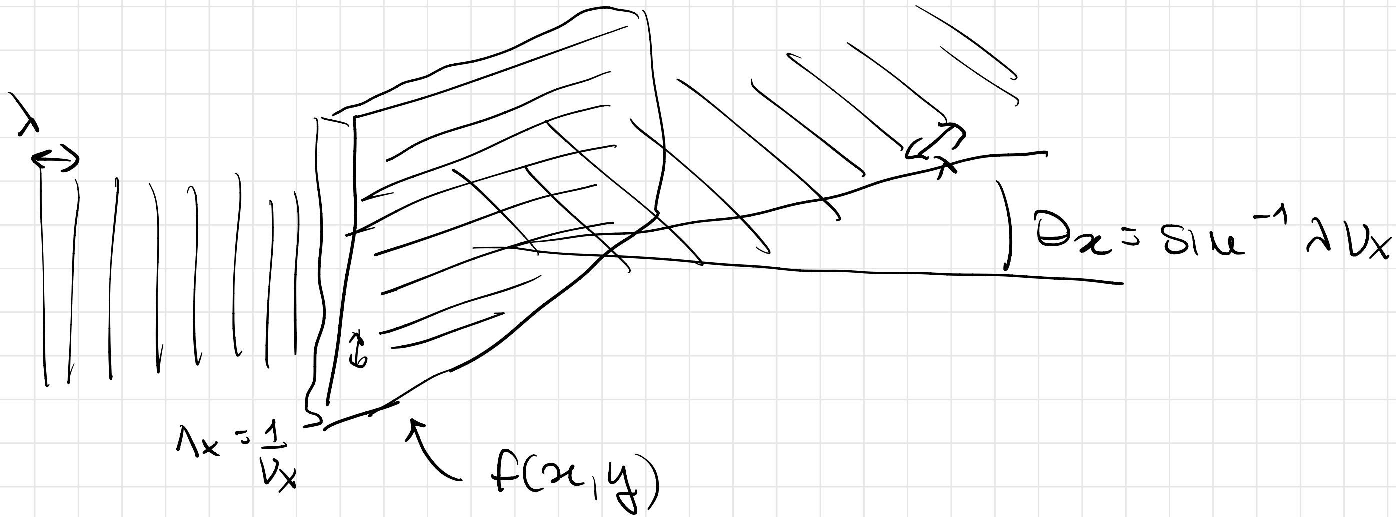
Photonics

Lecture 10:

Fourier Optics

recap & problems



EXAMPLES OF OPTICAL ELEMENTS AND THEIR SPATIAL SPECTRAL PROPERTIES

ⓐ $f(x,y) = \cos(2\pi\nu_x x)$ $\Leftarrow$ this optical element shifts the plane wave in an upward and a downward direction.

$$f(x,y) = \cos(2\pi\nu_x x) = \frac{1}{2}\left[e^{-j2\pi\nu_x x} + e^{j2\pi\nu_x x}\right]$$

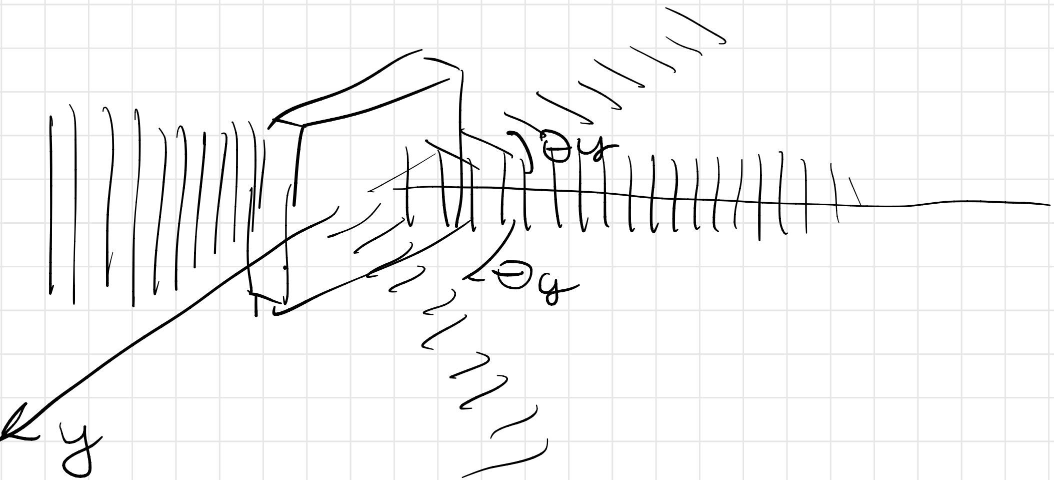
The generic spatial harmonic function

$$f(x,y) = A\, e^{-j2\pi(\nu_x x + \nu_y y)}$$

$$\theta_x = \sin^{-1} \lambda\nu_x \qquad \theta_y = \sin^{-1} \lambda\nu_y$$

From this follows that $f(x,y) = \cos(2\pi\nu_x x)$ deflects the plane wave of angles $\pm \sin^{-1}(\lambda\nu_x)$

(b) Optical element with $f(x,y) = 1 + \cos(2\pi \nu_y y)$



$$\pm \theta_y = \pm \sin^{-1}(\lambda \nu_y)$$

(c) Optical element $f(x,y) = \underline{u[\cos(2\pi \nu_x x)]}$

$\Downarrow$

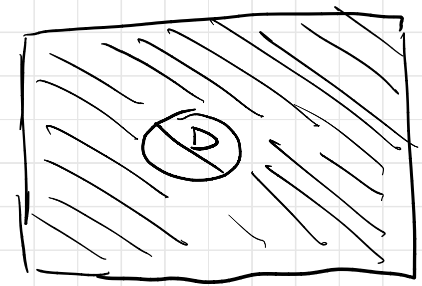
equivalent to a set of slits since it assumes either 1 or 0 values depending on x

$$\underline{u(x)} = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

EXAMPLES OF SPATIAL FILTERS:

(a) <u>Low-pass filter:</u>

$$H(\nu_x, \nu_y) = \begin{cases} 1 & \nu_x^2 + \nu_y^2 \leq \nu_s^2 \\ 0 & \nu_x^2 + \nu_y^2 > \nu_s^2 \end{cases}$$



CIRCULAR APERTURE

$$\frac{D}{2} = \nu_s \lambda f$$

For example: $D = 2\ cm$

$\lambda = 1\ \mu m$

$f = 100\ cm$

We can evaluate the cut-off frequency

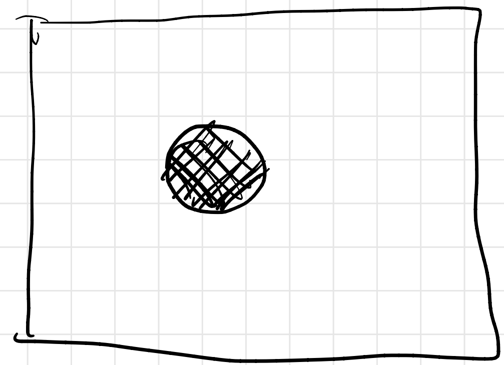
$$\nu_s = \frac{D}{2\lambda f} = \frac{\cancel{2} \cdot \cancel{10^{-2}}}{\cancel{2} \cdot 1 \cdot 10^{-6} \cdot 100 \cdot \cancel{10^{-2}}} = \frac{1}{10^{-4}} = 10 \text{ lines/mm}$$

The smallest discernible feature that could be detected

10 $\quad \dfrac{1}{\nu_0} = \underline{0.1\ \text{mm}}$

(b) <u>High-pass filter</u>

$$H(\nu_x, \nu_y) = \begin{cases} 0 & \nu_x^2 + \nu_y^2 \leq \nu_s^2 \\ 1 & \nu_x^2 + \nu_y^2 > \nu_s^2 \end{cases}$$



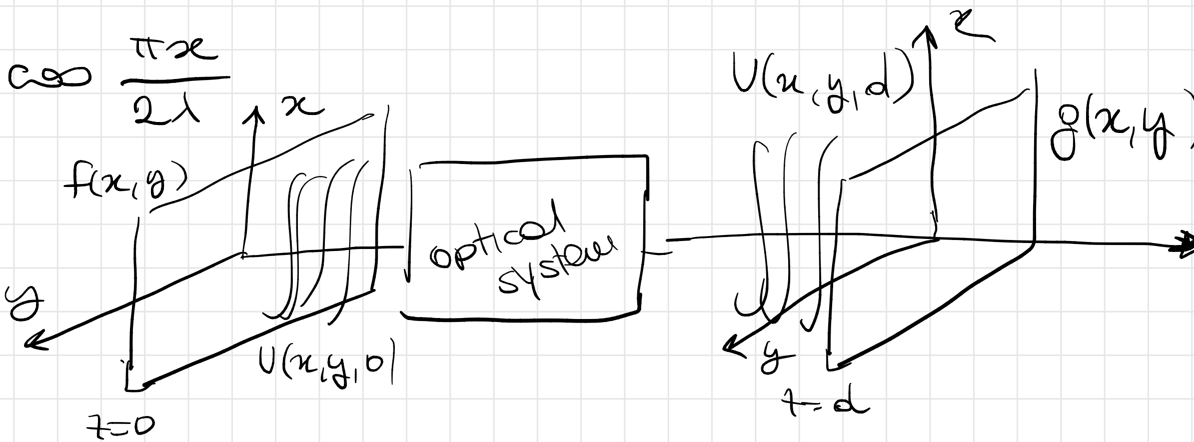
Useful for edge enhancement in image processing

<u>PROBLEM</u> : The complex amplitude of a monochromatic plane wave of wavelength λ in the $z=0$ and $z=d$ planes are $f(x,y)$ and $g(x,y)$. Assuming $d=10^4\lambda$ use harmonic analysis to find $g(x,y)$ when:

a) $f(x,y)=1$;

b) $f(x,y)=e^{-j\frac{\pi}{\lambda}(x+y)}$

c) $f(x,y)=\cos\dfrac{\pi x}{2\lambda}$



At $z=0$ $\quad U(x,y,0) = \underline{f(x,y)} = A\, e^{-j\,\underbrace{2\pi(\nu_x x}\,\underbrace{+\nu_y y)}}$

At a generic point z we have

$$U(x,y,z) = f(x,y)\, e^{-jk_z z} \qquad k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$k = \frac{2\pi}{\lambda} \qquad k_x = 2\pi\nu_x \qquad k_y = 2\pi\nu_y$$

At $z=d$

$$g(x,y) = U(x,y,d) = f(x,y)\, e^{-jk_z d}$$

a) If $f(x,y) = 1$ then at $z=d$

$$g(x,y) = e^{-jk_z d}$$

$$k_z = \sqrt{k^2 \underbrace{- k_x^2 - k_y^2}_{=\emptyset}} = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2} = \frac{2\pi}{\lambda}$$

So it follows that

$$g(x,y) = e^{-j \frac{2\pi}{\cancel{\lambda}} 10^4 \cancel{\lambda}} = \cos(2\pi \cdot 10^4) \cancel{- j \sin(2\pi 10^4)}^{\,0}$$

$$= \underline{1}$$

if $f(x,y)$ is a constant $\rightarrow$ $g(x,y)$ is a constant

(b) if $f(x,y) = e^{-j \frac{\pi}{\lambda}(x+y)}$

$$g(x,y) = f(x,y)\, e^{-j k_z d} = e^{-j \frac{\pi}{\lambda}(x+y)}\, e^{-j k_z d}$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{\pi}{\lambda}\right)^2 - \left(\frac{\pi}{\lambda}\right)^2} = \sqrt{\frac{4\pi^2 - \pi^2 - \pi^2}{\lambda^2}} = \underline{\frac{\pi\sqrt{2}}{\lambda}}$$

$$k_x = k_y = 2\pi \nu_{x,y}$$

$$\cancel{-j} 2\cancel{\pi} \nu_x \cancel{x} = \cancel{-j} \frac{\pi}{\lambda} \cancel{x}$$

$$\nu_x = \nu_y = \frac{1}{2\lambda}$$

$$\nu_x = \frac{1}{2\cancel{\lambda}}$$

$$k_x = k_y = \frac{2\pi}{2\lambda} = \frac{\pi}{\lambda}$$

So it follows:

$$g(x,y) = e^{-j\frac{\pi}{\lambda}(x+y)}$$

$$\underbrace{e^{-j\frac{\pi}{\cancel{\lambda}}\sqrt{2}\cdot 10^4 \cdot \cancel{\lambda}}}$$

c) if $f(x,y) = \cos\left(\frac{\pi x}{2\lambda}\right)$

then $g(x,y) = f(x,y)\, e^{-jk_z d} = \cos\left(\frac{\pi x}{2\pi}\right) e^{-jk_z d} =$

$$= \frac{1}{2}\left[e^{-j\frac{\pi x}{2\lambda}} + e^{j\frac{\pi x}{2\lambda}} \right] e^{-jk_z d} =$$

$$= \frac{1}{2}\, \underbrace{e^{-j\frac{\pi x}{2\lambda}}}_{k_x = \frac{\pi}{2\lambda} \quad k_y = 0} e^{-jk_z d} + \frac{1}{2}\, \underbrace{e^{j\frac{\pi x}{2\lambda}}}_{k_x = -\frac{\pi}{2\lambda} \quad k_y = 0} e^{-jk_z d}$$

$$\cancel{-j}\, 2\cancel{\pi}\, \nu_x \cancel{x} = \cancel{-j}\, \frac{\cancel{\pi} \cancel{x}}{2}$$

$$\nu_x = \frac{1}{4\lambda} \Rightarrow k_x = 2\pi \nu_x = \frac{\pi}{2\lambda}$$

$$k_z = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{\pi}{2\lambda}\right)^2} = \sqrt{\frac{4\pi^2}{\lambda^2} - \frac{\pi^2}{4\lambda^2}} = \sqrt{\frac{16\pi^2 - \pi^2}{4\lambda^2}} = \frac{\pi}{2\lambda}\sqrt{15}$$

It follows that:

$$\underline{g(x,y)} = \frac{1}{2} e^{-j\frac{\pi x}{2\lambda}} e^{-j\frac{\pi}{\cancel{\lambda}} \cdot 10^4 \cancel{\lambda} \sqrt{15}} + \frac{1}{2} e^{j\frac{\pi x}{2\lambda}} e^{-j\pi\sqrt{15}\cdot 10^4} =$$

$$= \cos\left(\frac{\pi x}{2\lambda}\right) e^{-j\pi\sqrt{15}\cdot 10^4}$$