C. Properties of Linear Time-Invariant Systems and Various Transforms

C.1. Continuous-Time LTI Systems

Unit impulse response: $h(t)$

Convolution: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$

Causality: $h(t) = 0, t < 0$

Stability: $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

C.2. The Laplace Transform

C.2.1. The Bilateral (or Two-Sided) Laplace Transform:

C.2.1.1. Definition:

$$x(t) \xleftrightarrow{\mathscr{L}} X(s)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\, dt$$

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\, ds$$

C.2.1.2. Properties of the Bilateral Laplace Transform:

Linearity: $a_1x_1(t) + a_2x_2(t) \leftrightarrow a_1X_1(s) + a_2X_2(s)$, $R' \supset R_1 \cap R_2$

Time shifting: $x(t - t_0) \leftrightarrow e^{-st_0}X(s)$, $R' = R$

Shifting in s: $e^{s_0t}x(t) \leftrightarrow X(s - s_0)$, $R' = R + \mathrm{Re}(s_0)$

Time scaling: $x(at) \leftrightarrow \frac{1}{|a|}X(s), R' = aR$

Time reversal: $x(-t) \leftrightarrow X(-s)$, $R' = -R$

Differentiation in t: $\frac{dx(t)}{dt} \leftrightarrow sX(s), R' \supset R$

Differentiation in s: $-tx(t) \leftrightarrow \frac{dX(s)}{ds}, R' = R$

Integration: $\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s} X(s),\ R' \supset R \cap \{\mathrm{Re}(s) > 0\}$

Convolution: $x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s),\ R' \supset R_1 \cap R_2$

C.2.1.3. Some Laplace Transforms Pairs:

$$
\begin{aligned}
&\delta(t) \leftrightarrow 1, \text{ all } s \\
&u(t) \leftrightarrow \frac{1}{s}, \mathrm{Re}(s) > 0 \\
&-u(-t) \leftrightarrow \frac{1}{s}, \mathrm{Re}(s) < 0 \\
&tu(t) \leftrightarrow \frac{1}{s^2}, \mathrm{Re}(s) > 0 \\
&t^k u(t) \leftrightarrow \frac{k!}{s^{k+1}}, \mathrm{Re}(s) > 0 \\
&e^{-at} u(t) \leftrightarrow \frac{1}{s+a}, \mathrm{Re}(s) > -\mathrm{Re}(a) \\
&-e^{-at} u(-t) \leftrightarrow \frac{1}{s+a}, \mathrm{Re}(s) < -\mathrm{Re}(a) \\
&te^{-at} u(t) \leftrightarrow \frac{1}{(s+a)^2}, \mathrm{Re}(s) > -\mathrm{Re}(a) \\
&-te^{-at} u(-t) \leftrightarrow \frac{1}{(s+a)^2}, \mathrm{Re}(s) < -\mathrm{Re}(a) \\
&\cos \omega_0 t u(t) \leftrightarrow \frac{s}{s^2 + \omega_0^2}, \mathrm{Re}(s) > 0 \\
&\sin \omega_0 t u(t) \leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2}, \mathrm{Re}(s) > 0 \\
&e^{-at} \cos \omega_0 t u(t) \leftrightarrow \frac{s+a}{(s+a)^2 + \omega_0^2}, \mathrm{Re}(s) > -\mathrm{Re}(a) \\
&e^{-at} \sin \omega_0 t u(t) \leftrightarrow \frac{\omega_0}{(s+a)^2 + \omega_0^2}, \mathrm{Re}(s) > -\mathrm{Re}(a)
\end{aligned}
$$

C.2.2. The Unilateral (or One-Sided) Laplace Transform:

C.2.2.1. Definition:

$$x(t) \xleftrightarrow{\mathcal{L}_I} X_I(s)$$

$$X_I(s) = \int_{0^-}^{\infty} x(t) e^{-st}\, dt \qquad 0^- = \lim_{\varepsilon \to \infty} (0 - \varepsilon)$$

C.2.3. Some Special Properties:

C.2.3.1. Differentiation in the Time Domain:

$$\frac{dx(t)}{dt} \leftrightarrow sX_I(s) - x(0^-)$$

$$\frac{d^2x(t)}{dt^2} \leftrightarrow s^2X_I(s) - sx(0^-) - x'(0^-)$$

$$\frac{d^nx(t)}{dt^n} \leftrightarrow s^nX_I(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) - \cdots - x^{(n-1)}(0^-)$$

C.2.4. Integration in the Time Domain:

$$\int_{0^-}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X_I(s)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X_I(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\,d\tau$$

$$\text{Initial value theorem}: x(0^+) = \lim_{s\to\infty} sX_I(s)$$

$$\text{Final value theorem}: \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX_I(s)$$

C.3. The Fourier Transform

C.3.1.1. Definition:

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

C.3.1.2. Properties of the Fourier Transform:

Linearity: $a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$

Time shifting: $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$

Frequency shifting: $e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0)$

Time scaling: $x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

Time reversal: $x(-t) \leftrightarrow X(-\omega)$

Duality: $X(t) \leftrightarrow 2\pi x(-\omega)$

Time differentiation: $\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$

Frequency differentiation: $(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega}$

Integraton: $\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \pi X(0)\, \delta(\omega) + \frac{1}{j\omega} X(\omega)$

Convolution: $x_1(t) * x_2(t) \leftrightarrow X_1(\omega)\, X_2(\omega)$

Multiplication: $x_1(t) x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Real signal: $x(\mathrm{t}) = x_e(t) + x_o(t) \leftrightarrow X(\omega) = A(\omega) + jB(\omega)$

$X(-\omega) = X^*(\omega)$

Even component: $x_e(t) \leftrightarrow \mathrm{Re}\{X(\omega)\} = A(\omega)$

Odd component: $x_o(t) \leftrightarrow j\,\mathrm{Im}\{X(\omega)\} = jB(\omega)$

C.3.1.3. Parseval's Relations:

$$\int_{-\infty}^{\infty} x_1(\lambda) X_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} X_1(\lambda) x_2(\lambda)\, d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t) x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

C.3.1.4. Common Fourier Transforms Pairs:

$$\delta(t) \leftrightarrow 1$$

$$\delta(t - t_0) \leftrightarrow e^{-j\omega t_0}$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\sin \omega_0 t \leftrightarrow j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$u(-t) \leftrightarrow \pi\delta(\omega) - \frac{1}{j\omega}$$

$$e^{-at}u(t) \leftrightarrow \frac{1}{j\omega + a}, a > 0$$

$$te^{-at}u(t) \leftrightarrow \frac{1}{(j\omega + a)^2}, \quad a > 0$$

$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2 + \omega^2}, \quad a > 0$$

$$\frac{1}{a^2 + t^2} \leftrightarrow e^{-a|\omega|}$$

$$e^{-at^2} \leftrightarrow \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}, \quad a > 0$$

$$p_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases} \leftrightarrow 2a\frac{\sin \omega a}{\omega a}$$

$$\frac{\sin at}{\pi t} \leftrightarrow p_a(\omega) = \begin{cases} 1 & |\omega| < a \\ 0 & |\omega| > a \end{cases}$$

$$\operatorname{sgn} t \leftrightarrow \frac{2}{j\omega}$$

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \leftrightarrow \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0), \ \omega_0 = \frac{2\pi}{T}$$

C.4. Discrete-Time LTI Systems

Unit sample response: $h[n]$

Convolution: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Causality: $h[n] = 0, \;\; n < 0$

Stability: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

C.5. The *z*-Transform

C.5.1. The Bilateral (or Two-Sided) *z*-Transform:

C.5.1.1. Definition:

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint_c X(z) z^{n-1}\, dz$$

C.5.1.2. Properties of the z-Transform:

Lineartity: $a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1(z) + a_2 X_2(z), \;\; R' \supset R_1 \cap R_2$

Time shifting: $x[n - n_0] \leftrightarrow z^{-n_0} X(z), \;\; R' \supset R_1 \cap \{0 < |z| < \infty\}$

Multiplication by z_0^n: $z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right), \;\; R' = |z_0| R$

Multiplication by $e^{j\Omega_0 N}$: $e^{j\Omega_0 n} X[n] \leftrightarrow X(e^{-j\Omega_0} z), \;\; R' = R$

Time reversal: $x[-n] \leftrightarrow X\left(\frac{1}{z}\right), \;\; R' = \frac{1}{R}$

Multiplication by n: $nx[n] \leftrightarrow -z\frac{dX(z)}{dz}, \;\; R' = R$

Accumulation: $\sum_{k=-\infty}^{n} x[n] \leftrightarrow \frac{1}{1 - z^{-1}} X(z), \;\; R' \supset R \cap \{|z| > 1\}$

Convolution: $x_1[n] * x_2[n] \leftrightarrow X_1(z) X_2(z), \;\; R' \supset R_1 \cap R_2$

C.5.1.3. Some Common z-Transforms Pairs:

C.5.1.3. Some Common z-Transforms Pairs:

$$\delta[n] \leftrightarrow 1, \text{ all } z$$

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}, |z|>1$$

$$-u[-n-1] \leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \quad |z|<1$$

$$\delta[n-m] \leftrightarrow z^{-m}, \text{ all } z \text{ except } 0 \text{ if } m>0, \text{ or } \infty \text{ if } m<0$$

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}} = \frac{z}{z-a}, |z|>|a|$$

$$-a^n u[-n-1] \leftrightarrow \frac{1}{1-az^{-1}} = \frac{z}{z-a}, |z|<|a|$$

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} = \frac{az}{(z-a)^2}, |z|>|a|$$

$$-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} = \frac{az}{(z-a)^2}, |z|<|a|$$

$$(n+1)a^n u[n] \leftrightarrow \frac{1}{(1-az^{-1})^2} = \left[\frac{z}{z-a}\right]^2, |z|>|a|$$

$$(\cos \Omega_0 n)u[n] \leftrightarrow \frac{z^2-(\cos\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}, |z|>1$$

$$(\sin \Omega_0 n)u[n] \leftrightarrow \frac{(\sin\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}, |z|>1$$

$$(r^n \cos \Omega_0 n)u[n] \leftrightarrow \frac{z^2-(r\cos\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}, |z|>r$$

$$(r^n \sin \Omega_0 n)u[n] \leftrightarrow \frac{(r\sin\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}, |z|>r$$

$$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}}, |z|>0$$

C.5.2. The Unilateral (or One-Sided) *z*-Transform:

$$x[n] \xleftrightarrow{\mathfrak{Z}_I} X_I(z)$$

$$X_I(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

C.5.3. Some Special Properties:

C.5.3.1. Time-Shifting Property:

$$x[n-m] \leftrightarrow z^{-m} X_I(z) + z^{-m+1} x[-1] + z^{-m+2} x[-2] + \cdots + x[-m]$$

$$x[n+m] \leftrightarrow z^{m} X_I(z) - z^{m} x[0] - z^{m-1} x[1] - \cdots - z x[m-1]$$

$$\text{Initial value theorem: } x[0] = \lim_{z \to \infty} X(z)$$

$$\text{Final value theorem: } \lim_{N \to \infty} x[N] = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

C.6. The Discrete-Time Fourier Transform

C.6.1.1. Definition:

$$x[n] \xleftrightarrow{\mathscr{F}} X(\Omega)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n}\, d\Omega$$

C.6.1.2. Properties of the Discrete-Time Fourier Transform:

Periodicity: $x[n] \leftrightarrow X(\Omega) = X(\Omega + 2\pi)$

Linearity: $a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega)$

Time shifting: $x[n - n_0] \leftrightarrow e^{-j\Omega n_0} X(\Omega)$

Frequency shifting: $e^{j\Omega_0 n} x[n] \leftrightarrow X(\Omega - \Omega_0)$

Conjugation: $x^*[n] \leftrightarrow X^*(-\Omega)$

Time reversal: $x[-n] \leftrightarrow X(-\Omega)$

Time scaling: $x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n = km \\ 0 & \text{if } n \neq km \end{cases} \leftrightarrow X(m\Omega)$

Frequency differentiation: $nx[n] \leftrightarrow j\dfrac{dX(\Omega)}{d(\Omega)}$

First difference: $x[n] - x[n-1] \leftrightarrow (1 - e^{-j\Omega}) X(\Omega)$

Accumulation: $\displaystyle\sum_{k=-\infty}^{n} x[k] \leftrightarrow \pi X(0)\,\delta(\Omega) + \frac{1}{1 - e^{-j\Omega}} X(\Omega)$

Convolution: $x_1[n] * x_2[n] \leftrightarrow X_1(\Omega)\, X_2(\Omega)$

Multiplication: $x_1[n] x_2[n] \leftrightarrow \dfrac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$

Real sequence: $x[n] = x_e[n] + x_o[n] \leftrightarrow X(\Omega) = A(\Omega) + jB(\Omega)$

$X(-\Omega) = X^*(\Omega)$

Even component: $x_e[n] \leftrightarrow \text{Re}\{x(\Omega)\} = A(\Omega)$

Odd component: $x_o[n] \leftrightarrow j\,\text{Im}\{X(\Omega)\} = jB(\Omega)$

C.6.1.3. Parseval's Relations:

$$\sum_{n=-\infty}^{\infty} x_1[n] x_2[n] = \frac{1}{2\pi} \int_{2\pi} X_1(\Omega) X_2(-\Omega)\, d\Omega$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2\, d\Omega$$

C.6.1.4. Some Common Fourier Transform Pairs:

$$\delta[n] \leftrightarrow 1$$

$$\delta[n-n_0] \leftrightarrow e^{-j\Omega n_0}$$

$$x[n]=1 \leftrightarrow 2\pi\,\delta(\Omega),\ \ |\Omega| \le \pi$$

$$e^{-j\Omega_0 n} \leftrightarrow 2\pi\,\delta(\Omega-\Omega_0),\ \ |\Omega|,|\Omega_0| \le \pi$$

$$\cos\Omega_0 n \leftrightarrow \pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)],\ \ |\Omega|,|\Omega_0| \le \pi$$

$$\sin\Omega_0 n \leftrightarrow -j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)],\ \ |\Omega|,|\Omega_0| \le \pi$$

$$u[n] \leftrightarrow \pi\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}},\ \ |\Omega_0| \le \pi$$

$$-u[-n-1] \leftrightarrow -\pi\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}},\ \ |\Omega| \le \pi$$

$$a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\Omega}},\ |a|<1$$

$$-a^n u[-n-1] \leftrightarrow \frac{1}{1-ae^{-j\Omega}},\ |a|>1$$

$$(n+1)\,a^n u[n] \leftrightarrow \frac{1}{(1-ae^{-j\Omega})^2},\ |a|<1$$

$$a^{|n|} \leftrightarrow \frac{1-a^2}{1-2a\cos\Omega+a^2},\ |a|<1$$

$$x[n]=\begin{cases}1 & |n| \le N_1\\ 0 & |n|>N_1\end{cases} \leftrightarrow \frac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$$

$$\frac{\sin W_n}{\pi n}(0<W<\pi) \leftrightarrow X(\Omega)=\begin{cases}1 & 0 \le |\Omega| \le W\\ 0 & W<|\Omega| \le \pi\end{cases}$$

$$\sum_{k=-\infty}^{\infty}\delta[n-kN_0] \leftrightarrow \Omega_0\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_0],\ \ \Omega_0=\frac{2\pi}{N_0}$$

C.7. Discrete Fourier Transform

C.7.1.1. Definition:

$$x[n] = 0 \qquad \text{outside the range } 0 \le n \le N-1$$

$$x[n] \xleftrightarrow{\text{DFT}} X[k]$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \qquad W_N = e^{-j(2\pi/N)}$$

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

C.7.1.2. Properties of the DFT:

$$\text{Linearity: } a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1[k] + a_2 X_2[k]$$

$$\text{Time shifting: } x[n-n_0]_{\text{mod } N} \leftrightarrow W_N^{kn_0} X[k]$$

$$\text{Frequency shifting: } W_N^{-kn_0} x[n] \leftrightarrow X[k-k_0]_{\text{mod } N}$$

$$\text{Conjugation: } x^*[n] \leftrightarrow X^*[-k]_{\text{mod } N}$$

$$\text{Time reversal: } x[-n]_{\text{mod } N} \leftrightarrow X[-k]_{\text{mod } N}$$

$$\text{Duality: } X[n] \leftrightarrow Nx[-k]_{\text{mod } N}$$

$$\text{Circular convolution: } x_1[n] \otimes x_2[n] \leftrightarrow X_1[k] X_2[k]$$

$$\text{Multiplication: } x_1[n] x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \otimes X_2[k]$$

$$\text{Real sequence: } x[n] = x_e[n] + x_o[n] \leftrightarrow X[k] = A[k] + jB[k]$$

$$X[-k]_{\text{mod } N} = X^*[k]$$

$$\text{Even component: } x_e[n] \leftrightarrow \text{Re}\{X[k]\} = A[k]$$

$$\text{Odd component: } x_o[n] \leftrightarrow j\,\text{Im}\{X[k]\} = jB[k]$$

C.7.1.3. Parseval's Relation:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X[k]|^2$$

Note

$$x_1[n] \otimes x_2[n] = \sum_{i=0}^{N-1} x_1[i] x_2[n-i]_{\text{mod } N}$$

C.8. Fourier Series

$$x(t + T_0) = x(t)$$

C.8.1.1. Complex Exponential Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t}\, dt$$

C.8.1.2. Trigonometric Fourier Series:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

$$a_k = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cos k\omega_0 t\, dt$$

$$b_k = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \sin k\omega_0 t\, dt$$

C.8.1.3. Harmonic Form Fourier Series:

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t - \theta_k) \qquad \omega_0 = \frac{2\pi}{T_0}$$

C.8.1.4. Relations among Various Fourier Coefficients:

$$\frac{a_0}{2} = c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k})$$

$$c_k = \frac{1}{2}(a_k - jb_k) \qquad c_{-k} = \frac{1}{2}(a_k + jb_k)$$

$$C_0 = \frac{a_0}{2} \qquad C_k = \sqrt{a_k^2 + b_k^2} \qquad \theta_k = \tan^{-1} \frac{b_k}{a_k}$$

C.8.1.5. Parseval's Theorem for Fourier Series:

$$\frac{1}{T_0} \int_{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

C.9. Discrete Fourier Series

$$x[n+N_0]=x[n]$$

$$x[n]=\sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n} \qquad \Omega_0=\frac{2\pi}{N_0}$$

$$c_k=\frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jk\Omega_0 n}$$

C.9.1.1. Parseval's Theorem for Discrete Fourier Series:

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}\left|x[n]\right|^2=\sum_{k=0}^{N_0-1}\left|c_k\right|^2$$