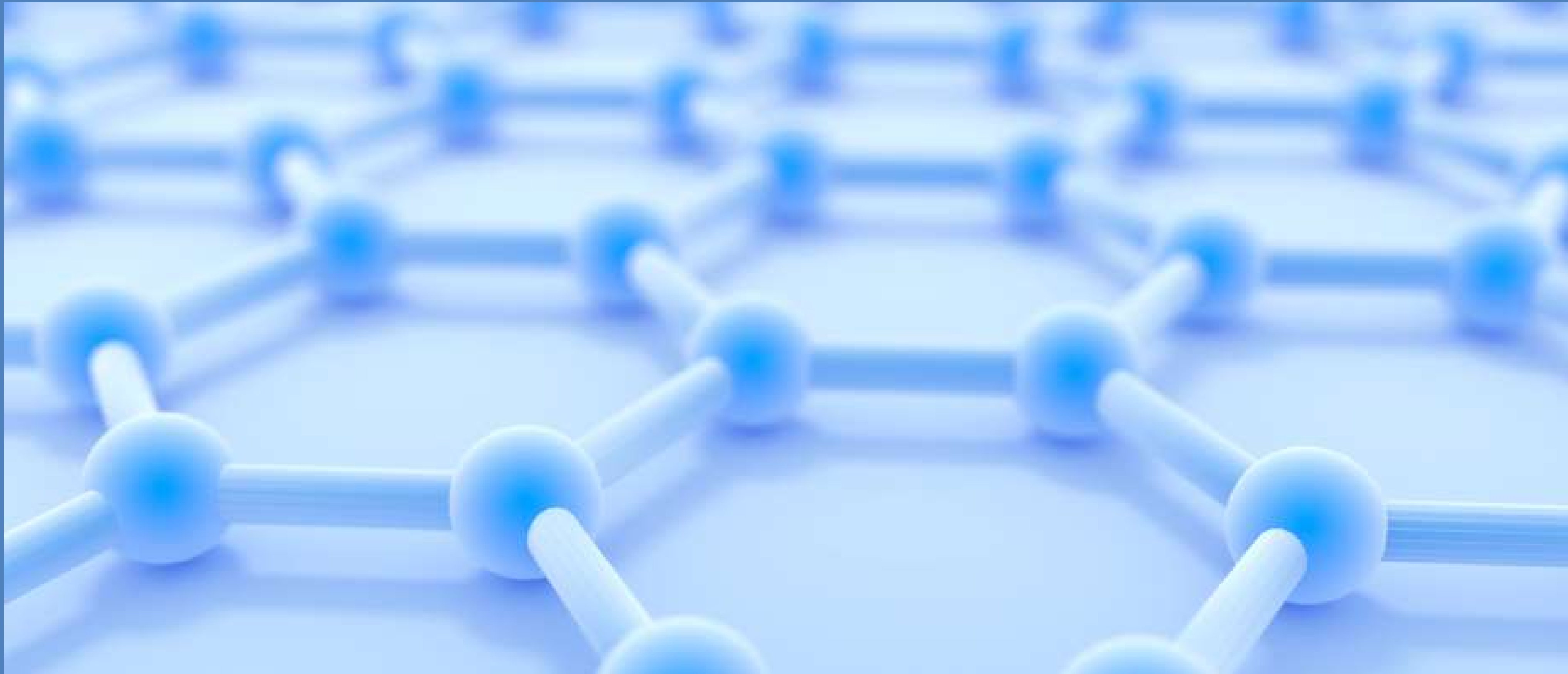

Photonics
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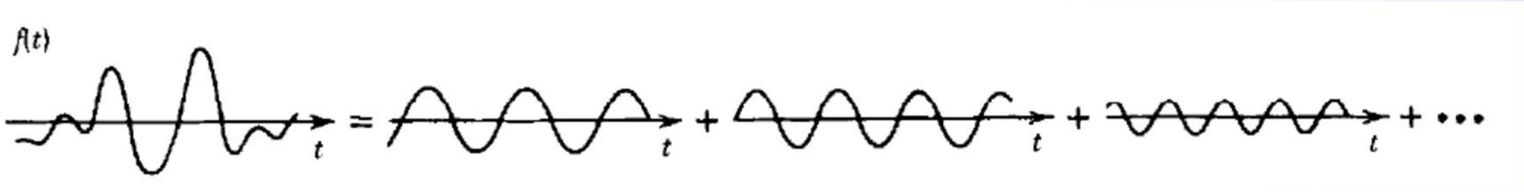
Fourier Optics:
Propagation in Free space and Optical Fourier Transform

Fourier Optics

Provides a description of the propagation of light waves based on the harmonic analysis, which is based on the expansion of an arbitrary function $f(t)$ in a superposition of harmonic functions of time of different frequencies $[F(\nu)exp(-j2\pi\nu t)]$, where $F(\nu)$ is the Fourier transform of $f(t)$.



Similar considerations can be applied to an arbitrary function of space $f(x,y)$, which can be written as a superposition of harmonic functions of x and y, each of the form $F(\nu_x, \nu_y)exp[-j2\pi(\nu_x x+ \nu_y y)]$, where ν_x and ν_y are the spatial frequencies.

These concepts can be easily transferred into wave optics considering the plane wave written as:

$$U(x,y,z) = Aexp[-j(k_x x + k_y y + k_z z)]$$

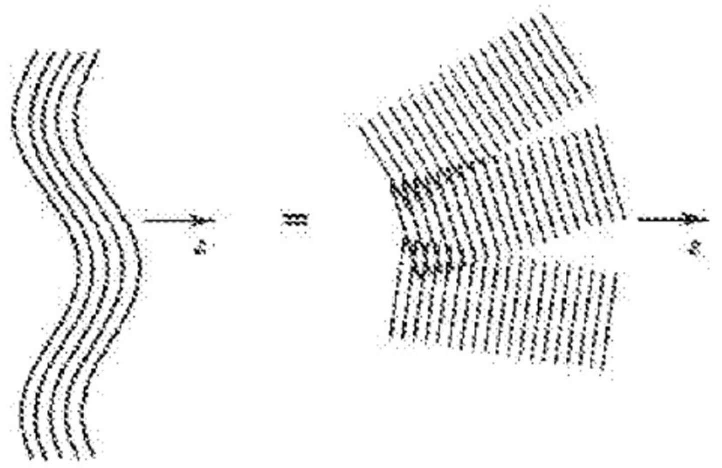
With k_x, k_y and k_z components of the wavevector ($k=2\pi/\lambda$) and A complex amplitude.

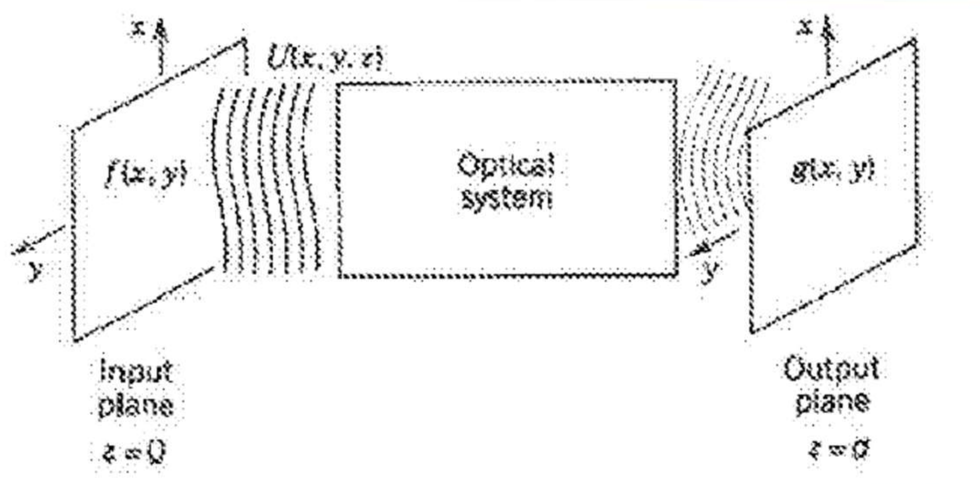
Fourier Optics

In other words, an **arbitrary traveling wave** can be analyzed as a **sum of plane waves**.

Moreover, if it's known how a system modifies the plane wave, the principle of superposition can be used to determine the effect of the system on an arbitrary wave. For this reason, is very useful to describe the propagation of light through linear optical components.

Any system will be characterized either by its **impulse response function** or by its **transfer function**.





Free space propagation

Let's consider the plane wave:

$$U(x, y, z) = Aexp[-j(k_x x + k_y y + k_z z)]$$

With wavevector **k**=(k_x,k_y, k_z), wavelength λ, wavenumber $k = \frac{2\pi}{\lambda} = \sqrt{k_x^2 + k_y^2 + k_z^2}$ and A complex envelope.

The vector **k** makes an angle $\theta_x = \sin^{-1}(k_x/k)$ and $\theta_y = \sin^{-1}(k_y/k)$ with the y-z and x-z planes. If $\theta_x = 0$, it means there is no component of **k** in the x direction.

Given a specific plane, for example z=0, the complex amplitude U(x,y,0) is a spatial harmonic function

$$f(x, y) = Aexp[-j2\pi(\nu_x x + \nu_y y)]$$

Where $\nu_x = \frac{k_x}{2\pi}$ and $\nu_y = \frac{k_y}{2\pi}$ are the spatial frequencies.

Free space propagation

The angles of the wavevector are related to the spatial frequencies so that:

$$\boxed{\theta_x = \sin^{-1}(\lambda\nu_x),\ \theta_y = \sin^{-1}(\lambda\nu_y)}$$

If we define the periods of the harmonic function as $\Lambda_x = 1/\nu_x$ and $\Lambda_y = 1/\nu_y$ we can see that the angles of the wavevector are therefore ruled by the ratio of the wavelength over the period of the harmonic function in each direction.
If k_x<<k and k_y<<k so that the wavevector is paraxial then the wavevector angles reduce to:

$$\boxed{\theta_x \approx \lambda/\Lambda_x, \quad \theta_y \approx \lambda/\Lambda_y}$$

Paraxial approximation

The one-on-one relation between the wave and harmonic function so that one can be constructed from the other according to the relation:

$$U(x,y,z) = f(x,y)exp[-jk_z z], \qquad k_z = \mp\sqrt{k^2 - k_x^2 - k_y^2}$$

NOTE: for this condition to be satisfied the root must be real.
NOTE 2: The sign of k_z depends on the direction of propagation.

Spatial spectral analysis

A thin optical element with complex amplitude transmittance :

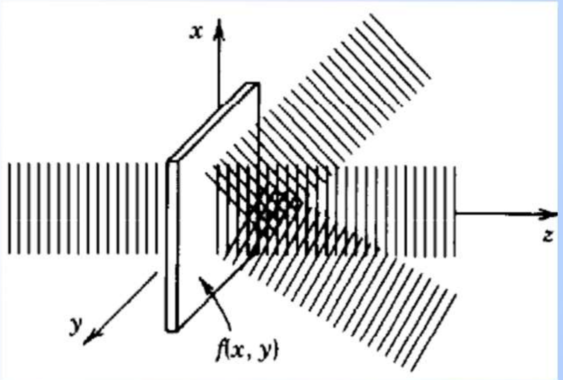
$$\boxed{f(x,y) = A exp[-j2\pi(\nu_x x + \nu_y y)]}$$

Transforms an incident plane wave of unitary amplitude in two plane waves traveling at angles $\theta_x = \sin^{-1}(\lambda\nu_x)$ and $\theta_y = \sin^{-1}(\lambda\nu_y)$.
Therefore, the thin optical element acts as a **prism**.

More generally, if the complex amplitude transmittance is a superposition integral of harmonic functions

Fourier transform of f(x,y)

$$f(x,y) = \iint_{-\infty}^{\infty} F(\nu_x,\nu_y)\, exp[-j2\pi(\nu_x x + \nu_y y)] d\nu_x d\nu_y$$



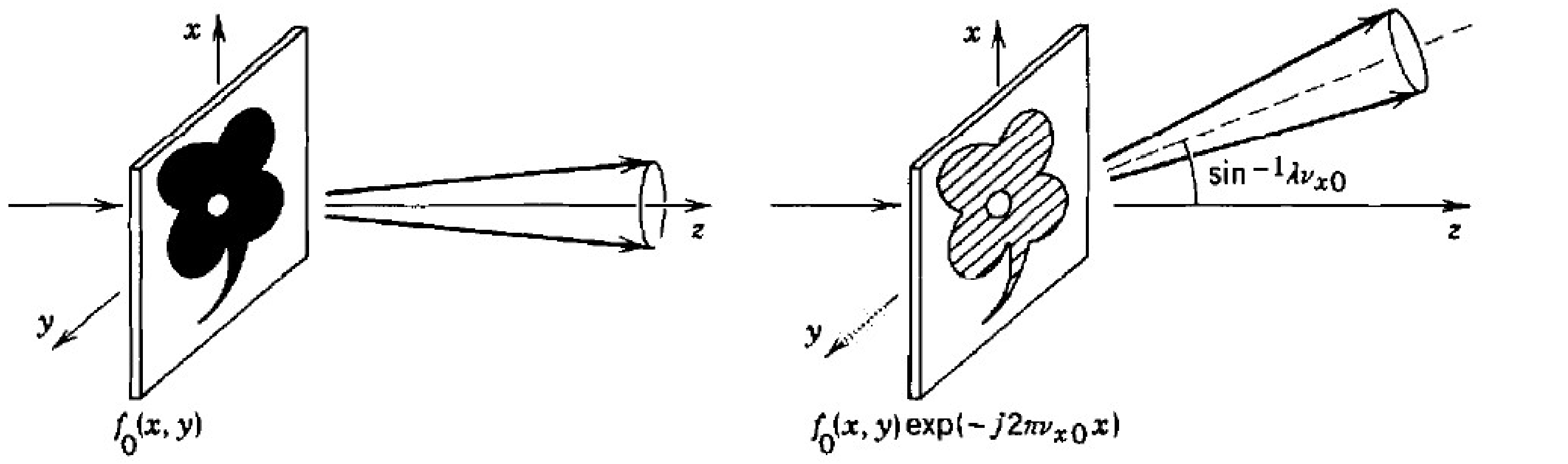
The transmitted waves that form from an incident plane wave of unitary amplitude traveling in the z direction will be the superposition of plane waves:

$$U(x,y,z) = \iint_{-\infty}^{\infty} F(\nu_x,\nu_y)\, exp[-j2\pi(\nu_x x + \nu_y y)] \exp(-jk_z z) d\nu_x d\nu_y$$

Amplitude Modulation

Let us now consider a transparency with complex amplitude transmittance $f_0(x, y)$. This optical element will deflect the plane wave according to its spatial frequencies in the range of angles $\Delta\theta_x = \sin^{-1}(\lambda\Delta\nu_x)$ and $\Delta\theta_y = \sin^{-1}(\lambda\Delta\nu_y)$.

If we now have a second transparency with complex amplitude transmittance $f(x, y) = f_0(x, y)exp[-j2\pi\nu_{x0}x]$ and assume that $f_0(x, y)$ is slowly varying (i.e., $\Delta\nu_x \ll \nu_{x0}$) we can consider $f(x, y)$ as an amplitude modulated function with carrier frequency ν_{x0} and modulation function $f_0(x, y)$. That means that the transparency will deflect the incoming plane wave by an angle $\theta_{x0} = \sin^{-1}(\lambda\nu_{x0})$.

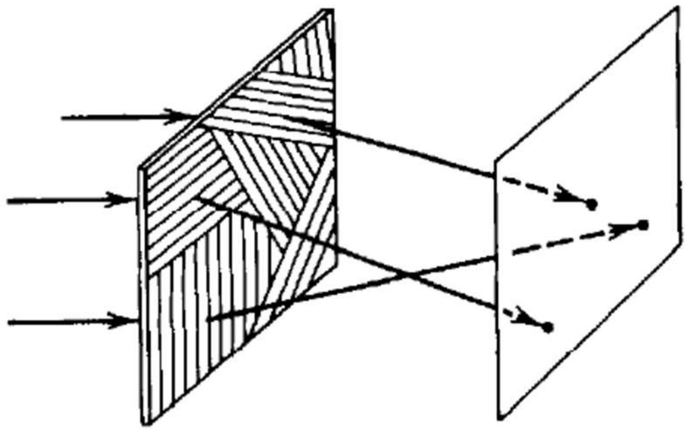


The idea of creating two images from the same optical element is called **spatial-frequency multiplexing** and is extremely useful to create **holograms**.

Frequency Modulation

If a plane wave hits a transparency where different regions have different complex transmittances and if the dimensions of those regions are much greater than the period, then each element act as a grating deflecting the plane wave of a different angle.

The variation in the transparency/optical elements can also be continuous and slow with position when compared to λ, in the same way the frequency of a FM signal varies with time.



Frequency Modulation

If we have for example a complex amplitude transmittance $f(x,y) = exp[-j2\pi\Phi(x,y)]$, in the neighborhood of a point (x_0, y_0) we can use the Taylor expansion for the function

$$\Phi(x,y) \approx \Phi(x_0,y_0) + (x - x_0)\nu_x + (y - y_0)\nu_y$$

Where $\nu_x = \frac{\delta\Phi}{\delta x}$ and $\nu_y = \frac{\delta\Phi}{\delta y}$ are evaluated at (x_0, y_0).

The local variation of $f(x,y)$ is therefore proportional to the quantity $exp[-j2\pi(\nu_x x + \nu_y y)]$.

The transparency $f(x,y) = exp[-j2\pi\Phi(x,y)]$ then deflects the portion of the wave at the position (x,y) by the position dependent angles:

$$\theta_x = \sin^{-1}(\lambda\frac{\delta\Phi}{\delta x}) \text{ and } \theta_y = \sin^{-1}(\lambda\frac{\delta\Phi}{\delta y})$$

Fresnel zone plate

A Fresnel zone plate is binary plate with circularly symmetric transparency of complex amplitude transmittance:

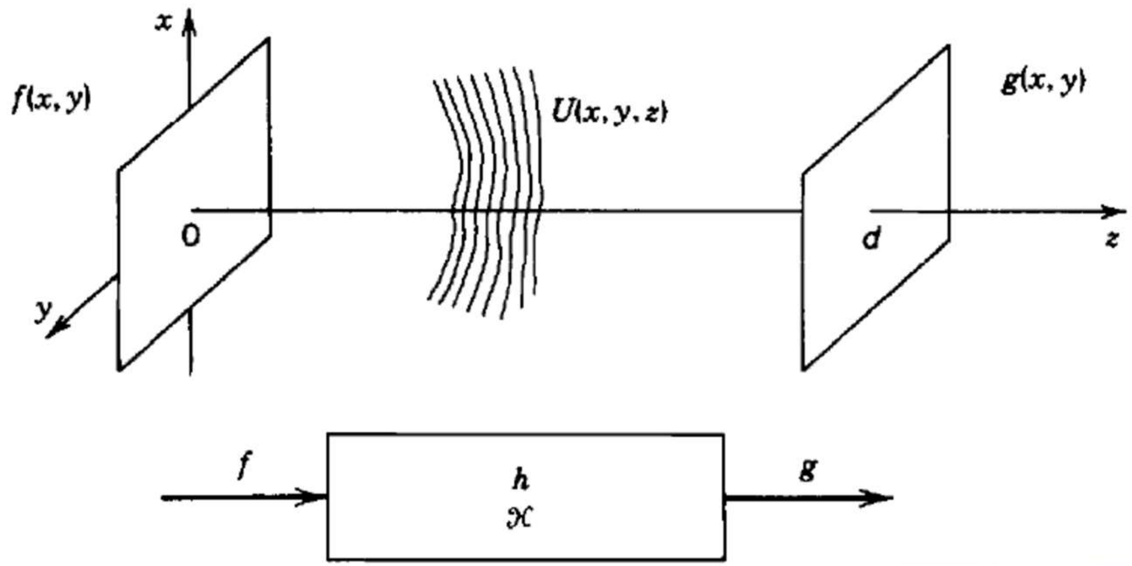
$$f(x,y) = \begin{cases} 1, & if \cos\left(\pi \dfrac{x^2+y^2}{\lambda f}\right) > 0 \\ 0, & otherwise \end{cases}$$

This structure serves as a spherical lens with multifocal lengths: a ray incident at one point is split into multiple rays and the transmitted rays meet at multiple foci with focal lengths $\pm f, \pm \frac{f}{2}, \ldots$

There is also one component that transmitted but it is not deflected.

Transfer function of free space

Let's try to determine how a monochromatic plane wave of wavelength λ propagates in free space between two planes z=0 and z=d. In other words, we want to find the function $g(x, y) = U(x, y, d)$ knowing that the input function is $f(x, y) = U(x, y, 0)$.
We assume that the system is linear and, therefore, $U(x, y, z)$ must satisfy the Helmholtz equation. Since this system is linear it can be characterized by the impulse response function $h(x, y)$ or by its transfer function $H(\nu_x, \nu_y)$.



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Transfer function of free space

Let us consider a harmonic input function $f(x,y) = Aexp[-j(k_x x + k_y y)]$.
Where $k_x = 2\pi\nu_x$, $k_y = 2\pi\nu_y$.

The output is $g(x,y) = Aexp[-j(k_x x + k_y y + k_z d)]$, where $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$.

From this follows that the **transfer function of free space** is:

$$\boxed{H(\nu_x,\nu_y) = \frac{g(x,y)}{f(x,y)} = exp(-jk_z d) = exp\left(-j2\pi d\sqrt{\lambda^{-2} - \nu_x^2 - \nu_y^2}\right)}$$

NOTE: $H(\nu_x,\nu_y)$ is a circularly symmetric complex function. The square root can be real, or imaginary depending on the value or the spatial frequencies and represents either a **propagating or an evanescent wave**, respectively.

NOTE 2: For this function **λ^{-1} represents a cutoff spatial frequency**. In other words, features contained in spatial frequencies greater than λ^{-1} (details finer than λ) cannot be transmitted by a wave of wavelength λ over a distance greater than λ.

Fresnel Approximation

If the input function $f(x, y)$ contains only spatial frequencies that are under much smaller than the cutoff λ^{-1}, that is $\lambda^{-2} \gg \nu_x^2 + \nu_y^2$, we can say that the **paraxial approximation is valid** since all components make small angles:

$$\theta_x \approx \lambda\nu_x \text{ and } \theta_y \approx \lambda\nu_y$$

So if we denote $\theta = \theta_x^2 + \theta_y^2 \approx \lambda^2 (\nu_x^2 + \nu_y^2)$ the free space transfer function can be re-written as:

$$H(\nu_x, \nu_y) = exp\left(-j2\pi d\sqrt{\lambda^{-2} - \nu_x^2 - \nu_y^2}\right) =$$

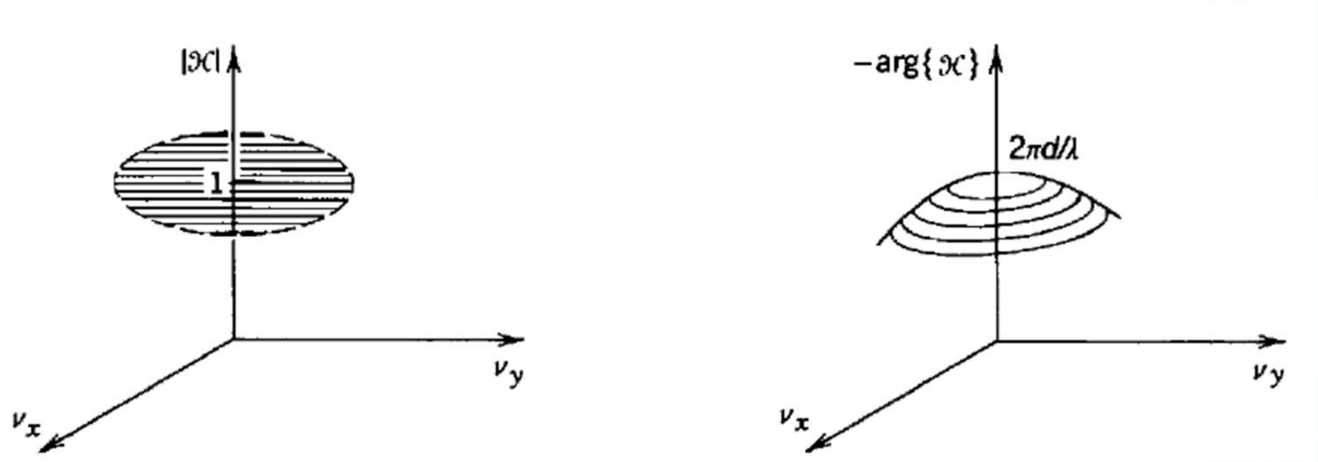
$$= exp\left(-j2\pi\frac{d}{\lambda}\sqrt{1-\theta^2}\right) \approx H_0 exp\left[-j\pi\lambda d\left(\nu_x^2 + \nu_y^2\right)\right].$$

From series expansion at the second term

Where $H_0 = exp[-jkd]$.

Fresnel Approximation

NOTE: The free space transfer function in the Fresnel approximation has constant magnitude and a quadratic phase.



NOTE 2: The approximation is valid only if the third term of the expansion is much smaller than π , which is equivalent to $(\theta^4 d)/4\lambda \ll 1$.
If a is the largest radial distance in the output plane, the largest angle $\theta_m \approx a/d$ and the condition of applicability of the series truncation become:

$$\boxed{\frac{N_F \theta_m^2}{4} \ll 1}$$

Fresnel approximation condition of validity

$$\boxed{N_F = \frac{a^2}{\lambda d}}$$

Fresnel number

Impulse response function of free space

The impulse response function $h(x, y)$ is the response $g(x, y)$ when the input $f(x, y)$ is a point at the origin (0,0) and it is, therefore, the inverse Fourier transform of $H(\nu_x, \nu_y)$, which is:

$$\boxed{h(x, y) \approx h_0 exp\left[-jk\frac{x^2+y^2}{2d}\right],}$$

Where $h_0 = \left(\frac{j}{\lambda d}\right)\exp(-jkd)$.

An alternative procedure to relate the complex amplitudes $g(x, y)$ and $f(x, y)$ is to regard the input as a superposition of delta functions, each producing a paraboloidal wave.
The sum of these contributions is the two-dimensional convolutions that returns the function $g(x, y)$.

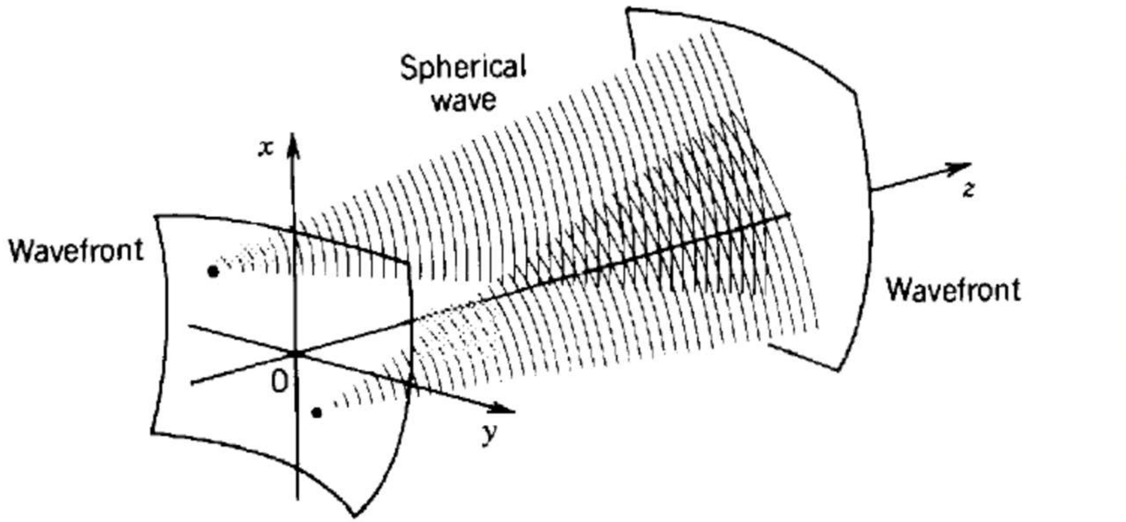
Huygens-Fresnel principle

The Huygens-Fresnel principle states that each point on a wave front generates a spherical wave. The envelope of this secondary waves constitutes a new wave front. Their superposition is a wave in another plane.

The system's impulse response function for propagation between the planes z=0 and z=d is:

$$h(x,y) \propto \frac{1}{r}\exp(-jkr), \qquad r = \sqrt{x^2+y^2+d^2}$$

In the paraxial approximation the spherical wave is approximated by the paraboloidal wave.



Optical Fourier Transform

As described in the spatial spectral analysis section we can use an optical element, such as a transparency, to compute the Fourier Transform with light.

More specifically, it can be shown that if a plane d is sufficiently far from the input the only plane wave that contributes to the complex amplitude at a point (x,y) in the output plane is the wave traveling with an angle $\theta_x \approx \frac{x}{d}$ and $\theta_y \approx \frac{y}{d}$ with the optical axis. This is the wave with wave vector components $k_x \approx k\frac{x}{d}$ and $k_y \approx k\frac{y}{d}$ and amplitudes $F(\nu_x, \nu_y)$, where $\nu_x = \frac{x}{\lambda d}$ and $\nu_y = \frac{y}{\lambda \mathrm{d}}$.

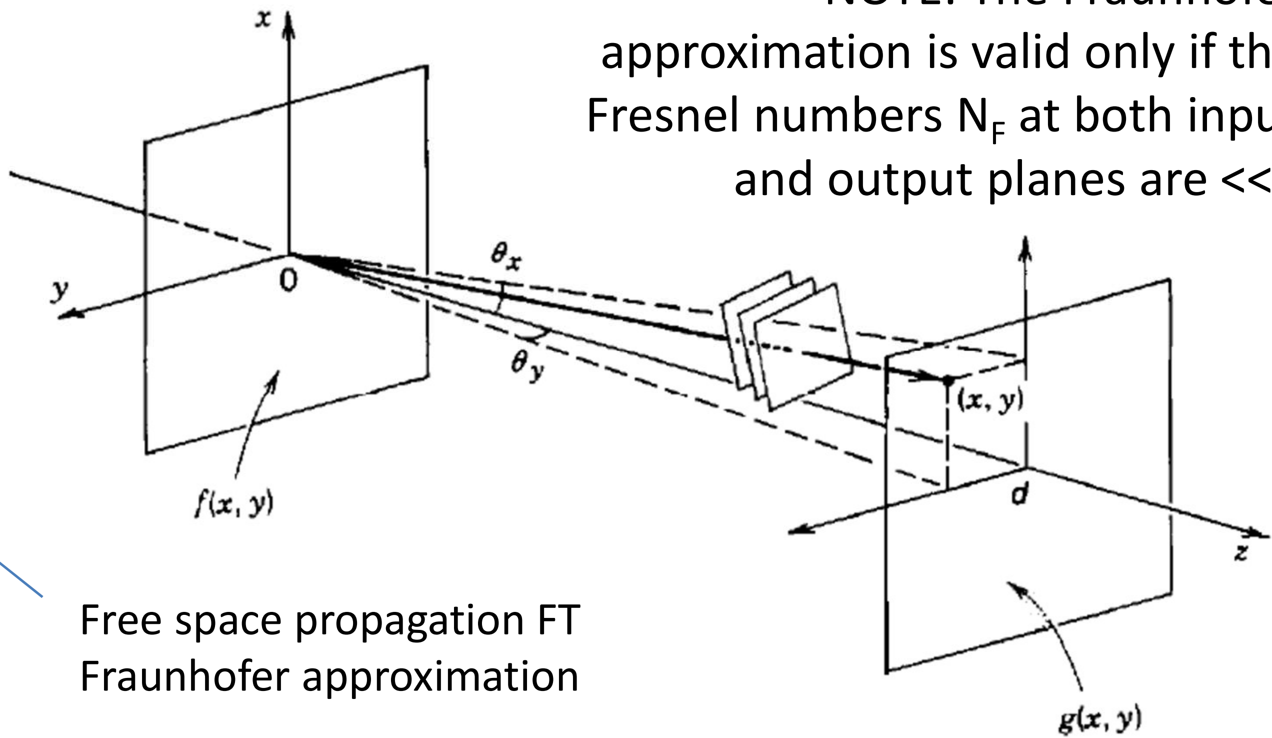
NOTE: The Fraunhofer approximation is valid only if the Fresnel numbers N_F at both input and output planes are <<1

The complex amplitudes at the Input and output plane are then Related by:

$$\boxed{g(x,y) \approx h_0 F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right),}$$

Where $F(\nu_x, \nu_y)$, is the Fourier transform of $f(x,y)$ and

$$h_0 = \frac{j}{\lambda d}\exp(-jkd)$$



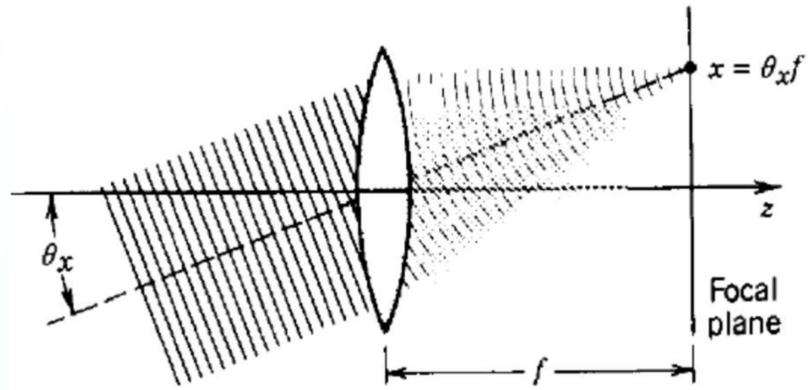
Free space propagation FT
Fraunhofer approximation

Optical Fourier Transform: Lenses

The plane wave components can also be separated using a lens: a think spherical lens transforms a plane wave into a paraboloidal wave focused on a point in the lens focal plane.

Therefore, one can say that the lens maps each direction into a single point on the focal plane thus separating the contribution of the different plane waves traveling in a different direction.

If we assume all the waves are paraxial, we can
Write the complex amplitude on the output plane
As the superposition of all contributions that pass through the lens.
Using the Fresnel approximation, at the focal plane in z=f we can write:



$$g(x,y) = h_l F\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right),$$

Where:

$$h_l = \left(\frac{j}{\lambda f}\right) exp(-j2kf)$$

Impulse response function of the lens