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2.161 Signal Processing: Continuous and Discrete  
Fall 2008

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
DEPARTMENT OF MECHANICAL ENGINEERING

2.161 *Signal Processing - Continuous and Discrete*  
Fall Term 2008

**Lecture 4**<sup>1</sup>

**Reading:**

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**1 Review of Development of Fourier Transform:**

We saw in Lecture 3 that the Fourier transform representation of *aperiodic* waveforms can be expressed as the limiting behavior of the Fourier series as the period of a *periodic extension* is allowed to become very large, giving the Fourier transform pair

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \quad (1)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t} d\Omega \quad (2)$$

Equation (1) is known as the *forward* Fourier transform, and is analogous to the analysis equation of the Fourier series representation. It expresses the time-domain function  $x(t)$  as a function of frequency, but unlike the Fourier series representation it is a continuous function of frequency. Whereas the Fourier series coefficients have units of amplitude, for example volts or Newtons, the function  $X(j\Omega)$  has units of amplitude density, that is the total “amplitude” contained within a small increment of frequency is  $X(j\Omega)\delta\Omega/2\pi$ .

Equation (2) defines the *inverse* Fourier transform. It allows the computation of the time-domain function from the frequency domain representation  $X(j\Omega)$ , and is therefore analogous to the Fourier series synthesis equation. Each of the two functions  $x(t)$  or  $X(j\Omega)$  is a complete description of the function and the pair allows the transformation between the domains.

We adopt the convention of using lower-case letters to designate time-domain functions, and the same upper-case letter to designate the frequency-domain function. We also adopt the nomenclature

$$x(t) \xLeftrightarrow{\mathcal{F}} X(j\Omega)$$

as denoting the bidirectional Fourier transform relationship between the time and frequency-domain representations, and we also frequently write

$$\begin{aligned} X(j\Omega) &= \mathcal{F}\{x(t)\} \\ x(t) &= \mathcal{F}^{-1}\{X(j\Omega)\} \end{aligned}$$

as denoting the operation of taking the forward  $\mathcal{F}\{\}$ , and inverse  $\mathcal{F}^{-1}\{\}$  Fourier transforms respectively.

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## 1.1 Alternate Definitions

Although the definitions of Eqs. (??) and (??) flow directly from the Fourier series, definitions for the Fourier transform vary from text to text and in different disciplines. The main objection to the convention adopted here is the asymmetry introduced by the factor  $1/2\pi$  that appears in the inverse transform. Some authors, usually in physics texts, define the so-called *unitary* Fourier transform pair as

$$\begin{aligned}X(j\Omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \\x(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t} d\Omega\end{aligned}$$

so as to distribute the constant symmetrically over the forward and inverse transforms.

Many engineering texts address the issue of the asymmetry by defining the transform with respect to frequency  $F = 2\pi\Omega$  in Hz, instead of angular frequency  $\Omega$  in radians/s. The effect, through the change in the variable in the inverse transform, is to redefine the transform pair as

$$\begin{aligned}X(F) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft} dt \\x(t) &= \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft} dF\end{aligned}$$

Some authors also adopt the notation of dropping the  $j$  from the frequency domain representation and write  $X(\Omega)$  or  $X(F)$  as above.

Even more confusing is the fact that some authors (particularly in physics) adopt a definition that reverses the sign convention on the exponential terms in the Fourier integral, that is they define

$$\begin{aligned}X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{j\Omega t} dt \\x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{-j\Omega t} d\Omega\end{aligned}$$

These various definitions of the transform pair mean that care must be taken to understand the particular definition adopted by texts and software packages. Throughout this course we will retain the definitions in Eqs. (??) and (??).

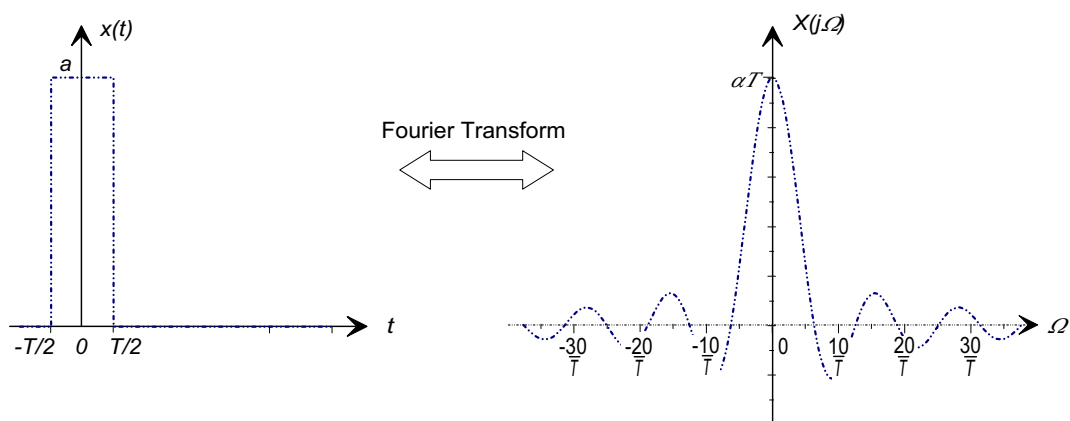
## 1.2 Fourier Transform Examples

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### ■ Example 1

Find the Fourier transform of the pulse function

$$x(t) = \begin{cases} a & |t| < T/2 \\ 0 & \text{otherwise.} \end{cases}$$



**Solution:** From the definition of the forward Fourier transform

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \quad (\text{i})$$

$$= a \int_{-T/2}^{T/2} e^{-j\Omega t} dt \quad (\text{ii})$$

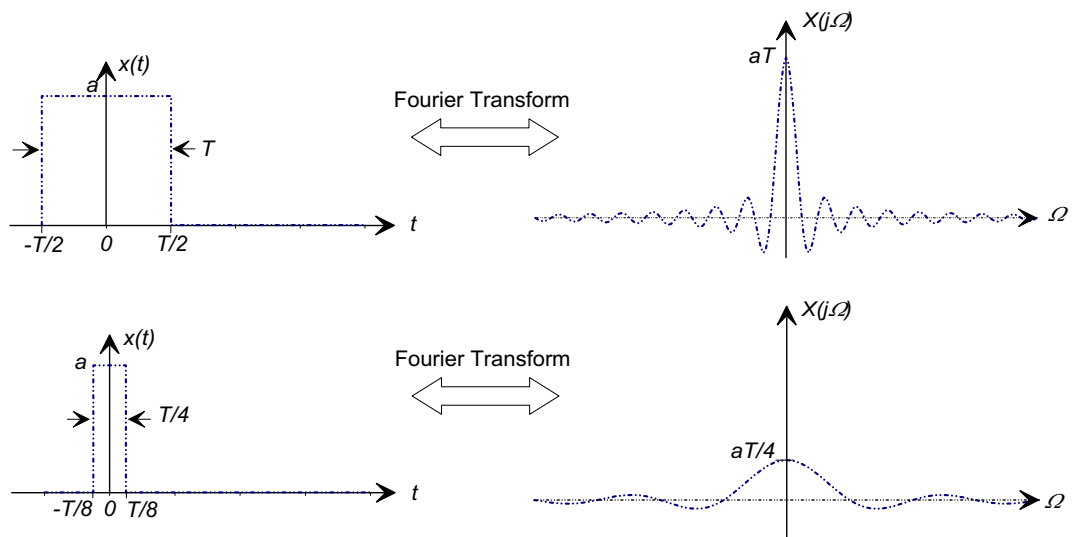
$$= a \left[ \frac{j}{\Omega} e^{-j\Omega t} \right]_{-T/2}^{T/2} \quad (\text{iii})$$

$$= \frac{ja}{\Omega} [e^{-j\Omega T/2} - e^{j\Omega T/2}] \quad (\text{iv})$$

$$= aT \frac{\sin(\Omega T/2)}{\Omega T/2}. \quad (\text{v})$$

The Fourier transform of the rectangular pulse is a real function, of the form  $(\sin x)/x$  centered around the  $j\Omega = 0$  axis. Because the function is real, it is sufficient to plot a single graph showing only  $|X(j\Omega)|$ . Notice that while  $X(j\Omega)$  is a generally decreasing function of  $\Omega$  it never becomes identically zero, indicating that the rectangular pulse function contains frequency components at all frequencies.

The function  $(\sin x)/x = 0$  when the argument  $x = n\pi$  for any integer  $n$  ( $n \neq 0$ ). The main peak or “lobe” of the spectrum  $X(j\Omega)$  is therefore contained within the frequency band defined by the first two zero-crossings  $|\Omega T/2| < \pi$  or  $|\Omega| < 2\pi/T$ . Thus as the pulse duration  $T$  is decreased, the spectral bandwidth of the pulse increases, indicating that short duration pulses have a relatively larger high frequency content.



## ■ Example 2

Find the Fourier transform of the Dirac delta function  $\delta(t)$ .

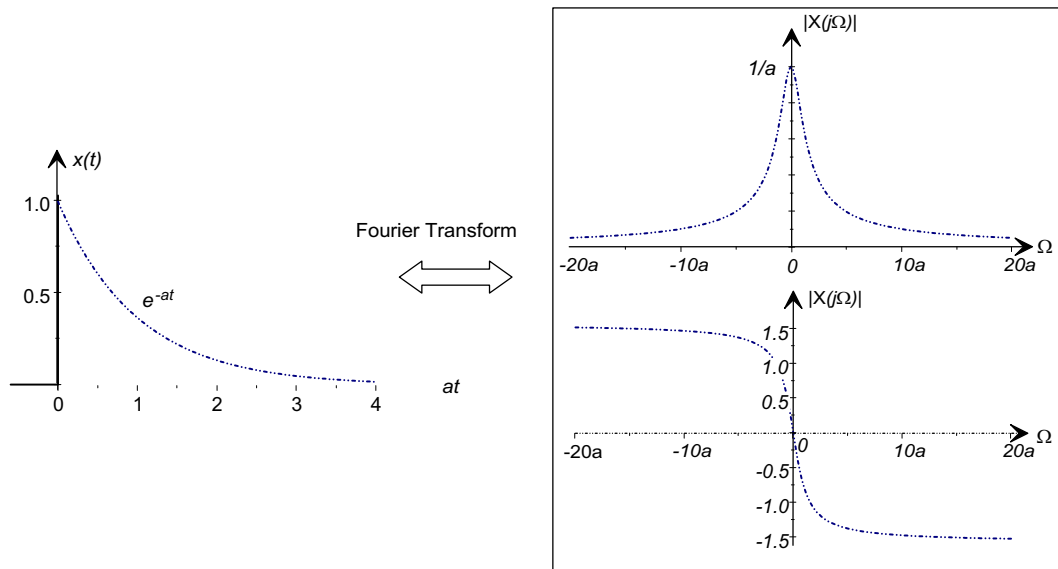
**Solution:** When substituted into the forward Fourier transform

$$\begin{aligned} \Delta(j\Omega) &= \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt \\ &= 1 \end{aligned} \tag{i}$$

by the sifting property. The spectrum of the delta function is therefore constant over all frequencies. It is this property that makes the impulse a very useful test input for linear systems.

## ■ Example 3

Find the Fourier transform of the causal real exponential function  $x(t) = u_s(t)e^{-at}$  ( for  $a > 0$  ).



**Solution:** From the definition of the forward Fourier transform

$$\begin{aligned}
 X(j\Omega) &= \int_0^{\infty} e^{-at} e^{-j\Omega t} dt \\
 &= \left[ \frac{-1}{a + j\Omega} e^{-(a+j\Omega)t} \right]_0^{\infty} \\
 &= \frac{1}{a + j\Omega}
 \end{aligned}$$

which is complex, and in terms of a magnitude and phase function is

$$|X(j\Omega)| = \frac{1}{\sqrt{a^2 + \Omega^2}} \quad (\text{i})$$

$$\angle X(j\Omega) = \tan^{-1} \left( \frac{-\Omega}{a} \right) \quad (\text{ii})$$

Other examples are given in the class handout.

### 1.3 Properties of the Fourier Transform

The properties of the Fourier transform are covered more fully in the class handout and are simply summarized here:

(1) **Existence of the Fourier Transform** The three Dirichlet conditions are sufficient conditions, but are not strictly necessary:

- The function  $x(t)$  must be *integrable in the absolute sense* over all time, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty.$$

- There must be at most a finite number of maxima and minima in the function  $x(t)$ . Notice that periodic functions are excluded by this and the previous condition.
- There must be at most a finite number of discontinuities in the function  $x(t)$ , and all such discontinuities must be finite in magnitude.

(2) **Linearity of the Fourier Transform** If

$$g(t) \xleftrightarrow{\mathcal{F}} G(j\Omega) \quad \text{and} \quad h(t) \xleftrightarrow{\mathcal{F}} H(j\Omega)$$

then for arbitrary constants  $a$  and  $b$ ,

$$ag(t) + bh(t) \xleftrightarrow{\mathcal{F}} aG(j\Omega) + bH(j\Omega). \quad (3)$$

(3) **Duality**

$$X(jt) \xleftrightarrow{\mathcal{F}} 2\pi x(-\Omega)$$

where  $X(jt)$  is  $X(j\Omega)$  where  $\Omega$  has been replaced by  $t$ , and  $x(-\Omega)$  is  $x(t)$  where  $t$  is replaced by  $-\Omega$ . Therefore if we know the Fourier transform of one function, we also know it for another.

(4) **Even and Odd Functions**

- The Fourier transform of a real even function of time is a real even function
- The Fourier transform of a real odd function is an imaginary odd function.

The same relationships hold for the inverse Fourier transform.

(5) **Time Shifting** The Fourier transform of  $x(t + \tau)$ , a time shifted version of  $x(t)$ , is

$$\mathcal{F}\{x(t + \tau)\} = e^{j\Omega\tau} X(j\Omega).$$

and in terms of a magnitude and phase

$$\mathcal{F}\{x(t + \tau)\} = |X(j\Omega)| e^{j(\angle X(j\Omega) + \Omega\tau)}.$$

(4) **Time Scaling**

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X(j\Omega/a), \quad a \neq 0$$

(9) **Time Reversal** If  $a = -1$ , the time scaling property gives

$$x(-t) \xleftrightarrow{\mathcal{F}} X(-j\Omega). \quad (4)$$

(5) **Waveform Energy** Parseval's theorem asserts the equivalence of the total waveform energy in the time and frequency domains by the relationship

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) \overline{X(j\Omega)} d\Omega.$$

In other words, the quantity  $|X(j\Omega)|^2$  is a measure of the energy of the function per unit bandwidth.

## (6) The Fourier Transform of the Derivative of a Function

$$\mathcal{F}\left\{\frac{dx}{dt}\right\} = j\Omega X(j\Omega),$$

The Fourier transform of the  $n$ th derivative of  $x(t)$  is

$$\mathcal{F}\left\{\frac{d^n x}{dt^n}\right\} = (j\Omega)^n X(j\Omega) \quad (5)$$

## (7) The Fourier Transform of the Integral of a Function

$$\mathcal{F}\left\{\int_{-\infty}^t x(\nu) d\nu\right\} = \pi X(0)\delta(j\Omega) + \frac{1}{j\Omega}X(j\Omega)$$

(7) Time Reversal If a function  $x(t)$  has a Fourier transform  $X(j\Omega)$  then

$$\mathcal{F}\{x(-t)\} = X(-j\Omega).$$

### 1.4 Extension of the Fourier Transform to Functions for which the Fourier Integral does not Converge.

The Dirichlet conditions are *sufficient* but not *necessary* conditions for the existence of the Fourier transform. If the use of the Dirac delta function  $\delta(x)$  is allowed, the Fourier transform of many functions with a non-convergent Fourier integral may be defined. This topic is covered in greater detail in the class handout (Sec. 4.4), and a simple example is given here

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#### ■ Example 4

Define the Fourier transform of the unit-step (Heaviside) function  $u_s(t)$ , where

$$u_s(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

Clearly the Fourier integral

$$U_s(j\Omega) = \int_{-\infty}^{\infty} u_s(t)e^{j\Omega t} dt = \int_0^{\infty} e^{-j\Omega t} dt$$

does not converge, and we seek an alternative approach.

Consider the one-sided real exponential function

$$x(t) = u_s(t)e^{-at}$$



as described in Example 3, and note that  $u_s(t) = \lim_{a \rightarrow 0} x(t)$  which implies  $U_s(j\Omega) = \lim_{a \rightarrow 0} X(j\Omega)$ . From Example 3

$$X(j\Omega) = \frac{1}{a + j\Omega} = \frac{a}{a^2 + \Omega^2} - j \frac{\Omega}{a^2 + \Omega^2}$$

The real part is in the form of a Cauchy distribution, and we note that for  $a > 0$

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + \Omega^2} d\Omega = \pi$$

and that as  $a \rightarrow 0$ ,  $a/(a^2 + \Omega^2)$  becomes impulse-like, Therefore, as  $a \rightarrow 0$ ,

$$\frac{a}{a^2 + \Omega^2} \rightarrow \pi\delta(\Omega) \quad \text{and} \quad -j \frac{\Omega}{a^2 + \Omega^2} \rightarrow -j \frac{1}{\Omega}$$

so that we may define the Fourier transform of the unit-step function as

$$U_s(j\Omega) = \pi\delta(\Omega) + \frac{1}{j\Omega}$$

## 2 The Frequency Response of a Linear System Defined Directly from the Fourier Transform

The system frequency response function  $H(j\Omega)$  may be defined directly using the transform property of derivatives. Consider a linear system described by the single input/output differential equation

$$\begin{aligned} a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = \\ b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \dots + b_1 \frac{du}{dt} + b_0 u \end{aligned}$$

and assume that the Fourier transforms of both the input  $u(t)$  and the output  $y(t)$  exist. Then the Fourier transform of both sides of the differential equation may be found by using the derivative property:

$$\mathcal{F} \left\{ \frac{d^n f}{dt^n} \right\} = (j\Omega)^n F(j\Omega)$$

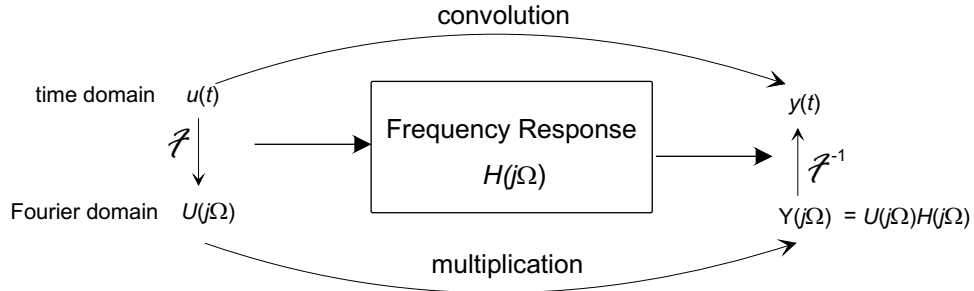
to give

$$\begin{aligned} \{a_n(j\Omega)^n + a_{n-1}(j\Omega)^{n-1} + \dots + a_1(j\Omega) + a_0\} Y(j\Omega) = \\ \{b_m(j\Omega)^m + b_{m-1}(j\Omega)^{m-1} + \dots + b_1(j\Omega) + b_0\} U(j\Omega), \end{aligned}$$

which has reduced the original differential equation to an algebraic equation in  $j\Omega$ . This equation may be rewritten explicitly in terms of  $Y(j\Omega)$  in terms of the frequency response  $H(j\Omega)$

$$\begin{aligned} Y(j\Omega) &= \frac{b_m(j\Omega)^m + b_{m-1}(j\Omega)^{m-1} + \dots + b_1(j\Omega) + b_0}{a_n(j\Omega)^n + a_{n-1}(j\Omega)^{n-1} + \dots + a_1(j\Omega) + a_0} U(j\Omega) \\ &= H(j\Omega) U(j\Omega), \end{aligned}$$

showing the multiplicative frequency domain relationship between input and output.



### 3 Relationship between the Frequency Response and the Impulse Response

The Dirac delta function  $\delta(t)$  has a unique property; its Fourier transform is unity for all frequencies

$$\mathcal{F}\{\delta(t)\} = 1,$$

The impulse response of a system  $h(t)$  is defined to be the response to an input  $u(t) = \delta(t)$ , the output spectrum is then  $Y_\delta(j\Omega) = \mathcal{F}\{h(t)\}$ ,

$$\begin{aligned} Y(j\Omega) &= \mathcal{F}\{\delta(t)\} H(j\Omega) \\ &= H(j\Omega). \end{aligned}$$

or

$$h(t) = \mathcal{F}^{-1}\{H(j\Omega)\}.$$

In other words, the system impulse response  $h(t)$  and its frequency response  $H(j\Omega)$  are a Fourier transform pair:

$$h(t) \xleftrightarrow{\mathcal{F}} H(j\Omega).$$

In the same sense that  $H(j\Omega)$  completely characterizes a linear system in the frequency response, the impulse response provides a complete system characterization in the time domain.

### 4 The Convolution Property

A system with an impulse response  $h(t)$ , driven by an input  $u(t)$ , responds with an output  $y(t)$  given by the convolution integral

$$\begin{aligned} y(t) &= h(t) \otimes u(t) \\ &= \int_{-\infty}^{\infty} u(\tau)h(t - \tau)d\tau \end{aligned}$$

In the frequency domain the input/output relationship for a linear system is multiplicative, that is  $Y(j\Omega) = U(j\Omega)H(j\Omega)$ . Because by definition

$$y(t) \xleftrightarrow{\mathcal{F}} Y(j\Omega),$$

we are lead to the conclusion that

$$h(t) \otimes u(t) \xLeftrightarrow{\mathcal{F}} H(j\Omega)U(j\Omega). \quad (6)$$

The computationally intensive operation of computing the convolution integral has been replaced by the operation of multiplication. This result, known as the convolution property of the Fourier transform, can be shown to be true for the product of any two spectra, for example  $F(j\Omega)$  and  $G(j\Omega)$

$$\begin{aligned} F(j\Omega)G(j\Omega) &= \int_{-\infty}^{\infty} f(\nu)e^{-j\Omega\nu}d\nu \cdot \int_{-\infty}^{\infty} g(\tau)e^{-j\Omega\tau}d\tau \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\nu)g(\tau)e^{-j\Omega(\nu+\tau)}d\tau d\nu, \end{aligned}$$

and with the substitution  $t = \nu + \tau$

$$\begin{aligned} F(j\Omega)G(j\Omega) &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \right\} e^{-j\Omega t} dt \\ &= \int_{-\infty}^{\infty} (f(t) \otimes g(t)) e^{-j\Omega t} dt \\ &= \mathcal{F}\{f(t) \otimes g(t)\}. \end{aligned}$$

A *dual* property holds: if any two functions,  $f(t)$  and  $g(t)$ , are multiplied together in the time domain, then the Fourier transform of their product is a convolution of their spectra. The dual convolution/multiplication properties are

$$f(t) \otimes g(t) \xLeftrightarrow{\mathcal{F}} F(j\Omega)G(j\Omega) \quad (7)$$

$$f(t)g(t) \xLeftrightarrow{\mathcal{F}} \frac{1}{2\pi} F(j\Omega) \otimes G(j\Omega). \quad (8)$$