Units

$\lambda = \lambda_0 / n(f_0)$

$\beta_0 = \pm 2\pi / \lambda_0$

$\beta_0 = \omega\sqrt{\mu_0 \varepsilon_0}$

$\beta = \frac{\omega}{c} n$

$\beta = \frac{2\pi}{\lambda} n$

$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$c = c_0 / n(\beta)$

$\hat{\chi}$ homogeneous $\rightarrow$ no depend on r

$\hat{\chi}$ isotropic $\rightarrow$ χ is a constant

$\bar{E}(\bar{r},t) = [V/m]$

$\varepsilon_0 = 8{,}854 \cdot 10^{-12} \, [F/m]$

$\bar{D}(\bar{r},t) = [C/m^2]$

$\mu_0 = 4\pi \cdot 10^{-7} \, [H/m]$

$\bar{H}(\bar{r},t) = [A/m]$

$k_0 = [m^{-1}]$

$\bar{B}(\bar{r},t) = [T]$

$k_0' = [s/m]$

$P(\bar{r},t) = [C/m^2]$

$k_0'' = [s^2/m]$

$\chi^{(1)} =$ unitless

$D = \left[\frac{ps}{nm \, km}\right]$

$\chi^{(n)} = [(m/V)^{n-1}]$

$\underset{\text{att. coef}}{\alpha} = [dB/km]$

$\vec{S} = [W/m^2]$

$F(z,t) = [V]$

$M(x,y) = [m^{-1}]$

Chapter 1

1.1 - Premise

Wavelength

UV: 1 nm to 400 nm

* Visible:
- Freq: $[7.5 \cdot 10^{14} - 4 \cdot 10^{14}]$ Hz
- Wavelength: $[400 - 750]$ nm

Infrared: 750 nm to 25 μm

* Communication systems:
- Guided: EM radiation confined in a guiding structure
- Non-guided: EM rad. freely propagates through space.

* EM propagation we consider: Slowly varying weakly non-linear dielectrics at optical frequencies.

1.2 - Propagation in weakly non-linear dielectrics

* Refractive index as a function of wavelength → $n(\lambda)$

This dispersion → Linear effects: Dispersion of the wavepacket.
Spectral components of the wavepacket travel at dif. speeds and reach the receiver at dif. times.
Wavepacket is distorted

$\beta_2 > 0$ → Normal (red first)

$\beta_2 < 0$ → Anomalous

* Refractive index changes with light intensity → $n(|E_0|^2)$

→ Kerr effect: Non-linear effect: Self phase modulation
Crossed phase modulation

1.3 - Maxwell equations and Wave equation

* Maxwell equations (with: $Q=0$, $\bar{J}=0$)

$$\nabla \times \bar{E}(\bar{r},t) = -\frac{\partial \bar{B}(\bar{r},t)}{\partial t} \qquad \nabla \cdot \bar{D}(\bar{r},t) = 0$$

$$\nabla \times \bar{H}(\bar{r},t) = \frac{\partial \bar{D}(\bar{r},t)}{\partial t} \qquad \nabla \cdot \bar{B}(\bar{r},t) = 0$$

* Constitutive relations

$$\bar{D}(\bar{r},t) = \varepsilon_0 \bar{E}(\bar{r},t) + \bar{P}(\bar{r},t)$$

$$\bar{B}(\bar{r},t) = \mu_0 \bar{H}(\bar{r},t) + \underbrace{\bar{M}(\bar{r},t)}_{0 \text{ non-mag. materials}}$$

* About $\bar{P}(\bar{r},t)$

Free space $\rightarrow$ $\bar{P}(\bar{r},t)=0$

Materials $\rightarrow$ E field distorts atomic structure
$\bar{P} \neq 0$

If E weak $\Rightarrow$ P is linear respect E

The reply of the media: Not simultaneous
Not anticipate
$\downarrow$
Causality relation

Induced polarization obtained by:
Convolution of External stimulus (Efield)
and function describing material (χ)

* Constitutive relations: Linear Case

temporal convolution

Relation between electric field and induced polarization: $\bar{P} = \varepsilon_0 \bar{\chi} \overset{t}{*} \bar{E}$

↳ Temporal convolution, time invariant media $\quad \vec{P} = \varepsilon_0 \int_{-\infty}^{+\infty} \bar{\chi}(\bar{r}, t-\tau)\, \vec{E}(\bar{r}, \tau)\, d\tau$

$$\boxed{\bar{\chi}(\bar{r}, t-\tau) = 0 \quad \text{if} \quad t-\tau \leq 0}$$ causality

↳ Another way to see it: As a filter 3in 3out

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{bmatrix} \overset{t}{*} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \Rightarrow \text{In general } \bar{\chi} \text{ not diagonal}$$

so, in general $\bar{P} \not\parallel \bar{E}$

convolution in time domain so temporal filter

↳ Isotropic media: Response <u>does not</u> depend on the orientation of the source.

$$\Rightarrow \chi_{jk}(\bar{r},t) = \chi(\bar{r},t)\, \delta_{jk} \quad \longrightarrow \quad \bar{\chi} = \chi(\bar{r},t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Isotropic} = \bar{\chi} \text{ diagonal}$$

↳ To proof: Rotation $\rightarrow E' = RE \Rightarrow RP = \varepsilon_0 \chi RE = \varepsilon_0 R \chi E \rightarrow \chi R = R\chi$

commutes with the rotation

↳ Consequences of isotropic media

$\Rightarrow$ Linear response described by a single function: $\bar{P}(\bar{r},t) = \varepsilon_0 \int_{-\infty}^{+\infty} \chi(\bar{r}, t-\tau)\, \bar{E}(\bar{r},\tau)\, d\tau$

$$\bar{P} \parallel \bar{E}$$

$\Rightarrow$ Electric displacement : $\bar{D}(\bar{r},t) = \varepsilon_0 \bar{E}(\bar{r},t) + \bar{P}(\bar{r},t) = \int_{-\infty}^{+\infty} \varepsilon_0 \left[\delta(t-\tau) + \chi(\bar{r}, t-\tau)\right] \bar{E}(\bar{r},\tau)\, d\tau$

$$\bar{D} \parallel \bar{E}$$

$\Rightarrow$ Fourier transform:
$$\hat{D}(\bar{r},f) = \left[\varepsilon_0 + \varepsilon_0 \hat{\chi}(\bar{r},f)\right] \hat{E}(\bar{r},f)$$
$$= \varepsilon_0 \varepsilon_r(\bar{r},f)\, \hat{E}(\bar{r},f)$$
$$= \varepsilon_0 n^2(\bar{r},f)\, \hat{E}(\bar{r},f)$$

$$\boxed{n(\bar{r},f) = \sqrt{\varepsilon_r(\bar{r},f)} = \sqrt{1 + \hat{\chi}(\bar{r},f)}}$$

↳ $\lim_{f \to \infty} n(\bar{r},f) \rightarrow 1$ $\quad$ Wave so small ($\lambda \to 0$) that the medium is like vacuum

* Constitutive relations: Non-linear case

when E field is strong, P is not longer proportional to E.

$\Rightarrow$ Expand $\bar{P}(\bar{E})$ with Taylor $\rightarrow$ $\bar{P}(\bar{r},t) = \underbrace{\bar{P}^{L}(\bar{r},t)}_{\text{linear term calculated previously}} + \underbrace{\bar{P}^{NL}(\bar{r},t)}_{\text{Non-lin. term Disappears if } \bar{E} \text{ is weak}}$

$\Rightarrow$ In freq. domain: $\hat{D}(\bar{r},f) = \varepsilon_0 \hat{\varepsilon}_r(\bar{r},f)\, \hat{E} + \hat{P}^{NL}(\bar{r},f)$

$\Rightarrow$ Volterra expansion to get $\bar{P}^{NL}$ $\bar{P} = \varepsilon_0 \bar{\bar{\chi}}^{(1)} \bar{E} + \varepsilon_0 \bar{\bar{\chi}}^{(2)} \bar{E}^2 + \varepsilon_0 \bar{\bar{\chi}}^{(3)} \bar{E}^3 + \dots$

↳ $\chi^{(2)}$: Induces: SHG, SFG, DFG, OPA, OPO

If centrosymmetric: $r \rightarrow -r \Rightarrow \chi^{(2)} = 0$ (i.e. silica)

↳ $\chi^{(3)}$: Responsible of: THG, Kerr effect, Raman & Brillouin scattering

In general 81 components but $\rightarrow$ Isotropic $\Rightarrow$ 21 components

If instantaneous, space independent $\rightarrow$ $P^{NL}(\bar{r},t) = \varepsilon_0 \hat{\chi}^{(3)} E(\bar{r},t)\, E(\bar{r},t)\, E(\bar{r},t)$

* Wave Equation

$\rightarrow$ start with:
$$\begin{cases} \bar{\nabla} \times \bar{E} = -\dfrac{\partial \bar{B}}{\partial t} & (1) \\ \bar{\nabla} \times \bar{H} = \dfrac{\partial \bar{D}}{\partial t} & (2) \end{cases}$$

a) $\nabla \times (1)$ $\rightarrow$ identity: $\nabla \times \nabla \times \bar{w} = -\nabla^2 \bar{w} + \nabla(\nabla \cdot \bar{w})$

b) $\nabla \cdot (\varepsilon E) = \underbrace{\cancel{\nabla \varepsilon}}_{0} \bar{E} + \varepsilon \cdot \underbrace{\cancel{\nabla} \bar{E}}_{0 \text{ not charges}}$

in homogeneous ε constant

in slowly varying almost constant, so $\rightarrow$ $\nabla \varepsilon \cdot E \simeq 0$

$\rightarrow$ end with: $\nabla^2 \bar{E} = \dfrac{1}{c^2} \dfrac{\partial^2}{\partial t^2}\left(\bar{E} + \dfrac{1}{\varepsilon_0} \bar{P}\right)$ ↳ If isotropic: $\bar{P} = \bar{P}^{L}_{(E)}$ no memory $\bar{P} = \varepsilon_0 \chi_0 \bar{E}$ $\nabla^2 E - \dfrac{n^2}{c^2} \dfrac{\partial^2 E}{\partial t^2}$ $n = \sqrt{1 + \chi_0}$

↳ P and E related so the unknown is E

1.4 - Ideal Reference Case

Assumptions:

① Linear material: $P \propto E$

② Homogenous material: $\hat{\chi}(\vec{r}, f) = \hat{\chi}(f)$

③ No losses material: $\mathrm{Im}\left[\hat{\chi}(f)\right] = 0$

④ Monochromatic wave, f_0

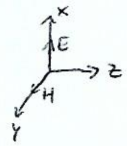
⑤ $\bar{E} \parallel \hat{x}$

⑥ No x, y dependence: $\bar{E} = \bar{E}(z,t)$

$$\bar{E}(z,t) = E_M \cos(\omega_0 t + \phi_0 + \beta_0 z)\ \vec{\imath}$$

↳ periodic in space and in time

↳ i.e.



→ Dispersion relation:

$$\beta_0^2 = \frac{\omega_0^2}{c^2}\left[1 + \hat{\chi}(f_0)\right] = \frac{\omega_0^2}{c^2} n^2(f_0)$$

⇒ β_0: prop. constant ⟶ Can be positive or negative

$\beta_0 > 0$ ⇒ wave prop. towards negative z

$\beta_0 < 0$ ⇒ Wave prop. towards positive z

→ Magnetic field from Maxwell: $\bar{H}(\vec{r},t) = -E_M \frac{\beta_0}{\omega_0 \mu_0} \cos(\omega_0 t + \phi_0 + \beta_0 z)\ \vec{J}$ η_0

↳ we get: $E \perp H$ and $E \perp \hat{k}$ $H \perp \hat{k}$

↳ lossless media: $n \in \mathbb{R}$

lossy media: $n \in \mathbb{C}$ ⟶ dissipative effects

These sols. are plane waves

$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi\ \Omega$

→ Poynting vector: $\bar{W} = \bar{E} \times \bar{H}$

↳ plane waves (previous): $\bar{W} = -\frac{E_M^2 \beta_0}{\omega_0 \mu_0} \cos^2(\omega_0 t + \phi_0 + \beta_0 z)\ \hat{k}$

↳ Temporal averge: $\langle \bar{W} \rangle = -\frac{E_M^2 \beta_0}{2\omega_0 \mu_0}\ \hat{k}$ $\begin{cases} \text{Energy flows towars positive z if } \beta_0 < 0 \\ \text{Energy " " negative z " } \beta_0 > 0 \end{cases}$

↳ Power: flux through surface S of Poynting vector

$$\langle P \rangle = \int_S -\frac{E_M^2 \beta_0}{2\omega_0 \mu_0}\ \hat{k} \cdot \hat{n}\ dS$$

1.5 Fourier transform and complex amplitude

* Fourier transform

$\bar{E}(\bar{r},t)$ function in 4D, so FT is in 4D

$\bar{f}_r = (f_x, f_y, f_z)$

$f_r \cdot \bar{r} = f_x x + f_y y + f_z z$

$$\hat{\bar{E}}(\bar{f}_r, f) = \int_{\mathbb{R}^3}\int_{\mathbb{R}} \bar{E}(\bar{r},t)\, e^{-j2\pi(\bar{f}_r\cdot\bar{r} + ft)}\, dt\, d\bar{r}$$

Derivation is: $\dfrac{\partial E(\bar{r},t)}{\partial t} \xrightarrow{\mathcal{F}} j2\pi f\, \hat{E}(\bar{f}_r, f)$

Wave equation: $\left(f_x^2 + f_y^2 + f_z^2\right)\hat{\bar{E}} = \dfrac{f^2}{c^2}\left(\hat{\bar{E}} + \dfrac{1}{\varepsilon_0}\hat{\bar{P}}\right)$

* Complex Envelope of a signal

Quasi-sinusoidal real signal: $v(t) = \frac{1}{2} C_v(t)\, e^{j2\pi f_0 t} + \frac{1}{2} C_v^*(t)\, e^{-j2\pi f_0 t}$ $C_v(t) \to$ complex envelope

Also: $v(t) = \mathrm{Re}\left\{C_v(t)\, e^{j2\pi f_0 t}\right\} = |C_v(t)| \cos(2\pi f_0 t + \phi(t))$ $\phi(t) = \arg[C_v(t)] \to$ instantaneous phase

To obtain complex envelope $C_v(t)$ from quasi-sin $C_v(t)$

↳ ① Remove from $v(t)$ negative freq. using a filter $H(f) = 2 \cdot 1(f)$ $1(f) = \begin{cases} 0 & f<0 \\ 1/2 & f=0 \\ 1 & f>0 \end{cases}$ Heaviside

↳ ② Translation the positive freq by $-f_0$ so the mode is in origin of freqs.

⇒ So we get instead of 2 spectral comps, complex envelope contains only one mode and it's a slowly varying and base band signal

Fourier transform: $\hat{v}(f) = \frac{1}{2} C_v(f - f_0) + \frac{1}{2} C_v^*(-f - f_0)$

* Complex amplitude

From perfect sinusoidal reference signal:

$$E_0(z,t) = E_M \cos(2\pi q_0 z + 2\pi f_0 t + \phi_0)$$

two spectral modes at the spatio-temporal freq.

(q_0, f_0) $(-q_0, -f_0)$

$$\downarrow$$

$$E_0(z,t) = \frac{1}{2} E_M e^{j\phi_0} e^{j(\beta_0 z + \omega_0 t)} + \frac{1}{2} E_M e^{-j\phi_0} e^{-j(\beta_0 z + \omega_0 t)}$$

$$\downarrow$$

$$\boxed{\mathrm{Re}\left\{E_M e^{j\phi_0} e^{j(\beta_0 z + \omega_0 t)}\right\}}$$

From quasi-sinusoidal:

$$E(z,t) = \frac{1}{2} A(z,t) e^{j(\beta_0 z + \omega_0 t)} + \frac{1}{2} A^*(z,t) e^{-j(\beta_0 z + \omega_0 t)}$$

$A(z,t)$ complex amplitude and slowly varying respect $_{z,t}$

$$\downarrow$$

$$E(z,t) = \mathrm{Re}\left\{A(z,t) e^{j2\pi(q_0 z + f_0 t)}\right\} = |A(z,t)| \cos\left[2\pi(q_0 z + f_0 t) + \phi(z,t)\right]$$

$|A(z,t)| \rightarrow$ instantaneous amplitude

$\phi(z,t) = \arg[A(z,t)] \rightarrow$ instant. phase

Fourier:

$$\hat{E}(f_z, f) = \frac{1}{2} \hat{A}(f_z - q_0, f - f_0) + \frac{1}{2} \left[\hat{A}(-f_z - q_0, -f - f_0)\right]^*$$

Magnetic field:

$$\bar{H}(z,t) \triangleq -|A| \frac{\beta_0}{\omega_0 \mu_0} \cos(\omega_0 t + \beta_0 z + \phi_0)\, \hat{j}$$

Average EM power passing through a surface

$$\langle P \rangle \simeq -\int_S \frac{|A|^2 \beta_0}{2\omega_0 \mu_0} \hat{k} \cdot \hat{n}\, dS$$

1.6 Wave equation for the complex amplitudes

We have:

$$\begin{cases} E(\bar{r},t) = \dfrac{A(\bar{r},t)}{2}\, e^{j(\beta_0 z + \omega_0 t)} + \dfrac{A^*(\bar{r},t)}{2}\, e^{-j(\beta_0 z + \omega_0 t)} \\ P(\bar{r},t) = \dfrac{A_p(\bar{r},t)}{2}\, e^{j(\beta_0 z + \omega_0 t)} + \dfrac{A_p^*(\bar{r},t)}{2}\, e^{-j(\beta_0 z + \omega_0 t)} \end{cases}$$

$$\boxed{A_1 = A + \frac{1}{\varepsilon_0} A_p}$$

Into the wave equation we get:

$$\nabla^2 A + 2j\beta_0 \frac{\partial A}{\partial z} - \beta_0^2 A = -\frac{\omega_0^2}{c^2} A_1 + \frac{2j\omega_0}{c^2}\frac{\partial A_1}{\partial t} + \frac{1}{c^2}\cdot\frac{\partial^2 A_1}{\partial t^2}$$

* Linear contribution of the induced polarization

↳ P and E are related through an isotropic linear medium as a filter:

$$P(\bar{r},t) = \varepsilon_0\, \mathcal{X} \overset{t}{*} E(\bar{r},t) = \varepsilon_0 \int_{-\infty}^{+\infty} \mathcal{X}(\bar{r},\tau)\, E(\bar{r}, t-\tau)\, d\tau \xrightarrow[\text{dom}]{\text{freq.}} \hat{P}(\bar{r},f) = \varepsilon_0\, \hat{\mathcal{X}}(\bar{r},f)\, \hat{E}(\bar{r},f)$$

It's an overkill bc it would require full knowledge of the response of the medium

↳ We rewrite A_p using susceptibility taylor series ⟶ $\hat{\mathcal{X}}(f+f_0) = \hat{\mathcal{X}}(f_0) + f\,\hat{\mathcal{X}}'(f_0) + \frac{f^2}{2}\hat{\mathcal{X}}''(f_0)$

$$A_p = \varepsilon_0 \sum_{n=0}^{\infty} \frac{\hat{\mathcal{X}}^{(n)}(\omega_0)}{(i)^n\, n!}\, \frac{\partial^n A(\bar{r},t)}{\partial t^n} \qquad \hat{\mathcal{X}}^{(n)}(\omega_0) = \left.\frac{\partial^n \hat{\mathcal{X}}(\omega)}{\partial \omega^n}\right|_{\omega=\omega_0}$$

* Nonlinear contribution of the induced polarization

↳ We restrict to:
3rd order nonlin. term
instantaneous response (no memory)

$$P^{NL}(\bar{r},t) = \varepsilon_0\, \hat{\mathcal{X}}^{(3)} E^3(\bar{r},t) \xrightarrow[\text{amplitude so that}]{\text{find complex}} P^{NL}(\bar{r},t) = \mathrm{Re}\left\{A_p^{NL}(\bar{r},t)\, e^{j\overbrace{(\omega_0 t + \beta_0 z)}^{=\alpha}}\right\}$$

↓ find with

$E(\bar{r},t) = \mathrm{Re}\, A e^{j\alpha} \xrightarrow{\text{in}} P^{NL} \sim E^3$

↳ We get: $A_p^{NL}(\bar{r},t) = \frac{1}{4}\varepsilon_0\, \hat{\mathcal{X}}^{(3)} \left[A^3(\bar{r},t)\, e^{j2\alpha} + 3A^2(\bar{r},t)\, A^*(\bar{r},t)\right]$

It contains two contributions → base band
↳ pulsation $2\omega_0$ ⟶ not very important

↳ Simplified: $A_p^{NL}(\bar{r},t) = \frac{3}{4}\varepsilon_0\, \hat{\mathcal{X}}^{(3)} |A(\bar{r},t)|^2 A(\bar{r},t)$

* Total contributions to the polarization

$$P = A + \sum_{n=0}^{\infty} \frac{\hat{\mathcal{X}}^{(n)}(\bar{r},\omega_0)}{i^n\, n!}\, \frac{\partial^n A(\bar{r},t)}{\partial t^n} + \frac{3}{4}\hat{\mathcal{X}}^{(3)} |A(\bar{r},t)|^3 A(\bar{r},t)$$

1.7 - Non Linear Schrödinger Equation (NLSE)

* NLSE

Using P in the wave eq. of A and, assumptions:

- → Complex amplitude of the field A is slowly varying
- → Neglect terms containing time derivatives of A higher than 2nd order
- → Neglect temporal deriv. of $|A|^2 A$

⇒ Wave number: $K^2(\bar{r},\omega) = \frac{\omega^2}{c^2}\left[1+\hat{\chi}(\bar{r},\omega)\right] = \frac{\omega^2}{c^2} n^2(\bar{r},\omega)$

⇒ NLSE:

$$\nabla^2 A + 2i\beta_0 \frac{\partial A}{\partial z} - \beta_0^2 A = -k_0^2 A + i(k_0^2)' \frac{\partial A}{\partial t} + \frac{1}{2}(k_0^2)'' \frac{\partial^2 A}{\partial t^2} - \frac{3}{4}\frac{\omega_0^2}{c^2}\hat{\chi}^{(3)}|A|^2 A$$

→ $k_0^2 = k^2(\bar{r},\omega_0)$ → $(k_0^2)' = \left.\frac{\partial(k^2(\bar{r},\omega))}{\partial\omega}\right|_{\omega=\omega_0}$

- There is a main direction which the wave propagates. We assume that it's parallel to z axis
- Between generic vector $\bar{k}$ and z is small angle
  Paraxial approx.
  ⇒ A slowly varying with spatial coord. x, y, z

* Discussion on the terms appearing in the NLSE

→ **First order**

Neglecting all derivatives $\beta_0^2 = \frac{\omega_0^2}{c^2}\left[1+\hat{\chi}(\omega_0)\right] = k_0^2$ → disp. relation

→ **Second order**

Neglecting 2nd order derivatives ⇒ $2j\beta_0 \frac{\partial A}{\partial z} = \overbrace{j(k_0^2)'}^{=2k_0k_0'} \frac{\partial A}{\partial t}$

with $k_0 \simeq \pm\beta_0$ ⇒ $\frac{\partial A}{\partial z} = \pm k_0' \frac{\partial A}{\partial t}$

→ group velocity $v = \frac{1}{k_0'} = \frac{c}{\sqrt{1+\hat{\chi}(\omega_0)}}$

the complex amplitude propagates with v

→ There are delay times due to finite propagation speed.

→ No signal distorsions

→ $\boxed{\text{Third order}}$

After some calculations and using: $\frac{\partial^2 A}{\partial z^2} = (k_0')^2 \frac{\partial^2 A}{\partial t^2}$

$A \to$ complex amplitude of electric field.

$$2j\beta_0 \frac{\partial A(\bar r,t)}{\partial z} + \overbrace{\frac{\partial^2 A(\bar r,t)}{\partial x^2} + \frac{\partial^2 A(\bar r,t)}{\partial y^2}}^{\text{diffractive terms}} + \left[\overbrace{\underbrace{\frac{3}{4}\frac{\omega_0^2}{c^2}\hat\chi^{(3)}|A(\bar r,t)|^2}_{\text{Kerr-effect}} + \underbrace{k_0^2 - \beta_0^2}_{\text{wave guiding}}}^{\text{changes in refractive index}}\right] A(\bar r,t)$$

$$-j2k_0k_0' \frac{\partial A(\bar r,t)}{\partial t} - \overbrace{k_0k_0'' \frac{\partial^2 A(\bar r,t)}{\partial t^2}}^{\text{dispersive term}} = 0$$

$$\hat\chi^{(1)} = \hat\chi^{(1)}_R + i\underbrace{\hat\chi^{(1)}_I}_{\text{losses}}$$

- $\beta_0 = 2\pi q_{f_0}$ · reference propagation constant
- $\omega_0 = 2\pi f_0$ reference pulsation
- $k_0^2 = \frac{\omega^2}{c^2}\left[1 + \hat\chi(\bar r,\omega)\right] = \frac{\omega^2}{c^2} n^2 \to k_0$ wavenumber
- $k_0' = \frac{\partial k}{\partial \omega}$ $\quad v = \frac{1}{k_0'} \to$ group velocity
- $k_0'' = \frac{\partial^2 k}{\partial \omega^2} \to$ related to group velocity dispersion
- Diffractive term responsible of spatial broadening of a beam propagating in a medium
- Dispersive term temporal broadening of a pulse propagating in a medium with $k_0'' \neq 0$
- $\frac{3}{4}\frac{\omega_0^2}{c^2}|A(\bar r,t)|^2$ This term contains refractive index depending on light intensity

$$\left(\frac{3}{4}\frac{\omega_0^2}{c^2}|A(\bar r,t)|^2 + k_0^2\right) A(\bar r,t) \simeq \frac{\omega_0^2}{c^2}\left[n_2 |A(\bar r,t)|^2 + n\right]^2 A(\bar r,t)$$

$$n_2 = \frac{3\hat\chi^{(3)}}{8n}$$

this nonlinearity refractive index in silica may yield effects on electric fields phases.

$\boxed{\text{Summary}}$

Changes <u>very</u> slow in time → 2nd order = 0

changes <u>very very</u> slow in time → 1st order = 0

Homogeneous medium → $\hat\chi$ constant

Diffraction: $\nabla_t^2 A + 2i\beta_0 \frac{\partial A}{\partial z} = 0$

Plane wave → $\frac{\partial^2 A}{\partial x^2} = 0, \frac{\partial^2 A}{\partial y^2} = 0$

* What was neglected

(1) Linear attenuation: (losses)

$$\hat{\chi} \longrightarrow \hat{\chi} = \hat{\chi}_R + i\,\hat{\chi}_I \qquad \text{and using first order: } \beta_0^2 = \frac{\omega_0^2}{c^2}\left[1 + \hat{\chi}_R(\omega_0)\right]$$

we obtain after calculations:

$$\frac{dA(z)}{dz} = \pm \frac{\omega_0}{2c} \frac{\hat{\chi}_I(\omega_0)}{\sqrt{1+\hat{\chi}_R(\omega_0)}} A(z) = \pm \alpha A(z) \xrightarrow{\text{Sol}} A(z) = A(0)\, e^{\pm \alpha/2\, z}$$

$\alpha \rightarrow$ attenuation coefficient.

In terms of intensity we have:

$$I(z) = I(0)\, e^{-\alpha z} \Rightarrow \text{Lambert-Beer law}$$

Intensity within a lossy medium decays exponentially

@ $z = \frac{1}{\alpha} \longrightarrow$ penetration length depends on wavelength

---

(2) Effects due to the electric field polarization

→ Linear:

↳ Birifrangence: Medium is not isotropic anymore. We have different susceptibility along x and y

$$\hat{\chi}_x \neq \hat{\chi}_y$$

↳ So the constitutive relation are dif.

$$\left.\begin{aligned} P_x &= \varepsilon_0\, \chi_x \overset{t}{*} E_x \\ P_y &= \varepsilon_0\, \chi_y * E_y \end{aligned}\right\} \xrightarrow[\text{complex ampl.}]{\text{two}} \left.\begin{aligned} A_x \\ A_y \end{aligned}\right\} \xrightarrow[\text{prop. dist.}]{\text{two}} \left.\begin{aligned} \beta_{0x} \\ \beta_{0y} \end{aligned}\right\} \xrightarrow[\text{wave numbers}]{\text{two}} \begin{aligned} k_{0x} \\ k_{0y} \end{aligned}$$

↳ restricting to:

Homogeneous
Linear
Non-dispersive media

$\Rightarrow$ neglect diffractive effects $\Rightarrow$

$$\left.\begin{aligned} \frac{\partial A_x}{\partial z} &= \pm k_{0x}' \frac{\partial A_x}{\partial t} \longrightarrow v_{gx} \\ \frac{\partial A_y}{\partial z} &= \pm k_{0y}' \frac{\partial A_y}{\partial t} \longrightarrow v_{gy} \end{aligned}\right\} \text{Dispersion of Polarization}$$

→ Non-Linear

If instantaneous response ⇒ E_z negligible

So:

$$\left.\begin{aligned} P_x^{NL} &= \varepsilon_0 \hat{\chi}^{(3)}\left(E_x^3 + E_x E_y^2\right) \\ P_y^{NL} &= \varepsilon_0 \hat{\chi}^{(3)}\left(E_y^3 + E_y E_x^2\right) \end{aligned}\right\} \xrightarrow[\text{neglecting } 3\omega_0]{}$$

also for $A_{P_y}^{NL}$

$$A_{P_x}^{NL} = \frac{3}{4}\varepsilon_0 \hat{\chi}^{(3)}\left[\left(|A_x|^2 + \frac{2}{3}|A_y|^2\right)A_x + \underbrace{\frac{1}{3}A_x^* A_y^2\, e^{2j(\beta_{0y}-\beta_{0x})z}}_{\text{incoherent term}}\right]$$

Self Phase Modulation
SPM
electric field polarized along x(y) modify its phase

Cross Phase Modulation
XPM
electric field polarized along x (or y) modifies the phase of the other electric field

(3) High Order Dispersion

Truncation the dispersion relation to 2nd order may not be enough.

Can be issues where pass to normal dispersion (V_g increase with λ) to anomalous disp. (V_g decrease with λ)

(4) Nonlinear attenuation

$$\hat{\chi}^{(3)} \in \mathbb{C}$$

(5) Nonlinear dispersion

$$\hat{\chi}^{(3)} \longrightarrow \hat{\chi}^{(3)}(\omega)$$

1.8 - Considerations on the importance of terms in the NLSE

* Linear polarization

In a perfect medium

Linear
No losses
Non dispersive: $k_0''=0$

} the best we can get is $\frac{\partial A}{\partial z} - k_0' \frac{\partial A}{\partial t} = 0 \longrightarrow k_0'(\vec{r}) = \frac{n(\vec{r})}{c} = \frac{1}{V_g(\vec{r})}$ Velocity V_g for energy transport

The eq. of k_0' is useful even in dispersive medium ($k_0'' \neq 0$)

The truncation in 2nd derivatives is reasonable if spectral extent of wave packet is much smaller than the main freq. ($f_0 \gg |\Delta f|$)

* Sellmeier equation

Law describing refractive index variations in silica as a function of wavelength.

→ Far from resonance freqs. ⇒ we can neglect losses.

$$n^2(\lambda) - 1 = \sum_i \Delta_i \frac{\lambda^2}{\lambda^2 - \lambda_i^2}$$

Normal dispersion: fastest comp. higher λ (red)

Anomalous dispersion: fastest comp. lower λ (blue)

Figure 1.11

* Nonlinear polarization

Interaction length: Length over which the nonlinear effect is active

Bulk optics (no guided) → IL limited by diffraction bc it causes continous broadening, broadening causes continous reduction of peak intensity ⟶ all nonlin. effects are negligible

Guided optics → Diffraction totally compensated by guiding effect → IL only limited by optical losses

→ [Kerr effect]: Non-lin effect that occurs in optical fiber, is the dependence of the refractive index on the intensity. Negligible of the norm
In phase not negligible → phase shift

Diffraction

$$2i\beta_0 \frac{\partial A}{\partial z} = -\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right) \qquad \beta_0 = \pm \frac{\omega_0}{c} n(\omega_0)$$

Homogeneous medium
$\beta_0 \simeq k_0$

NLSE 3D if Continuous Wave, linear and homogeneous medium

$\Rightarrow$ Plane waves $\rightarrow$ Solutions

$\Rightarrow$ Paraxial appr. $\vec{k} \parallel \hat{z}$

Gaussian beam scheme

$$\begin{tikzcd}
A(x,y,z=0) \arrow[r, "F", "\text{compute spectra}"'] & \hat{A}(x,y,z=0) \arrow[d, "\text{Propagation}"] \\
A(x,y,z=L) & \hat{A}(x,y,z=L) \arrow[l, "F^{-1}"']
\end{tikzcd}$$

Gaussian

$W_0 \rightarrow$ initial spot size

$W_z \rightarrow$ spot size everywhere

$$L_Z = \frac{\pi W_0^2}{\lambda}$$ Fresnel length

Distance which separates ray optics and wave optics

Near field: $z \ll L_z$

Far field: $z \gg L_z$

$$W^2(z) = W^2(0)\left[1 + \left(\frac{z}{L_F}\right)^2\right] \rightarrow \text{spot size}$$

$$R(z) = z\left[1 + \left(\frac{L_F}{z}\right)^2\right] \rightarrow \text{radious of curvature}$$

Dispersion Parameter

$$D = \frac{\partial\left(\frac{\partial k}{\partial \omega}\right)}{\partial \lambda} = -\frac{2\pi c}{\lambda^2}\frac{\partial^2 k}{\partial \omega^2}$$

Shorter pulses broad faster because they have more freq. components.

Chapter 2

2.1 - Propagation in the optical fiber

*General features

Core: Inner part, higher refractive index

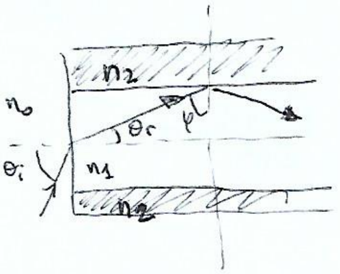
Cladding: Covers the core, lower refractive index

Jacket: Outer part, made of plastic to protect the transmitting part.

We restrict to → step optical fibers
↳ core refr. index constant
↳ cladding refr. index constant

Core dimensions →
- If large enough: several modes can be prop: Multimodal
- If small enough only one mode propagates: Monomodal

Scheme:



① Snell: $n_0 \sin\theta_i = n_1 \sin\theta_r$

② Ligh ray encounter the separation surface (core/cladding)
To trap the ray we need total internal refraction

$$\phi > \phi_c = a\sin\left(\frac{n_2}{n_1}\right)$$

③ We can determinate the maximum incidence angle allowing to the incoming ray to be trapped. inside the guide

$$NA = n_0 \sin\theta_{i_{max}} = n_1 \cos\phi_c = \sqrt{n_1^2 - n_2^2}$$

→ About multimodal

Limitation in the transmissive capacity → intermodal dispersion
A ray trapped inside the core may cover several paths, if propagation speed is the same the time to cover the distance (between beginning and end) could be very different.
To mitigate it use a refractive index $n(r)$ with smooth spatial profile

*Main factors affecting the propagation

Op. fiber are: dispersive, nonlinear and dissipative systems

① Attenuation of the fiber
bc is propagating in a medium

att. coef: $\alpha = -\frac{10}{D} \log_{10}\left[e^{-\alpha D}\right]$

Two contributions
↳ Intrinsic (medium): absorption, difusion, Rayleigth diffusion effect

↳ Extrinsic: (external): tensions, deforming, bending, ...

② Fiber dispersion: Different spectral comp. propagate with different group velocities (V_g)

*other factors affecting propagation

→ Material dispersion: Material weakly dispersive

→ Intramodal dispersion: ∀ prop. mode: Ratio of power propagating in cladding and power in the core depends on freq.

→ Intermodal dispersion:
Only in multimodal due to diff. group vel. of modes

→ Polarization dispersion:
Geometric. imperf. may imply coupling of two orthogonal polarization states travelling with diff. group vel.

→ Transmitted power signal
Above certain power distorsion of the signal

2.2 - Guided modes

Electromagnetic spatial confinement may be seen as a trade-off of diffraction and guiding effect.

We can study separately
→ spatial problem (variables: x, y)
→ temporal problem (variable: t)

$$A(x,y,z,t) = F(z,t)\, \underbrace{\mathcal{M}(x,y)}_{\text{modal profile}}\, e^{j\delta\beta z} \longrightarrow \mathcal{M}(r,\phi)$$

$\beta = \beta_0 + \delta\beta$: prop. constant of guided mode

β_0 → sol. prop. const. in the reference ideal case

$\delta\beta$ → due to the presence of the guide

Weakly guiding: $k_0 \simeq \beta_0$ ($n_1 \simeq n_2$, $n_1 > n_2$)

Introducing $A(x,y,z,t)$ in NLSE 3D with:
- weakly guiding → $k_0 \simeq \beta_0$
- Nonlin. term is small, dependence in x, y disregarded

$$\begin{cases} 2j\beta_0 \dfrac{\partial F(z,t)}{\partial z} + 2\dfrac{k_0^2}{n} n_2 \left| F(z,t)\,\mathcal{M}(x,y)\right|^2 F(z,t) - 2j\beta_0 k_0' \dfrac{\partial F(z,t)}{\partial z} - \beta_0 k_0'' \dfrac{\partial^2 F(z,t)}{\partial t^2} = 0 \\ \boxed{\dfrac{\partial^2 \mathcal{M}(x,y)}{\partial x^2} + \dfrac{\partial^2 \mathcal{M}(x,y)}{\partial y^2} + \left(k_0^2 - \beta_0^2 - 2\beta_0\delta\beta\right)\mathcal{M}(x,y) = 0} \end{cases}$$

↳ $n_2 = \dfrac{3\hat{\chi}^{(3)}}{8n}$

↳ This eq. allows us to determinate the guided modes $\xrightarrow{\text{through}}$ Linearly Polarized (LP) modes approximation

↳ Solving this eq. we obtain:
→ Profile of guided modes: $\mathcal{M}(x,y)$ → eigenfunctions
→ Their propag. constant: β → eigenvalues

~~* Guided modes of the optic~~

↘

* Guided modes of the optical fiber: LP approximation

→ Guiding dielectric structure with circular symmetry:

$$K_0^2(r) = \begin{cases} n_1^2 \omega^2 \mu_0 \varepsilon_0 & 0 \le r < a \rightarrow \text{core} \\ n_2^2 \omega^2 \mu_0 \varepsilon_0 & a \le r < +\infty \rightarrow \text{cladding} \end{cases}$$

$n_1 > n_2$

$n_1 \rightarrow$ core
$n_2 \rightarrow$ cladding
$a \rightarrow$ core radius

→ Using cilindrical coords. and using Fourier series of $\mathcal{M}(r,\phi)$

↳ $\mathcal{M}(r,\phi) = \sum_{l=-\infty}^{+\infty} \mathcal{M}_l(r)\, e^{il\phi}$

We set the eq:

$$\frac{\partial^2 \mathcal{M}_l(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{M}_l(r)}{\partial r} + \left[q_l^2(r) - \frac{l^2}{r^2}\right]\mathcal{M}_l(r) = 0$$

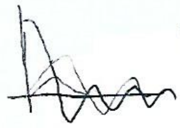
for every l we have an eq. for $\mathcal{M}_l(r)$ and associated $\delta\beta$

↳ $q_l^2(r) = K_0^2(r) - \beta_0^2 - 2\beta_0 \delta\beta_l$

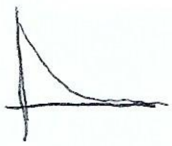
→ Impose the continutity of electric field at the interface between core and cladding
We can do it bc $n_1 \simeq n_2$. To remeber this cheating we call them LP modes

↳ The solution is:

$$\mathcal{M}_l(r) = \begin{cases} A_1 J_l(k_t r) & 0 \le r < a \rightarrow \text{core} \\ A_2 K_l(\gamma r) & a \le r < +\infty \rightarrow \text{cladding} \end{cases}$$



Bessel function 1st kind



modified Bessel func. 2nd kind

$k_t^2 = n_1^2 \omega^2 \mu_0 \varepsilon_0 - \beta_0^2 - 2\beta_0 \delta\beta_l \longrightarrow k_t^2 > 0$

$\gamma^2 = \beta_0^2 + 2\beta_0\delta\beta_l - n_2^2 \omega^2 \mu_0 \varepsilon_0 \longrightarrow \gamma^2 > 0$

↳ positive in guided propagation due to $n_1 > n_2$

↳ $\left.\begin{matrix} \mathcal{M}_l(r) \\ \frac{d\mathcal{M}_l(r)}{dr} \end{matrix}\right\}$ continuous in $r = a$.

⇓

$$-\frac{X J_{l-1}(X)}{J_l(X)} = \frac{Y K_{l-1}(Y)}{K_l(Y)}$$

$X = k_t a$

$Y = \gamma a$

$X^2 + Y^2 = \omega^2 \mu_0 \varepsilon_0 (n_1^2 - n_2^2) a^2 = \left(\frac{2\pi a}{\lambda}\right)^2 (n_1^2 - n_2^2) = V^2$

normalized freq.

trascendental eq. (dispersion relation) ⇒ Solving we get propagation constant and transvorse profile of LP mode

Steps

① Solve trascendental eq. we get $X \xrightarrow[\text{we get}]{\text{so}} K_t$

② Choosing β_0 we get $\beta_\ell \longrightarrow \beta_\ell = \beta_0 + \delta\beta_\ell = \dfrac{n_1^2 w^2 \mu_0 \varepsilon_0 - (X/a)^2}{2\beta_0} + \dfrac{\beta_0}{2}$

③ With β_0 we get effective refractive index of the mode $\rightarrow n_e(w) = \dfrac{\beta(w)}{w\sqrt{\mu_0\varepsilon_0}}$

↳ n_e increases with $V \Rightarrow$ If V increases the mode becomes more confined in the core

↳ V affects both: Modal profile (eigen func) and propagation constant (eigenvalue)

* Gaussian approximations

Profile mode LP_{01} can be approximated by a Gaussian function

$$M(r) = e^{-(r/W_0)^2}$$

W_0 modal radius: $\dfrac{W_0}{a} \simeq 0{,}65 + 1{,}619\, V^{-3/2} + 2{,}879\, V^{-6}$ — Very good approx. for $2 < V < 2{,}4$

Effective refractive index: $n_e \simeq n_2 + (n_1 - n_2)\left(1{,}1428 - 0{,}996/V\right)^2$

* Propagation equation