Discrete Fourier Transform (DFT)

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1. The Fourier Transform

- Basic observation (continuous time):
  A periodic signal can be decomposed into sinusoids at integer multiples of the fundamental frequency
- i.e. if $\tilde{x}(t) = \tilde{x}(t+T)$
  we can approach $\tilde{x}$ with

$$\tilde{x}(t) \approx \sum_{k=0}^{M} a_k \boxed{\cos\left(\frac{2\pi k}{T}t\right.} \left. + \phi_k \right)$$

Harmonics *of the fundamental*

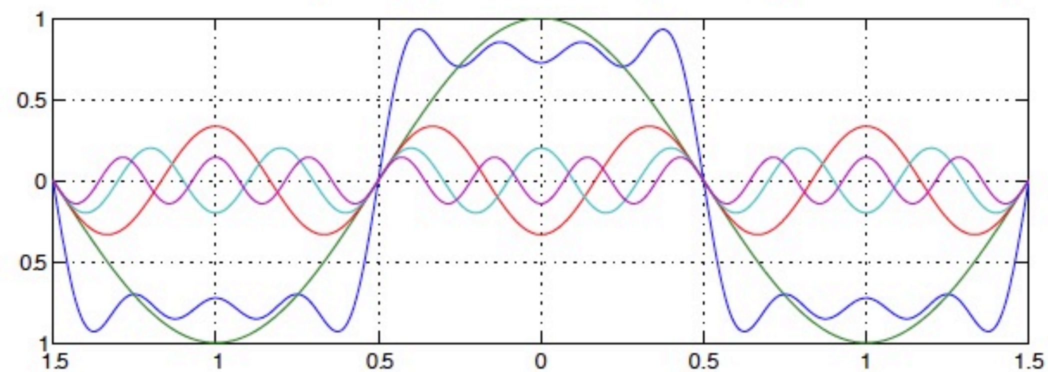
Fourier Series

$$\sum_{k=0}^{M} a_k \cos\left(\frac{2\pi k}{T}t + \phi_k\right)$$

- For a square wave,

$$\phi_k = 0; \quad a_k = \begin{cases} (-1)^{\frac{k-1}{2}} \frac{1}{k} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

i.e. $x(t) = \cos\left(\frac{2\pi}{T}t\right) - \frac{1}{3}\cos\left(\frac{2\pi}{T}3t\right) + \frac{1}{5}\cos\left(\frac{2\pi}{T}5t\right) - \ldots$



Fourier Analysis

- Thus, $c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt$

  because real & imag sinusoids in $e^{-j\frac{2\pi k}{T}t}$ pick out the corresponding sinusoidal components linearly combined

  in $x(t) = \sum_{k=-M}^{M} c_k e^{j\frac{2\pi k}{T}t}$

Fourier Transform

- Fourier series for periodic signals extends naturally to Fourier Transform for any (CT) signal (not just periodic):

$$X(j\Omega) = \int_{\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

Fourier Transform (FT)

$$x(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$$

Inverse Fourier Transform (IFT)

- Discrete index k → continuous freq. Ω

2. Discrete Time FT (DTFT)

- FT defined for discrete sequences:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{DTFT}$$

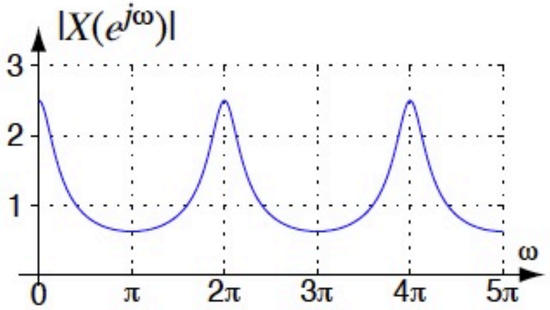
- Summation (not integral)
- Discrete (normalized) frequency variable ω
- Argument is $e^{j\omega}$, not $j\omega$

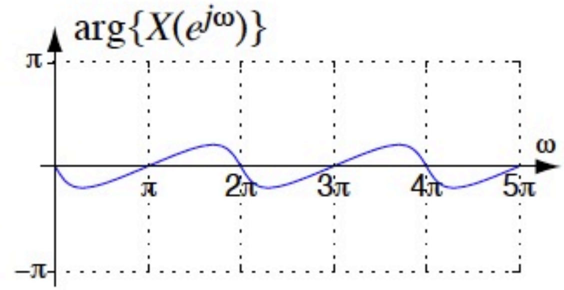
Periodicity of $X(e^{j\omega})$

- $X(e^{j\omega})$ has periodicity 2π in ω :

$$X(e^{j(\omega+2\pi)}) = \sum x[n]e^{-j(\omega+2\pi)n}$$

$$= \sum x[n]e^{-j\omega n}e^{-j2\pi n} = X(e^{j\omega})$$





Inverse DTFT (IDTFT)

- Same basic form as other IFTs:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \quad \textit{IDTFT}$$

- Note: continuous, periodic $X(e^{j\omega})$

  discrete, infinite $x[n]$ ...

- IDTFT is actually forward Fourier Series (except for sign of ω)

DTFT properties

- Linear:

$$\alpha g[n] + \beta h[n] \quad \leftrightarrow \quad \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$$

- Time shift:

$$g[n - n_0] \quad \leftrightarrow \quad e^{-j\omega n_0} G(e^{j\omega})$$

- Frequency shift:

$$e^{j\omega_0 n} g[n] \quad \leftrightarrow \quad G\left(e^{j(\omega-\omega_0)}\right)$$

'delay' in frequency

DTFT and convolution

- Convolution: $x[n] = g[n] \circledast h[n]$

$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (g[n] \circledast h[n]) e^{-j\omega n}$$
$$= \sum_n \left(\sum_k g[k] h[n-k] \right) e^{-j\omega n}$$
$$= \sum_k \left(g[k] e^{-j\omega k} \sum_n h[n-k] e^{-j\omega (n-k)} \right)$$
$$= G(e^{j\omega}) \cdot H(e^{j\omega})$$

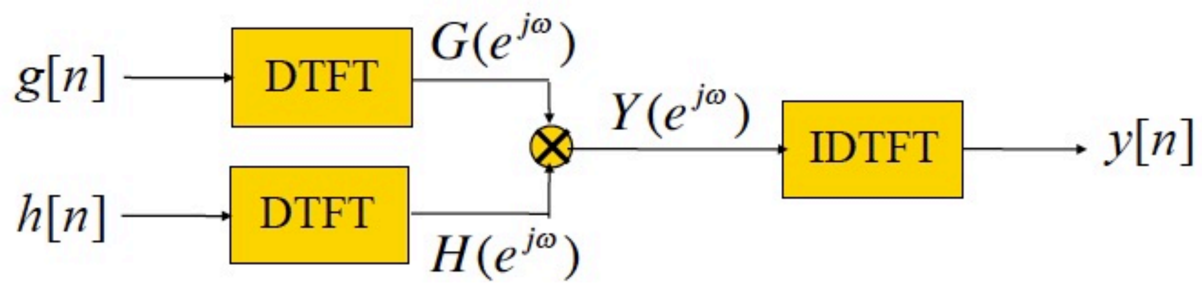
$$g[n] \circledast h[n] \quad \leftrightarrow \quad G(e^{j\omega}) H(e^{j\omega})$$

Convolution *becomes* **multiplication**

Convolution with DTFT

- Since $g[n] \circledast h[n] \quad \leftrightarrow \quad G(e^{j\omega})H(e^{j\omega})$

  we can calculate a convolution by:
  - finding DTFTs of $g, h \rightarrow G, H$
  - multiply them: $G \cdot H$
  - IDTFT of product is result, $g[n] \circledast h[n]$



3. Discrete FT (DFT)

Discrete FT (DFT)	Discrete finite/pdc $x[n]$	Discrete finite/pdc $X[k]$

- A *finite* or *periodic* sequence has only N unique values, $x[n]$ for $0 \leq n < N$
- Spectrum is completely defined by N distinct frequency samples
- Divide $0..2\pi$ into N equal steps,

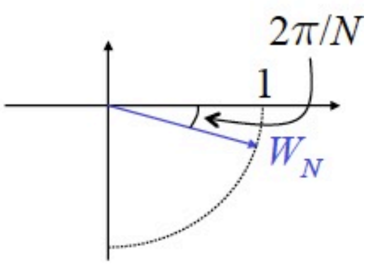
  $\{\omega_k\} = 2\pi k/N$

DFT and IDFT

- Uniform sampling of DTFT spectrum:

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}$$

- **DFT:** $X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$



where $W_N = e^{-j\frac{2\pi}{N}}$ i.e. $1/N^{th}$ of a revolution

IDFT

- Inverse DFT **IDFT** $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-nk}$
- Check:

$$x[n] = \frac{1}{N}\sum_k \left(\sum_l x[l] W_N^{kl}\right) W_N^{-nk}$$

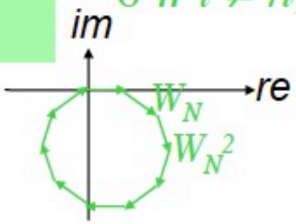
$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l] \boxed{\sum_{k=0}^{N-1} W_N^{k(l-n)}}$$

$$= x[n] \checkmark$$

$0 \leq n < N$

Sum of complete set of rotated vectors

$= 0$ *if* $l \neq n$; $= N$ *if* $l = n$



DFT: Matrix form

- $X[k]=\sum_{n=0}^{N-1}x[n]\cdot W_N^{kn}$ as a matrix multiply:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

- i.e. $\mathbf{X}=\mathbf{D}_N\cdot\mathbf{x}$

Matrix IDFT

- If $\mathbf{X} = \mathbf{D}_N \cdot \mathbf{x}$

  then $\mathbf{x} = \mathbf{D}_N^{-1} \cdot \mathbf{X}$
- i.e. inverse DFT is also just a matrix,

$$\mathbf{D}_N^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

$= {}^{1}/_{N} D_N^{*}$

DFT and MATLAB

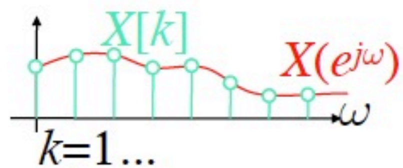
- MATLAB is concerned with *sequences* not continuous functions like $X(e^{j\omega})$
- Instead, we use the DFT to sample $X(e^{j\omega})$ on an (arbitrarily-fine) grid:
  - `X = freqz(x,1,w);` samples the DTFT of sequence `x` at angular frequencies in `w`
  - `X = fft(x);` calculates the N-point DFT of an N-point sequence `x`

DFT and DTFT

DTFT	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$	• *continuous freq* ω • *infinite* $x[n]$, $-\infty < n < \infty$
DFT	$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$	• *discrete freq* $k = N\omega/2\pi$ • *finite* $x[n]$, $0 \le n < N$

- DFT ‘samples’ DTFT at discrete freqs:

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$



- Discrete time Fourier transform (DTFT)
  - Taking the expression of the Fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$, the DTFT can be derived by numerical integration

$$X(e^{j\hat{\omega}}) = \sum_{-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

    - where $x[n] = x(nT_S)$ and $\hat{\omega} = 2\pi F/F_S$
- Discrete Fourier transform (DFT)
  - The DFT is obtained by "sampling" the DTFT at N discrete frequencies $\omega_k = 2\pi F_s/N$, which yields the transform

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

Properties: Circular time shift

- DFT properties mirror DTFT, with twists:
- Time shift must stay within N-pt 'window'

$$g[\langle n - n_0 \rangle_N] \quad \leftrightarrow \quad W_N^{kn_0} G[k]$$

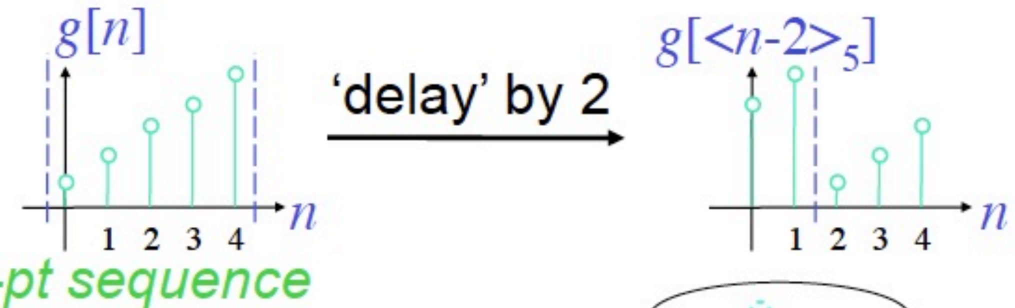
- Modulo-N indexing keeps index between 0 and N-1:

$$g[\langle n - n_0 \rangle_N] = \begin{cases} g[n - n_0] & n \geq n_0 \\ g[N + n - n_0] & n < n_0 \end{cases}$$

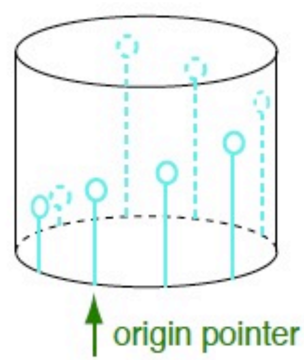
$0 \leq n_0 < N$

Circular time shift

- Points shifted out to the right don't disappear – they come in from the left



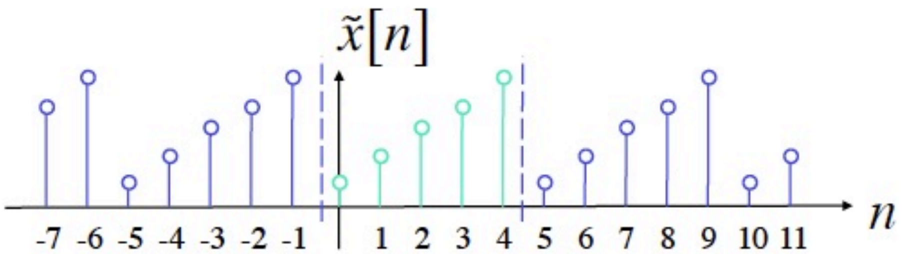
- Like a 'barrel shifter':



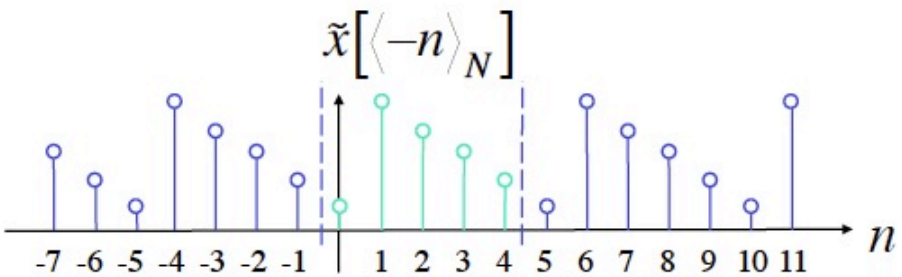
Circular time reversal

- Time reversal is tricky in 'modulo-N' indexing - **not** reversing the sequence:

5-pt sequence made periodic



Time-reversed periodic sequence



- Zero point stays fixed; remainder flips

Duality

- DFT and IDFT are very similar
  - both map an N-pt vector to an N-pt vector
- Duality:

$$\text{if} \quad g[n] \leftrightarrow G[k]$$

$$\text{then} \quad \boxed{G[n] \leftrightarrow N \cdot g[\langle -k \rangle_N]}$$

Circular time reversal

- i.e. if you treat DFT sequence as a time sequence, result is almost symmetric

4. Convolution with the DFT

- IDTFT of product of DTFTs of two N-pt sequences is their $2N$-1 pt convolution
- IDFT of the product of two N-pt DFTs can only give N points!
- Equivalent of $2N$-1 pt result time aliased:
  - i.e. $y_c[n] = \sum_{r=-\infty}^{\infty} y_l[n+rN]$ $(0 \leq n < N)$
  - must be, because $G[k]H[k]$ are exact samples of $G(e^{j\omega})H(e^{j\omega})$
- This is known as **circular convolution**

Circular convolution

- Can also do entire convolution with modulo-N indexing
- Hence, **Circular Convolution**:

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N] \leftrightarrow G[k]H[k]$$

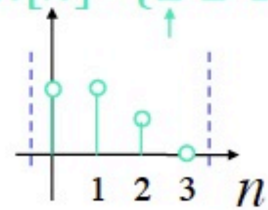
- Written as $g[n] \textcircled{\scriptsize N} h[n]$

Circular convolution example

- 4 pt sequences:

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$$

$g[n]=\{\underset{\uparrow}{1}\ 2\ 0\ 1\}$ $h[n]=\{\underset{\uparrow}{2}\ 2\ 1\ 0\}$

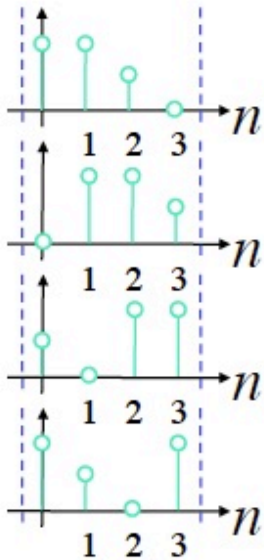


$h[\langle n-0\rangle_4]$ $\cdot 1$

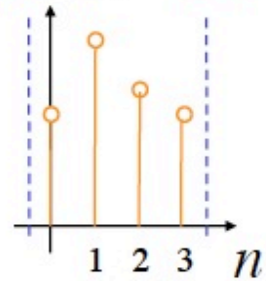
$h[\langle n-1\rangle_4]$ $\cdot 2$

$h[\langle n-2\rangle_4]$ $\cdot 0$

$h[\langle n-3\rangle_4]$ $\cdot 1$



$g[n] \circledast_4 h[n]=\{4\ 7\ 5\ 4\}$



check: $g[n] \circledast h[n]$
$=\{2\ 6\ 5\ 4\ 2\ 1\ 0\}$

DFT properties summary

- Circular convolution

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N] \leftrightarrow G[k]H[k]$$

- Modulation

$$g[n]\cdot h[n] \leftrightarrow \tfrac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m\rangle_N]$$

- Duality

$$G[n] \leftrightarrow N\cdot g[\langle -k\rangle_N]$$

- Parseval

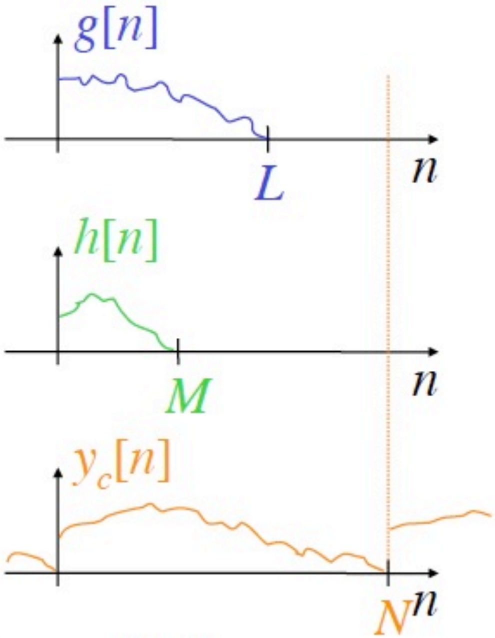
$$\sum_{n=0}^{N-1} \left|x[n]\right|^2 = \tfrac{1}{N}\sum_{k=0}^{N-1}\left|X[k]\right|^2$$

Linear convolution w/ the DFT

- DFT → fast circular convolution
- .. but we need linear convolution
- Circular conv. is time-aliased linear conv.; can aliasing be avoided?
- e.g. convolving L-pt $g[n]$ with M-pt $h[n]$: $y[n] = g[n] \circledast h[n]$ has $L+M-1$ nonzero pts
- Set DFT size $N \geq L+M-1$ → **no aliasing**

Linear convolution w/ the DFT

- Procedure ($N = L + M - 1$):
  - pad L-pt $g[n]$ with (at least) M-1 zeros
    $\rightarrow$ N-pt DFT $G[k], k = 0..N\text{-}1$
  - pad M-pt $h[n]$ with (at least) L-1 zeros
    $\rightarrow$ N-pt DFT $H[k], k = 0..N\text{-}1$
  - $Y[k] = G[k]\cdot H[k],\ k = 0..N\text{-}1$
  - IDFT{$Y[k]$} $= \sum_{r=-\infty}^{\infty} y_L[n+rN] = y_L[n]$ $\quad (0 < n < N)$



Overlap-Add convolution

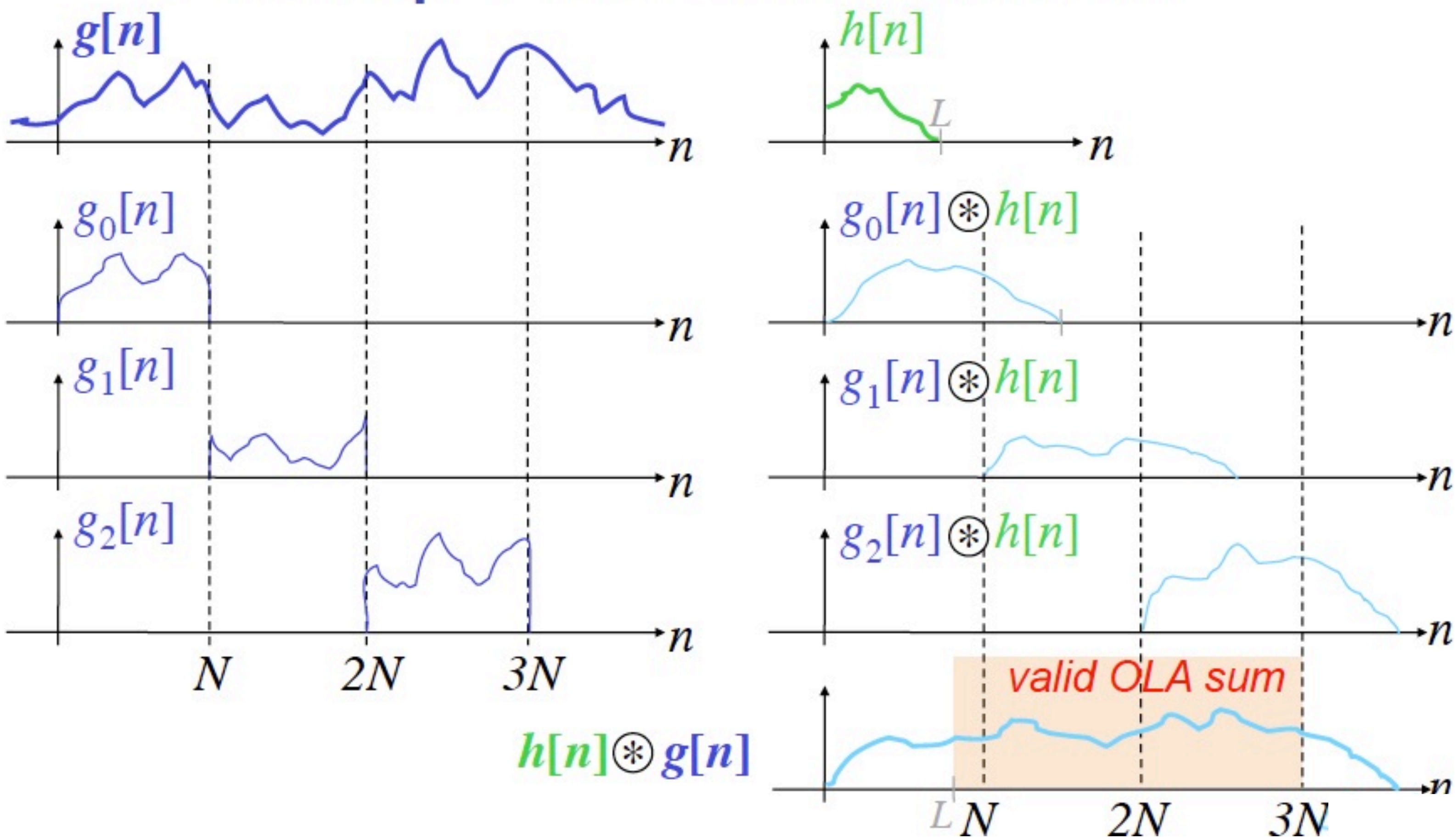
- Very long $g[n]$ → break up into segments, convolve piecewise, overlap
  → bound size of DFT, processing delay
- Make $g_i[n] = \begin{cases} g[n] & i \cdot N \le n < (i+1) \cdot N \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \qquad g[n] = \sum_i g_i[n]$$

$$\Rightarrow \quad h[n] \circledast g[n] = \sum_i h[n] \circledast g_i[n]$$

- Called Overlap-Add (**OLA**) convolution

Overlap-Add convolution



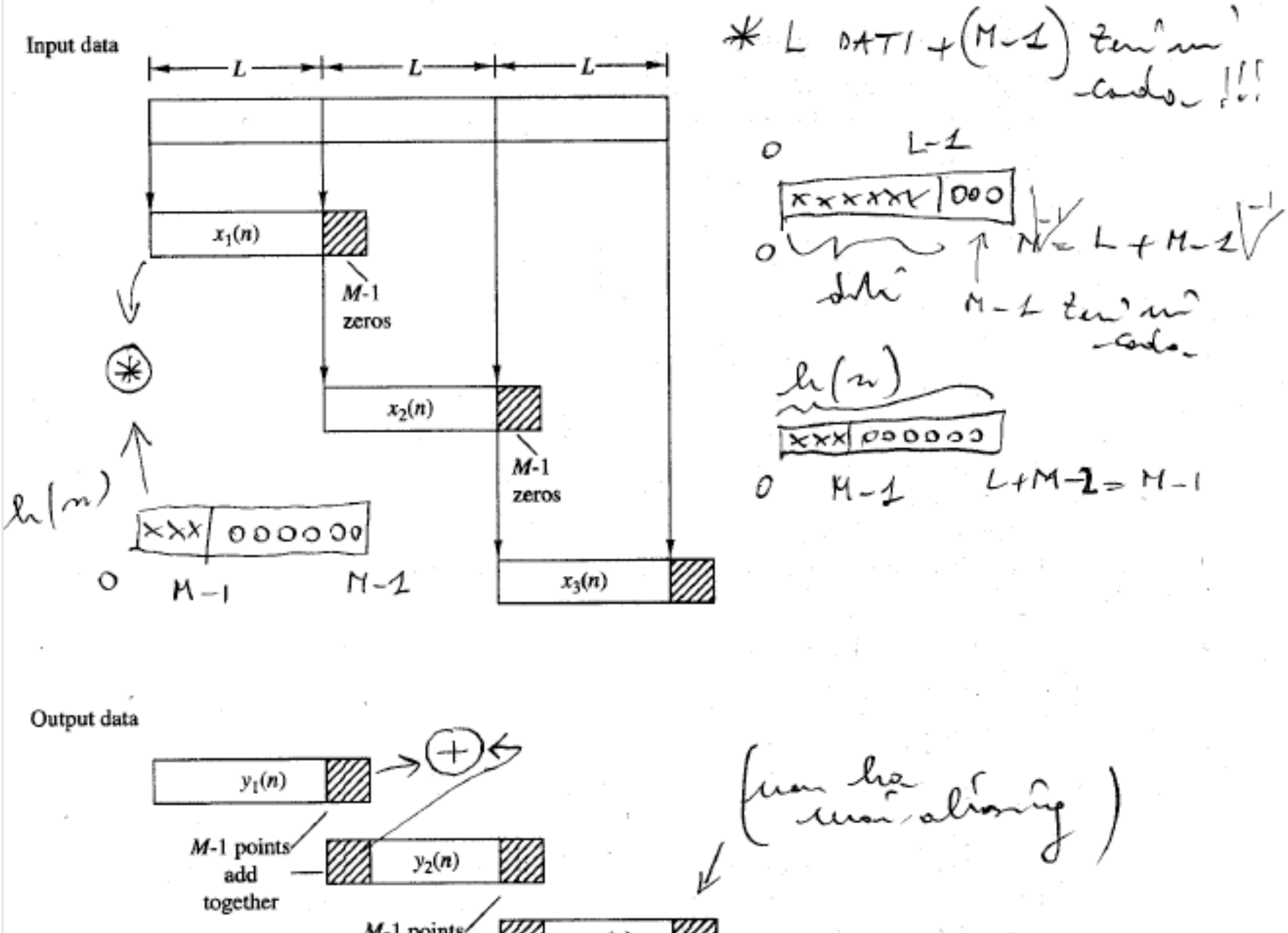


Figure 5.11 Linear FIR filtering by the overlap-add method.

Sequenza di lunghezza finita (N)	*DFT*
1. $x(n)$	$X(k)$
2. $y(n)$	$Y(k)$
3. $ax(n) + by(n)$	$aX(k) + bY(k)$
4. $x((n+m))_N \mathcal{R}_N(n)$	$W_N^{-km} X(k)$
5. $W_N^{ln} x(n)$	$X((k+l))_N \mathcal{R}_N(k)$
6. $\left[\sum_{m=0}^{N-1} x((m))_N y((n-m))_N\right] \mathcal{R}_N(n)$	$X(k)Y(k)$
7. $x(n)y(n)$	$\frac{1}{N}\left[\sum_{l=0}^{N-1} X((l))_N Y((k-l))_N\right] \mathcal{R}_N(k)$
8. $x^*(n)$	$X^*((-k))_N \mathcal{R}_N(k)$
9. $x^*((-n))_N \mathcal{R}_N(n)$	$X^*(k)$
10. $\mathrm{Re}\,[x(n)]$	$X_{ep}(k) = \frac{1}{2}[X((k))_N + X^*((-k))_N]\mathcal{R}_N(k)$
11. $j\,\mathrm{Im}\,[x(n)]$	$X_{op}(k) = \frac{1}{2}[X((k))_N - X^*((-k))_N]\mathcal{R}_N(k)$
12. $x_{ep}(n)$	$\mathrm{Re}\,[X(k)]$
13. $x_{op}(n)$	$j\,\mathrm{Im}\,[X(k)]$

Le proprietà seguenti valgono solo quando $x(n)$ é reale:

14. $x(n)$ reale qualsiasi	$\begin{cases} X(k) = X^*((-k))_N \mathcal{R}_N(k) \\ \mathrm{Re}\,[X(k)] = \mathrm{Re}\,[X((-k))_N]\mathcal{R}_N(k) \\ \mathrm{Im}\,[X(k)] = -\mathrm{Im}\,[X((-k))_N]\mathcal{R}_N(k) \\ \lvert X(k)\rvert = \lvert X((-k))_N\rvert \mathcal{R}_N(k) \\ \arg\,[X(k)] = -\arg\,[X((-k))_N]\mathcal{R}_N(k) \end{cases}$
15. $x_{ep}(n)$	$\mathrm{Re}\,[X(k)]$
16. $x_{op}(n)$	$j\,\mathrm{Im}\,[X(k)]$


"That's
all
folks!"