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Optical Communication Components

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Contents

1 Free and guided propagation of the optical waves **5**
1.1 Premise . . . 5
1.2 Propagation in weakly non-linear dielectrics . . . 7
1.3 Maxwell equations and wave equation . . . 8
1.3.1 Short review of Maxwell Equations . . . 8
1.3.2 Constitutive relations: linear case . . . 9
1.3.3 Constitutive relations: non-linear case . . . 12
1.3.4 Wave equation . . . 14
1.4 Ideal reference case . . . 16
1.5 Fourier transform and complex amplitude . . . 19
1.5.1 Fourier transform . . . 19
1.5.2 Complex envelope of a signal . . . 21
1.5.3 Complex amplitude . . . 23
1.6 Wave equation for the complex amplitudes . . . 25
1.6.1 Linear contribution of the induced polarization . . . 26
1.6.2 Nonlinear contribution of the induced polarization . . . 28
1.6.3 Total contributions to the polarization . . . 29
1.7 Non Linear Schrödinger Equation(NLSE) . . . 29
1.7.1 NLSE . . . 29
1.7.2 Discussion on the terms appearing in the NLSE . . . 31
1.7.3 What was neglected . . . 33
1.8 Considerations on the importance of the terms in the NLSE . . . 37
1.8.1 Linear polarization . . . 37
1.8.2 Nonlinear polarization . . . 39

2 Guided propagation of the optical waves **42**
2.1 Propagation in the optical fiber . . . 42
2.1.1 General features . . . 42
2.1.2 Main factors affcting the propagation . . . 45
2.1.3 Other factors affecting the propagation . . . 46
2.2 Guided modes . . . 47
2.2.1 Guided modes of the optical fiber: LP approximation . . . 49
2.2.2 Gaussian approximations . . . 56

2.2.3 The propagation equation 57
2.3 Parametri tipici di una fibra ottica monomodale per TLC 59
2.3.1 Fibre monomodali “standardizzate” I.T.U. 67
2.3.2 Fibre non standard 68

Chapter 1

Free and guided propagation of the optical waves

1.1 Premise

All communication systems are designed to allow the transfer of information from a source to one ore more users. The performance of a communication system can be evaluated by its transmission capacity - i.e. the quantity of information that can be transferred per unit time - and the maximum distance over which the information can be broadcast in between the source and the user.
In order to evaluate the system performance, it is necessary to know all the physical phenomena involved in the information transfer (a.k.a. the signal propagation) from the source to the user.

Name	Frequency Interval (Hz)	Wavelength interval
Gamma rays	$> 3 \times 10^{20}$	< 1 pm
X rays	3×10^{20} to 3×10^{17}	1 pm to 1 nm
Ultraviolet (UV)	3×10^{17} to 7.5×10^{14}	1 nm to 400 nm
Visible	7.5×10^{14} to 4×10^{14}	400 nm to 750 nm
Near Infrared (NIR)	4×10^{14} to 1.2×10^{14}	750 nm to 2.5 µm
Infrared (IR)	1.2×10^{14} to 1.2×10^{13}	2.5 µm to 25 µm
Microwaves	1.2×10^{13} to 3×10^{11}	25 µm to 1 mm
Radio waves	$< 3 \times 10^{11}$	> 1 mm

Table 1.1: Electromagnetic spectrum (in literature, the adopted definitions for the lower and upper boundaries of each range present slight differences).

The information transfer from one point to another typically occurs through a wisely modulated electromagnetic signal. One first classification concerns the frequency of the electromagnetic wave. In particular, we have radiofrequency systems, microwaves systems, millimetric waves systems and optical systems. In the latter case, the electromagnetic wave lies in the optical region of the electromagnetic spectrum, i.e. in the ultraviolet, in the visible or in the infrared (Table 1.1).

In this chapter we present the main properties of the modern communication systems. It is useful to introduce another classification for the communication systems: *guided* or *non-guided* propagation. In guided systems, the electromagnetic radiation is confined in a guiding structure (for instance, an optical fiber or a coaxial cable), that connects physically the source and the user. On the other hand, in non-guided systems the electromagnetic radiation emitted by the source freely propagates through space. Hence, it is evident that non-guided systems may be convenient for broadcasting applications, when the signal is transmitted from one source to many users, even if the performance and the costs of this solution have to be evaluated and may not be optimal[1].

The two problems show remarkable differences, both from the point of view of the analysis they need and their applications. As a consequence we would rather study them separately. We will tackle each problem with the momenta method and we will show applications of technological interest, such as the propagation in fiber or in space.

In this chapter we will first derive the equations ruling electromagnetic propagation in *slowly varying weakly non-linear dielectrics at optical frequencies.* We will try to keep the approach as general as possible and we will get eventually to the three-dimensional Nonlinear Schrödinger Equation (3DNLSE), describing the evolution of the electric field envelope. Then, we will apply the latter results for a few relevant cases (guided and non-guided propagation), where specific hypotheses may be introduced to simplify the approach. For instance, as we will see in this chapter, for the case of free space propagation we can neglect all the nonlinear and the dispersive terms (purely diffractive propagation). On the other hand, when we tackle the propagation in guiding structures, the spatio-temporal evolution of the guided mode is affected by the nonlinearity, whereas the diffractive and guiding effects (the latter due to the refractive index variation in the transverse plane with respect to the propagation direction) compensate each other. In the latter case, the two problems may be solved separately, thus considerably simplifying the study.

[1]We point out that also the guided systems may be used for broadcasting, realizable through clever technologies.

1.2 Propagation in weakly non-linear dielectrics

First we work out the expression of the refractive index as a function of the wavelength starting from the electromagnetic properties of any material. This dispersion is due to linear effects. Then we introduce the relation describing the refractive index changes as a function of the intensity - the so-called Kerr effect. This is the main nonlinear effect treated in this book. The refractive index variation with the light wavelength is well known since long times and it is a very important phenomenon for communication systems, yielding the dispersion of a wave packet. The spectral components of a wave packet travel at different speeds and reach the receiver at different times. Hence the wave packet is distorted, intersymbol interference may occur and the transmission capacity may be degraded.

The refractive index variations as function of the light intensity has been studied only recently. It yields relevant effects, such as self phase modulation or crossed phase modulation, only on modern communications systems based on optical fibers covering large distances.

The final target of the course is the derivation of the scalar equation of an optical pulse propagating in a lossless, weakly nonlinear, dielectric. To achieve this target:

- We recall *Maxwell equations*;
- We consider the nonzero response time upon application of an external electric field;
- We derive the *wave equation* starting from Maxwell equations;
- We consider the superposition of plane waves whose wave vectors and frequencies occupy a small band with respect to the central wave vector and frequency, respectively. We prove that this situation may be described with a plane wave modulated by a function slowly varying in time and space (the *complex* amplitude of the signal);
- We derive the general equation ruling the complex amplitude of the signal, the *three-dimensional Nonlinear Schrödinger Equation* (3DNLSE).

1.3 Maxwell equations and wave equation

1.3.1 Short review of Maxwell Equations

Figure 1.1: James Clerk Maxwell. Scottish mathematician and physicist born in Edinburgh on 13th June 1832 and dead in Cambridge on 5th November 1879.

We recall Maxwell equations written in the MSK system, without sources (without free charges and currents).

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r},t) &= -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \\
\nabla \times \mathbf{H}(\mathbf{r},t) &= \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} \\
\nabla \cdot \mathbf{D}(\mathbf{r},t) &= 0, \\
\nabla \cdot \mathbf{B}(\mathbf{r},t) &= 0,
\end{aligned} \tag{1.1}$$

where $\mathbf{E},(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ are the electric field, electric displacement field, magnetic flux density and magnetic field vectors. Furthermore, the constitutive relations hold:

$$\begin{aligned}
\mathbf{D}(\mathbf{r},t) &= \varepsilon_0 \mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t) \\
\mathbf{B}(\mathbf{r},t) &= \mu_0 \mathbf{H}(\mathbf{r},t) + \mathbf{M}(\mathbf{r},t)
\end{aligned} \tag{1.2}$$

where $\mathbf{P}(\mathbf{r},t)$, $\mathbf{M}(\mathbf{r},t)$, are the polatization and magnetization vectors, whereas $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ H/m are the permittivity and permeability of the free space. In this booklet we will work with non magnetic materials, hence from now on we set $\mathbf{M}(\mathbf{r},t) = \mathbf{0}$.

The polarization $\mathbf{P}$ is function of the applied electric field and of the material. In free space, the induced polarization is $\mathbf{0}$, whereas in real materials the electric field causes the distortion of the atomic structures generating local dipoles momenta yielding an induced polarization. For weak electric fields and far from resonances between electric field and material, the polarization is linear with the electric field.

The reply of the media cannot be simultaneous nor anticipate an external electric field. Instead the electric field must be established before and the reply occurs later. This statement is called *causality relation*. The induced polarization may in general be obtained as a convolution between the external stimulus (i.e. the electric field) and a function describing the material. This function is called electric susceptibility and has to fulfill the causality relation.

1.3.2 Constitutive relations: linear case

Under the hypothesis of linearity, the relation between the electric field and the induced polarization may be written in the compact form:

$$\mathbf{P} = \varepsilon_0 \, \boldsymbol{\chi} \overset{t}{*} \mathbf{E}, \tag{1.3}$$

where"$\overset{t}{*}$" indicates the temporal convolution and $\boldsymbol{\chi} = \boldsymbol{\chi}(\mathbf{r}, \tau)$ is the electric susceptibility. Here we consider only time-invariant media, i.e. media whose properties do not change with time. Since $\mathbf{P}$, $\boldsymbol{\chi}$ and $\mathbf{E}$ are functions of space $\mathbf{r}$ and time t, we may write:

$$\mathbf{P}(\mathbf{r}, t) = \varepsilon_0 \, \boldsymbol{\chi}(\mathbf{r}, t) \overset{t}{*} \mathbf{E}(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{+\infty} \boldsymbol{\chi}(\mathbf{r}, t - \tau) \mathbf{E}(\mathbf{r}, \tau) d\tau, \tag{1.4}$$

with $\boldsymbol{\chi}(\mathbf{r}, t - \tau) = \mathbf{0}$ for $t - \tau < 0$, due to the causality relation.

We may notice that the argument of $\mathbf{P}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ is the space-time $(\mathbf{r}, t)$, hence the domain of these functions is $\mathbb{R}^4$. Furthermore, $\mathbf{P}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ have 3 components and we may write them in cartesian coordinates as:

$$\begin{aligned} \mathbf{E} &= E_x \mathbf{i} + E_y \mathbf{j} + E_z \mathbf{k}, \\ \mathbf{P} &= P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k}, \end{aligned} \tag{1.5}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the versors.

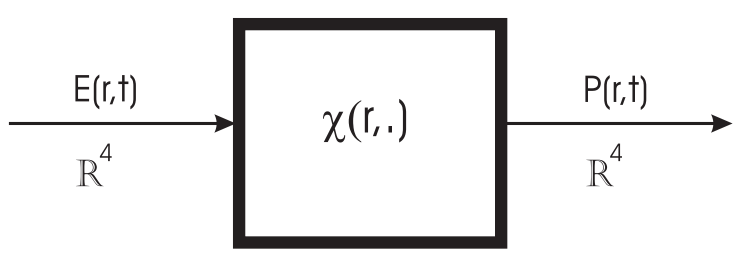

Figure 1.2: Relation between electric field and induced polarization interpreted as filter.

Equation (1.3) may be seen as a filter with three inputs and three outputs (as schematized in Fig. 1.2), whose impulsive response, represented by the electric susceptibility, is a 3×3 matrix[2].
Hence, Eq. (1.3) may be rewritten as:

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{bmatrix} \overset{t}{*} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{1.6}$$

which can be split into three scalar equations:

$$P_j(\mathbf{r},t) = \varepsilon_0 \int_{-\infty}^{+\infty} \chi_{jx}(\mathbf{r},t-\tau)E_x(\mathbf{r},\tau)d\tau + \varepsilon_0 \int_{-\infty}^{+\infty} \chi_{jy}(\mathbf{r},t-\tau)E_y(\mathbf{r},\tau)d\tau +$$

$$+ \varepsilon_0 \int_{-\infty}^{+\infty} \chi_{jz}(\mathbf{r},t-\tau)E_z(\mathbf{r},\tau)d\tau, \qquad j \in \{x,y,z\}. \tag{1.7}$$

For instance, an electric field linearly polarized along x may imply a polarization with component along x (if $\chi_{xx} \neq 0$), along y (if $\chi_{yx} \neq 0$), or along z (if $\chi_{zx} \neq 0$).

In general, any component of $\mathbf{P}$ depends on all the three components of $\mathbf{P}$. This is because, in general, the electric susceptibility is a non-diagonal 3×3 matrix. As a consequence, $\mathbf{P}$ has not to be necessarily parallel to $\mathbf{E}$. We point out that the convolution is performed only in the time domain (and not in space domain), hence we have a temporal filter.

Now we restrict ourselves to the case of isotropic media, in which the response does not depend on the orientation of the source.

Exercise 1. Prove that in isotropic media $\chi_{jk}(\mathbf{r},t) = \chi(\mathbf{r},t)\delta_{jk}$, where δ_{jk} is the Kronecker delta. In other words, the filter related to the susceptibility is diagonal with the same filter for each component. The 3×3 matrix of the susceptibility is:

$$\begin{bmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{bmatrix} = \chi(\mathbf{r},t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof. We apply an electric field inducing the polarization $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$. We suppose to apply an arbitrary rotation (described by the matrix R) to the electric field, to obtain $\mathbf{E}' = R\mathbf{E}$. The hypothesis of isotropy imply that the polarization $\mathbf{P}'$ induced by the electric field $\mathbf{E}'$ satisfies $\mathbf{P}' = R\mathbf{P}$. Hence, from the latter relation we must have:

$$\varepsilon_0 \chi R\mathbf{E} = \mathbf{P}' = R\mathbf{P} = \varepsilon_0 R \chi \mathbf{E}.$$

[2]The electric susceptibility $\chi(r,t)$ is a second rank tensor, yielding 3^2 components. In general, a tensor of rank n is completely specified by 3^n components. The scalars are tensors of rank 0 and the vectors are tensors of rank 1.

For the arbitrariness of the electric field, we obtain $\chi R = R\chi$ for any arbitrary rotation R. In particular, this should hold also for the rotation around the z axis of an arbitrary angle θ:

$$\begin{bmatrix} \chi_{11}\cos\theta + \chi_{12}\sin\theta & \chi_{12}\cos\theta - \chi_{11}\sin\theta & \chi_{13} \\ \chi_{21}\cos\theta + \chi_{22}\sin\theta & \chi_{22}\cos\theta - \chi_{21}\sin\theta & \chi_{23} \\ \chi_{31}\cos\theta + \chi_{32}\sin\theta & \chi_{32}\cos\theta - \chi_{31}\sin\theta & \chi_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \chi R_z(\theta) =$$

$$= R_z(\theta)\chi = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} =$$

$$\begin{bmatrix} \chi_{11}\cos\theta - \chi_{21}\sin\theta & \chi_{12}\cos\theta - \chi_{22}\sin\theta & \chi_{13}\cos\theta - \chi_{23}\sin\theta \\ \chi_{21}\cos\theta + \chi_{11}\sin\theta & \chi_{22}\cos\theta + \chi_{12}\sin\theta & \chi_{23}\cos\theta + \chi_{13}\sin\theta \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix}.$$

By equalizing component per component of the first and the last matrix of the latter relation, and invoking the arbitrariness of the angle θ we get:

$$\begin{cases} \chi_{12} = -\chi_{21} = \alpha \\ \chi_{11} = \chi_{22} = \chi \\ \chi_{13} = \chi_{23} = 0 \\ \chi_{31} = \chi_{32} = 0. \end{cases}$$

Summarizing, the electric susceptibility matrix reduces to:

$$\chi = \begin{bmatrix} \chi & \alpha & 0 \\ -\alpha & \chi & 0 \\ 0 & 0 & \chi_{33} \end{bmatrix}. \tag{1.8}$$

The relation $\chi R = R\chi$ should hold also for a rotation around x axis of an arbitrary angle ϕ. Hence:

$$\begin{bmatrix} \chi & \alpha\cos\phi & -\alpha\sin\phi \\ -\alpha & \chi\cos\phi & -\chi\sin\phi \\ 0 & \chi_{33}\sin\phi & \chi_{33}\cos\phi \end{bmatrix} = \begin{bmatrix} \chi & \alpha & 0 \\ -\alpha & \chi & 0 \\ 0 & 0 & \chi_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \chi R_x(\phi) =$$

$$= R_x(\phi)\chi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \chi & \alpha & 0 \\ -\alpha & \chi & 0 \\ 0 & 0 & \chi_{33} \end{bmatrix} = \begin{bmatrix} \chi & \alpha & 0 \\ -\alpha\cos\phi & \chi\cos\phi & -\chi_{33}\sin\phi \\ -\alpha\sin\phi & \chi\sin\phi & \chi_{33}\cos\phi \end{bmatrix}.$$

Again, by equalizing component per component of the first and the last matrix of the latter relation, and invoking the arbitrariness of the angle ϕ we get $\alpha = 0$ and $\chi_{33} = \chi$.

Hence, Eq. (1.8) reduces to:

$$\chi = \begin{bmatrix} \chi & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \chi \end{bmatrix}.$$

$\square$

As a consequence, in the case of isotropic media the linear response is described by one single function of time and space $\chi(\mathbf{r}, t)$, to obtain:

$$\mathbf{P}(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{+\infty} \chi(\mathbf{r}, t-\tau)\, \mathbf{E}(\mathbf{r}, \tau) d\tau, \tag{1.9}$$

where $\mathbf{P}$ and $\mathbf{E}$ are parallel. The electric displacement field, defined in the first equation of (1.2), becomes:

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0\, \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \varepsilon_0\, [\delta(t-\tau) + \chi(\mathbf{r}, t-\tau)]\, \mathbf{E}(\mathbf{r}, \tau) d\tau, \tag{1.10}$$

where $\delta(t)$ is the Dirac delta. As we can see, $\mathbf{D}$ is parallel to $\mathbf{E}$.

We take the Fourier transform with respect to time, indicated as $\widehat{\mathbf{E}}$, $\widehat{\chi}$, etc., (see further on), Eq. (1.10) becomes:

$$\begin{aligned} \widehat{\mathbf{D}}(\mathbf{r}, f) &= [\varepsilon_0 + \varepsilon_0\, \widehat{\chi}(\mathbf{r}, f)]\, \widehat{\mathbf{E}}(\mathbf{r}, f) = \\ &= \varepsilon_0\, \varepsilon_r(\mathbf{r}, f)\, \widehat{\mathbf{E}}(\mathbf{r}, f) = \varepsilon_0\, n^2(\mathbf{r}, f)\, \widehat{\mathbf{E}}(\mathbf{r}, f) \end{aligned} \tag{1.11}$$

where $\varepsilon_r(\mathbf{r}, f) = 1 + \widehat{\chi}(\mathbf{r}, f)$ is the relative dielectric constant and its square root

$$n(\mathbf{r}, f) = \sqrt{\varepsilon_r(\mathbf{r}, f)} = \sqrt{1 + \widehat{\chi}(\mathbf{r}, f)} \tag{1.12}$$

is called *linear refractive index*.

Remark 1. Often in the books the linear reply in the medium is written as $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$, without specifying if this relation holds for the time or the frequency domain. In this case, one has to interpret it in the frequency domain.

$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$ is valid also in the time domain only if the medium replies instantaneously to the applied electric field. Indeed, if $\boldsymbol{\chi}(\mathbf{r}, \tau) = \chi \delta(\tau)$, relation (1.9) reduces to $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$.

Exercise 2. Prove that for every physically realizable medium we must necessarily have a frequency dependent and complex $\widehat{\chi}$. Furthermore, prove that for every medium the relation $\widehat{\chi}^*(\mathbf{r}, f) = \widehat{\chi}(\mathbf{r}, -f)$ holds.

1.3.3 Constitutive relations: non-linear case

If the applied electric fields are very intense, the induced polarization may be no longer proportional to the electric field (at variance with section 1.3.2), but it can be a more complicated function of the electric field. By expanding the function $\mathbf{P}(\mathbf{E})$ in Taylor series with respect to $\mathbf{E}$, we obtain a linear term (simply proportional to $\mathbf{E}$) plus another nonlinear term:

$$\mathbf{P}(\mathbf{r},t) = \mathbf{P}^{L}(\mathbf{r},t) + \mathbf{P}^{NL}(\mathbf{r},t),$$

where $\mathbf{P}^{L}(\mathbf{r},t)$ is the linear term, calculated in the previous section. On the other hand, $\mathbf{P}^{NL}(\mathbf{r},t)$ is the nonlinear term. The latter disappears in the limit of weak electric field, thus retrieving the results of section 1.3.2.

Passing to frequency domain and using relation(1.11), we have:

$$\widehat{\mathbf{D}}(\mathbf{r},f) = \varepsilon_0\,\widehat{\varepsilon}_r(\mathbf{r},f)\,\widehat{\mathbf{E}} + \widehat{\mathbf{P}}^{NL}(\mathbf{r},f). \tag{1.13}$$

In order to account for the fact that the reply of the medium is not instantaneous, we may use the Volterra[3] series expansion, generalizing the filtering operation to the nonlinear case. The Volterra series expansion for the induced polarization reads[4]:

$$\begin{aligned} P_j(\mathbf{r},t) = \varepsilon_0 \sum_{k=x,y,z} \int_{-\infty}^{+\infty} \chi_{jk}(\mathbf{r},t-\tau)E_k(\mathbf{r},\tau)\,d\tau + \\ +\varepsilon_0 \sum_{k=x,y,z}\sum_{l=x,y,z} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \chi^{(2)}_{jkl}(\mathbf{r},t-\tau_1,t-\tau_2)E_k(\mathbf{r},\tau_1)E_l(\mathbf{r},\tau_2)\,d\tau_2\,d\tau_1 + \\ +\varepsilon_0 \sum_{k=x,y,z}\sum_{l=x,y,z}\sum_{m=x,y,z} \Big[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \chi^{(3)}_{jklm}(\mathbf{r},t-\tau_1,t-\tau_2,t-\tau_3)\times \\ \times E_k(\mathbf{r},\tau_1)E_l(\mathbf{r},\tau_2)E_m(\mathbf{r},\tau_3)\,d\tau_3\,d\tau_2\,d\tau_1\Big] + ... \end{aligned} \tag{1.14}$$

where the functions $\chi^{(2)}_{jkl}$ e $\chi^{(3)}_{jklm}$ correspond to the nonlinear response of the second and third order, respectively[5].

The nonlinear quadratic term of the nonlinear induced polarization is responsible of phenomena like the second-harmonic generation (SHG), sum frequency generation (SFG), difference frequency generation (DFG), optical parametric amplification (OPA) and optical parametric oscillation (OPO).

For centrosymmetric materials (materials whose structure does not change upon the transformation $\mathbf{r} \to -\mathbf{r}$, i.e. the inversion with respect to the origin), such as silica, the quadratic term is 0. Indeed, for symmetry reasons, the centrosymmetric materials cannot

[3]Vito Volterra, Italian mathematician born in Ancona (Italy) on 3rd May 1860 and dead in Rome on 11th October 1940.

[4]Here we truncate the series up to the third order, but one may also add higher order terms. Furthermore, for the sake of generality, we write the series for anisotropic media. However, in the case of isotropic materials, symmetry considerations reduce the non-zero components of the χ coefficients, as already shown in Section 1.3.2.

[5]In general, to know the nonlinear response of the second (third) order, 27 $\chi^{(2)}_{jkl}$ functions (81 $\chi^{(3)}_{jklm}$ functions) are required, corresponding to all the possible index permutations. Hence, the second (third) order nonlinear response is described by a tensor of rank 3 (of rank 4).

discern one sense with respect to the opposite one. In other words, if an external field $\mathbf{E}$ is applied and the polarization $\mathbf{P}$ is produced, upon application of the transformation $\mathbf{r} \to -\mathbf{r}$, $\mathbf{E}$ is mapped into $-\mathbf{E}$ and $\mathbf{P}$ into $-\mathbf{P}$. However, the quadratic nonlinear polarization is proportional to the square of the electric field, which does not change sign upon the transformation $\mathbf{r} \to -\mathbf{r}$. Hence, in order to avoid a contradiction it is necessary that $\chi^{(2)}_{jkl} = 0, \forall(j,k,l)$.

The nonlinear cubic term is responsible, for instance, of the third-harmonic generation (THG), of Kerr effect and of Raman and Brillouin scattering. In general, the nonlinear third order reply requests the knowledge of 81 coefficients. However, for symmetry reasons, only a few of these coefficients are non-zero and some of them must be equal. For instance, in an isotropic medium, 21 elements of $\chi^{(3)}$ matrix are nonzero and only 3 of them are independent (say $\chi^{(3)}_{xxyy}$, $\chi^{(3)}_{xyxy}$ and $\chi^{(3)}_{xyyx}$). The other coefficients are obtained as a combination of the 3 independent elements, as summarized in Table 1.2.

For the sake of simplicity, in this booklet we restrict ourselves to the case an instan-

$\chi^{(3)}_{yyzz} = \chi^{(3)}_{zzyy} = \chi^{(3)}_{zzxx} = \chi^{(3)}_{xxzz} = \chi^{(3)}_{yyxx} = \chi^{(3)}_{xxyy}$
$\chi^{(3)}_{yzyz} = \chi^{(3)}_{zyzy} = \chi^{(3)}_{zxzx} = \chi^{(3)}_{xzxz} = \chi^{(3)}_{yxyx} = \chi^{(3)}_{xyxy}$
$\chi^{(3)}_{yzzy} = \chi^{(3)}_{zyyz} = \chi^{(3)}_{zxxz} = \chi^{(3)}_{xzzx} = \chi^{(3)}_{yxxy} = \chi^{(3)}_{xyyx}$
$\chi^{(3)}_{xxxx} = \chi^{(3)}_{yyyy} = \chi^{(3)}_{zzzz} = \chi^{(3)}_{xxyy} + \chi^{(3)}_{xyxy} + \chi^{(3)}_{xyyx}$

Table 1.2: Nonzero components of $\chi^{(3)}$ for an isotropic medium. Three components are independent and all the others may be obtained as a combination of the independent ones.

taneous, space independent third-order nonlinear response:

$$\chi^{(3)}_{jklm}(\mathbf{r}, t-\tau_1, t-\tau_2, t-\tau_3) \approx \widehat{\chi}^{(3)}_{jklm}\ \delta(t-\tau_1)\delta(t-\tau_2)\delta(t-\tau_3).$$

With this assumption, $\widehat{\chi}^{(3)}_{jklm}$ is frequency independent and the nonlinear dispersion is neglected.

If the fields involved in the third-order nonlinear process are along the same direction (say all the electric fields linearly polarized along x), the nonlinear polarization reads:

$$P^{NL}(\mathbf{r},t) = \varepsilon_0\ \widehat{\chi}^{(3)}\ E(\mathbf{r},t)\, E(\mathbf{r},t)\, E(\mathbf{r},t) \tag{1.15}$$

in which the pedex have omitted, supposing that we refer to the components along x.

1.3.4 Wave equation

We take the curl of the first equation of (1.1), then we exploit the second equation of (1.1) and the second equation of (1.2) (written for non-magnetic materials) to obtain:

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} = -\mu_0 \frac{\partial}{\partial t}\nabla \times \mathbf{H} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}. \tag{1.16}$$

We use the vectorial identity $\nabla \times \nabla \times \mathbf{w} = -\nabla^2 \mathbf{w} + \nabla(\nabla \cdot \mathbf{w})$ (where $\mathbf{w}$ is an arbitrary vector) and the first equation of (1.2), to obtain:

$$-\nabla^2 \mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{1.17}$$

By introducing the speed of light in free space $c = (\varepsilon_0 \mu_0)^{-1/2} = 2.998 \times 10^8$ m/s, Eq. (1.17) can be recast as:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{E}) = \frac{1}{\varepsilon_0 c^2} \frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{1.18}$$

This approach is very general and can be used both for linear or nonlinear processes. Hence, in general, the induced polarization may be written as $\mathbf{P} = \mathbf{P}^L(\mathbf{E}) + \mathbf{P}^{NL}(\mathbf{E})$.

In optics, very often the term $\nabla(\nabla \cdot \mathbf{E})$ is negligible. The latter term is different from $\mathbf{0}$ when the linear susceptibility has a strong space dependence. However, in many applications the spatial variations of the linear susceptibility occurs on a space scale much longer than the working wavelength. For this reason, for the sake of simplicity, in this booklet the term $\nabla(\nabla \cdot \mathbf{E})$ will not be considered. With this assumption, we achieve the *wave equation*:

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(\mathbf{E} + \frac{1}{\varepsilon_0} \mathbf{P} \right). \tag{1.19}$$

Upon specification of the relation linking $\mathbf{P}$ and $\mathbf{E}$, the wave equation becomes an equation in 1 unknown, i.e. the electric field. Hence, after providing proper boundary conditions and initial conditions, it should in principle admit exactly one solution, even if it may be not easy to work it out analytically (numerical methods can prove useful in some cases). However, once the electric field has been worked out, all the problem is solved. In particular, the magnetic field can be derived by applying Maxwell equations to the electric field obtained before.

In the next sections we consider isotropic media only and hence we can limit the study to one component for the electric field and for the induced polarization. The equations reduce to scalar equations. First, we treat the ideal case in which the electromagnetic field follows a sinusoidal behaviour. Then, we introduce the concept of complex amplitude. Finally, we write the wave equation for the electric field complex amplitude and we derive Schrödinger equation.

Exercise 3. Prove that for linear and isotropic media (i.e. where $\mathbf{P} = \mathbf{P}^L(\mathbf{E})$) and without memory (i.e. satisfying $\mathbf{P} = \varepsilon_0 \chi_0 \mathbf{E}$) the following relation holds:

$$\nabla^2 \mathbf{E} - \frac{n^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \text{with} \qquad n = \sqrt{1 + \chi_0}$$

Exercise 4. Prove that in a linear and homogeneous medium the relation $\nabla \cdot \mathbf{E} = 0$ holds.

1.4 Ideal reference case

For the sake of simplicity, now we treat a simplified, ideal case. In particular the following assumptions are adopted:

1. linear material (i.e. the polarization is proportional to the electric field);
2. homogeneous material (i.e. the susceptibility does not depend on the spatial coordinates);
3. lossless material (i.e. the Fourier transform of the electric susceptibility is real, $\Im m(\widehat{\chi}(f)) = 0)$);
4. a perfectly sinusoidal electric field, with frequency f_0;
5. the electric field with only one nonzero component, say the component along x, with respect to a Cartesian reference frame (depicted in Fig. 1.3);

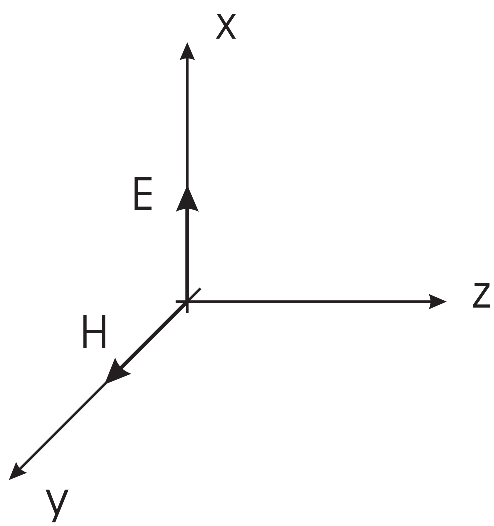

Figure 1.3: Electric and magnetic fields in the reference ideal case, when $\beta_0 < 0$.

6. the electromagnetic field depending only on the z spatial coordinate, i.e. $\mathbf{E}(\mathbf{r}, t) = E(z, t)\ \mathbf{i}$, where $\mathbf{i}$ is the versor along x and E is the scalar quantity of the electric field.

In summary:

$$E(z, t) = E_M \cos(\omega_0 t + \beta_0 z + \phi_0), \qquad \omega_0 = 2\pi f_0, \qquad \beta_0 = 2\,\pi q_0, \tag{1.20}$$

where, E_M, β_0, ω_0, are the electric field amplitude, the propagation constant, and the angular frequency, respectively.

The electric field is a periodic function of t and z. The spatial period in z, that is called *wavelength*, is $\lambda = q_0^{-1} = 2\pi/\beta_0$.

Exercise 5. Prove that under the hypotheses of the current section the wave equation

(1.19) reduces to:

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E + \frac{1}{\varepsilon_0}P\right), \tag{1.21}$$

where P is the x component of the induced polarization.
Tip. The gradient of a vector $\mathbf{v}$ is a matrix whose components are $(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}$. The divergence of a matrix S is a vector whose components are $(\nabla \cdot S)_i = \sum_{j=1}^{3} \frac{\partial S_{ij}}{\partial x_j}$.

Exercise 6. Prove that under the hypotheses of the current section the propagation constant satisfies the relation:

$$\beta_0^2 = \frac{\omega_0^2}{c^2}\left[1 + \widehat{\chi}(f_0)\right] = \frac{\omega_0^2}{c^2}\, n^2(f_0), \tag{1.22}$$

Proof. Combining Eq.s (1.21) and (1.3) we have

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E + \frac{1}{\varepsilon_0}P\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E + \ \chi \overset{t}{*} E\right) = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \frac{1}{c^2}\ \frac{\partial^2}{\partial t^2}(\chi \overset{t}{*} E). \tag{1.23}$$

Upon substitution of Eq. (1.20) into the left-hand-side of the latter expression we obtain:

$$\frac{\partial^2 E}{\partial z^2} = -\beta_0^2 E. \tag{1.24}$$

Analogously, the first term in the right-hand-side becomes:

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = -\frac{\omega_0^2}{c^2}E, \tag{1.25}$$

while the second:

$$\frac{1}{c^2}\ \frac{\partial^2}{\partial t^2}(\chi \overset{t}{*} E) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\int_{-\infty}^{+\infty}\chi(\tau)E(t-\tau)d\tau\right) = \frac{1}{c^2}\int_{-\infty}^{+\infty}\chi(\tau)\frac{\partial^2}{\partial t^2}\left[E(t-\tau)\right]d\tau =$$

$$= -\frac{\omega_0^2}{c^2}\int_{-\infty}^{+\infty}\chi(\tau)E(t-\tau)d\tau = -\frac{\omega_0^2}{c^2}(\chi \overset{t}{*} E) = -\frac{\omega_0^2}{c^2}\mathcal{F}^{-1}\left\{\mathcal{F}(\chi \overset{t}{*} E)\right\} = -\frac{\omega_0^2}{c^2}\mathcal{F}^{-1}\left\{\widehat{\chi}(f)\widehat{E}(f)\right\} =$$

$$= -\frac{\omega_0^2}{c^2}\frac{E_M}{2}\mathcal{F}^{-1}\left\{\widehat{\chi}(f)\exp\left[j(\beta_0 z + \phi)\right]\delta(f - f_0) + \chi(f)\exp\left[-j(\beta_0 z + \phi)\right]\delta(f + f_0)\right\} =$$

$$= -\frac{\omega_0^2}{c^2}\frac{E_M}{2}\mathcal{F}^{-1}\left\{\widehat{\chi}(f_0)\exp\left[j(\beta_0 z + \phi)\right]\delta(f - f_0) + \chi(-f_0)\exp\left[-j(\beta_0 z + \phi)\right]\delta(f + f_0)\right\} =$$

$$= -\frac{\omega_0^2}{c^2}\frac{E_M}{2}\mathcal{F}^{-1}\left\{\widehat{\chi}(f_0)\exp\left[j(\beta_0 z + \phi)\right]\delta(f - f_0) + \chi^*(-f_0)\exp\left[-j(\beta_0 z + \phi)\right]\delta(f + f_0)\right\} =$$

$$= -\frac{\omega_0^2}{c^2}\chi(f_0)\frac{E_M}{2}\mathcal{F}^{-1}\left\{\exp\left[j(\beta_0 z + \phi)\right]\delta(f - f_0) + \exp\left[-j(\beta_0 z + \phi)\right]\delta(f + f_0)\right\} = -\frac{\omega_0^2}{c^2}\chi(f_0)E.$$

To obtain the last passages we used Eq. (1.20), the fact that $\Im m(\widehat{\chi}(f)) = 0$, the Fourier transform ($\mathcal{F}$) and its inverse ($\mathcal{F}^{-1}$). We refer to Section 1.5.1 for a short survey of the

Fourier transform properties we have used.

Putting together the last expression with Eq.s (1.24) and (1.25), and invoking the arbitrariness of the electric field, we can get (1.22). □

Eq. (1.22) is the dispersion relation linking the propagation constant β_0 and the pulsation $\omega_0 = 2\pi f_0$. The dispersion relation depends on the properties of the media in which the light propagates.

While the frequency is the same in every medium, the wavelength depends on the particular medium. The wavelength λ of a wave propagating in a medium with a certain refractive index $n(f_0)$ may be related to the wavelength λ_0 of a wave with the same frequency, but propagating in the free space, through the relation $\lambda = \lambda_0/n(f_0)$. In table 1.3 we summarize the wavelength, wave number and propagation constant of a wave with a frequency f_0, obtained from the dispersion relation. We point out the fact that the propagation constant β_0 may be either positive or negative. If $\beta_0 > 0$ ($\beta_0 < 0$) the wave propagates towards negative (positive) z coordinate.

	free space	medium
Refractive index	$n = 1$	$n(f_0)$
Speed of light	c	$c/n(f_0)$
Wavelength	$\lambda_0 = c/f_0$	$\lambda = c/[f_0 n(f_0)] = \lambda_0/n(f_0)$
Propagation constant	$\beta_0 = \pm 2\pi/\lambda_0$	$\beta_0 = \pm 2\pi/\lambda = \pm 2\pi n(f_0)/\lambda_0$

Table 1.3: Summary of the properties of a wave with frequency f_0. The speed of light in free space is $c = (\varepsilon_0 \mu_0)^{-1/2} = 2.998 \times 10^8$ m/s.

The expression for the magnetic field can be derived by applying the first of Maxwell equations (1.1) to the expression of the electric field (1.20):

$$\mathbf{H}(\mathbf{r}, t) = -E_M \frac{\beta_0}{\omega_0 \mu_0} \cos(\omega_0 t + \beta_0 z + \phi_0)\, \mathbf{j}, \tag{1.26}$$

where $\mathbf{j}$ is the versor along y direction. As we can see, the magnetic field is cosinusoidal as well.

In this ideal reference case the obtained electromagnetic field satisfies Maxwell Equations and it is known as *uniform plane wave*. The electric and magnetic fields are mutually orthogonal and are also perpendicular to the z direction, i.e. the direction along which the wave propagates. In the real applications, such as telecommunications, it is unlikely that the signals are perfectly sinusoidal. Nevertheless, this simple ideal case is a sapid case-study to grasp the physics behind more complicated situations.

Remark 2. The hypothesis of lossless medium implies that the refractive index is purely real. The refractive index imaginary part is not negligible in media with losses, yielding dissipative effects related to the electromagnetic wave propagation.

Remark 3. The modulus of the Poynting vector $\mathbf{W} = \mathbf{E} \times \mathbf{H}$ allows to quantify the intensity of the electromagnetic fields, whose unit of measure is W/m^2.

For a uniform plane wave we have:

$$\mathbf{W} = -\frac{E_M^2\,\beta_0}{\omega_0\mu_0}\cos^2(\omega_0 t + \beta_0 z + \phi_0)\,\mathbf{k}, \tag{1.27}$$

where $\mathbf{k}$ is the versor along z direction. In a fixed point in space, the temporal average of the Poynting vector is :

$$< \mathbf{W} >= -\frac{E_M^2\,\beta_0}{2\omega_0\mu_0}\,\mathbf{k}. \tag{1.28}$$

This indicates that in our case, the energy carried by the electromagnetic wave flows towards the positive (negative) z coordinate if $\beta_0 < 0$ ($\beta_0 > 0$). The average electromagnetic power $< P >$ passing through a surface S is obtained as the flux of the average Poynting vector across that surface. In particular, for a uniform plane wave we have

$$< P >= \int_S -\frac{E_M^2\,\beta_0}{2\omega_0\mu_0}\,\mathbf{k}\cdot\mathbf{N}\;dS, \tag{1.29}$$

where $\mathbf{N}$ is a unit vector normal to the surface.

1.5 Fourier transform and complex amplitude

In order to represent a “quasi-sinusoidal” or a “narrow-band” signal in an efficient way it is necessary to introduce a complex representation to simplify the study, as done for the Modulation theory.

We start recalling the Fourier transform.

1.5.1 Fourier transform

For the sake of simplicity, we consider a scalar electric field $E(\mathbf{r}, t)$ (as in Section 1.4), and we express the spatial coordinate in Cartesian components: $\mathbf{r} = (x, y, z)$. The electric field is a real quantity. As we have already seen, $E(\mathbf{r}, t)$ is a function with a 4D domain. Hence, its Fourier transform is 4D:

$$\widehat{E}(\mathbf{f_r}, f) = \int_{\mathbb{R}^3}\int_{\mathbb{R}} E(\mathbf{r}, t)\; e^{-j2\pi(\mathbf{f_r}\mathbf{r}+ft)}\; dtd\mathbf{r} \tag{1.30}$$

where $\mathbf{f_r} = (f_x, f_y, f_z)$ are *spatial* frequencies, whereas f is a *temporal* frequency. The product in the exponent corresponds to the dot product: $\mathbf{f_r}\mathbf{r} = f_x x + f_y y + f_z z$.

The unit of measure of f is Hertz (Hz) or *cycles per second* (cps); the unit of measure of the spatial frequencies is *cycles per meter* (cpm).

In Eq. (1.30) we applied the Fourier transform to a scalar quantity. However, we can

Figure 1.4: Jean Baptiste Joseph Fourier. French mathematician and physicist born in Auxerre on 21st March 1768 and dead in Paris on 16th November 1830.

extend the Fourier transform also to vectorial quantities.

The transformation to the frequency domain proves very useful for handling the derivatives and the integrals, the latter becoming simple algebraic operations upon application of the Fourier transform. For instance:

$$\begin{aligned}
\frac{\partial E(\mathbf{r},t)}{\partial t} &\xrightarrow{\mathcal{F}} j2\pi f\,\widehat{E}(\mathbf{f_r},f),\\
\frac{\partial E(\mathbf{r},t)}{\partial x} &\xrightarrow{\mathcal{F}} j2\pi f_x\,\widehat{E}(\mathbf{f_r},f),\\
\frac{\partial E(\mathbf{r},t)}{\partial y} &\xrightarrow{\mathcal{F}} j2\pi f_y\,\widehat{E}(\mathbf{f_r},f),\\
\frac{\partial E(\mathbf{r},t)}{\partial z} &\xrightarrow{\mathcal{F}} j2\pi f_z\,\widehat{E}(\mathbf{f_r},f),
\end{aligned} \tag{1.31}$$

where the symbol "$\mathcal{F}$" denotes the Fourier transform.

Upon application of the Fourier transform, the wave equation (1.19) becomes:

$$\left(f_x^2+f_y^2+f_z^2\right)\widehat{\mathbf{E}}=\frac{f^2}{c^2}\left(\widehat{\mathbf{E}}+\frac{1}{\varepsilon_0}\widehat{\mathbf{P}}\right), \tag{1.32}$$

where $\widehat{\mathbf{E}}=\widehat{\mathbf{E}}\left(f_x,f_y,f_z,f\right)$ and $\widehat{\mathbf{P}}=\widehat{\mathbf{P}}\left(f_x,f_y,f_z,f\right)$ are the Fourier transform of the electric field and induced polarization, respectively.

Very often, instead of applying the Fourier transform $\widehat{E}(\mathbf{f_r},f)=\widehat{E}(f_x,f_y,f_z,f)$, performed with respect to all the coordinates, as done for Eq. (1.30), it is possible to perform the Fourier transform with respect to time only[6]

$$\widehat{E}(\mathbf{r}, f) = \int_{\mathbb{R}} E(\mathbf{r}, t)\, e^{-j2\pi f t}\, dt, \tag{1.33}$$

or only with respet to two coordinates, such as z and t:

$$\widehat{E}(x, y, f_z, f) = \int_{\mathbb{R}} \int_{\mathbb{R}} E(\mathbf{r}, t)\, e^{-j2\pi(f_z z + f t)}\, dt dz. \tag{1.34}$$

We pinpoint the fact that when we perform the partial Fourier transform, the derivation rule is applied only with respect to the coordinates over which the transformation occurs. For instance, if we apply the transformation (1.34) to the wave equation, we obtain:

$$\frac{\partial^2 \widehat{\mathbf{E}}}{\partial x^2} + \frac{\partial^2 \widehat{\mathbf{E}}}{\partial y^2} + (j2\pi f_z)^2 \widehat{\mathbf{E}} = \frac{(j2\pi f)^2}{c^2} \left(\widehat{\mathbf{E}} + \frac{1}{\varepsilon_0} \widehat{\mathbf{P}} \right) \tag{1.35}$$

instead of Eq. (1.32).

1.5.2 Complex envelope of a signal

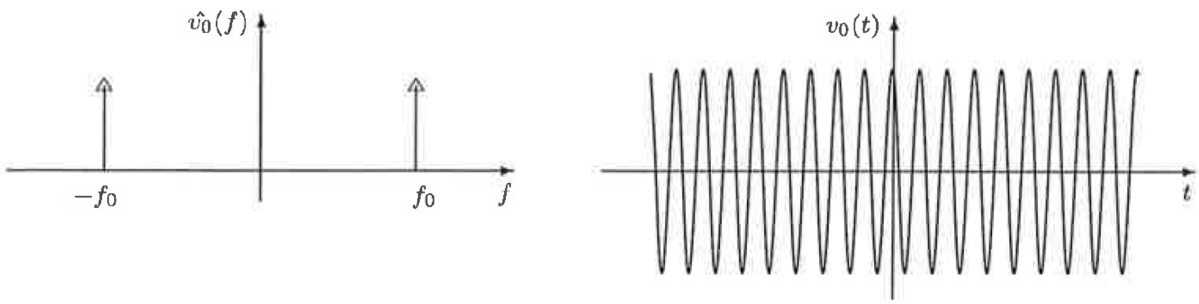

Figure 1.5: Perfectly sinusoidal signal.

We recall the modulation theory, where a "quasi sinusoidal" or "narrow band" signal is represented with the *complex envelope*. The reference signal is perfectly sinusoidal (Fig. 1.5):

$$v_0(t) = V_0 \cos(2\pi f_0 t + \phi_0)$$

and its Fourier transform contains two frequencies: $\widehat{v}_0(f) = \frac{1}{2} V_0\, e^{j\phi_0} \delta(f - f_0) + \frac{1}{2} V_0\, e^{-j\phi_0} \delta(f + f_0)$.

According to Euler formula, the sinusoidal signal can be written as:

$$v_0(t) = \frac{1}{2} V_0\, e^{j\phi_0}\, e^{j2\pi f_0 t} + \frac{1}{2} V_0\, e^{-j\phi_0}\, e^{-j2\pi f_0 t}, \tag{1.36}$$

[6]We use the same notation $\widehat{E}$, since it is clear what is the version of the Fourier transform we are using.

shownig two *spectral modes* around the frequencies $\pm f_0$. From this relation we have:

$$v_0(t) = \Re e\left[V_0\, e^{j\phi_0}\, e^{j2\pi f_0 t}\right], \tag{1.37}$$

where $V_0\, e^{j\phi_0}$ is the complex amplitude (or complex envelope). Once we fix the frequency f_0, the sinusoidal signal can be represented by its complex amplitude $V_0\, e^{j\phi_0}$.

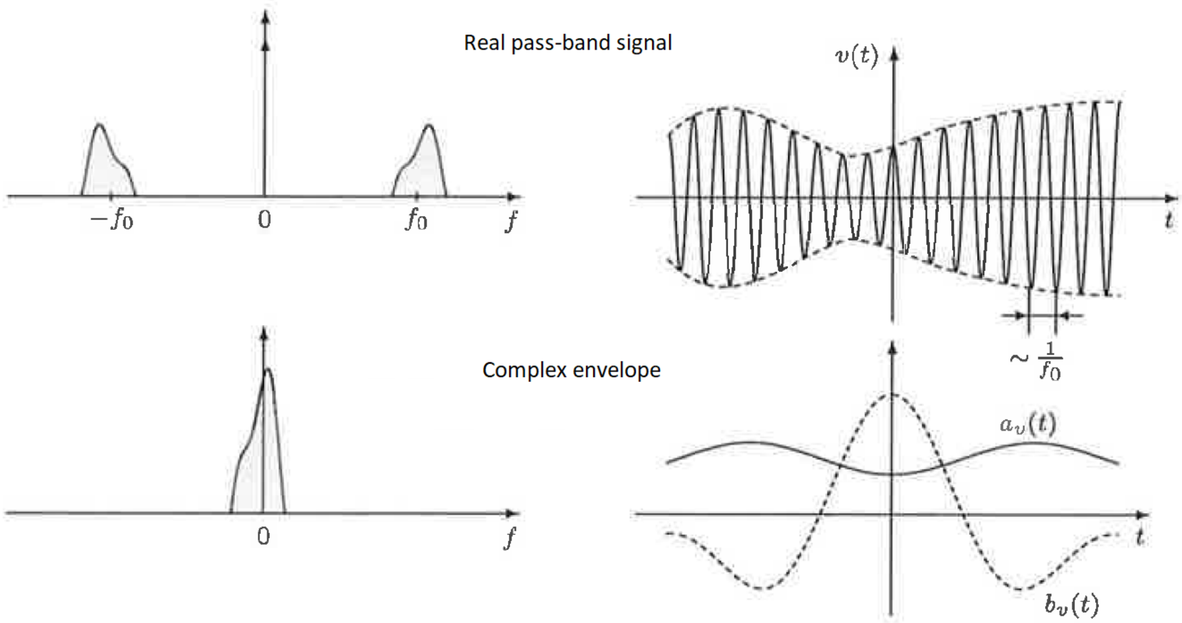

Figure 1.6: Real band-pass signal $v(t)$ and corresponding complex envelope $c_v(t) = a_v(t) + jb_v(t)$.

A "quasi-sinusoidal" real signal $v(t)$ (Fig. 1.6), that can be modulated in amplitude or frequency, shows two *spectral modes*, one at positive frequencies (around the frequency f_0) and one at negative frequencies (around the frequency $-f_0$). This signal can always be written as:

$$v(t) = \frac{1}{2}\, c_v(t)\, e^{j2\pi f_0 t} + \frac{1}{2}\, c_v^*(t)\, e^{-j2\pi f_0 t}, \tag{1.38}$$

which is the generalization of Eq. (1.36). The signal $c_v(t)$ is called *complex envelope* or *complex amplitude* of $v(t)$. Once we fix the reference frequency f_0, from $c_v(t)$ it is possible to derive $v(t)$ in the following way:

$$v(t) = \Re e\left[c_v(t)\, e^{j2\pi f_0 t}\right] = |c_v(t)| \cos\left[2\pi f_0 t + \phi(t)\right] \tag{1.39}$$

where $\phi(t) = \arg\left[c_v(t)\right]$ is the instantaneous phase.

By exploiting the rules of the Fourier transform on the translation and complex conjugation, we can Fourier transform Eq. (1.38), to obtain:

$$\widehat{v}(f) = \frac{1}{2}\,\widehat{c}_v(f - f_0) + \frac{1}{2}\,\widehat{c}_v^{*}(-f - f_0). \tag{1.40}$$

The latter relation proves that there are two spectral modes, as highlighted in Fig. 1.6. We point out that it is possible to obtain the complex envelope $c_v(t)$ from the "quasi sinusoidal" signal $v(t)$. To do that:

1. Remove from $v(t)$ the negative frequency mode by using a frequency filter $H(f) = 2 \cdot 1(f)$, where $1(f)$ is the Heaviside function[7].
2. Translate the positive frequency mode by the frequency $-f_0$, to have this mode around the origin of the frequencies.

Instead of having two spectral modes, the complex envelope contains only one mode. Furthermore, it is a slowly varying and base band signal. These are the advantages of using the complex envelope instead of the original signal.

1.5.3 Complex amplitude

In this section we conjugate the results presented in the previous section to the case of the electric field.

As in Section 1.4, we take a perfectly sinusoidal electric field in the coordinates z and t as the reference signal (Fig. 1.7).

$$E_0(z,t) = E_M \cos(2\pi q_0 z + 2\pi f_0 t + \phi_0), \tag{1.41}$$

where q_0 is a spatial frequency and f_0 is a temporal frequency.
The decomposition:

$$\begin{aligned} E_0(z,t) &= \frac{1}{2}\,E_M\,e^{j\phi_0}\,e^{j(\beta_0 z+\omega_0 t)} + \frac{1}{2}\,E_M\,e^{-j\phi_0}\,e^{-j(\beta_0 z+\omega_0 t)} = \\ &= \Re e\left[E_M\,e^{j\phi_0}\,e^{j(\beta_0 z+\omega_0 t)}\right]. \end{aligned} \tag{1.42}$$

shows that the 2D sinusoidal signal has *two spectral modes*, one at the spatio-temporal frequency (q_0, f_0) and one at the frequency $(-q_0, -f_0)$, as displayed in Fig. 1.7 on the plane (f_z, f). Also in the current case the signal can be simply obtained from the amplitude $A_0 = E_M\,e^{j\phi_0}$. Indeed, if we know the reference frequency (q_0, f_0), we can derive $E_0(z,t)$ from A_0 as:

$$E_0(z,t) = \Re e\left[A_0\,e^{j2\pi(q_0 z+f_0 t)}\right]. \tag{1.43}$$

Passing to a "quasi-sinusoidal" electric field $E(z,t)$, on the plane (f_z, f) we find other two spectral modes around the frequencies $(\pm q_0, \pm f_0)$ (Fig. 1.8), as we can see from the

[7]The Heaviside function is 0 for $f < 0$, 1/2 for $f = 0$ and 1 for $f > 0$.

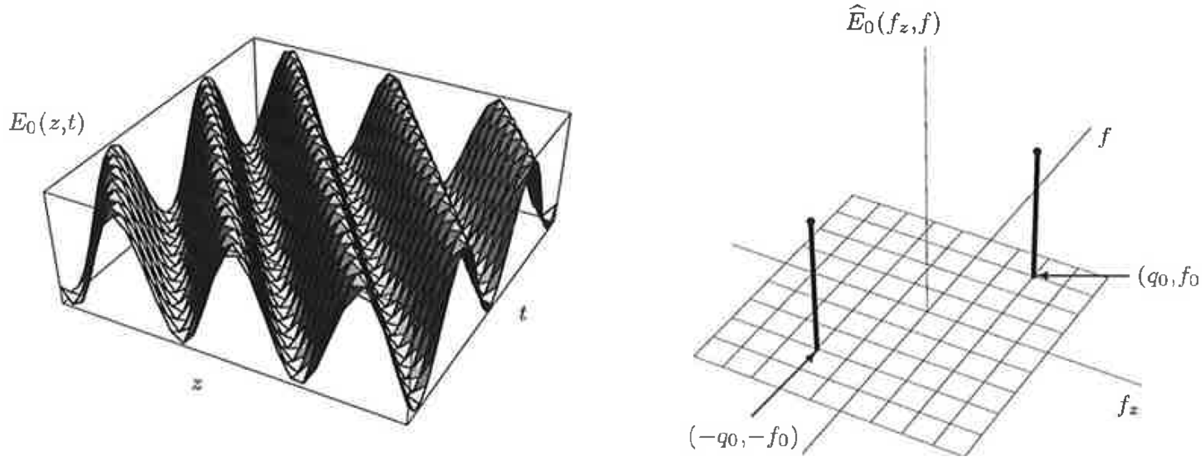

Figure 1.7: Perfectly sinusoidal electric field and its Fourier transform.

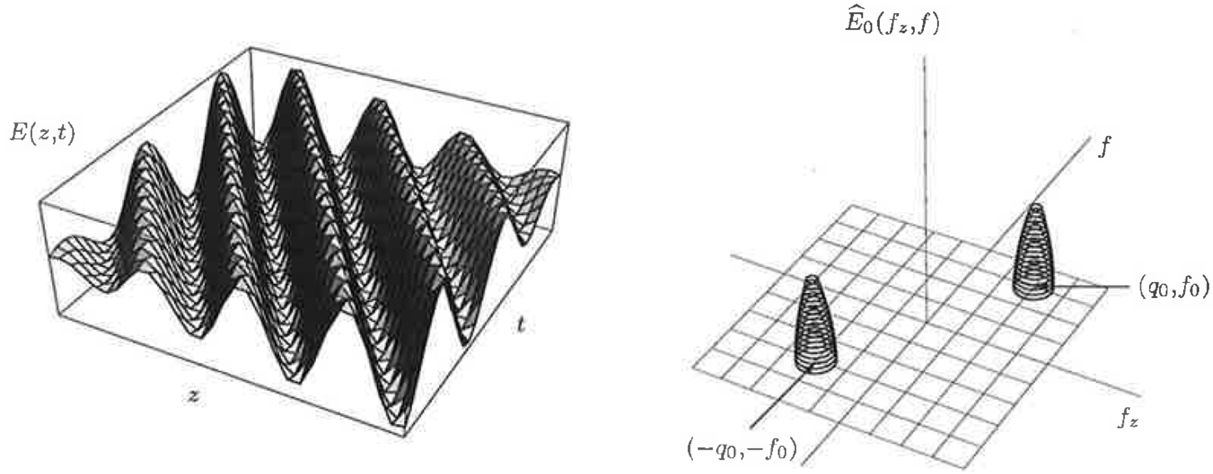

Figure 1.8: "Quasi-sinusoidal" electric field and its Fourier transform.

generalization of Eq. (1.42):

$$E(z,t) = \frac{1}{2}\,A(z,t)\,e^{j(\beta_0 z+\omega_0 t)} + \frac{1}{2}\,A^*(z,t)\,e^{-j(\beta_0 z+\omega_0 t)}. \tag{1.44}$$

Now, the *complex amplitude* $A(z,t)$ is not constant anymore as for the perfectly sinusoidal case, but rather *slowly varying* with respect to z e t.

Again, from $A(z,t)$ we can obtain the total electric field as:

$$E(z,t) = \Re e\left[A(z,t)e^{j2\pi(q_0 z+f_0 t)}\right] = |A(z,t)|\,\cos\left[2\pi(q_0 z + f_0 t) + \phi(z,t)\right], \tag{1.45}$$

where $|A(z,t)|$ and $\phi(z,t) = \arg(A(z,t))$ are the instantaneous amplitude and phase, respectively.

Exercise 7. Prove that the 2D Fourier transform of Eq. (1.44) reads:

$$\widehat{E}(f_z, f) = \frac{1}{2}\widehat{A}(f_z - q_0, f - f_0) + \frac{1}{2}\left[\widehat{A}(-f_z - q_0, -f - f_0)\right]^*.$$

We here summarize the algorithm to work out the full electric field from the complex amplitude:

1. Apply a filter to remove the negative frequencies $(-q_0, -f_0)$.
2. Apply a frequuency translation of the quantities $(-q_0, -f_0)$ to bring the mode with positive frequencies around the origin.

The electric field may be represented in the base band thanks to the complex amplitude $A(z, t)$. This allows to work with with a slowly-varying and mono-modal quantity.

This formalism may be generalized to other "quasi-sinusoidal" quantities, either scalar or vectorial, such as the magnetic field, the induced polarization, etc. The general spatial dependence on the position r (i.e. the dependence on x, y and z coordinates) may be included as well in this formalism.

Exercise 8. Prove that under the hypothesis of slowly varying amplitude (i.e. the temporal and spatial derivatives are much smaller than the amplitude itself), the magnetic field can be written as:

$$\mathbf{H}(z, t) \approx -|A|\frac{\beta_0}{\omega_0\mu_0}\cos(\omega_0 t + \beta_0 z + \phi_0)\,\mathbf{j}.$$

Furthermore, for slowly varying amplitudes the average electromagnetic power passing through a surface can be obtained from (1.29), as:

$$< P > \approx -\int_S \frac{|A|^2\,\beta_0}{2\omega_0\mu_0}\,\mathbf{k}\cdot\mathbf{N}\,dS. \tag{1.46}$$

1.6 Wave equation for the complex amplitudes

Here we assume that the electric field depends on the general spatial coordinate $\mathbf{r}$. Without losing generality, we assume that the wave propagates along the z coordinate. Now we express the scalar electric field[8] and the induced polarization in terms of their amplitudes ($A = A(\mathbf{r}, t)$ and $A_P = A_P(\mathbf{r}, t)$, respectively) as:

$$\begin{cases} E(\mathbf{r}, t) = \dfrac{A(\mathbf{r}, t)}{2}e^{j(\beta_0 z + \omega_0 t)} + \dfrac{A^*(\mathbf{r}, t)}{2}e^{-j(\beta_0 z + \omega_0 t)} \\[2ex] P(\mathbf{r}, t) = \dfrac{A_P(\mathbf{r}, t)}{2}e^{j(\beta_0 z + \omega_0 t)} + \dfrac{A_P^*(\mathbf{r}, t)}{2}e^{-j(\beta_0 z + \omega_0 t)}. \end{cases} \tag{1.47}$$

[8]In case of a vectorial electric field $\mathbf{E}$, the current approach may be used for each component, separately.

By substituting the latter expressions into the wave equation (1.19) and separating the terms with positive and negative frequencies, we get the wave equation for the complex amplitudes:

$$\nabla^2 A + 2j\beta_0 \frac{\partial A}{\partial z} - \beta_0^2 A =$$
$$-\frac{\omega_0^2}{c^2}\left(A + \frac{1}{\varepsilon_0}A_P\right) + \frac{2j\omega_0}{c^2}\frac{\partial}{\partial t}\left(A + \frac{1}{\varepsilon_0}A_P\right) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(A + \frac{1}{\varepsilon_0}A_P\right). \tag{1.48}$$

By introducing

$$A_1 = A + \frac{1}{\varepsilon_0}A_P, \tag{1.49}$$

we may rewrite Eq. (1.48) in the more compact form:

$$\nabla^2 A + 2j\beta_0 \frac{\partial A}{\partial z} - \beta_0^2 A = -\frac{\omega_0^2}{c^2}A_1 + \frac{2j\omega_0}{c^2}\frac{\partial A_1}{\partial t} + \frac{1}{c^2}\frac{\partial^2 A_1}{\partial t^2} \tag{1.50}$$

We pinpoint the fact that Eq.s (1.48) and (1.50) are totally equivalent to the wave equation for the electric field. In other words, if we solve (1.48) we also get the solution of Eq. (1.19). Furthermore, the amplitudes-based formulation allows to recognize in a more straight-forward way what is the relative weight of the terms appearing in the equation and thus if simplifications may be intruduced.

Now we calculate the complex amplitude of the complex induced polarization. First we tackle the linear case and then we pass to the nonlinear one.

1.6.1 Linear contribution of the induced polarization

The relation linking the induced polarization to the electric field, for a linear and isotropic medium, is a filtering (a scalar electric field is assumed):

$$P(\mathbf{r}, t) = \varepsilon_0\, \chi \overset{t}{*} E(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{+\infty} \chi(\mathbf{r}, \tau) E(\mathbf{r}, t - \tau)\, d\tau.$$

In the frequency domain it reads:

$$\widehat{P}(\mathbf{r}, f) = \varepsilon_0\, \widehat{\chi}(\mathbf{r}, f)\widehat{E}(\mathbf{r}, f).$$

The latter expressions are quite unfriendly to be used for the general case. Indeed, they would require the full knowledge of the response of the medium, i.e. of $\widehat{\chi}(\mathbf{r}, f)$, $\forall f$. This request is often overkill and too demanding in the practical cases, where the electric fields are not zero only in a narrow region around the main frequency.

Another complication is related to the evaluation of the second order derivative of the induced polarization, which may be quite involved. To overcome this problem, the complex amplitude of the induced polarization can be rewritten as (we refer to the appendix in the end of the current section for the proof):

$$A_P(\mathbf{r},t) = \varepsilon_0 \sum_{n=0}^{\infty} \frac{\widehat{\chi}^{(n)}(\mathbf{r},f_0)}{(j2\pi)^n n!} \frac{\partial^n A(\mathbf{r},t)}{\partial t^n}. \tag{1.51}$$

The latter relation links the complex amplitudes of the induced polarization $A_P(\mathbf{r},t)$ and of the electric field $A(\mathbf{r},t)$ through the derivatives of $\widehat{\chi}(\mathbf{r},f)$ evaluated at the reference frequency f_0.

Thanks to relation (1.51) we may rewrite expression (1.49) as:

$$A_1(\mathbf{r},t) = A(\mathbf{r},t) + \frac{1}{\epsilon_0} A_P(\mathbf{r},t) = \sum_{n=0}^{\infty} \frac{\widehat{Q}^{(n)}(\mathbf{r},f_0)}{(j2\pi)^n n!} \frac{\partial^n A(\mathbf{r},t)}{\partial t^n}$$

$$= \sum_{n=0}^{\infty} \frac{\widehat{Q}^{(n)}(\mathbf{r},\omega_0)}{j^n n!} \frac{\partial^n A(\mathbf{r},t)}{\partial t^n}, \tag{1.52}$$

where $\widehat{Q}(\mathbf{r},f) = 1 + \widehat{\chi}(\mathbf{r},f)$ and

$$\widehat{Q}^{(n)}(\mathbf{r},f_0) = \frac{\partial^n \widehat{Q}(\mathbf{r},f)}{\partial f^n}\bigg|_{f=f_0}, \qquad \widehat{Q}^{(n)}(\mathbf{r},\omega_0) = \frac{\partial^n \widehat{Q}(\mathbf{r},\omega)}{\partial \omega^n}\bigg|_{\omega=\omega_0}.$$

As we will see, it is often enough to retain the first terms in the series only.

Appendix

Dalla teoria della modulazione, il filtro in banda passante con risposta impulsiva $\chi(t)$ ha come rappresentazione in banda base $\frac{1}{2}c_\chi(t)$ che in frequenza comporta la relazione (Fig. 1.9):

$$\widehat{c}_P(f) = \varepsilon_0\, \widehat{\chi}(f+f_0)\, \widehat{c}_E(f). \tag{1.53}$$

Figure 1.9: Filter in passing-band and its equivalent in base band.

We expand the susceptibility in Taylor series to obtain:

$$\widehat{\chi}(f+f_0) = \widehat{\chi}(f_0) + \widehat{\chi}'(f_0)\, f + \frac{1}{2}\widehat{\chi}''(f_0)\, f^2 + \ldots,$$

and Eq. (1.53) becomes:

$$\widehat{c}_P(f) = \varepsilon_0 \left(\sum_{n=0}^{\infty} \widehat{\chi}^{(n)}(f_0) \frac{f^n}{n!} \right) \widehat{c}_E(f).$$

Passing to the time domain we have

$$c_P(t) = \varepsilon_0 \sum_{n=0}^{\infty} \frac{\widehat{\chi}^{(n)}(f_0)}{(j2\pi)^n n!} \frac{\partial^n c_E(t)}{\partial t^n}.$$

Finally, we multiply the latter expression by $e^{j\beta_0 z}$ and we obtain the relation linking the complex envelopes of the electric field and of the polarization.

1.6.2 Nonlinear contribution of the induced polarization

As we did in Section 1.3.3, we here restrict ourselves to the third order nolinear term. Furthermore, we assume an instantaneous reply (i.e. without memory). Hence, we can recall Eq. (1.15) to find the relation between the induced polarization and the electric field:

$$P^{NL}(\mathbf{r}, t) = \varepsilon_0 \, \widehat{\chi}^{(3)} \, E^3(\mathbf{r}, t),$$

where $\widehat{\chi}^{(3)}$ is a constant coefficient. To determine the complex amplitude of the nonlinear polarization we have to find a function $A_P^{NL}(\mathbf{r}, t)$ so that:

$$P^{NL}(\mathbf{r}, t) = \Re e \left[A_P^{NL}(\mathbf{r}, t) e^{j(\omega_0 t + \beta_0 z)} \right]. \tag{1.54}$$

We rewrite $E(\mathbf{r}, t)$ as:

$$E(\mathbf{r}, t) = \Re e \left[A e^{j\alpha} \right], \qquad \text{with} \qquad \alpha = \omega_0 t + \beta_0 z$$

and we exploit the complex identity $(\Re e(z))^3 = \frac{1}{4} \Re e(z^3 + 3z^2 z^*)$. Hence:

$$P^{NL}(\mathbf{r}, t) = \varepsilon_0 \widehat{\chi}^{(3)} E^3(\mathbf{r}, t) = \frac{1}{4} \varepsilon_0 \widehat{\chi}^{(3)} \Re e \left([A^3(\mathbf{r}, t) e^{j2\alpha} + 3A^2(\mathbf{r}, t) A^*(\mathbf{r}, t)] e^{j\alpha} \right). \tag{1.55}$$

Comparing Eqs (1.54) and (1.55) we find

$$A_P^{NL}(\mathbf{r}, t) = \frac{1}{4} \varepsilon_0 \widehat{\chi}^{(3)} \left(A^3(\mathbf{r}, t) e^{j2\alpha} + 3A^2(\mathbf{r}, t) A^*(\mathbf{r}, t) \right).$$

We pinpoint the fact that the complex amplitude of the induced polarization contains two contributions, one in the base band and one at the pulsation $2\omega_0$. However, usually the latter is not very important in dispersive media (indeed in general $n(3\omega_0) \neq n(\omega_0)$ and the phase matching condition is not satisfied[9]).

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[9]The interaction between the induced polarization at $3\omega_0$ (a.k.a. the source) and the electric field

On the other hand, the base band contribution is naturally granted, no matter of the dispersive behaviour of the medium. As a consequence, we may simplify the complex amplitude for the nonlinear induced polarization as:

$$A_P^{NL}(\mathbf{r},t) = \frac{3}{4}\varepsilon_0\, \widehat{\chi}^{(3)}|A(\mathbf{r},t)|^2 A(\mathbf{r},t). \tag{1.56}$$

1.6.3 Total contributions to the polarization

Recapping, the induced polarization is written as:

$$P = \varepsilon_0 \left(\widehat{\chi}E + \widehat{\chi}^{(3)}E^3\right) = \left(A_P^L + A_P^{NL}\right)e^{j\alpha} = A_P e^{j\alpha}.$$

Hence:

$$\begin{aligned} A_1 &= A + \frac{1}{\varepsilon_0}A_P = A + \frac{1}{\varepsilon_0}\left(A_P^L + A_P^{NL}\right) = \\ &= A + \sum_{n=0}^{\infty}\frac{\widehat{\chi}^{(n)}(\mathbf{r},\omega_0)}{j^n n!}\frac{\partial^n A(\mathbf{r},t)}{\partial t^n} + \frac{3}{4}\widehat{\chi}^{(3)}|A(\mathbf{r},t)|^2 A(\mathbf{r},t). \end{aligned} \tag{1.57}$$

1.7 Non Linear Schrödinger Equation(NLSE)

From the results reported in the previous sections we may write the equation ruling the electric field complex amplitude (a.k.a. Nonlinear Schrödinger equation, NLSE).

1.7.1 NLSE

We substitute relation (1.57) into Eq. (1.50). We assume that the complex amplitude of the electric field A is slowly varying and hence we can neglect all the terms containing the time derivatives of A higher than the second order[10]. By doing this we get to the NLSE:

$$\begin{aligned} \nabla^2 A + 2j\beta_0\frac{\partial A}{\partial z} - \beta_0^2 A &= -\frac{\omega_0^2}{c^2}\left(1+\widehat{\chi}\right)A + j\left(\frac{\omega_0^2}{c^2}\widehat{\chi}\,' + \frac{2\omega_0}{c^2}(1+\widehat{\chi})\right)\frac{\partial A}{\partial t} + \\ &+ \left(\frac{\omega_0^2}{2c^2}\widehat{\chi}\,'' + \frac{2\omega_0}{c^2}\widehat{\chi}\,' + \frac{1}{c^2}(1+\widehat{\chi})\right)\frac{\partial^2 A}{\partial t^2} - \frac{3}{4}\frac{\omega_0^2}{c^2}\widehat{\chi}^{(3)}|A|^2 A, \end{aligned} \tag{1.58}$$

at $3\omega_0$ is efficient if the two fields are in phase. The phase of the electric field at ω_0 is $\Phi(\omega_0, z, t) = \beta_0(\omega_0)z + \omega_0 t + \phi(\omega_0) = n(\omega_0)\omega_0 z/c + \omega_0 t + \phi(\omega_0)$. To grasp the physical meaning of this statement, we make an example: the condition of phase-matching requires that at a fixed coordinate z_0 the phases of the electric fields are the same at the initial time $t = 0$ and also after one cycle $t = 2\pi/\omega_0$, i.e. $\Phi(3\omega_0, z_0, t = 0) = \Phi(\omega_0, z_0, t = 0)$ and $\Phi(3\omega_0, z_0, t = 2\pi/\omega_0) = \Phi(\omega_0, z_0, t = 2\pi/\omega_0)$. This is possible only if $n_r(3\omega_0) = n_r(\omega_0)$. The latter condition is not true in general and in real materials it occurs seldom.

[10]On the same spirit, we may neglect also the temporal derivatives of $|A|^2A$.

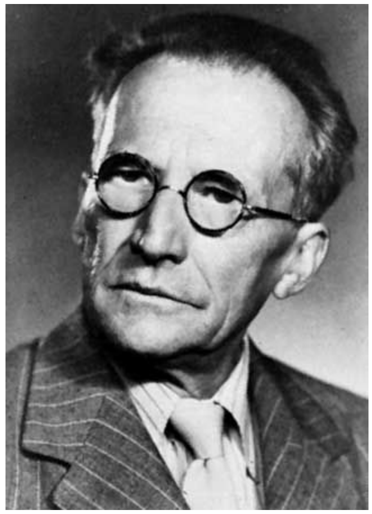

Figure 1.10: Erwin Schrödinger. Austrian physicist, born in Vienna on 12th August 1887 and dead in Vienna on 4th January 1961. He was awarded with the Nobel Prize for his fundamental studies on the wave mechanics and quantum mechanics.

with $\widehat{\chi}' = \partial\widehat{\chi}/\partial\omega \Big|_{\omega=\omega_0}$ and $\widehat{\chi}'' = \partial^2\widehat{\chi}/\partial\omega^2 \Big|_{\omega=\omega_0}$.

We define the *wave number* $k(\mathbf{r}, \omega)$ as:

$$k^2(\mathbf{r}, \omega) = \frac{\omega^2}{c^2}(1 + \widehat{\chi}(\mathbf{r}, \omega)) = \frac{\omega^2}{c^2} n^2(\mathbf{r}, \omega) \tag{1.59}$$

and we obtain:

$$\nabla^2 A + 2j\beta_0 \frac{\partial A}{\partial z} - \beta_0^2 A = -k_0^2 A + j(k_0^2)' \frac{\partial A}{\partial t} + \frac{1}{2}(k_0^2)'' \frac{\partial^2 A}{\partial t^2} - \frac{3}{4}\frac{\omega_0^2}{c^2}\widehat{\chi}^{(3)}|A|^2 A, \tag{1.60}$$

with $k_0^2 = k^2(\mathbf{r}, \omega_0)$, $(k_0^2)' = \partial[k^2(\mathbf{r}, \omega)]/\partial\omega \Big|_{\omega=\omega_0}$, and so on.

There is a main direction in space, along which the wave propagates. More correctly, this main direction can be any direction almost parallel to the direction along with the barycenter of the pulse propagates. For instance, in the communications between two satellites, the main direction in the space is parallel to the imaginary segment linking the two satellites. Here, for the sake of simplicity, we assume that the main direction is parallel to z axis of the reference frame. In this framework, it is reasonable to assume that the angle between the generic vector $\mathbf{k} = (k_x, k_y, k_z)$ and z axis is small (the so-called *paraxial approximation*). The latter approximation implies that A is slowly varying with the spatial coordinates x, y and z.

Example 1. Just to grasp the physics behind the paraxial approximation, let's suppose that the electric field can be written as $E = E_0 e^{j(\mathbf{k}\cdot\mathbf{r}+\omega_0 t)}$, where $\mathbf{k}$ is the wave vector with norm equal to β_0 and with a small angle θ with respect to the z axis. If we rewrite this electric field in terms of the complex amplitude, we have $E = A(\mathbf{r})e^{j(\beta_0 z+\omega_0 t)}$, with $A(\mathbf{r}) = E_0 e^{j(\mathbf{k}\cdot\mathbf{r}-\beta_0 z)} = E_0 e^{j(\beta_0 \sin\theta\cos\phi x+\beta_0\sin\theta\sin\phi y+\beta_0\cos\theta z-\beta_0 z)}$, where ϕ is the angle between the x axis and the projection of $\mathbf{k}$ on the x-y plane. As a consequence, $|\partial A(\mathbf{r})/\partial z| = |\beta_0(1-\cos\theta)A(\mathbf{r})| \ll |\beta_0 A(\mathbf{r})|$, since θ is small. The same conclusion holds also for the derivatives with respect to x and y.

Finally, the nonlinear contribution in Eq. (1.60) (last term in the right-hand-side) is often a small perturbation.

1.7.2 Discussion on the terms appearing in the NLSE

According to the speculations reported before, now we classify the terms appearing in Eq. (1.60) according to their importance.

First order. Neglecting all the terms containing the derivatives of the slowly varying amplitude E and the dependence of the electric susceptibility on the spatial coordinate, we recover the dispersion relation:

$$\beta_0^2 = k_0^2 = \frac{\omega_0^2}{c^2}[1+\widehat{\chi}(f_0)].$$

Second order. Neglecting all the terms involving the second order derivatives of A and the nonlinear contribution we have:

$$2j\beta_0\frac{\partial A}{\partial z} = j(k_0^2)'\frac{\partial A}{\partial t}.$$

At the second order, we assume again that the spatial dependence of the electris susceptibility are negligible (hence $k_0 \simeq \pm\beta_0$). As a consequence:

$$\frac{\partial A}{\partial z} = \pm k_0'\frac{\partial A}{\partial t}, \tag{1.61}$$

and iterating for another time the relation linking the derivatives reported in the latter relation we get to:

$$\frac{\partial^2 A}{\partial z^2} = (k_0')^2\frac{\partial^2 A}{\partial t^2}. \tag{1.62}$$

From Eq. (1.61) we can see that the complex amplitude propagates with a speed $1/(k_0{}')$, called *group velocity*. Indeed, the solution of Eq. (1.61) is:

$$A(x,y,z,t) = A(x,y,0,t\pm k_0' z).$$

In other words, the complex amplitude at the coordinate z is equal to the complex amplitude at the coordinate $z = 0$, translated in time by $\pm k_0' z$ (addressed as propagation time). At this perturbation order we account for the delay times due to the finite propagation speed of the electromagnetic radiation, but signal distortions are not present.

Third order By substituting relation (1.62) into Eq. (1.60), and developing the derivatives of k_0 we can get to:

$$2j\beta_0 \frac{\partial A(\mathbf{r},t)}{\partial z} + \frac{\partial^2 A(\mathbf{r},t)}{\partial x^2} + \frac{\partial^2 A(\mathbf{r},t)}{\partial y^2} +$$
$$+ \left(\frac{3}{4} \frac{\omega_0^2}{c^2} \widehat{\chi}^{(3)} \, |A(\mathbf{r},t)|^2 + k_0^2 - \beta_0^2 \right) A(\mathbf{r},t) - j2k_0 k_0' \, \frac{\partial A(\mathbf{r},t)}{\partial t} - k_0 k_0'' \, \frac{\partial^2 A(\mathbf{r},t)}{\partial t^2} = 0. \quad (1.63)$$

The latter equation is called 3D Nonlinear Schrödinger Equation (3DNLSE).

Now we recall the meaning of the quantities appearing in Eq. (1.63): $A(\mathbf{r},t)$ is the complex amplitude of the electric field $E(\mathbf{r},t)$. Furthermore, there are 5 constants:

- $\beta_0 = 2\,\pi q_0$ is the reference propagation constant.
- $\omega_0 = 2\pi f_0$ is the reference pulsation,
- k_0, k_0' and k_0'' are obtained as:

$$k_0^2 = \frac{\omega^2}{c^2}[1 + \widehat{\chi}(\mathbf{r},\omega)], \qquad k_0' = \frac{\partial k}{\partial \omega}, \qquad k_0'' = \frac{\partial^2 k}{\partial \omega^2},$$

evaluated for $\omega = \omega_0$ (see Eq. (1.59)).

The terms in Eq. (1.63) containing the second order derivatives with respect to x and y are the so-called *diffractive terms*, responsible of the spatial broadening of a beam propagating in a medium. On the other hand, the term containing the second order time derivative is the so-called *dispersive term*, corresponding to a temporal broadening of a pulse propagating in a medium with $k_0'' \neq 0$.

We pinpoint the fact that the term $(3/4)(\omega_0/c)^2\widehat{\chi}^{(3)}|A(\mathbf{r},t)|^2$ is similar to the term containing the linear refractive index, because they act on the complex amplitude in the same way. Indeed, this term implies a refractive index depending on the light intensity through the third order nonlinearity, according to the relation:

$$n_t(\omega, |A(\mathbf{r},t)|^2) = n(\omega) + n_2|A(\mathbf{r},t)|^2, \qquad \text{with} \quad n_2 = \frac{3\widehat{\chi}^{(3)}}{8n}.$$

The latter relation can be obtained from the first two terms in the second line of Eq. (1.63):

$$\left(\frac{3}{4} \frac{\omega_0^2}{c^2} \widehat{\chi}^{(3)} \, |A(\mathbf{r},t)|^2 + k_0^2 \right) A(\mathbf{r},t) = \frac{\omega_0^2}{c^2}(2nn_2|A(\mathbf{r},t)|^2 + n^2)A(\mathbf{r},t) \simeq \frac{\omega_0^2}{c^2}[n_2|A(\mathbf{r},t)|^2 + n]^2 A(\mathbf{r},t).$$

In the silica (SiO_2), $n_2 \simeq 1.2\times10^{-22}$ $(\mathrm{m/V})^2$, which is a small value. Hence, it induces a negligible variation on the electric fields norm. On the other hand, the nonlinear refractive index in the silica may yield appreciable effects on the electric fields phases.

From now on we will study the 3DNLSE in the limit of propagation in free space (or, more generally, in homogeneous media) or guided propagation. In each case we will introduce some simplifications to facilitate the study.

1.7.3 What was neglected

Now we revise the approximations introduced in the equations derived in the previous sections and we point out what are the possible improvements.

1. Linear attenuation.
   We introduce it by taking an electric susceptibility with a non-zero imaginary part $\hat{\chi} = \hat{\chi}_R + j\hat{\chi}_I$. We suppose that $\hat{\chi}$ does not depend on the spatial coordinates. We take the 3DNLSE (1.63) and we assume that the complex amplitude depends only on z coordinate (we neglect also its time dependence). We neglect also the nonlinear part of the electric susceptibility. At the zero-order, the dispersion reads:

$$\beta_0^2 = \frac{\omega_0^2}{c^2}[1 + \hat{\chi}_R(\omega_0)], \tag{1.64}$$

   and adding $j\frac{\omega_0^2}{c^2}\hat{\chi}_I(\omega_0)$ on both sides we get to:

$$\frac{\omega_0^2}{c^2}[1 + \hat{\chi}_R(\omega_0) + j\hat{\chi}_I(\omega_0)] = \beta_0^2 + j\frac{\omega_0^2}{c^2}\hat{\chi}_I(\omega_0).$$

   With the approximations adopted in this case, Eq. (1.63) reduces to:

$$0 = 2j\beta_0\frac{\partial A(\mathbf{r},t)}{\partial z} + \left(k_0^2 - \beta_0^2\right) A(\mathbf{r},t) = 2j\beta_0\frac{\partial A(\mathbf{r},t)}{\partial z} + \left\{\frac{\omega^2}{c^2}[1 + \hat{\chi}_R(\omega_0) + j\hat{\chi}_I(\omega_0)] - \beta_0^2\right\} A(\mathbf{r},t).$$

   By substituting Eq. (1.64) into the latter expression and rearranging terms we obtain:

$$\frac{\partial A(\mathbf{r},t)}{\partial z} = -\frac{\omega^2}{2c^2\beta_0}\hat{\chi}_I(\omega_0)A(\mathbf{r},t). \tag{1.65}$$

   Finally, from Eq. (1.64) we have that $\beta_0 = \pm\frac{\omega_0}{c}\sqrt{1 + \hat{\chi}_R(\omega_0)}$. Hence, Eq. (1.65) may be rewritten as:

$$\frac{dA(z)}{dz} = \pm\frac{\omega_0}{2c}\frac{\hat{\chi}_I(\omega_0)}{\sqrt{1 + \hat{\chi}_R(\omega_0)}}A(z) = \pm\frac{\alpha}{2}A(z), \tag{1.66}$$

   where α is called *attenuation coefficient*. Putting everything together, we have:

$$A(z) = A(0)e^{\pm\frac{\alpha}{2}z}, \tag{1.67}$$

and the complex amplitude is exponentially attenuated along z.

Exercise 9. Prove that the intensity of the electromagnetic field propagating towards positive z in a medium with an attenuation coefficient α satisfies the relation:

$$I(z) = I(0)e^{-\alpha z}, \tag{1.68}$$

where $I_0 = I(z = 0)$ is the light intensity at the beginning of the propagation.
Tip. In analogy with Exercise 8, evaluate the Poynting vector, the latter scaling as the square of the electric field amplitude.

Equation (1.68) is known as *Lambert-Beer* law, stating that the intensity within a lossy medium decays exponentially. In particular, after the distance $L = \alpha^{-1}$ the light intensity is reduced by a factor e. In other words, $I(z = L)$ is less than 37 % of the incident radiation I_0 and after the propagation for a distance longer than few L the intensity is too low to be easily detected. For this reason, L is known as *penetration length*. We point out that α and L depends of the light wavelength.

2. Effects due to the electric field polarization.
   In order to highlight what are the main phenomena influenced by the electric field polarization, we separate the linear and the nonlinear case. For both cases, we will enforce the paraxial approximation, allowing to neglect the z components of the electric field with respect to the transverse ones (i.e. x and y).

   Linear Case. In case of birifrangence[11], the constitutive relations read:

$$\begin{aligned} P_x(\mathbf{r}, t) &= \varepsilon_0\, \chi_x \overset{t}{*} E_x(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{+\infty} d\tau\, \chi_x(\mathbf{r}, \tau) E_x(\mathbf{r}, t-\tau) \\ P_y(\mathbf{r}, t) &= \varepsilon_0\, \chi_y \overset{t}{*} E_y(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{+\infty} d\tau\, \chi_y(\mathbf{r}, \tau) E_y(\mathbf{r}, t-\tau). \end{aligned} \tag{1.69}$$

   and the reply of the medium depends on the direction of the applied electric field.
   If we repeat all the steps adopted to derive the 3DNLSE also for the current case, we have to introduce two complex amplitudes, A_x for the x component of the electric field and A_y for the y component of the electric field. Analogously, we have two propagation constants (i.e. β_{0x} and β_{0y}), generalizing relation (1.22), and two wave numbers (i.e. k_{0x} and k_{0y}), generalizing Eq. (1.59).
   If we restrict ourselves to homogeneous, linear, non-dispersive media and neglect

[11] Birifrangence means that the medium is not isotropic anymore, but the optical susceptibility along x is different with respect to the one along y.

the diffractive effects, we obtain:

$$\begin{aligned}\frac{\partial A_x}{\partial z} &= \pm k'_{0x}\,\frac{\partial A_x}{\partial t}\\ \frac{\partial A_y}{\partial z} &= \pm k'_{0y}\,\frac{\partial A_y}{\partial t}.\end{aligned} \tag{1.70}$$

The latter relations show that the two complex amplitudes may admit different group velocities. Hence, we can conclude that the birifrangence induces a dispersion for the group velocities (*dispersion of polarization*), potentially limiting the transmissive capacity of a telecommunication optical system.
Although the silica is isotropic, the polarization dispersion may be achieved in optical fiber, when the guiding structure may introduce birifrangence effects (a.k.a. shape birifrangence).

Nonlinear Case. As stated in Table 1.2, only 21 of the 81 elements of $\chi^{(3)}$ are not 0 in isotropic media. Furthermore, as indicated before, the components of the electromagnetic field along z have a negligible amplitude with respect to the transverse ones only. As a consequence, only 8 components of the tensor $\chi^{(3)}$ play a role in this analysis. In silica we have:

$$\widehat{\chi}^{(3)}_{xxxx} = \widehat{\chi}^{(3)}_{yyyy} = \widehat{\chi}^{(3)}.$$

$$\widehat{\chi}^{(3)}_{xxyy} = \widehat{\chi}^{(3)}_{xyyx} = \widehat{\chi}^{(3)}_{xyxy} = \widehat{\chi}^{(3)}_{yyxx} = \widehat{\chi}^{(3)}_{yxxy} = \widehat{\chi}^{(3)}_{yxyx} = \frac{1}{3}\widehat{\chi}^{(3)}. \tag{1.71}$$

Hence, in case of instantaneous reply the nonlinear contribution of the induced polarization reads:

$$\begin{cases} P_x^{NL} = \varepsilon_0\,\widehat{\chi}^{(3)}\left[E_x^3 + \frac{1}{3}(E_xE_y^2 + E_yE_xE_y + E_y^2E_x)\right] = \varepsilon_0\,\widehat{\chi}^{(3)}\left(E_x^3 + E_xE_y^2\right), \\ P_y^{NL} = \varepsilon_0\,\widehat{\chi}^{(3)}\left[E_y^3 + \frac{1}{3}(E_yE_x^2 + E_xE_yE_x + E_x^2E_y)\right] = \varepsilon_0\,\widehat{\chi}^{(3)}\left(E_y^3 + E_yE_x^2\right). \end{cases} \tag{1.72}$$

We write the complex amplitudes of the electric fields and induced polarizations:

$$\begin{cases} E_x = \dfrac{1}{2}\left[A_x e^{j\alpha_x} + A_x^* e^{-j\alpha_x}\right] \\ \\ E_y = \dfrac{1}{2}\left[A_y e^{j\alpha_y} + A_y^* e^{-j\alpha_y}\right] \\ \\ P_x^{NL} = \dfrac{1}{2}\left[A_{Px}^{NL} e^{j\alpha_x} + A_{Px}^{NL*} e^{-j\alpha_x}\right] \\ \\ P_y^{NL} = \dfrac{1}{2}\left[A_{Py}^{NL} e^{j\alpha_y} + A_{Py}^{NL*} e^{-j\alpha_y}\right], \end{cases} \tag{1.73}$$

with $\alpha_x = \omega_0 t + \beta_{0x} z$ and $\alpha_y = \omega_0 t + \beta_{0y} z$.
By substituting Eq. (1.73) into Eq. (1.72) and neglecting the components of the induced polarization oscillating at pulsation $3\omega_0$, we get:

$$A_{Px}^{NL} = \frac{3}{4}\varepsilon_0\, \widehat{\chi}^{(3)} \left[(|A_x|^2 + \frac{2}{3}|A_y|^2)A_x + \frac{1}{3}A_x^* A_y^2 e^{2j(\beta_{0y}-\beta_{0x})z}\right],$$

$$A_{Py}^{NL} = \frac{3}{4}\varepsilon_0\, \widehat{\chi}^{(3)} \left[(|A_y|^2 + \frac{2}{3}|A_x|^2)A_y + \frac{1}{3}A_y^* A_x^2 e^{-2j(\beta_{0y}-\beta_{0x})z}\right].$$

In both the expressions for A_{Px}^{NL} and A_{Py}^{NL} we find three terms. The first one is the *self-phase modulation* (SPM), with which the electric field polarized along x (or y) modify its phase.
The second term is the *cross-phase modulation* (XPM), with which the electric field polarized along x (or y) modifies the phase of the other electric field.
The last term is the "incoherent" term, with which the electric field linearly polarized along x can generate electric field along y and vice versa. For the incoherent nature, its contribution is often small. Hence, this coupling term is often negligible.

3. Higher order dispersion.
   In some problems, the truncation of the dispersion to the second order is not enough. This is true, for instance, in the cases where the second order dispersion is zero, as occurring at the transition wavelength from the normal to the anomalous dispersion. At the latter wavelength, it is false that the wave-like propagation is not dispersive, as one could argue from the 3DNLSE. Instead, the first term contributing to the dispersion (and, hence, to the signal distortion) is the term proportional to the third order time derivative of the complex amplitude of the signal (weighed by the third order derivative of the propagation constant with respect to the pulsation and evaluated at the main pulsation).

4. Nonlinear attenuation.

5. Nonlinear dispersion.

In most of the practical cases, the contributions discussed at the points 4) and 5) are not relevant for the wave-like propagation in the telecommunications. This is not true for the linear attenuation (point 1). Indeed, the presence of losses is a major fact that must always be taken into account. The effects related to the attenuation can be taken into account by simply adding a dissipative term to the propagation equation (hypothesis of low losses).
As for point 2), this is a contribution potentially relevant, especially for the telecommunication systems on long distances and at high speed. Indeed, the polarization of the electric field carrying information to be transmitted may play an important role to describe the properties of the the transmitting medium. Without going in quantitative details, we can qualitatively say that an optical fiber guiding structure has never a perfect cylindrical symmetry, due to fabrication imperfections. If we consider linear polarization, the asymmetry in the optical fiber imply a different reply to electromagnetic fields oscillating along x or y direction. From the point of view of the telecommunication system, the more pronounced difference between the two responses is the difference in the group velocities between the two orthogonal polarizations, yielding an additional dispersion (called polarization dispersion) reducing the system performance. Furthermore, the polarization dispersion is random and hence, at variance with the chromatic dispersion, it is not easily equalizable in reception.

1.8 Considerations on the importance of the terms in the NLSE

1.8.1 Linear polarization

We point out that in a non-dispersive medium (i,e. with a frequency-independent refractive index), the absolute value of the wave number depends linearly on the frequency. Hence, its first derivative with respect to the pulsation is pulsation independent and it is equal to the speed of light in the medium:

$$k_0'(\mathbf{r}) = \frac{n(\mathbf{r})}{c} = \frac{1}{v_g(\mathbf{r})}. \tag{1.74}$$

Obviously, in this case we have $k_0'' = 0$.

Eq. (1.74) may be used to define the velocity v_g for the energy transport (the group velocity or the propagation velocity) also when the medium is dispersive. In the latter case k_0' is not constant with respect to the pulsation and k_0'' (which is not 0) allows to evaluate (in the first approximation) what are the effects caused by the dispersive properties in the medium.

The truncation to the second derivatives (second order dispersion) in Eq. (1.63) is reasonable when the spectral extent of the wave packet is much smaller than the main frequency ($f_0 \gg |\Delta f|$). If this hypothesis is not well satisfied, it is also necessary to consider higher order terms, proportional to the third order derivative (third order dispersion, weighted by the coefficient k_0'''), to the forth order derivative (forth order dispersion, weighted by the coefficient k_0''''), and so on.

Example 2. We consider the law describing the refractive index variations in silica, as a function of the light wavelength. If we suppose to be far from the resonance frequencies, we can neglect the losses (or, at least, we can consider them constant) in a certain spectral region. In this case, the Sellmeier equation describes in a good approximation the variations of the refractive index real part as function of certain easily measurable experimental parameters:

$$n^2(\lambda) - 1 = \sum_i A_i \frac{\lambda^2}{\lambda^2 - \lambda_i^2},$$

where λ_i are resonant wavelengths and A_i are constants characteristic of each materials and experimentally accessible. For the case of pure silica and silica doped with germanium dioxide, the values of some of the constants are reported in Table 1.4.

	100% SiO_2	8% GeO_2, 92% SiO_2
A_1	0.6961663	0.7136824
A_2	0.4079426	0.4254807
A_3	0.8974994	0.8964226
λ_1	0.0684043 µm	0.0617167 µm
λ_2	0.1162414 µm	0.1270814 µm
λ_3	9.8961609 µm	9.8961614 µm

Table 1.4: Parameters appearing in the Sellmeier equation for the pure silica and silica doped with germanium dioxide.

The behavior of the refractive index in the region from 1 µm to 1.6 µm are reported in Fig. 1.11. As we can see, in that wavelength region, the refractive index is a decreasing function of the wavelength. In particular, we can distinguish between two dispersion regions: one where v_g increases with the wavelength (normal dispersion) and another region where v_g decreases with the wavelength (anomalous dispersion). Furthermore, the propagation speed in silica, $v_g = 1/k_0' = [(\partial k/\partial\omega)|_{\omega=\omega_0}]^{-1}$, is function of the wavelength (and the material is dispersive). In the normal dispersion (occurring for $\lambda < 1.27$ µm in silica), the fastest components of the spectrum are the ones with higher wavelength. On the other hand, in the anomalous dispersion (occurring for $\lambda > 1.27$ µm in silica), the fastest components of the spectrum are the ones with lower wavelength.

Exercise 10. By using Fig. 1.12, prove that in silica, for complex amplitudes $A(\mathbf{r}, t)$ of duration longer than $T_0 = 100$ fs, the values of k_0'' and k_0''' allow to neglect third order dispersion.
Tip. $\partial/\partial t$ can be approximated with $1/T_0$.

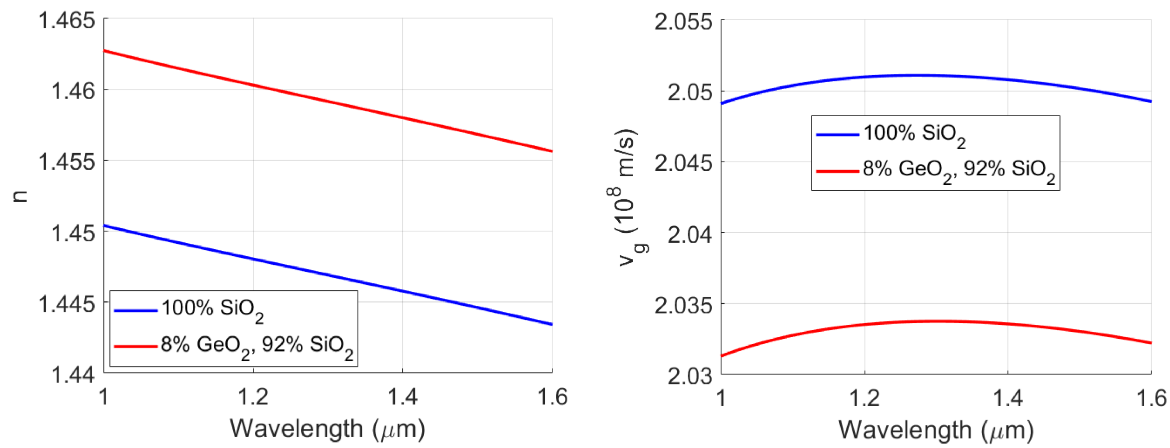

Figure 1.11: Refractive index (left panel) and group velocity (right panel) for the pure silica (blue curve) and for silica doped with germanium dioxide (red curve), obtained from Sellmeier formula.

1.8.2 Nonlinear polarization

Thanks to the availability of scientific data on the nonlinear properties of dielectrics, reported in literature in the last decades, it is possible to evaluate the magnitude of the effects induced by the nonlinear polarization. Indeed, the optical propagation in the nonlinear regime is subject of extensive research since the 60s of the 20th century, when the invention of the laser allowed to have radiation sources with high intensity and coherence. In particular, we can mention the pioneering works of Prof. Nicolaas Bloembergen, addressing the light-matter interaction in bulk devices[12]. At the beginning of the 70s of the 20th century, the availability of guiding optical structures opened the door to a new and reach scenario for the nonlinear optics.

We now define the concept of *interaction length*, a very important parameter to describe every nonlinear process. The interaction length is the length over which the nonlinear effect is active.

Example 3. Just to grasp the physical meaning of the interaction length, let's suppose that the electric field is attenuated while it propagates in a medium, according to the relation $\mathbf{E}(x, y, z) = \mathbf{E}_0(x, y)e^{-\alpha z}$. If at the beginning of the propagation (i.e. $z = 0$) the electric field was intense enough to have nonlinear effects, at the coordinate $z = 1/\alpha$ the electric field intensity drops by a factor e. As a consequence, at $z = 1/\alpha$ the nonlinear effects become negligible. For this reason, we may conclude that in this example the interaction length is $\sim 1/\alpha$.

In bulk optics, the interaction length is mainly limited by the diffraction causing a continuous broadening of the optical beams propagating from one point to the other. This broadening causes a continuous reduction of the peak intensity and at a certain point

[12] When we mention the bulk devices we refer to homogeneous (or almost homogeneous) media. Hence, the electromagnetic propagation is not guided.

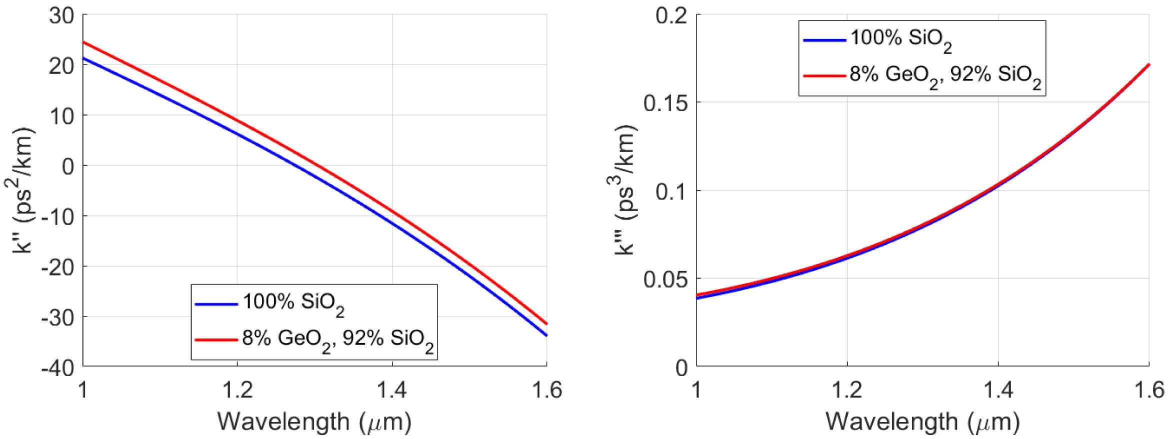

Figure 1.12: Second (left panel) and third (right panel) order dispersion for the pure silica (blue curve) and for silica doped with germanium dioxide (red curve), obtained from Sellmeier formula.

all the nonlinear effects become negligible. Just to give numbers, we consider a gaussian beam at the operative wavelength of 1.55 µm, propagating in glass. If the initial spot size of the beam has radius 4 µm, after the propagation of 2 cm the spot size radius becomes 1.6 mm, yielding a drop of the electric field peak value of a factor 400[13]. Hence, if the initial intensity were enough to yield an appreciable nonlinear response, after propagating for 2 centimeters the nonlinear phenomena is considerably reduced. This fact is not ascribable to the material losses. Even if the latter hinder further the nonlinear effects, in this treatment they are neglected because considered a higher order perturbation.

A totally different situation occurs for the guided optics. In this case, the diffraction is completely compensated by the guiding effect (i.e. refractive index variations on the plane perpendicular to the propagation direction) and the interaction length is only limited by the optical losses, due to light absorption. For instance, a monomodal fiber may have a modal radius W_0 of around 4 µm, a case similar to the initial condition of the free propagation of a Gaussian beam. At the wavelength of 1.55 µm, the losses are quantified with an absorption coefficient of around circa 0.2 dB/km. Hence, the intensity reduction obtained in the case of the aforementioned example for the bulk optics is achieved only after the propagation in fiber over a distance of 480 km. Accordingly, the interaction length achieved in the guided optics is 7 orders of magnitude higher than the one reported in the example for the bulk optics. We point out that the interaction length in the guided optics may in principle be extended to infinity if optical amplificators are employed.

We point out that the silica is a material with very weak nonlinear properties, hence no one would have used it for a bulk nonlinear experiment in the 60s. However, strong nonlinear effects may be achieved in silica optical fibers, since light-matter interaction length can be considerably enhanced in such a system. A nonlinear effect occurring in

[13]We refer to the free space wave-like propagation for a quantitative justification of these numbers, which will be treated in the following chapters.

the optical fiber is the so-called Kerr effect, i.e. the dependence of the refractive index on the intensity I according to the relation: $n = n_e + n_k I$ ($n_k = 3.2 \times 10^{-20}$ m^2/W). As we can see, n$_k$ is very small and the effects are negligible of the norm of the electromagnetic fields. Indeed n$_e$ is of the order of 1.5 and varies of a few per thousand over the linear guiding structure. For a power P of 10 mW, in a monomodal fiber, we find a peak intensity $I = 2P/(\pi W_0^2) = 4 \times 10^8$ W/m^2, implying a negligible variation of the refractive index.

A different situation occurs for the phase accumulated by the electromagnetic field during the propagation in a nonlinear medium. Indeed, the phase variation results $\phi = (2\pi/\lambda)(n_e + n_k I)z$ and the nonlinear contribution to the phase $\phi_{nl} = (2\pi/\lambda)(n_k I)z$ is not negligible. In particular, this term can introduce a phase shift up to π after only 60 km, a propagation distance of the order of the distance in between two optical amplificators employed in a optical transmission system based on fiber.

Chapter 2

Guided propagation of the optical waves

2.1 Propagation in the optical fiber

2.1.1 General features

The optical fiber is a very long and thin cylindrical structures composed by three concentric domains (as schematized in Fig. 2.1):

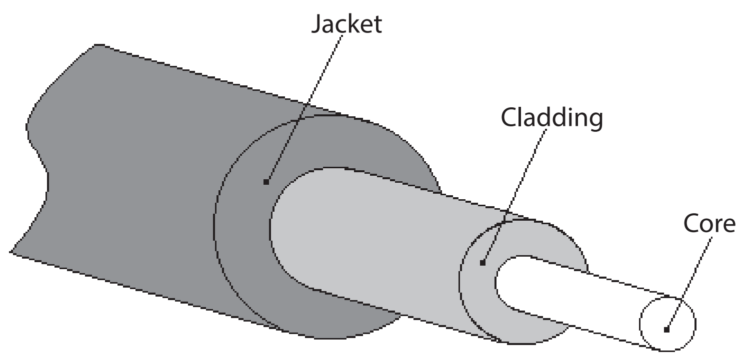

Figure 2.1: Structure of an optical fiber.

- The *core*: it's the inner part, with higher refractive index with respect to the rest. Usually, it is composed by silica (SiO_2), opportunely doped with other materials (such as P_2O_5, TiO_2, GeO_2 or Al_2O_3) allowing to increase the core refractive index.
- The *cladding*: it covers the core and it has a lower refractive index with respect to the core. It is made by silica (SiO_2), opportunely doped with other materials (such as B_2O_3) to lower its refractive index;
- The *jacket*: its the outer part of the fiber. It is usually made of plastic and it is useful to protect the transmitting part. Furthermore, it is engineered to achieve the desired mechanical properties for the fiber.

At the beginning, we restrict ourselves to the case in which the core refractive index is constant in space (step optical fibers). The same is assumed for the cladding. If the core dimension is large enough (with respect to the working wavelength) to allow the propagation of several guided modes[1], we have a *multimodal* fiber. The core diameter of this kind of fibers is of the order of 50 µm. On the other hand, if the core dimension is small enough to allow the propagation of only one guided mode, we have a *monomodal* fiber. In this case, the core diameter is of the order of 10 µm.
An important parameter for both the monomodal and the multimodal optical fibers is the numerical aperture (NA). We here explain the meaning of NA for the case of a step refractive index fiber.

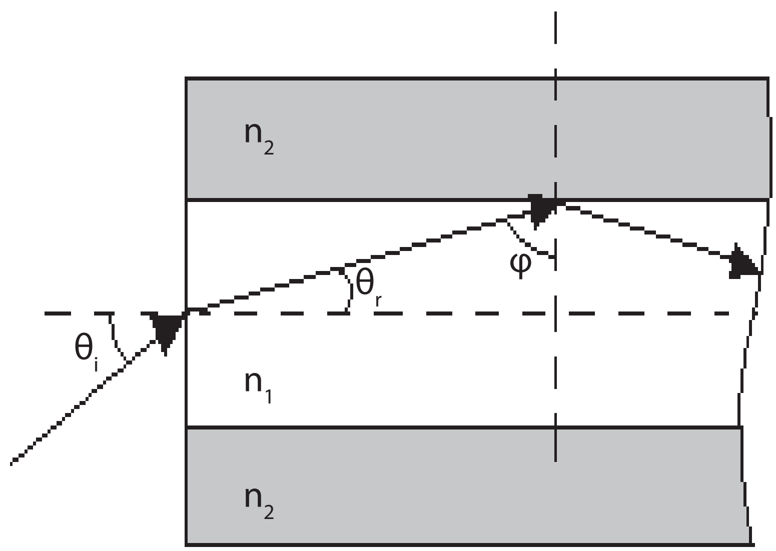

Figure 2.2: Light confinement in a step refractive index optical fiber.

We refer to Fig. 2.2 and we consider a light ray illuminating from one side of the fiber with an angle θ_i with respect to the geometrical axis of the fiber. The ray is refracted in the core and the angle of the ray is obtained from *Snell law*:

$$n_0 \sin \theta_i = n_1 \sin \theta_r \tag{2.1}$$

where n_0 and n_1 are the refractive indexes of the medium around the optical fiber and of the core, respectively.

[1]The formal definition of guided mode will be presented later on in this Section. For now it suffices to interpret a guided mode as an eigenfunction of the propagation problem: if the electric field profile at the beginning of the guide is a mode, the electric field profile at the outlet has the same shape (the phase could be the only difference).

Afterwards, the light ray encounters the separation surface in between the core and the cladding and hence it is refracted again. In order to completely trap the ray inside the core, it is necessary to have total internal reflection at the interface core-cladding, occurring if

$$\phi > \phi_c = \text{asin}\left(\frac{n_2}{n_1}\right), \tag{2.2}$$

where n_2 is the cladding refractive index.

Combining Eq.s (2.1) and (2.2) we can determine the maximum incidence angle allowing to the incoming ray to be trapped inside the structure. Since $\theta_r = \frac{\pi}{2} - \phi_c$, we have:

$$NA = n_0 \sin\theta_{im} = n_1 \cos\phi_c = \sqrt{(n_1{}^2 - n_2{}^2)}, \tag{2.3}$$

where θ_{im} is the maximum angle for an incident ray to be completely trapped into the fiber.

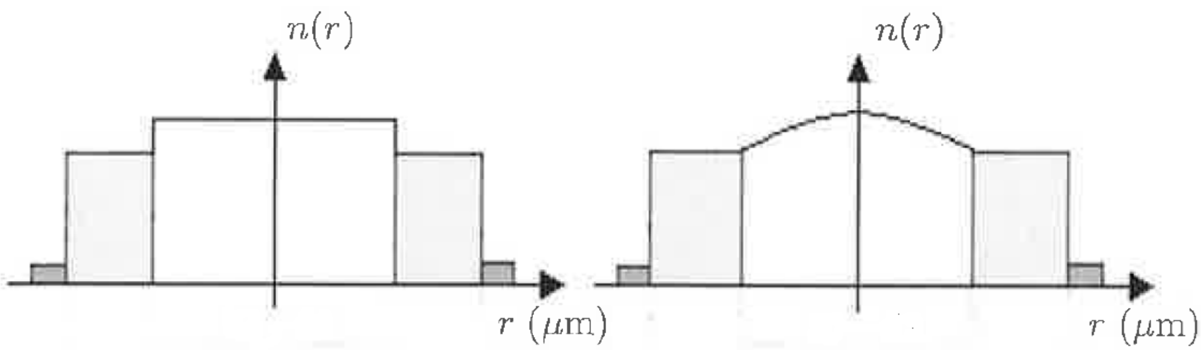

Figure 2.3: Step (left panel) and smooth (right panel) refractive index spatial profile.

Even if our main interest concerns the monomodal fibers, it is necessary to spend a few lines discussing on the multimodal fibers. The latter are nowadays still used in applications such as communications over limited distances or sensors. However, the main limit of the multimodal fibers is due to their dispersion, due to the multimodes (a.k.a. intermodal dispersion), importantly limiting the transmissive capacity. In the frame of the ray propagation, a ray trapped inside the core may cover several paths. If the propagation speed is the same, the time necessary to cover the distance in between the beginning and the end of the fiber could be very different, depending on the chosen path.

To mitigate the issues introduced by this effect, it it possible to use a refractive index $n(r)$ with a smooth spatial profile, i.e. with smooth decrease from the maximum value (n_1) in the center of the core to a minimum value (n_2) at the interface core-cladding, as schematized in Fig. 2.3. From Fig. 2.4 we can see that the ray along the geometrical axis of the core is the one covering the shortest path and with the slowest speed, because it encounters the highest refractive index. On the other hand, the transverse rays pass in regions with a bit lower refractive index, hence they propagate faster. Based on this

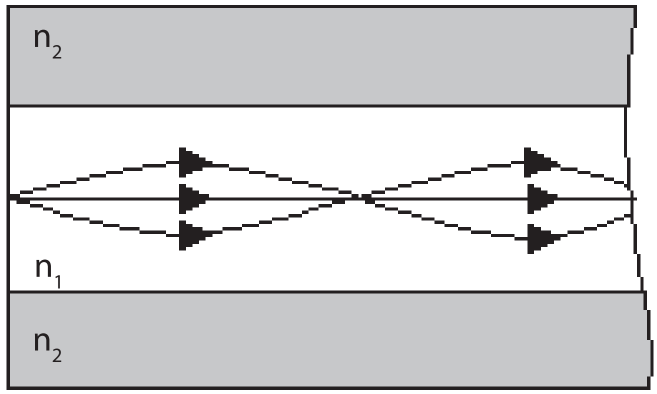

Figure 2.4: Rays in a fiber with smooth refractive index spatial profile.

rationale, by tailoring the refractive index spatial profile it is possible to minimize the intermodal dispersion. The performances (in terms of bit rate per distance) of the multimodal fibers with smooth spatial profile refractive index exceed of roughly three orders of magnitude those of the step refractive index fibers. Further improvements may be achieved by using **monomodal** fibers. The latter, that are described in the following Sections, are widely spread for long scale and high speed telecommunications, since they are characterized by a very low attenuation and are not plagued by the intermodal dispersion.

2.1.2 Main factors affcting the propagation

The optical fiber is a dispersive, nonlinear and dissipative system. Hence, the signal propagating in the fiber is subject to several phenomena altering the form, attenuating or distorting it. The transmittive capacity depends on two main phenomena, the attenuation and the dispersion.

Attenuation of the fiber. As we have already seen previously (see Eq. (1.67)) a signal propagating in a dielectric medium is subject to attenuation. Since the optical power P is proportional to the square norm of the complex amplitude, from Eq. (1.67), we get:

$$P(D) = P(0)\exp(-\alpha D). \tag{2.4}$$

Very often the attenuation coefficient α is expressed in dB/km through the relazione:

$$\alpha\ [\text{dB/km}] = -\frac{10}{D}\log_{10}\left(\exp(-\alpha D)\right).$$

The attenuation may be split into two contributions, the *intrinsic* (due to the properties of the medium) and *extrinsic* (due to all the other reasons, such as the tensions, deforming the transmittive medium) attenuation.

The intrinsic attenuation is due to the absorption and diffusion of the electromagnetic radiation. Every material has the maximum absorption in proximity of its own resonance frequencies. The silica has an electronic resonance for $\lambda < 0.4$ µm (ultraviolet) and vibrational for $\lambda > 7$ µm (infrared). On the other hand, in the wavelength of interest (i.e. 0.8µm $< \lambda <$1.6 µm) the absorption is mild (below 0.1 dB/km almost everywhere). The main cause for the absorption, implying the so called *transmission windows*, is due to the presence of hydroxide ions (OH^-). Due to the latter, silica shows absorption peaks at the wavelengths $\lambda = 1.39$ µm, $\lambda = 1.24$ µm and $\lambda = 0.95$ µm.

The intrinsic attenuation is also affected by the *Rayleigh diffusion* effect; microscopic fluctuations of the medium density diffuse the light in all the directions attenuating the propagating signal (the attenuation due to this phenomenon originates an attenuation proportional to λ^{-4}).
Fiber bending, asymmetry and microjunctions can contribute to extrinsic damping. In particular, junctions and connectors present in a real setup may lead to a strong attenuation (see next Sections).

Fiber dispersion. Due to the dispersion, the light pulse is distorted during the propagation. The different spectral components of the signal propagate with different group velocities and in the numerical transmission intersymbol interference may occur.

2.1.3 Other factors affecting the propagation

The transmission quality depends also on the following factors:

Material dispersion. The material composing the fiber is weakly dispersive (its electromagnetic parameters, in particular the dielectric permittivity, are functions of the frequency).

Intramodal dispersion. For every propagating mode, the ratio between the power propagating in the cladding and the power in the core depends on the frequency. Indeed, the mode group velocity is a average (weighted by the fraction of the power propagating in each medium) between the one of the core and the one of the cladding. Hence, the group velocity depends on the fraction of the power in the core and, consequently, on the frequency.

Intermodal dispersion. It is present only in the multimodal fibers and it is due to the different group velocities of the involved modes.

Polarization dispersion. Geometrical imperfections or small curvatures of the fiber may imply a coupling between two orthogonal polarization states travelling with different group velocities This effect is called *polarization dispersion.*

Transmitted signal power. Above a certain transmitted power threshold, some linear phenomena distorting the signal may occur. The consequences are particularly relevant in wavelength multiplexing systems.

2.2 Guided modes

In any guiding structure (both optical fiber or guide for an integrated optical circuit) the refractive index shows a spatial dependence along the transverse direction with respect to the propagation direction. This spatial dependence is the element allowing the guiding effect, that is the spatial confinement of the electromagnetic radiation in the space (see Fig. 2.5). For instance, in an optical fiber the electric field is mainly confined in a

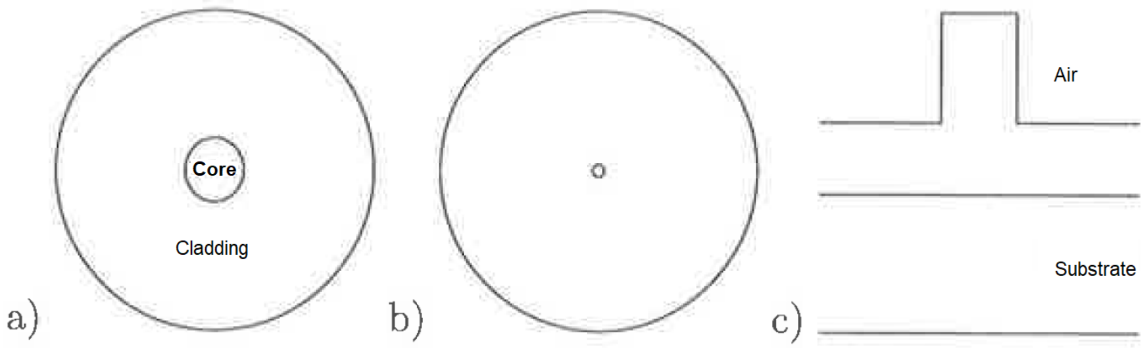

Figure 2.5: Examples of dielectric guiding structures: a multimodal optical fiber (a), a monomodal optical fiber (b) and a rib guide (c)

central region (a.k.a. core), where the refractive index is higher. Depending on the core dimension, we may be in the monomodal or multimodal propagation regime.

The electromagnetic spatial confinement may be seen as a trade-off between the diffraction and the guiding effect. For this reason, the spatial problem (i.e. the propagation of the electromagnetic field along z, as function of the spatial variables x and y) and the temporal problem (i.e. the evolution of the electromagnetic field as a function of the temporal variable t) may be studied separately. In this way, the analysis of the propagation considerably simplifies.

We suppose that the electric field complex amplitude is separable in the form[2]:

$$A(x, y, z, t) = F(z, t)\, M(x, y)\, e^{j\delta\beta z}, \tag{2.5}$$

where the parameter $\delta\beta$ has been introduced to account for the modifications of the propagation constant of the wave due to the presence of the guiding structure.
$M(x, y)$ is called *modal profile* and it is often written in cylindrical coordinates as $M(r, \phi)$. $\beta = \beta_0 + \delta\beta$ is *the propagation constant* of the guided mode. The propagation constant

[2]We recall that $A(x, y, z, t)$ has the dimensions of V/m and we can assign to $F(z, t)$ the dimension of a V and to $M(x, y)$ the dimension of a m^{-1}.

is the sum of two contributions. The first contribution is the propagation constant of the solution in the reference ideal case, defined in Eq. (1.13), and hence it is known. The second contribution is due to the presence of the guide and has to be determined.
The *weakly guiding* hypothesis allows to use the approximation $k_0 \simeq \beta_0$. Hence, from 3DNLSE (1.63) we get:

$$\begin{aligned}
&2j\beta_0 M(x,y)\frac{\partial F(z,t)}{\partial z} + 2\frac{k_0^2}{n}n_2|F(z,t)M(x,y)|^2 F(z,t)M(x,y) \\
&-2j\beta_0 k_0^{'} M(x,y)\frac{\partial F(z,t)}{\partial t} - \beta_0 k_0^{''} M(x,y)\frac{\partial^2 F(z,t)}{\partial t^2} \\
&+ F(z,t)\left[\frac{\partial^2 M(x,y)}{\partial x^2} + \frac{\partial^2 M(x,y)}{\partial y^2} + (k_0^2 - \beta_0^2 - 2\beta_0\delta\beta)M(x,y)\right] = 0
\end{aligned} \tag{2.6}$$

where we introduced:

$$n_2 = \frac{3\hat{\chi}^{(3)}}{8n}.$$

If we divide both sides of Eq. (2.6) by $F(z,t)M(x,y)$ we get to:

$$\begin{aligned}
&2j\beta_0\frac{\partial F(z,t)}{\partial z}\frac{1}{F(z,t)} + 2\frac{k_0^2}{n}n_2|F(z,t)M(x,y)|^2 \\
&-2j\beta_0 k_0^{'}\frac{\partial F(z,t)}{\partial t}\frac{1}{F(z,t)} - \beta_0 k_0^{''}\frac{\partial^2 F(z,t)}{\partial t^2}\frac{1}{F(z,t)} = \\
&-\frac{1}{M(x,y)}\left[\frac{\partial^2 M(x,y)}{\partial x^2} + \frac{\partial^2 M(x,y)}{\partial y^2} + (k_0^2 - \beta_0^2 - 2\beta_0\delta\beta)M(x,y)\right].
\end{aligned} \tag{2.7}$$

Since the right-hand-side does not depend of z and t, also the left-hand side must not depend on z and t. In analogy, the left-hand-side does not depend of x and y[3], also the right-hand side must not depend on x and y. Hence, both sides must be a constant C and we may rewrite the last equation as:

$$\begin{cases}
2j\beta_0\dfrac{\partial F(z,t)}{\partial z} + 2\dfrac{k_0^2}{n}n_2|F(z,t)M(x,y)|^2 F(z,t) - 2j\beta_0 k_0^{'}\dfrac{\partial F(z,t)}{\partial t} - \beta_0 k_0^{''}\dfrac{\partial^2 F(z,t)}{\partial t^2} = CF(z,t) \\
\dfrac{\partial^2 M(x,y)}{\partial x^2} + \dfrac{\partial^2 M(x,y)}{\partial y^2} + (k_0^2 - \beta_0^2 - 2\beta_0\delta\beta)M(x,y) = -CM(x,y).
\end{cases} \tag{2.8}$$

The solution with a constant electric field in time and space is solution of Maxwell equations and it must be solution of Eq. (2.8) as well. As a consequence, we must have $C = 0$,

[3] We recall that the nonlinear term is small. Hence, the dependence of x and y on the left-hand-side may be disregarded.

and hence:

$$\begin{cases} 2j\beta_0 \dfrac{\partial F(z,t)}{\partial z} + 2\dfrac{k_0^2}{n} n_2 |F(z,t)M(x,y)|^2 F(z,t) - 2j\beta_0 k_0' \dfrac{\partial F(z,t)}{\partial t} - \beta_0 k_0'' \dfrac{\partial^2 F(z,t)}{\partial t^2} = 0 \\ \dfrac{\partial^2 M(x,y)}{\partial x^2} + \dfrac{\partial^2 M(x,y)}{\partial y^2} + (k_0^2 - \beta_0^2 - 2\beta_0 \delta\beta) M(x,y) = 0. \end{cases} \tag{2.9}$$

The second equation of (2.9) allows to the determination of the guided modes in the dielectric structure, within the so-called linearly polarized (LP) modes approximation[4]. Now we solve that equation. This allows to evaluate the *profile* of the guided modes $M(x, y)$ and their propagation constant β. After determining these quantities, it is possible to analyze the propagation.

The reader who is already familiar with the dielectric structures guided modes features may skip next paragraph. For a complete and detailed analysis of the guided modes evaluation in dielectric structures (such as optical fibers), we refer to specific textbooks. Here, we tackle the problem in a simplified way, to highlight the relevant aspects of the applications of interest in this course.

2.2.1 Guided modes of the optical fiber: LP approximation

We consider a guiding dielectric structure with circular symmetry (in particular we refer to "step index" or "matched-cladding" geometries, see Fig. 2.6).

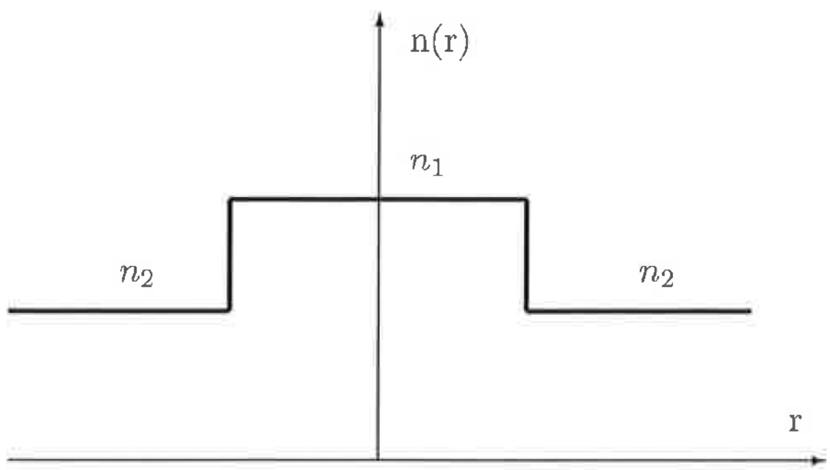

Figure 2.6: Step index optical fiber: profile of the refractive index.

[4]Since from the beginning we restricted ourselves to the case where the fields are scalar, it was implicitly obvious that only LP modes would have been obtained. We pinpoint the fact that in most of the practical problems, this approximation is reasonable.

Hence, we have:

$$k_0^2(r) = \begin{cases} n_1^2\omega^2\mu_0\epsilon_0 & 0 \leq r < a \\ \\ n_2^2\omega^2\mu_0\epsilon_0 & a \leq r < +\infty, \end{cases}$$

where a is the core radius, whereas n_1 and n_2 are the refractive indexes of the core and cladding, respectively, with $n_1 > n_2$.

The second equation of (2.9) is rewritten in cylindrical coordinates, to describe the optical fiber geometry in an easier way:

$$\frac{\partial^2 M(r,\phi)}{\partial r^2} + \frac{1}{r}\frac{\partial M(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M(r,\phi)}{\partial \phi^2} + \left[k_0^2(r) - \beta_0^2 - 2\beta_0\delta\beta\right] M(r,\phi) = 0, \qquad (2.10)$$

and $M(r,\phi)$ may be developed in Fourier series with respect to ϕ:

$$M(r,\phi) = \sum_{\ell=-\infty}^{+\infty} M_\ell(r) e^{j\ell\phi}.$$

Then, from Eq (2.10), for every integer ℓ, we obtain an equation for $M_\ell(r)$ and its associated $\delta\beta$ (to be referred to as $\delta\beta_\ell$ in the following):

$$\frac{\partial^2 M_\ell(r)}{\partial r^2} + \frac{1}{r}\frac{\partial M_\ell(r)}{\partial r} + \left[q_\ell^2(r) - \frac{\ell^2}{r^2}\right] M_\ell(r) = 0, \qquad (2.11)$$

where $q_\ell^2(r) = k_0^2(r) - \beta_0^2 - 2\beta_0\delta\beta_\ell$. The latter equation is known as Bessel function. If $q_\ell^2(r)$ is constant, as in the core and in the cladding of a step index optical fiber, Eq. (2.11) becomes a well-known differential equation, whose general solutions are the Bessel functions.

To solve the problem we have to compute the eigenvalue β - representing the propagation constant of the guided mode - and the eigenfunction $M(r)$ - i.e. the transverse profile of the guided mode. To proceed in the solution process we impose the continuity of the electromagnetic field at the interface between the core and the cladding. We are under the hypothesis of small variations of the refractive index (the so-called weakly guiding approximation). As a consequence, the electromagnetic fields are almost TEM (transverse electromagnetic fields). Under these hypotheses, it is possible to prove that at the interface between core and cladding the continuity of the electromagnetic fields can be replaced (in first approximation) by the condition of the continuity of the electric field (proportional to $M(r)$) and its spatial derivative. In this way we determine the guided modes of the optical fiber in the approximation of linearly polarized field (a.k.a. LP modes). Hence we consider the general solution of Eq. (2.11) in the core and in the cladding separately.

In the core, the general solution of Eq. (2.11) reads[5] $M_\ell(r) = A_1 J_\ell(k_t r) + B_1 Y_\ell(k_t r)$,

[5]The solution of the Bessel equation is well known in Mathematics. We refer to textbooks on the topic for further details.

where J_ℓ and Y_ℓ are called Bessel and Neumann functions, respectively[6], and $k_t^2 = n_1^2\omega^2\mu_0\epsilon_0 - \beta_0^2 - 2\beta_0\delta\beta_\ell$. Neumann functions diverge for $r \to 0$. Hence, since it is non-physical that $M_\ell(r)$ diverges in the core center, B_1 must be 0.

Analogously, in the cladding the solution may be written as $M_\ell(r) = C_1 J_\ell(j\gamma r) + C_1 Y_\ell(j\gamma r)$, with $\gamma^2 = \beta_0^2 + 2\beta_0\delta\beta_\ell - n_2^2\omega^2\mu_0\epsilon_0$. The imaginary unit appearing in the argument of J_ℓ and Y_ℓ comes from the fact that γ^2 is defined with the opposite sign with respect to k_t^2. In this case, it is possible to express the Bessel and Neumann functions in terms of the modified Bessel functions of the first and second kind, defined respectively as:

$$\begin{cases} I_\ell(x) = j^{-\ell} J_\ell(jx), \\ \\ K_\ell(x) = \dfrac{\pi}{2} j^{\ell+1} H_\ell^{(1)}(jx) = \dfrac{\pi}{2} j^{\ell+1} \left[J_\ell(jx) + jY_\ell(jx)\right], \end{cases} \tag{2.12}$$

where $H_\ell^{(1)}$ is known as Hankel function (or Bessel function of the third kind). By expressing the Bessel and Neumann functions in terms of the modified Bessel functions, the general solution in the cladding becomes $M_\ell(r) = A_2 K_\ell(j\gamma r) + B_2 I_\ell(j\gamma r)$. $I_\ell(j\gamma r)$ diverges exponentially as $r \to +\infty$, hence to obtain a physically sound solution for $M_\ell(r)$, we have to impose $B_2 = 0$.

Putting everything together, the solution of the problem is:

$$M_\ell(r) = \begin{cases} A_1 J_\ell(k_t r) & for \qquad 0 \le r < a \\ \\ A_2 K_\ell(\gamma r) & for \qquad a \le r < +\infty, \end{cases} \tag{2.13}$$

with

$$k_t^2 = n_1^2\omega^2\mu_0\epsilon_0 - \beta_0^2 - 2\beta_0\delta\beta_\ell, \qquad \gamma^2 = \beta_0^2 + 2\beta_0\delta\beta_\ell - n_2^2\omega^2\mu_0\epsilon_0. \tag{2.14}$$

We point out again that in our definition the terms appearing in the expressions for k_t^2 contribute with the opposite sign to the expression for γ^2. However, since $n_1 > n_2$, the adopted definition ensures that k_t^2 and γ^2 are both positive in the case of a guided propagation.

From Eq. (2.13), we understand that the electric field in the core is described by a Bessel function of the first kind of order ℓ, that is a function oscillating like cosinusoidal functions and asymptotically decaying as $1/\sqrt{r}$ for large r. The electric field in the cladding is described by a modified Bessel function of the second kind of order ℓ, exponentially decaying for large r.

Now we impose the boundary conditions at the interface core-cladding. Hence we request that $M_\ell(r)$ and dM_ℓ/dr are continuous for $r = a$. This implies that:

$$\frac{XJ'_\ell(X)}{J_\ell(X)} = \frac{YK'_\ell(Y)}{K_\ell(Y),} \tag{2.15}$$

[6]Sometimes, J_ℓ and Y_ℓ are also called as Bessel functions of the first and second kind, respectively.

with

$$X = k_t a, \quad Y = \gamma a \tag{2.16}$$

and

$$X^2 + Y^2 = \omega^2 \mu_0 \epsilon_0 (n_1^2 - n_2^2) a^2 = \left(\frac{2\pi a}{\lambda}\right)^2 (n_1^2 - n_2^2) = V^2 \tag{2.17}$$

Eq. (2.15) is a transcendental equation (a.k.a. dispersion relation) and by solving it we can obtain the propagation constant and the transverse profile of the LP mode. In Eq. (2.17) V is the normalized frequency containing the main guiding features of the dielectric structure. Once ℓ and V are assigned (the former depending on the mode we are interested in and the latter depending on the working frequency and on the properties of the guiding structure), the dispersion relation becomes one equation in only one unknown X (since $Y^2 = V^2 - X^2$ is blocked).
The first derivatives of Bessel functions are not very friendly, since they are not tabulated. Hence, we had better to express in terms of other Bessel functions through the relations:

$$\begin{cases} J'_\ell(z) = -\ell \dfrac{J_\ell(z)}{z} + J_{\ell-1}(z) \\ \\ K'_\ell(z) = -\ell \dfrac{K_\ell(z)}{z} - K_{\ell-1}(z). \end{cases} \tag{2.18}$$

Upon substitution of Eq. (2.18) into relation (2.15) we get to:

$$-\frac{X J_{\ell-1}(X)}{J_\ell(X)} = \frac{Y K_{\ell-1}(Y)}{K_\ell(Y)}. \tag{2.19}$$

We first consider the case ℓ=0, corresponding to the modes not depending on the azimuthal coordinate ϕ (i.e. $M(r,\phi) = M_0(r)$). In this case, the dispersion relation becomes:

$$-\frac{X J_{-1}(X)}{J_0(X)} = \frac{Y K_{-1}(Y)}{K_0(Y)} \tag{2.20}$$

The left-hand-side (LHS) and the right-hand-side (RHS) of Eq. (2.20) are drawn in the left panel of Fig. 2.7 for $V = 6$ (blue and orange curve, respectively). The term on the RHS is a decreasing function of X, defined for X between 0 and V[7]. On the other hand, the LHS term is an increasing function with asymptotes in correspondence of the zeros of the function $J_0(X)$ (occurring at $X \simeq 2.405$, $X \simeq 5.52$, ...). Hence, there are several intersections between the LHS and the RHS, corresponding to guided modes. The first guided mode of this set (a.k.a. LP_{01} mode) always exists (for any V there is always an intersection in the left panel of Fig. 2.7). Furthermore, the second guided mode of this family (a.k.a. LP_{02} mode) exists only for $V > 3.832$. $V = 3.832$ is the normalized cutoff frequency for the mode LP_{02}.

[7]If $X > V$, Y becomes imaginary and the RHS is not defined.

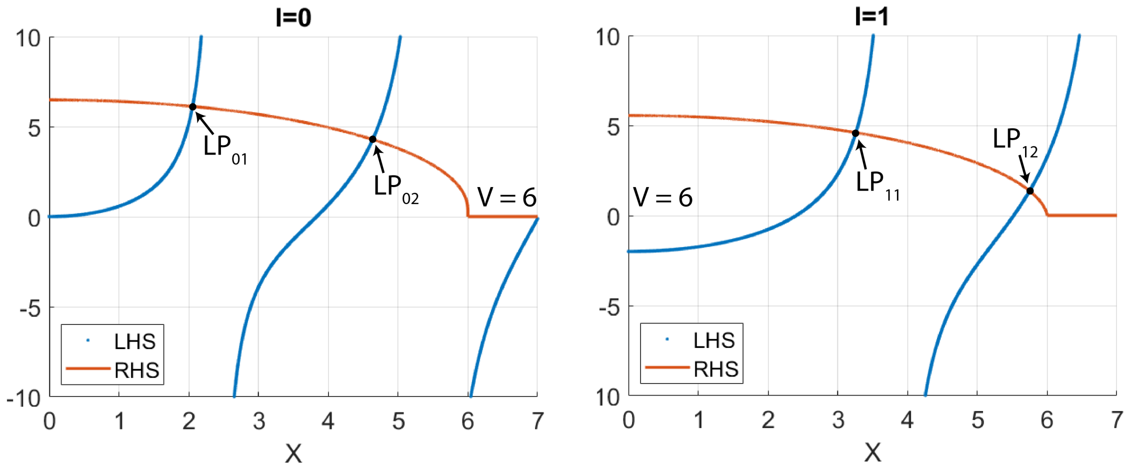

Figure 2.7: Graphical solution of the dispersion relation for the modes LP_{0m} (left panel) and LP_{1m} (right panel).

Now we turn to the case ℓ=1. The dispersion relation now becomes:

$$-\frac{XJ_0(X)}{J_1(X)} = \frac{YK_0(Y)}{K_1(Y)}.$$

The LHS and RHS of the latter equation (evaluated for $V = 6$) are plotted in Fig. 2.7. Also for the current case we have that the RHS is a DEcreasing function of X, defined for X between 0 and V. On the other hand, the LHS is an increasing function with asymptotes in correspondence of the zeros of $J_1(X)$ ($X \simeq 3.832$, $X \simeq 7.016$, ...). The intersections between the two curves is not possible for small X, because the LHS is negative for $X < 2.405$ (the latter being the first zero of the function $J_0(X)$). Hence, the first guided mode of this family (i.e the mode LP_{11}) is possible only for normalized frequencies bigger than 2.405 ($V = 2.405$ is the normalized cutoff frequency).

It is possible to prove that, excluding the fundamental mode (LP_{01}), the mode LP_{11} is the one with the lowest cutoff frequency[8]. As a consequence, the condition $V < 2.405$ implies the monomodality: for normalized frequencies smaller than 2.405, the optical fiber sustains only one guided mode (i.e. the mode LP_{01}). We address this case as monomodal waveguide. The latter condition is of paramount importance for the next part of the course, since we are treating high speed transmission systems over long distances, exploiting monomodal optical fibers with mode falling at the system operative wavelength. In order to complete that short review on the guided modes in a step optical fiber, we further comment on the fundamental mode. As we have already seen, the fundamental mode LP_{01} propagates for arbitrarily low normalized frequencies. Furthermore, it is the only admitted mode for $V < 2.405$. Now, we illustrate the main properties of the fundamental mode for $0 < V < 2.405$, studying the dispersion relation as a function of V.

[8]By plotting the dispersion relation for an arbitrary $\ell > 1$ we can notice that the cutoff frequency for the modes $\mathrm{LP}_{\ell 1}$ is always greater than the one of the mode LP_{11}.

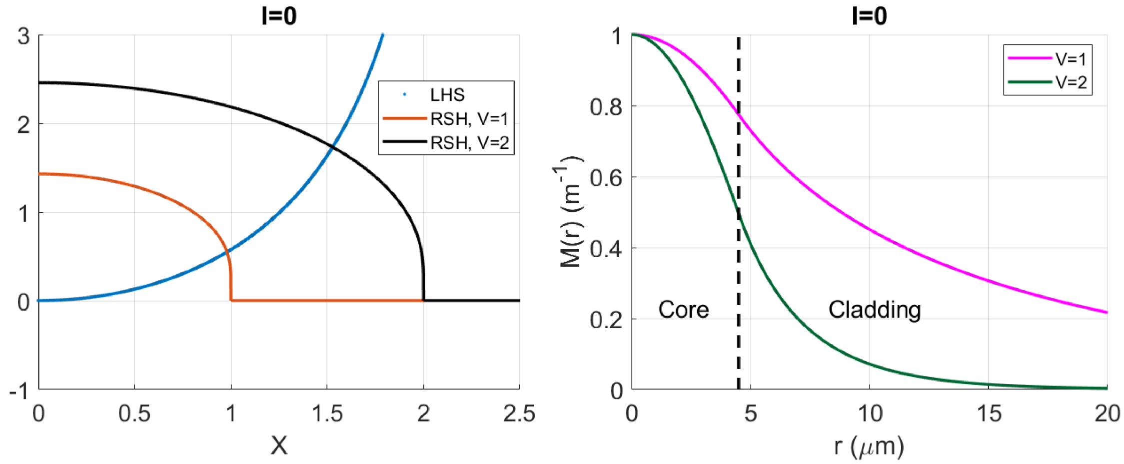

Figure 2.8: Graphical solution of the dispersion relation for the fundamental mode LP_{01} (left panel) and mode profiles for two values of normalized frequency (right panel). The core radius is $a = 4.5$ µm.

In the left panel of Fig. 2.8 we report the graphical solution of the dispersion relation for two different values of the normalized frequency, i.e. $V = 1$ and $V = 2$; in the former (latter) case, the dispersion relation admits solution for $X \simeq 0.9793$ (for $X \simeq 1.5282$).

Once the solution of the dispersion relation X is found, from Eq. (2.16) it is possible to quantify k_t as X/a. Finally, from the first of Eqs. (2.14) it is possible to work out the propagation constant as:

$$\beta_\ell = \beta_0 + \delta\beta_\ell = \frac{n_1^2\omega^2\mu_0\epsilon_0 - (X/a)^2}{2\beta_0} + \frac{\beta_0}{2}. \tag{2.21}$$

β_0 is an initial guess of the correct propagation constant and may be chosen arbitrarily. Depending on the choice of β_0, $\delta\beta_\ell$ adapts itself accordingly in order to obtain the correct propagation constant β_ℓ. Provided that β_0 is not too far from the correct propagation constant of the optical fiber[9], the value of the obtained β_ℓ is not affected from the particular choice of β_0.

Once the correct propagation constant is found, it is possible to introduce the effective refractive index n_e of the mode as: $n_e(\omega) = \beta(\omega)/(\omega\sqrt{\mu_0\varepsilon_0})$. From this point of view, the optical fiber may be seen, in first approximation, as a homogeneous material with refractive index n_e.

Exercise 11. Consider an optical fiber with $a = 4.5$ µm, $n_1 = 1.447$ and $n_2 = 1.443$. Prove that for normalized frequencies $V = 1$ and $V = 2$ the effective index of the LP_{01} modes are $n_e = 1.4432$ and $n_e = 1.4447$, respectively.

[9]The propagation constant of the guide is expected to be in between the propagation constants of homogeneous materials with the core and the cladding refractive indexes. Hence, it is reasonable to choose an initial guess in the range $\omega n_2\sqrt{\mu_0\varepsilon_0} \leq \beta_0 \leq \omega n_1\sqrt{\mu_0\varepsilon_0}$.

As we can see n_e increases with V. In the right panel of Fig. 2.8 we report the corresponding transverse modal profile $M(r)$[10]. If V increases, the mode becomes more confined in the core (or, in other words, the mode is more guided). Hence, the energy is more confined in the core and the optical fiber properties are closer to the core ones. In particular, for increasing frequencies n_e gets close to the core refractive index.
In general, the value of V affects both the modal profile (i.e. the eigenfunction) and the propagation constant (i.e. the eigenvalue) of the mode. However, the the eigenvalue is much more sensitive to the normalized frequencies variations with respect to the modal profile. Hence, in first approximation we may assume that the modal profile is not affected by small frequency changes whereas the variations of the propagation constant are not negligible.

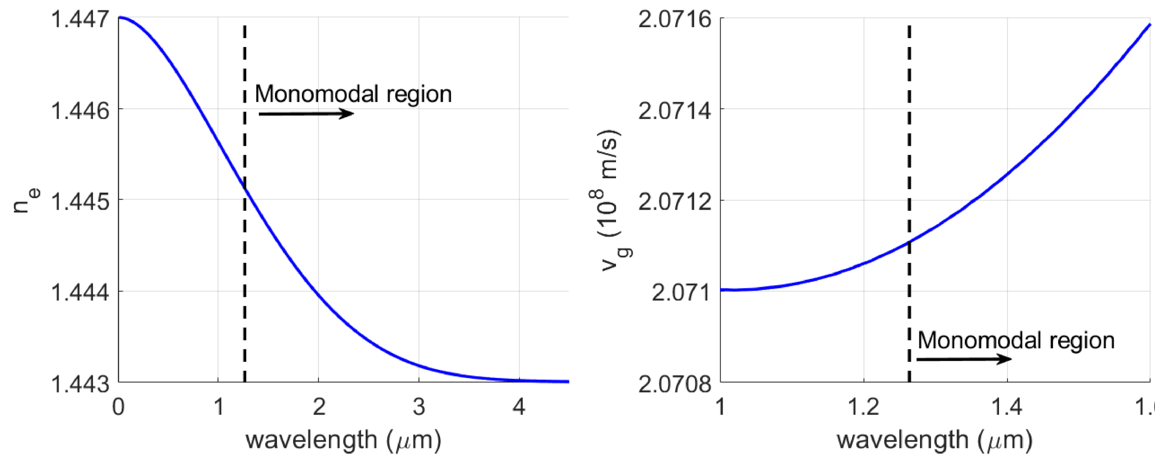

Figure 2.9: Effective refractive index (left panel) and group velocity (right panel) as functions of the wavelength, for the fundamental mode.

If we solve the dispersion relation by tuning the normalized frequency, we can generalize the results. In Fig. 2.9 we report the effective refractive index of the guided mode (left panel) and the group velocity (right panel) as functions of the wavelength. The group velocity is calculated as $v_g = [(\partial\beta/\partial\omega)|_{\omega=\omega_0}]^{-1}$ We can notice that the guide contribution to the dispersion yields a group velocity increasing with the wavelength (the so called "normal dispersion"). This is a general result: indeed all the guiding structures show a normal dispersion of the group velocity.

We remind that from the practical point of view, what matters is the total dispersion. The latter has in general two contributions, one due to the dispersion of the material (neglected in this section to focus on the guiding effects) and one induced by the guide.

[10] We point out that in Fig. 2.8 we assumed $A = 1$. Furthermore, the continuity of $M(r)$ requires that the coefficient A_2 appearing in Eq. (2.13) is related to A_1 through the relation $A_2 = A_1 J_\ell(k_t a)/K_\ell(\gamma a)$.

2.2.2 Gaussian approximations

Even if the expression for the mode profile (Eq. (2.13)) is analytical, it is still complicated, because it implies the evaluation of Bessel functions. Furthermore, the evaluation of the parameters requires the solution of a transcendental equation (Eq. (2.15)). For all these reasons, it may be difficult to handle it and the introduction of approximate expressions may prove very useful to treat the problem.

The profile of the mode LP_{01} may be approximated with the Gaussian function:

$$M(r) = \exp(-(r/W_0)^2),$$

where W_0 is called *modal radius*. A careful approximation of W_0 as a function of the normalized frequency V is:

$$\frac{W_0}{a} \simeq 0.65 + 1.619V^{-(3/2)} + 2.879V^{-6}.$$

The left panel of Fig. 2.10 reports the comparison between the exact expression of the fundamental mode (full line) and the approximate one (dashed line), for $V = 2$.

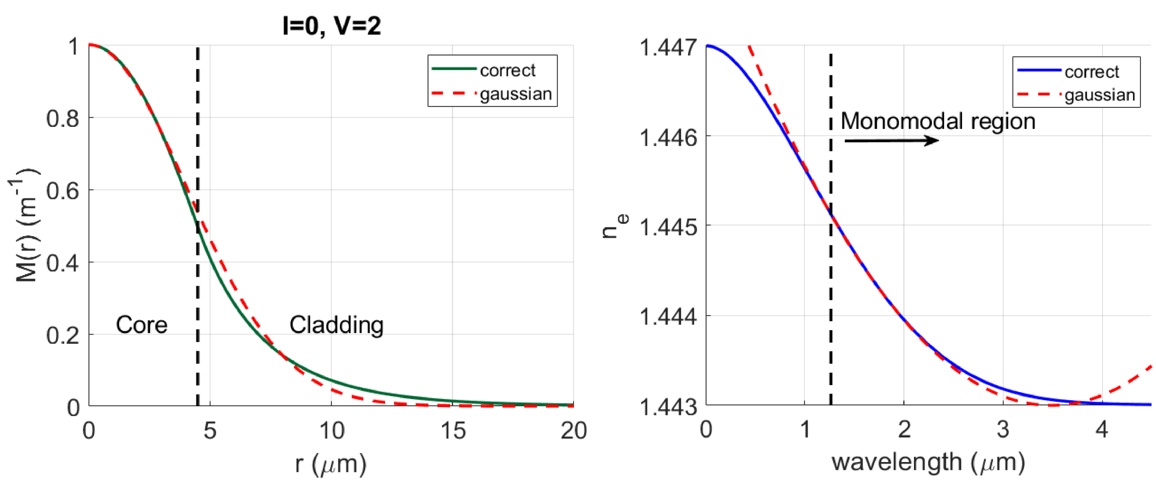

Figure 2.10: Mode profile for $V = 2$ (left panel) and effective refractive index (right panel). The full lines correspond to the correct solution, whereas the dashed lines is obtained under the Gaussian approximation. The fundamental mode is considered.

Also for the other important parameter of the guided mode, i.e. the eigenvalue, there are good approximations. Here, we report a careful approximations for the effective refractive index: $n_e \simeq n_2 + (n_1 - n_2)(1.1428 - 0.996/V)^2$. In the right panel of Fig. 2.10 we report the comparison between the exact effective refractive index n_e and the approximate one (full and dashed line, respectively).

It is worth to note that the aforementioned approximate expressions are very accurate for $2.0 < V < 2.4$, the latter being the normalized frequencies interval commonly adopted in fibers communication systems[11].

2.2.3 The propagation equation

From what we have seen in the previous section, now we have already computed the modal profile $M(x,y)$ of the guided field in the optical fiber and the corresponding propagation constant β.

We multiply the second equation of (2.9) by $M(x,y)$ and we integrate over the entire transverse plane (i.e. we integrate over x and y from $-\infty$ to $+\infty$). In this way, we get to:

$$2\beta_0\delta\beta = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[(k_0^2-\beta_0^2)M(x,y)+\frac{\partial^2 M(x,y)}{\partial x^2}+\frac{\partial^2 M(x,y)}{\partial y^2}\right]M(x,y)\ dx\ dy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}M(x,y)^2\ dx\ dy} \tag{2.22}$$

and for the *weakly guiding* hypothesis we have $(k_0^2-\beta_0^2)=(k_0-\beta_0)(k_0+\beta_0)\approx(k_0-\beta_0)2\beta_0$. Hence, Eq. (2.22) reduces to:

$$2\beta_0\delta\beta = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[2\beta_0(k_0-\beta_0)M(x,y)+\frac{\partial^2 M(x,y)}{\partial x^2}+\frac{\partial^2 M(x,y)}{\partial y^2}\right]M(x,y)\ dx\ dy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}M(x,y)^2\ dx\ dy} \tag{2.23}$$

The latter equation may be rewritten as:

$$\delta\beta = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(k_0-\beta_0)M(x,y)^2\ dx\ dy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}M(x,y)^2\ dx\ dy}+\frac{1}{2\beta_0}\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\frac{\partial^2 M(x,y)}{\partial x^2}+\frac{\partial^2 M(x,y)}{\partial y^2}\right]M(x,y)\ dx\ dy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}M(x,y)^2\ dx\ dy}. \tag{2.24}$$

$\delta\beta$, β_0 and k_0 are functions of ω_0. In principle, also the mode field profile is function of ω_0. However, in first approximation we may assume that the variations of M with respect to ω_0 are much smaller than the variations of the propagation constant with respect to ω_0[12]. Hence, in the following part of this section we neglect the angular frequency dependence of M.

We derive Eq. (2.24) with respect to the angular frequency and we obtain (the derivative of the second term in the right-hand-side of Eq. (2.24) with respect to ω_0 is here neglected):

$$\beta' = \beta_0' + \delta\beta' = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}k_0'(x,y,\omega)M(x,y)^2\ dx\,dy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}M(x,y)^2\ dx\,dy} \tag{2.25}$$

and

$$\beta'' = \beta_0'' + \delta\beta'' = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}k_0''(x,y,\omega)M(x,y)^2\ dx\,dy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}M(x,y)^2\ dx\,dy}. \tag{2.26}$$

[11]This interval is the one commonly used in communication systems due to the contained curvature losses.

[12]This fact derives from the perturbation theory: small variations of the guiding structure (due, for instance, to the change of the main angular frequency) at the first order do affect the eigenvalue (i.e. the propagation constant) but do not perturb the eigenfunction (i.e. the transverse profile M of the guided mode).

We get back again to the first Eq. of (2.9). We multiply it by $M(x,y)^2/(2\beta_0)$ and we integrate over the transverse plane. In this way, we get to:

$$j\frac{\partial F(z,t)}{\partial z}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M(x,y)^2\ dx\,dy + \frac{k_0^2}{n\beta_0}n_2|F(z,t)|^2F(z,t)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M(x,y)^4\ dx\,dy +$$

$$-j\frac{\partial F(z,t)}{\partial t}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_0' M(x,y)^2\ dx\,dy - \frac{1}{2}\frac{\partial^2 F(z,t)}{\partial t^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_0'' M(x,y)^2\ dx\,dy = 0.$$

Now we divide both sides by $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M(x,y)^2\ dx\,dy$ and we substitute Eqs. (2.25) and (2.26) to obtain:

$$j\frac{\partial F(z,t)}{\partial z} + \left(\frac{k_0^2 n_2}{n\beta_0 S_{eff}}\right)|F(z,t)|^2F(z,t) - j\beta'\frac{\partial F(z,t)}{\partial t} - \frac{\beta''}{2}\frac{\partial^2 F(z,t)}{\partial t^2} = 0, \tag{2.27}$$

with:

$$S_{eff} = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M(x,y)^2\ dx\ dy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M(x,y)^4\ dx\ dy}. \tag{2.28}$$

Finally, by using $\beta_0 \approx k_0$ and the definition $k_0 = \omega_0 n/c$, Eq. (2.27) turns to:

$$j\frac{\partial F(z,t)}{\partial z} + \left(\frac{\omega_0 n_2}{cS_{eff}}\right)|F(z,t)|^2F(z,t) - j\beta'\frac{\partial F(z,t)}{\partial t} - \frac{\beta''}{2}\frac{\partial^2 F(z,t)}{\partial t^2} = 0, \tag{2.29}$$

which is the 2D Nonlinear Schrödinger Equation (2DNLSE).
Once we have solved the problem for the transverse field we know M and β. With this information, we can study the evolution of the electric field amplitude along z direction and in time by tackling Eq. (2.29).