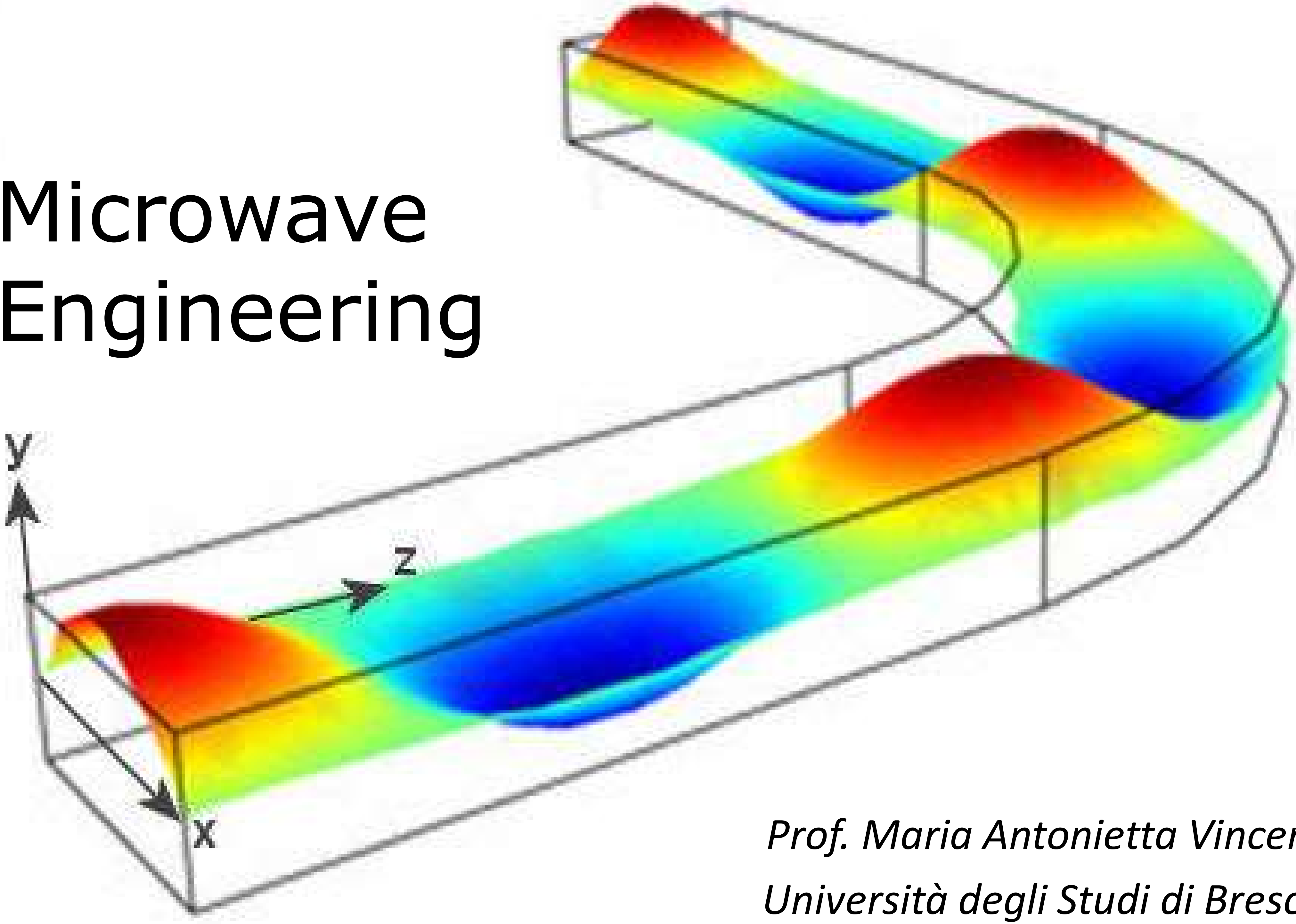
Microwave Engineering



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Maxwell's Equations
and
Boundary Conditions

- A ***scalar*** is a quantity having only an amplitude (and possibly phase).

Examples: voltage, current, charge, energy, temperature

- A ***vector*** is a quantity having direction in addition to amplitude (and possibly phase).

Examples: velocity, acceleration, force

Introduction to Electromagnetic Fields

- Electric and magnetic fields:
  - Are vector fields with three spatial components.
  - Vary as a function of position in 3D space as well as time.
  - Are governed by partial differential equations derived from Maxwell’s equations.

Fundamental vector field quantities in electromagnetics:

- Electric field **E** [V/m]

- Electric flux density (electric displacement) **D** [C/m^2] (ε [F/m])

- Magnetic field **H** [A/m]

- Magnetic flux density (magnetic induction) **B** [T] (μ[H/m])

Universal constants in electromagnetics:

- Velocity of light in free space (vacuum): $c \approx 3\times 10^{8}$ [m/s]
- Permeability of free space: $\mu_0 = 4\pi\times 10^{-7}$ [H/m]
- Permittivity of free space: $\varepsilon_0 \approx 8.854\times 10^{-12}$ [F/m]
- Intrinsic impedance of free space: $\eta_0 \approx 120\pi\,[\Omega]$

Relationships involving the universal constants:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Fundamentals of Vector Calculus

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Divergence $$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Curl $$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Laplace operator $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{SCALAR}$$

$$\nabla^2 \mathbf{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z \quad \text{VECTOR}$$

Useful identities

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A} \qquad \nabla \bullet (\nabla \times \mathbf{A}) = 0$$

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Maxwell's Equations

- ***Maxwell's equations*** are the <u>fundamental postulates of classical electromagnetics.</u> All the **classical** electromagnetic phenomena are explained by these equations.
- Phenomena: Propagation in free-space, interactions with matter, behavior of waves at interfaces.
- Valid to describe wave effects for signals in the whole spectrum, from radio frequencies and below up to optical frequencies and beyond.

Maxwell's Equations

In a generic medium the Maxwell's equations in the time domain are…

Description	Differential form	Integral form
Gauss' Law	$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_f$	$\oint_S \boldsymbol{D}\cdot d\boldsymbol{S}=\int_V \rho_f dV=Q$
Gauss' law for magnetism	$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$	$\oint_S \boldsymbol{B}\cdot d\boldsymbol{S}=0$
Faraday's Law	$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{\partial\boldsymbol{B}}{\partial t}-\boldsymbol{M}$	$\oint \boldsymbol{E}\cdot d\boldsymbol{l}=-\frac{d}{dt}\int_S \boldsymbol{B}\cdot d\boldsymbol{S}-\int_S \boldsymbol{M}\cdot d\boldsymbol{S}$
Ampere's law	$\boldsymbol{\nabla}\times\boldsymbol{H}=\frac{\partial\boldsymbol{D}}{\partial t}+\boldsymbol{J}_f$	$\oint \boldsymbol{H}\cdot d\boldsymbol{l}=\frac{d}{dt}\int_S \boldsymbol{D}\cdot d\boldsymbol{S}+\int_S \boldsymbol{J}_f\cdot d\boldsymbol{S}$

Maxwell’s Equations

Gauss’ Law	$\nabla \cdot \boldsymbol{D} = \rho_f$	$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_V \rho_f dV = Q$

Gauss’ law expresses the connection between the displacement field and the free change density.

Integral form: the displacement field is visualized by space-filling lines that are tangent to the displacement vector field at each point; the field lines start on positive charges (the sources) and end on negative charges (the sinks). The free charges are sources and sinks for the displacement field lines.

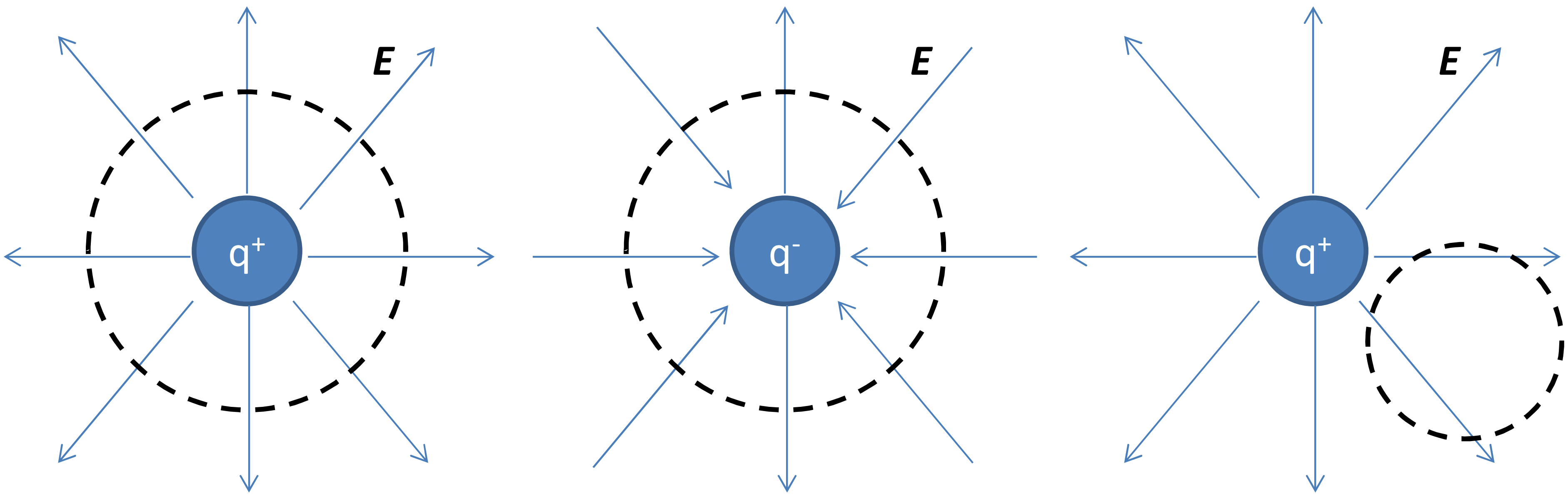
Differential form: when the divergence operator is applied on the displacement vector field, the charge density is recovered. It is physically visualized as an operation related to fluid-like flow through space. In the case of Gauss’ law, the flow through space is the flux of D-field lines spreading through space.

Gauss’ law expresses the fact that D-field lines begin or end on free charges.

NOTE: Absence of charges does not mean absence of electric field. The fields pass through the space with the same number of lines passing into and out of the surface.

Maxwell's Equations

Gauss' Law	$\nabla \cdot \boldsymbol{D} = \rho_f$	$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_V \rho_f dV = Q$



$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{q^+}{\varepsilon_0}$$

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = 0$$

Maxwell's Equations

Gauss' Law for magnetism	$\nabla \cdot \boldsymbol{B} = 0$	$\oint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0$

Gauss' law for magnetism expresses the ***lack of magnetic monopoles***. In other words there are no magnetic sources or sinks. This expression can be physically visualized as B-field lines closing in on themselves.

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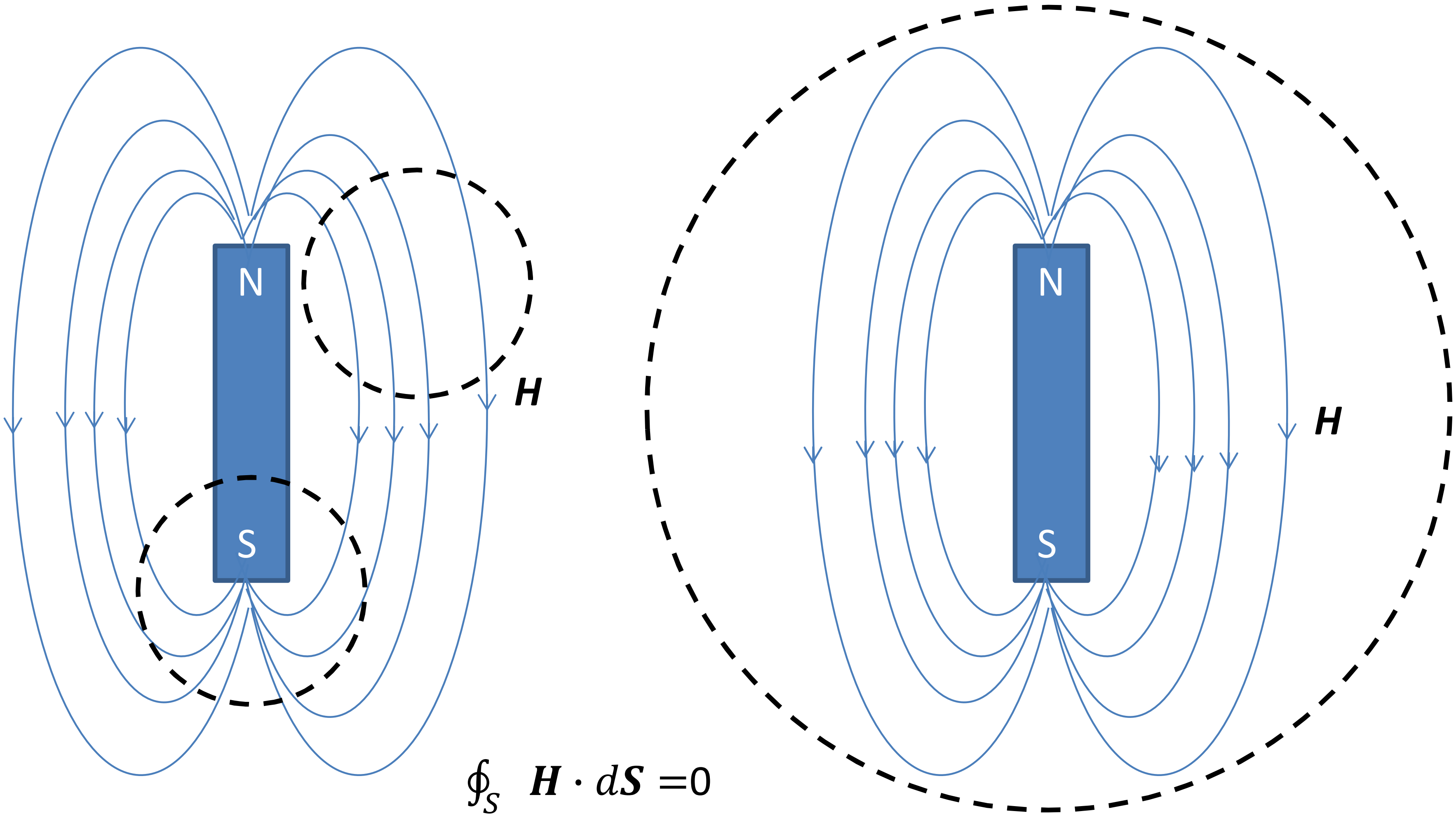
Maxwell's Equations

Gauss' Law for magnetism	$\nabla \cdot \boldsymbol{B} = 0$	$\oint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0$



$$\oint_S \boldsymbol{H} \cdot d\boldsymbol{S} = 0$$

Maxwell's Equations

Faraday's Law	$\nabla \times \mathbf{E} = -\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{M}$	$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{S} - \int_{\boldsymbol{S}} \boldsymbol{M} \cdot d\boldsymbol{S}$

Faraday's law of induction in integral form shows the close relationship between a time-varying magnetic flux (magnetic flux is defined by $\Phi_{\mathrm{B}} = \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$) through a closed loop and an induced potential around the loop called the electromotive force (EMF: $\oint \boldsymbol{E} \cdot d\boldsymbol{l}$). The curl operator $\nabla \times$… is connected with rotational flow of a "fluid" in space.

NOTE: The EMF is not a conserved quantity in general as is the case for the electrostatic potential; a charged particle under the influence of this electric field may gain or lose energy by going around the closed path.

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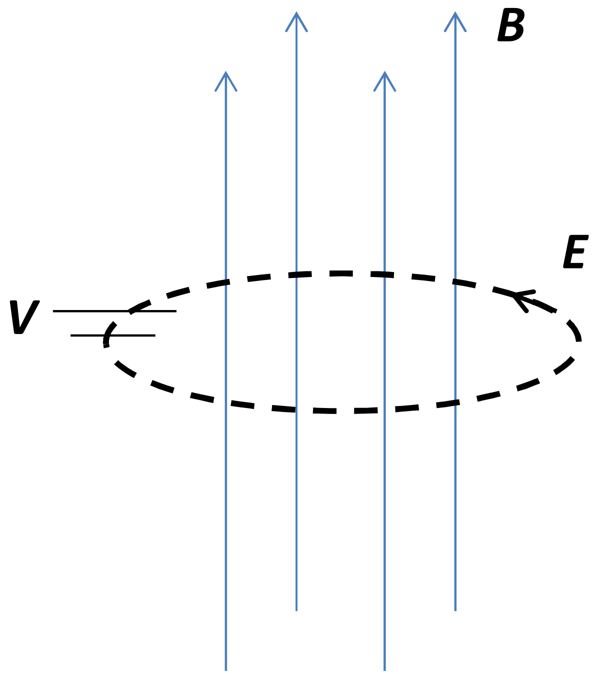
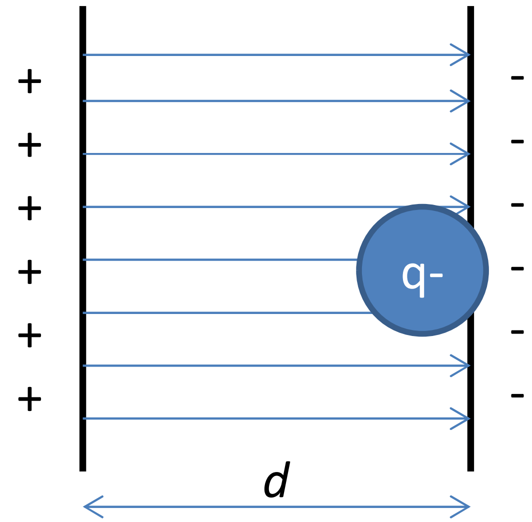
Maxwell's Equations

Faraday's Law	$\nabla \times \mathbf{E} = -\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{M}$	$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{S} - \int_S \boldsymbol{M} \cdot d\boldsymbol{S}$



EMF: $\oint \boldsymbol{E} \cdot d\boldsymbol{l}$ equivalent to $\mathrm{V} = \mathrm{E}d$



To move the charge from – plate to + plate we need to do a certain work $W = Fd = Eqd$ → dividing by q we get the potential between the plates.

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Maxwell’s Equations

Ampere’s Law	$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J_f}$	$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \frac{d}{dt}\int_S \boldsymbol{D} \cdot d\boldsymbol{S} + \int_{\boldsymbol{S}} \boldsymbol{J_f} \cdot d\boldsymbol{S}$

Ampere’s law expresses a deep connection between a quantity called the magneto-motive force (MMF: $\oint \boldsymbol{H} \cdot d\boldsymbol{l}$) and the current flowing through a surface. In the equation, the current has two contributions: one from the free charge current ($\int_{\boldsymbol{S}} \boldsymbol{J_f} \cdot d\boldsymbol{S}$) flowing through the materials and one from the displacement current $\int_S \boldsymbol{D} \cdot d\boldsymbol{S}$ that may exist in space outside of electrical conductors. The displacement current that Maxwell added to prior existing electromagnetic laws ensures that charges are conserved.

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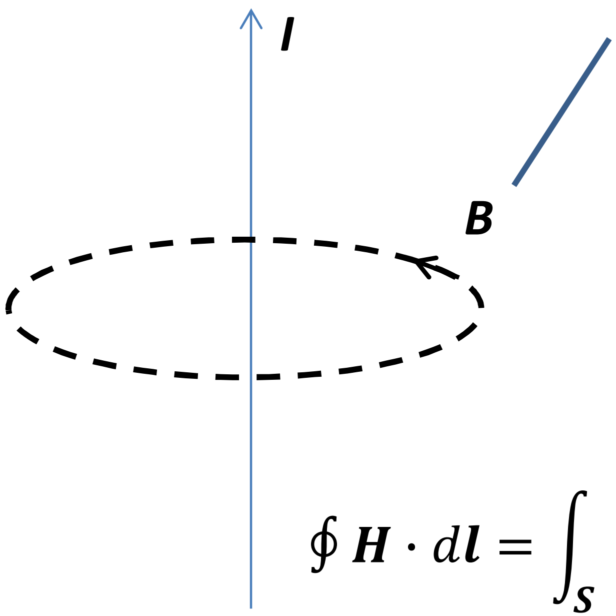
Maxwell's Equations

Ampere's Law	$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_f$	$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \frac{d}{dt}\int_S \boldsymbol{D} \cdot d\boldsymbol{S} + \int_S \boldsymbol{J}_f \cdot d\boldsymbol{S}$



Strength of the magnetic field is constant around the loop if we consider points equidistant from the wire

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \int_S \boldsymbol{J}_f \cdot d\boldsymbol{S}$$

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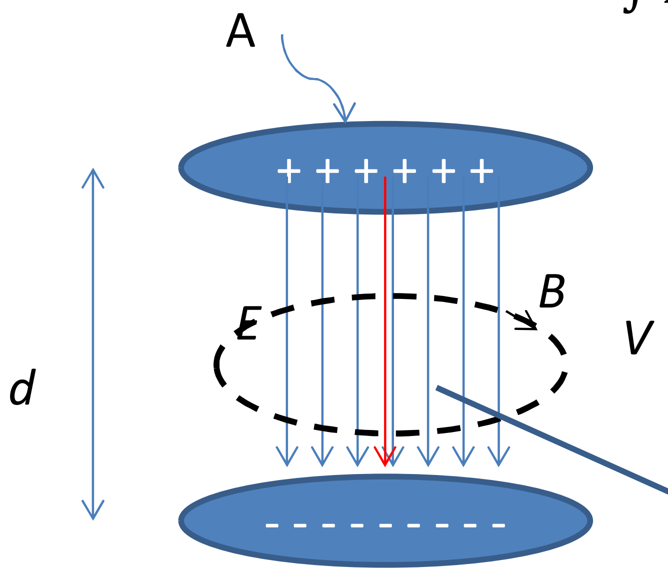
Maxwell’s Equations

Ampere’s Law	$\nabla \times \boldsymbol{H} = \dfrac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J_f}$	$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \dfrac{d}{dt}\displaystyle\int_S \boldsymbol{D} \cdot d\boldsymbol{S} + \int_S \boldsymbol{J_f} \cdot d\boldsymbol{S}$

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \frac{d}{dt}\int_S \boldsymbol{D} \cdot d\boldsymbol{S}$$



$$C = \frac{\varepsilon_0 A}{d}$$

$$C = \frac{Q}{V}$$

If we change the voltage across the plates there is an electric flux greater than zero. A electric flux causes a magnetic induction around the loop that in turn generates a virtual current (called DISPLACEMENT CURRENT) just like in the other part of the equation.

Continuity Equation

The *continuity equation* is implicit in Maxwell's equations when taking the divergence of the Ampere's law

$$\nabla\cdot\mathbf{J} = -\frac{\partial \rho_f}{\partial t}$$

This means that *__the flux of current from a closed surface represents a decrease of the charge inside the surface__*

Maxwell's Equations in the frequency domain

So far we have written Maxwell's equations in the time domain. For many situations, it is very common to deal with fields having a sinusoidal or harmonic time dependence. In this case is very practical to write the fields in phasor notation, therefore assuming all quantities to be complex vectors with an implicit $e^{j\omega t}$ time dependence.

For example, a sinusoidal electric field in the x direction can be written as:

$$\mathbf{E}(x,y,z,t)=\hat{x}A(x,y,z,t)\cos\left(\omega t+\phi\right)$$

With *A* the (real) amplitude, ω the angular frequency and ϕ the phase reference of the wave at t=0; The relation between the phasor and the time-domain quantity is:

$$\mathbf{E}(x,y,z,t)=\mathrm{Re}\left[\mathbf{E}(x,y,z)e^{j\omega t}\right]$$

Why it may be so important to write Maxwell's equations in the frequency domain?

Reason #1: The constitutive relations can be written in a simple and compact way.

Reason #2: If we know the fields in the frequency domain, we can apply the inverse Fourier transform in order to retrieve their time evolution of the electric field.

Maxwell's Equations in the frequency domain

If the fields are *monochromatic*, then we can use the complex-field amplitude representation

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r},\omega)e^{j\omega t}\right] = \frac{1}{2}\left[\mathbf{E}(\mathbf{r},\omega)e^{j\omega t} + \mathbf{E}^*(\mathbf{r},\omega)e^{-j\omega t}\right]$$

$$\partial/\partial t \to j\omega$$

IMPORTANT: $\mathbf{E}(\mathbf{r},t)$ is a real vector, $\mathbf{E}(\mathbf{r},\omega)$ is a complex vector

Time-harmonic Maxwell's equations

$$\nabla \times \mathbf{E}(\mathbf{r},\omega) = -j\omega\mathbf{B}(\mathbf{r},\omega) - \mathbf{M}(\mathbf{r},\omega)$$
$$\nabla \times \mathbf{H}(\mathbf{r},\omega) = j\omega\mathbf{D}(\mathbf{r},\omega) + \mathbf{J}(\mathbf{r},\omega)$$
$$\nabla \cdot \mathbf{D}(\mathbf{r},\omega) = \rho(\mathbf{r},\omega)$$
$$\nabla \cdot \mathbf{B}(\mathbf{r},\omega) = 0$$

Constitutive Relations

Maxwell's equations become complete with the constitutive relations that are in charge of describing the electromagnetic response of the media, which may have temporal and spatial dispersion.

They relate the "primary" fields **E** and **H** to the "secondary" fields **D** and **B** by including the properties of the medium in which the fields propagate.

In free space:

$$\boxed{\boldsymbol{D} = \epsilon_0 \boldsymbol{E}}$$

$$\boxed{\boldsymbol{B} = \mu_0 \boldsymbol{H}}$$

$$\boxed{\boldsymbol{J} = 0}$$

Fields in media

In a generic (linear) medium we can write the constitutive relations as:

$$\boxed{\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{Pe}}$$

$$\boxed{\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{Pm}}$$

Where:

$$\boldsymbol{P_e} = \varepsilon_0 \chi_e \boldsymbol{E}$$

$$\boldsymbol{P_m} = \mu_0 \chi_m \boldsymbol{H}$$

Where χ_e is the electric susceptibility, which may be complex. We can generally write:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{Pe} = \varepsilon_0 (1 + \chi_e) \boldsymbol{E} = \varepsilon \boldsymbol{E}$$

Where the complex permittivity of the medium is:

$$\varepsilon = \varepsilon_0 (1 + \chi_e) = \varepsilon' - \mathrm{j}\varepsilon''$$

Losses (heat)

Fields in media

Losses in a dielectric can be also quantified as the losses in an equivalent conductor with conductivity σ and conduction current density:

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

$$\Downarrow$$

$$\begin{aligned}\nabla \times \mathbf{H} &= j\omega\mathbf{D} + \mathbf{J} \\ &= j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} \\ &= j\omega\left(\varepsilon' \underbrace{- j\varepsilon'' - j\frac{\sigma}{\omega}}_{\text{Total effective losses}}\right)\mathbf{E}\end{aligned}$$

Loss tangent:

$$\tan\delta = \frac{\omega\varepsilon'' + \sigma}{\omega\varepsilon'}$$

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Fields in media

Analogous considerations can be made for magnetic materials where we can write the constitutive relations as:

$$\boxed{\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{Pm}}$$

Where:

$$\boldsymbol{P_m} = \mu_0 \chi_m \boldsymbol{H}$$

Where χ_m is the magnetic susceptibility, which may be complex. We can generally write:

$$\boldsymbol{B} = \mu_0 \boldsymbol{B} + \boldsymbol{Pm} = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu\boldsymbol{H}$$

Where the complex permeability of the medium is:

$$\mu = \mu_0(1 + \chi_m) = \mu' - j\mu''$$

Losses (damping forces)

Note that in general we will deal with materials with no magnetic response (**M**=0) and magnetic conductivity is absent since there is no real magnetic current.

Materials Properties

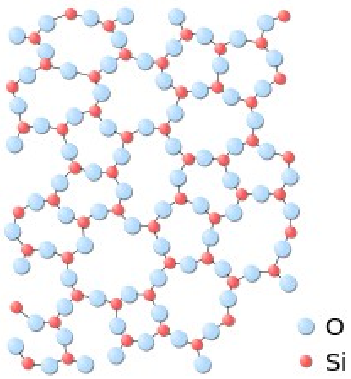
Materials may respond differently depending on the direction of propagation.

Isotropic

$$\mathbf{D} = \varepsilon \mathbf{E}$$

ε is a scalar

Glass

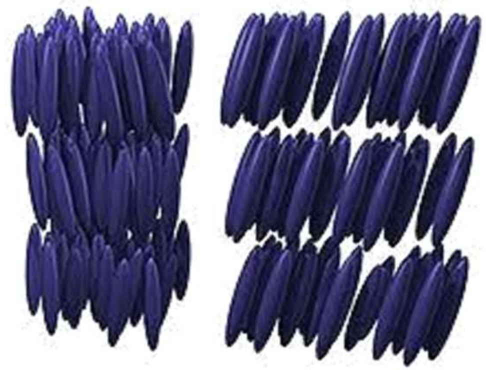


Anisotropic

$$\mathbf{D} = \boldsymbol{\varepsilon} \cdot \mathbf{E}$$

ε is a tensor

Liquid crystal



$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- Gas, liquids and amorphous solids are isotropic because molecules are oriented randomly in space, so macroscopically, the material behave isotropically.
- On the other hand, if the molecules/particles have a preferred orientation, then the material shows anisotropy.

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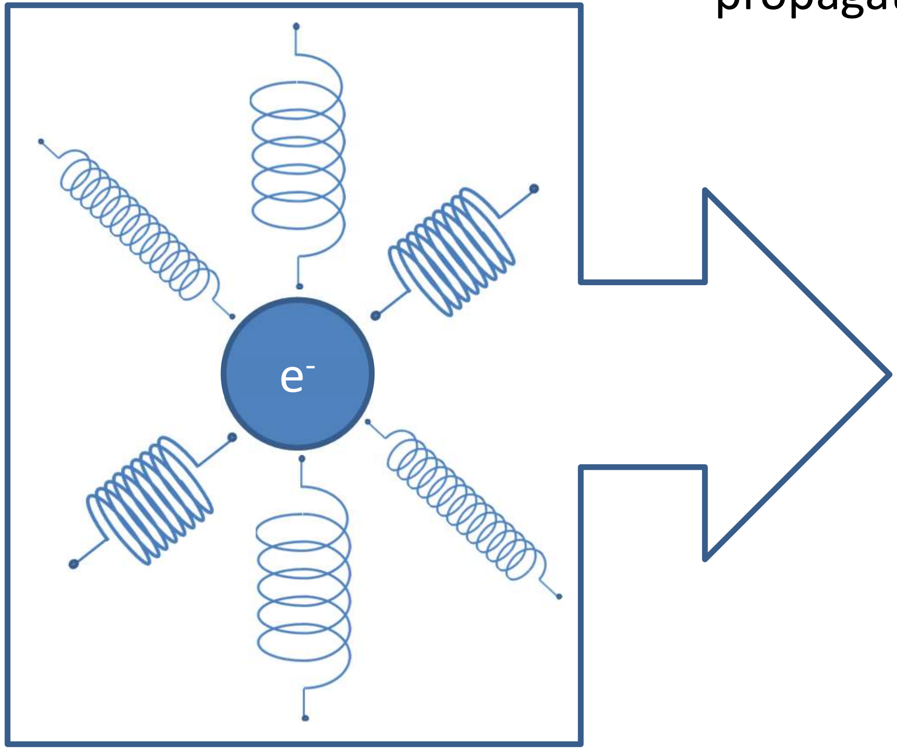
Anisotropic materials

What is the meaning of ***anisotropy***?

The **polarization** in a crystal by a given electric field **varies** depending on the direction of the applied field with respect to the crystal lattice.
This also implies that **phase velocity** can assume a **different value** depending on the direction of propagation and polarization.



Bound electron connected to a fictitious set of springs

The displacement of an electron under the action of an external field E depends on the direction of the field and its magnitude. Therefore we can write:

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$$

or

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

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Anisotropic materials

For symmetry we can write:

$$\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$$

or

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

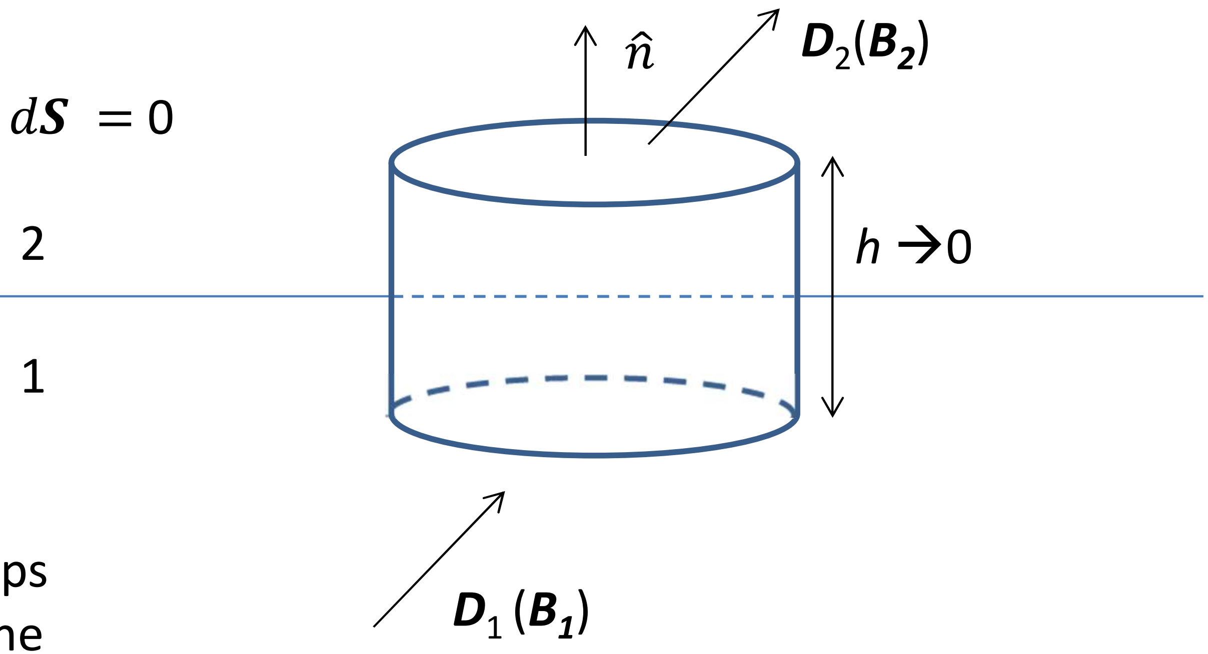
An important example of anisotropic magnetic materials in microwave engineering is the class of ferrimagnetic materials known as ferrites.

Boundary conditions

We can extract the boundary conditions connecting the electromagnetic field between two regions applying the integral form of Maxwell's equations.

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_V \rho_f \, dV = Q \qquad \oint_S \boldsymbol{B} \cdot d\boldsymbol{S} = 0$$

We can consider a Gaussian pillbox that straddles the surface under study. As the cylinder is made smaller and smaller the surface integrals containing the transverse field components vanish and the surface integrals over the end caps containing field components normal to the boundary satisfy the following equations:



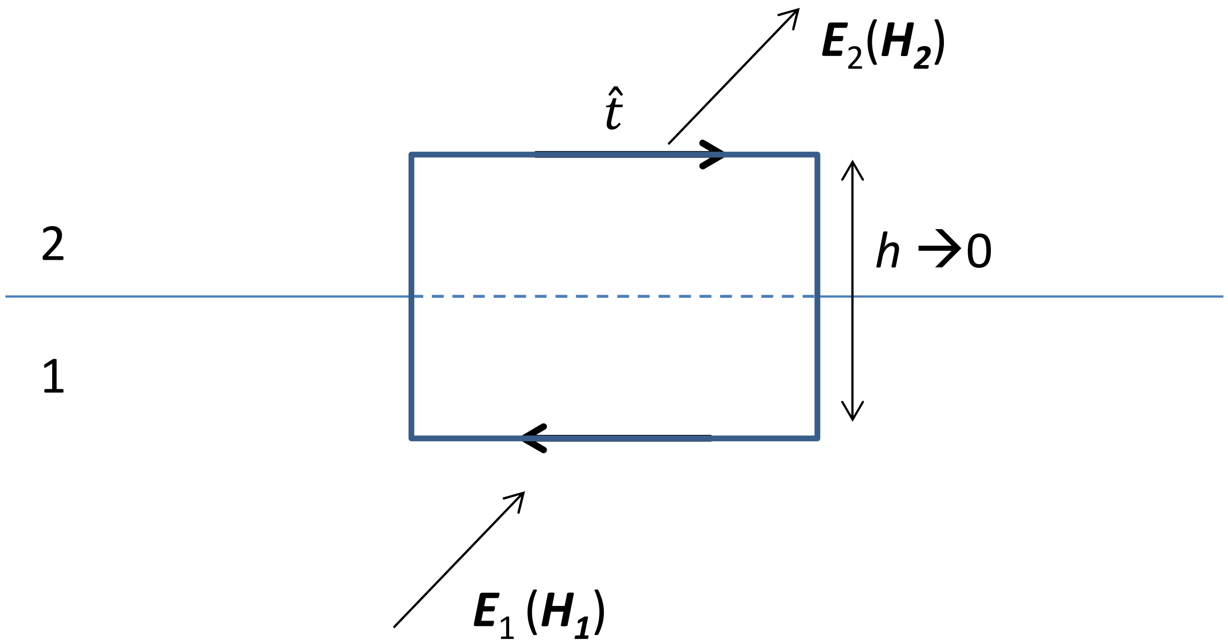
$$\boxed{\hat{n} \cdot (\boldsymbol{D_2} - \boldsymbol{D_1}) = \rho_s} \qquad \boxed{\hat{n} \cdot (\boldsymbol{B_2} - \boldsymbol{B_1}) = 0}$$

In other words the normal components of the electric displacement field and the magnetic induction are continuous across an interface.

Boundary conditions

The second set of boundary conditions connecting the two media is derived from the integral forms of Ampere's law and Faraday's law.

The line integral is traced over a closed loop that straddles the interface. As the size of the loop is reduced to an infinitesimal size, the field contributions tangential to the boundary are the leading terms and the line segments perpendicular to the boundary cancel one another. The resulting boundary conditions can be recast into a vector form as:



$$\boxed{(\boldsymbol{E_2} - \boldsymbol{E_1}) \times \hat{n} = \boldsymbol{M}_s}$$

$$\boxed{\hat{n} \times (\boldsymbol{H_2} - \boldsymbol{H_1}) = \boldsymbol{J_s}}$$

In other words the tangential components of the electric field and the magnetic field are continuous across an interface.

Boundary conditions

Interface between two dielectrics

$$\hat{n} \cdot \boldsymbol{D_1} = \hat{n} \cdot \boldsymbol{D_2}$$

$$\hat{n} \cdot \boldsymbol{B_1} = \hat{n} \cdot \boldsymbol{B_2}$$

$$\hat{n} \times \boldsymbol{E_1} = \hat{n} \times \boldsymbol{E_2}$$

$$\hat{n} \times \boldsymbol{H_1} = \hat{n} \times \boldsymbol{H_2}$$

Interface with PEC (Electric Wall)

$$\hat{n} \cdot \boldsymbol{D} = \rho_s$$

$$\hat{n} \cdot \boldsymbol{B} = 0$$

$$\hat{n} \times \boldsymbol{E} = 0$$

$$\hat{n} \times \boldsymbol{H} = \boldsymbol{J}_s$$

Interface with PMC (Magnetic Wall)

$$\hat{n} \cdot \boldsymbol{D} = 0$$

$$\hat{n} \cdot \boldsymbol{B} = 0$$

$$\hat{n} \times \boldsymbol{E} = -\boldsymbol{M}_s$$

$$\hat{n} \times \boldsymbol{H} = 0$$