Set #4

12.

****Problem 2.6** A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)].$$

(a) Normalize $\Psi(x,0)$. (That is, find A. This is very easy if you exploit the orthonormality of ψ_1 and ψ_2. Recall that, having normalized Ψ at $t = 0$, you can rest assured that it *stays* normalized—if you doubt this, check it explicitly after doing part b.)

(b) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. (Express the latter in terms of sinusoidal functions of time, eliminating the exponentials with the help of **Euler's formula**: $e^{i\theta} = \cos\theta + i\sin\theta$.) Let $\omega \equiv \pi^2\hbar/2ma^2$.

(c) Compute $\langle x \rangle$. Notice that it oscillates in time. What is the frequency of the oscillation? What is the amplitude of the oscillation? (If your amplitude is greater than $a/2$, go directly to jail.)

(d) Compute $\langle p \rangle$. (As Peter Lorre would say, "Do it ze *kveek* vay, Johnny!")

(e) Find the expectation value of H. How does it compare with E_1 and E_2?

(f) A *classical* particle in this well would bounce back and forth between the walls. If its energy is equal to the expectation value you found in (e), what is the frequency of the classical motion? How does it compare with the quantum frequency you found in (c)?

13.

For the asymmetric quantum well worked out in class (E=4.85 μeV, V_1= 5 μeV and a = 1 μm):

a. compute the electron wavevector k inside the well

b. compute the electron wavevector k_1 inside the left barrier

c. compute the shift δ and the probability densities C and B_1

d. write the specific form of the eigenfunctions (i) inside the well and (ii) inside the left barrier

e. compute the probability of finding the electron in the "whole" region [0,-d] on the left of the well

f. compute and plot the probability of finding the electron at a "specific" distance $-d_0$ to the left of the well, as d_0 varies from 0 to 5 μm.

Superposition State Evolution

∗∗Problem 2.6 A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

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(e) Find the expectation value of H. How does it compare with E_1 and E_2?

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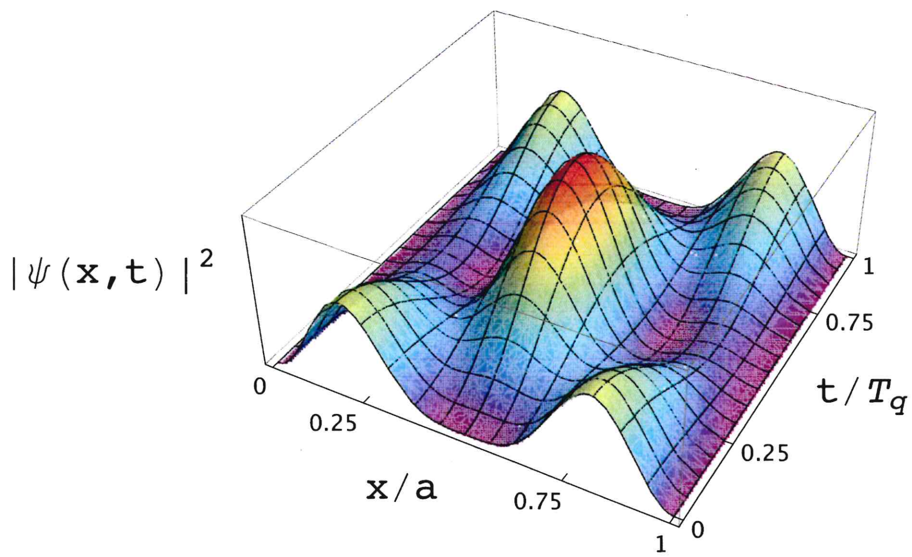


2.6 a $|\psi|^2 = |\psi_1+\psi_2|^2 = |A|^2 (\psi_1^*+\psi_2^*)(\psi_1+\psi_2) = |A|^2 (\psi_1\psi_1^* + \psi_1\psi_2^* + \psi_1^*\psi_2 + \psi_2\psi_2^*)$

$$1 = \int_{\text{well}} |\psi|^2 dx = |A|^2 \left\{ \underbrace{\int |\psi_1|^2 dx}_{1} + \underbrace{\int |\psi_2|^2 dx}_{1} + \underbrace{\int \psi_1\psi_2^* dx}_{0} + \underbrace{\int \psi_1^*\psi_2 dx}_{0} \right\} = 2|A|^2$$

$$A = \frac{1}{\sqrt{2}}$$

2.6 b w/ function $\psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x) e^{-iE_1t/\hbar} + \psi_2(x) e^{-iE_2t/\hbar}\right]$ $E_n/\hbar = \frac{\hbar\pi^2}{2m\underset{\downarrow\ \text{width}}{a^2}} n^2$

write: $\frac{E_n}{\hbar} \equiv \omega \cdot n^2$ or $\omega = \frac{\hbar\pi^2}{2ma^2}$

$$\psi(x,t) = \frac{1}{\sqrt{2}} \times \sqrt{\frac{2}{a}} \left[\sin\frac{\pi}{a}x\, e^{-i\omega t} + \sin\frac{2\pi}{a}x\, e^{-i\omega t 4}\right] = \frac{1}{\sqrt{a}} e^{-i\omega t}\left[\sin\frac{\pi x}{a} + \sin\frac{2\pi}{a}x\, e^{-i\omega t 3}\right]$$

$$|\psi(x,t)|^2 = \frac{1}{a} \times 1 \times \left[\sin^2\frac{\pi x}{a} + \sin^2\frac{2\pi x}{a} + \sin\frac{\pi}{a}x \sin\frac{3\pi x}{a} \underbrace{(e^{3i\omega t} + e^{-3i\omega t})}_{2\cos 3\omega t}\right]$$

$$= \frac{1}{a} \times \left[\sin^2\frac{\pi x}{a} + \sin^2\frac{3\pi x}{a} + 2\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}\cos 3\omega t\right]$$

2.6 c $\langle x(t)\rangle \equiv \int_{\text{well}} x|\psi(x,t)|^2 dx = \frac{1}{a}\int_0^a x\left[\underbrace{\sin^2\frac{\pi x}{a}}_{a^2/4} + \underbrace{\sin^2\frac{2\pi}{a}x}_{a^2/4} + \underbrace{2\cos\sin\cos 3\omega t}_{-8a^2/9\pi^2 \times \cos 3\omega t}\right]dx$

$$= \frac{1}{a}\left[\frac{a^2}{4}\times 2 - \frac{16a^2}{9\pi^2}\cos 3\omega t\right]$$

$$= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos 3\omega t\right]$$

Oscillation freq.: $\nu_q = \frac{3\omega}{2\pi} = \frac{3\pi\hbar}{4ma^2}$ Amplitude: $A_q = \frac{32}{9\pi^2}(a/2)$ $\overset{\text{period}}{T_q} = \frac{1}{\nu_q}$

2.6 d $\langle p\rangle = m\frac{d}{dt}\langle x\rangle = \frac{8\hbar}{3a}\sin 3\omega t$

2.6 e $\underbrace{\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \overset{0}{V(x)}\right]}\psi_1(x) = E_1\psi_1(x)$; $\underbrace{\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \overset{0}{V(x)}\right]}\psi_2(x) = E_2\psi_2(x)$

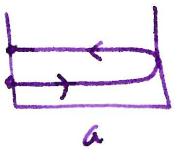
Now: $H\psi = H\frac{1}{\sqrt{2}}\left(\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right) = \frac{1}{\sqrt{2}}\left(E_1\psi_1 e^{-iE_1t/\hbar} + E_2\psi_2 e^{-iE_2t/\hbar}\right)$

$$\langle H\rangle \equiv \int \psi^*(x,t)\, H\psi(x,t) = \int \frac{dx}{\sqrt{2}}\left(\psi_1^* e^{iE_1t/\hbar} + \psi_2^* e^{iE_2t/\hbar}\right)\frac{1}{\sqrt{2}}\left(E_1\psi_1 e^{-iE_1t/\hbar} + E_2\psi_2 e^{-iE_2t/\hbar}\right)$$

$$= \frac{1}{2}\left\{ E_1\underbrace{\int|\psi_1|^2dx}_{1} + E_2\underbrace{\int|\psi_2(x)|^2dx}_{1} + E_1 e^{i(E_2-E_1)t/\hbar}\underbrace{\int\psi_2^*\psi_1\,dx}_{0} + \underbrace{\cdots}_{0}\right\}$$

$$= E_1+E_2/2 = \frac{5\pi^2\hbar^2}{4ma^2} \quad (\text{avg of } E_1 \,\&\, E_2)$$

2.6 f



classical motion: $2a = \overset{\text{cl. speed}}{v_c}\cdot\underset{\text{period}}{T}$

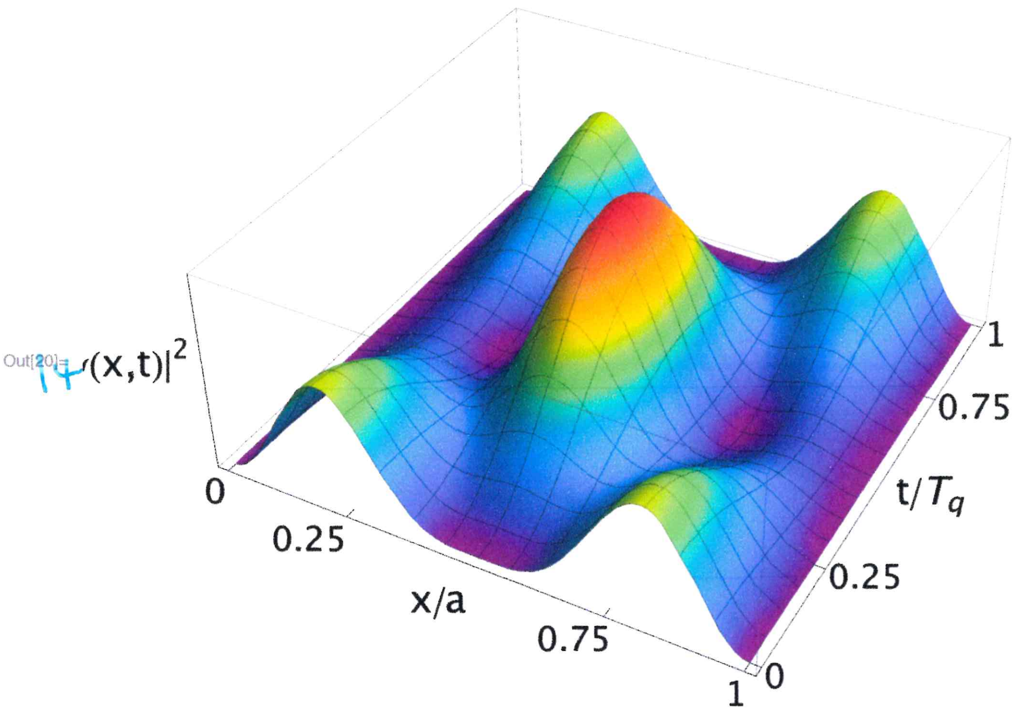
$$T = \frac{2\pi}{\omega_c} = \frac{1}{\nu_c} \quad \text{or} \quad \nu_c = \frac{v_c}{2a} = \frac{\sqrt{\frac{5}{2}}\,\frac{\hbar\pi}{ma}}{2a} = \sqrt{\frac{10}{9}}\,\nu_q$$

fairly close

$$\frac{1}{2}mv_c^2 = \frac{E_1+E_2}{2} = \frac{5\pi^2\hbar^2}{4ma^2}$$

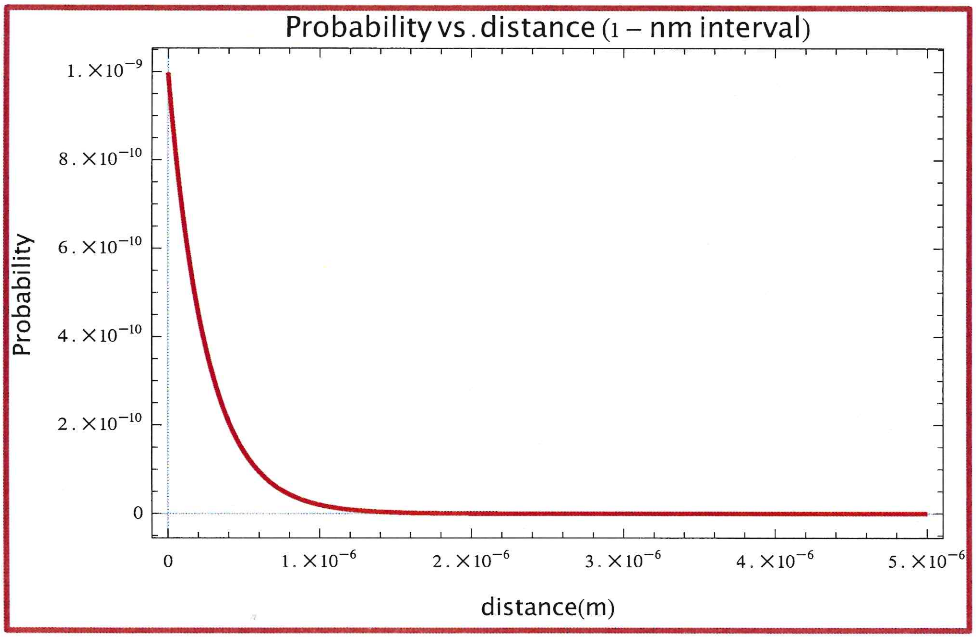
$$v_c = \sqrt{\frac{5}{2}}\,\frac{\pi\hbar}{ma}$$

Out[20]=



✦ For the asymmetric quantum well worked out in class (E=8.45 μeV, V_I= 5 μeV and a=1 μm):

a. compute the electron wavevector k inside the well
b. compute the electron wavevector k_I inside the left barrier
c. compute the shift δ and the probability densities C and B_I
d. write the specific form of the eigenfunctions (i) inside the well and (ii) inside the left barrier
e. compute the probability of finding the electron in the "whole" region [0,-d] on the left of the well
f. compute and plot the probability of finding the electron at a "specific" distance $-d_o$ to the left of the well, as d_o varies from 0 to 5 μm.



```
Clear[K, K1, δ, c, B1, P, a, V1, d, hbc]
```

$$\text{par} = \{a \to 10^{-6},\ V1 \to 5\times10^{-6},\ d \to 10^{-3}\,10^{-6},\ hbc \to 1973\times10^{-10}\};$$

```
(*wavevector inside the well*)
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$$K = \sqrt{2\times0.5\times10^{6}\,x}\,\Big/\,hbc;$$

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(*wavevector inside the left barrier*)
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$$K1 = \sqrt{2\times0.5\times10^{6}\,(V1-x)}\,\Big/\,hbc;$$

```
(*shift δ and probability densities C and B1*)
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$$\delta = \text{ArcSin}\left[\frac{x}{V1}\right];$$

$$\%\,\frac{360}{2\pi}\ /.\ \text{par}\ /.\ \{x \to 4.85\times10^{-6}\};$$

$$c = \sqrt{\frac{4\,K}{2\,K\,a + \text{Sin}[2\,\delta] - \text{Sin}[2\,K\,a + 2\,\delta]}};$$

$$B1 = c\,\text{Sin}[\delta];$$

```
(*Probability of finding the electron within
 a region of width "d" on the left of the barrier*)
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$$P = B1^{2} * \frac{(1-\text{Exp}[-2\,K1\,d])}{2\,K1};$$

$$\{K, K1, \delta, c, B1, P\}\ /.\ \text{par}\ /.\ \{x \to 4.85\times10^{-6}\}$$

$$\{1.1162\times10^{7},\ 1.96299\times10^{6},\ 1.32523,\ 1394.7,\ 1352.86,\ 0.00182663\}$$

```
(*Probability of finding the electron
 at points (-d0) on the left of the barrier*)
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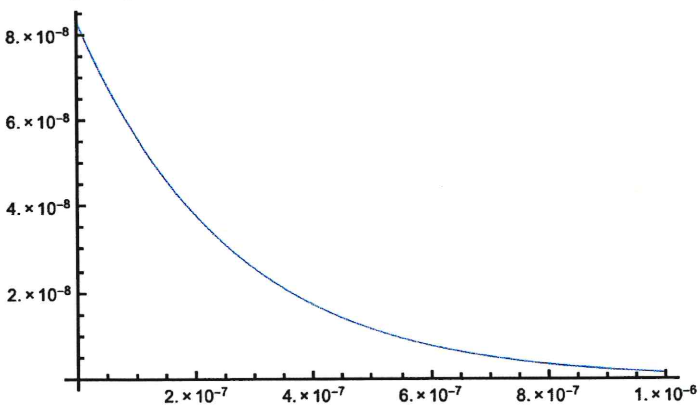
$$P = \int_{-(d0+10^{-1}\,10^{-6})}^{-d0} \text{Exp}\left[2\,(1.96\times10^{6})\,x\right]\,dx$$

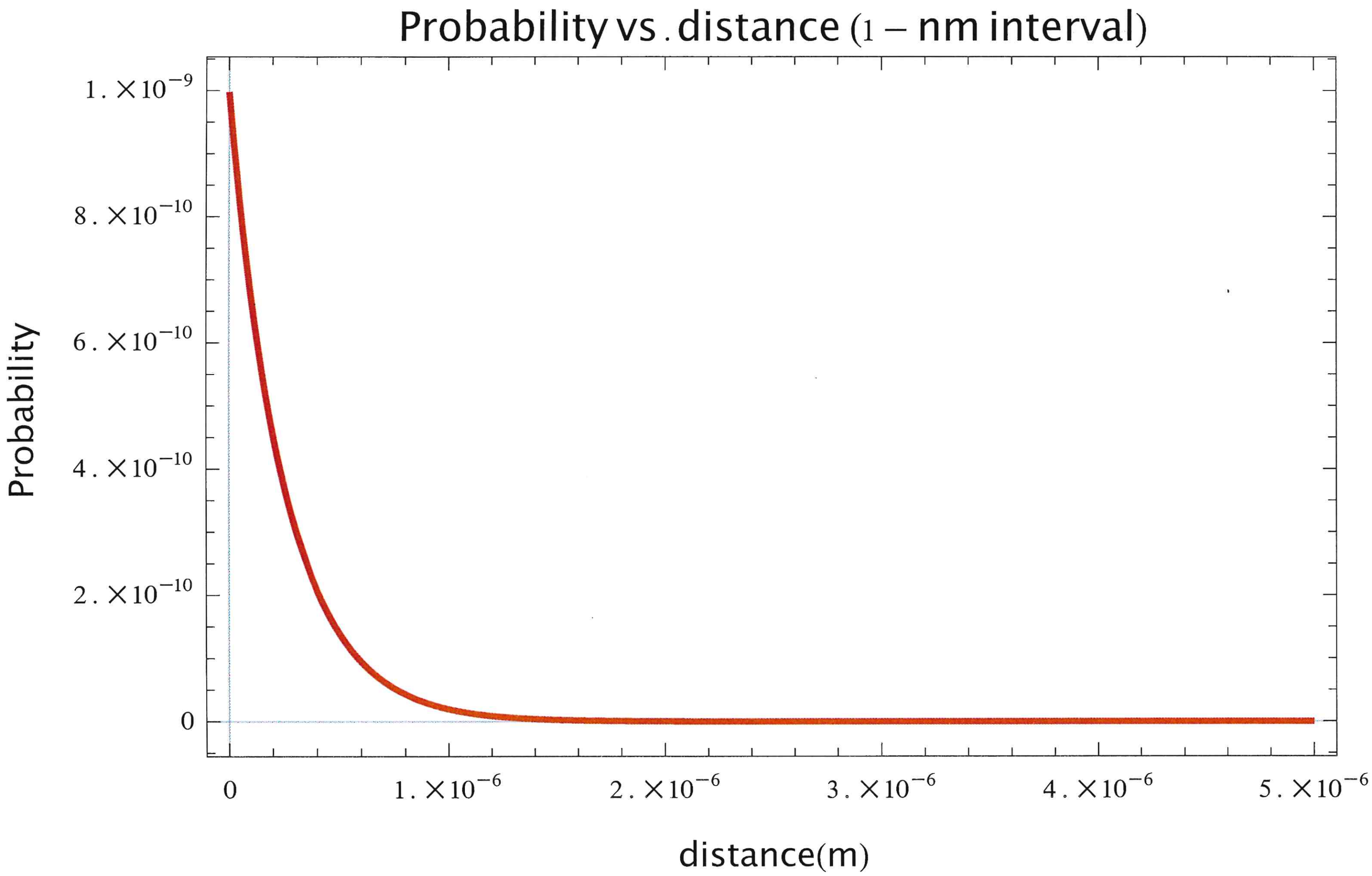
$$P\ /.\ \{d0 \to 1\times10^{-6}\}$$

$$8.27285\times10^{-8}\,e^{-3.92\times10^{6}\,d0}$$

$$1.64142\times10^{-9}$$

$$\text{Plot}\left[P,\ \{d0,\ 0,\ 1\times10^{-6}\},\ \text{PlotRange} \to \text{All}\right]$$





Probability vs. distance (1 – nm interval)
Probability
1. × 10^-9
8. × 10^-10
6. × 10^-10
4. × 10^-10
2. × 10^-10
0
0
1. × 10^-6
2. × 10^-6
3. × 10^-6
4. × 10^-6
5. × 10^-6
distance(m)