MICROWAVE ENGINEERING

Lecture 17:
Impedance
Matching part
II

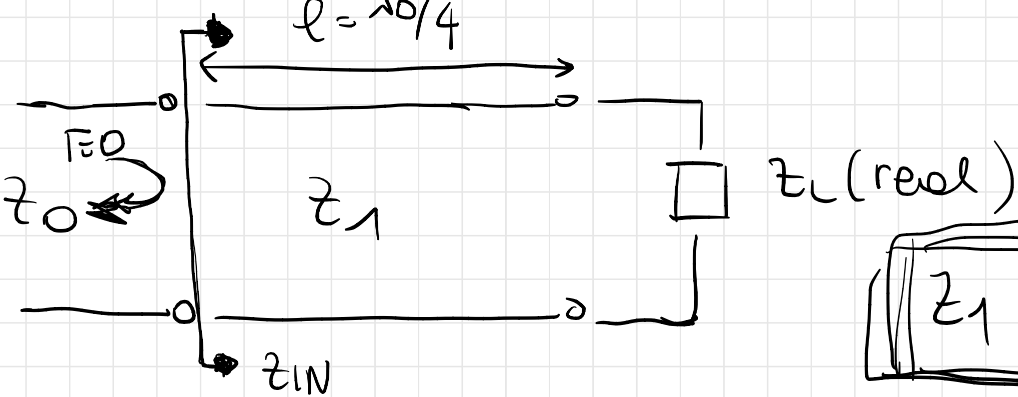
QUARTER-WAVE TRANSFORMER

PROS

- Simple Design
- Can be modified to multi-section

CONS

- Matching for real loads
- Single frequency design



$$\boxed{Z_1 = \sqrt{Z_0 Z_L}}$$

$$Z_{IN} = Z_1 \frac{Z_L + j Z_1 \tan \beta l}{Z_1 + j Z_L \tan \beta l}$$

$$\beta l = \frac{2\pi}{\cancel{\lambda_0}} \cdot \frac{\cancel{\lambda_0}}{4} = \frac{\pi}{2} = \theta$$

$$\Gamma = \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0} = \frac{Z_1 (Z_L - Z_0) + j \tan \beta l \,(Z_1^2 - Z_0 Z_L)}{Z_1 (Z_L + Z_0) + j \tan \beta l \,(Z_1^2 + Z_0 Z_L)}$$

$$Z_1^2 = Z_0 Z_L \qquad \text{if } \theta = \frac{\pi}{2} \Rightarrow \Gamma = 0$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0 + 2j \tan\beta l \left(\sqrt{Z_0 Z_L}\right)}$$

The magnitude of Γ

$$|\Gamma| = \frac{|Z_L - Z_0|}{\sqrt{(Z_L+Z_0)^2 + 4 Z_0 Z_L \tan^2 \beta l}} = \quad t = \tan\beta l$$

$$= \frac{1}{\left\{ \frac{(Z_L+Z_0)^2}{(Z_L-Z_0)^2} + 4t^2 \frac{Z_0 Z_L}{(Z_L-Z_0)^2} \right\}^{\frac{1}{2}}} =$$

$$= \frac{1}{\left\{ 1 + \frac{4 z_0 z_L}{(z_L - z_0)^2} + \frac{4 t^2 z_0 z_L}{(z_L - z_0)^2} \right\}^{\frac{1}{2}}} =$$

$$\theta = \beta \ell$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$= \frac{1}{\left\{ 1 + \frac{4 z_0 z_L}{(z_L - z_0)^2} \sec^2\theta \right\}^{\frac{1}{2}}}$$

Assuming $f \simeq f_0$ $\quad l \simeq \frac{\lambda_0}{4}$ $\quad \theta \simeq \frac{\pi}{2}$

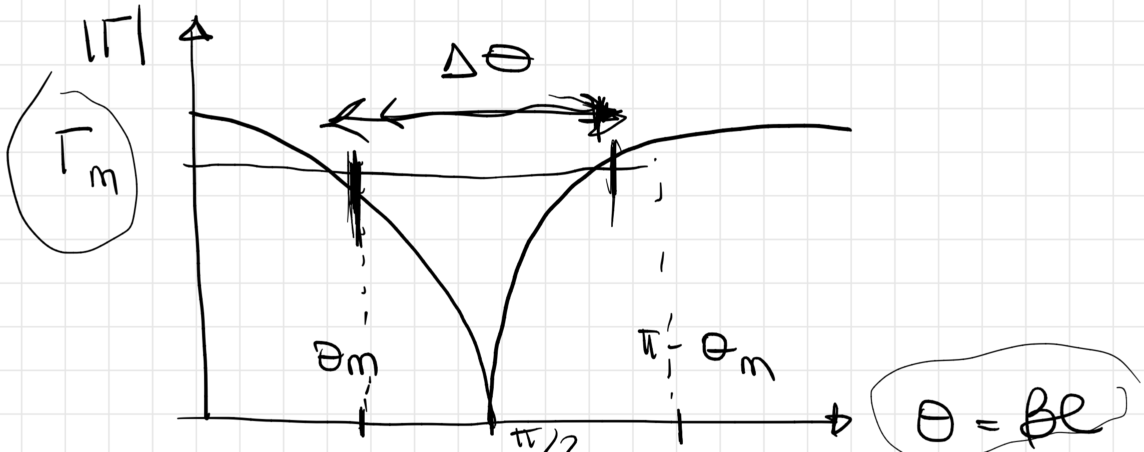
$$\Downarrow$$

$$\sec^2\theta \gg 1$$

then

$$\boxed{|\Gamma| \simeq \frac{|Z_L - Z_0|}{2\sqrt{Z_0 Z_L}} \, |\cos\theta|}$$

for $\theta \sim \frac{\pi}{2}$



We can define a max deviation we can accept that is Γ_m that defines the Bandwidth of the transformer

$$\Delta\theta = 2\left(\frac{\pi}{2} - \theta_m\right)$$

if $\Gamma = \Gamma_m$ $\theta = \theta_m$ $\theta = \pi - \theta_m$

$$\frac{1}{\Gamma_m^2} = 1 + \left(\frac{2\sqrt{Z_0 Z_L}}{Z_L - Z_0}\sec\theta_m\right)^2$$

or

$$\cos\theta_m = \frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|} \Leftarrow$$

If we assume to have a TEM line

$$\theta = \beta \ell = \frac{2\pi f}{\cancel{v_p}} \cdot \frac{\cancel{v_p}}{4 f_0} = \frac{\pi f}{2 f_0}$$

at $\theta = \theta_m$

$$\boxed{f_m = \frac{2\theta_m f_0}{\pi}}$$

the fractional bandwidth

$$BW = \frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{2f_m}{f_0}$$

$$= 2 - \frac{4\theta_m}{\pi} =$$

$$= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\,\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right]$$

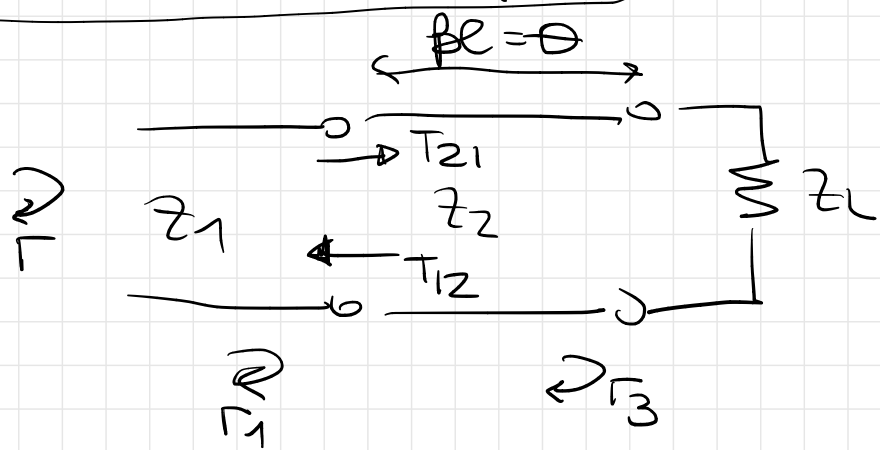
usually given in percentage

NOTE

- Formulas are valid for TEM lines where there is no dispersion

THEORY OF SMALL REFLECTIONS

- Single section transformer

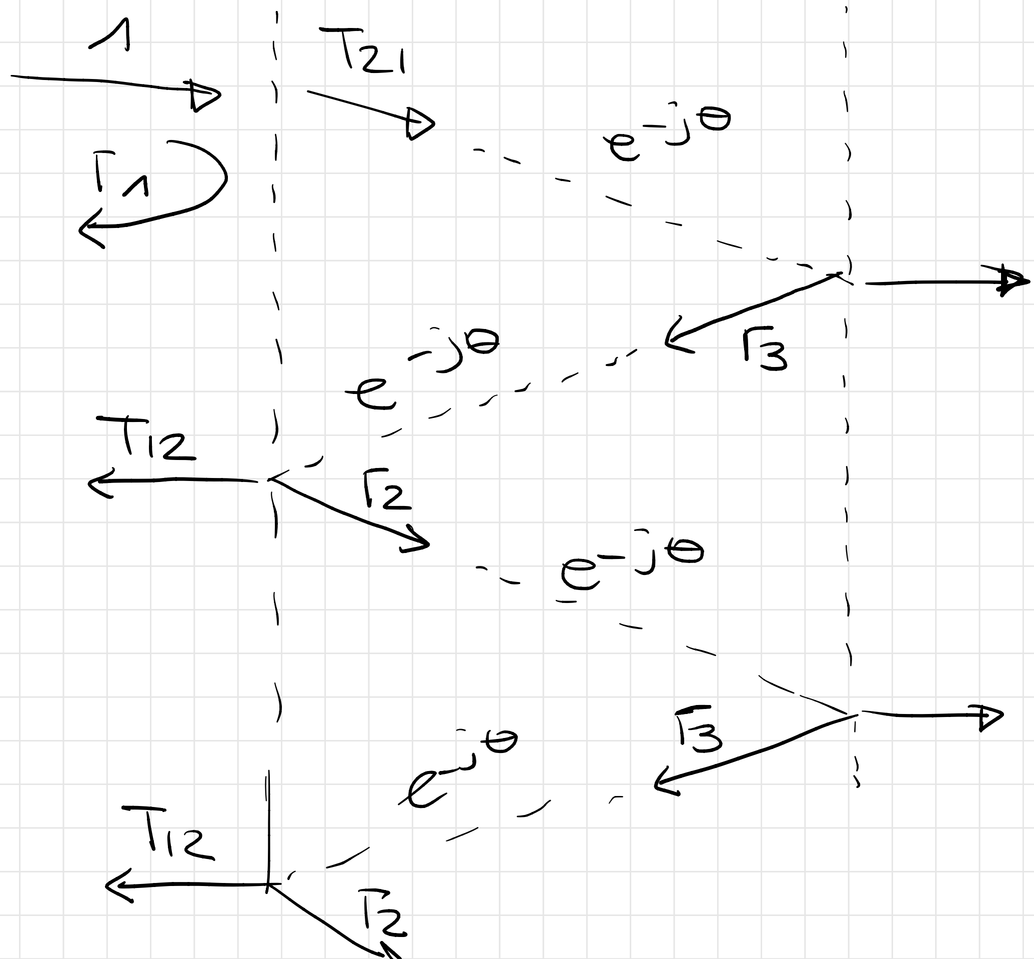


$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad \Gamma_2 = -\Gamma_1$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}$$

$$T_{21} = 1 + \Gamma_1 = \frac{2 Z_2}{Z_1 + Z_2}$$

$$T_{12} = 1 + \Gamma_2 = \frac{2 Z_1}{Z_1 + Z_2}$$



1
T21
Γ1
e-jθ
Γ3
e-jθ
T12
Γ2
e-jθ
Γ3
e-jθ
T12
Γ2

$$\Gamma = \Gamma_1 + \underbrace{T_{12} T_{21} \Gamma_3 e^{-2j\theta}} + T_{12} T_{21} \Gamma_3^2 \Gamma_2 e^{-4j\theta} + \ldots$$

$$= \Gamma_1 + T_{12} T_{21} \Gamma_3 e^{-j2\theta} \sum_{n=0}^{\infty} \Gamma_2^n \Gamma_3^n e^{-j2n\theta}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x| < 1$$

expanding Γ we get $\Gamma = \Gamma_1 + \dfrac{T_{12} T_{21} \Gamma_3 e^{-2j\theta}}{1 - \Gamma_2 \Gamma_3 e^{-2j\theta}}$

Replacing $\Gamma_2 = -\Gamma_1$ $\qquad T_{21} = 1 + \Gamma_1$

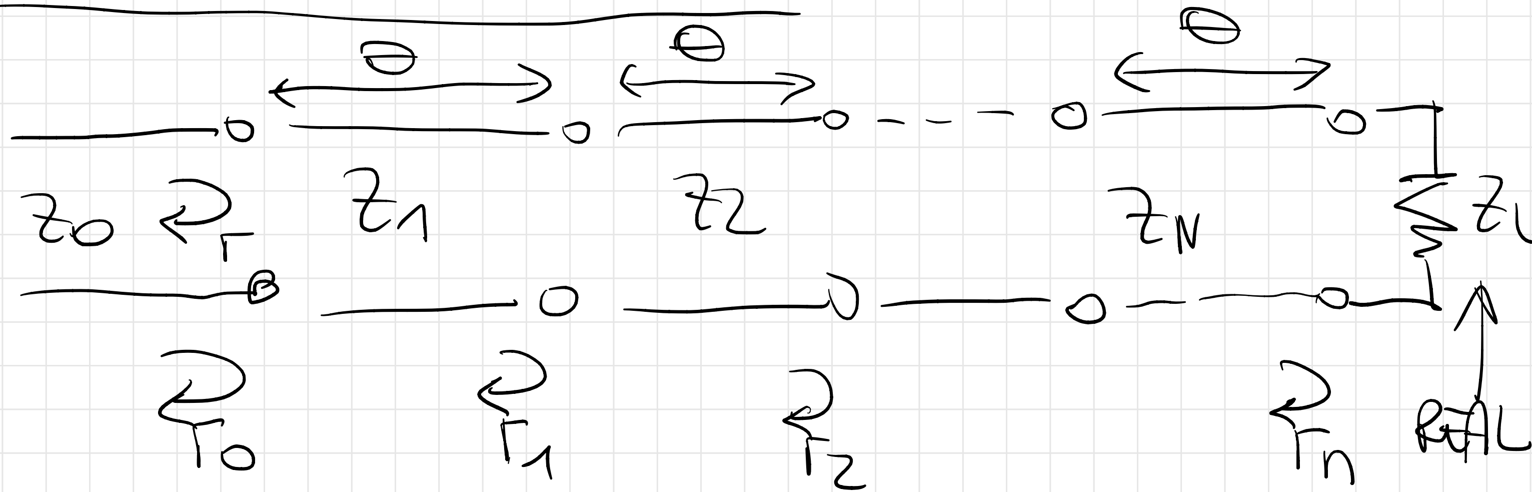
$$T_{12} = 1 + \Gamma_2 = 1 - \Gamma_1$$

$$\boxed{\Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-2j\theta}}{1 + \underline{\Gamma_1 \Gamma_3 e^{-2j\theta}}}}$$

If the discontinuities are <u>small</u> ($|Z_1 - Z_2|$ and $|Z_2 - Z_L|$ are small) then

$$|\Gamma_1 \Gamma_3| << 1 \Rightarrow \boxed{\Gamma \simeq \Gamma_1 + \Gamma_3 e^{-2j\theta}}$$

- Multisection transformer



NOTE: line sections are commensurate

$$\boxed{\theta = \beta l}$$

Partial reflections are:

$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

$$\Gamma_m = \frac{Z_{m+1} - Z_m}{Z_{m+1} + Z_m}$$

$$\Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}$$

Assume

Z_L is real; Z_m increases or decreases monotonically

So Γ_n is Real

$\Gamma_n > 0$ if $Z_L > Z_0$

$\Gamma_n < 0$ if $Z_L < Z_0$

The overall Γ is

$$\boxed{\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \dots + \Gamma_N e^{-j2N\theta}}$$

If we assume the transformer is symmetrical
($\Gamma_0 = \Gamma_N$, $\Gamma_1 = \Gamma_{N-1}$, $\Gamma_2 = \Gamma_{N-2}$ - - -)
↳ Does not imply that Z_n are symmetrical!

In this case

$$\Gamma(\theta) = e^{-jN\theta} \left\{ \Gamma_0 \left[e^{-jN\theta} + e^{jN\theta} \right] + \Gamma_1 \left[e^{j(N-2)\theta} + e^{j(N-2)\theta} \right] - - - - \right\}$$

Finite Fourier series of cosines

$$
\begin{cases}
\Gamma(\theta) = 2e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots \Gamma_n \cos(N-2n\theta) + \cdots \frac{1}{2}\Gamma_{N/2}\right] & \text{N is even} \\[2em]
\Gamma(\theta) = 2e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots + \Gamma_n \cos(N-2n\theta) + \cdots + \Gamma_{(N-1)/2} \cos\theta\right] & \text{N is odd}
\end{cases}
$$

BINOMIAL MULTISECTION TRANSFORMER

↳ provides the flattest response
(MAXIMALLY FLAT)

$$\Gamma(\theta) = A\left(1 + e^{-2j\theta}\right)^N$$

We have to set to zero the first $N-1$ derivatives of $\Gamma(\theta)$

The magnitude

$$|\Gamma(\theta)| = |A|\,|e^{-j\theta}|^N\,|e^{j\theta} + e^{-j\theta}|^N =$$

$$= 2^N \, |A| \cos\theta^N$$

for $\theta = \frac{\pi}{2} \Rightarrow \quad |\Gamma(\theta)| = 0$

for $\theta = \frac{\pi}{2} \Rightarrow \quad \frac{d^n \Gamma(\theta)}{d\theta^n} = 0 \qquad n = 1, 2, \ldots N$

We can determine A by letting $f \to 0$ or $\theta \to 0$

$$\Gamma(0) = 2^N A = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$A = 2^{-N} \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma(\theta) = A \left(1 + e^{-2j\theta}\right)^N =$$

$$= A \sum_{n=0}^{N} C_n^N e^{-2jn\theta}$$

$$C_0^N = 1$$

$$C_n^N = \frac{N!}{(N-n)!\, n!}$$

$$C_1^N = N = C_{N-1}^N$$

$$C_n^N = C_{N-n}^N$$

$$\Gamma(\theta) = A \sum_{n=0}^{N} C_n^N e^{-2jn\theta} =$$

$$= \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \ldots$$

$$+ \Gamma_N e^{-2jN\theta}$$

$$\Downarrow$$

$$\boxed{\Gamma_N = A\, C_n^N} \longleftarrow \underline{\text{Tabulated}}$$

Bandwidth calculated by imposing the max tolerable Γ_m

$$\Gamma_m = 2^N |A| \cos^N \theta_m$$

where $\theta_m < \frac{\pi}{2}$

$$\theta_m = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{\frac{1}{N}}\right]$$

Fractional bandwidth

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$$= 2 - \frac{4}{\pi} \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{\frac{1}{N}}\right]$$

CHEBISHEV MULTISECTION TRANSFORMER

↳ optimizes bandwidth at the expense of the passband ripples

We equate $\Gamma(\theta)$ to the Chebishev polinomials

$$\begin{cases} T_1(x) = x \\ T_2(x) = 2x^2 - 1 \end{cases}$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$\vdots$

$$T_n(x) = 2x\, T_{n-1}(x) - T_{n-2}(x)$$

<u>Properties</u>

- For $-1 \leq x \leq 1$, $|T_n(x)| \leq 1 \leftarrow$ INSIDE PASSBAND

- For $|x|<1$, $|T_n(x)|>1$ ← OUTSIDE PASS BAND
- For $|x|>1$, $|T_n(x)|$ increases faster with x as n increases

If we assume $x = \cos(\theta)$ then $|x|<1$

$$T_n(\cos\theta) = \cos n\theta$$

In general

$$T_m(x) = \cos\left(m\cos^{-1} x\right), \quad |x| \leq 1$$

$$T_m(x) = \cosh\left(m\cosh^{-1} x\right), \quad |x| > 1$$

For example if we have 4 sections:

$$T_1(\sec\theta_m \cos\theta) = \sec\theta_m \cos\theta$$

$$T_2(\sec\theta_m \cos\theta) = \sec^2\theta_m\left(1 + \cos 2\theta - 1\right)$$

$$T_3(\sec\theta_m \cos\theta) = \sec^3\theta_m\left[\cos(3\theta) + 3\cos\theta\right) - 3\sec\theta_m \cos\theta$$

$$T_4(\sec\theta_m \cos\theta) = \sec^4\theta_m(\cos 4\theta + 4\cos 2\theta + 3) - 4\sec^2\theta_m(\cos 2\theta + 1) + 1$$

To summarize:

$$\Gamma(\theta) = A e^{-jN\theta}\, T_N(\sec\theta_m \cos\theta)$$

The constant A is calculated for $\theta = 0$

$$\Rightarrow \quad \Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = A\, T_N(\sec\theta_m)$$

$$A = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \frac{1}{T_N(\sec\theta_m)}$$

if the max allowed Γ is Γ_m

since the max $T_m(\sec\theta_m \cos\theta) = 1$

Then

$$T_N(\sec\theta_m) = \frac{1}{\Gamma_m} \left| \frac{Z_L - Z_0}{Z_L + Z_0} \right| \simeq \frac{1}{2\Gamma_m} \left| \ln \frac{Z_L}{Z_0} \right|$$

The fractional bandwidth is

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi}$$