CHAPTER 2
Reminders on modes in optical waveguides (example of the slab waveguide)

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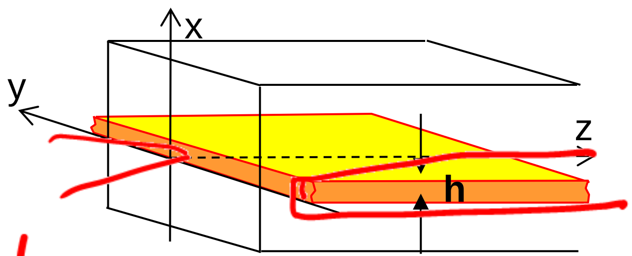
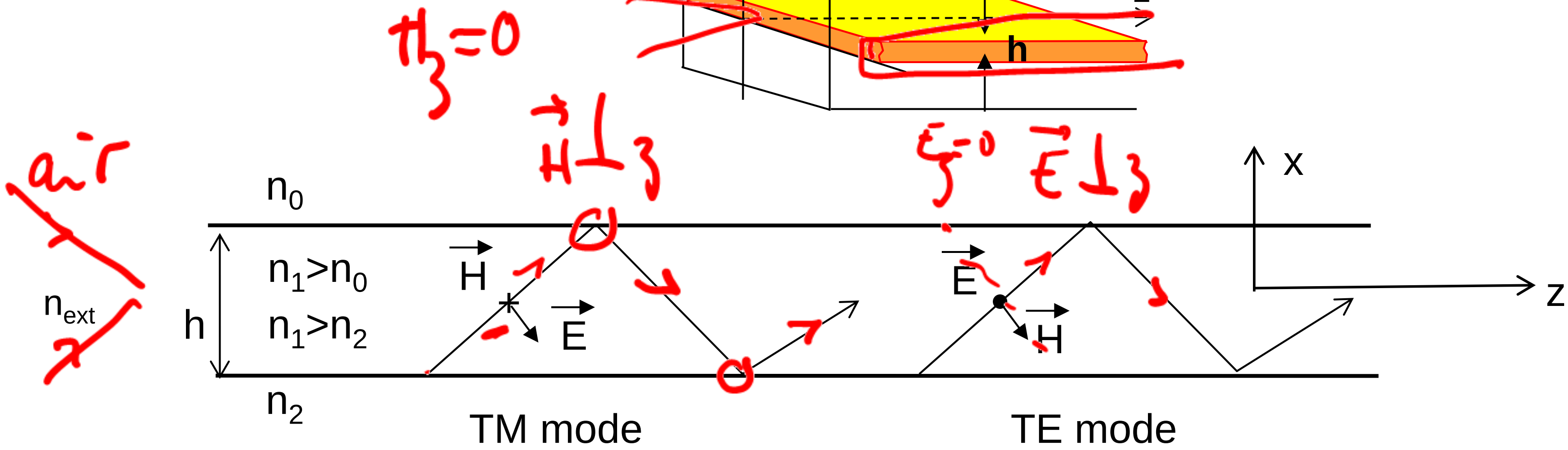
With the support of the Erasmus+ Programme of the European Union

Slab waveguide





- ➢ slab waveguide → 3 layers of transparent dielectric materials with indices : n_0, n_1, et n_2
- ➢ constitution : substrate (index n_2), confinement waveguide (index n_1), superstrate (indice n_0).
- ➢ **guiding conditions : $n_1>n_2$ et $n_1> n_0$** (if $n_0 = n_2$: symmetrical waveguide)

In practical case, we often have: $n_0 = n_{ext} = 1$ (air)

For an optical fiber: $n_0 = n_2$

Guiding principle of light (1)

guiding if :

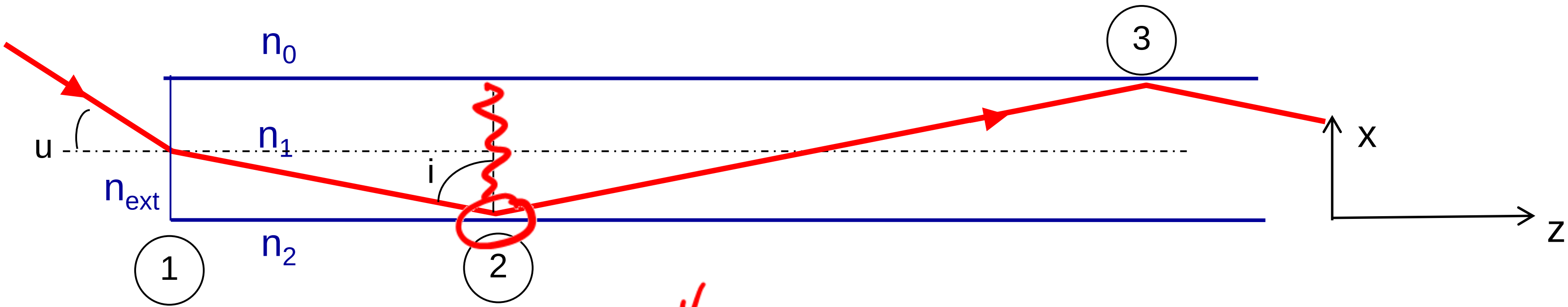
- réfraction at interface (1) (input face)
- total reflections at interfaces (2) , en (3)

totale reflection if : $i > i_l = Arc\sin(n_2 / n_1)$

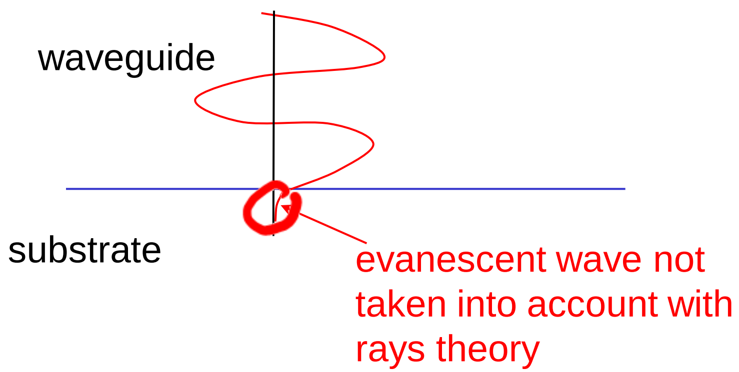
if $i=i_l$, u= u_{max} with sin u_{max} = numerical aperture(NA)

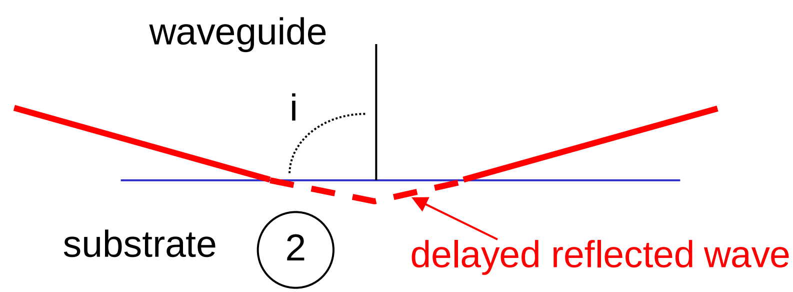
$$\boxed{NA = \frac{\sqrt{n_1^2 - n_2^2}}{n_{ext}}}$$

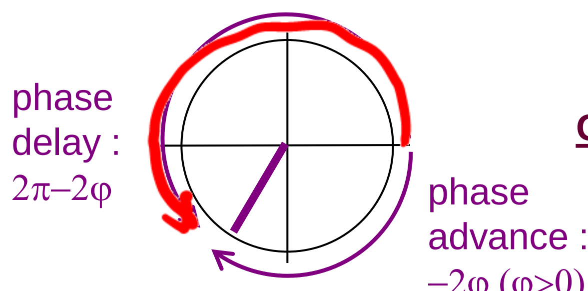
Guiding principle of light (2)



Warning : at the reflection points 2 and 3, ~~te~~ the field does not cancel !

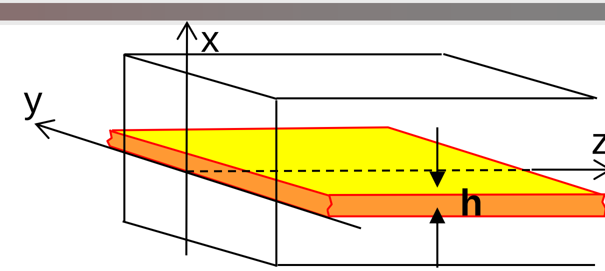




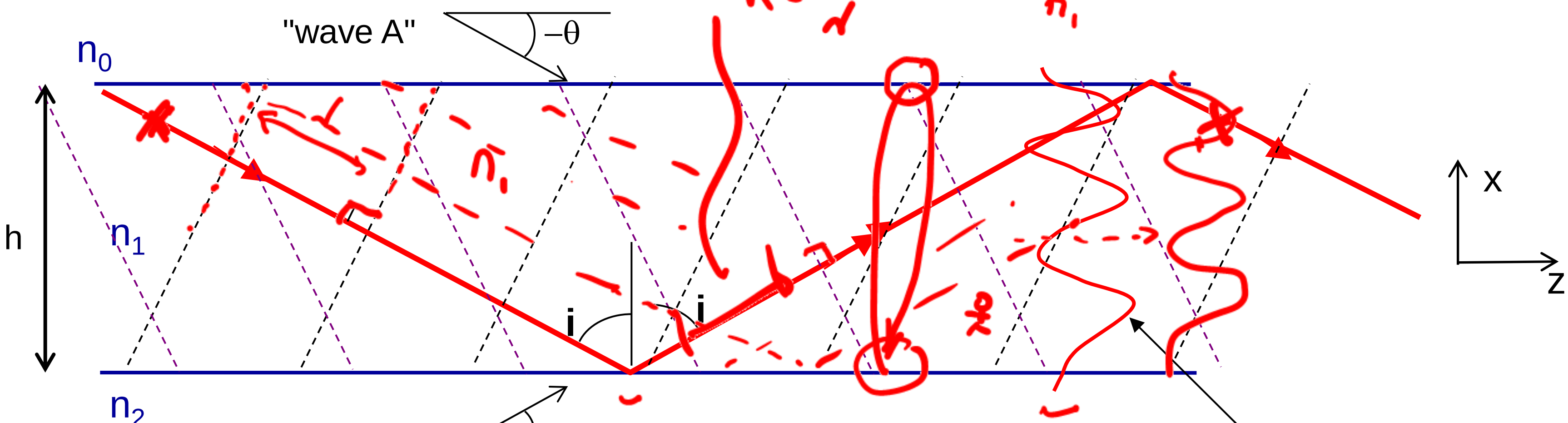


Goos Hänchen phase shift

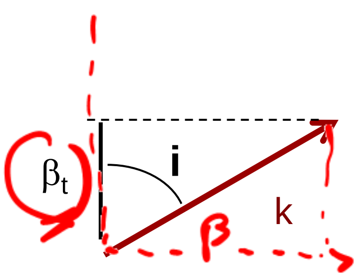
Modes of a slab waveguide (1)



$k = \frac{2\pi}{\lambda}$ $\lambda = \frac{\lambda_0}{n_1}$



interference pattern unchanged with z
= <u>transverse mode</u>



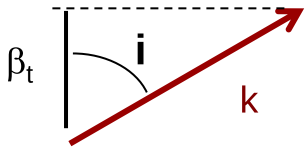
$$\beta_t = k\cos i$$

condition of existence of a mode : $\beta_t .2h - 2\varphi_{10} - 2\varphi_{12} = 2m\pi$

0
1
2
:

$-2\varphi_{10}$ and $-2\varphi_{12}$: Goos-Hänchen phase shifts at interfaces n_1/n_0 and n_1/n_2 , respectively

Modes of a slab waveguide (2)



$$\beta_t = k\cos i$$

One mode can exist if: $\beta_t.2h - 2\varphi_{10} \text{-} 2\varphi_{12} = 2m\pi$

thus : $\boxed{\cos i = \left(m\pi + \varphi_{10} + \varphi_{12}\right)\dfrac{1}{k_0 n_1 h}}$ given h, $\varphi_{10} = f(i)$, $\varphi_{12} = g(i)$

For each value of m → one value of i → one interference pattern → one transverse mode

➔ angles i are discretised

The guiding condition *i*> i_l must be verified : $i > i_l \Rightarrow \cos i_l = \dfrac{\sqrt{n_1^2 - n_2^2}}{n_1} > \cos i = \left(m\pi + \varphi_{10} + \varphi_{12}\right)\dfrac{1}{k_0 n_1 h}$

$$\Leftrightarrow \boxed{m < \frac{1}{\pi}\left(k_0.h.NA - \varphi_{10} - \varphi_{12}\right)}$$

➔ the number of guided modes is limited

Modes of a slab waveguide (3)

$2\pi \times \frac{1}{d}$

$$m < \frac{1}{\pi}\left(k_0.h.NA - \varphi_{10} - \varphi_{12}\right) \qquad k_0 = \frac{2\pi}{\lambda_0}$$

$V = k_0.h.NA$ V= <u>normalised spatial frequency</u> of the guide, at λ_0 T

t

$\nu = \frac{1}{T}$ temp freq

<u>Remarks</u>

➔ **the number of guided modes increases if V increases, i.e. :**

- ➔ if h ↗
- ➔ if NA ↗
- ➔ if, for a given waveguide, λ ↘

d

$\frac{1}{d}$ spatial freq

➔ if $h < \frac{\varphi_{10} + \varphi_{12}}{k_0.NA} = h_{\lim}$ then $m < 0$ no guided mode

➔ if 0 < m < 1 : only one guided mode (fundamental mode) → <u>single mode regime</u>

structure of the modes → EM approach : case of TE modes of a slab waveguide

$\frac{\partial}{\partial t} \rightarrow \times j\omega$

Maxwell equations

Electric field, in harmonic regime, in a waveguide: $\vec{\mathcal{E}}(x,y,z) = \Re e\left[\vec{E}(x,y).e^{j(\omega t-\beta z)}\right]$ (V/m)

$$\vec{E}(x,y) = \left| \begin{array}{l} E_x(x,y).\overrightarrow{ex} \\ E_y(x,y).\overrightarrow{ey} \\ E_z(x,y).\overrightarrow{ez} \end{array} \right.$$

$\vec{E}(x,y)$: one mode of the guide

β= axial propagation constant (along z)

Associated magnetic field: $\vec{\mathcal{H}}(x,y,z) = \Re e\left[\vec{H}(x,y).e^{j(\omega t-\beta z)}\right]$ (A/m)

Maxwell equations, in a linear, isotropic homogeneous with no electric charge nor current densities :

$$\begin{cases} curl\vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t} \text{ with } \vec{\mathcal{B}} = \mu\vec{\mathcal{H}} \quad \mu = \mu_0 = 4\pi\ 10^{-7} \text{ H/m} & (1) \\ curl\vec{H} = \varepsilon\frac{\partial \vec{E}}{\partial t} \text{ with } \varepsilon = \varepsilon_0\varepsilon_r \text{ et } \varepsilon_0 = \frac{1}{36\pi}10^{-9} \text{ F/m} & (2) \end{cases}$$

$curl\vec{U} = \nabla \wedge \vec{U}$

($\vec{U}$ being any vector)

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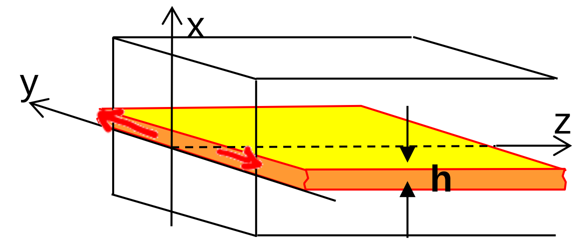
one starts from Maxwell equations

- harmonic form of the fields ➔ $\dfrac{\partial(X)}{\partial t} = j\omega X$ and $\dfrac{\partial(X)}{\partial z} = -j\beta X$



- slab waveguide supposed to have translation symmetry along y (infinite extension in this direction)

  ➔ components of the fields independent of y ➔ $\dfrac{\partial(X)}{\partial y} = 0$

- seak of TE modes ➔ $E_z = 0$

In these conditions, (1) and (2) lead to :

$$\vec{E}(x,y) = \vec{E}(x)\left|\begin{array}{l} E_x = 0 \\ E_y \neq 0 \\ E_z = 0 \end{array}\right. \quad \text{et} \quad \vec{H}(x,y) = \vec{H}(x)\left|\begin{array}{l} H_x = \dfrac{-\beta}{\omega\mu_0} E_y \\ H_y = 0 \\ H_z = \dfrac{j}{\omega\mu_0}\dfrac{\partial E_y}{\partial x} \end{array}\right.$$

$$\vec{\mathcal{E}}(x,y,z) = E_y(x).e^{j(\omega t-\beta z)}.\vec{e}_y$$

structure of the modes → EM approach : case of TE modes of the slab waveguide

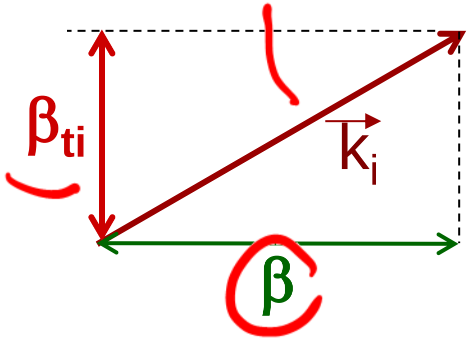
Expression of the fields :

From (1) and (2) ➔ propagation equation (= or Helmotz equation) : $\Delta\vec{\mathcal{E}} + k_0^2 n_i^2 \vec{\mathcal{E}} = \vec{0}$ (3)

with Δ vectorial laplacian : $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

Using (3) the expression of $\vec{\mathcal{E}}$ ➔ $\dfrac{\partial^2 E_y}{\partial x^2} + \underbrace{\left(k_0^2 n_i^2 - \beta^2\right)}_{\beta_{ti}^2} E_y = 0$ (4)

$\|\vec{k_i}\| = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda} n_i = k_0 n_i$



$\vec{k_i} = n_i \vec{k_0}$

$$\boxed{\beta_{ti}^2 = k_0^2 n_i^2 - \beta^2}$$

β_{ti} = **transverse propagation constant**

$\boxed{\text{one can write } \beta = k_0 . n_e}$ with n_e the effective index of the mode

➔ $\beta_{ti}^2 = k_0^2 (n_i^2 - n_e^2)$

structure of the modes → EM approach : case of TE modes of the slab waveguide

Expression of the fields :

solution of $\frac{\partial^2 E_y}{\partial x^2} + \beta_{ti}^2 E_y = 0$ with $\beta_{ti}^2 = k_0^2 (n_i^2 - n_e^2)$:

→ $\boxed{E_y = A_i . e^{-\gamma_i x} + B_i . e^{+\gamma_i x}}$ and $\gamma_i = j\beta_{ti} \Rightarrow \beta_{ti}^2 = -\gamma_i^2$

- if γ_i real (avec $B_i = 0$) → decreasing exponential solution (media n_0 et n_2)

  $E_y = A_i . e^{-\gamma_i x}$ then $\beta_{ti}^2 < 0$ → $n_0 < n_e$ and $n_2 < n_e$ cond 1

- if γ_i pure imaginary → sinusoidal solution (medium n_1)

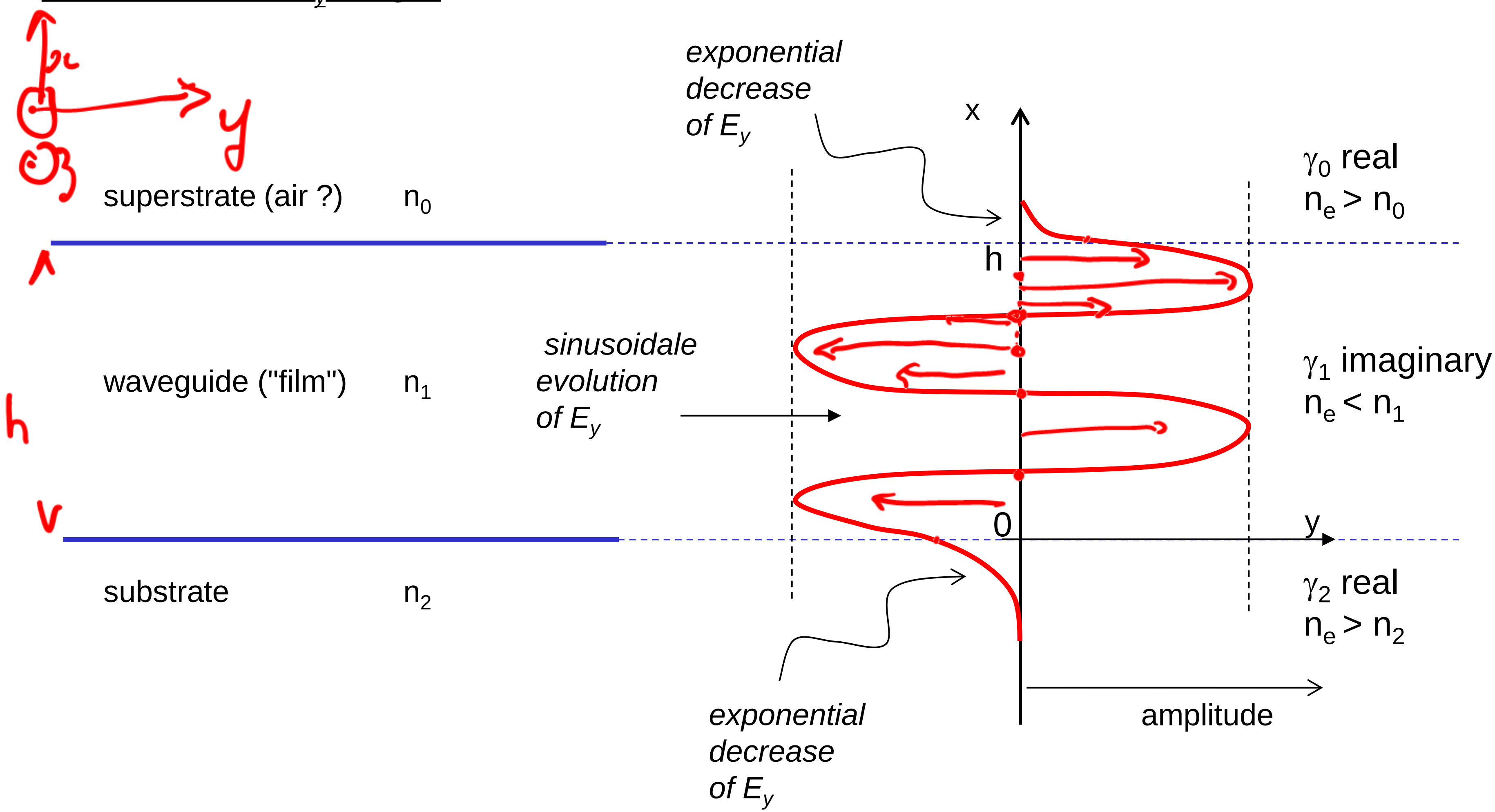
  $E_y = C . \cos(\beta_{ti} x + \Phi)$ then $\beta_{ti}^2 > 0$ → $n_1 > n_e$ cond 2

→ guiding condition : max $(n_0, n_2) < n_e < n_1$

if $n_e < n_0$ and/or $n_e < n_2$: non guided superstrate and/or substrate modes

structure of the modes → EM approach : case of TE modes of the slab waveguide

field distribution of E_y along x:



dispersion relationship

$\frac{2\pi}{\lambda}$ $\quad$ $\beta = k_0 n_e$

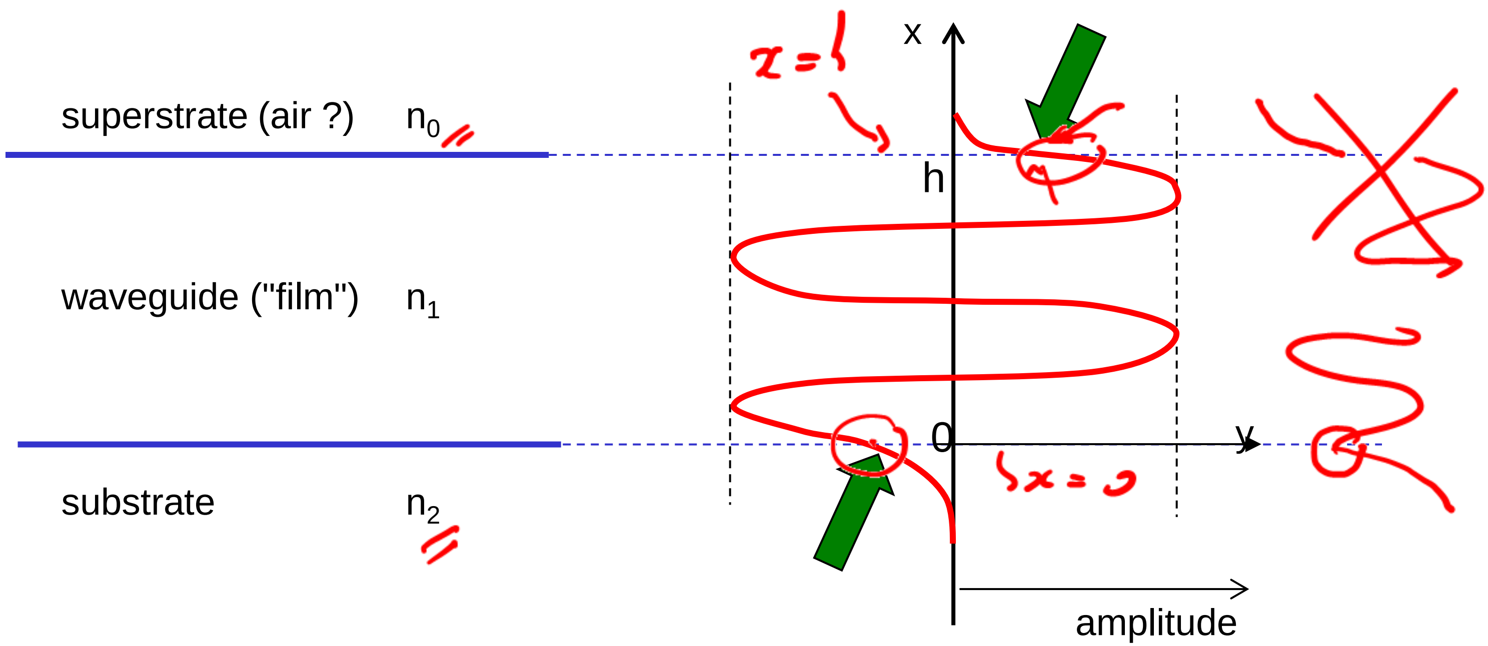
for a given mode, in a given waveguide : for each λ (or ν, or ω…) ➔ an associated value of β (or n_e)

$V = k_0 h NA$

curves β=f(ω) or n_e =f(ω) or ω=f(β) or β=f(V)…

= DISPERSION CURVES of the mode

for obtaining the dispersion relationship of a mode
➔ one must write the <u>continuity conditions</u> of the tangential components of the fields and of their derivatives at the interfaces



dispersion relationship

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= DISPERSION CURVES of the mode

for obtaining the dispersion relationship of a mode
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In the example of TE modes of the considered infinite slab waveguide, one must write :

$$E_y(x=0)\Big|_{\text{in the substrate}} = E_y(x=0)\Big|_{\text{in the waveguide}} \qquad E_y(x=h)\Big|_{\text{in the waveguide}} = E_y(x=h)\Big|_{\text{in the superstrate}}$$

$$\frac{\partial E_y}{\partial x}(x=0)\Big|_{\text{in the substrate}} = \frac{\partial E_y}{\partial x}(x=0)\Big|_{\text{in the waveguide}} \qquad \frac{\partial E_y}{\partial x}(x=h)\Big|_{\text{in the waveguide}} = \frac{\partial E_y}{\partial x}(x=h)\Big|_{\text{in the superstrate}}$$

➔ This leads to : $\beta_t.h = \varphi_{10} + \varphi_{12} + m\pi$ with :

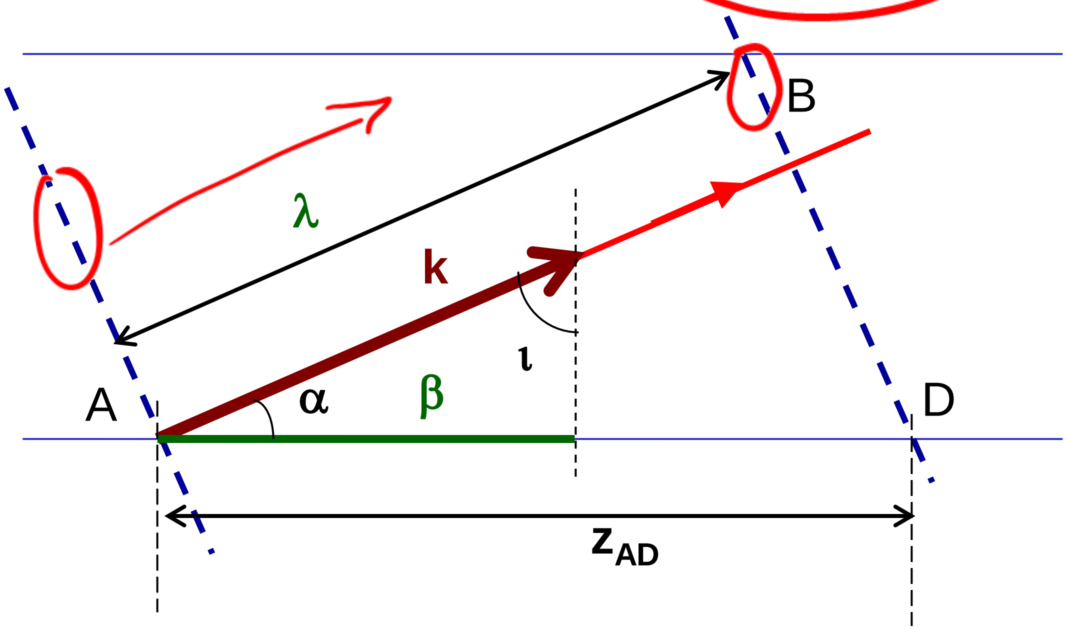
$$\beta_t = k_0\sqrt{n_1^2 - n_e^2} \qquad \varphi_{10} = \mathrm{Atan}\sqrt{\frac{n_e^2 - n_0^2}{n_1^2 - n_e^2}} \qquad \varphi_{12} = \mathrm{Atan}\sqrt{\frac{n_e^2 - n_2^2}{n_1^2 - n_e^2}}$$

$\beta(\omega)$

numerical resolution ➔ n_e=f(λ) or β=f(ω) or …

phase velocity vφ

propagation velocity of a WAVE FRONT, in the z direction



$$AB = \lambda$$

$$\rightarrow \quad v_\varphi = \frac{z_{AD}}{T}$$

with : $\cos\alpha = \dfrac{\lambda}{z_{AD}} = \sin i = \dfrac{\beta}{k} \Rightarrow z_{AD} = \dfrac{\lambda k}{\beta} = \dfrac{2\pi}{\beta}$

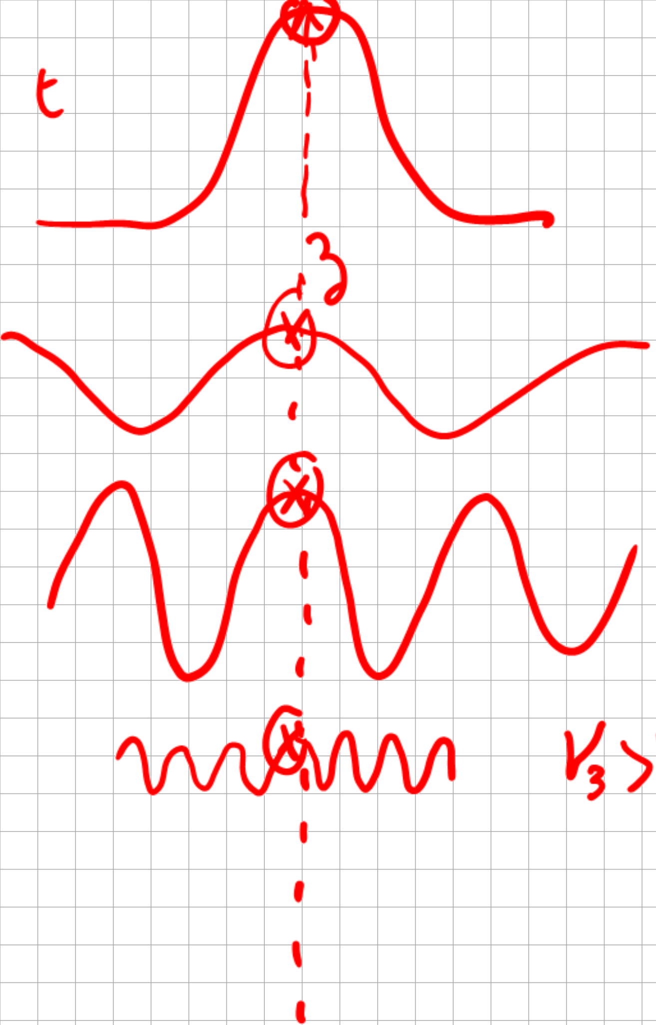
$$T = \frac{2\pi}{\omega}$$

$$\boxed{v_\varphi = \frac{\omega}{\beta}}$$

$k_0.c$

$k_0.n_e$

$$\rightarrow \quad \boxed{v_\varphi = \frac{c}{n_e}}$$



$\nu_1 \quad \varphi_1 = \omega_1 t - \beta_1 z$

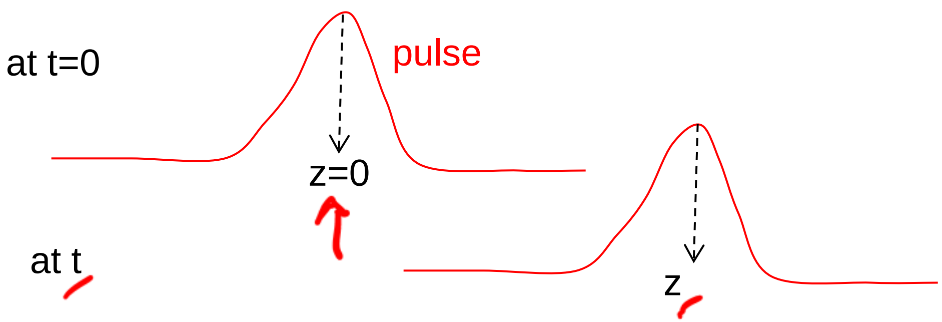
$\nu_2 > \nu_1 \quad \varphi_2 = \omega_2 t - \beta_2 z$

$\nu_3 > \nu_2 \quad \varphi_3 = \omega_3 t - \beta_3 z$

$$\varphi_1 = \varphi_2 = \varphi_3 \Rightarrow \frac{d\varphi}{d\omega} = 0$$

Group velocity v_g

propagation velocity of a WAVE PACKET, in the z direction (velocity of energy)



the peak of the pulse propagates at the speed :

$$v_g = \frac{z}{t}$$

at the peak of the pulse → all the chromatic components are in phase ➔ ωt-βz =cte ∀ω

$$\frac{d}{d\omega}(\omega t - \beta z) = 0 \quad \Rightarrow \quad \omega \underbrace{\frac{dt}{d\omega}}_{0} + t \underbrace{\frac{d\omega}{d\omega}}_{1} - \left(\beta \frac{dz}{d\omega} + z \frac{d\beta}{d\omega} \right) = 0$$

where $\frac{dz}{d\omega} \rightarrow 0$

$$\Rightarrow t - z \frac{d\beta}{d\omega} = 0 \Leftrightarrow \frac{d\omega}{d\beta} = \frac{z}{t} = v_g$$

$$\boxed{v_g = \frac{d\omega}{d\beta} = \frac{c}{N_g}}$$

N_g : group index

$$N_g = \frac{c}{v_g}$$

calculation of N_g versus the wavelength λ_0 (in the vacuum)

$$\frac{1}{v_g}=\frac{N_g}{c}=\frac{d\beta}{d\omega}=\frac{d}{d\omega}\overbrace{(k_0 n_e)}^{\beta}$$

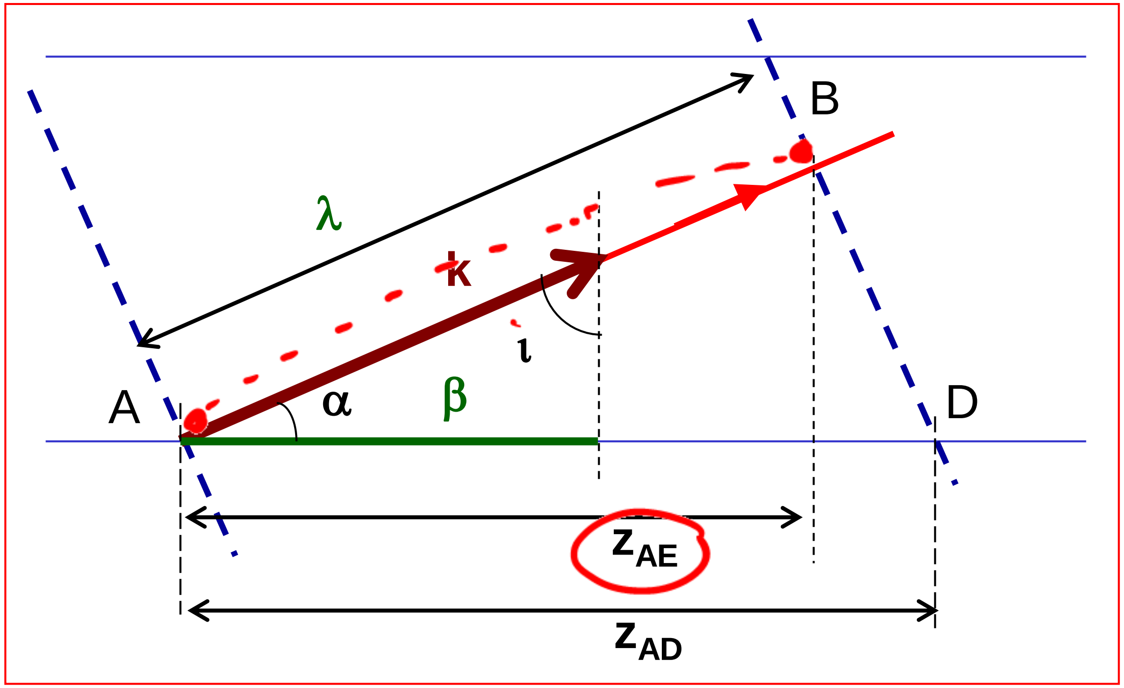
$$=\frac{dk_0}{d\omega}.n_e+k_0\frac{dn_e}{d\omega} \quad \text{avec } k_0=\frac{\omega}{c}$$

$$=\frac{1}{c}.n_e+\frac{2\pi}{\lambda_0}\frac{dn_e}{d\lambda_0}\frac{d\lambda_0}{d\omega} \qquad (1)$$

with $\omega=\dfrac{2\pi.c}{\lambda_0}$, one obtains : $\dfrac{d\lambda_0}{d\omega}=-\dfrac{\lambda_0^2}{2\pi.c}$ and (1) becomes $\dfrac{\mathrm{N_g}}{c}=\dfrac{n_e}{c}-\dfrac{\lambda_0}{c}\dfrac{dn_e}{d\lambda_0}$

$$\boxed{\mathrm{N_g}=n_e-\lambda_0\frac{dn_e}{d\lambda_0}}$$

Approximative comparison between v_φ and v_g versus modes orders



$$v_\varphi \approx: \frac{z_{AD}}{T} = \frac{AB/\cos\alpha}{\lambda_0/c} = \frac{(\lambda_0/n_1)/\cos\alpha}{\lambda_0/c} = \frac{c}{n_1.\sin i}$$

$$v_g \approx: \frac{z_{AE}}{T} = \frac{(\lambda_0/n_1).\sin i}{\lambda_0/c} = \frac{c}{n_1}.\sin i$$

$$v_\varphi.v_g \approx: \left(\frac{c}{n_1}\right)^2 = v^2 = \text{cte}$$

$$v_g = \frac{c}{N_g} \approx \frac{c}{n_1}.\sin i < \frac{c}{n_1} < v_\varphi = \frac{c}{n_e} \approx \frac{c}{n_1.\sin i} \Rightarrow N_g > n_1 > n_e$$

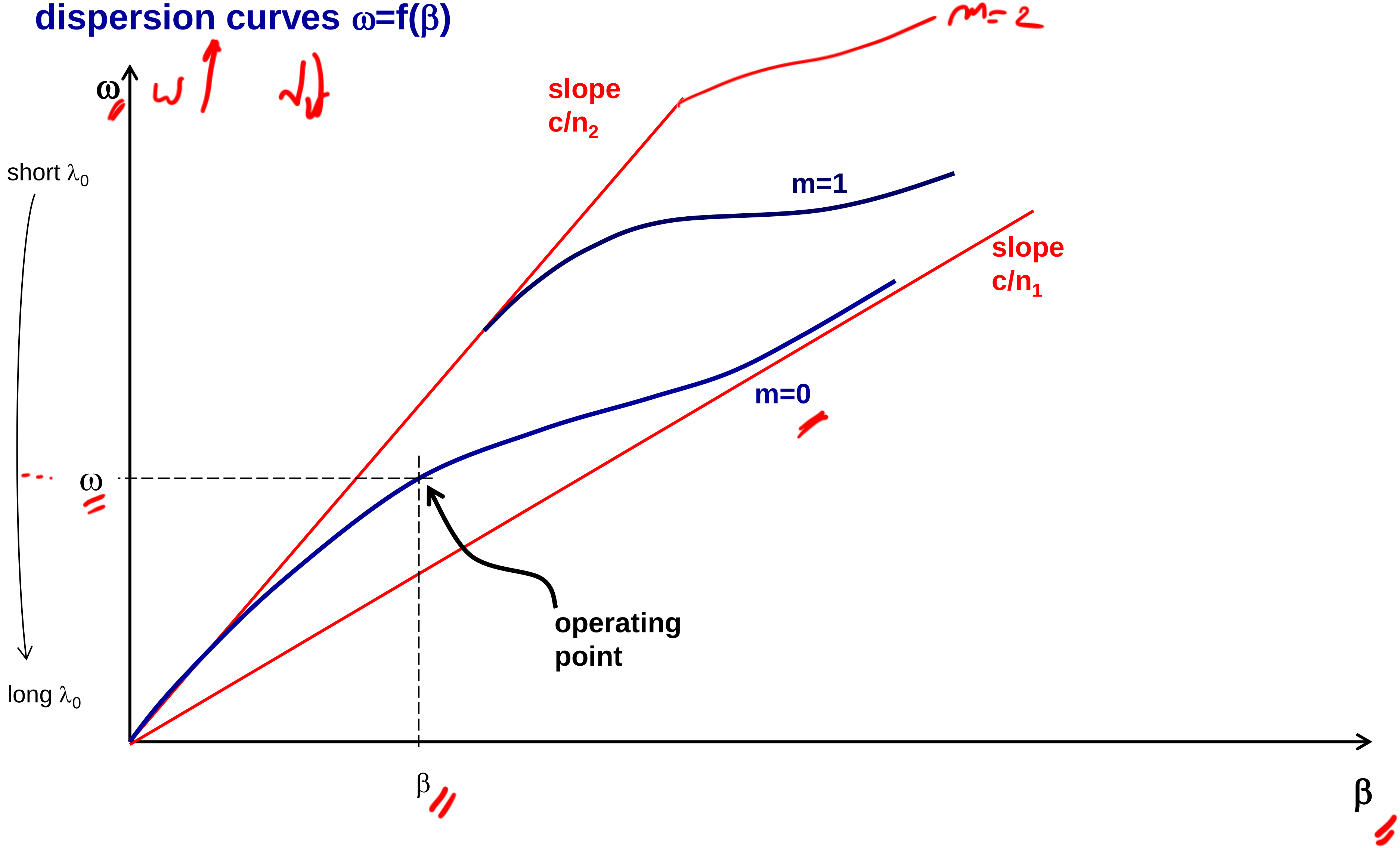
$$N_g > n_e \Rightarrow \frac{dn_e}{d\lambda} < 0$$

$$N_g = n_e - \lambda\frac{dn_e}{d\lambda}$$

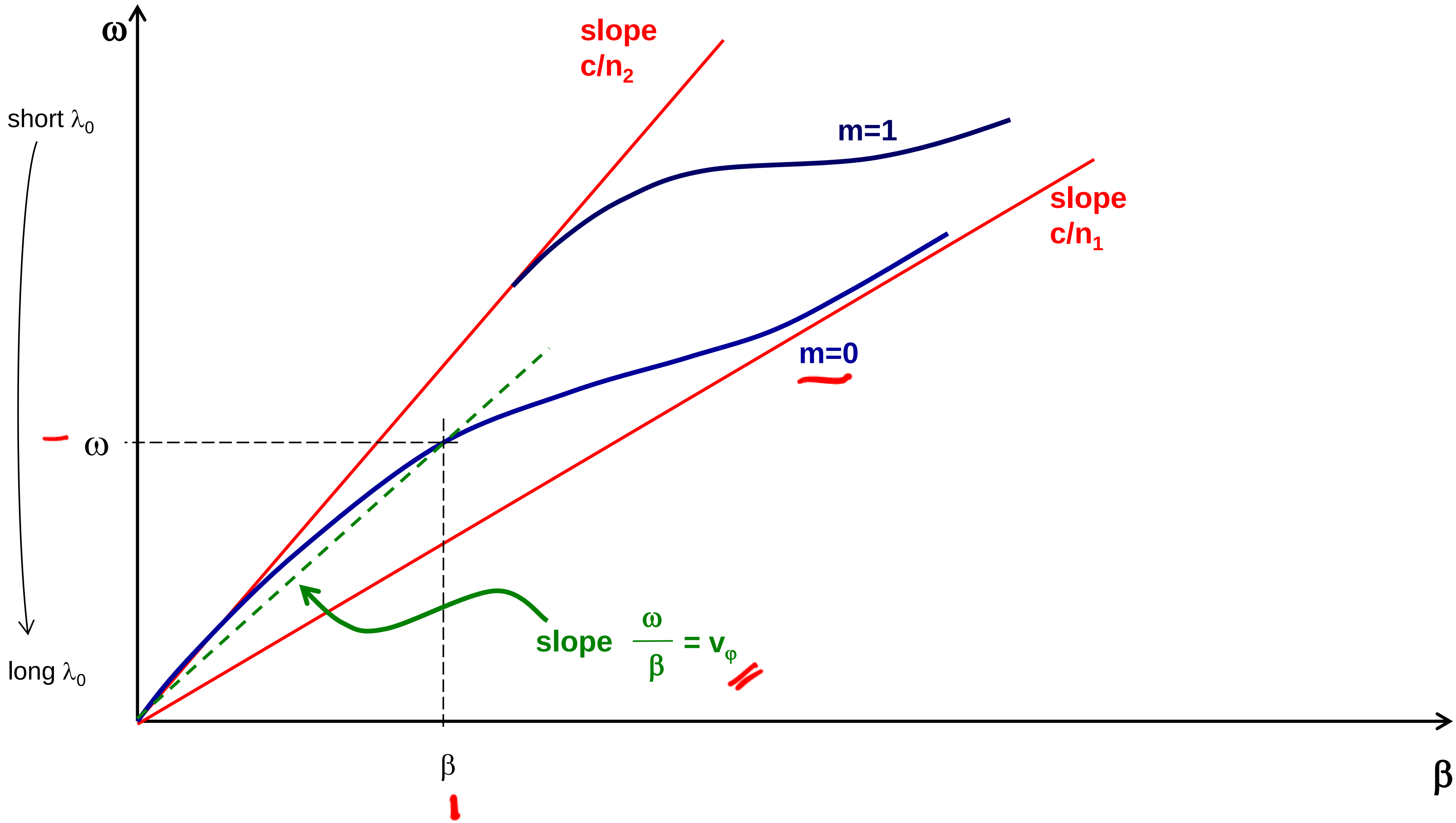
$$N_e > n_e$$

if i ↗ (very inclined light ray = low order modes) => v_φ ↘ et v_g ↗

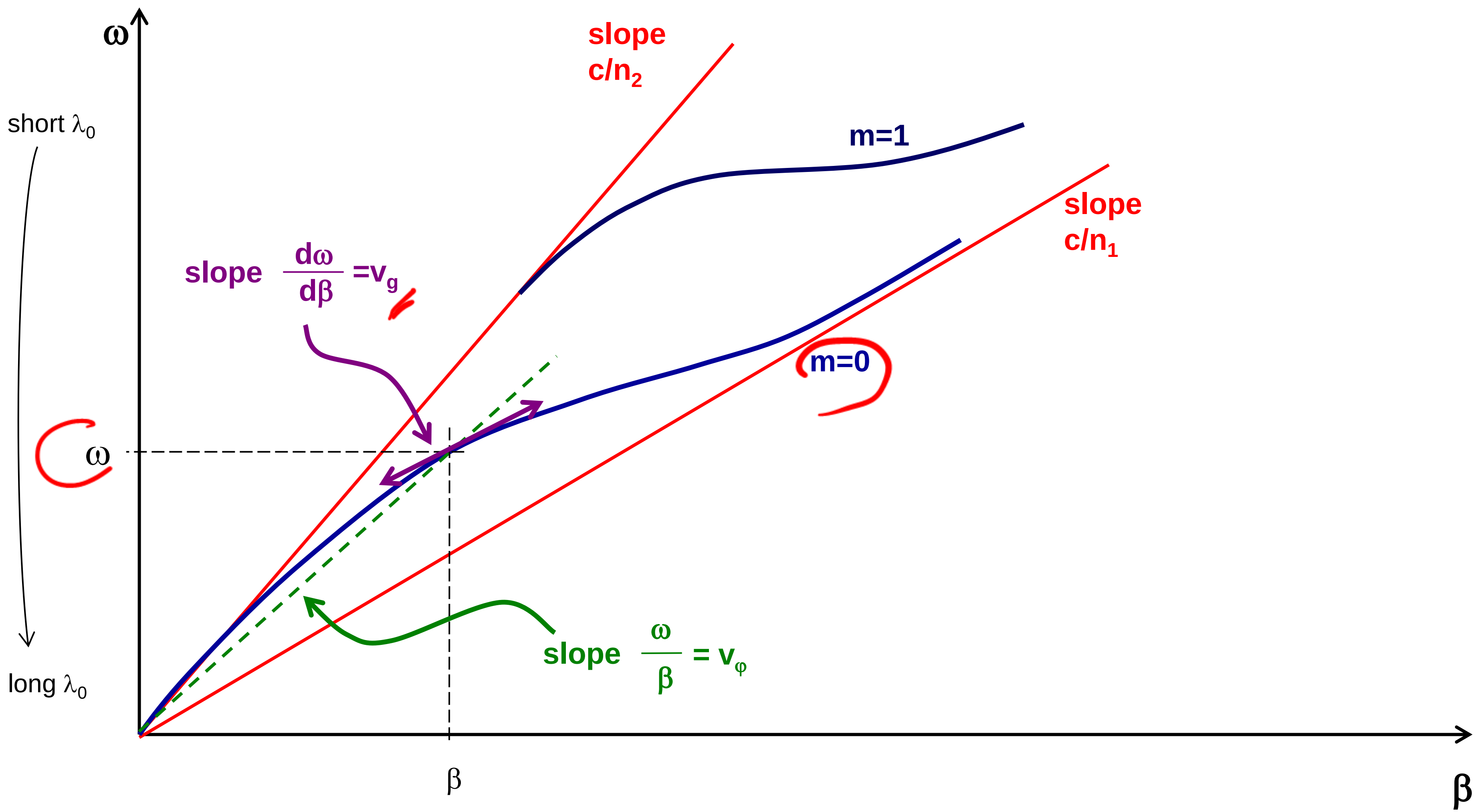
dispersion curves $\omega = f(\beta)$



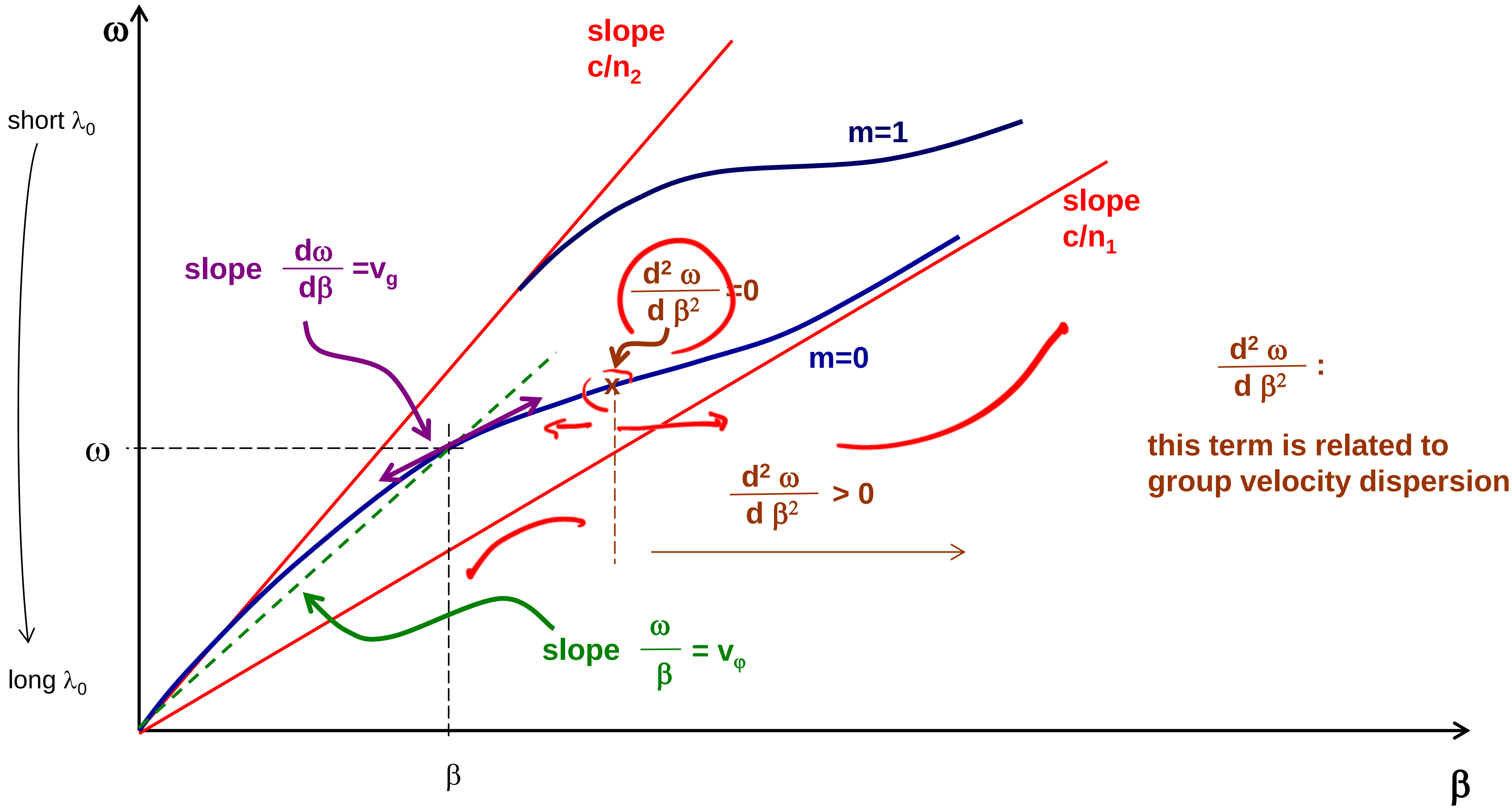
dispersion curves $\omega=f(\beta)$



dispersion curves $\omega=f(\beta)$

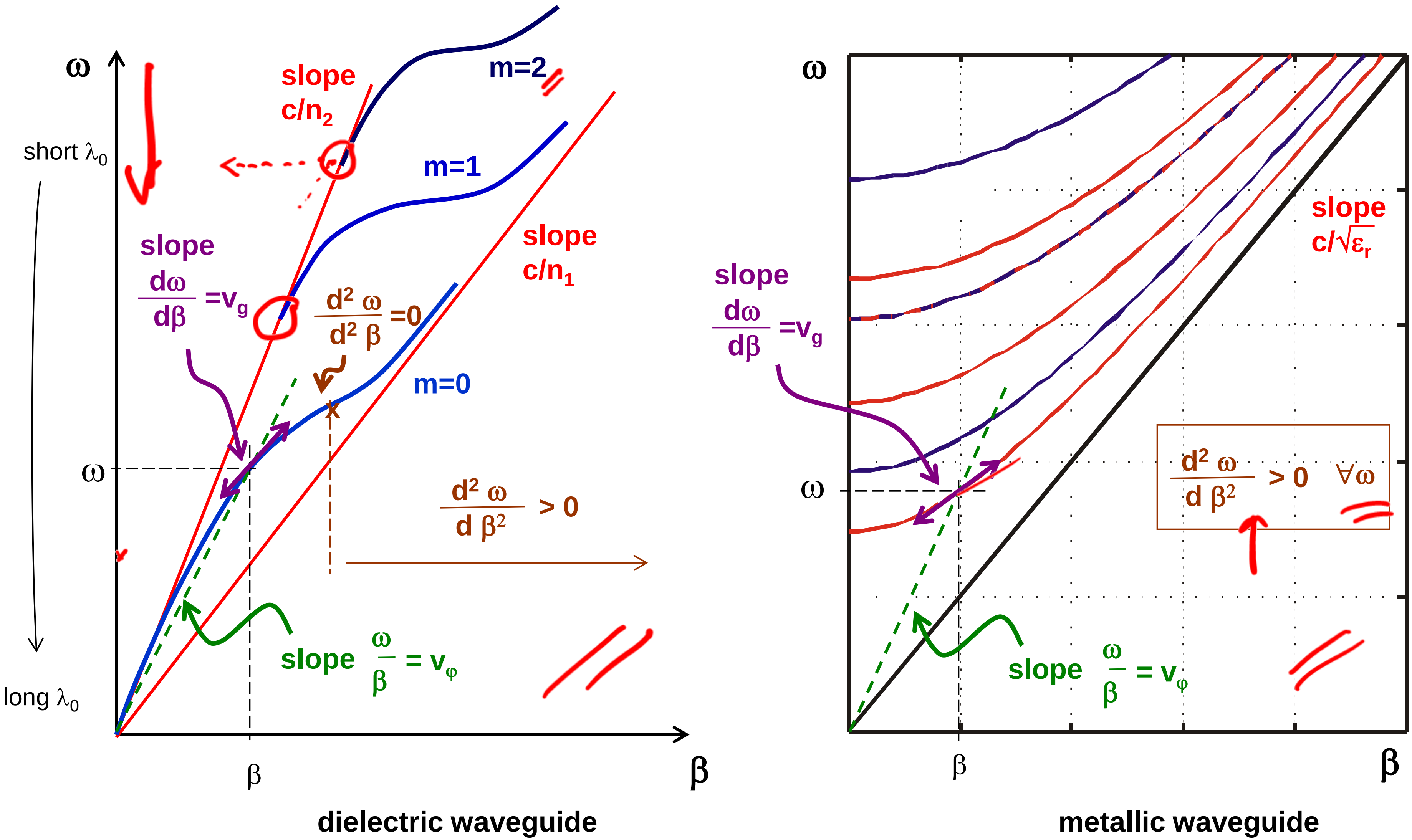


dispersion curves $\omega=f(\beta)$

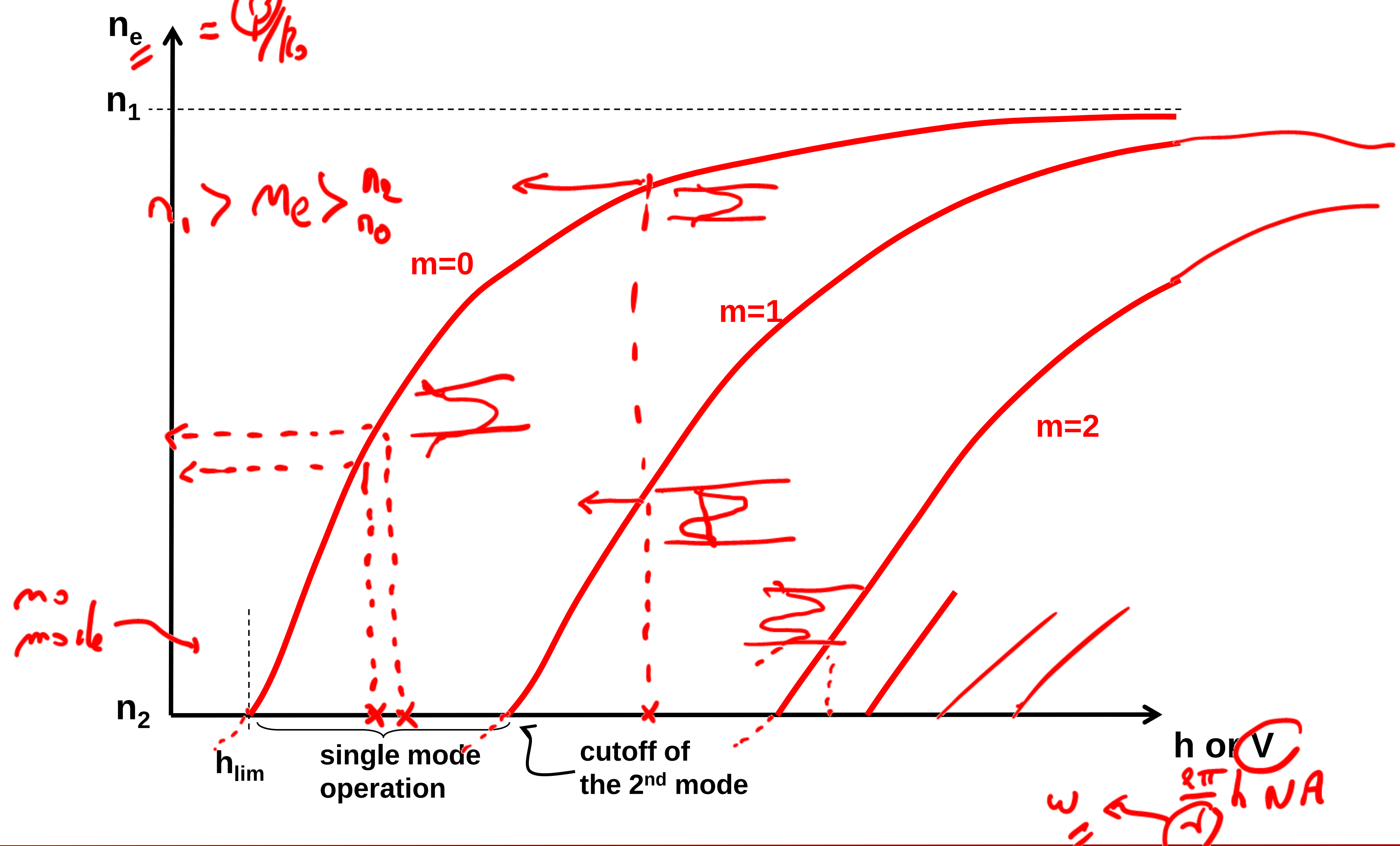


remark : the sign of $\frac{d^2\omega}{d\beta^2}$ is the opposite of that of $\frac{d^2\beta}{d\omega^2}$

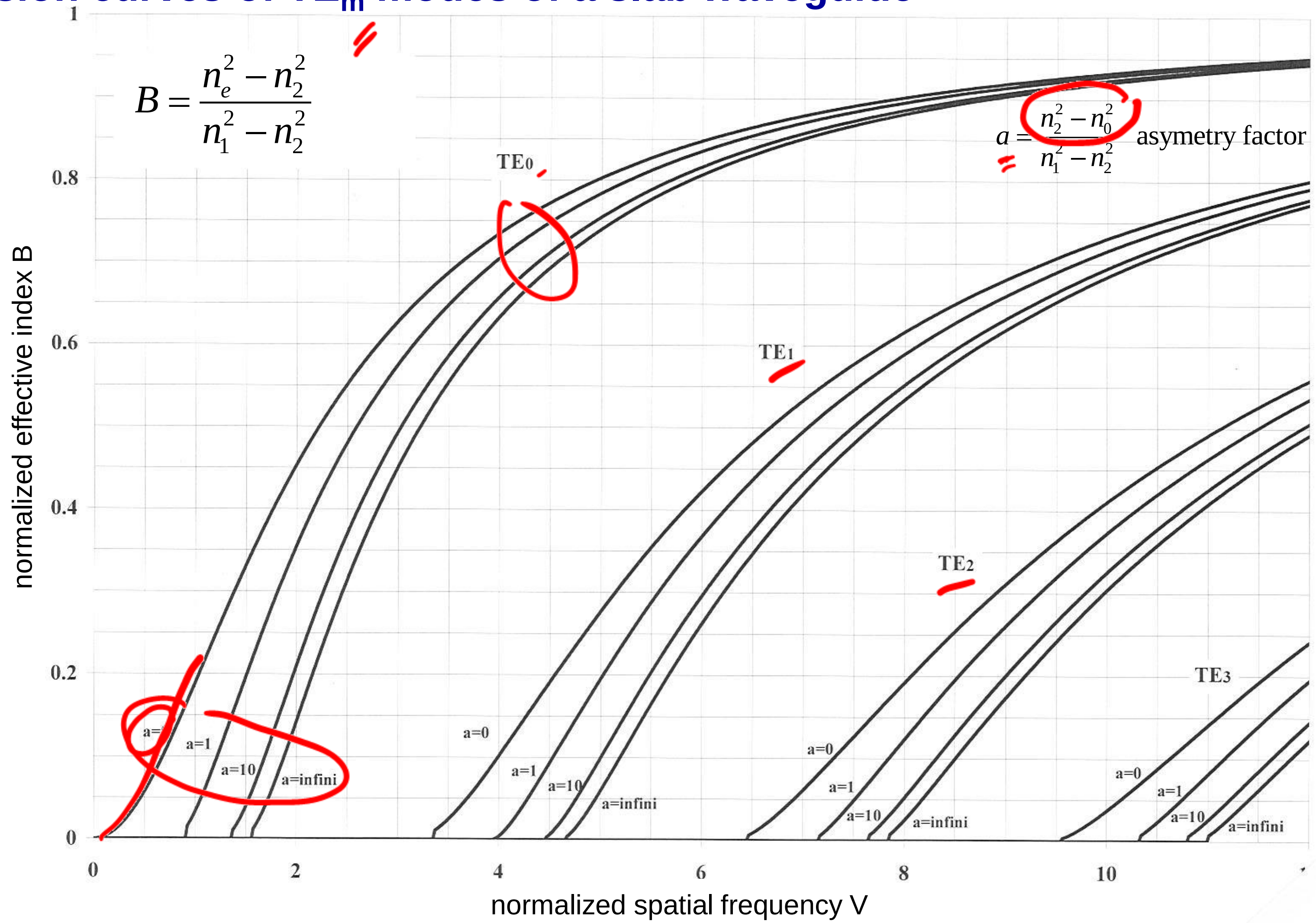
dispersion curves ω=f(β) : comparison with the case of a metallic waveguide



other representations of dispersion curves for a dielectric waveguide



dispersion curves of TE_m modes of a slab waveguide



case of TM modes

Previous calculations can be conducted as well for TM modes (H_z=0)

→ calculation of components E_z and H_z

→ continuity conditions for tangential components

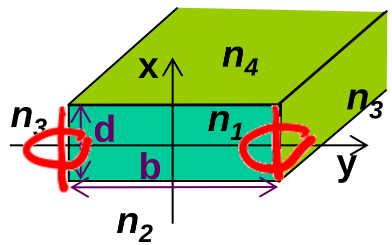
➔ dispersion equation : $$\beta_t.h = \varphi_{10} + \varphi_{12} + m\pi$$

$$\beta_t = k_0\sqrt{n_1^2 - n_e^2} \qquad \varphi_{10} = \mathrm{Atan}\boxed{\left(\frac{n_1}{n_0}\right)^2}\sqrt{\frac{n_e^2 - n_0^2}{n_1^2 - n_e^2}} \qquad \varphi_{12} = \mathrm{Atan}\boxed{\left(\frac{n_1}{n_2}\right)^2}\sqrt{\frac{n_e^2 - n_2^2}{n_1^2 - n_e^2}}$$

➔ **effective index of the TM_m mode different from that of the TE_m mode**

BUT if $n_0 \to n_1$ and if $n_2 \to n_1$ then $\varphi_{10}(TE) \to \varphi_{10}(TM)$ and $\varphi_{12}(TE) \to \varphi_{12}(TM)$
➔very similar dispersion curves
➔ <u>quasi degenerated modes</u>

case of modes of rectangular dielectric waveguides (non infinite in the y direction)



… not addressed in this course

End of chapter 2