20. Use the following property:

$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$$

to show that $[L_x, L_y] = i\hbar L_z$ e that $[L^2, L_x] = 0$.

a) $[L_x, L_y] = i\hbar L_z$

$L_x = -i\hbar(y\partial_z - z\partial_y) = yP_z - zP_y$

$L_y = -i\hbar(z\partial_x - x\partial_z) = zP_x - xP_z$

$L_z = -i\hbar(x\partial_y - y\partial_x) = xP_y - yP_x$

$[L_x, L_y] = [yP_z - zP_y, zP_x - xP_z] = (yP_z - zP_y)(zP_x - xP_z) - (zP_x - xP_z)(yP_z - zP_y) =$

$= yP_z zP_x - yP_z xP_z - zP_y zP_x + zP_y xP_z - (zP_x yP_z - zP_x zP_y - xP_z yP_z + xP_z zP_y) =$

$= yP_z zP_x - \cancel{xyP_z^2} - \cancel{z^2 P_x P_y} + zP_y xP_z - zP_x yP_z + \cancel{z^2 P_x P_y} + \cancel{xyP_z^2} - xP_z zP_y =$

$= yP_x P_z z + xzP_y P_z - yzP_x P_z - xP_y P_z z = yP_x(P_z z - zP_z) + xP_y(zP_z - P_z z) =$

$[A,B] = -[B,A] \longrightarrow zP_z - P_z z = [z, P_z], \quad P_z z - zP_z = -[z, P_z]$

$= -yP_x[z, P_z] + xP_y[z, P_z] = [z, P_z](xP_y - yP_x) = [z, P_z]L_z \underset{[z,P_z] = i\hbar}{=} i\hbar L_z$

$\boxed{[L_x, L_y] = i\hbar L_z}$

a.1) Alternative way

$[A+B, C+D] = [A,C] + [A,D] + [B,C] + [B,D]$

$[L_x, L_y] = [yP_z - zP_y, zP_x - xP_z] = [yP_z, zP_x] - [yP_z, xP_z] - [zP_y, zP_x] + [zP_y, xP_z] =$

$= yP_x[P_z, z] - xy\cancel{[P_z, P_z]}^{0} - P_x P_y \cancel{[z, z]}^{0} + xP_y[z, P_z] = [z, P_z](xP_y - yP_x) =$

$= i\hbar L_z$

b) $[L^2, L_x] = 0$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z$$

$$[L^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x] = [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] =$$

$$= [L_x L_x, L_x] + [L_y L_y, L_x] + [L_z L_z, L_x] = L_x \cancel{[L_x, L_x]}^{0} + \cancel{[L_x, L_x]}^{0} L_x +$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$+ L_y [L_y, L_x] + [L_y, L_x] L_y + L_z [L_z, L_x] + [L_z, L_x] L_z = L_y(-i\hbar L_z) + (-i\hbar L_z) L_y +$$

$$+ L_z(i\hbar L_y) + (i\hbar L_y) L_z = -i\hbar L_y L_z - i\hbar L_y L_z + i\hbar L_y L_z + i\hbar L_y L_z = 0$$

$$\boxed{[L^2, L_x] = 0}$$

21. Quantum numbers for macroscopic systems (*Bohr's Correspondence Principle: classical physics is recovered in the limit of large quantum #s*).

a. Estimate the angular momentum quantum number "n" for a Ferries wheel using the Bohr condition for the quantisation of the angular momentum. (Use your own parameters set for the estimate.)

b. The energy of a harmonic oscillator (i.e. mass and spring) are quantised via

$$E_n = \hbar\omega_o(n + \frac{1}{2})$$

where $\omega = \sqrt{k/m}$ is the natural frequency of oscillation. If you push down the back end of a car, it will spring back. Estimate the quantum number associated with this classical motion. (Use your own parameters set for the estimate).

a) We need to calculate the classical angular momentum.

During Christmas season in the capital of my region Donostia (basque name) or San Sebastián (spanish) the city council puts a Ferris wheel in Alderdi Eder square, in front of La Concha beach.

The parameters I have chosen are:

- diameter = 40m
- 1 round per minute
- weight = 60 kg

$$v = \omega \cdot R = \cancel{2}\pi \cdot \frac{1}{6\cancel{0}} \cdot \frac{4\cancel{0}}{\cancel{2}} = \frac{2\pi}{3} \text{ m/s}$$

$$\bar{L} = \vec{r} \wedge \vec{p} = |r||mv| \sin\left(\frac{\pi}{2}\right) \hat{u} = \frac{40}{\cancel{2}} \cdot 60 \cdot \frac{2\pi}{3} \hat{u} =$$

$$\Rightarrow \bar{L} = 800\pi \, \hat{u}$$

Now, according to Bohr: $L_\theta = n\hbar$

$$L_\theta = |\vec{L}| \Rightarrow n = \frac{800\pi}{\hbar} = \frac{800\pi}{\frac{h}{2\pi}} = \frac{1600\pi^2}{h} = \frac{1600\pi^2}{6{,}63 \cdot 10^{-34}} \Rightarrow \boxed{n = 2{,}4 \cdot 10^{38}}$$

b) The best-selling car in Italy is the Fiat Pand and the pushing force of a person is the equivalent of 80% of their body weight (according to a poorly checked Google search). We have:

Medidas Fiat Panda 2021.

$m_{person} = 80\ kg$ $\quad m_{car} = 1110\ kg$

wheels compression $= 0{,}5cm$

$F = kx \Rightarrow 80 \cdot 0{,}8 \cdot 9{,}8 = K \cdot 0{,}05 \Rightarrow K = 12544\ N/m$

In the classical harmonic oscillator the maximum energy corresponds to the maximum potential energy, therefore:

$$E_{P_{max}} = \frac{1}{2} K A^2 = \frac{1}{2} 12544 \cdot 0{,}05^2 = 15{,}68\ J$$

Using the result of the energy in the quantized harmonic oscillator:

$$E_n = \hbar \omega_0 \left(n + \frac{1}{2}\right) \Rightarrow n = \frac{E_n}{\hbar \omega_0} - \frac{1}{2} = \frac{E_n}{\hbar}\sqrt{\frac{m_{car}}{K}} - \frac{1}{2} = \frac{15{,}68}{1{,}054 \cdot 10^{-34}}\sqrt{\frac{1110}{12544}} - \frac{1}{2}$$

$$\boxed{n = 4{,}43 \cdot 10^{34}}$$

22. How many possible orientations may the angular momentum of a rotating wheel (m=10 g, r=50 cm, T=1 sec) take?

How many possible angular momentum states do we span when turning the wheel's hub by 10 degrees. Compare with the above exercise (22) and briefly explain.

a) First we calculate the classical angular momentum:

$$v = \omega \cdot r \Rightarrow |\bar{L}| = |\bar{r}||\bar{p}| = m \omega r^2 = m \frac{2\pi}{T} r^2 = 0{,}01 \cdot \frac{2\pi}{1} 0{,}5^2$$

$$|\vec{L}| = 0{,}016 \ \text{kg m}^2/\text{s}$$

The quantized L is: $L = \hbar \sqrt{\ell(\ell+1)}$

$$\Rightarrow \ell(\ell+1) = \frac{L}{\hbar} = \frac{0{,}016}{1{,}054 \cdot 10^{-34}} = 1{,}52 \cdot 10^{32} \Rightarrow \ell^2 + \ell - 1{,}52 \cdot 10^{32} = 0$$

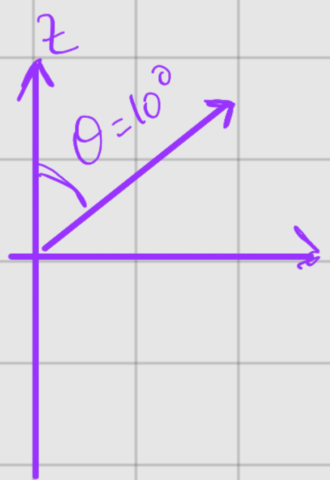
ℓ goes from 0 to n-1 so it cannot be negative

$$\ell = \frac{1}{2}\left(-1 + \sqrt{1^2 + 4 \cdot 1 \cdot 1{,}52 \cdot 10^{32}}\right) \simeq 1{,}23 \cdot 10^{16}$$

The number of possible values for m: # integers of m = $2\ell + 1$

$$\boxed{\# \text{Orientations} = 2{,}5 \cdot 10^{16}}$$

b)



For 180° we have all the states so for 10° we will have:

$$\# \text{span states} = \frac{10 \cdot \# \text{orientations}}{180}$$

$$\boxed{\# \text{span states} = 1{,}4 \cdot 10^{15}}$$

23. Show all possible orientations of the electron orbital angular momentum in the energy level n=3 (l=0,1,2). Give all the angles the orbital magnetic moment μ forms about a given direction (z).

We have that $l = 0, \dots, n-1 \Rightarrow n=3 \Rightarrow l = 0, 1, 2$

And the electron orbital angular momentum:

$|\bar{\mu}| = \mu_B\sqrt{l(l+1)}$ and $\bar{\mu} \cdot \hat{z} = -\mu_B m_l = \mu_z$

- For $l=0 \Rightarrow m_l = 0$

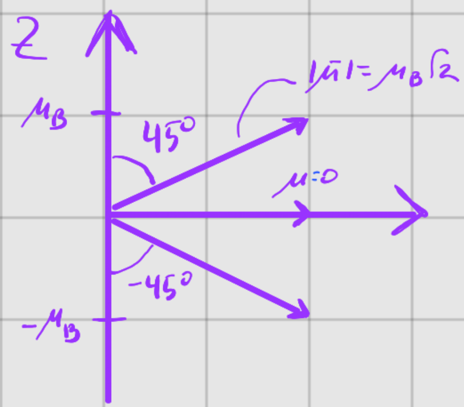
$|\bar{\mu}| = \mu_B\sqrt{l(l+1)} \Rightarrow |\bar{\mu}| = 0$

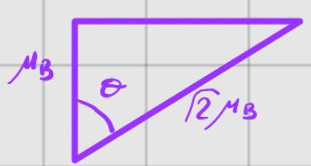
$\vec{\mu}_z = 0$

- For $l=1 \Rightarrow m_l = +1, 0, -1$

$|\bar{\mu}| = \mu_B\sqrt{2}$

$$\mu_z = \begin{cases} -\mu_B \\ 0 \\ \mu_B \end{cases}$$



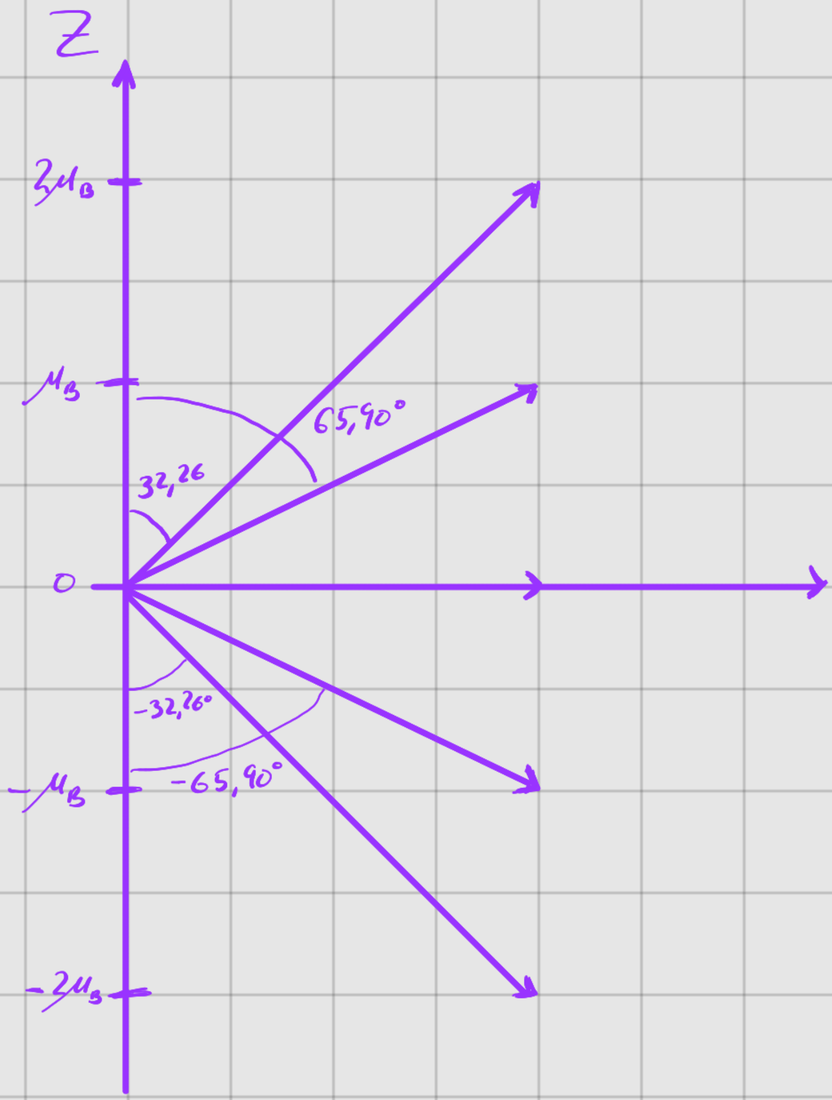


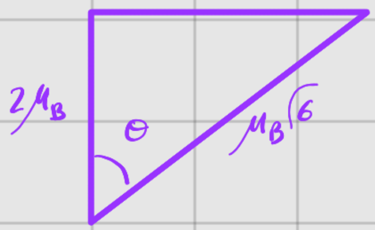
$\cos\theta = \frac{\mu_B}{\sqrt{2}\mu_B} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = 45^\circ}$

o) For $l = 2 \Rightarrow m_l = +2, +1, 0, -1, -2$

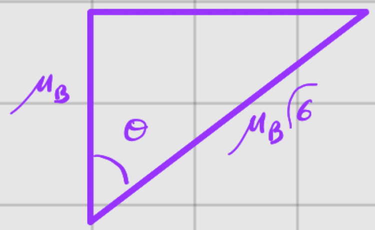
$|\vec{\mu}| = \mu_B \sqrt{6}$

$$\mu_z = \begin{cases} -2\mu_B \\ -\mu_B \\ 0 \\ +\mu_B \\ +2\mu_B \end{cases}$$





$$\cos\theta = \frac{2\mu_B}{\sqrt{6}\,\mu_B} \Rightarrow \boxed{\theta = 32{,}26^\circ}$$



$$\cos\theta = \frac{\mu_B}{\sqrt{6}\,\mu_B} \Rightarrow \boxed{\theta = 65{,}90^\circ}$$