Optimal receiver

Principles of digital communication in presence of AWGN (1/5)

$$\begin{tikzcd} {} \arrow[r, "s_i(t)"] & \oplus \arrow[r, "r(t)=s_i(t)+w(t)"] & \boxed{\text{Rec}} \\ & w(t) \arrow[u] & \end{tikzcd}$$

$P(\varepsilon)$

The receiver chooses the message which maximize the «a posteriori» probability

BAYES:

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B)}$$

$$P(A_i \mid X) = \frac{f_x(x \mid A_i)P(A_i)}{f_x(x)}$$

$f_x(x) \longrightarrow$ Independent from i

$A_i \rightarrow s_i(t) \quad B \rightarrow r(t) = s_i(t) + n(t) \longrightarrow$ Infinit dimension (n(t))

Principles of digital communication in presence of AWGN (2/5)

If n(t) is AWGN, then:

$$E[r_k / s_i] = \begin{cases} s_{ik} & n \leq N \\ 0 & n > N \end{cases}$$

Where N is the signal space dimension

Considering n components:

$$f(r_1, r_2, \ldots, r_n / s_i) = \prod_{k=1}^{N} \left(\frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-\frac{(r_k - s_{ik})^2}{2\frac{N_0}{2}}} \right) \cdot \prod_{k=N+1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma_k^2}} e^{-\frac{r_k{}^2}{2\sigma_k^2}} \right)$$

Principles of digital communication in presence of AWGN (3/5)

We are looking for the maximum with respect to «i», therefore the terms that are independent from «i» are irrelevant.
We can say that the component of the received signal in the signal space represent a «sufficient statistic» for the optimal detection, therefore the component of the received signal out of the signal space are «irrelevant» (beeing orthogonal and therefore (beeing gaussian) statistically independent.

If we consider:

$$f\left(\frac{r}{s_i}\right) \equiv e^{-\frac{1}{N_0}\sum_{k=1}^{N}(r_k - s_{ik})^2} = e^{-\frac{1}{N_0}\|r - s_i\|^2}$$

Square distance between r(t) and si(t), in the signal space (N dimensional)

Proportional to

Principles of digital communication in presence of AWGN (4/5)

$$\left\| r - s_i \right\|^2 = \int (r(t) - s_i(t))^2 dt = \sum_{k=1}^{\infty} (r_k - s_{ik})^2 = \sum_{k=1}^{N} (r_k - s_{ik})^2 + \sum_{k=N+1}^{\infty} r_k{}^2$$

Returning on Bayes rule:

$$f(r/s_i)P(s_i) = e^{-\frac{1}{N_0}\left\| r - s_i \right\|^2} P(s_i)$$

A priori probability

Likelihood function

Using log(…) $\left\| r - s_i \right\|^2 - N_0 \log P(s_i)$

I can define some «decision regions», associated to each possible transmitted signal

Principles of digital communication in presence of AWGN (5/5)

In general: $\|r - s_i\|^2 = \|r\|^2 + \|s_i\|^2 - 2 < r, s_i >$

We have to evaluate the maximum $< r, s_i > - \frac{1}{2}\|s_i\|^2 \longrightarrow < r, s_i >= \sum_{k=1}^{N} r_k s_{ik}$

$$r_k = \int r(t)\phi_k(t)dt$$

$$< r, s_i >= \int r(t)s_i(t)dt$$

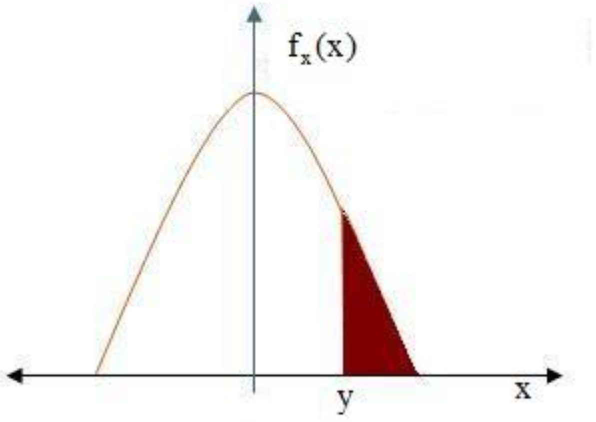
In case of pass-band signal $\longrightarrow r_k = \frac{1}{2}\int z(t)\overline{z_k}(t)dt$

Error probabilities (1/2)

$$P(E) = \sum_{i=1}^{M} P(s_i) P(E / s_i) = \sum_{i=1}^{M} P(s_i) \sum_{j \neq i} P(s_j / s_i)$$

$$P(s_2 / s_1) = P(s_1 / s_2) = P(E) = Q\left(\frac{d / r}{\sqrt{N_0 / 2}} \right) = Q\left(\frac{d}{\sqrt{2N_0}} \right)$$

$$Q(y) = \int_{y}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \longrightarrow$$



Error probabilities (2/2)

$$P(E) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

If $s_2 = -s_1$

$P(E)$	10^{-3}	10^{-5}	10^{-7}	10^{-10}	10^{-13}
$E_b / N_0[dB]$	6,79	9,59	11,31	13,06	14,41

$$Q(y) \approx \frac{1}{\sqrt{2\pi y}} e^{-\frac{y^2}{2}} \qquad y > 3$$

$$\log_{10} Q(y) \approx -0{,}22\gamma^2 - 1{,}04$$

Number of bits differing between the bit mapping associated to s_i and s_j

$$P(E) = \frac{1}{\log_2 M} \sum_{i=1}^{M} P(s_i) \sum_{j \neq i} n_{ij} P(s_i / s_j)$$

$$\frac{P(E)}{\log_2 M} \leq P_b(E) \leq P(E)$$

Symbol error

Bit error

Union bound

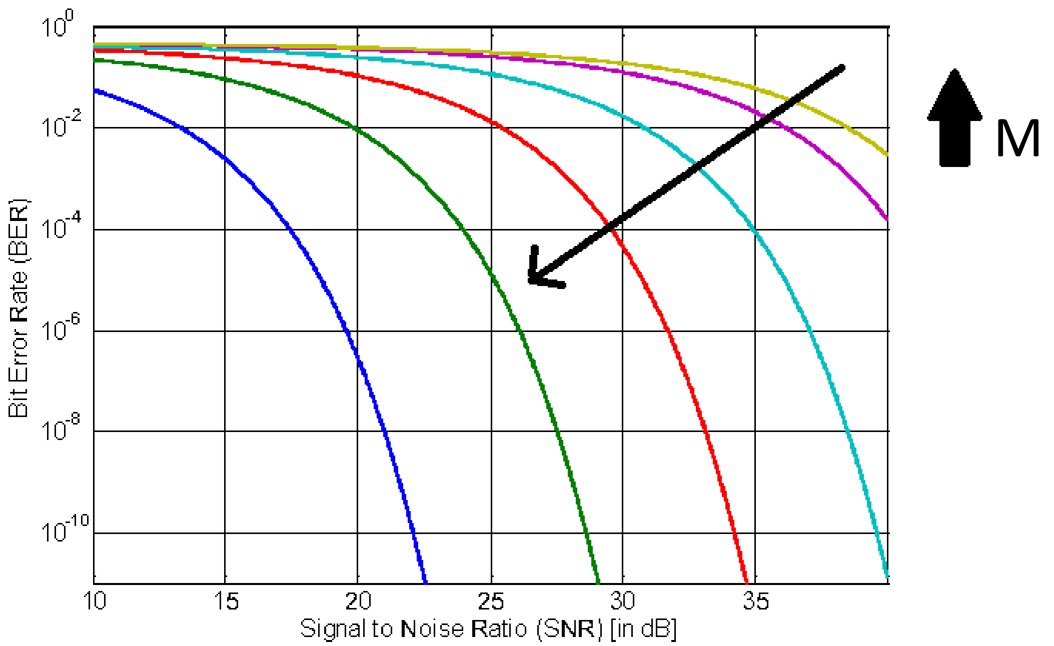
$$P(E) = \sum_{i=1}^{M} P(s_i) \sum_{j \neq i} P(s_i / s_j) \leq \sum_{i=1}^{M} P(s_i) \sum_{j \neq i} Q\left(\frac{d_{ij}}{\sqrt{2N_0}} \right)$$

$$P_b(E) = \frac{1}{\log_2 M} \sum_{i=1}^{M} P(s_i) \sum_{j \neq i} n_{ij} P(s_i / s_j) \leq \frac{1}{\log_2 M} \sum_{i=1}^{M} P(s_i) \sum_{j \neq i} n_{ij} Q\left(\frac{d_{ij}}{\sqrt{2N_0}} \right)$$

Orthogonal signal (1/2)

$$P(E) \le (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right) = (M-1)Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$

$$P_b(E) \le \frac{M}{2}Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$



Orthogonal signal (2/2)

$$Q(y) \le \frac{1}{2} e^{-\frac{y^2}{2}} \quad \text{If} \quad y \ge 0$$

$$P(E) < M e^{\left(-\frac{E_b \log_2 M}{2N_0}\right)} = e^{\left(-\log_2 M\left(\frac{E_b}{2N_0} - \log 2\right)\right)}$$

$$\lim_{M\to\infty} e^{\left(-\log_2 M\left(\frac{E_b}{2N_0} - \log 2\right)\right)} \to 0 \quad \text{If}$$

$$\frac{E_b}{N_0} > 2\log 2 \qquad (1{,}41 \quad dB)$$

$$\frac{E_b}{N_0} = 9{,}59 \quad dB \Rightarrow P(E) = 10^{-5}$$ (bi antipodal)

In reality it's enough

$$\frac{E_b}{N_0} > \log 2 \qquad (-1{,}59 \quad dB)$$

$$P(E) = 0 \qquad M \to \infty$$