NUMERICAL TECHNIQUES

In particular, we consider the split-step Fourier method. To understand the philosophy behind the split-step Fourier method, it is useful to write the NLSE in the form

$$\frac{\partial F}{\partial z} = \left(\hat{D} + \hat{N} \right) F$$

where $\hat{D}$ is a differential operator that accounts for dispersion (but it can take also into account for absorption and/or higher order dispersions) in a linear medium; $\hat{N}$ is a nonlinear operator that governs the effect of fiber nonlinearity on pulse propagation, i.e. Kerr nonlinearity (but it can take also into account other nonlinear phenomena, like Raman effects, etc.).

Thus

$$\boxed{\hat{D} = -i\,\frac{B''}{2}\,\frac{\partial^2}{\partial t^2}}$$

$$\boxed{\hat{N} = i\,\gamma\,|F|^2}$$

In general, dispersion and nonlinearity act toghether along The fiber

The split-step Fourier Method obtains an approximate solution by assuming That in propagating The envelope over a small distance h, The dispersive and nonlinear effects can be considered To act independently.

More specifically, propagation From (z) To $(z+h)$ is carried out in Two steps.

In the first step, the nonlinearity acts alone, and $\hat{D} = 0$.

In the second step, dispersion acts alone, and $\hat{N} = 0$.

Mathematically:

$$F(z+h, t) \approx \exp(h\hat{D}) \exp(h\hat{N}) F(z,t)$$

It's interesting to underline that the exponential operator $\exp(h\hat{D})$ can be evaluated in the Fourier domain using the prescription

$$\exp(h\hat{D})\, F(z,t) = F_T^{-1} \exp\left[h\hat{D}(i\omega) \right] F_T\, F(z,t)$$

where F_T denotes the Fourier-Transform operation, $\hat{D}(i\omega)$ is obtained by replacing the differential operator $\frac{\partial}{\partial t} \longrightarrow i\omega$

$$\frac{\partial}{\partial t} \xrightarrow{\text{by}} i\omega \quad \text{and} \quad \frac{\partial^2}{\partial t^2} \xrightarrow{\text{by}} (i\omega)^2 = -\omega^2$$

where ω is the frequency in the Fourier domain.

As $\hat{D}(i\omega)$ is just a number in the Fourier space, the evaluation of the dispersive term is simple.

The use of the FFT algorithm makes numerical evaluation fast.

To estimate the accuracy of the split-step method, we note that formally exact solution of the NLSE is given by the expression

$$F(z+h,t) = \exp\left[h\left(\hat{D}+\hat{N}\right)\right] F(z,t)$$

assuming $\hat{N}$ z-independent.

At This point, it's useful To recall The Baker-Husdorff Formula for operators $\hat{a}$ and $\hat{b}$:

$$\exp(\hat{a})\exp(\hat{b}) = \exp\left(\hat{a} + \hat{b} + \frac{1}{2}[\hat{a},\hat{b}] + \frac{1}{12}[\hat{a}-\hat{b},[\hat{a},\hat{b}]] + \dots\right)$$

where $[\hat{a},\hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a}$.

IF we have two commuting operators $[\hat{a},\hat{b}] = 0$

IF NOT, $[\hat{a},\hat{b}] \neq 0$

A comparison of the expression

$$F(z+h,t) \approx \exp(h\hat{D})\exp(h\hat{N})\,F(z,t)$$ *

and the expression

$$F(z+h,t) = \exp\left[h(\hat{D}+\hat{N})\right]F(z,t)$$

shows that the split-step method[a] ignores the noncommutating nature of operators $\hat{D}$ and $\hat{N}$.

By using the equation

$$\exp(\hat{a})\exp(\hat{b}) = \exp\left(\hat{a} + \hat{b} + \frac{1}{2}[\hat{a}, \hat{b}] \dots\right)$$

with $\hat{a} = h\hat{D}$ and $\hat{b} = h\hat{N}$, the dominant error is found to result from the term

$$\frac{1}{2} h^2 [\hat{D}, \hat{N}].$$

Thus, the split-step Fourier method is accurate to second order in the step size h.

I want to emphasize that taking h small, we have small errors

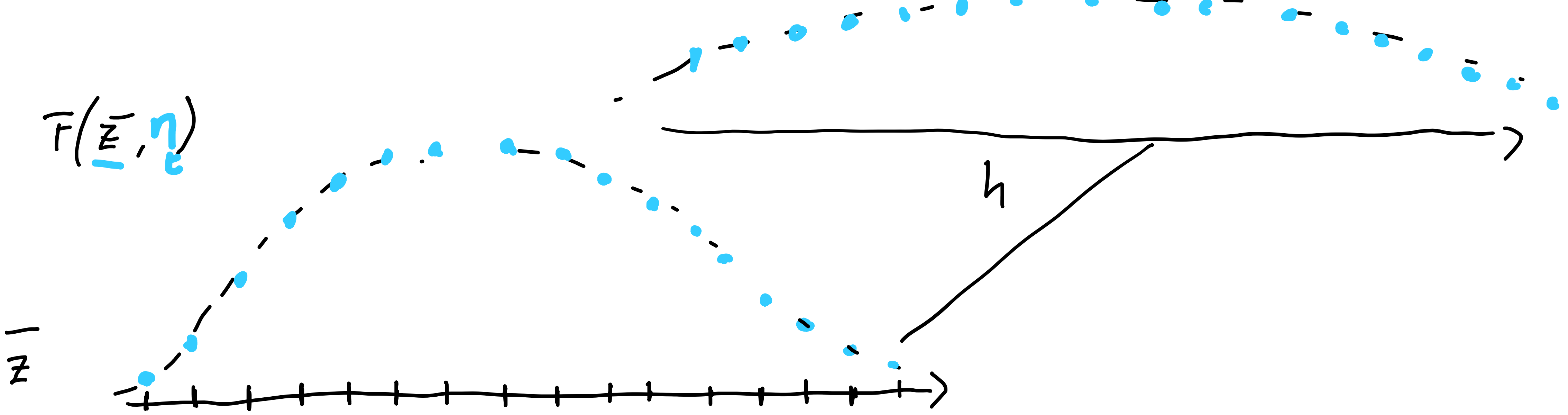
In general, the implementation of the method is

$$F(z+h,t) = \exp\left[h(\hat{D}+\hat{N})\right] F(z,t) \boxed{\simeq} \exp(h\hat{D})\exp(h\hat{N}) F(z,t)$$

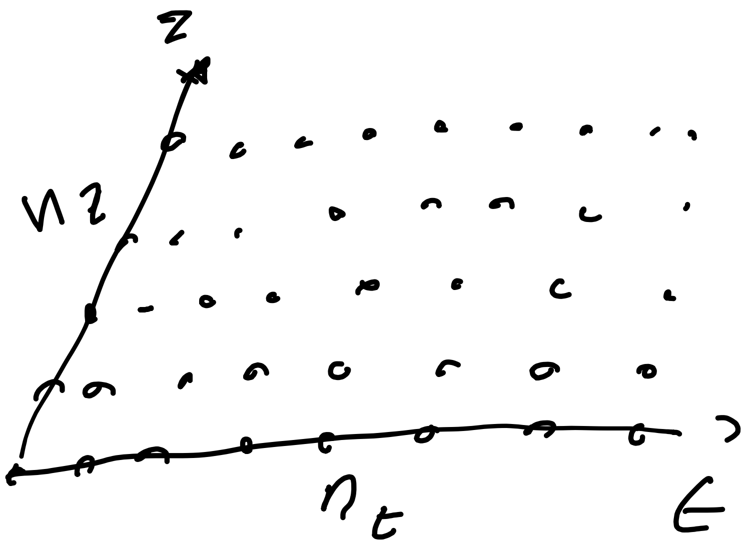
To propagate from (z) to $(z+h)$ we consider

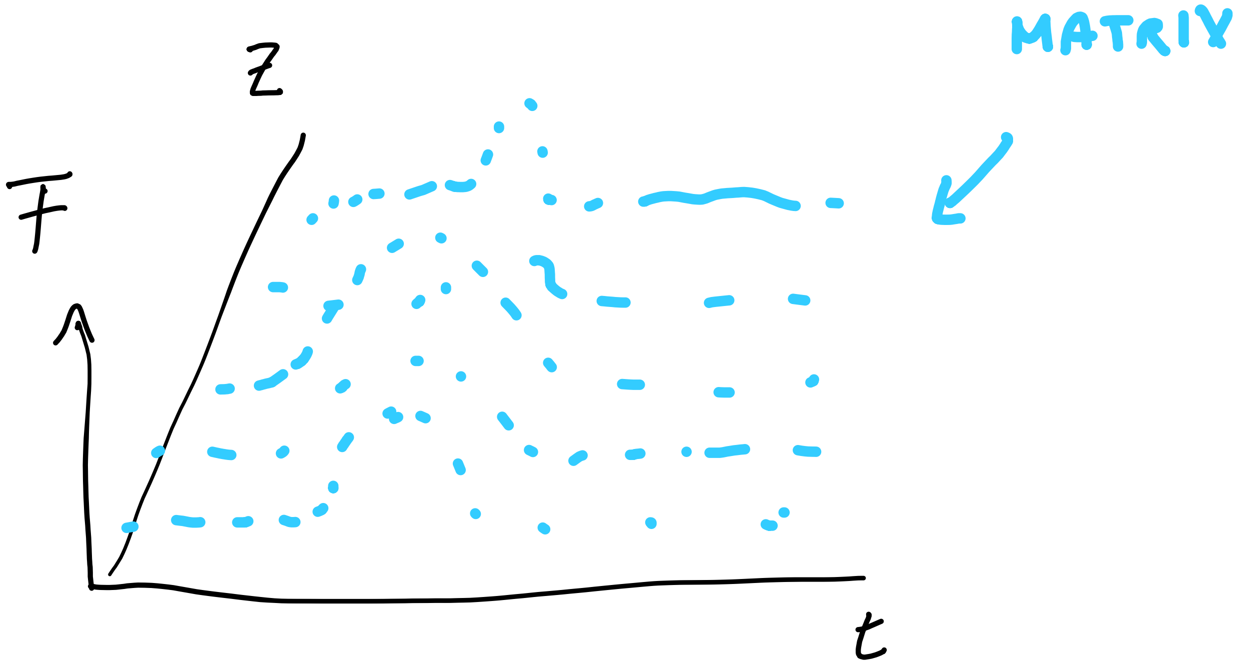
1) First step: dispersion (or viceversa)
2) second step: nonlinear effect

As to the envelope $F(z,t)$, take into account that the profile is usually recorded in a vector or matrix, by sampling F in a good manner (i.e., ref. to Shannon Theorem on sampling).



In general I want to represent and elaborate $F(z,t)$ in the space (z,t), thus we have to take into account matrices





$F(z,t) \xrightarrow{\text{REPRESENTED BY}} F(n_z, n_t)$

In details, as to the dispersive step, how we solve the equation

$$\frac{\partial F(z,t)}{\partial z} = -i \frac{\beta''}{2} \frac{\partial^2 F(z,t)}{\partial t^2}$$

We consider the Fourier Transform of F

$$\hat{F}(z,\omega) = \int_{-\infty}^{+\infty} F(z,t)\, e^{-i\omega}\, dt$$

Thus

$$\frac{\partial \hat{F}(z,w)}{\partial z} = \left(i \frac{w^2}{2} \beta'' \right) \hat{F}(z,w)$$

The equation can be expressed analytically

$$\hat{F}(z+h,w) = \hat{F}(z,w) \cdot \exp\left[+ i \frac{w^2}{2} \beta'' \cdot h \right]$$

$$F(z+h,t) = \mathcal{F}^{-1}\left[\hat{F}(z+h,w) \right]$$

As To nonlinear term, how to solve

$$\frac{\partial F(z,t)}{\partial z} = i\gamma |F|^2 F$$

It's possible to demonstrate that $\underline{|F|}^2$ is z-independent

$$F(z+h,t) = F(z,t) \cdot \exp\left[i\gamma |F(z,t)|^2 \cdot h \right]$$

step in z.

SCHEME FOR NUMERICAL ALGORITHM

1) INITIALIZATON

2) DEFINITION OF TEMPORAL AND SPATIAL COORDINATES
DEFINITION OF FREQUENCY DOMAIN
DEFINITION OF QUANTITIES (ENVELOPES ETC.)

3) FOR : $m : 1 : N_z$
- dispersive step
- nonlinearr step
- save data $F(n_t)$, every n_z

4) DISPLAY THE RESULTS

In next lectures our goal will be the implementation, the numerical implementation of this technique.