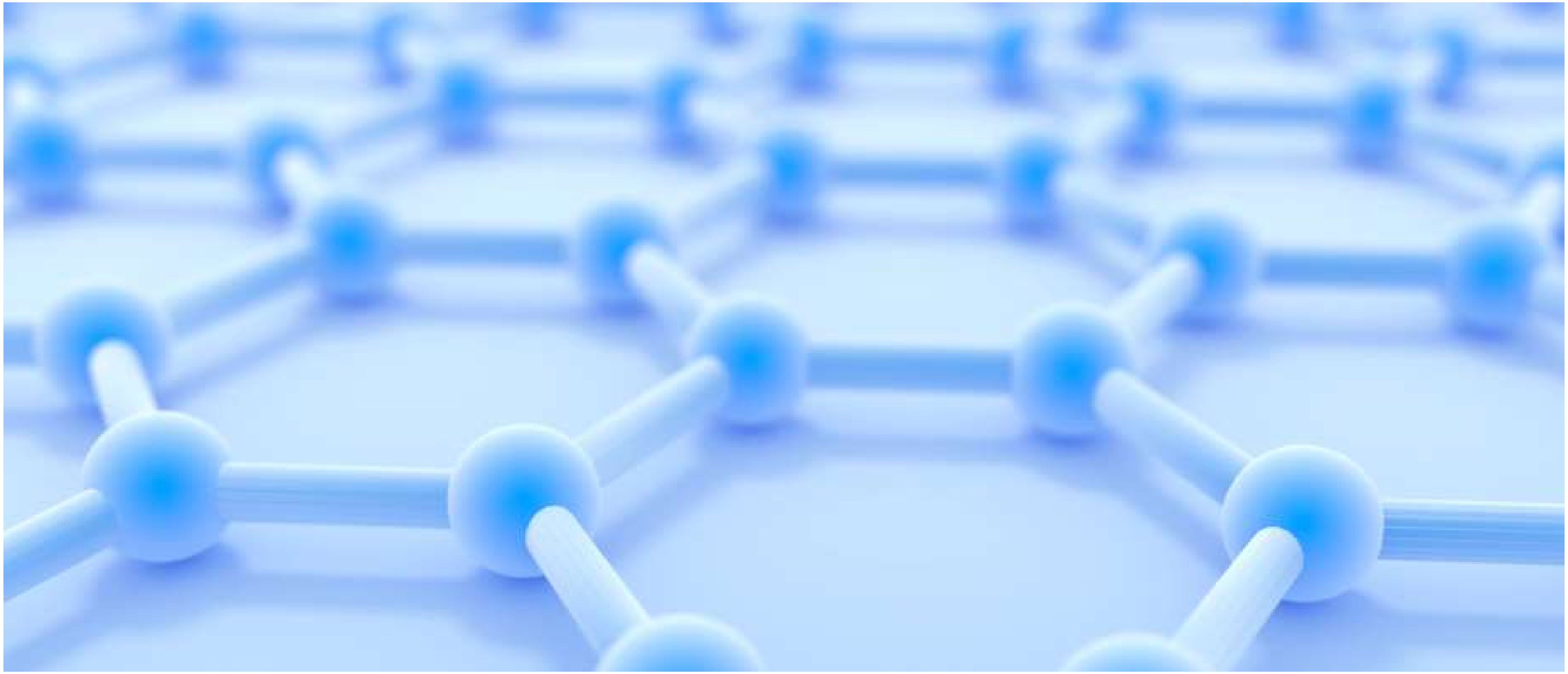

Photonics
aa 2021/2022

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Electromagnetic Optics:
Maxwell's equations in free space and in dielectric media, boundary conditions and constitutive relations, Wave equation

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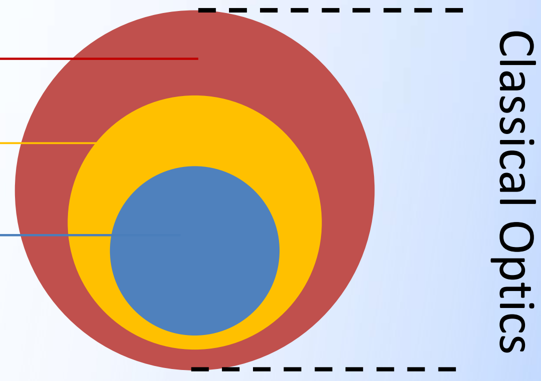
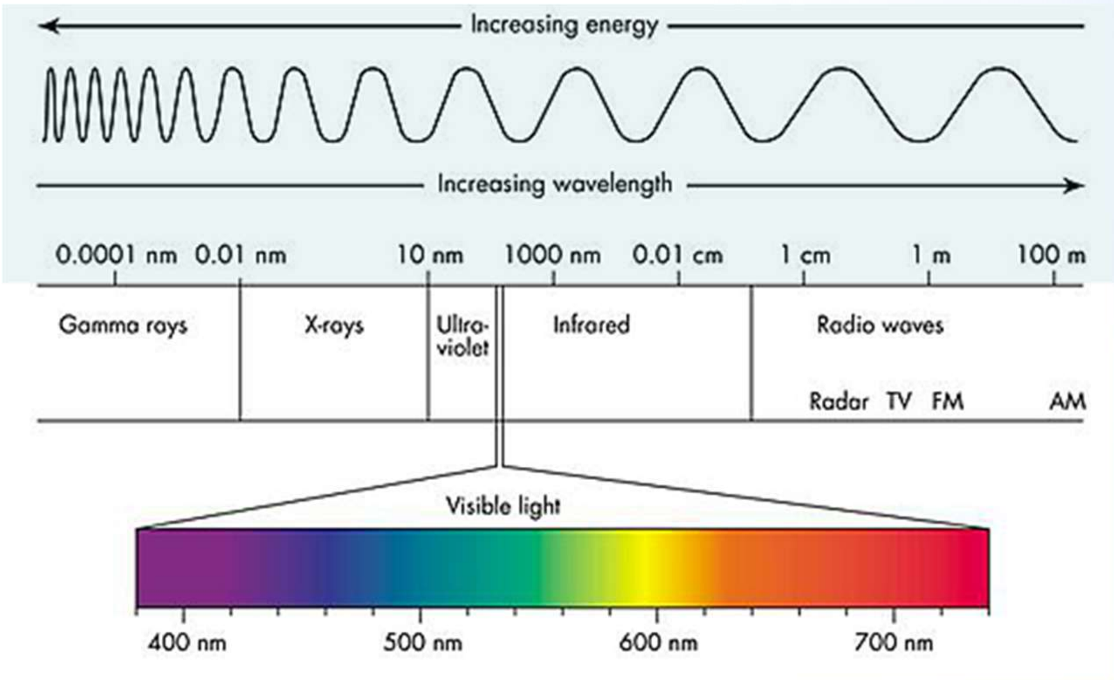
Introduction



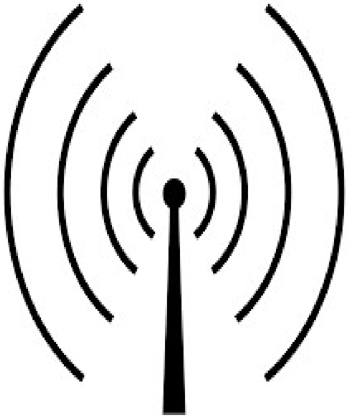
As much as **wave optics** and **ray optics** can describe many phenomena, including interference and diffraction, they still cannot explain simple phenomena and give quantitative answers to some experiments such as the division of light at a beamsplitter.

For these kind of phenomena we need to treat light as a vector and, therefore, we need to step forward with a different approach: **electromagnetic optics**.



Introduction to Electromagnetic Fields

transmitter



field

receiver

When an event in one place has an effect on something at a different location, we talk about the events as being connected by a “field”.

A ***field*** is a spatial distribution of a quantity; in general, it can be either *scalar* or *vector* in nature.

- A ***scalar*** is a quantity having only an amplitude (and possibly phase).

Examples: voltage, current, charge, energy, temperature

- A ***vector*** is a quantity having direction in addition to amplitude (and possibly phase).

Examples: velocity, acceleration, force

Introduction to Electromagnetic Fields

- Electric and magnetic fields:
  - Are vector fields with three spatial components.
  - Vary as a function of position in 3D space as well as time.
  - Are governed by partial differential equations derived from Maxwell’s equations.

Fundamental vector field quantities in electromagnetics:

- Electric field **E** [V/m]
- Electric flux density (electric displacement) **D** [C/m^2] (ε [F/m])
- Magnetic field **H** [A/m]
- Magnetic flux density (magnetic induction) **B** [T] (μ[H/m])

Universal constants in electromagnetics:

– Velocity of light in free space (vacuum): $c \approx 3\times10^{8}$ [m/s]

– Permeability of free space: $\mu_0 = 4\pi\times10^{-7}$ [H/m]

– Permittivity of free space: $\varepsilon_0 \approx 8.854\times10^{-12}$ [F/m]

– Intrinsic impedance of free space: $\eta_0 \approx 120\pi\,[\Omega]$

Relationships involving the universal constants:

$$\boxed{c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}} \qquad \boxed{\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}}$$

Fundamentals of Vector Calculus

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Divergence

$$\nabla \bullet \mathbf{A} = \frac{\partial \mathrm{A}_x}{\partial x} + \frac{\partial \mathrm{A}_y}{\partial y} + \frac{\partial \mathrm{A}_z}{\partial z}$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathrm{A}_x & \mathrm{A}_y & \mathrm{A}_z \end{vmatrix}$$

Laplace operator

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{SCALAR}$$

$$\nabla^2 \mathbf{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z \quad \text{VECTOR}$$

Useful identities

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla\left(\nabla \bullet \mathbf{A}\right) - \nabla^2 \mathbf{A} \qquad \nabla \bullet \left(\nabla \times \mathbf{A}\right) = 0$$

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Maxwell’s Equations

- ***Maxwell’s equations*** are the fundamental postulates of classical electromagnetics. All the **classical** electromagnetic phenomena are explained by these equations.
- Phenomena: Propagation in free-space, interactions with matter, behavior of waves at interfaces.
- Valid to describe wave effects for signals in the whole spectrum, from radio frequencies and below up to optical frequencies and beyond.

Maxwell's Equations

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In a generic medium the Maxwell's Equations in the time domain are…

Description	Differential form	Integral form
Gauss' Law	$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f$	$\oiint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho_f dV$
Gauss' law for magnetism	$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$	$\oiint \boldsymbol{B} \cdot d\boldsymbol{S} = 0$
Faraday's Law	$\boldsymbol{\nabla} \times \mathbf{E} = -\partial \boldsymbol{B}/\partial t\text{-}\boldsymbol{M}$	$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\iint \boldsymbol{B} \cdot d\boldsymbol{S} - \iint \boldsymbol{M} \cdot d\boldsymbol{S}$
Ampere's law	$\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_f$	$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \frac{d}{dt}\iint \boldsymbol{D} \cdot d\boldsymbol{S} + \iint \boldsymbol{J}_f \cdot d\boldsymbol{S}$

Continuity Equation

The *continuity equation* is implicit in Maxwell's equations when taking the divergence of the Ampere's law

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_f}{\partial t}}$$

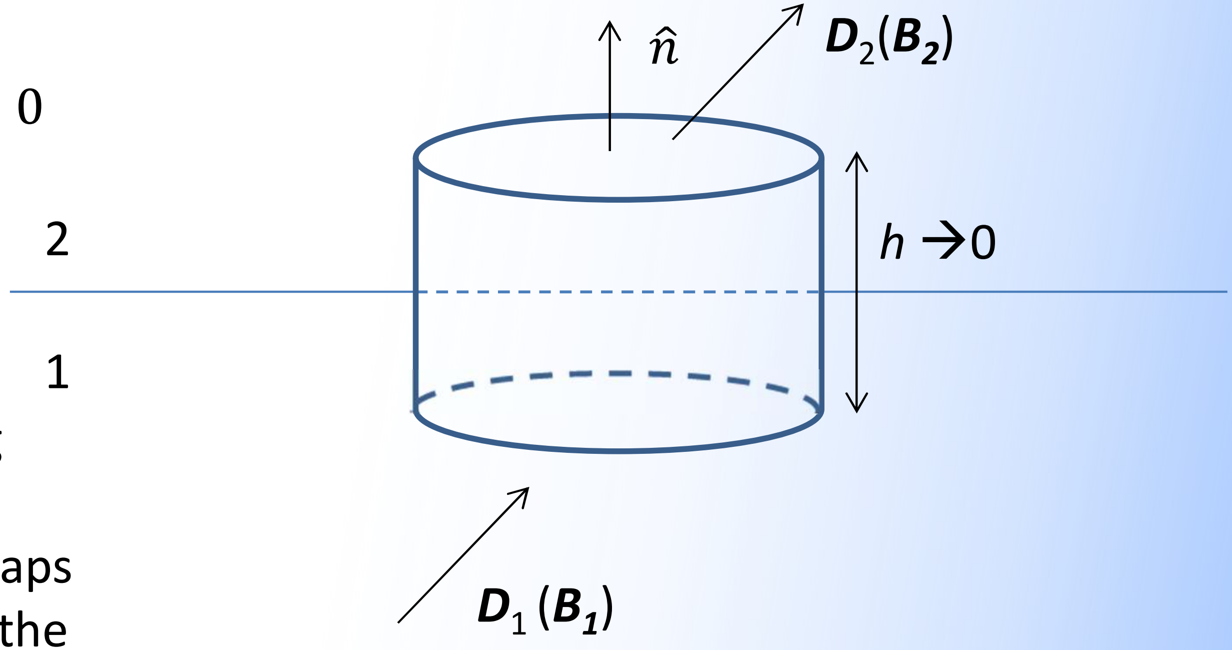
This means that *<u>the flux of current from a closed surface represents a decrease of the charge inside the surface</u>*

Boundary conditions

We can extract the boundary conditions connecting the electromagnetic field between two regions applying the integral form of Maxwell's equations.

$$\oiint \boldsymbol{D} \cdot d\boldsymbol{S} = \iiint \rho_f dV \qquad \oiint \boldsymbol{B} \cdot d\boldsymbol{S} = 0$$

We can consider a Gaussian pillbox that straddles the surface under study. As the cylinder is made smaller and smaller the surface integrals containing the transverse field components vanish and the surface integrals over the end caps containing field components normal to the boundary satisfy the following equations:



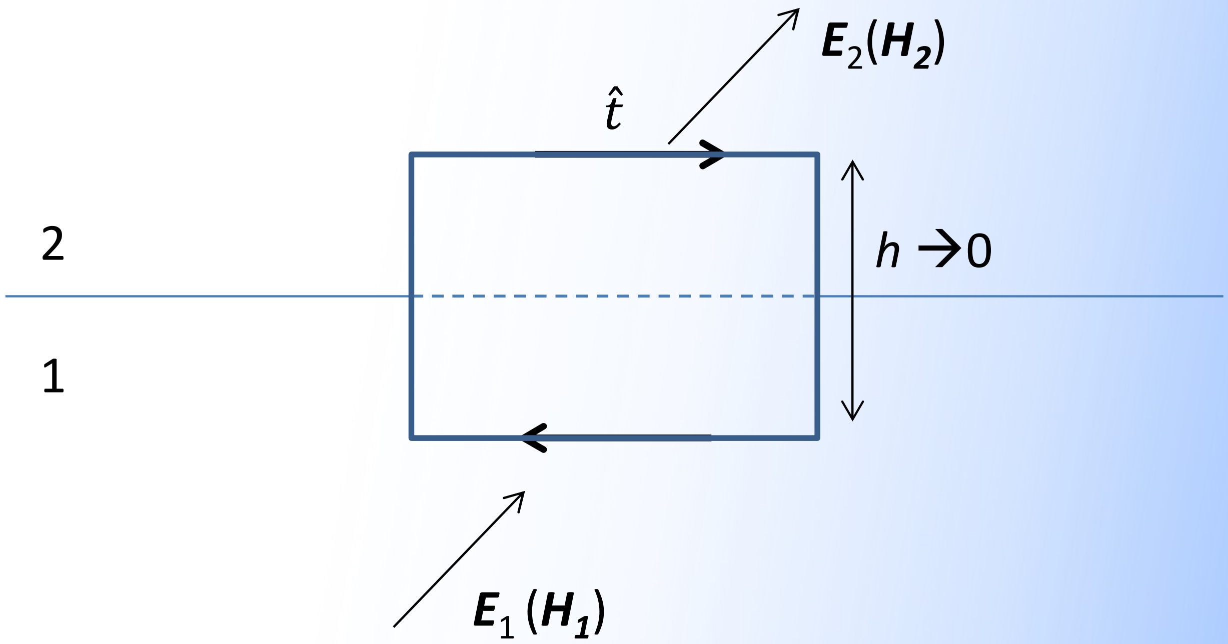
$$\boxed{\hat{n} \cdot (\boldsymbol{D_2} - \boldsymbol{D_1}) = \rho_s} \qquad \boxed{\hat{n} \cdot (\boldsymbol{B_2} - \boldsymbol{B_1}) = 0}$$

In other words the normal components of the electric displacement field and the magnetic induction are continuous across an interface.

Boundary conditions

The second set of boundary conditions connecting the two media is derived from the integral forms of Ampere's law and Faraday's law.

The line integral is traced over a closed loop that straddles the interface. As the size of the loop is reduced to an infinitesimal size, the field contributions tangential to the boundary are the leading terms and the line segments perpendicular to the boundary cancel one another. The resulting boundary conditions can be recast into a vector form as:



$$\boxed{\hat{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0}$$

$$\boxed{\hat{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s}$$

In other words the tangential components of the electric field and the magnetic field are continuous across an interface.

Constitutive Relations

Maxwell's equations become complete with the constitutive relations that are in charge of describing the electromagnetic response of the media, which may have temporal and spatial dispersion.

They relate the "primary" fields **E** and **H** to the "secondary" fields **D** and **B** by including the properties of the medium in which the fields propagate.

In free space:

$$\boxed{\boldsymbol{D} = \epsilon_0 \boldsymbol{E}}$$

$$\boxed{\boldsymbol{B} = \mu_0 \boldsymbol{H}}$$

$$\boxed{\boldsymbol{J} = 0}$$

Constitutive Relations

In a generic medium we can write the constitutive relations as:

$$\boxed{\boldsymbol{D} = \varepsilon \boldsymbol{E}}$$

$$\boxed{\boldsymbol{B} = \mu \boldsymbol{H}}$$

Where:

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$\mu = \mu_0 \mu_r$$

Where ε_r is the relative permittivity and μ_r is the relative permeability.

The response of conduction electrons to the electric field is given by the Ohm's law (or current equation):

$$\boxed{\boldsymbol{J} = \sigma \boldsymbol{E}}$$

Maxwell's Equations in the frequency domain

So far we have written Maxwell's equations in the time domain. For many situations, it is very common to deal with fields having a sinusoidal or harmonic time dependence. In this case is very practical to write the fields in phasor notation, therefore assuming all quantities to be complex vectors with an implicit $e^{j\omega t}$ time dependence.

For example, a sinusoidal electric field in the x direction can be written as:

$$\mathbf{E}(x, y, z, t) = \hat{x} A(x, y, z, t) \cos\left(\omega t + \phi\right)$$

With *A* the (real) amplitude, ω the angular frequency and ϕ the phase reference of the wave at t=0; The relation between the phasor and the time-domain quantity is:

$$\mathbf{E}(x, y, z, t) = \mathrm{Re}\left[\mathbf{E}(x, y, z) e^{j\omega t}\right]$$

Why it may be so important to write Maxwell's equations in the frequency domain?

Reason #1: The constitutive relations can be written in a simple and compact way.

Reason #2: If we know the fields in the frequency domain, we can apply the inverse Fourier transform in order to retrieve their time evolution of the electric field.

Maxwell's Equations in the frequency domain

If the fields are *monochromatic*, then we can use the complex-field amplitude representation

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r},\omega)e^{j\omega t}\right] = \frac{1}{2}\left[\mathbf{E}(\mathbf{r},\omega)e^{j\omega t} + \mathbf{E}^*(\mathbf{r},\omega)e^{-j\omega t}\right]$$

$$\partial / \partial t \rightarrow j\omega$$

IMPORTANT: $\mathbf{E}(\mathbf{r},t)$ is a real vector, $\mathbf{E}(\mathbf{r},\omega)$ is a complex vector

Time-harmonic Maxwell's equations

$$\boxed{\begin{aligned}
&\nabla \times \mathbf{E}(\mathbf{r},\omega) = -j\omega\mathbf{B}(\mathbf{r},\omega)\\
&\nabla \times \mathbf{H}(\mathbf{r},\omega) = j\omega\mathbf{D}(\mathbf{r},\omega) + \mathbf{J}(\mathbf{r},\omega)\\
&\nabla \cdot \mathbf{D}(\mathbf{r},\omega) = \rho(\mathbf{r},\omega)\\
&\nabla \cdot \mathbf{B}(\mathbf{r},\omega) = 0
\end{aligned}}$$

Wave equation

In general, the wave equations are vector equations that incorporate Maxwell's equations into a single expression for a single field and they are obtained using the constitutive relations.

Let's start from simplicity from the Maxwell's equations in the frequency domain and in the monochromatic approximation:

$$\nabla \times \mathbf{E}(\mathbf{r},\omega) = -j\omega\mathbf{B}(\mathbf{r},\omega)$$

$$\nabla \times \mathbf{H}(\mathbf{r},\omega) = j\omega\mathbf{D}(\mathbf{r},\omega) + \mathbf{J}(\mathbf{r},\omega)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},\omega) = \rho(\mathbf{r},\omega)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},\omega) = 0$$

The constitutive relations are:

$$\mathbf{B}(\mathbf{r},\omega) = \mu(\mathbf{r},\omega)\mathbf{H}(\mathbf{r},\omega),\ \mathbf{D}(\mathbf{r},\omega) = \varepsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega)\ ,\ \mathbf{J}(\mathbf{r},\omega) = \sigma(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega)$$

Wave equations
(Inhomogeneous Helmholtz equations)

Taking the curl of the Faraday's law and Ampere's law and replacing the magnetic/electric fields and the constitutive relations we have:

$$\nabla \times \mathbf{E}(\mathbf{r},\omega) = -j\omega\mathbf{B}(\mathbf{r},\omega)$$

$$\boxed{\nabla \times \left(\frac{\nabla \times \mathbf{E}(\mathbf{r},\omega)}{\boldsymbol{\mu}(\mathbf{r},\omega)} \right) - \omega^2 \boldsymbol{\varepsilon}(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = -j\omega\mathbf{J}(\mathbf{r},\omega)}$$

$$\nabla \times \mathbf{H}(\mathbf{r},\omega) = j\omega\mathbf{D}(\mathbf{r},\omega) + \mathbf{J}_f(\mathbf{r},\omega)$$

$$\boxed{\nabla \times \left(\frac{\nabla \times \mathbf{H}(\mathbf{r},\omega)}{\boldsymbol{\varepsilon}(\mathbf{r},\omega)} \right) - \omega^2 \boldsymbol{\mu}(\mathbf{r},\omega)\mathbf{H}(\mathbf{r},\omega) = \nabla \times \left(\frac{\mathbf{J}(\mathbf{r},\omega)}{\boldsymbol{\varepsilon}(\mathbf{r},\omega)} \right)}$$

Plane wave equations
(Homogeneous Helmholtz equations)

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In a homogeneous, isotropic medium the material coefficients are independent of the spatial coordinates (details on material properties will be discussed later on).
We will further assume that current density and free charge are zero. Under these assumptions we can write:

$$\nabla \times \mathbf{E}(\mathbf{r},\omega) = -j\omega \mathbf{B}(\mathbf{r},\omega)$$

$$\Downarrow$$

$$\boxed{\nabla^2 \mathbf{E}(\mathbf{r},\omega) + k^2 \mathbf{E}(\mathbf{r},\omega) = 0}$$

$$\nabla \times \mathbf{H}(\mathbf{r},\omega) = j\omega \mathbf{D}(\mathbf{r},\omega) + \mathbf{J}_f(\mathbf{r},\omega)$$

$$\Downarrow$$

$$\boxed{\nabla^2 \mathbf{H}(\mathbf{r},\omega) + k^2 \mathbf{H}(\mathbf{r},\omega) = 0}$$

Plane wave equations
(Homogeneous Helmholtz equations)

Solution of these homogeneous equations is a plane wave that can be written as follows:

$$\boxed{\nabla^2\mathbf{E}(\mathbf{r},\omega)+k^2\mathbf{E}(\mathbf{r},\omega)=0} \Rightarrow \boxed{\left(\nabla^2+k^2\right)\mathbf{E}(\mathbf{r},\omega)=0}$$

$$\nabla^2\mathbf{A}=\hat{x}\nabla^2 A_x+\hat{y}\nabla^2 A_y+\hat{z}\nabla^2 A_z \quad \text{VECTOR}$$

If $\mathbf{E}=(E_x,E_y,E_z)$

The vector equation is equivalent to three scalar equations that can be written as:

$$\left\{\begin{array}{l}\boxed{\left(\nabla^2+k^2\right)E_x=0}\\ \boxed{\left(\nabla^2+k^2\right)E_y=0}\\ \boxed{\left(\nabla^2+k^2\right)E_z=0}\end{array}\right.$$

For the separation of variables we can write:

$$E_{i,i=x,y,z}=X(x)Y(y)Z(z)$$

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Plane wave equations
(Homogeneous Helmholtz equations)

Applying the second derivative and dividing by XYZ we get:

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}+k^2=0$$

Since the three function in the equations are independent we can write:

$$\frac{X''}{X}+k_x^{\ 2}=0$$

$$\frac{Y''}{Y}+k_y^{\ 2}=0$$

$$\frac{Z''}{Z}+k_z^{\ 2}=0$$

Therefore: $\boxed{k_x^{\ 2}+k_y^{\ 2}+k_z^{\ 2}=k^2}$

DISPERSION RELATION

The solution of these differential equations is:

$$X=X_0 e^{-jk_x x}+cc$$

$$Y=Y_0 e^{-jk_y y}+cc$$

$$Z=Z_0 e^{-jk_z z}+cc$$

Plane wave equations
(Homogeneous Helmholtz equations)

Therefore, we can assume the field to have the following form:

$$\mathbf{E} = \mathbf{E_0} e^{-j\left(k_x x + k_y y + k_z z\right)} + cc = E_0 e^{-j\mathbf{kr}} + cc$$

Where the modulus of the wavevector **k** is the wavenumber: $\left|\mathbf{k}\right| = k = \omega\sqrt{\mu\varepsilon}$

$$\boxed{k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = k_0 n}$$

where *n* is the (complex) refractive index of the medium and

$$\boxed{k_0 = \frac{\omega}{c}}$$

is the free-space wavenumber

NOTE: the wavenumber can be complex, and the plane wave is attenuated as it propagates through a lossy medium

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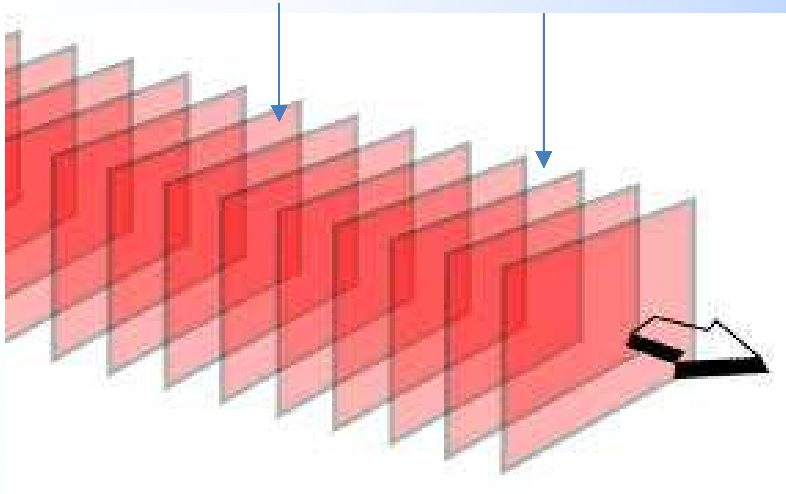
Wave velocities

The velocity of an electromagnetic wave needs to be carefully examined to avoid pitfalls that lead to confusing and even nonphysical interpretations of experimental results.
An optical pulse carries carrier frequency oscillations and an envelope function of the field.

The **phase velocity** is the velocity at which the phase of any frequency component of the wave travels. In other words is the ratio between the space travelled by a plane of the wave over the time it takes to travel that space. If we consider the temporal solution of the wave equation assuming the wave propagates only in the z direction:

$$E(t) = E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r})$$

$$\omega t_1 - kz_1 = \omega t_2 - kz_2$$



The surface of constant phase are planes:

$$\omega t - \mathbf{k} \cdot \mathbf{r} = \text{constant}$$

Let’s assume **k** is in the z direction, then we can write the velocity as

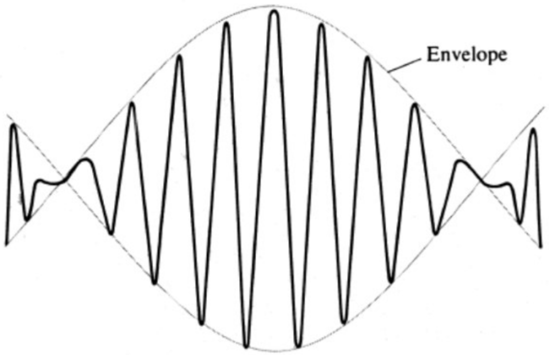
$$\boxed{v_p = \Delta z / \Delta t = \omega / k = 1 / \sqrt{\mu\varepsilon} = c/n}$$

Wave velocities

What happens if a wave packet, i.e., a collection of waves with different frequencies travels through a dispersive medium?

Let's consider only two waves traveling in the z direction with same amplitude and slightly different frequencies: $\omega+\Delta\omega$ and $\omega-\Delta\omega$. The two waves have different wave numbers, $k+\Delta k$ and $k-\Delta k$. Then the superposition of the two waves is

$$E = E_0 e^{-j[(\omega+\Delta\omega)t-(k+\Delta k)z]} + E_0 e^{-j[(\omega-\Delta\omega)t-(k-\Delta k)z]}$$



$$E = 2\underbrace{E_0 e^{-j(\omega t-kz)}}_{\text{plane wave}} \underbrace{\cos(t\Delta\omega - z\Delta k)}_{\text{modulation envelope}}$$

The modulation envelope travels at the **group velocity**:

$$\boxed{v_g = \Delta\omega / \Delta k}$$

For a medium with dispersive index $n(\omega)$, the group velocity is:

$$\boxed{v_g = \partial(ck/n)/\partial k = c/n - ck/n^2 \partial n/\partial k = v_p\left(1 - \frac{k}{n}\frac{\partial n}{\partial k}\right)}$$

Phase vs Group Velocity