Set #8

24.

Electrons, protons and neutrons have spin 1/2. There are two values of m_s for s=1/2. These are m_s=1/2 and m_s=-1/2. The corresponding eigenstates $\chi_\uparrow$ (+$\hbar$/2) and $\chi_\downarrow$ (-$\hbar$\2) obey the following eigenvalue eq.s

$$S^2\chi_\uparrow = \frac{3\hbar^2}{4}\chi_\uparrow \quad S_z\chi_\uparrow = \hbar/2\chi_\uparrow$$

One can conveniently represent the two eigenstates as column matrices

$$\chi_\uparrow = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \chi_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$$

a. Compute the matrix form of S^{2}.
b. Compute the matrix form of S_z.
c. Are these two eigenstates normalized?

Set #3

25. One can conveniently represent the two spin-eigenstates as column matrices

$$\chi_\uparrow = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \chi_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$$

Using $\chi_\uparrow$ and $\chi_\downarrow$ as spin basis states, the most general state of a spin 1/2 particle can be expressed as two-element column matrix

$$\chi = \begin{pmatrix}a\\b\end{pmatrix} = a\ \chi_\uparrow + b\ \chi_\downarrow$$

a. If you measured S_z on a particle in the state χ what is the probability of measuring +$\hbar$/2 or -$\hbar$/2?

b. What is the sum of the probabilities?

c. What does it mean?

Suppose the electron is a classical solid "sphere" with radius $r \sim 5 \times 10^{-17} m$, what would be the speed a point placed on the equator of the sphere? Is such a spherical model for the electron sound?

1. a  Mom. ang. sfera $\quad S = I\omega$

$I = \frac{2}{5} m r^2 \quad$ Mom. inertia

$v = \omega r \quad$ (vel. all'equatore)

Per mom. ang. intrinseco (spin) di un el. uno ha: $\quad S = \frac{\sqrt{3}}{2}\hbar \quad (s = 1/2)$

quindi: $\quad \frac{\sqrt{3}}{2}\hbar = \frac{2}{5} m r^{\cancel{2}} \frac{v}{\cancel{r}} \Rightarrow v = \frac{\hbar}{mr}\frac{5\sqrt{3}}{4} = \frac{\hbar c}{mc^2}\frac{1}{r}\frac{5\sqrt{3}}{4}c$

$$= \frac{1973\,eV\,Å}{0.5\times10^{6}\,eV}\;\frac{5\sqrt{3}/4}{5\times10^{-17}m}\;c = 1.7\times10^{4}\,c$$

b. Il modello a sfera dell'elettrone "non funziona" poiché otteniamo una velocità $v > c$. Note: model independent of charge.

Electrons, protons and neutrons have spin 1/2. There are two values of m_s for s=1/2. These are m_s=1/2 and m_s=-1/2. The corresponding eigentstates $\chi_\uparrow$ ($\hbar/2$) and $\chi_\downarrow$ ($-\hbar/2$) obey the following eigenvalue eq.s:

$$S^2\chi_\uparrow = \frac{3\hbar^2}{4}\chi_\uparrow \quad S_z\chi_\uparrow = \hbar/2\chi_\uparrow$$
$$S^2\chi_\downarrow = \frac{3\hbar^2}{4}\chi_\downarrow \quad S_z\chi_\downarrow = -\hbar/2\chi_\downarrow$$

One can conveniently represent the two eigenstates as column matrices

$$\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

a. Compute the matrix form of S^2.

b. Compute the matrix form of S_z.

c. Are these two eigenstates normalized?

4.

a) Write $S^2 = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$

Eq. $S^2 \chi_\uparrow = \frac{3\hbar^2}{4} \chi_\uparrow \rightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3\hbar^2}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} c \\ e \end{pmatrix} = \begin{pmatrix} 3\hbar^2/4 \\ 0 \end{pmatrix}$

Eq. $S^2 \chi_\downarrow = \frac{3\hbar^2}{4} \chi_\downarrow \rightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3\hbar^2}{4}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 3\hbar^2/4 \end{pmatrix}$

Thus:

$$S^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b.

a) Write $S_z = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$

Eq. $S_z \chi_\uparrow = \frac{\hbar}{2} \chi_\uparrow \rightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} c \\ e \end{pmatrix} = \begin{pmatrix} \hbar/2 \\ 0 \end{pmatrix}$

Eq. $S_z \chi_\downarrow = -\frac{\hbar}{2} \chi_\downarrow \rightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ -\hbar/2 \end{pmatrix}$

Thus:

$$S_z = \hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

c. orthogonality cond. $\int d\vec{r}\, \psi^*_{E_1}(\vec{r})\, \psi_{E_2}(\vec{r}) = \delta_{E_1,E_2}$ $\nearrow 1$ (Spatial eigenstates) $\searrow 0$

Similarly $(1,0)^*\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ ok 'orthogonal'

$\boxed{\text{Transpose} \oplus \text{C.C.}}$

$(1,0)^*\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ ok Normalized.

We only have two states "$\chi_\uparrow$" and "$\chi_\downarrow$", so this is the only orthogonality condition.

One can conveniently represent the two spin-igenstates as column matrices

$$\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Using $\chi_\uparrow$ and $\chi_\downarrow$ as spin basis states, the general state of a spin 1/2 particle can be expressed as two-element column matrix

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\ \chi_\uparrow + b\ \chi_\downarrow$$

a. If you measure S_z on a particle in the general state χ what is the probability of measuring $+\hbar/2$ or $-\hbar/2$?

b. What is the sum of the probabilities?

c. What does it mean?

a. $$\underbrace{\left[a^* (1,0)^* + b^* (0,1)^*\right]}_{\chi^{T*}} \underbrace{\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}}_{S_z} \underbrace{\left[a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]}_{\chi} =$$

= expectation value of S_z in the general state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$= \left[a^* (1,0) + b^* (0,1)\right] \left[\begin{pmatrix} a\hbar/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -b\hbar/2 \end{pmatrix}\right] = |a|^2 \frac{\hbar}{2} + |b|^2 \left(-\frac{\hbar}{2}\right)$$

$|a|^2$ = prob. of measuring $+\hbar/2$ (spin-up)

$|b|^2$ = " " " $-\hbar/2$ (spin-down)

b. Since for an electron we only have two possible orientations along z, it follows that $|a|^2 + |b|^2$ is the sum of the probabilities, hence

$$|a|^2 + |b|^2 = 1$$

c. If I measured S_z on the state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, the prob. of observing the electron in the spin-up state is $|a|^2$, while the prob. of observing the el. in the spin-down state is $|b|^2$.

a. $\underbrace{\left[a^* (1,0)^* + b^* (0,1)^*\right]}_{\chi^{T*}} \underbrace{\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}}_{S_z} \underbrace{\left[a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]}_{\chi} =$

= expectation value of S_z in the general state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$= \left[a^* (1,0) + b^* (0,1)\right] \left[\begin{pmatrix} a\hbar/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -b\hbar/2 \end{pmatrix}\right] = |a|^2 \frac{\hbar}{2} + |b|^2 \left(-\frac{\hbar}{2}\right)$$

$|a|^2$ = prob. of measuring $+\hbar/2$ (spin-up)

$|b|^2$ = " " " $-\hbar/2$ (spin-down)

b. Since for an electron we only have two possible orientations along z, it follows that $|a|^2 + |b|^2$ is the sum of the probabilities, hence

$$|a|^2 + |b|^2 = 1$$

c. If I measured S_z on the state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, the prob. of observing the electron in the spin-up state is $|a|^2$, while the prob. of observing the el. in the spin-down state is $|b|^2$.

• Compute the angles that the intrinsic angular momentum (spin) forms about a given direction z. These angles define the two eigenstates $\chi_\uparrow$ e $\chi_\downarrow$ of the operator S_z.

Write the relevant eigenvalue eq. for $\chi_\uparrow$ e $\chi_\downarrow$ and the orthogonality condition.

Write the total electron spin wavefunction: this is a spin q-bit (spin quantum-bit)

Write the relation the coefficients of the spin wavefunction have to satisfy.

Briefly explain how such a q-bit differ from a classical bit.

- Angles → done in class.
- Eigenvalue Eq. $S_z \chi_\uparrow = \frac{\hbar}{2} \chi_\uparrow$; $S_z \chi_\downarrow = -\frac{\hbar}{2} \chi_\downarrow$
- ORTHOG. $(\chi_\uparrow, \chi_\downarrow) = 0$; $(\chi_\uparrow, \chi_\uparrow) = (\chi_\downarrow, \chi_\downarrow) = 1$

  KET: $\langle \chi_\uparrow | \chi_\downarrow \rangle = 0$; $\langle \chi_\uparrow | \chi_\uparrow \rangle = \langle \chi_\downarrow | \chi_\downarrow \rangle = 1$.
- We have only two state $|\chi_\uparrow\rangle$ and $|\chi_\downarrow\rangle$ (two-dim. discrete space)

  TOTAL w/f: $|\psi\rangle = \alpha |\chi_\uparrow\rangle + \beta |\chi_\downarrow\rangle$ $\{\alpha, \beta\}$ → complex
- Normalization: $\langle \psi | \psi \rangle = 1$ implies

$$(\langle \chi_\uparrow | \alpha^* + \langle \chi_\downarrow | \beta^*)(\alpha |\chi_\uparrow\rangle + \beta |\chi_\downarrow\rangle) =$$

$$= \alpha^* \alpha \langle \uparrow | \uparrow \rangle + \beta^* \beta \langle \downarrow | \downarrow \rangle + \dots = |\alpha|^2 + |\beta|^2$$

- Compare TOTAL w/f $|\psi\rangle$ (quantum bit) w/ a two-bit STATE $(0,1)$ ....

Use suitable numerics (Python, Mathematica, Matlab etc.) to:

a. show that "common" eigenstates of L^2 and L_z are spherical harmonics $Y_l^m(\theta, \phi)$ i.e., the Hilbert space for the orbital angular momentum.

b. find the corresponding eigenvalues.

Hint: Re-write the cartesian components expressions for L^2 and L_z (in class) in terms of spherical polar coordinates.

For convenience, we list the spherical harmonics for $\ell = 0, 1, 2$ and non-negative values of m.

$$\ell = 0\,, \qquad Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$

$$\ell = 1\,, \qquad \begin{cases} Y_1^1(\theta, \phi) = -\sqrt{\dfrac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y_1^0(\theta, \phi) = \sqrt{\dfrac{3}{4\pi}} \cos\theta \end{cases}$$

$$\ell = 2\,, \qquad \begin{cases} Y_2^2(\theta, \phi) = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}} \sin^2\theta e^{2i\phi} \\ Y_2^1(\theta, \phi) = -\sqrt{\dfrac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi} \\ Y_2^0(\theta, \phi) = \dfrac{1}{2}\sqrt{\dfrac{5}{4\pi}} (3\cos^2\theta - 1) \end{cases}$$

```
Clear[f]

(*SQUARE LX2 OF THE x-COMPONENT OF THE ANGULAR MOMENTUM - POLAR COORDINATES*)
```

In[1]:=
```
Lx1 = -Sin[ϕ] ∂θ f[θ, ϕ] - Cos[ϕ] Cot[θ] ∂ϕ f[θ, ϕ]
Lx2 = -Sin[ϕ] ∂θ Lx1 - Cos[ϕ] Cot[θ] ∂ϕ Lx1
```

Out[1]= $-\mathrm{Cos}[\phi]\,\mathrm{Cot}[\theta]\,f^{(0,1)}[\theta,\phi]-\mathrm{Sin}[\phi]\,f^{(1,0)}[\theta,\phi]$

Out[2]= $$-\mathrm{Cos}[\phi]\,\mathrm{Cot}[\theta]\,\big(\mathrm{Cot}[\theta]\,\mathrm{Sin}[\phi]\,f^{(0,1)}[\theta,\phi]-\mathrm{Cos}[\phi]\,\mathrm{Cot}[\theta]\,f^{(0,2)}[\theta,\phi]-\mathrm{Cos}[\phi]\,f^{(1,0)}[\theta,\phi]-\mathrm{Sin}[\phi]\,f^{(1,1)}[\theta,\phi]\big)-\mathrm{Sin}[\phi]\,\big(\mathrm{Cos}[\phi]\,\mathrm{Csc}[\theta]^2\,f^{(0,1)}[\theta,\phi]-\mathrm{Cos}[\phi]\,\mathrm{Cot}[\theta]\,f^{(1,1)}[\theta,\phi]-\mathrm{Sin}[\phi]\,f^{(2,0)}[\theta,\phi]\big)$$

```
(*SQUARE LY2 OF THE y-COMPONENT OF THE ANGULAR MOMENTUM - POLAR COORDINATES*)
```

In[3]:=
```
Ly1 = Cos[ϕ] ∂θ f[θ, ϕ] - Sin[ϕ] Cot[θ] ∂ϕ f[θ, ϕ];
Ly2 = Cos[ϕ] ∂θ Ly1 - Sin[ϕ] Cot[θ] ∂ϕ Ly1;

(*SQUARE LZ2 OF THE z-COMPONENT OF THE ANGULAR MOMENTUM - POLAR COORDINATES*)
```

In[5]:=
```
Lz1 = ∂ϕ f[θ, ϕ];
Lz2 = ∂ϕ Lz1;

(*TOTAL ANGULAR MOMENTUM SQUARE - POLAR COORDINATES*)
L2 = -ℏ^2 (Lx2 + Ly2 + Lz2);
Simplify[L2]
```

Out[8]= $-\hbar^2\,\big(\mathrm{Csc}[\theta]^2\,f^{(0,2)}[\theta,\phi]+\mathrm{Cot}[\theta]\,f^{(1,0)}[\theta,\phi]+f^{(2,0)}[\theta,\phi]\big)$

In[9]:=
```
(*z-COMPONENT OF THE ANGULAR MOMENTUM - POLAR COORDINATES*)
Lz = -I ℏ Lz1;
Simplify[Lz]
```

Out[10]= $-i\,\hbar\,f^{(0,1)}[\theta,\phi]$

(*APPLY to spherical harmonic $Y_l^m(\theta,\phi)\to Y_1^{-1}(\theta,\phi)$ *)

In[17]:=
```
f[θ_, ϕ_] = SphericalHarmonicY[1, -1, θ, ϕ]
Simplify[L2]
Simplify[Lz]
```

Out[17]= $\frac{1}{2}\,e^{-i\phi}\,\sqrt{\frac{3}{2\pi}}\,\mathrm{Sin}[\theta]$

Out[18]= $e^{-i\phi}\,\sqrt{\frac{3}{2\pi}}\,\hbar^2\,\mathrm{Sin}[\theta]$

Out[19]= $-\frac{1}{2}\,e^{-i\phi}\,\sqrt{\frac{3}{2\pi}}\,\hbar\,\mathrm{Sin}[\theta]$

```
(*FIND THE EIGENVALUES*)
```

In[15]:= $\text{Simplify}\left[\dfrac{\text{L2}}{\frac{1}{2}\,e^{-i\phi}\sqrt{\frac{3}{2\pi}}\,\text{Sin}[\theta]}\right]$

$\text{Simplify}\left[\dfrac{\text{Lz}}{\frac{1}{2}\,e^{-i\phi}\sqrt{\frac{3}{2\pi}}\,\text{Sin}[\theta]}\right]$

Out[15]= $2\,\hbar^2$

Out[16]= $-\hbar$

```
(*APPLY to spherical harmonic  Y_l^m(θ,ϕ)->Y_2^-1(θ,ϕ)*)
```

In[20]:=
```
f[θ_, ϕ_] = SphericalHarmonicY[2, -1, θ, ϕ]
Simplify[L2]
Simplify[Lz]
```

Out[20]= $\frac{1}{2}\,e^{-i\phi}\sqrt{\frac{15}{2\pi}}\,\text{Cos}[\theta]\,\text{Sin}[\theta]$

Out[21]= $3\,e^{-i\phi}\sqrt{\frac{15}{2\pi}}\,\hbar^2\,\text{Cos}[\theta]\,\text{Sin}[\theta]$

Out[22]= $-\frac{1}{2}\,e^{-i\phi}\sqrt{\frac{15}{2\pi}}\,\hbar\,\text{Cos}[\theta]\,\text{Sin}[\theta]$

```
(*FIND THE EIGENVALUES*)
```

In[23]:= $\text{Simplify}\left[\dfrac{\text{L2}}{\frac{1}{2}\,e^{-i\phi}\sqrt{\frac{15}{2\pi}}\,\text{Cos}[\theta]\,\text{Sin}[\theta]}\right]$

$\text{Simplify}\left[\dfrac{\text{Lz}}{\frac{1}{2}\,e^{-i\phi}\sqrt{\frac{15}{2\pi}}\,\text{Cos}[\theta]\,\text{Sin}[\theta]}\right]$

Out[23]= $6\,\hbar^2$

Out[24]= $-\hbar$

In[15]:= $\text{Simplify}\left[\dfrac{\text{L2}}{\frac{1}{2}\, e^{-i\,\phi} \sqrt{\frac{3}{2\,\pi}}\ \text{Sin}[\theta]}\right]$

$\text{Simplify}\left[\dfrac{\text{Lz}}{\frac{1}{2}\, e^{-i\,\phi} \sqrt{\frac{3}{2\,\pi}}\ \text{Sin}[\theta]}\right]$

Out[15]= $2\,\hbar^2$

Out[16]= $-\hbar$

```
(*APPLY to spherical harmonic  Y_l^m(θ,ϕ)->Y_2^-1(θ,ϕ)*)
```

In[20]:= f[θ_, ϕ_] = SphericalHarmonicY[2, -1, θ, ϕ]
Simplify[L2]
Simplify[Lz]

Out[20]= $\frac{1}{2}\, e^{-i\,\phi} \sqrt{\frac{15}{2\,\pi}}\ \text{Cos}[\theta]\ \text{Sin}[\theta]$

Out[21]= $3\, e^{-i\,\phi} \sqrt{\frac{15}{2\,\pi}}\ \hbar^2\, \text{Cos}[\theta]\ \text{Sin}[\theta]$

Out[22]= $-\frac{1}{2}\, e^{-i\,\phi} \sqrt{\frac{15}{2\,\pi}}\ \hbar\, \text{Cos}[\theta]\ \text{Sin}[\theta]$

```
(*FIND THE EIGENVALUES*)
```

In[23]:= $\text{Simplify}\left[\dfrac{\text{L2}}{\frac{1}{2}\, e^{-i\,\phi} \sqrt{\frac{15}{2\,\pi}}\ \text{Cos}[\theta]\ \text{Sin}[\theta]}\right]$

$\text{Simplify}\left[\dfrac{\text{Lz}}{\frac{1}{2}\, e^{-i\,\phi} \sqrt{\frac{15}{2\,\pi}}\ \text{Cos}[\theta]\ \text{Sin}[\theta]}\right]$

Out[23]= $6\,\hbar^2$

Out[24]= $-\hbar$