Set #8

24.

Electrons, protons and neutrons have spin 1/2. There are two values of m_s for s=1/2. These are m_s=1/2 and m_s=-1/2. The corresponding eigenstates $\chi_\uparrow$ (+$\hbar$/2) and $\chi_\downarrow$ (-$\hbar$\2) obey the following eigenvalue eq.s

$$S^2\chi_\uparrow = \frac{3\hbar^2}{4}\chi_\uparrow \quad S_z\chi_\uparrow = \hbar/2\chi_\uparrow$$

One can conveniently represent the two eigenstates as column matrices

$$\chi_\uparrow = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \chi_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$$

a. Compute the matrix form of S^2.
b. Compute the matrix form of S_z.
c. Are these two eigenstates normalized?

ⓐ For S^2 we have $s=\frac{1}{2}$, so in the eigenvalues equation:

$$S^2\chi_\uparrow = \frac{3\hbar^2}{4}\chi_\uparrow \quad \text{and} \quad S^2\chi_\downarrow = \frac{3\hbar^2}{4}\chi_\downarrow$$

If we write S^2 as a generic 2×2 matrix:

$$S^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Through the two previous equations we get the values:

$$\circ\ S^2\chi_\uparrow = \frac{3\hbar^2}{4}\chi_\uparrow \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}\frac{3}{4}\hbar^2\\0\end{pmatrix}$$

$$\circ\ S^2\chi_\downarrow = \frac{3\hbar^2}{4}\chi_\downarrow \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}0\\\frac{3}{4}\hbar^2\end{pmatrix}$$

Therefore:

$$\boxed{S^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}$$

(b) For S_z the eigenvalues are $m=+\frac{1}{2}$ for $\chi_\uparrow$ and $m=-\frac{1}{2}$ for $\chi_\downarrow$

So the equations we get are:

$$S_z\,\chi_\uparrow = \frac{\hbar}{2}\chi_\uparrow \quad \text{and} \quad S_z\,\chi_\downarrow = -\frac{\hbar}{2}\chi_\downarrow$$

Again we use a generic matrix and get the values

- $S_z\,\chi_\uparrow = +\frac{\hbar}{2}\chi_\uparrow \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \frac{\hbar^2}{2} \\ 0 \end{pmatrix}$

- $S_z\,\chi_\downarrow = -\frac{\hbar}{2}\chi_\downarrow \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\hbar}{2} \end{pmatrix}$

Therefore:

$$\boxed{S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

(c) We know that $\chi_\uparrow$ and $\chi_\downarrow$ form a basis. To see if they are nomalized we have to see if the vector length is 1.

$$\|\chi_\uparrow\| = \sqrt{1^2+0^2} = 1$$

$$\|\chi_\downarrow\| = \sqrt{0^2+1^2} = 1$$

Therefore, $\boxed{\chi_\uparrow \text{ and } \chi_\downarrow \text{ are normalized}}$

25. One can conveniently represent the two spin-eigenstates as column matrices

$$\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Using $\chi_\uparrow$ and $\chi_\downarrow$ as spin basis states, the most general state of a spin 1/2 particle can be expressed as two-element column matrix

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\,\chi_\uparrow + b\,\chi_\downarrow$$

a. If you measured S_z on a particle in the state χ what is the probability of measuring $+\hbar/2$ or $-\hbar/2$?

b. What is the sum of the probabilities?

c. What does it mean?

(a) If we measure S_z we can get the value $+\frac{\hbar}{2}$ with a probability of $|a|^2$ and the value $-\frac{\hbar}{2}$ with a probability of $|b|^2$ since they are the only possible values.

(b) The sum of all of probabilities must be 1. Therefore:

$$\boxed{|a|^2 + |b|^2 = 1}$$

(c) It means that we will always find a value of the spin

26. Compute the angles that the intrinsic angular momentum (*spin*) forms about a given direction z. These angles define the two eigenstates $\chi_\uparrow$ e $\chi_\downarrow$ of the operator S_z.

Write the relevant eigenvalue eq. for $\chi_\uparrow$ e $\chi_\downarrow$ and the orthogonality condition.

Write the total electron spin wavefunction: this is often used an archetype of a spin q-bit (spin quantum-bit)

Write the relation the coefficients of the spin wavefunction have to satisfy. Briefly explain how such a q-bit differ from a classical bit.

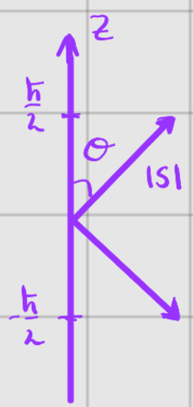
Angles of the spin

$$\begin{cases} S^2 \chi_{s,m_s} = \hbar^2 s(s+1) \chi_{s,m_s} \\ S_z \chi_{s,m_s} = \hbar m_s \chi_{s,m_s} \end{cases}$$

We know that for $e^- \Rightarrow s = \frac{1}{2} \Rightarrow S^2 = \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3\hbar^2}{4}$

$|S| = \hbar \frac{\sqrt{3}}{2}$

We also know that: $m_s = -s, \dots, 0, \dots, s \Rightarrow m_s = -\frac{1}{2}, +\frac{1}{2}$



$$\cos\theta = \frac{\cancel{\hbar}/\cancel{2}}{\cancel{\hbar}\sqrt{3}/\cancel{2}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 54{,}73^\circ$$

So, we have:

$$\boxed{\begin{array}{l} \chi_\uparrow, \; m_s = +\frac{1}{2} \Rightarrow \theta_\uparrow = 54{,}73^\circ \\ \chi_\downarrow, \; m_s = -\frac{1}{2} \Rightarrow \theta_\downarrow = -54{,}73^\circ \end{array}}$$

The orthogonality condition, taking into account we use spherical coordinates:

$$\boxed{\int \chi^*_{s,m_s} \chi_{s',m_s'} r^2 dr \sin\theta d\theta d\varphi = \delta_{ss'} \delta_{m_s m_s'}}$$

To get the total wavefunction we have 3 different parts that contribute, we put the case of the infinite well in one dimension

space $\rightarrow$ $\psi_{(x)} = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$

orbital $\rightarrow$ φ_{ℓ, m_ℓ}

spin $\rightarrow$ χ_{s, m_s}

$$\boxed{\psi_{(x)} = A \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \varphi_{\ell, m_\ell} \chi_{s, m_s}}$$

Extra.

Suppose the electron is a classical solid "sphere" with radius $r \sim 5 \times 10^{-17} m$, what would be the speed a point placed on the equator of the sphere? Is such a spherical model for the electron sound?

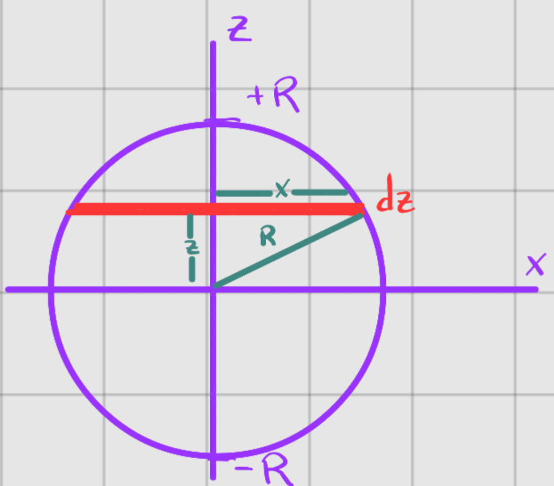
According to what we saw in class the spin angular momentum of the electron is: $L_z = \frac{\hbar}{2}$

We also know that the angular momentum can be write as:

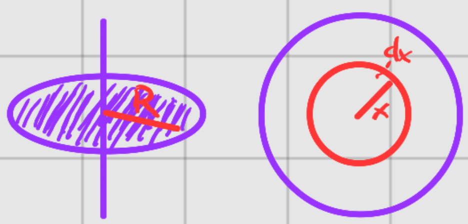
$L = I\omega$, with $\omega = \frac{v}{R}$ because we focus on a point on the equator and I is the moment of intertial of a sphere.

$$L = I\omega \Rightarrow \frac{\hbar}{2} = I\frac{v}{R} \Rightarrow v = \frac{\hbar R}{2I}$$

We now calculate the moment of inertia of a sphere. First, we divide the sphere in discs of radious x and width dz. So we will add all of the contiributions of each disc through an integral: $I_{sphere} = \int_{-R}^{R} dI_{disc}$



Thus, we have to calculate the moment of inertia of a disc respect a perpendicular axis of the plane of the disc through its centre



$$I_{disc} = \int_0^R x^2 dm = \frac{2m}{R^2}\int_0^R x \cdot x^2 dx = \frac{2m}{R^2}\left[\frac{x^4}{4}\right]_0^R =$$

dm = surface density · area = $\frac{m}{\cancel{\pi} R^2} \cdot 2\cancel{\pi} x \cdot dx = \frac{2m}{R^2} x dx$

mass: m, radious: R

$$= \frac{2m}{R^2}\frac{R^4}{4} \Rightarrow I_{disc} = \frac{1}{2} m R^2$$

Coming back to the calculation of the moment of inertia of the sphere, we now know I_{disc}. In this case we see that the radious of the different is changing:

$$dI_{disc} = \frac{1}{2} x^2 dm$$

The mass of each disc is:

$$dm = \text{volume density} \cdot \text{volume} = \rho \cdot dV = \frac{m}{\frac{4}{3}\pi R^3} \cdot \overbrace{\pi x^2}^{\text{area}} \cdot \overbrace{dz}^{\text{height}} = \frac{3m}{4R^3} x^2 dz$$

$$I_{sphere} = \int_{-R}^{R} dI_{disc} = \int_{-R}^{R} \frac{1}{2} x^2 \cdot \frac{3m}{4R^3} x^2 dz = \frac{3m}{8R^3} \int_{-R}^{R} x^4 dz = \frac{3m}{8R^3} \int_{-R}^{R} (R^2 - z^2)^2 dz =$$

x and z are related $\Rightarrow R^2 = x^2 + z^2 \Rightarrow x^2 = R^2 - z^2$

$$= \frac{3m}{8R^3} \int_{-R}^{R} (R^4 + z^4 - 2R^2 z^2) dz = \frac{3m}{8R^3} \left[R^4 z + \frac{z^5}{5} - 2R^2 \frac{z^3}{3} \right]_{-R}^{R} =$$

$$= \frac{3m}{8R^3} \left[R^5 + \frac{1}{5} R^5 - \frac{2}{3} R^5 - \left(-R^5 - \frac{1}{5} R^5 + \frac{2}{3} R^5 \right) \right] = \frac{3m}{8R^3} \left(2R^5 + \frac{2}{5} R^5 - \frac{4}{3} R^5 \right) =$$

$$= \frac{3m}{8R^3} \cdot \frac{16}{15} R^5 \Rightarrow I_{sphere} = \frac{2}{5} m R^2$$

Finally, we can calculate the speed:

$$v = \frac{\hbar R}{2I} = \frac{\hbar \cancel{R}}{2 \cdot \frac{2}{5} m R^{\cancel{2}}} = \frac{5\hbar}{4mR} = \frac{5 \cdot 1{,}054 \cdot 10^{-34}}{4 \cdot 9{,}1 \cdot 10^{-31} \cdot 5 \cdot 10^{-17}} \simeq 3 \cdot 10^{12}$$

$$\boxed{v = 3 \cdot 10^{12} \text{ m/s}}$$

It is higher than the speed of light so the model is not sound.