PLANE WAVES

LET'S CONSIDER A VOLUME V FILLED WITH A HOMOGENEOUS MEDIUM WITH PARAMETERS ε AND μ AND WITHOUT ANY SOURCE

WE CAN EASILY PROVE THAT PLANE WAVES ARE SOLUTIONS OF MAXWELL'S EQUATIONS IN THE VOLUME V (A VERY LARGE VOLUME OR THE FULL 3D SPACE)

PLANE WAVE
$$\begin{cases} E = E_0 \, e^{-jk \cdot z} \\ H = H_0 \, e^{-jk \cdot z} \end{cases}$$
WHERE E_0 AND H_0 ARE CONSTANT VECTORS

WE INSERT THE PLANE WAVE IN THE TWO MAXWELL'S CURL EQUATION

$$\nabla \times E = -j\omega\mu H \qquad \nabla \times \left(E_0 e^{-jk \cdot z} \right) = -j\omega\mu H_0 e^{-jk \cdot z}$$

$$\nabla \times H = j\omega\varepsilon E \qquad \nabla \times \left(H_0 e^{-jk \cdot z} \right) = j\omega\varepsilon \left(E_0 e^{-jk \cdot z} \right)$$

LET'S RECALL THE CURL OPERATOR IN RECTANGULAR COORDINATES

$$\nabla \times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \times \left(F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \right)$$

CROSS (OR VECTOR) PRODUCT

$$\nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$$

LET'S APPLY THE SAME RULE TO CALCULATE THE CURL OF $E_0 \, e^{-jk \cdot z}$

$$\nabla \times \left(E_0 e^{-jk \cdot z} \right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{0x} e^{-jk \cdot z} & E_{0y} e^{-jk \cdot z} & E_{0z} e^{-jk \cdot z} \end{vmatrix}$$

WHERE THE CONSTANT VECTOR E_0 IS GIVEN BY $E_0 = E_{0x} \hat{x} + E_{0y} \hat{y} + E_{0z} \hat{z}$

WE OBSERVE THAT

$$\frac{\partial}{\partial x} A e^{-jk\cdot z} = -jk_x A e^{-jk\cdot z}$$

WHERE A IS A CONSTANT AND THE PROPAGATION VECTOR IS

$$k = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

SIMILARLY WE CAN WRITE

$$\frac{\partial}{\partial y} A e^{-jk\cdot z} = -jk_y A e^{-jk\cdot z} \qquad \frac{\partial}{\partial z} A e^{-jk\cdot z} = -jk_z A e^{-jk\cdot z}$$

AND BY APPLYING THESE RESULTS IN OUR PROBLEM, WE OBTAIN

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{ox} e^{-jk\cdot z} & E_{oy} e^{-jk\cdot z} & E_{oz} e^{-jk\cdot z} \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -jk_x & -jk_y & -jk_z \\ E_{ox} e^{-jk\cdot z} & E_{oy} e^{-jk\cdot z} & E_{oz} e^{-jk\cdot z} \end{vmatrix} =$$

$$= \left(-jk_x \hat{x} - jk_y \hat{y} - jk_z \hat{z}\right) \underset{\text{VECTOR PRODUCT}}{\times} \left(E_{ox} e^{-jk\cdot z} \hat{x} + E_{oy} e^{-jk\cdot z} \hat{y} + E_{oz} e^{-jk\cdot z} \hat{z}\right)$$

$$= -jk \times E_o e^{-jk\cdot z}$$

THE FINAL RESULT READS AS

$$\nabla \times \left(E_o e^{-jk\cdot z}\right) = -jk \times E_o e^{-jk\cdot z}$$

A SIMILAR RESULT CAN BE WRITTEN FOR THE MAGNETIC FIELD, AS WELL

$$\nabla \times \left(H_o e^{-jk\cdot z}\right) = -jk \times H_o e^{-jk\cdot z}$$

$$-jk \times E_0 e^{-jk \cdot z} = -j\omega\mu H_0 e^{-jk \cdot z}$$
$$-jk \times H_0 e^{-jk \cdot z} = j\omega\varepsilon E_0 e^{-jk \cdot z}$$

BY CHANGING THE SIGN IN THE FIRST EQUATION AND AFTER DIVIDING BY $je^{-jk \cdot z}$ WE CAN WRITE

$$\begin{cases} k \times E_0 = \omega\mu H_0 \\ k \times H_0 = -\omega\varepsilon E_0 \end{cases}$$

AND IF WE MULTIPLY BY $e^{-jk \cdot z}$ WE OBTAIN

$$\begin{cases} k \times E = \omega\mu H \\ k \times H = -\omega\varepsilon E \end{cases}$$ WHERE $E = E_0 e^{-jk \cdot z}$ $H = H_0 e^{-jk \cdot z}$