SMALL DIPOLES AND SMALL LOOPS

Electrically small dipoles and loops are analyzed in detail and their practical use is discussed. Since their dimensions must be smaller than about a tenth of a wavelength, those elemental antennas find application only at frequencies below 30 MHz.

IDEAL DIPOLE

In the case of the ideal dipole the current is uniform over the full length of the dipole and the wire radius is so small that the current distribution can be considered as a filament of current.

Let's recapitulate the mathematical steps that led us to the solution for the electromagnetic field

$$\mathbf{A} = \iiint_{v'} \mu \mathbf{J} \frac{e^{-j\beta R}}{4\pi R} dv' = \mu \int_{-\frac{\Delta z}{2}}^{+\frac{\Delta z}{2}} I(z')\hat{\mathbf{z}} \frac{e^{-j\beta R}}{4\pi R} dz' = \mu \frac{e^{-j\beta r}}{4\pi r} \hat{\mathbf{z}} \int_{-\frac{\Delta z}{2}}^{+\frac{\Delta z}{2}} I(z') dz' = \mu \frac{e^{-j\beta r}}{4\pi r} I\Delta z \, \hat{\mathbf{z}}$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} = \frac{I\Delta z}{4\pi} j\beta \left[1 + \frac{1}{j\beta r}\right] \frac{e^{-j\beta r}}{r} \sin\theta \, \hat{\boldsymbol{\phi}}$$

$$\mathbf{E} = \frac{1}{j\omega\varepsilon} \nabla \times \mathbf{H} = \frac{I\Delta z}{4\pi} j\omega\mu \left[1 + \frac{1}{j\beta r} - \frac{1}{(\beta r)^2}\right] \frac{e^{-j\beta r}}{r} \sin\theta \, \hat{\boldsymbol{\theta}} + \frac{I\Delta z}{2\pi} \sqrt{\frac{\mu}{\varepsilon}} \left[\frac{1}{r} - j \frac{1}{\beta r^2}\right] \frac{e^{-j\beta r}}{r} \cos\theta \, \hat{\mathbf{r}}$$

Under the assumption that the observation point is in the far-field ($r > 5\lambda$), the formulas for the electric and magnetic fields reduce to the following ones

$$\mathbf{E} = E_\theta \hat{\boldsymbol{\theta}} = j\eta \frac{\beta}{4\pi} I\Delta z \frac{e^{-j\beta r}}{r} \sin\theta \;\; \hat{\boldsymbol{\theta}} = E_\theta \hat{\boldsymbol{\theta}}$$

$$\mathbf{H} = H_\phi \hat{\boldsymbol{\phi}} = j \frac{\beta}{4\pi} I\Delta z \frac{e^{-j\beta r}}{r} \sin\theta \;\; \hat{\boldsymbol{\phi}} = H_\phi \hat{\boldsymbol{\phi}}$$

The fields can also be otained by using a general procedure valid for wire antennas (i.e. linear current distributions along the z-axis)

$$\mathbf{E} = -j\omega\left(A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}}\right) = -j\omega A_\theta \hat{\boldsymbol{\theta}} = j\omega \sin\theta \, A_z \hat{\boldsymbol{\theta}} = j\omega \sin\theta \, \mu \frac{e^{-j\beta r}}{4\pi r} I\Delta z \hat{\boldsymbol{\theta}} = j\eta \frac{\beta}{4\pi} I\Delta z \frac{e^{-j\beta r}}{r} \sin\theta \, \hat{\boldsymbol{\theta}}$$

$$\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{r}} \times \mathbf{E} = -\frac{j\beta}{\mu} A_\theta \hat{\boldsymbol{\phi}} = \frac{j\beta}{\mu} \sin\theta \, A_z \hat{\boldsymbol{\phi}} = \frac{j\beta}{\mu} \sin\theta \, \mu \frac{e^{-j\beta r}}{4\pi r} I\Delta z \hat{\boldsymbol{\phi}} = j \frac{\beta}{4\pi} I\Delta z \frac{e^{-j\beta r}}{r} \sin\theta \, \hat{\boldsymbol{\phi}}$$

Starting from the fields $\mathbf{E}$ and $\mathbf{H}$, the real radiated power P_R, the radiation resistance R_R and the directivity D can be easily obtained

$$P_R = \iint_{\Sigma} \mathbf{S} \cdot d\mathbf{s} = \iint_{\Omega} |\mathbf{S}| \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, r^2 \sin\theta \; d\theta \; d\phi = \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \eta\beta^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta \; d\theta = \frac{\pi}{3}\eta \frac{I^2 \Delta z^2}{\lambda^2}$$

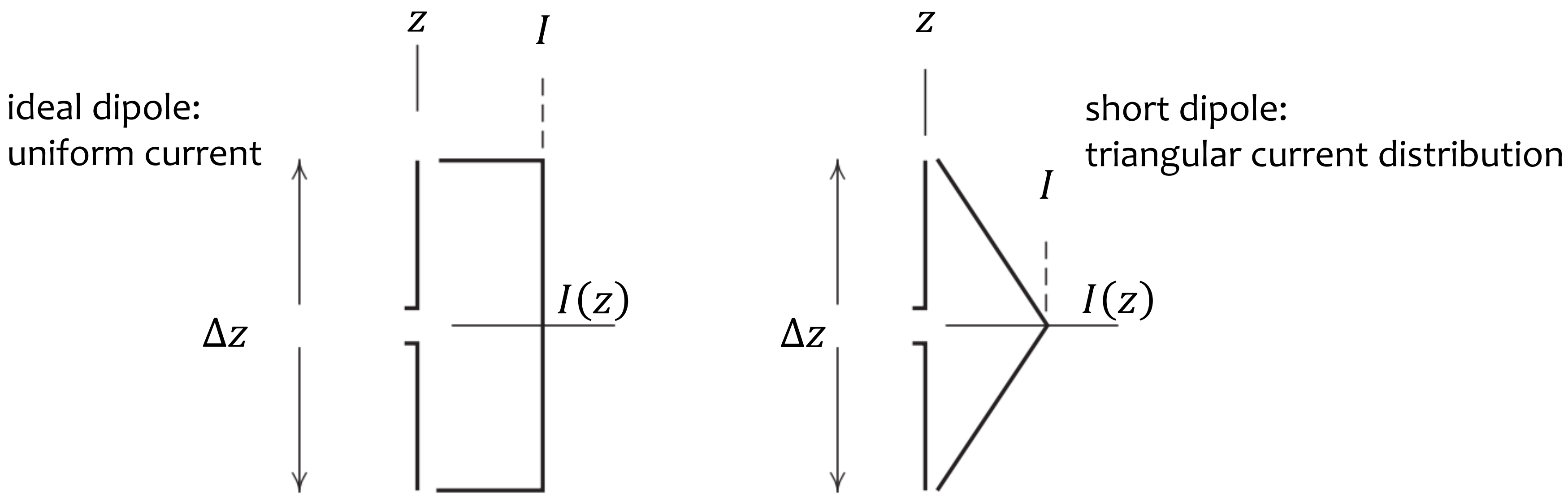
$$R_R = \frac{2P_R}{I^2} = 2\frac{\pi}{3}\eta\frac{\Delta z^2}{\lambda^2} \cong 80\pi^2\left(\frac{\Delta z}{\lambda}\right)^2$$

$$|F(\theta,\varphi)| = |\sin\theta| \qquad D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\iint_{4\pi} |F(\theta,\varphi)|^2 d\Omega} = \frac{4\pi}{\int_0^{2\pi}\int_0^{\pi} |F(\theta,\varphi)|^2 \sin\theta \; d\theta d\varphi} = \frac{3}{2}$$

Moreover, by assuming that the conductivity of the metal is infinity ($\sigma = \infty$), the antenna is lossless and thus the radiation efficiency is unity and gain equals directivity: $e_r = 1$, $G = D$

SHORT DIPOLE

The current on a straight wire antenna cannot be uniform but must smoothly go to zero at the wire ends. We call short dipole a center-fed wire dipole of length $\Delta z \ll \lambda$ supporting a current distribution which is triangular; the wire is made of real metal and the effect of ohmic losses can be accounted for.



The far-field can be calculated by following the same method used for the ideal dipole; once again the difference in magnitude and phase between rays coming from different points on the wire can be neglected (it is a consequence of the fact that $\Delta z \ll \lambda$).

Due to the fact that the uniform current of the ideal dipole is replaced by the triangular current of the short dipole, we can use the same formulas obtained for the ideal dipole but the factor $I\Delta z$ (obtained by integrating the current $I(z)$) must be replaced by $I\Delta z/2$

$$\mathbf{A} = \mu \int_{-\frac{\Delta z}{2}}^{+\frac{\Delta z}{2}} I(z')\hat{\mathbf{z}} \frac{e^{-j\beta R}}{4\pi R} dz' = \mu \frac{e^{-j\beta r}}{4\pi r} \hat{\mathbf{z}} \int_{-\frac{\Delta z}{2}}^{+\frac{\Delta z}{2}} I(z') dz' = \mu \frac{e^{-j\beta r}}{4\pi r} \frac{1}{2} I\Delta z \hat{\mathbf{z}}$$

$$\mathbf{E} = E_\theta \widehat{\boldsymbol{\theta}} = j\eta \frac{\beta}{4\pi} \frac{1}{2} I\Delta z \frac{e^{-j\beta r}}{r} \sin\theta \ \widehat{\boldsymbol{\theta}} = E_\theta \widehat{\boldsymbol{\theta}}$$

$$\mathbf{H} = H_\phi \widehat{\boldsymbol{\phi}} = j \frac{\beta}{4\pi} \frac{1}{2} I\Delta z \frac{e^{-j\beta r}}{r} \sin\theta \ \widehat{\boldsymbol{\phi}} = H_\phi \widehat{\boldsymbol{\phi}}$$

The short dipole has the same radiation pattern of the ideal dipole $|F(\theta, \varphi)| = |\sin\theta|$ and the directivity is $D = 1.5$

The magnitude of the field radiated by the short dipole is one half of the magnitude radiated by the ideal dipole, the Poynting vector is one fourth, and so **the radiation resistance is one fourth that of the ideal dipole**.

$$P_R = \frac{1}{4}\frac{\pi}{3}\eta\frac{I^2\Delta z^2}{\lambda^2} \qquad R_R = \frac{2P_R}{I^2} = \frac{1}{2}\frac{\pi}{3}\eta\frac{\Delta z^2}{\lambda^2} \cong 20\pi^2\left(\frac{\Delta z}{\lambda}\right)^2$$

We want to estimate the dissipation resistance R_D, as well, and let's first consider again the uniform current distribution of an ideal dipole. The wire is a cylinder with radius a and length L and the current density must flow in the axial direction through the "skin" of the conductor having conductivity σ: **the dissipation resistance is proportional to the wire length and inversely proportional to the "skin" cross-section** that can be approximated by $2\pi a\delta$, where δ is the skin depth

Skin depth $$\boxed{\delta = \frac{1}{\sqrt{\pi\mu f\sigma}}} \qquad \boxed{R_D = \frac{L}{2\pi a\delta}\frac{1}{\sigma}} = \frac{L}{2\pi a}\sqrt{\frac{\pi\mu f}{\sigma}}$$

Surface resistance $$R_s = \sqrt{\frac{\pi\mu f}{\sigma}} \qquad R_D = \frac{L}{2\pi a}R_s = \frac{L}{2\pi a}\sqrt{\frac{\pi\mu f}{\sigma}}$$

The dissipation (or ohmic) resistance R_D of the short dipole is found by first determining the power dissipation from ohmic losses, which at any point along the antenna is proportional to the current squared. In fact, in general the total power is evaluated by integrating the current squared over the wire antenna.

$$R_D = \frac{2P_D}{I^2} = \frac{1}{I^2}\frac{1}{2\pi a}\frac{1}{\delta\sigma}\int_{-\frac{\Delta z}{2}}^{+\frac{\Delta z}{2}} I(z')^2 dz' = \frac{1}{I^2}\frac{1}{2\pi a}\frac{1}{\delta\sigma}\frac{\Delta z}{3}I^2 = \frac{1}{3}\frac{\Delta z}{2\pi a}\frac{1}{\delta\sigma}$$

$$\int_{-\frac{\Delta z}{2}}^{+\frac{\Delta z}{2}} I(z')^2 dz' = \int_{-\frac{\Delta z}{2}}^{0} I^2\left[1+2\frac{z'}{\Delta z}\right]^2 dz' + \int_{0}^{+\frac{\Delta z}{2}} I^2\left[1-2\frac{z'}{\Delta z}\right]^2 dz' = \frac{\Delta z}{6}I^2 + \frac{\Delta z}{6}I^2 = \frac{\Delta z}{3}I^2$$

The dissipation resistance of the short dipole is one third that of an ideal dipole of the same length Δz

and the radiation efficiency of the short dipole is given by

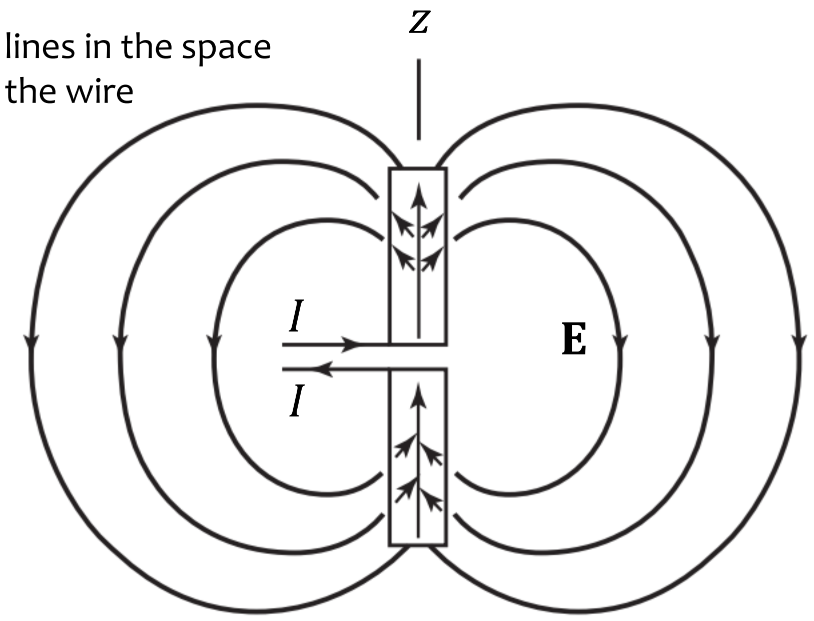
$$e_r = \frac{R_R}{R_R + R_D} = \frac{20\pi^2\left(\frac{\Delta z}{\lambda}\right)^2}{20\pi^2\left(\frac{\Delta z}{\lambda}\right)^2 + \frac{1}{3}\frac{\Delta z}{2\pi a}\frac{1}{\delta\sigma}}$$

We cannot neglect that the input impedance has an imaginary part, as well: the reactive part of the input impedance represents power stored in the near field. The short dipole has a capacitive reactance, which is approximated by

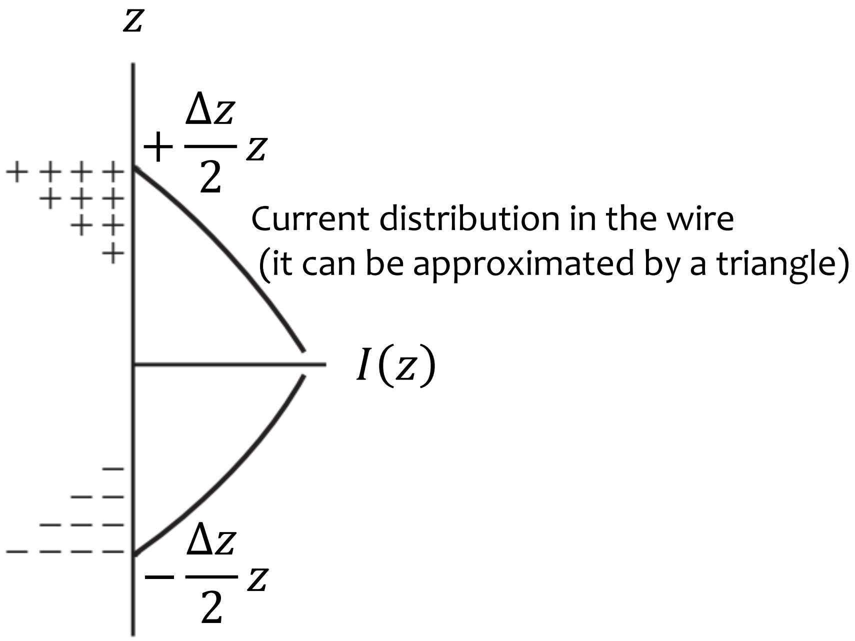
$$X_A = -\frac{120}{\pi \frac{\Delta z}{\lambda}}\left[\ln\left(\frac{\Delta z}{2a}\right) - 1\right]$$

In a short dipole, since the current decreases towards the wire ends, some charges accumulate on the wire surface. Because the input (feeding) current is changing sinusoidally with time, the current and charge distribution on the dipole vary sinusoidally at the same angular frequency ω. The following two figures show current and charge distribution for an instant of time when the input current is maximum:

Electric field lines in the space surrounding the wire



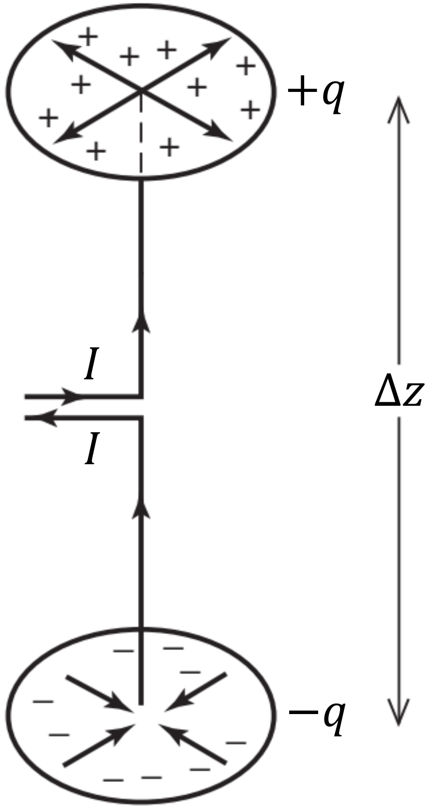
Charge distribution



In the ideal dipole (i.e. by assuming a uniform current) all charge accumulates at the end of the antenna. In fact, the ideal dipole can be analyzed as either a uniform current $\boldsymbol{I}$ (as we did) or two point-charges $\boldsymbol{q}$ oscillating at angular frequency $\boldsymbol{\omega}$ (and $\boldsymbol{I = j\omega q}$).

A uniform current distribution along the wire can be realized in practice by providing a mechanism for charge accumulation at the wire ends (otherwise one would obtain $I(\pm\Delta z/2) = 0$). One method of accomplishing this is to place large metal plates (or large metallic objects) at the ends of the wire: this is called a capacitor-plate antenna or a top-hat-loaded dipole antenna.

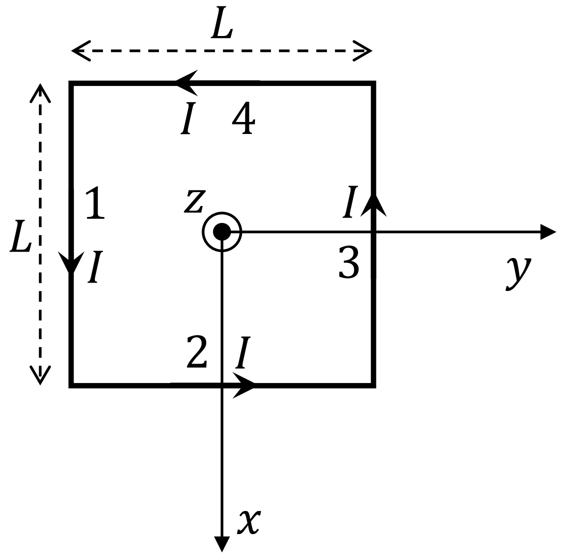
Since $\Delta z \ll \lambda$ the radial currents on the plates produce fields that cancel in the far-field: in fact, the currents on the two plates are opposite-directed and the phase difference due to separation is small. In addition, if the radius (or transverse dimensions) of the plates is much larger than Δz, the plates provide for charge storage such that the current on the wire is constant. Frequently, in practice, radial wires are used for the top loading in place of the solid plates.



SMALL LOOP

It can be proved that the radiation fields of small loops are independent of the shape of the loop and depend only on the area of the loop. In order to simplify the mathematics we select a square loop of side L: each side is a short uniform current filament having amplitude I that is modeled as an ideal dipole.

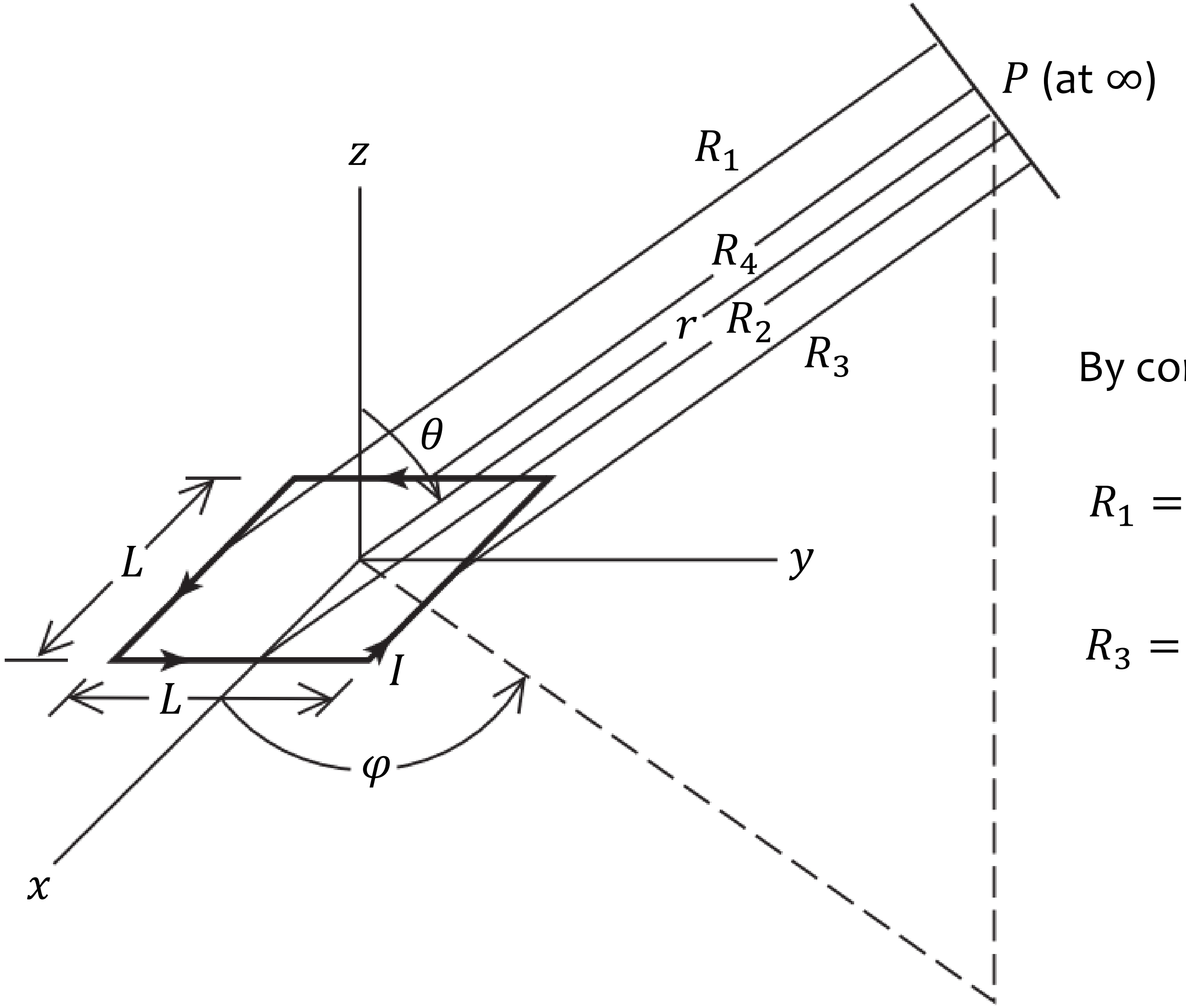
Small square loop in the Cartesian reference system



The two sides parallel to the x-axis (1 and 3) give rise to the x-directed component of the vector potential A_x, similarly the two sides parallel to the y-axis (2 and 4) generate the y-directed component of the vector potential A_y (and R_k is the distance between the observation point P and the square side $k = 1,2,3,4$)

$$A_x = \frac{\mu I L}{4\pi}\left(\frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_3}}{R_3}\right) \qquad A_y = \frac{\mu I L}{4\pi}\left(\frac{e^{-j\beta R_2}}{R_2} - \frac{e^{-j\beta R_4}}{R_4}\right)$$

To evaluate the magnetic vector potential in the far-field we refer to the following 3D drawing, where r is the distance between the observation point P and the origin



By comparing the parallel path lengths we find

$$R_1 = r + \frac{L}{2}\sin\theta\sin\varphi \quad R_2 = r - \frac{L}{2}\sin\theta\cos\varphi$$

$$R_3 = r - \frac{L}{2}\sin\theta\sin\varphi \quad R_4 = r + \frac{L}{2}\sin\theta\cos\varphi$$

By inserting the new expressions for the path lengths in the formulas for A_x and A_y (and neglecting the small correction proportional to L in the denominator), we have

$$A_x = \frac{\mu IL}{4\pi}\frac{e^{-j\beta r}}{r}\left(e^{-j\beta\frac{L}{2}\sin\theta\sin\varphi} - e^{+j\beta\frac{L}{2}\sin\theta\sin\varphi}\right) = -2j\frac{\mu IL}{4\pi}\frac{e^{-j\beta r}}{r}\sin\left(\beta\frac{L}{2}\sin\theta\sin\varphi\right)$$

$$A_y = \frac{\mu IL}{4\pi}\frac{e^{-j\beta r}}{r}\left(e^{+j\beta\frac{L}{2}\sin\theta\cos\varphi} - e^{-j\beta\frac{L}{2}\sin\theta\cos\varphi}\right) = 2j\frac{\mu IL}{4\pi}\frac{e^{-j\beta r}}{r}\sin\left(\beta\frac{L}{2}\sin\theta\cos\varphi\right)$$

Since $L \ll \lambda$, $\beta\, L/2 \ll 1$ and the sine functions of the previous formulas can be replaced by their arguments

$$A_x = -2j\frac{\mu IL}{4\pi}\frac{e^{-j\beta r}}{r}\beta\frac{L}{2}\sin\theta\sin\varphi = -j\frac{\mu I}{4\pi}\frac{e^{-j\beta r}}{r}\beta L^2\sin\theta\sin\varphi$$

$$A_y = 2j\frac{\mu IL}{4\pi}\frac{e^{-j\beta r}}{r}\beta\frac{L}{2}\sin\theta\cos\varphi = j\frac{\mu I}{4\pi}\frac{e^{-j\beta r}}{r}\beta L^2\sin\theta\cos\varphi$$

Combining components to form the total vector potential gives:

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} = j\beta L^2 \frac{\mu I}{4\pi} \frac{e^{-j\beta r}}{r} \sin\theta \,(-\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}}) = j\beta S \frac{\mu I}{4\pi} \frac{e^{-j\beta r}}{r} \sin\theta\, \hat{\boldsymbol{\varphi}}$$

where S is the area of the loop. The electric and magnetic fields are finally obtained

$$\mathbf{E} = -j\omega\mathbf{A} = \eta\beta^2 S \frac{I}{4\pi} \frac{e^{-j\beta r}}{r} \sin\theta\, \hat{\boldsymbol{\varphi}}$$

$$\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{r}} \times \mathbf{E} = -\beta^2 S \frac{I}{4\pi} \frac{e^{-j\beta r}}{r} \sin\theta\, \hat{\boldsymbol{\theta}}$$

IS is the magnetic moment, and **it can be proved that the fields radiated by a small loop depends only on the magnetic moment and not on the loop shape**.

The small loop has the same radiation pattern as the ideal dipole $|\boldsymbol{F}(\boldsymbol{\theta},\boldsymbol{\varphi})| = |\mathbf{sin}\,\boldsymbol{\theta}|$, but the loop electric field is horizontally polarized (i.e. it lies in the $\boldsymbol{xy}$ plane) $\mathbf{E} = E_\varphi \widehat{\boldsymbol{\varphi}}$, whereas the dipole electric field is vertically polarized $\mathbf{E} = E_\theta \widehat{\boldsymbol{\theta}}$.

The radiation resistance R_R is calculated from the Poynting theorem

$$P_{OUT} = \iint_\Sigma \mathbf{S}\cdot d\mathbf{s} = \iint_\Omega |\mathbf{S}|\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\, r^2 \sin\theta\, d\theta\, d\phi = \iint_\Omega \frac{1}{2\eta}\left|E_\varphi\right| r^2 \sin\theta\, d\theta\, d\phi$$

$$P_R = \frac{1}{2\eta}\frac{\eta^2\left(\beta^2 S\right)^2}{(4\pi)^2} I^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta\, d\theta = \frac{\eta\left(\beta^2 S\right)^2}{32\pi^2} I^2 2\pi \frac{4}{3} \cong \frac{120\pi}{32\pi^2}\pi\frac{8}{3}\left(\beta^2 S\right)^2 I^2 \cong 10\left(\beta^2 S\right)^2 I^2$$

$$R_R = \frac{2P_R}{I^2} = 20\left(\beta^2 S\right)^2 \cong 31200\left(\frac{S}{\lambda^2}\right)^2$$

The radiation resistance of the loop antenna can be increased by using multiple turns: the magnetic moment of an N turn loop is NIS, the radiated field is increased by the same factor N and the radiation resistance reads

$$R_R = 20\left(\beta^2 NS\right)^2 \cong 31200\left(\frac{NS}{\lambda^2}\right)^2$$

We underline that, provided the total wire length of the multiturn loop is small compared to the wavelength (less than about 0.1λ), the current remain nearly constant over the wire length and previous small loop analysis applies so that the pattern is the same as a single turn loop.

Another popular way to enhance the radiation resistance of a loop antenna is to wind the turns around a ferrite core forming the ferrite rod antenna (also called loop-stick antennas), which is used up to about 30 MHz, where usually losses become excessive. Due to the finite extent of the ferrite core the relative permeability μ_r of the bulk material (inside β) must be replaced by an effective relative permeability μ_{eff}, which depends on the relative core length R (i.e. the length to diameter ratio) .

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_{\text{eff}}} \qquad \mu_{\text{eff}} = \frac{\mu_r}{1 + 0.37R^{-1.44}(\mu_r - 1)} \qquad R_R \cong 31200\left(\mu_{\text{eff}}\frac{NS}{\lambda^2}\right)^2$$

This equation assumes the core is much longer than the winding on it and reveals that μ_{eff} is always less than μ_r but for longer cores (larger R values) μ_{eff} approaches μ_r. Receiver for AM broadcast around 1 MHz are a popular application for the ferrite rod antenna with typical values in a wide range around 50 for μ_{eff} and 10 for R. Ferrite being lossy eliminates them for use in transmitters.

Small loop antennas also have ohmic resistance R_D. For a square loop (with side L) made of a thin wire ($L \gg a$, where a is the wire radius) we obtain

$$R_D = \frac{4L}{2\pi a\delta}\frac{1}{\sigma} = \frac{L_p}{2\pi a\delta}\frac{1}{\sigma} \qquad\qquad \delta = \frac{1}{\sqrt{\pi\mu f\sigma}}$$

Where L_p is the length of the loop (i.e. the perimeter) and the last expression can be used for any shape of the loop.

The small loop is inherently inductive, and the inductance L_A of a square loop is given by

$$L_A = 2\frac{\mu}{\pi} L \cosh^{-1}\frac{L}{2a}$$

The ohmic resistance R_D of an N-turn loop must be multiplied by N (since the wire is N times longer) and the inductance L of an N-turn loop must be multiplied by N^2 (as it happens in a solenoid where the inductance is proportional to the number of turns squared).

For a rectangular loop (having side lengths $L_1$and L_2) the formulas become

$$R_D = \frac{L_p}{2\pi a\delta}\frac{1}{\sigma} = \frac{2(L_1 + L_2)}{2\pi a\delta}\frac{1}{\sigma} \qquad L_A = \frac{\mu}{\pi}\left(L_2 \cosh^{-1}\frac{L_1}{2a} + L_1 \cosh^{-1}\frac{L_2}{2a}\right)$$

For a circular loop of radius b the formulas become

$$R_D = \frac{L_p}{2\pi a\delta}\frac{1}{\sigma} = \frac{2\pi b}{2\pi a\delta}\frac{1}{\sigma} = \frac{b}{a}\frac{1}{\delta\sigma} \qquad L_A = \mu b\left[\ln\left(\frac{8b}{a}\right) - 2\right]$$