Exercise 1

a) Formula of spectral radiance of Black Body as function of wavelength. Units of measurement.
Why this formula is important in Remote Sensing.

- Spectral radiance of black body : $L_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left[\frac{hc}{\lambda K_B T}\right] - 1}$

* Meaning: For a given temperature we have many wavelengths
To take into account all the λ's we use a spectral range
$L = \int_{\lambda_1}^{\lambda_2} L_\lambda \, d\lambda$

- Units: spectral radiance density of wavelength λ per unit wavelength

$$\frac{\text{Power}/[\text{area} * \text{solid angle}]}{\text{wavelength unit}} \Rightarrow \left[\frac{W}{m^2 \cdot sr \cdot m}\right]$$

* What is Black Body
It is an object absorbing all the radiation incident on it. It can be seen as cavity with a small hole. Through hole radiation is absorbed
Temperature is not changing.
The emitted radiation is constant.

- Black Body model is important in remote sensing because the radiance of any body can be obtained adding a correction factor, the emissivity ε.

$$L_{\lambda,\varepsilon} = \varepsilon(\lambda) L_\lambda$$

b) Write the Stefan's Law and explain its meaning

It is obtained by:

→ Integrating the spectral radiance of black body across all wavelengths, obtaining total radiance L
$L = \int_0^\infty L_\lambda \, d\lambda$

→ The radiation from black-body is isotropic, therefore the total radiant exitance: $M = \pi L$

$M = \sigma T^4$ $[W/m^2]$. The radiant exitance of black body is proportional to the fourth power of the temperature.

c) Write Wien's Law

Plot qualitatively the spectral radiance of black body for $T = 30°C$ and $T = 5000°C$

Wien's Law gives us the wavelength at which the spectral radiance reaches the maximum.

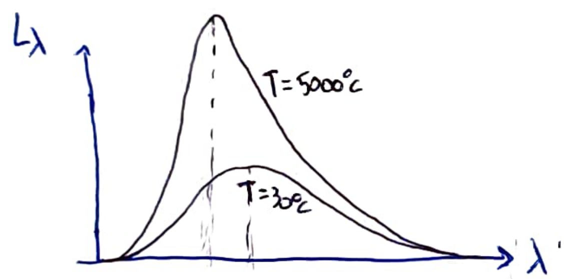
Increasing T shifts to a shorter wavelengths

$$\lambda_{max} = \frac{A}{T} \quad [m]$$

$0°C \rightarrow 273\,K$

$T_1 = 30°C = 303K \longrightarrow \frac{\lambda_{max}}{A} = \frac{1}{303} = 3{,}3 \cdot \; mK^{-1}$

$T_2 = 5000°C = 5273K \longrightarrow \frac{\lambda_{max}}{A} = \frac{1}{5273} = 0{,}19 \; mK^{-1}$



d) Spectral radiance of black body as a function of frequency

$$\lambda = \frac{c}{f}, \quad L_\lambda d\lambda = L_\lambda d\left(\frac{c}{f}\right) = L_\lambda(-cf^{-2})df \Rightarrow L_f = L_\lambda(c/f) \cdot cf^{-2}$$

$$L_f = \frac{2hc^2}{\frac{c^5}{f^5}} \cdot \frac{1}{\exp\left[\frac{hc}{\frac{c}{f}k_BT}\right] - 1} \cdot cf^{-2} = \frac{2hc^3f^5}{c^5 \cdot f^2} \cdot \frac{1}{\exp\left[\frac{hf}{k_BT}\right] - 1}$$

$$L_f = \frac{2hf^3}{c^2} \frac{1}{\exp\left[\frac{hf}{k_BT}\right] - 1}$$

Exercise 2

a) Structure of atmosphere.
Where airplanes and satellites can fly.

The atmosphere is composed by troposphere, stratosphere, mesosphere, thermosphere and exosphere.

The airplanes flies within the troposphere because in this layer (0 - 12 km) is where there is more gas density allowing planes to fly.

The satellites are located ~~mainly~~ in the thermosphere (80 - 700 km) where the density of gases is low (too much friction would make satellite to fall)
But satellites can be also located in the exosphere. (700 - 10 000 km)

b) Law of gravitation, explain.
General features of satellite orbits

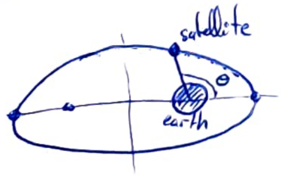
The law of gravitation express the force between two masses. For instance, between a planet and a satellite. This force is inversely proportional to the square of distance

$$\vec{F} = - G \frac{Mm}{r^2} \hat{r}$$

G → gravitational constant
M → mass of one body
m → mass of the other body
r → distance between bodies

The minus in the equation express that the force is attractive

From this law we know that a satellite follows a elliptical orbit being the Earth one of the focal points of the ellipse



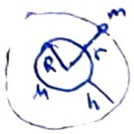
Perigee → the closest distance between Earth and satellite
Apogee → The furthest distance between Earth and satellite

The distance depends on the angle : $r = r(\theta)$

c) Assuming circular satellite orbit

→ Formula for orbital velocity

→ Formula for period



$r = R + h$

Balance of forces

$F_c = m \frac{v^2}{r}$

$F_G = G \frac{Mm}{r^2}$

$F_c = F_G$

$\Rightarrow \cancel{m}\frac{v^2}{\cancel{r}} = \frac{GM\cancel{m}}{r^{\cancel{2}}} \Rightarrow v = \sqrt{\frac{GM}{r}}$

•) Velocity

$v = \sqrt{\frac{GM}{R^2} \cdot \frac{R^2}{r}} = \sqrt{g\frac{R^2}{r}} = \sqrt{g\frac{R^2}{R+h}} \Rightarrow \boxed{v = \sqrt{g\frac{R^2}{R+h}}}$

•) Period

$T = \frac{\text{distance of one round}}{\text{velocity}} = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^2}{\frac{gR^2}{R+h}}} = 2\pi\sqrt{\frac{r^2(R+h)}{gR^2}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}}$

d) Period of two realistic satellite orbits

•) $h_1 = 400$ km (ISS)

$gR^2 = \frac{GM}{R^2} \cdot R^2 = GM$

$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} = 2\pi\sqrt{\frac{(6371+400)^3}{6{,}67\cdot10^{-11}\cdot5{,}974\cdot10^{24}}}\left(10^3\right)^{3/2} \Rightarrow T = 5545{,}8 \text{ sec} \Rightarrow \boxed{T = 92{,}43 \text{ min}}$

•) $h_2 = 35\,000$ Km

$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} = 2\pi\sqrt{\frac{(6371+35000)^3}{6{,}67\cdot10^{-11}\cdot5{,}974\cdot10^{24}}}\cdot\left(10^3\right)^{3/2} \Rightarrow T = 83{,}76\cdot10^3 \text{ sec} \Rightarrow T = 1396 \text{ min}$

$\Rightarrow \boxed{T = 23{,}26 \text{ hours}}$

Exercise 3

a) Define radiometric quantities

- **Radiance**: In radiometry, it is the radiant ~~flux~~ power emitted, reflected, transmitted or received incident by a given surface per unit solid angle per unit projected area. In other words, the incident radiation in the direction given by θ and φ

$d\Omega = \sin\theta \, d\theta \, d\varphi$ and θ is the angle between the propagation direction and the normal of the surface

$$L = \frac{dP}{\cos\theta \, dA \, d\Omega} \quad \left[\frac{W}{m^2 \, sr}\right]$$

→ **Irradiance**: It is the total incident power per unit area

$$E = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} L_{in} \cos\theta \, d\Omega \quad \left[\frac{W}{m^2}\right]$$

→ **Radiant Exitance**: It is the total emitted power per unit area.

$$M = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} L_{out} \cos\theta \, d\Omega \quad \left[\frac{W}{m^2}\right]$$

↳ If isotropic radiation $L = constant$

$$M = \pi L$$

b)

c) Explain how wavelength of max radiance depends on the body temperature

Through the Wien's displacement law we have the wavelength at which the spectral radiance reaches its maximum

$$\lambda_{max} = \frac{A}{T} \quad [m]$$

It is obtained simply doing $\frac{\partial L_\lambda}{\partial \lambda} = 0$ and solving.

d) Starting from ⓑ derive Rayleigh-Jeans formula. Explain when and why it can be useful.

$$L_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left[\frac{hc}{\lambda k_B T}\right] - 1}$$

If $\lambda \to \infty$ then: $\frac{hc}{\lambda k_B T} << 1$

So by Taylor: $e^x \simeq 1 + x \Rightarrow e^{\frac{hc}{\lambda k_B T}} \simeq 1 + \frac{hc}{\lambda k_B T}$

$$L_\lambda = \frac{2hc^2}{\lambda^5} \cdot \frac{1}{1 + \frac{hc}{\lambda k_B T} - 1} = \frac{2\cancel{h}c^{\cancel{2}}}{\lambda^5} \cdot \frac{\lambda k_B T}{\cancel{hc}} = \frac{2 k_B T c}{\lambda^4} \Rightarrow \boxed{L_\lambda \simeq \frac{2 k_B T c}{\lambda^4}}$$

It is useful or valid when we have long wavelengths and it gives us a simple equation to use

e) Define emissivity and brigthness temperature of a body and explain importance of these parameters

◦) Emissivity: It is the effectiveness in emitting energy as thermal radiation. It is the correction factor we multiply to black body spectral radiance to obtain the spectral radiance of a material. The emissivity depends on wavelength.

$$L_{\lambda,\varepsilon} = \varepsilon(\lambda)\, L_\lambda$$

◦) Brigthness temperature of a body (T_b): It is the temperature of the equivalent black body that would give the same radiance at the wavelength under consideration.

$$\varepsilon L_\lambda(\lambda, T) = L_\lambda(\lambda, T_b) \xrightarrow{\text{if } \lambda \to \infty} T_b = \varepsilon T$$

◦) Importance: These parameters are important because we can model any material using the black body and obtain the radiance.

Exercise 5

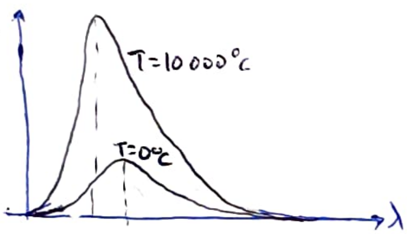
a).

b).

c) Plot spectral radiance of black body for a temperature of 0°c and 10 000°c

$\lambda_{max} = \frac{A}{T}$ $\quad T_1 = 0°c = 273\,K \Rightarrow \frac{\lambda_{max1}}{A} = \frac{1}{273} = 3,66 \ mK^{-1}$

$T_2 = 10\,000 = 10\,273 \Rightarrow \frac{\lambda_{max2}}{A} = \frac{1}{10\,273} = 0,097 \ mK^{-1}$



d).

e) starting from (b) obtain approximate formula for the wavelength of maximum radiance. Comment the result.

$$L_\lambda = \frac{2hc^2}{\lambda^5} \cdot \frac{1}{\exp\left[\frac{hc}{\lambda k_B T}\right] - 1}$$

Max at $\frac{\partial L_\lambda}{\partial \lambda} = 0$

$$\frac{\partial L_\lambda}{\partial \lambda} = 2hc^2 \left[-5\lambda^{-6} \cdot \frac{1}{\exp\left[\frac{hc}{\lambda k_B T}\right] - 1} + \lambda^{-5} \cdot (\cancel{-1}) \cdot \left(\exp\left[\frac{hc}{\lambda k_B T}\right] - 1\right)^{-2} \cdot \left((\cancel{-1}) \cdot \frac{hc}{k_B T} \lambda^{-2} \cdot \exp\left[\frac{hc}{\lambda k_B T}\right] \right) \right] =$$

$$= \frac{2hc^2}{\lambda^6} \left[\frac{hc}{k_B T} \cdot \frac{1}{\lambda} \cdot \frac{\exp\left[\frac{hc}{\lambda k_B T}\right]}{\left(\exp\left[\frac{hc}{\lambda k_B T}\right] - 1\right)^2} - \frac{5}{\exp\left[\frac{hc}{\lambda k_B T}\right] - 1} \right] = 0$$

$\frac{hc}{\lambda k_B T} = x$

$$\frac{hc}{k_B T} \frac{1}{\lambda} \frac{e^x}{(e^x - 1)^2} = \frac{5}{\cancel{e^x - 1}} \Rightarrow x e^x - 5(e^x - 1) = 0$$

We approximate it saying:

$$\frac{hc}{\lambda k_B T} >> 1 \Rightarrow e^x - 1 \simeq e^x$$

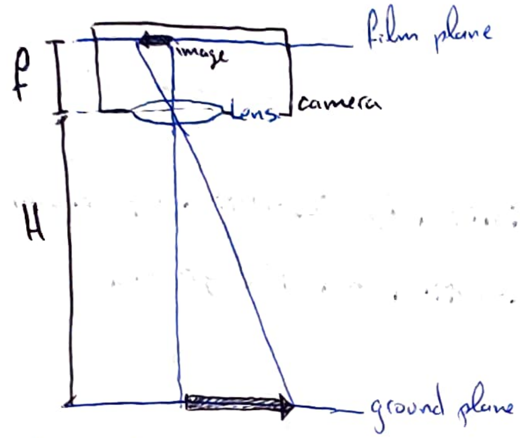
$$x e^x - 5 e^x = 0 \Rightarrow x = 5 \Rightarrow \frac{hc}{\lambda k_B T} = 5$$

$$\boxed{\lambda_{max} \simeq \frac{hc}{5 k_B T} = \frac{A}{T}}$$ Wien's displacement Law

We can see that according to this result when increasing T the maximum shifts to a shorter wavelengths.

Exercise 6

a) Describe a simple aerial photography system based on a single lens camera.



For the aerial photography we can have a scheme like is shown.
Where f is the focal point of the lens and H is the height.
Because H is much higher than f the distance from the lens to the image can be taken as focal length

$$\underset{\text{object}}{\frac{1}{u}} + \underset{\text{image}}{\frac{1}{v}} = \frac{1}{f} \xrightarrow[u \to \infty]{} v = f$$

b) Briefly explain the structure of a photographic film
Explain the meaning and importance of the parameters: Speed, Resolution.

- <u>Photographic film</u>: The film is composed by crystals of a salt embedded in a gelatin with a plastic base

Mechanism → If there is enough energy it absorbs photons and the salt grains become metallic silver.
Unexposed grains are removed and we obtain the negative (exposed areas appear dark)

- <u>Speed</u>: It is the time duration a film has to be exposed to light of a given illuminance to get a significant change of opacity after processing
  Grain size: Hige speed films ⇒ Large grains

- <u>Spatial Resolution</u>: It is the ability of a remote sensing system to distinguish two points.
  Line pairs per unit length (lp/mm) = It is the greatest number of lp per unit length that can be resolved

$$\delta_x = \frac{1}{2r}$$

δ_x → the smallest distance between two points
r → resolution [lp/length]

c) Define f/number of a lens
Explain why it is important in a photographic system.

It is the ratio between the focal length and diameter of the entrance pupil.

$N = \frac{f}{D}$ It accounts the brightness of the image. and lens size
The smaller f/number $\Rightarrow$ the larger the lens $\Rightarrow$ the brighter the image

d) Explain how resolution of photographic system is limited by resolution of film
Derive formula for film limited spatial resolution on the ground.

- Resolution of the film: We are limited to the material with the number of lines that it can distinguish or resolve. $\delta x = \frac{1}{2r}$

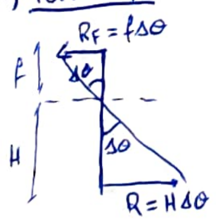
- Formula: Being the scale of the image: $s = \frac{f}{H}$
The spatial resolution on the ground δx_g would be: $s = \frac{\delta x}{\delta x_g} \rightarrow \delta x_g = \frac{\delta x}{s}$

$$\boxed{\delta x_g = \frac{1}{2r}\frac{H}{f}}$$

e) Explain how resolution of a phot. syst. is limited by diffraction
Obtain formula for the diffraction limited resolution on the ground.

- Resolution: The light can be treated as a wave therefore the phenomena of diffraction appears.
To resolve two points it's necessary to have a distance $\Delta\theta \simeq 1{,}22\frac{\lambda}{D}$

- Formula



For the resolution on the ground (R) $\rightarrow$ $s = \frac{R_F}{R} \rightarrow R = \frac{R_F}{s} = \Delta\theta f \cdot \frac{H}{f}$

$$\boxed{R = \Delta\theta f \frac{H}{f}}$$

f) Problem. Resolution ground limited by diffraction or film.

⑥

Data:

$H = 10\text{ km} = 10^4\text{ m}$ $\qquad$ $D = 5\text{cm} = 0{,}05\text{ m}$ $\qquad$ $\lambda_{visible}: [380 - 750]\text{ nm}$

$r = 200\ \frac{lp}{mm} = 200\cdot 10^3\ \frac{lp}{m}$ $\qquad$ $f = 150\text{ mm} = 0{,}15\text{ m}$

$$\delta x = \frac{1}{2r} = 2{,}5\cdot 10^{-6}\ m^{-1}$$

$$S = \frac{f}{H} = \frac{0{,}15}{10^4} = 1{,}5\cdot 10^{-5}$$

* $\lambda_1 = 380\text{ nm} \Rightarrow \Delta\theta_1 = 1{,}22\ \frac{\lambda_1}{D} = 1{,}22\ \frac{380\cdot 10^{-9}}{0{,}05} = 9{,}272\cdot 10^{-6}$

$\Rightarrow R_{F_1} = f\,\Delta\theta_1 = 1{,}4\cdot 10^{-6}\text{ m}$

* $\lambda_2 = 750\text{ nm} \Rightarrow \Delta\theta_2 = 18{,}3\cdot 10^{-6}$

$\Rightarrow R_{F_2} = f\,\Delta\theta_2 = 2{,}745\cdot 10^{-6}\text{ m}$

The results of resolution in the film or in the ground produce the same result. As we can see for short λ the limit of resolution is in the film, while, for long λ the limit is in diffraction.

We can calculate the corresponded λ to match the limits of both.

$$R_F = 2{,}5\cdot 10^{-6}\text{ m} \longrightarrow R_F = f\,\Delta\theta = f\ 1{,}22\ \frac{\lambda}{D} \Rightarrow \lambda = \frac{D\cdot R_F}{1{,}22\cdot f} = \frac{0{,}05\cdot 2{,}5\cdot 10^{-6}}{1{,}22\cdot 0{,}15} = 6{,}83\cdot 10^{-7}\text{ m}$$

$$\boxed{\lambda = 683\text{ nm}}$$

So the limitation goes like this