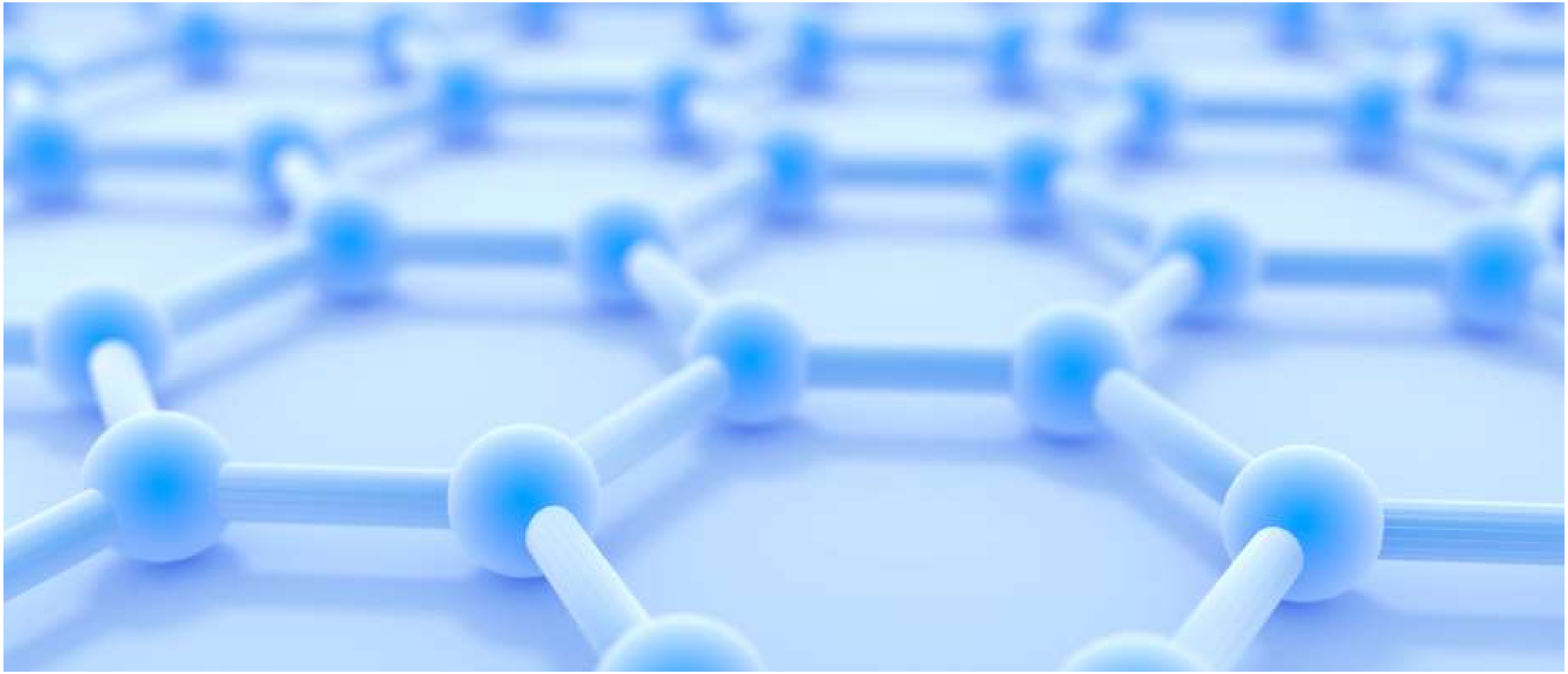

Photonics

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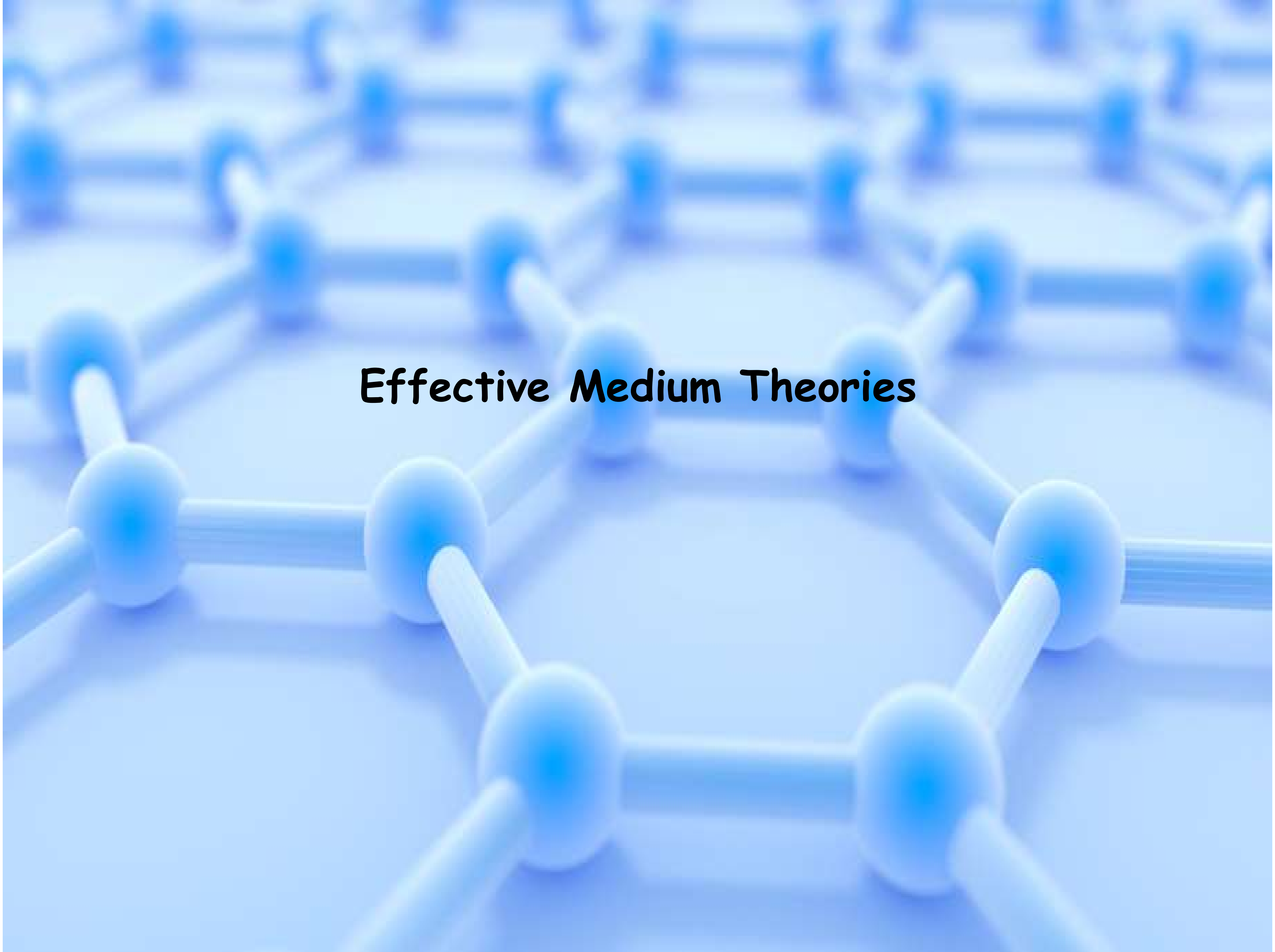
Effective Medium Theories



Effective Medium Theories

What are effective medium theories?

EMT provide macroscopic models of inhomogeneous media based on analytical, numerical or experimental techniques. A description of composite materials in terms of effective medium approximations is a valuable and versatile tool to investigate, predict and design the electromagnetic response of natural and structured materials.

Why effective medium theories are helpful?

Effective medium models allow to write very simple constitutive relations, eliminating the complexity of simulating light-matter interactions at the constituents' level.

What are their limits?

Pushing any effective medium theory beyond its limit of validity may lead to wrong predictions. EMT usually depend on the **electric and magnetic properties** of the constituent materials, the **volume fraction** of each constituent, and in some case the **geometry** of the structure at the constituent level. The fundamental assumption of EMT is that the wavelength of the field is much larger than the characteristic scale of the inhomogeneity. Depending on the size, permittivity and permeability of the constituents, as well as the index of the hosting medium, the limitations of the model may be more or less strict.

Effective Medium Theories

- ✓ Maxwell Garnett theory
- ✓ Bruggeman theory
- ✓ Quasi-static numerical approaches
- ✓ Nicolson-Ross-Weir technique

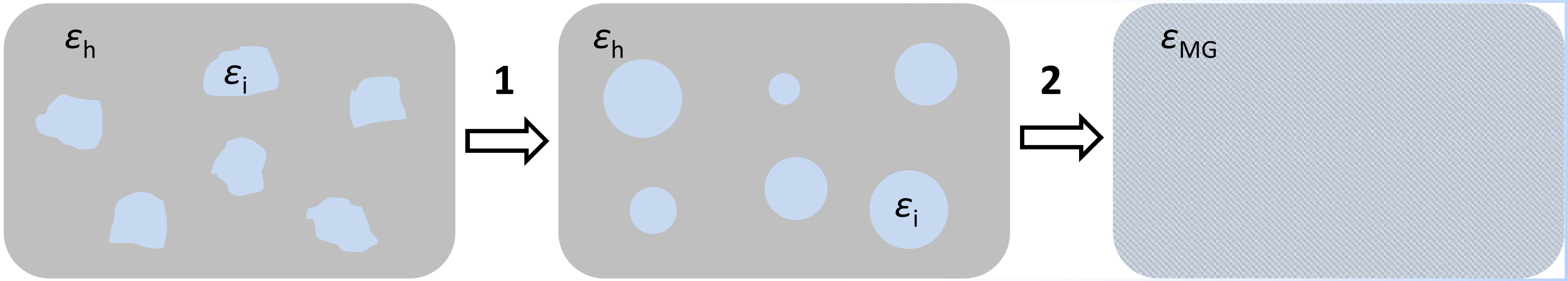
Maxwell Garnett Theory

Media with **small inclusions** dispersed in a **continuous host medium**

Useful for two-phase medium with grains of a guest material: the inclusions have relative permittivity ε_i and they are hosted by a background medium with relative permittivity ε_h.
If the inclusions are small enough, then a quasi-static approximation can be adopted. In absence of any information about the shape of the inclusions, the most natural approach is to assume that the inclusions are small spheres.

Limits of validity

If $\varepsilon_i > 0$ the particle size should not exceed one tenth of the effective wavelength.
If $\varepsilon_i < 0$ the limits of validity may be stricter especially near the localized surface plasmon resonances.



We consider non-magnetic and isotropic materials.

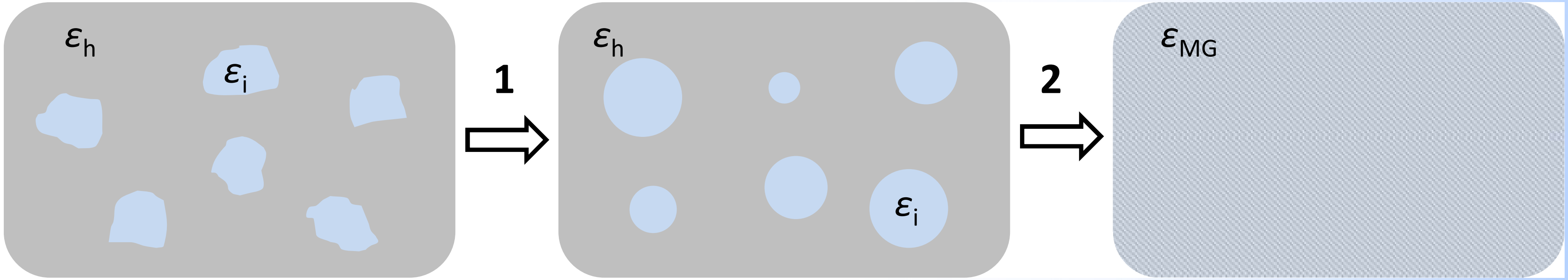
Maxwell Garnett Theory

Step 1

In the quasi-static approximation the external field applied to our material (**E**$_e$) can be considered constant on the length scale of each sphere.
If we consider each inclusion like a very small, isolated sphere, we can assume it behaves like a point source with electric dipole moment proportional to the applied field.
The response of an isolated sphere in the host medium is:

$$\mathbf{p}_\mathrm{h} = \varepsilon_0 \varepsilon_\mathrm{h} \alpha \mathbf{E}_\mathrm{e} \ ,$$

where ε_0 is the vacuum permittivity, $\mathbf{p}_\mathrm{h}$ is the induced dipole moment, $\alpha = 3V \frac{\varepsilon_\mathrm{i} - \varepsilon_\mathrm{h}}{\varepsilon_\mathrm{i} + 2\varepsilon_\mathrm{h}}$ is the static electric polarizability of the sphere and V is the sphere's volume. The field inside the sphere, $\mathbf{E}_\mathrm{i} = 3\varepsilon_\mathrm{h}/(\varepsilon_\mathrm{i} + 2\varepsilon_\mathrm{h})\mathbf{E}_\mathrm{e}$, is uniform and parallel to the external field. The polarizability of the sphere is isotropic since both the permittivity and shape of the inclusions are assumed isotropic.



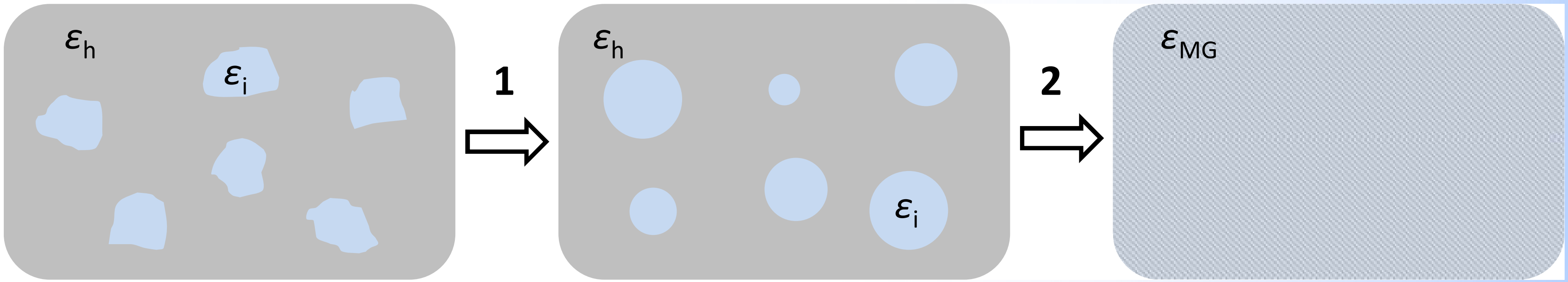
Maxwell Garnett Theory

Step 2

The next step is to create an effective model of the distribution of small spheres. The spheres are reduced to electric point dipoles and the field radiated by each dipole is now influenced by the presence of all the other dipoles. At this point the information required is the number of dipoles per unit volume *N*. The definition of the effective permittivity is based on the average, or macroscopic, constitutive relation that links the average electric field $<\mathbf{E}>$ to the average displacement field $<\mathbf{D}>$:

$$<\mathbf{D}> = \varepsilon_0 \varepsilon_{\mathrm{MG}} <\mathbf{E}>,$$

where $\varepsilon_{\mathrm{MG}}$ is the effective (relative) Maxwell Garnett permittivity that models the original mixture.



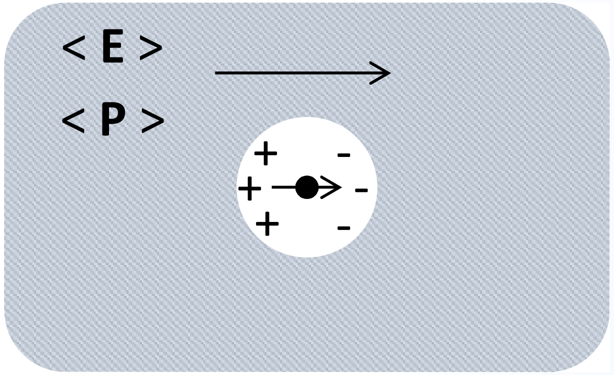
Maxwell Garnett Theory

Step 2

Average medium response = Average response of host medium + average response of dipoles:

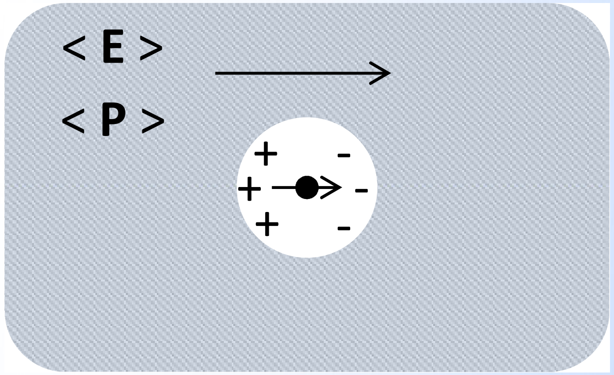
$$<\mathbf{D}> = \varepsilon_0 \varepsilon_\mathrm{h} <\mathbf{E}> + <\mathbf{P}>.$$

Average dipole response: $<\mathbf{P}> = N\mathbf{p}$, with $\mathbf{p} \neq \mathbf{p}_\mathrm{h}$ since it is now calculated in the presence of all the other dipoles. **p** is evaluated by means of the local electric field $\mathbf{E}_\mathrm{L}$, which is the field locally "felt" by each dipole. This field is the average field $<\mathbf{E}>$ augmented by a contribution due to the average polarization that surrounds each dipole, also known as Lorentz field.
The value $\mathbf{E}_\mathrm{L}$ acting on a single dipole is calculated assuming a fictitious spherical boundary that separates a macroscopic background with average polarization $<\mathbf{P}>$ from a microscopic spherical cavity surrounding the dipole at the center of the sphere. The field due to the polarized background can be easily calculated by considering that the charge density $\nabla \cdot \mathbf{P}$ is zero everywhere except on the cavity boundary where a surface charge distribution of density $-<\mathbf{P}>\cdot \hat{\mathbf{n}}$ induces an additional electric field at the dipole location, i.e., at the sphere's center.



Maxwell Garnett Theory

The local field can be straightforwardly written as:



$$\mathbf{E}_{\mathrm{L}} = <\mathbf{E}> + \frac{<\mathbf{P}>}{3\varepsilon_0\varepsilon_{\mathrm{h}}}$$

and the dipole moment as:

$$\mathbf{p} = \varepsilon_0\varepsilon_{\mathrm{h}}\alpha\mathbf{E}_{\mathrm{L}} = \varepsilon_0\varepsilon_{\mathrm{h}}\alpha <\mathbf{E}> + \alpha\frac{<\mathbf{P}>}{3}\,.$$

But since $<\mathbf{P}> = N\mathbf{p}$ we can write:

$$\mathbf{p} = \frac{<\mathbf{P}>}{N} = \varepsilon_0\varepsilon_{\mathrm{h}}\alpha <\mathbf{E}> + \alpha\frac{<\mathbf{P}>}{3}$$

$$<\mathbf{P}> = \frac{3N\alpha\;{}_0\varepsilon_{\mathrm{h}}}{3-N\alpha} <\mathbf{E}>$$

Maxwell Garnett Theory

Let's now make a step back to the definitions of the average displacement field we made:

$$< \mathbf{D} > = \varepsilon_0 \varepsilon_{\mathrm{MG}} \ < \mathbf{E} >$$
$$< \mathbf{D} > = \ \varepsilon_0 \varepsilon_{\mathrm{h}} \ < \mathbf{E} > \ + < \mathbf{P} >$$

It follows that:

$$\varepsilon_0 \varepsilon_{\mathrm{MG}} \ < \mathbf{E} > = \varepsilon_0 \varepsilon_{\mathrm{h}} \ < \mathbf{E} > \ + < \mathbf{P} >$$

We now can use the expression for the average polarization in the equation:

$$\varepsilon_0 \varepsilon_{\mathrm{MG}} \ < \mathbf{E} > = \varepsilon_0 \varepsilon_{\mathrm{h}} \ < \mathbf{E} > \ + \frac{3N\alpha\varepsilon_0\varepsilon_{\mathrm{h}}}{3-N\alpha} < \mathbf{E} >$$

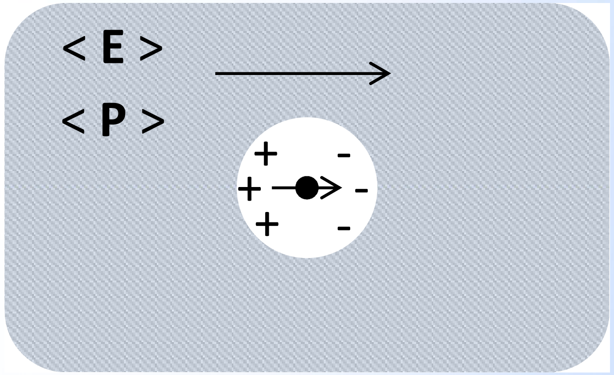
From which follows:

$$\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}} \left(1 + \frac{N\alpha}{1-\frac{N\alpha}{3}}\right).$$

For very diluted media $1 - \frac{N\alpha}{3} \approx 1$ and the effective permittivity is simply $\varepsilon_{\mathrm{MG}} \approx \varepsilon_{\mathrm{h}}(1 + N\alpha)$.
The same expression can be easily obtained when the local field is $\mathbf{E}_{\mathrm{L}} \ \approx < \mathbf{E} >$.
This approximation is fully justified in diluted mixtures where the interaction between dipoles is weak.

Maxwell Garnett Theory

The Maxwell Garnett permittivity $\varepsilon_{\mathrm{MG}}$ can be also written in the following form:



$$\frac{N\alpha}{3} = \frac{\varepsilon_{\mathrm{MG}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{MG}} + 2\varepsilon_{\mathrm{h}}}$$

This formula is known as Clausius-Mossotti formula, or Maxwell's formula, or Lorentz-Lorenz formula.
Substitution of the expression of the polarizability α in the Clausius-Mossotti relation gives the Rayleigh formula:

$$\frac{\varepsilon_{\mathrm{MG}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{MG}} + 2\varepsilon_{\mathrm{h}}} = f\,\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}}},$$

that relates the effective permittivity to the constituents' permittivities and to the parameter $f = NV$, which is the volume fraction of the inclusions (in this case spheres) in the medium.
The so-called Maxwell Garnett formula derives from the Rayleigh formula and it is written as follows:

$$\varepsilon_{\mathrm{MG}} = \varepsilon_{\mathrm{h}}\left[1 + 3f\,\frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + 2\varepsilon_{\mathrm{h}} - f(\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}})}\right].$$

Maxwell Garnett Theory

Limitations

- ✓ Predictions for large values of f are questionable because of the higher-order multipole effects triggered by decreased interparticle distances. Values of f larger than 0.5, can be included in the Maxwell Garnett formula with the introduction of fictitious depolarization factors

- ✓ Higher-order terms cannot be handled with the simple approach described here, which is strictly based on the quasi-static and dipole approximations.

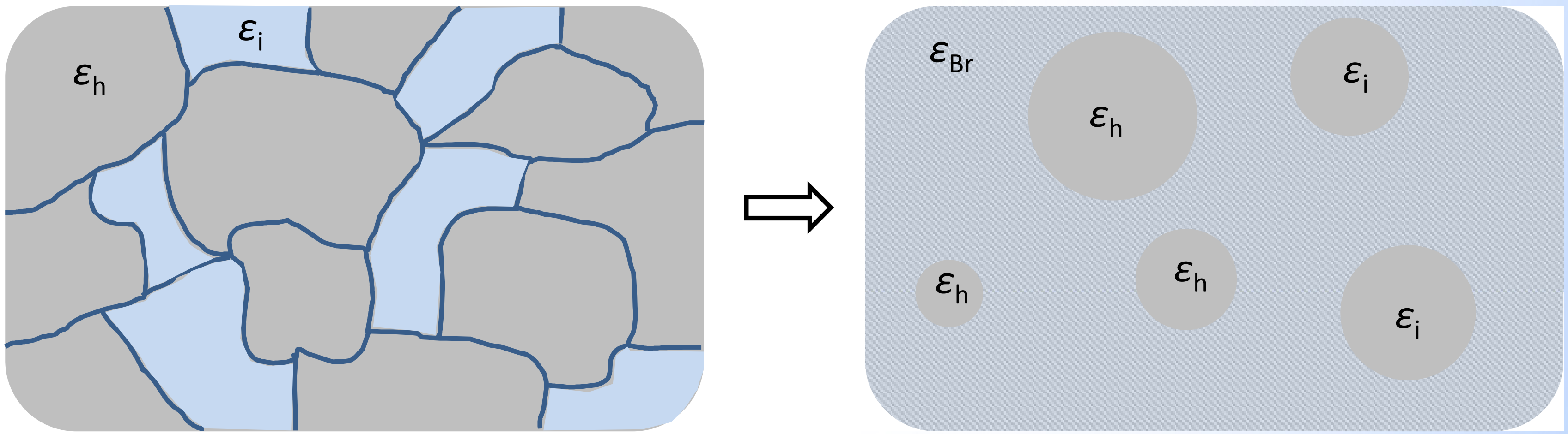
- ✓ Asymmetric behavior of the central formula: The roles of the inclusion phase and the host phase are not interchangeable, so that the validity of the Maxwell Garnett formula is limited to mixtures in which there is a clear determination of a host medium phase and a small-inclusions phase.

Bruggeman Theory

Aggregate mixture media

For aggregate mixtures with random distributions of two or more constituents, effective medium theories based on a statistical formulation are more suitable. The classic theory for this class of inhomogeneous mixtures is the Bruggeman theory.

For example, we can consider a two-phase microstructure where the constituent with permittivity ε_i has volume fill factor f and the constituent with permittivity ε_h has volume fill factor 1-f.



Bruggeman Theory

The mixture is modeled as a continuous medium hosting a distribution of small spherical inclusions of two different dielectric permittivities. The probabilities of finding spheres with permittivity ε_i and ε_h are f and 1-f, respectively, which correspond to the volume fill factors of the two phases in the original mixture.

We assume that the host medium of the Bruggeman mixture have the unknown effective permittivity ε_Br and invoke the transparency or "invisibility" condition for the distribution of the spherical inclusions.

In the quasi-static approximation, this condition can be retrieved by setting the scattering amplitude for small spherical particles of permittivity ε_p and radius R_p to zero, i.e.,

$$\left(\mathbf{S}\cdot\widehat{\mathbf{e}_p}\right)_{\vartheta=\vartheta_0} \approx -j\left(kR_\mathrm{p}\right)^3 \frac{\varepsilon_\mathrm{p}-\varepsilon_\mathrm{Br}}{\varepsilon_\mathrm{p}+2\varepsilon_\mathrm{Br}} = 0.$$

In our distribution of spheres ε_p can be either ε_i with probability f or ε_h with probability 1-f.

The resulting "averaged transparency" condition reads as follows

$$f\,\frac{\varepsilon_\mathrm{i}-\varepsilon_\mathrm{Br}}{\varepsilon_\mathrm{i}+2\varepsilon_\mathrm{Br}} + (1-f)\,\frac{\varepsilon_\mathrm{h}-\varepsilon_\mathrm{Br}}{\varepsilon_\mathrm{h}+2\varepsilon_\mathrm{Br}} = 0.$$

The Bruggeman formula is symmetric since the roles of inclusions and host are interchangeable.

Bruggeman Theory

The Bruggeman formula

$$f\,\frac{\varepsilon_{\mathrm{i}}-\varepsilon_{\mathrm{Br}}}{\varepsilon_{\mathrm{i}}+2\varepsilon_{\mathrm{Br}}}+(1-f)\,\frac{\varepsilon_{\mathrm{h}}-\varepsilon_{\mathrm{Br}}}{\varepsilon_{\mathrm{h}}+2\varepsilon_{\mathrm{Br}}}=0.$$

has two solutions for the effective permittivity $\varepsilon_{\mathrm{Br}}$, one of which is usually unphysical because it violates causality.

The formula for spherical inclusions can be easily extended to multi-phase aggregates by adding more terms in the formula, yielding

$$\sum_{m=1}^{M} f_m\,\frac{\varepsilon_m-\varepsilon_{\mathrm{Br}}}{\varepsilon_m+2\varepsilon_{\mathrm{Br}}}=0,$$

where f_m is the fill factor of the *m*-th constituent of the mixture and *M* is the number of phases.

Bruggeman Theory

Shape effects of the inclusions can be included in the Bruggeman theory.

The relative permittivity of a multi-phase distribution of randomly oriented ellipsoids with equal shape and dielectric constants ε_m dispersed in a Bruggeman-type mixture of background permittivity ε_h has been derived by Polden and van Stanten and reads as follows:

$$\varepsilon_\mathrm{Br} = \varepsilon_\mathrm{h} \left\{ 1 - \frac{1}{3} \sum_{m=1}^{M} \left[f_m (\varepsilon_\mathrm{m} - \varepsilon_\mathrm{h}) \sum_{j=1}^{3} \frac{1}{\varepsilon_\mathrm{Br} + (\varepsilon_\mathrm{m} - \varepsilon_\mathrm{Br}) L_j} \right] \right\}^{-1} .$$

The Bruggeman formula for fully aligned ellipsoidal inclusions reads as follows :

$$\sum_{m=1}^{M} f_m \frac{\varepsilon_m - \varepsilon_{\mathrm{Br},j}}{\varepsilon_m + K_j \varepsilon_{\mathrm{Br},j}} = 0,$$

where $\varepsilon_{\mathrm{Br},j}$ are the effective permittivities in the principal directions, and $K_j = (1 - L_j)/L_j$ are the screening parameters. For a two-phase mixture with constituents having permittivities ε_i and ε_h and filling ratios of *f* and 1-*f*, respectively, the relation reduces to:

$$f \frac{\varepsilon_\mathrm{i} - \varepsilon_{\mathrm{Br},j}}{\varepsilon_\mathrm{i} + K_j \varepsilon_{\mathrm{Br},j}} + (1 - f) \frac{\varepsilon_\mathrm{h} - \varepsilon_{\mathrm{Br},j}}{\varepsilon_\mathrm{h} + K_j \varepsilon_{\mathrm{Br},j}} = 0$$

Maxwell Garnett VS Bruggeman Theory

Bruggeman

$$\boxed{f\,\frac{\varepsilon_{\mathrm{i}}-\varepsilon_{\mathrm{Br},j}}{\varepsilon_{\mathrm{i}}+K_j\varepsilon_{\mathrm{Br},j}}+(1-f)\,\frac{\varepsilon_{\mathrm{h}}-\varepsilon_{\mathrm{Br},j}}{\varepsilon_{\mathrm{h}}+K_j\varepsilon_{\mathrm{Br},j}}=0}$$

Maxwell Garnett

$$\boxed{\frac{\varepsilon_{\mathrm{MG},j}-\varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{MG},j}+K_j\varepsilon_{\mathrm{h}}}=f\,\frac{\varepsilon_{\mathrm{i}}-\varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}}+K_j\varepsilon_{\mathrm{h}}}}$$

Maxwell Garnett theory does not handle well particles with very small depolarization factors (elongated shapes) since they show a stronger particle-particle interactions.
This situation is similar to the scenario of a mixture with large inclusions' fill factor.

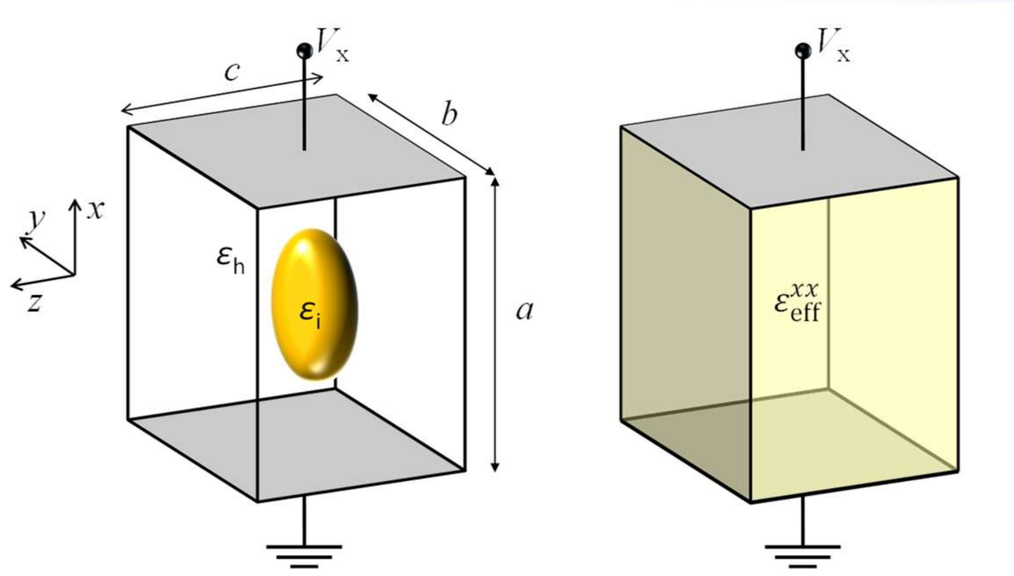
Another scenario in which Bruggeman theory provides a more realistic electromagnetic description is for mixtures with large differences in the permittivities of the constituents – e.g., metal-dielectric mixtures. The threshold for these structures is the critical metal filling factor above which there is formation of long-range connectivity between metal grains and the optical response of the mixture changes abruptly. In the Bruggeman theory for isotropic spherical, metallic inclusions dispersed in a dielectric host ($|\varepsilon_{\mathrm{i}}| \gg \varepsilon_{\mathrm{h}}$) the effective permittivity reduces to $\varepsilon_{\mathrm{h}}/(1-3f)$. The zero in the denominator of this expression gives the critical fill factor $f_c = 1/3$ for metal-based mixtures. In other words, for $f \le f_c$ the mixture behaves like an insulator, whereas for $f > f_c$ the mixture acts like a conductor. It is worth mentioning that Maxwell Garnett theory predicts an unrealistic percolation threshold ($f_c = 1$).

Quasi-static approaches

As for the Maxwell Garnet and the Bruggeman theories, these numerical techniques are based on quasi-static approximation, i.e., one assumes that the typical size of the inclusions is much smaller than the effective wavelength.

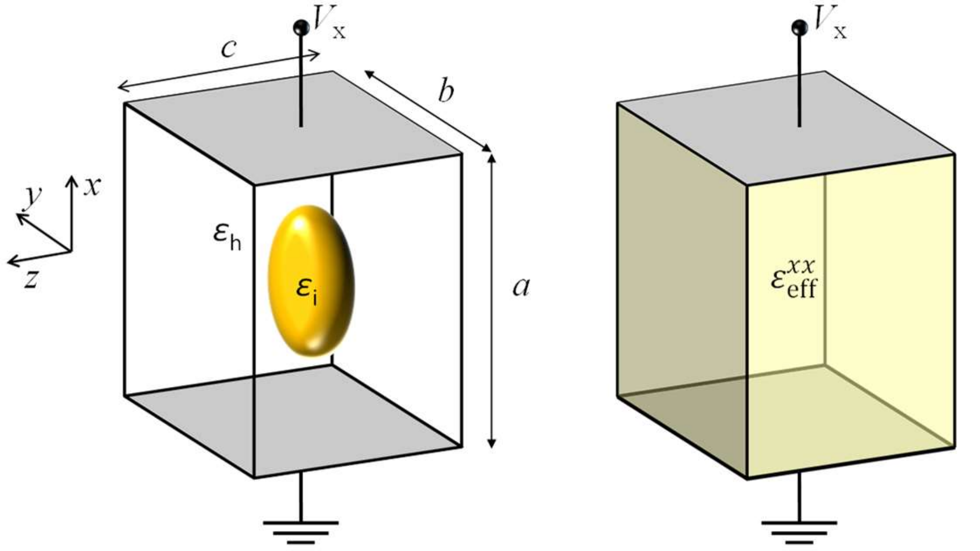
Quasi-static approaches have been applied extensively to metamaterials, in particular to periodic and sub-wavelength arrangements of plasmonic resonators.

We consider a mixture that can be modeled as an array of scatterers with relative permittivity ε_{i} immersed in a host medium with permittivity ε_{h} and divide the volume in rectangular-prism unit cells, similar to a three-dimensional metamaterial. In order to understand the macroscopic bulk response of this system we focus on the behavior of the unit cell centered at the origin of the Cartesian coordinates, i.e., the rectangular domain $\left[-\frac{a}{2},\frac{a}{2}\right]\times\left[-\frac{b}{2},\frac{b}{2}\right]\times\left[-\frac{c}{2},\frac{c}{2}\right]$.



Quasi-static approaches

- Constituent materials are nonmagnetic
- Symmetric scatterers (only non zero effective permittivities are $\varepsilon_{\mathrm{eff}}^{xx}$, $\varepsilon_{\mathrm{eff}}^{yy}$ and $\varepsilon_{\mathrm{eff}}^{zz}$)



When an external, static electric field $\mathbf{E}_0$ is applied to the unit cell, the local electric potential $\varphi(\mathbf{r})$ is found by solving the Poisson's equation

$$\nabla \cdot [\varepsilon_\omega(\mathbf{r}) \nabla \varphi(\mathbf{r})] = 0,$$

where $\varepsilon_\omega(\mathbf{r})$ is the frequency-dependent absolute permittivity at position **r** in the unit cell.
The boundary conditions depend on the permittivity component to retrieve.
If we are interested in calculating $\varepsilon_{\mathrm{eff}}^{xx}$ then it is useful to apply a field $\mathbf{E}_0 = \mathrm{E}_{0,x}\hat{\boldsymbol{x}}$ along the *x* direction. At this point the unit cell may be thought as a small capacitor with a static voltage $V_x = \varphi\left(-\frac{a}{2}, y, z\right) - \varphi\left(\frac{a}{2}, y, z\right) = \mathrm{E}_{0,x}a$ applied between two virtual plates located at $x = -\frac{a}{2}, \frac{a}{2}$. The boundary conditions at $y = -\frac{b}{2}, \frac{b}{2}$ and $z = -\frac{c}{2}, \frac{c}{2}$ are $\frac{\partial \varphi(\mathbf{r})}{\partial \mathrm{y}} = 0$ and $\frac{\partial \varphi(\mathbf{r})}{\partial \mathrm{z}} = 0$, respectively.
Using these boundary conditions, the problem can be solved numerically with different approaches. The most popular are based on finite-differences and finite-elements methods.

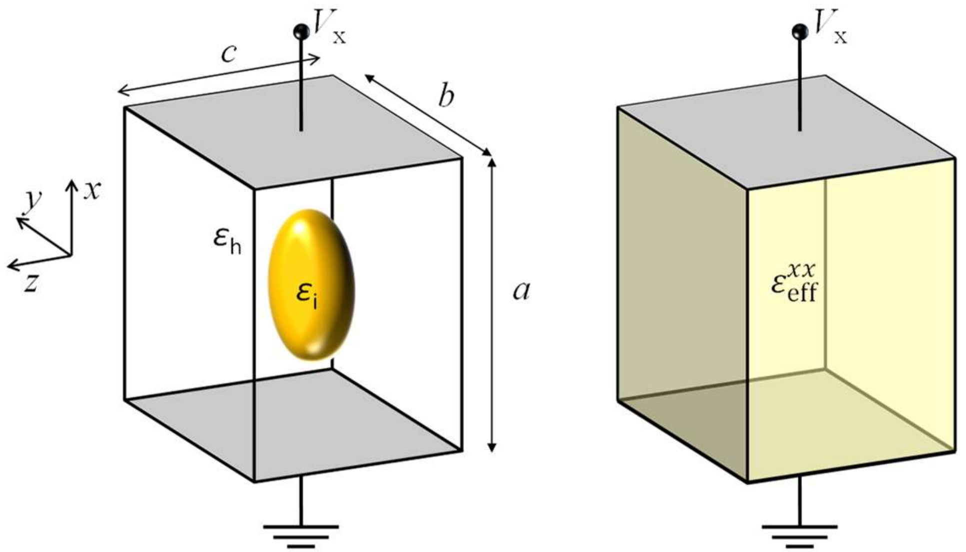
Quasi-static approaches

Once the potential $\varphi(\mathbf{r})$ is known, we impose that the inhomogeneous unit cell under investigation show the same capacitance of a parallel-plate capacitor filled with a homogeneous medium with relative permittivity $\varepsilon_{\text{eff}}^{xx}$. This capacitance is simply:

$$C_{\text{eff},x} = \varepsilon_0 \, \varepsilon_{\text{eff}}^{xx} \, bc/a.$$

On the other hand, the capacitance of the inhomogeneous unit cell is given by $C_x = q_0/V_x$, where q_0 is the charge per unit length on the virtual plate at $x = a/2$.
The charge is found straightforwardly by integrating the surface charge density $\sigma(y,z) = -\varepsilon_0 \varepsilon_h \frac{\partial \varphi(\mathbf{r})}{\partial \mathrm{x}}|_{x=a/2}$ over the surface of the virtual plate located at $x = a/2$. By equating $C_x = C_{\text{eff},x} = \varepsilon_0 \, \varepsilon_{\text{eff}}^{xx} \, bc/a$, the Quasi-static effective permittivity is retrieved and it reads as follows:

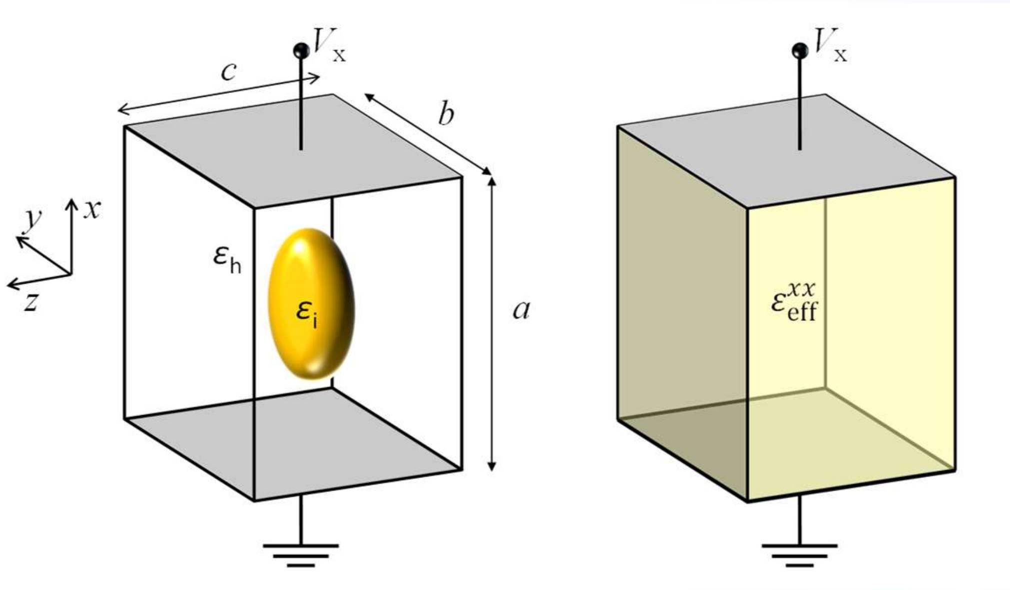
$$\varepsilon_{\text{eff}}^{xx} = -\varepsilon_h \frac{\int_{y=-b/2}^{b/2} \int_{z=-c/2}^{c/2} \frac{\partial \varphi(\mathbf{r})}{\partial \mathrm{x}}|_{x=a/2} dy dz}{\mathrm{E}_{0,x} bc}.$$



Quasi-static approaches

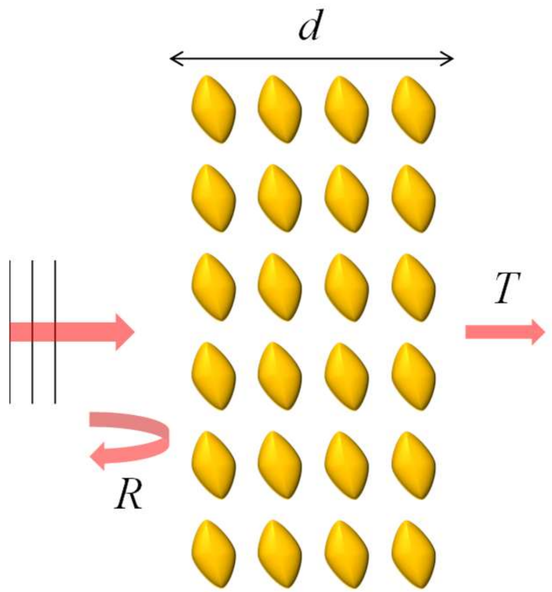
A similar procedure can be followed to retrieve $\varepsilon_{\mathrm{eff}}^{yy}$ and $\varepsilon_{\mathrm{eff}}^{zz}$ by applying static voltages across virtual parallel plates perpendicular to the *y* and *z* directions, respectively.

For constituents with chromatic dispersion, such as metal or semiconductor inclusions, the procedure described above must be repeated for each frequency so that a frequency-dependent, complex permittivity tensor is obtained.



Nicholson – Ross – Weir Method

Homogenization method **based on the inversion of Fresnel formulas** relative to the transmission and reflection coefficients through slabs of homogeneous media.
The technique was conceived to estimate the complex permittivity and permeability of an unknown material from the measured transmission and reflection spectra of a finite-thickness sample.
It was originally proposed in the time-domain for pulsed measurement systems and then adapted to higher-resolution, frequency domain systems. The transmission and reflection spectra may be retrieved with experiments or with numerical simulations.



A slab of thickness *d* of the unknown natural or artificial mixture is modeled as a slab of a homogeneous medium with effective permittivity $\varepsilon_{\mathrm{eff}}$ and permeability μ_{eff}.
It is supposed that the thickness of the homogeneous slab is equal to *d*.
The (complex) reflection and transmission coefficients *R* and *T* under normal incidence plane-wave excitation are somehow known, via an experiment or a theoretical prediction or a numerical simulation.

Nicholson – Ross – Weir Method

The Fresnel reflection (R) and transmission (T) coefficients may be written in the following form

$$R = \frac{\Gamma(1-e^{2jkn_{\mathrm{eff}}d})}{1-\Gamma^2 e^{2jkn_{\mathrm{eff}}d}},$$

$$T = \frac{(1-\Gamma^2)e^{jkn_{\mathrm{eff}}d}}{1-\Gamma^2 e^{2jkn_{\mathrm{eff}}d}},$$

which are dependent on the effective refractive index of the slab $n_{\mathrm{eff}} = \sqrt{\varepsilon_{\mathrm{eff}}\mu_{\mathrm{eff}}}$, free-space wavenumber $k = \omega/c$ and the reflection coefficient Γ across the first interface between the input medium and the semi-infinite homogeneous slab with relative parameters ($\varepsilon_{\mathrm{eff}}, \mu_{\mathrm{eff}}$). At normal incidence $\Gamma = \frac{\eta_{\mathrm{eff}}-\eta_0}{\eta_{\mathrm{eff}}+\eta_0}$, where $\eta_{\mathrm{eff}} = \sqrt{\mu_{\mathrm{eff}}/\varepsilon_{\mathrm{eff}}}$ is the effective intrinsic impedance of the homogeneous slab and η_0 is the intrinsic impedance of the input/output medium.

Nicholson – Ross – Weir Method

The inversion of the Fresnel formulas leads to the following expression of the effective impedance

$$\eta_{\text{eff}} = \pm\eta_0\sqrt{\frac{(1+R)^2-T^2}{(1-R)^2-T^2}},$$

and the following expression for the quantity $Q = e^{jkn_{\text{eff}}d}$,

$$Q = \frac{T}{1-R\frac{\eta_{\text{eff}}-\eta_0}{\eta_{\text{eff}}+\eta_0}}$$

From the expressions η_{eff} and Q one can write the effective refractive index as follows:

$$n_{\text{eff}} = \frac{-j}{kd}\log(Q) = \frac{1}{kd}\{-j\,\text{Log}|Q| - [\text{Arg}(Q) + 2m\pi]\}.$$

Here $\log(Q)$ is the complex, multiple-valued logarithm of (Q), $\text{Log}|Q|$ is the ordinary real logarithm of $|Q|$ and $\text{Arg}(Q)$ is the argument in the principal branch (m=0), and $m = \pm1, \pm2, \ldots$ indicates the branch of $\log(Q)$. The effective parameters can be finally written as follows:

$$\varepsilon_{\text{eff}} = \frac{kn_{\text{eff}}}{\omega\eta_{\text{eff}}}, \mu_{\text{eff}} = \frac{kn_{\text{eff}}\eta_{\text{eff}}}{\omega}.$$

Nicholson – Ross – Weir Method

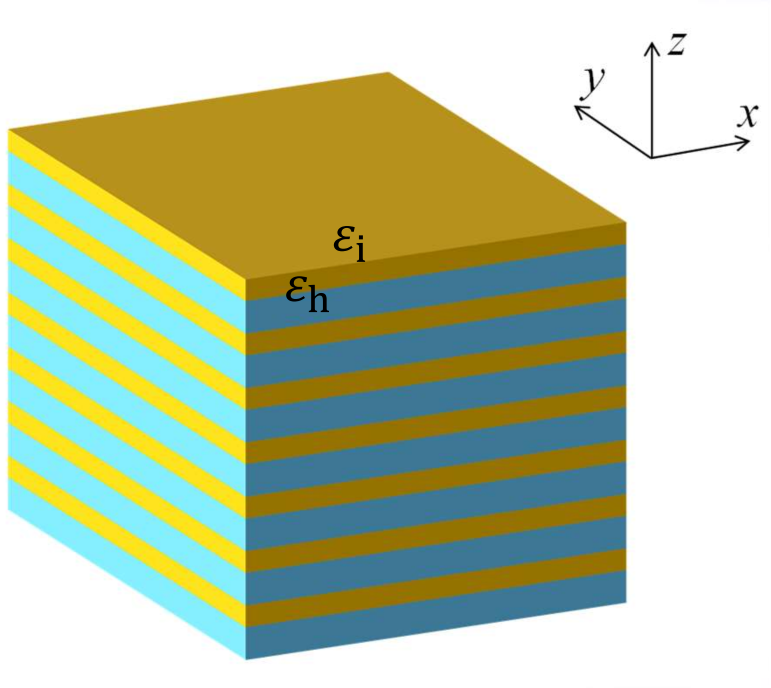
Limitations

- There is an intrinsic ambiguity in the definition of $\mathrm{Re}(n_{\mathrm{eff}})$ in equation
  $$n_{\mathrm{eff}} = -\frac{j}{kd}\log(Q) = \frac{1}{kd}\{-j\,\mathrm{Log}|Q| - [\mathrm{Arg}(Q) + 2m\pi]\}.,$$
  because of the multiple-valued complex logarithm $\log(Q)$ and the choice of the branch order m. This problem may be solved in very thin slabs in which the effective wavelength is larger than $2d$. In this case it is possible to consider only the principal branch (m=0) so that an unambiguous definition of the effective parameters is retrieved. However, the application of homogenization principles conceived for bulk natural or artificial mixtures may be misleading for the homogenization of very thin films.

- For structured films or artificial surfaces, more sophisticated techniques have been recently developed, based on the introduction of a nonlocal, effective surface susceptibility. In the case of electrically thick slabs, a common solution is to apply the retrieval method described above for two different values of slab thickness d_1 and d_2. The value of the effective refractive index should be unique once the frequency is fixed and must not depend on the sample thickness. As a consequence, the ambiguity is removed by identifying the correct branches that lead to the same value of $\mathrm{Re}(n_{\mathrm{eff}})$ for the two sample thicknesses ($d = d_1$ and $d = d_2$).

Example 1: Multilayer Media

A strong anisotropy can be engineered in artificial materials with inhomogeneous distributions of inclusions in one or two dimensions. In the figure below a multilayer structure is illustrated. This structure has two phases with inclusions of permittivities ε_{i} immersed in a host medium with permittivity ε_{h}. The inclusion filling factor is f.



Example 1: Multilayer Media

The multilayer shows two different responses for waves polarized parallel or perpendicular to the plane of the layers, so **that it is appropriate to assume an anisotropic uniaxial response** with effective permittivity $\bar{\boldsymbol{\varepsilon}} = \varepsilon_{\parallel}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \varepsilon_{\perp}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ are the unit vectors along the principal axes.

In order to find $\varepsilon_{\parallel}$, one should consider a wave with the electric field polarized in the parallel direction, i.e., in the plane *x*-*y*. For this kind of waves, the electric field is tangential to the interfaces hence it must be continuous, so that the average static electric field in the multilayer can be written as

$$< E_{\parallel} >= E_{\parallel,\mathrm{i}} = E_{\parallel,\mathrm{h}},$$

where $E_{\parallel,\mathrm{i}}$ and $E_{\parallel,\mathrm{h}}$ are the static electric fields in the inclusion and host layers. From the definition of the average displacement field,

$$< D_{\parallel} >= \varepsilon_0 \varepsilon_{\parallel} < E_{\parallel} >= \varepsilon_0 f \varepsilon_{\mathrm{i}} E_{\parallel,\mathrm{i}} + \varepsilon_0 (1-f) \varepsilon_{\mathrm{h}} E_{\parallel,\mathrm{h}},$$

directly follows the expression of the parallel permittivity:

$$\varepsilon_{\parallel} = f\varepsilon_{\mathrm{i}} + (1-f)\varepsilon_{\mathrm{h}}.$$

Example 1: Multilayer Media

For waves polarized in the direction perpendicular to the layers the boundary condition at the interfaces requires continuity of the displacement field, therefore the average displacement field is simply

$$< D_{\perp} >= \varepsilon_0 \varepsilon_{\perp} < E_{\perp} >= D_{\perp,\mathrm{i}} = D_{\perp,\mathrm{h}},$$

where $D_{\perp,\mathrm{i}}$ and $D_{\perp,\mathrm{h}}$ are the static displacement fields in the inclusion and host layers, respectively.

From the definition of the average electric field, $< E_{\perp} >= f E_{\perp,\mathrm{i}} + (1-f) E_{\perp,\mathrm{h}}$, where $E_{\perp,\mathrm{i}} = \frac{D_{\perp,\mathrm{i}}}{\varepsilon_0 \varepsilon_\mathrm{i}}$ and $E_{\perp,\mathrm{h}} = \frac{D_{\perp,\mathrm{h}}}{\varepsilon_0 \varepsilon_\mathrm{h}}$ are the static electric fields in the inclusion and host layers, in that order, we can derive the expression of the effective permittivity $\varepsilon_{\perp}$, which reads as follows:

$$\varepsilon_{\perp} = \frac{\varepsilon_\mathrm{i} \varepsilon_\mathrm{h}}{f \varepsilon_\mathrm{h} + (1-f) \varepsilon_\mathrm{i}}.$$

The permittivity tensor of the multilayer can be also retrieved by applying the Maxwell Garnett or the Bruggeman effective medium theories for ellipsoids. The depolarization factors in the case of multilayers are $L_x = L_y = 0$ and $L_z = 1$, equivalent to virtual disk-like particles with eccentricity $e \to \infty$. The corresponding screening parameters are $K_x = K_y \to \infty$ and $K_z \to 0$.

Example 2: Wire Media

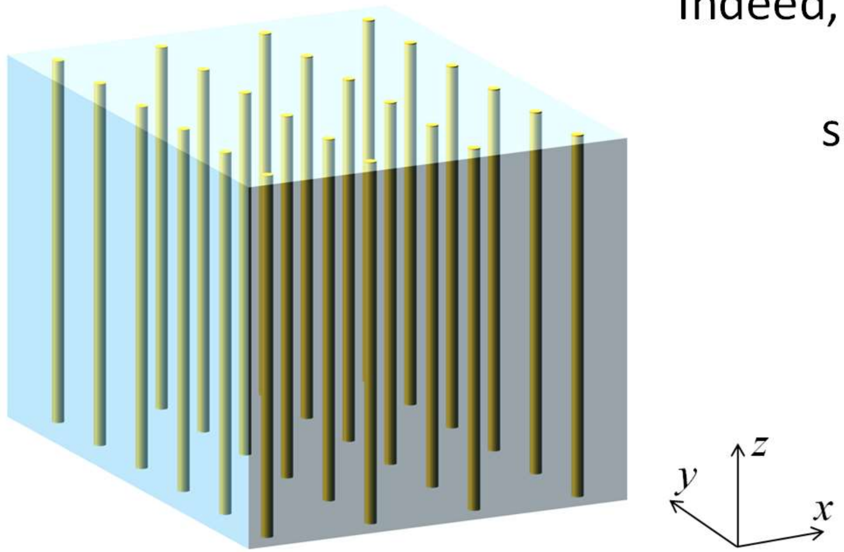
The material shows two different responses for waves polarized parallel or perpendicular to the wires, so that it is appropriate to assume an anisotropic uniaxial response with effective permittivity $\bar{\boldsymbol{\varepsilon}} = \varepsilon_{\perp}(\widehat{\boldsymbol{x}}\widehat{\boldsymbol{x}} + \widehat{\boldsymbol{y}}\widehat{\boldsymbol{y}}) + \varepsilon_{\parallel}\, \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$. Note that in this case the parallel component of the effective permittivity is related to waves polarized along the *z* direction, i.e., parallel to the wires' long axis.

For these waves the electric field is continuous and the average displacement field is

$$< D_{\parallel} >= \varepsilon_0 \varepsilon_{\parallel} < E_{\parallel} >= \varepsilon_0 f \varepsilon_{\mathrm{i}} E_{\parallel,\mathrm{i}} + \varepsilon_0 (1-f) \varepsilon_{\mathrm{h}} E_{\parallel,\mathrm{h}},$$

It follows that $\varepsilon_{\parallel} = f\varepsilon_{\mathrm{i}} + (1-f)\varepsilon_{\mathrm{h}}$. This result may be also derived from the Maxwell Garnett formula (with L_z=0).

For the permittivity component $\varepsilon_{\perp}$, associated with waves polarized in the plane perpendicular to the wires, the most natural homogenization strategy is again the Maxwell Garnett formula.



Indeed, the cross section view of the wire medium is very similar to the typical Maxwell Garnett mixture, in which small spherical inclusions are dispersed in a homogeneous host. The only difference with respect to three-dimensional mixtures is that the depolarization factor for cylinders is $L_x = L_y = 1/2$. The resulting expression for the quasi-static permittivity reads as follows

$$\varepsilon_{\perp} = \varepsilon_{\mathrm{h}} \left(1 + 2f \frac{\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}}}{\varepsilon_{\mathrm{i}} + \varepsilon_{\mathrm{h}} - f(\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{h}})}\right).$$