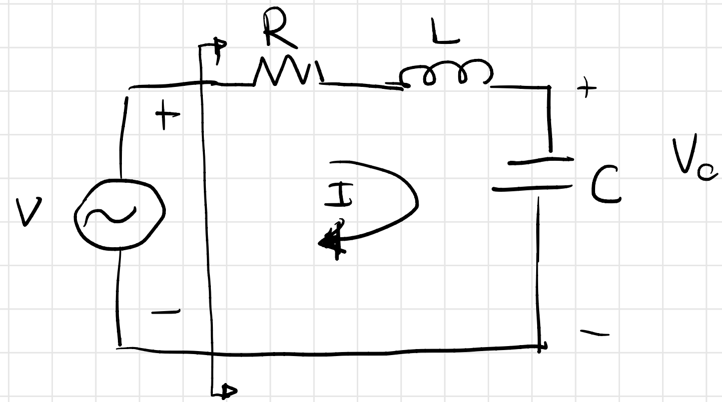
MICROWAVE ENGINEERING

Lecture 22:

Microwave Resonators

series and parallel resonant circuits

SERIES RESONANT CIRCUIT



The input impedance is:

$$Z_{IN} = R + j\omega L - j/\omega C = \underbrace{\frac{2P_{IN}}{|I|^2}}$$

Power delivered to the resonator is

$$\underline{P_{IN}} = \frac{1}{2} V I^* = \frac{1}{2} Z_{IN} |I|^2 = \frac{1}{2} Z_{IN} \left| \frac{V}{Z_{IN}} \right|^2 =$$

$$= \frac{1}{2} |I|^2 \left(\underline{R} + \underline{j\omega L - \frac{j}{\omega C}} \right) = P_{loss} + 2j\omega \left(W_m - W_e \right)$$

Power dissipated by the resistor is:

$$P_{LOSS} = \frac{1}{2}|I|^2 R$$

Stored magnetic energy $\quad W_m = \frac{1}{4}|I|^2 L$

Stored electric energy $\quad W_e = \frac{1}{4}|V_c|^2 C = \frac{1}{4}|I|^2 \frac{1}{\omega^2 C}$

The circuit is at resonance when $\boxed{W_m = W_e}$

$$\Downarrow$$

$$P_{IN} = P_{loss}$$

$$Z_{IN} = \frac{2 P_{IN}}{|I|^2} = \frac{2 P_{loss}}{|I|^2} = \frac{\cancel{2}\,\frac{1}{\cancel{2}}\cancel{|I|^2} R}{\cancel{|I|^2}} = R \quad \text{PURELY REAL}$$

When $W_m = W_e \Rightarrow \frac{1}{\cancel{4}} \cancel{|I|^2} L = \frac{1}{\cancel{4}} \cancel{|I|^2} \frac{1}{\omega^2 C} \Rightarrow \boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$

RESONANCE FREQUENCY

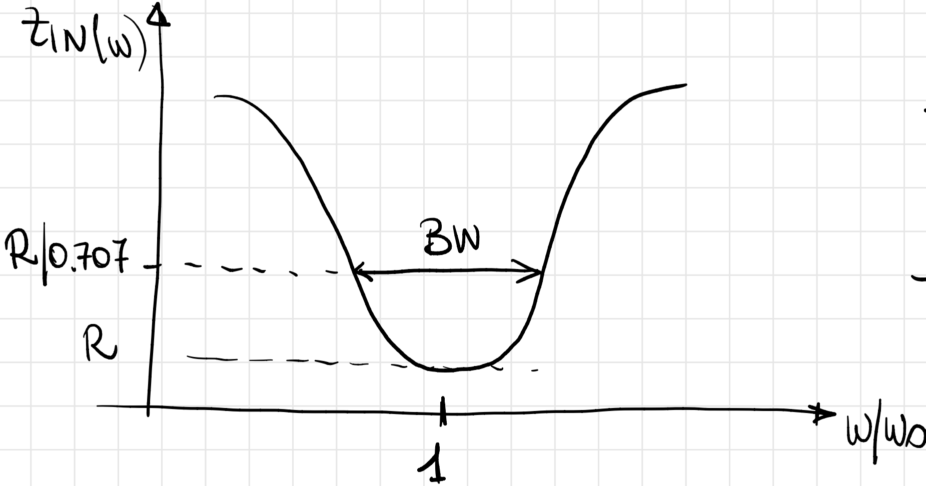
Quality Factor

$$Q = \omega \frac{\text{average energy stored}}{\text{energy loss / sec}} = \omega \frac{W_m + W_e}{P_{loss}}$$

AT resonance :

$$Q = \omega_0 \frac{2 W_m}{P_{loss}} = \omega_0 \frac{\cancel{2} \frac{1}{\cancel{4}} \cancel{|I|^2} L}{\cancel{\frac{1}{2}} \cancel{|I|^2} R} = \omega_0 \frac{L}{R} =$$

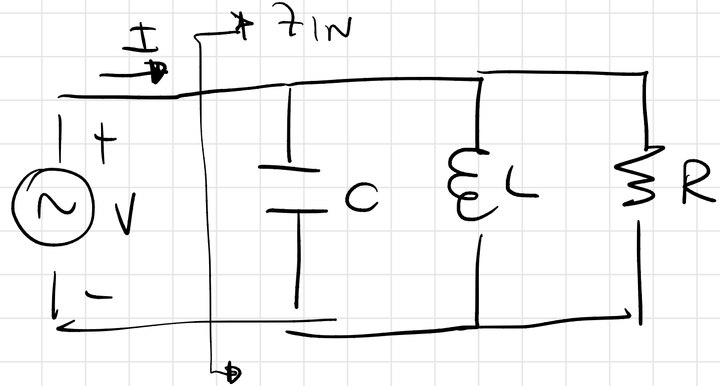
$$= \frac{1}{\sqrt{LC}} \cdot \frac{L}{R} = \frac{1}{R} \sqrt{\frac{L}{C}}$$



$$\boxed{BW = \frac{1}{Q}}$$

PARALLEL RESONANT CIRCUIT

$$Z_{IN} = \left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)^{-1}$$



Power delivered at the resonator is:

$$P_{IN} = \frac{1}{2} V I^* = \frac{1}{2} Z_{IN} |I|^2 = \frac{1}{2} \frac{|V|^2}{Z_{IN}^*} = \frac{1}{2} |V|^2 \left(\frac{1}{R} + \frac{j}{\omega L} - j\omega C \right)$$

$$= P_{loss} + 2j\omega \left(W_m - W_e \right)$$

Power loss $\quad P_{loss} = \frac{1}{2} \frac{|V|^2}{R}$

Stored electric energy $\quad W_e = \frac{1}{4} |V|^2 C$

Stored magnetic energy $\quad W_m = \frac{1}{4} |I|^2 L = \frac{1}{4} |V|^2 \frac{1}{\omega^2 L}$

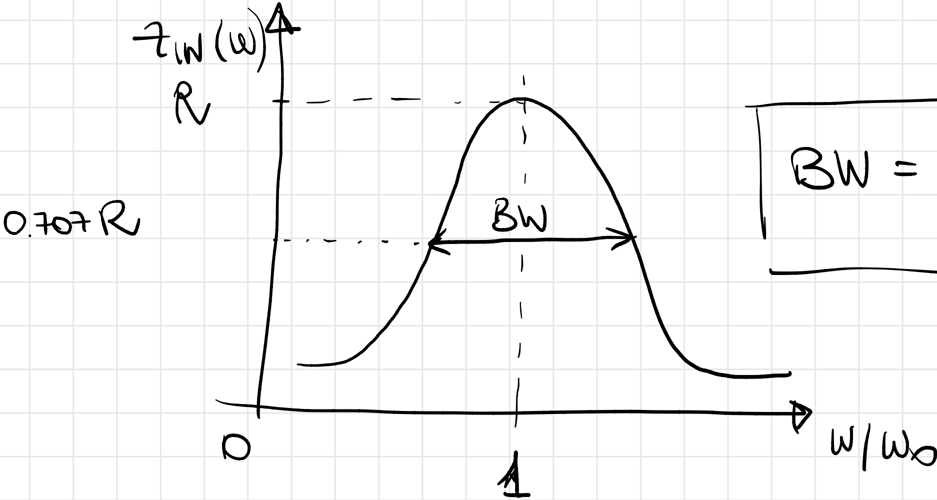
Resonance occurs when $W_m = W_e \Rightarrow \cancel{\frac{1}{4}} |\cancel{V}|^2 C = \cancel{\frac{1}{4}} |\cancel{V}|^2 \frac{1}{\omega^2 L}$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

Quality factor

$$Q = \omega_0 \frac{W_m + W_e}{P_{loss}} = \omega_0 \frac{2 W_m}{P_{loss}} = \omega_0 \frac{\cancel{2} \cancel{\frac{1}{4}} |\cancel{V}|^2 \frac{1}{\omega^2 L}}{\cancel{\frac{1}{2}} \frac{\cancel{|V|^2}}{R}} = \cancel{\omega_0} \frac{1}{\omega_0^{\cancel{2}} L} \cdot R =$$

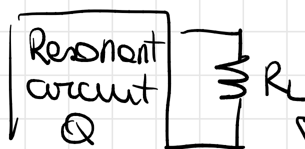
$$= \frac{R}{L} \sqrt{LC} = R \sqrt{\frac{C}{L}}$$



$$\boxed{BW = \frac{1}{Q}}$$

LOADED AND UNLOADED Q

UNLOADED Q $\rightarrow$ INTRINSIC PROPERTY OF THE RESONATOR



can be in series or parallel

we can define the EXTERNAL Q

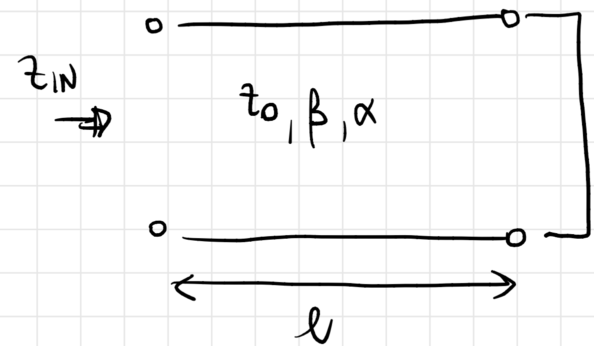
$$Q_E = \begin{cases} \dfrac{\omega_0 L}{R_L} & \text{SERIES} \\ \dfrac{R_L}{\omega_0 L} & \text{PARALLEL} \end{cases}$$

The LOADED Q FACTOR IS THEN:

$$1/Q_L = 1/Q_E + 1/Q$$

TRANSMISSION LINE RESONATORS

- **Short-circuit $\lambda/2$ line**



At $\omega = \omega_0$

$$\ell = \frac{\lambda}{2}$$

$$\lambda = \frac{2\pi}{\beta}$$

$$Z_{IN} = Z_0 \tanh(\alpha + j\beta)\ell = Z_0 \frac{\tanh \alpha\ell + j \tan\beta\ell}{1 + j \tan\beta\ell \tanh\alpha\ell}$$

If we assume to have small losses $\alpha\ell \ll 1 \Rightarrow \alpha\ell \approx \tanh\alpha\ell$

If we let $\omega = \omega_0 + \Delta\omega$ ($\Delta\omega$ very small)

for a TEM line

$$\beta l = \frac{\omega l}{v_p} = \frac{\omega_0 l}{v_p} + \frac{\Delta\omega l}{v_p}$$

we also know $l = \frac{\lambda}{2} = \frac{\pi}{\beta}$ ($\omega = \omega_0$)

so we can write

$$\beta l = \pi + \frac{\Delta\omega l}{v_p} = \pi + \frac{\Delta\omega \pi}{\omega_0}$$

it follows that

$$\tan \beta l = \tan\Big(\underbrace{\pi}_{0} + \frac{\Delta\omega\pi}{\omega_0}\Big) = \tan \frac{\Delta\omega\pi}{\omega_0} \simeq \frac{\Delta\omega\pi}{\omega_0}$$

Replacing in the expression of the input impedance:

$$z_{IN} = z_0 \frac{\tanh \alpha l + j \tan \beta l}{1 + j \tan \beta l \tanh \alpha l} \simeq$$

$$\simeq z_0 \frac{\alpha l + j\left(\frac{\Delta\omega\pi}{\omega_0}\right)}{1 + j\alpha l\left(\frac{\Delta\omega\pi}{\omega_0}\right)} \simeq z_0\left(\alpha l + j\frac{\Delta\omega\pi}{\omega_0}\right)$$

$$\frac{\Delta\omega\alpha l}{\omega_0} << 1$$

The input impedance is in a form of :

$$Z_{IN} = R + 2jL\Delta\omega$$

$\Leftarrow$ Impedance of an RLC resonant circuit

$$R = Z_0 \alpha \ell \qquad L = \frac{Z_0 \pi}{2\omega_0}$$

At resonance $\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{\omega_0^2 L}$

The $\frac{\lambda}{2}$ short-circuited TL resonates at $\Delta\omega = 0$ $\left(\ell = \frac{\lambda}{2}\right)$

$$\Rightarrow Z_{IN} = R + Z_0 \alpha \ell$$

circuit resonates for $\ell = n\frac{\lambda}{2}$

$n = 1, 2, 3, \dots$

Quality factor

$$Q = \frac{\omega_0 L}{R} = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha}$$

$\beta\ell = \pi$ at the first resonance

- **Short-circuited $\frac{\lambda}{4}$ line**

$$Z_{IN} = Z_0 \tanh(\alpha + j\beta)\ell =$$

$$Z_0 \frac{\tanh \alpha\ell + j \tan\beta\ell}{1 + j \tan\beta\ell \tanh\alpha\ell}$$

multiply Z_{IN} by $-j \cot\beta\ell$

$$Z_{IN} = Z_0 \frac{1 - j \tanh \alpha \ell \cot \beta \ell}{\tanh \alpha \ell - j \cot \beta \ell}$$

if $\ell = \frac{\lambda}{4}$ at $\omega = \omega_0$ and we let $\omega \doteq \omega_0 + \Delta\omega$ then

for a TEM line:

$$\beta \ell = \frac{\omega_0 \ell}{v_p} + \frac{\Delta\omega \ell}{v_p} = \frac{\pi}{2} + \frac{\pi \Delta\omega}{2\omega_0}$$

so that $\cot \beta \ell = \cot\left(\frac{\pi}{2} + \frac{\pi \Delta\omega}{2\omega_0}\right) = -\tan \frac{\pi \Delta\omega}{2\omega_0} \simeq$

$$\simeq -\frac{\pi \Delta\omega}{2\omega_0}$$

We are also assuming small losses $\alpha l \simeq \tanh \alpha l$

We can write

$$Z_{IN} = Z_0 \frac{1 + \overbrace{j \alpha l \frac{\pi \Delta \omega}{2\omega_0}}^{<<1}}{\alpha l + j\pi \frac{\Delta\omega}{2\omega_0}} \simeq \frac{Z_0}{\alpha l + j\pi \frac{\Delta \omega}{2\omega_0}}$$

This type of impedance is typical of parallel RLC circuits

$$Z_{IN} = \frac{1}{\frac{1}{R} + 2j\Delta\omega C}$$

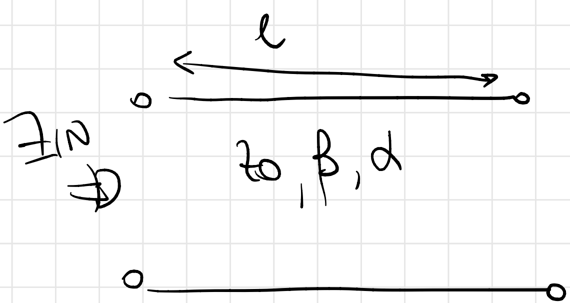
$$R = Z_0/\alpha l \qquad C = \frac{\pi}{4\omega_0 Z_0}$$

The value of inductance can be calculated as $L = \frac{1}{\omega_0^2 C}$

The Quality factor:

$$Q = \omega_0 RC = \frac{\pi}{4\alpha l} = \frac{\beta}{2\alpha}$$

• Open-circuited $\lambda/2$ line



$$Z_{IN} = Z_0 \coth(\alpha + j\beta)l =$$

$$= Z_0 \frac{1 + j\tan\beta l \tanh\alpha l}{\tanh\alpha l + j\tan\beta l}$$

We assume $l = \frac{\lambda}{2}$ and $\omega = \omega_0 + \Delta\omega \Rightarrow \beta l = \pi + \pi\frac{\Delta\omega}{\omega_0} \Rightarrow \tan\beta l \simeq \frac{\pi\Delta\omega}{\omega_0}$

$\alpha l \simeq \tanh\alpha l$ it follows:

$$Z_{IN} = \frac{Z_0}{\alpha l + j\left(\frac{\Delta \omega \pi}{\omega_0}\right)}$$

$$R = \frac{Z_0}{\alpha l} \qquad C = \frac{\pi}{2\omega_0 Z_0}$$

$$L = \frac{1}{\omega_0^2 C} = \frac{2\,\cancel{\omega_0}\,Z_0}{\pi\,\omega_0^{\cancel{2}}} = \frac{2Z_0}{\pi \omega_0}$$

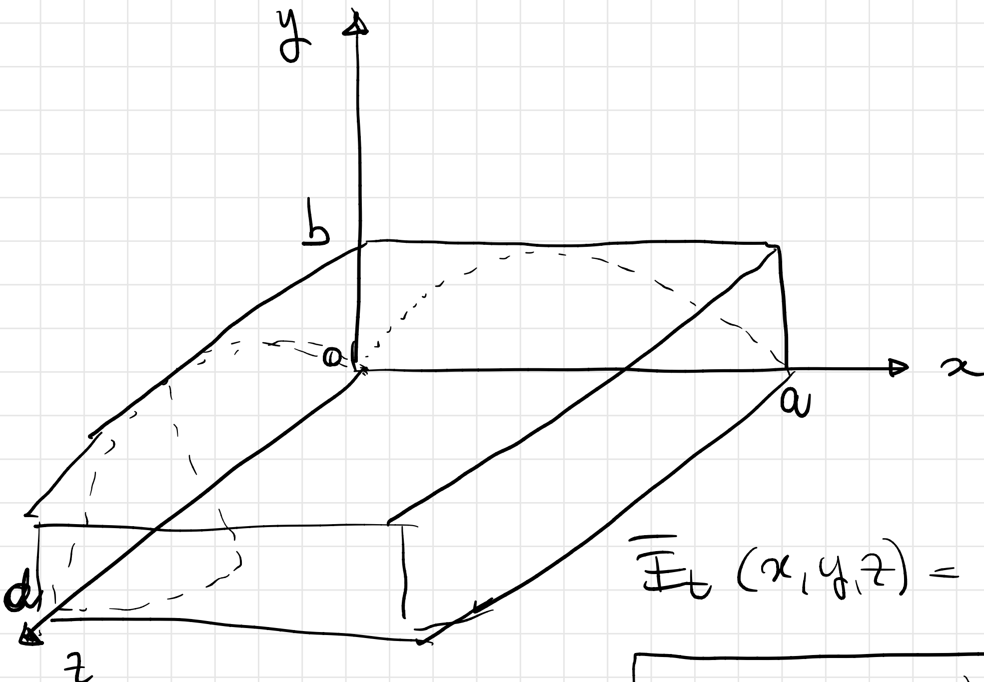
Quality factor Q:

$$Q = \omega_0 R C = \frac{\pi}{2\alpha l} = \frac{\beta}{2\alpha}$$

$l = \frac{\pi}{\beta}$ at resonance

RECTANGULAR WAVEGUIDE CAVITIES

Resonant frequencies are found assuming WG is lossless.



$$\bar{E}_t(x,y,z) = \bar{e}(x,y)\left[A^+ e^{-j\beta_{mn}z} + A^- e^{j\beta_{mn}z}\right]$$

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad k = \omega\sqrt{\mu\varepsilon}$$

By imposing that $\bar{E}_t = 0$ at $z = 0$ $\Rightarrow$ $A^+ = -A^-$

At $z = d$ we have

$$\bar{E}_t(x, y, d) = -\bar{e}(x, y)\, A^+ \, 2j \sin \beta_{mn} d = 0$$

$$\Downarrow$$

$$\beta_{mn} d = l\pi \qquad l = 1, 2, 3 \ldots$$

$$\Downarrow$$

The waveguide version of the $\frac{\lambda}{2}$ TL/SC

The resonant wave number $k_{mnl} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2}$

The resonant frequencies for $TE_{mn\ell}$ or $TM_{mn\ell}$ modes are:

$$f_{mn\ell} = \frac{c\,k_{mn\ell}}{2\pi\sqrt{\mu_r \varepsilon_r}} = \frac{c}{2\pi\sqrt{\mu_r \varepsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}$$

If $b < a < d$ → THE DOMINANT MODE IS TE_{101} for TE

TM_{110} for TM

Quality factor of $TE_{10\ell}$ mode

The stored electric energy is: $W_e = \frac{\varepsilon}{4}\int_V E_y E_y^* \, dV = \frac{\varepsilon abd}{16} E_0^2$

The <u>stored magnetic energy</u> is:

$$W_m = \frac{\mu}{4}\int_V \left(H_x H_x^* + H_z H_z^* \right) dV = \frac{\mu abd}{16} E_0^2 \left(\frac{1}{Z_{TE}^2} + \frac{\pi^2}{k^2\eta^2 a^2} \right)$$

$$Z_{TE} = \frac{k\eta}{\beta}$$

$$\beta = \beta_{10} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

$$W_m = \frac{\mu abd}{16} E_0^2 \left(\frac{\beta^2}{k^2\eta^2} + \frac{\pi^2}{k^2\eta^2 a^2} \right) =$$

$$\frac{\mu abd}{16} E_0^2 \left(\frac{\cancel{k^2} - \cancel{\left(\frac{\pi}{a}\right)^2} + \cancel{\left(\frac{\pi}{a}\right)^2}}{\cancel{k^2}\eta^2} \right) = \frac{\mu abd}{16} \frac{E_0^2}{\eta^2} = \frac{abd}{16} E_0^2 \varepsilon$$

$\eta^2 \to \mu/\varepsilon$

$= W_e!$

For small losses in the conducting walls we get:

$$P_c = \frac{R_s}{2} \int_{walls} |H_t|^2 \, dS \qquad R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}}$$

$H_x^2 + H_z^2$

$$\boxed{P_c = \frac{R_s E_0^2 \lambda^2}{8\eta^2} \left(\frac{l^2 ab}{d^2} + \frac{bd}{a^2} + \frac{l^2 a}{2d} + \frac{d}{2a} \right)}$$

Then: $Q_c = \dfrac{2\omega_0 W_e}{P_c}$

The power lost in the dielectric $\quad (\sigma = \omega \varepsilon'' = \omega \varepsilon_r \varepsilon_0 \tan\delta)$

$$P_d = \frac{1}{2} \int_V J \cdot E^* dV = \frac{\omega \varepsilon''}{2} \int_V |E|^2 dV =$$

$$= \frac{abd\, \omega \varepsilon'' |E_0|^2}{8}$$

then

$$Q_d = \frac{2\omega_0 W_e}{P_d} = \frac{\varepsilon'}{\varepsilon''} = \frac{1}{\tan\delta}$$

The total power loss is $P_{loss} = P_c + P_d$ and the quality factor is

$$Q = \left(\frac{1}{Q_c} + \frac{1}{Q_D} \right)^{-1}$$