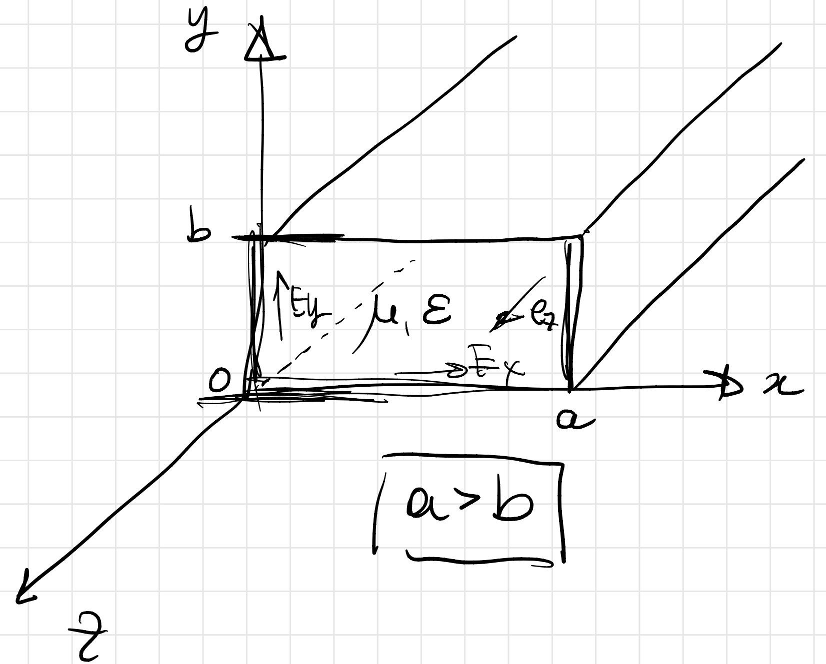
MICROWAVE ENGINEERING

Rectangular and circular waveguides

RECTANGULAR WAVEGUIDE



$$\boxed{a>b}$$

TE modes $\quad E_z = 0$

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right) h_z(x,y) = 0}$$

$$H_z(x,y,z) = h_z(x,y)\, e^{-j\beta z}$$

$$k_c^2 = k^2 - \beta^2 \quad \text{cutoff wavenumber}$$

$$h_z(x,y) = X(x)\,Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + k_c^2 = 0$$

$$\begin{cases} \dfrac{d^2X}{dx^2} + k_x^2 X = 0 \leftarrow \\ \dfrac{d^2Y}{dy^2} + k_y^2 Y = 0 \leftarrow \\ \underline{k_x^2 + k_y^2} = \underline{k_c^2} \end{cases}$$

$$h_z(x,y) = \underbrace{(A\cos k_x x + B\sin k_x x)}_{X(x)} \underbrace{(C\cos k_y y + D\sin k_y y)}_{Y(y)}$$

A, B, C, D will be found by imposing boundary conditions

$$e_x(x,y) = 0 \quad @ \quad y = 0, b$$

$$e_y(x,y) = 0 \quad @ \quad x = 0, a$$

The general solution for TE waves gave us

$$\begin{cases} E_x = -\dfrac{j\omega\mu}{k_c^2} \dfrac{\partial H_z}{\partial y} \\ E_y = \dfrac{j\omega\mu}{k_c^2} \dfrac{\partial H_z}{\partial x} \end{cases}$$

So we can write

$$e_x = \frac{-j\omega\mu}{k_c^2} \left(\underline{A} \cos k_x x + B \sin k_x x \right) k_y \left(-C \sin \underline{k_y y} + \underline{D} \cos k_y y \right)$$

$$e_y = \frac{j\omega\mu}{k_c^2} k_x \left(-A \sin \underline{k_x x} + B \cos k_x x \right) \left(\underline{C} \cos k_y y + D \sin k_y y \right)$$

$e_x = 0$ e $y = 0 \Rightarrow D = 0$

$y = b \Rightarrow \boxed{k_y = \frac{n\pi}{b}}$ $\quad n = 0, 1, 2 \dots$

$e_y = 0$ e $x = 0 \Rightarrow B = 0$

$x = a \Rightarrow \boxed{k_x = \frac{m\pi}{a}}$ $\quad m = 0, 1, 2 \dots$

$$\boxed{H_z(x, y, z) = \underline{A_{mn}} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}}$$

From general solution of waveguide transverse fields:

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x} \Rightarrow \boxed{H_x = \frac{j\beta m\pi}{k_c^2 a} A_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}}$$

$$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y} \Rightarrow \boxed{H_y = \frac{j\beta n\pi}{k_c^2 b} A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}}$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} \Rightarrow \boxed{E_x = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} \Rightarrow \boxed{E_y = \frac{-j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}}$$

The propagation constant

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad k_c^2 = k_x^2 + k_y^2$$

β is real (propagating mode) if

$$k > k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

The cutoff frequency $\quad k = \omega_c \sqrt{\mu\varepsilon} = 2\pi f_c \sqrt{\mu\varepsilon} > k_c$

$$f_{c_{mn}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

The LOWEST CUTOFF FREQUENCY identifies the DOMINANT MODE TE_{10} $(m=1, n=0)$

$$f_{c10} = \frac{1}{2\cancel{\pi}\sqrt{\mu\varepsilon}} \cdot \frac{m\cancel{\pi}}{a} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

NOTE: if $m=n=0 \Rightarrow$ all fields are zero, $\Rightarrow$ THERE IS NO TE_{00} mode

if $f < f_c \Rightarrow$ all modes are EVANESCENT (β imaginary)

<u>WAVE IMPEDANCE</u>

$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{k\eta}{\beta}$$

<u>GUIDE WAVELENGTH</u>

$$\underline{\lambda_g} = \frac{2\pi}{\beta} > \frac{2\pi}{k} = \underline{\lambda}$$

PHASE VELOCITY $\quad v_p = \frac{\omega}{\beta} > \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$

TM Modes $\quad H_z = 0$ and E_z should satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z(x, y) = 0$$

$$E_z(x, y, z) = e_z(x, y)\, e^{-j\beta z} \qquad k_c^2 = k^2 - \beta^2$$

$$e_z(x, y) = (A\cos k_x x + B\sin k_x x)(C\sin k_y y + D\cos k_y y)$$

Boundary conditions now apply on e_z

$e_z = 0 \quad @ \; x = 0, a \Rightarrow A = 0 \;, \; k_x = \frac{m\pi}{a}$

$e_z = 0 \quad @ \; y = 0, b \Rightarrow C = 0 \;, \; k_y = \frac{n\pi}{b}$

$$\boxed{E_z(x,y,z) = B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}}$$

Transverse field components are:

$$E_x = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial x} \Rightarrow E_x = -\frac{j\beta m\pi}{a k_c^2} B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

$$E_y = \frac{-j\beta}{k_c^2} \frac{\partial E_z}{\partial y} \Rightarrow E_y = -\frac{j\beta n\pi}{b k_c^2} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$H_x = \frac{j\omega\varepsilon}{K_c^2}\frac{\partial E_z}{\partial y} \Rightarrow H_x = \frac{j\omega\varepsilon n\pi}{b\,K_c^2} B_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}$$

$$H_y = \frac{-j\omega\varepsilon}{K_c^2}\frac{\partial E_z}{\partial x} \Rightarrow H_y = -\frac{j\omega\varepsilon m\pi}{a\,K_c^2} B_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}$$

Propagation constant

$$\beta = \sqrt{k^2 - K_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

f_c, λ_g, V_p are the same of TE modes.

NOTE There are no TM_{00}, TM_{10}, TM_{01} modes

↳ if either m or n are equal to ∅ then fields are zero!

The lowest order TM mode is TM_{11}

$$f_{c_{11}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}$$

$$f_{c_{TM_{11}}} > f_{c_{TE_{10}}}$$

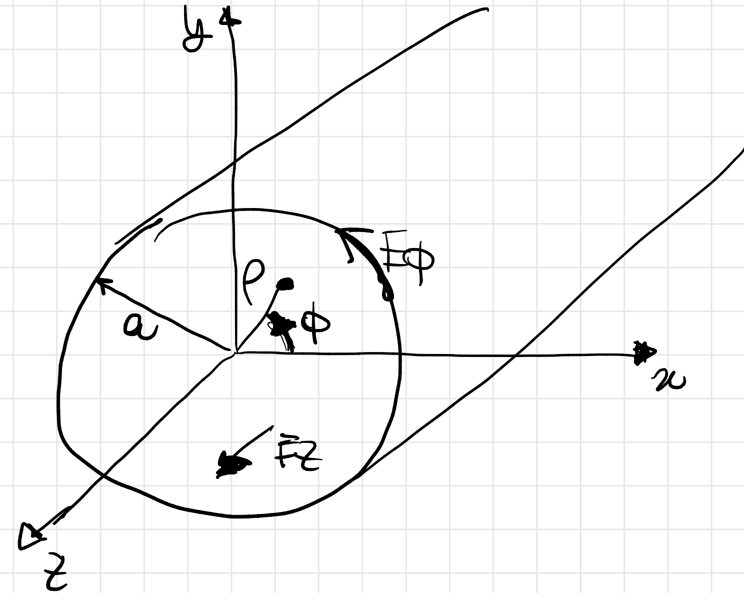
always the dominant mode

WAVE IMPEDANCE

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta\eta}{k}$$

CIRCULAR WAVEGUIDE (TM, TE)

$$\begin{cases} E_\rho = \dfrac{-j}{K_c^2}\left(\beta \dfrac{\partial E_z}{\partial \rho} + \dfrac{\omega\mu}{\rho}\dfrac{\partial H_z}{\partial \phi}\right) \\ E_\phi = \dfrac{-j}{K_c^2}\left(\dfrac{\beta}{\rho}\dfrac{\partial E_z}{\partial \phi} - \omega\mu \dfrac{\partial H_z}{\partial \rho}\right) \\ H_\rho = \dfrac{j}{K_c^2}\left(\dfrac{\omega\varepsilon}{\rho}\dfrac{\partial E_z}{\partial \phi} - \beta \dfrac{\partial H_z}{\partial \rho}\right) \\ H_\phi = \dfrac{-j}{K_c^2}\left(\omega\varepsilon \dfrac{\partial E_z}{\partial \rho} + \dfrac{\beta}{\rho}\dfrac{\partial H_z}{\partial \phi}\right) \end{cases}$$



$K_c^2 = k^2 - \beta^2$

$e^{-j\beta z}$ is implicit

TE Modes $E_z = 0$ $H_z(\rho, \phi, z) = h_z(\rho, \phi)\, e^{-j\beta z}$

$$\nabla^2 H_z + k^2 H_z = 0 \Rightarrow H_z(\rho, \phi, z) = (A \sin n\phi + B \cos n\phi) J_n$$

Imposing boundary condition $E_{tan} = 0 \Rightarrow E_\phi(\rho, \phi) = 0$ @ $\rho = a$

$$\boxed{K_{c_{mn}} = \frac{p'_{nm}}{a}}$$

p'_{nm} are the roots of the Bessel function $J'_n(a)$

The propagation constant

$$\beta_{nm} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{p'_{nm}}{a}\right)^2}$$

The cutoff frequency

$$f_{c_{nm}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{p'_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$$

The DOMINANT MODE for the circular waveguide is the $\boxed{TE_{11}}$

NOTE: There is NO TE_{10} mode but there is a TE_{01}

WAVE IMPEDANCE $\quad Z_{TE} = \frac{E_\rho}{H_\phi} = \frac{-E_\phi}{H_\rho} = \frac{\eta k}{\beta}$

TM Modes $\quad H_z = 0 \qquad E_z(\rho,\phi,z) = e_z(\rho,\phi)\, e^{-j\beta z}$

General solution of the wave equation is

$$e_z(\rho,\phi) = (A \sin n\phi + B \cos n\phi)\, J_n(k_c \rho)$$

The boundary condition is

$$E_z(\rho,\phi) = 0 \quad @ \ \rho = a$$

$$J_n(k_c a) = 0$$

$$\Downarrow$$

$$\boxed{k_c = \frac{p_{nm}}{a}}$$

p_{nm} is the m^{th} root of J_n

Propagation constant is

$$\beta_{nm} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{p_{nm}}{a}\right)^2}$$

Cutoff frequency is

$$f_{c\,nm} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{p_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$$

The 1st TM mode $\rightarrow$ TM_{01} mode

NOTE TM_{01} comes after TE_{11}

$$p_{01} = 2.405 > p'_{11} = 1.841$$

There is no TM_{10} mode $(m \geq 1)$

Wave impedance

$$Z_{TM} = \frac{E_\rho}{H_\phi} = \frac{-E_\phi}{H_\rho} = \frac{\eta\beta}{k}$$

FOR BOTH TE & TM modes attenuation is:

- DUE TO DIELECTRIC:

$$\alpha_d = \frac{k^2 \tan\delta}{2\beta}$$

- DUE TO CONDUCTOR (for 11 modes)

$$\alpha_c = \frac{R_s}{a k \eta \beta}\left(k_c^2 + \frac{k^2}{p_{11}'^2 - 1}\right)$$

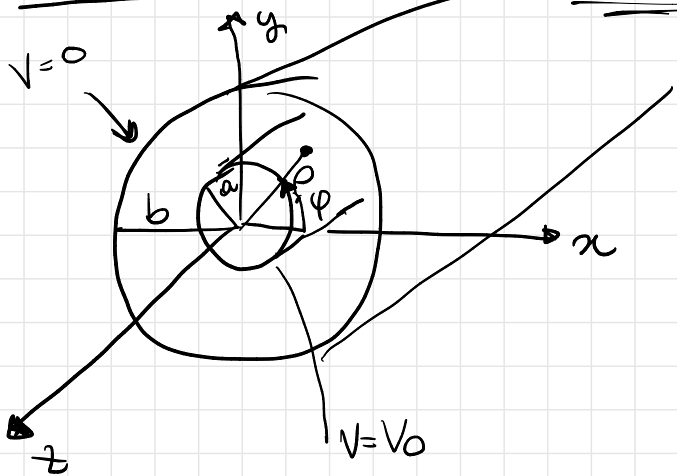
$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}}$$

TOTAL ATTENUATION:

$$\boxed{\alpha = \alpha_d + \alpha_c}$$

λ

COAXIAL LINES



TEM modes can propagate

$$\boxed{\Phi(\rho,\varphi) = \frac{V_0 \ln b/\rho}{\ln b/a}}$$

$$\downarrow$$

$$\boxed{\begin{aligned} \bar{e} &= -\nabla_t \, \Phi(x,y) \\ \bar{h} &= \frac{1}{Z_{TEM}} \, \hat{z} \times \bar{e} \end{aligned}}$$

TM and TE modes are typically cut-off!

If there were higher order mode the first one would be a TE11 ($k_c \simeq \frac{2}{a+b} \Rightarrow \beta$ and f_c)