Chapter 1

- What is the visible wavelength? Table 1.1

400-750nm approximately.

- Maxwell Equations... explain EF, MF
- How the magnetic field and electric field are related... the vectors, etc
- Poynting vector, etc
- Simple case in case of a plane wave... EF, MF, etc...

Maxwell equations are differential equations that provide a mathematical model for electromagnetic dynamics. They describe how EF and MF are generated by charges, currents, and changes of the fields, and demonstrate how fluctuations in EM fields propagate.

An EF is a physical field that either attracts or repels electrically charged particles.

A MF is a vector field that describes the magnetic influence on moving electric charges, electric currents, and magnetic materials.

Both EF and MF are associated with each other following Maxwell Equations (In case of plane monochromatic waves is E = mu H x k)

The Poynting vector W = E x H allows to quantify the instantaneous power density of the electromagnetic fields in the propagation direction (unit of measure is W/m2). However, it makes more sense to calculate the **average** Poynting vector <W> = ½ E x H

A plane monochromatic wave is a solution of the wave equation:
Ex(z,t)=E0 cos(w0 t + beta0 z + phi0) then using the relation above
Hy(z,t) = -E0 mu^-1 cos(w0 t + beta0 z + phi0) and Poynting vector
W = -E0^2 mu^-1 cos^2(w0 t + beta0 z + phi0) so
<W> = -½ E0^2 mu^-1

- Definition of polarization (instantaneous and non-instantaneous... isotropic/no isotropic)
-What is nonlinear polarization?

Polarization describes how a material reacts to an applied EF, as well as how the material changes the EF. It depends on the material electrical susceptibility tensor.

Linear case (Weak EF):
Plin(t) = Eps0 Chi(t) * EF(t) -> Plin(f) = Eps0 Chi(f) EF(f)
Then, D(f) = Eps0 (1+Chi(f)) EF = Eps0 EpsR EF = Eps0 n^2 EF
Where EpsR is the relative electric permittivity and "n" the linear refractive index.

Isotropic material: The material reaction doesn't depend on the source orientation (Chi becomes a diagonal matrix = Chi(r,t)), so Plin and EF parallel!

If the medium replies instantaneously: Chi(f) = Chi Dirac(f), then also Plin(t) = Eps0 Chi(t) EF(t) because an impulse in frequency is a constant in time.

Nonlinear case (Very Intense EF):
P = Plin + Pnl (due to 3rd order nonlinearity, causes Third Harmonic Generation, Kerr Effect, Raman scattering…)
If E = Ex -> Pnl = Eps0 Chi^3 Ex^3

- Wave equation -> Must know the equation

From Maxwell's equations when applying divergence, we obtain the Wave Equation. All the non-trivial solutions of this equation are called waves.

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\mathbf{E} + \frac{1}{\varepsilon_0}\mathbf{P}\right)$$

Ideal reference case:

- Linear medium -> Pnl = 0 -> P=Plin -> P and E parallel
- Homogeneous material -> Chi(r,f)=Chi(f)
- Lossless material -> Chi(f) is real
- Monochromatic E = E0 cos(w0 t) -> Chi(f)=Chi(f0)
- Only one nonzero component: E = Ex
- EF only z-dependent spatially: E(r,t) = E(z,t)

Basically: E(z,t) = E0 cos(w0 t + beta0 z + phi0), P(f0)=Eps0 Chi(f0) E(f0)

-Refractive index - Refractive index is a matrix, scalar, vector??

n(r,f) = sqrt(1+Chi(r,f)) = sqrt(EpsR(r,f))

The refractive index indicates how light interacts with a medium, for example the amount of light bending when crossing the interface, the light speed inside the medium… Depending on the material electrical susceptibility

(Chi(r,f)), the refractive index is a scalar or a matrix. If the material is isotropic, n is a scalar, otherwise is a matrix.

- How refractive index is related to c
- Table 1.3 - Quantities should be known

Dispersion relation relates c to n

$$\beta_0^2 = \frac{\omega_0^2}{c^2}\left[1 + \widehat{\chi}(f_0)\right] = \frac{\omega_0^2}{c^2}\, n^2(f_0),$$

	free space	medium
Refractive index	$n = 1$	$n(f_0)$
Speed of light	c	$c/n(f_0)$
Wavelength	$\lambda_0 = c/f_0$	$\lambda = c/[f_0 n(f_0)] = \lambda_0/n(f_0)$
Propagation constant	$\beta_0 = \pm 2\pi/\lambda_0$	$\beta_0 = \pm 2\pi/\lambda = \pm 2\pi n(f_0)/\lambda_0$

- Second harmonic generation/third harmonic generation
- Centrosymmetric material -> cannot have x(2)
- Optical parametric amplification (OPA)
- What is Kerr effect?
- We must be able to use table 1.2
- Self-phase modulation

When the EF is strong enough, P is no longer proportional to E. To analyse P, we expand it using Taylor series w.r.t. E. Then, we obtain P = Plin + Pnl. When signal is weak, Pnl disappears.

Using the Volterra series expansion (as the reply of the media is not instantaneous),

P(r,t) = Eps0 Chi E + Eps0 Chi2 E^2 + Eps0 Chi3 E^3…

The nonlinear **quadratic** term is responsible of the second-harmonic generation (SHG), optical parametric amplification (OPA) and other effects. OPA is the pump, signal amplification (absorption and stim. emission)
For centrosymmetric materials (materials whose structure does not change upon the transformation r -> -r, i.e., the inversion with respect to the origin), such as silica, the quadratic term is 0. That explains why lasers need nonlinear crystals (non-centrosymmetric) such as Erbium-doped crystals in OPA.

However, the nonlinear **cubic** term is responsible, of the third-harmonic generation (THG) and Kerr effect, which implies a variation on the refractive index depending on the field intensity that causes phenomena such as Self-phase modulation, which is basically a phase variation w.r.t time (and consequently a frequency shift and pulse broadening) that increases with propagation.

This cubic term has 81 coefficients, but for symmetry reasons only 21 are nonzero and only 3 are independent (Xxxyy, Xxyxy, Xxyyx) and the Xiiii = sum of these 3.

- Basics of Fourier Transform... no mathematical...
- What is Dirac

The Fourier transform is a mathematical transformation that helps us analyse both spatial and temporal frequency spectra. Due to its properties, it can transform temporal convolutions and derivatives into simple algebraic operations.

A Dirac function is a unit impulse whose value is zero except at $\delta(0)$, and its integral across all the spectrum is 1.

- The envelope function
- Complex amplitude... meaning, why is it useful...

The EF expression can be expressed as E(r,t)=EM(r,t) cos(w0 t + beta0 + phi0), where EM is the slowly varying envelope and EM(r,t) = Re(A(r,t) exp(j 2 pi f0 t + j 2 pi q0 z)), where A0 = E0 exp(phi0) is the complex amplitude. Basically, the complex amplitude changes much slower than the exponential phase term that depends on time and propagation (carrier).
The electric field can be represented in the base band thanks to the complex amplitude A(z, t). This allows to work with a slowly varying and mono-modal quantity.

- Wave equation from the amplitude envelope

Knowing that

$$E(\mathbf{r},t) = \frac{A(\mathbf{r},t)}{2} e^{j(\beta_0 z + \omega_0 t)} + \frac{A^*(\mathbf{r},t)}{2} e^{-j(\beta_0 z + \omega_0 t)}$$

$$P(\mathbf{r},t) = \frac{A_P(\mathbf{r},t)}{2} e^{j(\beta_0 z + \omega_0 t)} + \frac{A_P^*(\mathbf{r},t)}{2} e^{-j(\beta_0 z + \omega_0 t)}.$$

As P(r,f) = Eps0 Chi(r,f) E(r,f) is quite unfriendly to be used for the general case, we expand the susceptibility in Taylor series to rewrite the expression of A and consequently we obtain the NLSE 3D

Using the scalar wave equation $\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\mathbf{E} + \frac{1}{\varepsilon_0}\mathbf{P}\right)$ we get

$$\nabla^2 A + 2j\beta_0\frac{\partial A}{\partial z} - \beta_0^2 A = -\frac{\omega_0^2}{c^2}A_1 + \frac{2j\omega_0}{c^2}\frac{\partial A_1}{\partial t} + \frac{1}{c^2}\frac{\partial^2 A_1}{\partial t^2}$$

where A1 = A + 1/Eps0 Ap.

- NLSE

- Terms appearing in the NLSE: 1st order, 2nd order, 3rd order, ... (3dNLSE)....

The NLSE describes the dynamics of the EF complex amplitude.

$$\nabla^2 A + 2j\beta_0\frac{\partial A}{\partial z} - \beta_0^2 A = -\frac{\omega_0^2}{c^2}\left(1+\widehat{\chi}\right)A + j\left(\frac{\omega_0^2}{c^2}\widehat{\chi}' + \frac{2\omega_0}{c^2}(1+\widehat{\chi})\right)\frac{\partial A}{\partial t} +$$

$$+\left(\frac{\omega_0^2}{2c^2}\widehat{\chi}'' + \frac{2\omega_0}{c^2}\widehat{\chi}' + \frac{1}{c^2}(1+\widehat{\chi})\right)\frac{\partial^2 A}{\partial t^2} - \frac{3}{4}\frac{\omega_0^2}{c^2}\widehat{\chi}^{(3)}|A|^2 A,$$

When approximating with respect to derivative orders:

- 0-order approximation: No derivatives considered ->
  Dispersion relation
  beta0^2 = w0^2/c^2 (1 + Chi(w0))
- 1-order approximation: Only first order derivatives ->
  Translation wave with group velocity = +-1/k0 = +-vg'
  $A(x,y,z,t) = A(x,y,0,t \pm k_0' z).$
- 2-order approximation:
  Nonlinearity term
  ApNL = ¾ Eps0 Chi3 |A|^2 A

The next equation is the 3DNLSE rewritten with k0^2 = wo^2/c^2 n^2

$$2j\beta_0\frac{\partial A(\mathbf{r},t)}{\partial z} + \frac{\partial^2 A(\mathbf{r},t)}{\partial x^2} + \frac{\partial^2 A(\mathbf{r},t)}{\partial y^2} +$$

$$+\left(\frac{3}{4}\frac{\omega_0^2}{c^2}\widehat{\chi}^{(3)}|A(\mathbf{r},t)|^2 + k_0^2 - \beta_0^2\right)A(\mathbf{r},t) - j2k_0 k_0'\frac{\partial A(\mathbf{r},t)}{\partial t} - k_0 k_0''\frac{\partial^2 A(\mathbf{r},t)}{\partial t^2} = 0.$$

The second and third term are the diffractive terms that cause spatial broadening of the beam.
The Chi3 term is a change in the refractive index (**Kerr effect**) that causes a phase shift. This phase shift depends on the envelope intensity and the 3rd order nonlinear term.
The last term is the dispersive term, which causes temporal broadening of the pulse when travelling through a dispersive medium (k0'' not 0)

- What is the correction due to the 3rd order nonlinearity

Defining n2=3Chi3/8n we obtain a nonlinear refractive index correction equal to nNL = n2|A|^2 = 3 Chi3 |A|^2 /8n that depends on Chi3 and the envelope intensity.

- Neglections
- Lambert-Beer
- Cross-phase modulations

Some approximations and neglections were made in the analysis:

- Chi Real -> No losses
  When calculating losses with Chi being complex, the Lamber Beer equation appears to measure the intensity decrease: I(z)=I(0)e^-αz, where alpha depends mainly on ChiRe, ChiIm and the wavelength. The penetration length is known as the length where I(z) decreases an e^-1 factor, so L=α^-1.

- No effects due to EF polarization:
  Linear case: Birefringence -> Medium not isotropic anymore -> ChiX different from ChiY -> Polarization dispersion induced.
  Nonlinear case: Chi3 elements -> Medium not isotropic anymore -> ApxNL and ApyNL have 3 terms:
    1. Self-phase modulation: Ei modifies its phase
    2. Cross-phase modulation: Ei modifies the phase of Ej
    3. Incoherent term: Ei generates Ej, but often negligible.

- Higher order dispersion: The truncation at the second time derivative of A(r,t) might not be enough around k0''= 0 (when passing from normal dispersion to anomalous dispersion).

- NL attenuation and dispersion.

- Sellmeier equation
- Show the graphics and explain what were done... (MATLAB)

When losses can be neglected, the Sellmeier equation describes in a good approximation the variations of the refractive index real part as function of measurable experimental parameters, which are tabulated for each material and wavelength:

$$n^2(\lambda) - 1 = \sum_i A_i \frac{\lambda^2}{\lambda^2 - \lambda_i^2}$$

- NLSE 3D terms explanation

$$\nabla^2 A + 2j\beta_0 \frac{\partial A}{\partial z} - \beta_0^2 A = -\frac{\omega_0^2}{c^2}\left(1 + \widehat{\chi}\right) A$$

0-order (space dependent): Diffraction, propagation variation, waveguiding (beta0, k0)

$$+ j\left(\frac{\omega_0^2}{c^2}\widehat{\chi}' + \frac{2\omega_0}{c^2}(1 + \widehat{\chi})\right)\frac{\partial A}{\partial t} +$$

1-order (time dependent): Group Velocity

$$+ \left(\frac{\omega_0^2}{2c^2}\widehat{\chi}'' + \frac{2\omega_0}{c^2}\widehat{\chi}' + \frac{1}{c^2}(1 + \widehat{\chi})\right)\frac{\partial^2 A}{\partial t^2}$$

2-order (time derivative dependent): Group Velocity Dispersion

$$- \frac{3}{4}\frac{\omega_0^2}{c^2}\widehat{\chi}^{(3)}|A|^2 A$$

Nonlinearity (3^{rd} order time derivative): Kerr effect

Chapter 2
- Definitions of optical fibre

The optical fibre is a dispersive, nonlinear, and dissipative system. It normally is a very long and thin cylindrical structure composed by three concentric domains:

- The core: it's the inner part, with higher refractive index with respect to the rest. Usually, it is composed by SiO_2 doped with GeO_2, which increases the refractive index.
- The cladding: it covers the core, and it has a lower refractive index with respect to the core. It is made by silica (SiO_2), opportunely doped with other materials to lower its refractive index.
- The jacket: It is usually made of plastic, and it is useful to protect the transmitting part.

- Internal reflection

Due to Snell's law, we need to trap the input light inside the core, which happens when we have total internal reflection (angle of incidence > critical angle):

$$\phi > \phi_c = \operatorname{asin}\left(\frac{n_2}{n_1}\right)$$

, where ϕ is the angle of incidence with respect to the normal at the interface.

- Attenuation of the fibre in dB/km... examples

As previously seen in the previous chapter, attenuation throughout propagation follows the Lambert-Beer law:

$$P(D) = P(0)\exp(-\alpha D)$$

where D is the distance of the measurement and α the attenuation (normally expressed in dB/km)

$$\alpha\ [\mathrm{dB/km}] = -\frac{10}{D}\log_{10}\left(\exp(-\alpha D)\right)$$

Attenuation has two type of sources, intrinsic (due to the medium) and extrinsic (due to other reasons such as deforming…).

$\alpha SiO_2(\lambda=1550nm) = 0.2dB/km$, $\alpha SiO_2(\lambda=1300nm) = 0.5dB/km$

- Types of dispersion

Due to dispersion, the light pulse is distorted during the propagation.
The different spectral components of the signal propagate with different group velocities and inter-symbol interference may occur if pulses are overlapped.
The types of dispersion are:

- **Material dispersion:** The material composing the fibre is weakly dispersive (its electromagnetic parameters, in particular the dielectric permittivity, are functions of the frequency).
- **Intramodal dispersion:** For every propagating mode, the ratio between the power propagating in the cladding and the power in the core depends on the frequency. Moreover, the group velocity depends on this ratio (The ratio affects Neff and consquently affects group velocity). Consequently, group velocity changes with frequency -> GVD appears.
- **Intermodal dispersion:** Several modes -> Different velocities -> GVD
- **Polarization dispersion:** Geometrical imperfections can imply different velocities into orthogonal polarization states -> GVD

- Guided modes
- What is M, mode profile
- Propagation constant

Weakly guiding hypothesis valid only if “n” does not have great variations (n1 ≅ n2) -> **k0^2 ≅ β0^2**

Defining

$A(x, y, z, t) = F(z, t)\, M(x, y)\, e^{j\delta\beta z}$, where the parameter δβ has been introduced to account for the modifications of the propagation constant of the wave due to the presence of the guiding structure. M(x,y) is the transverse mode profile and it is often written in cylindrical coordinates as M(r,ϕ). β = β0 + δβ is the propagation constant of the guided mode. The weakly guiding hypothesis allows to use the approximation k0^2 = β0^2.
And deriving again the 3D NLSE we obtain:

$$
\begin{aligned}
&2j\beta_0 M(x,y)\frac{\partial F(z,t)}{\partial z} + 2\frac{k_0^2}{n}n_2|F(z,t)M(x,y)|^2 F(z,t)M(x,y) \\
&-2j\beta_0 k_0' M(x,y)\frac{\partial F(z,t)}{\partial t} - \beta_0 k_0'' M(x,y)\frac{\partial^2 F(z,t)}{\partial t^2} \\
&+ F(z,t)\left[\frac{\partial^2 M(x,y)}{\partial x^2} + \frac{\partial^2 M(x,y)}{\partial y^2} + (k_0^2 - \beta_0^2 - 2\beta_0\delta\beta)M(x,y)\right] = 0
\end{aligned}
$$

The last term allows the determination of the guided modes in the dielectric structure (only depends on x,y), within the so-called linearly polarized (LP) modes approximation -> Solve $M(r,\phi)$ and β (Taylor series -> Differential equations whose solution are Bessel functions)

$$
M_\ell(r) = \begin{cases} A_1 J_\ell(k_t r) & for \quad 0 \le r < a \\ A_2 K_\ell(\gamma r) & for \quad a \le r < +\infty, \end{cases}
$$

, with

$$
k_t^2 = n_1^2\omega^2\mu_0\epsilon_0 - \beta_0^2 - 2\beta_0\delta\beta_\ell, \qquad \gamma^2 = \beta_0^2 + 2\beta_0\delta\beta_\ell - n_2^2\omega^2\mu_0\epsilon_0.
$$

Using Bessel equations (continuity for Ml(r) and its derivative @r=a) ->

$$
\frac{X J_\ell'(X)}{J_\ell(X)} = \frac{Y K_\ell'(Y)}{K_\ell(Y),}
$$

with

$$
X = k_t a, \quad Y = \gamma a
$$

and

$$
X^2 + Y^2 = \omega^2\mu_0\epsilon_0(n_1^2 - n_2^2)a^2 = \left(\frac{2\pi a}{\lambda}\right)^2 (n_1^2 - n_2^2) = V^2
$$

- Solution in dispersion relation for monomode LP01
- Condition for single mode

For l=0, $M(r,\phi) = M(r)$. As the figures show, while $X < 2.405$ we'll only obtain one solution (LP01) due to the shape of the LHS part of the dispersion relation -> **Monomode region for V < 2.405.**

- Show the figure 2.7 in MATLAB... explain how it was done

Basically, to make that figure first we need to define V and X and derive Y from the latter variables.
Also, an arbitrary initial beta0 guess must be defined (n2*ω/c <= beta0 <= n1*ω/c) to calculate "Neff".
After, calculate LHS and RHS of the dispersion relation. Finally, the points where LHS and RHS are equal are the solutions of the modes.

- Effective refractive index, dependence with the wavelength

To be able to calculate the effective refractive index we first need the X,Y values of the solution and calculate the kt and γ values. Then the propagation constant is easily deduced and consequently the effective refractive index.
For instance, for LP01 mode, when the normalised frequency increases, the energy of the pulse is more confined into the core and consequently, the effective refractive index gets closer to the core refractive index (see Matlab file HW).

- Graphics and gaussian approximation... plot the graphics

Using the kt and γ of the solution previously calculated it is trivial to plot M(r).
However, the expression of the gaussian approximation is given. Be careful with the validity of the gaussian approximation (1.8<V<2.4).

- Propagation Equation for M

Finally, by using $\beta_0 \approx k_0$ and the definition $k_0 = \omega_0 n/c$, Eq. (2.27) turns to:

$$j\frac{\partial F(z,t)}{\partial z} + \left(\frac{\omega_0 n_2}{cS_{eff}}\right)|F(z,t)|^2 F(z,t) - j\beta' \frac{\partial F(z,t)}{\partial t} - \frac{\beta''}{2}\frac{\partial^2 F(z,t)}{\partial t^2} = 0,$$

which is the 2D Nonlinear Schrödinger Equation (2DNLSE).

$$S_{eff} = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M(x,y)^2 \, dx \, dy}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M(x,y)^4 \, dx \, dy}$$

, where Seff means how the transversal mode is distributed in space.
With this information, we can study the evolution of the electric field amplitude along z direction and in time.