**17.** *Radial Current Density.*

**Problem 4.6.** At instant $t = 0$ the electron behavior is described by the following wavefunction:

$$\Psi(r, 0) = Ae^{-r^2/\alpha^2 + ikr}. \tag{4.32}$$

Find the normalization constant, $A$, the most probable value $r_{pr}$, and the radial part of the probability current, $j$.

① Normalization

We are working here is spherical coordinates, so the normalization we have to use:

$$\int_V |\Psi|^2 dV = \int_V |\Psi|^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi = 1 \qquad r: 0 \to \infty, \; \theta: 0 \to \pi, \; \varphi: 0 \to 2\pi$$

$$\hookrightarrow |\Psi_{(r,0)}|^2 = \Psi^*_{(r,0)} \Psi_{(r,0)} = A^* e^{-r^2/\alpha^2 - ikr} \cdot A e^{-r^2/\alpha^2 + ikr} = |A|^2 e^{-2r^2/\alpha^2}$$

$$1 = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |\Psi_{(r,0)}|^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi = 4\pi \int_0^{\infty} |\Psi_{(r,0)}|^2 r^2 dr = 4\pi \int_0^{\infty} |A|^2 r^2 e^{-2r^2/\alpha^2} dr$$

We solve the integral in general:

$$① \quad I(a) = \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \implies \frac{dI(a)}{da} = -\frac{\sqrt{\pi}}{4} a^{-3/2}$$

$$② \quad \frac{dI(a)}{da} = \frac{d}{da}\int_0^{\infty} e^{-ax^2} dx = \int_0^{\infty} \frac{d}{da}\left(e^{-ax^2}\right) dx = \int_0^{\infty} -x^2 e^{-ax^2} dx$$

From ① and ② we get and taking into account that in our case: $a = \frac{2}{\alpha^2}$

$$\int_0^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4} a^{-3/2} = \frac{\sqrt{\pi}}{4}\left(\frac{2}{\alpha^2}\right)^{-3/2}$$

Therefore:

$$1 = \cancel{4}\pi |A|^2 \frac{\sqrt{\pi}}{\cancel{4}} \left(\frac{2}{\alpha^2}\right)^{-3/2} = |A|^2 \sqrt{\frac{\pi^3 \alpha^6}{8}} \implies \boxed{|A| = \sqrt[4]{\frac{8}{\pi^3 \alpha^6}}}$$

② $r_{pr}$

We have to calculate : $P(r)\,dV = |\psi(r)|^2 dV$

Where the volume is of a sphere: $V = \frac{4\pi}{3}r^3 \Rightarrow dV = 4\pi r^2 dr$

$$P(r)\,dV = |\psi(r,0)|^2\, 4\pi r^2 dr = \underbrace{4\pi|A|^2 r^2 e^{\overbrace{-2r^2/\alpha^2}^{=U}}}_{P(r)} dr$$

$\Rightarrow$ The most probable value will be the maximum of $P(r)$

$$\frac{dP(r)}{dr} = 0 \Rightarrow 4\pi|A|^2\left[2re^U + \frac{dU}{dr}r^2e^U\right] = 0 \Rightarrow 4\pi|A|^2 r e^U\left[2 - \frac{4r^2}{\alpha^2}\right] = 0$$

$$\Rightarrow 1 - \frac{2r^2}{\alpha^2} = 0 \Rightarrow r = \pm\frac{\alpha}{\sqrt{2}} \xrightarrow{\text{No negative distance}} \boxed{r = \frac{\alpha}{\sqrt{2}}}$$

③ Probability Current

The expression is: $j = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$

$$\nabla\psi(r) = \frac{\partial\psi}{\partial r} = \frac{\partial}{\partial r}A\cdot e^{\overbrace{-r^2/\alpha^2+ikr}^{=U}} = A\frac{dU}{dr}e^U = AU'e^U = A\left(-\frac{2}{\alpha^2}r + ik\right)e^U$$

$$\nabla\psi^* = A^*U'^*e^{U^*}$$

$$\psi = Ae^U \longrightarrow \psi^* = A^*e^{U^*} \longrightarrow |\psi|^2 = |A|^2 e^{\mathrm{Re}[U]}$$

$$\Rightarrow \psi^*\nabla\psi = A^*e^{U^*}\cdot AU'e^U = |A|^2U'e^{(U+U^*)} = |A|^2U'e^{\mathrm{Re}[U]} = U'|\psi|^2$$

$$\Rightarrow \psi\nabla\psi^* = Ae^U\cdot A^*U'^*e^{U^*} = |A|^2U'^*e^{\mathrm{Re}[U]} = U'^*|\psi|^2$$

$$j = \frac{\hbar}{2mi}\left(U'|\psi|^2 - U'^*|\psi|^2\right) = \frac{\hbar}{2mi}|\psi|^2\cdot\mathrm{Im}[U'] = \frac{\hbar}{2m\cancel{i}}|\psi|^2\cdot\cancel{i}k$$

$$\boxed{j = \frac{\hbar k}{2m}\sqrt{\frac{8}{\pi^3\alpha^6}}\,e^{-r^2/\alpha^2}}$$

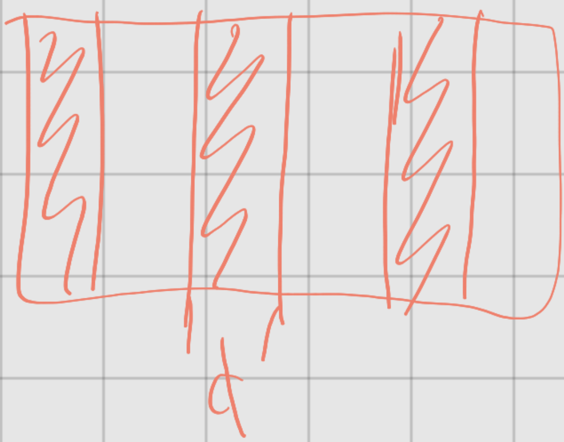
**18.**

The reflection and transmission coefficient of a dielectric slab of thickness "d" are:

$$r = \frac{(n^2-1)\,(e^{2ikdn}-1)}{(n+1)^2-(n-1)^2 e^{2ikdn}} \qquad t = \frac{4n\, e^{ikdn}}{(n+1)^2-(n-1)^2 e^{2ikdn}}$$

A non absorbing photonic crystal with n=2 (real) is made up of a periodic distribution of dielectric slabs whose thickness is $d = 0.1\, a$ alternated by vacuum regions. The structure periodicity is $a$.

a. Find the equation that describes the photonic band gap in terms of ωa/c.
b. Plot (Python, Mathematica, Matlab etc.) such an equation as a function of ωa/c.
c. Would an incident light beam with frequency such that ωa/c=3 e ωa/c=4 propagate or not?




**19.** Electrons may tunnel from a metal through the application of a suitable (constant) external electric field ε. After the application of the electric field ε the potential at the metal surface taken at x=a reads as (see class notes)

$$V(x) = E_F + \Phi - e\mathscr{E}(x - a)$$

Assuming that the tunneling electrons originates from a single-electronic state, estimate the field strength ε (volt/cm) needed to draw (*tunneling*) current densities of the order of mA/cm$^2$ from a potassium sample surface.