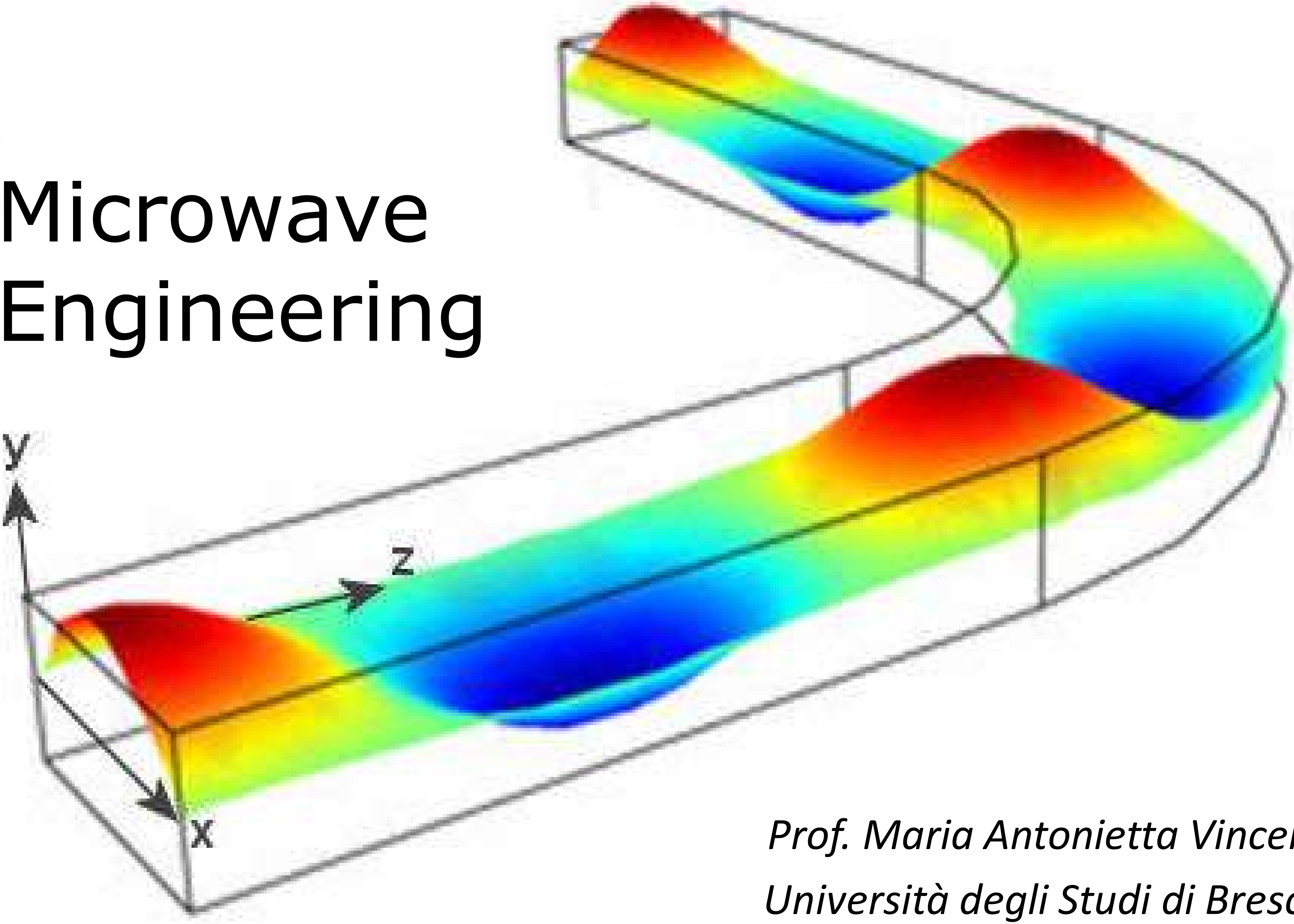
Microwave Engineering



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Wave Equations and plane wave solutions

Wave equations

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In general, the wave equations are vector equations that incorporate Maxwell's equations into a single expression for a single field and they are obtained using the constitutive relations.

Let's start from simplicity from the Maxwell's equations in the frequency domain and in the monochromatic approximation:

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

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Plane wave equations
(Homogeneous Helmholtz equations)

In a source-free, homogeneous, isotropic medium the material coefficients are independent of the spatial coordinates and Maxwell's equation reduce to:

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D}$$

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Taking the curl of the first equation…

$$\boxed{\nabla^2 \mathbf{E} + \omega^2 \mu\varepsilon \mathbf{E} = 0}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D}$$

$$\boxed{\nabla^2 \mathbf{H} + \omega^2 \mu\varepsilon \mathbf{H} = 0}$$

Wavenumber
(propagation constant)

$$k = \omega\sqrt{\mu\varepsilon}$$

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Plane wave in lossless medium

In a lossless medium *k* is a real number since ε and μ are real. Solution of the homogeneous wave equation is a plane wave that can be found by assuming a wave with only one electric field component, i.e. E_x. Then we can write:

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$

Whose solution is of the form:

$$E_x(z) = E_x^+ e^{-jkz} + E_x^- e^{jkz}$$

That can be transformed in the time domain as:

$$E_x(z,t) = E_x^+ \cos(\omega t - kz) + E_x^- \cos(\omega t + kz)$$

Propagating forward   Propagating backward

Plane wave in lossless medium

The magnetic field can be then calculated from the Maxwell's equation:

$$E_x(z) = E_x^+ e^{-jkz} + E_x^- e^{jkz}$$

$$\nabla \times \mathbf{E} = -i\omega\mu \mathbf{H}$$

Therefore:

$$H_x = H_z = 0$$

$$H_y(z) = \frac{1}{\eta}\left[E_x^+ e^{-jkz} - E_x^- e^{jkz}\right]$$

NOTE: E and H are orthogonal to each other and to the propagation direction.
This is a TEM wave.

Where:

$$\boxed{\eta = \omega\mu / k = \sqrt{\mu / \varepsilon}}$$

Wave impedance (medium impedance)

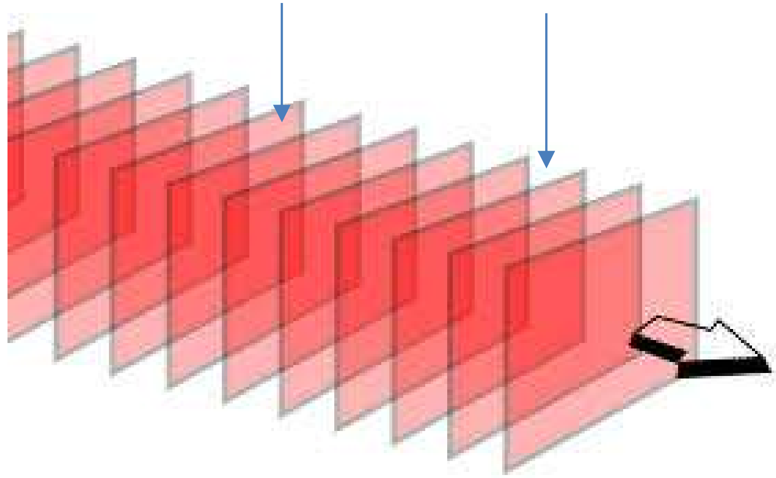
In free space: $\eta_0 = \sqrt{\mu_0 / \varepsilon_0} = 377\Omega$

Wave velocity

The **phase velocity** is the velocity at which the phase of any frequency component of the wave travels. In other words, it is the ratio between the space travelled by a plane of the wave over the time it takes to travel that space. If we consider the temporal solution of the wave equation assuming the wave propagates only in the z direction:

$$E_x\left(z,t\right)=E_x^+\cos\left(\omega t-kz\right)+E_x^-\cos\left(\omega t+kz\right)$$

$$\omega t_1 - kz_1 = \omega t_2 - kz_2$$



The surface of constant phase are planes:

$$\omega t - kz = \text{constant}$$

Let's assume *k* is in the *z* direction, then we can write the velocity as

$$\boxed{v_p = \Delta z/\Delta t = \omega / k = 1/\sqrt{\mu\varepsilon} = c/\sqrt{\varepsilon_r}}$$

In free space: $v_p = 1/\sqrt{\mu_0\varepsilon_0} = c = 2.998\times 10^8 \text{ m/s}$

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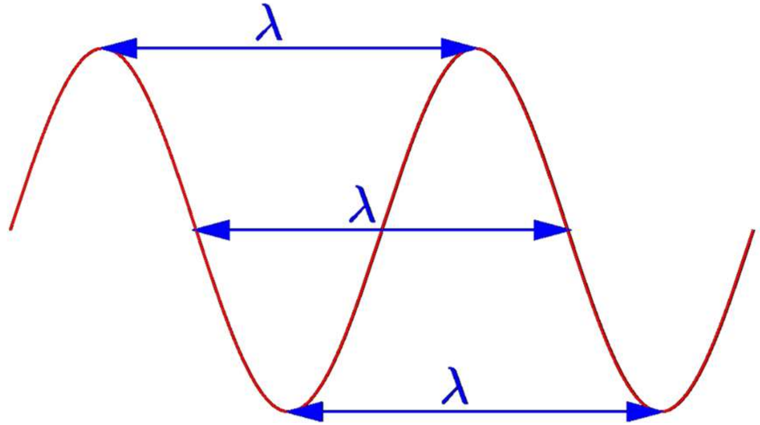
Wavelength

The **wavelength** λ is the distance between to maxima (or minima) on the wave at a certain time. Assuming the same solution of the wave equation in the time domain:



$$E_x(z,t) = E_x^+ \cos(\omega t - kz) + E_x^- \cos(\omega t + kz)$$

We need to solve:

$$[\omega t - kz] - [\omega t - k(z+\lambda)] = 2\pi$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}$$

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Plane wave in lossy medium

Maxwell's equation in a generic lossy medium can be written as:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E}$$

The wave equation therefore is:

$$\boxed{\nabla^2\mathbf{E} + \underbrace{\omega^2\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}_{\text{wavenumber}}\mathbf{E} = 0}$$

$$-\gamma^2 = \omega^2\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$$

$$\Downarrow$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$ Complex propagation constant

Plane wave in lossy medium

If we again assume a wave with only one electric field component, i.e. E_x. Then we can write:

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0$$

Whose solution is of the form:

$$E_x(z) = E_x^+ e^{-\gamma z} + E_x^- e^{\gamma z} \quad \Rightarrow \quad e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z}$$

That can be transformed in the time domain as:

$$E_x(z,t) = E_x^+ e^{-\alpha z}\cos(\omega t - \beta z) + E_x^- e^{\alpha z}\cos(\omega t + \beta z)$$

Propagating forward — Propagating backward

Phase velocity

$$v_p = \omega / \beta$$

Wavelength

$$\lambda = \frac{2\pi}{\beta}$$

Attenuation constant

$$\alpha$$

Plane wave in lossy medium

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The magnetic field can be then calculated from the Maxwell's equation:

$$E_x(z) = E_x^+ e^{-\gamma z} + E_x^- e^{\gamma z}$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

Therefore:

$$H_x = H_z = 0$$

$$H_y(z) = \frac{-j\gamma}{\omega\mu}\left[E_x^+ e^{-\gamma z} - E_x^- e^{\gamma z}\right]$$

Where:

$$\boxed{\eta = {j\omega\mu}/{\gamma}}$$

Wave impedance
(medium impedance)

Alternatively, one can assume σ=0 and consider a complex permittivity to take losses into account. Under this scenario we would have:

$$\gamma = j\omega\sqrt{\mu\varepsilon} = jk = j\omega\sqrt{\mu\varepsilon'(1 - j\tan\delta)}$$

Loss tangent

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'}$$

Plane wave in good conductor

Most metals are very good, but not perfect, conductors. For these materials it is safe to assume that:

$$\sigma \gg \omega\varepsilon \quad \text{or} \quad \varepsilon'' \gg \varepsilon'$$

From this approximation it follows that the propagation constant can be simplified to:

$$\gamma = \alpha + j\beta \simeq j\omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{j\omega\varepsilon}} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

For good conductors we can also define the skin depth (depth of penetration for which the fields amplitude decays to 1/e):

$$\delta_s = 1/\alpha = \sqrt{\frac{2}{\omega\mu\sigma}}$$

The wave impedance is:

$$\eta = j\omega\mu/\gamma \simeq (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta_s}$$

General Plane wave solution

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A general solution for the plane wave equations can be found using the method of the separation of variables.

$$\boxed{\nabla^2 \mathbf{E}(\mathbf{r},\omega) + k^2 \mathbf{E}(\mathbf{r},\omega) = 0} \Rightarrow \boxed{\left(\nabla^2 + k^2\right)\mathbf{E}(\mathbf{r},\omega) = 0}$$

$$\nabla^2 \mathbf{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z \quad \text{VECTOR}$$

If $\mathbf{E} = (E_x, E_y, E_z)$

The vector equation is equivalent to three scalar equations that can be written as:

$$\begin{cases} \boxed{\left(\nabla^2 + k^2\right)E_x = 0} \\ \boxed{\left(\nabla^2 + k^2\right)E_y = 0} \\ \boxed{\left(\nabla^2 + k^2\right)E_z = 0} \end{cases}$$

For the separation of variables we can write:

$$E_{i,i=x,y,z} = X(x)Y(y)Z(z)$$

General Plane wave solution

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Applying the second derivative and dividing by XYZ we get:

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}+k^2=0$$

Since the three function in the equations are independent they should be individually equal to a constant, therefore we can write:

$$\frac{X''}{X}+k_x{}^2=0$$

$$\frac{Y''}{Y}+k_y{}^2=0$$

$$\frac{Z''}{Z}+k_z{}^2=0$$

$$\longrightarrow \quad \boxed{k_x{}^2+k_y{}^2+k_z{}^2=k^2}$$

DISPERSION RELATION

The solution of these differential equations is: $X=X_0e^{-jk_xx}+cc$

$$Y=Y_0e^{-jk_yy}+cc$$

$$Z=Z_0e^{-jk_zz}+cc$$

General Plane wave solution

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The complete solution for the field can be written as:

$$E_x = Ae^{-j\left(k_x x + k_y y + k_z z\right)} + cc = Ae^{-j\mathbf{kr}} + cc$$

Where the modulus of the wavevector **k** is the wavenumber: $\left|\mathbf{k}\right| = k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = k_0 n$

where *n* is the (complex) refractive index of the medium (more generally **k**=k_0**n**)

And the vector **r** is: $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$

Similar solutions can be written for E_y and E_z

$$E_y = Be^{-j\left(k_x x + k_y y + k_z z\right)} + cc = Be^{-j\mathbf{kr}} + cc$$

$$E_z = Ce^{-j\left(k_x x + k_y y + k_z z\right)} + cc = Ce^{-j\mathbf{kr}} + cc$$

$$\Downarrow$$

$$\mathbf{E} = \mathbf{E_0}e^{-j\mathbf{kr}} + cc$$

Plane Wave Cont'd

Let's consider the exponential part of a plane wave, $e^{\mathrm{j}(\omega t - \mathbf{k}\cdot\mathbf{r})}$ which expresses the space-time variation of the wave.
It is obvious that derivatives in time and derivatives in space can be performed with the simple operator transformations:

$$\partial / \partial t \to j\omega$$

$$\nabla \to -j\mathbf{k}$$

This means that the vectors E, H, k are a mutually orthogonal triad

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\longrightarrow$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}$$

$$\mathbf{k} \times \mathbf{H} = -\omega\varepsilon\mathbf{E}$$

$$\mathbf{k} \cdot \mathbf{E} = 0$$

$$\mathbf{k} \cdot \mathbf{H} = 0$$