Digital Communications: Signals Spaces

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References: DIGITAL COMMUNICATION, Third Edition, John R. Barry, Edward A. Lee David G. Messerschmitt

2.6. Signals as Vectors

It is possible to abstractly represent the signals in a digital communication system as vectors in a *linear space* or *vector space*, much like the familiar three-dimensional vectors in our physical world. This representation does not allow us to solve any problems that cannot be solved by other methods, but it gives valuable intuition.

2.6.1. Linear Spaces and Subspaces

A *linear space* or *vector space* is a set of elements called *vectors* together with two operators, addition of vectors and multiplication by a scalar.

Example 2-15.

Ordinary *Euclidean space* is the most familiar example of a linear space. In n-dimensional Euclidean space, a vector is specified by its n coordinates,

$$\mathbf{X} = [x_1,\ x_2,\ \ldots,\ x_n]^T, \tag{2.70}$$

where the superscript $(\cdot)^T$ denotes a transpose. There are rules for adding two vectors (sum the individual components) and multiplying a vector by a scalar (multiply each of the components by that scalar).

Example 2-16.

A space of some importance in this book is the *Euclidean space of complex-valued vectors*. Vectors in this space are identical to (2.70) except that the components x_k of the vector are complex-valued. Ordinary (real) Euclidean space is a special case, in which the imaginary parts of the vectors are zero.

More abstract linear spaces may be defined, as long as the addition and scalar multiplication operations satisfy certain constraints. The addition rule must produce a new vector $\mathbf{X}+\mathbf{Y}$ that must be in the linear space. Addition must obey familiar rules of arithmetic, such as the commutative and associative laws,

$$\mathbf{X}+\mathbf{Y}=\mathbf{Y}+\mathbf{X}, \qquad \mathbf{X}+(\mathbf{Y}+\mathbf{Z})=(\mathbf{X}+\mathbf{Y})+\mathbf{Z}. \tag{2.71}$$

The direct sum of two vectors has the interpretation illustrated in Fig. 2-14(a) for the two-dimensional Euclidean space. A linear space must include a zero vector $\mathbf{0}$, and every vector must have an *additive inverse*, denoted $-\mathbf{X}$, such that

$$\mathbf{0}+\mathbf{X}=\mathbf{X}, \qquad \mathbf{X}+(-\mathbf{X})=\mathbf{0}. \tag{2.72}$$

Multiplication by a scalar α produces a new vector $\alpha\cdot\mathbf{X}$ that must be in the vector space. Multiplications must obey the associative law,

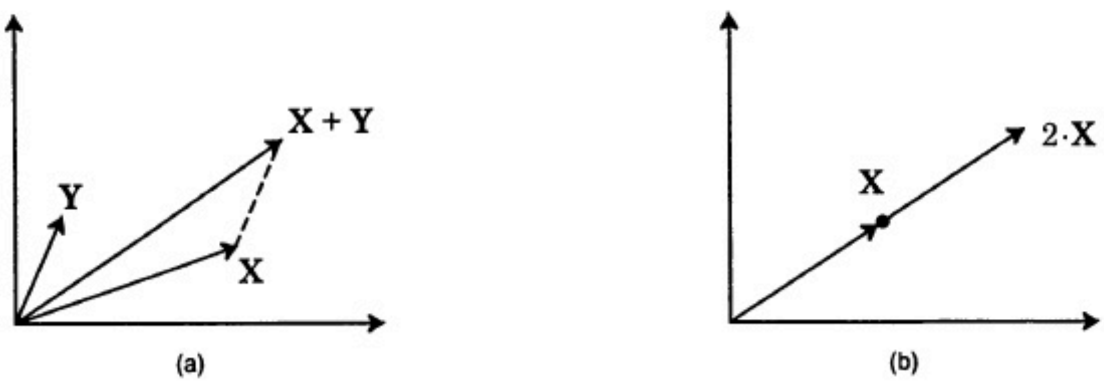


Fig. 2-14. Elementary operations in a two-dimensional linear space. (a) Sum of two vectors. (b) Multiplication of a vector by a scalar.

$$\alpha \cdot (\beta \cdot \mathbf{X}) = (\alpha\beta) \cdot \mathbf{X} \tag{2.73}$$

and also follow the rules

$$1 \cdot \mathbf{X} = \mathbf{X}, \qquad 0 \cdot \mathbf{X} = \mathbf{0}. \tag{2.74}$$

The geometric interpretation of multiplying a vector by a scalar is shown in Fig. 2-14(b). Finally, addition and multiplication must obey the distributive laws,

$$\alpha \cdot (\mathbf{X} + \mathbf{Y}) = \alpha \cdot \mathbf{X} + \alpha \cdot \mathbf{Y}, \qquad (\alpha + \beta) \cdot \mathbf{X} = \alpha \cdot \mathbf{X} + \beta \cdot \mathbf{X}. \tag{2.75}$$

Real linear spaces are defined in terms of real-valued scalars, while *complex* linear spaces have complex-valued scalars. We will encounter both types.

Euclidean space as defined earlier meets all of these requirements, and is therefore a linear space. There are less obvious examples.

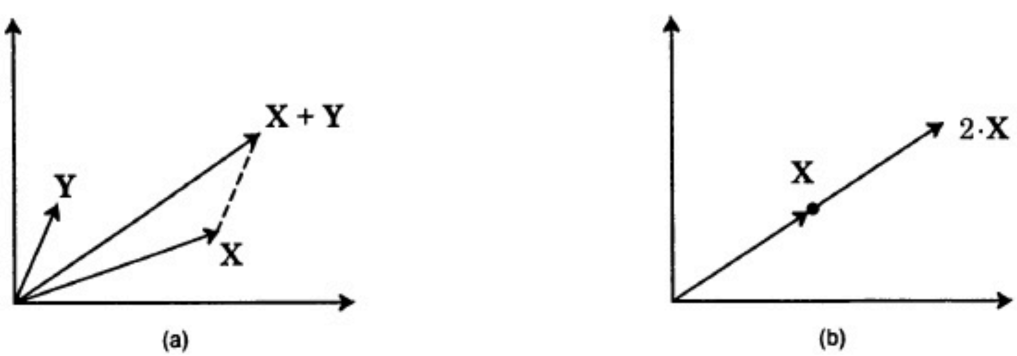


Fig. 2-14. Elementary operations in a two-dimensional linear space. (a) Sum of two vectors. (b) Multiplication of a vector by a scalar.

Example 2-17.

The set of all complex-valued discrete-time signals of the form $\{y_k\}$ that have finite energy, so that,

$$\Sigma_k |y_k|^2 < \infty, \tag{2.76}$$

is a linear space; it can be viewed as a generalization of the n-dimensional Euclidean space of (2.70) to the case in which the number of components is infinite. Scalar multiplication and vector addition are the same as for Euclidean spaces.

Example 2-18.

The set of complex-valued continuous-time signals $y(t)$ that have finite energy,

$$\int_{-\infty}^{\infty} |y(t)|^2 dt < \infty, \tag{2.77}$$

is a linear space. We can think of this space as a strange Euclidean space with a continuum of coordinates. The definition of multiplication of a signal vector by a scalar and the addition of two signal vectors are the natural and obvious, and the definition of a zero vector is the zero-valued signal.

Sub-Spaces

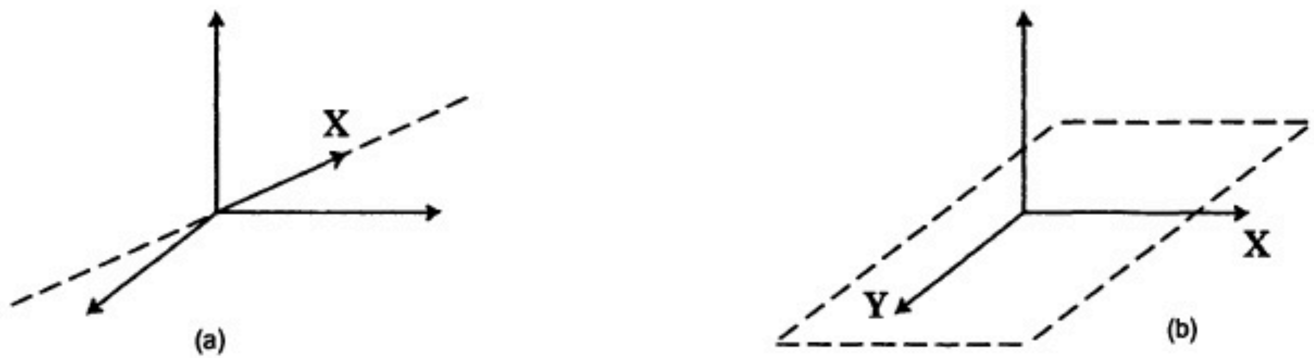


Fig. 2-15. Subspaces in three-dimensional Euclidean space.

A *subspace* of a linear space is a *subset* of the linear space that is itself a linear space. Roughly speaking this means that the sum of any two vectors in the subspace must also be in the subspace, and the product of any vector in the subspace by any scalar must also be in the subspace.

Example 2-19.

An example of a subspace in three-dimensional Euclidean space is either a line or a plane in the space, where in either case the vector **0** must be in the subspace.

Example 2-20.

A more general subspace is the set of vectors obtained by forming all possible weighted linear combinations of n vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$. The subspace so formed is said to be *spanned* by the set of n vectors. This is illustrated in Fig. 2-15 for three-dimensional Euclidean space. In Fig. 2-15(a), the subspace spanned by **X** is the dashed line, which is infinite in length and co-linear with the vector **X**. Any vector on this line can be obtained by multiplying **X** by the appropriate scalar. In Fig. 2-15(b), the subspace spanned by **X** and **Y** is the plane of infinite extent (depicted by the dashed lines) that is determined by the two vectors. Any vector in this plane can be formed as a linear combination of the two vectors multiplied by appropriate scalars.

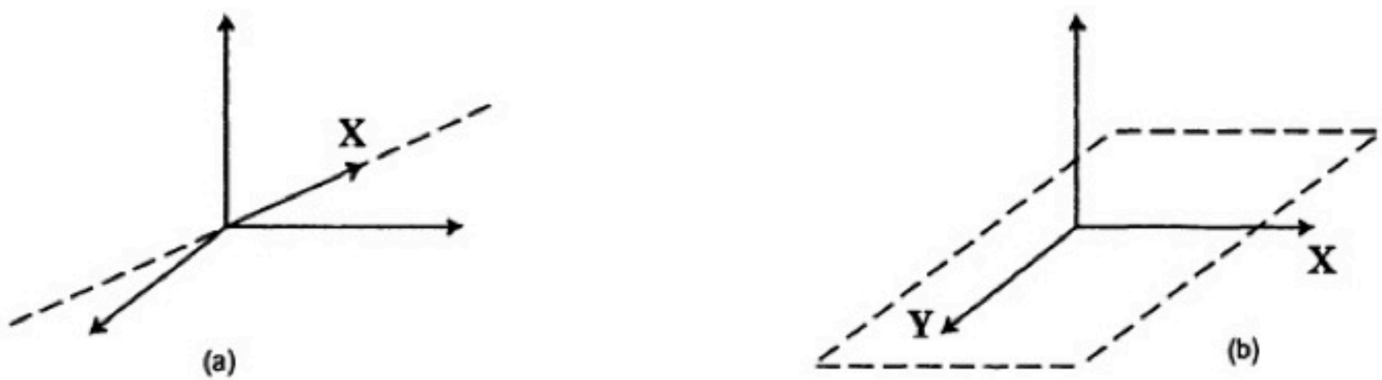


Fig. 2-15. Subspaces in three-dimensional Euclidean space.

The most famous subspace encountered in digital communications is called *signal space*, and is defined in the context of a digital transmitter that transmits one of M possible signals $\{s_1(t), s_2(t), \ldots s_M(t)\}$. In this setting, the *signal space* $\mathcal{S}$ is defined as the span of the M signals:

$$\mathcal{S} = \text{span}\{s_1(t), s_2(t), \ldots s_M(t)\} . \tag{2.78}$$

In other words, $\mathcal{S}$ is the set of all signals that can be expressed as a linear combination of $\{s_1(t), s_2(t), \ldots s_M(t)\}$.

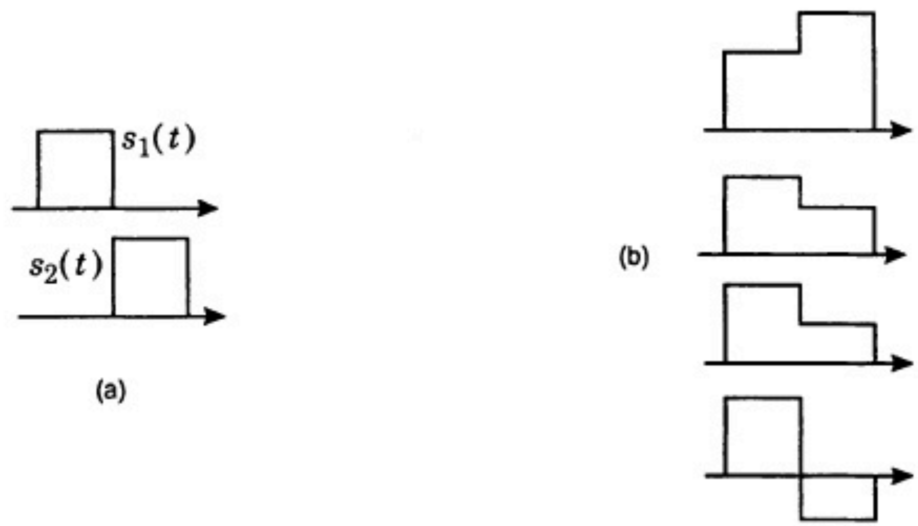


Fig. 2-16. (a) Two signals $s_1(t)$, $s_2(t)$; (b) four examples of signals that are in $\mathcal{S} = \text{span}\{s_1(t), s_2(t)\}$.

Inner Products

2.6.2. Inner Products

The definition of a linear space does not capture the most important properties of Euclidean space; namely, its *geometric* structure. This structure includes such concepts as the length of a vector, the distance between two vectors, and the angle between two vectors. All these properties of Euclidean space can be deduced from the definition of an *inner product* of two vectors, defined for n-dimensional Euclidean space by

$$\begin{aligned} \langle \mathbf{X}, \mathbf{Y} \rangle &= \sum_{i=1}^{n} x_i \, y_i^* \\ &= \mathbf{Y}^*\mathbf{X}\,, \end{aligned} \tag{2.79}$$

where y_i^* is the complex conjugate of the scalar y_i, and $\mathbf{Y}^*$ is the *conjugate transpose* of the vector $\mathbf{Y}$. The squared length of the vector is denoted $\|\mathbf{X}\|^2$, and is defined by $\|\mathbf{X}\|^2 = \langle \mathbf{X}, \mathbf{X} \rangle = \mathbf{X}^*\mathbf{X} = \sum_{i=1}^{n} |x_i|^2$.

Norm

The inner product as applied to Euclidean space can be generalized to the other linear spaces of interest. The important consequence is that the geometric concepts familiar in Euclidean space can be applied to these spaces as well. Let $\mathbf{X}$ and $\mathbf{Y}$ be elements of a linear space; they might represent vectors (Example 2-15), or discrete-time sequences (Example 2-15), or continuous-time waveforms (Example 2-15). Then the inner product $\langle \mathbf{X}, \mathbf{Y} \rangle$ is a mapping from two elements of the linear space to a scalar (real or complex, depending on the scalar field associated with the linear space), and it must obey *the rules*

$$\langle \mathbf{X} + \mathbf{Y}, \mathbf{Z} \rangle = \langle \mathbf{X}, \mathbf{Z} \rangle + \langle \mathbf{Y}, \mathbf{Z} \rangle$$
$$\langle \alpha \cdot \mathbf{X}, \mathbf{Y} \rangle = \alpha \langle \mathbf{X}, \mathbf{Y} \rangle, \qquad \langle \mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{Y}, \mathbf{X} \rangle^*$$
$$\langle \mathbf{X}, \mathbf{X} \rangle > 0, \quad \text{for } \mathbf{X} \neq \mathbf{0}. \tag{2.80}$$

We again adopt the shorthand notation $\|\mathbf{X}\|^2 = \langle \mathbf{X}, \mathbf{X} \rangle$, where $\|\mathbf{X}\|$ is called the *norm* of $\mathbf{X}$.

Inner Product (Scalar Product)

These rules are all obeyed by the familiar Euclidean space inner product of (2.79), as can be easily verified. For the other linear spaces of interest, analogous definitions of the inner product satisfying the rules can be made. In particular, for discrete-time signals, the inner product and norm are defined by

$$\langle x_k, y_k \rangle = \sum_{k=-\infty}^{\infty} x_k y_k^* \ , \qquad \| x_k \|^2 = \sum_{k=-\infty}^{\infty} | x_k |^2 \ , \tag{2.81}$$

which is a natural extension of the finite-dimensional case. For continuous-time signals, the summation becomes an integral, and the inner product is defined as

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt \ , \qquad \| x(t) \|^2 = \int_{-\infty}^{\infty} | x(t) |^2 \, dt \ . \tag{2.82}$$

Schwarz Inequality

Exercise 2-13.

Verify that the definitions of inner product of (2.81) and (2.82) satisfy the properties of (2.80).

All valid inner products satisfy the *Schwarz inequality*, which bounds their magnitudes:

Exercise 2-14.

(The Schwarz Inequality.) Show that for two elements $\mathbf{X}$ and $\mathbf{Y}$ of an inner product space,

$$|\langle \mathbf{X}, \mathbf{Y} \rangle| \leq \|\mathbf{X}\| \cdot \|\mathbf{Y}\| , \tag{2.83}$$

with equality if and only if $\mathbf{X} = K \cdot \mathbf{Y}$ for some scalar K.

Correlation

In the context of continuous-time signals, the inner product (2.82) is also called the *correlation* of $x(t)$ with $y(t)$, and its computation arises frequently. Intuitively, the correlation measures how closely the two signals resemble one another. At one extreme, when $x(t)$ and $y(t)$ are identical, the correlation reduces to the energy $\int |x(t)|^2 dt$. At the other extreme, the two signals are said to be *orthogonal* when the correlation is zero, $\langle x(t), y(t) \rangle = 0$, which is sometimes written as $x(t) \perp y(t)$. In general, two element signals or vectors $\mathbf{X}$ and $\mathbf{Y}$ are said to be orthogonal whenever $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$, in which case the shorthand $\mathbf{X} \perp \mathbf{Y}$ is used.

The geometric properties are so important that the special name *inner-product space* is given to a linear space on which an inner product is defined. Thus, both Example 2-17 and Example 2-18 defined earlier are inner-product spaces. If the inner product space has the additional property of *completeness*, then it is defined to be a *Hilbert space*. Intuitively the notion of completeness means that there are no "missing" vectors that are arbitrarily close to vectors in the space but are not themselves in the space. Since the spaces used in this book are all complete and hence formally Hilbert spaces, we will not dwell on this property further.

2.6.3. Projection onto a Subspace

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{S}$ be a subspace of $\mathcal{H}$. We are often interested in finding an element of $\mathcal{S}$ that best matches a given element of $\mathcal{H}$. We define the *projection on* $\mathcal{S}$ of any $\mathbf{X} \in \mathcal{H}$ as the unique element $\hat{\mathbf{X}} \in \mathcal{S}$ that is "closest" to $\mathbf{X}$, satisfying

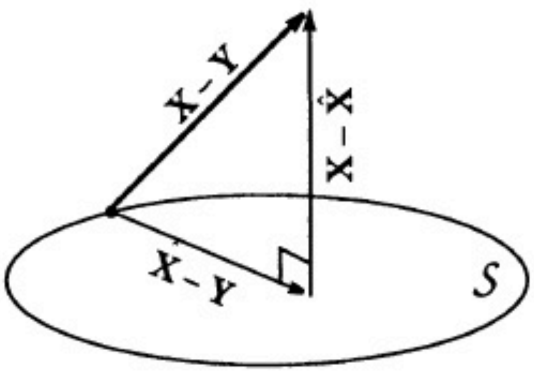
$$\| \mathbf{X} - \hat{\mathbf{X}} \| = \min_{\mathbf{Y} \in \mathcal{S}} \| \mathbf{X} - \mathbf{Y} \| . \tag{2.84}$$

Projection

Theorem 2-2. (Projection Theorem.) Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{S}$ be a subspace of $\mathcal{H}$. Then the *projection* $\hat{\mathbf{X}} \in \mathcal{S}$ of $\mathbf{X}$ onto $\mathcal{S}$ satisfies (2.84) if and only if

$$(\mathbf{X} - \hat{\mathbf{X}}) \perp \mathbf{Y}, \qquad \text{for all } \mathbf{Y} \in \mathcal{S}. \tag{2.85}$$

It is instructive to show how this orthogonality condition implies (2.84). Because $\mathbf{X} \in \mathcal{H}$ and $\mathbf{Y} \in \mathcal{S}$, we can decompose the difference $\mathbf{X} - \mathbf{Y}$ as the sum of $\mathbf{X} - \hat{\mathbf{X}} \perp \mathcal{S}$ and $\hat{\mathbf{X}} - \mathbf{Y} \in \mathcal{S}$; this decomposition is illustrated below:



As a consequence, we have

$$\begin{aligned}\|\mathbf{X}-\mathbf{Y}\|^2 &= \|\mathbf{X}-\hat{\mathbf{X}}+\hat{\mathbf{X}}-\mathbf{Y}\|^2 \\ &= \|\mathbf{X}-\hat{\mathbf{X}}\|^2+\|\hat{\mathbf{X}}-\mathbf{Y}\|^2+2\mathrm{Re}\{\langle \mathbf{X}-\hat{\mathbf{X}}, \hat{\mathbf{X}}-\mathbf{Y}\rangle\}.\end{aligned} \tag{2.86}$$

The projection theorem guarantees that $\hat{\mathbf{X}}-\mathbf{Y}\in\mathcal{S}$ is orthogonal to the projection error, so that the last term is zero, yielding:

$$\|\mathbf{X}-\mathbf{Y}\|^2 = \|\mathbf{X}-\hat{\mathbf{X}}\|^2+\|\hat{\mathbf{X}}-\mathbf{Y}\|^2. \tag{2.87}$$

This is a *Pythagorean theorem*: the left-hand side is the squared hypotenuse of a right-angle triangle, and the right-hand side adds the squared height to the squared base. The squared height is independent of $\mathbf{Y}\in\mathcal{S}$. Clearly, therefore, the sum is minimized by choosing $\mathbf{Y}=\hat{\mathbf{X}}$, which collapses the base to zero, thus implying (2.84).

Example 2-22.

A projection is illustrated in Fig. 2-17 for three-dimensional Euclidean space, where the subspace $\mathcal{S}$ is the plane formed by the x-axis and y-axis and $\mathbf{X}$ is an arbitrary vector. The projection is the result of dropping a perpendicular line from $\mathbf{X}$ down to the plane (this is the dashed line). The resulting vector $(\mathbf{X}-\hat{\mathbf{X}})$ is the vector shown parallel to the dashed line. It is orthogonal to the plane $\mathcal{S}$, and hence to every vector in $\mathcal{S}$.

Example 2-23.

Let $\mathcal{S} = \text{span}\{s_1(t), s_2(t)\}$ be the linear subspace spanned by two rectangular waveforms of Example 2-21. Then the projection $\hat{r}(t)$ of a truncated sinusoid $r(t)$ onto $\mathcal{S}$ is sketched in Fig. 2-18.

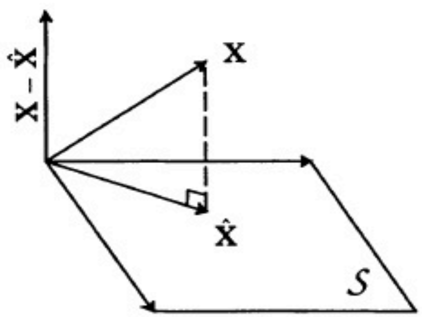


Fig. 2-17. The vector in $\mathcal{S}$ closest to $\mathbf{X}$ is evidently $\hat{\mathbf{X}}$; any other vector in $\mathcal{S}$ is farther from $\mathbf{X}$.

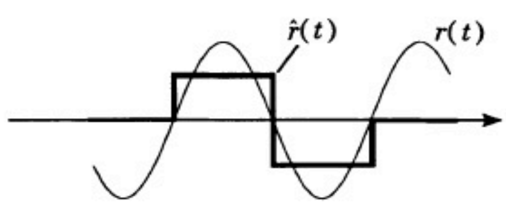


Fig. 2-18. The projection of a sinusoid onto the subspace spanned by two rectangular pulses.

2.6.4. Projection onto the Signal Space

When a digital transmitter transmits one of M signals $\{s_1(t), \dots s_M(t)\}$, it is common for the receiver to project the received signal $r(t)$ onto the signal space spanned by the M signals. (The motivation for this projection will be explained in later chapters.) In concept this projection can be found by searching in a brute-force manner the subspace for the signal closest to the $r(t)$, but in practice it is much more efficient to project $r(t)$ onto a collection of one-dimensional subspaces and combine the results, as described below.

A set of functions $\{\phi_1(t), \phi_2(t), \dots\}$ is said to be *orthonormal* when each signal has unit energy, and distinct signals are orthogonal, so that:

$$\int_{-\infty}^{\infty} \phi_i(t)\phi_k^*(t)dt = \delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}. \tag{2.88}$$

An *orthonormal basis* for the signal space $\mathcal{S} = \text{span}\{s_1(t), \dots s_M(t)\}$ is a minimal set of N orthonormal functions $\{\phi_1(t), \dots \phi_N(t)\}$ with the property that every element $s(t) \in \mathcal{S}$ can be expressed as a linear combination of basis functions:

$$s(t) = \sum_{i=1}^{N} s_i\phi_i(t) . \tag{2.89}$$

An *orthonormal basis* for the signal space $\mathcal{S} = \text{span}\{s_1(t), \ldots s_M(t)\}$ is a minimal set of N orthonormal functions $\{\phi_1(t), \ldots \phi_N(t)\}$ with the property that every element $s(t) \in \mathcal{S}$ can be expressed as a linear combination of basis functions:

$$s(t) = \sum_{i=1}^{N} s_i \phi_i(t) \,. \tag{2.89}$$

Any signal $r(t)$ can now be expressed as a sum $r(t) = \hat{r}(t) + e(t)$, where $\hat{r}(t) = \sum_{i=1}^{N} r_i \phi_i(t)$ is the projection of $r(t)$ onto $\mathcal{S}$, and $e(t) = r(t) - \hat{r}(t)$ is the projection error.The j-th expansion coefficient r_j of the projection is found by taking the inner product of $r(t)$ with the j-th basis function:

$$
\begin{aligned}
\int_{-\infty}^{\infty} r(t)\phi_j^*(t)\,dt &= \langle\, r(t), \phi_j(t) \,\rangle \\
&= \langle\, \hat{r}(t) + e(t), \phi_j(t) \,\rangle \\
&= \langle\, \hat{r}(t), \phi_j(t) \,\rangle + \langle\, e(t), \phi_j(t) \,\rangle \\
&= \langle\, \textstyle\sum_{i=1}^{N} r_i\phi_i(t), \phi_j(t) \,\rangle + 0 \\
&= r_j \,. \qquad (2.90)
\end{aligned}
$$

The second equality follows from the fact that the projection error $e(t)$ is orthogonal to everything in $\mathcal{S}$, including $\phi_j(t)$.

Since the k-th signal $s_k(t)$ is an element of $\mathcal{S}$, it too can be expanded in terms of a basis:

$$
s_k(t) = \sum_{i=1}^{N} s_{k,i}\phi_i(t) \,. \qquad (2.91)
$$

This expansion is most useful when the number of basis functions is smaller than the size of the original signal set, or $N < M$.

Gram - Schmidt

The Gram-Schmidt Orthonormalization Procedure

The Gram-Schmidt orthonormalization procedure is a systematic method for converting a set of M waveforms $\{s_1(t), \ldots s_M(t)\}$ into an orthonormal basis $\{\phi_1(t), \ldots \phi_N(t)\}$ of size N for their span, with N as small as possible. Let the first basis function be:

$$\phi_1(t) = \frac{s_1(t)}{\|s_1(t)\|}, \tag{2.92}$$

where the denominator is the square root of the energy of $s_1(t)$. (The signals may have to be reordered to ensure that this energy is nonzero.) This defines a one-dimensional subspace $\mathcal{S}_1 = \text{span}\{\phi_1(t)\}$. If we let $\hat{s}_2(t)$ denote the projection of $s_2(t)$ onto this subspace, then the projection error will be orthogonal to $\mathcal{S}_1$; in particular, $s_2(t) - \hat{s}_2(t) \perp \phi_1(t)$. Thus, if we normalize the projection error to have unit energy, we produce a second basis function:

$$\phi_2(t) = \frac{s_2(t) - \hat{s}_2(t)}{\|s_2(t) - \hat{s}_2(t)\|}. \tag{2.93}$$

Now $\phi_1(t)$ and $\phi_2(t)$ are unit-energy and orthogonal. But what if the denominator in (2.93) is zero? This would imply that already $s_2(t) \in \mathcal{S}_1$, and so the projection error is zero. In this case we do not use (2.93), but instead we move $s_2(t)$ to the end of the list and *renumber* the signals so that $s_2(t)$ becomes the new $s_M(t)$, $s_3(t)$ becomes the new $s_2(t)$, $s_4(t)$ becomes the new $s_3(t)$, etc.

The Gram-Schmidt procedure repeats the above process M times. Let $\hat{s}_k(t)$ denote the projection of $s_k(t)$ onto the span of $\{\phi_1(t), \ldots \phi_{k-1}(t)\}$ for $k \in \{2, \ldots, M\}$. If the projection error $s_k(t) - \hat{s}_k(t)$ is zero, renumber the signals by moving $s_k(t)$ to the end of the list. Otherwise, the k-th basis function can be found by normalizing the k-th projection error:

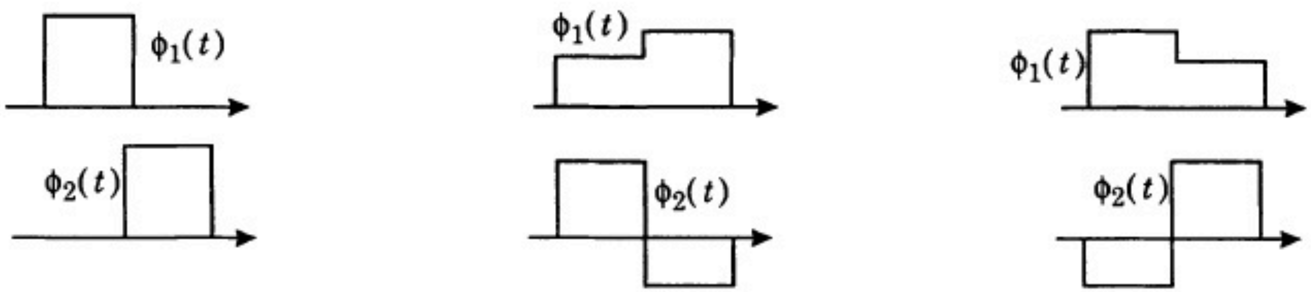
$$\phi_k(t) = \frac{s_k(t) - \hat{s}_k(t)}{\|s_k(t) - \hat{s}_k(t)\|}. \tag{2.94}$$

$$\phi_k(t) = \frac{s_k(t) - \hat{s}_k(t)}{\|s_k(t) - \hat{s}_k(t)\|}. \tag{2.94}$$

The Gram-Schmidt orthonormalization procedure is defined by the recursion (2.94), together with the initialization (2.92) and the renumbering strategy. If all $M-1$ of the projection errors are nonzero, then the signals are linearly independent and the final number of basis functions N will be the same as the number of signals, $N = M$. Each zero projection error would decrease the number of basis functions by one, yielding $N < M$. The *dimension* of $\mathcal{S}$ is the number N of basis functions. The basis functions themselves are not unique; reordering the signals $\{s_i(t)\}$ will generally lead to a different basis. However, the dimension N is fixed and independent of the basis.

Example 2-25.

Recall the signal space $\mathcal{S}$ spanned by the two rectangular pulses of Fig. 2-16(a). There are an infinite number of possible bases for this space. Three examples are shown below:



The basis on the left is found by normalizing $s_1(t)$ and $s_2(t)$, since they are already orthogonal. However, this basis is by no means unique. We may apply the Gram-Schmidt procedure to any pair of linearly independent signals in $\mathcal{S}$ to find a basis. For example, applying the procedure to the first two signals in Fig. 2-16(b) yields the middle basis shown above, and applying the procedure to the same two signals *but in reverse order* yields the last basis shown above.

2.6.5. The Geometry of Signal Space

Once we find an orthonormal basis $\{\phi_1(t), \ldots \phi_N(t)\}$ for the signal space $\mathcal{S}$ spanned by $\{s_1(t), \ldots s_M(t)\}$, the k-th signal $s_k(t)$ may be expressed as a linear combination of basis functions according to (2.91), where the coefficients are given by:

$$s_{k,i} = \int_{-\infty}^{\infty} s_k(t)\phi_i^*(t)dt . \qquad (2.95)$$

Hence, in the context of a basis $\{\phi_1(t), \ldots \phi_N(t)\}$, the k-th signal is uniquely specified by the N expansion coefficients. Thus, we may associate with each signal $s_k(t) \in \mathcal{S}$ the vector

$$\mathbf{s}_k = [s_{k,1}, s_{k,2}, \ldots s_{k,N}]^T, \tag{2.96}$$

which uniquely specifies the signal.

A remarkable consequence is that operations on $s_1(t) \ldots s_M(t)$ can be interpreted as like operations on the signal vectors $\mathbf{s}_1 \ldots \mathbf{s}_M$. In particular, observe that:

$$\begin{aligned}
\langle s_j(t), s_k(t) \rangle &= \int_{-\infty}^{\infty} s_j(t) s_k^*(t) dt \\
&= \int_{-\infty}^{\infty} \left(\sum_{i=1}^{N} s_{j,i} \phi_i(t) \right) \left(\sum_{m=1}^{N} s_{k,m}^* \phi_m^*(t) \right) dt \\
&= \sum_{i=1}^{N} s_{j,i} \sum_{m=1}^{N} s_{k,m}^* \int_{-\infty}^{\infty} \phi_i(t) \phi_m^*(t)\ dt \\
&= \sum_{i=1}^{N} s_{j,i} \sum_{m=1}^{N} s_{k,m}^* \delta_{im} \\
&= \sum_{i=1}^{N} s_{j,i} s_{k,i}^* \\
&= \mathbf{s}_k^* \mathbf{s}_j \\
&= \langle \mathbf{s}_j, \mathbf{s}_k \rangle.
\end{aligned} \tag{2.97}$$

Hence, the inner product between two signals in $\mathcal{S}$ is identical to the inner product of the corresponding coefficient vectors in N-dimensional complex Euclidean space. Furthermore, as a special case of (2.97) when $j = k$, we have:

$$\| s_k(t) \|^2 = \int_{-\infty}^{\infty} | s_k(t) |^2 \, dt = \sum_{i=1}^{N} | s_{k,i} |^2 = \| \mathbf{s}_k \|^2 . \tag{2.98}$$

In words, the energy of the signal is the squared length of its corresponding vector. Finally, if we apply this result to the difference between two signals, we find that:

$$\| s_j(t) - s_k(t) \|^2 = \int_{-\infty}^{\infty} | s_j(t) - s_k(t) |^2 \, dt = \| \mathbf{s}_j - \mathbf{s}_k \|^2 . \tag{2.99}$$

Hence, the energy in the error between two signals is equal to the squared distance between the two corresponding vectors.

Observe that (2.98) is a form of *Parseval's relationship*: the energy of a signal in one domain (the time domain) is equal to its energy in another. For this reason, (2.97) may be viewed as a *generalized* Parseval's relationship: the inner product in one domain is equal to the inner product in another. See Problem 2-4 and Problem 2-5.

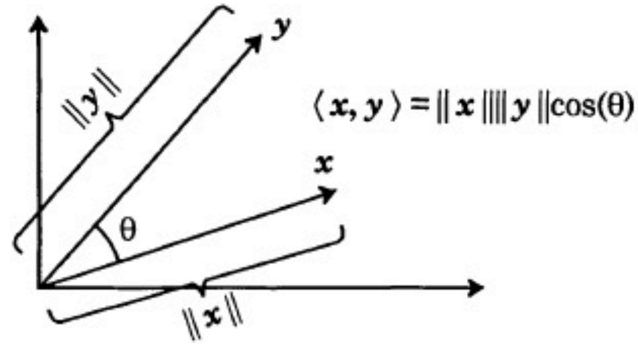


Fig. 2-19. Geometrical interpretation of inner product, illustrated in the plane spanned by x and y.

The implication of the equivalence (2.97) is that, by equating signals with their vector counterparts in Euclidean space, we can now adopt geometric intuition when thinking about continuous-time signals. The power of this tool stems from our considerable experience and intuition regarding Euclidean space. For example, we may think of signals as being close to one another when the distance between their vectors is small, i.e., when the energy of the error is small. Or we may think of two signals as being orthogonal, or at right angles, when their correlation is zero.

Angles

When the signals are real, the signal vectors $\{\mathbf{s}_1, \ldots \mathbf{s}_M\}$ live in *real* Euclidean space of N dimensions. In this case, the inner product defines the *angle* between two signals. Consider the correlation between two signals $x(t)$ and $y(t)$; as illustrated in Fig. 2-19, the correlation is equal to the product of the length of the first vector, the length of the second vector, and the cosine of the angle between the vectors. The angle θ between two signals $x(t)$ and $y(t)$ is the angle between the two signal vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Note that $\|\boldsymbol{x}\|\cos(\theta)$ is the length of the component of $\boldsymbol{x}$ in the direction of $\boldsymbol{y}$. Hence we get a particularly useful interpretation of correlation: $\int_{-\infty}^{\infty} x(t)y(t)\,dt/\left(\int_{-\infty}^{\infty}|y(t)|^2dt\right)^{1/2} = \langle x(t), y(t)\rangle/\|y(t)\|$ is the length of the component of $x(t)$ in the direction of $y(t)$, and $\langle y(t), x(t)\rangle/\|x(t)\|$ is the length of the component of $y(t)$ in the direction of $x(t)$. Here, the length of a signal is the square root of its energy.

Example 2-26.

Consider $s_1(t)$ and $s_2(t)$ as defined in Fig. 2-16(a), and define $s_3(t)$ through $s_6(t)$ as the four signals in Fig. 2-16(b) from top to bottom. Given a basis for the span $\mathcal{S}$ of these signals, each can be represented as a two-dimensional vector of expansion coefficients. For example, in terms of the first basis of Example 2-25, the vectors $\mathbf{s}_1$ through $\mathbf{s}_6$ corresponding to $s_1(t)$ through $s_6(t)$ are shown in Fig. 2-20(a). Fig. 2-20(b) shows how the same signals map to vectors when the second basis of Example 2-25 is used. Finally, Fig. 2-20(c) corresponds to the last basis of Example 2-25. Observe that the picture in (b) is a reflected and rotated version of that in (a), while the picture in (c) is a rotated version (by about 34°) of that in (a). As a result, the geometric relationships between signals is invariant to the choice of the basis: the distance between $\mathbf{s}_2$ and $\mathbf{s}_3$ (for example, or the angle between $\mathbf{s}_4$ and $\mathbf{s}_5$) is the same regardless of which basis is chosen. This invariance is a general result that follows from the generalized Parseval's relationship of (2.97): since the left-hand side of (2.97) is clearly independent of the basis, the right-hand side must be as well.

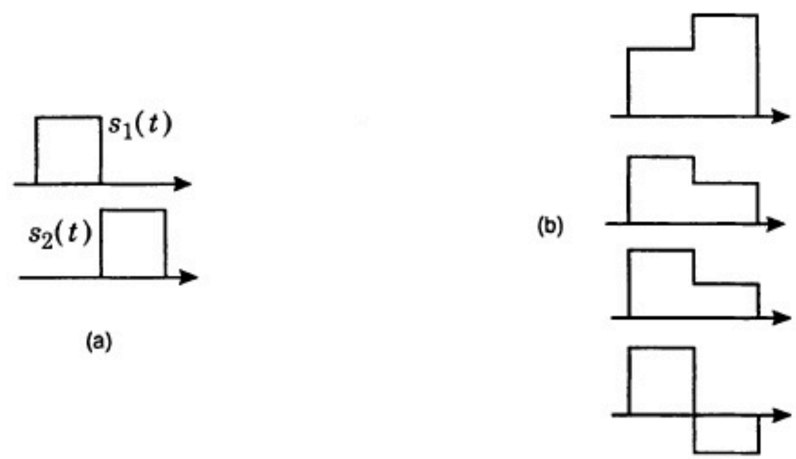


Fig. 2-16. (a) Two signals $s_1(t)$, $s_2(t)$; (b) four examples of signals that are in $\mathcal{S} = \mathrm{span}\{s_1(t), s_2(t)\}$.

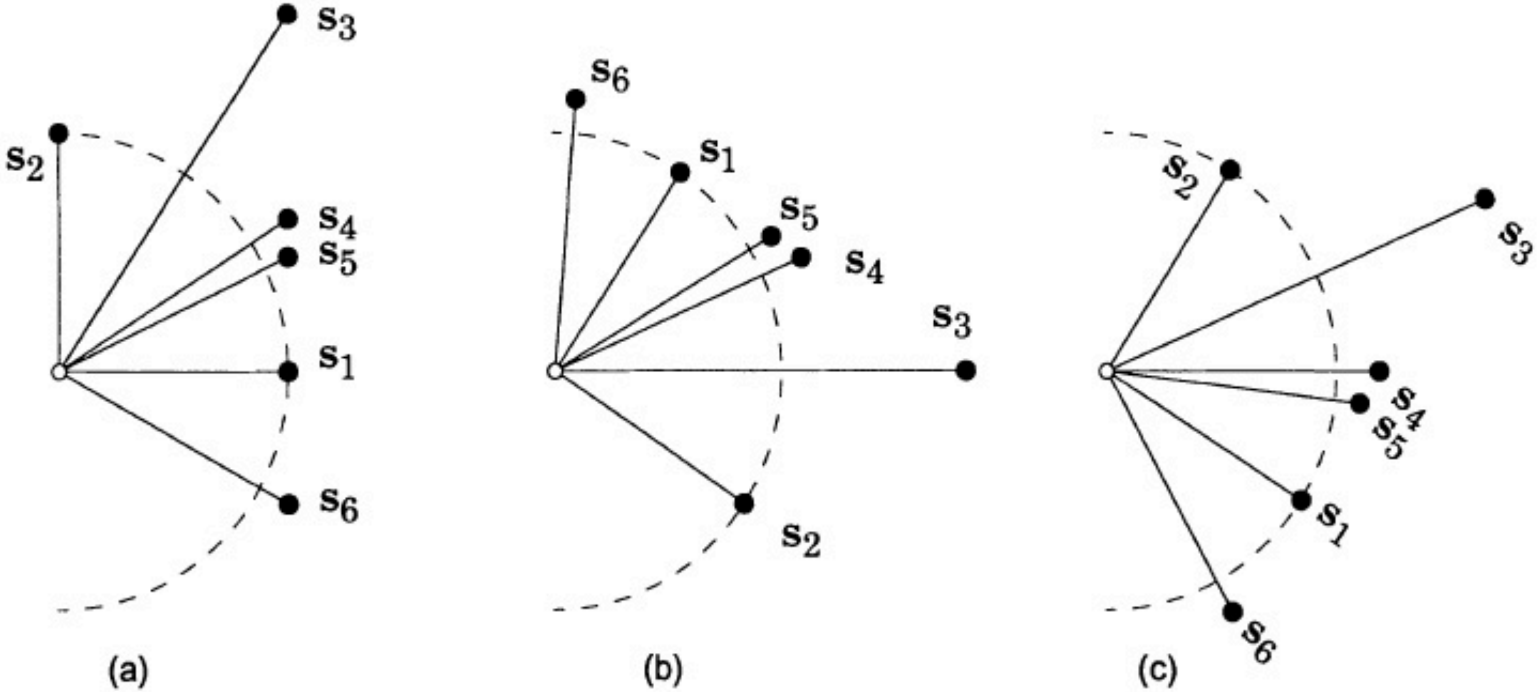
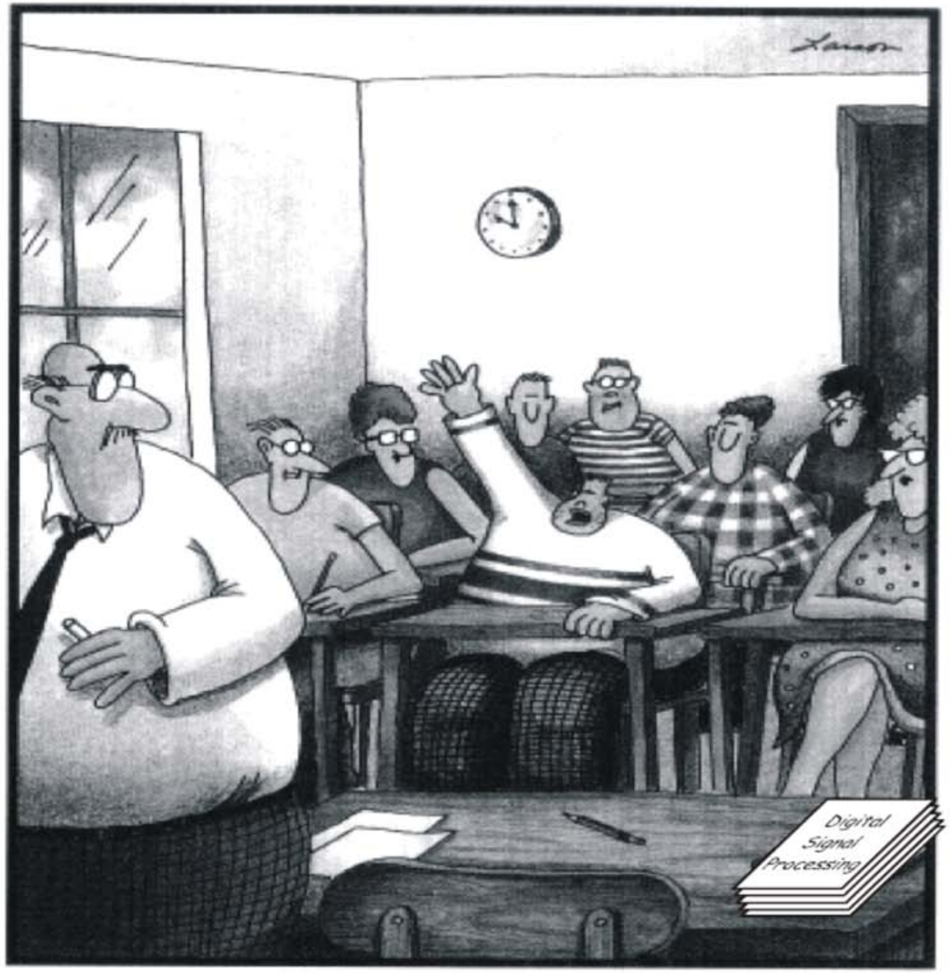


Fig. 2-20. A geometric representation of the six waveforms of Fig. 2-16. Regardless of whether basis (a), basis (b), or basis (c) is used, the geometric relationships between the different signal vectors remain the same.

That's all Folks



Professor harris, may I be excuse
My brain is full.