6. Fourier Analysis of Discrete-Time Signals and Systems

6.1. Introduction

In this chapter we present the Fourier analysis in the context of discrete-time signals (sequences) and systems. The Fourier analysis plays the same fundamental role in discrete time as in continuous time. As we will see, there are many similarities between the techniques of discrete-time Fourier analysis and their continuous-time counterparts, but there are also some important differences.

6.2. Discrete Fourier Series

6.2.1. A. Periodic Sequences:

In Chap. 1 we defined a discrete-time signal (or sequence) $x[n]$ to be periodic if there is a positive integer N for which

$$x[n+N] = x[n] \qquad \text{all } n$$

(6.1)

The fundamental period N_0 of $x[n]$ is the smallest positive integer N for which Eq. (6.1) is satisfied.

As we saw in Sec. 1.4, the complex exponential sequence

$$x[n] = e^{j(2\pi/N_0)n} = e^{j\Omega_0 n}$$

(6.2)

where $\Omega_0 = 2\pi/N_0$, is a periodic sequence with fundamental period N_0. As we discussed in Sec. 1.4C, one very important distinction between the discrete-time and the continuous-time complex exponential is that the signals $e^{j\omega_0 t}$ are distinct for distinct values of ω_0, but the sequences $e^{j\Omega_0 n}$, which differ in frequency by a multiple of 2π, are identical. That is,

$$e^{j(\Omega_0 + 2\pi k)n} = e^{j\Omega_0 n}\, e^{j2\pi kn} = e^{j\Omega_0 n}$$

(6.3)

Let

$$\Psi_k[n] = e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0} \qquad k = 0, \pm 1, \pm 2, \ldots$$

(6.4)

Then by Eq. (6.3) we have

$$\Psi_0[n] = \Psi_{N_0}[n] \qquad \Psi_1[n] = \Psi_{N_0+1}[n] \qquad \cdots \qquad \Psi_k[n] = \Psi_{N_0+k}[n] \qquad \cdots$$

(6.5)

and more generally,

$$\Psi_k[n] = \Psi_{k+mN_0}[n] \qquad m = \text{integer}$$

(6.6)

Thus, the sequences $\Phi_k[n]$ are distinct only over a range of N_0 successive values of k.

6.2.2. B. Discrete Fourier Series Representation:

The discrete Fourier series representation of a periodic sequence $x[n]$ with fundamental period N_0 is given by

$$x[n] = \sum_{k=0}^{N_0-1} c_k\, e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

(6.7)

where c_k are the Fourier coefficients and are given by (Prob. 6.2)

$$c_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n}$$

(6.8)

Because of Eq. (6.5) [or Eq. (6.6)], Eqs. (6.7) and (6.8) can be rewritten as

$$x[n] = \sum_{k=\langle N_0 \rangle} c_k\, e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

(6.9)

$$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n]\, e^{-jk\Omega_0 n}$$

(6.10)

where $\Sigma_{k=<N_0>}$ denotes that the summation is on k as k varies over a range of N_0 successive integers. Setting $k = 0$ in Eq. (6.10), we have

$$c_0 = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x[n]$$

(6.11)

which indicates that c_0 equals the average value of $x[n]$ over a period.

The Fourier coefficients c_k are often referred to as the *spectral coefficients* of $x[n]$.

6.2.3. C. Convergence of Discrete Fourier Series:

Since the discrete Fourier series is a finite series, in contrast to the continuous-time case, there are no convergence issues with discrete Fourier series.

6.2.4. D. Properties of Discrete Fourier Series:

6.2.4.1. 1. Periodicity of Fourier Coefficients:

From Eqs. (6.5) and (6.7) [or (6.9)], we see that

$$c_{k+N_0} = c_k$$

(6.12)

which indicates that the Fourier series coefficients c_k are periodic with fundamental period N_0.

6.2.4.2. 2. Duality:

From Eq. (6.12) we see that the Fourier coefficients c_k form a periodic sequence with fundamental period N_0. Thus, writing c_k as $c[k]$, Eq. (6.10) can be rewritten as

$$c[k] = \sum_{n=\langle N_0 \rangle} \frac{1}{N_0} x[n] e^{-jk\Omega_0 n}$$

(6.13)

Let $n = -m$ in Eq. (6.13). Then

$$c[k] = \sum_{m=\langle N_0 \rangle} \frac{1}{N_0} x[-m] e^{jk\Omega_0 m}$$

Letting $k = n$ and $m = k$ in the above expression, we get

$$c[n] = \sum_{k=\langle N_0 \rangle} \frac{1}{N_0} x[-k] e^{jk\Omega_0 n}$$

(6.14)

Comparing Eq. (6.14) with Eq. (6.9), we see that $(1/N_0)x[-k]$ are the Fourier coefficients of $c[n]$. If we adopt the notation

$$x[n] \xleftrightarrow{\text{DFS}} c_k = c[k]$$

(6.15)

to denote the discrete Fourier series pair, then by Eq. (6.14) we have

$$c[n] \xleftrightarrow{\text{DFS}} \frac{1}{N_0} x[-k]$$

(6.16)

Equation (6.16) is known as the *duality* property of the discrete Fourier series.

6.2.4.3. 3. Other Properties:

When $x[n]$ is real, then from Eq. (6.8) or [Eq. (6.10)] and Eq. (6.12) it follows that

$$c_{-k} = c_{N_0 - k} = c_k^*$$

(6.17)

where $*$ denotes the complex conjugate.

6.2.4.4. Even and Odd Sequences:

When $x[n]$ is real, let

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n]$ and $x_o[n]$, are the even and odd components of $x[n]$, respectively. Let

$$x[n] \xleftrightarrow{\text{DFS}} c_k$$

Then

$$x_e[n] \xleftrightarrow{\text{DFS}} \operatorname{Re}[c_k]$$

(6.18a)

$$x_o[n] \xleftrightarrow{\text{DFS}} j\operatorname{Im}[c_k]$$

(6.18b)

Thus, we see that if $x[n]$ is real and even, then its Fourier coefficients are real, while if $x[n]$ is real and odd, its Fourier coefficients are imaginary.

6.2.5. E. Parseval's Theorem:

If $x[n]$ is represented by the discrete Fourier series in Eq. (6.9), then it can be shown that (Prob. 6.10)

$$\frac{1}{N_0} \sum_{n=\langle N_0 \rangle} |x[n]|^2 = \sum_{k=\langle N_0 \rangle} |c_k|^2$$

(6.19)

Equation (6.19) is called *Parseval's identity* (or *Parseval's theorem*) for the discrete Fourier series.

6.3. The Fourier Transform

6.3.1. A. From Discrete Fourier Series to Fourier Transform:

Let $x[n]$ be a nonperiodic sequence of finite duration. That is, for some positive integer N_1,

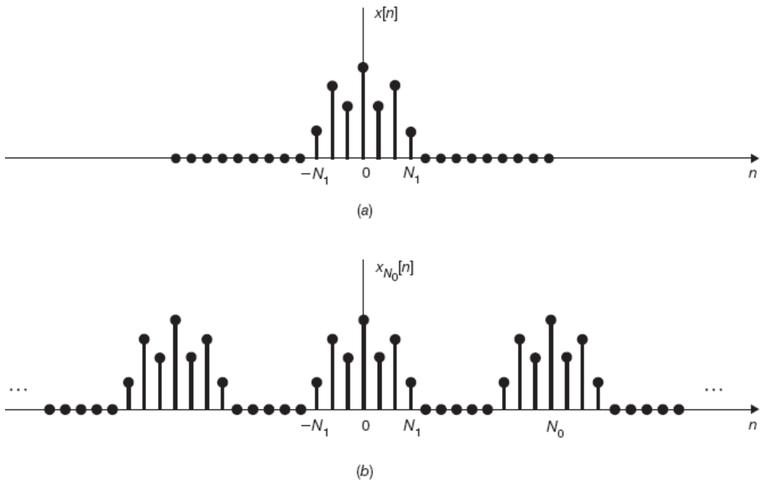
$$x[n] = 0 \qquad |n| > N_1$$

Such a sequence is shown in Fig. 6-1(*a*). Let $x_{N_0}[n]$ be a periodic sequence formed by repeating $x[n]$ with fundamental period N_0 as shown in Fig. 6-1(*b*). If we let $N_0 \to \infty$, we have

$$\lim_{N_0 \to \infty} x_{N_0}[n] = x[n]$$

(6.20)

Figure 6-1 (a) Nonperiodic finite sequence x[n]; (b) periodic sequence formed by periodic extension of x[n].



The discrete Fourier series of $x_{N_0}[n]$ is given by

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

(6.21)

where

$$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x_{N_0}[n]\, e^{-jk\Omega_0 n}$$

(6.22a)

Since $x_{N_0}[n]$ = $x[n]$ for $|n| \le N_1$ and also since $x[n] = 0$ outside this interval, Eq. (6.22a) can be rewritten as

$$c_k = \frac{1}{N_0} \sum_{n=-N_1}^{N_1} x[n]\, e^{-jk\Omega_0 n} = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\Omega_0 n}$$

(6.22b)

Let us define $X(\Omega)$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

(6.23)

Then, from Eq. (6.22b) the Fourier coefficients c_k can be expressed as

$$c_k = \frac{1}{N_0} X(k\Omega_0)$$

(6.24)

Substituting Eq. (6.24) into Eq. (6.21), we have

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} \frac{1}{N_0} X(k\Omega_0)\, e^{jk\Omega_0 n}$$

or

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=\langle N_0 \rangle} X(k\Omega_0)\, e^{jk\Omega_0 n} \Omega_0$$

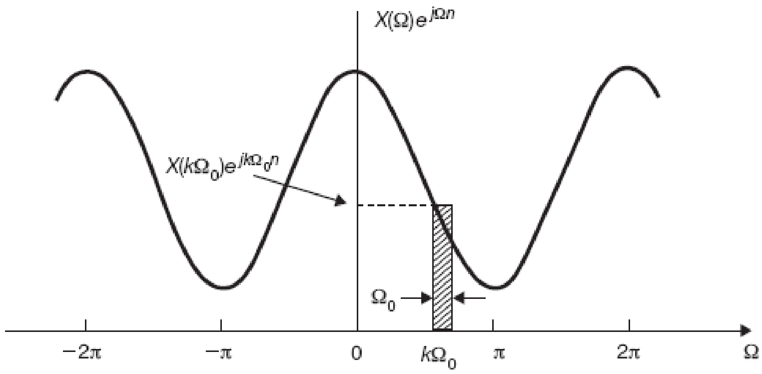
(6.25)

From Eq. (6.23), $X(\Omega)$ is periodic with period 2π and so is $e^{j\Omega n}$. Thus, the product $X(\Omega)\, e^{j\Omega n}$ will also be periodic with period 2π. As shown in Fig. 6-2, each term in the summation in Eq. (6.25) represents the area of a rectangle of height $X(k\Omega_0)e^{jk\Omega_0 n}$ and width Ω_0. As $N_0 \to \infty$, $\Omega_0 = 2\pi/N_0$ becomes infinitesimal ($\Omega_0 \to 0$) and Eq. (6.25) passes to an integral. Furthermore, since the summation in Eq. (6.25) is over N_0 consecutive intervals of width $\Omega_0 = 2\pi/N_0$, the total interval of integration will always have a width 2π. Thus, as $N_0 \to \infty$ and in view of Eq. (6.20), Eq. (6.25) becomes

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

(6.26)

Figure 6-2 Graphical interpretation of Eq. (6.25).



Since $X(\Omega)e^{j\Omega n}$ is periodic with period 2π, the interval of integration in Eq. (6.26) can be taken as any interval of length 2π.

6.3.2. B. Fourier Transform Pair:

The function $X(\Omega)$ defined by Eq. (6.23) is called the *Fourier transform* of $x[n]$, and Eq. (6.26) defines the *inverse Fourier transform* of $X(\Omega)$. Symbolically they are denoted by

$$X(\Omega) = \mathscr{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

(6.27)

$$x[n] = \mathscr{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

(6.28)

and we say that *x*[*n*] and *X*(Ω) form a Fourier transform pair denoted by

$$x[n] \leftrightarrow X(\Omega)$$

(6.29)

Equations (6.27) and (6.28) are the discrete-time counterparts of Eqs. (5.31) and (5.32).

6.3.3. C. Fourier Spectra:

The Fourier transform *X*(Ω) of *x*[*n*] is, in general, complex and can be expressed as

$$X(\Omega) = |X(\Omega)|\, e^{j\phi(\Omega)}$$

(6.30)

As in continuous time, the Fourier transform *X*(Ω) of a nonperiodic sequence *x*[*n*] is the frequency-domain specification of *x*[*n*] and is referred to as the *spectrum* (or *Fourier spectrum*) of *x*[*n*]. The quantity |*X*(Ω)| is called the *magnitude spectrum* of *x*[*n*], and ϕ(Ω) is called the *phase spectrum* of *x*[*n*]. Furthermore, if *x*[*n*] is real, the amplitude spectrum |*X*(Ω)| is an even function and the phase spectrum ϕ(Ω) is an odd function of Ω.

6.3.4. D. Convergence of *X*(Ω):

Just as in the case of continuous time, the sufficient condition for the convergence of *X*(Ω) is that *x*[*n*] is absolutely summable, that is,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

(6.31)

6.3.5. E. Connection between the Fourier Transform and the *z*-Transform:

Equation (6.27) defines the Fourier transform of *x*[*n*] as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

(6.32)

The *z*-transform of *x*[*n*], as defined in Eq. (4.3), is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(6.33)

Comparing Eqs. (6.32) and (6.33), we see that if the ROC of $X(z)$ contains the unit circle, then the Fourier transform $X(\Omega)$ of $x[n]$ equals $X(z)$ evaluated on the unit circle, that is,

$$X(\Omega) = X(z)\big|_{z=e^{j\Omega}}$$

(6.34)

Note that since the summation in Eq. (6.33) is denoted by $X(z)$, then the summation in Eq. (6.32) may be denoted as $X(e^{j\Omega})$. Thus, in the remainder of this book, both $X(\Omega)$ and $X(e^{j\Omega})$ mean the same thing whenever we connect the Fourier transform with the z-transform. Because the Fourier transform is the z-transform with $z = e^{j\Omega}$, it should not be assumed automatically that the Fourier transform of a sequence $x[n]$ is the z-transform with z replaced by $e^{j\Omega}$. If $x[n]$ is absolutely summable, that is, if $x[n]$ satisfies condition (6.31), the Fourier transform of $x[n]$ can be obtained from the z-transform of $x[n]$ with $z = e^{j\Omega}$ since the ROC of $X(z)$ will contain the unit circle; that is, $|e^{j\Omega}| = 1$. This is not generally true of sequences which are not absolutely summable. The following examples illustrate the above statements.

EXAMPLE 6.1 Consider the unit impulse sequence $\delta[n]$.

From Eq. (4.14) the z-transform of $\delta[n]$ is

$$\mathfrak{Z}\{\delta[n]\} = 1 \qquad \text{all } z$$

(6.35)

By definitions (6.27) and (1.45), the Fourier transform of $\delta[n]$ is

$$\mathscr{F}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1$$

(6.36)

Thus, the z-transform and the Fourier transform of $\delta[n]$ are the same. Note that $\delta[n]$ is absolutely summable and that the ROC of the z-transform of $\delta[n]$ contains the unit circle.

EXAMPLE 6.2 Consider the causal exponential sequence

$$x[n] = a^n u[n] \qquad a \text{ real}$$

From Eq. (4.9) the z-transform of $x[n]$ is given by

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

Thus, $X(e^{j\Omega})$ exists for $|a| < 1$ because the ROC of $X(z)$ then contains the unit circle. That is,

$$X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}} \qquad |a| < 1$$

(6.37)

Next, by definition (6.27) and Eq. (1.91) the Fourier transform of $x[n]$ is

$$
\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\Omega n} = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (ae^{-j\Omega})^n \\
&= \frac{1}{1-ae^{-j\Omega}} \qquad \left|ae^{-j\Omega}\right| = |a| < 1
\end{aligned}
$$

(6.38)

Thus, comparing Eqs. (6.37) and (6.38), we have

$$X(\Omega) = X(z)\big|_{z=e^{j\Omega}}$$

Note that $x[n]$ is absolutely summable.

EXAMPLE 6.3 Consider the unit step sequence $u[n]$.

From Eq. (4.16) the z-transform of $u[n]$ is

$$\mathcal{Z}\{u[n]\} = \frac{1}{1-z^{-1}} \qquad |z| > 1$$

(6.39)

The Fourier transform of $u[n]$ cannot be obtained from its z-transform because the ROC of the z-transform of $u[n]$ does not include the unit circle. Note that the unit step sequence $u[n]$ is not absolutely summable. The Fourier transform of $u[n]$ is given by (Prob. 6.28)

$$\mathscr{F}\{u[n]\} = \pi\,\delta(\Omega) + \frac{1}{1-e^{-j\Omega}} \qquad |\Omega| \le \pi$$

(6.40)

6.4. Properties of the Fourier Transform

Basic properties of the Fourier transform are presented in the following. There are many similarities to and several differences from the continuous-time case. Many of these properties are also similar to those of the z-transform when the ROC of $X(z)$ includes the unit circle.

6.4.1. A. Periodicity:

$$X(\Omega + 2\pi) = X(\Omega)$$

(6.41)

As a consequence of Eq. (6.41), in the discrete-time case we have to consider values of Ω (radians) only over the range $0 \le \Omega < 2\pi$ or $-\pi \le \Omega < \pi$, while in the continuous-time case we have to consider values of ω (radians/second) over the entire range $-\infty < \omega < \infty$.

6.4.2. B. Linearity:

$$a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega)$$

(6.42)

6.4.3. C. Time Shifting:

$$x[n-n_0] \leftrightarrow e^{-j\Omega n_0} X(\Omega)$$

(6.43)

6.4.4. D. Frequency Shifting:

$$e^{j\Omega_0 n} x[n] \leftrightarrow X(\Omega - \Omega_0)$$

(6.44)

6.4.5. E. Conjugation:

$$x^*[n] \leftrightarrow X^*(-\Omega)$$

(6.45)

where ∗ denotes the complex conjugate.

6.4.6. F. Time Reversal:

$$x[-n] \leftrightarrow X(-\Omega)$$

(6.46)

6.4.7. G. Time Scaling:

In Sec. 5.4D the scaling property of a continuous-time Fourier transform is expressed as [Eq. (5.52)]

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

(6.47)

However, in the discrete-time case, $x[an]$ is not a sequence if a is not an integer. On the other hand, if a is an integer, say $a = 2$, then $x[2n]$ consists of only the even samples of $x[n]$. Thus, time scaling in discrete time takes on a form somewhat different from Eq. (6.47).

Let m be a positive integer and define the sequence

$$x_{(m)}[n] = \begin{cases} x[n/m] = x[k] & \text{if } n = km, k = \text{integer} \\ 0 & \text{if } n \neq km \end{cases}$$

(6.48)

Then we have

$$x_{(m)}[n] \leftrightarrow X(m\Omega)$$

(6.49)

Equation (6.49) is the discrete-time counterpart of Eq. (6.47). It states again the inverse relationship between time and frequency. That is, as the signal spreads in time ($m > 1$), its Fourier transform is compressed (Prob. 6.22). Note that $X(m\Omega)$ is periodic with period $2\pi/m$ since $X(\Omega)$ is periodic with period 2π.

6.4.8. H. Duality:

In Sec. 5.4F the duality property of a continuous-time Fourier transform is expressed as [Eq. (5.54)]

$$X(t) \leftrightarrow 2\pi\, x(-\omega)$$

(6.50)

There is no discrete-time counterpart of this property. However, there is a duality between the discrete-time Fourier transform and the continuous-time Fourier series. Let

$$x[n] \leftrightarrow X(\Omega)$$

From Eqs. (6.27) and (6.41)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

(6.51)

$$X(\Omega + 2\pi) = X(\Omega)$$

(6.52)

Since Ω is a continuous variable, letting $\Omega = t$ and $n = -k$ in Eq. (6.51), we have

$$X(t) = \sum_{k=-\infty}^{\infty} x[-k]\, e^{jkt}$$

(6.53)

Since $X(t)$ is periodic with period $T_0 = 2\pi$ and the fundamental frequency $\omega_0 = 2\pi/T_0 = 1$, Eq. (6.53) indicates that the Fourier series coefficients of $X(t)$ will be $x[-k]$. This duality relationship is denoted by

$$X(t) \xleftrightarrow{\text{FS}} c_k = x[-k]$$

(6.54)

where FS denotes the Fourier series and c_k are its Fourier coefficients.

6.4.9. I. Differentiation in Frequency:

$$nx[n] \leftrightarrow j\frac{dX(\Omega)}{d\Omega}$$

(6.55)

6.4.10. J. Differencing:

$$x[n] - x[n-1] \leftrightarrow (1 - e^{-j\Omega})X(\Omega)$$

(6.56)

The sequence $x[n] - x[n-1]$ is called the *first difference* sequence. Equation (6.56) is easily obtained from the linearity property (6.42) and the time-shifting property (6.43).

6.4.11. K. Accumulation:

$$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \pi X(0)\delta(\Omega) + \frac{1}{1-e^{-j\Omega}} X(\Omega) \qquad |\Omega| \leq \pi$$

(6.57)

Note that accumulation is the discrete-time counterpart of integration. The impulse term on the right-hand side of Eq. (6.57) reflects the dc or average value that can result from the accumulation.

6.4.12. L. Convolution:

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega)\, X_2(\Omega)$$

(6.58)

As in the case of the *z*-transform, this convolution property plays an important role in the study of discrete-time LTI systems.

6.4.13. M. Multiplication:

$$x_1[n]\, x_2[n] \leftrightarrow \frac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$$

(6.59)

where ⊗ denotes the periodic convolution defined by [Eq. (2.70)]

$$X_1(\Omega) \otimes X_2(\Omega) = \int_{2\pi} X_1(\theta)\, X_2(\Omega - \theta)\, d\theta$$

(6.60)

The multiplication property (6.59) is the dual property of Eq. (6.58).

6.4.14. N. Additional Properties:

If $x[n]$ is real, let

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n]$ and $x_o[n]$ are the even and odd components of $x[n]$, respectively. Let

$$x[n] \leftrightarrow X(\Omega) = A(\Omega) + jB(\Omega) = |X(\Omega)|e^{j\theta(\Omega)}$$

(6.61)

Then

$$X(-\Omega) = X^*(\Omega)$$

(6.62)

$$x_e[n] \leftrightarrow \mathrm{Re}\{X(\Omega)\} = A(\Omega)$$

(6.63a)

$$x_o[n] \leftrightarrow j\mathrm{Im}\{X(\Omega)\} = jB(\Omega)$$

(6.63b)

Equation (6.62) is the necessary and sufficient condition for $x[n]$ to be real. From Eqs. (6.62) and (6.61) we have

$$A(-\Omega) = A(\Omega) \qquad B(-\Omega) = -B(\Omega)$$

(6.64a)

$$|X(-\Omega)| = |X(\Omega)| \qquad \theta(-\Omega) = -\theta(\Omega)$$

(6.64b)

From Eqs. (6.63a), (6.63b), and (6.64a) we see that if $x[n]$ is real and even, then $X(\Omega)$ is real and even, while if $x[n]$ is real and odd, $X(\Omega)$ is imaginary and odd.

6.4.15. O. Parseval's Relations:

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n] = \frac{1}{2\pi}\int_{2\pi} X_1(\Omega)\, X_2(-\Omega)\, d\Omega$$

(6.65)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2\, d\Omega$$

(6.66)

Equation (6.66) is known as *Parseval's identity* (or *Parseval's theorem*) for the discrete-time Fourier transform.

Table 6-1 contains a summary of the properties of the Fourier transform presented in this section. Some common sequences and their Fourier transforms are given in Table 6-2.

Table 6-1 Properties of the Fourier Transform

PROPERTY	SEQUENCE	FOURIER TRANSFORM
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega + 2\pi) = X(\Omega)$
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(\Omega) + a_2 X_2(\Omega)$
Time shifting	$x[n - n_0]$	$e^{-j\Omega n_0} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n} x[n]$	$X(\Omega - \Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n = km \\ 0 & \text{if } n \neq km \end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-j\Omega})X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega) + \dfrac{1}{1 - e^{-j\Omega}} X(\Omega)$ $\lvert\Omega\rvert \leq \pi$
Convolution	$x_1[n] * x_2[n]$	$X_1(\Omega) X_2(\Omega)$
Multiplication	$x_1[n] x_2[n]$	$\dfrac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$
Real sequence	$x[n] = x_e[n] + x_o[n]$	$X(\Omega) = A(\Omega) + jB(\Omega)$ $X(-\Omega) = X^*(\Omega)$
Even component	$x_e[n]$	$\text{Re}\{X(\Omega)\} = A(\Omega)$
Odd component	$x_o[n]$	$j\,\text{Im}\{X(\Omega)\} = jB(\Omega)$
Parseval's theorem		

$$\sum_{n=-\infty}^{\infty} x_1[n] x_2[n] = \frac{1}{2\pi} \int_{2\pi} X_1(\Omega) X_2(-\Omega)\, d\Omega$$

$$\sum_{n=-\infty}^{\infty} \lvert x[n] \rvert^2 = \frac{1}{2\pi} \int_{2\pi} \lvert X(\Omega) \rvert^2\, d\Omega$$

Table 6-2 Common Fourier Transform Pairs

$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta(n-n_0)$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \|\Omega\|\le\pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \|\Omega\|, \|\Omega_0\|\le\pi$
$\cos\Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \|\Omega\|, \|\Omega_0\|\le\pi$
$\sin\Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \|\Omega\|, \|\Omega_0\|\le\pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \|\Omega\|\le\pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \|\Omega\|\le\pi$
$a^n u[n], \|a\|<1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \|a\|>1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$(n+1)a^n u[n], \|a\|<1$	$\dfrac{1}{\left(1-ae^{-j\Omega}\right)^2}$
$a^{\|n\|}, \|a\|<1$	$\dfrac{1-a^2}{1-2a\cos\Omega+a^2}$
$x[n]=\begin{cases}1 & \|n\|\le N_1\\ 0 & \|n\|>N_1\end{cases}$	$\dfrac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$
$\dfrac{\sin Wn}{\pi n}, 0<W<\pi$	$X(\Omega)=\begin{cases}1 & 0\le\|\Omega\|\le W\\ 0 & W<\|\Omega\|\le\pi\end{cases}$
$\sum_{k=-\infty}^{\infty}\delta[n-kN_0]$	$\Omega_0\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_0), \Omega_0=\dfrac{2\pi}{N_0}$

6.5. The Frequency Response of Discrete-Time LTI Systems

6.5.1. A. Frequency Response:

In Sec. 2.6 we showed that the output $y[n]$ of a discrete-time LTI system equals the convolution of the input $x[n]$ with the impulse response $h[n]$; that is,

$$y[n] = x[n] * h[n]$$

(6.67)

Applying the convolution property (6.58), we obtain

$$Y(\Omega) = X(\Omega)H(\Omega)$$

(6.68)

where $Y(\Omega)$, $X(\Omega)$, and $H(\Omega)$ are the Fourier transforms of $y[n]$, $x[n]$, and $h[n]$, respectively. From Eq. (6.68) we have

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

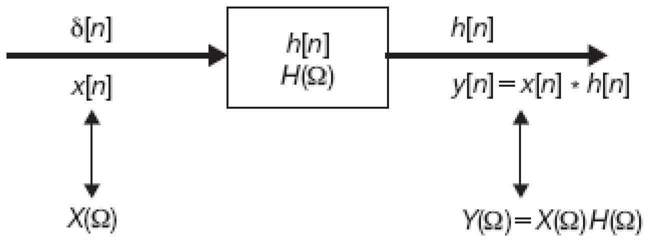
(6.69)

Relationship represented by Eqs. (6.67) and (6.68) are depicted in Fig. 6-3. Let

$$H(\Omega) = |H(\Omega)|e^{j\theta_H(\Omega)}$$

(6.70)

Figure 6-3 Relationships between inputs and outputs in an LTI discrete-time system.



As in the continuous-time case, the function $H(\Omega)$ is called the *frequency response* of the system, $|H(\Omega)|$ the *magnitude response* of the system, and $\theta_H(\Omega)$ the *phase response* of the system.

Consider the complex exponential sequence

$$x[n] = e^{j\Omega_0 n}$$

(6.71)

Then, setting $z = e^{j\Omega_0}$ in Eq. (4.1), we obtain

$$y[n] = H(e^{j\Omega_0})\, e^{j\Omega_0 n} = H(\Omega_0)\, e^{j\Omega_0 n}$$

(6.72)

which indicates that the complex exponential sequence $e^{j\Omega_0 n}$ is an eigenfunction of the LTI system with corresponding eigenvalue $H(\Omega_0)$, as previously observed in Chap. 2 (Sec. 2.8). Furthermore, by the linearity property (6.42), if the input $x[n]$ is periodic with the discrete Fourier series

$$x[n] = \sum_{k=\langle N_0 \rangle} c_k\, e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

(6.73)

then the corresponding output $y[n]$ is also periodic with the discrete Fourier series

$$y[n] = \sum_{k=\langle N_0 \rangle} c_k H(k\Omega_0)\, e^{jk\Omega_0 n}$$

(6.74)

If $x[n]$ is not periodic, then from Eqs. (6.68) and (6.28) the corresponding output $y[n]$ can be expressed as

$$y[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} H(\Omega)\, X(\Omega)\, e^{j\Omega n}\, d\Omega$$

(6.75)

6.5.2. B. LTI Systems Characterized by Difference Equations:

As discussed in Sec. 2.9, many discrete-time LTI systems of practical interest are described by linear constant-coefficient difference equations of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

(6.76)

with $M \le N$. Taking the Fourier transform of both sides of Eq. (6.76) and using the linearity property (6.42) and the time-shifting property (6.43), we have

$$\sum_{k=0}^{N} a_k\, e^{-jk\Omega}\, Y(\Omega) = \sum_{k=0}^{M} b_k\, e^{-jk\Omega}\, X(\Omega)$$

or, equivalently,

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} b_k\, e^{-jk\Omega}}{\sum_{k=0}^{N} a_k\, e^{-jk\Omega}}$$

(6.77)

The result (6.77) is the same as the z-transform counterpart $H(z) = Y(z)/X(z)$ with $z = e^{j\Omega}$ [Eq. (4.44)]; that is,

$$H(\Omega) = H(z)\big|_{z=e^{j\Omega}} = H(e^{j\Omega})$$

6.5.3. C. Periodic Nature of the Frequency Response:

From Eq. (6.41) we have

$$H(\Omega) = H(\Omega + 2\pi)$$

(6.78)

Thus, unlike the frequency response of continuous-time systems, that of all discrete-time LTI systems is periodic with period 2π. Therefore, we need observe the frequency response of a system only over the frequency range $0 \le \Omega < 2\pi$ or $-\pi \le \Omega < \pi$.

6.6. System Response to Sampled Continuous-Time Sinusoids

6.6.1. A. System Responses:

We denote by $y_c[n]$, $y_s[n]$, and $y[n]$ the system responses to $\cos \Omega n$, $\sin \Omega n$, and $e^{j\Omega n}$, respectively (Fig. 6-4). Since $e^{j\Omega n} = \cos \Omega n + j \sin \Omega n$, it follows from Eq. (6.72) and the linearity property of the system that

$$y[n] = y_c[n] + jy_s[n] = H(\Omega)\, e^{j\Omega n}$$

(6.79a)

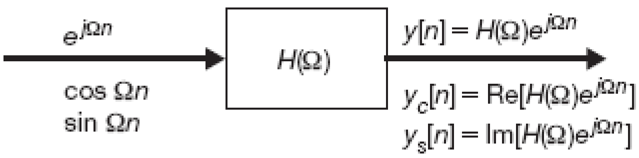
$$y_c[n] = \mathrm{Re}\{y[n]\} = \mathrm{Re}\{H(\Omega)\, e^{j\Omega n}\}$$

(6.79b)

$$y_s[n] = \mathrm{Im}\{y[n]\} = \mathrm{Im}\{H(\Omega)\, e^{j\Omega n}\}$$

(6.79c)

Figure 6-4 *System responses to $e^{j\Omega n}$, $\cos \Omega n$, and $\sin \Omega n$.*



When a sinusoid $\cos \Omega n$ is obtained by sampling a continuous-time sinusoid $\cos \omega t$ with sampling interval T_s, that is,

$$\cos \Omega n = \cos \omega t \big|_{t=nT_s} = \cos \omega T_s n$$

(6.80)

all the results developed in this section apply if we substitute ωT_s for Ω:

$$\Omega = \omega T_s$$

(6.81)

For a continuous-time sinusoid $\cos \omega t$ there is a unique waveform for every value of ω in the range 0 to ∞. Increasing ω results in a sinusoid of ever-increasing frequency. On the other hand, the discrete-time sinusoid $\cos \Omega n$ has a unique waveform only for values of Ω in the range 0 to 2π because

$$\cos[(\Omega + 2\pi m)n] = \cos(\Omega n + 2\pi mn) = \cos \Omega n \qquad m = \text{integer}$$

(6.82)

This range is further restricted by the fact that

$$\cos(\pi \pm \Omega)n = \cos \pi n \cos \Omega n \mp \sin \pi n \sin \Omega n$$
$$= (-1)^n \cos \Omega n$$

(6.83)

Therefore,

$$\cos(\pi + \Omega)n = \cos(\pi - \Omega)n$$

(6.84)

Equation (6.84) shows that a sinusoid of frequency (π + Ω) has the same waveform as one with frequency (π − Ω). Therefore, a sinusoid with any value of Ω outside the range 0 to π is identical to a sinusoid with Ω in the range 0 to π. Thus, we conclude that every discrete-time sinusoid with a frequency in the range $0 \le \Omega < \pi$ has a distinct waveform, and we need observe only the frequency response of a system over the frequency range $0 \le \Omega < \pi$.

6.6.2. B. Sampling Rate:

Let ω_M (= $2\pi f_M$) be the highest frequency of the continuous-time sinusoid. Then from Eq. (6.81) the condition for a sampled discrete-time sinusoid to have a unique waveform is

$$\omega_M T_s < \pi \rightarrow T_s < \frac{\pi}{\omega_M} \qquad \text{or} \qquad f_s > 2 f_M$$

(6.85)

where $f_s = 1/T_s$ is the sampling rate (or frequency). Equation (6.85) indicates that to process a continuous-time sinusoid by a discrete-time system, the sampling rate must not be less than twice the frequency (in hertz) of the sinusoid. This result is a special case of the sampling theorem we discussed in Prob. 5.59.

6.7. Simulation

Consider a continuous-time LTI system with input $x(t)$ and output $y(t)$. We wish to find a discrete-time LTI system with input $x[n]$ and output $y[n]$ such that

$$\text{if } x[n] = x(nT_s) \text{ then } y[n] = y(nT_s)$$

(6.86)

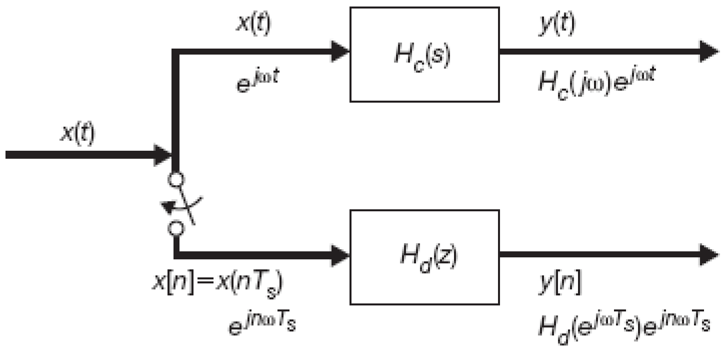
where T_s is the sampling interval.

Let $H_c(s)$ and $H_d(z)$ be the system functions of the continuous-time and discrete-time systems, respectively (Fig. 6-5). Let

$$x(t) = e^{j\omega t} \qquad x[n] = x(nT_s) = e^{jn\omega T_s}$$

(6.87)

Figure 6-5 *Digital simulation of analog systems.*



Then from Eqs. (3.1) and (4.1) we have

$$y(t) = H_c(j\omega)\, e^{j\omega t} \qquad y[n] = H_d(e^{j\omega T_s})\, e^{jn\omega T_s}$$

(6.88)

Thus, the requirement $y[n] = y(nT_s)$ leads to the condition

$$H_c(j\omega)\, e^{jn\omega T_s} = H_d(e^{j\omega T_s})\, e^{jn\omega T_s}$$

from which it follows that

$$H_c(j\omega) = H_d(e^{j\omega T_s})$$

(6.89)

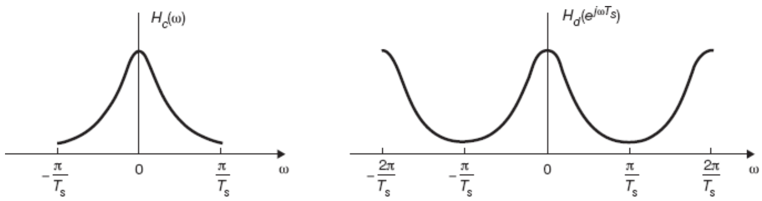
In terms of the Fourier transform, Eq. (6.89) can be expressed as

$$H_c(\omega) = H_d(\Omega) \qquad \Omega = \omega T_s$$

(6.90)

Note that the frequency response $H_d(\Omega)$ of the discrete-time system is a periodic function of ω (with period $2\pi/T_s$), but that the frequency response $H_c(\omega)$ of the continuous-time system is not. Therefore, Eq. (6.90) or Eq. (6.89) cannot, in general, be true for every ω. If the input $x(t)$ is band-limited [Eq. (5.94)], then it is possible, in principle, to satisfy Eq. (6.89) for every ω in the frequency range $(-\pi/T_s, \pi/T_s)$ (Fig. 6-6). However, from Eqs. (5.85) and (6.77), we see that $H_c(\omega)$ is a rational function of ω, whereas $H_d(\Omega)$ is a rational function of $e^{j\Omega}$ ($\Omega = \omega T_s$). Therefore, Eq. (6.89) is impossible to satisfy. However, there are methods for determining a discrete-time system so as to satisfy Eq. (6.89) with reasonable accuracy for every ω in the band of the input (Probs. 6.43 to 6.47).

Figure 6-6



6.8. The Discrete Fourier Transform

In this section we introduce the technique known as the *discrete Fourier transform* (DFT) for finite-length sequences. It should be noted that the DFT should not be confused with the Fourier transform.

6.8.1. A. Definition:

Let $x[n]$ be a finite-length sequence of length N, that is,

$$x[n] = 0 \qquad \text{outside the range } 0 \le n \le N-1$$

(6.91)

The DFT of $x[n]$, denoted as $X[k]$, is defined by

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

(6.92)

where W_N is the Nth root of unity given by

$$W_N = e^{-j(2\pi/N)}$$

(6.93)

The inverse DFT (IDFT) is given by

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k]\, W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

(6.94)

The DFT pair is denoted by

$$x[n] \leftrightarrow X[k]$$

(6.95)

Important features of the DFT are the following:

1. There is a one-to-one correspondence between $x[n]$ and $X[k]$.
2. There is an extremely fast algorithm, called the fast Fourier transform (FFT) for its calculation.
3. The DFT is closely related to the discrete Fourier series and the Fourier transform.
4. The DFT is the appropriate Fourier representation for digital computer realization because it is discrete and of finite length in both the time and frequency domains.

Note that the choice of N in Eq. (6.92) is not fixed. If $x[n]$ has length $N_1 < N$, we want to assume that $x[n]$ has length N by simply adding $(N - N_1)$ samples with a value of 0. This addition of dummy samples is known as *zero padding*. Then the resultant $x[n]$ is often referred to as an *N-point sequence*, and $X[k]$ defined in Eq. (6.92) is referred to as an *N-point DFT*. By a judicious choice of N, such as choosing it to be a power of 2, computational efficiencies can be gained.

6.8.2. B. Relationship between the DFT and the Discrete Fourier Series:

Comparing Eqs. (6.94) and (6.92) with Eqs. (6.7) and (6.8), we see that $X[k]$ of finite sequence $x[n]$ can be interpreted as the coefficients c_k in the discrete Fourier series representation of its periodic extension multiplied by the period N_0 and $N_0 = N$. That is,

$$X[k] = Nc_k$$

(6.96)

Actually, the two can be made identical by including the factor $1/N$ with the DFT rather than with the IDFT.

6.8.3. C. Relationship between the DFT and the Fourier Transform:

By definition (6.27) the Fourier transform of $x[n]$ defined by Eq. (6.91) can be expressed as

$$X(\Omega) = \sum_{n=0}^{N-1} x[n]\, e^{-j\Omega n}$$

(6.97)

Comparing Eq. (6.97) with Eq. (6.92), we see that

$$X[k] = X(\Omega)\big|_{\Omega = k2\pi/N} = X\left(\frac{k2\pi}{N}\right)$$

(6.98)

Thus, $X[k]$ corresponds to the sampled $X(\Omega)$ at the uniformly spaced frequencies $\Omega = k2\pi/N$ for integer k.

6.8.4. D. Properties of the DFT:

Because of the relationship (6.98) between the DFT and the Fourier transform, we would expect their properties to be quite similar, except that the DFT $X[k]$ is a function of a discrete variable while the Fourier transform $X(\Omega)$ is a function of a continuous variable. Note that the DFT variables n and k must be restricted to the range $0 \le n, k < N$, the DFT shifts $x[n - n_0]$ or $X[k - k_0]$ imply $x[n - n_0]_{\text{mod}N}$ or $X[k - k_0]_{\text{mod}N}$, where the modulo notation $[m]_{\text{mod}N}$ means that

$$[m]_{\text{mod } N} = m + iN$$

(6.99)

for some integer i such that

$$0 \le [m]_{\text{mod } N} < N$$

(6.100)

For example, if $x[n] = \delta[n - 3]$, then

$$x[n-4]_{\text{mod } 6} = \delta[n-7]_{\text{mod } 6} = \delta[n-7+6] = \delta[n-1]$$

The DFT shift is also known as a *circular shift*. Basic properties of the DFT are the following:

6.8.4.1. 1. Linearity:

$$a_1x_1[n] + a_2x_2[n] \leftrightarrow a_1X_1[k] + a_2X_2[k]$$

(6.101)

6.8.4.2. 2. Time Shifting:

$$x[n-n_0]_{\mathrm{mod}\,N} \leftrightarrow W_N^{kn_0}X[k] \qquad W_N = e^{-j(2\pi/N)}$$

(6.102)

6.8.4.3. 3. Frequency Shifting:

$$W_N^{-kn_0}\;x[n] \leftrightarrow X[k-k_0]_{\mathrm{mod}\,N}$$

(6.103)

6.8.4.4. 4. Conjugation:

$$x^*[n] \leftrightarrow X^*[-k]_{\mathrm{mod}\,N}$$

(6.104)

where ∗ denotes the complex conjugate.

6.8.4.5. 5. Time Reversal:

$$x[-n]_{\mathrm{mod}\,N} \leftrightarrow X[-k]_{\mathrm{mod}\,N}$$

(6.105)

6.8.4.6. 6. Duality:

$$X[n] \leftrightarrow Nx[-k]_{\mathrm{mod}\,N}$$

(6.106)

6.8.4.7. 7. Circular Convolution:

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1[k]\,X_2[k]$$

(6.107)

where

$$x_1[n] \otimes x_2[n] = \sum_{i=0}^{N-1} x_1[i]x_2[n-i]_{\mathrm{mod}\,N}$$

(6.108)

The convolution sum in Eq. (6.108) is known as the *circular convolution* of $x_1[n]$ and $x_2[n]$.

6.8.4.8. 8. Multiplication:

$$x_1[n]\,x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \otimes X_2[k]$$

(6.109)

where

$$X_1[k] \otimes X_2[k] = \sum_{i=1}^{N-1} X_1[i]\,X_2[k-i]_{\mathrm{mod}\,N}$$

6.8.4.9. 9. Additional Properties:

When $x[n]$ is real, let

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n]$ and $x_o[n]$ are the even and odd components of $x[n]$, respectively. Let

$$x[n] \leftrightarrow X[k] = A[k] + jB[k] = |X\,[k]|\,e^{j\theta\,[k]}$$

Then

$$X[-k]_{\mathrm{mod}\,N} = X^*\,[k]$$

(6.110)

$$x_e[n] \leftrightarrow \mathrm{Re}\{X[k]\} = A[k]$$

(6.111a)

$$x_o[n] \leftrightarrow j\,\mathrm{Im}\{X[k]\} = jB\,[k]$$

(6.111b)

From Eq. (6.110) we have

$$A[-k]_{\mathrm{mod}\,N} = A[k] \qquad B[-k]_{\mathrm{mod}\,N} = -B[k]$$

(6.112a)

$$|X[-k]|_{\mathrm{mod}\,N} = |\,X[k]| \qquad \theta[-k]_{\mathrm{mod}\,N} = -\theta[k]$$

(6.112b)

6.8.4.10. 10. Parseval's Relation:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X[k]|^2$$

(6.113)

Equation (6.113) is known as *Parseval's identity* (or *Parseval's theorem*) for the DFT.

6.9. SOLVED PROBLEMS

6.9.1. Discrete Fourier Series

6.1. We call a set of sequences $\{\Phi_k[n]\}$ orthogonal on an interval $[N_1, N_2]$ if any two signals $\Phi_m[n]$ and $\Phi_k[n]$ in the set satisfy the condition

$$\sum_{n=N_1}^{N_2} \Psi_m[n]\Psi_k^*[n] = \begin{cases} 0 & m \neq k \\ \alpha & m = k \end{cases}$$

(6.114)

where $*$ denotes the complex conjugate and $\alpha \neq 0$. Show that the set of complex exponential sequences

$$\Psi_k[n] = e^{jk(2\pi/N)n} \qquad k = 0, 1, \ldots, N-1$$

(6.115)

is orthogonal on any interval of length N.

From Eq. (1.90) we note that

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \end{cases}$$

(6.116)

Applying Eq. (6.116), with $\alpha = e^{jk(2\pi/N)}$, we obtain

$$\sum_{n=0}^{N-1} e^{jk(2\pi/N)n} = \begin{cases} N & k = 0, \pm N, \pm 2N, \ldots \\ \dfrac{1-e^{jk(2\pi/N)N}}{1-e^{jk(2\pi/N)}} = 0 & \text{otherwise} \end{cases}$$

(6.117)

since $e^{jk(2\pi/N)N} = e^{jk2\pi} = 1$. Since each of the complex exponentials in the summation in Eq. (6.117) is periodic with period N, Eq. (6.117) remains valid with a summation carried over any interval of length N. That is,

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N & k = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(6.118)

Now, using Eq. (6.118), we have

$$\sum_{n=\langle N\rangle} \Psi_m[n]\Psi_k^*[n] = \sum_{n=\langle N\rangle} e^{jm(2\pi/N)n} e^{-jk(2\pi/N)n}$$

$$= \sum_{n=\langle N\rangle} e^{j(m-k)(2\pi/N)n} = \begin{cases} N & m = k \\ 0 & m \neq k \end{cases}$$

(6.119)

where m, $k < N$. Equation (6.119) shows that the set $\{e^{jk(2\pi/N)n};\ k = 0, 1, \ldots, N-1\}$ is orthogonal over any interval of length N. Equation (6.114) is the discrete-time counterpart of Eq. (5.95) introduced in Prob. 5.1.

6.2. Using the orthogonality condition Eq. (6.119), derive Eq. (6.8) for the Fourier coefficients.

Replacing the summation variable k by m in Eq. (6.7), we have

$$x[n] = \sum_{m=0}^{N-1} c_m\, e^{jm(2\pi/N_0)n}$$

(6.120)

Using Eq. (6.115) with $N = N_0$, Eq. (6.120) can be rewritten as

$$x[n] = \sum_{m=0}^{N_0-1} c_m \Psi_m[n]$$

(6.121)

Multiplying both sides of Eq. (6.121) by $\Psi_k^*[n]$ and summing over $n = 0$ to $(N_0 - 1)$, we obtain

$$\sum_{n=0}^{N_0-1} x[n]\Psi_k^*[n] = \sum_{n=0}^{N_0-1}\left(\sum_{m=0}^{N_0-1} c_m \Psi_m[n]\right)\Psi_k^*[n]$$

Interchanging the order of the summation and using Eq. (6.119), we get

$$\sum_{n=0}^{N_0-1} x[n]\Psi_k^*[n] = \sum_{m=0}^{N_0-1} c_m\left(\sum_{n=0}^{N_0-1} \Psi_m[n]\,\Psi_k^*[n]\right) = N_0 c_k$$

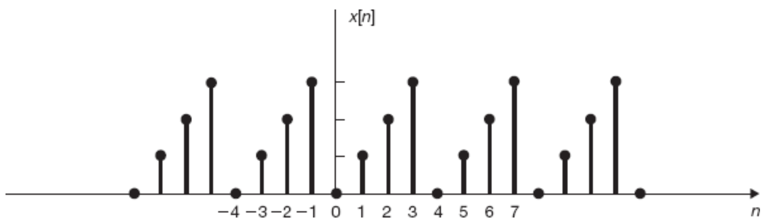
(6.122)

Thus,

$$c_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]\Psi_k^*[n] = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]\, e^{-jk(2\pi/N_0)n}$$

6.3. Determine the Fourier coefficients for the periodic sequence $x[n]$ shown in Fig. 6-7.

Figure 6-7



From Fig. 6-7 we see that $x[n]$ is the periodic extension of $\{0, 1, 2, 3\}$ with fundamental period $N_0 = 4$. Thus,

$$\Omega_0 = \frac{2\pi}{4} \qquad \text{and} \qquad e^{-j\Omega_0} = e^{-j2\pi/4} = e^{-j\pi/2} = -j$$

By Eq. (6.8) the discrete-time Fourier coefficients c_k are

$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = \frac{1}{4}(0+1+2+3) = \frac{3}{2}$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^n = \frac{1}{4}(0-j1-2+j3) = -\frac{1}{2}+j\frac{1}{2}$$

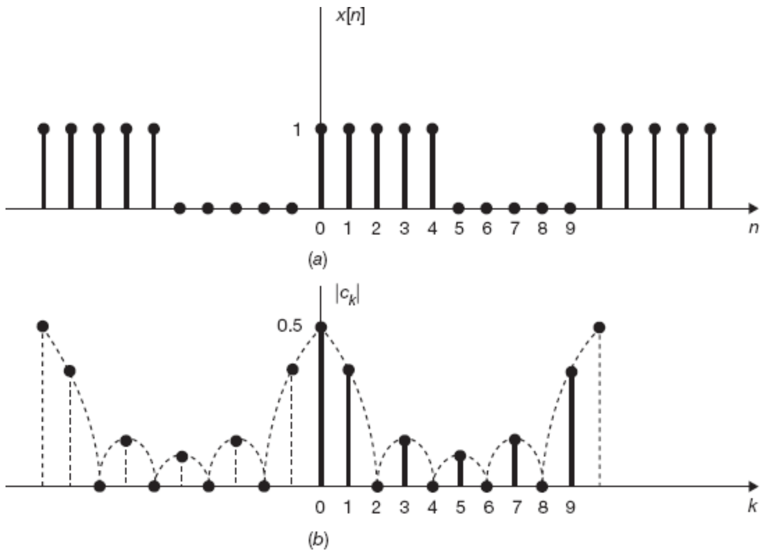
$$c_2 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^{2n} = \frac{1}{4}(0-1+2-3) = -\frac{1}{2}$$

$$c_3 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^{3n} = \frac{1}{4}(0+j1-2-j3) = -\frac{1}{2}-j\frac{1}{2}$$

Note that $c_3 = c_{4-1} = c_1^*$ [Eq. (6.17)].

6.4. Consider the periodic sequence $x[n]$ shown in Fig. 6-8(a). Determine the Fourier coefficients c_k and sketch the magnitude spectrum $|c_k|$.

Figure 6-8



From Fig. 6-8(a) we see that the fundamental period of $x[n]$ is $N_0 = 10$ and $\Omega_0 = 2\pi/N_0 = \pi/5$. By Eq. (6.8) and using Eq. (1.90), we get

$$\begin{aligned} c_k &= \frac{1}{10}\sum_{n=0}^{4} e^{-jk(\pi/5)n} = \frac{1}{10}\frac{1-e^{-jk\pi}}{1-e^{-jk(\pi/5)}} \\ &= \frac{1}{10}\frac{e^{-jk\pi/2}(e^{jk\pi/2}-e^{-jk\pi/2})}{e^{-jk\pi/10}(e^{jk\pi/10}-e^{-jk\pi/10})} \\ &= \frac{1}{10}e^{-jk(2\pi/5)}\frac{\sin(k\pi/2)}{\sin(k\pi/10)} \qquad k = 0, 1, 2, \dots, 9 \end{aligned}$$

The magnitude spectrum $|c_k|$ is plotted in Fig. 6-8(b)

6.5. Consider a sequence

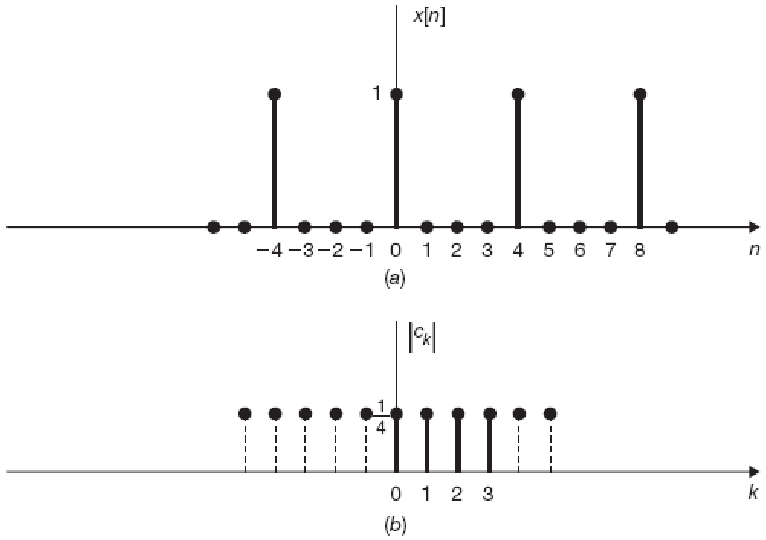
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$$

a. Sketch $x[n]$.

b. Find the Fourier coefficients c_k of $x[n]$.

a. The sequence $x[n]$ is sketched in Fig. 6-9(a). It is seen that $x[n]$ is the periodic extension of the sequence {1, 0, 0, 0} with period $N_0 = 4$.

Figure 6-9



b. From Eqs. (6.7) and (6.8) and Fig. 6-9(a) we have

$$x[n] = \sum_{k=0}^{3} c_k\, e^{jk(2\pi/4)n} = \sum_{k=0}^{3} c_k\, e^{jk(\pi/2)n}$$

and

$$c_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-jk(2\pi/4)n} = \frac{1}{4}x[0] = \frac{1}{4} \qquad \text{all } k$$

since $x[1] = x[2] = x[3] = 0$. The Fourier coefficients of $x[n]$ are sketched in Fig. 6-9(b).

6.6. Determine the discrete Fourier series representation for each of the following sequences:

$$(a)\quad x[n] = \cos\frac{\pi}{4}n$$

$$(b)\quad x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n$$

$$(c)\quad x[n] = \cos^2\left(\frac{\pi}{8}n\right)$$

a. The fundamental period of $x[n]$ is $N_0 = 8$, and $\Omega_0 = 2\pi/N_0 = \pi/4$. Rather than using Eq. (6.8) to evaluate the Fourier coefficients c_k, we use Euler's formula and get

$$\cos\frac{\pi}{4}n = \frac{1}{2}\left(e^{j(\pi/4)n} + e^{-j(\pi/4)n}\right) = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n}$$

Thus, the Fourier coefficients for $x[n]$ are $c_1 = \frac{1}{2}, c_{-1} = c_{-1+8} = c_7 = \frac{1}{2}$, and all other $c_k = 0$. Hence, the discrete Fourier series of $x[n]$ is

$$x[n] = \cos\frac{\pi}{4}n = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{j7\Omega_0 n} \qquad \Omega_0 = \frac{\pi}{4}$$

b. From Prob. 1.16(i) the fundamental period of $x[n]$ is $N_0 = 24$, and $\Omega_0 = 2\pi/N_0 = \pi/12$. Again by Euler's formula we have

$$\begin{aligned} x[n] &= \frac{1}{2}(e^{j(\pi/3)n} + e^{-j(\pi/3)n}) + \frac{1}{2j}(e^{j(\pi/4)n} - e^{-j(\pi/4)n}) \\ &= \frac{1}{2}e^{-j4\Omega_0 n} + j\frac{1}{2}e^{-j3\Omega_0 n} - j\frac{1}{2}e^{j3\Omega_0 n} + \frac{1}{2}e^{j4\Omega_0 n} \end{aligned}$$

Thus, $c_3 = -j(\frac{1}{2}), c_4 = \frac{1}{2}, c_{-4} = c_{-4+24} = c_{20} = \frac{1}{2}, c_{-3} = c_{-3+24} = c_{21} = j(\frac{1}{2})$, and all other $c_k = 0$. Hence, the discrete Fourier series of $x[n]$ is

$$x[n] = -j\frac{1}{2}e^{j3\Omega_0 n} + \frac{1}{2}e^{j4\Omega_0 n} + \frac{1}{2}e^{j20\Omega_0 n} + j\frac{1}{2}e^{j21\Omega_0 n} \qquad \Omega_0 = \frac{\pi}{12}$$

c. From Prob. 1.16(j) the fundamental period of $x[n]$ is $N_0 = 8$, and $\Omega_0 = 2\pi/N_0 = \pi/4$. Again by Euler's formula we have

$$\begin{aligned} x[n] &= \left(\frac{1}{2}e^{j(\pi/8)n} + \frac{1}{2}e^{-j(\pi/8)n}\right)^2 = \frac{1}{4}e^{j(\pi/4)n} + \frac{1}{2} + \frac{1}{4}e^{-j(\pi/4)n} \\ &= \frac{1}{4}e^{j\Omega_0 n} + \frac{1}{2} + \frac{1}{4}e^{-j\Omega_0 n} \end{aligned}$$

Thus, $c_0 = \frac{1}{2}, c_1 = \frac{1}{4}, c_{-1} = c_{-1+8} = c_7 = \frac{1}{4}$, and all other $c_k = 0$. Hence, the discrete Fourier series of $x[n]$ is

$$x[n] = \frac{1}{2} + \frac{1}{4}e^{j\Omega_0 n} + \frac{1}{4}e^{j7\Omega_0 n} \qquad \Omega_0 = \frac{\pi}{4}$$

6.7. Let $x[n]$ be a real periodic sequence with fundamental period N_0 and Fourier coefficients $c_k = a_k + jb_k$, where a_k and b_k are both real.

a. Show that $a_{-k} = a_k$ and $b_{-k} = -b_k$.

b. Show that $c_{N_0/2}$ is real if N_0 is even.

c. Show that $x[n]$ can also be expressed as a discrete trigonometric Fourier series of the form

$$x[n] = c_0 + 2\sum_{k=1}^{(N_0-1)/2}(a_k \cos k\Omega_0 n - b_k \sin k\Omega_0 n) \qquad \Omega_0 = \frac{2\pi}{N_0}$$

(6.123)

if N_0 is odd or

$$x[n] = c_0 + (-1)^n c_{N_0/2} + 2\sum_{k=1}^{(N_0-2)/2}(a_k \cos k\Omega_0 n - b_k \sin k\Omega_0 n)$$

(6.124)

if N_0 is even.

a. If $x[n]$ is real, then from Eq. (6.8) we have

$$c_{-k} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{jk\Omega_0 n} = \left(\frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n} \right)^* = c_k^*$$

Thus,

$$c_{-k} = a_{-k} + jb_{-k} = (a_k + jb_k)^* = a_k - jb_k$$

and we have

$$a_{-k} = a_k \qquad \text{and} \qquad b_{-k} = -b_k$$

b. If N_0 is even, then from Eq. (6.8)

$$\begin{aligned} c_{N_0/2} &= \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-j(N_0/2)(2\pi/N_0)n} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-j\pi n} \\ &= \frac{1}{N_0} \sum_{n=0}^{N_0-1} (-1)^n x[n] = \text{real} \end{aligned}$$

(6.125)

c. Rewrite Eq. (6.7) as

$$x[n] = \sum_{k=0}^{N_0-1} c_k\, e^{jk\Omega_0 n} = c_0 + \sum_{k=1}^{N_0-1} c_k\, e^{jk\Omega_0 n}$$

If N_0 is odd, then $(N_0 - 1)$ is even and we can write $x[n]$ as

$$x[n] = c_0 + \sum_{k=1}^{(N_0-1)/2} \left(c_k\, e^{jk\Omega_0 n} + c_{N_0-k}\, e^{j(N_0-k)\Omega_0 n} \right)$$

Now, from Eq. (6.17)

$$c_{N_0-k} = c_k^*$$

and

$$e^{j(N_0-k)\Omega_0 n} = e^{jN_0\Omega_0 n} e^{-jk\Omega_0 n} = e^{j2\pi n} e^{-jk\Omega_0 n} = e^{-jk\Omega_0 n}$$

Thus,

$$\begin{aligned} x[n] &= c_0 + \sum_{k=1}^{(N_0-1)/2} (c_k e^{jk\Omega_0 n} + c_k^* e^{-jk\Omega_0 n}) \\ &= c_0 + \sum_{k=1}^{(N_0-1)/2} 2\,\text{Re}(c_k e^{jk\Omega_0 n}) \\ &= c_0 + 2 \sum_{k=1}^{(N_0-1)/2} \text{Re}(a_k + jb_k)(\cos k\Omega_0 n + j \sin k\Omega_0 n) \\ &= c_0 + 2 \sum_{k=1}^{(N_0-1)/2} (a_k \cos k\Omega_0 n - b_k \sin k\Omega_0 n) \end{aligned}$$

If N_0 is even, we can write $x[n]$ as

$$x[n]=c_0+\sum_{k=1}^{N_0-1} c_k e^{jk\Omega_0 n}$$

$$=c_0+\sum_{k=1}^{(N_0-2)/2}\left(c_k e^{jk\Omega_0 n}+c_{N_0-k}e^{j(N_0-k)\Omega_0 n}\right)+c_{N_0/2}e^{j(N_0/2)\Omega_0 n}$$

Again from Eq. (6.17)

$$c_{N_0-k}=c_k^* \qquad \text{and} \qquad e^{j(N_0-k)\Omega_0 n}=e^{-jk\Omega_0 n}$$

and

$$e^{j(N_0/2)\Omega_0 n}=e^{j(N_0/2)(2\pi/N_0)n}=e^{j\pi n}=(-1)^n$$

Then

$$x[n]=c_0+(-1)^n c_{N_0/2}+\sum_{k=1}^{(N_0-2)/2} 2\,\mathrm{Re}(c_k e^{jk\Omega_0 n})$$

$$=c_0+(-1)^n c_{N_0/2}+2\sum_{k=1}^{(N_0-2)/2}(a_k\cos k\Omega_0 n-b_k\sin k\Omega_0 n)$$

6.8. Let $x_1[n]$ and $x_2[n]$ be periodic sequences with fundamental period N_0 and their discrete Fourier series given by

$$x_1[n]=\sum_{k=0}^{N_0-1} d_k e^{jk\Omega_0 n} \qquad x_2[n]=\sum_{k=0}^{N_0-1} e_k\, e^{jk\Omega_0 n} \qquad \Omega_0=\frac{2\pi}{N_0}$$

Show that the sequence $x[n] = x_1[n]x_2[n]$ is periodic with the same fundamental period N_0 and can be expressed as

$$x[n]=\sum_{k=0}^{N_0-1} c_k\, e^{jk\Omega_0 n} \qquad \Omega_0=\frac{2\pi}{N_0}$$

where c_k is given by

$$c_k=\sum_{m=0}^{N_0-1} d_m e_{k-m}$$

(6.126)

Now note that

$$x[n+N_0]=x_1[n+N_0]x_2[n+N_0]=x_1[n]x_2[n]=x[n]$$

Thus, $x[n]$ is periodic with fundamental period N_0. Let

$$x[n]=\sum_{k=0}^{N_0-1} c_k\, e^{jk\Omega_0 n} \qquad \Omega_0=\frac{2\pi}{N_0}$$

Then

$$c_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jk\Omega_0 n} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x_1[n]x_2[n]\,e^{-jk\Omega_0 n}$$
$$= \frac{1}{N_0}\sum_{n=0}^{N_0-1}\left(\sum_{m=0}^{N_0-1} d_m e^{jm\Omega_0 n}\right)x_2[n]\,e^{-jk\Omega_0 n}$$
$$= \sum_{m=0}^{N_0-1} d_m\left(\frac{1}{N_0}\sum_{n=0}^{N_0-1} x_2[n]e^{-j(k-m)\Omega_0 n}\right) = \sum_{m=0}^{N_0-1} d_m e_{k-m}$$

since

$$e_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x_2[n]e^{-jk\Omega_0 n}$$

and the term in parentheses is equal to e_{k-m}.

6.9. Let $x_1[n]$ and $x_2[n]$ be the two periodic signals in Prob. 6.8. Show that

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} x_1[n]x_2[n] = \sum_{k=0}^{N_0-1} d_k\, e_{-k}$$

(6.127)

Equation (6.127) is known as *Parseval's relation* for periodic sequences.

From Eq. (6.126) we have

$$c_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x_1[n]x_2[n]\,e^{-jk\Omega_0 n} = \sum_{m=0}^{N_0-1} d_m e_{k-m}$$

Setting $k = 0$ in the above expression, we get

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} x_1[n]x_2[n] = \sum_{m=0}^{N_0-1} d_m e_{-m} = \sum_{k=0}^{N_0-1} d_k e_{-k}$$

6.10.

a. Verify Parseval's identity [Eq. (6.19)] for the discrete Fourier series; that is,

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}\left|x[n]\right|^2 = \sum_{k=0}^{N_0-1}\left|c_k\right|^2$$

b. Using $x[n]$ in Prob. 6.3, verify Parseval's identity [Eq. (6.19)].

a. Let

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$$

and

$$x^*[n] = \sum_{k=0}^{N_0-1} d_k e^{jk\Omega_0 n}$$

Then

$$d_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x^*[n]\,e^{-jk\Omega_0 n} = \left(\frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{jk\Omega_0 n}\right)^* = c_{-k}^*$$

(6.128)

Equation (6.128) indicates that if the Fourier coefficients of $x[n]$ are c_k, then the Fourier coefficients of $x*[n]$ are c_{-k}^*. Setting $x_1[n] = x[n]$ and $x_2[n] = x*[n]$ in Eq. (6.127), we have $d_k = c_k$ and $e_k = c_{-k}^*$ (or $e^{-k} = c_k^*$) and we obtain

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]\,x^*[n] = \sum_{k=0}^{N_0-1} c_k c_k^*$$

(6.129)

and

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} |x[n]|^2 = \sum_{k=0}^{N_0-1} |c_k|^2$$

b. From Fig. 6-7 and the results from Prob. 6.3, we have

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} |x[n]|^2 = \frac{1}{4}(0+1^2+2^2+3^2) = \frac{14}{4} = \frac{7}{2}$$

$$\sum_{n=0}^{N_0-1} |c_k|^2 = \left(\frac{3}{2}\right)^2 + \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] + \left(-\frac{1}{2}\right)^2 + \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] = \frac{14}{4} = \frac{7}{2}$$

and Parseval's identity is verified.

6.9.2. Fourier Transform

6.11. Find the Fourier transform of

$$x[n] = -a^n u[-n-1] \qquad a \text{ real}$$

From Eq. (4.12) the z-transform of $x[n]$ is given by

$$X(z) = \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

Thus, $X(e^{j\Omega})$ exists for $|a| > 1$ because the ROC of $X(z)$ then contains the unit circle. Thus,

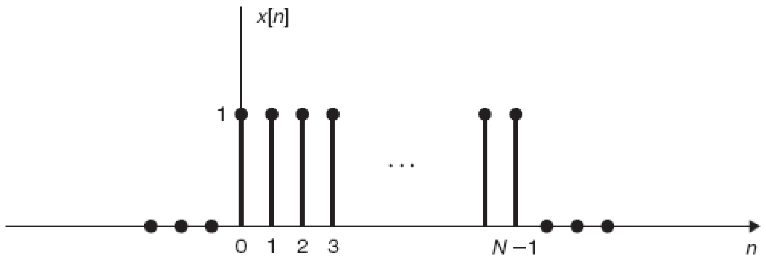
$$X(\Omega) = X(e^{j\Omega}) = \frac{1}{1-ae^{-j\Omega}} \qquad |a| > 1$$

(6.130)

6.12. Find the Fourier transform of the rectangular pulse sequence (Fig. 6-10)

$$x[n] = u[n] - u[n-N]$$

Figure 6-10



Using Eq. (1.90), the z-transform of $x[n]$ is given by

$$X(z) = \sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z} \qquad |z| > 0$$

(6.131)

Thus, $X(e^{j\Omega})$ exists because the ROC of $X(z)$ includes the unit circle. Hence,

$$\begin{aligned} X(\Omega) = X(e^{j\Omega}) &= \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}} = \frac{e^{-j\Omega N/2}\,(e^{j\Omega N/2} - e^{j\Omega N/2})}{e^{-j\Omega/2}\,(e^{j\Omega/2} - e^{-j\Omega/2})} \\ &= e^{-j\Omega(N-1)/2}\,\frac{\sin(\Omega N/2)}{\sin(\Omega/2)} \end{aligned}$$

(6.132)

6.13. Verify the time-shifting property (6.43); that is,

$$x[n - n_0] \leftrightarrow e^{-j\Omega n_0} X(\Omega)$$

By definition (6.27)

$$\mathscr{F}\{x[n - n_0]\} = \sum_{n=-\infty}^{\infty} x[n - n_0]\, e^{-j\Omega n}$$

By the change of variable $m = n - n_0$, we obtain

$$\begin{aligned} \mathscr{F}\{x[n - n_0]\} &= \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega(m+n_0)} \\ &= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega m} = e^{-j\Omega n_0} X(\Omega) \end{aligned}$$

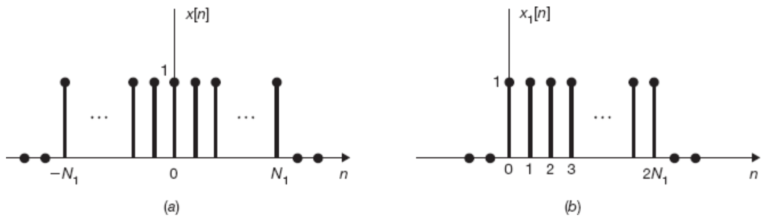
Hence,

$$x[n - n_0] \leftrightarrow e^{-j\Omega n_0} X(\Omega)$$

6.14.

a. Find the Fourier transform $X(\Omega)$ of the rectangular pulse sequence shown in Fig. 6-11(a).

Figure 6-11



b. Plot $X(\Omega)$ for $N_1 = 4$ and $N_1 = 8$.

a. From Fig. 6-11 we see that

$$x[n] = x_1[n + N_1]$$

where $x_1[n]$ is shown in Fig. 6-11(b). Setting $N = 2N_1 + 1$ in Eq. (6.132), we have

$$X_1(\Omega) = e^{-j\Omega N_1} \frac{\sin\left[\Omega\left(N_1 + \frac{1}{2}\right)\right]}{\sin(\Omega/2)}$$

Now, from the time-shifting property (6.43) we obtain

$$X(\Omega) = e^{j\Omega N_1} X_1(\Omega) = \frac{\sin\left[\Omega\left(N_1 + \frac{1}{2}\right)\right]}{\sin(\Omega/2)}$$

(6.133)

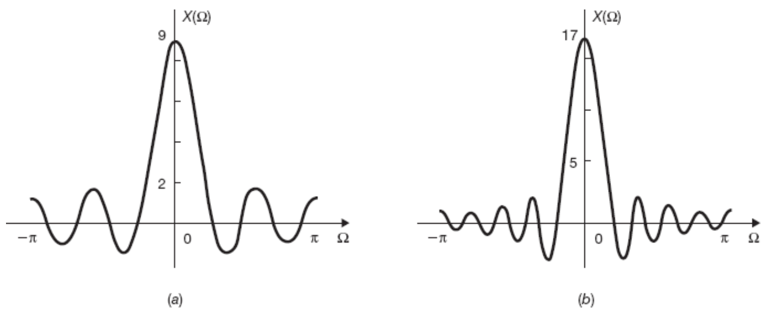
b. Setting $N_1 = 4$ in Eq. (6.133), we get

$$X(\Omega) = \frac{\sin(4.5\Omega)}{\sin(0.5\Omega)}$$

which is plotted in Fig. 6-12(a). Similarly, for $N_1 = 8$ we get

$$X(\Omega) = \frac{\sin(8.5\Omega)}{\sin(0.5\Omega)}$$

Figure 6-12



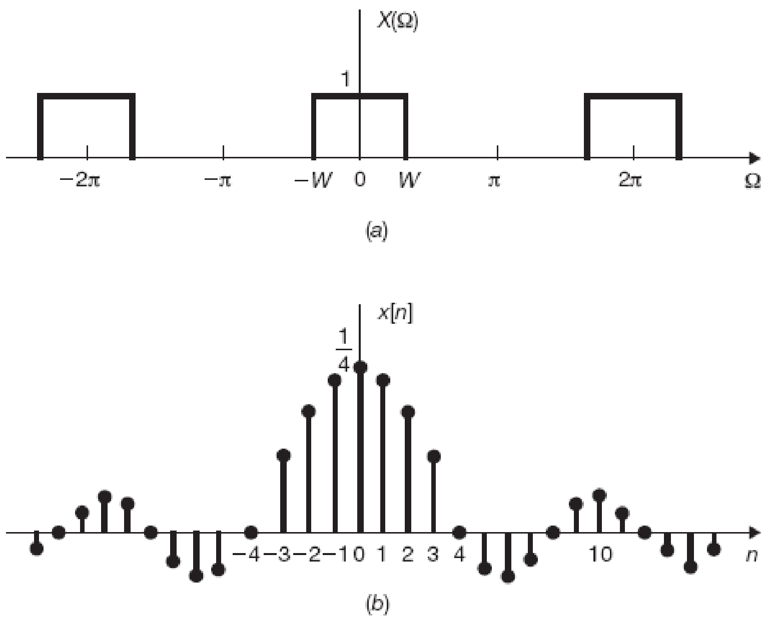
which is plotted in Fig. 6-12(b).

6.15.

a. Find the inverse Fourier transform $x[n]$ of the rectangular pulse spectrum $X(\Omega)$ defined by [Fig. 6-13(*a*)]

$$X(\Omega)=\begin{cases}1 & |\Omega|\le W\\ 0 & W<|\Omega|\le\pi\end{cases}$$

Figure 6-13



b. Plot $x[n]$ for $W = \pi/4$.

a. From Eq. (6.28)

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\Omega)\,e^{j\Omega n}\,d\Omega=\frac{1}{2\pi}\int_{-W}^{W}e^{j\Omega n}\,d\Omega=\frac{\sin Wn}{\pi n}$$

Thus, we obtain

$$\frac{\sin Wn}{\pi n}\leftrightarrow X(\Omega)=\begin{cases}1 & |\Omega|\le W\\ 0 & W<|\Omega|\le\pi\end{cases}$$

(6.134)

b. The sequence $x[n]$ is plotted in Fig. 6-13(*b*) for $W = \pi/4$.

6.16. Verify the frequency-shifting property (6.44); that is,

$$e^{j\Omega_0 n}x[n]\leftrightarrow X(\Omega-\Omega_0)$$

By Eq. (6.27)

$$\mathscr{F}\{e^{j\Omega_0 n}x[n]\}=\sum_{n=-\infty}^{\infty}e^{j\Omega_0 n}x[n]\,e^{-j\Omega n}$$

$$=\sum_{n=-\infty}^{\infty}x[n]\,e^{-j(\Omega-\Omega_0)n}=X(\Omega-\Omega_0)$$

Hence,

$$e^{j\Omega_0 n}x[n] \leftrightarrow X(\Omega-\Omega_0)$$

6.17. Find the inverse Fourier transform *x*[*n*] of

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0) \qquad |\Omega|, |\Omega_0| \le \pi$$

From Eqs. (6.28) and (1.22) we have

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\Omega-\Omega_0)\,e^{j\Omega n}\,d\Omega=e^{j\Omega_0 n}$$

Thus, we have

$$e^{j\Omega_0 n} \leftrightarrow 2\pi\,\delta(\Omega-\Omega_0) \qquad |\Omega|,|\Omega_0| \le \pi$$

(6.135)

6.18. Find the Fourier transform of

$$x[n] = 1 \qquad \text{all } n$$

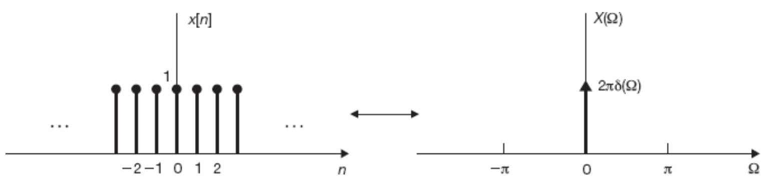
Setting $\Omega_0 = 0$ in Eq. (6.135), we get

$$x[n] = 1 \leftrightarrow 2\pi\delta(\Omega) \qquad |\Omega| \le \pi$$

(6.136)

Equation (6.136) is depicted in Fig. 6-14.

Figure 6-14 A constant sequence and its Fourier transform.



6.19. Find the Fourier transform of the sinusoidal sequence

$$x[n] = \cos \Omega_0 n \qquad |\Omega_0| \le \pi$$

From Euler's formula we have

$$\cos \Omega_0 n = \frac{1}{2}(e^{j\Omega_0 n} + e^{-j\Omega_0 n})$$

Thus, using Eq. (6.135) and the linearity property (6.42), we get

$$X(\Omega) = \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \qquad |\Omega|, |\Omega_0| \le \pi$$

which is illustrated in Fig. 6-15. Thus,

$$\cos \Omega_0 n \leftrightarrow \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \qquad |\Omega|, |\Omega_0| \le \pi$$

(6.137)

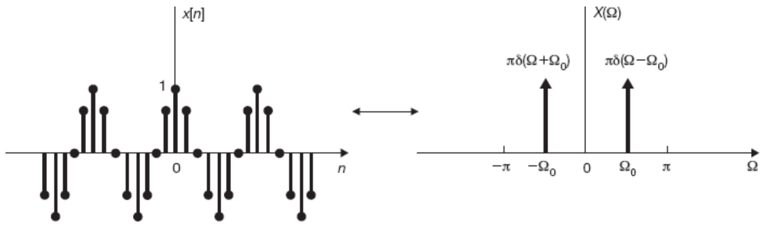


Figure 6-15 A cosine sequence and its Fourier transform.

6.20. Verify the conjugation property (6.45); that is,

$$x^*[n] \leftrightarrow X^*(-\Omega)$$

From Eq. (6.27)

$$\mathscr{F}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{-j\Omega n} = \left(\sum_{n=-\infty}^{\infty} x[n]\, e^{j\Omega n}\right)^*$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]\, e^{-j(-\Omega)n}\right)^* = X^*(-\Omega)$$

Hence,

$$x^*[n] \leftrightarrow X^*(-\Omega)$$

6.21. Verify the time-scaling property (6.49); that is,

$$x_{(m)}[n] \leftrightarrow X(m\Omega)$$

From Eq. (6.48)

$$x_{(m)}[n] = \begin{cases} x[n/m] = x[k] & \text{if } n = km,\ k = \text{integer} \\ 0 & \text{if } n \neq km \end{cases}$$

Then, by Eq. (6.27)

$$\mathscr{F}\{x_{(m)}[n]\} = \sum_{n=-\infty}^{\infty} x_{(m)}[n]\, e^{-j\Omega n}$$

Changing the variable $n = km$ on the right-hand side of the above expression, we obtain

$$\mathscr{F}\{x_{(m)}[n]\} = \sum_{k=-\infty}^{\infty} x_{(m)}[km]\, e^{-j\Omega km} = \sum_{k=-\infty}^{\infty} x[k]\, e^{-j(m\Omega)k} = X(m\Omega)$$

Hence,

$$x_{(m)}[n] \leftrightarrow X(m\Omega)$$

6.22. Consider the sequence $x[n]$ defined by

$$x[n] = \begin{cases} 1 & |n| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

a. Sketch $x[n]$ and its Fourier transform $X(\Omega)$.

b. Sketch the time-scaled sequence $x_{(2)}[n]$ and its Fourier transform $X_{(2)}(\Omega)$.

c. Sketch the time-scaled sequence $x_{(3)}[n]$ and its Fourier transform $X_{(3)}(\Omega)$.

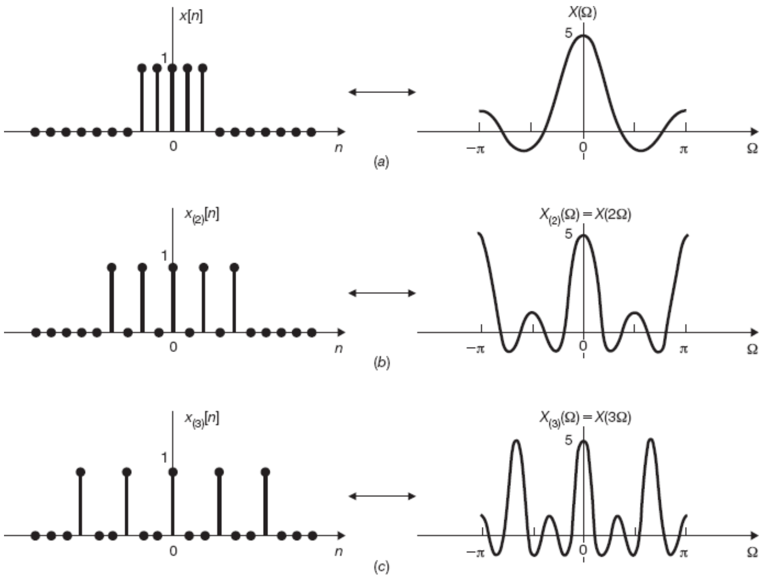
a. Setting $N_1 = 2$ in Eq. (6.133), we have

$$X(\Omega)=\frac{\sin(2.5\Omega)}{\sin(0.5\Omega)}$$

(6.138)

The sequence $x[n]$ and its Fourier transform $X(\Omega)$ are sketched in Fig. 6-16(a).

Figure 6-16



b. From Eqs. (6.49) and (6.138) we have

$$X_{(2)}(\Omega)=X(2\Omega)=\frac{\sin(5\Omega)}{\sin(\Omega)}$$

The time-scaled sequence $x_{(2)}[n]$ and its Fourier transform $X_{(2)}(\Omega)$ are sketched in Fig. 6-16(b).

c. In a similar manner we get

$$X_{(3)}(\Omega)=X(3\Omega)=\frac{\sin(7.5\Omega)}{\sin(1.5\Omega)}$$

The time-scaled sequence $x_{(3)}[n]$ and its Fourier transform $X_{(3)}(\Omega)$ are sketched in Fig. 6-16(c).

6.23. Verify the differentiation in frequency property (6.55); that is,

$$nx[n]\leftrightarrow j\frac{dX(\Omega)}{d\Omega}$$

From definition (6.27)

$$X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]\,e^{-j\Omega n}$$

Differentiating both sides of the above expression with respect to Ω and interchanging the order of differentiation and summation, we obtain

$$\frac{dX(\Omega)}{d\Omega} = \frac{d}{d\Omega}\left(\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}\right) = \sum_{n=-\infty}^{\infty} x[n] \frac{d}{d\Omega}(e^{-j\Omega n})$$
$$= -j \sum_{n=-\infty}^{\infty} nx[n]\, e^{-j\Omega n}$$

Multiplying both sides by j, we see that

$$\mathscr{F}\{nx[n]\} = \sum_{n=-\infty}^{\infty} nx[n]\, e^{-j\Omega n} = j\frac{dX(\Omega)}{d\Omega}$$

Hence,

$$nx[n] \leftrightarrow j\frac{dX(\Omega)}{d\Omega}$$

6.24. Verify the convolution theorem (6.58); that is,

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega)\, X_2(\Omega)$$

By definitions (2.35) and (6.27), we have

$$\mathscr{F}\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right) e^{-j\Omega n}$$

Changing the order of summation, we get

$$\mathscr{F}\{x_1[n] * x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k]\left(\sum_{n=-\infty}^{\infty} x_2[n-k]\, e^{-j\Omega n}\right)$$

By the time-shifting property Eq. (6.43)

$$\sum_{n=-\infty}^{\infty} x_2[n-k]\, e^{-j\Omega n} = e^{-j\Omega k} X_2(\Omega)$$

Thus, we have

$$\mathscr{F}\{x_1[n] * x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k]\, e^{-j\Omega k} X_2(\Omega)$$
$$= \left(\sum_{k=-\infty}^{\infty} x_1[k]\, e^{-j\Omega k}\right) X_2(\Omega) = X_1(\Omega)\, X_2(\Omega)$$

Hence,

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega)\, X_2(\Omega)$$

6.25. Using the convolution theorem (6.58), find the inverse Fourier transform $x[n]$ of

$$X(\Omega) = \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} \qquad |a| < 1$$

From Eq. (6.37) we have

$$a^n u[n] \leftrightarrow \frac{1}{1 - ae^{-j\Omega}} \qquad |a| < 1$$

Now

$$X(\Omega) = \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} = \left(\frac{1}{1 - ae^{-j\Omega}}\right)\left(\frac{1}{1 - ae^{-j\Omega}}\right)$$

Thus, by the convolution theorem Eq. (6.58) we get

$$x[n] = a^n u[n] * a^n u[n] = \sum_{k=-\infty}^{\infty} a^k u[k] a^{n-k} u[n-k]$$

$$= a^n \sum_{k=0}^{\infty} 1 = (n+1)\, a^n u[n]$$

Hence,

$$(n+1)\, a^n u[n] \leftrightarrow \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} \qquad |a| < 1$$

(6.139)

6.26. Verify the multiplication property (6.59); that is,

$$x_1[n] x_2[n] \leftrightarrow \frac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$$

Let $x[n] = x_1[n]x_2[n]$. Then by definition (6.27)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n]\, e^{-j\Omega n}$$

By Eq. (6.28)

$$x_1[n] = \frac{1}{2\pi} \int_{2\pi} X_1(\theta)\, e^{j\theta n}\, d\theta$$

Then

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{2\pi} X_1(\theta)\, e^{j\theta n}\, d\theta\right] x_2[n]\, e^{-j\Omega n}$$

Interchanging the order of summation and integration, we get

$$X(\Omega) = \frac{1}{2\pi} \int_{2\pi} X_1(\theta) \left(\sum_{n=-\infty}^{\infty} x_2[n]\, e^{-j(\Omega-\theta)n}\right) d\theta$$

$$= \frac{1}{2\pi} \int_{2\pi} X_1(\theta)\, X_2(\Omega - \theta)\, d\theta = \frac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$$

Hence,

$$x_1[n]x_2[n] \leftrightarrow \frac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$$

6.27. Verify the properties (6.62), (6.63a), and (6.63b); that is, if $x[n]$ is real and

$$x[n] = x_e[n] + x_o[n] \leftrightarrow X(\Omega) = A(\Omega) + jB(\Omega)$$

(6.140)

where $x_e[n]$ and $x_o[n]$ are the even and odd components of $x[n]$, respectively, then

$$X(-\Omega) = X^*(\Omega)$$
$$x_e[n] \leftrightarrow \mathrm{Re}\{X(\Omega)\} = A(\Omega)$$
$$x_o[n] \leftrightarrow j\,\mathrm{Im}\{X(\Omega)\} = jB(\Omega)$$

If $x[n]$ is real, then $x*[n]$, = $x[n]$, and by Eq. (6.45) we have

$$x^*[n] \leftrightarrow X^*(-\Omega)$$

from which we get

$$X(\Omega) = X^*(-\Omega) \qquad \text{or} \qquad X(-\Omega) = X^*(\Omega)$$

Next, using Eq. (6.46) and Eqs. (1.2) and (1.3), we have

$$x[-n] = x_e[n] - x_o[n] \leftrightarrow X(-\Omega) = X^*(\Omega) = A(\Omega) - jB(\Omega)$$

(6.141)

Adding (subtracting) Eq. (6.141) to (from) Eq. (6.140), we obtain

$$x_e[n] \leftrightarrow A(\Omega) = \mathrm{Re}\{X(\Omega)\}$$
$$x_o[n] \leftrightarrow jB(\Omega) = j\,\mathrm{Im}\{X(\Omega)\}$$

6.28. Show that

$$u[n] \leftrightarrow \pi\delta(\Omega) + \frac{1}{1-e^{-j\Omega}} \qquad |\Omega| \le \pi$$

(6.142)

Let

$$u[n] \leftrightarrow X(\Omega)$$

Now, note that

$$\delta[n] = u[n] - u[n-1]$$

Taking the Fourier transform of both sides of the above expression and by Eqs. (6.36) and (6.43), we have

$$1 = (1 - e^{-j\Omega})\, X(\Omega)$$

Noting that $(1 - e^{-j\Omega}) = 0$ for $\Omega = 0, X(\Omega)$ must be of the form

$$X(\Omega)=A\delta(\Omega)+\frac{1}{1-e^{-j\Omega}} \qquad |\Omega|\le\pi$$

where *A* is a constant. To determine *A* we proceed as follows. From Eq. (1.5) the even component of $u[n]$ is given by

$$u_e[n]=\frac{1}{2}+\frac{1}{2}\delta[n]$$

Then the odd component of $u[n]$ is given by

$$u_o[n]=u[n]-u_e[n]=u[n]-\frac{1}{2}-\frac{1}{2}\delta[n]$$

and

$$\mathscr{F}\{u_o[n]\}=A\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}-\pi\,\delta(\Omega)-\frac{1}{2}$$

From Eq. (6.63b) the Fourier transform of an odd real sequence must be purely imaginary. Thus, we must have $A = \pi$, and

$$u[n]\leftrightarrow\pi\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}} \qquad |\Omega|\le\pi$$

6.29. Verify the accumulation property (6.57); that is,

$$\sum_{k=-\infty}^{n}x[k]\leftrightarrow\pi X(0)\,\delta\,(\Omega)+\frac{1}{1-e^{-j\Omega}}X(\Omega) \qquad |\Omega|\le\pi$$

From Eq. (2.132)

$$\sum_{k=-\infty}^{n}x[k]=x[n]*u[n]$$

Thus, by the convolution theorem (6.58) and Eq. (6.142) we get

$$\begin{aligned}\sum_{k=-\infty}^{n}x[k]&\leftrightarrow X(\Omega)\left[\pi\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}\right] \qquad |\Omega|\le\pi\\&=\pi X(0)\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}X(\Omega)\end{aligned}$$

since $X(\Omega)\delta(\Omega) = X(0)\delta(\Omega)$ by Eq. (1.25).

6.30. Using the accumulation property (6.57) and Eq. (1.50), find the Fourier transform of $u[n]$.

From Eq. (1.50)

$$u[n]=\sum_{k=-\infty}^{n}\delta[k]$$

Now, from Eq. (6.36) we have

$$\delta[n]\leftrightarrow 1$$

Setting $x[k] = \delta[k]$ in Eq. (6.57), we have

$$x[n]=\delta[n]\leftrightarrow X(\Omega)=1 \qquad \text{and} \qquad X(0)=1$$

and

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] \leftrightarrow \pi\,\delta(\Omega) + \frac{1}{1-e^{-j\Omega}} \qquad |\Omega| \leq \pi$$

6.9.3. Frequency Response

6.31. A causal discrete-time LTI system is described by

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

(6.143)

where $x[n]$ and $y[n]$ are the input and output of the system, respectively (Prob. 4.32).

a. Determine the frequency response $H(\Omega)$ of the system.

b. Find the impulse response $h[n]$ of the system.

a. Taking the Fourier transform of Eq. (6.143), we obtain

$$Y(\Omega) - \frac{3}{4}e^{-j\Omega}\,Y(\Omega) + \frac{1}{8}e^{-j2\Omega}Y(\Omega) = X(\Omega)$$

or

$$\left(1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}\right)Y(\Omega) = X(\Omega)$$

Thus,

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}} = \frac{1}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}$$

b. Using partial-fraction expansions, we have

$$H(\Omega) = \frac{1}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)} = \frac{2}{1 - \frac{1}{2}e^{-j\Omega}} - \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}$$

Taking the inverse Fourier transform of $H(\Omega)$, we obtain

$$h[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]$$

which is the same result obtained in Prob. 4.32(*b*).

6.32. Consider a discrete-time LTI system described by

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{2}x[n-1]$$

(6.144)

a. Determine the frequency response $H(\Omega)$ of the system.

b. Find the impulse response $h[n]$ of the system.

c. Determine its response $y[n]$ to the input

$$x[n] = \cos\frac{\pi}{2}n$$

a. Taking the Fourier transform of Eq. (6.144), we obtain

$$Y(\Omega) - \frac{1}{2}e^{-j\Omega}Y(\Omega) = X(\Omega) + \frac{1}{2}e^{-j\Omega}X(\Omega)$$

Thus,

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1+\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

$$H(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{2}\frac{e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

Taking the inverse Fourier transform of $H(\Omega)$, we obtain

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1] = \begin{cases} 1 & n=0 \\ \left(\frac{1}{2}\right)^{n-1} & n \geq 1 \end{cases}$$

c. From Eq. (6.137)

$$X(\Omega) = \pi\left[\delta\left(\Omega - \frac{\pi}{2}\right) + \delta\left(\Omega + \frac{\pi}{2}\right)\right] \qquad |\Omega| \leq \pi$$

Then

$$\begin{aligned}
Y(\Omega) = X(\Omega)\,H(\Omega) &= \pi\left[\delta\left(\Omega - \frac{\pi}{2}\right) + \delta\left(\Omega + \frac{\pi}{2}\right)\right]\frac{1+\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}} \\
&= \pi\left(\frac{1+\frac{1}{2}e^{-j\pi/2}}{1-\frac{1}{2}e^{-j\pi/2}}\right)\delta\left(\Omega - \frac{\pi}{2}\right) + \pi\left(\frac{1+\frac{1}{2}e^{+j\pi/2}}{1-\frac{1}{2}e^{+j\pi/2}}\right)\delta\left(\Omega + \frac{\pi}{2}\right) \\
&= \pi\left(\frac{1-j\frac{1}{2}}{1+j\frac{1}{2}}\right)\delta\left(\Omega - \frac{\pi}{2}\right) + \pi\left(\frac{1+j\frac{1}{2}}{1-j\frac{1}{2}}\right)\delta\left(\Omega + \frac{\pi}{2}\right) \\
&= \pi\,\delta\left(\Omega - \frac{\pi}{2}\right)e^{-j2\tan^{-1}(1/2)} + \pi\,\delta\left(\Omega + \frac{\pi}{2}\right)e^{j2\tan^{-1}(1/2)}
\end{aligned}$$

Taking the inverse Fourier transform of $Y(\Omega)$ and using Eq. (6.135), we get

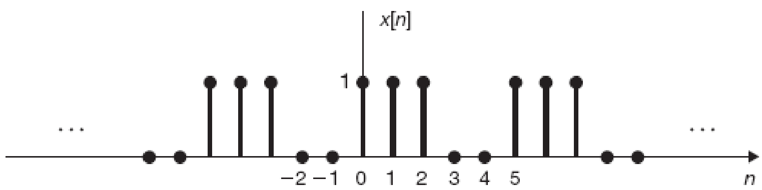
$$y[n] = \frac{1}{2} e^{j(\pi/2)n} e^{-j2\tan^{-1}(1/2)} + \frac{1}{2} e^{-j(\pi/2)} e^{j2\tan^{-1}(1/2)}$$
$$= \cos\left(\frac{\pi}{2} n - 2\tan^{-1}\frac{1}{2}\right)$$

6.33. Consider a discrete-time LTI system with impulse response

$$h[n] = \frac{\sin(\pi n/4)}{\pi n}$$

Find the output $y[n]$ if the input $x[n]$ is a periodic sequence with fundamental period $N_0 = 5$ as shown in Fig. 6-17.

Figure 6-17



From Eq. (6.134) we have

$$H(\Omega) = \begin{cases} 1 & |\Omega| \le \pi/4 \\ 0 & \pi/4 < |\Omega| \le \pi \end{cases}$$

Since $\Omega_0 = 2\pi/N_0 = 2\pi/5$ and the filter passes only frequencies in the range $|\Omega| \le \pi/4$, only the dc term is passed through. From Fig. 6-17 and Eq. (6.11)

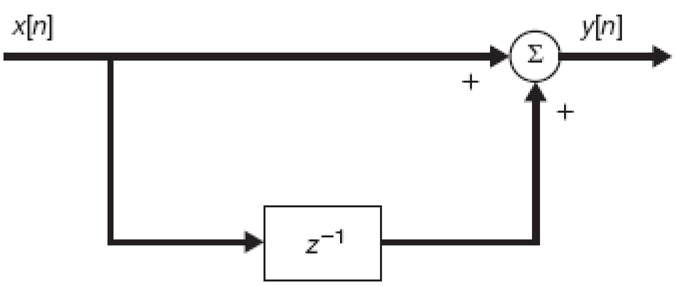
$$c_0 = \frac{1}{5} \sum_{n=0}^{4} x[n] = \frac{3}{5}$$

Thus, the output $y[n]$ is given by

$$y[n] = \frac{3}{5} \qquad \text{all } n$$

6.34. Consider the discrete-time LTI system shown in Fig. 6-18.

Figure 6-18



a. Find the frequency response $H(\Omega)$ of the system.

b. Find the impulse response $h[n]$ of the system.

c. Sketch the magnitude response $|H(\Omega)|$ and the phase response $\theta(\Omega)$.

d. Find the 3-dB bandwidth of the system.

a. From Fig. 6-18 we have

$$y[n] = x[n] + x[n-1]$$

(6.145)

Taking the Fourier transform of Eq. (6.145) and by Eq. (6.77), we have

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = 1 + e^{-j\Omega} = e^{-j\Omega/2}\left(e^{j\Omega/2} + e^{-j\Omega/2}\right)$$
$$= 2e^{-j\Omega/2}\cos\left(\frac{\Omega}{2}\right) \qquad |\Omega| \leq \pi$$

(6.146)

b. By the definition of $h[n]$ [Eq. (2.30)] and Eq. (6.145) we obtain

$$h[n] = \delta[n] + \delta[n-1]$$

or

$$h[n] = \begin{cases} 1 & 0 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

c. From Eq. (6.146)

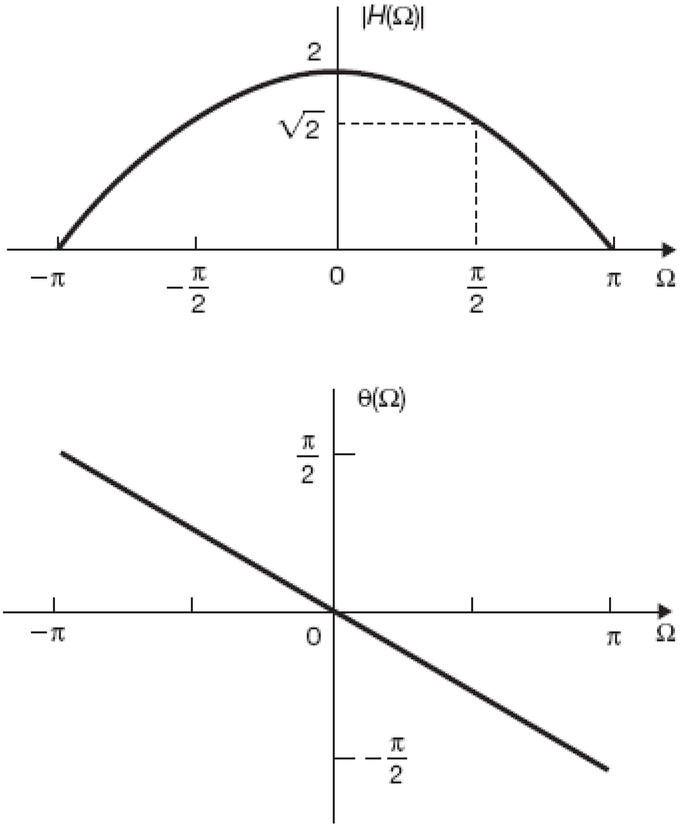
$$|H(\Omega)| = 2\cos\left(\frac{\Omega}{2}\right) \qquad |\Omega| \leq \pi$$

and

$$\theta(\Omega) = -\frac{\Omega}{2} \qquad |\Omega| \leq \pi$$

which are sketched in Fig. 6-19.

Figure 6-19



d. Let $\Omega_{3\text{ db}}$ be the 3-dB bandwidth of the system. Then by definition (Sec. 5.7)

$$\left|H(\Omega_{3\,\text{dB}})\right| = \frac{1}{\sqrt{2}}\left|H(\Omega)\right|_{\max}$$

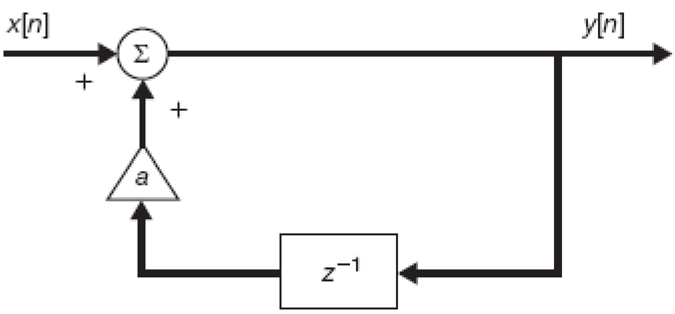
we obtain

$$\cos\left(\frac{\Omega_{3\,\text{dB}}}{2}\right) = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \Omega_{3\,\text{dB}} = \frac{\pi}{2}$$

We see that the system is a discrete-time wideband low-pass finite impulse response (FIR) filter (Sec. 2.9C).

6.35. Consider the discrete-time LTI system shown in Fig. 6-20, where a is a constant and $0 < a < 1$.

Figure 6-20



a. Find the frequency response $H(\Omega)$ of the system.

b. Find the impulse response $h[n]$ of the system.

c. Sketch the magnitude response $|H(\Omega)|$ of the system for $a = 0.9$ and $a = 0.5$.

a. From Fig. 6-20 we have

$$y[n] - ay[n-1] = x[n]$$

(6.147)

Taking the Fourier transform of Eq. (6.147) and by Eq. (6.77), we have

$$H(\Omega) = \frac{1}{1 - ae^{-j\Omega}} \qquad |a| < 1$$

(6.148)

b. Using Eq. (6.37), we obtain

$$h[n] = a^n u[n]$$

c. From Eq. (6.148)

$$H(\Omega) = \frac{1}{1 - ae^{-j\Omega}} = \frac{1}{1 - a\cos\Omega + ja\sin\Omega}$$

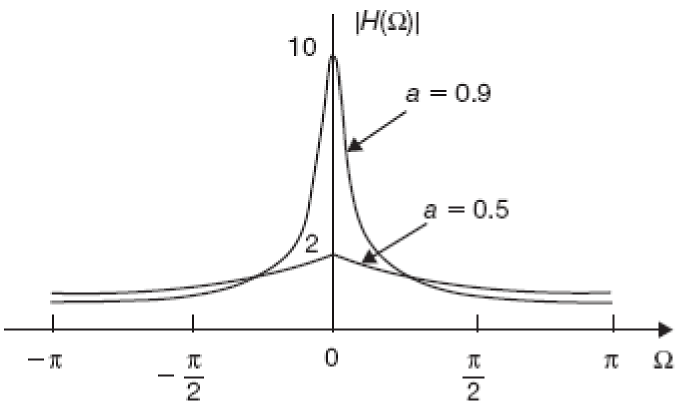
and

$$|H(\Omega)| = \frac{1}{\left[(1 - a\cos\Omega)^2 + (a\sin\Omega)^2\right]^{1/2}} = \frac{1}{(1 + a^2 - 2a\cos\Omega)^{1/2}}$$

(6.149)

which is sketched in Fig. 6-21 for $a = 0.9$ and $a = 0.5$.

Figure 6-21



We see that the system is a discrete-time low-pass infinite impulse response (IIR) filter (Sec. 2.9C).

6.36. Let $h_{LPF}[n]$ be the impulse response of a discrete-time low-pass filter with frequency response $H_{LPF}(\Omega)$. Show that a discrete-time filter whose impulse response $h[n]$ is given by

$$h[n] = (-1)^n h_{LPF}[n]$$

(6.150)

is a high-pass filter with the frequency response

$$H(\Omega) = H_{\text{LPF}}(\Omega - \pi)$$

(6.151)

Since $-1 = e^{j\pi}$, we can write

$$h[n] = (-1)^n h_{\text{LPF}}[n] = e^{j\pi n} h_{\text{LPF}}[n]$$

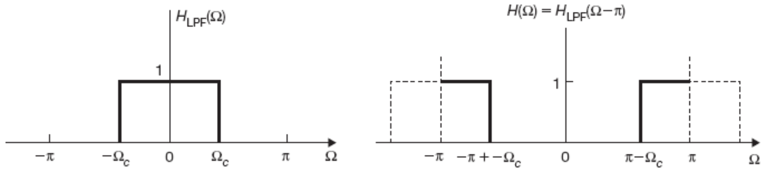
(6.152)

Taking the Fourier transform of Eq. (6.152) and using the frequency-shifting property (6.44), we obtain

$$H(\Omega) = H_{\text{LPF}}(\Omega - \pi)$$

which represents the frequency response of a high-pass filter. This is illustrated inFig. 6-22.

Figure 6-22 Transformation of a low-pass filter to a high-pass filter.



6.37. Show that if a discrete-time low-pass filter is described by the difference equation

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

(6.153)

then the discrete-time filter described by

$$y[n] = -\sum_{k=1}^{N} (-1)^k a_k\, y[n-k] + \sum_{k=0}^{M} (-1)^k b_k\, x[n-k]$$

(6.154)

is a high-pass filter.

Taking the Fourier transform of Eq. (6.153), we obtain the frequency response $H_{\text{LPF}}(\Omega)$ of the low-pass filter as

$$H_{\text{LPF}}(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\Omega}}{1 + \sum_{k=1}^{N} a_k e^{-jk\Omega}}$$

(6.155)

If we replace Ω by (Ω − π) in Eq. (6.155), then we have

$$H_{\mathrm{HPF}}(\Omega)=H_{\mathrm{LPF}}(\Omega-\pi)=\frac{\sum_{k=0}^{M} b_k e^{-jk(\Omega-\pi)}}{1+\sum_{k=1}^{N} a_k e^{-jk(\Omega-\pi)}}=\frac{\sum_{k=0}^{M} b_k(-1)^k e^{-jk\Omega}}{1+\sum_{k=1}^{N}(-1)^k a_k e^{-jk\Omega}}$$

(6.156)

which corresponds to the difference equation

$$y[n]=-\sum_{k=1}^{N}(-1)^k a_k\, y[n-k]+\sum_{k=0}^{M}(-1)^k b_k\, x[n-k]$$

6.38. Convert the discrete-time low-pass filter shown in Fig. 6-18 (Prob. 6.34) to a high-pass filter.

From Prob. 6.34 the discrete-time low-pass filter shown in Fig. 6-18 is described by [Eq. (6.145)]

$$y[n] = x[n] + x[n-1]$$

Using Eq. (6.154), the converted high-pass filter is described by

$$y[n] = x[n] - x[n-1]$$

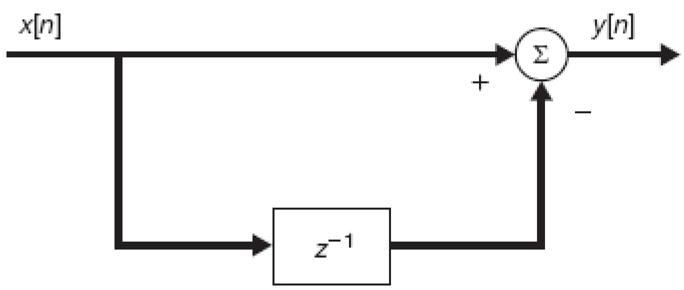
(6.157)

which leads to the circuit diagram in Fig. 6-23. Taking the Fourier transform of Eq. (6.157) and by Eq. (6.77), we have

$$\begin{aligned} H(\Omega)&=1-e^{-j\Omega}=e^{-j\Omega/2}\left(e^{j\Omega/2}-e^{-j\Omega/2}\right)\\ &=j2e^{-j\Omega/2}\sin\frac{\Omega}{2}=2e^{j(\pi-\Omega)/2}\sin\frac{\Omega}{2} \qquad |\Omega|\le\pi \end{aligned}$$

(6.158)

Figure 6-23



From Eq. (6.158)

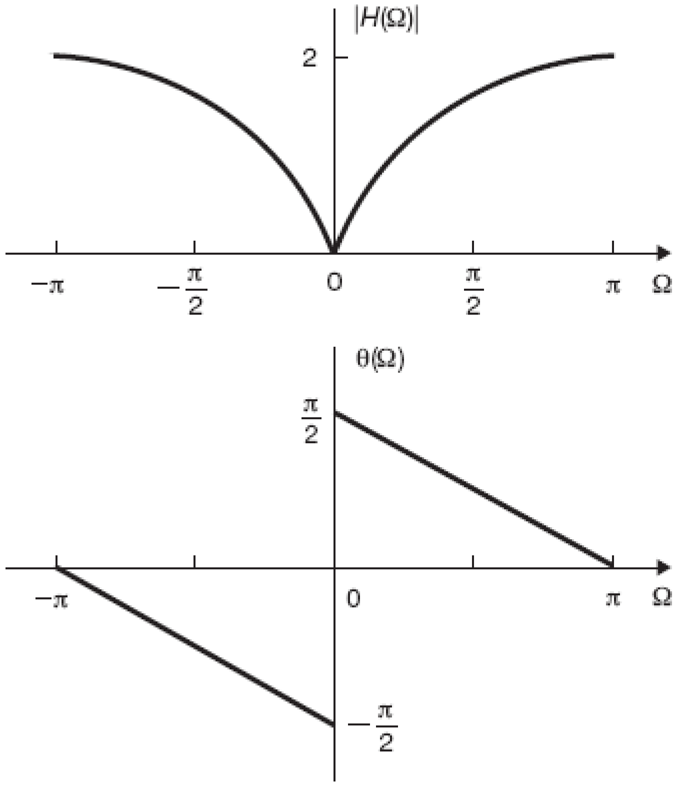
$$|H(\Omega)|=2\left|\sin\left(\frac{\Omega}{2}\right)\right| \qquad |\Omega|\le\pi$$

and

$$\theta(\Omega)=\begin{cases}(\pi-\Omega)/2 & 0<\Omega<\pi\\ (-\pi-\Omega)/2 & -\pi\le\Omega<0\end{cases}$$

which are sketched in Fig. 6-24. We see that the system is a discrete-time high-pass FIR filter.

Figure 6-24



6.39. The system function $H(z)$ of a causal discrete-time LTI system is given by

$$H(z) = \frac{b + z^{-1}}{1 - az^{-1}}$$

(6.159)

where a is real and $|a| < 1$. Find the value of b so that the frequency response $H(\Omega)$ of the system satisfies the condition

$$|H(\Omega)| = 1 \qquad \text{all } \Omega$$

(6.160)

Such a system is called an *all-pass* filter.

By Eq. (6.34) the frequency response of the system is

$$H(\Omega) = H(z)\big|_{z=e^{j\Omega}} = \frac{b + e^{-j\Omega}}{1 - ae^{-j\Omega}}$$

(6.161)

Then, by Eq. (6.160)

$$|H(\Omega)| = \left|\frac{b + e^{-j\Omega}}{1 - ae^{-j\Omega}}\right| = 1$$

which leads to

$$|b + e^{-j\Omega}| = |1 - ae^{-j\Omega}|$$

or

$$\left|b + \cos\Omega - j\sin\Omega\right| = \left|1 - a\cos\Omega + ja\sin\Omega\right|$$

or

$$1 + b^2 + 2b\cos\Omega = 1 + a^2 - 2a\cos\Omega$$

(6.162)

and we see that if $b = -a$, Eq. (6.162) holds for all Ω and Eq. (6.160) is satisfied.

6.40. Let $h[n]$ be the impulse response of an FIR filter so that

$$h[n] = 0 \qquad n < 0, n \geq N$$

Assume that $h[n]$ is real and let the frequency response $H(\Omega)$ be expressed as

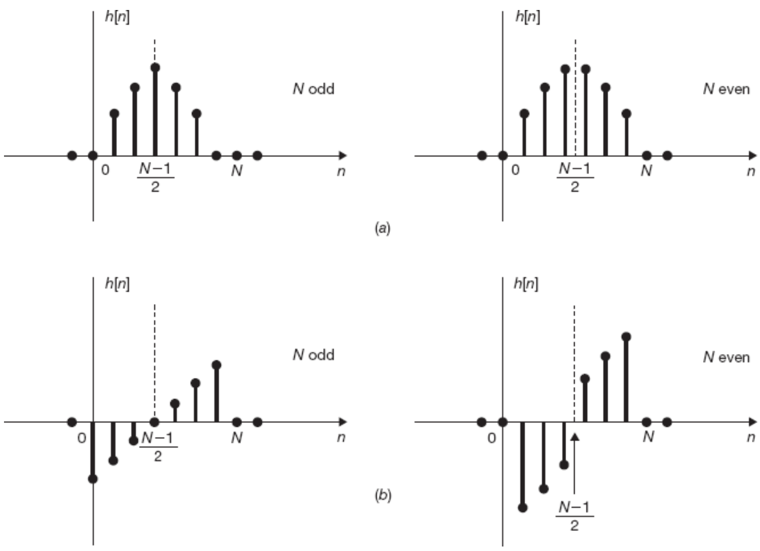
$$H(\Omega) = \left|H(\Omega)\right| e^{j\theta(\Omega)}$$

a. Find the phase response $\theta(\Omega)$ when $h[n]$ satisfies the condition [Fig. 6-25(*a*)]

$$h[n] = h[N - 1 - n]$$

(6.163)

Figure 6-25



b. Find the phase response $\theta(\Omega)$ when $h[n]$ satisfies the condition [Fig. 6-25(*b*)]

$$h[n] = -h[N - 1 - n]$$

(6.164)

a. Taking the Fourier transform of Eq. (6.163) and using Eqs. (6.43), (6.46), and (6.62), we obtain

$$H(\Omega) = H^*(\Omega)\, e^{-j(N-1)\Omega}$$

or

$$\left|H(\Omega)\right|\, e^{j\theta(\Omega)} = \left|H(\Omega)\right|\, e^{-j\theta(\Omega)}\, e^{-j(N-1)\Omega}$$

Thus,

$$\theta(\Omega) = -\theta(\Omega) - (N-1)\Omega$$

and

$$\theta(\Omega) = -\frac{1}{2}(N-1)\Omega$$

(6.165)

which indicates that the phase response is linear.

b. Similarly, taking the Fourier transform of Eq. (6.164), we get

$$H(\Omega) = -H^*(\Omega)\, e^{-j(N-1)\Omega}$$

or

$$|H(\Omega)|\, e^{j\theta(\Omega)} = |H(\Omega)|\, e^{j\pi}\, e^{-j\theta(\Omega)}\, e^{-j(N-1)\Omega}$$

Thus,

$$\theta(\Omega) = \pi - \theta(\Omega) - (N-1)\Omega$$

and

$$\theta(\Omega) = \frac{\pi}{2} - \frac{1}{2}(N-1)\Omega$$

(6.166)

which indicates that the phase response is also linear.

6.41. Consider a three-point moving-average discrete-time filter described by the difference equation

$$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

(6.167)

a. Find and sketch the impulse response *h*[*n*] of the filter.

b. Find the frequency response *H*(Ω) of the filter.

c. Sketch the magnitude response |*H*(Ω)| and the phase response θ(Ω) of the filter.

a. By the definition of *h*[*n*] [Eq. (2.30)] we have

$$h[n] = \frac{1}{3}\{\delta[n] + \delta[n-1] + \delta[n-2]\}$$

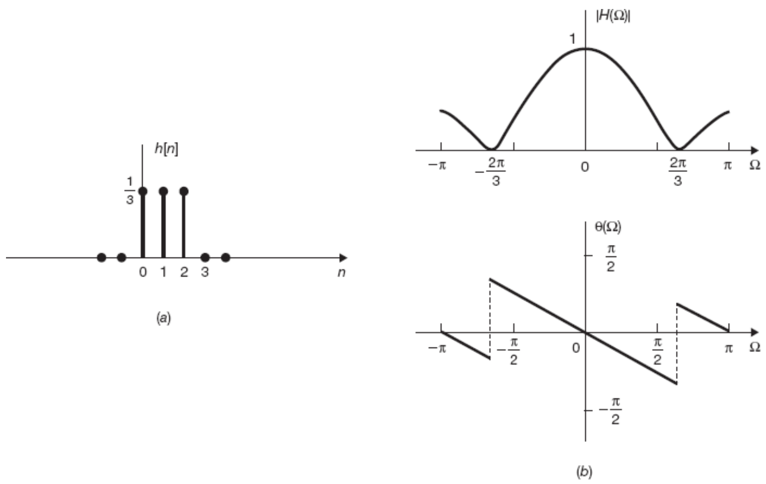
(6.168)

or

$$h[n] = \begin{cases} \dfrac{1}{3} & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

which is sketched in Fig. 6-26(a). Note that $h[n]$ satisfies the condition (6.163) with $N = 3$.

Figure 6-26



b. Taking the Fourier transform of Eq. (6.168), we have

$$H(\Omega)=\frac{1}{3}\{1+e^{-j\Omega}+e^{-2j\Omega}\}$$

By Eq. (1.90), with $\alpha = e^{-j\Omega}$, we get

$$H(\Omega)=\frac{1}{3}\frac{1-e^{-j3\Omega}}{1-e^{-j\Omega}}=\frac{1}{3}\frac{e^{-j3\Omega}(e^{j3\Omega/2}-e^{-j3\Omega/2})}{e^{-j\Omega/2}(e^{j\Omega/2}-e^{-j\Omega/2})}$$
$$=\frac{1}{3}e^{-j\Omega}\frac{\sin(3\Omega/2)}{\sin(\Omega/2)}=H_r(\Omega)\,e^{-j\Omega}$$

(6.169)

where

$$H_r(\Omega)=\frac{1}{3}\frac{\sin(3\Omega/2)}{\sin(\Omega/2)}$$

(6.170)

c. From Eq. (6.169)

$$|H(\Omega)|=|H_r(\Omega)|=\frac{1}{3}\left|\frac{\sin(3\Omega/2)}{\sin(\Omega/2)}\right|$$

and

$$\theta(\Omega)=\begin{cases}-\Omega & \text{when } H_r(\Omega)>0\\ -\Omega+\pi & \text{when } H_r(\Omega)<0\end{cases}$$

which are sketched in Fig. 6-26(b). We see that the system is a low-pass FIR filter with linear phase.

6.42. Consider a causal discrete-time FIR filter described by the impulse response

$$h[n] = \{2, 2, -2, -2\}$$

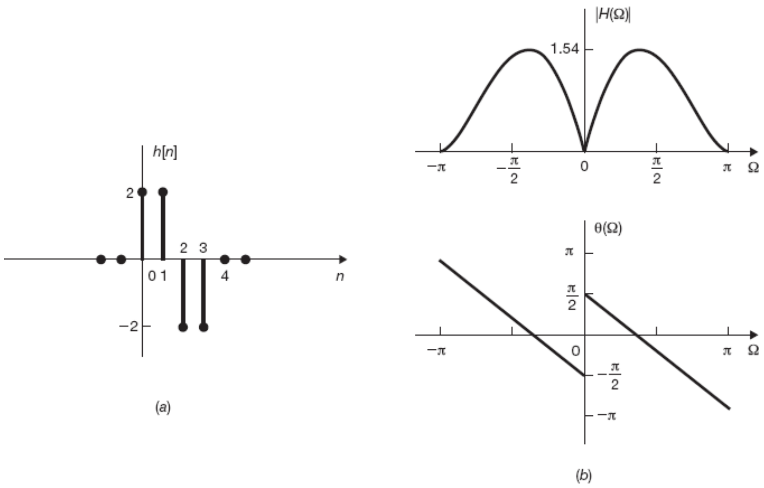
a. Sketch the impulse response $h[n]$ of the filter.

b. Find the frequency response $H(\Omega)$ of the filter.

c. Sketch the magnitude response $|H(\Omega)|$ and the phase response $\theta(\Omega)$ of the filter.

a. The impulse response $h[n]$ is sketched in Fig. 6-27(a). Note that $h[n]$ satisfies the condition (6.164) with $N = 4$.

Figure 6-27



b. By definition (6.27)

$$H(\Omega)=\sum_{n=-\infty}^{\infty} h[n]\,e^{-j\Omega n}=2+2e^{-j\Omega}-2e^{-j2\Omega}-2e^{-j3\Omega}$$
$$=2(1-e^{-j3\Omega})+2(e^{-j\Omega}-e^{-j2\Omega})$$
$$=2e^{-j3\Omega/2}(e^{j3\Omega/2}-e^{-j3\Omega/2})+2e^{-j3\Omega/2}(e^{j\Omega/2}-e^{-j\Omega/2})$$
$$=je^{-j3\Omega/2}\left(\sin\frac{\Omega}{2}+\sin\frac{3\Omega}{2}\right)=H_r(\Omega)\,e^{j[(\pi/2)-(3\Omega/2)]}$$

(6.171)

where

$$H_r(\Omega)=\sin\left(\frac{\Omega}{2}\right)+\sin\left(\frac{3\Omega}{2}\right)$$

c. From Eq. (6.171)

$$|H(\Omega)|=|H_r(\Omega)|=\left|\sin\left(\frac{\Omega}{2}\right)+\sin\left(\frac{3\Omega}{2}\right)\right|$$

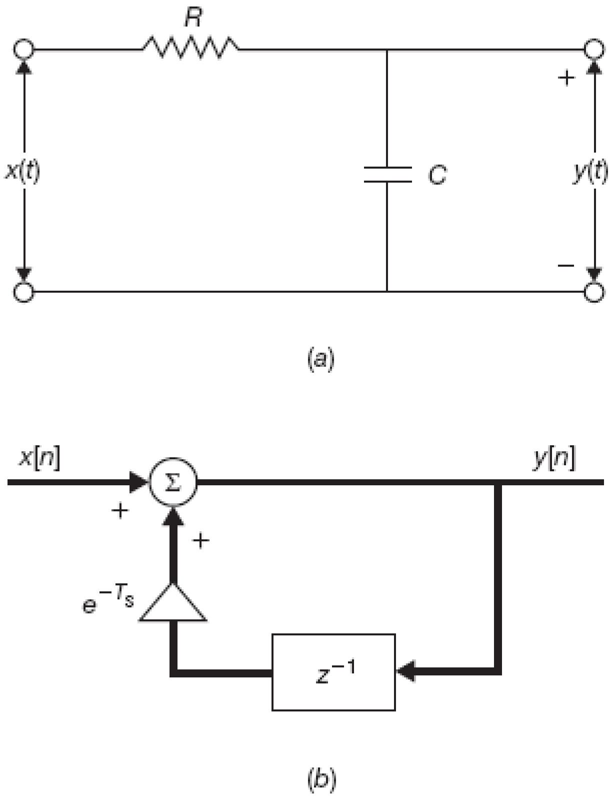
$$\theta(\Omega)=\begin{cases}\pi/2-\dfrac{3}{2}\Omega & H_r(\Omega)>0\\[2mm] -\pi/2-\dfrac{3}{2}\Omega & H_r(\Omega)<0\end{cases}$$

which are sketched in Fig. 6-27(b). We see that the system is a bandpass FIR filter with linear phase.

6.9.4. Simulation

6.43. Consider the *RC* low-pass filter shown in Fig. 6-28(*a*) with *RC* = 1.

Figure 6-28 Simulation of an RC filter by the impulse invariance method.



a. Construct a discrete-time filter such that

$$h_d[n] = h_c(t)\big|_{t=nT_s} = h_c(nT_s)$$

(6.172)

where $h_c(t)$ is the impulse response of the *RC* filter, $h_d[n]$ is the impulse response of the discrete-time filter, and T_s is a positive number to be chosen as part of the design procedures.

b. Plot the magnitude response $|H_c(\omega)|$ of the *RC* filter and the magnitude response $|H_d(\omega T_s)|$ of the discrete-time filter for $T_s = 1$ and $T_s = 0.1$.

a. The system function $H_c(s)$ of the *RC* filter is given by (Prob. 3.23)

$$H_c(s) = \frac{1}{s+1}$$

(6.173)

and the impulse response $h_c(t)$ is

$$h_c(t) = e^{-t}u(t)$$

(6.174)

By Eq. (6.172) the corresponding $h_d[n]$ is given by

$$h_d[n] = e^{-nT_s}u[n] = (e^{-T_s})^n\, u[n]$$

(6.175)

Then, taking the z-transform of Eq. (6.175), the system function $H_d(z)$ of the discrete-time filter is given by

$$H_d(z) = \frac{1}{1 - e^{-T_s} z^{-1}}$$

from which we obtain the difference equation describing the discrete-time filter as

$$y[n] - e^{-T_s} y[n-1] = x[n]$$

(6.176)

from which the discrete-time filter that simulates the RC filter is shown in Fig. 6-28(b).

b. By Eq. (5.40)

$$H_c(\omega) = H_c(s)\big|_{s=j\omega} = \frac{1}{j\omega + 1}$$

Then

$$|H_c(\omega)| = \frac{1}{(1+\omega^2)^{1/2}}$$

By Eqs. (6.34) and (6.81)

$$H_d(\omega T_s) = H_d(z)\big|_{z=e^{j\omega T_s}} = \frac{1}{1 - e^{-T_s} e^{-j\omega T_s}}$$

From Eq. (6.149)

$$|H_d(\omega T_s)| = \frac{1}{\left[1 + e^{-2T_s} - 2e^{-T_s}\cos(\omega T_s)\right]^{1/2}}$$

From $T_s = 1$,

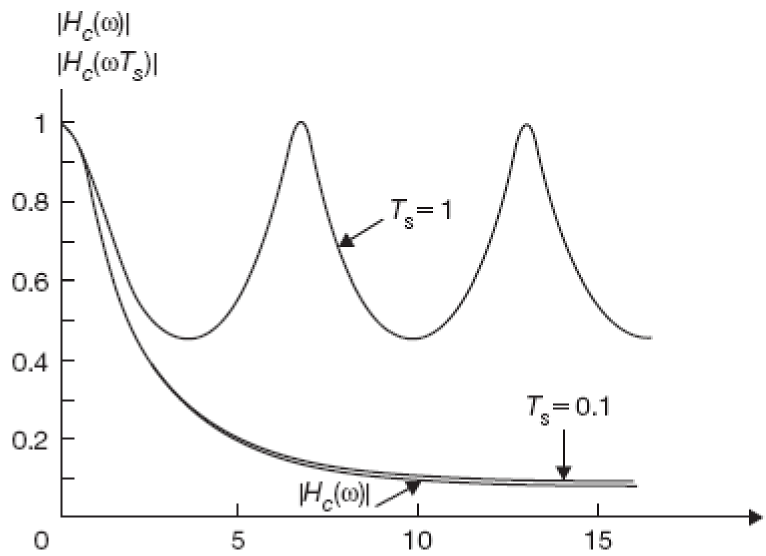
$$|H_d(\omega T_s)| = \frac{1}{\left[1 + e^{-2} - 2e^{-1}\cos(\omega)\right]^{1/2}}$$

For $T_s = 0.1$,

$$|H_d(\omega T_s)| = \frac{1}{\left[1 + e^{-0.2} - 2e^{-0.1}\cos(0.1\omega)\right]^{1/2}}$$

The magnitude response $|H_c(\omega)|$ of the RC filter and the magnitude response $|H_d(\omega T_s)|$ of the discrete-time filter for T_s = 1 and T_s = 0.1 are plotted in Fig. 6-29. Note that the plots are scaled such that the magnitudes at ω = 0 are normalized to 1.

Figure 6-29



The method utilized in this problem to construct a discrete-time system to simulate the continuous-time system is known as the *impulse-invariance* method.

6.44. By applying the impulse-invariance method, determine the frequency response $H_d(\Omega)$ of the discrete-time system to simulate the continuous-time LTI system with the system function

$$H_c(s) = \frac{1}{(s+1)(s+2)}$$

Using the partial-fraction expansion, we have

$$H_c(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Thus, by Table 3-1 the impulse response of the continuous-time system is

$$h_c(t) = (e^{-t} - e^{-2t})u(t)$$

(6.177)

Let $h_d[n]$ be the impulse response of the discrete-time system. Then, by Eq. (6.177)

$$h_d[n] = h_c(nT_s) = (e^{-nT_s} - e^{-2nT_s})\,u[n]$$

and the system function of the discrete-time system is given by

$$H_d(z) = \frac{1}{1 - e^{-nT_s} z^{-1}} - \frac{1}{1 - e^{-2nT_s} z^{-1}}$$

(6.178)

Thus, the frequency response $H_d(\Omega)$ of the discrete-time system is

$$H_d(\Omega) = H_d(z)\big|_{z=e^{j\Omega}} = \frac{1}{1 - e^{-nT_s} e^{-j\Omega}} - \frac{1}{1 - e^{-2nT_s} e^{-j\Omega}}$$

(6.179)

Note that if the system function of a continuous-time LTI system is given by

$$H_c(s)=\sum_{k=1}^{N}\frac{A_k}{s+\alpha_k}$$

(6.180)

then the impulse-invariance method yields the corresponding discrete-time system with the system function $H_d(z)$ given by

$$H_d(z)=\sum_{k=1}^{N}\frac{A_k}{1-e^{-\alpha_k n T_s}z^{-1}}$$

(6.181)

6.45. A differentiator is a continuous-time LTI system with the system function [Eq. (3.20)]

$$H_c(s) = s$$

(6.182)

A discrete-time LTI system is constructed by replacing s in $H_c(s)$ by the following transformation known as the *bilinear transformation*:

$$s=\frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}$$

(6.183)

to simulate the differentiator. Again T_s in Eq. (6.183) is a positive number to be chosen as part of the design procedure.

a. Draw a diagram for the discrete-time system.

b. Find the frequency response $H_d(\Omega)$ of the discrete-time system and plot its magnitude and phase responses.

a. Let $H_d(z)$ be the system function of the discrete-time system. Then, from Eqs. (6.182) and (6.183) we have

$$H_d(z)=\frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}$$

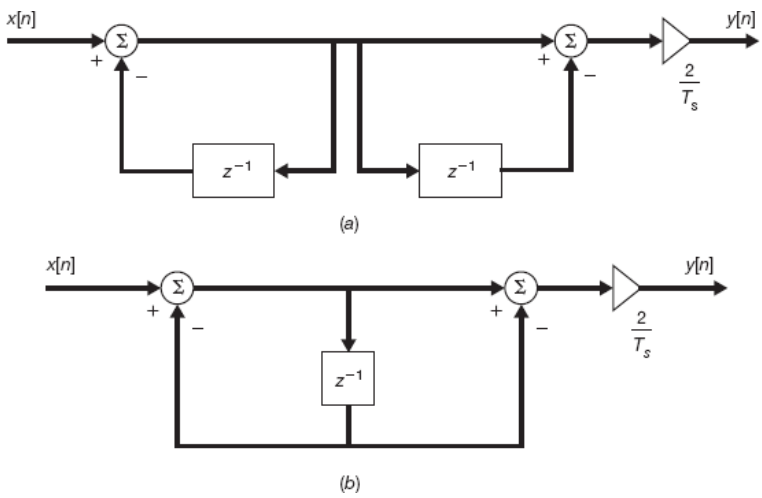
(6.184)

Writing $H_d(z)$ as

$$H_d(z)=\frac{2}{T_s}\left(\frac{1}{1+z^{-1}}\right)(1-z^{-1})$$

then, from Probs. (6.35) and (6.38) the discrete-time system can be constructed as a cascade connection of two systems as shown in Fig. 6-30(*a*). From Fig. 6-30(*a*) it is seen that we can replace two unit-delay elements by one unit-delay element as shown in Fig. 6-30(*b*).

Figure 6-30 *Simulation of a differentiator.*



b. By Eq. (6.184) the frequency response $H_d(\Omega)$ of the discrete-time system is given by

$$H_d(\Omega)=\frac{2}{T_s}\frac{1-e^{-j\Omega}}{1+e^{-j\Omega}}=\frac{2}{T_s}\frac{e^{-j\Omega/2}(e^{j\Omega/2}-e^{-j\Omega/2})}{e^{-j\Omega/2}(e^{j\Omega/2}+e^{-j\Omega/2})}$$
$$=j\frac{2}{T_s}\frac{\sin\Omega/2}{\cos\Omega/2}=j\frac{2}{T_s}\tan\frac{\Omega}{2}=\frac{2}{T_s}\tan\frac{\Omega}{2}e^{j\pi/2}$$

(6.185)

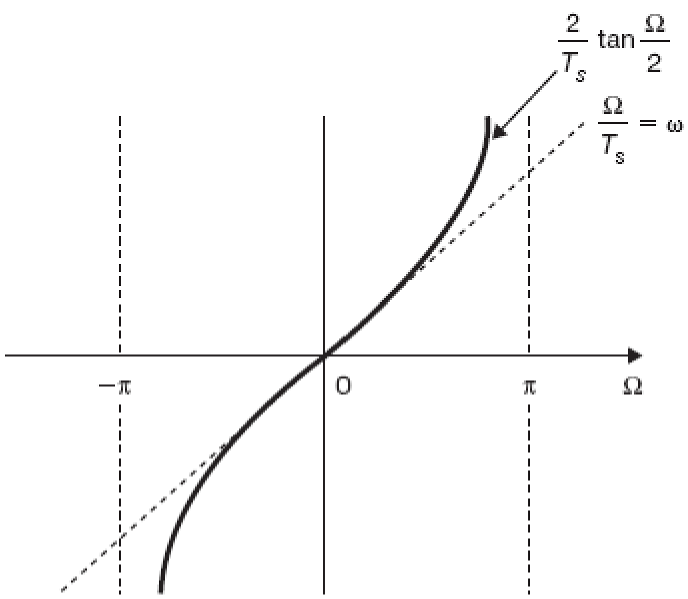
Note that when Ω << 1, we have

$$H_d(\Omega)=j\frac{2}{T_s}\tan\frac{\Omega}{2}\approx j\frac{\Omega}{T_s}=j\omega$$

(6.186)

If $\Omega=\omega T_s$(Fig. 6-31).

Figure 6-31



6.46. Consider designing a discrete-time LTI system with system function $H_d(z)$ obtained by applying the bilinear transformation to a continuous-time LTI system with rational system function $H_c(s)$. That is,

$$H_d(z) = H_c(s)\big|_{s = (2/T_s)(1 - z^{-1})/(1 + z^{-1})}$$

(6.187)

Show that a stable, causal continuous-time system will always lead to a stable, causal discrete-time system.

Consider the bilinear transformation of Eq. (6.183)

$$s = \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}}$$

(6.188)

Solving Eq. (6.188) for z, we obtain

$$z = \frac{1 + (T_s/2)s}{1 - (T_s/2)s}$$

(6.189)

Setting $s = j\omega$ in Eq. (6.189), we get

$$|z| = \left|\frac{1 + j\omega(T_s/2)}{1 - j\omega\,(T_s/2)}\right| = 1$$

(6.190)

Thus, we see that the $j\omega$-axis of the s-plane is transformed into the unit circle of the z-plane. Let

$$z = re^{j\Omega} \qquad \text{and} \qquad s = \sigma + j\omega$$

Then from Eq. (6.188)

$$s = \frac{2}{T_s}\frac{z-1}{z+1} = \frac{2}{T_s}\frac{re^{j\Omega} - 1}{re^{j\Omega} + 1}$$
$$= \frac{2}{T_s}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\Omega} + j\frac{2r\sin\Omega}{1 + r^2 + 2r\cos\Omega}\right)$$

Hence,

$$\sigma = \frac{2}{T_s}\frac{r^2 - 1}{1 + r^2 + 2r\cos\Omega}$$

(6.191a)

$$\omega = \frac{2}{T_s}\frac{2r\sin\Omega}{1 + r^2 + 2r\cos\Omega}$$

(6.191b)

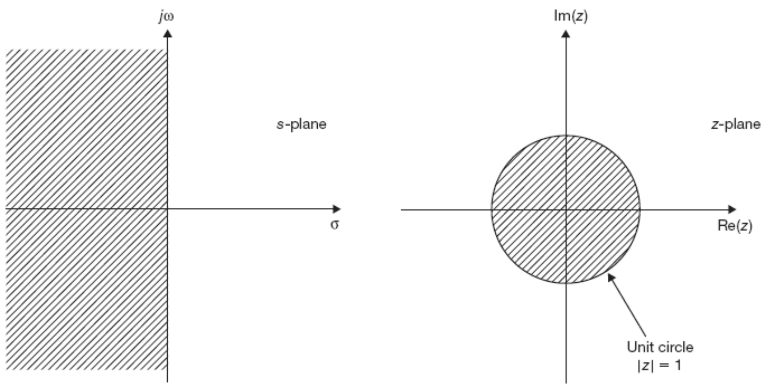
From Eq. (6.191a) we see that if $r < 1$, then $\sigma < 0$, and if $r > 1$, then $\sigma > 0$. Consequently, the left-hand plane (LHP) in s maps into the inside of the unit circle in the z-plane, and the right-hand plane (RHP) in s maps into the outside of the unit circle

(Fig. 6-32). Thus, we conclude that a stable, causal continuous-time system will lead to a stable, causal discrete-time system with a bilinear transformation (see Sec. 3.6B and Sec. 4.6B). When $r = 1$, then $\sigma = 0$ and

$$\omega = \frac{2}{T_s}\frac{\sin\Omega}{1+\cos\Omega} = \frac{2}{T_s}\tan\frac{\Omega}{2}$$

(6.192)

Figure 6-32 Bilinear transformation.



or

$$\Omega = 2\tan^{-1}\frac{\omega T_s}{2}$$

(6.193)

From Eq. (6.193) we see that the entire range $-\infty < \omega < \infty$ is mapped only into the range $-\pi \le \Omega \le \pi$.

6.47. Consider the low-pass *RC* filter in Fig. 6-28(*a*). Design a low-pass discrete-time filter by the bilinear transformation method such that its 3-dB bandwidth is $\pi/4$.

Using Eq. (6.192), $\Omega_{3\,\mathrm{dB}} = \pi/4$ corresponds to

$$\omega_{3\,\mathrm{dB}} = \frac{2}{T_s}\tan\frac{\Omega_{3\,\mathrm{dB}}}{2} = \frac{2}{T_s}\tan\frac{\pi}{8} = \frac{0.828}{T_s}$$

(6.194)

From Prob. 5.55(*a*), $\omega_{3\,\mathrm{dB}} = 1/RC$. Thus, the system function $H_c(s)$ of the *RC* filter is given by

$$H_c(s) = \frac{0.828/T_s}{s + 0.828/T_s}$$

(6.195)

Let $H_d(z)$ be the system function of the desired discrete-time filter. Applying the bilinear transformation (6.183) to Eq. (6.195), we get

$$H_d(z) = \frac{0.828/T_s}{\dfrac{2}{T_s}\dfrac{1-z^{-1}}{1+z^{-1}} + \dfrac{0.828}{T_s}} = \frac{0.293\,(1+z^{-1})}{1-0.414z^{-1}}$$

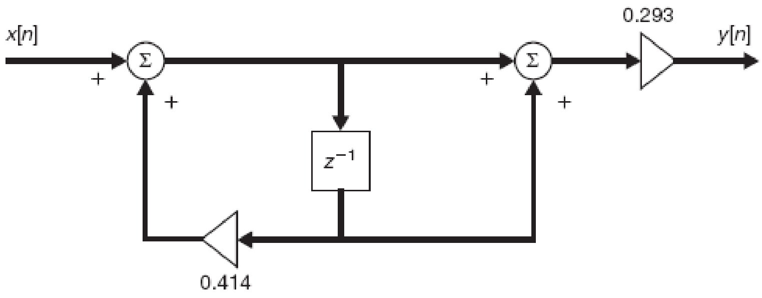
(6.196)

from which the system in Fig. 6-33 results. The frequency response of the discrete-time filter is

$$H_d(\Omega) = \frac{0.293\,(1 + e^{-j\Omega})}{1 - 0.414\,e^{-j\Omega}}$$

(6.197)

Figure 6-33 Simulation of an RC filter by the bilinear transformation method.



At Ω = 0, $H_d(0) = 1$, and at Ω = π/4, $|H_d(\pi/4)| = 0.707 = 1/\sqrt{2}$, which is the desired response.

6.48. Let $h[n]$ denote the impulse response of a desired IIR filter with frequency response $H(\Omega)$ and let $h_o[n]$ denote the impulse response of an FIR filter of length N with frequency response $H_o(\Omega)$. Show that when

$$h_o[n] = \begin{cases} h[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

(6.198)

the mean-square error ε^2 defined by

$$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\Omega) - H_o(\Omega)|^2 \, d\Omega$$

(6.199)

is minimized.

By definition (6.27)

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]\,e^{-j\Omega n} \qquad \text{and} \qquad H_o(\Omega) = \sum_{n=-\infty}^{\infty} h_o[n]\,e^{-j\Omega n}$$

Let

$$E(\Omega) = H(\Omega) - H_o(\Omega) = \sum_{n=-\infty}^{\infty} (h[n] - h_o[n])\,e^{-j\Omega n}$$
$$= \sum_{n=-\infty}^{\infty} e[n]\,e^{-j\Omega n}$$

(6.200)

where $e[n] = h[n] - h_o[n]$. By Parseval's theorem (6.66) we have

$$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|E(\Omega)|^2\,d\Omega = \sum_{n=-\infty}^{\infty}|e[n]|^2 = \sum_{n=-\infty}^{\infty}|h[n]-h_o[n]|^2$$
$$= \sum_{n=0}^{N-1}|h[n]-h_0[n]|^2 + \sum_{n=-\infty}^{-1}|h[n]|^2 + \sum_{n=N}^{\infty}|h[n]|^2$$

(6.201)

The last two terms in Eq. (6.201) are two positive constants. Thus, ε^2 is minimized when

$$h[n] - h_o[n] = 0 \qquad 0 \le n \le N-1$$

that is,

$$h[n] = h_o[n] \qquad 0 \le n \le N-1$$

Note that Eq. (6.198) can be expressed as

$$h_o[n] = h[n]w[n]$$

(6.202)

where $w[n]$ is known as a rectangular *window function* given by

$$w[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

(6.203)

6.9.5. Discrete Fourier Transform

6.49. Find the N-point DFT of the following sequences $x[n]$:

a. $x[n] = \delta[n]$

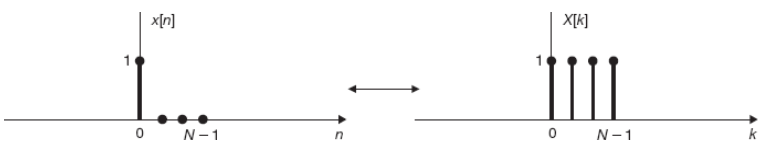
b. $x[n] = u[n] - u[n - N]$

a. From definitions (6.92) and (1.45), we have

$$X[k] = \sum_{n=0}^{N-1}\delta[n]w_N^{kn} = 1 \qquad k = 0, 1, \ldots, N-1$$

Fig. 6-34 shows $x[n]$ and its N-point DFT $X[k]$.

Figure 6-34



b. Again from definitions (6.92) and (1.44) and using Eq. (1.90), we obtain

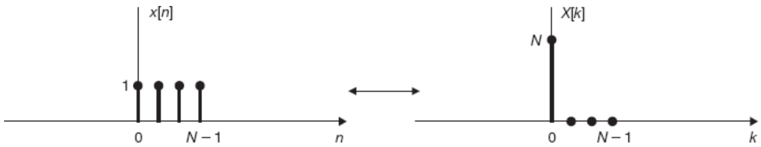
$$X[k] = \sum_{n=0}^{N-1}W_N^{kn} = \frac{1-W_N^{kN}}{1-W_N^{k}} = 0 \qquad k \ne 0$$

since $W_N^{kN} = e^{-j(2\pi/N)kN} = e^{-jk2\pi} = 1$.

$$X[0] = \sum_{n=0}^{N-1} W_N^0 = \sum_{n=0}^{N-1} 1 = N$$

Fig. 6-35 shows $x[n]$ and its N-point DFT $X[k]$.

Figure 6-35



6.50. Consider two sequences $x[n]$ and $h[n]$ of length 4 given by

$$x[n] = \cos\left(\frac{\pi}{2}n\right) \qquad n = 0, 1, 2, 3$$

$$h[n] = \left(\frac{1}{2}\right)^n \qquad n = 0, 1, 2, 3$$

a. Calculate $y[n] = x[n] \otimes h[n]$ by doing the circular convolution directly.

b. Calculate $y[n]$ by DFT.

a. The sequences $x[n]$ and $h[n]$ can be expressed as

$$x[n] = \{1, 0, -1, 0\} \qquad \text{and} \qquad h[n] = \{1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}\}$$

By Eq. (6.108)

$$y[n] = x[n] \otimes h[n] = \sum_{i=0}^{3} x[i]h[n-i]_{\text{mod}4}$$

The sequences $x[i]$ and $h[n-i]_{\text{mod}4}$ for $n = 0, 1, 2, 3$ are plotted in Fig. 6-36(a). Thus, by Eq. (6.108) we get

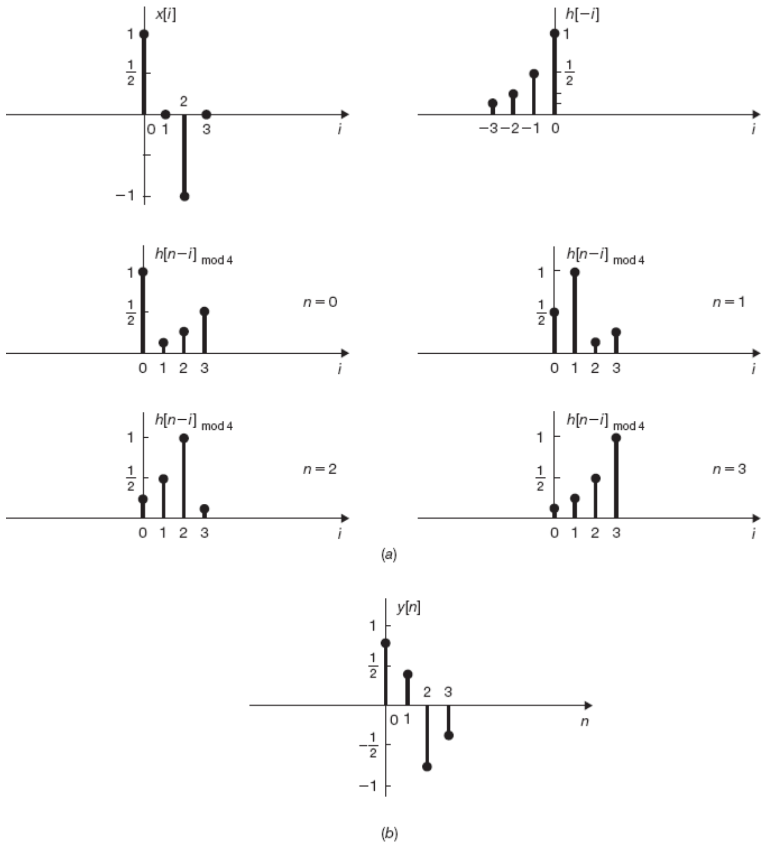
$$n = 0 \qquad y[0] = 1(1) + (-1)\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$n = 1 \qquad y[1] = 1\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{8}\right) = \frac{3}{8}$$

$$n = 2 \qquad y[2] = 1\left(\frac{1}{4}\right) + (-1)(1) = -\frac{3}{4}$$

$$n = 3 \qquad y[3] = 1\left(\frac{1}{8}\right) + (-1)\left(\frac{1}{2}\right) = -\frac{3}{8}$$

Figure 6-36



and

$$y[n]=\left\{\frac{3}{4},\frac{3}{8},-\frac{3}{4},-\frac{3}{8}\right\}$$

which is plotted in Fig. 6-36(*b*).

b. By Eq. (6.92)

$$X[k]=\sum_{n=0}^{3}x[n]W_4^{kn}=1-W_4^{2k}\qquad k=0,1,2,3$$

$$H[k]=\sum_{n=0}^{3}h[n]W_4^{kn}=1+\frac{1}{2}W_4^{k}+\frac{1}{4}W_4^{2k}+\frac{1}{8}W_4^{3k}\qquad k=0,1,2,3$$

Then by Eq. (6.107) the DFT of *y*[*n*] is

$$Y[k]=X[k]H[k]=\left(1-W_4^{2k}\right)\left(1+\frac{1}{2}W_4^{k}+\frac{1}{4}W_4^{2k}+\frac{1}{8}W_4^{3k}\right)$$

$$=1+\frac{1}{2}W_4^{k}-\frac{3}{4}W_4^{2k}-\frac{3}{8}W_4^{3k}-\frac{1}{4}W_4^{4k}-\frac{1}{8}W_4^{5k}$$

Since $W_4^{4k}=(W_4^{4})^k=1^k$ and $W_4^{5k}=W_4^{(4+1)k}=W_4^{k}$, we obtain

$$Y[k]=\frac{3}{4}+\frac{3}{8}W_4^{k}-\frac{3}{4}W_4^{2k}-\frac{3}{8}W_4^{3k}\qquad k=0,1,2,3$$

Thus, by the definition of DFT [Eq. (6.92)] we get

$$y[n] = \left\{ \frac{3}{4}, \frac{3}{8}, -\frac{3}{4}, -\frac{3}{8} \right\}$$

6.51. Consider the finite-length complex exponential sequence

$$x[n] = \begin{cases} e^{j\Omega_0 n} & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the Fourier transform $X(\Omega)$ of $x[n]$.

b. Find the N-point DFT $X[k]$ of $x[n]$.

a. From Eq. (6.27) and using Eq. (1.90), we have

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{j\Omega_0 n} e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{-j(\Omega-\Omega_0)n} \\ &= \frac{1-e^{-j(\Omega-\Omega_0)N}}{1-e^{-j(\Omega-\Omega_0)}} = \frac{e^{-j(\Omega-\Omega_0)N/2}\left(e^{j(\Omega-\Omega_0)N/2} - e^{-j(\Omega-\Omega_0)N/2}\right)}{e^{-j(\Omega-\Omega_0)/2}\left(e^{j(\Omega-\Omega_0)/2} - e^{-j(\Omega-\Omega_0)/2}\right)} \\ &= e^{j(\Omega-\Omega_0)(N-1)/2} \frac{\sin[(\Omega-\Omega_0)N/2]}{\sin[(\Omega-\Omega_0)/2]} \end{aligned}$$

b. Note from Eq. (6.98) that

$$X[k] = X(\Omega)\big|_{\Omega = k2\pi/N} = X\left(\frac{k2\pi}{N}\right)$$

we obtain

$$X[k] = e^{j[(2\pi/N)k-\Omega_0][(N-1)/2]} \frac{\sin\left[\left(\frac{2\pi}{N}k-\Omega_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi}{N}k-\Omega_0\right)\frac{1}{2}\right]}$$

6.52. Show that if $x[n]$ is real, then its DFT $X[k]$ satisfies the relation

$$X[N-k] = X^*[k]$$

(6.204)

where $*$ denotes the complex conjugate.

From Eq. (6.92)

$$X[N-k] = \sum_{n=0}^{N-1} x[n] W_N^{(N-k)n} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)(N-k)n}$$

Now

$$e^{-j(2\pi/N)(N-k)n} = e^{-j2\pi n} e^{j(2\pi/N)kn} = e^{j(2\pi/N)kn}$$

Hence, if $x[n]$ is real, then $x*[n] = x[n]$ and

$$X[N-k]=\sum_{n=0}^{N-1} x[n]\, e^{j(2\pi/N)kn}=\left[\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}\right]^{*}=X^{*}[k]$$

6.53. Show that

$$x[n]=\mathrm{IDFT}\{X[k]\}=\frac{1}{N}[\mathrm{DFT}\{X^{*}[k]\}]^{*}$$

(6.205)

where $*$ denotes the complex conjugate and

$$X[k]=\mathrm{DFT}\{x[n]\}$$

We can write Eq. (6.94) as

$$x[n]=\frac{1}{N}\left[\sum_{n=0}^{N-1} X[k]\, e^{j(2\pi/N)kn}\right]=\frac{1}{N}\left[\sum_{n=0}^{N-1} X^{*}[k]\, e^{-j(2\pi/N)nk}\right]^{*}$$

Noting that the term in brackets in the last term is the DFT of $X*[k]$, we get

$$x[n]=\mathrm{IDFT}\{X[k]\}=\frac{1}{N}[\mathrm{DFT}\{X^{*}[k]\}]^{*}$$

which shows that the same algorithm used to evaluate the DFT can be used to evaluate the IDFT.

6.54. The DFT definition in Eq. (6.92) can be expressed in a matrix operation form as

$$\mathbf{X}=\mathbf{W}_N\mathbf{x}$$

(6.206)

where

$$\mathbf{x}=\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \qquad \mathbf{X}=\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\mathbf{W}_N=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

(6.207)

The $N \times N$ matrix $\mathbf{W}_N$ is known as the DFT matrix. Note that $\mathbf{W}_N$ is symmetric; that is, $\mathbf{W}_N^T=\mathbf{W}_N$, where $\mathbf{W}_N^T$ is the transpose of $\mathbf{W}_N$.

a. Show that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^*$$

(6.208)

where $\mathbf{W}_N^{-1}$ is the inverse of $\mathbf{W}_N$ and $\mathbf{W}_N$ and $\mathbf{W}_N^*$ is the complex conjugate of $\mathbf{W}_N$.

b. Find $\mathbf{W}_4$ and $\mathbf{W}_4^{-1}$ explicitly.

a. If we assume that the inverse of $\mathbf{W}_N$ exists, then multiplying both sides of Eq. (6.206) by $\mathbf{W}_N^{-1}$, we obtain

$$\mathbf{x} = \mathbf{W}_N^{-1}\mathbf{X}$$

(6.209)

which is just an expression for the IDFT. The IDFT as given by Eq. (6.94) can be expressed in matrix form as

$$\mathbf{x} = \frac{1}{N}\mathbf{W}_N^*\mathbf{X}$$

(6.210)

Comparing Eq. (6.210) with Eq. (6.209), we conclude that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^*$$

b. Let $W_{n+1,\,k+1}$ denote the entry in the (n + 1)st row and (k + 1)st column of the $\mathbf{W}_4$ matrix. Then, from Eq. (6.207)

$$W_{n+1,\,k+1} = W_4^{nk} = e^{-j(2\pi/4)nk} = e^{-j(\pi/2)nk} = (-j)^{nk}$$

(6.211)

and we have

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathbf{W}_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

(6.212)

6.55.

a. Find the DFT $X[k]$ of $x[n]$ = {0, 1, 2, 3}.

b. Find the IDFT $x[n]$ from $X[k]$ obtained in part (a).

a. Using Eqs. (6.206) and (6.212), the DFT $X[k]$ of $x[n]$ is given by

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix}$$

b. Using Eqs. (6.209) and (6.212), the IDFT $x[n]$ of $X[k]$ is given by

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 6 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

6.56. Let $x[n]$ be a sequence of finite length N such that

$$x[n] = 0 \qquad n < 0, n \geq N$$

(6.213)

Let the N-point DFT $X[k]$ of $x[n]$ be given by [Eq. (6.92)]

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad W_N = e^{-j(2\pi/N)} \qquad k = 0, 1, \ldots, N-1$$

(6.214)

Suppose N is even and let

$$f[n] = x[2n]$$

(6.215a)

$$g[n] = x[2n+1]$$

(6.215b)

The sequences $f[n]$ and $g[n]$ represent the even-numbered and odd-numbered samples of $x[n]$, respectively.

a. Show that

$$f[n] = g[n] = 0 \qquad \text{outside } 0 \leq n \leq \frac{N}{2} - 1$$

(6.216)

b. Show that the N-point DFT $X[k]$ of $x[n]$ can be expressed as

$$X[k] = F[k] + W_N^k G[k] \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

(6.217a)

$$X\left[k + \frac{N}{2}\right] = F[k] - W_N^k G[k] \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

(6.217b)

where

$$F[k] = \sum_{n=0}^{(N/2)-1} f[n]\, W_{N/2}^{kn} \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

(6.218a)

$$G[k]=\sum_{n=0}^{(N/2)-1} g[n]\,W_{N/2}^{kn} \qquad k=0,1,\ldots,\frac{N}{2}-1$$

(6.218b)

c. Draw a flow graph to illustrate the evaluation of $X[k]$ from Eqs. (6.217a) and (6.217b) with N = 8.

d. Assume that $x[n]$ is complex and W_N^{nk} have been precomputed. Determine the numbers of complex multiplications required to evaluate $X[k]$ from Eq. (6.214) and from Eqs. (6.217a) and (6.217b) and compare the results for $N = 2^{10}$ = 1024.

a. From Eq. (6.213)

$$f[n]=x[2n]=0,\, n<0 \qquad \text{and} \qquad f\left[\frac{N}{2}\right]=x[N]=0$$

Thus

$$f[n]=0 \qquad n<0,\, n\geq\frac{N}{2}$$

Similarly

$$g[n]=x[2n+1]=0,\, n<0 \qquad \text{and} \qquad g\left[\frac{N}{2}\right]=x[N+1]=0$$

Thus,

$$g[n]=0 \qquad n<0,\, n\geq\frac{N}{2}$$

b. We rewrite Eq. (6.214) as

$$\begin{aligned} X[k] &= \sum_{n\,\text{even}} x[n]\,W_N^{kn} + \sum_{n\,\text{odd}} x[n]\,W_N^{kn} \\ &= \sum_{m=0}^{(N/2)-1} x[2m]\,W_N^{2mk} + \sum_{m=0}^{(N/2)-1} x[2m+1]\,W_N^{(2m+1)k} \end{aligned}$$

(6.219)

But

$$W_N^2=(e^{-j(2\pi/N)})^2=e^{-j(4\pi/N)}=e^{-j(2\pi/N/2)}=W_{N/2}$$

(6.220)

With this substitution Eq. (6.219) can be expressed as

$$\begin{aligned} X[k] &= \sum_{m=0}^{(N/2)-1} f[m]\,W_{N/2}^{mk} + W_N^k \sum_{m=0}^{(N/2)-1} g[m]\,W_{N/2}^{mk} \\ &= F[k]+W_N^k\,G[k] \qquad k=0,1,\ldots,N-1 \end{aligned}$$

(6.221)

where

$$F[k] = \sum_{n=0}^{(N/2)-1} f[n] W_{N/2}^{kn} \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

$$G[k] = \sum_{n=0}^{(N/2)-1} g[n] W_{N/2}^{kn} \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

Note that $F[k]$ and $G[k]$ are the $(N/2)$-point DFTs of $f[n]$ and $g[n]$, respectively. Now

$$W_N^{k+N/2} = W_N^k W_N^{N/2} = -W_N^k$$

(6.222)

since

$$W_N^{N/2} = (e^{-j(2\pi/N)})^{(N/2)} = e^{-j\pi} = -1$$

(6.223)

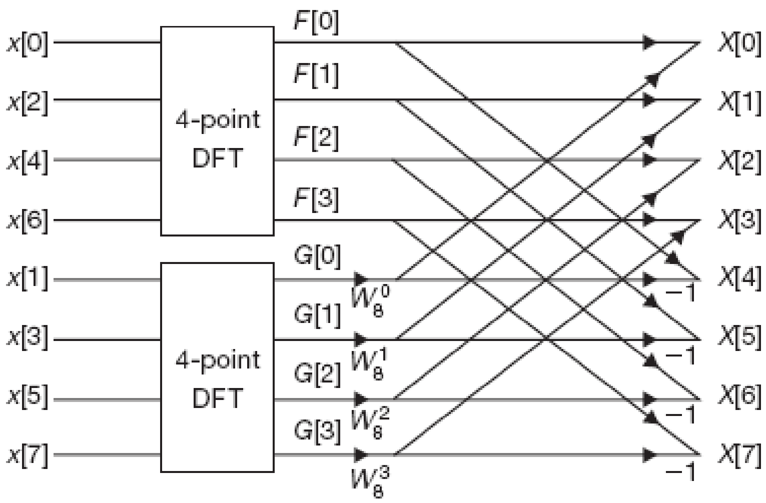
Hence, Eq. (6.221) can be expressed as

$$X[k] = F[k] + W_N^k G[k] \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

$$X\left[k + \frac{N}{2}\right] = F[k] - W_N^k G[k] \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

c. The flow graph illustrating the steps involved in determining $X[k]$ by Eqs. (6.217a) and (6.217b) is shown in Fig. 6-37.

Figure 6-37 Flow graph for an 8-point decimation-in-time FFT algorithm.



d. To evaluate a value of $X[k]$ from Eq. (6.214) requires N complex multiplications. Thus, the total number of complex multiplications based on Eq. (6.214) is N^2. The number of complex multiplications in evaluating $F[k]$ or $G[k]$ is $(N/2)^2$. In addition, there are N multiplications involved in the evaluation of $W_N^k G[k]$. Thus, the total number of complex multiplications based on Eqs. (6.217a) and (6.217b) is $2(N/2)^2 + N = N^2/2 + N$. For $N = 2^{10} = 1024$ the total number of complex multiplications based on Eq. (6.214) is $2^{20} \approx 10^6$ and is $10^6/2 + 1024 \approx 10^6/2$ based on Eqs. (6.217a) and (6.217b). So we see that the number of multiplications is reduced approximately by a factor of 2 based on Eqs. (6.217a) and (6.217b).

The method of evaluating $X[k]$ based on Eqs. (6.217a) and (6.217b) is known as the *decimation-in-time fast Fourier transform* (FFT) algorithm. Note that since $N/2$ is even, using the same procedure, $F[k]$ and $G[k]$ can be found by first determining the (N/4)-point DFTs of appropriately chosen sequences and combining them.

6.57. Consider a sequence

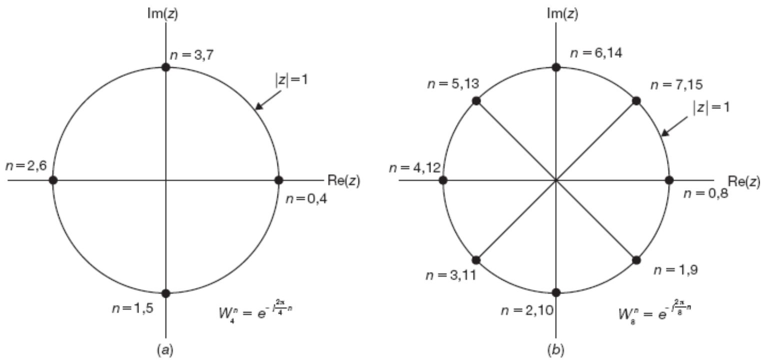
$$x[n] = \{1, 1, -1, -1, -1, 1, 1, -1\}$$

Determine the DFT $X[k]$ of $x[n]$ using the decimation-in-time FFT algorithm.

From Figs. 6-38(a) and (b), the phase factors W_4^k and W_8^k are easily found as follows:

$$W_4^0 = 1 \qquad W_4^1 = -j \qquad W_4^2 = -1 \qquad W_4^3 = j$$

Figure 6-38 Phase factors W_4^n and W_8^n.



and

$$W_8^0 = 1 \qquad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \qquad W_8^2 = -j \qquad W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^4 = -1 \qquad W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \qquad W_8^6 = j \qquad W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

Next, from Eqs. (6.215a) and (6.215b)

$$f[n] = x[2n] = \{x[0], x[2], x[4], x[6]\} = \{1, -1, -1, 1\}$$
$$g[n] = x[2n+1] = \{x[1], x[3], x[5], x[7]\} = \{1, -1, 1, -1\}$$

Then, using Eqs. (6.206) and (6.212), we have

$$\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2+j2 \\ 0 \\ 2-j2 \end{bmatrix}$$

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

and by Eqs. (6.217a) and (6.217b) we obtain

$$X[0] = F[0] + W_8^0 G[0] = 0 \qquad X[4] = F[0] - W_8^0 G[0] = 0$$
$$X[1] = F[1] + W_8^1 G[1] = 2 + j2 \qquad X[5] = F[1] - W_8^1 G[1] = 2 + j2$$
$$X[2] = F[2] + W_8^2 G[2] = -j4 \qquad X[6] = F[2] - W_8^2 G[2] = j4$$
$$X[3] = F[3] + W_8^3 G[3] = 2 - j2 \qquad X[7] = F[3] - W_8^3 G[3] = 2 - j2$$

Noting that since $x[n]$ is real and using Eq. (6.204), $X[7]$, $X[6]$, and $X[5]$ can be easily obtained by taking the conjugates of $X[1]$, $X[2]$, and $X[3]$, respectively.

6.58. Let $x[n]$ be a sequence of finite length N such that

$$x[n] = 0 \qquad n < 0, n \geq N$$

Let the N-point DFT $X[k]$ of $x[n]$ be given by [Eq. (6.92)]

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad W_N = e^{-j(2\pi/N)} \qquad k = 0, 1, \ldots, N-1$$

(6.224)

Suppose N is even and let

$$p[n] = x[n] + x\left[n + \frac{N}{2}\right] \qquad 0 \leq n < \frac{N}{2}$$

(6.225a)

$$q[n] = \left(x[n] - x\left[n + \frac{N}{2}\right]\right) W_N^n \qquad 0 \leq n < \frac{N}{2}$$

(6.225b)

a. Show that the N-point DFT $X[k]$ of $x[n]$ can be expressed as

$$X[2k] = P[k] \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

(6.226a)

$$X[2k+1] = Q[k] \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

(6.226b)

where

$$P[k] = \sum_{n=0}^{(N/2)-1} p[n] W_{N/2}^{kn} \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

(6.227a)

$$Q[k] = \sum_{n=0}^{(N/2)-1} q[n] W_{N/2}^{kn} \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

(6.227b)

b. Draw a flow graph to illustrate the evaluation of $X[k]$ from Eqs. (6.226a) and (6.226b) with N = 8.

a. We rewrite Eq. (6.224) as

$$X[k]=\sum_{n=0}^{(N/2)-1} x[n]\,W_N^{kn}+\sum_{n=N/2}^{N-1} x[n]\,W_N^{kn}$$

(6.228)

Changing the variable $n = m + N/2$ in the second term of Eq. (6.228), we have

$$X[k]=\sum_{n=0}^{(N/2)-1} x[n]\,W_N^{kn}+W_N^{(N/2)k}\sum_{m=0}^{(N/2)-1} x\left[m+\frac{N}{2}\right]W_N^{km}$$

(6.229)

Noting that [Eq. (6.223)]

$$W_N^{(N/2)k}=(-1)^k$$

Eq. (6.229) can be expressed as

$$X[k]=\sum_{n=0}^{(N/2)-1}\left\{x[n]+(-1)^k\,x\left[n+\frac{N}{2}\right]\right\}W_N^{kn}$$

(6.230)

For k even, setting $k = 2r$ in Eq. (6.230), we have

$$X[2r]=\sum_{m=0}^{(N/2)-1} p[n]\,W_N^{2rn}=\sum_{n=0}^{(N/2)-1} p[n]\,W_{N/2}^{rn}\qquad r=0,1,\ldots,\frac{N}{2}-1$$

(6.231)

where the relation in Eq. (6.220) has been used. Similarly, for k odd, setting $k = 2r + 1$ in Eq. (6.230), we get

$$X[2r+1]=\sum_{m=0}^{(N/2)-1} q[n]\,W_N^{2rn}=\sum_{n=0}^{(N/2)-1} q[n]\,W_{N/2}^{rn}\qquad r=0,1,\ldots,\frac{N}{2}-1$$

(6.232)

Equations (6.231) and (6.232) represent the (N/2)-point DFT of $p[n]$ and $q[n]$, respectively. Thus, Eqs. (6.231) and (6.232) can be rewritten as

$$X[2k]=P[k]\qquad k=0,1,\ldots,\frac{N}{2}-1$$
$$X[2k+1]=Q[k]\qquad k=0,1,\ldots,\frac{N}{2}-1$$

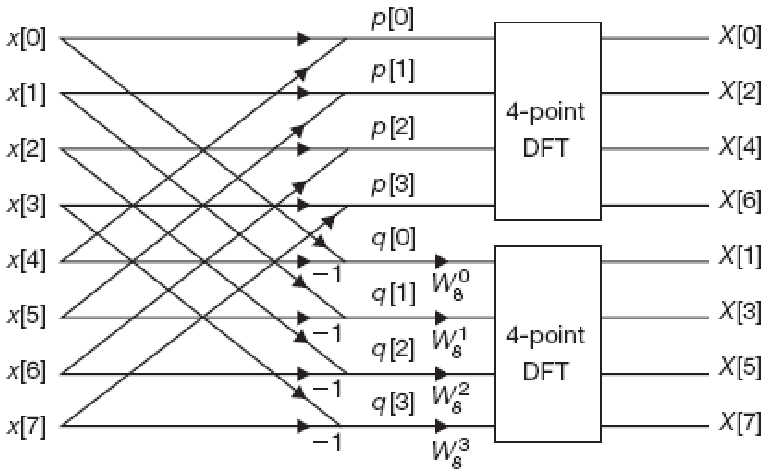
where

$$P[k]=\sum_{n=0}^{(N/2)-1} p[n]\,W_{N/2}^{kn} \qquad k=0,1,\ldots,\frac{N}{2}-1$$

$$Q[k]=\sum_{n=0}^{(N/2)-1} q[n]\,W_{N/2}^{kn} \qquad k=0,1,\ldots,\frac{N}{2}-1$$

b. The flow graph illustrating the steps involved in determining $X[k]$ by Eqs. (6.227a) and (6.227b) is shown in Fig. 6-39.

Figure 6-39 Flow graph for an 8-point decimation-in-frequency FFT algorithm.



The method of evaluating $X[k]$ based on Eqs. (6.227a) and (6.227b) is known as the *decimation-in-frequency fast Fourier transform* (FFT) algorithm.

6.59. Using the decimation-in-frequency FFT technique, redo Prob. 6.57.

From Prob. 6.57

$$x[n] = \{1, 1, -1, -1, -1, 1, 1, -1\}$$

By Eqs. (6.225a) and (6.225b) and using the values of W_8^n obtained in Prob. 6.57, we have

$$\begin{aligned}
p[n] &= x[n]+x\left[n+\frac{N}{2}\right] \\
&= \{(1-1),(1+1),(-1+1),(-1-1)\} = \{0,2,0,2\} \\
q[n] &= \left(x[n]-x\left[n+\frac{N}{2}\right]\right)W_8^n \\
&= \{(1+1)\,W_8^0,(1-1)\,W_8^1,(-1-1)\,w_8^2,(-1+1)\,W_8^3\} \\
&= \{2,0,j2,0\}
\end{aligned}$$

Then using Eqs. (6.206) and (6.212), we have

$$\begin{bmatrix} P[0] \\ P[1] \\ P[2] \\ P[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -j4 \\ 0 \\ j4 \end{bmatrix}$$

$$\begin{bmatrix} Q[0] \\ Q[1] \\ Q[2] \\ Q[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ j2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+j2 \\ 2-j2 \\ 2+j2 \\ 2-j2 \end{bmatrix}$$

and by Eqs. (6.226a) and (6.226b) we get

$$\begin{aligned} X[0] &= P[0] = 0 & X[4] &= P[2] = 0 \\ X[1] &= Q[0] = 2 + j2 & X[5] &= Q[2] = 2 + j2 \\ X[2] &= P[1] = -j4 & X[6] &= P[3] = j4 \\ X[3] &= Q[1] = 2 - j2 & X[7] &= Q[3] = 2 - j2 \end{aligned}$$

which are the same results obtained in Prob. 6.57.

6.60. Consider a causal continuous-time band-limited signal $x(t)$ with the Fourier transform $X(\omega)$. Let

$$x[n] = T_s x(nT_s)$$

(6.233)

where T_s is the sampling interval in the time domain. Let

$$X[k] = X(k\,\Delta\,\omega)$$

(6.234)

where $\Delta\omega$ is the sampling interval in the frequency domain known as the *frequency resolution*. Let T_1 be the record length of $x(t)$, and let ω_M be the highest frequency of $x(t)$. Show that $x[n]$ and $X[k]$ form an N-point DFT pair i f

$$\frac{T_1}{T_s} = \frac{2\omega_M}{\Delta\omega} = N \qquad \text{and} \qquad N \geq \frac{\omega_M T_1}{\pi}$$

(6.235)

Since $x(t) = 0$ for $t < 0$, the Fourier transform $X(\omega)$ of $x(t)$ is given by [Eq. (5.31)]

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

(6.236)

Let T_1 be the total recording time of $x(t)$ required to evaluate $X(\omega)$. Then the above integral can be approximated by a finite series as

$$X(\omega) = \Delta t \sum_{n=0}^{N-1} x(t_n)\, e^{-j\omega t_n}$$

where $t_n = n\,\Delta t$ and $T_1 = N\,\Delta t$. Setting $\omega = \omega_k$ in the above expression, we have

$$X(\omega_k) = \Delta t \sum_{n=0}^{N-1} x(t_n)\, e^{-j\omega_k t_n}$$

(6.237)

Next, since the highest frequency of $x(t)$ is ω_M, the inverse Fourier transform of $X(\omega)$ is given by [Eq. (5.32)]

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} X(\omega)\, e^{j\omega t}\, d\omega$$

(6.238)

Dividing the frequency range $-\omega_M \le \omega \le \omega_M$ into N (even) intervals of length $\Delta\omega$, the above integral can be approximated by

$$x(t) = \frac{\Delta\omega}{2\pi} \sum_{k=-N/2}^{(N/2)-1} X(\omega_k)\, e^{j\omega_k t}$$

where $2\omega_M = N\,\Delta\omega$. Setting $t = t_n$ in the above expression, we have

$$x(t_n) = \frac{\Delta\omega}{2\pi} \sum_{k=-N/2}^{(N/2)-1} X(\omega_k)\, e^{j\omega_k t_n}$$

(6.239)

Since the highest frequency in $x(t)$ is ω_M, then from the sampling theorem (Prob. 5.59) we should sample $x(t)$ so that

$$\frac{2\pi}{T_s} \ge 2\omega_M$$

where T_s is the sampling interval. Since $T_s = \Delta t$, selecting the largest value of Δt (the Nyquist interval), we have

$$\Delta t = \frac{\pi}{\omega_M}$$

and

$$\omega_M = \frac{\pi}{\Delta t} = \frac{\pi N}{T_1}$$

(6.240)

Thus, N is a suitable even integer for which

$$\frac{T_1}{T_s} = \frac{2\omega_M}{\Delta\omega} = N \qquad \text{and} \qquad N \ge \frac{\omega_M T_1}{\pi}$$

(6.241)

From Eq. (6.240) the frequency resolution $\Delta\omega$ is given by

$$\Delta\omega = \frac{2\omega_M}{N} = \frac{2\pi N}{N T_1} = \frac{2\pi}{T_1}$$

(6.242)

Let $t_n = n\,\Delta t$ and $\omega_k = k\,\Delta\omega$. Then

$$t_n \omega_k = (n\Delta t)(k\Delta\omega) = nk\frac{T_1}{N}\frac{2\pi}{T_1} = \frac{2\pi}{N}nk$$

(6.243)

Substituting Eq. (6.243) into Eqs. (6.237) and (6.239), we get

$$X(k\Delta\omega) = \sum_{n=0}^{N-1} \Delta t\, x(n\Delta t)\, e^{-j(2\pi/N)nk}$$

(6.244)

and

$$x(n\Delta t) = \frac{\Delta\omega}{2\pi}\sum_{k=-N/2}^{(N/2)-1} X(k\Delta\omega)\, e^{(2\pi/N)nk}$$

(6.245)

Rewrite Eq. (6.245) as

$$x(n\Delta t) = \frac{\Delta\omega}{2\pi}\left[\sum_{k=0}^{(N/2)-1} X(k\Delta\omega)\, e^{j(2\pi/N)nk} + \sum_{k=-N/2}^{-1} X(k\Delta\omega)\, e^{j(2\pi/N)nk}\right]$$

Then from Eq. (6.244) we note that $X(k\Delta\omega)$ is periodic in k with period N. Thus, changing the variable $k = m - N$ in the second sum in the above expression, we get

$$\begin{aligned} x(n\Delta t) &= \frac{\Delta\omega}{2\pi}\left[\sum_{k=0}^{(N/2)-1} X(k\Delta\omega)\, e^{j(2\pi/N)nk} + \sum_{m=N/2}^{N-1} X(m\Delta\omega)\, e^{j(2\pi/N)nm}\right] \\ &= \frac{\Delta\omega}{2\pi}\sum_{k=0}^{N-1} X(k\Delta\omega)\, e^{j(2\pi/N)nk} \end{aligned}$$

(6.246)

Multiplying both sides of Eq. (6.246) by Δt and noting that $\Delta\omega\, \Delta t = 2\pi/N$, we have

$$x(n\Delta t)\Delta t = \frac{1}{N}\sum_{n=0}^{N-1} X(k\Delta\omega)\, e^{j(2\pi/N)nk}$$

(6.247)

Now if we define

$$x[n] = \Delta t\, x(n\Delta t) = T_s x(nT_s)$$

(6.248)

$$X[k] = X(k\Delta\omega)$$

(6.249)

then Eqs. (6.244) and (6.247) reduce to the DFT pair; that is,

$$X[k]=\sum_{n=0}^{N-1} x[n]\,W_N^{kn} \qquad k=0,1,\ldots,N-1$$

$$x[n]=\frac{1}{N}\sum_{n=0}^{N-1} X[k]\,W_N^{-kn} \qquad n=0,1,\ldots,N-1$$

6.61.

a. Using the DFT, estimate the Fourier spectrum $X(\omega)$ of the continuous-time signal

$$x(t)=e^{-t}u(t)$$

Assume that the total recording time of $x(t)$ is T_1 = 10 s and the highest frequency of $x(t)$ is ω_M = 100 rad/s.

b. Let $X[k]$ be the DFT of the sampled sequence of $x(t)$. Compare the values of $X[0]$, $X[1]$, and $X[10]$ with the values of $X(0)$, $X(\Delta\omega)$, and $X(10\Delta\omega)$.

a. From Eq. (6.241)

$$N \geq \frac{\omega_M T_1}{\pi}=\frac{100(10)}{\pi}=318.3$$

Thus, choosing N = 320, we obtain

$$\Delta\omega=\frac{200}{320}=\frac{5}{8}=0.625 \text{ rad}$$

$$\Delta t=\frac{10}{320}=\frac{1}{32}=0.031 \text{ s}$$

and

$$W_N=W_{320}=e^{-j(2\pi/230)}$$

Then from Eqs. (6.244), (6.249), and (1.92), we have

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} \Delta t\, x(n\,\Delta t)\, e^{-j(2\pi/N)nk} \\ &= \frac{1}{32}\sum_{n=0}^{319} e^{-n(0.031)}e^{-j(2\pi/320)nk}=\frac{1}{32}\,\frac{1-e^{320(0.031)}}{1-e^{-0.031}e^{-j(2\pi/320)k}} \\ &= \frac{0.031}{[1-0.969\cos(k\pi/160)]+j0.969\sin(k\pi/160)} \end{aligned}$$

(6.250)

which is the estimate of $X(k\Delta\,\omega)$.

b. Setting k = 0, k = 1, and k = 10 in Eq. (6.250), we have

$$X[0]=\frac{0.031}{1-0.969}=1$$

$$X[1]=\frac{0.031}{0.0312+j0.019}=0.855e^{-j0.547}$$

$$X[10]=\frac{0.031}{0.0496-j0.189}=0.159e^{-j1.314}$$

From Table 5-2

$$x(t)=e^{-t}u(t)\leftrightarrow X(\omega)=\frac{1}{j\omega+1}$$

and

$$X(0)=1$$
$$X(\Delta\omega)=X(0.625)=\frac{1}{1+j0.625}=0.848e^{-j0.559}$$
$$X(10\,\Delta\,\omega)=X(6.25)=\frac{1}{1+j6.25}=0.158e^{-j1.412}$$

Even though $x(t)$ is not band-limited, we see that $X[k]$ offers a quite good approximation to $X(\omega)$ for the frequency range we specified.

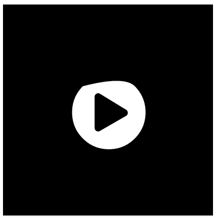
6.10. SUPPLEMENTARY PROBLEMS

6.62. Find the discrete Fourier series for each of the following periodic sequences:

a. $x[n] = \cos(0, 1\pi n)$

b. $x[n] = \sin(0, 1\pi n)$

c. $x[n] = 2\cos(1.6\pi n) + \sin(2.4\pi n)$



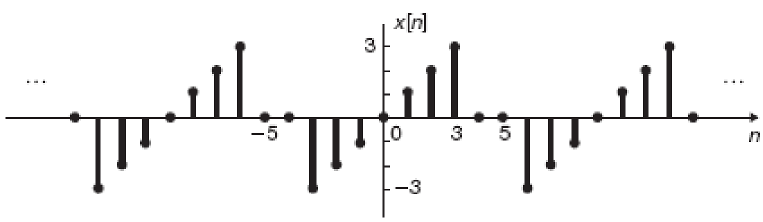
Schaum's Signals and Systems Supplementary Problem 6.62: Discrete Fourier Series

This video demonstrates how to find the discrete Fourier series for a periodic sequence.

Carlotta A. Berry, Associate Professor, Electrical and Computer Engineering, Rose-Hulman Institute of Technology
2013

6.63. Find the discrete Fourier series for the sequence $x[n]$ shown in Fig. 6-40.

Figure 6-40



6.64. Find the trigonometric form of the discrete Fourier series for the periodic sequence $x[n]$ shown in Fig. 6-7 in Prob. 6.3.

6.65. Find the Fourier transform of each of the following sequences:

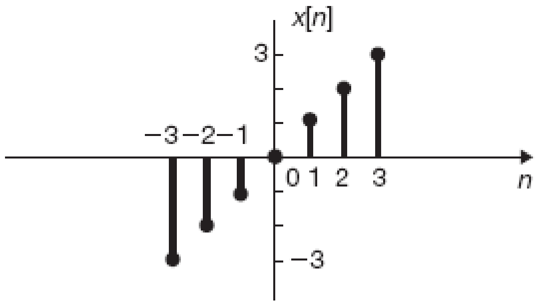
a. $x[n] = a^{|n|}$, $|a| < 1$

b. $x[n] = \sin(\Omega_0 n)$, $|\Omega_0| < \pi$

c. $x[n] = u[-n-1]$

6.66. Find the Fourier transform of the sequence $x[n]$ shown in Fig. 6-41.

Figure 6-41



6.67. Find the inverse Fourier transform of each of the following Fourier transforms:

a. $X(\Omega) = \cos(2\Omega)$

b. $X(\Omega) = j\Omega$

6.68. Consider the sequence $y[n]$ given by

$$y[n] = \begin{cases} x[n] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Express $y(\Omega)$ in terms of $X(\Omega)$.

6.69. Let

$$x[n] = \begin{cases} 1 & |n| \leq 2 \\ 0 & |n| > 2 \end{cases}$$

a. Find $y[n] = x[n] * x[n]$.

b. Find the Fourier transform $Y(\Omega)$ of $y[n]$.

6.70. Verify Parseval's theorem [Eq. (6.66)] for the discrete-time Fourier transform, that is,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2 \, d\Omega$$

6.71. A causal discrete-time LTI system is described by

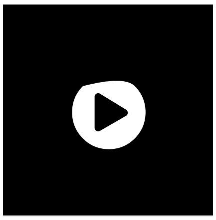
$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

where $x[n]$ and $y[n]$ are the input and output of the system, respectively.

a. Determine the frequency response $H(\Omega)$ of the system.

b. Find the impulse response $h[n]$ of the system.

c. Find $y[n]$ if $x[n] = (\frac{1}{2})^n u[n]$.



Schaum's Signals and Systems Supplementary Problem 6.71: Frequency and Impulse Response of a Discrete-Time System

This video demonstrates how to find the frequency and impulse response of a causal discrete-time LTI system.

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2013

6.72. Consider a causal discrete-time LTI system with frequency response

$$H(\Omega) = \mathrm{Re}\{H(\Omega)\} + j\,\mathrm{Im}\{H(\Omega)\} = A(\Omega) + jB(\Omega)$$

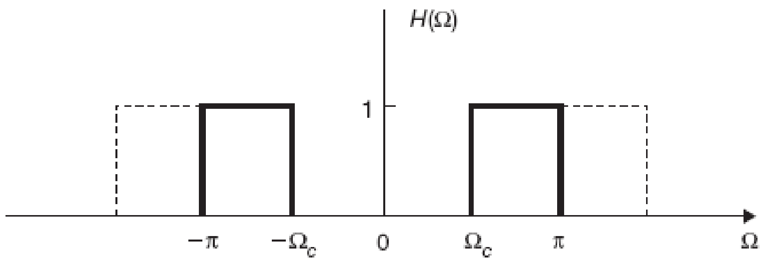
a. Show that the impulse response $h[n]$ of the system can be obtained in terms of $A(\Omega)$ or $B(\Omega)$ alone.

b. Find $H(\Omega)$ and $h[n]$ if

$$\mathrm{Re}\{H(\Omega)\} = A(\Omega) = 1 + \cos\Omega$$

6.73. Find the impulse response $h[n]$ of the ideal discrete-time HPF with cutoff frequency Ω_c ($0 < \Omega_c < \pi$) shown in Fig. 6-42.

Figure 6-42



6.74. Show that if $H_{LPF}(z)$ is the system function of a discrete-time low-pass filter, then the discrete-time system whose system function $H(z)$ is given by $H(z) = H_{LPF}(-z)$ is a high-pass filter.

6.75. Consider a continuous-time LTI system with the system function

$$H_c(s) = \frac{1}{(s+1)^2}$$

Determine the frequency response $H_d(\Omega)$ of the discrete-time system designed from this system based on the impulse invariance method.

6.76. Consider a continuous-time LTI system with the system function

$$H_c(s) = \frac{1}{s+1}$$

Determine the frequency response $H_d(\Omega)$ of the discrete-time system designed from this system based on the step response invariance; that is,

$$s_d[n] = s_c(nT_s)$$

where $s_c(t)$ and $s_d[n]$ are the step response of the continuous-time and the discrete-time systems, respectively.

6.77. Let $H_p(z)$ be the system function of a discrete-time prototype low-pass filter. Consider a new discrete-time low-pass filter whose system function $H(z)$ is obtained by replacing z in $H_p(z)$ with $(z - \alpha)/(1 - \alpha z)$, where α is real.

a. Show that

$$H_p(z)\big|_{z=1+j0} = H(z)\big|_{z=1+j0}$$
$$H_p(z)\big|_{z=-1+j0} = H(z)\big|_{z=-1+j0}$$

b. Let Ω_{p1} and Ω_1 be the specified frequencies (< π) of the prototype low-pass filter and the new low-pass filter, respectively. Then show that

$$\alpha = \frac{\sin\left[\left(\Omega_{p1} - \Omega_1\right)/2\right]}{\sin\left[\left(\Omega_{p1} + \Omega_1\right)/2\right]}$$

6.78. Consider a discrete-time prototype low-pass filter with system function

$$H_p(z) = 0.5(1 + z^{-1})$$

a. Find the 3-dB bandwidth of the prototype filter.

b. Design a discrete-time low-pass filter from this prototype filter so that the 3-dB bandwidth of the new filter is 2π/3.

6.79. Determine the DFT of the sequence

$$x[n] = a^n \qquad 0 \le n \le N-1$$

6.80. Evaluate the circular convolution

$$y[n] = x[n] \otimes h[n]$$

where

$$x[n] = u[n] - u[n-4]$$
$$h[n] = u[n] - u[n-3]$$

a. Assuming $N = 4$.

b. Assuming $N = 8$.

6.81. Consider the sequences $x[n]$ and $h[n]$ in Prob. 6.80.

a. Find the 4-point DFT of $x[n]$, $h[n]$, and $y[n]$.

b. Find $y[n]$ by taking the IDFT of $Y[k]$.

6.82. Consider a continuous-time signal $x(t)$ that has been prefiltered by a low-pass filter with a cutoff frequency of 10 kHz. The spectrum of $x(t)$ is estimated by use of the N-point DFT. The desired frequency resolution is 0.1 Hz. Determine the required value of N (assuming a power of 2) and the necessary data length T_1.

6.11. ANSWERS TO SUPPLEMENTARY PROBLEMS

6.62.

$$(a)\quad x[n]=\frac{1}{2}e^{j\Omega_0 n}+\frac{1}{2}e^{j19\Omega_0 n},\ \Omega_0=0.1\pi$$

$$(b)\quad x[n]=\frac{1}{2j}e^{j\Omega_0 n}-\frac{1}{2j}e^{j19\Omega_0 n},\ \Omega_0=0.1\pi$$

$$(c)\quad x[n]=(1-j0.5)\,e^{j\Omega_0 n}+(1+j0.5)\,e^{j4\Omega_0 n},\ \Omega_0=0.4\pi$$

6.63.

$$x[n]=\sum_{k=0}^{8} c_k\, e^{j\Omega_0 kn},\ \Omega_0=\frac{2\pi}{9}$$

$$c_k=-j\frac{2}{9}\left[\sin\left(\frac{2\pi}{9}\right)k+2\sin\left(\frac{4\pi}{9}\right)k+3\sin\left(\frac{6\pi}{9}\right)k\right]$$

6.64. $x[n]=\frac{3}{2}-\cos\frac{\pi}{2}n-\sin\frac{\pi}{2}n-\frac{1}{2}\cos\pi n$

6.65.

$$(a)\quad X(\Omega)=\frac{1-a^2}{1-2a\cos\Omega+a^2}$$

$$(b)\quad X(\Omega)=-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega-\Omega_0)],|\Omega|,|\Omega_0|\leq\pi$$

$$(c)\quad X(\Omega)=\pi\,\delta(\Omega)-\frac{1}{1-e^{-j\Omega}},|\Omega|\leq\pi$$

6.66. $X(\Omega) = j2(\sin\Omega + 2\sin 2\Omega + 3\sin 3\Omega)$

6.67.

$$(a)\quad x[n]=\frac{1}{2}\delta[n-2]+\frac{1}{2}\delta[n+2]$$

$$(b)\quad x[n]=\begin{cases}(-1)^n/n & n\neq 0\\ 0 & n=0\end{cases}$$

6.68. $Y(\Omega)=\frac{1}{2}X(\Omega)+\frac{1}{2}X(\Omega-\pi)$

6.69.

$$(a)\quad y[n]=\begin{cases}5(1-|n|/5) & |n|\leq 5\\ 0 & |n|>5\end{cases}$$

$$(b)\quad Y(\Omega)=\left(\frac{\sin(2.5\Omega)}{\sin(0.5\Omega)}\right)^2$$

6.70. *Hint:* Proceed in a manner similar to that for solving Prob. 5.38.

6.71.

$$(a)\quad H[\Omega]=\frac{1}{1-\frac{3}{4}e^{-j\Omega}+\frac{1}{8}e^{-2j\Omega}}$$

$$(b)\quad h[n]=\left[2\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n\right]u[n]$$

$$(c)\quad y[n]=\left[\left(\frac{1}{4}\right)^n+n\left(\frac{1}{2}\right)^{n-1}\right]u[n]$$

6.72.

a. *Hint:* Process in a manner similar to that for Prob. 5.49.

b. $H(\Omega) = 1 + e^{-j\Omega}$, $h[n] = \delta[n] + \delta[n-1]$

6.73. $h[n]=\delta[n]-\dfrac{\sin\Omega_c n}{\pi n}$

6.74. *Hint:* Use Eq. (6.156) in Prob. 6.37.

6.75. $H(\Omega)=T_s\, e^{-T_s}\dfrac{e^{-j\Omega}}{(1-e^{-T_s}\,e^{-j\Omega})^2}$, where T_s is the sampling interval of $h_c(t)$.

6.76. *Hint:* $h_d[n]=s_d[n]-s_d[n-1]$

$$H_d(\Omega)=\frac{(1-e^{-T_s})\,e^{-j\Omega}}{1-e^{-T_s}\,e^{-j\Omega}}$$

6.77. *Hint:* Set Set $e^{j\Omega_{p1}}=\dfrac{e^{j\Omega_1}-\alpha}{1-\alpha\, e^{j\Omega_1}}$ and solve for α.

6.78. *Hint:* Use the result from Prob. 6.77.

$$(a)\quad \Omega_{3\,\mathrm{db}}=\frac{\pi}{2}$$

$$(b)\quad H(z)=0.634\frac{1+z^{-1}}{1+0.268z^{-1}}$$

6.79. $X[k]=\dfrac{1-a^N}{1-ae^{-j(2\pi/N)k}}\qquad k=0,1,\ldots,N-1$

6.80.

a. $y[n]$ = {3, 3, 3, 3}

b. $y[n]$ = {1, 2, 3, 3, 2, 1, 0, 0}

6.81.

a. $[X[0], X[1], X[2], X[3]]$ = [4, 0, 0, 0]

$[H[0], H[1], H[2], H[3]] = [3, -j, 1, j]$

$[Y[0], Y[1], Y[2], Y[3]]$ = [12, 0, 0, 0]

b. $y[n]$ = {3, 3, 3, 3}

6.82. $N = 2^{18}$ and T_1 = 13.1072 s