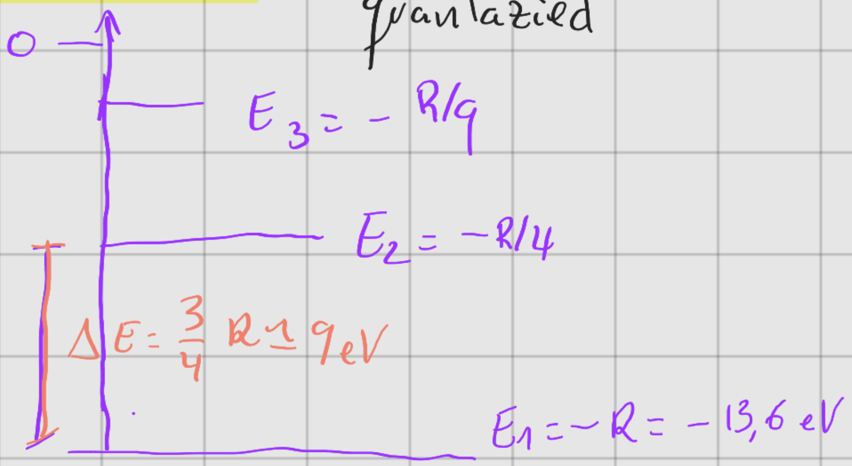
$$= \frac{1}{2} m \frac{e^4}{\hbar^2}\frac{1}{n^2} - \frac{m e^4}{\hbar^2}\frac{1}{n^2} = \frac{1}{n^2}\left[-\frac{1}{2} m \frac{e^4}{\hbar^2}\right] = -\frac{R}{n^2}$$

$$E_n = -\frac{R}{n^2}$$

Energy is quantazied

$$R = \frac{me^4}{2\hbar^2} = \text{Rydberg constant}$$

$R = 13{,}6\,eV$



Doing the difference between E_1 and E_2:

$$\Delta E = E_2 - E_1 = h\nu_{21} \rightarrow \nu_{21} = \frac{E_2 - E_1}{h} = \frac{9 \cdot 1{,}9\cdot 10^{-19}\,J}{6{,}6\cdot 10^{-34}\,J\cdot s}$$

and it's the exact freq. of the photon after extracting it

Recap

27/09/2022

- Planck

$$\varepsilon = h\nu = \hbar \omega$$

- Fotoelectric efect

$$\nu_{th} = \Phi / h$$

$$h\nu = E_K + \Phi$$

↳ $E_K = eV_{stop}$

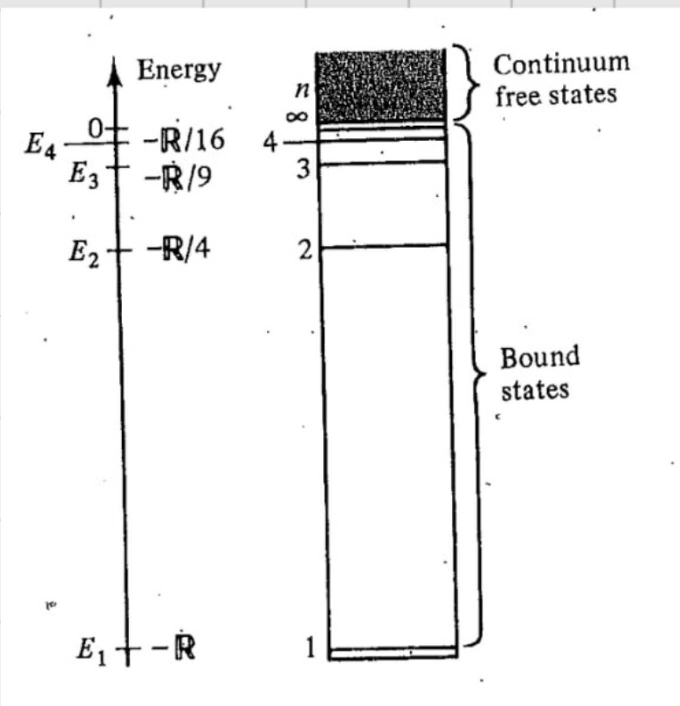
- Bohr's Atom

$$r_n = r_0 n^2 \quad , \quad n = 1, 2, 3, \dots$$

$$v_{R_n} = \alpha c \frac{1}{n} \quad , \quad \alpha = \frac{1}{137}$$

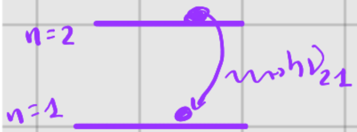
$$E_n = -\frac{R}{n^2} \quad , \quad R = 13.6 \text{ eV}$$

⊛ We use these results for the Hydrogen



If we are not at E_1 (i.e. E_2), we cannot stay there forever. There is a limited time we can stay at that other level.

The time we can stay: τ: lifetime



$$\lambda_{21} = E_2 - E_1 = h\nu_{21} = \frac{hc}{\lambda_{21}}$$

$$\rightarrow \lambda_{21} = \frac{hc}{-\frac{1}{4} + R} = \frac{12.41 \cdot 10^3 \text{ eV} \cdot \text{Å}}{3/4 \cdot 13,6 \text{ eV}}$$

$$\Rightarrow \lambda_{21} = 0,12 \, \mu m$$

Each decay level corresponds to a series for λ_{31}, λ_{41} and so on....

Balmer formula: $\frac{1}{\lambda} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$, R : Rydberg constant $= 1,097 \cdot 10^7 \text{ m}^{-1}$

n=1 → Lyman , n=2 → Balmer , n=3 → Paschen

We can see that the potential is negative, therefore the e^- is attracted to the nucleus

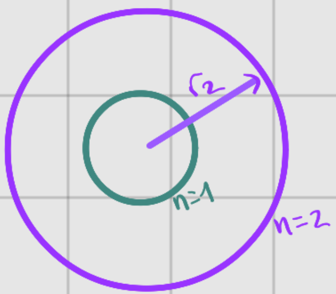
As the energy level increases the energy required is less until the e^- is free and is not longer bounded to the proton.

The energy required to release the e^- from the atom is: 13,6 eV

Now we are going to calculate how long it will take to the e^- to fall from E_2 to E_1 (fund. state).

$\tau_{21} = 10^{-8}$ sec

$$\left.\begin{array}{l} v_2 = \alpha c \frac{1}{n=2} = \alpha c \frac{1}{2} \\ r_2 = r_0 (n=2)^2 = 4 r_0 \end{array}\right\} \longrightarrow \omega_2 = \frac{v_2}{r_2}$$



angular velocity: $v = \omega r$

ω_2 or better to call ω_{21} will be related to the period we spend in E_2

$$\omega_{21} \cdot T_{21} = 2\pi \longrightarrow T_{21} = \frac{2\pi}{\omega_{21}} = \frac{r_2 \cdot 2\pi}{v_2} = \frac{4 r_0 \cdot 2\pi \cdot 2}{\alpha c} = 1{,}15 \cdot 10^{-15} \text{ sec}$$

T_{21} is the time an e^- expend doing a rotation. Thus, to decay it will need: $N = \frac{\tau_{21}}{T_{21}} = \frac{10^{-8} \text{sec}}{1{,}15 \cdot 10^{-15} \text{sec/rotation}} \simeq 10^{7}$ sec to decay

Wave Nature of Matter

Planck stated that the energy of one photon is: $E = h\nu$

The kinetic energy is: $E = \frac{1}{2} m v^2 = \frac{p^2}{2m}$

Now, we want to find the momentum of a single photon. According to Newton the momentum would be zero because photon has no matter. But, according to Einstein and relativity we know that:

$\boxed{E^2 = E_0^2 + (pc)^2}$ $\quad E \rightarrow$ total energy, $E_0 \rightarrow$ rest energy

$E_0 = m_0 c^2 \longrightarrow$ In the case of photon: $m_0 = 0 \Rightarrow E_0 = 0$

Therefore, $E^2 = (pc)^2 \Rightarrow p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}$

Momentum of a single photon: $\boxed{p = \frac{h}{\lambda} \qquad p = \hbar K}$, $K = \frac{2\pi}{\lambda}$ wavenumber

We saw the corpuscular nature of light, is there a wave nature of particles?

$\Rightarrow$ De Broglie.

EM radiation has a dual particle-wave nature. De Broglie stated that matter has a dual particle-wave nature also.

$\boxed{\lambda_{DB} = \frac{h}{p} = \frac{h}{m \cdot v}}$

- Wavelength of a ball

$\left.\begin{array}{l} m = 0{,}01\ kg \\ v = 10\ m/s \end{array}\right\} \lambda_{DB} = \frac{h}{m \cdot v} \simeq 6{,}6 \cdot 10^{-33}\ m$

$\Rightarrow \lambda_{DB_{Ball}} \sim 0$

- Wavelength of a neutron

$E = 0{,}05\ eV/\text{neutron} \rightarrow \lambda_{DB} = \frac{h}{p} = \frac{h}{\sqrt{2 m_n E}} \simeq 1{,}28\ Å$

$\lambda_{DB_{Neutron}} \simeq 1{,}28 \cdot 10^{-10}\ m$

Conclusions: Massive objects have no wavelength or it is so small we can never measure it. But particles have a measurable wavelength

We use this for electronical microscopes. For instance, if the object is $d = 2.5 \mathring{A}$

$$E = h\nu \simeq \frac{hc}{\lambda} = \frac{12{,}4 \cdot 10^3 \, eV \, \mathring{A}}{2{,}5 \, \mathring{A}} \simeq 5 \, KeV$$

?

Used wavelength of e^-: $\lambda_{DB} = \frac{h}{\sqrt{2mE_K}} \Rightarrow E_K = \frac{h^2}{\lambda_{DB}^2 \cdot 2m_e} = 24{,}2 \, eV$

$\lambda_{DB} \to 2{,}5 \mathring{A}$

We need to use way way less energy and if we want to see objects with different sizes we just need to change the energy

o) Example: We want to create an e^- with de Broglie wavelength $\lambda_{DB} = 1 \mathring{A}$ how fast should we accelerate it.

$$E_K = \frac{1}{2} m v^2 = \frac{1}{2} \frac{p^2}{m} = \frac{1}{2m} \left(\frac{h}{\lambda_{DB}} \right)^2 \simeq 150 \, eV$$

How do we give such an energy to an e^-? We know that a given electrostatic energy is required to cause this, a plate such that it imparts this energy

$e \cdot V = E_K \Rightarrow \text{Voltage} = \frac{E_K}{e^- \text{charge}}$, $eV = 1{,}6 \cdot 10^{-19}$ Coulomb

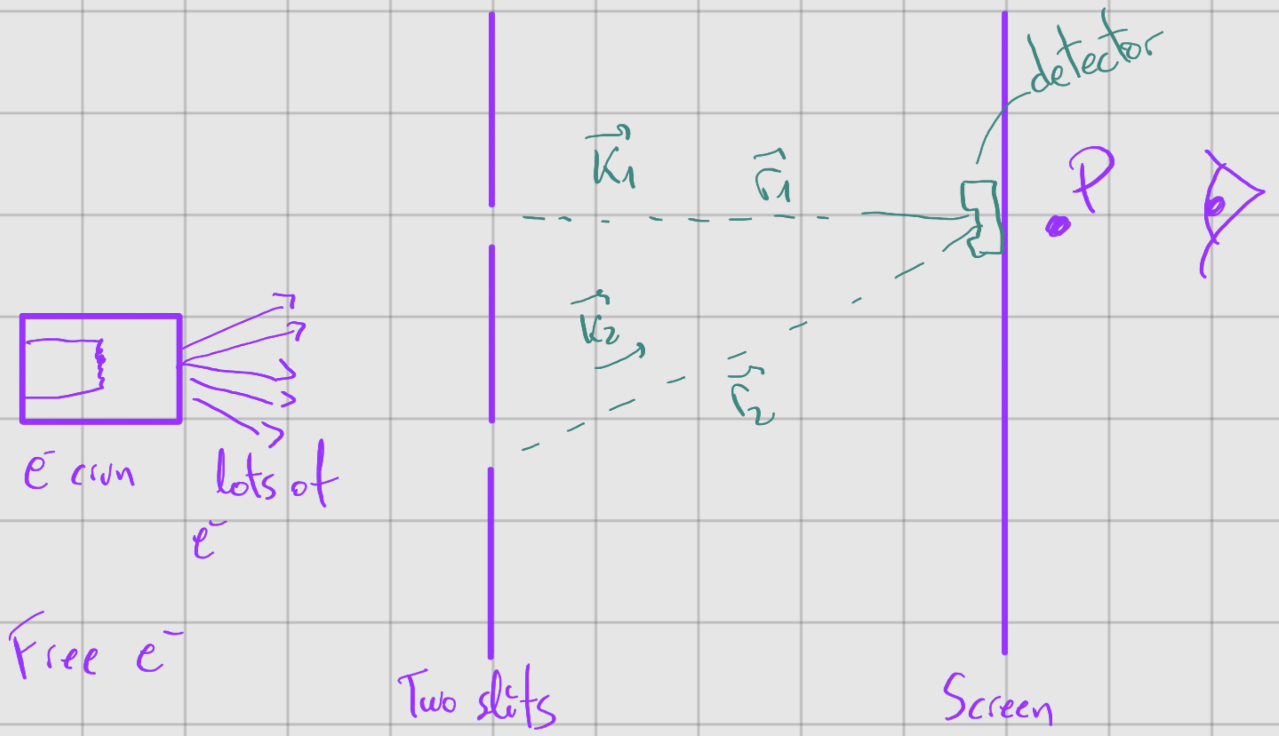
$$V = \frac{E_K}{[e]} = \frac{150 \, eV \cdot 1.6 \cdot 10^{-19} \, J/eV}{1{,}6 \, C} = 150 \, J/C = 150 \text{ volt}$$

This shows that is very easy to have a de Broglie wavelength very small with particles with a very low energy

For free particles: $E_K = \frac{p^2}{2m}$, $\lambda_{DB} = \frac{h}{p}$ $\Rightarrow$ $\boxed{\lambda_{DB} = \frac{h}{\sqrt{2m E_K}}}$

Particle Interference

Let's see what happens with the double slit experiment using e^-.



The e^- being fired are free e^- described by plane waves:

$A \cdot e^{i\vec{k}\cdot\vec{r}}$, $\vec{k} = \vec{k}_{DB} = k_{DB} \cdot \hat{k} = \frac{2\pi}{\lambda_{DB}} \hat{k}$, $\vec{k}_{DB}$ is $\perp$ to the screen and slits

versor (unit vector)

At point P we observe any e^- that hit the screen. As the e^- travel, they will eventually hit the slits and on the other side will be described by two equations: $A_1 e^{i\vec{k}_1\cdot\vec{r}_1}$, $A_2 e^{i\vec{k}_2\cdot\vec{r}_2}$

The result we will observe at the screen will be the superposition of these two equations: $A_1 e^{i\vec{k}_1\cdot\vec{r}_1} + A_2 e^{i\vec{k}_2\cdot\vec{r}_2}$

We can measure the intensity: $\left| A_1 e^{i\vec{k}_1\cdot\vec{r}_1} + A_2 e^{i\vec{k}_2\cdot\vec{r}_2} \right|^2 =$ Intensity at P

If the slits are exactly the same $\rightarrow$ $A_1 = A_2$

$$I_P = |A|^2 \left| e^{i\vec{k}_1\cdot\vec{r}_1} + e^{i\vec{k}_2\cdot\vec{r}_2} \right|^2$$

Another aprox. the screen is far enough from the slits and the two slits are close enough to each other $\Rightarrow$ $\vec{k}_1 \simeq \vec{k}_2 \simeq \vec{k}$

$$I_P = |A|^2 \left| e^{i\vec{k}\cdot\vec{r}_1} + e^{i\vec{k}\cdot\vec{r}_2} \right|^2 = 2|A|^2 \left[1 + \cos\left[\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)\right]\right]$$

We can get max. and mins.

↳ $I_P\ max = 4|A|^2$ when $\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) = 2n\pi$, $n = 0, 1, 2, \ldots$ (even)

↳ $I_P\ min = 0$ when $\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) = (2n+1)\pi$, $n = 0, 1, 2 \ldots$ (odd)

What we have to see is the phase difference

$$\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) = \vec{k}\cdot\vec{d} = k \cdot d \cdot \cos\left[\frac{\pi}{2} - \theta\right] = kd\sin\theta$$

↳ max: $2n\pi$, $k = \frac{2\pi}{\lambda_{DB}}$ $\Rightarrow$ $\frac{2\pi}{\lambda_{DB}} d\sin\theta = 2n\pi$ $\Rightarrow$ $n\lambda_{DB} = d\sin\theta_n$

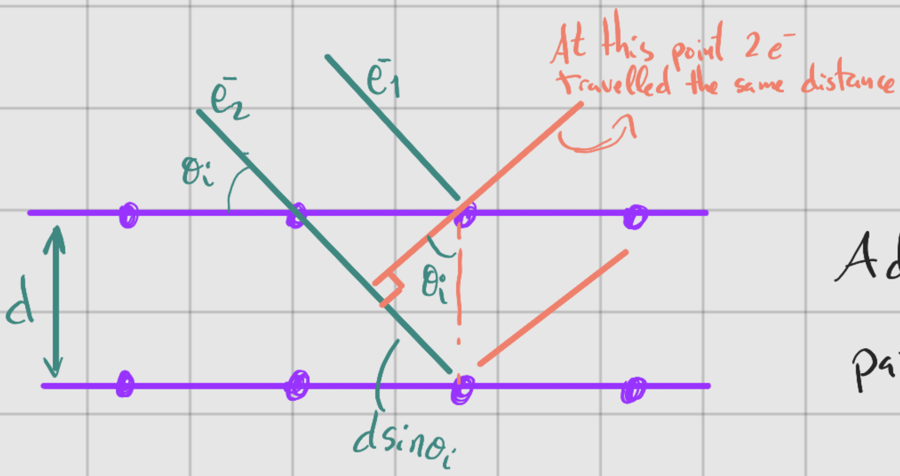
↳ min: $(2n+1)\pi$, $k = \frac{2\pi}{\lambda_{DB}}$ $\Rightarrow$ $(n + \frac{1}{2})\lambda_{DB} = d\sin\theta_n$

Finally we obtain:

$$\boxed{\begin{array}{ll} \text{Constructive interference} \xrightarrow{\text{Max}} & d\sin\theta_n = n\lambda_{DB} \\ \text{Destructive interference} \xrightarrow{\text{Min}} & d\sin\theta_n = (n+\frac{1}{2})\lambda_{DB} \end{array}}$$

Diffraction

The diffraction of ligth can be described by a reflective gratting



Adding both sides of the extra path (optical path difference)

$2d\sin\theta$

To see the maximum of interference at the output we can use previous results.

$$k \cdot 2d\sin\theta_i = 2\pi n \longrightarrow \frac{2\pi}{\lambda} 2d\sin\theta_i = 2\pi n$$

$$\boxed{2d\sin\theta_i = n\lambda_{DB}}$$

d → separation of planes known as bragg planes

n → diffraction order

There is a relationship between diffraction order n and angle of incidence θ_i.

If we fix d and θ ther will instead have a relation between n and λ_{DB}. Thus, we only need to fix λ_{DB} to see different diffraction orders

04/10/2022

Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t}\Psi(\bar{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\bar{r},t)\right]\Psi(\bar{r},t)$$

$\Psi(\bar{r},t)$: wavefunction → quantum state

$V(\bar{r},t)$: potential

↳ If it doesn't depend on time

$V(\bar{r},t) = V(\bar{r})$ → conservative

when the potential is non time dependent we can split:

$$\underbrace{\Psi(\bar{r},t)}_{\text{wave func.}} = \underbrace{\psi(\bar{r})}_{\text{eigenfunct.}} e^{-i\frac{E}{\hbar}t}$$

$$\boxed{\nabla^2\psi(\bar{r}) + \frac{2m}{\hbar^2}\left(E - V(\bar{r})\right)\psi(\bar{r}) = 0}$$

S.E. stationary states

↳ $\{E, \psi_E(\bar{r})\}$ Energy

$\psi_E(\bar{r}) \longrightarrow E$

$$\sum_{\{E\}} C_E\,\psi_E(\bar{r})\,e^{-i\frac{E t}{\hbar}} = \Psi(\bar{r},t)$$

→ Most general solution

If we have two solutions $\{E_1, \psi_{E_1}(\bar{r})\}$, $\{E_2, \psi_{E_2}(\bar{r})\}$ $\Rightarrow \Psi(\bar{r},t) = C_1\,\psi_{E_1}(\bar{r})\,e^{-i\frac{E_1}{\hbar}t} + C_2\,\psi_{E_2}(\bar{r})\,e^{-i\frac{E_2}{\hbar}t}$

superposition

eigenfunction Properties

① $\psi_E(\bar{r})$ must be: Continous, Differenciable, Finite → $|\bar{r}| \to \infty$

② $\int \psi^*_{E_1}(\bar{r})\,\psi_{E_2}(\bar{r})\,d\bar{r} = \delta_{E_1,E_2}$ (Kronicker: $\delta_{\alpha\beta} = \begin{cases} 1 & \alpha=\beta \\ 0 & \alpha\neq\beta \end{cases}$)

$\delta(E_1 - E_2)$ (Dirac when E_1, E_2 continous)

Orthogonality condition

Wave function Properties

① Probability of finding a particle: $|\psi(\vec{r},t)|^2 d\vec{r}$ prob. to find a part. in: $\vec{r} \to \vec{r}+d\vec{r}$

$\int |\psi(\vec{r},t)|^2 d\vec{r} = 1$ → Normalization condition

From all the results we obtain we care about whos $\psi(\vec{r},t) \to$ square: $\int$

② $\int \psi^*(\vec{r},t)\, \hat{O}\, \psi(\vec{r},t)\, d\vec{r} = \langle \hat{O}(t) \rangle$

$\hat{O}$: observable

↳ $\hat{p}$ = momentum $\longrightarrow -i\hbar \frac{d}{dx}$

↳ $\hat{x}$ = position $\longrightarrow x$

example: $E_1 \longrightarrow \psi_1 e^{-i\frac{E_1}{\hbar}t}$

$$\int \psi^*_{1(x)} e^{+i\frac{E_1}{\hbar}t} \left[-i\hbar \frac{d}{dx}\right] \psi_{1(x)} e^{-i\frac{E}{\hbar}t} dx = \langle p(t) \rangle$$

value of the momentum at a given time

Free particle "m"

$\nabla^2 \psi(\vec{r}) + \frac{2m}{\hbar^2} E\, \psi(\vec{r}) = 0$ S.E. stationary states of a free p.r.e. (m)

$$\boxed{\frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2} E\, \psi(x) = 0 \qquad \text{sol: } \psi(x) = A e^{\pm i\eta x}}$$

Demos. for (+) : $-A\eta^2 e^{+i\eta x} + \frac{2m}{\hbar^2} E A e^{+i\eta x} = 0$

η

If: $\eta^2 = \frac{2m}{\hbar^2} E \rightarrow E = \frac{\hbar^2}{2m}\eta^2$

$\left\{ \frac{\hbar^2}{2m}k^2,\ A e^{\pm ikx} \right\}$ $\quad \eta \to k = \frac{2\pi}{\lambda_{DB}}$

De Broglie

$E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m} = \frac{\hbar^2}{2m}k^2$

$k = \frac{2\pi}{\lambda_{DB}}$

$p = \frac{h}{\lambda_{DB}} = \hbar k$

e.f. ⟶ free p.t.e

e.v. ⟶ $E = \frac{\hbar^2}{2m}k^2$ $\longrightarrow$ $\psi(x) = A e^{\pm i k x}$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

Demos for $\oplus$ A $\rightarrow$ $|A|^2 \int dx\, e^{-ik'x} e^{+ikx} = \delta(k-k')$ Dirac $(k,k')\rightarrow$ continous

$$|A|^2 \int_{-\infty}^{+\infty} dx\, e^{i(k-k')x} = \delta(x-x')$$

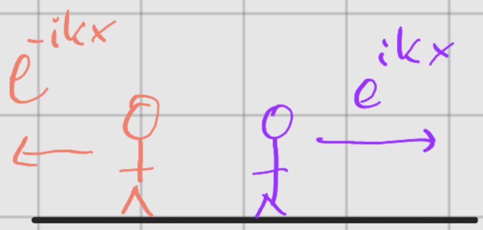
$$\hookrightarrow \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx\, e^{i(k-k')x}$$

$$\Rightarrow |A|^2 = \frac{1}{2\pi} \longrightarrow A = \sqrt{\frac{1}{2\pi}}$$

Solution of Sch eq. for stationary state for free part.

$\hookrightarrow$ 2 sets of sols. $\left\{ \frac{\hbar^2}{2m} k^2, \frac{1}{\sqrt{2\pi}} e^{+ikx} \right\}$

$\left\{ \frac{\hbar^2}{2m} k^2, \frac{1}{\sqrt{2\pi}} e^{-ikx} \right\}$



General sol. of Schö for a free particle:

$$\boxed{\psi_k(x,t) = \frac{1}{\sqrt{2\pi}} e^{ikx} e^{-i\frac{\hbar}{2m}k^2 t} + \frac{1}{\sqrt{2\pi}} e^{-ikx} \cdot e^{-i\underbrace{\left(\frac{\hbar^2}{2m}k^2\right)}_{E}\frac{1}{\hbar} t}}$$

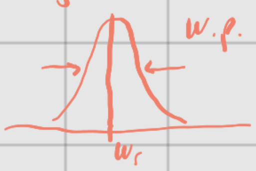
Superposition

Calculation of the momentum:

$$\langle p \rangle = \int dx \frac{1}{\sqrt{2\pi}} e^{-ikx} e^{+i\frac{\hbar}{2m}k^2 t} \left[-i\hbar \frac{d}{dx}\right] \frac{1}{\sqrt{2\pi}} e^{+ikx - i\frac{\hbar}{2m}k^2 t} = -i\hbar\,(ik)\, \frac{1}{2\pi} \int dx\, e^{i(k-k)x} =$$

$= +\hbar k\, \delta(\,)$ $\longrightarrow$ To solve this singularity of the delta we take 2 very close states k, k' and do the limit $\lim_{k \to k'}$

what is happening is that a plane wave with a single k doesn't exist. We have a wave packet that is just weighted with an exact plane wave



$$\int e^{ik \cdot x} f(k)\, dk = \text{wave packet}$$

$\boxed{05/10/2022}$

Particle confined

Potential Well : $V = \begin{cases} \infty & x<0 \quad x>L \\ 0 & 0 \le x \le L \end{cases}$

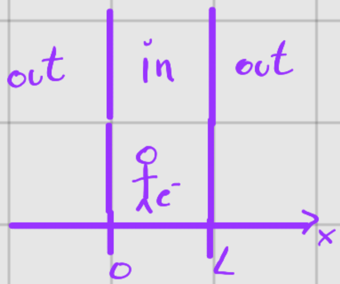
Remember:

$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right]\Psi(x,t)$

Prob. of finding e in $x \to x+dx$: $|\Psi(x,t)|^2 dx$

$|\Psi(x,t)|^2 dx = |\Psi(x)|^2 dx \longrightarrow 0$

free part. $\to \left\{ \frac{1}{\sqrt{2\pi}} e^{\pm ikx}, \frac{\hbar^2}{2m}k^2 \right\}$



"OUT" $\longrightarrow \Psi(x,t) = \Psi(x) e^{-i\frac{E}{\hbar}t} \equiv 0$

"IN" $\longrightarrow \Psi(x,t) \xrightarrow{\text{for free } e^-} e^{\pm ikx} e^{-i\frac{E}{\hbar}t}$

$e^{+ikx},\ e^{-ikx},\ \underbrace{\sin kx,\ \cos kx}$

eigenfunctions

obtained by a linear comb. of eigenfunc.
A linear comb. of solution is a solution.

There are 4 math. valid solutions but there is only one physical valid solution $\longrightarrow \sin kx$

$\sin kL = 0 \Rightarrow kL = n \cdot \pi \Rightarrow k_n = \frac{n\pi}{L}$

The energy : $E_n = \frac{\hbar^2}{2m} k_n^2 = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$

Confined particle

$\left\{ A \sin\left(\frac{n\pi}{L}x\right),\ E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2 \right\}$

In order to calculate the constant $\longrightarrow$ Orthogonality

$$\int_0^L A^* \sin\left(\frac{\pi}{L}nx\right) A \sin\left(\frac{\pi}{L}n'x\right) dx = \delta_{n,n'} \begin{cases} 1 \\ 0 \end{cases}$$

discrete (kronicker)

↳ Case ① $n = n'$: $|A|^2 \int_0^L \sin^2\left(\frac{\pi}{L}nx\right) dx = |A|^2 \int_0^L \frac{1-\cos\left(2\cdot\frac{\pi}{L}nx\right)}{2} dx =$

$= \frac{|A|^2}{2}\int_0^L dx + \frac{|A|^2}{2}\int_0^L \cos\left(\frac{2\pi n}{L}x\right) dx = |A|^2 \frac{L}{2} + \frac{|A|^2}{2} \cdot \frac{L}{4\pi n} \cancel{\int_0^L \sin\left(\frac{2\pi n}{L}x\right) dx} \to 0$

$= |A|^2 \frac{L}{2} \Rightarrow |A|^2 \frac{L}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{L}}$

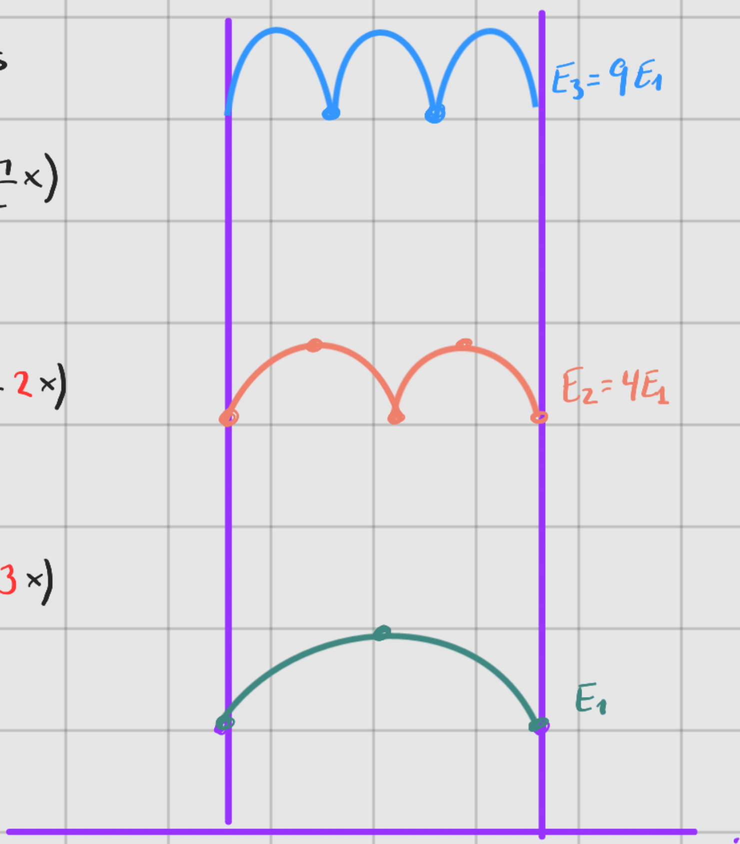
The result for confined particle: $\boxed{\left\{\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right),\ E_n=\frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2\right\}}$

Prob. to find the part. for different energies

$n=1 \rightarrow \psi_1(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right) \longrightarrow |\psi_1(x)|^2=\frac{2}{L}\cdot\sin^2\left(\frac{\pi}{L}x\right)$

$n=2 \rightarrow \psi_2(x)=\sqrt{\frac{2}{L}}\sin\left(2\frac{\pi}{L}x\right) \longrightarrow |\psi_2(x)|^2=\frac{2}{L}\cdot\sin^2\left(\frac{\pi}{L}2x\right)$

$n=3 \rightarrow \psi_3(x)=\sqrt{\frac{2}{L}}\sin\left(3\frac{\pi}{L}x\right) \longrightarrow |\psi_3(x)|^2=\frac{2}{L}\cdot\sin^2\left(\frac{\pi}{L}3x\right)$



Recap

11/10/2022

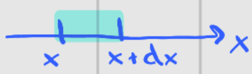
① free part. $\rightarrow$ $E \rightarrow \psi_E(x) = \frac{1}{\sqrt{2\pi}} e^{\pm ikx}$ $\quad K = \frac{\sqrt{2mE}}{\hbar}$ $\quad \{E, \psi_E(x)\}$

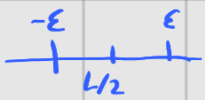
Wave func. $\rightarrow$ $\psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{\pm ikx - i\frac{E}{\hbar}t}$

3D $\rightarrow$ $\psi(\bar{r},t) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{\pm i\vec{k}\cdot\vec{r} - i\frac{E}{\hbar}t}$

② Potential well $\rightarrow$ $E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$ $\quad \psi_n(x) = \sqrt{\frac{2}{L}} \cdot \sin \frac{\pi}{L} n x$

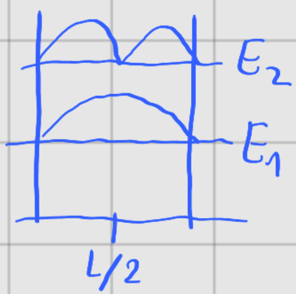
$|\psi_n(x)|^2 dx \longrightarrow$ prob. finding a particle



↳ 

Prob. finding in a range

$$Prob = \int_{L/2-\varepsilon}^{L/2+\varepsilon} dx\, |\psi_2(x)|^2$$



↳ If we have a quantum well (Q.W.) of: $L \sim 1cm$, energy?

$$E_1 = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 \cdot 1 = \frac{(\hbar c)^2}{2mc^2} \simeq \frac{(12{,}4 \cdot 10^3 eV \cdot \mathring{A})^2 \cdot 10}{10^{-4}\, m^2\; 10^6\, eV} = \frac{144 \cdot 10^6 \cdot 10^{-20}\, m^2 \cdot 10\, eV^2}{10^2\, m^2\; eV} \simeq 10^{-12} eV$$

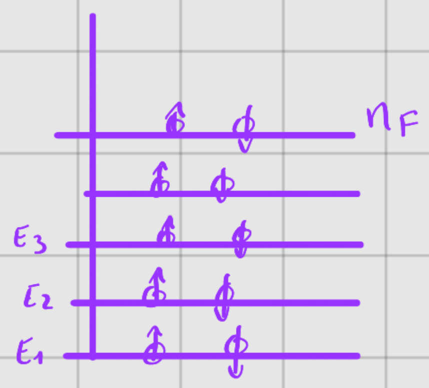
$$E_2 = 4 \cdot E_1$$

$\Delta E_{21} = E_2 - E_1 = 3E_1 = 3 \cdot 10^{-12}\, eV$ $\Rightarrow$ super tiny!!

↳ If we do with $L \sim 1nm$ $\Rightarrow$ $\Delta E_{21} \simeq 1{,}2\, eV$

{Pauli exclusion principle}

How can we put 10 e^- in energy levels



There can only be 2 e^- (with diff spin) at the same level.

Fermi level (n_F): Is the last level filled by e^-

$\mathcal{N}$ = total # of e^- $2 \times n_F$

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$$

free p. $\Rightarrow E = \frac{\hbar^2}{2m} k^2$

density

$$E_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n_F^2 = \frac{\hbar^2}{2m}\underbrace{\left(\frac{\pi}{L}\right)^2\left(\frac{\mathcal{N}}{2}\right)^2}_{k_F^2} = \frac{\hbar^2}{2m}\left(\frac{\mathcal{N}}{L}\right)^2\left(\frac{\pi}{2}\right)^2$$

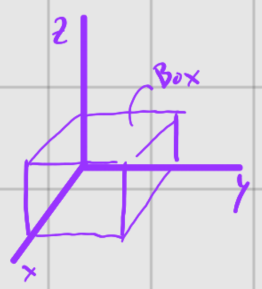
momentum

$$k_F = \frac{\pi}{2}\cdot\frac{\mathcal{N}}{L}$$

$$V_F = \frac{P_F}{m} = \frac{\hbar k_F}{m} = \frac{\hbar}{m}\left(\frac{\pi}{2}\cdot\frac{\mathcal{N}}{L}\right)$$

Fermi velocity

{3D confined particle}



$$V(\vec{r},t) = V(\vec{r}) = \begin{cases} 0 & \text{inside} \\ \infty & \text{out} \end{cases}$$

$$i\hbar \frac{\partial}{\partial t}\Psi(\vec{r},t) = [H]\Psi(\vec{r},t)$$

$$\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r}) + E\,\Psi(\vec{r}) = 0$$

Conditions:

outside: $\Psi(\vec{r}) = 0$

inside: $\Psi(\vec{r}) = \Psi(x,y,z) = A \sin(k_x x)\cdot\sin(k_y y)\cdot\sin(k_z \cdot z)$

↳ Applyin $\Psi(\vec{r})$ in the eq. we get the energy: $\frac{\hbar^2}{2m}\left[k_x^2 + k_y^2 + k_z^2\right] = E$

↳ Now we have conditions for the limits:

$$x: \; k_x L = \pi n_x \rightarrow k_x = \frac{\pi}{L} n_x$$
$$y: \; k_y L = \pi n_y \rightarrow k_y = \frac{\pi}{L} n_y$$
$$z: \; k_z L = \pi n_z \rightarrow k = \frac{\pi}{L} n_z$$

$$\boxed{E_{n_x,n_y,n_z} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2\left[n_x^2+n_y^2+n_z^2\right]}$$

Orthogonality to compute the constant $\int d\bar{r}\; \psi^*_E(\bar{r})\, \psi_{E'}(\bar{r}) = \delta_{E,E'}$ $\begin{cases} 0 & E \neq E \\ 1 & E = E \end{cases}$

$$\int dx \int dy \int dz \; \underbrace{A^* \sin(k_x x)\sin(k_y y)\sin(k_z z)}_{\psi_E}\; \underbrace{A \sin(k_x x)\sin(k_y y)\sin(k_z z)}_{\psi_{E'}} = 1$$

$$|A|^2 \int_0^L dx\, \sin^2(k_x x) \int_0^L dy\, \sin^2(k_y y) \int_0^L dz\, \sin^2(k_z z) = 1$$

$$|A|^2\left(\frac{L}{2}\right)^3 = 1 \longrightarrow |A| = \left(\frac{2}{L}\right)^{3/2}$$

The wavefunction then → $\boxed{\psi_{n_x,n_y,n_z}(\bar{r}) = \left(\sqrt{\frac{2}{L}}\right)^3 \sin(k_x x)\, \sin(k_y y)\, \sin(k_z z)}$

For example for $\psi_{1,2,3}$ and $\psi_{2,1,3}$ have the same energy $E_{1,2,3} = E_{2,1,3}$ but the prob. distribution is different. In 1D this does not happen but in 2D and 3D. This is called: Quantum Degeneracy

3D $E_{\{n_x,n_y,n_z\}} = \begin{cases} \psi \\ \neq \\ \psi \end{cases}$] Quantum Degenaracy

12/10/2022

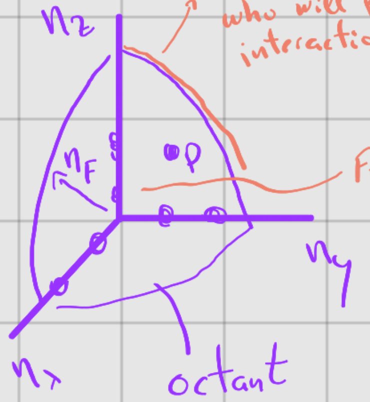
Recap

3D box

$$E_{n_x,n_y,n_z} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (n_x^2+n_y^2+n_z^2)$$

$$\psi_{n_x,n_y,n_z}(\bar{r}) = \left(\sqrt{\frac{2}{L}}\right)^3 \sin\left(\frac{\pi}{L}n_x x\right)\sin\left(\frac{\pi}{L}n_y y\right)\sin\left(\frac{\pi}{L}n_z z\right)$$

We can say: $n^2 = (n_x^2+n_y^2+n_z^2)$



$P = \{n_x, n_y, n_z\}$ each point is a state

All points <u>on</u> the sphere surface → states with same energy

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$$

$n \to n_F$ = Fermi level $\to E_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n_F^2$

$$\boxed{E_n \le E_F}$$

$\frac{4}{3}\pi n_F^3 \times \frac{1}{8}$ = Total n° of points inside sphere with radius n_F $(1/8)$

↳ we consider only positive values of n_x, n_y, n_z

$2\times \frac{4}{3}\pi n_F^3 \frac{1}{8}$ = Tot. n° e⁻ when $E \le E_F$ $\doteq N$

↳ bc of Pauli: there are at max. two e⁻ in each state

With the previous results we can compute:

$$\boxed{n_F = \left(\frac{3N}{\pi}\right)^{1/3}}$$ 3D

$$E_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n_F^2 = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2\left(\frac{3N}{\pi}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{\pi^2}{L^3}\,3N\right)^{2/3} = \frac{\hbar^2}{2m}\left(\overbrace{3\pi^2\frac{N}{\underbrace{L^3}_{\text{volume}}}}^{k_F}\right)^{2/3}$$

$$\boxed{E_F = \frac{\hbar^2}{2m}\,k_F^{2/3}}$$

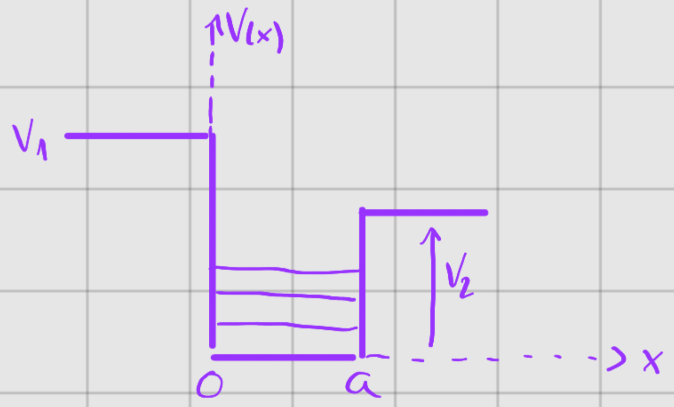
$$\boxed{k_F = \left[3\pi^2\frac{N}{L^3}\right]^{1/3}} \qquad \boxed{p_F = \hbar k_F = \hbar\left[3\pi^2\frac{N}{L^3}\right]^{1/3}} \qquad \boxed{v_F = \frac{p_F}{m} = \frac{\hbar}{m}\left[3\pi^2\frac{N}{L^3}\right]^{1/3}}$$

Finite Well

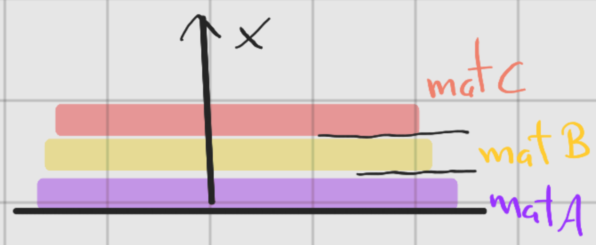
What we have done so far is using Schödinger eq. to solve:

free part. ⟶  ⟶ 

Now finite well:



In materials we have layers



Potential $V(x) = \begin{cases} V_1 & x<0 \\ 0 & 0 \leq x \leq a \\ V_2 & x>a \end{cases}$

Schö Eq for each part

$$\begin{cases} \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + (E-V_1)\psi(x) = 0 & x<0 \\ \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + E\psi(x) = 0 & 0 \leq x \leq a \\ \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + (E-V_2)\psi(x) = 0 & x>a \end{cases}$$

At left side we'll have: $\psi(x) \sim e^{\frac{i\sqrt{2m(E-V_1)}}{\hbar}x} = e^{-\frac{\sqrt{2m(V_1-E)}}{\hbar}x}$

$k = \frac{\sqrt{2mE}}{\hbar}$

$k_1 = \frac{\sqrt{2m(V_1-E)}}{\hbar}$

$k_2 = \frac{\sqrt{2m(V_2-E)}}{\hbar}$

General Solutions

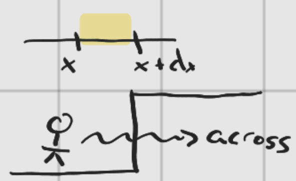
$$\begin{cases} \psi(x) = A_1 e^{-k_1 x} + B_1 e^{+k_1 x} & x < 0 \\ \psi(x) = A e^{-ikx} + B e^{ikx} & 0 \leq x \leq a \\ \psi(x) = A_2 e^{-k_2 x} + B_2 e^{+k_2 x} & x > a \end{cases}$$

•) $\psi(x)$ has to be finite so: $A_1 = 0$ and $B_2 = 0$

•) There is a small probability of e^- to be found after the barrier (tunneling)

Prob of finding the part. $|\psi(x)|^2 dx$ in

↳ for right side $|A_2|^2 e^{-2k_2 x}$



•) Rewriting $\psi(x)$, $0 \leq x \leq a$:

$$A \cos kx - iA \sin kx + B \cos kx + iA \sin kx = \overbrace{(A+B)}^{c \sin\delta} \cos kx + i \overbrace{(B-A)}^{c \cos\delta} \sin kx =$$

$$= c\,(\sin\delta \cos kx + \cos\delta \sin kx) = c \sin(kx+\delta)$$

↳ Orthogonality cond. for inside region

$$\int_0^a dx\, c^* \sin(\bar{k}x+\bar{\delta}) \cdot c \sin(kx+\delta) = \delta_{(k,\bar{k})(\delta,\bar{\delta})}$$

$$|c|^2 \int_0^a dx \sin^2(kx+\delta) = 1 \Rightarrow |c|^2 = \frac{1}{\int_0^a \dots} = \frac{4k}{2ka + \sin 2\delta - \sin(2ka+2\delta)}$$

matlab

related with E

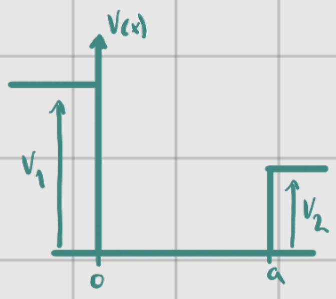
$\frac{\sqrt{2mE}}{\hbar}$

Now, the constant is not longer a constant it's a function of k

$$c \longrightarrow c(k)$$

18/10/2022

Recap



$$\psi(x) = \begin{cases} B_1 e^{+K_1 x} & x<0 \\ C(K)\sin(Kx+\delta) & 0\leq x\leq a \\ A_2 e^{-K_2 x} & x>a \end{cases}$$

$$K = \frac{\sqrt{2mE}}{\hbar}, \quad K_1 = \frac{\sqrt{2m(V_1-E)}}{\hbar}, \quad K_2 = \frac{\sqrt{2m(V_2-E)}}{\hbar}$$

Orthogonality: $|C(K)|^2 = \dfrac{4K}{2Ka + \sin 2\delta - \sin(2Ka+2\delta)}$

⊛ K is the energy so by changing the energy we change the value of $C(K)$

So, every energy level has its own coefficient.

Now we have to impose the two conditions: 1 continuity of the function at the boundaries $[x=0, x=a]$ and 2 continuity of the derivative also at the boundaries.

In general: $f_1 = f_2$ and $f_1' = f_2'$ $\Longrightarrow$ $f_1/f_1' = f_2/f_2'$ which is logarithm derivative. Therefore, it is an easy way to verify impose our conditions and do the calculations.

$\boxed{x=0}$

$$\frac{d}{dx}\left[\ln\left(B_1 e^{K_1 x}\right)\right]\Bigg|_{x=0} = \frac{d}{dx}\left[\ln\left(C(K)\sin(Kx+\delta)\right)\right]\Bigg|_{x=0} \Rightarrow \frac{K_1 \cancel{B_1 e^{K_1 x}}}{\cancel{B_1 e^{K_1 x}}}\Bigg|_{x=0} = \frac{K\,\cancel{C(K)}\cos(Kx+\delta)}{\cancel{C(K)}\sin(Kx+\delta)}\Bigg|_{x=0}$$

$$\Rightarrow K_1 = K\cdot\frac{\cos\delta}{\sin\delta}$$

$\boxed{x=a}$

$$\frac{d}{dx}\left[\ln\left(A_2 e^{-k_2 x}\right)\right]\Bigg|_{x=a} = \frac{d}{dx}\left[\ln\left(C(k)\sin(kx+\delta)\right)\right]\Bigg|_{x=a} \Rightarrow \frac{-k_2 A_2 e^{-k_2 x}}{A_2 e^{-k_2 x}}\Bigg|_{x=a} = \frac{k\,\cancel{C(k)}\cos(kx+\delta)}{\cancel{C(k)}\sin(kx+\delta)}\Bigg|_{x=a}$$

$$\Rightarrow k_2 = -k\,\frac{\cos(ka+\delta)}{\sin(ka+\delta)}$$

Now we get the value of δ

$$(1) \rightarrow \operatorname{cotg}\delta = \frac{k_1}{k} = \sqrt{\frac{\cancel{2m}(V_1-E)}{\cancel{2m}E}} = \sqrt{\frac{V_1}{E}-1} = \sqrt{\frac{2mV_1}{\hbar^2k^2}-1}$$

$$\operatorname{cotg}^2\delta = \frac{\cos^2\delta}{\sin^2\delta} = \frac{2mV_1}{\hbar^2k_1^2}-1 \Rightarrow \frac{\cos^2\delta}{\sin^2\delta}+1 = \frac{2mV_1}{\hbar^2k^2} \Rightarrow \frac{1}{\sin^2\delta} = \frac{2mV_1}{\hbar^2k^2}$$

$$\Rightarrow \sin^2\delta = \frac{\hbar^2k^2}{2mV_1} = \underbrace{\sin^2(\delta-\pi n_1)}$$ → due to periodicity of $\sin^2$

$$\Rightarrow \boxed{\delta(k) = \pi n_1 + \sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV_1}}\right)}$$

$$(2) \rightarrow \operatorname{cotg}(ka+\delta) = -\frac{k_2}{k}$$

$$\Rightarrow \boxed{\delta(k) + ka = \pi n_2 - \sin^{-1}\left(\frac{\hbar k}{2mV_2}\right)}$$

Now if we do (1) = (2) we get:

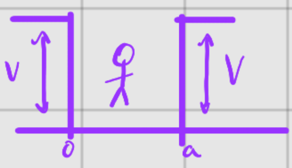
$$\boxed{ka = -\sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV_1}}\right) - \sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV_2}}\right) - \pi(n_2-n_1)}$$ → Trancendental equation

Or implicit equation. Solving this, the solutions give us the energy.

$f(k) = 0 = f(E)$ $\quad n \rightarrow$

n	k
1	$k_1(E_1)$
2	$k_2(E_2)$

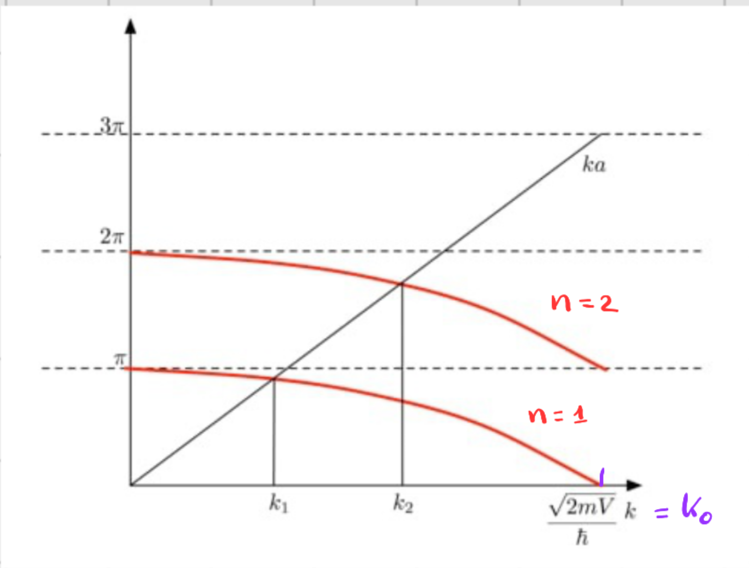
There is a special case when the both potential barriers have the same value.



So $V_1 = V_2 = V \Rightarrow$ Symmetric Quantum Well

In this case we have the next equation; with $n_1 - n_2 = n$:

$$\boxed{Ka = -2\sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV}}\right) + \pi n}$$



$n_1 \to \quad -2\sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV}}\right) + \pi$

$n_2 \to \quad -2\sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV}}\right) + 2\pi$

Finite number of energy levels inside barrier, after that particle could be free.

Now we are interested in knowing what happens when $E << V$

$$Ka = -2\sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV}}\right) + \pi n \xrightarrow[V \text{ big} \quad \sin^{-1}x \sim x]{\text{Taylor}} Ka = -2\frac{\hbar k}{\sqrt{2mV}} + n\pi$$

$$K = \frac{\pi n}{a} - \frac{2}{a}\cdot\frac{\hbar k}{\sqrt{2mV}} \Rightarrow K\left(1 + \frac{2\hbar}{a\sqrt{2mV}}\right) = \frac{\pi}{a}n \Rightarrow K = \frac{\pi}{a}n\left(1 + \frac{2\hbar}{a\sqrt{2mV}}\right)^{-1}$$

↳ when V big:

$$\left(1 + \frac{a}{x}\right)^{-1} \underset{\text{Taylor } x\to\infty}{\approx} 1 - \frac{a}{x}$$

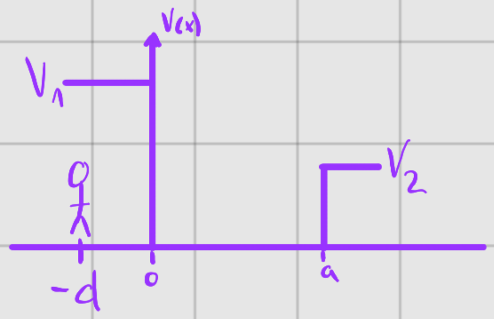
$$\boxed{K_n = \frac{\pi}{a}\left[1 - \frac{\hbar\sqrt{2}}{a\sqrt{mV}}\right]n}$$

$$E_n = \frac{\hbar^2}{2m}k_n^2 \Rightarrow \boxed{E_n = \underbrace{\frac{\hbar^2\pi^2}{2ma^2}}_{E_1}\left(1 - \frac{\hbar\sqrt{2}}{a\sqrt{mV}}\right)^2 n^2}$$

This is the energy of finite well

If $V \to \infty \Rightarrow$ we have infinite well

We now want to calculate the probability of finding a particle outside from the well, on the left.



$$\psi(x) = \begin{cases} B_1 e^{+K_1 x} & x<0 \\ C(k)\sin(kx+\delta) & 0 \leq x \leq a \\ A_2 e^{-K_2 x} & x>a \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$K_1 = \frac{\sqrt{2m(V_1-E)}}{\hbar}$$

$$|C(k)|^2 = \frac{4k}{2ka + \sin 2\delta - \sin(2ka+2\delta)}$$

$$\sin\delta(k) = \frac{\hbar k}{\sqrt{2mV_1}} = \sqrt{\frac{E}{V_1}}$$

We can write the prob as:

$$\int_{-d}^{0} |\psi(x,t)|^2 dx = |B_1|^2 \int_{-d}^{0} e^{-2K_1 x} dx = |B_1|^2 \frac{1-e^{2K_1 d}}{2K_1} + \dots$$

at $x=0$ the eigenfunction has to be continous $\Rightarrow B_1 e^{K_1 x}\Big|_{x=0} = C(k)\sin(kx+\delta)\Big|_{x=0}$

$$\Rightarrow \boxed{B_1 = C(k)\sin\delta(k)}$$

Example:

$E = 4{,}85\,eV$ $\quad V_1 = 5\,\mu eV$ $\quad a = 1\,\mu m$ $\quad d = 10^{-2}\,\mu m$

$$k = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2mc^2E}}{\hbar c} = \frac{\sqrt{2 \cdot 0{,}5 \cdot 10^6 eV \cdot 4{,}85 \cdot 10^{-6} eV}}{1973\, eV \cdot 10^{-10} m} = 11{,}16 \cdot 10^{-6}\, m^{-1}$$

$\Rightarrow ka = 11{,}16$

$$K_1 = \frac{\sqrt{2mc^2(V_1-E)}}{\hbar c} = 1{,}96 \cdot 10^6\, m^{-1}$$

$\sin(\delta) = \sqrt{\frac{4{,}85}{5}} = 0{,}98 \rightarrow \delta = 78{,}5°$ $\Rightarrow C^2 = 1{,}9 \cdot 10^6\, m^{-1}$ $\Rightarrow B_1 = 1{,}3 \cdot 10^3\, m^{-1/2}$

Probability $\sim 0{,}02 \Rightarrow 2\%$

19/10/2022

Recap

① Infinite Well $\Rightarrow E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 n^2$

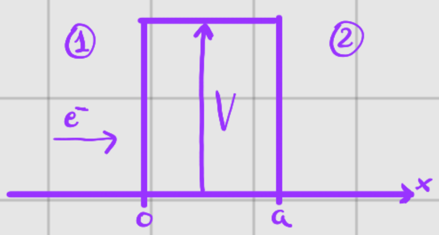
② Finite Well $\Rightarrow Ka = -\sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV_1}}\right) - \sin^{-1}\left(\frac{\hbar k}{\sqrt{2mV_2}}\right) + \pi n \Rightarrow f(k_n) = 0$ solve for each n

$\hookrightarrow E_n = \frac{\hbar^2}{2m} k_n^2$

n	1	2	...
k_n	k_1	k_2	....

(2.1) When the potential level is the same $=$ $E_n \simeq E_n^0 (1 + \text{Correction})$, Correction $\propto \frac{1}{\sqrt{V}}$

Inverse Barrier



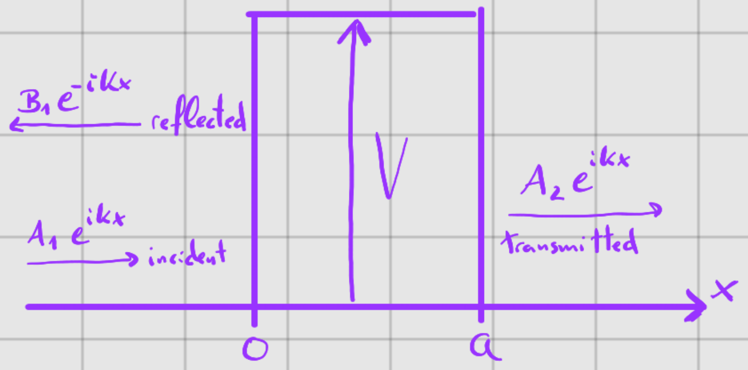
$$V(x) \begin{cases} V & 0 \leq x \leq a \\ 0 & x < 0 \,;\; x > a \end{cases}$$

Schrödinger Equations:

$$\begin{cases} \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + (E-V)\psi(x) = 0 & 0 \leq x \leq a \\ \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + E\psi(x) = 0 & x < 0 \end{cases}$$

We have two situations here: Ⓐ outside the barrier where the solution of the free particle is a plane wave and Ⓑ inside the barrier.

(A) Outside



$$\psi_1(x) = A_1 e^{ikx} + B_1 e^{-ikx}$$

$$\psi_2(x) = A_2 e^{ikx} + \cancel{B_2 e^{-ikx}}$$

this one makes no sense because out of barrier there is no reflection from anywhere

$$\begin{cases} \psi_1(x) = A_1 e^{ikx} + B_1 e^{-ikx} \\ \psi_2(x) = A_2 e^{ikx} \end{cases} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

(B) Inside

$$\psi(x) = A e^{k_1 x} + B e^{-k_1 x} \qquad k_1 = \frac{\sqrt{2m(V-E)}}{\hbar}$$

As usual we have to impose continuity of the function and of its derivatives.

$\boxed{x=0}$

(1) $\rightarrow$ $A_1 + B_1 = A + B$

(2) $\rightarrow$ $ik(A_1 - B_1) = k_1(A - B)$

$\boxed{x=a}$

(3) $\rightarrow$ $A e^{k_1 a} + B e^{-k_1 a} = A_2 e^{ika}$

(4) $\rightarrow$ $A k_1 e^{k_1 a} - B k_1 e^{-k_1 a} = ik A_2 e^{ika}$

If we substitute (1) on (2):

(I) $\rightarrow$ $2A_1 = A\left(1 + \frac{k_1}{ik}\right) + B\left(1 - \frac{k_1}{ik}\right)$

Sum of (3) and (4):

$$(\mathrm{II}) \to 2A\, e^{k_1 a} = A_2 e^{ika}\left(1+\frac{ik}{k_1}\right)$$

Diff of (3) and (4)

$$(\mathrm{III}) \to 2B\, e^{-k_1 a} = A_2 e^{ika}\left(1-\frac{ik}{k_1}\right)$$

$$\left(1-\frac{a}{b}\right)\left(1-\frac{b}{a}\right) = 2-\frac{a^2+b^2}{ab}$$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{a}\right) = 2+\frac{a^2+b^2}{ab}$$

Now, $2\cdot(\mathrm{I})$ and substitute (II) and (III) in (I)

$$4A_1 = A_2 e^{ika}\cdot e^{-k_1 a}\left(1+\frac{ik}{k_1}\right)\cdot\left(1+\frac{k_1}{ik}\right) + A_2 e^{ika} e^{k_1 a}\left(1-\frac{ik}{k_1}\right)\left(1-\frac{k_1}{ik}\right) =$$

$$= A_2 e^{ika}\left[e^{-k_1 a}\left(2+\frac{k_1^2-k^2}{ikk_1}\right) + e^{k_1 a}\left(2+\frac{k^2-k_1^2}{ikk_1}\right)\right]$$

$$= A_2 \frac{1}{ikk_1} e^{ika}\left[e^{-k_1 a}\left(k_1^2-k^2+2ikk_1\right) + e^{k_1 a}\left(k^2-k_1^2+2ikk_1\right)\right] =$$

$$= A_2 \frac{1}{ikk_1} e^{ika}\left[e^{-k_1 a}(k_1+ik)^2 + e^{k_1 a}(k+ik_1)^2\right]$$

$$\frac{A_2}{A_1} = \frac{4ikk_1 e^{-ika}}{e^{-k_1 a}(k_1+ik)^2 + e^{k_1 a}(k+ik_1)^2} = \left\{\begin{array}{l} \sinh x = \dfrac{e^x-e^{-x}}{2} \\ \cosh x = \dfrac{e^x+e^{-x}}{2}\end{array}\right\} =$$

$$= \frac{4ikk_1 e^{-ika}}{(k^2-k_1^2)\,2\sinh(k_1 a) + 4ikk_1\cosh(k_1 a)} = \frac{2ikk_1 e^{-ika}}{\underbrace{(k^2-k_1^2)\cdot\sinh(k_1 a)}_{a} + \underbrace{2ikk_1\cosh(k_1 a)}_{ib}}$$

Transmittance is defined as: $\left|\frac{A_2}{A_1}\right|^2$. So, if we get the modulous square of the previous equation:

$$\left|\frac{A_2}{A_1}\right|^2 = \frac{4k^2k_1^2}{(k^2-k_1)^2\sinh^2(k_1 a)+4k^2k_1^2\cosh^2(k_1 a)} = \frac{4k^2k_1^2}{(k^2-k_1)^2\sinh^2(k_1 a)+4k^2k_1^2\left(1+\sinh^2(k_1 a)\right)} =$$

$$= \frac{4k^2k_1^2}{(k^2+k_1^2)^2\sinh^2(k_1 a)+4k^2k_1^2} = \left[1+\frac{(k^2+k_1^2)^2\sinh^2(k_1 a)}{4k^2k_1^2}\right]^{-1} = T$$

Using now the expression of k and k_1 in terms of E and V:

$$T = \left[1 + \frac{V^2}{4E(V-E)} \sinh^2(k_1 a) \right]^{-1}$$

Transmission function of a particle across the barrier (probability)

•) Check that the behaviour of a transmittance correspons with what we obtained

- $a \to 0 \Rightarrow T = 1$
- $V \to 0 \Rightarrow T = 1$
- $V \to \infty \Rightarrow T = 0$
- $a \to \infty \Rightarrow T = 0$

Let's focus now on the reflectance, we need to find $\left|\frac{B_1}{A_1}\right|^2$

$$\frac{|B_1|^2}{|A_1|^2} = \frac{(k^2+k_1)^2 \sinh^2(k_1 a)}{(k^2+k_1^2)^2 \sinh^2(k_1 a) + 4k^2 k_1^2} \Rightarrow$$

$$R = \left[1 + \frac{4E(V-E)}{V^2} \cdot \frac{1}{\sinh^2(k_1 a)} \right]^{-1}$$

To see everything more clear let's define: $\xi = \frac{V^2}{4E(V-E)} \sinh^2\left(\frac{\sqrt{2m(V-E)}}{\hbar} \cdot a\right)$

So we have:

$$T = (1+\xi)^{-1} \qquad R = \left(1 + \frac{1}{\xi}\right)^{-1}$$

Therefore:

$$T + R = \frac{1}{1+\xi} + \frac{1}{1+\frac{1}{\xi}} = 1 \Rightarrow \boxed{T + R = 1}$$

⊛ Note: In photonics $T+R=1$ is the ideal case because we have to take into consideration that the photons can be absorbed.

In our case $T+R=1$ is real because particles are not lost.

We consider now the particular case where a particle has an energy slightly above the barrier $\boxed{E > V}$

- $ak_1 = a\frac{\sqrt{2m(V-E)}}{\hbar} = a\frac{\sqrt{2m(E-V)}}{\hbar}\, i$
- $\frac{\sinh(ix)}{i} = \frac{e^{ix} - e^{-ix}}{2i} = \sin(x)$

$$\Rightarrow \quad R = \left[1 + \frac{4E(E-V)}{V^2} \cdot \frac{1}{\sin^2\left(\frac{\sqrt{2m(E-V)}}{\hbar}a\right)}\right]^{-1}$$

If we have: $k_1 a = \frac{a\sqrt{2m(E-V)}}{\hbar} = \pi n \Rightarrow R = 0 \ , \ T = 1$

Transmission Resonance

and the energy $\Rightarrow \frac{2m(E-V)}{\hbar^2} = \frac{\pi^2 n^2}{e} \longrightarrow E_{III} = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 n^2 + V$

energy of resonance where the incident e^- with an energy above the barrier behaves as a free electron.

The reflection across a barrier is very small when we are slightly above the barrier. It is zero when we are far above the barrier

25/10/2022

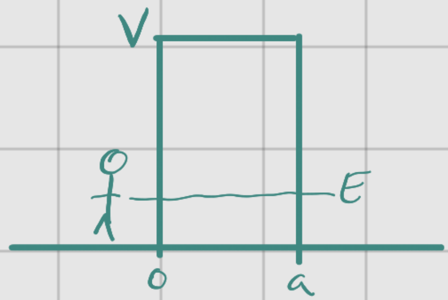
Recap

Reflectance R:

- E < V
- E > V (unusual)

Transmittance T:

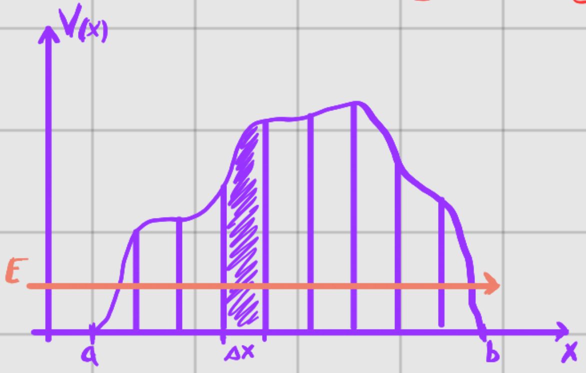
- E < V
- E > V



$$T = \frac{|A_2|^2}{|A_1|^2} = \left[1 + \frac{V^2}{4E(V-E)} \cdot \sinh^2(k_1 a)\right]^{-1}$$

$$k_1 = \frac{\sqrt{2m(V-E)}}{\hbar}$$

Generic Barrier



We study now the case where an e^- travels from the left to the right as is shown in the drawing. We can divide this barrier in several thin barriers of Δx width.

We use the previous result of T we had and also we impose that the potential V(x) is big enough so:

$$V - E \gg \frac{\hbar^2}{2ma^2} \Rightarrow k_1 a \gg 1$$

Therefore: $\sinh^2(k_1 a) = \left(\frac{e^{k_1 a} + e^{-k_1 a}}{2}\right)^2 \simeq \frac{e^{2k_1 a}}{4}$

The for the transmittance:

$$T \simeq \frac{16 k^2 k_1^2}{(k^2 + k_1^2)^2} e^{-2k_1 a} \simeq \underbrace{T_0 e^{-2k_1 a}}_{\text{prob of crossing the barrier}}$$

$$\boxed{T_0 = \frac{16 k^2 k_1^2}{(k^2 + k_1^2)^2}}$$

This quantity doesn't depend on a

① For a generic situation V is not longer a constant it depends on x.
Because V changes at different positions. Now the width of a single barrier is ΔX and we indicate the position of the barrier X_i

- $K_1 \longrightarrow K_1(x_i) = \frac{\sqrt{2m(V(x_i)-E)}}{\hbar}$
- $e^{-2K_1 a} \longrightarrow e^{-2K_1(x_i)\Delta X} \equiv P(x_i)$ Prob. of e^- crossing the barrier x_i

② Probability of crossing boxes is not correlated if the events are independent as individual boxes.

$$P_{Total} = n\text{- independent events} = \prod_i P(x_i) = \prod_i e^{-2K_1(x_i)\Delta X} = e^{-2\sum_i K_1(x_i)\Delta X}$$

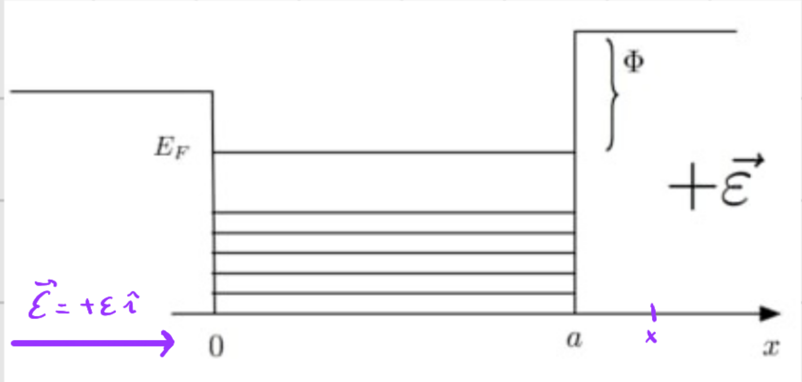
when $\Delta X \to 0$ we integrate

$$\exp\left\{-2\int_a^b K(x)dx\right\} = \exp\left\{-2\int_a^b \frac{\sqrt{2m(V(x)-E)}}{\hbar}dx\right\}$$

$$\boxed{P_{Total} = \exp\left\{-2\sqrt{\frac{2m}{\hbar^2}}\int_a^b \sqrt{V(x)-E}\,dx\right\}}$$

where $V(x)$ is a generic potential that can take any value.

Lets apply the result we got to a finite potential well immersed in an electric field $\vec{\mathcal{E}}$ with an electron with a charge $q = -|e|$



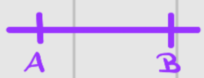
$$V_0 = E_F + \phi$$

$$V(x) = \begin{cases} 0 & 0 \leq x \leq a \quad \text{(inside)} \\ V_0 & x > a,\ x < 0 \quad \text{(outside)} \end{cases}$$

The electric force is : $\vec{F} = q\vec{\mathcal{E}} = q\,\mathcal{E}\,\vec{\imath}$

The work done by the electric field $\vec{\mathcal{E}}$ on the charge "q" from A to B is:

$$W = \int_A^B \vec{F}\cdot d\vec{s} = \int_A^B \vec{F}\cdot d\vec{x} = \int_A^B -|e|\,\mathcal{E}\,dx = q\,\mathcal{E}\,(x_B - x_A)$$

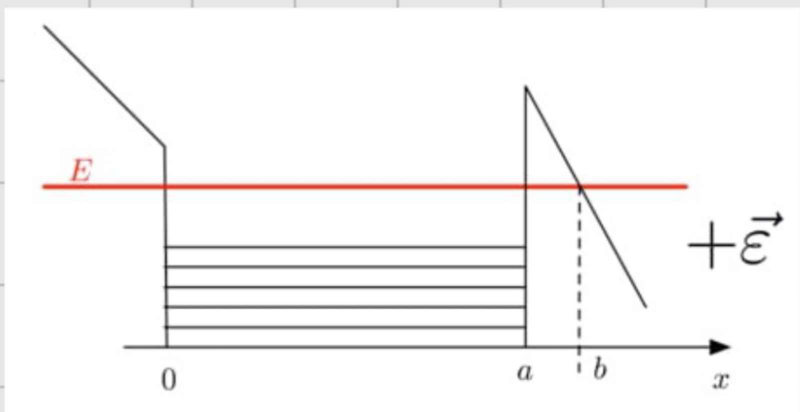


So, the general form of the potential we obtain is:

$V(x) = V_0 + q\,\mathcal{E}(x-a)$ → when $x>a$

$V(x) = V_0 + q\,\mathcal{E}(-x)$ → when $x<0$

Taking int account that $q = -|e|$ what we obtain is the next:



$$V(x) \begin{cases} V_0 - |e|\,\mathcal{E}(x-a) & x>a \\ 0 & 0 \leq x \leq a \\ V_0 + |e|\,\mathcal{E}\,x & x<0 \end{cases}$$

To calculate the probability of the e^- to tunneling we have to compute the integral $\int_a^b \sqrt{V(x) - E}\,dx$, but first some considerations. If we take a generic energy E we will have the next:

$V(a) = V_0 = E_F + \phi$

$V(b) = E \Rightarrow V_0 - |e|\mathcal{E}(b-a) = E \Rightarrow b = a + \frac{V_0 - E}{|e|\mathcal{E}}$

So we calculate the integral:

$$\int_a^{a+\frac{V_0-E}{|e|\mathcal{E}}} \sqrt{V_0 - |e|\mathcal{E}(x-a) - E} \rightarrow \left\{ \begin{matrix} V_0 - E - |e|\mathcal{E}(x-a) = y \\ -|e|\mathcal{E}\,dx = dy \end{matrix} \right\} \rightarrow -\frac{1}{|e|\mathcal{E}} \int_{V_0-E}^{0} \sqrt{y}\,dy = \frac{1}{|e|\mathcal{E}} \int_0^{V_0-E} \sqrt{y}\,dy =$$

$$= \frac{2}{3|e|\mathcal{E}} (V_0 - E)^{3/2}$$

Therefore:

$$\boxed{P_T = \exp\left\{ -\frac{4}{3|e|\mathcal{E}} \sqrt{\frac{2m}{\hbar^2}} (V_0 - E)^{3/2} \right\} = e^{-\mathcal{E}_0/\mathcal{E}}}$$ with $\mathcal{E}_0 = \frac{4}{3|e|}\sqrt{\frac{2m}{\hbar^2}}(V_0-E)^{3/2}$

By turning the E field we change the probability of the tunnelling. So if we increase the strenght of the E field $\Rightarrow$ We increase the probability of tunnelling

Also, modifying the E-field will change the slope of $V(x)$.

① If we use Fermi Level energy:

$E = E_F \Rightarrow V_0 - E = V_0 - E_F \equiv \phi$

$\mathcal{E}_0 = \frac{4}{3}\sqrt{\frac{2m\phi^3}{\hbar^2 |e|^2}}$

② Tunnelling condition : $\rightarrow \boxed{V_0 - E = (b-a)|e|\mathcal{E}}$

$P_T = \exp -\frac{4}{3}\sqrt{\frac{2m|e|\mathcal{E}}{\hbar^2}}\,(\underbrace{b-d}_{})^{3/2}$ $\longrightarrow$ Scanning Tunnelling Microscop (STM)

↳ d = distance!!

③ Using Fermi Level energy and tunnelling condition

$V_0 - E_F = (b-a)|e|\mathcal{E} = d|e|\mathcal{E}$

$\phi = |e|\mathcal{E}d$

$\sqrt{\frac{2m|e|\mathcal{E}}{\hbar^2}}(b-d)^{3/2} = \sqrt{\frac{2m|e|\mathcal{E}d^2 d}{\hbar^2}} = \sqrt{\frac{2m\phi}{\hbar^2}}\,d$

$$\boxed{P_T = \exp -\frac{4}{3}\sqrt{\frac{2m\phi}{\hbar^2}}\,d}$$

26/10/2022

Recap

Probability of tunnelling in presence of an electric field

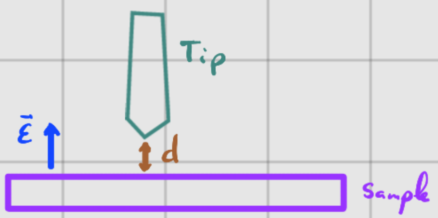
- Depending on the energy: $P_T(E) = \exp\left\{-\frac{4}{3}\sqrt{\frac{2m}{\hbar^2 |e|^2 \varepsilon^2}}(V_0 - E)^{3/2}\right\}$

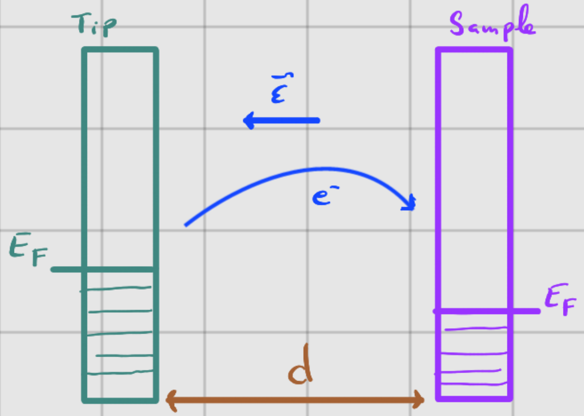
- Depending on Tunnelling distance: $\exp -\frac{4}{3}\sqrt{\frac{2m|e|\varepsilon}{\hbar^2}}\, d^{3/2}$

If we are in the Fermi Level $\Rightarrow$ $E = E_F$

- $P_T(E_F) = \exp\left\{-\frac{4}{3}\sqrt{\frac{2m\phi^3}{\hbar^2 |e|^2 \varepsilon^2}}\right\}$
- $P_T(d) = \exp\left\{-\frac{4}{3}\sqrt{\frac{2m\phi}{\hbar^2}}\, d\right\} \longrightarrow$ STM

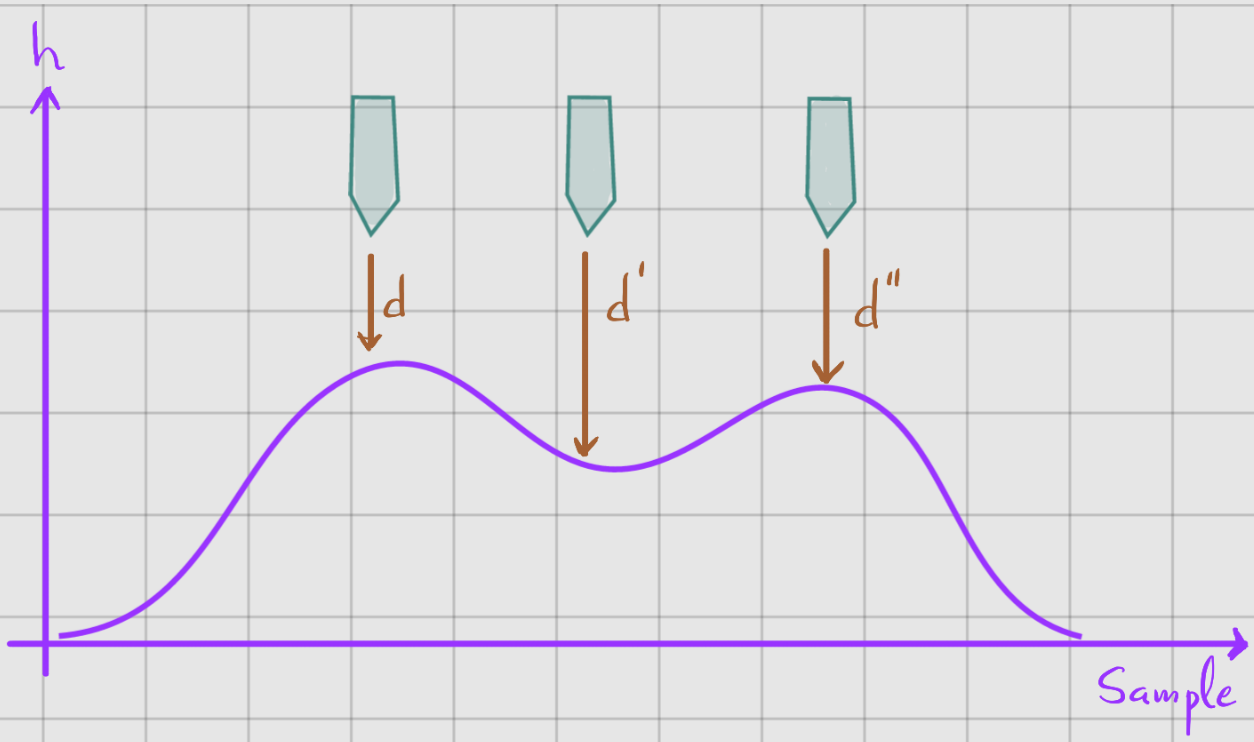
STM



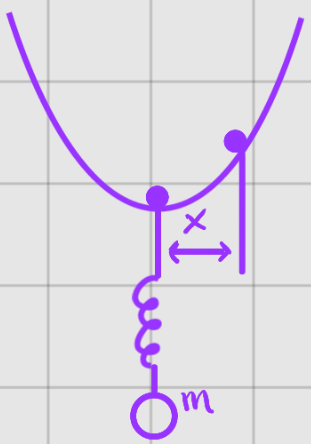


The distance between the tip and the sample is very small, in the range of nanometers. The electric field is applied in the direction of the tip and the electron tunnel from the tip the sample.

When the tunnelling gets to saturation, that is the sample has been changed such that it repels electron and no election can tunnel from the tip to the sample. We change the direction of the electric field such that the e^- will tunnel backwards.



Confinement of a particle in a Parabolic Well



The potential of the particle: $V(x) = \frac{1}{2} K x^2$

The force: $F(x) = K x$

The Schrödinger Equation: $\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right]\psi(x) = E\,\psi(x)$

↳ spring constant $k \Rightarrow \omega_0 = \sqrt{\frac{K}{m}}$

We rearrange the equation as:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2\right]\psi = E\psi \Longrightarrow \left[-\frac{d^2}{dx^2} + \left(\frac{m\omega_0}{\hbar}\right)^2 x^2\right]\psi = \frac{2mE}{\hbar^2}\psi \Rightarrow$$

$$\Rightarrow \left[-\frac{d^2}{dx^2} + \left(\frac{x}{\frac{\hbar}{m\omega_0}}\right)^2\right]\psi = \frac{2mE}{\hbar^2}\psi \longrightarrow \bar{x} = \frac{x}{\sqrt{\hbar/m\omega_0}} \Rightarrow dx = d\bar{x}\sqrt{\frac{\hbar}{m\omega_0}}, \quad \bar{E} = \frac{E}{\hbar\omega_0}$$

$$\Rightarrow \left[-\frac{d^2}{d\bar{x}^2} + \bar{x}^2\right]\psi(\bar{x}) = 2\bar{E}\,\psi(\bar{x})$$

Finally we get the S. eq.

$$\left[\frac{d^2}{dx^2} + (2\bar{E} - \bar{x}^2)\right]\psi(\bar{x}) = 0$$

We can try a solution like: $\psi(\bar{x}) = e^{-\frac{\bar{x}^2}{2}}\, u(\bar{x})$ and we get;

$$\frac{d^2}{dx^2}u(\bar{x}) - 2\bar{x}\frac{du(\bar{x})}{dx} + (2\bar{E} - 1)\, u(\bar{x}) = 0 \longrightarrow \text{Hermite Equation}$$

- It can only be solved if $2\bar{E} - 1 = 2\cdot n$
- The solutions are Hermite polynomials $H_0 = 1$, $H_1 = 2\bar{x}$, $H_2 = 4\bar{x} - 2$

Therefore

$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) \longrightarrow$ The energy level is equidistance

(unlike hydrogen atom: $E_n = -\frac{2}{n^2}$ and finite well: $E_n = E_1 \cdot n^2$)

$$\psi(\bar{x}) = A e^{-\bar{x}^2/2} H_n(\bar{x}) \longrightarrow \psi(x) = A\, e^{-\frac{x^2}{2\hbar/m\omega_0}}\, H\left(\sqrt{\frac{m\omega_0}{\hbar}}\,x\right)$$

The solution is:

$$\boxed{\begin{aligned} E_n &= \hbar\omega_0\left(n + \frac{1}{2}\right) \\ \psi_n(x) &= A_n\, \underbrace{e^{-\frac{m\omega_0}{2\hbar}x^2}}_{\text{gaussian} \to e^{-x^2/2\sigma^2}}\; \underbrace{H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}\,x\right)}_{\text{polynomial}} \end{aligned}}$$

$\sigma = \sqrt{\frac{\hbar}{m\omega_0}}$

First electronic state: $\psi_0(x) = A_0\, e^{-\frac{m\omega_0}{2\hbar}x^2}$

Second electronic state: $\psi_1(x) = A_1\, e^{-\frac{m\omega_0^2}{2\hbar}x^2}\, 2x\sqrt{\frac{m\omega_0}{\hbar}}$

In general we have:

↳ Eigenfunction ($\psi(x)$) = Hermite Polynomial * Gaussian function

↳ Energy has equal spacing

The subsequent states after the first electronic state are multiples of a gaussian state. The second state is like multiplying the gaussian by straight line.

Let's compute the constant A_0 for the fundamental state.

$$\psi_0(x) = A_0\, e^{-\frac{m\omega_0}{2\hbar}x^2}$$

Orthonormality : $\displaystyle\int_{-\infty}^{+\infty} \psi_0^*(x)\,\psi_0(x)\,dx = 1$

$$\int_{-\infty}^{+\infty} A_0^*\, e^{-\frac{m\omega_0}{2\hbar}x^2} A_0\, e^{-\frac{m\omega_0}{2\hbar}x^2}\,dx = |A_0|^2 \int_{-\infty}^{+\infty} e^{-\frac{m\omega_0}{\hbar}x^2}\,dx = \left\{\begin{array}{l} \sqrt{\frac{m\omega_0}{\hbar}}\,x = y \\ \sqrt{\frac{m\omega_0}{\hbar}}\,dx = dy \end{array}\right\}$$

$$= |A_0|^2 \sqrt{\frac{\hbar}{m\omega_0}} \underbrace{\int_{-\infty}^{+\infty} e^{-y^2}\,dy}_{=\sqrt{\pi}} = |A_0|^2 \sqrt{\frac{\hbar}{m\omega_0}}\sqrt{\pi}$$

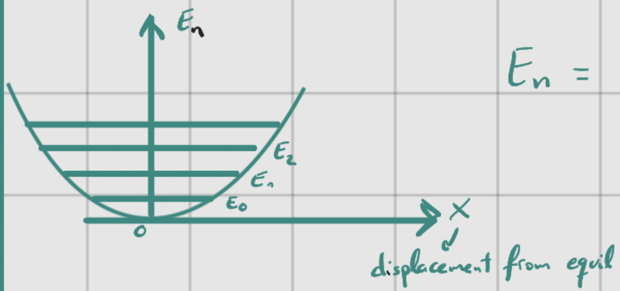
$$|A_0|^2 \sqrt{\frac{\hbar}{m\omega_0}}\sqrt{\pi} = 1 \quad\Rightarrow\quad A_0 = \left(\frac{m\omega_0}{\hbar\pi}\right)^{1/4}$$

08/11/2022

Recap

We saw the very important parabolic confiment

$V(x) = \frac{1}{2} k x^2$



$$E_n = \hbar \omega_0 \left(\frac{1}{2} + n\right) \rightarrow \psi_n(x) = A_n e^{-\frac{m\omega_0}{2\hbar}x^2} H_n\left(x\sqrt{\frac{m\omega_0}{\kappa}}\right)$$

Eigenfunctions

gaussian × Pol. Hermite

$$E_n = -\frac{R}{n^2}$$

$H_0(x) = 1$

$H_1(x) = 2x$

$$\psi_0(x) = \text{ground state} = A_0 e^{-\frac{m\omega_0}{2\hbar}x^2} = \left(\frac{m\omega_0}{\hbar\pi}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}x^2}$$

using: $\int dx\, |\psi_0(x)|^2 = 1$

$$\psi_0(x,t) = \psi_0(x)\, e^{-i\frac{E_0}{\hbar}t}$$

Position expectation value: at ground state E_0

$$\langle x(t) \rangle_0 = \int dx\, \psi^*(x,t)\, x\, \psi(x,t) = \int dx\, |\psi(x)|^2 x = 0 \rightarrow \text{at origin!}$$

Expectation value of x^2 at ground state E_0

$$\langle x^2(t) \rangle_0 = \int dx\, \psi^*(x,t)\, x^2\, \psi(x,t) = \int_{-\infty}^{\infty} dx\, |\psi_0(x)|^2 x^2 = \sqrt{\frac{m\omega_0}{\hbar\pi}} \int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{m\omega_0}{\hbar}x^2} =$$

$$= \frac{\hbar}{2m\omega_0} \longrightarrow x^2 \text{ account of the fluctuations of the wavefunction at the origin}$$

Variance: $\Delta x(t) = \sqrt{\langle x^2(t) \rangle - \langle x(t) \rangle^2} = \sqrt{\frac{\hbar}{2m\omega_0}}$

Now we compute $\langle p \rangle$ for E_0

$$\langle p(t) \rangle_0 = \int dx \, \psi_0^*(x) \left[-i\hbar \frac{d}{dx} \right] \psi_0(x) = 0$$

$$\langle p^2(t) \rangle = \int dx \, \psi_0^*(x) \left[-i\hbar \frac{d}{dx} \right]^2 \psi_0(x) = \frac{\hbar \omega_0 m}{2}$$

Variance $= \Delta p(t) = \sqrt{\langle p^2(t) \rangle - \langle p(t) \rangle^2} = \sqrt{\frac{\hbar \omega_0 m}{2}}$

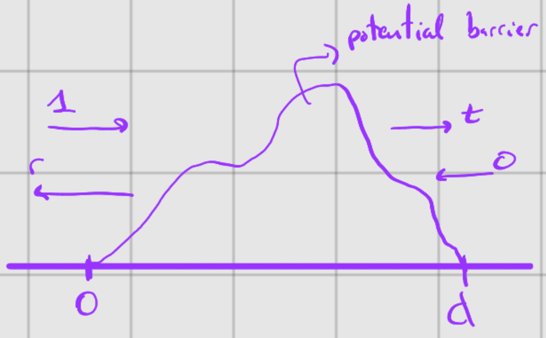
Now we take the product of these two variances:

$$\Delta x(t) \cdot \Delta p(t) = \sqrt{\frac{\hbar}{2m\omega_0}} \cdot \sqrt{\frac{\hbar \omega_0 m}{2}} = \frac{\hbar}{2}$$

This is the Heisenberg uncertainty principle: $\boxed{\Delta x \, \Delta p \geq \frac{\hbar}{2}}$

In the ground state where we are, we have MUS (minimum uncertaintly state) that is where we have the equal in the Heiserberg relation. We cannot go below that value.

Transfer Matrix



moving direction →

$$\bar{u}(0) = \begin{pmatrix} 1 \\ r \end{pmatrix} \qquad \bar{u}(d) = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

moving direction: ←

$$\boxed{\bar{u}(d) = \mathcal{M}\ \bar{u}(0)}$$

→ Transfer Matrix (2x2)

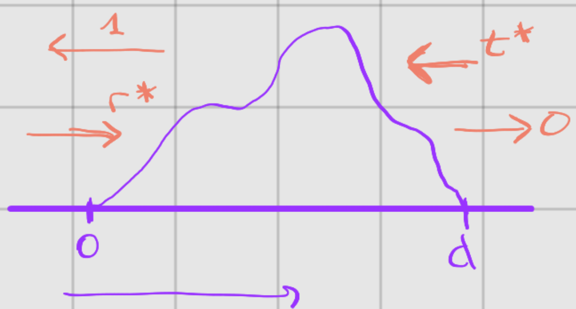
r = complex reflection coefficient = $|r| e^{i\phi}$

t = complex transfer coefficient

So we have expandid the matrix. $\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix}$

This matrix should describe the direct process and also time reverse process ⇒ **Transpose + Complex Conjugate**

For the time reverse process:



$$\bar{u}(d) = \begin{pmatrix} 0 \\ t^* \end{pmatrix} \qquad \bar{u}(0) = \begin{pmatrix} r^* \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ t^* \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} r^* \\ 1 \end{pmatrix}$$

In the end we have 4 eqs. and 4. unknowns

$$t = m_{11} + m_{12} r$$

$$0 = m_{21} + m_{22} r$$

$$0 = m_{11} r^* + m_{12}$$

$$t^* = m_{21} r^* + m_{22}$$

$$\mathcal{M} = \begin{pmatrix} \dfrac{1}{1-|r|^2} & -\dfrac{t r^*}{1-|r|^2} \\ -\dfrac{t^* r}{1-|r|^2} & \dfrac{t^*}{1-|r|^2} \end{pmatrix} = \frac{1}{1-|r|^2} \begin{pmatrix} t & -t r^* \\ -t^* r & t^* \end{pmatrix}$$

The transfer Matrix

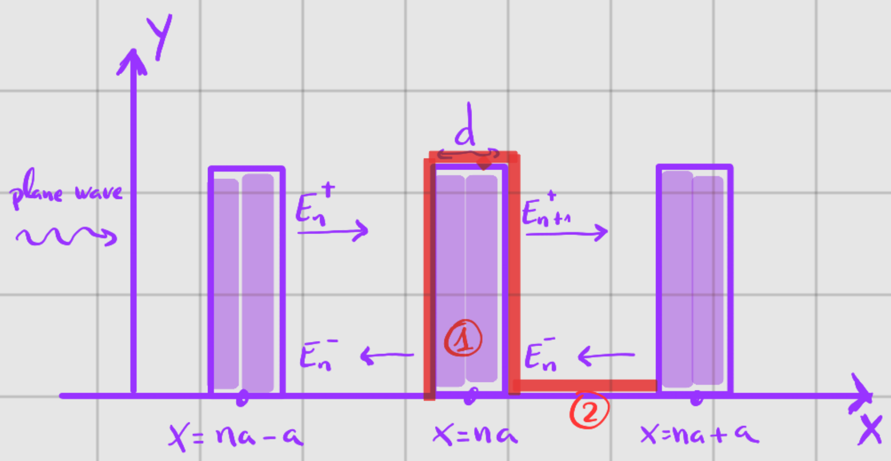
$$\boxed{\mathcal{M} = \frac{1}{1-|r|^2}\begin{pmatrix} t & -tr^* \\ -t^*r & t^* \end{pmatrix}}$$

The determinant

$$\det \mathcal{M} = \frac{|t|^2}{(1-|r|^2)^2} - \frac{|t|^2|r|^2}{(1-|r|^2)^2} \Rightarrow \boxed{\det \mathcal{M} = \frac{|t|^2}{1-|r|^2}}$$

1. No Absorption : $1 = |r|^2 + |t|^2 \Rightarrow \det \mathcal{M} = 1$
2. Absorption : $1 \neq |r|^2 + |t|^2 \Rightarrow \det \mathcal{M} \neq 1$

Block Modes in a Periodic Structure



a = period

$\omega = ck$

$k_n = \frac{\omega}{c} = k$

a = period

$$\vec{E}(\vec{r},t) = \vec{E}(x,t) = \hat{y}\left[E_n^+(\vec{r}) + E_n^-(\vec{r})\right]e^{-i\omega t} = \hat{y}\left[\mathcal{E}_n^+ e^{ikx} + \mathcal{E}_n^- e^{-ikx}\right]e^{-i\omega t}$$

$\hookrightarrow$ field amplitude $\quad\quad \hookrightarrow$ plane wave

① Through the barrier

$$\begin{pmatrix} E_{n+1}^+(x) \\ E_{n+1}^-(x) \end{pmatrix}_{x=na+\frac{d}{2}} = \frac{1}{1-|r|^2}\begin{pmatrix} t & -r^*t \\ -rt^* & t^* \end{pmatrix}\begin{pmatrix} E_n^*(x) \\ E_n^-(x) \end{pmatrix}_{x=na-\frac{d}{2}}$$

② Travelling from the end of barrier to the beginning of the next barrier

$$\begin{pmatrix} E^{+}_{n+1}(x) \\ E^{-}_{n+1}(x) \end{pmatrix}_{x=na+a-\frac{d}{2}} = \begin{pmatrix} e^{ik(a-d)} & 0 \\ 0 & e^{-ik(a-d)} \end{pmatrix} \begin{pmatrix} E^{*}_{n}(x) \\ E^{-}_{n}(x) \end{pmatrix}_{x=na+\frac{d}{2}}$$

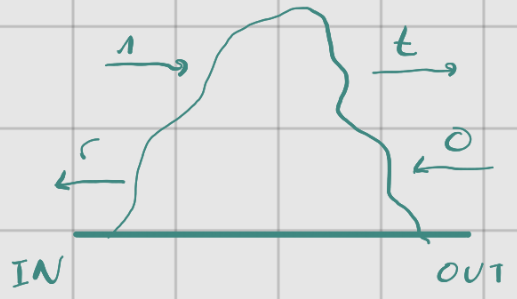
Computing the chain of matrix:

$$\begin{pmatrix} E^{+}_{n+1}(x) \\ E^{-}_{n+1}(x) \end{pmatrix}_{x=na+a-\frac{d}{2}} = \begin{pmatrix} e^{ik(a-d)} & 0 \\ 0 & e^{-ik(a-d)} \end{pmatrix} \frac{1}{1-|r|^2} \begin{pmatrix} t & -r^{*}t \\ -rt^{*} & t^{*} \end{pmatrix} \begin{pmatrix} E^{*}_{n}(x) \\ E^{-}_{n}(x) \end{pmatrix}_{x=na-\frac{d}{2}}$$

Recap

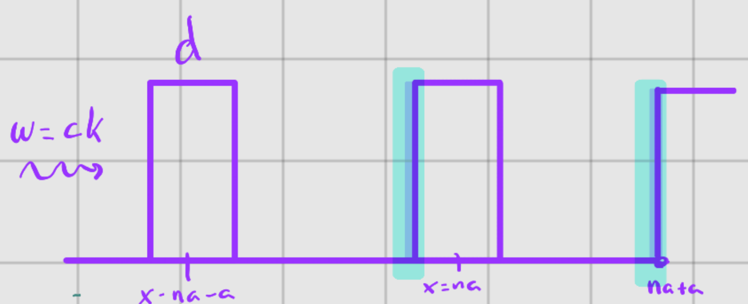
09/11/2022

$$M = \frac{1}{1-|r|^2}\begin{pmatrix} t & -r t^* \\ -r^* t & t^* \end{pmatrix}$$



r and t complex functions

Block Theorem



$\underbrace{w/c = 2\pi/\lambda}_{}\neq k$ → straight wavevector

(1)
$$\begin{pmatrix} E^+_{n+1}(x) \\ E^-_{n+1}(x) \end{pmatrix}_{x=na+a-\frac{d}{2}} = \begin{pmatrix} e^{ik(a-d)} & 0 \\ 0 & e^{-ik(a-d)} \end{pmatrix} [M] \begin{pmatrix} E^+_n(x) \\ E^-_n(x) \end{pmatrix}_{x=na-\frac{d}{2}}$$

$$\boxed{E_{n+1}(x+a) = e^{iKa} E_n(x)}$$

Block wavevector → it is the wavevector inside the structure

Using block theorem we get:

(2)
$$\begin{pmatrix} E^+_{n+1}(x) \\ E^-_{n+1}(x) \end{pmatrix}_{x=na+a-\frac{d}{2}} = \begin{pmatrix} e^{iKa} & 0 \\ 0 & e^{iKa} \end{pmatrix} \cdot \begin{pmatrix} E^+_n(x) \\ E^-_n(x) \end{pmatrix}_{x=na-\frac{d}{2}}$$

$$\left[\begin{pmatrix} e^{ik(a-d)} & 0 \\ 0 & e^{-ik(a-d)} \end{pmatrix} \frac{1}{D} \begin{pmatrix} t & -tr^* \\ -t^* r & t^* \end{pmatrix} - \begin{pmatrix} e^{ika} & 0 \\ 0 & e^{ika} \end{pmatrix}\right] \begin{pmatrix} E_n^+(x) \\ E_n^-(x) \end{pmatrix} = 0$$

Determinant of $[\;\;]$

$$\boxed{\frac{1-|r|^2}{|t|^2} e^{2ika} - 2\,\mathrm{Re}\left[\frac{e^{ik(a-d)}}{t^*}\right] e^{ika} + 1 = 0}$$

Equation for the structure

$f(k, k) = 0$

We can solve it analitically for some specific cases:

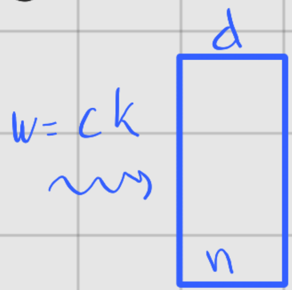
(A) Vacuum:

$$e^{2ika} - 2\,\mathrm{Re}\left[e^{ik(a-d)}\right] e^{ika} + 1 = 0$$

$$e^{ika} - 2\cos(ka) + e^{-ika} = 0$$

$$2\cos ka = 2\cos ka \longrightarrow k = k = \frac{w}{c}$$

(B) When we have $\left[\,\xi\,|\,\xi\,\right]$ the same medium (refractive index n) so: $d \rightsquigarrow a$



Fresnel:

$$r = \frac{(n^2-1)\left(e^{2ika}-1\right)}{(n+1)^2 - (n-1)^2 e^{2ikdn}}$$

$$t = \frac{4n\, e^{ikdn}}{\text{same}} \longrightarrow \frac{1}{t} = \frac{(n+1)^2 e^{-ikdn}}{4n} - \frac{(n-1)^2 e^{+ikdn}}{4n}$$

We also take $n=$ real and No absorption

$$e^{2ika} - 2\,\mathrm{Re}\left[\frac{-(n+1)^2 e^{ikdn}}{4n} - \frac{(n-1)^2 e^{-ikdn}}{4n}\right] e^{ika} + 1 = 0$$

$$-2\cos kdn \left(\frac{(n+1)^2}{4n} - \frac{(n-1)^2}{4n}\right)$$

$$e^{2ika} - 2\cos kdn\, e^{ika} + 1 = 0$$

$$e^{ika} - 2\cos kdn + e^{-ika} = 0$$

$$\cos ka = \cos kdn \longrightarrow \boxed{k = kn} \quad \boxed{w = \frac{c}{n} k}$$

(C) No absorption

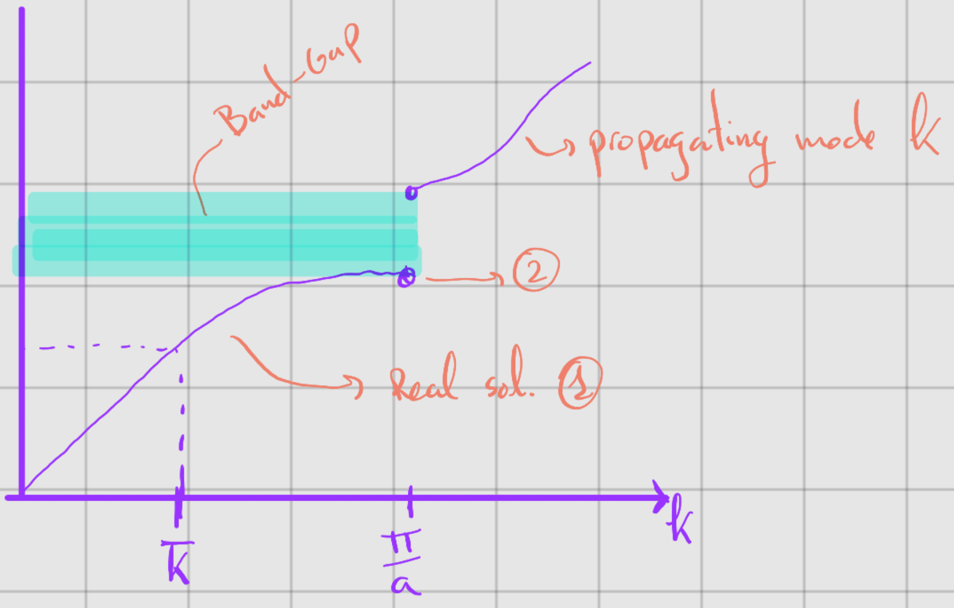
$$\cos ka = \mathrm{Re}\left[\frac{e^{ik(a-d)}}{t^*}\right]$$

Case 1 $\quad -1 < \mathrm{Re}[\;] < 1 \Rightarrow$ Real solution k

Case 2 $\quad \mathrm{Re}[\;] = 1 \Rightarrow$ Real sol. k

Case 3 $\quad \mathrm{Re}[\;] > 1$, $< -1 \Rightarrow$ Imaginary Solution k

Bloch dispersion for a photonic periodic structure



Bloch Theorem for $e^- \Rightarrow \boxed{\psi(x+a) = e^{ika}\psi(x)}$ here k is de Broglie wavevector

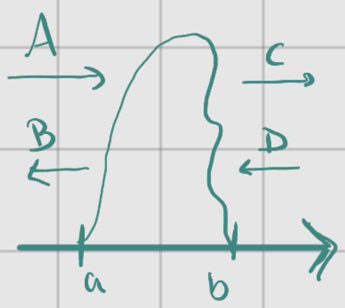
Creating deffects in the structure. we can trap a photon. In the place of the deffect the photon will see a periodic structure but choosing the correct frequency we trap the photon

Recap

S.E.

$$i\hbar\,\partial_t \psi(x,t) = \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x,t) + V(x)\,\psi(x,t) \quad \rightarrow \quad \psi(x,t) = \psi(x)\, e^{-i\frac{E}{\hbar}t}$$

$\boxed{\text{S.S.} \mid \text{S.E.}}$



$$\psi(x) = A e^{+ikx} + B e^{-ikx} \qquad x<a$$

$$\psi(x) = \psi_{ab}(x)$$

$$\psi(x) = C e^{ikx} + D e^{-ikx} \qquad x>b$$

Boundary Conds. $\rightarrow \{A, B, C, D\}$

$\vec{u}(b) = M\, \vec{u}(a)$

M: transfer matrix

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \qquad \begin{matrix} \text{wave direct} \rightarrow \\ \text{wave direct.} \leftarrow \end{matrix}$$

Applying time reversal: $t \rightarrow -t$

Now $\psi(x,-t)$ doesn't satify the original S.E. $\quad i\hbar\partial_t\psi \rightarrow -i\hbar\partial_t\psi$

We use: $\quad \psi^*(x,-t)$

Original

$$\psi(x,t) = e^{-i\frac{E}{\hbar}t} \cdot \begin{cases} A e^{+ikx} + B e^{-ikx} & x<a \\ \psi_{ab}(x) & \\ C e^{ikx} + D e^{-ikx} & x>b \end{cases}$$

Time reversal

$$\psi^*_{(x,-t)} = e^{-i\frac{E}{\hbar}t} \cdot \begin{cases} A^* e^{-ikx} + B^* e^{+ikx} & x<a \\ \psi^*_{ab}(x) & \\ C^* e^{-ikx} + D^* e^{+ikx} & x>b \end{cases}$$

$$\begin{pmatrix} D^* \\ C^* \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} B^* \\ A^* \end{pmatrix}$$

Expanding the matrix and doing the complex conjugate

$$C = m_{21}^* B + m_{22}^* A$$

$$D = m_{11}^* B + m_{12}^* A$$

$$\Rightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \underbrace{\begin{pmatrix} m_{22}^* & m_{21}^* \\ m_{12}^* & m_{11}^* \end{pmatrix}}_{M} \begin{pmatrix} A \\ B \end{pmatrix}$$

So now we know:

$$m_{11} = m_{22}^*$$

$$m_{21}^* = m_{12}$$

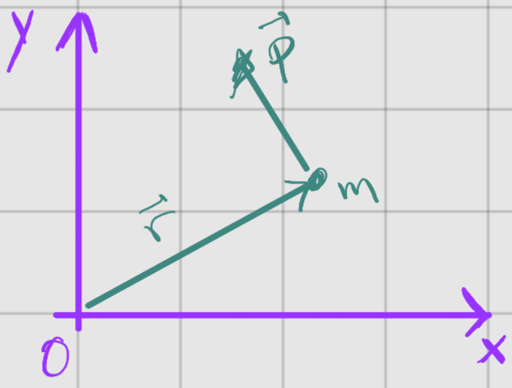
$$\Rightarrow \boxed{M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12}^* & m_{11}^* \end{pmatrix}}$$

Previously in the notes we said that transmission and reflection were connected. We did in the notes: transpose and conjugate.

Here in time reversal we did it.

Angular Momentum

$\vec{L} = \vec{r} \wedge \vec{p}$



① Highschool level to compute $\vec{L}$

Magnitude: $|L| = |r|\,|p| \sin\theta$

Direction: pulgar: r, índice: p, corazón: L

② Bachelor's level to compute $\vec{L}$

$\vec{r} = (x, y, z)$

$\vec{p} = (p_x, p_y, p_z)$ $\Rightarrow$ $\vec{L} = \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix}$

$L_x = y p_z - z p_y$

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$

We have to go to quantum world: $p_x = m v_x \longrightarrow \left(-i\hbar \frac{\partial}{\partial x}\right)$

$$\boxed{\begin{aligned} \hat{L}_x &= -i\hbar\,(y\,\partial_z - z\,\partial_y) \\ \hat{L}_y &= -i\hbar\,(z\,\partial_x - x\,\partial_z) \\ \hat{L}_z &= -i\hbar\,(x\,\partial_y - y\,\partial_x) \end{aligned}}$$

Some operators will commute but not all

Commutator: $[\hat{A}, \hat{B}] = \hat{A} \cdot \hat{B} - \hat{B}\hat{A}$

$\boxed{Ex 1}$: $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = x(-i\hbar\partial_x) - (-i\hbar\partial_x)x$

$$[\hat{x}, \hat{p}]f = x(-i\hbar)\partial_x f - (-i\hbar)\partial_x(xf) = -i\hbar\left[x\partial_x f - f - x\partial_x f\right] = i\hbar f(x)$$

$$\boxed{[\hat{x}, \hat{p}] = i\hbar}$$

$\boxed{Ex 2}$: $[\hat{L}_x, \hat{L}_y] = \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x = (y\hat{p}_z - z\hat{p}_y)(-x\hat{p}_z + z\hat{p}_x) - (-x\hat{p}_z + z\hat{p}_x)(yp_z - zp_y) =$

$$= i\hbar\,(xp_y - yp_x)\; i\hbar\hat{L}_z$$

$$\boxed{[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z}$$

We can see the cyclick property of the commutator of the angular moment

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$$
$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$$
$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

x → y, z

Theorem: If two operators commute then there is always a common set of eigenfunctions that describes both

$[\hat{A}, \hat{B}] = 0 \longrightarrow \begin{matrix}\hat{A}\psi = a\psi \\ \hat{B}\psi = b\psi\end{matrix}\Big\}$ same ψ (eigenfunc) set of eigenfunc $\{\psi, \phi, \varphi\}$

I cannot measure simultaneously two observables when they don't commute.

$\boxed{\text{Ex}}$ $\hat{K} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

$\hat{A} \rightarrow \hat{P}$

$\hat{B} \rightarrow \hat{K}$ (kinetik energy)

$\left.\right] \; [\hat{P}, \hat{K}] = 0 \longrightarrow$ The common eigenfunction is the one of free particle (plane wave)

So we get $\psi = A e^{ikx} \Rightarrow$ We can measure both

$$\left.\begin{aligned} \hat{p}\, e^{ikx} &= a\, e^{ikx} \\ \hat{k}\, e^{ikx} &= b\, e^{ikx} \end{aligned}\right\} \qquad \begin{aligned} a &= \hbar k \\ b &= \frac{\hbar^2}{2m} k^2 \end{aligned}$$

More commutators:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2, \hat{L}_x] + \underbrace{[\hat{L}_y^2, \hat{L}_x]}_{L_y [L_y, L_x] + [L_y, L_x] L_y} + [\hat{L}_z^2, \hat{L}_x]$$

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

$$\boxed{[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0}$$

So we can find a common base

$\{\varphi_{lm}\}$

$\downarrow$

common set of eigenfunct.

L^2, L_z

$$\boxed{\begin{aligned} L^2 \varphi_{lm} &= \hbar^2 l(l+1)\varphi_{lm} \\ L_z \varphi_{lm} &= \hbar m\, \varphi_{lm} \end{aligned}}$$

16/11/2022

We will start with this equation

$$\begin{cases} L^2 \psi_{lm} = \hbar^2 l(l+1)\psi_{lm} & l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, n-1 \\ L_z \psi_{lm} = \hbar m_l \psi_{lm} & m_l = -l, \ldots, 0, \ldots, l-1, l \end{cases}$$

energy level minus one

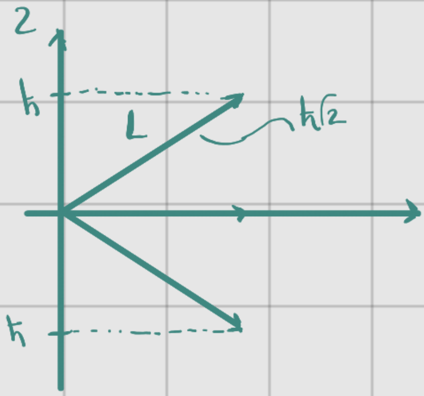
So we can see that we can only have a certain values of the angular momentum. It is discrete or quantized $\Rightarrow$ $L = \hbar\sqrt{l(l+1)}$

The value $l=0 \Rightarrow m=0 \Rightarrow$ Is a particle with no angular momentum

Example:

- $l = 1 \Rightarrow m = \hbar \cdot 1,\ \hbar \cdot 0,\ \hbar(-1)$

  $L = \hbar\sqrt{2}$



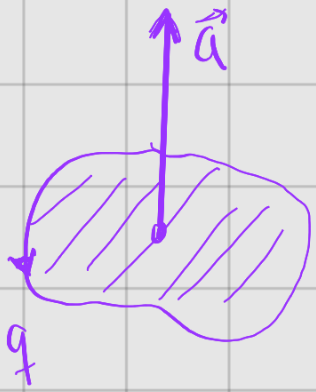
- $l = \frac{1}{2} \Rightarrow m = -\frac{1}{2}, \frac{1}{2}$

  $L = \frac{\hbar}{2}\sqrt{3}$

So in the nanoworld we cannot change the angular momentum continously, it will change in units of $\hbar$ $(\hbar, 2\hbar, 3\hbar, \ldots)$

Orbital Angular Momentum

No $\frac{1}{2}$ integer values : $\cancel{\frac{1}{2}}$, $\cancel{\frac{3}{2}}$, . . .



I = current $= q \cdot \nu$ $\quad (\omega = 2\pi\nu)$

ν = frequency

Magnetic momentum : $\vec{\mu} = I\vec{a}$ = Orbital magnetic momentum

$$\vec{\mu} = I\vec{a} = I\underbrace{a}_{\text{area enclosed by loop}} \cdot \hat{a} = q\nu \cdot a\hat{a} \overset{a \to \circ}{=} \underbrace{q}_{e^-}\nu\pi r^2\hat{a} =$$

$$= -|e|\nu\pi r^2\hat{a}$$

Calculating $\vec{L}$ we know that $\vec{r} \perp \vec{p}$

$$\vec{L} = \vec{r} \wedge \vec{p} = rp = rmv = rm\omega r\ \hat{a} = 2\pi\nu m r^2\hat{a}$$

Using both results:

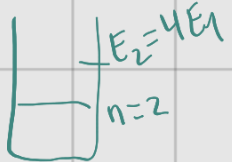
$$\boxed{\vec{\mu} = -\frac{|e|}{2m}\vec{L}}$$

Now applying quantum we get that the directions of $\vec{\mu}$ are also discrete.

$$|\vec{\mu}| = \frac{|e|}{2m}|\vec{L}| = \frac{|e|}{2m}\hbar\sqrt{\ell(\ell+1)} = \mu_B\sqrt{\ell(\ell+1)} \qquad \boxed{|\vec{\mu}| = \mu_B\sqrt{\ell(\ell+1)}}$$

Bohr's magneton: $\mu_B = 5{,}79 \cdot 10^{-5}\ eV/T$

Example



$n = 2 \Rightarrow \ell = 0, 1$

- $\ell = 0 \Rightarrow \mu = 0$
- $\ell = 1 \Rightarrow |\vec{\mu}| = \mu_B\sqrt{(1+1)\cdot 1} = \sqrt{2}\mu_B \neq 0$

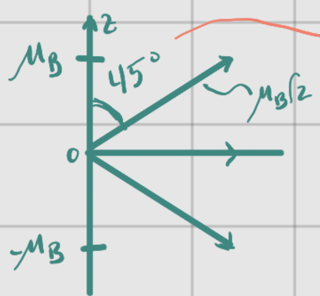
$$\vec{\mu} = -\frac{|e|}{2m}\vec{L}$$

$$\underbrace{\vec{\mu}\cdot\hat{z}}_{\text{proyection on z direction}} = -\frac{|e|}{2m}L_z = -\frac{|e|\hbar m_\ell}{2m} = -\mu_B m_\ell$$

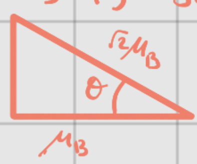
$m = 1, 0, -1 \rightarrow -\mu_B \cdot 1$

$-\mu_B \cdot 0$

$-\mu_B \cdot (-1)$



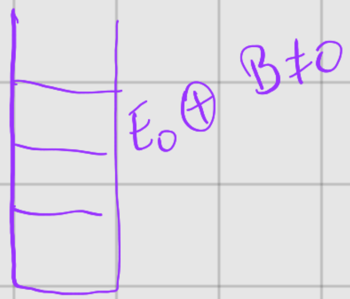
It is 45° because



$\Rightarrow \cos\theta = \frac{\mu_B}{\sqrt{2}\mu_B}$

$\theta = 45°$

Now if we put a magnetic field

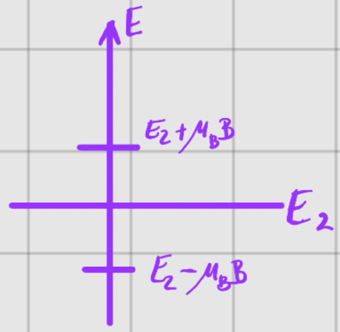


Interaction $\rightarrow$ $E_0 - \vec{\mu} \cdot \vec{B} = E_0 + \frac{|e|}{2m} \vec{L} \cdot \vec{B} = E_0 + \frac{|e|}{2m} \hbar m_\ell B =$

$$= E_0 + \mu_B B m_\ell$$

$$\boxed{E = E_0 + \mu_B B m_\ell}$$

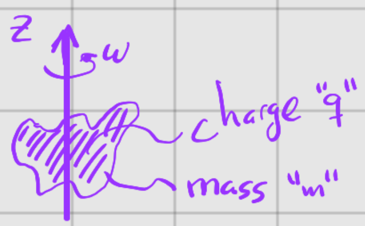
Zeeman

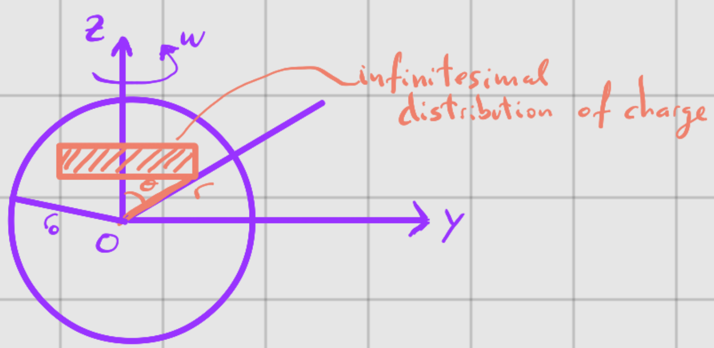


22/11/2022

Electron Intrinsic Angular Momentum
SPIN

Each particle has his own orbital angular momentum. The e^- is a distribution of charge rotating about some direction





Rotation $\Rightarrow$ $d\vec{\mu} = (\nu q) \cdot$ surface $\hat{k} = (\nu dq) \cdot (\pi y^2)\, \hat{k}$

$dq = \rho \cdot dV = \rho\, r^2 dr \sin\theta\, d\theta\, d\varphi$ $\qquad \rho = \frac{q}{\frac{4}{3}\pi r_0^3}$ → tot. charge sphere

Magnetic moment due rotation ▨ $\Rightarrow$ $d\vec{\mu} = \nu dq (\pi y^2)\, \hat{k} =$

$= \pi \nu \rho\, r^4 dr \sin^3\theta\, d\theta\, d\varphi\, \hat{k}$

Total is: $\vec{\mu} = \int d\vec{\mu} = \nu\pi\rho \frac{r_0^5}{5} 2\pi \underbrace{\int_0^{\pi} d\theta \sin^3\theta}_{4/3} = \frac{8}{3}\pi^2 \nu \rho \cdot \frac{r_0^5}{5}\, \hat{k}$

Magnetic momentum induced by ROTATION in z axis

$\Rightarrow \vec{\mu} = \frac{8}{3}\pi^2 \nu \rho \frac{r_0^5}{5}\, \hat{k}$

Now, we have to compute the angular moment of the sphere

$$\vec{S} = I\omega = \frac{2}{5} m r_0^2 \, 2\pi\nu \, \hat{k}$$

magnetic moment due to the intrinsic rotation

If we compare both we get: $\boxed{\vec{\mu} = \frac{q}{2m}\vec{S}}$

angular momentum due to rotation

Remember: $\vec{\mu} = \frac{q}{2m} \cdot \vec{L}$ → magnetic moment due to orbital momentum

IMPORTANT: The right value is $\vec{\mu}_s = -\frac{|e|}{m}\vec{S}$ for the electron

the discrepancy comes from the fact tha the e^- is not a ball but a wavefunction. Also, to find the right solution we have to take into account that the e^- is a relativistic particle.

Right values:

$$\boxed{\vec{\mu} = -\frac{|e|}{2m}\vec{L}} \qquad \boxed{\vec{\mu}_s = -\frac{|e|}{m}\vec{S}}$$

Coming back to the orbital equations:

$$\begin{cases} L^2 \varphi_{lm} = \hbar^2 \ell(\ell+1)\varphi_{lm} \\ L_z \varphi_{lm} = \hbar m_\ell \varphi_{lm} \end{cases}$$

For the electron

$$\begin{cases} S^2 \chi_{s,m_s} = \hbar^2 s(s+1)\chi_{s,m_s} \\ S_z \chi_{s,m_s} = \hbar m_s \chi_{s,m_s} \end{cases}$$

the value of $s = \frac{1}{2}$ (experiments)

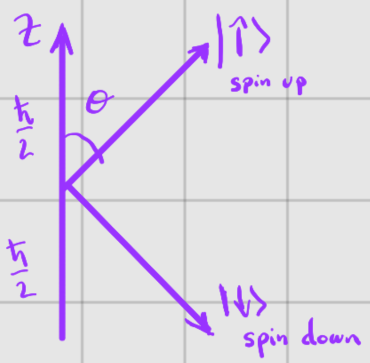
$\Rightarrow m_s = -\frac{1}{2} \quad m_s = +\frac{1}{2}$

$\chi_{\frac{1}{2},\frac{1}{2}} \rightarrow |\uparrow\rangle$

$\chi_{\frac{1}{2},-\frac{1}{2}} \rightarrow |\downarrow\rangle$

Calculating the angles of orientation:

$S = \hbar\sqrt{s(s+1)} = \hbar\frac{\sqrt{3}}{2}$



$\frac{\hbar}{2} = \frac{\hbar}{2}\sqrt{3}\cos\theta \Rightarrow \theta =$

Therefore the full wavefunction of an electron inside the quantum well is:

spatial part, orbital part, spin part

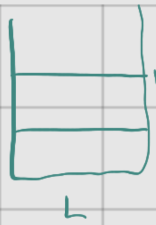
space $\rightarrow \psi_{(x)} = \sqrt{\frac{2}{L}}\sin\frac{\pi}{L}2x$

Orbital: φ_{ℓ,m_ℓ}

spin: χ_{s,m_s}

$$\boxed{\psi = A\sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}2x\right)\varphi_{\ell,m_\ell}\chi_{s,m_s}}$$

Example:



$\longrightarrow E_2 = 4E_1$

Orbital angular momentum: $l=0$, $l=1$

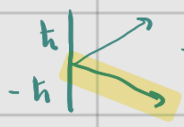
$l=0 \;\downarrow\; L=0 \;\downarrow\; |\vec{\mu}|=0$

$l=1 \;\downarrow\; L=\hbar\sqrt{2} \;\downarrow\; |\vec{\mu}| = \frac{|e|}{2m}\hbar\sqrt{2} = \sqrt{2}\,\mu_B$

$\llcorner\!\!\to$

3 possible orientation $\vec{\mu}_z$

$S = \frac{1}{2}$ $\to \chi_{\frac{1}{2},\frac{1}{2}}$, $\to \chi_{\frac{1}{2},-\frac{1}{2}}$



$\longrightarrow$ 2 possible spin

The wavefunction in case of ——— is: $\psi \sim \sqrt{\frac{2}{L}}\,\sin\;\varphi_{1,1}\;\chi_{\frac{1}{2},-\frac{1}{2}}$

23/11/2022

Recap

Spin

$$S^2 \chi_{s,m_s} = \hbar^2 s(s+1)\chi_{s,m_s}$$
$$S_z \chi_{s,m_s} = \hbar m_s \chi_{s,m_s}$$

$\Rightarrow$ spin $e^- = \frac{1}{2}$, $m_s = -\frac{1}{2}, \frac{1}{2}$ $\Rightarrow$ $\chi_{\frac{1}{2},\frac{1}{2}}$ spin up, $\chi_{\frac{1}{2},-\frac{1}{2}}$ spin down

$\Rightarrow \vec{\mu}_s = -\frac{|e|}{m}\cdot\vec{S}$ $\Rightarrow$ $\vec{\mu} = -\frac{|e|}{2m}\vec{L}$

We have now an electron with no orbital angular momentum ($l=0$)

Ⓐ No magnetic field $B=0$

$$E = E_0$$

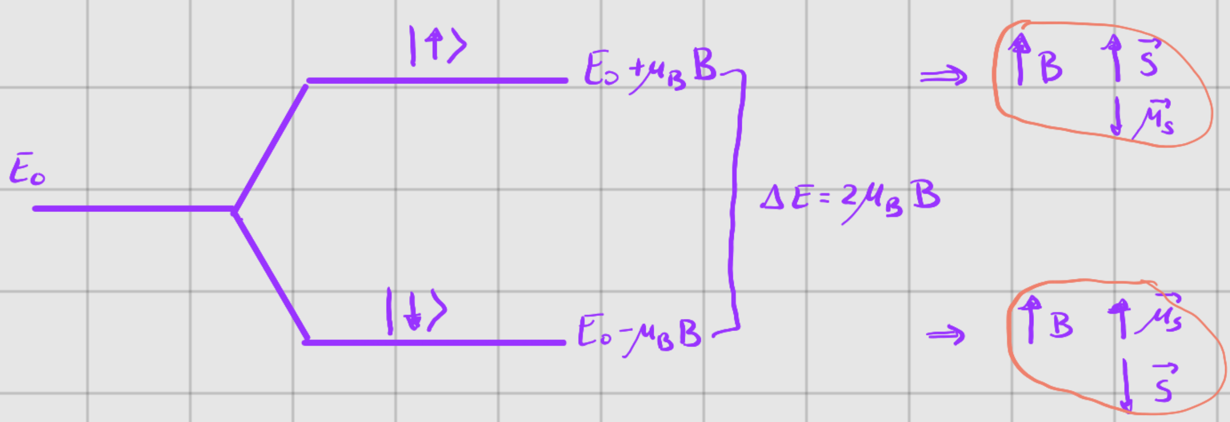
Ⓑ Constant magnetic field $\vec{B} = B\hat{k}$

$$E = E_0 - \vec{\mu}\cdot\vec{B} = E_0 - \left(-\frac{|e|}{m}\vec{S}\right)\cdot\vec{B} = E_0 + \frac{|e|}{m}[S_z]B =$$
$$= E_0 + \frac{|e|}{m}\hbar m_s B = E_0 + \frac{\hbar|e|}{2m} 2 m_s B = \text{2 possibilities}$$

$\boxed{1} \Rightarrow E_0 + \frac{\hbar|e|}{2m}B = E_0 + \mu_B B$

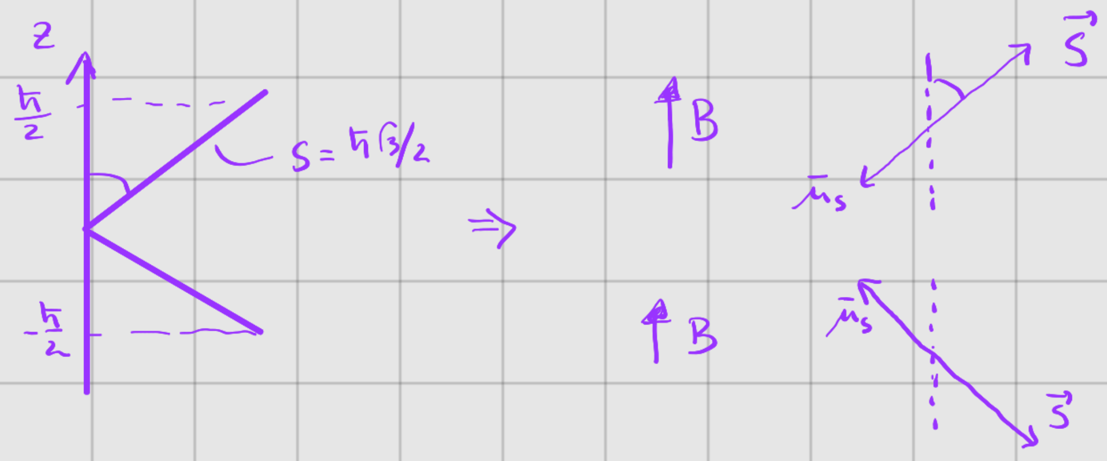
$\boxed{2} \Rightarrow E_0 - \frac{\hbar|e|}{2m}B = E_0 - \mu_B B$

So, we will have the energy level splited in two levels



These are not the correct picture

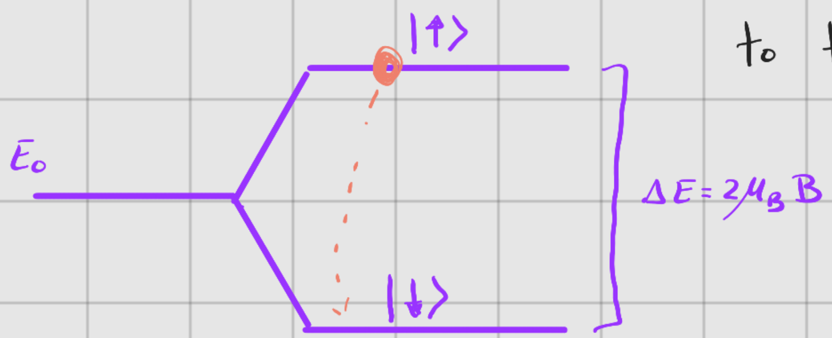
In fact, the spin has this direction



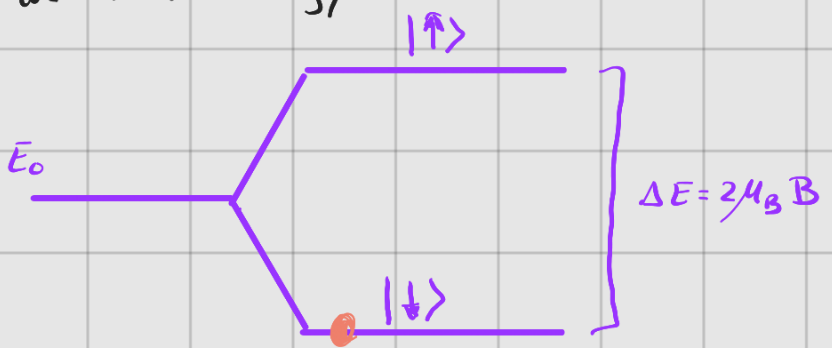
If we have an e^- in the level shown, after a while it can decay to the lower level $\Rightarrow$ spin-flip transition

or

spin relaxation process



If we are in the lower state and we want to go back to the upper level we need energy



$h\nu = \Delta E = 2\mu_B B$

$\boxed{\nu_{SF} = \frac{2\mu_B B}{h}}$ for the e^-

$|\uparrow\rangle \longrightarrow |\downarrow\rangle \Rightarrow$ Release photon

Using some numbers:

$$\nu_{SF} = \frac{2 \cdot 5{,}79 \cdot 10^{-5}\,\frac{eV}{T} \cdot 1\,T}{10^{-15}\,eV \cdot s} = 3 \cdot 10^{10}\,Hz \longrightarrow \text{microwaves frequency}$$

The photon is a microwave photon when we put 1 Tesla

{Nuclear Magnetic Resonance}

NMR is the previous situation but for the proton. Also, the proton does not have orbital ang. moment and its spin is like e^-

- $\ell = 0$, $s = \frac{1}{2}$

So everything is like with the e^- in the previous case. We can introduce the

gyro magnetic factor : g_s

electron $\rightarrow$ $\vec{\mu} = -\frac{|e|}{m}\vec{S} = -g_s \frac{|e|}{2m}\vec{S} \Rightarrow g_s = 2$

proton $\rightarrow$ $\vec{\mu} = \ldots\ldots = +g_p \frac{|e|}{2m_p}\vec{S} \Rightarrow g_p = 5{,}597$ $\rightarrow$ Nobel prize for measuring

This very strange number appears for the fact of the proton is made by 3 quarks.

If we compare both magnetic moments:

$$\frac{|\mu_p|}{|\mu|} = \frac{g_p}{m_p}\cdot\frac{m_e}{g_e} \simeq \frac{6}{2}\,\frac{1}{2\cdot 10^3} = 1{,}5\cdot 10^{-3}$$

Now we put a nucleous in a magnetic field

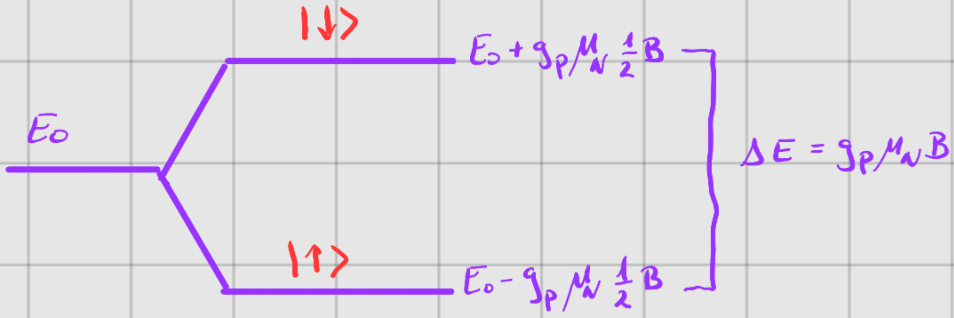
$$E = E_0 - \vec{\mu}_p\cdot\vec{B} = E_0 - \left(g_p \frac{|e|}{2m_p}\vec{S}\right)\cdot\vec{B} = E_0 - g_p\frac{|e|}{2m_p}[S_z]B = E_0 - \frac{g_p|e|}{2m_p}\hbar m_s B =$$

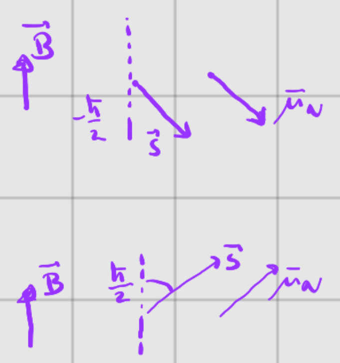
$$= E_0 - g_p\frac{\hbar|e|}{2m_p}m_s B = E_0 - g_p \mu_N m_s B = \text{2 possibilities}$$

$\frac{\hbar|e|}{2m_p} = \mu_N = \text{nuclear magneton} = 3{,}15\cdot 10^{-8}\,\frac{eV}{T}$

$\boxed{1} \Rightarrow E_0 + g_p \mu_N \frac{1}{2} B$ $\quad m_s = -\frac{1}{2}$ (spin down)

$\boxed{2} \Rightarrow E_0 - g_p \mu_N \frac{1}{2} B$ $\quad m_s = \frac{1}{2}$ (spin-up)





The angle is the same as we had with the electron

If we do the same calculation as before of the spin-flip frequency:

$$\Delta E = g_p \mu_N B \simeq 6 \cdot 3{,}15 \cdot 10^{-8} \frac{eV}{T} \cdot 1\,T \simeq 10^{-7}\,eV = h\,\nu_{SF}$$

$$\nu_{SF} = \frac{\Delta E}{h} \simeq 40\ MHz$$ → Radio waves

$$\boxed{\nu_{SF} = \frac{g_p \mu_N B}{h}}$$ for the proton

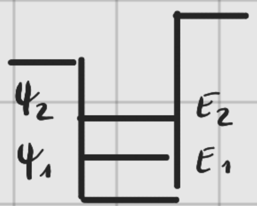
An application of this is to find if a biological tissue is healthy or not.

The spin-flip transition is different when the tissue is healthy.

Quantum BIT

q-bit = quantum system with 2 distinct states

•) i.e. quantum well with 2 energy levels:



The two states are orthogonal

•) i.e. single photon → $|V\rangle$, $|H\rangle$ } flying q. bit

V → vertical polarization

H → Horizontal polariz.

So we have: $\{\psi_0, \psi_1\}$ → classical encoding: $\{\psi_0\} \to 0$ $\{\psi_1\} \to 1$

→ quantum encoding: $a_0\,\psi_0(x) + a_1\,\psi_1(x) = a_0|0\rangle + a_1|1\rangle$

superposition

Operation with q-bit

$$\boxed{I = \underbrace{|0\rangle}_{\psi_0}\underbrace{\langle 0|}_{\psi_0^*} + \underbrace{|1\rangle}_{\psi_1}\underbrace{\langle 1|}_{\psi_1^*}}$$ → Identity operator

$$\boxed{\begin{aligned} \sigma_x &= |0\rangle\langle 1| + |1\rangle\langle 0| \\ \sigma_y &= i\left(|1\rangle\langle 0| - |0\rangle\langle 1|\right) \\ \sigma_z &= |0\rangle\langle 0| - |1\rangle\langle 1| \end{aligned}}$$

q-bit basis states

$\{|0\rangle, |1\rangle\}$

Example:

$$\sigma_x|0\rangle = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|0\rangle = |0\rangle\langle 1|0\rangle + |1\rangle\overbrace{\langle 0|0\rangle}^{\text{BRAKET}} = |1\rangle$$

$$\Rightarrow \langle 0|0\rangle = \int \psi_0^*(x)\,\psi_0(x)\,dx = \int |\psi(x)|^2\,dx = 1$$

$$\Rightarrow \langle 1|1\rangle = 1$$

$$\Rightarrow \langle 0|1\rangle = \int \psi_0^*(x)\,\psi_1(x)\,dx = \delta_{10} = 0$$ → the two states are orthogonal

σ_x is the flip operator

$$\sigma_x|0\rangle = |1\rangle$$

$$\sigma_x|1\rangle = |0\rangle$$

29/12/2022

Recap

A q-bit is having two states system. These states have to be orthogonal

Important Operators

- Identity: $I = |0\rangle\langle 0| + |1\rangle\langle 1|$
- Flip: $\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$
- $\hat{\sigma}_y = i\left(|1\rangle\langle 0| - |0\rangle\langle 1|\right)$
- $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

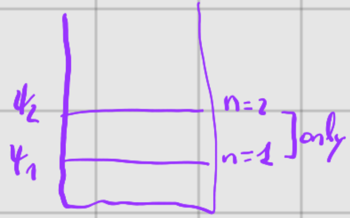
Notation: Dirac representation

$|\alpha\rangle \rightarrow$ Dirac quantum state $\rightarrow$ ket

Formal q-bit representation: $a_0|0\rangle + a_1|1\rangle$

with $|0\rangle, |1\rangle$ the two basis states and $\{a_0, a_1\} \in \mathbb{C}$

Also $a_0\psi_0(x) + a_1\psi_1(x)$ = Solution of Schrödinger Equation



Computation: $\hat{\sigma}_y|\alpha\rangle$

$\sigma_y|0\rangle = i\left(|1\rangle\langle 0| - |0\rangle\langle 1|\right)|0\rangle = i\left(|1\rangle\right) = i|1\rangle = e^{i\pi/2}|1\rangle$

$\sigma_y|1\rangle = \dots = -i|0\rangle = e^{-i\pi/2}|0\rangle$

Flip and add a phase

Computation: $\hat{\sigma}_z|\alpha\rangle$

$\sigma_z|0\rangle = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)|0\rangle = |0\rangle$ $\rightarrow$ Maintain the state

$\sigma_z|1\rangle = \dots = -|1\rangle = e^{i\pi}|1\rangle$ $\rightarrow$ add a phase

Commutation:

•) $[\sigma_x, \sigma_y] = 2i\sigma_z$ → They don't commute, they will **NOT** have common set of eigen states

$$\left.\begin{aligned} \sigma_x \sigma_y &= (|0\rangle\langle 1| + |1\rangle\langle 0|) \cdot i(|1\rangle\langle 0| - |0\rangle\langle 1|) = i(|0\rangle\langle 0| - |1\rangle\langle 1|) \\ \sigma_y \sigma_x &= i(|1\rangle\langle 0| - |0\rangle\langle 1|) \cdot (|0\rangle\langle 1| + |1\rangle\langle 0|) = i(|1\rangle\langle 1| - |0\rangle\langle 0|) \end{aligned}\right\} 2i\sigma_z$$

•) $\hat{O}\,\psi = \lambda\,\psi$

$$\underbrace{\hat{\sigma}_x}_{\hat{O}} \underbrace{\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]}_{\psi} = \frac{1}{\sqrt{2}}(|0\rangle\langle 1| + |1\rangle\langle 0|)(|0\rangle + |1\rangle) = \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{\psi}\underbrace{(+1)}_{\lambda}$$

→ Eigenvalue for THIS eigenstate $\lambda = 1$

$$\hat{\sigma}_y \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \cdots = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)(+1)$$

So we have:

$$\left.\begin{aligned} &\text{Eigenstate of } \sigma_x \rightarrow \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \\ &\text{Eigenstate of } \sigma_y \rightarrow \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle) \end{aligned}\right\} \text{Different set of eigenstates}$$

Operation with q-bit "matrices"

•) $$\left.\begin{aligned} |0\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |1\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \text{matrix of single state}$$

$$a_0|0\rangle + a_1|1\rangle = a_0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

matrix of the superposition q-bit state!

•) $\sigma_x [a_0|0\rangle + a_1|1\rangle] = a_0|1\rangle + a_1|0\rangle$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \Rightarrow \begin{aligned} a_0 A + a_1 B &= a_1 \\ a_0 C + a_1 D &= a_0 \end{aligned} \Rightarrow \begin{aligned} A = 0 &, \ B = 1 \\ C = 1 &, \ D = 0 \end{aligned}$$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ Matrix representation of $\hat{\sigma}_x$ operator

•) $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$ $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Matrix form of $\sigma_x, \sigma_y, \sigma_z$ are called Pauli spin matrices

Computation:

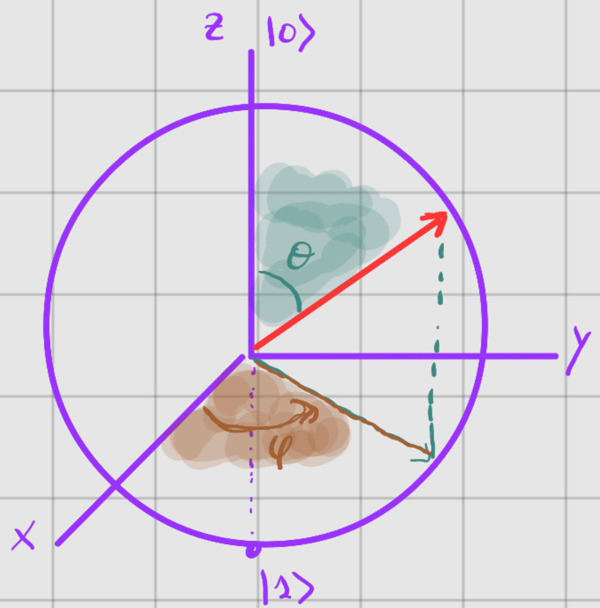
$\langle 0|\sigma_x|0\rangle = \langle 0|\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)|0\rangle = 0 \longrightarrow \text{matrix}(\sigma_x)_{00}$

$\langle 0|\sigma_y|1\rangle = \langle 0|\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)|1\rangle = 1 \longrightarrow \text{matrix}(\sigma_x)_{01}$

$\langle 1|\sigma_x|0\rangle = \dots 1 \longrightarrow \text{matrix}(\sigma_x)_{10}$

$\langle 1|\sigma_x|1\rangle = \dots 0 \longrightarrow \text{matrix}(\sigma_x)_{11}$

Bloch Sphere



The most general form of q-bit is:

$$\boxed{|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle}$$

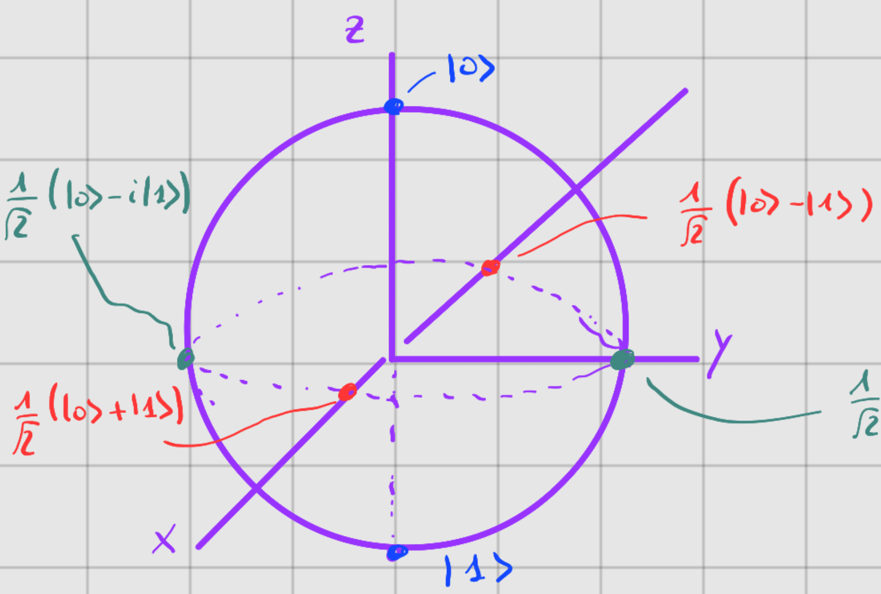
If we rembember: $\int dx\,|\psi(x)|^2 \longrightarrow \langle\psi|\psi\rangle = \big(a_0^*\langle 0| + a_1^*\langle 1|\big)\big(a_0|0\rangle + a_1|1\rangle\big) =$

We have to normalise $\Longrightarrow$ $= |a_0|^2 + |a_1|^2 = 1$

Using bloch representation: $\langle\psi|\psi\rangle = \big(\cos\langle 0| + e^{-i\varphi}\sin\langle 1|\big)\big(\cos|0\rangle + e^{i\varphi}|1\rangle\big) = 1$

It is naturally normalised!

The different states of $\sigma_x, \sigma_y, \sigma_z$ are located in the sphere like this

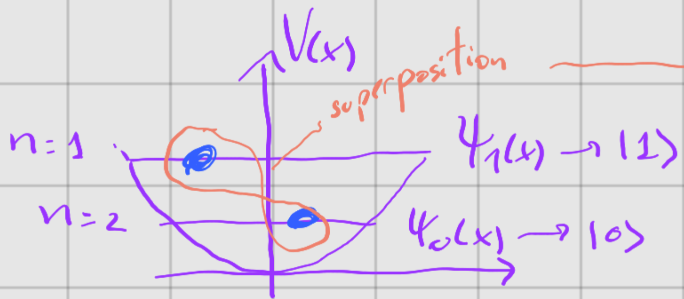


$\left.\begin{array}{l}\sigma_x \\ \sigma_y\end{array}\right]$ Lay on the equatorean plane

$\sigma_z \rightarrow$ Lay on poles

The points in the opposite position with respect to the centre are always orthogonal.

Let's take for instance the harmonic oscillator



$$|\psi\rangle = \cos\frac{\theta}{2}\,\psi_0(x) + e^{i\varphi}\sin\frac{\theta}{2}\,\psi_1(x)$$

$$\psi_0(x) \rightarrow A_0 e^{-\frac{m\omega_0}{2\hbar}x^2} \cdot \underbrace{H_0(x)}_{=1} \qquad \psi_1(x) \rightarrow A_1 e^{-\frac{m\omega_0}{2\hbar}x^2} H_1(x), \quad H_1(x) = 2x\sqrt{\frac{m\omega_0}{2\hbar}}$$

Here we have that the e^- is in a superposition of two states so we don't initialise the e^- in $\psi_0 \rightarrow$ (bit 0) or $\psi_1 \rightarrow$ (bit 1), that is classical.

30/12/2022

Entanglement

Particles at some point will decay, what means they will break. If it breaks in two and it has neutral charge, the two obtained particles can have neutral charges or opposite charges. The spin is also maintain: π-on has spin = 0 when it decay the particles will have spin up and the other spin down

$$\text{General particle with spin} = 0 \xrightarrow{\text{decay}} \underbrace{|\uparrow\rangle_R |\downarrow\rangle_L + |\downarrow\rangle_R |\uparrow\rangle_L}_{2 \text{ possibilities}}$$

Making it more formal

$$|\psi\rangle = \underbrace{|\lambda\rangle_a |\phi\rangle_b}_{\text{two particle system}}$$

$\lambda \perp \phi$ states

$|\lambda\rangle_a$ = state system "a"

$|\phi\rangle_b$ = state system "a"

For instance

$$|\psi\rangle = \frac{1}{2}\left(|0\rangle_a|0\rangle_b + |0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b + |1\rangle_a|1\rangle_b\right) =$$

$$= \frac{1}{2}\Big(|0\rangle_b\left(|0\rangle_a + |1\rangle_a\right) + |1\rangle_b\left(|0\rangle_a + |1\rangle_a\right)\Big) = \frac{1}{\sqrt{2}}\left(|0\rangle_b + |1\rangle_b\right)\frac{1}{\sqrt{2}}\left(|0\rangle_a + |1\rangle_a\right)$$

$\Rightarrow$ Not an entangled state

4 Bell states:

2 states

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_a|0\rangle_b \pm |1\rangle_a|1\rangle_b\right) \neq |\lambda\rangle_a \otimes |\phi\rangle_b \Rightarrow \text{Definition of entangled state}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_a|1\rangle_b \pm |1\rangle_a|0\rangle_b\right) \neq |\lambda\rangle_a|\phi\rangle_b$$

2 states

The superposition state is said to be entangled when you can't split the superposition into the product of two independent states.

The general form of an entangled state:

$$\boxed{|\psi\rangle = \alpha\,\underbrace{|\phi_1\rangle_a|\phi_2\rangle_b}_{2\text{ particles}} + \beta\,\underbrace{|\phi_2\rangle_a|\phi_1\rangle_b}_{2\text{ particles}}}$$

$\{|\phi_1\rangle_a ; |\phi_2\rangle_a\} = 2 \perp$ states $\rightarrow$ part. "a"

$\{|\phi_1\rangle_b ; |\phi_2\rangle_b\} = 2 \perp$ states $\rightarrow$ part. "b"

SPDC [II]

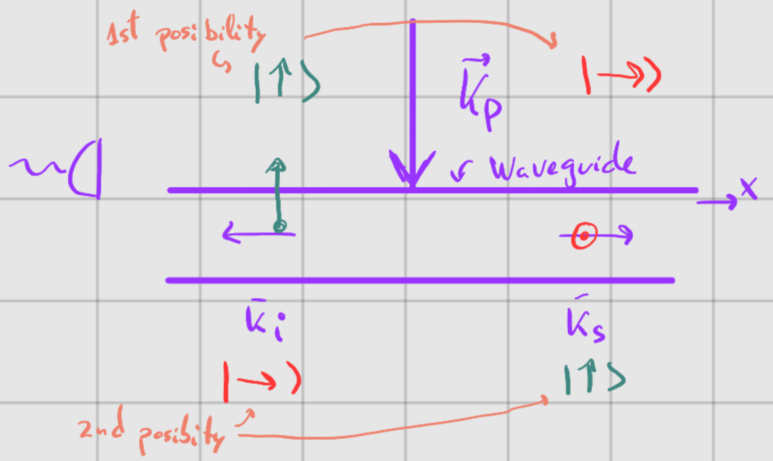
$\omega_p \longrightarrow \omega_i, \omega_s$

$\omega_p = \omega_i + \omega_s$

$\bar{k}_p = \bar{k}_s + \bar{k}_i$

If I pump with $\vec{k}_p$ the momentum will conserve. Here the initial momentum is zero.

Also this can of material conserve the polarization so if one of them have vertical polarization the other has horizontal



$|\uparrow\rangle$ = photon Vertical Polariz. (Linear) $\bar{E}\uparrow$

$|\rightarrow\rangle$ = photon Horizontal Polariz. (Linear) $\bar{E}\rightarrow$

$a \rightarrow$ signal photon
$b \rightarrow$ idler photon

$|0\rangle \rightarrow$ Vertical $|\uparrow\rangle$
$|1\rangle \rightarrow$ Horizontal $|\rightarrow\rangle$

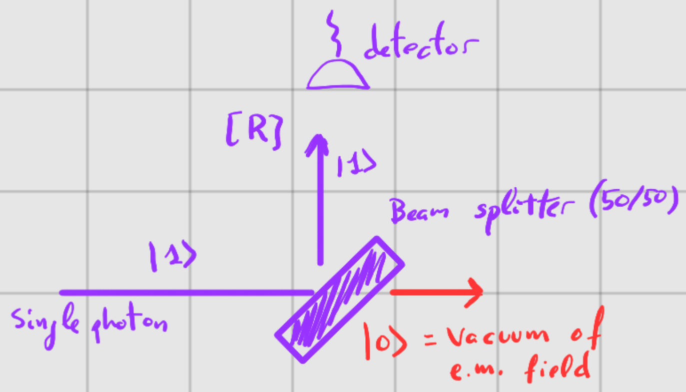
State of whole system: $\boxed{|\psi\rangle = \alpha\,|\uparrow\rangle_i|\rightarrow\rangle_s + \beta\,|\rightarrow\rangle_i|\uparrow\rangle_s} \neq |\lambda\rangle_a \otimes |\phi\rangle_b$

This is an entangled state of two photons: idler and signal with two polarization states Vertical and Horizontal

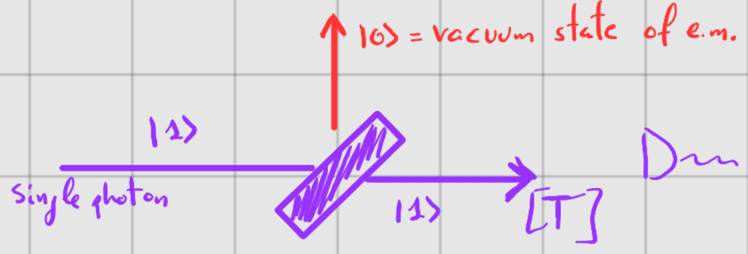
Known as: Two photon polarization entangled state

We have a beam splitter, 50% reflected and 50% transmitted. If we have a single photon it cannot break up so it will be either transmitted or reflected

Possibility 1



Possibility 2



This is: one photon entangled state

$$\boxed{|\psi\rangle = \underbrace{\alpha\,|1\rangle_R|0\rangle_T}_{①} \oplus \underbrace{\beta\,|0\rangle_R|1\rangle_T}_{②}}$$

No-cloning Theory

$|\psi\rangle \rightarrow$ particle quantum state to be cloned

$|x\rangle \rightarrow$ Register $\rightarrow$ clone quantum state

$$|\psi\rangle\,|x\rangle \xrightarrow{\text{"cloning"}} |\psi\rangle|\psi\rangle$$

If this cloning process is true for two q-states it should work for the superposition $\{|\psi_1\rangle, |\psi_2\rangle\} \longrightarrow [\alpha|\psi_1\rangle + \beta|\psi_2\rangle]$

$$① \rightarrow (\alpha|\psi_1\rangle + \beta|\psi_2\rangle)\,|x\rangle \overset{\text{linearity}}{=} \alpha|\psi_1\rangle|x\rangle + \beta|\psi_2\rangle\,|x\rangle \overset{\text{cloning}}{=} \alpha\,|\psi_1\rangle\,|\psi_1\rangle + \beta\,|\psi_2\rangle\,|\psi_2\rangle$$

We can also write:

$$② \rightarrow (\alpha\,|\psi_1\rangle + \beta\,|\psi_2\rangle)|x\rangle \overset{\text{cloning}}{=} (\alpha|\psi_1\rangle + \beta|\psi_2\rangle)\,(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) =$$

$$= \alpha^2\,|\psi_1\rangle|\psi_1\rangle + \beta^2\,|\psi_2\rangle|\psi_2\rangle + \alpha\beta\,(|\psi_1\rangle|\psi_2\rangle + |\psi_2\rangle|\psi_1\rangle)$$

1 and 2 should be equivalent, the conditions for that to happen are:

$\alpha^2 = \alpha \qquad \beta^2 = \beta$

$\alpha\beta = 0$

But if we have $\alpha = 0$ or $\beta = 0$ we don't have superposition, we have a single state

Therefore superposition is not consistently cloned $\Rightarrow$ **We cannot clone a superposition state**

You CAN copy a single particle state

This is a very interesting because we can encode a message in a superposition quantum state and be sure that it cannot be cloned.

$\text{Alice} \xrightarrow[\text{Eve}]{} \text{Bob}$

If Alice sends a message to Bob encoded in a superposition then Eve cannot clone (copy) and it will be secret.