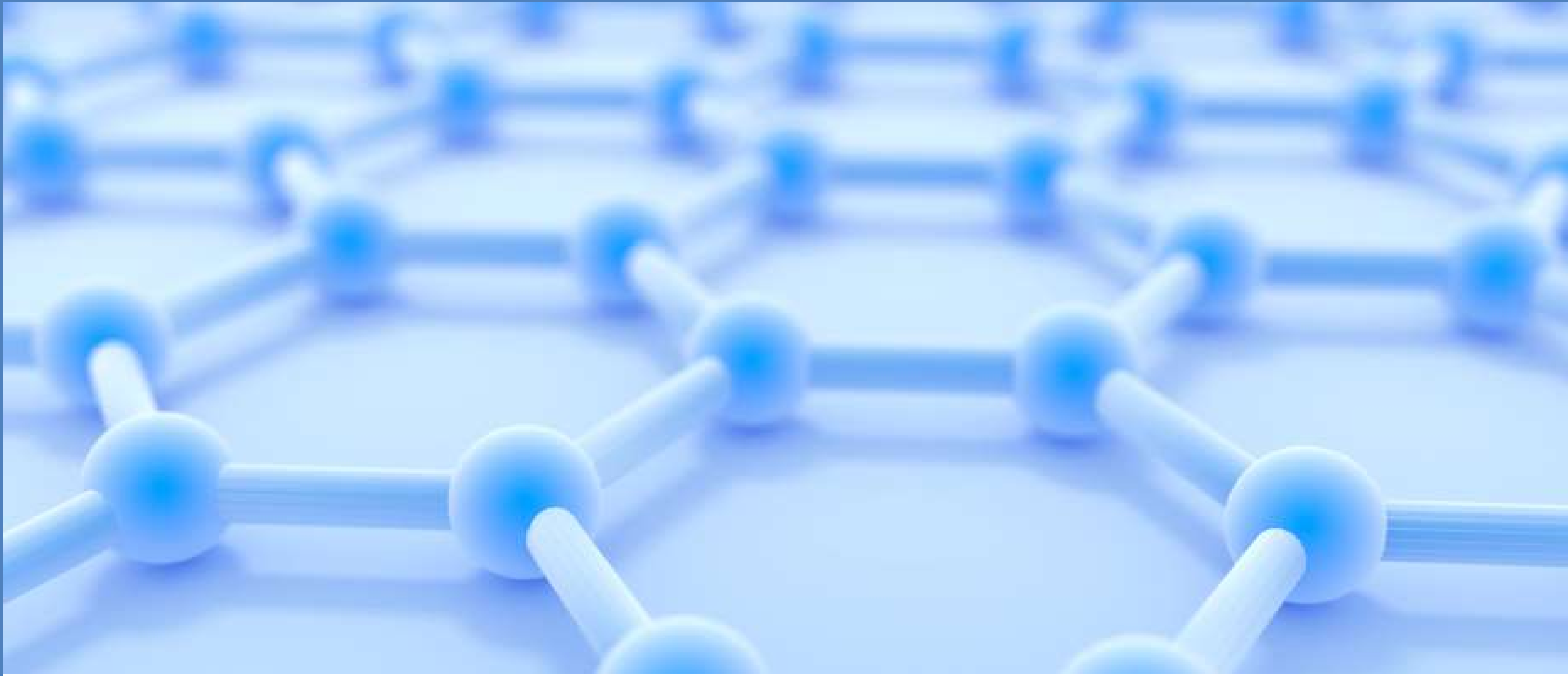

Photonics
aa 2021/2022

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Wave and Beam Optics:
Interference and pulsed light
Gaussian and other beams

Interference

When two or more waves are present at the same time in a location the total wavefunction will be the sum of each individual wavefunction.

IMPORTANT: The superposition principles applies to the complex amplitudes but DOES not apply to the optical intensity.

INTERFERENCE
depends on the relative phase between the waves

Monochromatic waves

When two monochromatic waves are superimposed, the result is another monochromatic wave with the same frequency and complex amplitude:

$$U(\boldsymbol{r}) = U_1(\boldsymbol{r}) + U_2(\boldsymbol{r})$$

The intensity of the resulting wave is not simply the sum of the two intensities of the constituent waves but should be calculated as:

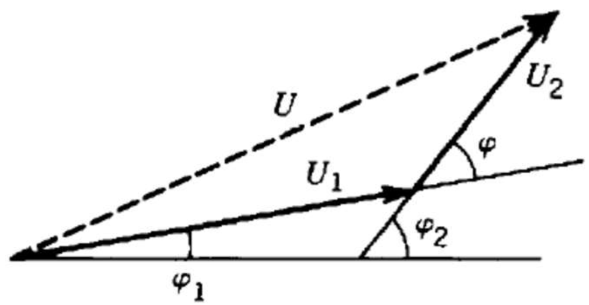
$$I = |U|^2 = |U_1 + U_2|^2 = |U_1|^2 + |U_2|^2 + U_1^* U_2 + U_1 U_2^*$$

Since the two original wavefunctions can be written as $U_1 = \sqrt{I_1}\exp(j\varphi_1)$ and $U_2 = \sqrt{I_2}\exp(j\varphi_2)$ we can write the total intensity of the resulting wave as:

Interference Equation

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2} cos\varphi}$$

Where $\varphi = \varphi_2 - \varphi_1$.



Monochromatic waves

In particular if the two waves have the same intensity $I_1=I_2=I_0$ the interference equation reduces to:

$$\boxed{I = 2I_0(1 + cos\varphi) = 4I_0 cos^2(\varphi/2)}$$

It follows that if:

- $\varphi = 0$, then $I = 4I_0$; - constructive interference
- $\varphi = \pi$, then $I = 0$; - destructive interference
- $\varphi = \frac{\pi}{2}, \frac{3}{2}\pi$, then $I = 2I_0$.

Numerous optical systems evaluate the resulting intensity to infer the phase difference between two interfering waves.

NOTE: Interference can be observed only for *coherent* light. Typically, light is only *partially coherent* , i.e., phase fluctuates randomly, and therefore interference is not observable.

Interferometers

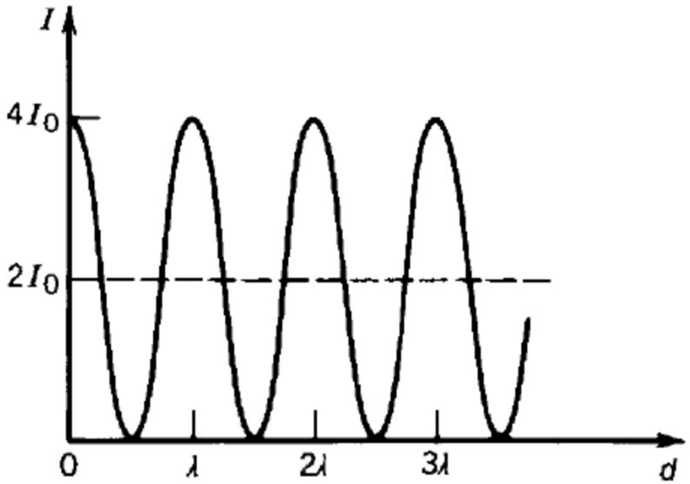
When two waves of the same intensity propagate in the same direction z but one of the two is delayed by a distance *d* we can write the two wavefunction as:

$$U_1 = \sqrt{I_0}\exp(-jkz) \text{ and } U_2 = \sqrt{I_0}\exp[-jk(z-d)]$$

So that the intensity of the resulting wave is:

$$\boxed{I = 2I_0[1 + \cos\left(\frac{2\pi d}{\lambda}\right)]}$$

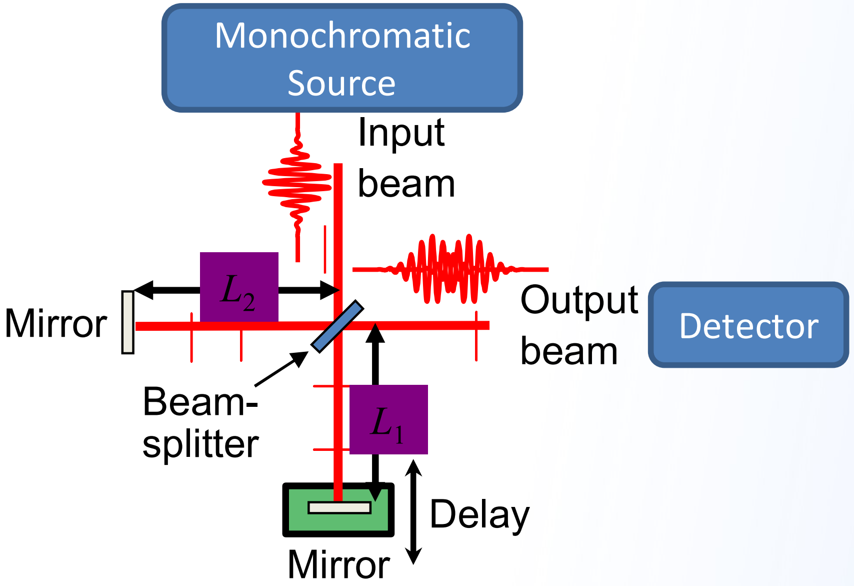
Since $kd = \frac{2\pi d}{\lambda} = \frac{2\pi n d}{\lambda_0}$



An **INTERFEROMETER** is a device that splits a wave into two waves, delays them by unequal distances, and measures the intensity of their superposition after they have been recombined

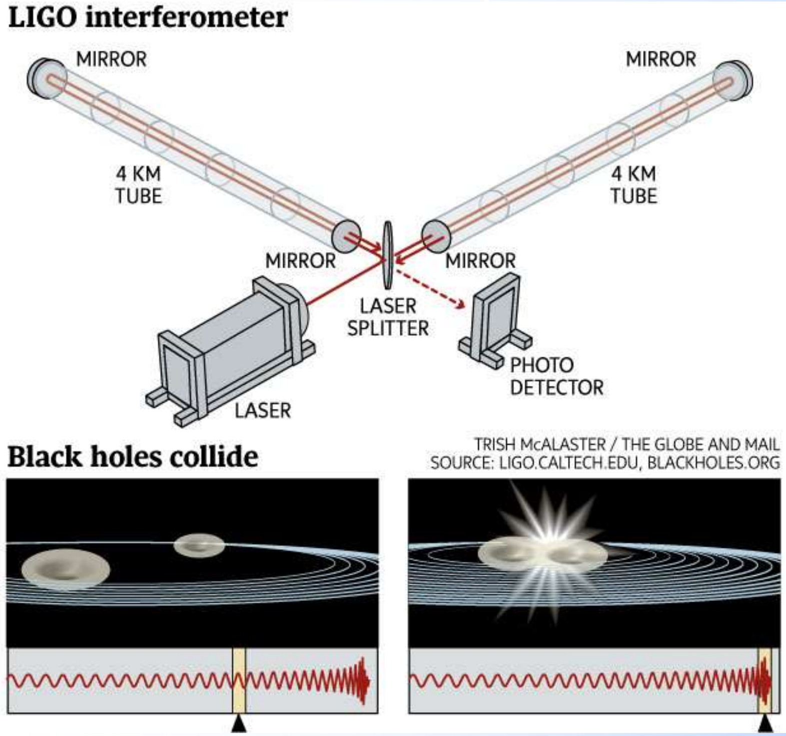
Interferometers

Interferometers can detect very small path changes; therefore, they can be used to detect change in refractive index, wavelength or frequency, but also strains, temperature, and gravitational waves.



LIGO Interferometers

CalTech LIGO



Oblique (plane) waves

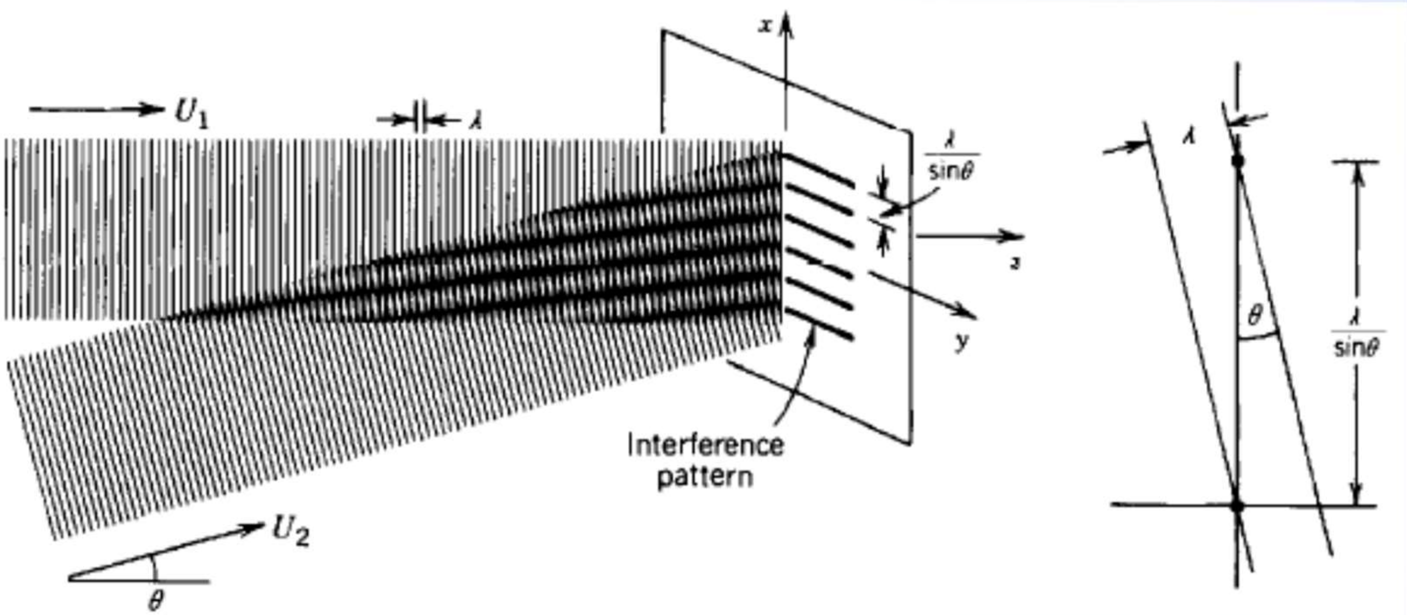
When two plane waves of equal intensities travel with two different angles so that:

$$U_1 = \sqrt{I_0}\exp(-jkz) \text{ and } U_2 = \sqrt{I_0}\exp[-j(kcos\theta\, z + ksin\theta\, x)]$$

At the plane z=0 the intensity that results from the interference is:

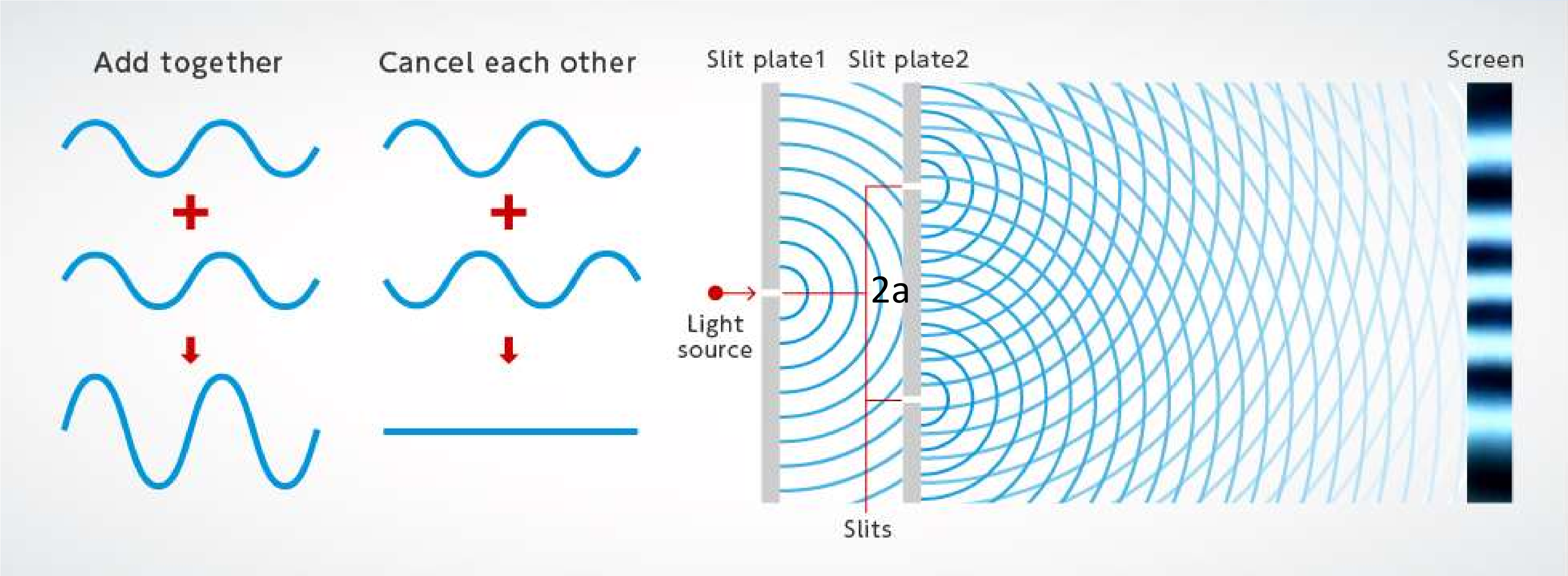
$$\boxed{I = 2I_0[1 + \cos(ksin\theta\, x)]}$$

The interference pattern is a sinusoidal signal with period $\lambda/\sin\theta$.



Spherical waves

Young's interference experiment



In 1807, the Young's interference experiment showed that waves passing through two slits add together or cancel each other depending on their relative phase. Using the paraboloidal approximation we can demonstrate that the intensity at a plane placed at z=d is:

$$I(x, y, d) \approx 2I_0\left[1 + \cos\left(\frac{2\pi x\theta}{\lambda}\right)\right]$$

Where θ~2a/d. The intensity pattern is periodic with period λ/θ.

Multiple-wave Interference

The superposition of M monochromatic waves of the same frequency is the sum of the complex amplitudes:

$$U = U_1 + U_2 + U_3 + \cdots + U_M$$

Knowledge of the intensity of each wave is not sufficient and the relative phase between each wave must be known.

We can identify few special cases where the total intensity can be calculated.

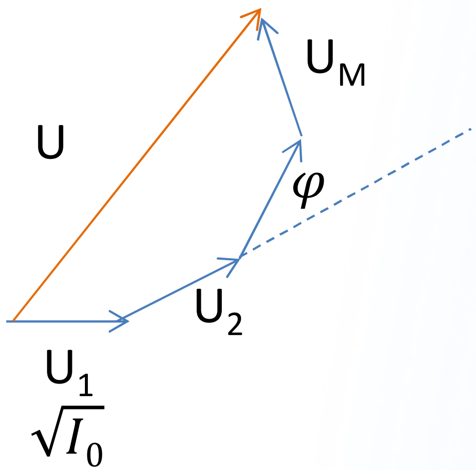
Multiple-wave Interference

M WAVES WITH EQUAL AMPLITUDE AND EQUAL PHASE DIFFERENCE

Let's start by supposing that all waves have complex amplitude:

$$U_m = \sqrt{I_0}\,\exp[j(m-1)\varphi]$$

The waves have equal intensities I_0 and phase difference φ between successive waves. The intensity of the resulted wave can be "seen" graphically as follows:



The total intensity can be calculated as the sum of a finite series from which the complex amplitude of the superimposed wave is:

$$U = \sqrt{I_0}\,\frac{1-\exp[jM\varphi]}{1-\exp[j\varphi]}$$

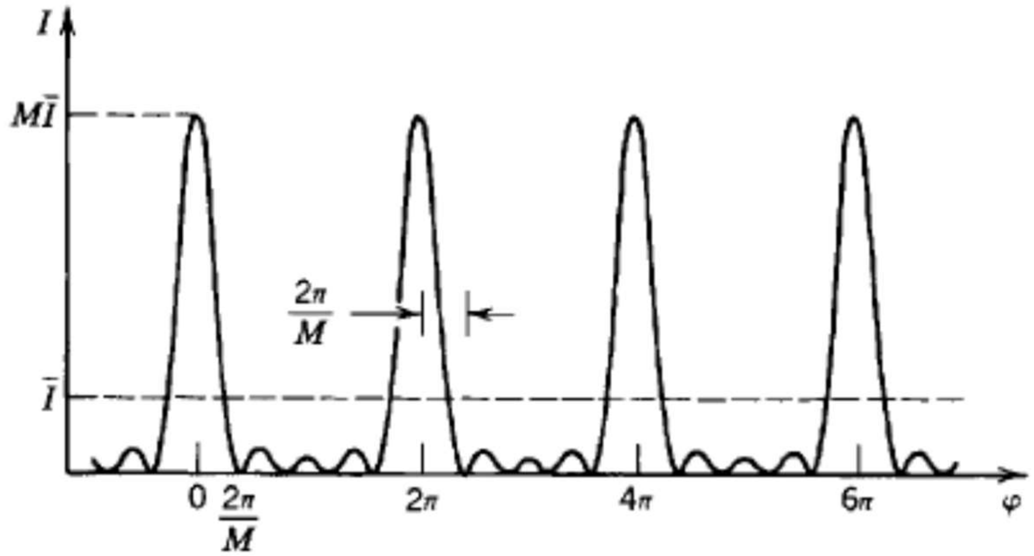
Multiple-wave Interference

The corresponding intensity is:

$$I = |U|^2 = I_0 \left| \frac{\exp[-jM\varphi/2] - \exp[jM\varphi/2]}{\exp[-j\varphi/2] - \exp[j\varphi/2]} \right|^2$$

From which follows

$$\boxed{I = I_0 \frac{sin^2(M\varphi/2)}{sin^2(\varphi/2)}}$$



Multiple-wave Interference

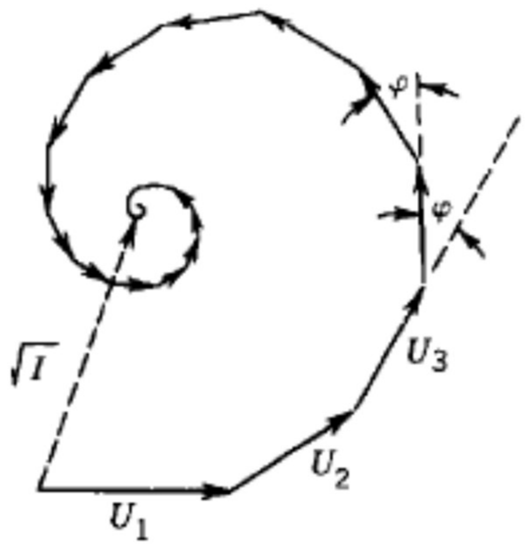
INFINITE NUMBER OR WAVES OF PROGRESSIVELY SMALLER AMPLITUDES AND EQUAL PHASE DIFFERENCE

This time we must calculate the intensity of the superposition of an infinite number or waves with equal phase differences and amplitudes that decrease at a geometric rate:

$$U_1 = \sqrt{I_0}\,,\ U_2 = hU_1,\ U_3 = hU_2 = h^2 U_1, \dots,$$

Where h=|h|$e^{j\varphi}$, |h|<1

The waves have progressively small intensities and phase difference φ between successive waves. The intensity of the resulted wave can be “seen” graphically with the phasor diagram.



The total intensity can be calculated as the sum of an infinite series from which the complex amplitude of the superimposed wave is:

$$U = \frac{\sqrt{I_0}}{1 - |h|\exp[j\varphi]}$$

Multiple-wave Interference

The corresponding intensity is:

$$I = |U|^2 = I_0 \left| \frac{1}{1 - |h| \exp[j\varphi]} \right|^2 = \frac{I_0}{(1 - |h|)^2 + 4|h| sin^2(\varphi/2)}$$

The intensity expression can be more conveniently expressed in the following form

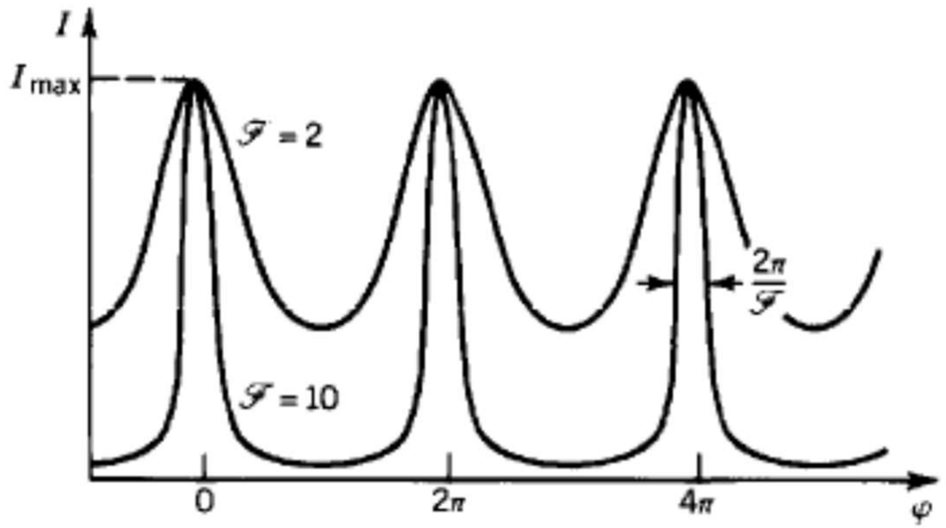
$$\boxed{I = \frac{I_{max}}{1 + \left(\frac{2F}{\pi}\right)^2 sin^2(\varphi/2)}}$$

Where :

$$\boxed{I_{max} = \frac{I_0}{(1 - |h|)^2}, \qquad F = \frac{\pi\sqrt{|h|}}{1 - |h|}}$$

F defines the **finesse**. The larger F, the sharper are the peaks of the Intensity function.
The width of the interference pattern is:

$$\Delta\varphi \approx 2\pi/F$$



Polychromatic light

Monochromatic light is an ideal approximation for a signal that is extended from -∞ to ∞. In reality, waves have a time dependence and finite duration and therefore they are always polychromatic.

A **polychromatic wave**, which is described by a function u(**r**, t) may be expanded as a **superposition of monochromatic waves.**
An arbitrary function u(**r**,t) can therefore be expanded at a fixed position **r** as a superposition integral of harmonic functions of different frequencies, amplitudes and phases:

$$u(t) = \int_{-\infty}^{\infty} v(f)\exp(j2\pi f t)\, df$$

Where $v(f)$ is determined by performing the Fourier Transform:

$$v(f) = \int_{-\infty}^{\infty} u(t)\exp(-j2\pi f t)\, dt$$

Polychromatic light

It's still convenient to represent the real function u(t) by a complex function that includes only the positive frequency components:

$$U(t) = 2\int_0^{\infty} v(f)\exp(j2\pi f t)\,df$$

It follows that $V(f) = 2v(f)$ for positive frequencies and 0 otherwise.
We can extract the real part of the function

$$u(t) = Re\{U(t)\} = \frac{1}{2}\,[U(t) + U^*(t)]$$

Where $U(t)$ is called complex analytic function, while $U(\boldsymbol{r}, t)$ **is the complex wavefunction**. The complex wavefunction, being a superposition of monochromatic waves that satisfy the wave equation, also satisfies the wave equation:

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0$$

Polychromatic light

The optical intensity is related to the wavefunction as follows:

$$I(\boldsymbol{r},t) = 2\langle u^2(\boldsymbol{r},t)\rangle = 2\left\langle\left\{\frac{1}{2}\,[U(t)+U^*(t)]\right\}^2\right\rangle$$
$$= \frac{1}{2}\langle U^2(\boldsymbol{r},t)\rangle + \frac{1}{2}\langle U^{*2}(\boldsymbol{r},t)\rangle + \langle U(\boldsymbol{r},t)U^*(\boldsymbol{r},t)\rangle$$

If the wave is quasi-monochromatic (has a spectral width $\Delta f \ll f_0$) then the average operation washes out the first two terms that oscillate at frequencies $\sim 2f_0$ and $\sim -2f_0$. The only surviving term is the third one, which contains frequency differences of the order of $\Delta f \ll f_0$. In other words, the time average operation does not affect this term since it varies slowly in time.
From this follows that the light intensity becomes:

$$\boxed{I(\boldsymbol{r},t) = |U\,(\boldsymbol{r},t)|^2}$$

Pulsed light

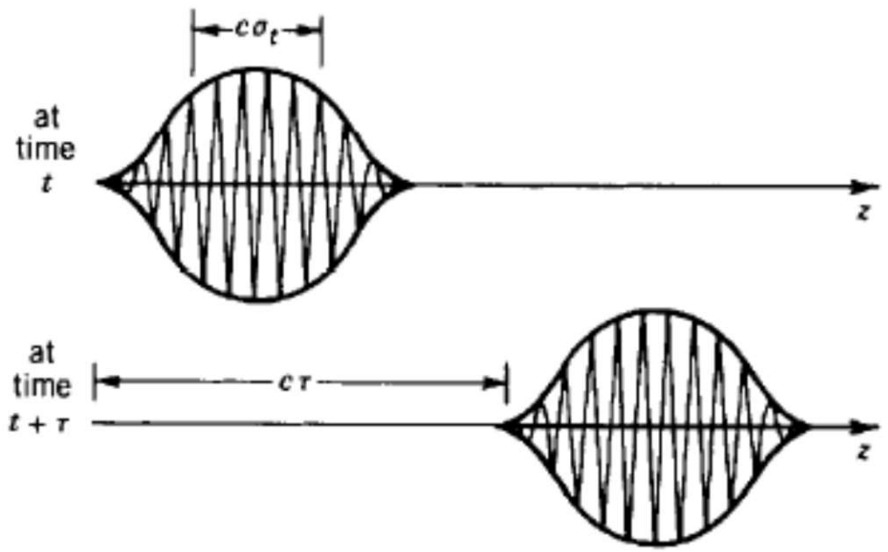
The simplest example of pulsed light is a pulsed plane wave, whose complex wavefunction is:

$$U(\boldsymbol{r},t) = A\left(t-\frac{z}{c}\right)\exp[j2\pi f_0\left(t-\frac{z}{c}\right)]$$

Where *A(t)* is the complex envelope, and it is a time-varying function and f_0 is the central frequency. The monochromatic wave is a special case where *A(t)= constant.*

Regardless the form of the function A(t) the complex wavefunction U(**r**,t) satisfies the wave equation.

If *A(t)* is of finite duration σ_τ then at any point z the wave extends over a space $c\sigma_\tau$. We can say that this is a **wavepacket** that travels in the z direction.



Pulsed light

The Fourier transform of the complex wavefunction is:

$$V(\boldsymbol{r}, f) = A(f - f_0) \exp\left[-j2\pi f \frac{z}{c}\right]$$

Where *A(f)* is Fourier Transform of *A(t)*. Typically *A(t)* is slowly varying and therefore *A(f)* has a spectral bandwidth much smaller then f_0.

NOTE: The spectral width is inversely proportional to the temporal width.

NOTE 2: The Fourier transform of a Gaussian pulse is still a Gaussian pulse.

Light Beating

INTERFERENCE OF TWO MONOCHROMATIC WAVES WITH DIFFERENT FREQUENCIES

An optical wave composed of two monochromatic waves has a complex wavefunction at a generic point in space:

$$U(t) = \sqrt{I_1}\exp(j2\pi f_1 t) + \sqrt{I_2}\exp(j2\pi f_2 t)$$

Where the phases are assumed to be zero. Using the interference equation, we find that the intensity is:

$$\boxed{I(t) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos[2\pi(f_2 - f_1)t]}$$

Where $|f_2 - f_1|$ is the **beat frequency**.
This phenomenon is known as **light beating**, **optical mixing**, **photomixing**, or **optical heterodyning**.

NOTE: This intensity expression is similar to the one that allows to determine the direction of a wave.
NOTE 2: Optical detectors are only sensitive only to the difference frequency.

Gaussian Beam

Among all possible solutions of paraxial Helmholtz equation, the Gaussian beam is probably the most important one.

A Gaussian beam has the following characteristics:

- The **beam power** is mostly concentrated within a small cylinder that surrounds the beam axis;

- The **intensity distribution** in any transverse plane is a circularly symmetric Gaussian function centered on the beam axis;

- The **width** of the function is minimum at the beam waist gradually becomes larger;

- The **angular divergence** of the wavefront normal assume the minimum value permitted by the wave equation;

Gaussian Beam

A paraxial wave can be made of a plane wave traveling along the z direction e^{-jkz} modulated by a slowly varying complex envelope A(**r**):

$$U(\boldsymbol{r}) = A(\boldsymbol{r}) \exp(-jkz)$$

Where the wavenumber is k=2π/λ. The envelope is approximately constant around λ. If the complex amplitude U(**r**) has to satisfy the Helmholtz equation, so does the complex envelope A(**r**):

$$\nabla_T^2 A - j2k \frac{\partial A}{\partial z} = 0$$

One of the solution of the Helmholtz equation is the paraboloidal wave. Another possible solution is the Gaussian beam, which is obtained from the paraboloidal wave with a simple transformation obtained by replacing the spatial coordinate z with z-ξ, where ξ is a constant:

$$A(\boldsymbol{r}) = \frac{A_1}{q(z)} \exp\left(-jk \frac{\rho^2}{2q(z)}\right), \qquad q(z) = z - \xi, \qquad \rho^2 = x^2 + y^2$$

Gaussian Beam

Whatever the value of ξ the expression of A(**r**) remains a solution of the Helmholtz equation. However, when $\xi = -jz_0$, i.e., the value is purely imaginary, we found the complex envelope of the **Gaussian beam**:

$$A(\boldsymbol{r}) = \frac{A_1}{q(z)} \exp\left(-jk\frac{\rho^2}{2q(z)}\right), \qquad q(z) = z - jz_0, \qquad \rho^2 = x^2 + y^2$$

The quantity q(z) is the **q-parameter** and z_0 is known as the **Rayleigh range**.

To find the amplitude and phase of the complex envelope we can re-write the function 1/q(z) in terms of real and imaginary parts R(z) and W(z) as follows:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi W^2(z)}$$

Gaussian Beam

From which the complex amplitude is:

$$U(\boldsymbol{r}) = A_0 \frac{W_0}{W(z)} \exp\left(-\frac{\rho^2}{W^2(z)}\right) \exp\left(-jkz - jk\frac{\rho^2}{2R(z)} + j\varsigma(z)\right)$$

Where:

$$A_0 = \frac{A_1}{jz_0},$$

$$W_0 = \sqrt{\frac{\lambda z_0}{\pi}},$$

$$W(z) = W_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2},$$

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right],$$

$$\varsigma(z) = \tan^{-1}\frac{z}{z_0}$$

Gaussian Beam Properties

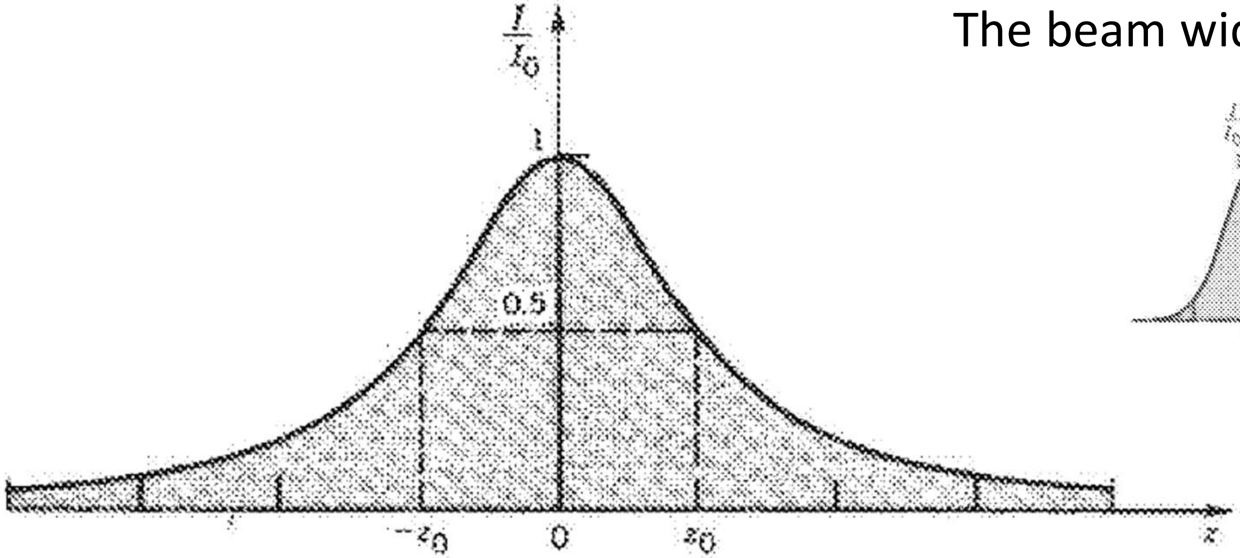
Intensity

$$I(\boldsymbol{r}) = |U(\boldsymbol{r})|^2$$

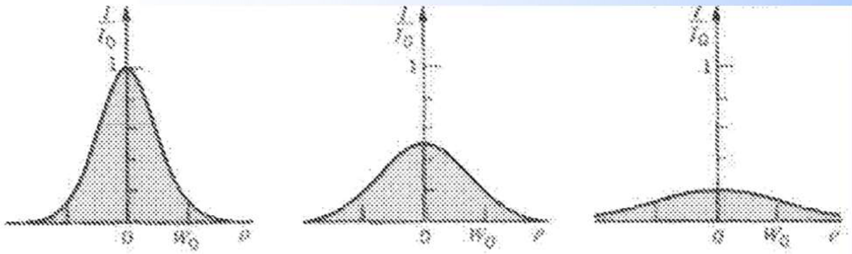
$$\boxed{I(\rho, z) = I_0 \left[\frac{W_0}{W(z)}\right]^2 \exp\left[-\frac{2\rho^2}{W^2(z)}\right]}$$

Where $I_0 = |A_0|^2$

At any value z the intensity is a Gaussian function: it's peak is at ρ=0 and decreases monotonically as ρ increases.



The beam width W(z) increases with increasing z.



Gaussian Beam Properties

Power

$$\boxed{P = \int_0^\infty I(\rho, z)\, 2\pi\rho d\rho = \frac{1}{2} I_0 \underbrace{(\pi W_0^2)}_{\text{Beam Area}}}$$

Power does not depend on z

Since very often the power is known and we need to calculate the intensity, it is useful to write down the relation between I and P in the following form:

$$I(\rho, z) = \frac{2P}{\pi W^2(z)} \exp\left[-\frac{2\rho^2}{W^2(z)}\right]$$

NOTE: About 99% of the power is carried in a circle of radius 1.5W(z).

Gaussian Beam Properties

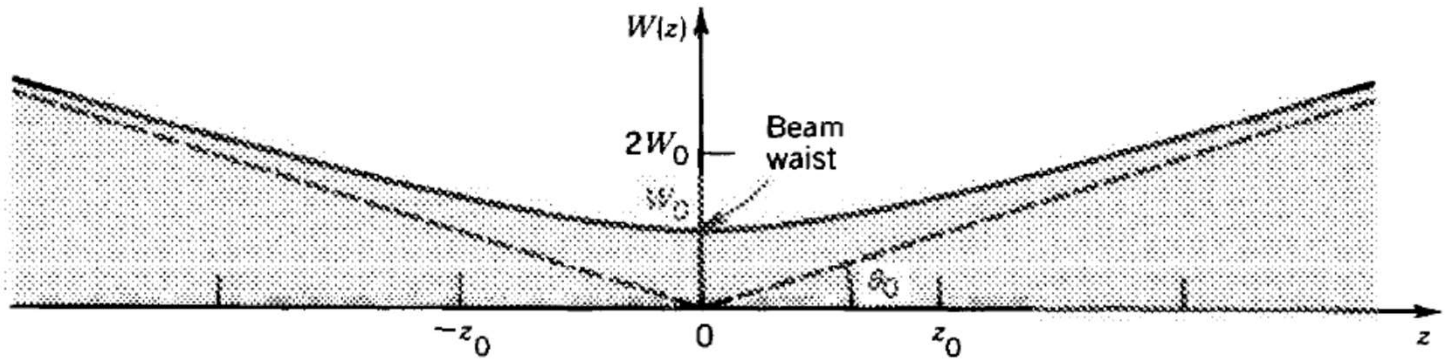
Beam Width

At any transverse plane the beam intensity is maximum on the beam axis and decreases by $1/e^2$ at a radial distance $\rho = W(z)$. Since approximately 86% of the power is carried within a radius $W(z)$ we will regard **W(z)** as the **beam width.**

The dependence of the beam width on z is:

$$\boxed{W(z) = W_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}}$$

W_0 is also called **waist radius**. $2W_0$ is the **spot size**.



Gaussian Beam Properties

Beam Divergence

If $z >> z_0$ then the beam waist becomes:

$$W(z) \approx \frac{W_0}{z_0} z = \theta_0 z$$

In other words when z is sufficiently big the beam diverges as a cone with half-angle:

$$\theta_0 = \frac{W_0}{z_0} = \frac{\lambda}{\pi W_0}$$

Approximately 86% of the power is concentrated within this cone.
We can define the **angular divergence** of the beam as:

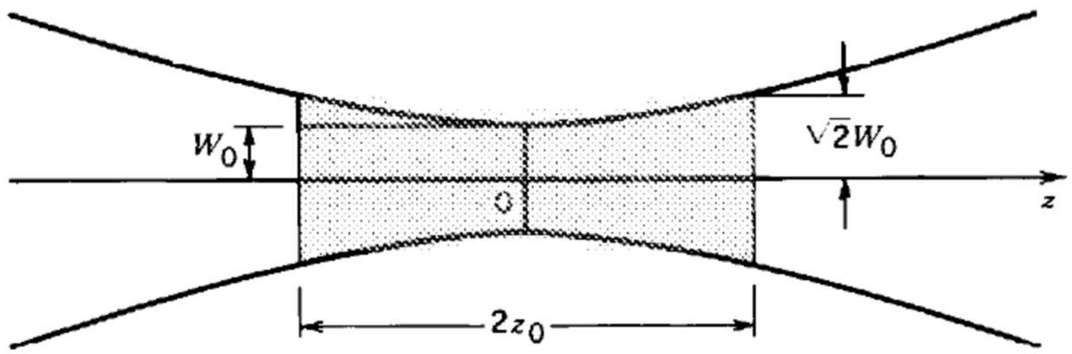
$$\boxed{2\theta_0 = \frac{4\lambda}{2\pi W_0}}$$

Gaussian Beam Properties

Depth of focus

Since the beam has a minimum at z=0, here is also when it achieves the best focus. Then the beam increases in size going in both direction. The distance at which the beam radius is no greater than $\sqrt{2}$ of its minimum value is known as **depth-of-focus or confocal parameter**. The depth of focus is also twice the Rayleigh range:

$$\boxed{2z_0 = \frac{2\pi W_0^2}{\lambda}}$$



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Gaussian Beam Properties

Phase

From the complex amplitude expression:

$$U(\boldsymbol{r}) = A_0 \frac{W_0}{W(z)} \exp\left(-\frac{\rho^2}{W^2(z)}\right) \exp\left(-jkz - jk\frac{\rho^2}{2R(z)} + j\varsigma(z)\right)$$

We can extract the phase of the gaussian beam:

$$\varphi(\rho, z) = kz + k\frac{\rho^2}{2R(z)} - \varsigma(z)$$

On the beam axis this reduces to:

$$\varphi(0, z) = \underbrace{kz}_{\text{Plane Wave}} - \underbrace{\varsigma(z)}_{\text{Phase Retardation}}$$

The phase retardation is an excess delay of the wavefront in relation to a plane or spherical wave, which is equal to π going from z=-∞ to z=∞. This phenomenon is known as **Gouy effect.**

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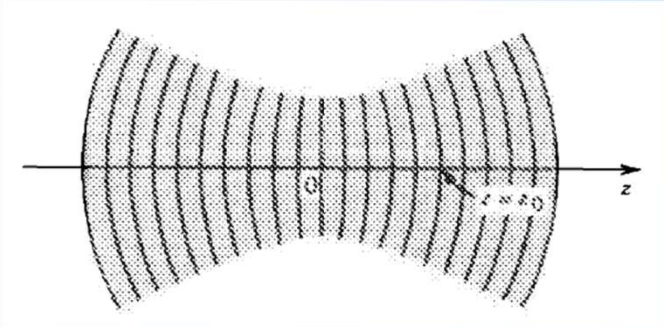
Gaussian Beam Properties

Wavefronts

There is another term in the phase of a gaussian beam that is responsible for wavefront bending:

$$\varphi(\rho, z) = kz + k\underbrace{\frac{\rho^2}{2R(z)}}_{\text{Bending}} - \varsigma(z)$$



This term represents a deviation of the phase at off-axis points.
The surface of constant phase satisfies the equation:

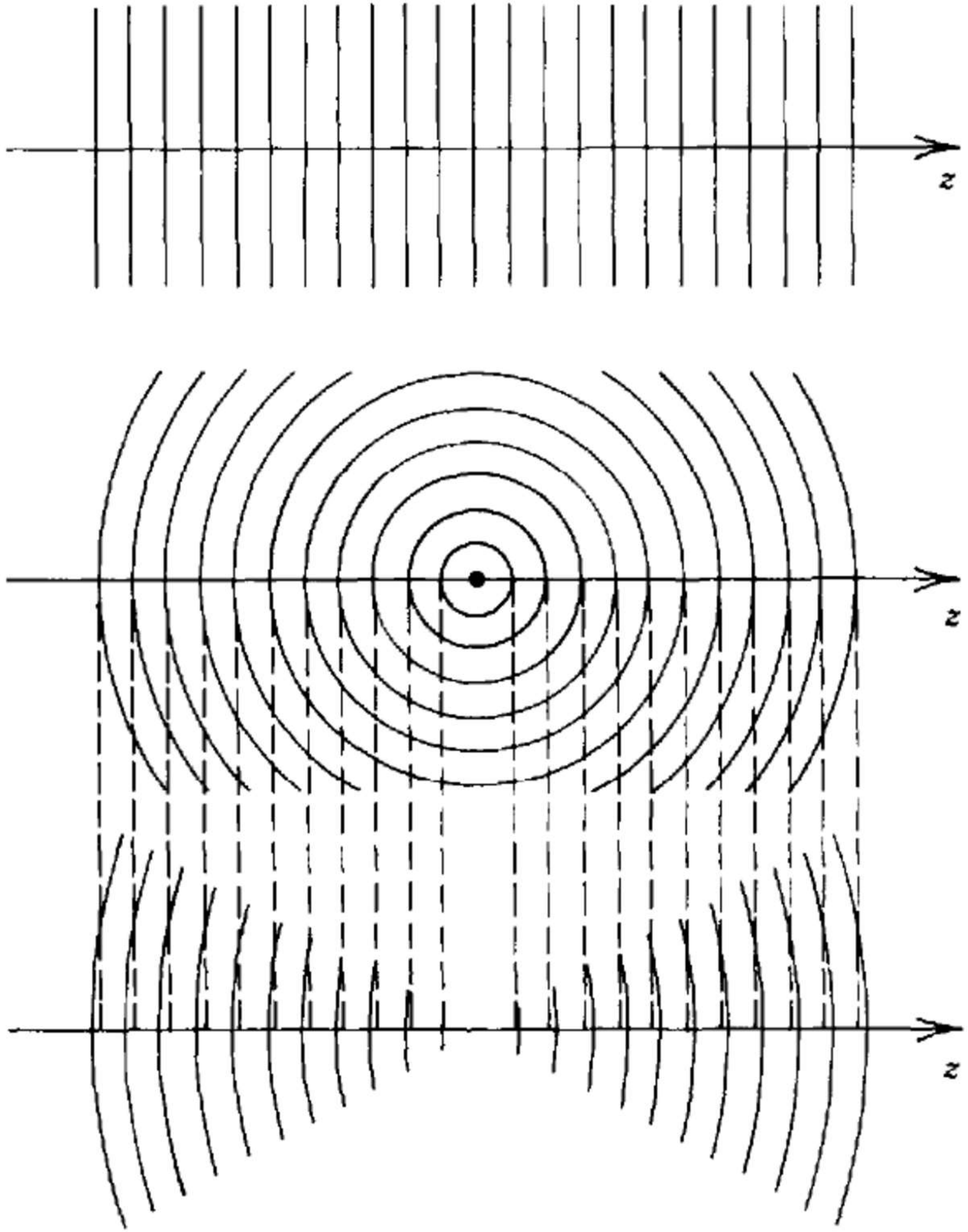
$$k\left[z + \frac{\rho^2}{2R(z)}\right] - \varsigma(z) = 2\pi q$$

Since R(z) and $\varsigma(z)$ are slowly varying we can consider them constant at points within the beam width of each front so that we can write:

$$\boxed{z + \frac{\rho^2}{2R} \approx q\lambda + \varsigma\frac{\lambda}{2\pi}}$$

PARABOLOIDAL SURFACE
with radius of curvature R

Gaussian Beam Properties



PLANE WAVE

SPHERICAL WAVE

GAUSSIAN BEAM

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Gaussian Beam Properties

Beam Quality

The deviation from the ideal Gaussian Beam can be quantified with the following parameter that depends on the beam waist, diameter and divergence as follows:

$$M^2 = \frac{2W_m \cdot 2\theta_m}{\underbrace{4\lambda/\pi}}$$

Waist-diameter-divergence parameter of a gaussian beam $\frac{4\lambda}{\pi} = 2W_0 \cdot 2\theta_0$

Where the Gaussian beam exhibits the smallest possible beam quality parameter M=1.

The Beam quality parameter is often used to quantify the specification of laser beams. For example, a He-Ne laser tipically has M<1.1. Ion lasers have M=1.1-1.3, collimated diode lasers have M=1.1-1.7, while multimode lasers have higher M parameters of the order of 3 or 4.

Other Beams

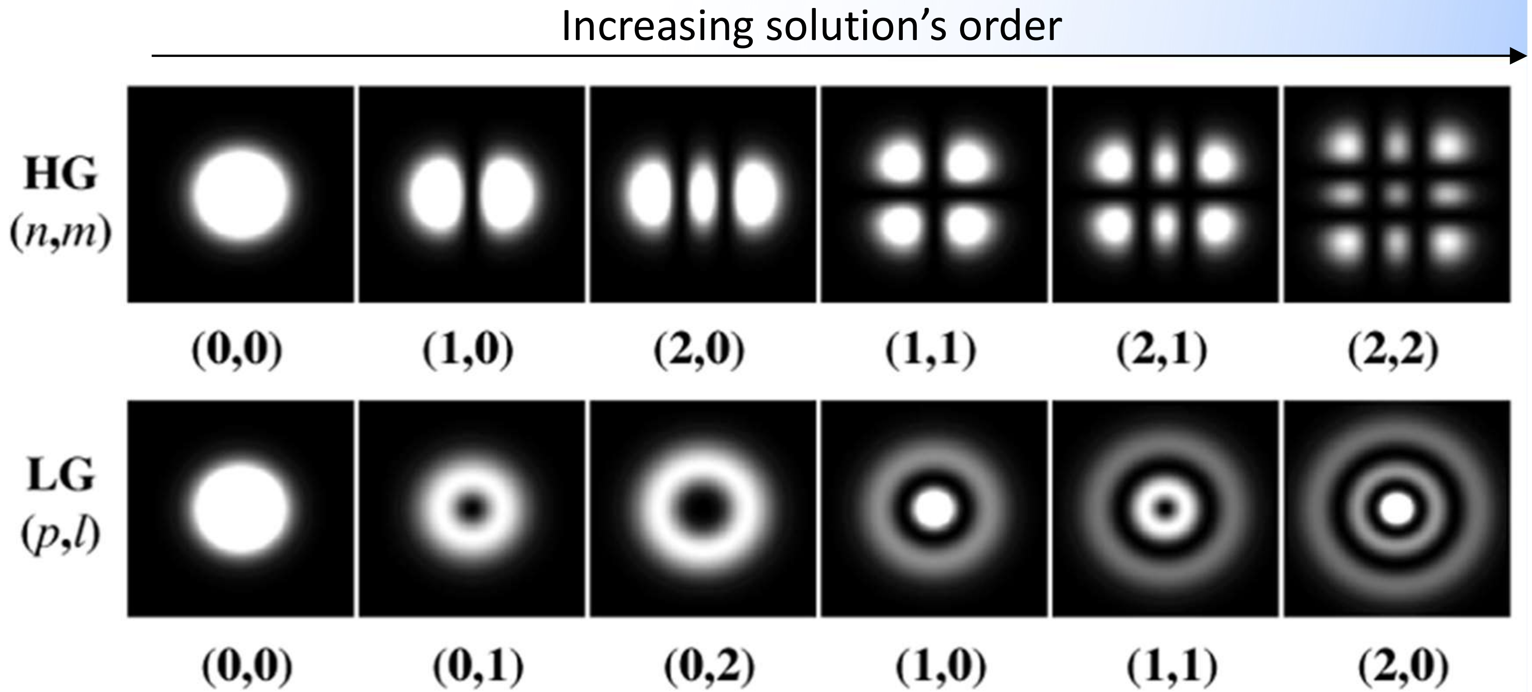
The Gaussian beam is not the only possible solution of the paraxial Helmholtz equation. Other solutions exhibit non-Gaussian intensity distributions but share the wavefronts of Gaussian beams: these are called **Hermite-Gaussian beams.**

The Hermite-Gaussian beams form a complete set of solution of the paraxial Helmholtz equation. Any other solution can be written as a superposition of those beams.

Another full set of solution is known as **Laguerre-Gaussian beams**, and it is obtained by writing the paraxial Helmholtz equation in cylindrical coordinates.

Hermite-Gauss solutions
Rectangular symmetry

Laguerre-Gauss solutions
Cylindrical symmetry



The (0,0) solution, the lowest-order, is the gaussian solution