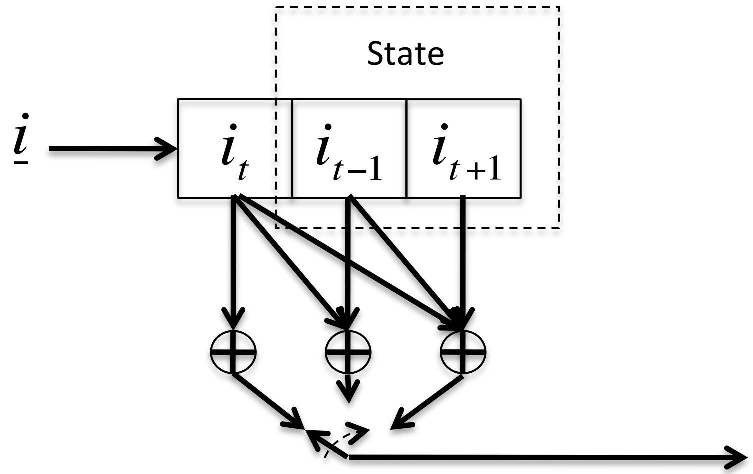
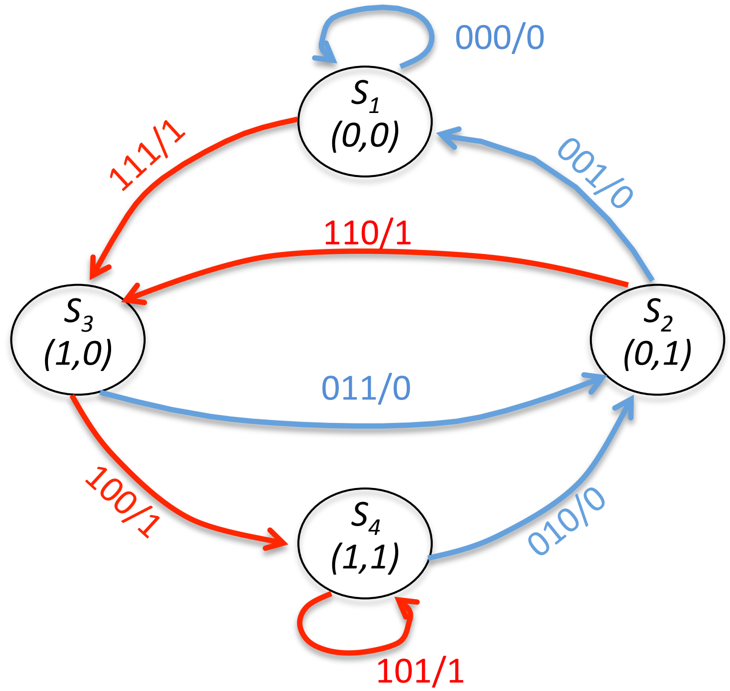
Viterbi Algorithm

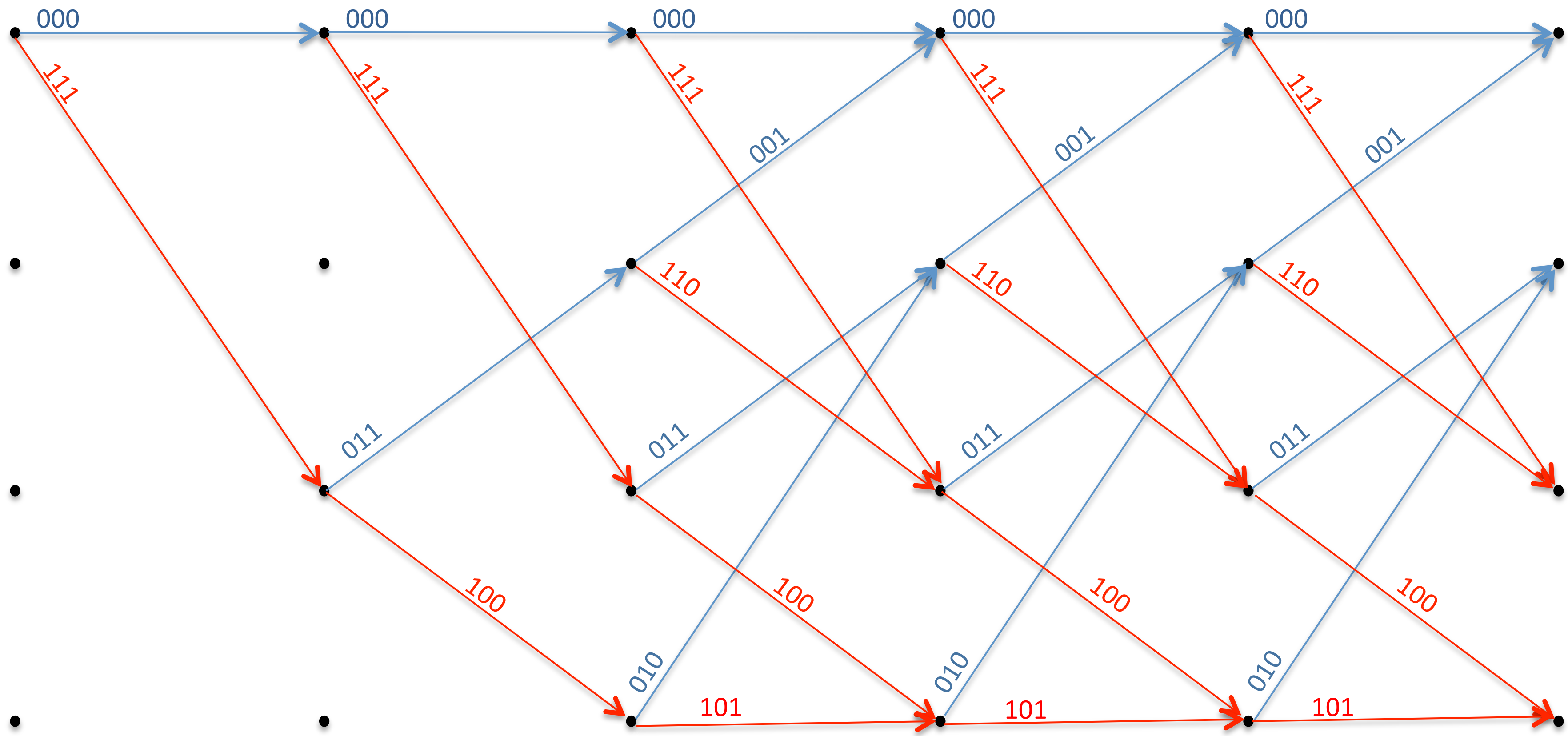
Example



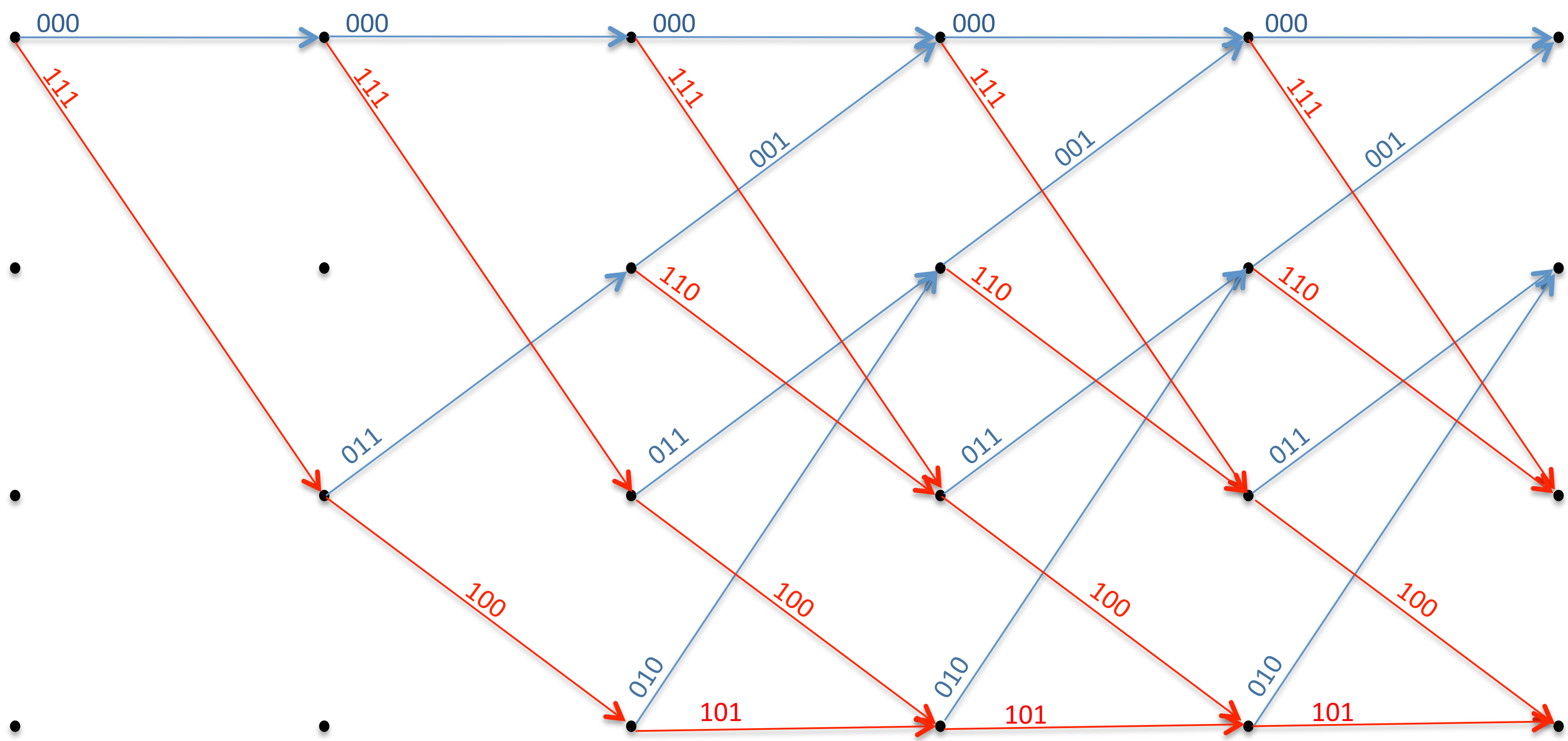


State diagram

Trellis

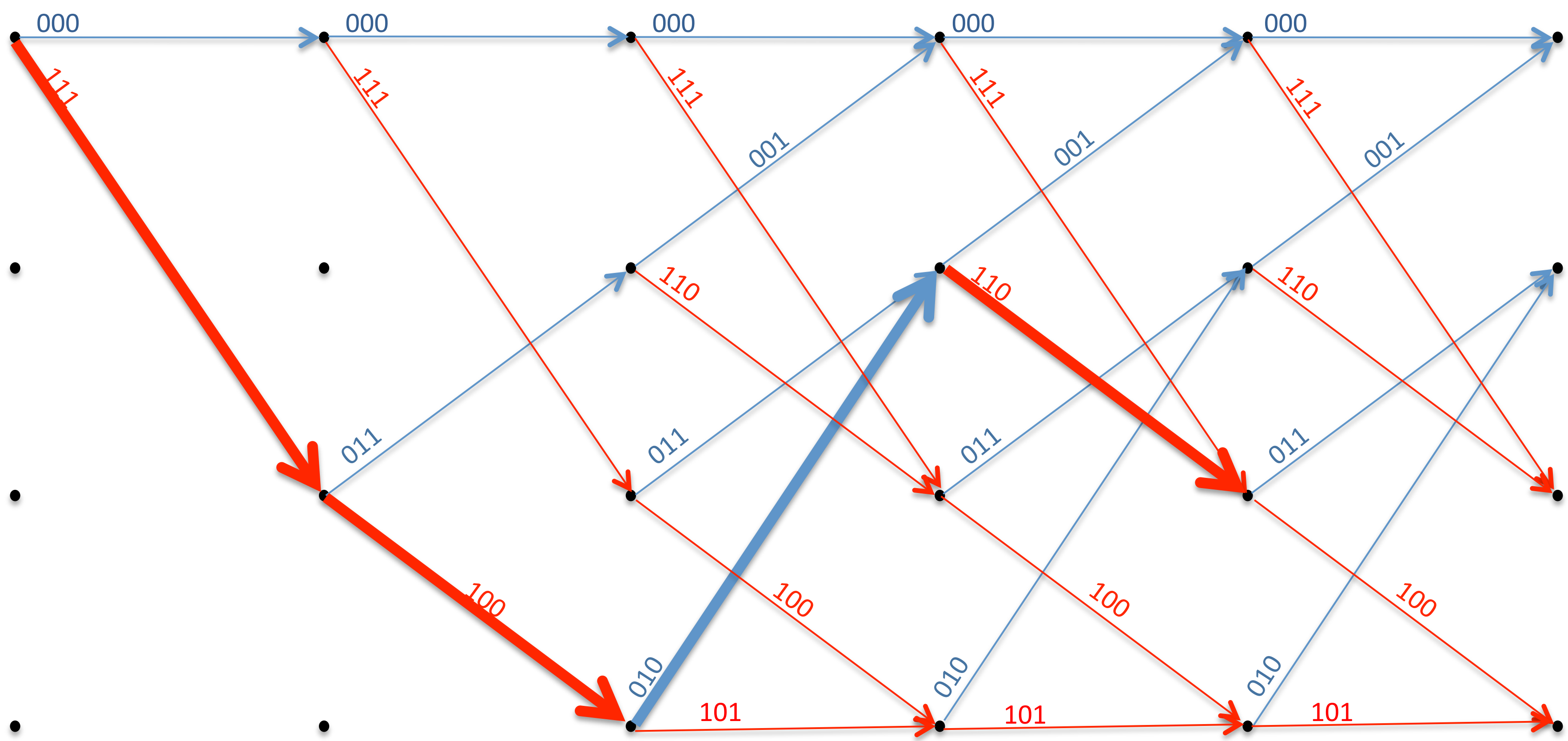


Trellis



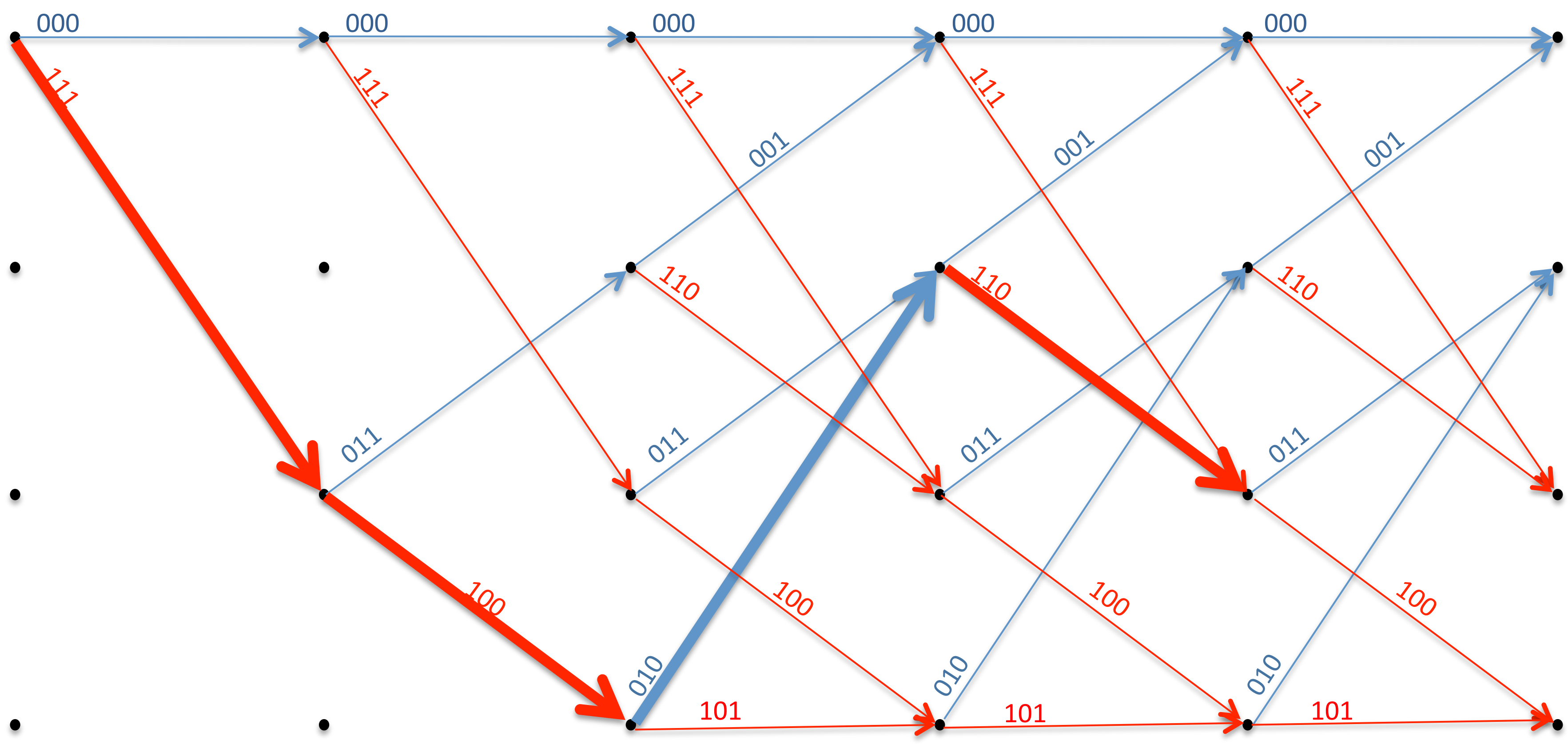
We send the sequence $\underline{i}$=1101

Trellis



We send the sequence *i*=1101 → *c*=111100010110

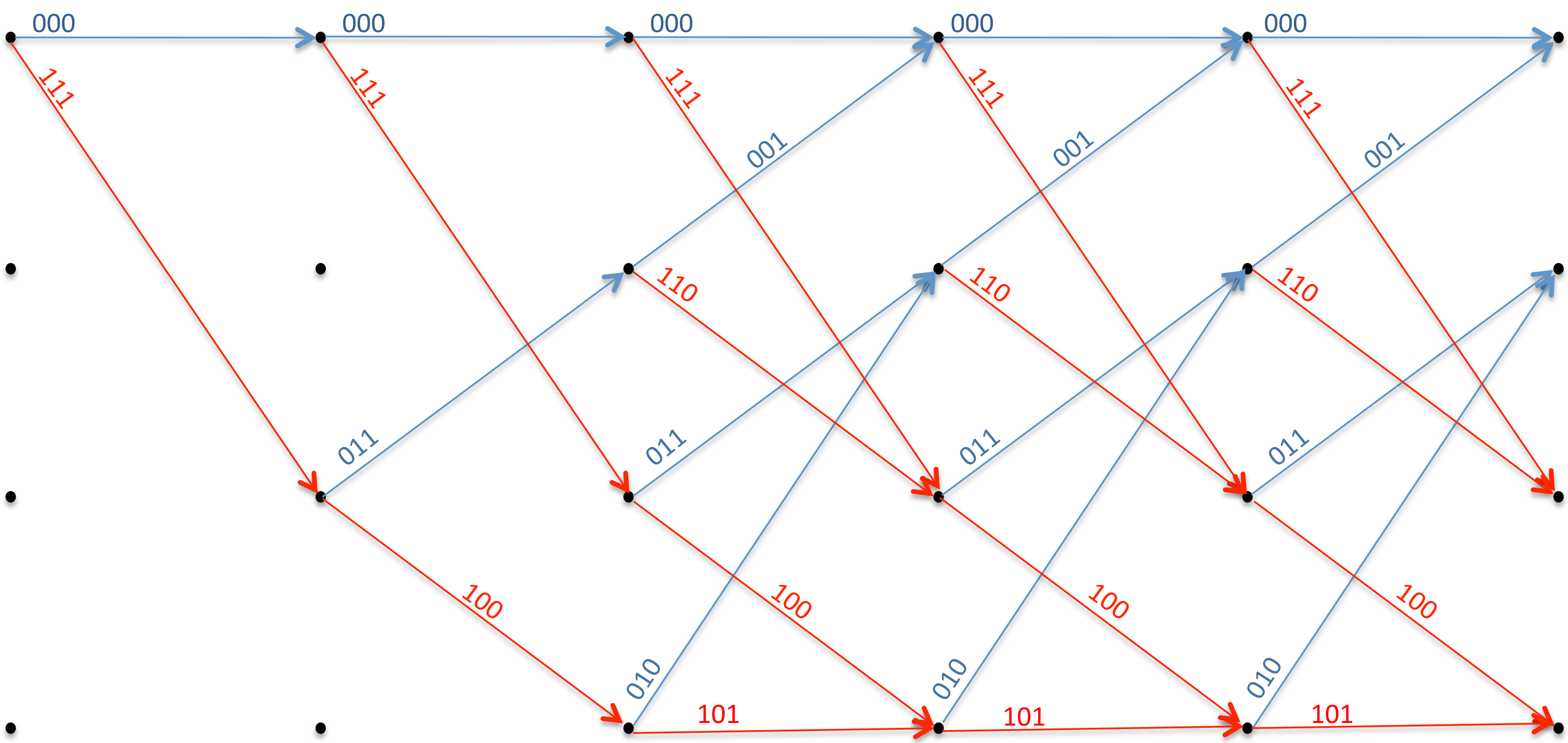
Trellis

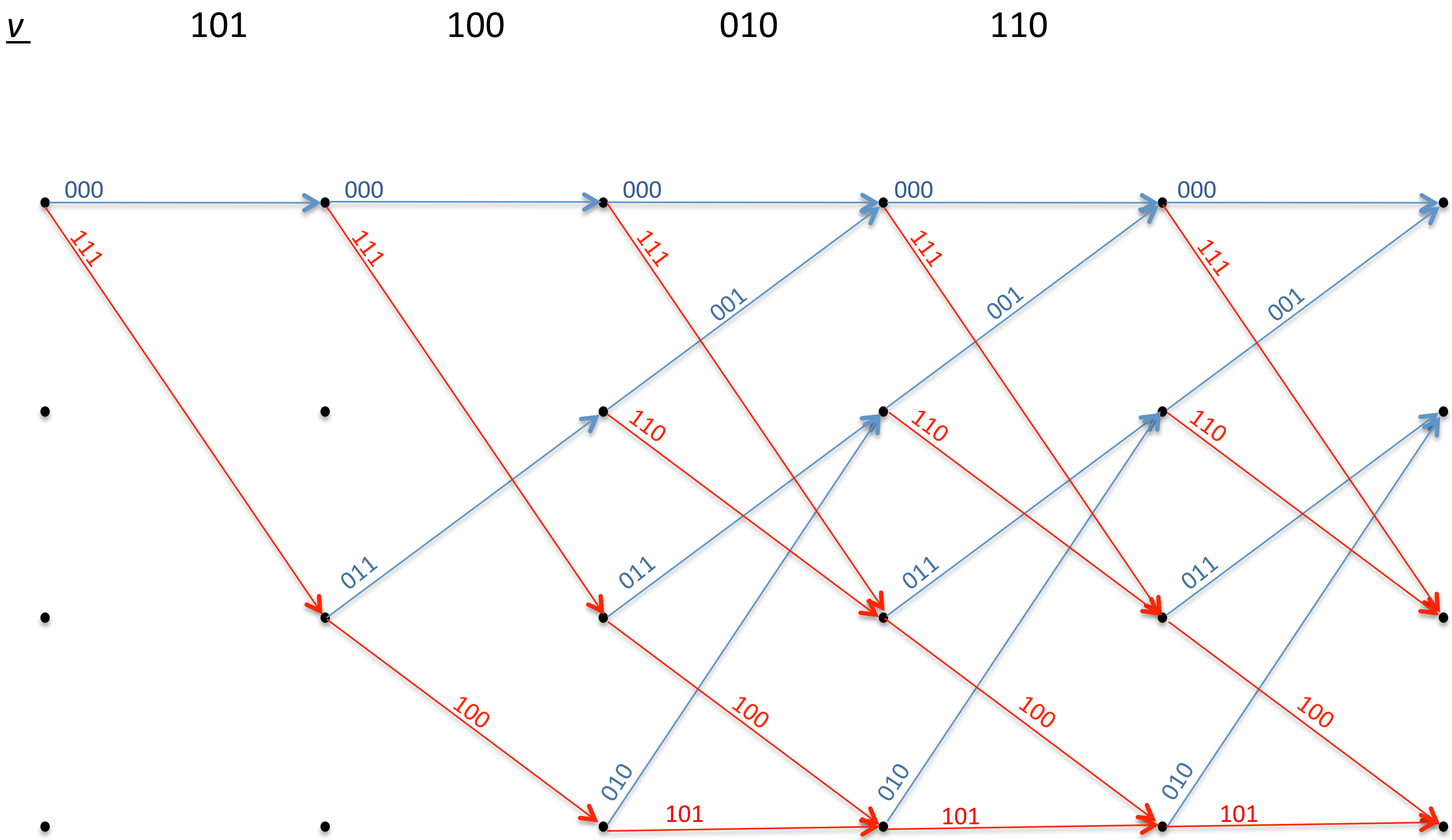


$\underline{c}$=111100010110 → ⊕ → $\underline{v}$=101100010110

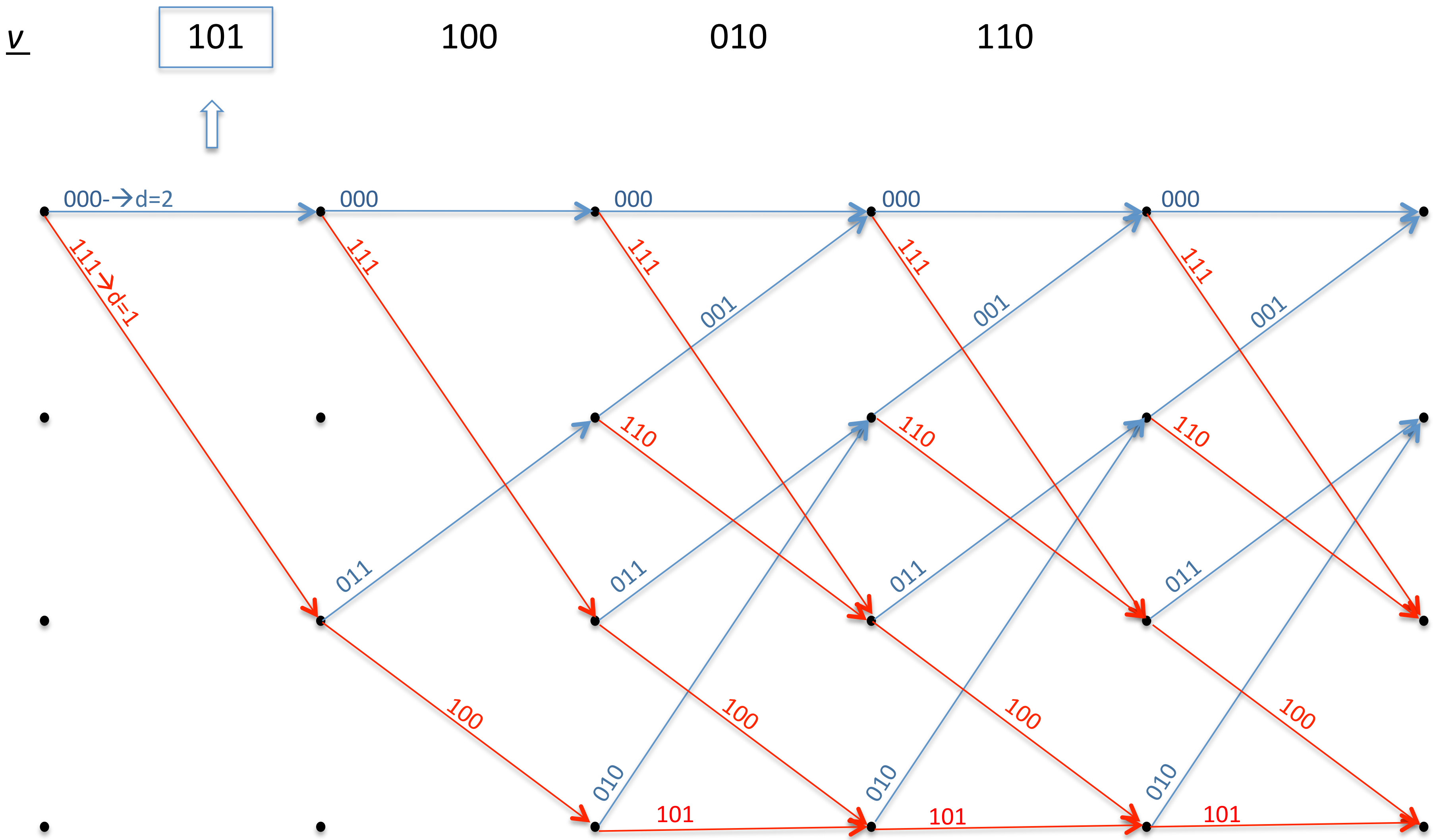
$\underline{e}$=010000000000

Viterbi Algorithm: decoding of string $\underline{v}$=101100010110

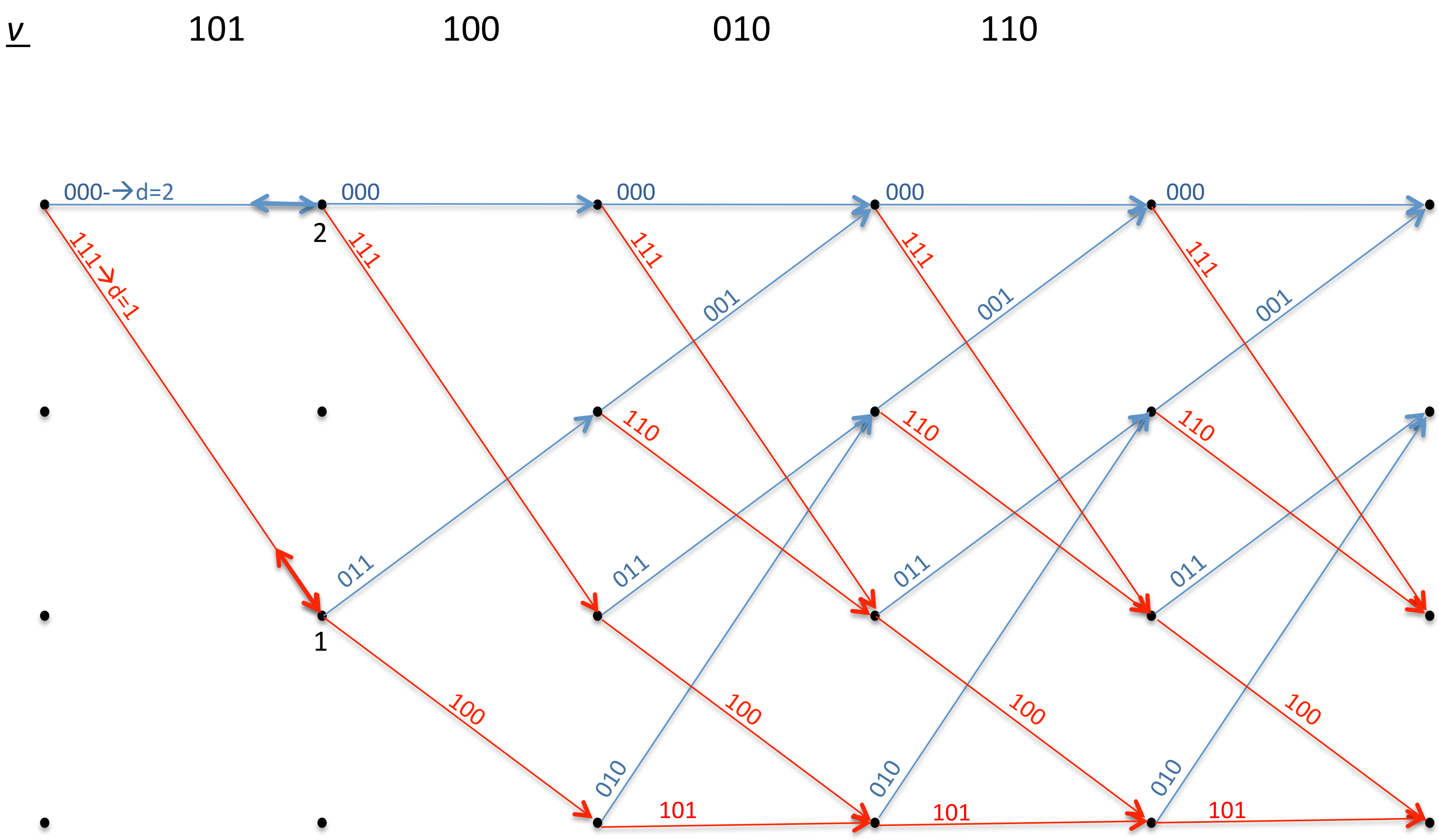




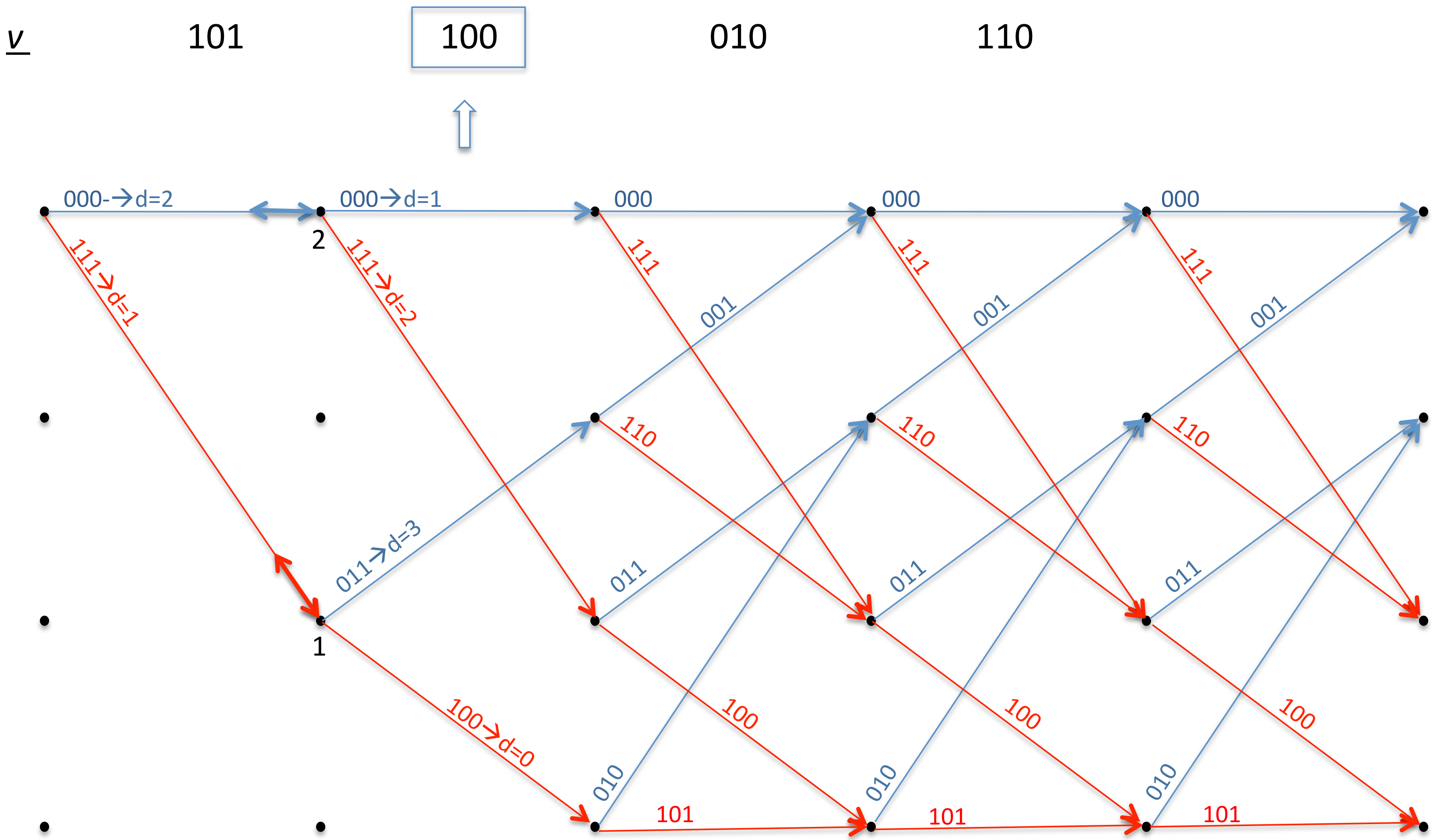
v
101
100
010
110
000
111
001
110
011
100
010
101



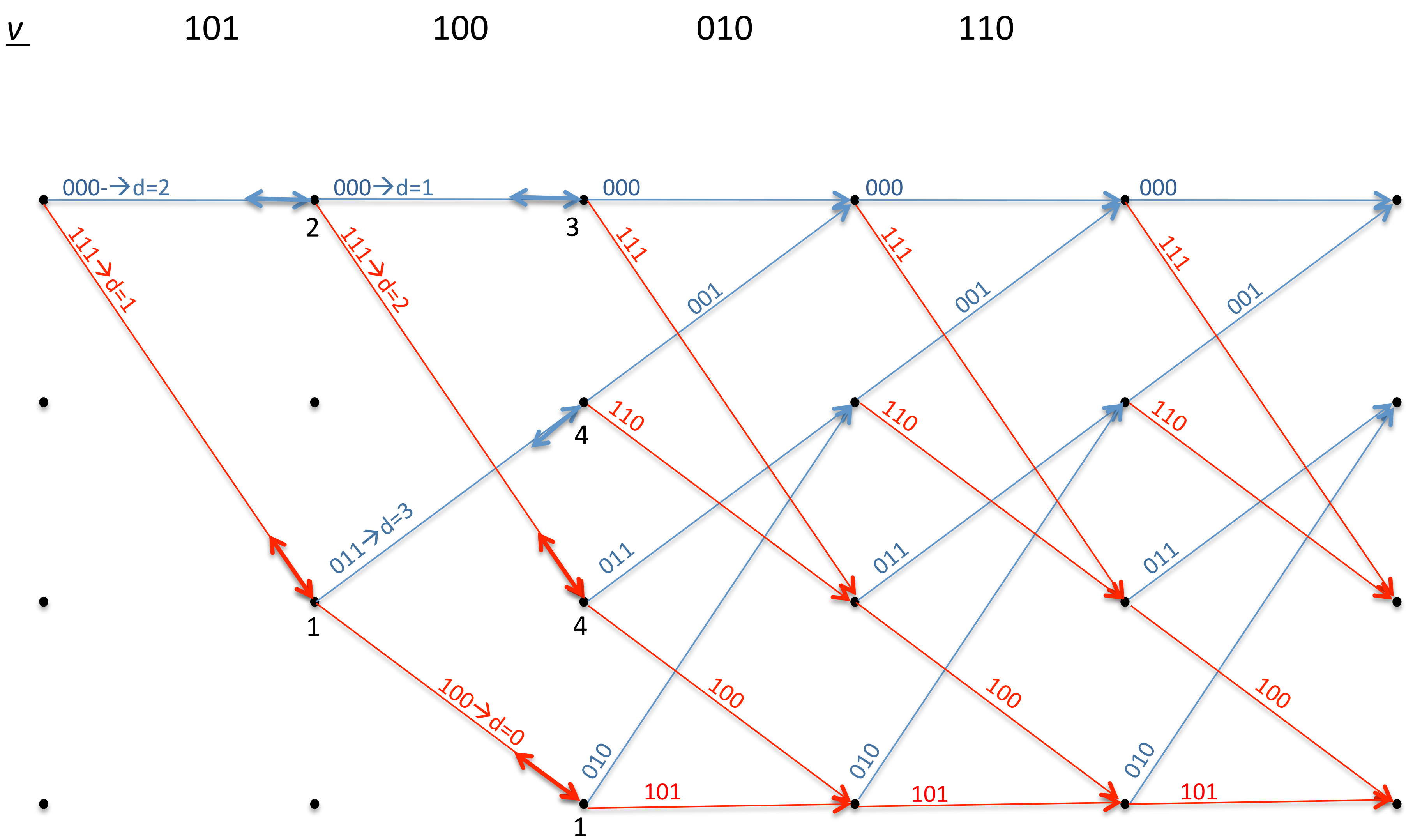
v
101
100
010
110
000-→d=2
111→d=1
000
111
001
110
011
100
010
101



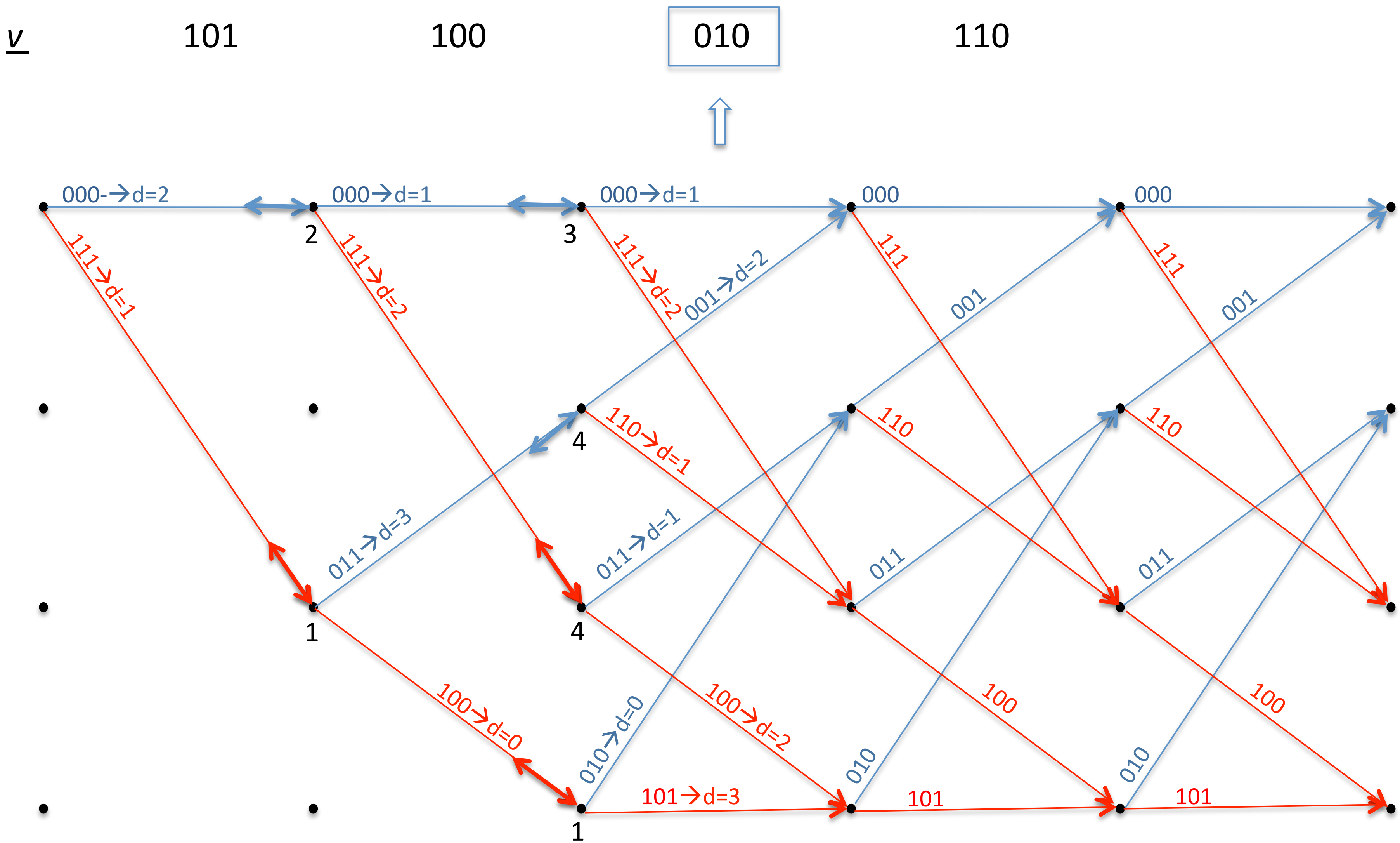
v
101
100
010
110
000-→d=2
111→d=1
2
1
000
111
001
110
011
100
010
101



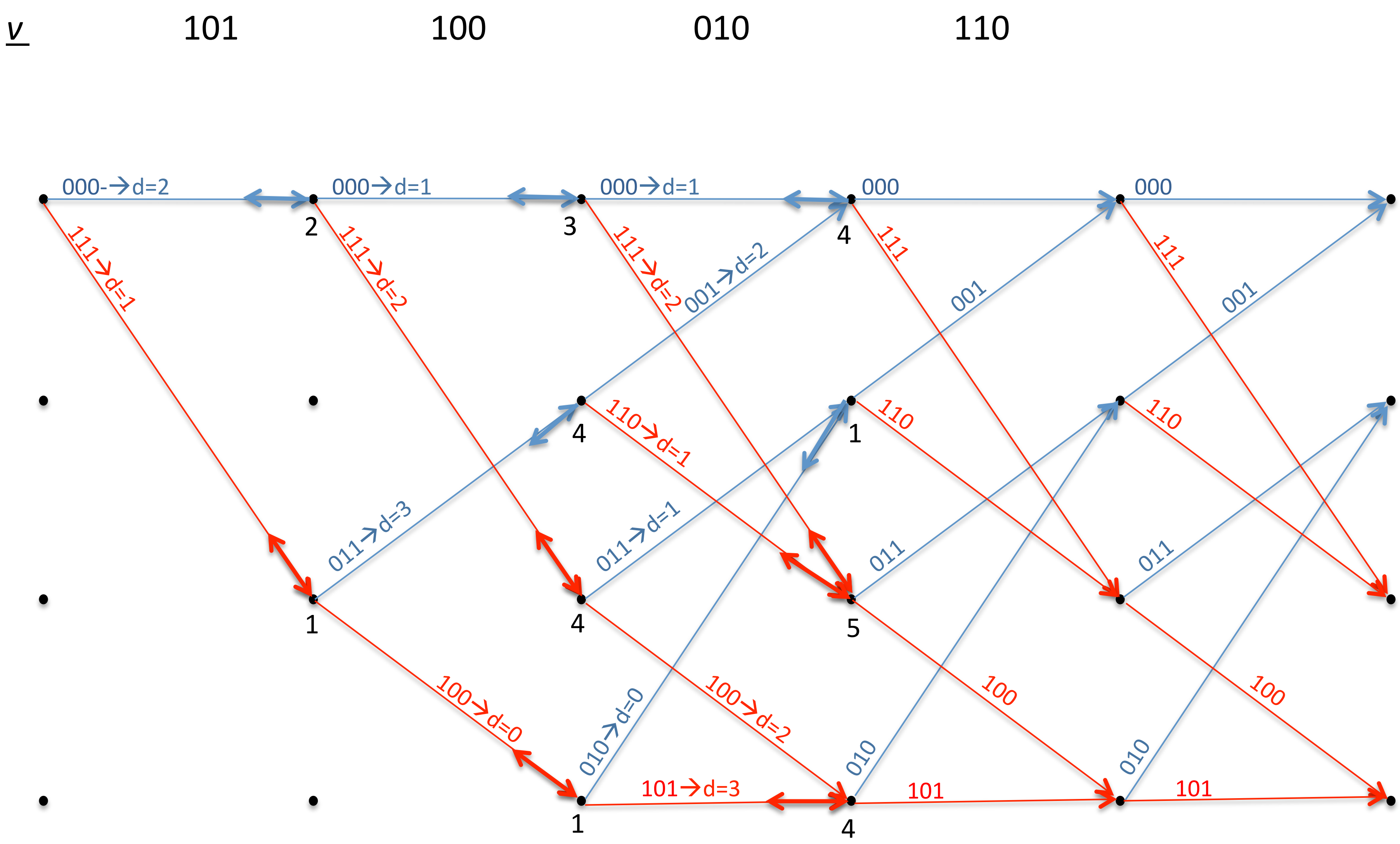
v
101
100
010
110
000-→d=2
000→d=1
000
000
000
2
111→d=1
111→d=2
111
111
111
001
001
001
110
110
110
011→d=3
011
011
011
1
100→d=0
100
100
100
010
010
010
101
101
101

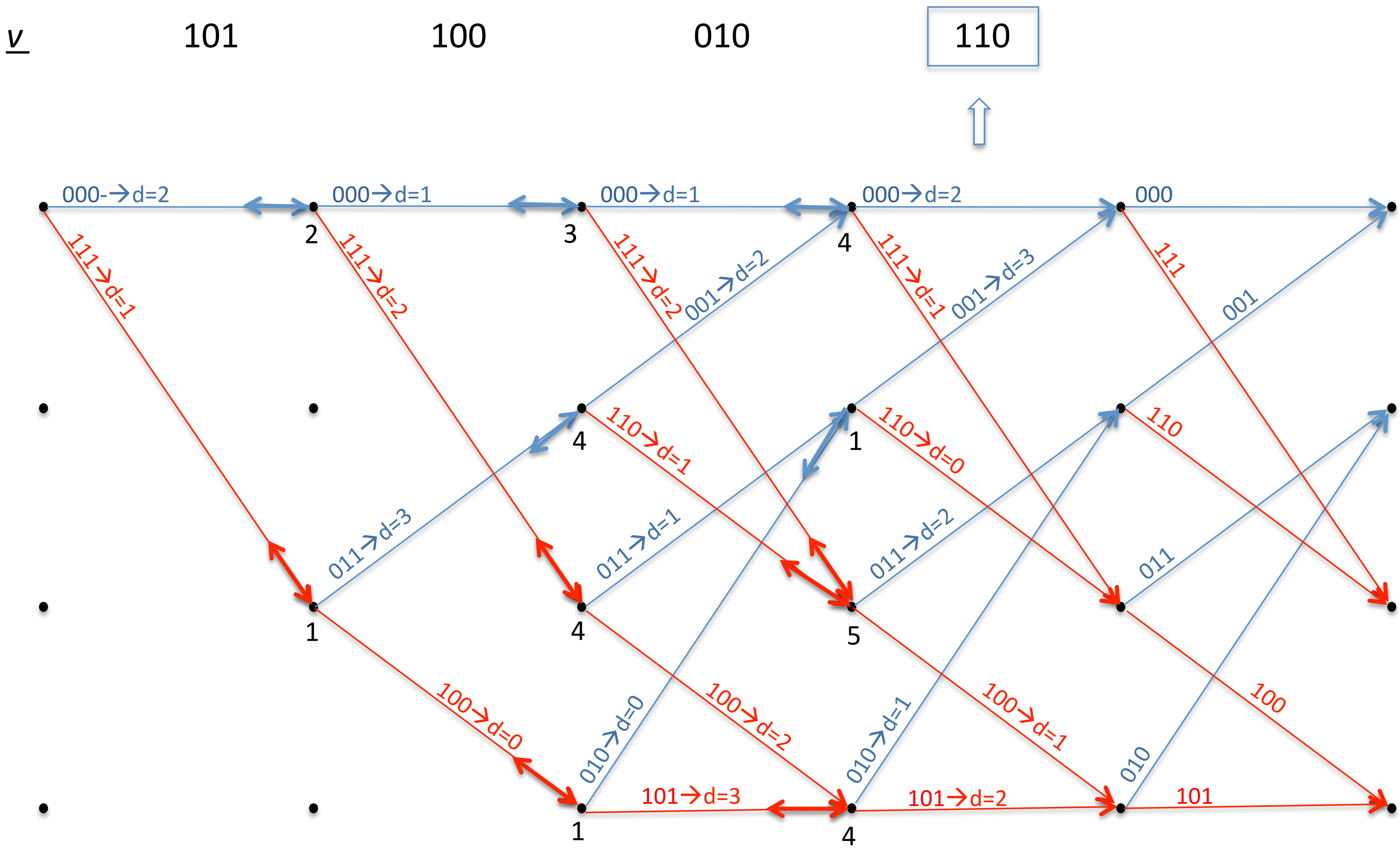


v
101
100
010
110
000-→d=2
000→d=1
000
000
000
111→d=1
111→d=2
111
111
111
001
001
001
110
110
110
011→d=3
011
011
011
100→d=0
100
100
100
010
010
010
101
101
101
2
3
4
1
4
1

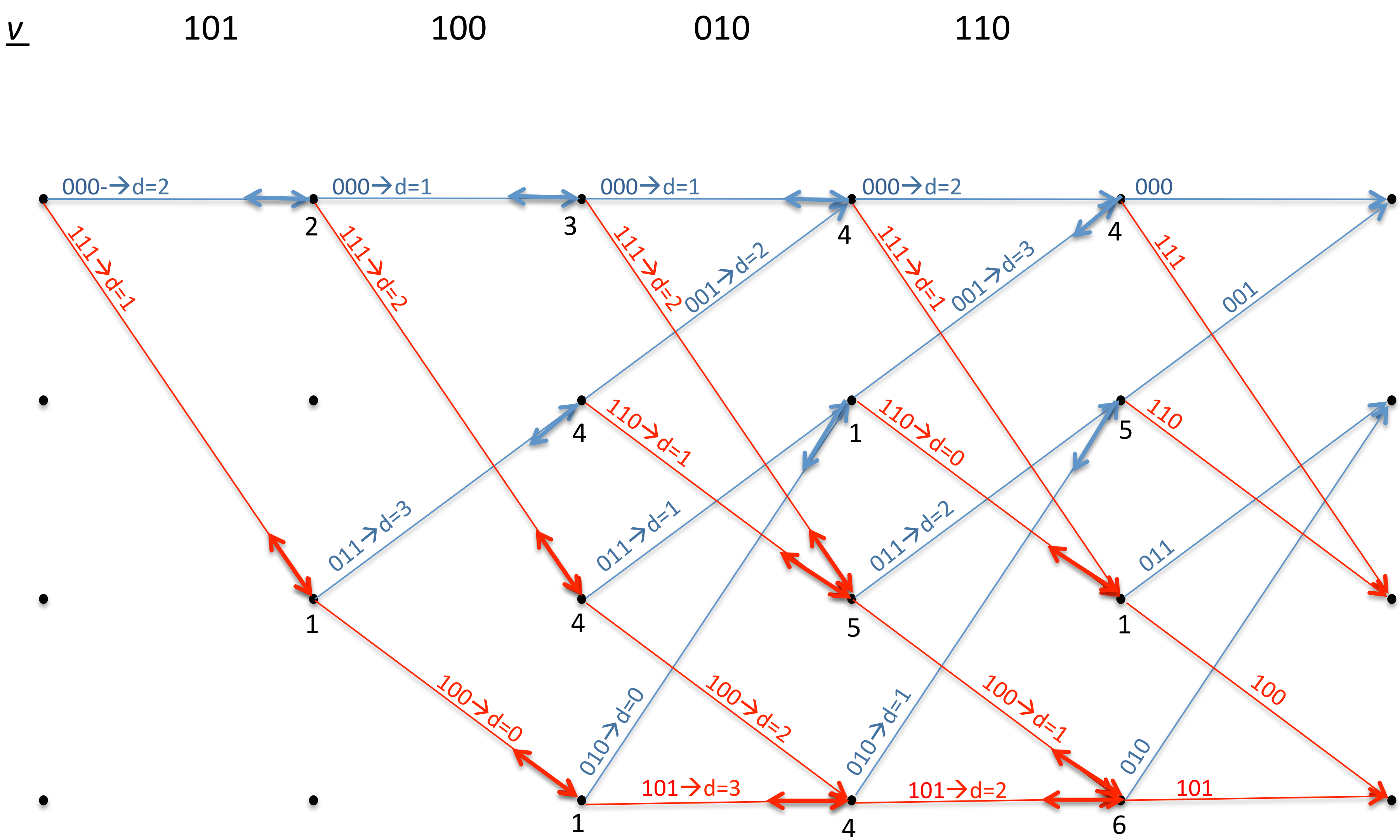


v
101
100
010
110
000-→d=2
000→d=1
000→d=1
000
000
111→d=1
111→d=2
111→d=2
111
111
001→d=2
001
001
110→d=1
110
110
011→d=3
011→d=1
011
011
100→d=0
100→d=2
100
100
010→d=0
010
010
101→d=3
101
101
2
3
4
1
4
1

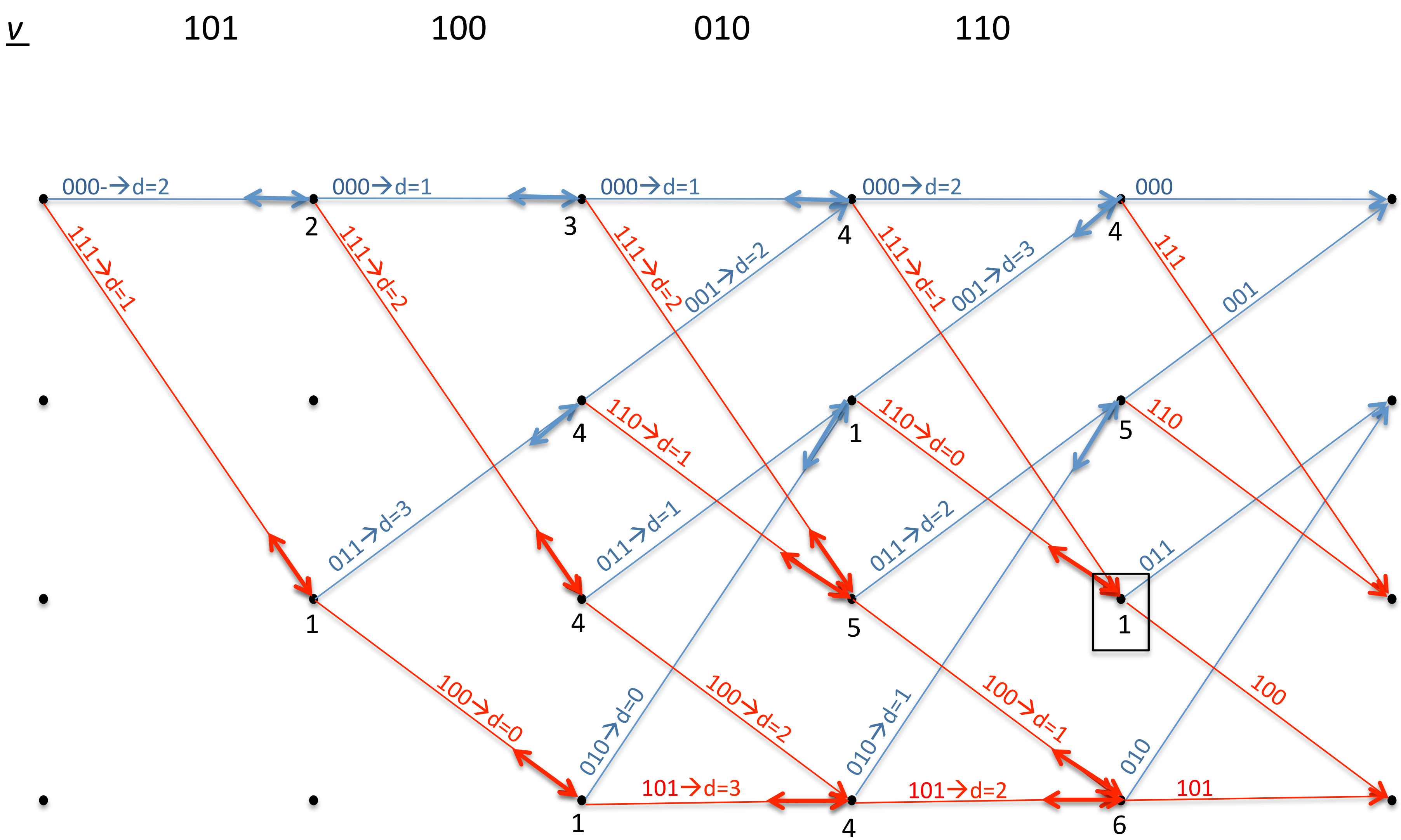

v
101
100
010
110
000-→d=2
000→d=1
000→d=1
000
000
111→d=1
111→d=2
111→d=2
111
111
001→d=2
001
001
110→d=1
110
110
011→d=3
011→d=1
011
011
100→d=0
100→d=2
100
100
010→d=0
010
010
101→d=3
101
101
2
3
4
4
1
1
4
4
5
1
1
4



v
101
100
010
110
000-→d=2
000→d=1
000→d=1
000→d=2
000
111→d=1
111→d=2
111→d=2
111→d=1
111
001→d=2
001→d=3
001
110→d=1
110→d=0
110
011→d=3
011→d=1
011→d=2
011
100→d=0
100→d=2
100→d=1
100
010→d=0
010→d=1
010
101→d=3
101→d=2
101
2
3
4
4
1
1
4
4
5
1
4



v
101
100
010
110
000-→d=2
000→d=1
000→d=1
000→d=2
000
111→d=1
111→d=2
111→d=2
111→d=1
111
001→d=2
001→d=3
001
110→d=1
110→d=0
110
011→d=3
011→d=1
011→d=2
011
100→d=0
100→d=2
100→d=1
100
010→d=0
010→d=1
010
101→d=3
101→d=2
101
2
3
4
4
4
1
5
4
4
1
1
5
1
4
6



v
101
100
010
110
000-→d=2
000→d=1
000→d=1
000→d=2
000
111→d=1
111→d=2
111→d=2
111→d=1
111
001→d=2
001→d=3
001
011→d=3
011→d=1
011→d=2
011
110→d=1
110→d=0
110
100→d=0
100→d=2
100→d=1
100
010→d=0
010→d=1
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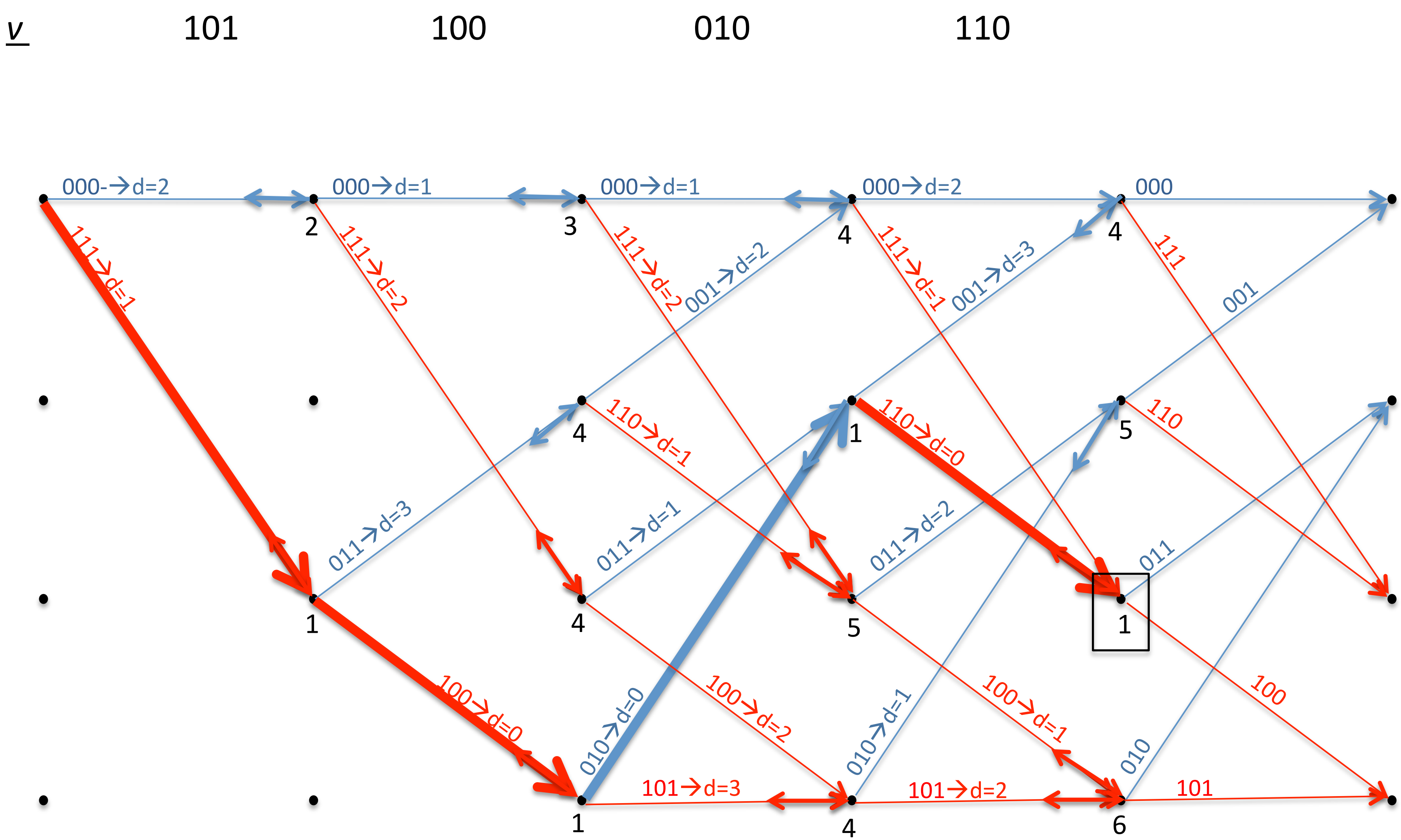
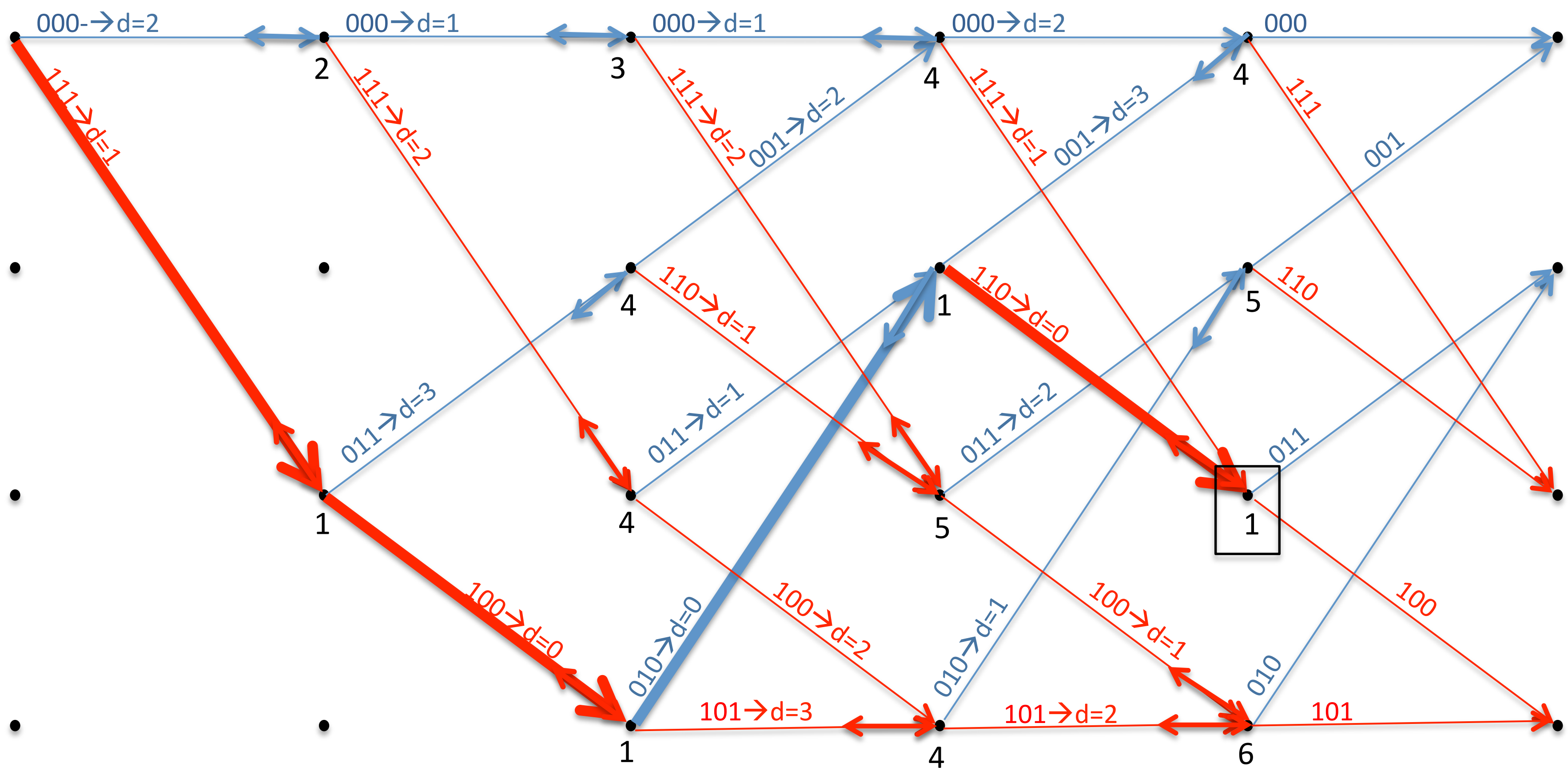
4
1
5
1
4
5
1
1
4
6



v
101
100
010
110
000-→d=2
000→d=1
000→d=1
000→d=2
000
111→d=1
111→d=2
111→d=2
111→d=1
111
001→d=2
001→d=3
001
110→d=1
110→d=0
110
011→d=3
011→d=1
011→d=2
011
100→d=0
100→d=2
100→d=1
100
010→d=0
010→d=1
010
101→d=3
101→d=2
101
2
3
4
4
4
1
5
1
4
5
1
1
4
6

$\underline{v}$	101	100	010	110
$\underline{\hat{c}}$	111	100	010	110
$\underline{\hat{I}}$	1	1	0	1



Probability of error

• A decoding error leads to a wrong path in the trellis

• The error probability over a sequence goes to one as N increases without bounds

• We need to normalize the probability of error to the sequence length. We compute the probability that, at instant *t*, an “error event” begins

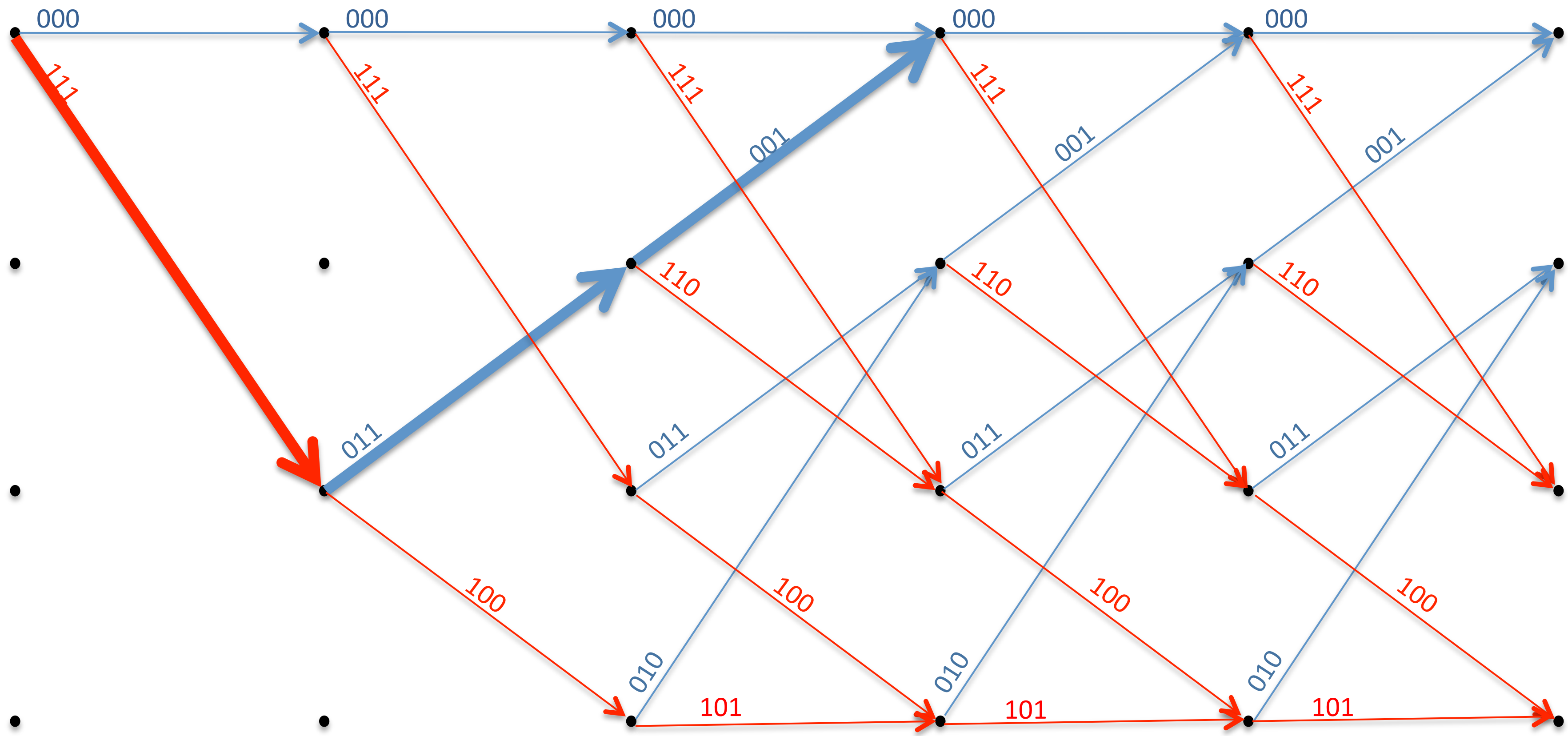
• We then consider error events of finite length, that is, paths that deviate from the correct one only for a short period (more probable errors are between words at small length, which are associated to paths with only short deviations)

• The code is linear, so we assume we send the all-zeros word. The correct path in the trellis is the paths that always stays in the first state

• We study error events that start from the first state and end in the first state, from short ones to longer ones

• We compute the number of errors on the information bits for each error event

$\underline{c}$	000	000	000	000	…
$\underline{\hat{c}}$	111	011	001	000	…
$\underline{\hat{\imath}}$	1	0	0	0	…

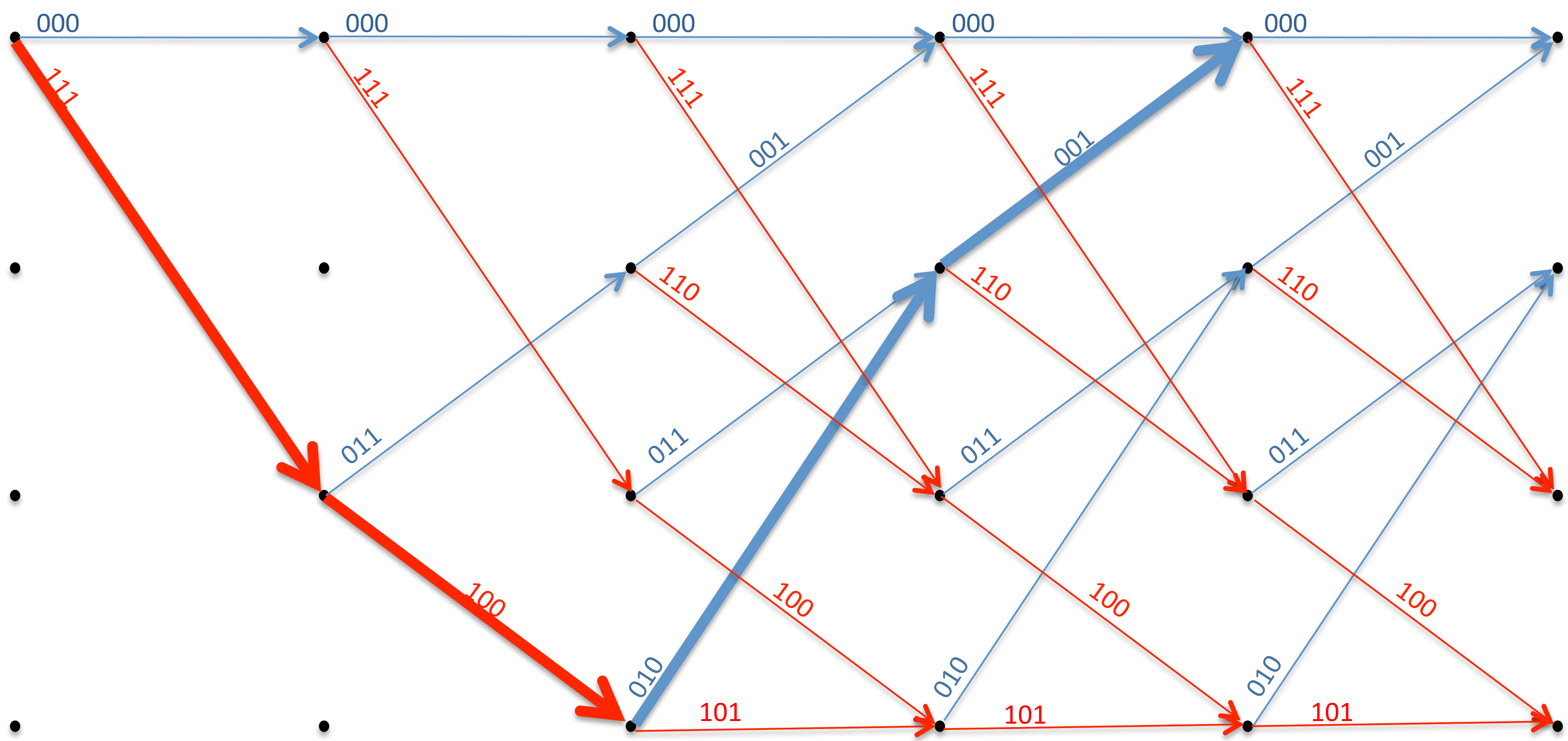


Error event E_1

- Length L=3
- Distance d=6
- Bit errors: 1

$$\Rightarrow Q\left(\sqrt{\frac{2E_s}{N_0}d}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}6R}\right)$$

$\underline{c}$	000	000	000	000	…
$\underline{\hat{c}}$	111	100	010	001	…
$\underline{\hat{i}}$	1	1	0	0	…

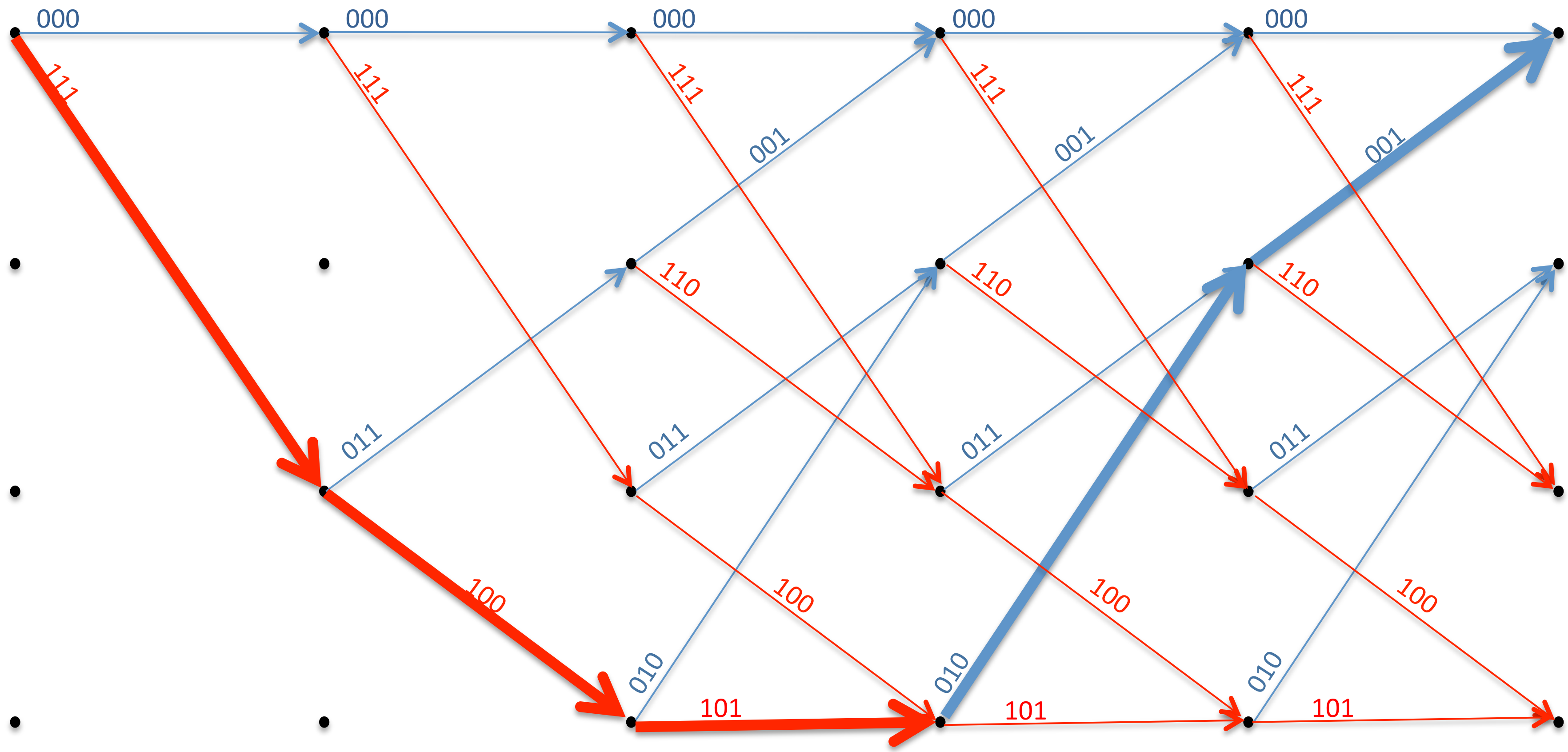


Error event E_2

- Length L=4
- Distance d=6
- Bit errors: 2

$$\Rightarrow Q\left(\sqrt{\frac{2E_s}{N_0}d}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}6R}\right)$$

$\underline{c}$	000	000	000	000	001
$\underline{\hat{c}}$	111	100	101	010	001
$\underline{\hat{\imath}}$	1	1	1	0	0

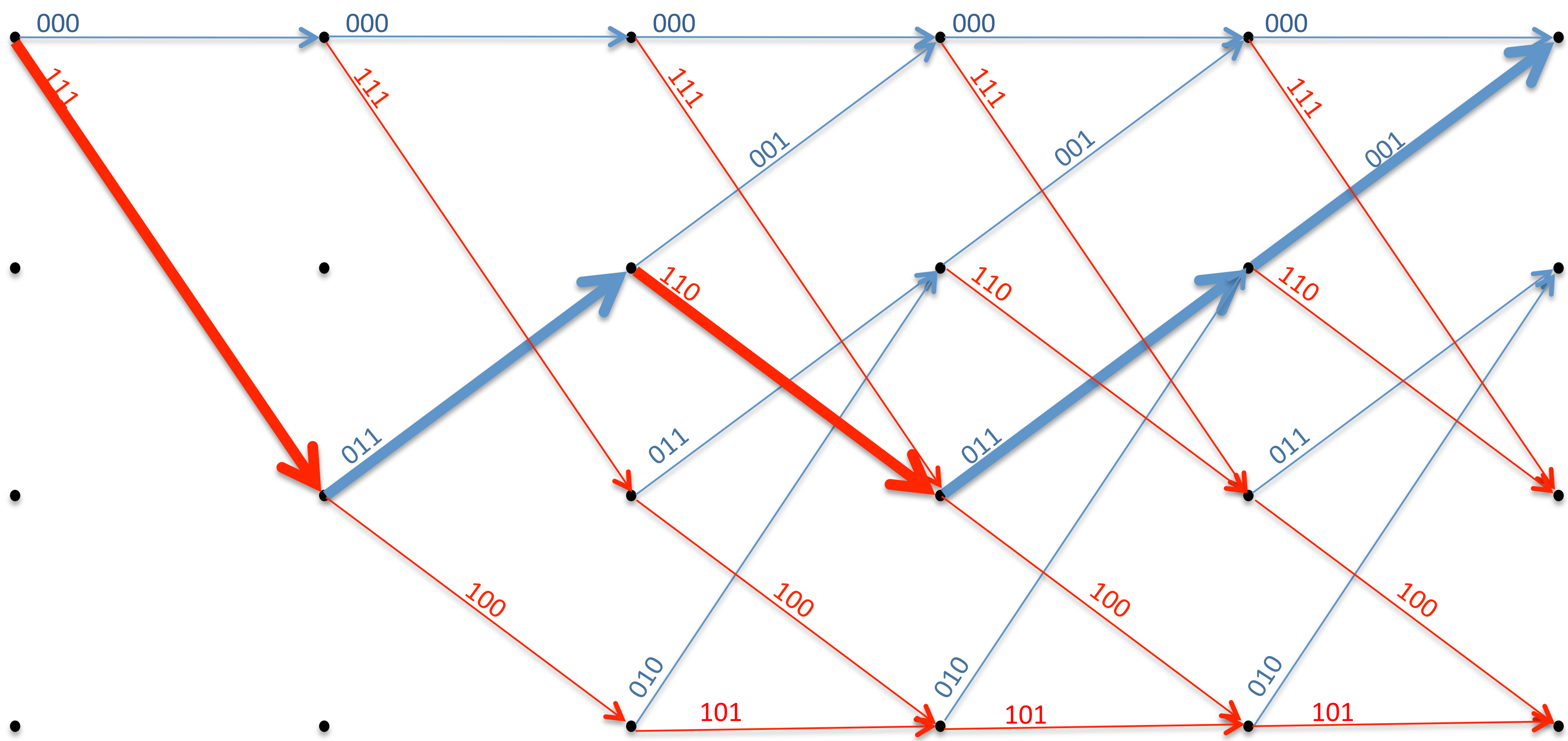


Error event E_3

- Lenght L=5
- Distance d=8
- Bit errors: 3

$$\Rightarrow Q\left(\sqrt{\frac{2E_s}{N_0}d}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}8R}\right)$$

$\underline{c}$	000	000	000	000	001
$\underline{\hat{c}}$	111	011	110	011	001
$\underline{\hat{\imath}}$	1	0	1	0	0



Error event E_4

- Length L=5
- Distance d=10
- Bit errors: 2

$$\Rightarrow Q\left(\sqrt{\frac{2E_s}{N_0}d}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}10R}\right)$$

Error probability

Probability that an error event starts at instant *t* (Union Bound)

$$P(E) \le 2Q\left(\sqrt{\frac{2E_b}{N_0}\cdot 6\cdot\frac{1}{3}}\right)+Q\left(\sqrt{\frac{2E_b}{N_0}\cdot 8\cdot\frac{1}{3}}\right)+Q\left(\sqrt{\frac{2E_b}{N_0}\cdot 10\cdot\frac{1}{3}}\right)\cdots$$

Bit error probability?

For the information bit at instant *t,* we have to consider all error events that lead to an error on that information bit

$$\begin{array}{cccccc}
 & \hat{i}_{t-2} & \hat{i}_{t-1} & \hat{i}_{t} & \hat{i}_{t+1} & \hat{i}_{t+2} \\
E_1(t) & 0 & 0 & 1 & 0 & 0 \\
E_2(t) & 0 & 0 & 1 & 1 & 0 \\
E_2(t-1) & 0 & 1 & 1 & 0 & 0 \\
E_3(t) & 0 & 0 & 1 & 1 & 1 \;\cdots \\
E_3(t-1) & 0 & 1 & 1 & 1 & 0 \\
E_3(t-2) & 1 & 1 & 1 & 0 & 0 \\
E_4(t) & 0 & 0 & 1 & 0 & 1 \\
E_4(t-2) & 1 & 0 & 1 & 0 & 0 \\
 & & \vdots & & &
\end{array}$$

- We have to sum, for the bit at instant *t,* the probability of these error events
- If we consider all the information bits, each error event E_i will be considered as many times as the number of bit errors caused by E_i itself
- The probability of the error events E_i can be bounded using their distance from the correct codeword. Thus, error events with the same distance are estimated with the same bound
- We can group all error events E_i on the base of their distance and compute the number of bit errors caused by them together
- If *w(d)* is the total number of bit errors caused by events at distance *d*, we have

$$P_b(E) \leq \sum_d w(d) Q\left(\sqrt{\frac{2E_b}{N_0} \cdot d \cdot \frac{1}{3}}\right)$$

In our example

$$P_b(E) \leq 3Q\left(\sqrt{\frac{2E_b}{N_0} \cdot 6 \cdot \frac{1}{3}}\right) + Q\left(\sqrt{\frac{2E_b}{N_0} \cdot 8 \cdot \frac{1}{3}}\right) + \cdots$$