FUNDAMENTALS OF ELECTROMAGNETICS AND RADIATION

Mathematical models suitable for describing the current induced electromagnetic radiation and for explaining the behaviour of any antenna are introduced and commented.

MAXWELL’S EQUATIONS

The equations describing the electromagnetic fields in the time domain can be expressed in the following integral form, which is suitable to be verified experimentally:

Faraday’s law
$$\oint_C \boldsymbol{\mathcal{E}} \cdot d\boldsymbol{l} = -\iint_{S_C} \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \cdot d\boldsymbol{s}$$

Ampère’s law
$$\oint_C \boldsymbol{\mathcal{H}} \cdot d\boldsymbol{l} = \iint_{S_C} \left(\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} + \boldsymbol{\mathcal{J}_T} \right) \cdot d\boldsymbol{s}$$

Gauss’ law of electricity
$$\iint_S \boldsymbol{\mathcal{D}} \cdot d\boldsymbol{s} = \iiint_{V_S} \rho_T dv = Q$$

Gauss’ law of magnetism
$$\iint_S \boldsymbol{\mathcal{B}} \cdot d\boldsymbol{s} = 0$$

Continuity equation
$$\iint_S \boldsymbol{\mathcal{J}_T} \cdot d\boldsymbol{s} = -\frac{\partial}{\partial t} \iiint_{V_S} \rho_T dv$$

There are 5 vectorial fields
Electric field $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}(\mathbf{r}, t)$
Magnetic induction $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}(\mathbf{r}, t)$
Magnetic field $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}(\mathbf{r}, t)$
Electric displacement $\boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{D}}(\mathbf{r}, t)$
Total current density $\boldsymbol{\mathcal{J}_T} = \boldsymbol{\mathcal{J}_T}(\mathbf{r}, t)$

The total charge carrier density $\rho_T = \rho_T(\mathbf{r}, t)$ is a scalar field

$\mathbf{r}$: position vector
t: time

By using the Stokes' theorem and the divergence theorem, the differential form of Maxwell's equations can be obtained:

Integral form of Maxwell's equations

$$\oint_C \boldsymbol{\mathcal{E}} \cdot d\boldsymbol{l} = -\iint_{S_C} \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \cdot d\boldsymbol{s}$$

$$\oint_C \boldsymbol{\mathcal{H}} \cdot d\boldsymbol{l} = \iint_{S_C} \left(\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} + \boldsymbol{\mathcal{J}_T} \right) \cdot d\boldsymbol{s}$$

$$\iint_S \boldsymbol{\mathcal{D}} \cdot d\boldsymbol{s} = \iiint_{V_S} \rho_T dv = Q$$

$$\iint_S \boldsymbol{\mathcal{B}} \cdot d\boldsymbol{s} = 0$$

Differential form of Maxwell's equations

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}$$

$$\nabla \times \boldsymbol{\mathcal{H}} = \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} + \boldsymbol{\mathcal{J}_T}$$

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \rho_T$$

$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0$$

$$\iint_S \boldsymbol{\mathcal{J}_T} \cdot d\boldsymbol{s} = -\frac{\partial}{\partial t} \iiint_{V_S} \rho_T dv$$

$$\nabla \cdot \boldsymbol{\mathcal{J}_T} = -\frac{\partial \rho_T}{\partial t}$$

If the sources $\mathcal{J}_T(\mathbf{r}, t)$ and $\rho_T(\mathbf{r}, t)$ vary sinusoidally with time at angular frequency $\omega = 2\pi f$, the fields will also vary sinusoidally and are called time-harmonic fields. The fundamental electromagnetic equations and their solutions are considerably simplified if phasor fields are introduced as follows:

$$\boxed{\mathcal{E} = \mathrm{Re}\left\{\mathbf{E}\, e^{j\omega t}\right\}} \qquad \mathcal{H} = \mathrm{Re}\left\{\mathbf{H}\, e^{j\omega t}\right\}$$

where phasor quantities $\mathbf{E}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$, $\mathbf{D}(\mathbf{r})$, $\boldsymbol{J_T}(\mathbf{r})$, $\rho_T(\mathbf{r})$ are complex-valued functions of spatial coordinates $\mathbf{r}$ only. Using the phasor definitions of the electromagnetic quantities and eliminating the $e^{i\omega t}$ factors that appear on both sides of the equations yields:

$$\begin{aligned}
\nabla \times \mathbf{E} &= -j\omega\mathbf{B} \\
\nabla \times \mathbf{H} &= j\omega\mathbf{D} + \mathbf{J_T}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho_T \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

$$\nabla \cdot \mathbf{J_T} = -j\omega\rho_T$$

The time derivatives have been replaced by a $j\omega$ factor and time-varying electromagnetic quantities have been replaced by their phasor counterpart. Since in the most basic antenna problems the bandwidth of the signal is quite small (compared to the carrier frequency), we can assume that there is a single frequency (i.e. the carrier frequency f) and we will assume time-harmonic fields.

The total current density $\mathbf{J_T}$ is composed of an impressed, or source, current $\mathbf{J}$ and a conduction current density term $\sigma\mathbf{E}$, which occurs in response to the electric field (σ is the electrical conductivity):

$$\mathbf{J_T} = \sigma\mathbf{E} + \mathbf{J}$$

The impressed current density is a known quantity: it is quite frequently an assumed current density on an antenna and, as far as the field equations are concerned, it is a known function. The current density $\sigma\mathbf{E}$ is a current density flowing on nearby conductors due to the fields created by source $\mathbf{J}$ and can be computed after the field equations are solved for $\mathbf{E}$.

A dielectric material is characterized by permittivity ε and permeability μ. The following constitutive relations are instrumental in describing the interaction of the electromagnetic fields with matter and for isotropic materials they read as:

$$\mathbf{D} = \varepsilon\,\mathbf{E} \qquad\qquad \mathbf{B} = \mu\,\mathbf{H}$$

If the dielectric is lossy (i.e. electromagnetic power is dissipated inside the dielectric) the permittivity is a complex quantity.

If we solve the equations for the fields in air surrounding an antenna, we can assume that the conductivity is zero $\sigma = 0$ and that the permittivity ε is real (we neglect the ohmic losses in nearby dielectrics).

Let ρ be the source charge corresponding to the source current density $\mathbf{J}$, Maxwell's equations can be expressed in terms of the electric field $\mathbf{E}$ and of the magnetic field $\mathbf{H}$:

Frequency domain Maxwell's equations

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} &= j\omega\varepsilon\mathbf{E} + \mathbf{J} \end{aligned}$$

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon} \\ \nabla \cdot \mathbf{H} &= 0 \end{aligned}$$

$$\nabla \cdot \mathbf{J} = -j\omega\rho$$

Sometimes, it is convenient to introduce a fictitious magnetic current density $\mathbf{M}$, which can be useful as equivalent source that replaces complicated electric fields, and the first curl Maxwell's equation becomes

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{M}$$

POYNTING THEOREM

Starting from Maxwell's curl equations and after multiplying the first equation by the conjugate magnetic field and the conjugate of the second equation by the electric field, we obtain

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \qquad \mathbf{H}^* \cdot (\nabla \times \mathbf{E}) = -j\omega\mu\mathbf{H}^* \cdot \mathbf{H} = -j\omega\mu|\mathbf{H}|^2$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{J} \qquad \mathbf{E} \cdot (\nabla \times \mathbf{H}^*) = -j\omega\varepsilon\mathbf{E} \cdot \mathbf{E}^* + \sigma\mathbf{E} \cdot \mathbf{E}^* + \mathbf{E} \cdot \mathbf{J}^* = -j\omega\varepsilon|\mathbf{E}|^2 + \sigma|\mathbf{E}|^2 + \mathbf{E} \cdot \mathbf{J}^*$$

By subtracting the second equation from the first, using the identity $\nabla \cdot (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot \nabla \times \mathbf{U} - \mathbf{U} \cdot \nabla \times \mathbf{V}$, integrating over the volume V bounded by the closed surface S, and using the divergence theorem, we finally obtain

$$\boxed{-\frac{1}{2}\iiint_V \mathbf{E} \cdot \mathbf{J}^* dv = \frac{1}{2}\iint_S \mathbf{E} \times \mathbf{H}^* \cdot d\boldsymbol{s} + \frac{1}{2}\iiint_V \sigma|\mathbf{E}|^2 dv + j\frac{1}{2}\omega \iiint_V (-\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2) dv}$$

The complex power P_{IN} delivered by the source in the volume V equals the sum of the power P_{OUT} flowing out of S, the time-average power P_D dissipated in V, plus the time-average stored power in V

$$\boxed{P_{IN} = P_{OUT} + P_D + j2\omega(W_m - W_e)}$$

- The source complex power can be calculated from the imposed current density

$$P_{IN} = -\frac{1}{2}\iiint_V \mathbf{E} \cdot \mathbf{J}^* dv$$

- The time-average dissipated power due to the conductivity σ is

$$P_D = \frac{1}{2}\iiint_V \sigma|\mathbf{E}|^2 dv$$

- The time-average stored magnetic energy is

$$W_m = \frac{1}{2}\iiint_V \frac{1}{2}\mu|\mathbf{H}|^2 dv$$

- The time-average stored electric energy is

$$W_e = \frac{1}{2}\iiint_V \frac{1}{2}\varepsilon|\mathbf{E}|^2 dv$$

From the power budget, we can claim that the complex power flowing out through closed surface S is

$$P_{OUT} = \frac{1}{2}\iint_S \mathbf{E} \times \mathbf{H}^* \cdot d\boldsymbol{s}$$

and the integrand inside this integral is defined as the Poynting vector, which is a power density with units of W/m^2

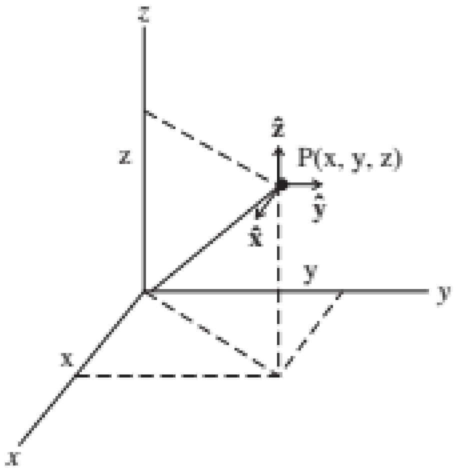
$$\boxed{\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*}$$

The integral of the complex Poynting vector $\mathbf{S}$ over a closed surface S gives the total complex power flowing out through the surface S. It is assumed that $\mathbf{S}$ represents the complex power density in watts per square meter at a point. Then the complex power through any surface S (not necessarily closed) can be found by integrating the complex Poynting vector over that surface. The real power flowing through surface S is

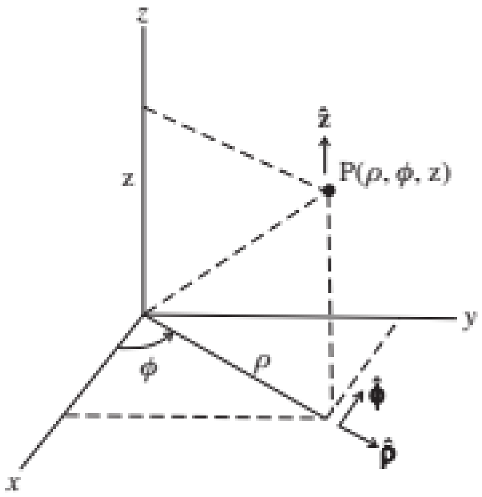
$$\boxed{P = \mathrm{Re}\left\{\iint_S \frac{1}{2}\mathbf{E} \times \mathbf{H}^* \cdot d\boldsymbol{s}\right\} = \mathrm{Re}\left\{\iint_S \mathbf{S} \cdot d\boldsymbol{s}\right\}}$$

REFERENCE SYSTEMS

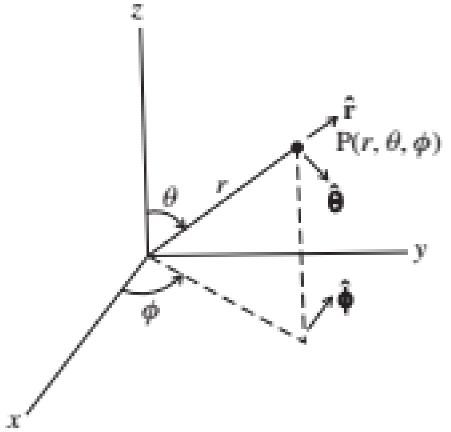
Rectangular coordinates



Cylindrical coordinates

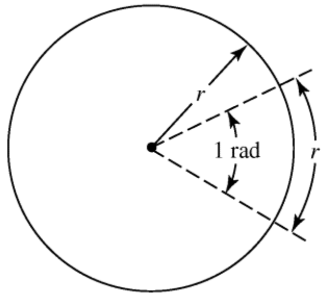


Spherical coordinates

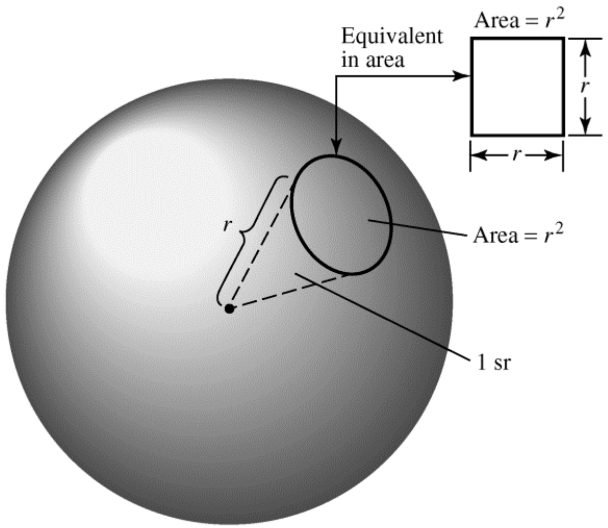


(r, θ, ϕ) spherical coordinates
r radial distance
θ polar angle
ϕ azimuth angle

SPHERICAL COORDINATES AND STERADIAN



Radian: the planar angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle.
A circle subtends 2π rad at its center.

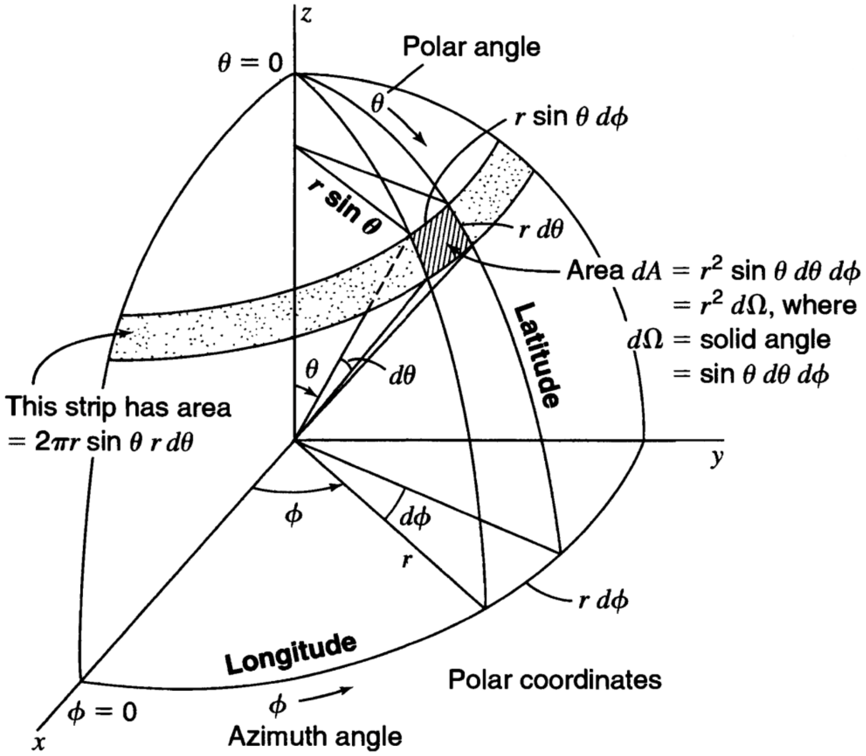


Steradian: the solid angle subtended at the center of a sphere by a surface having an area equal to the radius of the sphere squared.
A sphere subtends 4π sr at its center.

$d\Omega$ is the incremental solid angle subtended by the incremental area dA on the surface of the sphere having radius r

$$dA = (rd\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,d\theta d\phi$$

$$d\Omega = \frac{dA}{r^2} = \sin\theta\,d\theta d\phi$$



PLANE WAVES

The forward (and backward) travelling plane waves $\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$ ($\mathbf{E}_0 e^{+j\mathbf{k}\cdot\mathbf{r}}$) are solutions of Maxwell'equations in a homogeneous medium and in the absence of sources

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \qquad \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \quad \text{position vector}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \qquad \mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}} \quad \text{propagation vector}$$

We use a Cartesian coordinate system and insert the forward travelling wave in Maxwell's curl equations

$$\nabla \times \left(\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\right) = -j\omega\mu\mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \qquad -j\mathbf{k} \times \left(\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\right) = -j\omega\mu\mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \qquad -j\mathbf{k} \times \mathbf{E}_0 = -j\omega\mu\mathbf{H}_0$$

$$\nabla \times \left(\mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\right) = j\omega\varepsilon\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \qquad -j\mathbf{k} \times \left(\mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\right) = j\omega\varepsilon\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \qquad -j\mathbf{k} \times \mathbf{H}_0 = j\omega\varepsilon\mathbf{E}_0$$

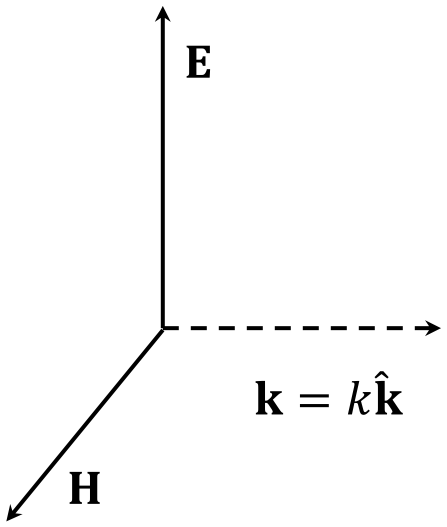
The vectors $\mathbf{E}$ and $\mathbf{H}$ are transverse to the propagation vector $\mathbf{k}$ (and to the propagation direction) , moreover $\mathbf{E}$ and $\mathbf{H}$ are mutually orthogonal: the 3 vectors form a right-hand triplet.

The vectors **E, H** and **k** form a right-hand triplet

$$\mathbf{H}_0 = \frac{k}{\omega\mu}\hat{\mathbf{k}} \times \mathbf{E}_0$$

$$\mathbf{E}_0 = -\frac{k}{\omega\varepsilon}\hat{\mathbf{k}} \times \mathbf{H}_0$$

$$\frac{k}{\omega\mu} = \frac{\omega\varepsilon}{k} \qquad k = \omega\sqrt{\mu\varepsilon}$$



The magnitude of the propagation vector is $|\mathbf{k}| = k = \omega\sqrt{\mu\varepsilon} = 2\pi/\lambda$

and $\eta = \sqrt{\mu/\varepsilon}$ is the wave impedance of the medium

$$\mathbf{H}_0 = \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E}_0 \qquad |\mathbf{H}_0| = \frac{|\mathbf{E}_0|}{\eta}$$

$$\mathbf{E}_0 = -\eta\hat{\mathbf{k}} \times \mathbf{H}_0 \qquad |\mathbf{E}_0| = \eta|\mathbf{H}_0|$$

RADIATION

The aim is to obtain the fields created by an impressed current density $\mathbf{J}$. The current distribution can be obtained during the solution process, but for the moment let's suppose we have the current distribution and wish to determine the fields $\mathbf{E}$ and $\mathbf{H}$. In principle, we have to solve simultaneously the two curl Maxwell's equations; in order to simplify the problem, it is customary to introduce the scalar and vector potential functions Φ and $\mathbf{A}$.

The divergence of $\mathbf{B}$ is zero and, as a consequence, it can be represented by the curl of some other vector function as follows:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \qquad \mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

where $\mathbf{A}$ is the magnetic vector potential.

Let's observe that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for any $\mathbf{A}$, and for a complete definition of $\mathbf{A}$ its divergence must be specified.

In order to express the electric field as a function of the magnetic vector potential, we have to rearrange the first curl Maxwell's equation:

$$\nabla \times \mathbf{E} \quad = \quad -j\omega\mu\mathbf{H} = -j\omega\nabla \times \mathbf{A} \qquad\qquad \nabla \times (\mathbf{E} + j\omega\mathbf{A}) \quad = 0$$

and since the curl of the gradient of scalar function f is identically zero $\nabla \times (\nabla f) = 0$, the electric scalar potential Φ can be defined by means of the following formula:

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$$

Let's observe that we already know form electrostatics (i.e. $\omega = 0$) that the electric field is minus the gradient of the electric potential $\mathbf{E} = -\nabla\Phi$.

The field $\mathbf{E}$ and $\mathbf{H}$ are now expressed in terms of the potential functions

$$\nabla \times \mathbf{H} = \mathrm{j}\omega\varepsilon\mathbf{E} + \mathbf{J} \qquad\qquad \nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{A}\right) = \frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = \frac{1}{\mu}[\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}] = \mathrm{j}\omega\varepsilon\mathbf{E} + \mathbf{J}$$

$$\nabla \times \nabla \times \mathbf{A} = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mathrm{j}\omega\mu\varepsilon\mathbf{E} + \mu\mathbf{J} = \mathrm{j}\omega\mu\varepsilon(-j\omega\mathbf{A} - \nabla\Phi) + \mu\mathbf{J}$$

By rearranging the equation, we obtain

$$\nabla^2 \mathbf{A} + \omega^2 \mu\varepsilon \mathbf{A} - \nabla(j\omega\mu\varepsilon\Phi + \nabla \cdot \mathbf{A}) = -\mu \mathbf{J}$$

The divergence of $\mathbf{A}$ is yet to be specified. A convenient choice would be the one that eliminates the third term of the previous equation: this can be achieved by choosing the Lorentz condition $\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi$

$$\boxed{\nabla^2 \mathbf{A} + \omega^2 \mu\varepsilon \mathbf{A} = -\mu \mathbf{J}}$$

It is a vector wave equation (inhomogeneous Helmholtz equation) that can be solved for $\mathbf{A}$ after the impressed current $\mathbf{J}$ is specified, and no knowledge of Φ is required. The fields are then easily found

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi = -j\omega\mathbf{A} - j\frac{\nabla(\nabla \cdot \mathbf{A})}{\omega\mu\varepsilon}$$

We want to better understand the meaning of the Lorentz condition.

We start from $\nabla \times \nabla \times \mathbf{A} - \omega^2 \mu\varepsilon \mathbf{A} + j\omega\mu\varepsilon\nabla\Phi = \mu\mathbf{J}$ and, by calculating the divergence, we obtain:

$$-\omega^2 \mu\varepsilon \nabla \cdot \mathbf{A} + j\omega\mu\varepsilon\nabla^2\Phi = \mu\nabla \cdot \mathbf{J} \qquad \nabla \cdot \mathbf{A} = -\frac{1}{\omega^2 \varepsilon} \nabla \cdot \mathbf{J} + j\frac{1}{\omega}\nabla^2\Phi$$

We recall the Lorentz condition $\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi$ and from the second Maxwell's equation, and after calculating the divergence, we can write

$$j\omega\varepsilon\nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{J} = 0 \qquad\qquad j\omega\rho + \nabla \cdot \mathbf{J} = 0$$

$$\nabla \cdot \mathbf{A} = -\frac{1}{\omega^2\varepsilon}\nabla \cdot \mathbf{J} + j\frac{1}{\omega}\nabla^2\Phi \qquad\qquad -j\omega\mu\varepsilon\Phi = j\frac{1}{\omega\varepsilon}\rho + j\frac{1}{\omega}\nabla^2\Phi$$

and the result is the equation for the scalar electric potential $\nabla^2\Phi + \omega^2\mu\varepsilon\Phi = -\frac{\rho}{\varepsilon}$

we also observe that for $\omega = 0$ we obtain the Poisson's equation of electrostatics $\nabla^2\Phi = -\frac{\rho}{\varepsilon}$

which could also be obtained from the first curl Maxwell's equation $(\omega = 0)$

$$\nabla \times \mathbf{E} = 0 \qquad\qquad \mathbf{E} = -\nabla\Phi$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \qquad\qquad \nabla \cdot (\nabla\Phi) = \nabla^2\Phi = -\frac{\rho}{\varepsilon}$$

In rectangular coordinates, the Laplacian of the vector $\mathbf{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}$ reads as

$$\nabla^2\mathbf{A} = \nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$$

If $\mathbf{A}$ is decomposed into rectangular components, the vector wave equation (inhomogeneous Helmholtz equation) is equivalent to 3 scalar equations.

$$\mathbf{J} = J_x\hat{\mathbf{x}} + J_y\hat{\mathbf{y}} + J_z\hat{\mathbf{z}} \qquad \nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J} \qquad \begin{aligned} \nabla^2 A_x + \beta^2 A_x &= -\mu J_x \\ \nabla^2 A_y + \beta^2 A_y &= -\mu J_y \\ \nabla^2 A_z + \beta^2 A_z &= -\mu J_z \end{aligned}$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ is the propagation constant for a propagating wave. The speed of light is $c = 1/\sqrt{\mu\varepsilon}$ and thus $\beta = 2\pi f/c = 2\pi/\lambda$ where λ is the wavelength, and it is related to frequency by $\lambda = c/f$.

It must be underlined that the Laplacian of each component can be expressed in a coordinate system different from the rectangular coordinates and appropriate to the geometry of the problem.

The 3 scalar wave equations are identical in form. We find the solution for a point source and this unit impulse response solution can then be used to form a general solution by describing an arbitrary source as a collection of point sources. We start by finding the solution for just one component (for instance, A_z).

The differential equation for the point source is

$$\nabla^2\psi + \beta^2\psi = -\delta(x)\delta(y)\delta(z)$$

where ψ is the response to a point source in the origin and δ is the unit impulse function (Dirac delta function). Although the point source is of infinitesimal extent, its associated current has a direction (for instance, z).

Because the point source is zero everywhere except at the origin, we start by solving the homogeneous Helmholtz equation $\nabla^2\psi + \beta^2\psi = 0$. Due to the physical spherical symmetry, the Laplacian is conveniently written in spherical coordinates and ψ has only radial dependence $\psi(r,\theta,\phi) = \psi(r)$.

$$\nabla^2\psi + \beta^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \beta^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \beta^2\psi = 0$$

This homogeneous Helmholtz equation has two general solutions corresponding to waves propagating radially outward and inward

$$\psi(r,\theta,\phi) = \psi(r) = C_1\frac{e^{-j\beta r}}{r} + C_2\frac{e^{+j\beta r}}{r}$$

It the case of the point source, the only physically meaningful solution is the one for the wave travelling away from the source, and it can be proved that the constant factor is $C_1 = 1/4\pi$

$$\nabla^2\psi + \beta^2\psi = -\delta(x)\delta(y)\delta(z) \qquad\qquad \psi(r) = \frac{e^{-j\beta r}}{4\pi\, r}$$

If the source were positioned at an arbitrary location, in the solution r must be replaced by the distance R between the source location and the observation point P.

By applying the superposition principle, if we consider the source to be a collection of point sources weighted by the distribution J_z , the response A_z is a sum of the point source responses, and this can be expressed by the integral over the source volume v':

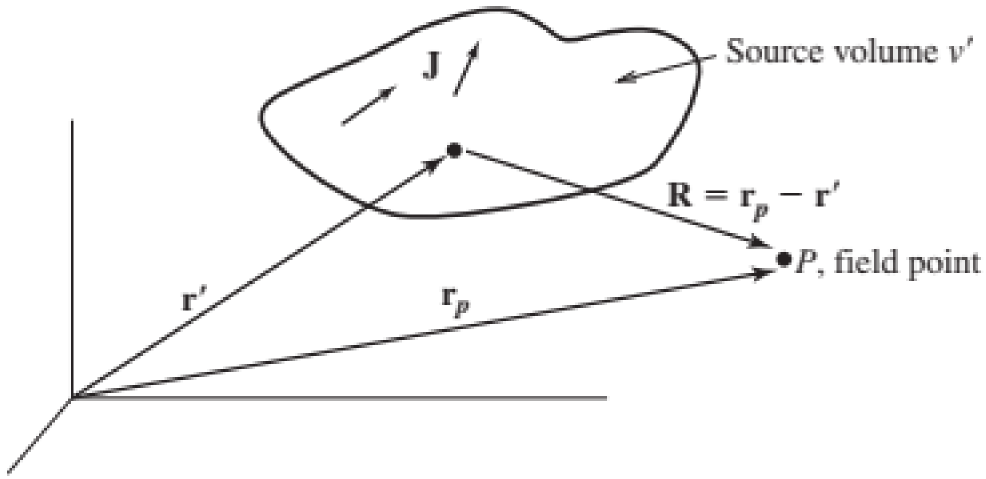
$$A_z = \iiint_{v'} \mu J_z \frac{e^{-j\beta R}}{4\pi R} dv'$$

Similar equations hold for the x and y components, as well, and the total solution is the sum of all components

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}} = \iiint_{v'} \mu\big(J_x\hat{\mathbf{x}} + J_y\hat{\mathbf{y}} + J_z\hat{\mathbf{z}}\big) \frac{e^{-j\beta R}}{4\pi R} dv'$$

$$\nabla^2 \mathbf{A} + \omega^2 \mu\varepsilon \mathbf{A} = -\mu \mathbf{J}$$

$$\mathbf{A} = \iiint_{v'} \mu \mathbf{J} \frac{e^{-j\beta R}}{4\pi R} dv'$$



$\mathbf{r}'$ is the vector from the coordinate origin to the source point
$\mathbf{r}_p$ is the vector from the coordinate origin to the field point P
$\mathbf{R} = \mathbf{r}_p - \mathbf{r}'$ is the vector from the source point to the field point

Once $\mathbf{A}$ is known the electric and magnetic fields can be finally calculated

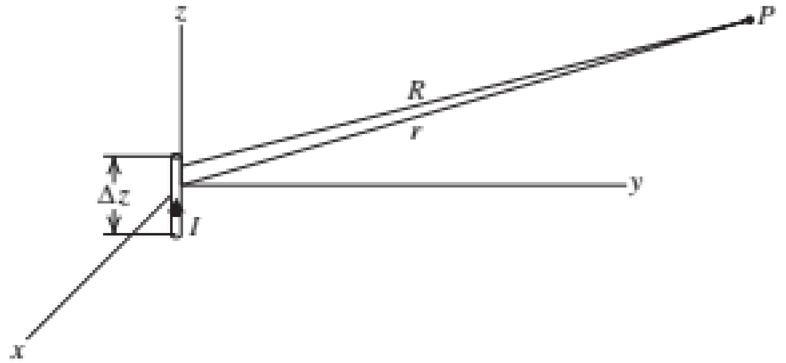
$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \qquad \mathbf{E} = \frac{1}{j\omega\varepsilon} (\nabla \times \mathbf{H} - \mathbf{J})$$

THE IDEAL DIPOLE

We shall use the term ideal dipole for a uniform amplitude current whose spatial extention is electrically small compared to the wavelength ($\Delta z \ll \lambda$) . It is ideal in the sense that the current is uniform in both magnitude and phase over the radiating element extent. In practice, it is very difficult to make an antenna that behaves like an ideal dipole, but any practical antenna can be decomposed into filaments of current that are then subdivided into ideal dipoles, and the fields from the antenna are then found by summing contributions from the ideal dipoles. In the literature there are also other terms to refer to the ideal dipole: Hertzian dipole, electric dipole, infinitesimal dipole.

Consider a current element of length Δz along the z axis centered on the coordinate origin (in a Cartesian coordinate system) and having a constant amplitude I ($\mathbf{I} = |\mathbf{I}|\hat{\mathbf{z}} = I\hat{\mathbf{z}}$); the volume integral to evaluate the magnetic vector potential reduces to a one-dimensional integral:

$$\mathbf{A} = \iiint_{v'} \mu \mathbf{J} \frac{e^{-j\beta R}}{4\pi R} dv' \qquad \mathbf{A} = \hat{\mathbf{z}} \mu I \int_{-\frac{\Delta z}{2}}^{+\frac{\Delta z}{2}} \frac{e^{-j\beta R}}{4\pi R} dz'$$



Since Δz is very small, we assume that the distance R from points on the current element to the field point P approximately equals the distance r from the origin to the field point (or observation point): $R \cong r$

$$\mathbf{A} = \frac{\mu I e^{-j\beta r}}{4\pi\, r} \Delta z\, \hat{\mathbf{z}} = A_z \hat{\mathbf{z}}$$

The magnetic field is found by using the definition of magnetic vector potential

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} = \frac{1}{\mu} \nabla \times (A_z \hat{\mathbf{z}}) = \frac{1}{\mu} (\nabla A_z) \times \hat{\mathbf{z}} + \frac{1}{\mu} A_z (\nabla \times \hat{\mathbf{z}}) = \frac{1}{\mu} (\nabla A_z) \times \hat{\mathbf{z}}$$

$$\mathbf{H} = \nabla \left(\frac{I \Delta z\, e^{-j\beta r}}{4\pi\, r} \right) \times \hat{\mathbf{z}} = \frac{I \Delta z}{4\pi} \left[\hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right] \left(\frac{e^{-j\beta r}}{r} \right) \times \hat{\mathbf{z}} = \frac{I \Delta z}{4\pi} \left[-j\beta \frac{e^{-j\beta r}}{r} - \frac{e^{-j\beta r}}{r^2} \right] \hat{\mathbf{r}} \times \hat{\mathbf{z}}$$

and by observing that $\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\sin\theta\, \hat{\boldsymbol{\phi}}$ the magnetic field reads as

$$\mathbf{H} = \frac{I \Delta z}{4\pi} j\beta \left[1 + \frac{1}{j\beta r} \right] \frac{e^{-j\beta r}}{r} \sin\theta\, \hat{\boldsymbol{\phi}}$$

Also the electric field can be obtained by means of differential calculus

$$\mathbf{E} = \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H} = \frac{1}{j\omega\varepsilon}\left\{\hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(H_\phi\sin\theta\right) - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial r}\left(rH_\phi\right)\right\}$$

$$\mathbf{E} = \frac{I\Delta z}{4\pi}j\omega\mu\left[1+\frac{1}{j\beta r}-\frac{1}{(\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\boldsymbol{\theta}} + \frac{I\Delta z}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\left[\frac{1}{r}-j\frac{1}{\beta r^2}\right]\frac{e^{-j\beta r}}{r}\cos\theta\,\hat{\mathbf{r}}$$

RADIATION FIELDS OF THE IDEAL DIPOLE

If the distance between P and the origin is very large with respect to the wavelength, we can conclude that $\beta r \gg 1$ and then all terms in $\mathbf{E}$ and $\mathbf{H}$ having inverse powers of $j\beta r$ are small compared to one, and the fields read as

$$\mathbf{E} = \frac{I\Delta z}{4\pi} j\omega\mu \frac{e^{-j\beta r}}{r} \sin\theta \ \hat{\boldsymbol{\theta}}$$

$$\mathbf{H} = \frac{I\Delta z}{4\pi} j\beta \frac{e^{-j\beta r}}{r} \sin\theta \ \hat{\boldsymbol{\phi}}$$

$$\boxed{\begin{aligned} \mathbf{E} &= E_\theta \hat{\boldsymbol{\theta}} = j\eta \frac{\beta}{4\pi} I\Delta z \frac{e^{-j\beta r}}{r} \sin\theta \ \hat{\boldsymbol{\theta}} = E_\theta \hat{\boldsymbol{\theta}} \\ \mathbf{H} &= H_\phi \hat{\boldsymbol{\phi}} = j \frac{\beta}{4\pi} I\Delta z \frac{e^{-j\beta r}}{r} \sin\theta \ \hat{\boldsymbol{\phi}} = H_\phi \hat{\boldsymbol{\phi}} \end{aligned}}$$

The ratio of the electric field to the magnetic field is given by the intrinsic impedance η of the medium (as in the case of plane waves)

$$\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

The Poynting vector can be easily evaluated starting from its definition

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\frac{I\Delta z}{4\pi} j\eta\beta \frac{e^{-j\beta r}}{r} \sin\theta\ \hat{\boldsymbol{\theta}} \times \frac{I\Delta z}{4\pi}(-j\beta)\frac{e^{+j\beta r}}{r}\sin\theta\ \hat{\boldsymbol{\phi}} = \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \eta\beta^2 \frac{1}{r^2}\sin^2\theta\,\hat{\mathbf{r}}$$

which is real-valued and radially directed. The total real power emitted by the ideal dipole can be obtained by calculating the total power flowing out through a sphere of arbitrary radius r surrounding the ideal dipole

$$P_{OUT} = \iint_{\Sigma} \mathbf{S} \cdot d\mathbf{s} = \iint_{\Omega} |\mathbf{S}|\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, r^2 \sin\theta\, d\theta\, d\phi = \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \eta\beta^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta\, d\theta = \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \eta\beta^2 2\pi \frac{4}{3}$$

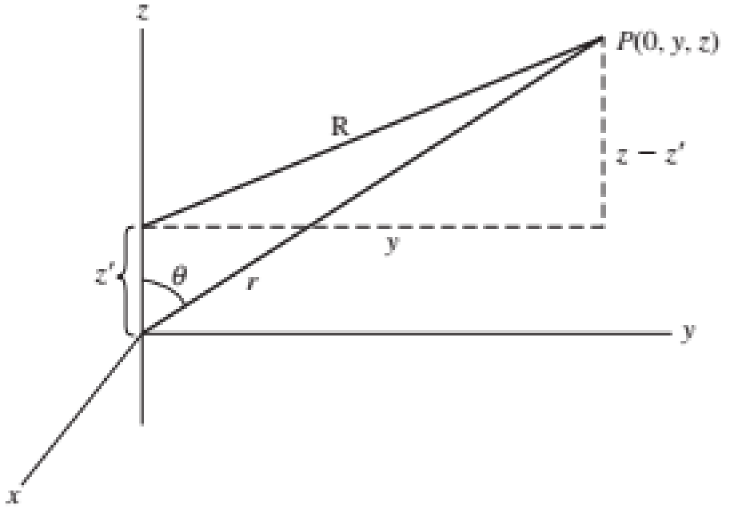
$$\boxed{P_{OUT} = \frac{\pi}{3}\eta\frac{I^2\Delta z^2}{\lambda^2}}$$

RADIATION FROM LINE AND VOLUME CURRENTS

Consider a filament of current along the z axis and located near the origin: straight wire antennas can be modeled by this line source. The magnetic vector potential has only a z component and can be calculated from

$$A_z = \mu \int_l I(z') \frac{e^{-j\beta R}}{4\pi R} dz'$$

Due to the axial symmetry of the source, the radiated field does not vary with ϕ, therefore for simplicity we will confine the observation point to a fixed ϕ in the yz plane (i.e. $\phi = \pi/2$)



$$r^2 = y^2 + z^2$$
$$z = r\cos\theta$$
$$y = r\sin\theta$$

$$\mathbf{r}_p = \mathbf{r} = y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad \mathbf{r}' = z'\hat{\mathbf{z}}$$

$$\mathbf{R} = \mathbf{r}_p - \mathbf{r}' = y\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$

The distance R between the observation point P and a point on the line source can be rearranged in the following approximated form

$$R = |\mathbf{R}| = \sqrt{y^2 + (z - z')^2} = \sqrt{y^2 + z^2 - 2zz' + z'^2} = \sqrt{r^2 + \left[-2r\cos\theta\, z' + z'^2\right]} = r\sqrt{1 + \frac{-2r\cos\theta\, z' + z'^2}{r^2}}$$

Remembering the following Taylor series $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$ we can write

$$R = r + \frac{1}{2}\frac{-2r\cos\theta\, z' + z'^2}{r} - \frac{1}{8}\frac{\left(-2r\cos\theta\, z' + z'^2\right)^2}{r^3} + \cdots$$

and if z' is very small compared to r (i.e. $z'/r \ll 1$) the zero order and the first order approximations for R read as

$$R \cong r \qquad\qquad R \cong r - z'\cos\theta$$

In the integral for A_z we use the zero order approximation for the amplitude factor, but the first order approximation for the phase factor

$$A_z = \mu \int_l I(z') \frac{e^{-j\beta R}}{4\pi R} dz' = \mu \int_l I(z') \frac{e^{-j\beta\left(r - z' \cos\theta\right)}}{4\pi\, r} dz' = \mu \frac{e^{-j\beta r}}{4\pi\, r} \int_l I(z') e^{j\beta z' \cos\theta} dz'$$

It is convenient to calculate the magnetic field in a spherical coordinate system

$$H = \frac{1}{\mu} \nabla \times (A_z \hat{z}) = \frac{1}{\mu} \nabla \times \left(-A_z \sin\theta\, \widehat{\boldsymbol{\theta}} + A_z \cos\theta\, \hat{\mathbf{r}}\right) = \widehat{\boldsymbol{\phi}} \frac{1}{\mu} \frac{1}{r} \left[\frac{\partial}{\partial r}(-r A_z \sin\theta) - \frac{\partial}{\partial\theta}(A_z \cos\theta)\right]$$

$$H = \widehat{\boldsymbol{\phi}} \frac{1}{\mu} \left\{-\mu \sin\theta \frac{1}{4\pi\, r} \frac{\partial}{\partial r}\left(e^{-j\beta r}\right) \int_l I(z') e^{j\beta z' \cos\theta} dz' - \mu \frac{e^{-j\beta r}}{4\pi\, r^2} \frac{\partial}{\partial\theta}\left(\cos\theta \int_l I(z') e^{j\beta z' \cos\theta} dz'\right)\right\}$$

$$H = \widehat{\boldsymbol{\phi}} \frac{e^{-j\beta r}}{4\pi\, r} \left\{j\beta \sin\theta \int_l I(z') e^{j\beta z' \cos\theta} dz' - \frac{1}{r} \frac{\partial}{\partial\theta}\left(\cos\theta \int_l I(z') e^{j\beta z' \cos\theta} dz'\right)\right\}$$

We observe that the ratio of the first term in braces to the second term is of the order of βr. If we assume that $\beta r \gg 1$ the second term can be neglected, and the only component of the magnetic field reads as

$$\mathbf{H} = \widehat{\boldsymbol{\phi}} \frac{j\beta}{\mu} \sin\theta \, \mu \frac{e^{-j\beta r}}{4\pi \, r} \int_l I(z') e^{j\beta z' \cos\theta} dz' = \frac{j\beta}{\mu} \sin\theta \, A_z \widehat{\boldsymbol{\phi}}$$

The electric field can be found from the magnetic potential

$$\mathbf{E} = -j\omega\mathbf{A} - j\frac{\nabla(\nabla \cdot \mathbf{A})}{\omega\mu\varepsilon} = -j\omega(-A_z \sin\theta \, \widehat{\boldsymbol{\theta}} + A_z \cos\theta \, \hat{\mathbf{r}}) - j\frac{\nabla\left(\nabla \cdot \left(-A_z \sin\theta \, \widehat{\boldsymbol{\theta}} + A_z \cos\theta \, \hat{\mathbf{r}}\right)\right)}{\omega\mu\varepsilon}$$

By assuming that $\beta r \gg 1$, retaining the term proportional to $1/r$ and neglecting the smaller terms $1/r^2, 1/r^3, \cdots$ we could prove that the electric field depends on the component of the magnetic vector potential which is transverse to the propagation direction $\hat{\mathbf{r}}$.

$$\mathbf{E} = -j\omega A_\theta \widehat{\boldsymbol{\theta}} = j\omega \sin\theta \, A_z \widehat{\boldsymbol{\theta}}$$

To sum up, the solution for the far-field (i.e. $\beta r \gg 1$) is given by

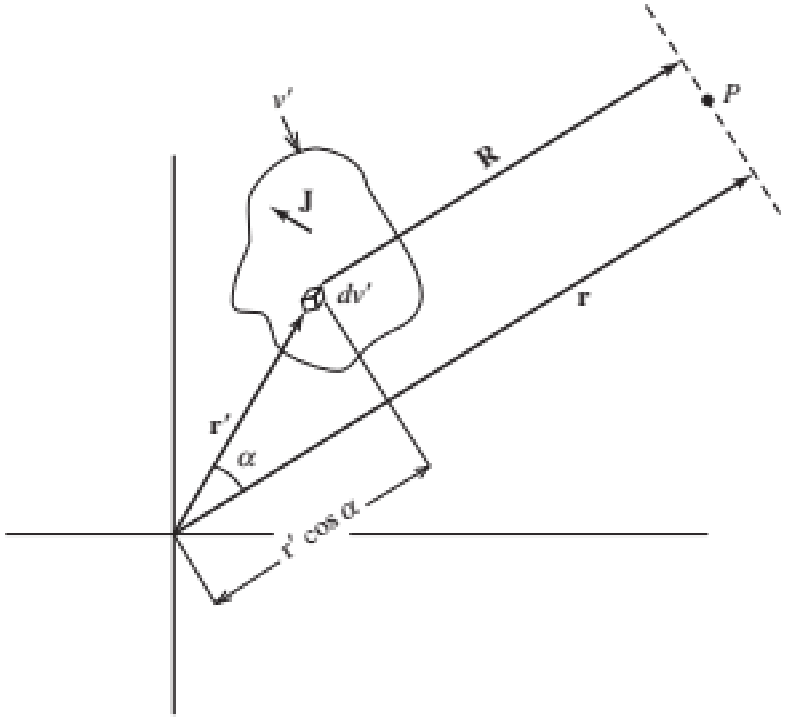
$$A_z = \mu \int_l I(z') \frac{e^{-j\beta R}}{4\pi R} dz' = \mu \int_l I(z') \frac{e^{-j\beta\left(r - z' \cos\theta\right)}}{4\pi\, r} dz' = \mu \frac{e^{-j\beta r}}{4\pi\, r} \int_l I(z') e^{j\beta z' \cos\theta} dz'$$

$$\mathbf{E} = -j\omega A_\theta \hat{\boldsymbol{\theta}} = j\omega \sin\theta\, A_z \hat{\boldsymbol{\theta}}$$

$$\mathbf{H} = -\frac{j\beta}{\mu} A_\theta \hat{\boldsymbol{\phi}} = \frac{j\beta}{\mu} \sin\theta\, A_z \hat{\boldsymbol{\phi}}$$

We observe that the fields are perpendicular to each other and to the propagation direction and that the ratio of their amplitudes is given by the medium intrinsic impedance $E_\theta / H_\phi = \eta = \sqrt{\mu/\epsilon}$.

The previous results can be generalized to an arbitrary volume current density **J**



$$R = r - r'\cos\alpha = r - r'\frac{\mathbf{r}\cdot\mathbf{r}'}{rr'} = r - \hat{\mathbf{r}}\cdot\mathbf{r}'$$

$$\mathbf{A} = \iiint_{v'} \mu\mathbf{J}\frac{e^{-j\beta R}}{4\pi R}dv' \cong \int_{v'} \mu\mathbf{J}\frac{e^{-j\beta\left(r-\hat{\mathbf{r}}\cdot\mathbf{r}'\right)}}{4\pi\, r}dv' = \mu\frac{e^{-j\beta r}}{4\pi\, r}\int_{v'}\mathbf{J}e^{j\beta\hat{\mathbf{r}}\cdot\mathbf{r}'}dv'$$

EVALUATION OF RADIATION FIELDS (AT LARGE DISTANCE FROM THE SOURCE)

The derivation of the far-fields radiated by a volume current density $\mathbf{J}$ can be reduced to a three-step procedure

1. Find the magnetic vector potential $\mathbf{A}$

$$\boxed{\mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi\, r} \int_{v'} \mathbf{J} e^{j\beta \hat{\mathbf{r}}\cdot\mathbf{r}'} dv'}$$

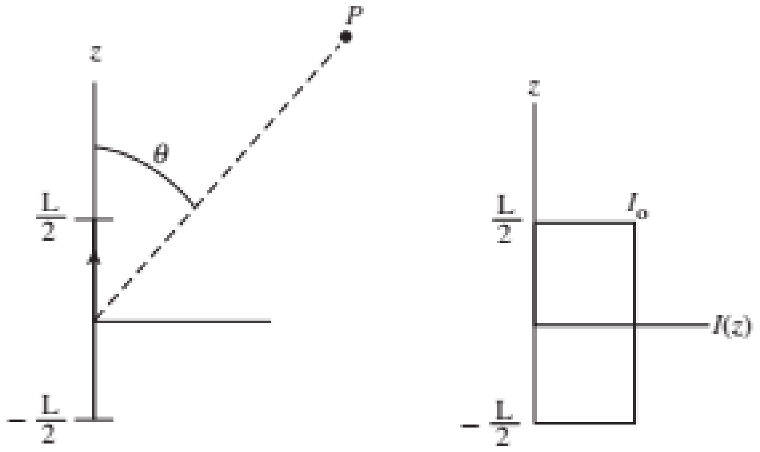
2. Find the electric field $\mathbf{E}$ from the components of $\mathbf{A}$ transverse to the direction of propagation $\hat{\mathbf{r}}$

$$\boxed{\mathbf{E} = -j\omega\left(A_\theta \widehat{\boldsymbol{\theta}} + A_\phi \widehat{\boldsymbol{\phi}}\right)}$$

3. Find the magnetic field $\mathbf{H}$ by taking advantage of the fact that the radiated field is locally a spherical wave (and $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to each other and to the direction of propagation)

$$\boxed{\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{r}} \times \mathbf{E}}$$

Example: uniform line source having a current I_0 and a length L



$$I(z') = \begin{cases} I_0 & x' = 0, y' = 0, |z'| \leq \dfrac{L}{2} \\ 0 & elsewhere \end{cases}$$

in such a case the volume integral reduces to a line integral

$$\mathbf{A} = \mu \frac{e^{-j\beta r}}{4\pi\, r} \int_{v'} \mathbf{J} e^{j\beta \hat{\mathbf{r}} \cdot \mathbf{r}'} dv' = \mu \frac{e^{-j\beta r}}{4\pi\, r} \int_{-\frac{L}{2}}^{+\frac{L}{2}} I_0 e^{j\beta z' \cos\theta} dz' \,\hat{\mathbf{z}} = \mu \frac{e^{-j\beta r}}{4\pi\, r} I_0 \frac{e^{j\beta \frac{L}{2}\cos\theta} - e^{-j\beta \frac{L}{2}\cos\theta}}{j\beta \cos\theta} \hat{\mathbf{z}}$$

The fields are given by

$$\mathbf{A} = A_z\hat{\mathbf{z}} = \mu\frac{e^{-j\beta r}}{4\pi\, r} I_0 L \frac{\sin\left(\frac{\beta L}{2}\cos\theta\right)}{\frac{\beta L}{2}\cos\theta}\hat{\mathbf{z}}$$

$$\mathbf{E} = j\omega\sin\theta\, A_z\widehat{\boldsymbol{\theta}} = j\omega\mu\frac{e^{-j\beta r}}{4\pi\, r} I_0 L\sin\theta\frac{\sin\left(\frac{\beta L}{2}\cos\theta\right)}{\frac{\beta L}{2}\cos\theta}\widehat{\boldsymbol{\theta}}$$

$$\mathbf{H} = H_\phi\widehat{\boldsymbol{\phi}} = \frac{E_\theta}{\eta}\widehat{\boldsymbol{\phi}}$$