The soliton solution given in the last lecture is not the only possible solution of the NLSE.
Many other kinds of solitons have been discovered in past decades.

- HIGHER-ORDER SOLITONS

Higher-order solitons are described by general analytical solutions of the NLSE. A special role is played by solitons whose initial shape at $z=0$ is given by the expression:

$$F(0,t) = N \operatorname{sech}(t)$$

where N is an integer and define the soliton order.

For the second-order soliton ($N=2$), the envelope distribution in (z,t) plane can be obtained, using IST, and results:

$$F(z,t) = \frac{4\left[\cosh(3t) + 3\exp(4iz)\cosh(t)\right]\exp\left(i\frac{z}{2}\right)}{\left[\cosh(4t) + 4\cosh(2t) + 3\cos(4z)\right]}$$

$$F(0,t) = 2\,\mathrm{sech}(t)$$

We can say an interesting property of the aforementioned solution: $|F(z,t)|$ is periodic in z with the period $z_0 = \pi/2$.

I remember that we are considering this NLSE:

$$i\frac{\partial F}{\partial z} + \frac{1}{2}\frac{\partial^2 F}{\partial t^2} + |F|^2 F = 0 \qquad \begin{aligned}\beta_2 &= -1\\ \gamma &= 1\end{aligned}$$

Higher-order solitons behave periodically. As the the pulse at the input propagates along the fiber, it first contracts to a fraction of its initial width, splits, broadens recovering the original shape at the end of the period $z = \frac{\pi}{2}$.
This pattern is repeated over each section of lenght $z = \frac{\pi}{2}$.

To understand the complete dynamics, we consider also the spectral evolution. We see a broadening, we see two peaks and then a spectral narrowing till its original spectral shape.

In the case of a fundamental soliton ($N=1$), GVD and SPM balance each other in such a way that neither the pulse shape nor the pulse spectrum changes along the fiber lenght. In the case of higher-order solitons, SPM dominates initially but GVD soon catches up and leads to pulse contraction.

soliton theory shows that for pulses with a hyperbolic-secant shape, the two effects can cooperate in such a way that the pulse follows a periodic evolution pattern with original shape recurring at multiples of the soliton period $z = \frac{\pi}{2}$.

DARK SOLITONS

Dark solitons correspond to the solution of NLSE with $B_2 = 1$, they occur in the normal-GVD regime. The intensity profile associated with such soliton solutions exhibits a dip in a uniform background, hence the name of dark soliton. They were discovered in 1973 and have attracted significant attention. Pulse-like solitons discussed up to now (the fundamental soliton, and the higher-order solitons) are called bright to make the distinction clear.

The NLSE describing dark solitons is :

$$i \frac{\partial F}{\partial z} - \frac{1}{2} \frac{\partial^2 F}{\partial t^2} + |F|^2 F = 0$$

$\beta_2 = +1$

$\gamma = 1$

Similar to the case of bright solitons, the IST method can be used to find dark soliton solutions.

The main difference compared with the case of bright solitons is that the solution becomes a constant (rather than being zero) a $|t| \to \infty$.

The general solution can be written as

$$|F(z,t)| = \eta \left\{ 1 - B^2 \operatorname{sech}^2 [\eta B (t - T_s)] \right\}^{1/2}$$

with the phase

$$\phi(z,t) = \frac{1}{2} \eta^2 (3 - B^2) z + \eta \sqrt{1 - B^2}\, t + \tan^{-1} \left(\frac{B \tanh(\eta B t)}{\sqrt{1 - B^2}} \right)$$

The parameters η and t_s represent the soliton amplitude and the dip location. T_s can be chosen to be zero, without loss of generality.

In contrast with the bright soliton case, the dark soliton has a new parameter B.

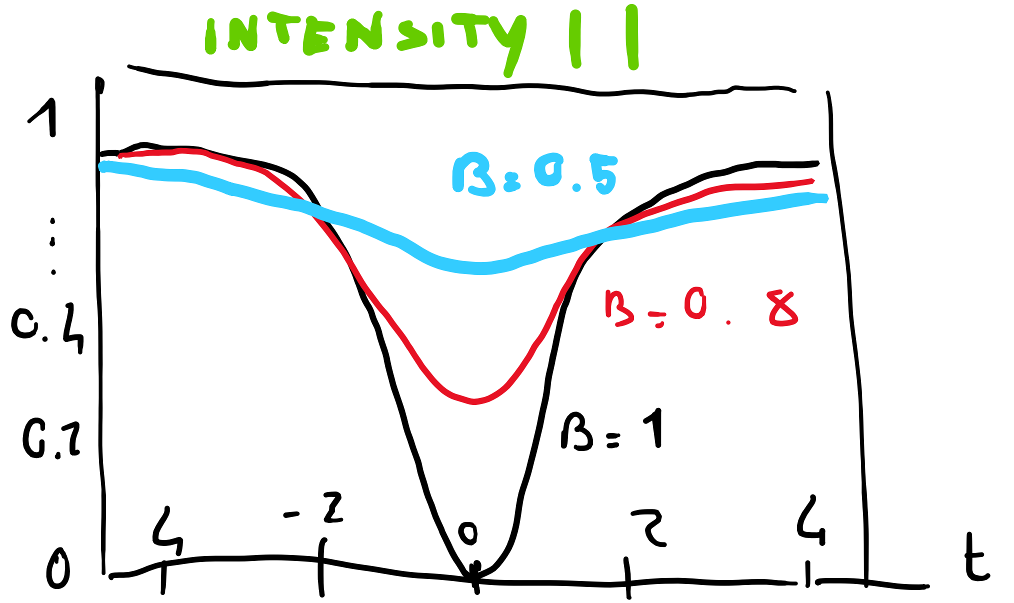
B governs the depth of the dip ($|B| \leq 1$).

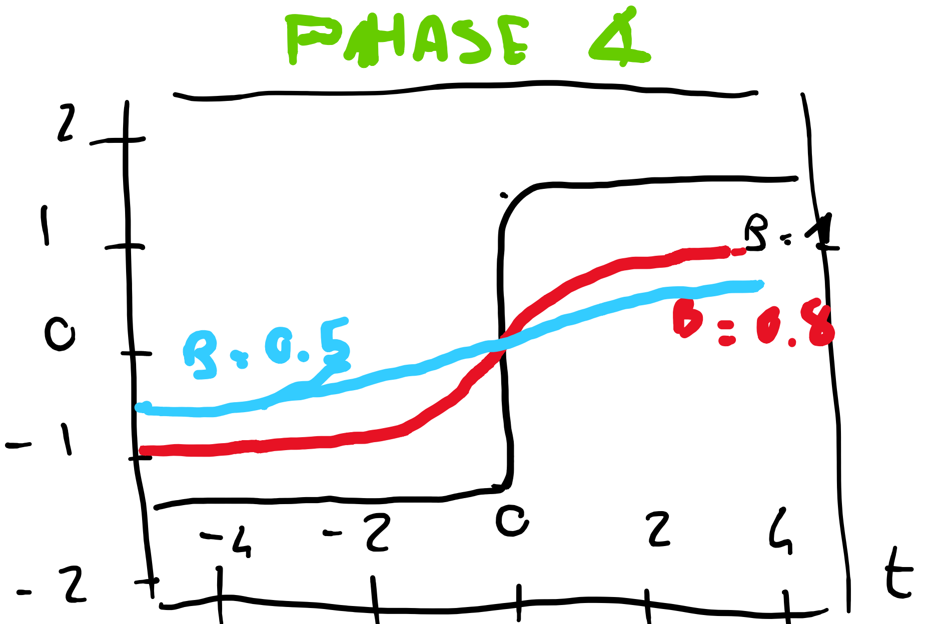
For $|B| = 1$, the intensity at the dip center falls to zero. For other values of B, the dip does not go to zero.

Dark solitons for which $|B| < \underline{1}$ are called gray solitons. The $|B|=1$ case corresponds to a black soliton.

For a given value of η, The expression $|F(z,t)|$ describes a family of dark solitons whose width increases inversely with B.

I want to plot the intensity and phase profile of such solitons for several values of B.





Whereas the phase of bright solitons remains constant across the entire pulse, the phase of dark soliton changes, i.e. dark solitons are chirped. For black soliton ($|B|=1$) the chirp is such that the phase changes abrupty by π in the center.

Dark solitons exhibit several interesting features. We consider a black soliton in its canonical form, choosing $\eta = 1$ and $B = 1$ and given by

$$F(z,t) = \tanh(T)\exp(iZ)$$

where the phase jump of π at $t=0$ is included in the amplitude $\tanh(t)$ part.

This dark soliton would propagate unchanged in the normal dispersion region of optical fibers.

→ NUMERICS