DERIVATION OF POYNTING THEOREM

MAXWELL'S EQUATIONS

$$\nabla \times E = -j\omega\mu H$$
$$\nabla \times H = j\omega\varepsilon E + \sigma E + J$$ WHERE J IS THE IMPOSED CURRENT

SCALAR PRODUCT BETWEEN H^* AND THE FIRST EQUATION

$$H^* \cdot \nabla \times E = H^*(-j\omega\mu H) = -j\omega\mu |H|^2$$

SCALAR PRODUCT BETWEEN E AND THE COMPLEX CONJUGATE OF THE SECOND EQUATION

$$E \cdot (\nabla \times H)^* = E \cdot (j\omega\varepsilon E + \sigma E + J)^* = E \cdot (-j\omega\varepsilon E^* + \sigma E^* + J^*)$$
$$E \cdot \nabla \times H^* = -j\omega\varepsilon |E|^2 + \sigma |E|^2 + E \cdot J^*$$

THE TWO EQUATIONS READ AS

$$H^* \cdot \nabla \times E = -j\omega\mu |H|^2$$
$$E \cdot \nabla \times H^* = -j\omega\varepsilon |E|^2 + \sigma |E|^2 + E \cdot J^*$$

WE SUBTRACT THE SECOND EQUATION FROM THE FIRST

$$H^* \cdot \nabla \times E - E \cdot \nabla \times H^* = -j\omega\mu |H|^2 + j\omega\varepsilon |E|^2 - \sigma |E|^2 - E \cdot J^*$$

SINCE $\nabla \cdot (U \times V) = V \cdot \nabla \times U - U \cdot \nabla \times V$

WE CAN REARRANGE THE LEFT-HAND SIDE TERM BY OBSERVING THAT:

$$\nabla \cdot (E \times H^*) = H^* \cdot \nabla \times E - E \cdot \nabla \times H^*$$

THE EXPRESSION BECOMES

$$\nabla\cdot\left(E\times H^*\right) = -j\omega\mu|H|^2 + j\omega\varepsilon|E|^2 - \sigma|E|^2 - E\cdot J^*$$

WE MOVE THE "SOURCE" TERM $-E\cdot J^*$ TO THE LEFT-HAND SIDE AND THE $\nabla\cdot\left(E\times H^*\right)$ TERM TO THE RIGHT HAND SIDE

$$E\cdot J^* = -\nabla\cdot\left(E\times H^*\right) - j\omega\mu|H|^2 + j\omega\varepsilon|E|^2 - \sigma|E|^2$$

WE CHANGE THE SIGN EVERYWHERE AND WE MULTIPLY BY 1/2 TO OBTAIN TIME-AVERAGE VALUES (SINCE WE ARE WORKING WITH PHASORS)

$$-\frac{1}{2}E\cdot J^* = \frac{1}{2}\nabla\cdot\left(E\times H^*\right) + \frac{1}{2}\sigma|E|^2 + \frac{1}{2}j\omega\varepsilon|E|^2 + \frac{1}{2}j\omega\mu|H|^2$$

WE INTEGRATE THE PREVIOUS EXPRESSION OVER A VOLUME V BOUNDED BY A CLOSED SURFACE S

$$-\frac{1}{2}\int_V E\cdot J^* dV = \frac{1}{2}\int_V \nabla\cdot\left(E\times H^*\right)dV + \frac{1}{2}\int_V \sigma|E|^2 dV + \frac{1}{2}j\omega\int_V\left(\mu|H|^2 - \varepsilon|E|^2\right)dV$$

THANKS TO THE GAUSS THEOREM WE CAN WRITE THAT

$$\frac{1}{2}\int_V \nabla\cdot\left(E\times H^*\right)dV = \frac{1}{2}\int_S E\times H^*\cdot d\vec{S} \qquad d\vec{S} = \hat{n}\,dS$$

UNIT VECTOR NORMAL TO THE SURFACE AND DIRECTED OUT FROM THE VOLUME

INCREMENT OF THE AREA

AND FINALLY WE CAN WRITE

$$-\frac{1}{2}\int_V E\cdot J^* dV = \frac{1}{2}\int E\times H^*\cdot d\vec{S} + \frac{1}{2}\int_V \sigma|E|^2 dV + j2\omega\int_V\left(\frac{1}{4}\mu|H|^2 - \frac{1}{4}\varepsilon|E|^2\right)dV$$