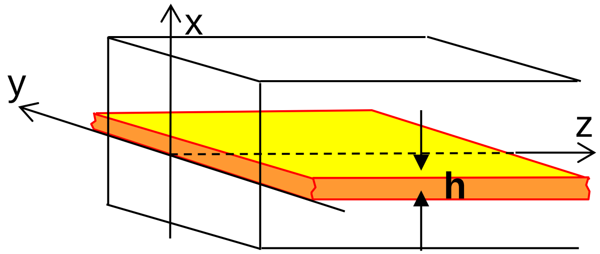
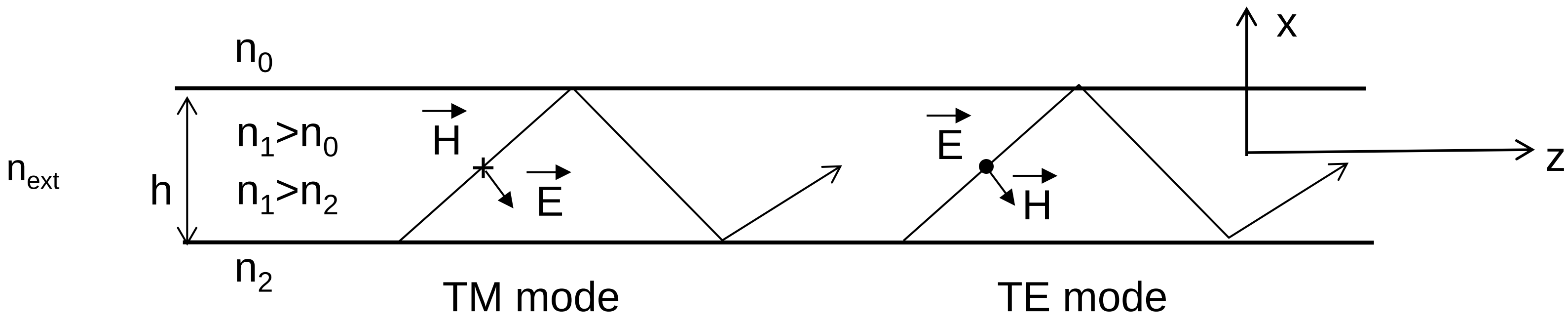
CHAPTER 2
Reminders on modes in optical waveguides (example of the slab waveguide)

Dominique PAGNOUX

With the support of the Erasmus+ Programme of the European Union

Slab waveguide



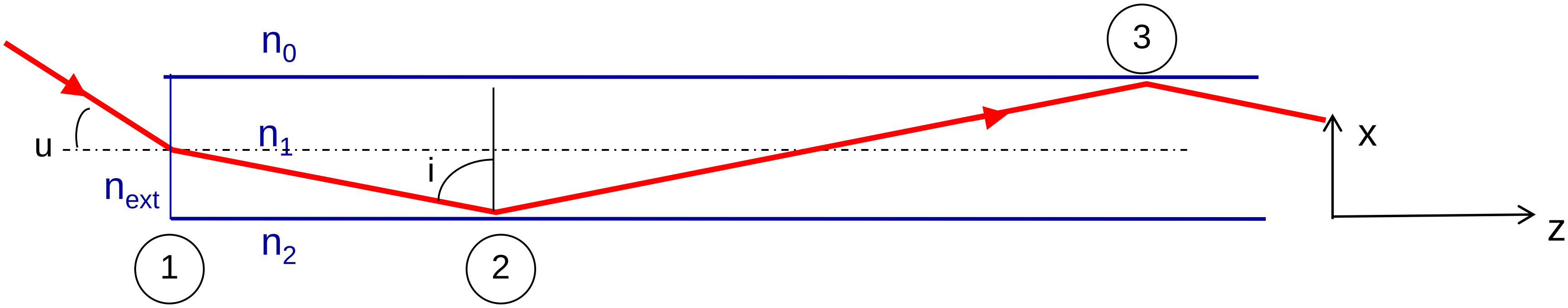


- slab waveguide → 3 layers of transparent dielectric materials with indices : n_0, n_1, et n_2
- constitution : substrate (index n_2), confinement waveguide (index n_1), superstrate (indice n_0).
- **guiding conditions : $n_1>n_2$ et $n_1> n_0$** (if $n_0 = n_2$: symmetrical waveguide)

In practical case, we often have: $n_0 = n_{ext} = 1$ (air)

For an optical fiber: $n_0 = n_2$

Guiding principle of light (1)



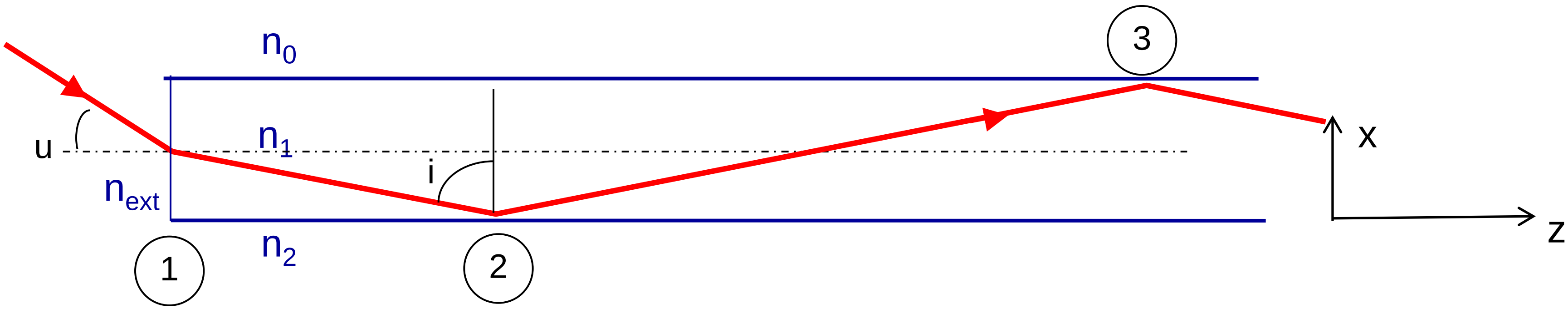
guiding if :
- réfraction at interface 1 (input face)
- total reflections at interfaces 2 , en 3

totale reflection if : $i > i_l = Arc\sin(n_2 / n_1)$

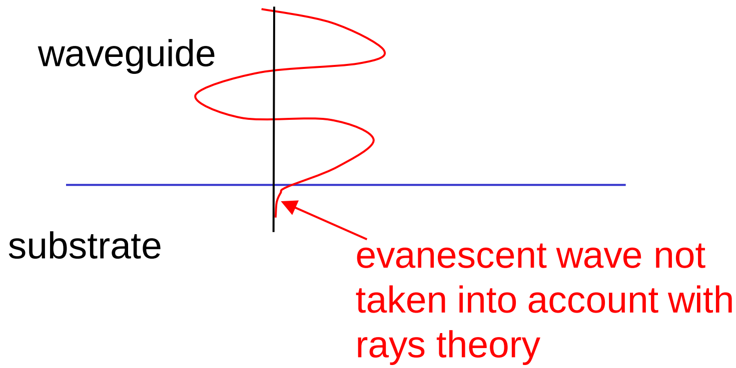
if $i=i_l$, u= u_{max} with sin u_{max} = numerical aperture(NA)

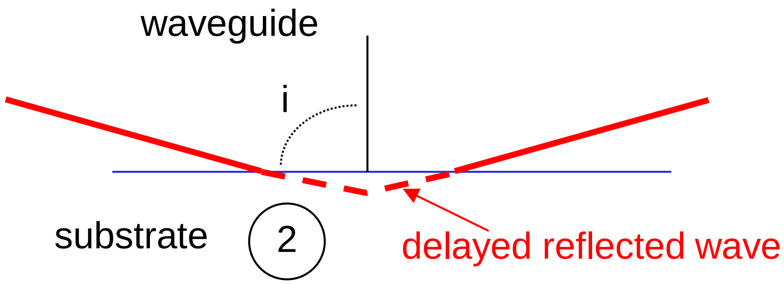
$$\boxed{NA = \frac{\sqrt{n_1^2 - n_2^2}}{n_{ext}}}$$

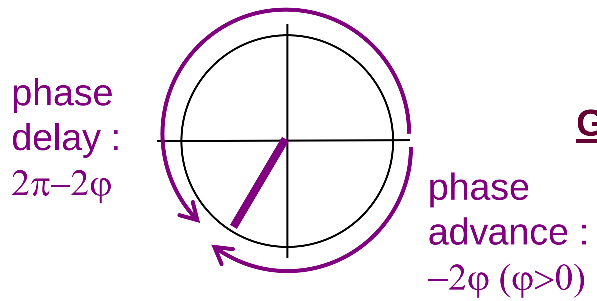
Guiding principle of light (2)



Warning : at the reflection points 2 and 3, the field does not cancel !

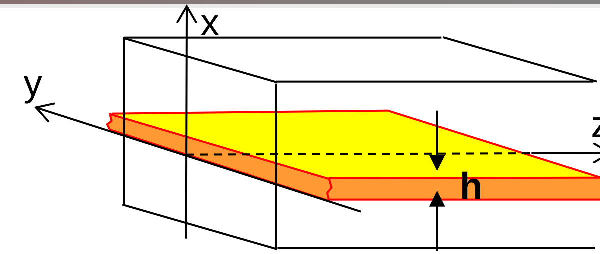


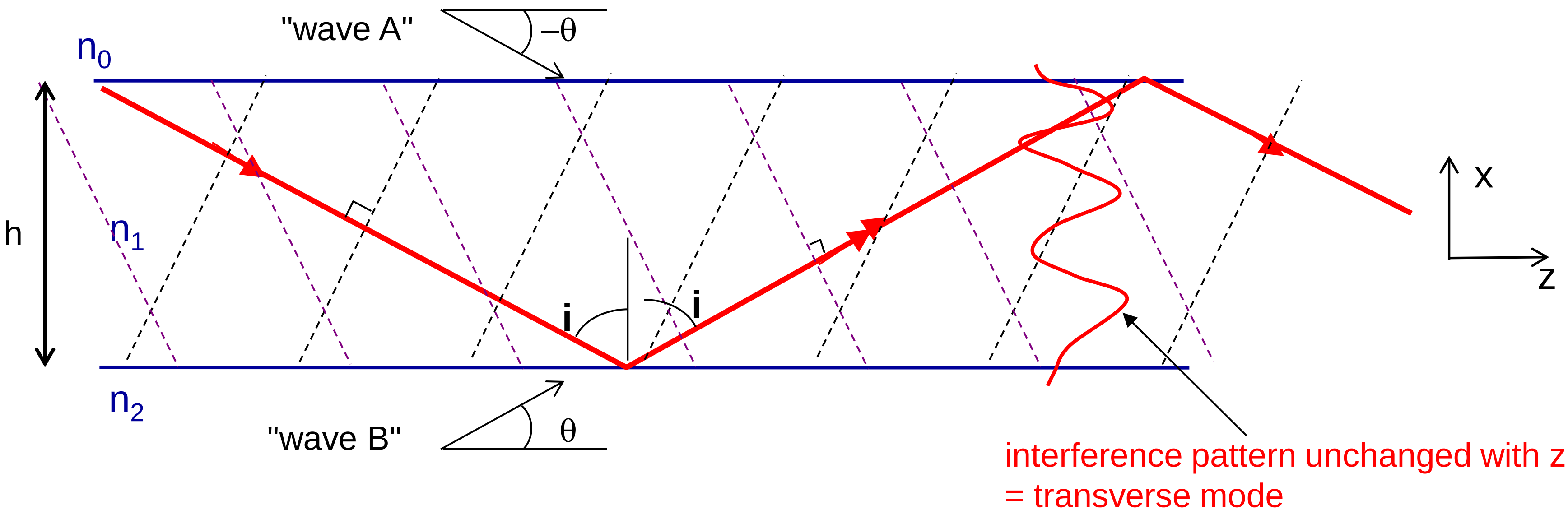


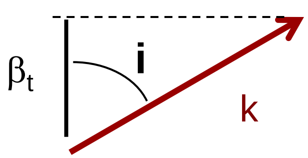


Goos Hänchen phase shift

Modes of a slab waveguide (1)





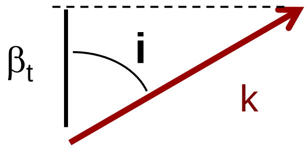


$$\beta_t = k\cos i$$

condition of existence of a mode : $\beta_t.2h - 2\varphi_{10} - 2\varphi_{12} = 2m\pi$

$-2\varphi_{10}$ and $-2\varphi_{12}$: Goos-Hänchen phase shifts at interfaces n_1/n_0 and n_1/n_2 , respectively

Modes of a slab waveguide (2)



$$\beta_t = k\cos i$$

One mode can exist if: $\beta_t .2h - 2\varphi_{10} \text{-} 2\varphi_{12} = 2m\pi$

thus : $\boxed{\cos i = \left(m\pi + \varphi_{10} + \varphi_{12}\right)\dfrac{1}{k_0 n_1 h}}$ given h, $\varphi_{10} = f(i)$, $\varphi_{12} = g(i)$

For each value of m → one value of i → one interference pattern → one transverse mode

➔ <u>angles i are discretised</u>

The guiding condition $i > i_l$ must be verified : $i > i_l \Rightarrow \cos i_l = \dfrac{\sqrt{n_1^2 - n_2^2}}{n_1} > \cos i = \left(m\pi + \varphi_{10} + \varphi_{12}\right)\dfrac{1}{k_0 n_1 h}$

$$\Leftrightarrow \boxed{m < \frac{1}{\pi}\left(k_0 .h.NA - \varphi_{10} - \varphi_{12}\right)}$$

➔ <u>the number of guided modes is limited</u>

With the support of the Erasmus+ Programme of the European Union

Modes of a slab waveguide (3)

$$m < \frac{1}{\pi}\left(k_0.h.NA - \varphi_{10} - \varphi_{12}\right) \qquad k_0 = \frac{2\pi}{\lambda_0}$$

$V = k_0.h.NA$

V= normalised spatial frequency of the guide, at λ_0

Remarks

➔ **the number of guided modes increases if V increases, i.e. :**

- ➔ if h ↗
- ➔ if NA ↗
- ➔ if, for a given waveguide, λ ↘

➔ $\text{if} \quad h < \frac{\varphi_{10} + \varphi_{12}}{k_0.NA} = h_{\lim} \quad \text{then} \quad m < 0$ no guided mode

➔ if 0 < m < 1 : only one guided mode (fundamental mode) → single mode regime

structure of the modes → EM approach : case of TE modes of a slab waveguide

Maxwell equations

Electric field, in harmonic regime, in a waveguide: $\vec{\mathcal{E}}(x,y,z) = \Re e\left[\vec{E}(x,y).e^{j(\omega t-\beta z)}\right]$ (V/m)

$$\vec{E}(x,y) = \left| \begin{array}{l} E_x(x,y).\overrightarrow{ex} \\ E_y(x,y).\overrightarrow{ey} \\ E_z(x,y).\overrightarrow{ez} \end{array} \right.$$

$\boxed{\vec{E}(x,y) \text{ : one mode of the guide}}$

β= axial propagation constant (along z)

Associated magnetic field: $\vec{\mathcal{H}}(x,y,z) = \Re e\left[\vec{H}(x,y).e^{j(\omega t-\beta z)}\right]$ (A/m)

Maxwell equations, in a linear, isotropic homogeneous medium with no electric charge nor current densities :

$$curl\vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t} \quad \text{with } \vec{\mathcal{B}} = \mu\vec{\mathcal{H}} \qquad \mu = \mu_0 = 4\pi\ 10^{-7}\ \text{H/m} \qquad (1)$$

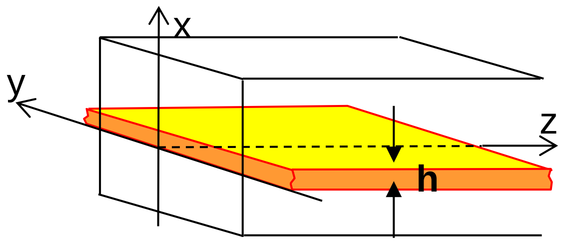
$$curl\vec{H} = \varepsilon\frac{\partial \vec{E}}{\partial t} \quad \text{with } \varepsilon = \varepsilon_0\varepsilon_r \text{ et } \varepsilon_0 = \frac{1}{36\pi}10^{-9}\ \text{F/m} \qquad (2)$$

$curl\vec{U} = \nabla \wedge \vec{U}$

($\vec{U}$ being any vector)

one starts from Maxwell equations

- harmonic form of the fields ➔ $\dfrac{\partial(X)}{\partial t} = j\omega X$ and $\dfrac{\partial(X)}{\partial z} = -j\beta X$



- slab waveguide supposed to have translation symmetry along y (infinite extension in this direction)

  ➔ components of the fields independent of y ➔ $\dfrac{\partial(X)}{\partial y} = 0$

- seak of TE modes ➔ $E_z = 0$

In these conditions, (1) and (2) lead to : $\vec{E}(x,y) = \vec{E}(x) \left| \begin{array}{l} E_x = 0 \\ E_y \neq 0 \\ E_z = 0 \end{array} \right.$ et $\vec{H}(x,y) = \vec{H}(x) \left| \begin{array}{l} H_x = \dfrac{-\beta}{\omega\mu_0} E_y \\ H_y = 0 \\ H_z = \dfrac{j}{\omega\mu_0} \dfrac{\partial E_y}{\partial x} \end{array} \right.$

$$\vec{\mathcal{E}}(x,y,z) = E_y(x).e^{j(\omega t - \beta z)}.\vec{e}_y$$

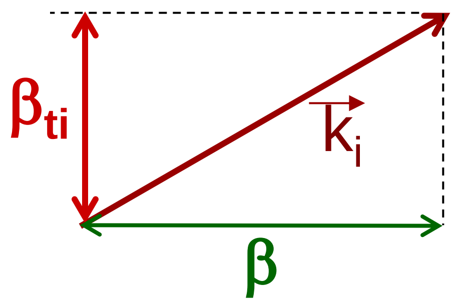
structure of the modes → EM approach : case of TE modes of the slab waveguide

Expression of the fields :

From (1) and (2) → propagation equation (= or Helmotz equation) : $\Delta\vec{\mathcal{E}} + k_0^2 n_i^2 \vec{\mathcal{E}} = \vec{0}$ (3)

with Δ vectorial laplacian : $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Using (3) the expression of $\vec{\mathcal{E}}$ → $\frac{\partial^2 E_y}{\partial x^2} + \underbrace{\left(k_0^2 n_i^2 - \beta^2\right)}_{\beta_{ti}^2} E_y = 0$ (4)



$\vec{k_i} = n_i \vec{k_0}$

$$\boxed{\beta_{ti}^2 = k_0^2 n_i^2 - \beta^2}$$

β_{ti} = <u>transverse</u> propagation constant

$\boxed{\text{one can write } \beta = k_0.n_e}$ with <u>n_e the effective index</u> of the mode

→ $\beta_{ti}^2 = k_0^2 (n_i^2 - n_e^2)$

structure of the modes → EM approach : case of TE modes of the slab waveguide

Expression of the fields :

solution of $\dfrac{\partial^2 E_y}{\partial x^2} + \beta_{ti}^2 E_y = 0$ with $\beta_{ti}^2 = k_0^2 (n_i^2 - n_e^2)$:

→ $\boxed{E_y = A_i.e^{-\gamma_i x} + B_i.e^{+\gamma_i x}}$ and $\gamma_i = j\beta_{ti} \Rightarrow \beta_{ti}^2 = -\gamma_i^2$

- if γ_i real (avec $B_i = 0$) → decreasing exponential solution (media n_0 et n_2)

  $E_y = A_i.e^{-\gamma_i x}$ then $\beta_{ti}^2 < 0$ → $n_0 < n_e$ and $n_2 < n_e$

- if γ_i pure imaginary → sinusoidal solution (medium n_1)

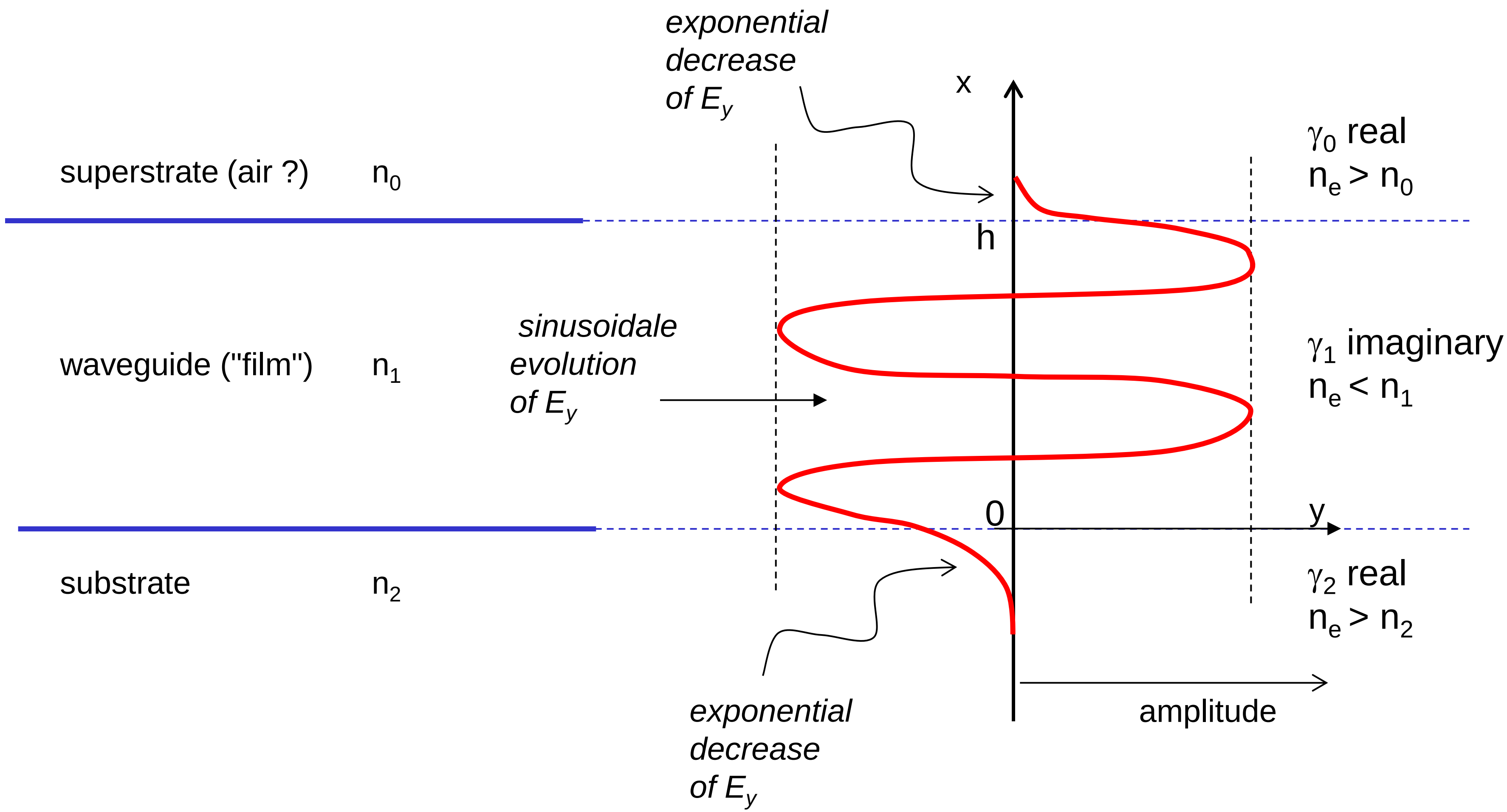
  $E_y = C.\cos(\beta_{ti} x + \Phi)$ then $\beta_{ti}^2 > 0$ → $n_1 > n_e$

→ guiding condition : max (n_0, n_2) < n_e < n_1

if $n_e < n_0$ and/or $n_e < n_2$: non guided superstrate and/or substrate modes

structure of the modes → EM approach : case of TE modes of the slab waveguide

field distribution of E_y along x:



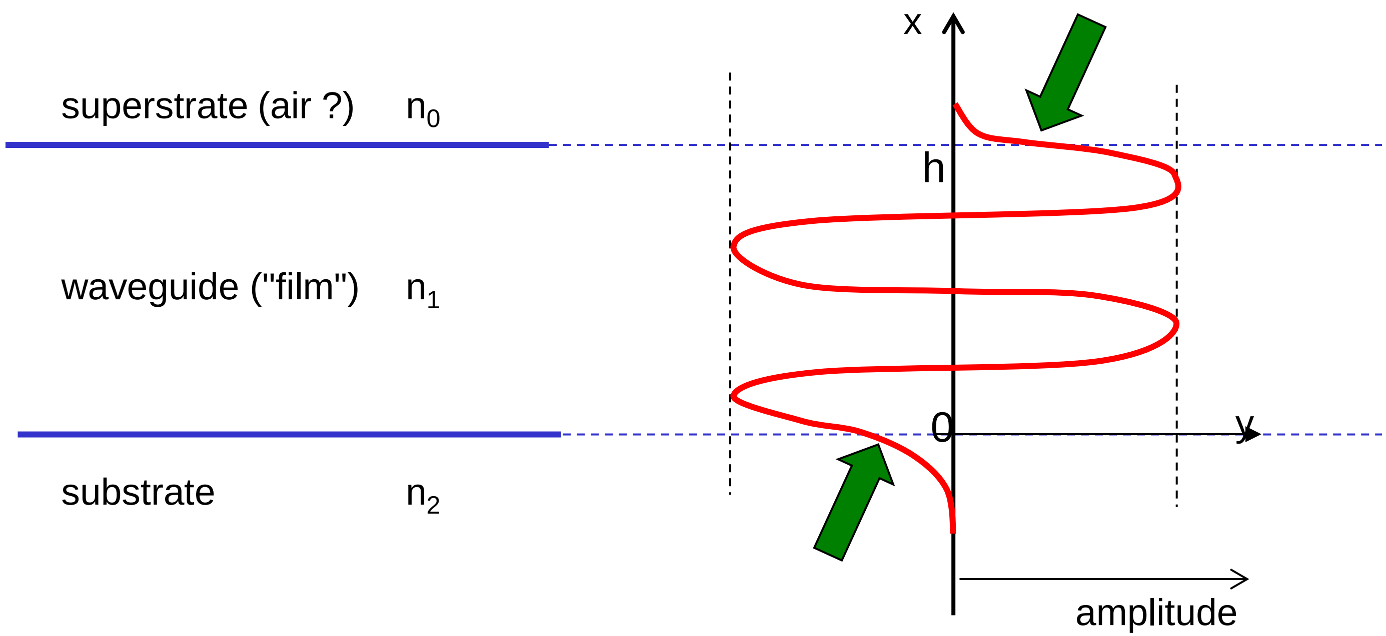
dispersion relationship

for a given mode, in a given waveguide : for each λ (or ν, or ω…) ➔ an associated value of β (or n_e)

curves β=f(ω) or n_e =f(ω) or ω=f(β) or β=f(V)…

= DISPERSION CURVES of the mode

for obtaining the dispersion relationship of a mode
➔ one must write the continuity conditions of the tangential components of the fields and of their derivatives at the interfaces



With the support of the Erasmus+ Programme of the European Union
Université de Limoges

dispersion relationship

for a given mode, in a given waveguide : for each λ (or ν, or ω…) ➔ an associated value of β (or n_e)

curves β=f(ω) or n_e =f(ω) or ω=f(β) or β=f(V)…

= DISPERSION CURVES of the mode

for obtaining the dispersion relationship of a mode
➔ one must write the continuity conditions of the tangential components of the fields and of their derivatives at the interfaces

In the example of TE modes of the considered infinite slab waveguide, one must write :

$$E_y(x{=}0)\Big|_{\text{in the substrate}} = E_y(x{=}0)\Big|_{\text{in the waveguide}} \qquad E_y(x{=}h)\Big|_{\text{in the waveguide}} = E_y(x{=}h)\Big|_{\text{in the superstrate}}$$

$$\frac{\partial E_y}{\partial x}(x{=}0)\Bigg|_{\text{in the substrate}} = \frac{\partial E_y}{\partial x}(x{=}0)\Bigg|_{\text{in the waveguide}} \qquad \frac{\partial E_y}{\partial x}(x{=}h)\Bigg|_{\text{in the waveguide}} = \frac{\partial E_y}{\partial x}(x{=}h)\Bigg|_{\text{in the superstrate}}$$

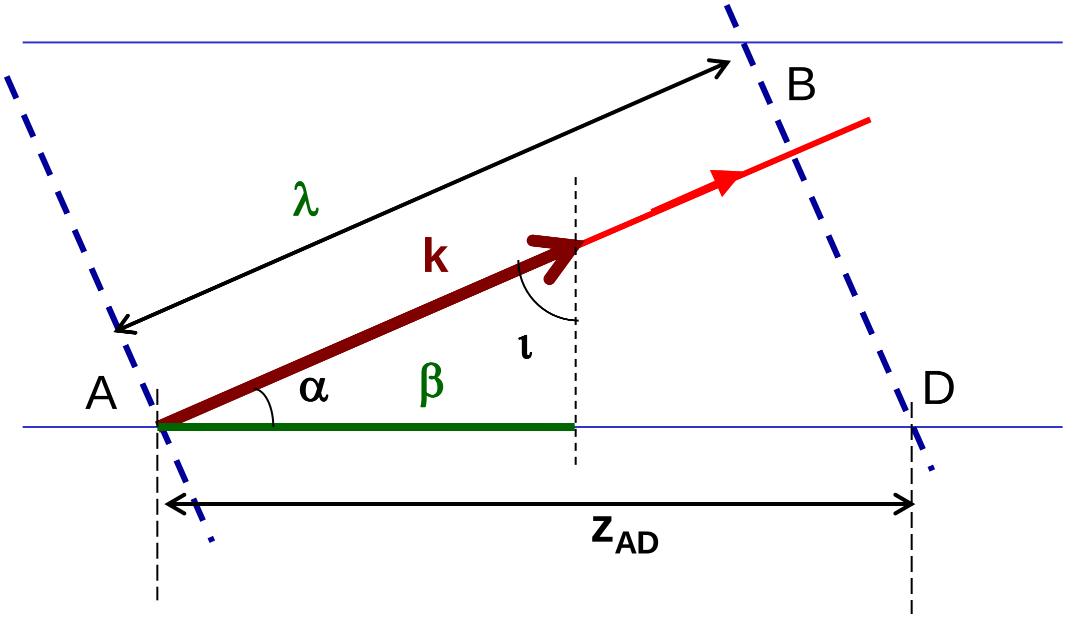
➔ This leads to : $\beta_t.h = \varphi_{10} + \varphi_{12} + m\pi$ with :

$$\beta_t = k_0\sqrt{n_1^2 - n_e^2} \qquad \varphi_{10} = \mathrm{Atan}\sqrt{\frac{n_e^2 - n_0^2}{n_1^2 - n_e^2}} \qquad \varphi_{12} = \mathrm{Atan}\sqrt{\frac{n_e^2 - n_2^2}{n_1^2 - n_e^2}}$$

numerical resolution ➔ n_e=f(λ) or β=f(ω) or …

phase velocity vφ

propagation velocity of a WAVE FRONT, in the z direction



$$AB = \lambda$$

$$\rightarrow \quad v_\varphi = \frac{z_{AD}}{T}$$

with : $\cos\alpha = \dfrac{\lambda}{z_{AD}} = \sin i = \dfrac{\beta}{k} \quad \Rightarrow z_{AD} = \dfrac{\lambda k}{\beta} = \dfrac{2\pi}{\beta}$

$T = \dfrac{2\pi}{\omega}$

$$\boxed{v_\varphi = \frac{\omega}{\beta}}$$

ω: $k_0.c$

β: $k_0.n_e$

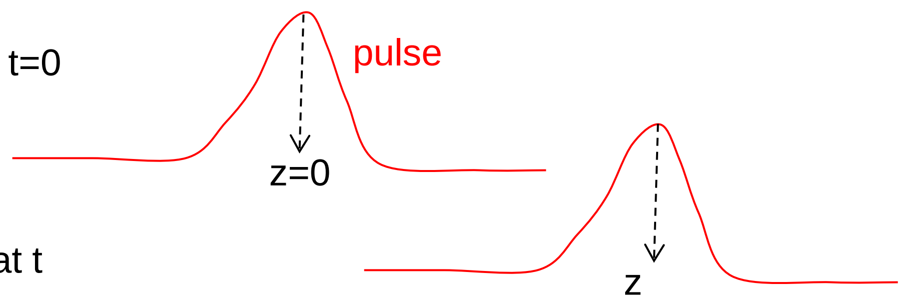
$$\rightarrow \quad \boxed{v_\varphi = \frac{c}{n_e}}$$

Group velocity v_g

propagation velocity of a WAVE PACKET, in the z direction (velocity of energy)

at t=0

at t



the peak of the pulse propagates at the speed :

$$v_g = \frac{z}{t}$$

at the peak of the pulse → all the chromatic components are in phase ➔ $\omega t-\beta z = cte \quad \forall\omega$

$$\frac{d}{d\omega}(\omega t-\beta z)=0 \quad \Rightarrow \quad \omega\underbrace{\frac{dt}{d\omega}}_{0}+t\underbrace{\frac{d\omega}{d\omega}}_{1}-\left(\beta\underbrace{\frac{dz}{d\omega}}_{0}+z\frac{d\beta}{d\omega}\right)=0$$

$$\Rightarrow \quad t-z\frac{d\beta}{d\omega}=0 \quad \Leftrightarrow \quad \frac{d\omega}{d\beta}=\frac{z}{t}=v_g$$

$$\boxed{v_g=\frac{d\omega}{d\beta}=\frac{c}{N_g}}$$

N_g : group index

calculation of N_g versus the wavelength λ_0 (in the vacuum)

$$\frac{1}{v_g} = \frac{N_g}{c} = \frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(k_0 n_e\right)$$

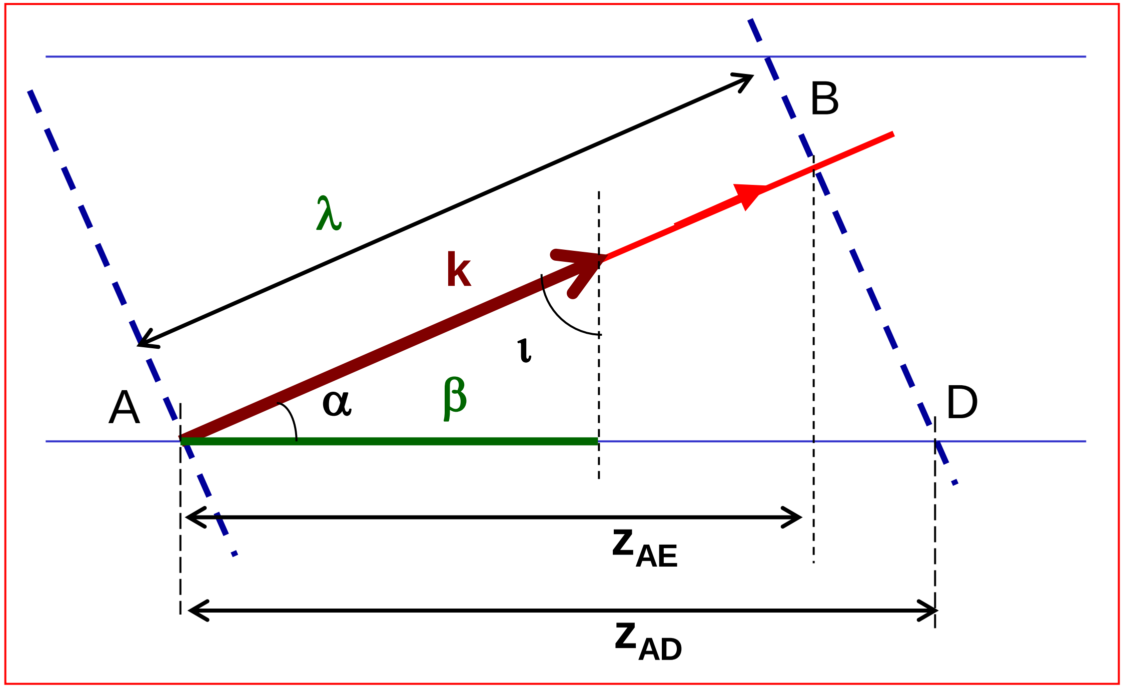
$$= \frac{dk_0}{d\omega}.n_e + k_0 \frac{dn_e}{d\omega} \quad \text{avec } k_0 = \frac{\omega}{c}$$

$$= \frac{1}{c}.n_e + \frac{2\pi}{\lambda_0}\frac{dn_e}{d\lambda_0}\frac{d\lambda_0}{d\omega} \qquad (1)$$

with $\omega = \dfrac{2\pi.c}{\lambda_0}$, one obtains : $\dfrac{d\lambda_0}{d\omega} = -\dfrac{\lambda_0^2}{2\pi.c}$ and (1) becomes $\dfrac{\mathrm{N_g}}{c} = \dfrac{n_e}{c} - \dfrac{\lambda_0}{c}\dfrac{dn_e}{d\lambda_0}$

$$\boxed{\mathrm{N_g} = n_e - \lambda_0 \frac{dn_e}{d\lambda_0}}$$

Approximative comparison between v_φ and v_g versus modes orders



$$v_\varphi \approx \frac{z_{AD}}{T} = \frac{AB/\cos\alpha}{\lambda_0/c} = \frac{(\lambda_0/n_1)/\cos\alpha}{\lambda_0/c} = \frac{c}{n_1.\sin i}$$

$$v_g \approx \frac{z_{AE}}{T} = \frac{(\lambda_0/n_1).\sin i}{\lambda_0/c} = \frac{c}{n_1}.\sin i$$

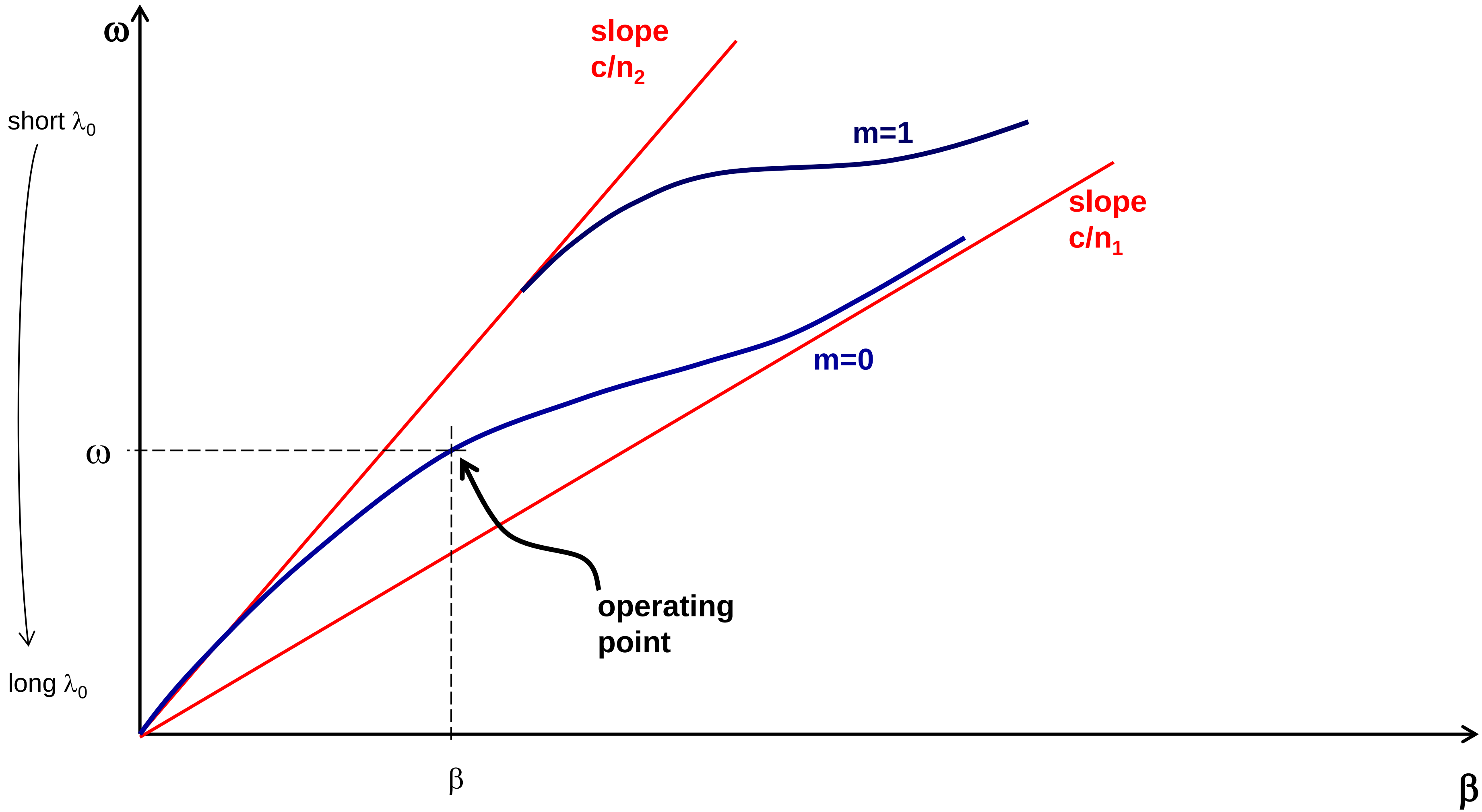
$$v_\varphi.v_g \approx \left(\frac{c}{n_1}\right)^2 = v^2 = \text{cte}$$

$$v_g = \frac{c}{N_g} \approx \frac{c}{n_1}.\sin i \quad < \quad \frac{c}{n_1} \quad < \quad v_\varphi = \frac{c}{n_e} \approx \frac{c}{n_1.\sin i} \quad \Rightarrow N_g > n_1 > n_e$$

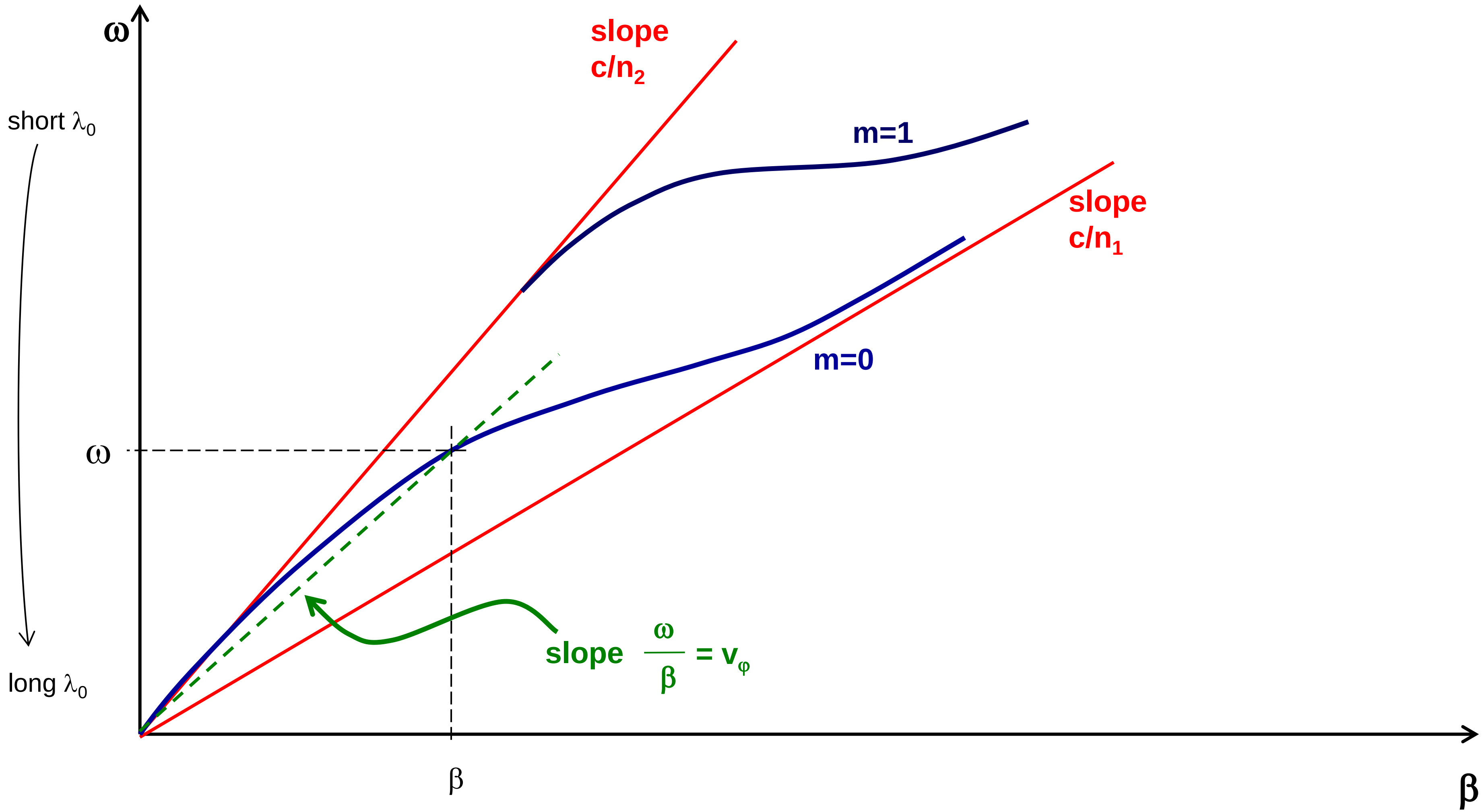
$$N_g > n_e \quad \Rightarrow \quad \frac{dn_e}{d\lambda} < 0$$

if i ↗ (very inclined light ray = low order modes) => v_φ ↘ et v_g ↗

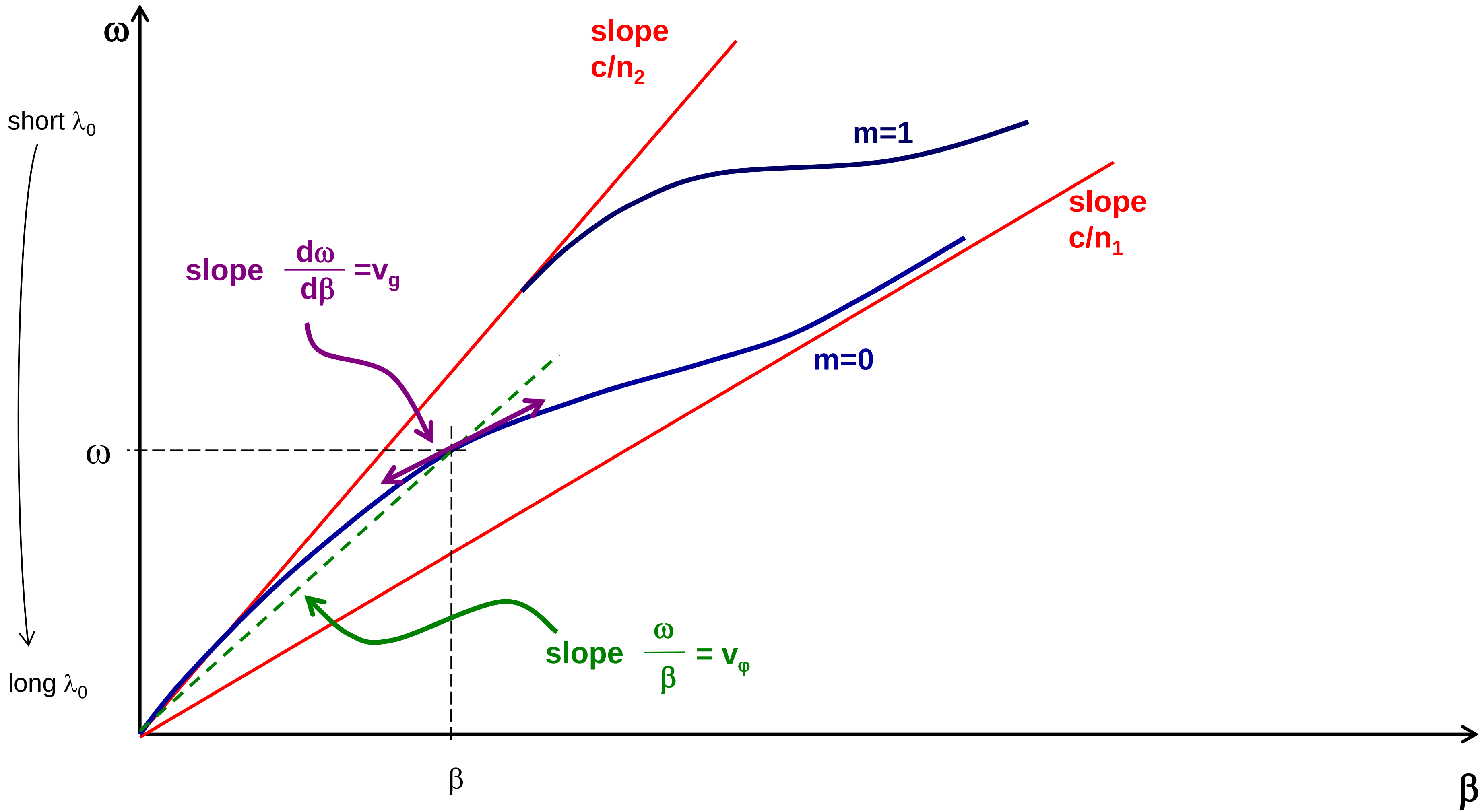
dispersion curves ω=f(β)



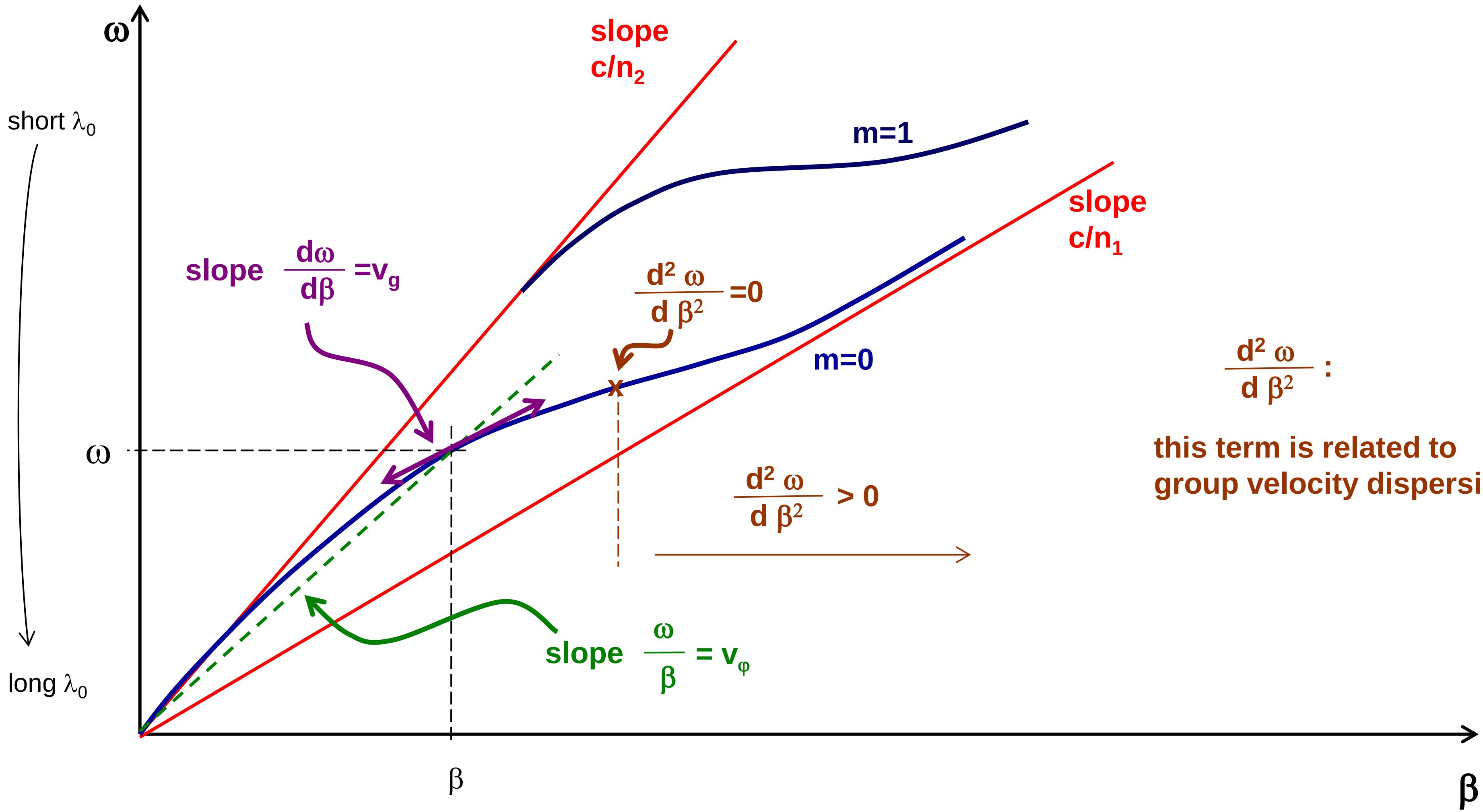
dispersion curves $\omega=f(\beta)$



dispersion curves ω=f(β)



dispersion curves $\omega=f(\beta)$

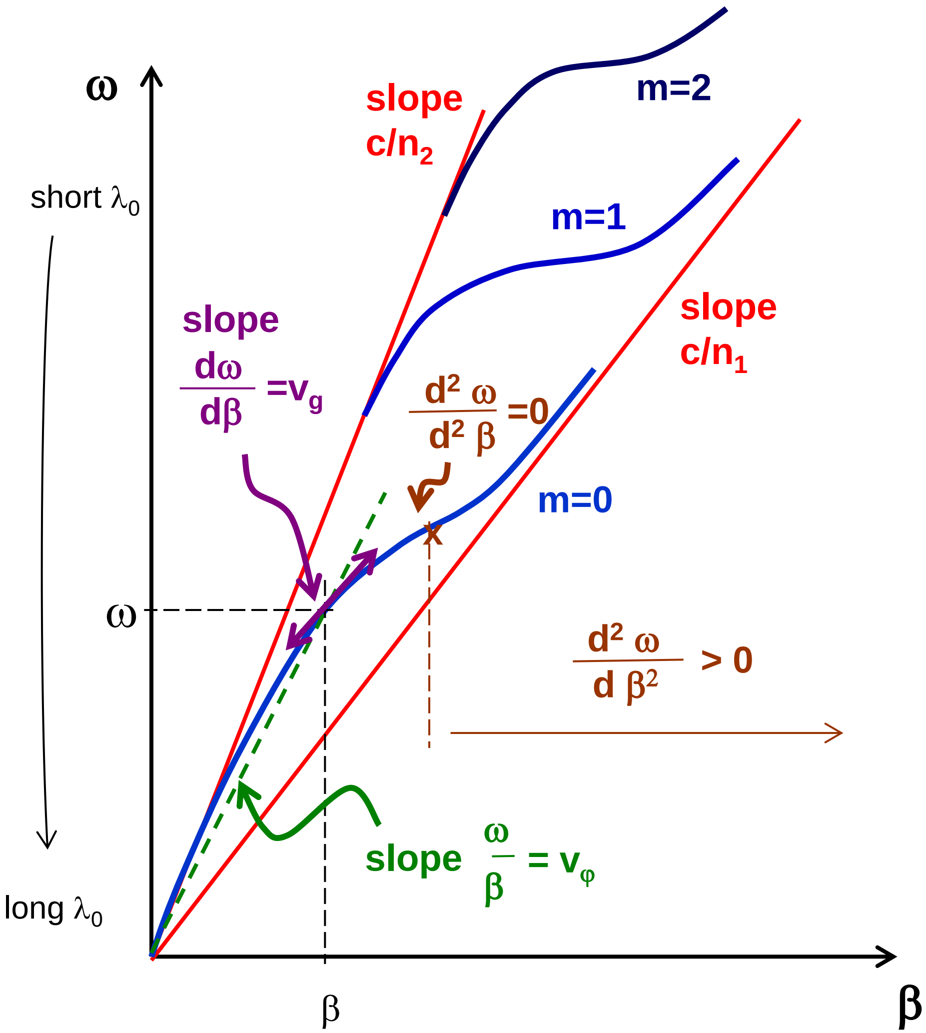


$\frac{d^2\omega}{d\beta^2}$:

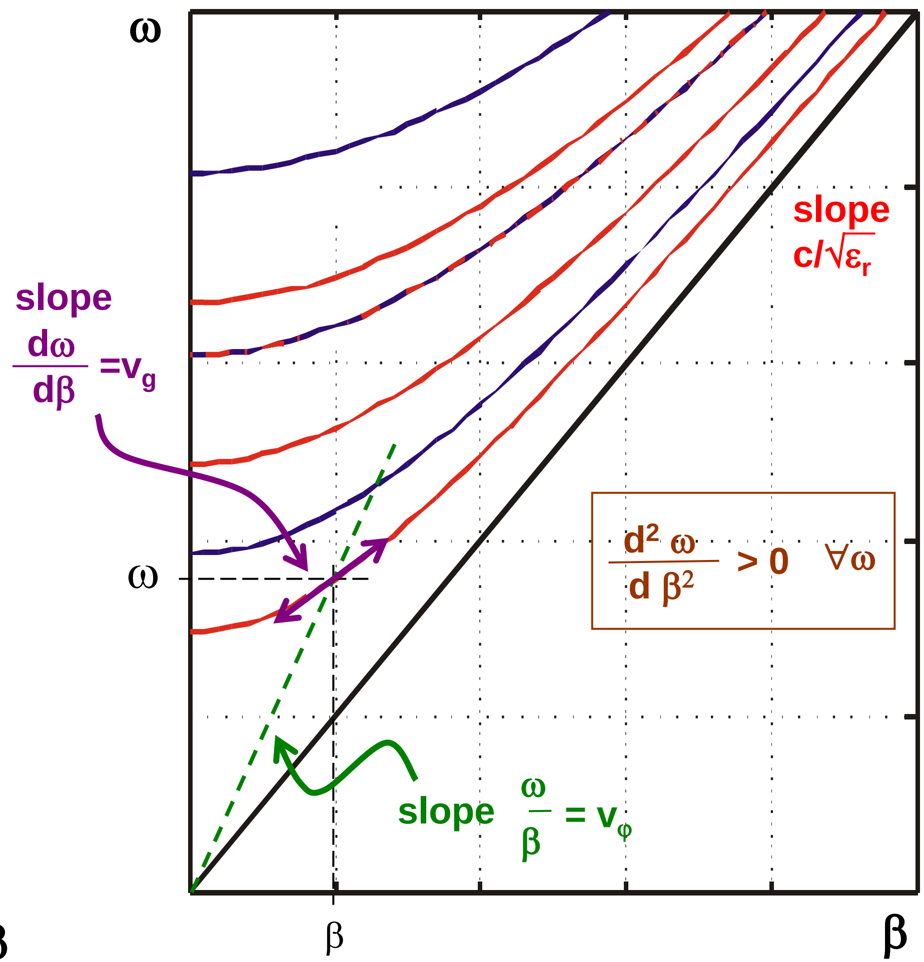
this term is related to group velocity dispersion

remark : the sign of $\frac{d^2\omega}{d\beta^2}$ is the opposite of that of $\frac{d^2\beta}{d\omega^2}$

dispersion curves ω=f(β) : comparison with the case of a metallic waveguide

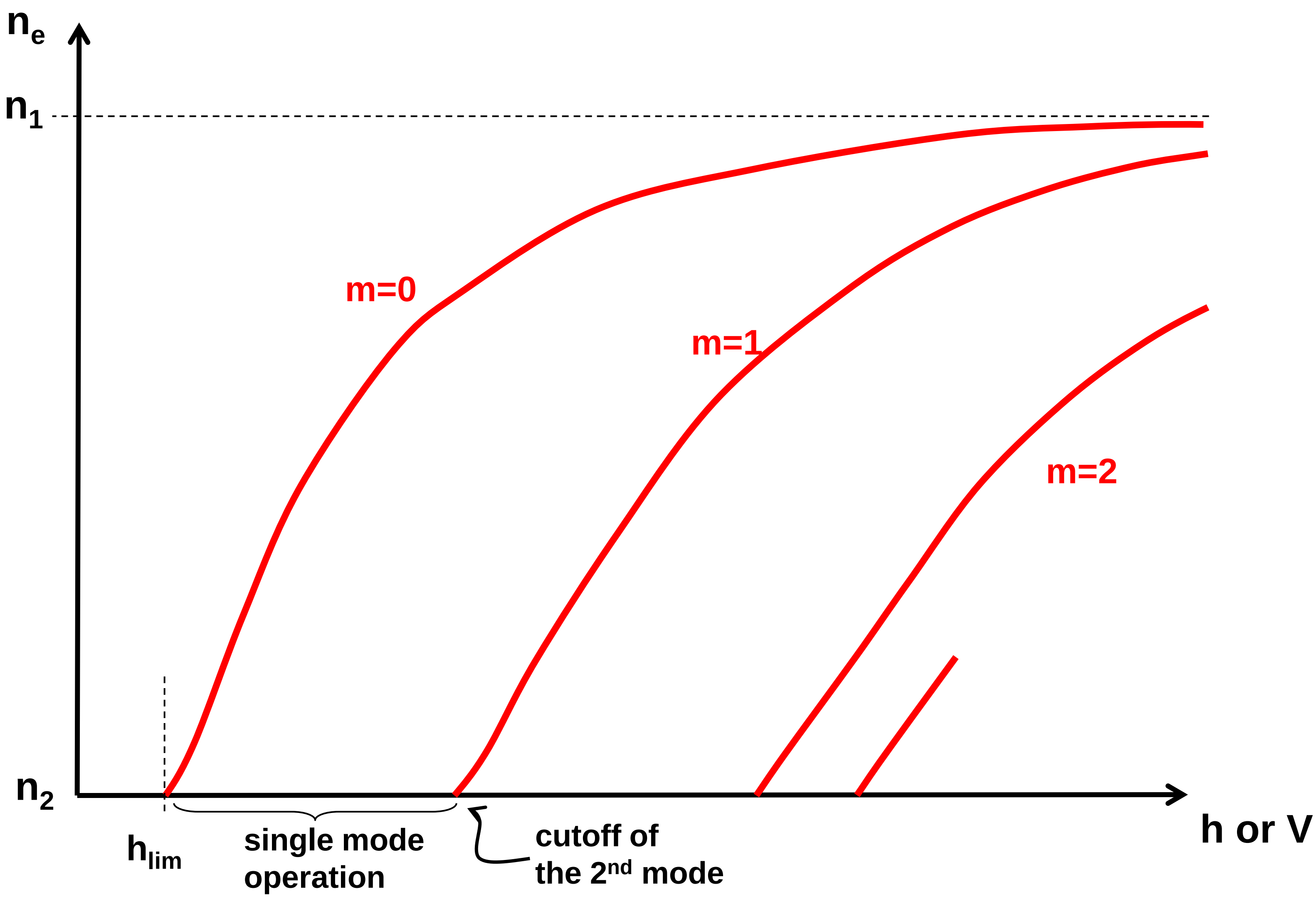


dielectric waveguide

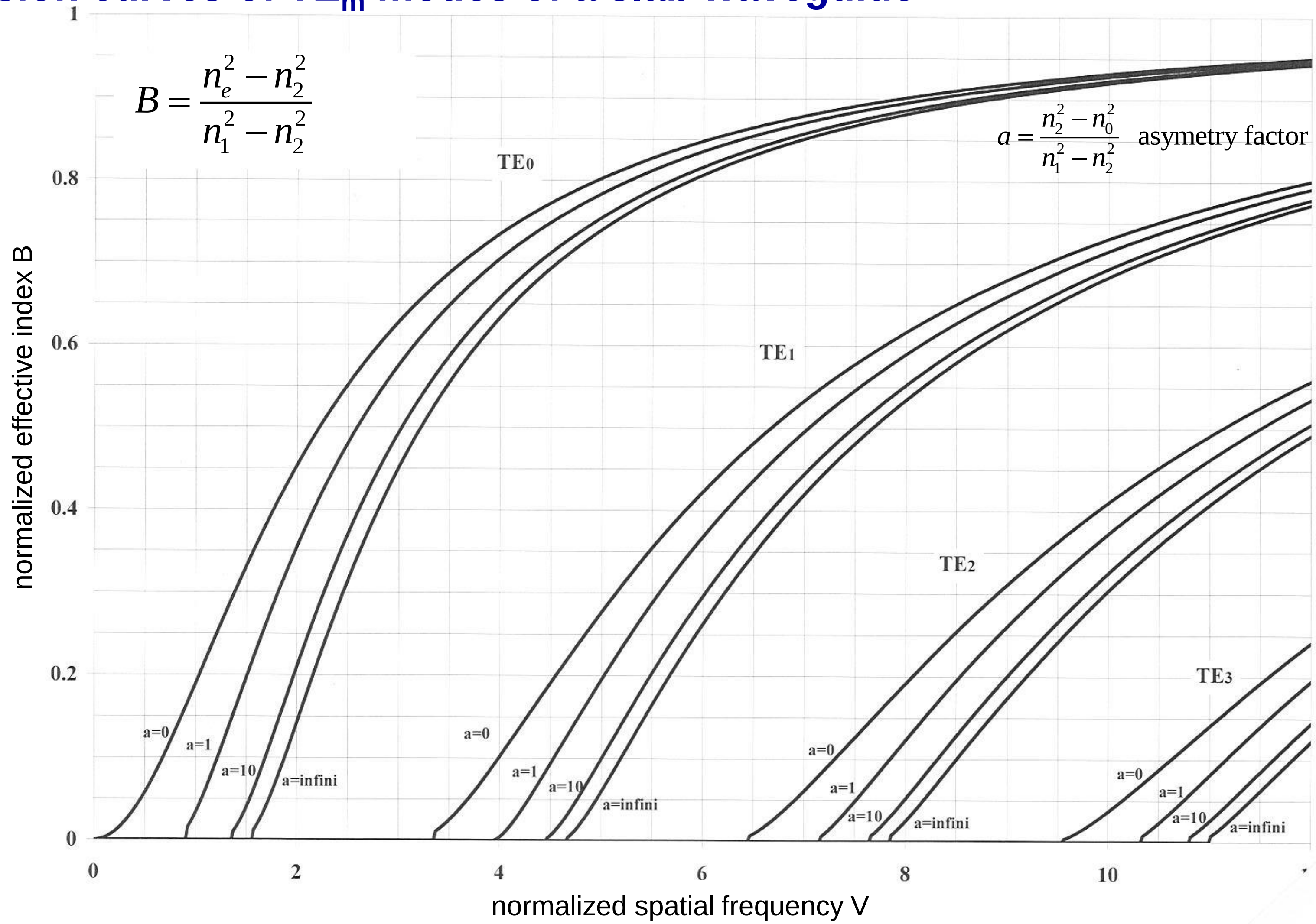


metallic waveguide

other representations of dispersion curves for a dielectric waveguide



dispersion curves of TE_m modes of a slab waveguide



case of TM modes

Previous calculations can be conducted as well for TM modes (H_z=0)

→ calculation of components E_z and H_z

→ continuity conditions for tangential components

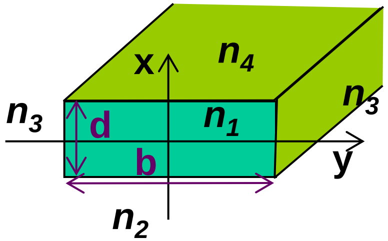
→ dispersion equation : $$\beta_t.h = \varphi_{10} + \varphi_{12} + m\pi$$

$$\beta_t = k_0\sqrt{n_1^2 - n_e^2} \qquad \varphi_{10} = \mathrm{Atan}\left[\boxed{\left(\frac{n_1}{n_0}\right)^2}\sqrt{\frac{n_e^2 - n_0^2}{n_1^2 - n_e^2}}\right] \qquad \varphi_{12} = \mathrm{Atan}\left[\boxed{\left(\frac{n_1}{n_2}\right)^2}\sqrt{\frac{n_e^2 - n_2^2}{n_1^2 - n_e^2}}\right]$$

→ **effective index of the TM_m mode different from that of the TE_m mode**

BUT if n_0 → n_1 and if n_2 → n_1 then φ_{10}(TE) → φ_{10}(TM) and φ_{12}(TE) → φ_{12}(TM)
→very similar dispersion curves
→ quasi degenerated modes

case of modes of rectangular dielectric waveguides (non infinite in the y direction)



… not addressed in this course

End of chapter 2