8.11 Construct the standard array for the (7, 3) code with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and determine the correctable patterns and their corresponding syndromes.

8.12 Determine the correctable error patterns (of least weight) and their syndromes for the systematic (7, 4) cyclic Hamming code.

8.13 Prove that if the sum of two error patterns $\mathbf{e}_1$ and $\mathbf{e}_2$ is a valid code word $\mathbf{C}_j$, then each pattern has the same syndrome.

Ex.

8.14 Let $g(p) = p^8 + p^6 + p^4 + p^2 + 1$ be a polynomial over the binary field.

a) Find the lowest-rate cyclic code whose generator polynomial is $g(p)$. What is the rate of this code?

b) Find the minimum distance of the code found in (a).

c) What is the coding gain for the code found in (a)?

8.15 The polynomial $g(p) = p + 1$ over the binary field is considered.

a) Show that this polynomial can generate a cyclic code for any choice of n. Find the corresponding k.

b) Find the systematic form of **G** and **H** for the code generated by $g(p)$.

c) Can you say what type of code this generator polynomial generates?

Ex.

8.16 Design a (6, 2) cyclic code by choosing the shortest possible generator polynomial.

a) Determine the generator matrix **G** (in the systematic form) for this code and find all possible code words.

b) How many errors can be corrected by this code?

8.17 Prove that any two n-tuples in the same row of a standard array add to produce a valid code word.

8.18 Beginning with a (15, 7) BCH code, construct a shortened (12, 4) code. Give the generator matrix for the shortened code.

8.19 In Section 8.1.2, it was indicated that when an (n, k) Hadamard code is mapped into waveforms by means of binary PSK, the corresponding $M = 2^k$ waveforms are orthogonal. Determine the bandwidth expansion factor for the M orthogonal waveforms and compare this with the bandwidth requirements of orthogonal FSK detected coherently.

8.20 Show that the signaling waveforms generated from a maximum-length shift-register code by mapping each bit in a code word into a binary PSK signal are equicorrelated with correlation coefficient $\rho_r = -1/(M-1)$, i.e., the M waveforms form a simplex set.

8.21 Compute the error probability obtained with a (7, 4) Hamming code on an AWGN channel, both for hard-decision and soft-decision decoding. Use Equations 8.1–50, 8.1–52, 8.1–82, 8.1–90, and 8.1–91.

(c) A vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$ is a codeword of the $(n, n-1)$ cyclic code if it satisfies the condition $\mathbf{c}\mathbf{H}^t = 0$. But,

$$\mathbf{c}\mathbf{H}^t = 0 = \mathbf{c}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = c_1 + c_2 + \cdots c_n$$

Thus, the vector $\mathbf{c}$ belongs to the code if it has an even weight. Therefore, the cyclic code generated by the polynomial $p+1$ is a simple parity check code.

Problem 8.16 :

(a) The generator polynomial of degree $4 = n - k$ should divide the polynomial $p^6 + 1$. Since the polynomial $p^6 + 1$ assumes the factorization

$$p^6 + 1 = (p+1)^3(p+1)^3 = (p+1)(p+1)(p^2+p+1)(p^2+p+1)$$

we find that the shortest possible generator polynomial of degree 4 is

$$g(p) = p^4 + p^2 + 1$$

The i^{th} row of the generator matrix $\mathbf{G}$ has the form

$$\mathbf{g}_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & p_{i,1} & \cdots & p_{i,4} \end{bmatrix}$$

where the 1 corresponds to the i-th column (to give a systematic code) and the $p_{i,1}, \ldots, p_{i,4}$ are obtained from the relation

$$p^{6-i} + p_{i,1}p^3 + p_{i,2}p^2 p_{i,3}p + p_{i,4} = p^{6-i}(\bmod p^4 + p^2 + 1)$$

Hence,

$$\begin{aligned} p^5 \bmod p^4 + p^2 + 1 &= (p^2+1)p \bmod p^4 + p^2 + 1 = p^3 + p \\ p^4 \bmod p^4 + p^2 + 1 &= p^2 + 1 \bmod p^4 + p^2 + 1 = p^2 + 1 \end{aligned}$$

and therefore,

$$\mathbf{G} = \left(\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

The codewords of the code are

$$\begin{aligned} \mathbf{c}_1 &= [\,0\ 0\ 0\ 0\ 0\ 0\,] \\ \mathbf{c}_2 &= [\,1\ 0\ 1\ 0\ 1\ 0\,] \\ \mathbf{c}_3 &= [\,0\ 1\ 0\ 1\ 0\ 1\,] \\ \mathbf{c}_4 &= [\,1\ 1\ 1\ 1\ 1\ 1\,] \end{aligned}$$

(b) The minimum distance of the linear $(6, 2)$ cyclic code is $d_{\min} = w_{\min} = 3$. Therefore, the code can correct

$$e_c = \frac{d_{\min} - 1}{2} = 1 \text{ error}$$

Problem 8.17 :

Consider two n-tuples in the same row of a standard array. Clearly, if $\mathbf{Y}_1, \mathbf{Y}_2$ denote the n-tuples, $\mathbf{Y}_1 = \mathbf{C}_j + \mathbf{e}$, $\mathbf{Y}_2 = \mathbf{C}_k + \mathbf{e}$, where $\mathbf{C}_k, \mathbf{C}_j$ are two valid codewords, and the error pattern e is the same since they are in the same row of the standard array. Then :

$$\mathbf{Y}_1 + \mathbf{Y}_2 = \mathbf{C}_j + \mathbf{e} + \mathbf{C}_k + \mathbf{e} = \mathbf{C}_j + \mathbf{C}_k = \mathbf{C}_m$$

where $\mathbf{C}_m$ is another valid codeword (this follows from the linearity of the code).

Problem 8.18 :

From Table 8-1-6 we find that the coefficients of the generator polynomial for the (15,7) BCH code are $721 \rightarrow 111010001$ or $g(p) = p^8 + p^7 + p^6 + p^4 + 1$. Then, we can determine the l-th row of the generator matrix $\mathbf{G}$, using the modulo $R_l(p)$: $p^{n-l} = Q_l(p)g(p) + R_l(p)$, $l = 1, 2, ..., 7$. Since the generator matrix of the shortened code is obtained by removing the first three rows of $\mathbf{G}$, we perform the above calculations for $l = 4, 5, 6, 7$, only :

$$\begin{aligned} p^{11} &= (p^3 + p^2 + 1)g(p) + p^4 + p^3 + p^2 + 1 \\ p^{10} &= (p^2 + p)g(p) + p^7 + p^6 + p^5 + p^2 + p \\ p^9 &= (p + 1)g(p) + p^6 + p^5 + p^4 + p + 1 \\ p^8 &= (p + 1)g(p) + p^7 + p^6 + p^4 + 1 \end{aligned}$$

Hence :

$$\mathbf{G}_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 8.19 :

For M-ary FSK detected coherently, the bandwidth expansion factor is :

$$\left(\frac{W}{R}\right)_{FSK} = \frac{M}{2\log_2 M}$$