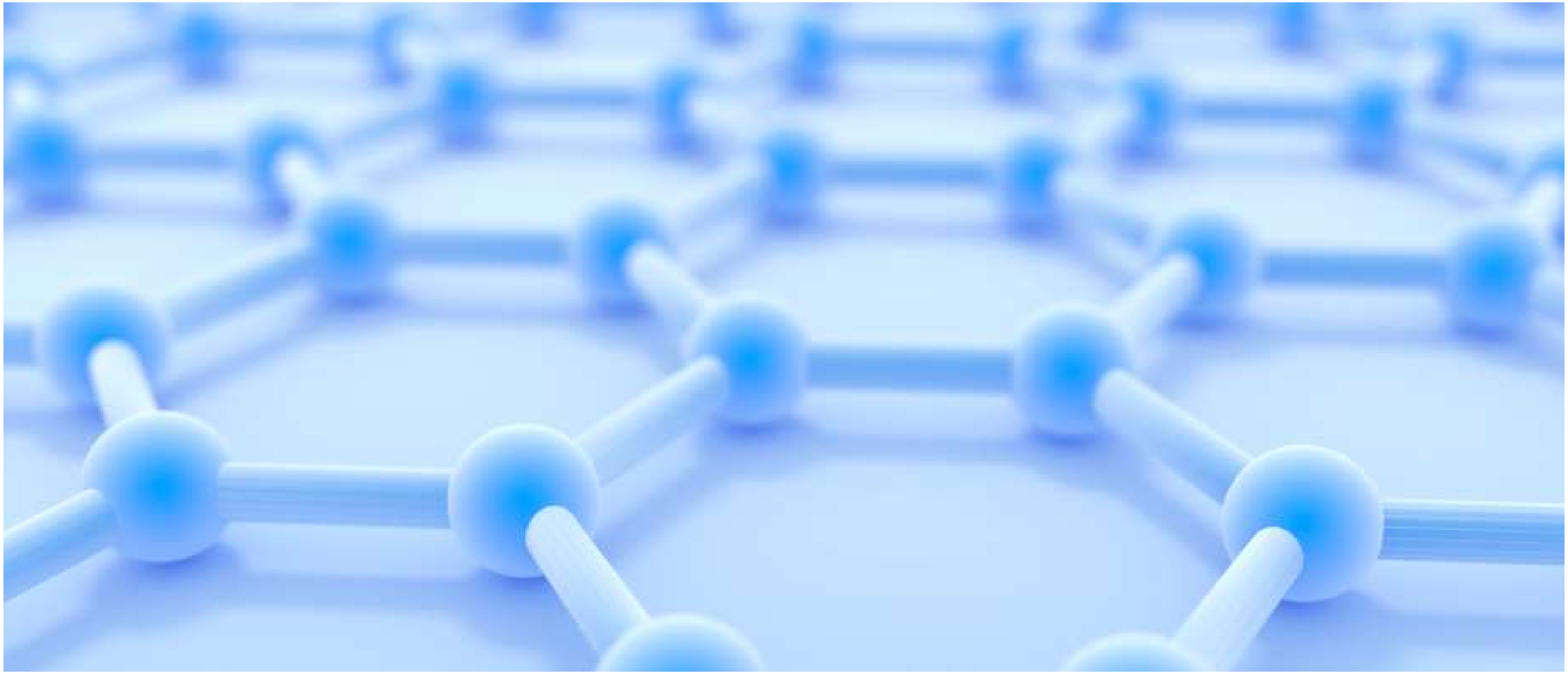

Photonics
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Introduction to Periodic Nanostructures:

-Thin film Multilayers

- Transfer Matrix Method

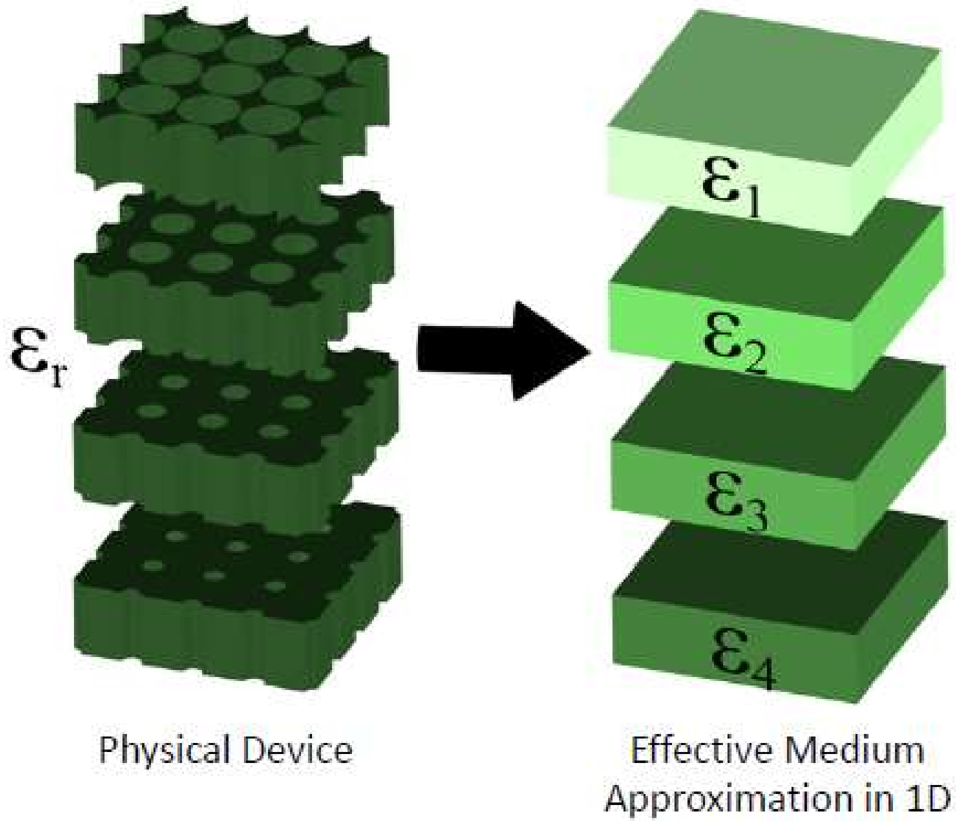
- Multiple Reflections

Transfer Matrix Method (TMM)

The TRANSFER MATRIX METHOD is used to calculate transmission and reflection from a planar layered structure in which the optical properties (the refractive index) vary in one dimension – the dimension normal to the planar interfaces between layers.

In general, it is often possible to describe a physical device using just one dimension. Doing so in fact dramatically reduces the numerical complexity of the problem and is ALWAYS GOOD PRACTICE!

In the example of the figure, thanks to the EFFECTIVE MEDIUM APPROXIMATION, we can often simplify a 3D structure in a layered medium, so that the "effective structure" shows index inhomogeneity only in the dimension orthogonal to the interfaces. In this way, the TMM can be applied to easily solve Maxwell's equations and find Reflectance, Transmittance and Absorption of the multilayer.



Transfer Matrix Method (TMM)

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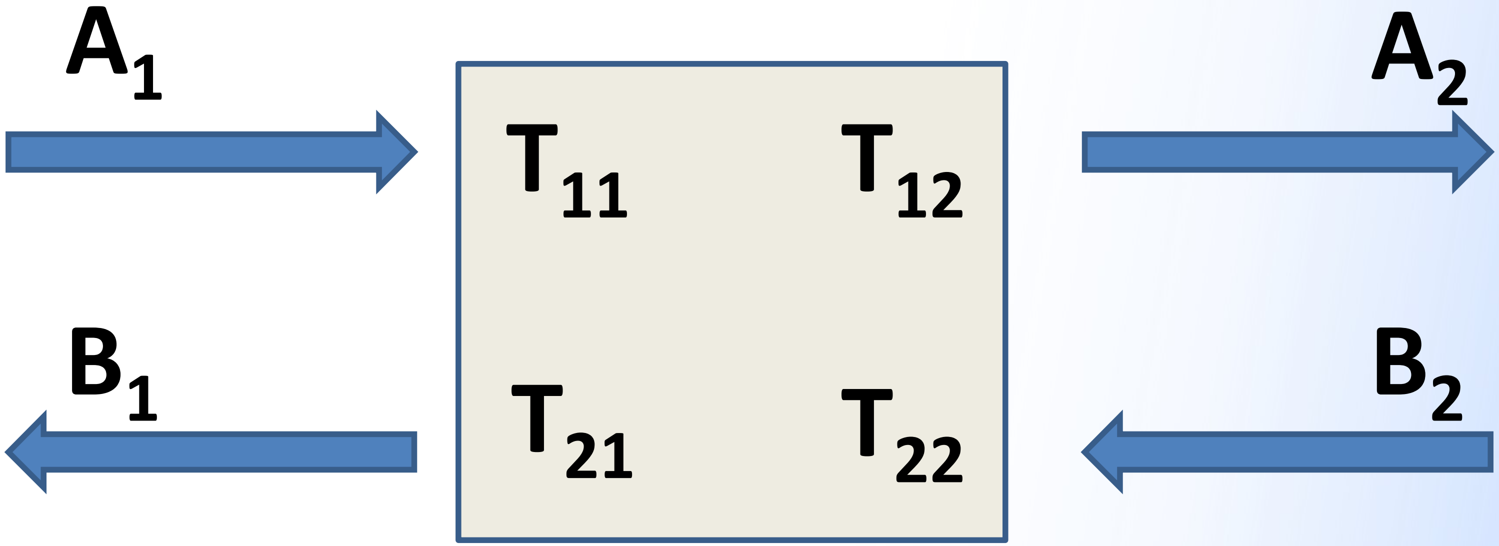
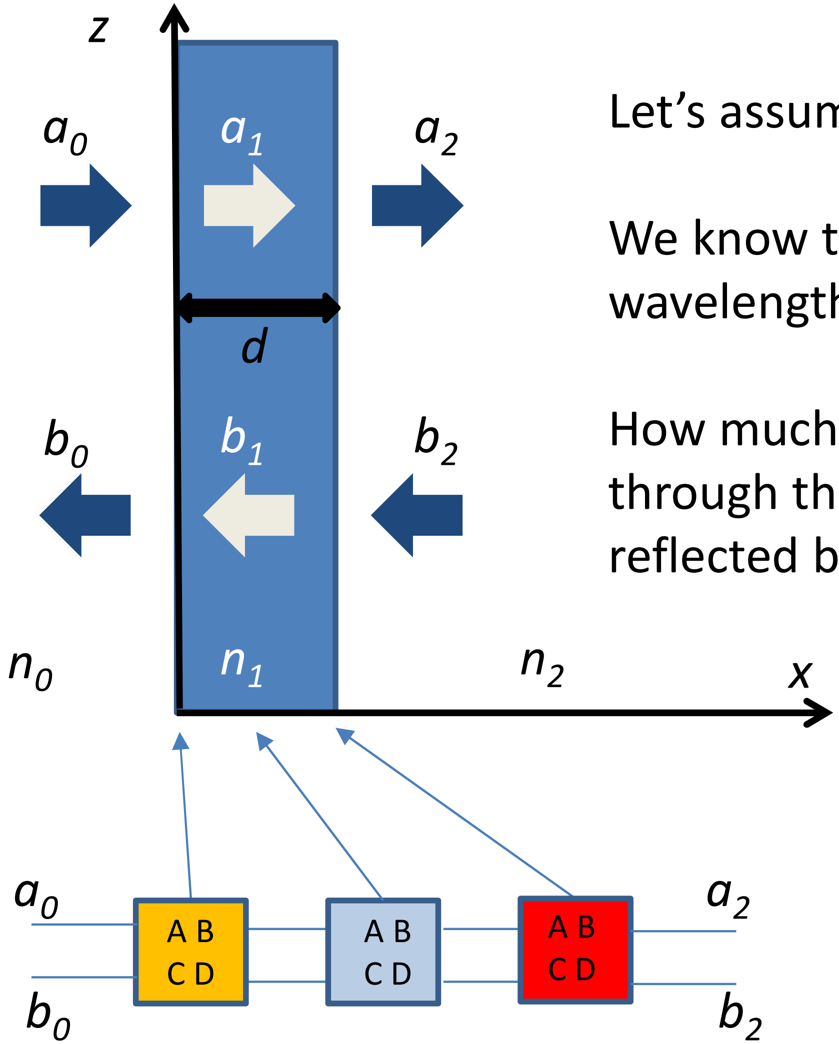
One way to look at the transfer matrix method is to take it as the analysis of a two-port system, where we can define the quantities at the exit port, and viceversa, as a function of the quantities at the entry port by means of the so-called ABCD matrix, or TRANSFER MATRIX, or TRANSMISSION MATRIX.

This method is useful to describe periodic and not periodic structures, since the transfer matrix of a chain of two-port systems like the one depicted below is straightforwardly obtained by multiplying the matrices associated with the single systems.



$$\begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} \begin{vmatrix} A_2 \\ B_2 \end{vmatrix}$$

Application of TMM to a simple problem
*Light interaction with a **thin film (slab)***



Let’s assume light comes from the left (medium 0).

We know the angle of incidence, the polarization, the wavelength, and the material parameters (n_0, n_1 and n_2).

How much of the incoming light power is transmitted through the film on medium 2? How much light is reflected back?

D_{10} A B C D **Transmission matrix of the first interface**

P_1^{-1} A B C D **Transmission matrix of the film**

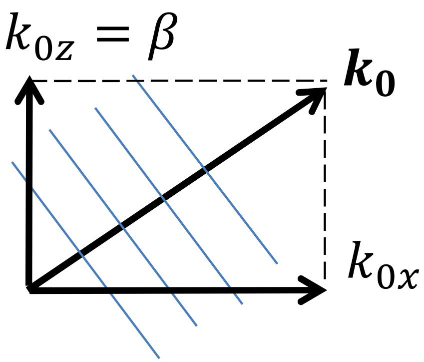
D_{21} A B C D **Transmission matrix of the second interface**

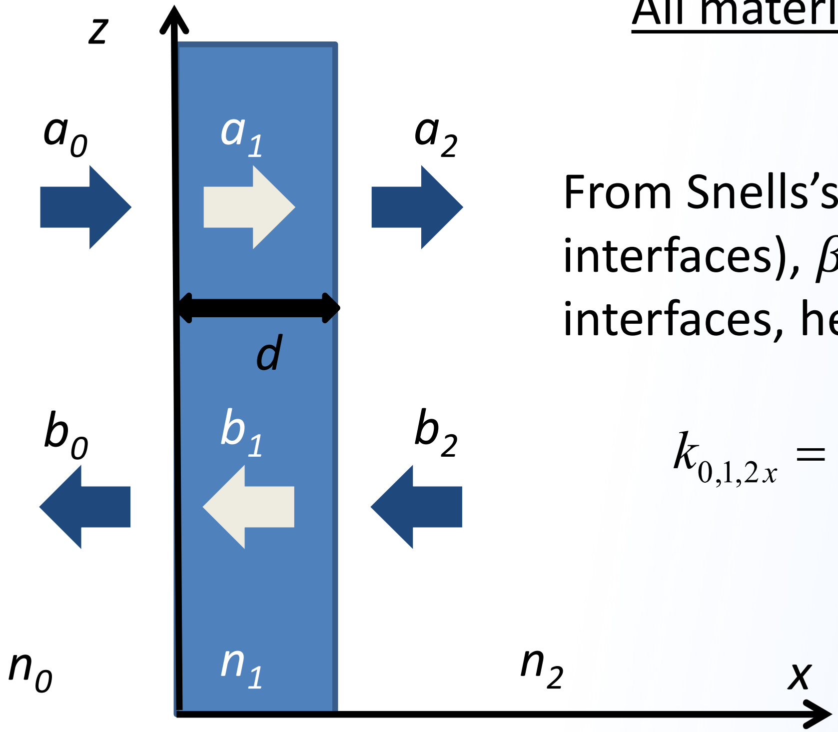
By properly multiplying the 3 matrices, we obtain the transmission matrix of the entire system, similarly to what we can do with Jones matrices in polarization optics. In order to find the 2×2 matrices, we need to solve Maxwell’s equations and impose the continuity of the tangential fields at the interfaces. However, this tool is extremely helpful: Once we know the transmission matrices for interfaces and for homogeneous films, we can solve a large number of problems involving thin films, even with many films (or multilayers), and for any polarization and any combination of materials.

Single Slab – **TE polarization**

Let's consider a slab of material with refractive index n_1 illuminated with TE-polarized light (E field in the y direction) from a medium with refractive index n_0. The output medium has index n_2. Here we will find the transfer matrices for TE polarization only, and for nonmagnetic materials. Extending the method to TM polarization and to nonmagnetic materials is straightforward.

<u>All materials are assumed nonmagnetic</u>





From Snells's law (phase-matching at interfaces), β is a conserved quantity at interfaces, hence

$$k_{0,1,2x} = \sqrt{\left(\frac{\omega}{c} n_{0,1,2}\right)^2 - \beta^2}$$

a_0 and b_0 FW and BW waves amplitudes at x=0$^-$

a_2 and b_2 FW and BW waves amplitudes at x=d$^+$

E-field in the three regions:

$$\mathbf{E}_0 = E_{0y}\hat{y} = (a_0 e^{-jk_{0x}x} + b_0 e^{jk_{0x}x})e^{-j\beta z}\hat{y}$$

$$\mathbf{E}_1 = E_{1y}\hat{y} = (a_1 e^{-jk_{1x}x} + b_1 e^{jk_{1x}x})e^{-j\beta z}\hat{y}$$

$$\mathbf{E}_2 = E_{2y}\hat{y} = (a_2 e^{-jk_{2x}(x-d)} + b_2 e^{jk_{2x}(x-d)})e^{-j\beta z}\hat{y}$$

Single Slab – **TE polarization**

In order to calculate the amplitude of the fields in reflection/transmission, boundary conditions need to be satisfied at both boundaries *x* = 0, *x* = *d* for the electric field and its derivative in the *x* direction. We need to apply continuity of the E_y field and the H_z field. Let's find the expression of the H_z field in the three regions. From Faraday's law:

$$\frac{\partial}{\partial z} = -j\beta$$

$$\mathbf{H} = \frac{1}{-j\omega\mu_0}\nabla\times\mathbf{E} = \frac{1}{-j\omega\mu_0}\left(j\beta E_y\hat{x} + \frac{\partial E_y}{\partial x}\hat{z}\right)$$

$$H_{0z} = \frac{1}{-j\omega\mu_0}\frac{\partial E_{0y}}{\partial x} = \frac{k_{0x}}{\omega\mu_0}(a_0 e^{-jk_{0x}x} - b_0 e^{jk_{0x}x})e^{-j\beta z}$$

$$H_{1z} = \frac{1}{-j\omega\mu_0}\frac{\partial E_{1y}}{\partial x} = \frac{k_{1x}}{\omega\mu_0}(a_0 e^{-jk_{1x}x} - b_0 e^{jk_{1x}x})e^{-j\beta z}$$

$$H_{2z} = \frac{1}{-j\omega\mu_0}\frac{\partial E_{2y}}{\partial x} = \frac{k_{2x}}{\omega\mu_0}(a_2 e^{-jk_{2x}(x-d)} - b_2 e^{jk_{2x}(x-d)})e^{-j\beta z}$$

Single Slab – **TE polarization**

In order to calculate the amplitude of the fields at the exit of the slab as a function of the input fields, boundary conditions need to be satisfied at both boundaries $x = 0$, $x = d$ for the electric field and its derivative in the x direction.

For $x = 0$ we have:

$$\begin{cases} a_0 + b_0 = a_1 + b_1 \\ k_{0x}(a_0 - b_0) = k_{1x}(a_1 - b_1) \end{cases} \qquad D_0 \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = D_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \qquad (1a)$$

While for x = d we have:

$$\begin{cases} a_1 e^{-jk_{1x}d} + b_1 e^{jk_{1x}d} = a_2 + b_2 \\ k_{1x}\left(a_1 e^{-jk_1 d} - b_1 e^{jk_1 d}\right) = k_{2x}\left(a_2 - b_2\right) \end{cases} \qquad D_1 \begin{pmatrix} a_1 e^{-jk_{1x}d} \\ b_1 e^{jk_{1x}d} \end{pmatrix} = D_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \qquad (1b)$$

$$D_n = \begin{pmatrix} 1 & 1 \\ k_{nx} & -k_{nx} \end{pmatrix}$$

Single Slab – **TE polarization**

From the first set of equations (1a) we can write a_0 and b_0 as a function of a_1 and b_1 :

So we can write the matrix problem that describes the first interface as:

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = D_0^{-1} D_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = D_{10} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 1 + k_{1x} / k_{0x} & 1 - k_{1x} / k_{0x} \\ 1 - k_{1x} / k_{0x} & 1 + k_{1x} / k_{0x} \end{bmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

The second set of equations for the boundary conditions at x=d can be rewritten similarly as

$$\begin{cases} a_1^+ + b_1^+ = a_2 + b_2 \\ k_{1x}\left(a_1^+ - b_1^+\right) = k_{2x}\left(a_2 - b_2\right) \end{cases} \xrightarrow{\text{Matrix form}} \begin{pmatrix} a_1^+ \\ b_1^+ \end{pmatrix} = D_1^{-1} D_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 1 + k_{2x} / k_{1x} & 1 - k_{2x} / k_{1x} \\ 1 - k_{2x} / k_{1x} & 1 + k_{2x} / k_{1x} \end{bmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

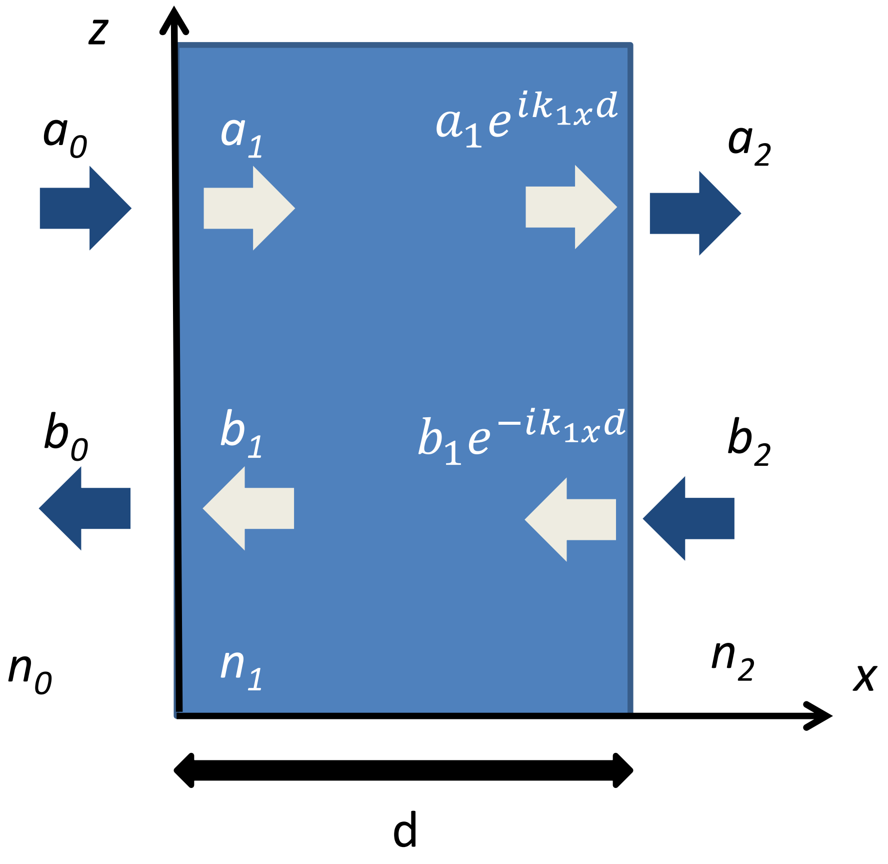
where $a_1^+ = a_1 e^{-jk_{1x}d}$ and $b_1^+ = b_1 e^{jk_{1x}d}$

DISCONTINUITY MATRIX $\quad D_{n+1,n} = D_{\mathrm{n}}^{-1} D_{\mathrm{n+1}} = \dfrac{1}{2} \begin{bmatrix} 1 + k_{\mathrm{n+1}x}/k_{\mathrm{n}x} & 1 - k_{\mathrm{n+1}x}/k_{\mathrm{n}x} \\ 1 - k_{n+1x}/k_{\mathrm{n}x} & 1 + k_{\mathrm{n+1}x}/k_{nx} \end{bmatrix}$

Single Slab – **TE polarization**

Important

The amplitudes $a_1^+ = a_1 e^{-jk_{2x}d}$ and $b_1^+ = b_1 e^{jk_{2x}d}$ are the "phase-shifted version" of a_1 and b_1



The fields at x=d are shifted by $k_{1x}d$ with respect to the fields at x=0. The sign of the phase-shift is positive for the forward (a) wave and negative for the backward (b) wave.

In matrix form:

$$\begin{pmatrix} a_1^+ \\ b_1^+ \end{pmatrix} = \begin{bmatrix} e^{-jk_{1x}d} & 0 \\ 0 & e^{jk_{1x}d} \end{bmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = P_1 \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

or

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = P_1^{-1} \begin{pmatrix} a_1^+ \\ b_1^+ \end{pmatrix}$$

PROPAGATION MATRIX $P_n = \begin{bmatrix} e^{-jk_{nx}d} & 0 \\ 0 & e^{jk_{nx}d} \end{bmatrix}$

Single Slab – **TE polarization**

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} 1+k_{1x}/k_{0x} & 1-k_{1x}/k_{0x} \\ 1-k_{1x}/k_{0x} & 1+k_{1x}/k_{0x} \end{bmatrix}}_{\text{Interface 0/1}} \underbrace{\begin{bmatrix} e^{jk_{1x}d} & 0 \\ 0 & e^{-jk_{1x}d} \end{bmatrix}}_{\text{Propagation matrix in 1}} \underbrace{\frac{1}{2}\begin{bmatrix} 1+k_{2x}/k_{1x} & 1-k_{2x}/k_{1x} \\ 1-k_{2x}/k_{1x} & 1+k_{2x}/k_{1x} \end{bmatrix}}_{\text{Interface 1/2}} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

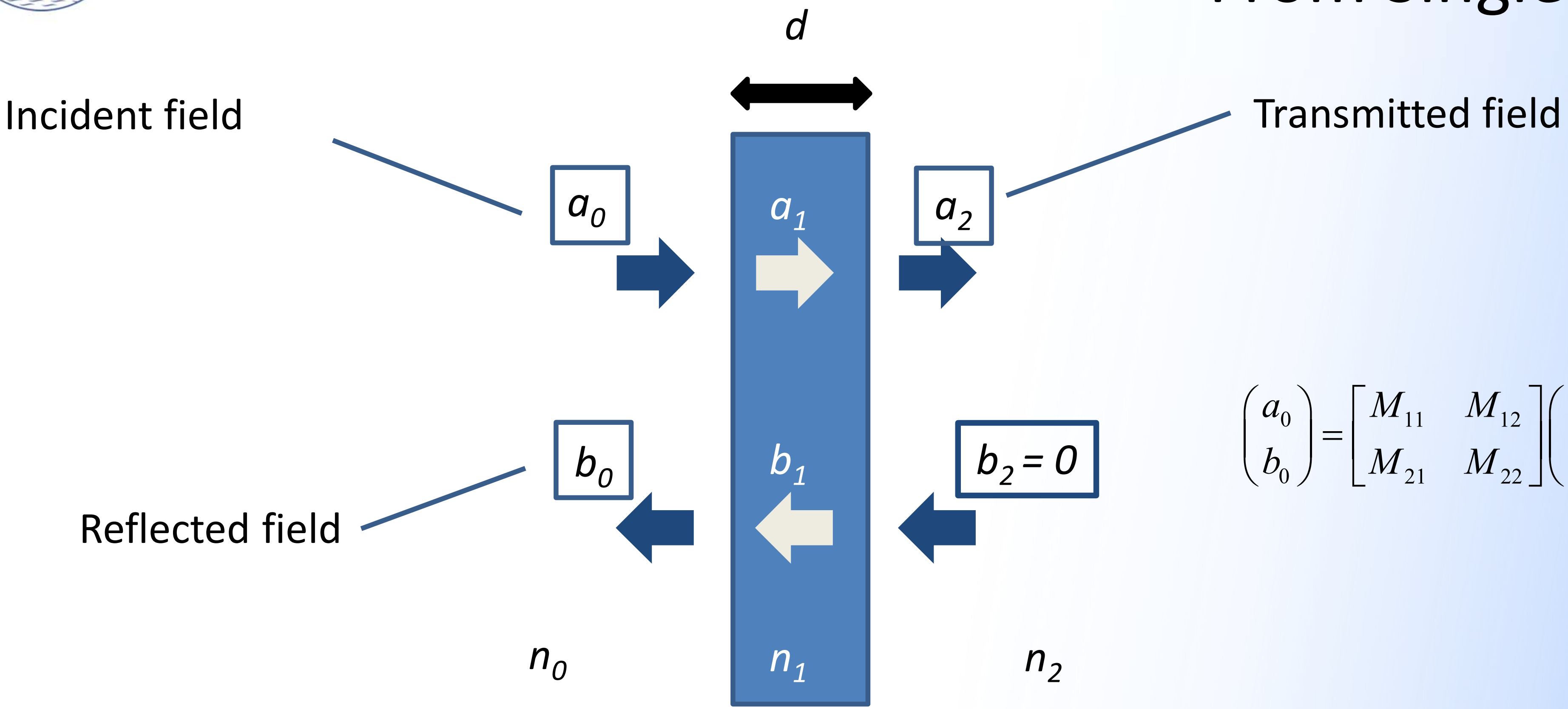
$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = D_{10} P_1^{-1} D_{21} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = M \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$M = D_{10} P_1^{-1} D_{21}$ **TRANSFER MATRIX**

Of course…

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = M^{-1} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = (D_{10} P_1^{-1} D_{21})^{-1} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = D_{21}^{\ -1} P_1 D_{10}^{\ -1} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

Reflectance and Transmittance From Single Slab



$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

Reflection and Transmission coefficients

$$t = \left.\frac{a_2}{a_0}\right|_{b_2=0} = 1/M_{11}$$

$$r = \left.\frac{b_0}{a_0}\right|_{b_2=0} = M_{21}/M_{11}$$

REFLECTANCE, TRANSMITTANCE, ABSORPTION

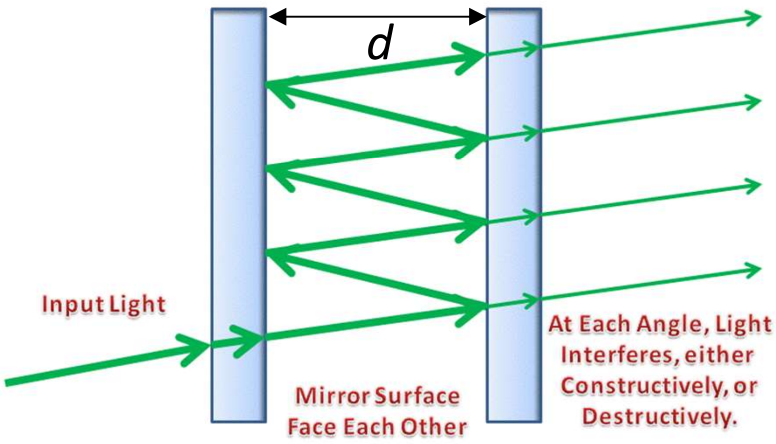
$$T = |t|^2 \frac{n_2 \cos\theta_2}{n_0 \cos\theta_0}$$

$$R = |r|^2$$

$$A = 1 - R - T$$

Example 1: FABRY-PEROT Resonator

The simplest photonic resonance, the Fabry-Perot resonator or etalon, can be designed by considering a region of space (a dielectric, or even vacuum) surrounded by two mirrors.

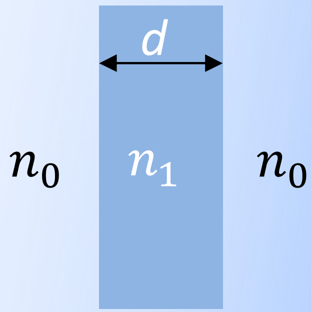


If the phase-shift of the round-trip is a multiple of 2π (i.e., the distance between the two mirrors is a multiple of $\frac{\lambda}{2}$) then the system sustains a resonance

$$\boldsymbol{2kd = 2m\pi}$$

In a similar way, a single slab of material with index n_1 and thickness d, surrounded by a medium with a different index n_0 supports Fabry-Perot resonances when the following condition for the free-space wavelength, λ_0 is met:

$$d = m\frac{\lambda_0}{2n_1}, m = 1,2, \dots$$



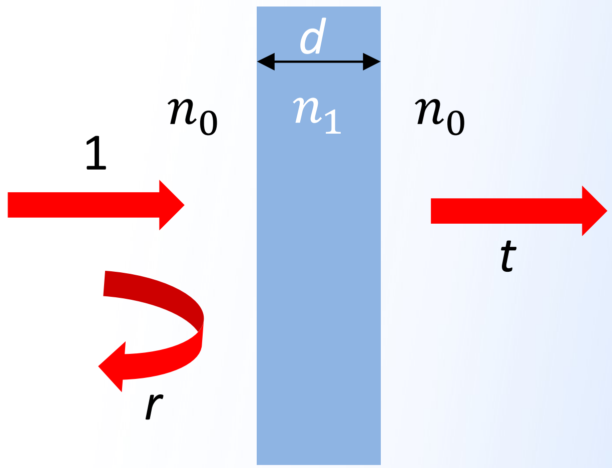
In this case, instead of physical mirrors, we only have reflections from the two interfaces

Example 1: FABRY-PEROT Resonator

Fabry-Perot (FP) resonance condition

$$n_1 d = m\frac{\lambda_0}{2}, m = 1,2, \dots$$

Phase accumulated in *d* is $m\pi$



$n_1 d$ is known as the **OPTICAL LENGTH OF THE FILM**

What does it happen to the incoming light under the FP resonance condition?

Does light get mostly reflected or transmitted?

There is a simple way to understand what happens, and it is based on looking at the problem as a multiple-reflections problem. Let’s assume m=1 (first-order FP resonance).

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Example 1: FABRY-PEROT Resonator

$$n_1 d = m\frac{\lambda_0}{2}$$

OBS: Reflection coefficient *r* is between -1 and 1. Transmission coefficient *t* is between 0 and 2.

Example:

$$r_{01} = -0.5 < 0$$

reflection coeff. at the first interface

$$r_{10} = -r_{01} = 0.5 > 0$$

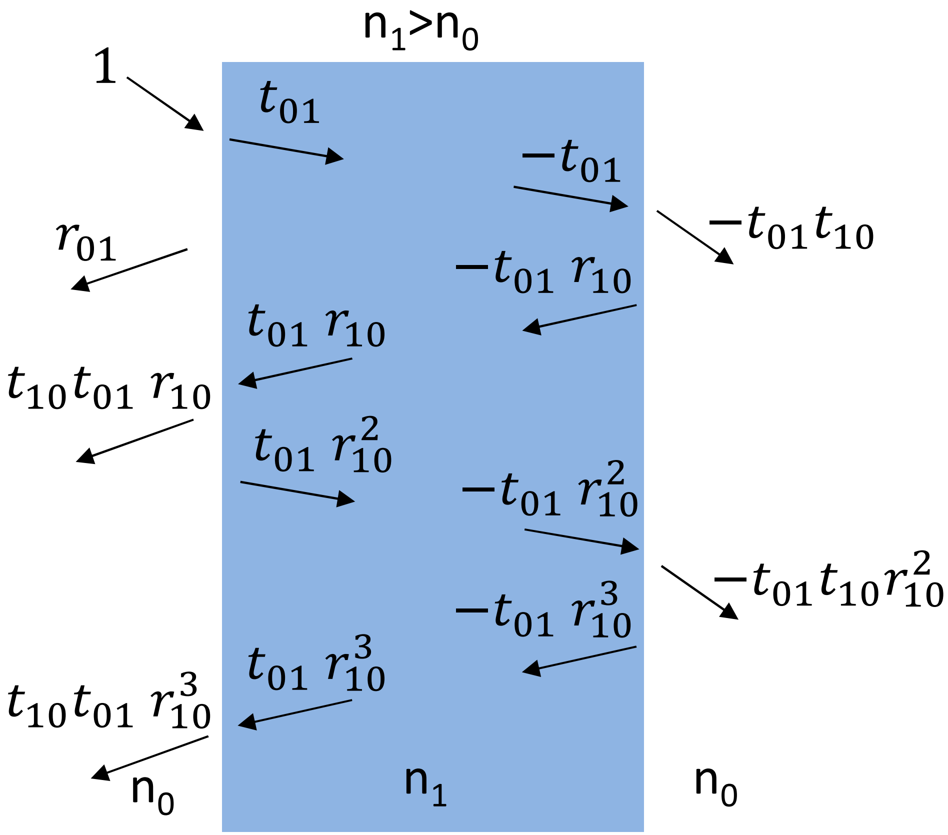
reflection coeff. at the second interface

$$t_{01} = 1 + r_{01} > 0$$

transmission coeff. at the first interface

$$t_{10} = 1 - r_{01} > 0$$

transmission coeff. at the second interface



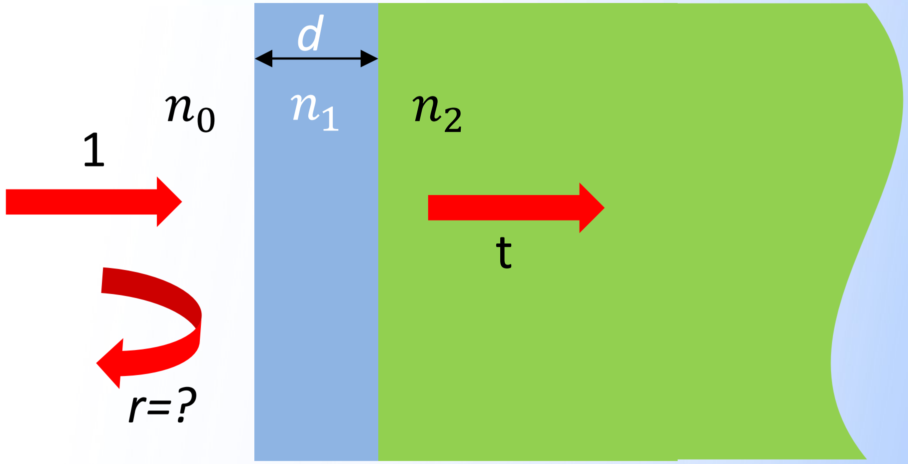
$$r = r_{01} + t_{01}t_{10}r_{10} + t_{01}t_{10}r_{10}^3 + \cdots$$
$$= r_{01} + t_{01}t_{10}r_{10}(1 + r_{10}^2 + \cdots r_{10}^{2n}) =$$
$$= r_{01} + \frac{t_{01}t_{10}r_{10}}{(1 - r_{10}^2)} = 0$$

Because $t_{01}t_{10} = (1 + r_{01})(1 - r_{01})$
And $r_{10} = -r_{01}$. The multiple reflections are out of phase!

Hence, R=0, and, in absence of absorption losses, T=1.

In conclusion: on FP resonances, the slab becomes transparent (or highly transmissive)

Generalization of multiple-reflections to an arbitrary slab

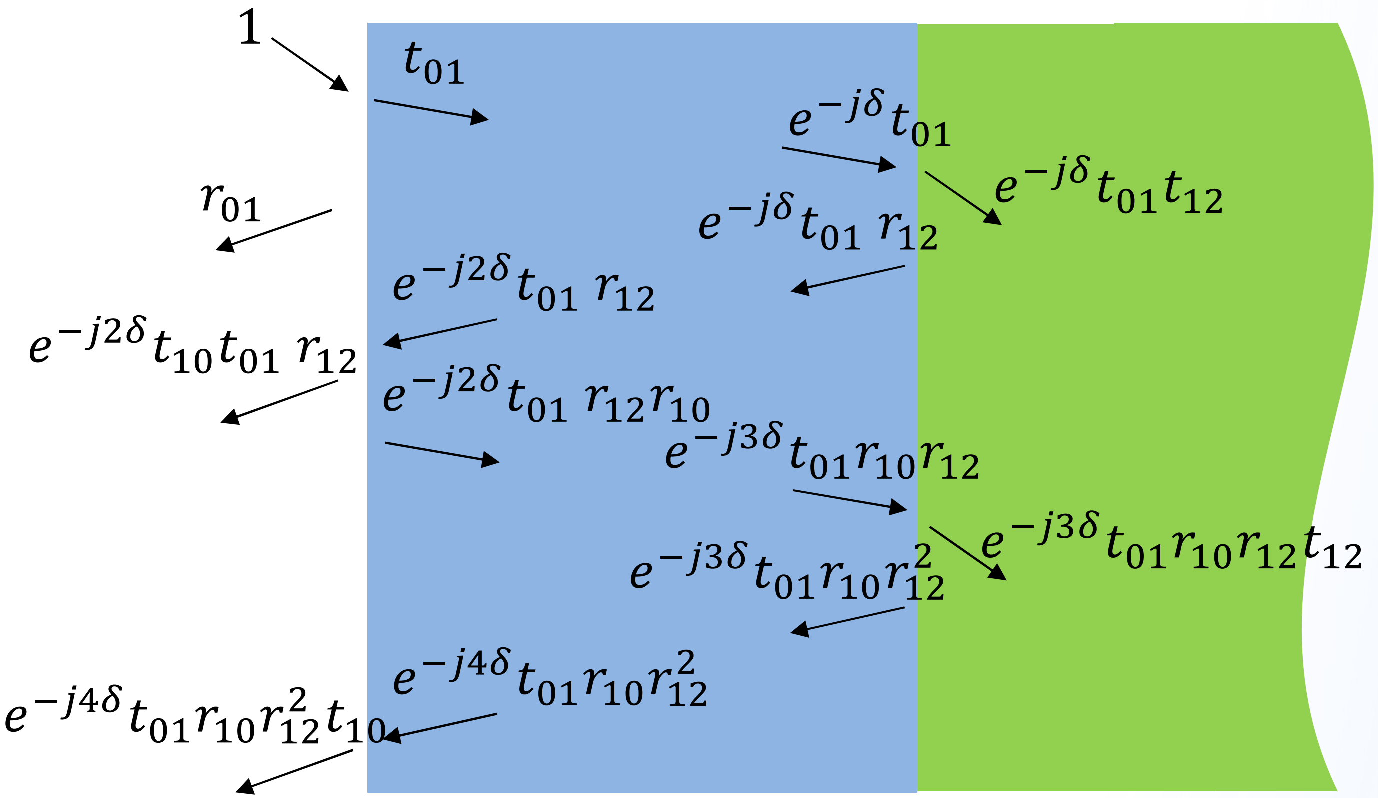


<u>Generic case</u>

Phase accumulated in *d* is $\boldsymbol{\delta = k_{1x} d}$

Round-trip phase term is $e^{-j2\delta}$

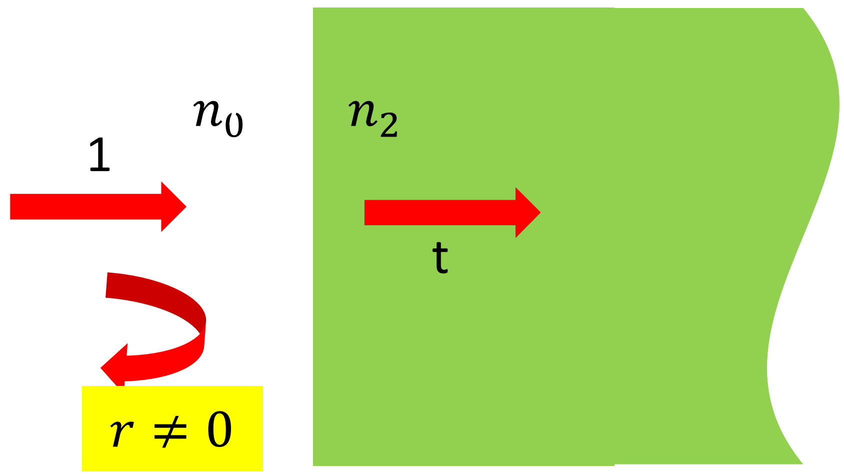
Single trip phase term is $e^{-j\delta}$

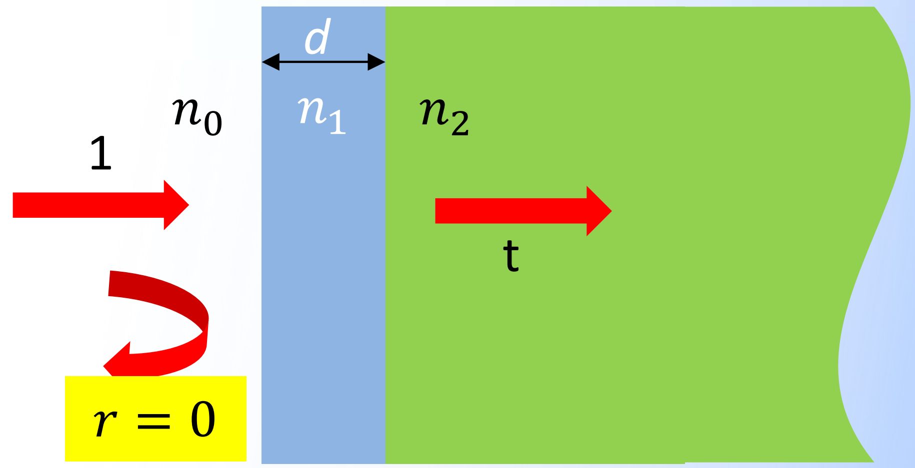


$$r = r_{01} + \frac{t_{01} t_{10} r_{12} e^{-j2\delta}}{(1 - r_{10} r_{12} e^{-j2\delta})}$$

$$t = \frac{t_{01} t_{12} e^{-j\delta}}{(1 - r_{10} r_{12} e^{-j2\delta})}$$

An important application: Antireflection coating





Since $n_0 \neq n_2$, there is always some reflection from medium 2 back to medium 0, because of impedance (or index) mismatch. For simplicity, let's assume normal incidence.
We now demonstrate that, by adding a so-called antireflection coating, i.e., a single layer with optical length $n_1 d = \lambda_{AR}/4$, one can inhibit reflection at least in a wavelength bandwidth near λ_{AR}. How? Let's find a way to zero the reflection coefficient r at λ_{AR} . At λ_{AR}, we know that $\delta = k_1 d = \frac{\pi}{2}$, hence we write:

$$r = r_{01} + \frac{t_{01}t_{10}r_{12}e^{-j2\delta}}{(1 - r_{10}r_{12}e^{-j2\delta})} = r_{01} - \frac{t_{01}t_{10}r_{12}}{(1 + r_{10}r_{12})} = \frac{r_{01} - r_{01}r_{12}r_{10} - t_{01}t_{10}r_{12}}{1 + r_{10}r_{12}} = 0$$

$$r_{01} - r_{12}(r_{01}^2 + t_{01}t_{10}) = 0 \quad \Longrightarrow \quad r_{01} = r_{12} \quad \Longrightarrow \quad \boxed{n_1 = \sqrt{n_0 n_2}}$$

Example: we have $n_0 = 1, n_2 = 3$ and we want zero reflection at the wavelength of green ($532\ nm$). Then, $n_1 = \sqrt{n_0 n_2} \sim 1.73$ and $d = \frac{532}{4n_1} \sim 77\ nm$

… back to the FABRY-PEROT Resonator

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Let's now assume we are not necessarily under FP resonance condition, and let's proceed to find a general expression for reflectance and transmittance as a function of an arbitrary phase delay in the cavity, i.e., an arbitrary wavelength λ. We now know the general expressions of transmission and reflection coefficients. Let's assume that medium 0 and medium 2 are identical, hence the symmetric situation we have earlier described.

$$r = r_{01} + \frac{t_{01}t_{10}r_{12}e^{-j2\delta}}{(1 - r_{10}r_{12}e^{-j2\delta})} = r_{01} - \frac{(1 - r_{01}^2)r_{01}e^{-j2\delta}}{(1 - r_{01}^2 e^{-j2\delta})}$$

$$t = \frac{t_{01}t_{12}e^{-j\delta}}{(1 - r_{10}r_{12}e^{-j2\delta})} = \frac{(1 - r_{01}^2)e^{-j\delta}}{(1 - r_{01}^2 e^{-j2\delta})}$$

Now we restrict the problem to the case in which there are no phase shifts in the reflection coefficients of the two mirrors (the interfaces), e.g., when n_0 and n_1 are real quantities. In fact, general mirrors do introduce some phase-shift. In our restricted case, $\arg(r_{01}) = 0$ and the reflectance and transmittance of the interface are simply $R = r_{01}^2 = r_{10}^2$ and $T = t_{01}^2 = t_{10}^2$. Be careful: the transmission/reflection coefficient of the film do instead have a phase term generally different from zero (see the δ in r and t)!

… back to the FABRY-PEROT Resonator

The transmittance of the film is then

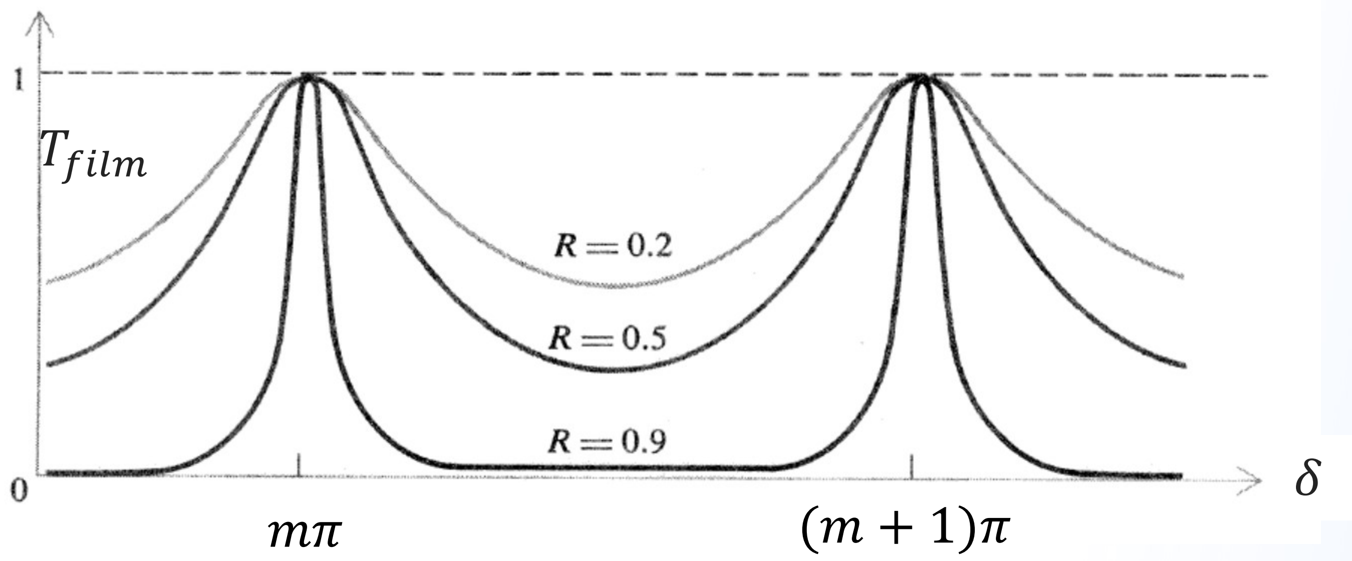
$$T_{film} = |t|^2 = tt^* = \frac{(1-R)e^{-j\delta}}{(1-Re^{-j2\delta})}\frac{(1-R)e^{j\delta}}{(1-Re^{j2\delta})} = \frac{T^2}{1+R^2-2R\cos(2\delta)}$$

Using the identity $\cos(2\delta) = 1 - 2\sin^2\delta$,

$$T_{film} = \frac{T^2}{1+R^2-2R+4R\sin^2\delta} = \frac{T^2}{(1-R)^2+4R\sin^2\delta}$$

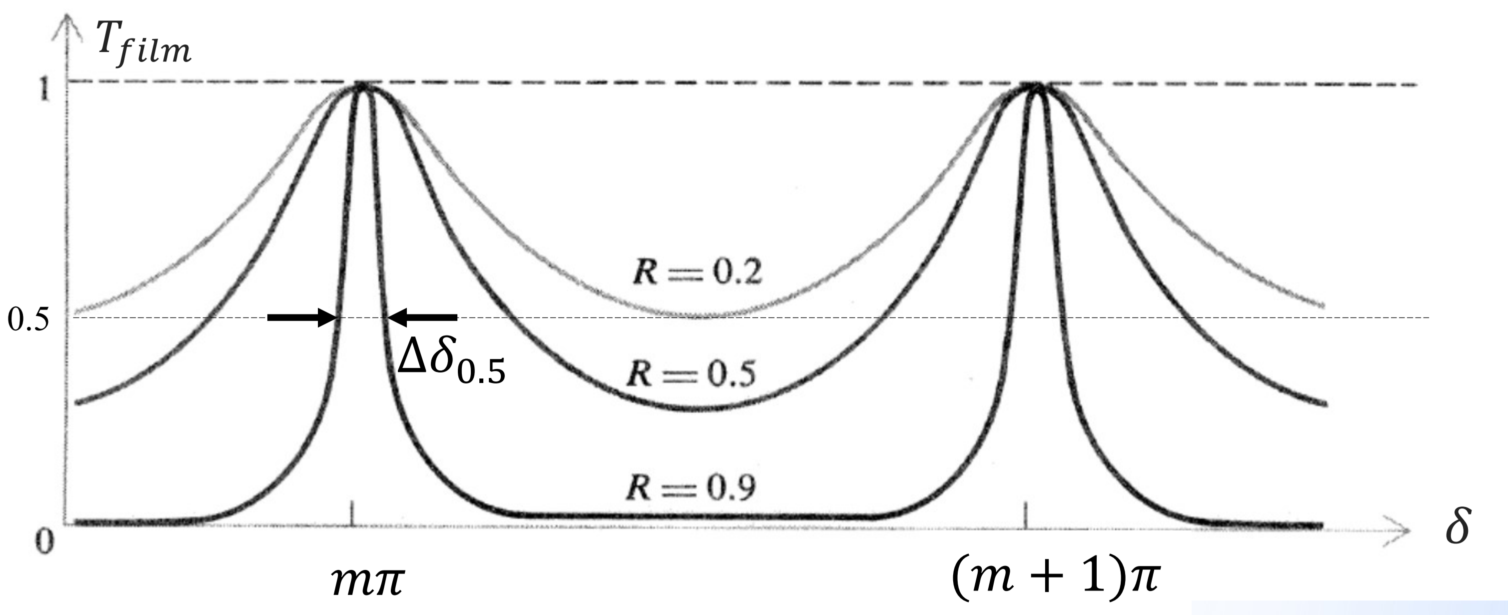
So, the transmittance can be written, as a function of the so-called finesse $F = \frac{4R}{(1-R)^2}$ as

$$T_{film} = \frac{T^2/(1-R)^2}{1+F\sin^2\delta}$$



- **The finesse F is proportional to the QUALITY FACTOR of the cavity, $F \propto Q$**
- **A small R induces big leakage from the cavity, hence poor F and poor Q**
- **A big R inhibits radiation leakage from the cavity, therefore it indices high F and high Q**

Quality factor and photon lifetime



FP Bandwidth

$\Delta\omega = FWHM = c/(n_1 d)\ \Delta\delta_{0.5}$

Quality factor

$Q = \omega_{FP}/\Delta\omega \sim F$

Photon lifetime

$\tau = 1/\Delta\omega = Q/\omega_{FP}$

Quality factor and photon lifetime

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- At the FP condition, i.e., $\delta = m\pi$ and , $\omega_{FP} = \frac{m\pi c}{n_1 d}$, $T = 1, R = 0$, therefore $T_{film} = 1$.

- The higher R, the larger the finesse F and the narrower the FP resonances.

- For a fixed cavity length d and incident angle and index of the film n_1, the quantity δ depends only on the frequency, e.g., at normal incidence, $\delta = k_1 d = \frac{\omega}{c} n_1 d$, this means that a narrow resonance in the δ domain (see figure with a **big R value**, e.g., 0.9) is a **narrowband resonance** in the freq. domain. In Fig. $\Delta\delta_{0.5} = \frac{\Delta\omega}{c} n_1 d$

- The bandwidth $\Delta\omega = c/(n_1 d)\,\Delta\delta_{0.5}$ is usually defined as the full-width-at-half-maximum (FWHM).

- The reflectivities of the two mirrors (i.e., the interfaces) define the quality factor of the FP cavity, $Q = \omega_{FP}/\Delta\omega$, which is a measure of the photon lifetime.

- A photon in the FP cavity will survive in the cavity without being radiated out for a time $\boldsymbol{\tau} = \frac{\mathbf{1}}{\boldsymbol{\Delta\omega}} = \boldsymbol{Q}/\boldsymbol{\omega_{FP}}$