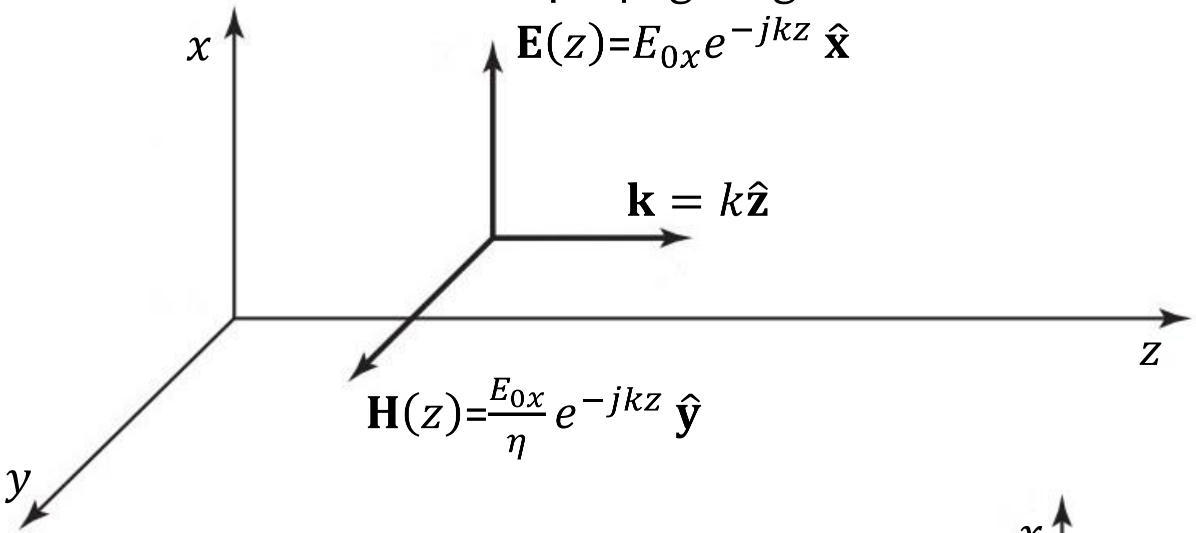
POLARIZATION OF ELECTROMAGNETIC WAVES

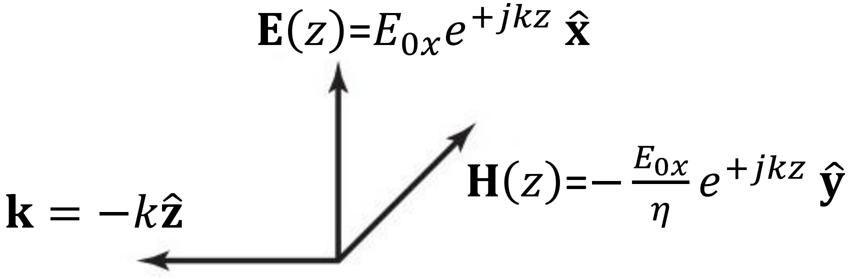
The phase front of a wave radiated by a finite-sized radiator becomes nearly planar over small observation regions. The polarization of a plane wave is the figure the instantaneous electric field traces out with time at a fixed observation point.

PLANE WAVE: PHASOR DOMAIN

Forward propagating wave

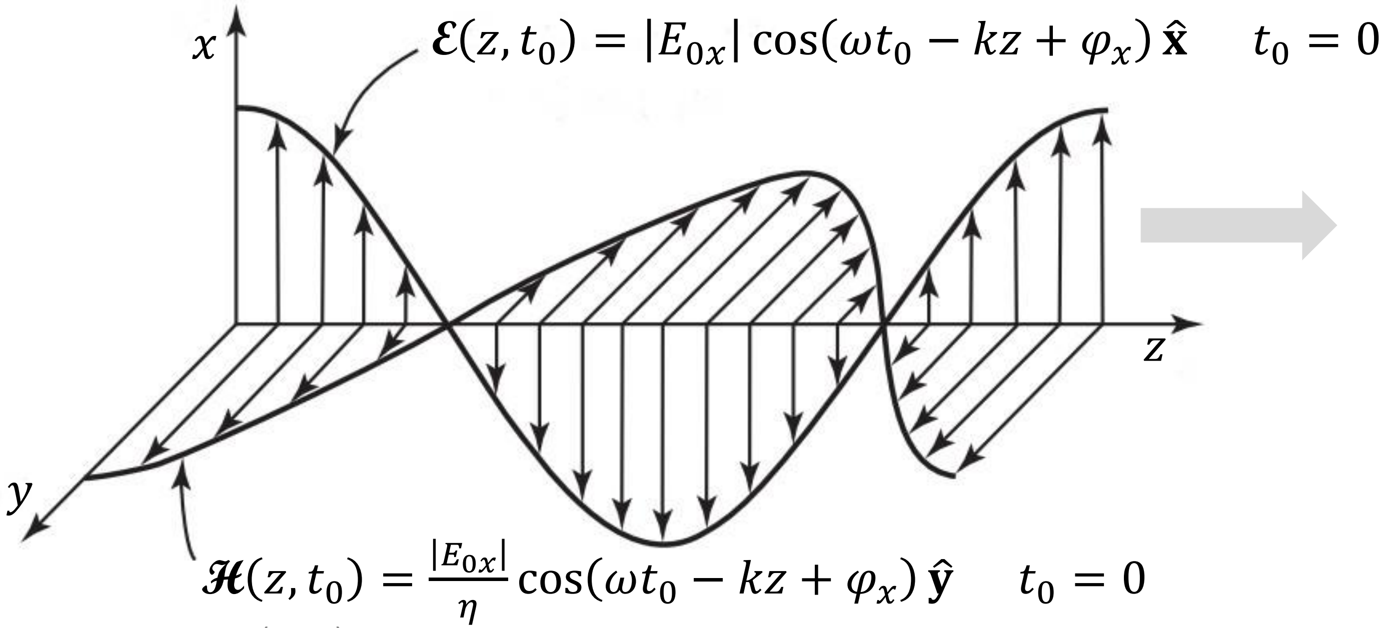


Backward propagating wave



PLANE WAVE: TIME DOMAIN

The forward propagating wave is travelling to the right at the speed of light. The electric field $\mathcal{E}$ oscillates along x and the magnetic field $\mathcal{H}$ is perpendicular to $\mathcal{E}$ (and in this case it oscillates along y)
… linearly polarized field ?!?



In general, the polarization of a plane wave refers to the orientation of the electric field vector, which may be in a fixed direction or may change with time.

A plane wave travelling in the $+z$ direction reads as

$$\mathbf{E}(z)=\mathbf{E}_0 e^{-jkz} = \left(E_{0x}\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}}\right)e^{-jkz} = E_{0x}e^{-jkz}\hat{\mathbf{x}} + E_{0y}e^{-kz}\hat{\mathbf{y}}$$

This wave can be considered as the superposition of a wave having only a component of the electric field along the x-axis $E_{0x}e^{-jkz}\hat{\mathbf{x}}$, and a wave having only a component of the electric field along the y-axis $E_{0y}e^{-jkz}\hat{\mathbf{y}}$: if considered separately, those two waves are linearly polarized because the electric field in the time domain moves forth and back along the x and y-axes, respectively.

$$\boldsymbol{\mathcal{E}}(z,t) = \mathrm{Re}\left\{E_{0x}e^{-jkz}\hat{\mathbf{x}}e^{j\omega t}\right\} = \mathrm{Re}\left\{|E_{0x}|e^{j\varphi_x}e^{-jkz}e^{j\omega t}\right\}\hat{\mathbf{x}} = |E_{0x}|\cos(\omega t - kz + \varphi_x)\,\hat{\mathbf{x}}$$

In the phasor domain the amplitudes of the electric field components are complex quantities and the formula for the total field can be rearranged in the following form

$$\mathbf{E}(z)=\mathbf{E}_0 e^{-jkz} = \left(|E_{0x}|e^{j\varphi_x}\hat{\mathbf{x}} + \left|E_{0y}\right|e^{j\varphi_y}\hat{\mathbf{y}}\right)e^{-jkz} = e^{j\varphi_x}\left(|E_{0x}|\hat{\mathbf{x}} + \left|E_{0y}\right|e^{j\left(\varphi_y - \varphi_x\right)}\hat{\mathbf{y}}\right)e^{-jkz}$$

The phase difference between the y and x-component is $\boxed{\delta = \varphi_y - \varphi_x}$

$$\mathbf{E}(z) = \mathbf{E}_0 e^{-jkz} = e^{j\varphi_x}\left(|E_{0x}|\hat{\mathbf{x}} + |E_{0y}|e^{j(\varphi_y - \varphi_x)}\hat{\mathbf{y}}\right)e^{-jkz} = e^{j\varphi_x}\left(|E_{0x}|\hat{\mathbf{x}} + |E_{0y}|e^{j\delta}\hat{\mathbf{y}}\right)e^{-jkz}$$

and in the time domain the electric field is given by

$$\mathcal{E}(z,t) = \mathrm{Re}\left\{e^{j\varphi_x}\left(|E_{0x}|\hat{\mathbf{x}} + |E_{0y}|e^{j\delta}\hat{\mathbf{y}}\right)e^{-jkz}e^{j\omega t}\right\} = |E_{0x}|\cos(\omega t - kz + \varphi_x)\,\hat{\mathbf{x}} + |E_{0y}|\cos(\omega t - kz + \delta + \varphi_x)\,\hat{\mathbf{y}}$$

The commont intial phase term φ_x is equivalente to a translation of the time axis and therefore its presence does not change the essence of the field evolution over time: from now on we can assume that $\varphi_x = 0$ and we are going to study how the amplitudes of the components along the x and y-axes ($|E_{0x}|$, $|E_{0y}|$) and the phase difference δ determine the polarization state

$$\boxed{\mathbf{E}(z) = \left(|E_{0x}|\hat{\mathbf{x}} + |E_{0y}|e^{j\delta}\hat{\mathbf{y}}\right)e^{-jkz}}$$

$$\mathcal{E}(z,t) = |E_{0x}|\cos(\omega t - kz)\,\hat{\mathbf{x}} + |E_{0y}|\cos(\omega t - kz + \delta)\,\hat{\mathbf{y}}$$

The magnitude of the electric field in the time domain is given by

$$|\boldsymbol{\mathcal{E}}(z,t)| = \sqrt{|E_{0x}|^2 \cos^2(\omega t - kz) + \left|E_{0y}\right|^2 \cos^2(\omega t - kz + \delta)}$$

whereas the angle ψ between the x-axis and the electric field vector $\boldsymbol{\mathcal{E}}$ (measured counterclockwise from the x-axis) reads as

$$\psi = \tan^{-1}\left\{\frac{\left|E_{0y}\right| \cos(\omega t - kz + \delta)}{|E_{0x}| \cos(\omega t - kz)}\right\}$$

The polarization state can be analyzed by tracing the time varying electric field vector $\boldsymbol{\mathcal{E}}$ in the (x, y) plane for a given value of z, or by tracing the same vector in the (x, y, z) space for a given value of t.

LINEAR POLARIZATION

If $E_{0x}$and E_{0y} are in phase ($\delta = 0$) or in phase opposition ($\delta = \pi$), the electric field is linearly polarized.

$\boxed{\delta = 0}$

$$\mathbf{E}(z) = \left(|E_{0x}|\hat{\mathbf{x}} + \left|E_{0y}\right|\hat{\mathbf{y}}\right)e^{-jkz}$$

$$\mathcal{E}(z,t) = \left(|E_{0x}|\hat{\mathbf{x}} + \left|E_{0y}\right|\hat{\mathbf{y}}\right)\cos(\omega t - kz)$$

$$|\mathcal{E}(z,t)| = \sqrt{|E_{0x}|^2 + \left|E_{0y}\right|^2}\cos(\omega t - kz) \qquad \psi = \tan^{-1}\left\{\frac{\left|E_{0y}\right|}{|E_{0x}|}\right\}$$

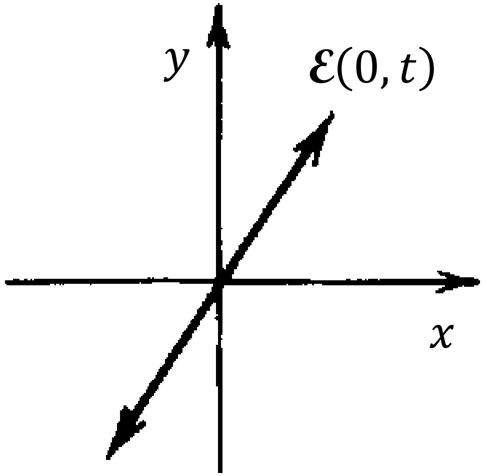
$\boxed{\delta = \pi}$

$$\mathbf{E}(z) = \left(|E_{0x}|\hat{\mathbf{x}} - \left|E_{0y}\right|\hat{\mathbf{y}}\right)e^{-jkz}$$

$$\mathcal{E}(z,t) = \left(|E_{0x}|\hat{\mathbf{x}} - \left|E_{0y}\right|\hat{\mathbf{y}}\right)\cos(\omega t - kz)$$

$$|\mathcal{E}(z,t)| = \sqrt{|E_{0x}|^2 + \left|E_{0y}\right|^2}\cos(\omega t - kz) \qquad \psi = \tan^{-1}\left\{-\frac{\left|E_{0y}\right|}{|E_{0x}|}\right\}$$

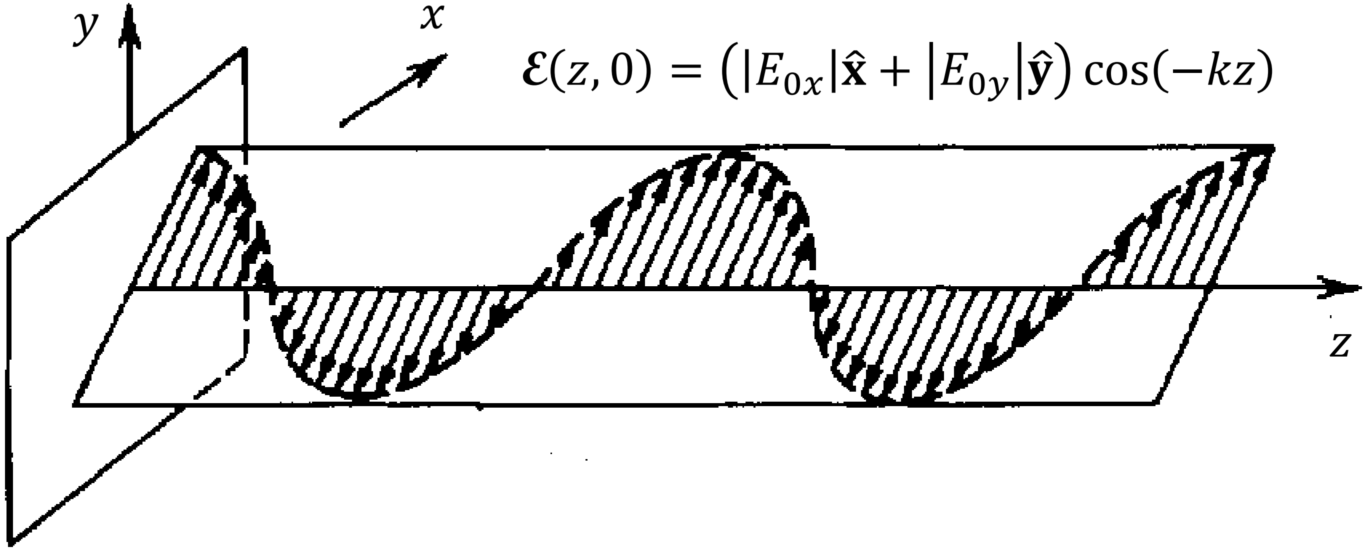
As time progresses, the electric field $\mathcal{E}(z,t)$ at a fixed point z oscillates forth and back along a line.



$z = 0$ plane

Electric field vector at $t = 0$

$$\mathcal{E}(z,0) = \left(|E_{0x}|\hat{\mathbf{x}} + |E_{0y}|\hat{\mathbf{y}}\right)\cos(-kz)$$



CIRCULAR POLARIZATION

If E_{0x} and E_{0y} have the same magnitude and are in quadrature $\left(\delta = \pm\frac{\pi}{2}\right)$ the electric field is circularly polarized.

$$\delta = \pm\frac{\pi}{2}$$

$$|E_{0x}| = |E_{0y}| = |E_0|$$

$$\mathbf{E}(z) = \left(|E_0|\hat{\mathbf{x}} + |E_0|e^{\pm j\frac{\pi}{2}}\hat{\mathbf{y}}\right)e^{-jkz} = |E_0|\left(\hat{\mathbf{x}} + e^{\pm j\frac{\pi}{2}}\hat{\mathbf{y}}\right)$$

$$\boldsymbol{\mathcal{E}}(z,t) = |E_0|\left[\cos(\omega t - kz)\,\hat{\mathbf{x}} + \cos\left(\omega t - kz \pm \frac{\pi}{2}\right)\hat{\mathbf{y}}\right]$$

$$|\boldsymbol{\mathcal{E}}(z,t)| = |E_0|\sqrt{\cos^2(\omega t - kz) + \cos^2\left(\omega t - kz \pm \frac{\pi}{2}\right)}$$

$$\psi = \tan^{-1}\left\{\frac{\cos\left(\omega t - kz \pm \frac{\pi}{2}\right)}{\cos(\omega t - kz)}\right\}$$

Right-hand circular polarization: the electric field vector remains constant in length but rotates around in a circular path and the vector rotates **counterclockwise** at the uniform angular velocity ω (from the point of view of an observer oriented as $+z$)

$$\boxed{\begin{aligned}&\delta = -\frac{\pi}{2}\\ &|E_{0x}| = |E_{0y}| = |E_0|\end{aligned}}$$

$$\mathbf{E}(z) = |E_0|\left(\hat{\mathbf{x}} + e^{-j\frac{\pi}{2}}\hat{\mathbf{y}}\right)e^{-jkz} = \boxed{|E_0|(\hat{\mathbf{x}} - j\hat{\mathbf{y}})e^{-jkz}}$$

$$\mathcal{E}(z,t) = |E_0|[\cos(\omega t - kz)\,\hat{\mathbf{x}} + \sin(\omega t - kz)\,\hat{\mathbf{y}}]$$

$$|\mathcal{E}(z,t)| = |E_0| \qquad \psi = \tan^{-1}\left\{\frac{\sin(\omega t - kz)}{\cos(\omega t - kz)}\right\} = \omega t - kz$$

Left-hand circular polarization: the electric field vector remains constant in length but rotates around in a circular path and the vector rotates **clockwise** at the uniform angular velocity $-\omega$ (from the point of view of an observer oriented as $+z$)

$$\boxed{\begin{aligned}&\delta = +\frac{\pi}{2}\\ &|E_{0x}| = |E_{0y}| = |E_0|\end{aligned}}$$

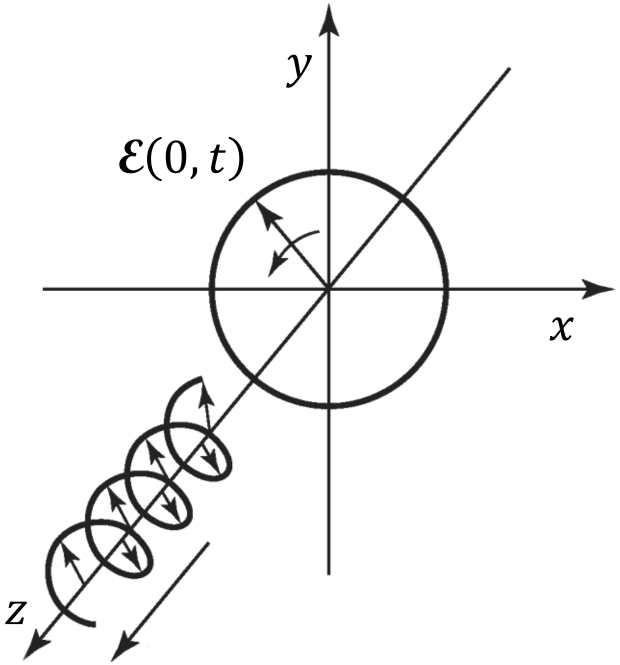
$$\mathbf{E}(z) = |E_0|\left(\hat{\mathbf{x}} + e^{+j\frac{\pi}{2}}\hat{\mathbf{y}}\right)e^{-jkz} = \boxed{|E_0|(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jkz}}$$

$$\mathcal{E}(z,t) = |E_0|[\cos(\omega t - kz)\,\hat{\mathbf{x}} - \sin(\omega t - kz)\,\hat{\mathbf{y}}]$$

$$|\mathcal{E}(z,t)| = |E_0| \qquad \psi = \tan^{-1}\left\{-\frac{\sin(\omega t - kz)}{\cos(\omega t - kz)}\right\} = -\omega t + kz$$

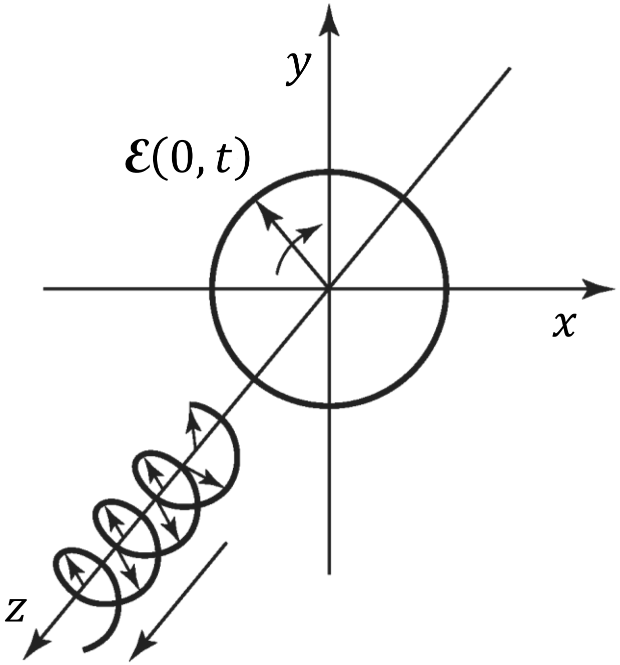
Right-hand circular polarization (RHCP): with the thumb of the right hand in the direction of propagation, the fingers will curl in the direction of rotation (counterclockwise) of the instantaneous electric field $\boldsymbol{\mathcal{E}}$

$$\boldsymbol{\mathcal{E}}(0,t) = |E_0|[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$$



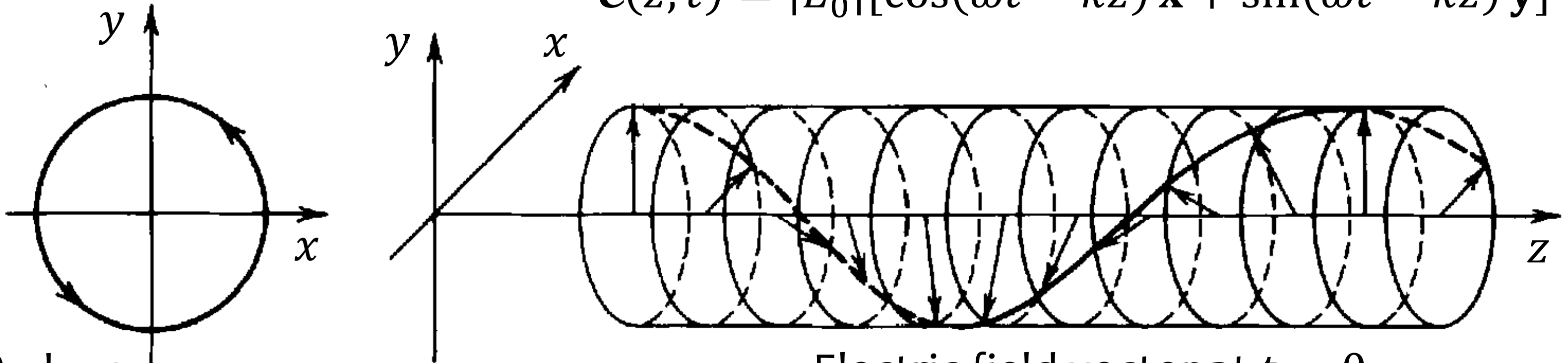
Left-hand circular polarization (LHCP): with the thumb of the left hand in the direction of propagation, the fingers will curl in the direction of rotation (clockwise) of the instantaneous electric field $\boldsymbol{\mathcal{E}}$

$$\boldsymbol{\mathcal{E}}(0,t) = |E_0|[\cos(\omega t)\,\hat{\mathbf{x}} - \sin(\omega t)\,\hat{\mathbf{y}}]$$



Right-hand circular polarization (RHCP)

$$\boldsymbol{\mathcal{E}}(z,t) = |E_0|[\cos(\omega t - kz)\,\hat{\mathbf{x}} + \sin(\omega t - kz)\,\hat{\mathbf{y}}]$$

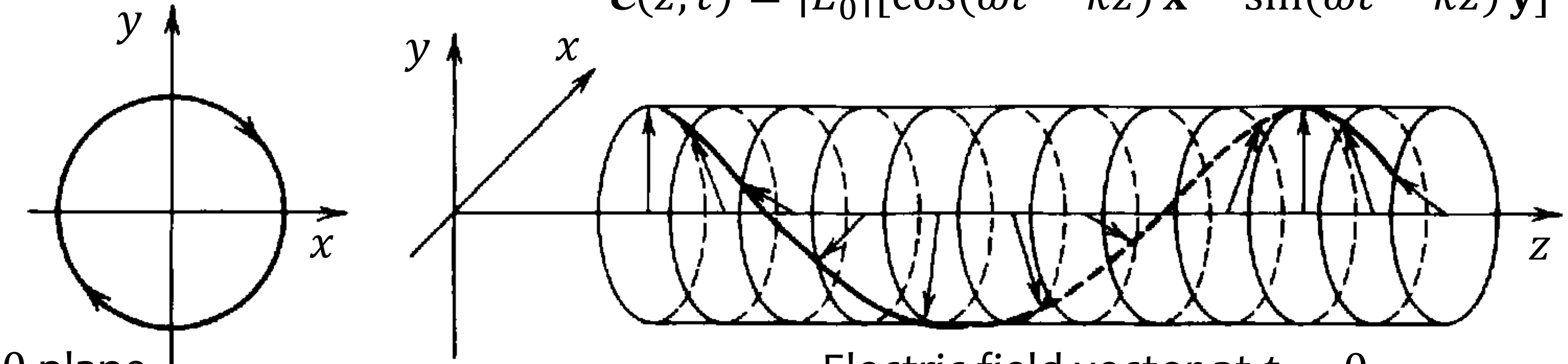


$z = 0$ plane

Electric field vector at $t = 0$

Left-hand circular polarization (LHCP)

$$\boldsymbol{\mathcal{E}}(z,t) = |E_0|[\cos(\omega t - kz)\,\hat{\mathbf{x}} - \sin(\omega t - kz)\,\hat{\mathbf{y}}]$$



$z = 0$ plane

Electric field vector at $t = 0$

ELLIPTICAL POLARIZATION

In the most general case the two components of $\mathcal{E}$ have different magnitudes and are not in phase or in quadrature and the vector traces out an ellipse (the sense of rotation can be counterclockwise or clockwise)

$$\mathbf{E}(z) = \left(|E_{0x}|\hat{\mathbf{x}} + \left|E_{0y}\right|e^{j\delta}\hat{\mathbf{y}}\right)e^{-jkz}$$

$$\mathcal{E}(z,t) = |E_{0x}|\cos(\omega t - kz)\,\hat{\mathbf{x}} + \left|E_{0y}\right|\cos(\omega t - kz + \delta)\,\hat{\mathbf{y}} = \mathcal{E}_x(z,t)\hat{\mathbf{x}} + \mathcal{E}_y(z,t)\hat{\mathbf{y}}$$

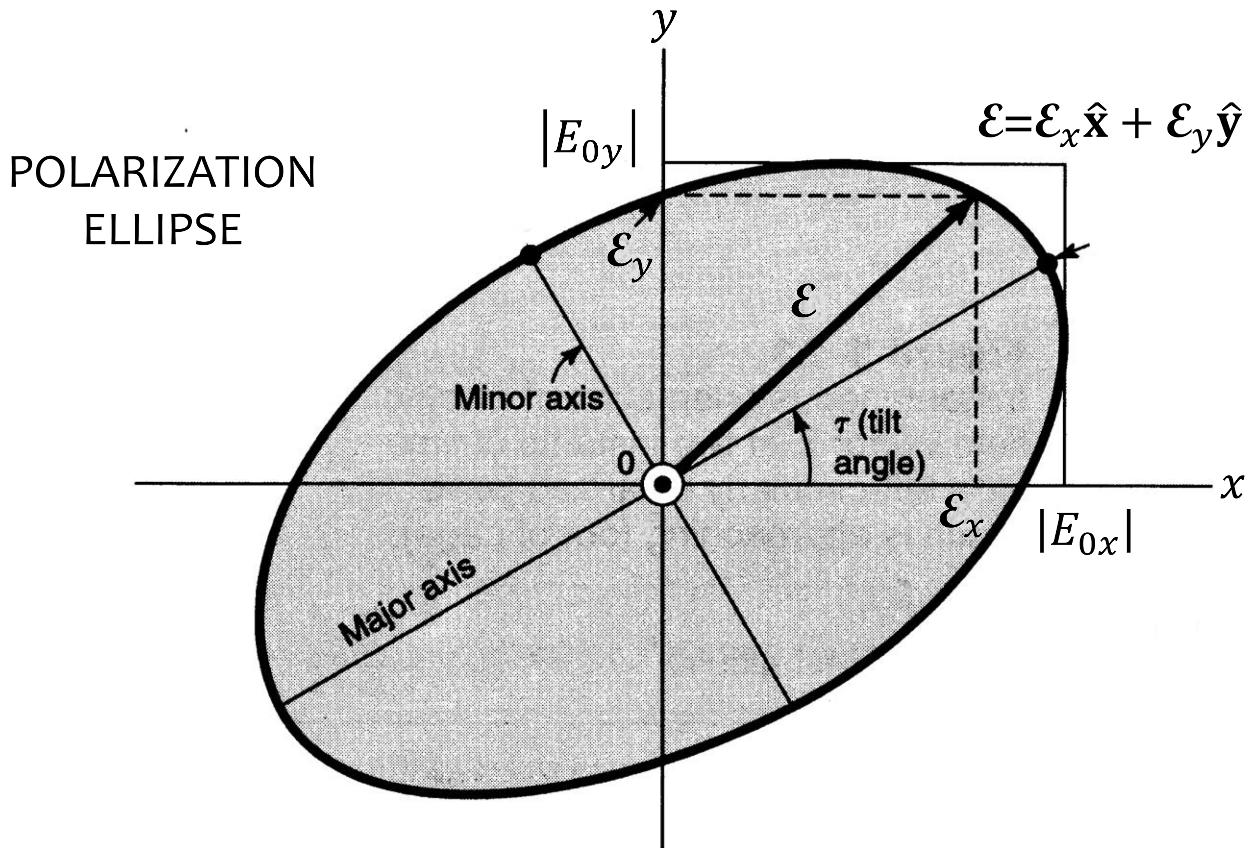
We can prove that the geometrical shape traced out by the rotating vector is an ellipse

$$\frac{\mathcal{E}_x}{|E_{0x}|} = \cos(\omega t - kz) \qquad \sqrt{1 - \left(\frac{\mathcal{E}_x}{|E_{0x}|}\right)^2} = \sin(\omega t - kz)$$

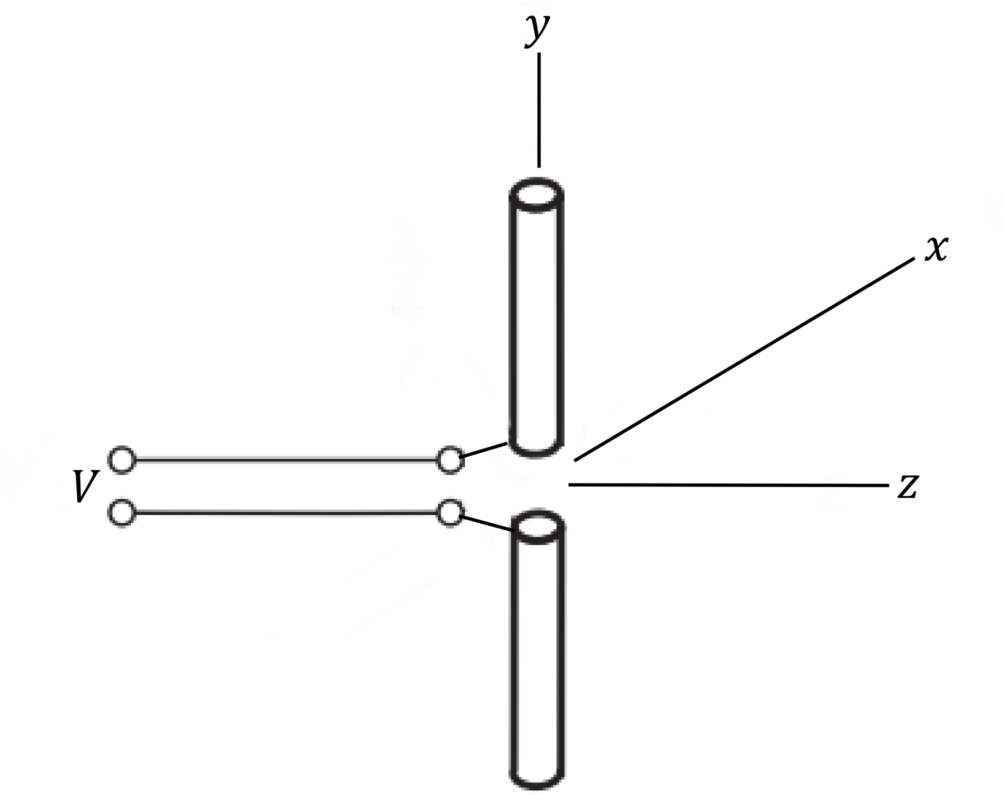
$$\frac{\mathcal{E}_y}{\left|E_{0y}\right|} = \cos(\omega t - kz)\cos\delta - \sin(\omega t - kz)\sin\delta \qquad \frac{\mathcal{E}_y}{\left|E_{0y}\right|} = \frac{\mathcal{E}_x}{|E_{0x}|}\cos\delta - \sqrt{1 - \left(\frac{\mathcal{E}_x}{|E_{0x}|}\right)^2}\sin\delta$$

By squaring and rearranging the last formula, we obtain an equation describing an ellipse

$$\left(\frac{\mathcal{E}_x}{|E_{0x}|}\right)^2 + \left(\frac{\mathcal{E}_y}{|E_{0y}|}\right)^2 - 2\frac{\mathcal{E}_x\,\mathcal{E}_y}{|E_{0x}||E_{0y}|}\cos\delta = \sin^2\delta$$



Source of a LINEARLY POLARIZED field: ideal DIPOLE



Source of a RIGHT-HAND CIRCULARLY POLARIZED field: CROSSED ideal DIPOLES with a QUARTER-WAVE DELAY LINE