

## 4. The z-Transform and Discrete-Time LTI Systems

### 4.1. Introduction

In [Chap. 3](#) we introduced the Laplace transform. In this chapter we present the z-transform, which is the discrete-time counterpart of the Laplace transform. The z-transform is introduced to represent discrete-time signals (or sequences) in the z-domain (z is a complex variable), and the concept of the system function for a discrete-time LTI system will be described. The Laplace transform converts integrodifferential equations into algebraic equations. In a similar manner, the z-transform converts difference equations into algebraic equations, thereby simplifying the analysis of discrete-time systems.

The properties of the z-transform closely parallel those of the Laplace transform. However, we will see some important distinctions between the z-transform and the Laplace transform.

### 4.2. The z-Transform

In [Sec. 2.8](#) we saw that for a discrete-time LTI system with impulse response  $h[n]$ , the output  $y[n]$  of the system to the complex exponential input of the form  $z^n$  is

$$y[n] = \mathbf{T}\{z^n\} = H(z)z^n \quad (4.1)$$

where

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad (4.2)$$

#### 4.2.1. A. Definition:

The function  $H(z)$  in [Eq. \(4.2\)](#) is referred to as the z-transform of  $h[n]$ . For a general discrete-time signal  $x[n]$ , the z-transform  $X(z)$  is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (4.3)$$

The variable z is generally complex-valued and is expressed in polar form as

$$z = re^{j\Omega} \quad (4.4)$$

where  $r$  is the magnitude of  $z$  and  $\Omega$  is the angle of  $z$ . The z-transform defined in [Eq. \(4.3\)](#) is often called the *bilateral* (or *two-sided*) z-transform in contrast to the *unilateral* (or *one-sided*) z-transform, which is defined as

$$X_I(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

(4.5)

Clearly the bilateral and unilateral z-transforms are equivalent only if  $x[n] = 0$  for  $n < 0$ . The unilateral z-transform is discussed in [Sec. 4.8](#). We will omit the word "bilateral" except where it is needed to avoid ambiguity.

As in the case of the Laplace transform, [Eq. \(4.3\)](#) is sometimes considered an operator that transforms a sequence  $x[n]$  into a function  $X(z)$ , symbolically represented by

$$X(z) = \mathcal{Z}\{x[n]\}$$

(4.6)

The  $x[n]$  and  $X(z)$  are said to form a z-transform pair denoted as

$$x[n] \leftrightarrow X(z)$$

(4.7)

## 4.2.2. B. The Region of Convergence:

As in the case of the Laplace transform, the range of values of the complex variable  $z$  for which the z-transform converges is called the region of convergence. To illustrate the z-transform and the associated ROC let us consider some examples.

**EXAMPLE 4.1** Consider the sequence

$$x[n] = a^n u[n] \quad a \text{ real}$$

(4.8)

Then by [Eq. \(4.3\)](#) the z-transform of  $x[n]$  is

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For the convergence of  $X(z)$  we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

Thus, the ROC is the range of values of  $z$  for which  $|az^{-1}| < 1$  or, equivalently,  $|z| > |a|$ . Then

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

(4.9)

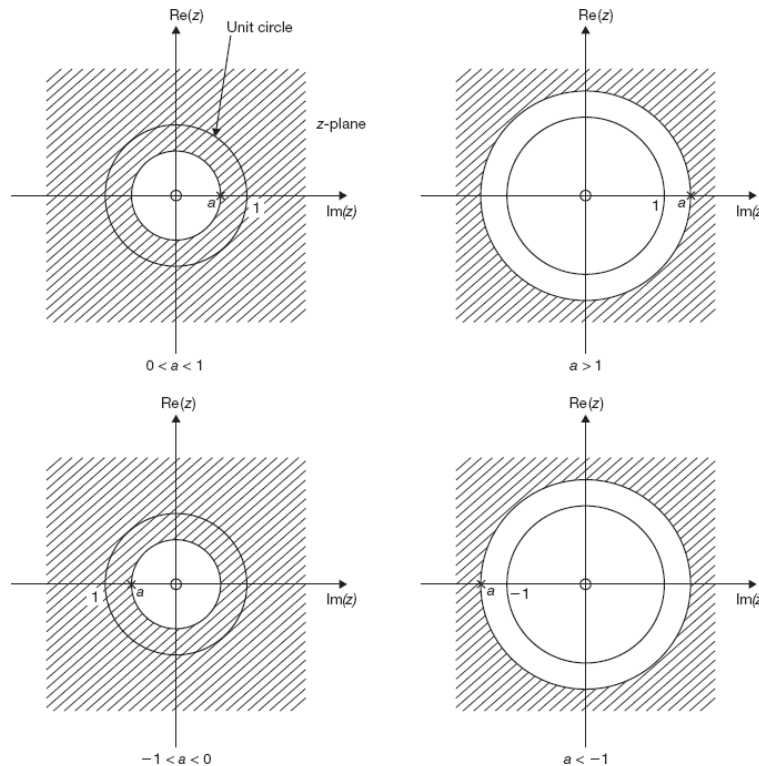
Alternatively, by multiplying the numerator and denominator of [Eq. \(4.9\)](#) by  $z$ , we may write  $X(z)$  as

$$X(z) = \frac{z}{z - a} \quad |z| > |a|$$

(4.10)

Both forms of  $X(z)$  in Eqs. (4.9) and (4.10) are useful depending upon the application. From Eq. (4.10) we see that  $X(z)$  is a rational function of  $z$ . Consequently, just as with rational Laplace transforms, it can be characterized by its zeros (the roots of the numerator polynomial) and its poles (the roots of the denominator polynomial). From Eq. (4.10) we see that there is one zero at  $z = 0$  and one pole at  $z = a$ . The ROC and the pole-zero plot for this example are shown in Fig. 4-1. In  $z$ -transform applications, the complex plane is commonly referred to as the  $z$ -plane.

Figure 4-1 ROC of the form  $|z| > |a|$ .



**EXAMPLE 4.2** Consider the sequence

$$x[n] = -a^n u[-n-1] \quad (4.11)$$

Its  $z$ -transform  $X(z)$  is given by (Prob. 4.1)

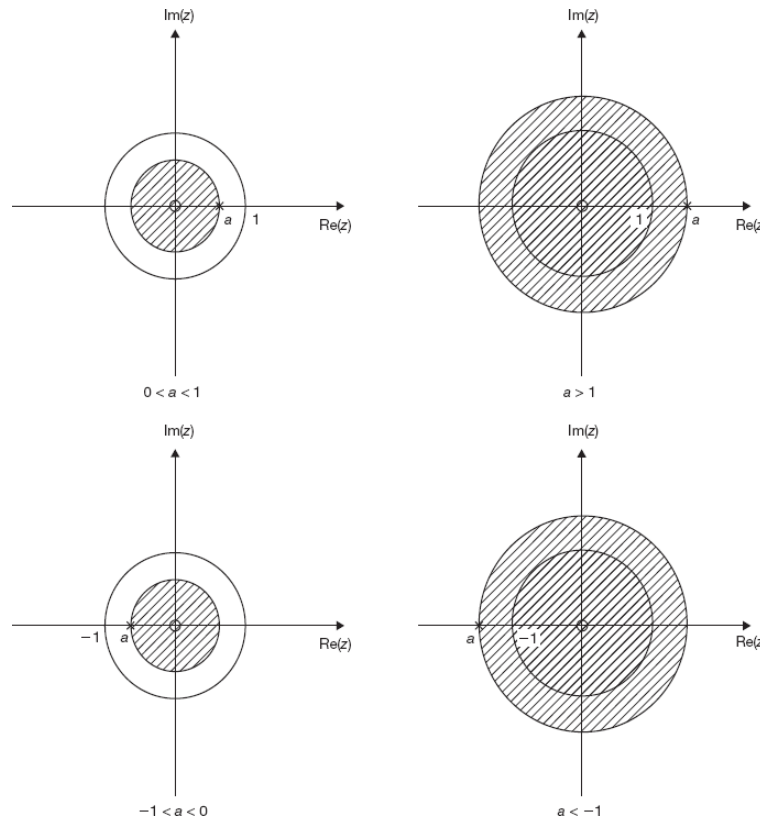
$$X(z) = \frac{1}{1 - az^{-1}} \quad |z| < |a| \quad (4.12)$$

Again, as before,  $X(z)$  may be written as

$$X(z) = \frac{z}{z - a} \quad |z| < |a| \quad (4.13)$$

Thus, the ROC and the pole-zero plot for this example are shown in Fig. 4-2. Comparing Eqs. (4.9) and (4.12) [or Eqs. (4.10) and (4.13)], we see that the algebraic expressions of  $X(z)$  for two different sequences are identical except for the ROCs. Thus, as in the Laplace transform, specification of the  $z$ -transform requires both the algebraic expression and the ROC.

Figure 4-2 ROC of the form  $|z| < |a|$ .



### 4.2.3. C. Properties of the ROC:

As we saw in Examples 4.1 and 4.2, the ROC of  $X(z)$  depends on the nature of  $x[n]$ . The properties of the ROC are summarized below. We assume that  $X(z)$  is a rational function of  $z$ .

**Property 1:** The ROC does not contain any poles.

**Property 2:** If  $x[n]$  is a finite sequence (that is,  $x[n] = 0$  except in a finite interval  $N_1 \leq n \leq N_2$ , where  $N_1$  and  $N_2$  are finite) and  $X(z)$  converges for some value of  $z$ , then the ROC is the entire  $z$ -plane except possibly  $z = 0$  or  $z = \infty$ .

**Property 3:** If  $x[n]$  is a right-sided sequence (that is,  $x[n] = 0$  for  $n < N_1 < \infty$ ) and  $X(z)$  converges for some value of  $z$ , then the ROC is of the form

$$|z| > r_{\max} \quad \text{or} \quad \infty > |z| > r_{\max}$$

where  $r_{\max}$  equals the largest magnitude of any of the poles of  $X(z)$ . Thus, the ROC is the exterior of the circle  $|z| = r_{\max}$  in the  $z$ -plane with the possible exception of  $z = \infty$ .

**Property 4:** If  $x[n]$  is a left-sided sequence (that is,  $x[n] = 0$  for  $n > N_2 > -\infty$ ) and  $X(z)$  converges for some value of  $z$ , then the ROC is of the form

$$|z| < r_{\min} \quad \text{or} \quad 0 < |z| < r_{\min}$$

where  $r_{\min}$  is the smallest magnitude of any of the poles of  $X(z)$ . Thus, the ROC is the interior of the circle  $|z| = r_{\min}$  in the  $z$ -plane with the possible exception of  $z = 0$ .

**Property 5:** If  $x[n]$  is a two-sided sequence (that is,  $x[n]$  is an infinite-duration sequence that is neither right-sided nor left-sided) and  $X(z)$  converges for some value of  $z$ , then the ROC is of the form

$$r_1 < |z| < r_2$$

where  $r_1$  and  $r_2$  are the magnitudes of the two poles of  $X(z)$ . Thus, the ROC is an annular ring in the  $z$ -plane between the circles  $|z| = r_1$  and  $|z| = r_2$  not containing any poles.

Note that Property 1 follows immediately from the definition of poles; that is,  $X(z)$  is infinite at a pole. For verification of the other properties, see Probs. 4.2 and 4.5.

## 4.3. z-Transforms of Some Common Sequences

### 4.3.1. A. Unit Impulse Sequence $\delta[n]$ :

From definitions (1.45) and (4.3)

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^{-0} = 1 \quad \text{all } z$$

(4.14)

Thus,

$$\delta[n] \leftrightarrow 1 \quad \text{all } z$$

(4.15)

### 4.3.2. B. Unit Step Sequence $u[n]$ :

Setting  $a = 1$  in Eqs. (4.8) to (4.10), we obtain

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \quad |z| > 1$$

(4.16)

### 4.3.3. C. z-Transform Pairs:

The  $z$ -transforms of some common sequences are tabulated in (Table 4-1).

**Table 4-1** Some Common  $z$ -Transform Pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All $z$
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$ z  > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$ z  < 1$
$\delta[n-m]$	$z^{-m}$	All $z$ except 0 if $(m > 0)$ or $\infty$ if $(m < 0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$ z  >  a $
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$ z  <  a $
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$ z  >  a $
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$ z  <  a $
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}, \left[ \frac{z}{z-a} \right]^2$	$ z  >  a $
$(\cos \Omega_0 n)u[n]$	$\frac{z^2 - (\cos \Omega_0)z}{z^2 - (2 \cos \Omega_0)z + 1}$	$ z  > 1$
$(\sin \Omega_0 n)u[n]$	$\frac{(\sin \Omega_0)z}{z^2 - (2 \cos \Omega_0)z + 1}$	$ z  > 1$
$(r^n \cos \Omega_0 n)u[n]$	$\frac{z^2 - (r \cos \Omega_0)z}{z^2 - (2r \cos \Omega_0)z + r^2}$	$ z  > r$
$(r^n \sin \Omega_0 n)u[n]$	$\frac{(r \sin \Omega_0)z}{z^2 - (2r \cos \Omega_0)z + r^2}$	$ z  > r$
$\begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$ z  > 0$

## 4.4. Properties of the z-Transform

Basic properties of the z-transform are presented in the following discussion. Verification of these properties is given in Probs. 4.8 to 4.14.

### 4.4.1. A. Linearity:

If

$$\begin{aligned} x_1[n] &\leftrightarrow X_1(z) & \text{ROC} = R_1 \\ x_2[n] &\leftrightarrow X_2(z) & \text{ROC} = R_2 \end{aligned}$$

then

$$a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1(z) + a_2 X_2(z) \quad R' \supset R_1 \cap R_2$$

(4.17)

where  $a_1$  and  $a_2$  are arbitrary constants.

### 4.4.2. B. Time Shifting:

If

$$x[n] \leftrightarrow X(z) \quad \text{ROC} = R$$

then

$$x[n - n_0] \leftrightarrow z^{-n_0} X(z) \quad R' = R \cap \{0 < |z| < \infty\}$$

(4.18)

#### 4.4.2.1. Special Cases:

$$x[n - 1] \leftrightarrow z^{-1} X(z) \quad R' = R \cap \{0 < |z| < \infty\}$$

(4.19)

$$x[n + 1] \leftrightarrow z X(z) \quad R' = R \cap \{|z| < \infty\}$$

(4.20)

Because of these relationships [Eqs. (4.19) and (4.20)],  $z^{-1}$  is often called the *unit-delay operator* and  $z$  is called the *unit-advance operator*. Note that in the Laplace transform the operators  $s^{-1} = 1/s$  and  $s$  correspond to time-domain integration and differentiation, respectively [Eqs. (3.22) and (3.20)].

### 4.4.3. C. Multiplication by $z^{n_0}$ :

If

$$x[n] \leftrightarrow X(z) \quad \text{ROC} = R$$

then

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \quad R' = |z_0| R$$

(4.21)

In particular, a pole (or zero) at  $z = z_k$  in  $X(z)$  moves to  $z = z_0 z_k$  after multiplication by  $z_0^n$  and the ROC expands or contracts by the factor  $|z_0|$ .

#### 4.4.3.1. Special Case:

$$e^{j\Omega_0 n} x[n] \leftrightarrow X(e^{-j\Omega_0} z) \quad R' = R$$

(4.22)

In this special case, all poles and zeros are simply rotated by the angle  $\Omega_0$  and the ROC is unchanged.

#### 4.4.4. D. Time Reversal:

If

$$x[n] \leftrightarrow X(z) \quad \text{ROC} = R$$

then

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \quad R' = \frac{1}{R}$$

(4.23)

Therefore, a pole (or zero) in  $X(z)$  at  $z = z_k$  moves to  $1/z_k$  after time reversal. The relationship  $R' = 1/R$  indicates the inversion of  $R$ , reflecting the fact that a right-sided sequence becomes left-sided if time-reversed, and vice versa.

#### 4.4.5. E. Multiplication by $n$ (or Differentiation in $z$ ):

If

$$x[n] \leftrightarrow X(z) \quad \text{ROC} = R$$

then

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz} \quad R' = R$$

(4.24)

#### 4.4.6. F. Accumulation:

If



$$x[n] \leftrightarrow X(z) \quad \text{ROC} = R$$

then

$$\sum_{k=-\infty}^n x[k] \leftrightarrow \frac{1}{1-z^{-1}} X(z) = \frac{z}{z-1} X(z) \quad R' \supset R \cap \{|z| > 1\}$$

(4.25)

Note that  $\sum_{k=-\infty}^n x[k]$  is the discrete-time counterpart to integration in the time domain and is called the *accumulation*. The comparable Laplace transform operator for integration is  $1/s$ .

### 4.4.7. G. Convolution:

If

$$\begin{aligned} x_1[n] &\leftrightarrow X_1(z) & \text{ROC} = R_1 \\ x_2[n] &\leftrightarrow X_2(z) & \text{ROC} = R_2 \end{aligned}$$

then

$$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \quad R' \supset R_1 \cap R_2$$

(4.26)

This relationship plays a central role in the analysis and design of discrete-time LTI systems, in analogy with the continuous-time case.

### 4.4.8. H. Summary of Some z-transform Properties:

For convenient reference, the properties of the z-transform presented above are summarized in (Table 4-2).

Table 4-2 Some Properties of the z-Transform

PROPERTY	SEQUENCE	TRANSFORM	ROC
	$x[n]$	$X(z)$	$R$
	$x_1[n]$	$X_1(z)$	$R_1$
	$x_2[n]$	$X_2(z)$	$R_2$
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	$R' \supset R_1 \cap R_2$
Time shifting	$x[n - n_0]$	$z^{-n_0} X(z)$	$R' \supset R \cap \{0 <  z  < \infty\}$
Multiplication by $z_0^n$	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$R' =  z_0  R$
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n} x[n]$	$X(e^{-j\Omega_0} z)$	$R' = R$
Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$R' = \frac{1}{R}$
Multiplication by $n$	$nx[n]$	$-z \frac{dX(z)}{dz}$	$R' = R$
Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1-z^{-1}} X(z)$	$R' \supset R \cap \{ z  > 1\}$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	$R' \supset R_1 \cap R_2$

## 4.5. The Inverse z-Transform

Inversion of the z-transform to find the sequence  $x[n]$  from its z-transform  $X(z)$  is called the inverse z-transform, symbolically denoted as

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} \quad (4.27)$$

### 4.5.1. A. Inversion Formula:

As in the case of the Laplace transform, there is a formal expression for the inverse z-transform in terms of an integration in the z-plane; that is,

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \quad (4.28)$$

where  $C$  is a counterclockwise contour of integration enclosing the origin. Formal evaluation of Eq. (4.28) requires an understanding of complex variable theory.

## 4.5.2. B. Use of Tables of z-Transform Pairs:

In the second method for the inversion of  $X(z)$ , we attempt to express  $X(z)$  as a sum

$$X(z) = X_1(z) + \cdots + X_n(z) \quad (4.29)$$

where  $X_1(z), \dots, X_n(z)$  are functions with known inverse transforms  $x_1[n], \dots, x_n[n]$ . From the linearity property (4.17) it follows that

$$x[n] = x_1[n] + \cdots + x_n[n] \quad (4.30)$$

## 4.5.3. C. Power Series Expansion:

The defining expression for the z-transform [Eq. (4.3)] is a power series where the sequence values  $x[n]$  are the coefficients of  $z^{-n}$ . Thus, if  $X(z)$  is given as a power series in the form

$$\begin{aligned} X[z] &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \end{aligned} \quad (4.31)$$

we can determine any particular value of the sequence by finding the coefficient of the appropriate power of  $z^{-1}$ . This approach may not provide a closed-form solution but is very useful for a finite-length sequence where  $X(z)$  may have no simpler form than a polynomial in  $z^{-1}$  (see Prob. 4.15). For rational z-transforms, a power series expansion can be obtained by long division as illustrated in Probs. 4.16 and 4.17.

## 4.5.4. D. Partial-Fraction Expansion:

As in the case of the inverse Laplace transform, the partial-fraction expansion method provides the most generally useful inverse z-transform, especially when  $X(z)$  is a rational function of  $z$ . Let

$$X(z) = \frac{N(z)}{D(z)} = k \frac{(z - z_1) \cdots (z - z_m)}{(z - p_1) \cdots (z - p_n)} \quad (4.32)$$

Assuming  $n \geq m$  and all poles  $p_k$  are simple, then

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \cdots + \frac{c_n}{z - p_n} = \frac{c_0}{z} + \sum_{k=1}^n \frac{c_k}{z - p_k} \quad (4.33)$$

where

$$c_0 = X(z) \Big|_{z=0} \quad c_k = (z - p_k) \frac{X(z)}{z} \Big|_{z=p_k}$$

(4.34)

Hence, we obtain

$$X(z) = c_0 + c_1 \frac{z}{z - p_1} + \cdots + c_n \frac{z}{z - p_n} = c_0 + \sum_{k=1}^n c_k \frac{z}{z - p_k}$$

(4.35)

Inferring the ROC for each term in Eq. (4.35) from the overall ROC of  $X(z)$  and using (Table 4-1), we can then invert each term, producing thereby the overall inverse z-transform (see Probs. 4.19 to 4.23).

If  $m > n$  in Eq. (4.32), then a polynomial of  $z$  must be added to the right-hand side of Eq. (4.35), the order of which is  $(m - n)$ . Thus for  $m > n$ , the complete partial-fraction expansion would have the form

$$X(z) = \sum_{q=0}^{m-n} b_q z^q + \sum_{k=1}^n c_k \frac{z}{z - p_k}$$

(4.36)

If  $X(z)$  has multiple-order poles, say,  $p_i$  is the multiple pole with multiplicity  $r$ , then the expansion of  $X(z)/z$  will consist of terms of the form

$$\frac{\lambda_1}{z - p_i} + \frac{\lambda_2}{(z - p_i)^2} + \cdots + \frac{\lambda_r}{(z - p_i)^r}$$

(4.37)

where

$$\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{dz^k} \left[ (z - p_i)^r \frac{X(z)}{z} \right] \Big|_{z=p_i}$$

(4.38)

## 4.6. The System Function of Discrete-Time LTI Systems

### 4.6.1. A. The System Function:

In Sec. 2.6 we showed that the output  $y[n]$  of a discrete-time LTI system equals the convolution of the input  $x[n]$  with the impulse response  $h[n]$ ; that is [Eq. (2.35)],

$$y[n] = x[n] * h[n]$$

(4.39)

Applying the convolution property (4.26) of the z-transform, we obtain

$$Y(z) = X(z)H(z)$$

(4.40)

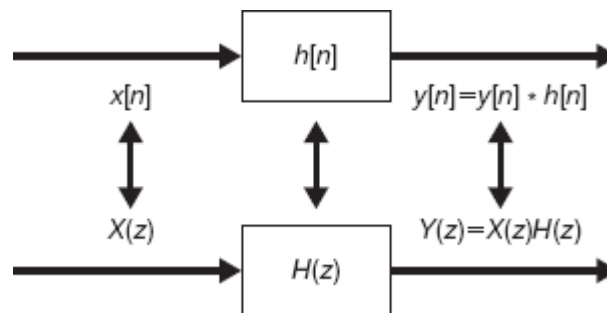
where  $Y(z)$ ,  $X(z)$ , and  $H(z)$  are the  $z$ -transforms of  $y[n]$ ,  $x[n]$ , and  $h[n]$ , respectively. Equation (4.40) can be expressed as

$$H(z) = \frac{Y(z)}{X(z)}$$

(4.41)

The  $z$ -transform  $H(z)$  of  $h[n]$  is referred to as the *system function* (or the *transfer function*) of the system. By Eq. (4.41) the system function  $H(z)$  can also be defined as the ratio of the  $z$ -transforms of the output  $y[n]$  and the input  $x[n]$ . The system function  $H(z)$  completely characterizes the system. Fig. 4-3 illustrates the relationship of Eqs. (4.39) and (4.40).

Figure 4-3 Impulse response and system function.



## 4.6.2. B. Characterization of Discrete-Time LTI Systems:

Many properties of discrete-time LTI systems can be closely associated with the characteristics of  $H(z)$  in the  $z$ -plane and in particular with the pole locations and the ROC.

### 4.6.2.1. 1. Causality:

For a causal discrete-time LTI system, we have [Eq. (2.44)]

$$h[n] = 0 \quad n < 0$$

since  $h[n]$  is a right-sided signal, the corresponding requirement on  $H(z)$  is that the ROC of  $H(z)$  must be of the form

$$|z| > r_{\max}$$

That is, the ROC is the exterior of a circle containing all of the poles of  $H(z)$  in the  $z$ -plane. Similarly, if the system is anticausal, that is,

$$h[n] = 0 \quad n \geq 0$$

then  $h[n]$  is left-sided and the ROC of  $H(z)$  must be of the form

$$|z| < r_{\min}$$

That is, the ROC is the interior of a circle containing no poles of  $H(z)$  in the  $z$ -plane.

### 4.6.2.2. 2. Stability:

In [Sec. 2.7](#) we stated that a discrete-time LTI system is BIBO stable if and only if [Eq. \(2.49\)](#)

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

The corresponding requirement on  $H(z)$  is that the ROC of  $H(z)$  contains the unit circle (that is,  $|z| = 1$ ). (See Prob. 4.30.)

### 4.6.2.3. 3. Causal and Stable Systems:

If the system is both causal and stable, then all of the poles of  $H(z)$  must lie inside the unit circle of the  $z$ -plane because the ROC is of the form  $|z| > r_{\max}$ , and since the unit circle is included in the ROC, we must have  $r_{\max} < 1$ .

## 4.6.3. C. System Function for LTI Systems Described by Linear Constant-Coefficient Difference Equations:

In [Sec. 2.9](#) we considered a discrete-time LTI system for which input  $x[n]$  and output  $y[n]$  satisfy the general linear constant-coefficient difference equation of the form

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

(4.42)

Applying the  $z$ -transform and using the time-shift property [\(4.18\)](#) and the linearity property [\(4.17\)](#) of the  $z$ -transform, we obtain

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

or

$$Y(z) \sum_{k=0}^N a_k z^{-k} = X(z) \sum_{k=0}^M b_k z^{-k}$$

(4.43)

Thus,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

(4.44)

Hence,  $H(z)$  is always rational. Note that the ROC of  $H(z)$  is not specified by [Eq. \(4.44\)](#) but must be inferred with additional requirements on the system such as the causality or the stability.

## 4.6.4. D. Systems Interconnection:

For two LTI systems (with  $h_1[n]$  and  $h_2[n]$ , respectively) in cascade, the overall impulse response  $h[n]$  is given by

$$h[n] = h_1[n] * h_2[n]$$

(4.45)

Thus, the corresponding system functions are related by the product

$$H(z) = H_1(z)H_2(z) \quad R \supset R_1 \cap R_2$$

(4.46)

Similarly, the impulse response of a parallel combination of two LTI systems is given by

$$h[n] = h_1[n] + h_2[n]$$

(4.47)

and

$$H(z) = H_1(z) + H_2(z) \quad R \supset R_1 \cap R_2$$

(4.48)

## 4.7. The Unilateral z-Transform

### 4.7.1. A. Definition:

The *unilateral* (or *one-sided*) z-transform  $X_I(z)$  of a sequence  $x[n]$  is defined as [Eq. (4.5)]

$$X_I(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

(4.49)

and differs from the bilateral transform in that the summation is carried over only  $n \geq 0$ . Thus, the unilateral z-transform of  $x[n]$  can be thought of as the bilateral transform of  $x[n]u[n]$ . Since  $x[n]u[n]$  is a right-sided sequence, the ROC of  $X_I(z)$  is always outside a circle in the z-plane.

### 4.7.2. B. Basic Properties:

Most of the properties of the unilateral z-transform are the same as for the bilateral z-transform. The unilateral z-transform is useful for calculating the response of a causal system to a causal input when the system is described by a linear constant-coefficient difference equation with nonzero initial conditions. The basic property of the unilateral z-transform that is useful in this application is the following time-shifting property which is different from that of the bilateral transform.

#### 4.7.2.1. Time-Shifting Property:

If  $x[n] \leftrightarrow X_I(z)$ , then for  $m \geq 0$ ,

$$x[n - m] \leftrightarrow z^{-m} X_I(z) + z^{-m+1} x[-1] + z^{-m+2} x[-2] + \dots + x[-m]$$

(4.50)

$$x[n + m] \Leftrightarrow z^m X_I(z) - z^m x[0] - z^{m-1} x[1] - \cdots - z x[m - 1]$$

(4.51)

The proofs of Eqs. (4.50) and (4.51) are given in Prob. 4.36.

### 4.7.3. D. System Function:

Similar to the case of the continuous-time LTI system, with the unilateral z-transform, the system function  $H(z) = Y(z)/X(z)$  is defined under the condition that the system is relaxed; that is, all initial conditions are zero.

## 4.8. SOLVED PROBLEMS

### 4.8.1. The z-Transform

4.1. Find the z-transform of

a.  $x[n] = -a^n u[-n - 1]$

b.  $x[n] = a^{-n} u[-n - 1]$

a. From Eq. (4.3)

$$\begin{aligned} X(z) &= - \sum_{n=-\infty}^{\infty} a^n u[-n - 1] z^{-n} = - \sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= - \sum_{n=1}^{\infty} (a^{-1} z)^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n \end{aligned}$$

By Eq. (1.91)

$$\sum_{n=0}^{\infty} (a^{-1} z)^n = \frac{1}{1 - a^{-1} z} \quad \text{if } |a^{-1} z| < 1 \text{ or } |z| < |a|$$

Thus,

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{-a^{-1} z}{1 - a^{-1} z} = \frac{z}{z - a} = \frac{1}{1 - az^{-1}} \quad |z| < |a|$$

(4.52)

b. Similarly,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} a^{-n} u[-n - 1] z^{-n} = \sum_{n=-\infty}^{-1} (az)^{-n} \\ &= \sum_{n=1}^{\infty} (az)^n = \sum_{n=0}^{\infty} (az)^n - 1 \end{aligned}$$

Again by Eq. (1.91)



$$\sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az} \quad \text{if } |az| < 1 \text{ or } |z| < \frac{1}{|a|}$$

Thus,

$$X(z) = \frac{1}{1-az} - 1 = \frac{az}{1-az} = -\frac{z}{z-1/a} \quad |z| < \frac{1}{|a|}$$

(4.53)

**4.2.** A finite sequence  $x[n]$  is defined as

$$x[n] = \begin{cases} \neq 0 & N_1 \leq n \leq N_2 \\ = 0 & \text{otherwise} \end{cases}$$

where  $N_1$  and  $N_2$  are finite. Show that the ROC of  $X(z)$  is the entire  $z$ -plane except possibly  $z = 0$  or  $z = \infty$ .

From Eq. (4.3)

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}$$

(4.54)

For  $z$  not equal to zero or infinity, each term in Eq. (4.54) will be finite and thus  $X(z)$  will converge. If  $N_1 < 0$  and  $N_2 > 0$ , then Eq. (4.54) includes terms with both positive powers of  $z$  and negative powers of  $z$ . As  $|z| \rightarrow 0$ , terms with negative powers of  $z$  become unbounded, and as  $|z| \rightarrow \infty$ , terms with positive powers of  $z$  become unbounded. Hence, the ROC is the entire  $z$ -plane except for  $z = 0$  and  $z = \infty$ . If  $N_1 \geq 0$ , Eq. (4.54) contains only negative powers of  $z$ , and hence the ROC includes  $z = \infty$ . If  $N_2 \leq 0$ , Eq. (4.54) contains only positive powers of  $z$ , and hence the ROC includes  $z = 0$ .

**4.3.** A finite sequence  $x[n]$  is defined as

$$x[n] = \{5, 3, -2, 0, 4, -3\}$$

↑

Find  $X(z)$  and its ROC.

From Eq. (4.3) and given  $x[n]$  we have

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-2}^3 x[n]z^{-n} \\ &= x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} \\ &= 5z^2 + 3z - 2 + 4z^{-2} - 3z^{-3} \end{aligned}$$

For  $z$  not equal to zero or infinity, each term in  $X(z)$  will be finite and consequently  $X(z)$  will converge. Note that  $X(z)$  includes both positive powers of  $z$  and negative powers of  $z$ . Thus, from the result of Prob. 4.2 we conclude that the ROC of  $X(z)$  is  $0 < |z| < \infty$ .

**4.4.** Consider the sequence

$$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1, a > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find  $X(z)$  and plot the poles and zeros of  $X(z)$ .

By Eq. (4.3) and using Eq. (1.90), we get

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a} \quad (4.55)$$

From Eq. (4.55) we see that there is a pole of  $(N - 1)$ th order at  $z = 0$  and a pole at  $z = a$ . Since  $x[n]$  is a finite sequence and is zero for  $n < 0$ , the ROC is  $|z| > 0$ . The  $N$  roots of the numerator polynomial are at

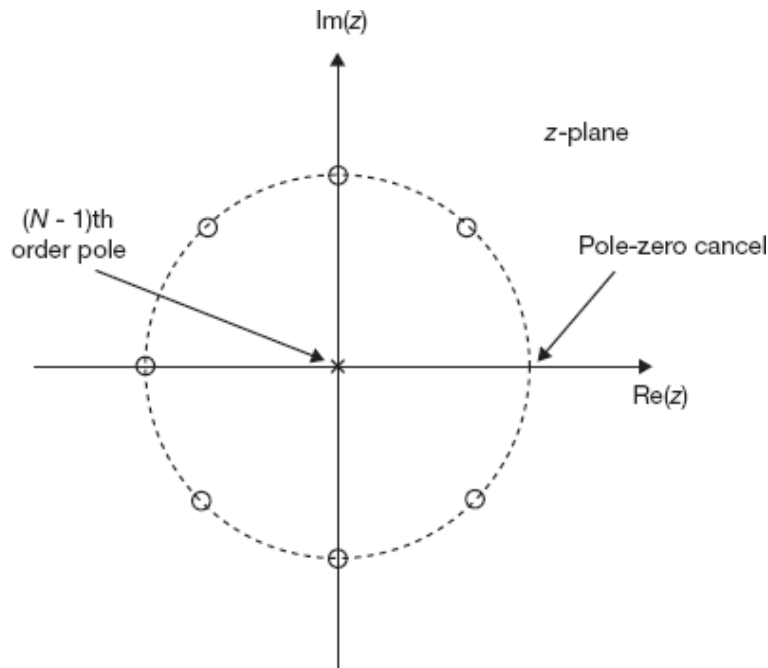
$$z_k = ae^{j(2\pi k/N)} \quad k = 0, 1, \dots, N - 1 \quad (4.56)$$

The root at  $k = 0$  cancels the pole at  $z = a$ . The remaining zeros of  $X(z)$  are at

$$z_k = ae^{j(2\pi k/N)} \quad k = 1, \dots, N - 1 \quad (4.57)$$

The pole-zero plot is shown in Fig. 4-4 with  $N = 8$ .

Figure 4-4 Pole-zero plot with  $N = 8$ .



4.5. Show that if  $x[n]$  is a right-sided sequence and  $X(z)$  converges for some value of  $z$ , then the ROC of  $X(z)$  is of the form

$$|z| > r_{\max} \quad \text{or} \quad \infty > |z| > r_{\max}$$

where  $r_{\max}$  is the maximum magnitude of any of the poles of  $X(z)$ .

Consider a right-sided sequence  $x[n]$  so that

$$x[n] = 0 \quad n < N_1$$

and  $X(z)$  converges for  $|z| = r_0$ . Then from Eq. (4.3)

$$|X(z)| \leq \sum_{n=-\infty}^{\infty} |x[n]| r_0^{-n} = \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} < \infty$$

Now if  $r_1 > r_0$ , then

$$\begin{aligned} \sum_{n=N_1}^{\infty} |x[n]| r_1^{-n} &= \sum_{n=N_1}^{\infty} |x[n]| \left( r_0 \frac{r_1}{r_0} \right)^{-n} = \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} \left( \frac{r_1}{r_0} \right)^{-n} \\ &\leq \left( \frac{r_1}{r_0} \right)^{-N_1} \sum_{n=N_1}^{\infty} |x[n]| r_0^{-n} < \infty \end{aligned}$$

since  $(r_1/r_0)^{-n}$  is a decaying sequence. Thus,  $X(z)$  converges for  $r = r_1$  and the ROC of  $X(z)$  is of the form

$$|z| > r_0$$

Since the ROC of  $X(z)$  cannot contain the poles of  $X(z)$ , we conclude that the ROC of  $X(z)$  is of the form

$$|z| > r_{\max}$$

where  $r_{\max}$  is the maximum magnitude of any of the poles of  $X(z)$ .

If  $N_1 < 0$ , then

$$X(z) = \sum_{n=N_1}^{\infty} x[n] z^{-n} = x[N_1] z^{-N_1} + \dots + x[-1] z + \sum_{n=0}^{\infty} x[n] z^{-n}$$

That is,  $X(z)$  contains the positive powers of  $z$  and becomes unbounded at  $z = \infty$ . In this case the ROC is of the form

$$\infty > |z| > r_{\max}$$

From the above result we can tell that a sequence  $x[n]$  is causal (not just right-sided) from the ROC of  $X(z)$  if  $z = \infty$  is included. Note that this is not the case for the Laplace transform.

**4.6.** Find the  $z$ -transform  $X(z)$  and sketch the pole-zero plot with the ROC for each of the following sequences:

$$(a) \quad x[n] = \left( \frac{1}{2} \right)^n u[n] + \left( \frac{1}{3} \right)^n u[n]$$

$$(b) \quad x[n] = \left( \frac{1}{3} \right)^n u[n] + \left( \frac{1}{2} \right)^n u[-n-1]$$

$$(c) \quad x[n] = \left( \frac{1}{2} \right)^n u[n] + \left( \frac{1}{3} \right)^n u[-n-1]$$

a. From (Table 4-1)

$$\left( \frac{1}{2} \right)^n u[n] \leftrightarrow \frac{z}{z - \frac{1}{2}} \quad |z| > \frac{1}{2}$$

(4.58)

$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{z}{z - \frac{1}{3}} \quad |z| > \frac{1}{3}$$

(4.59)

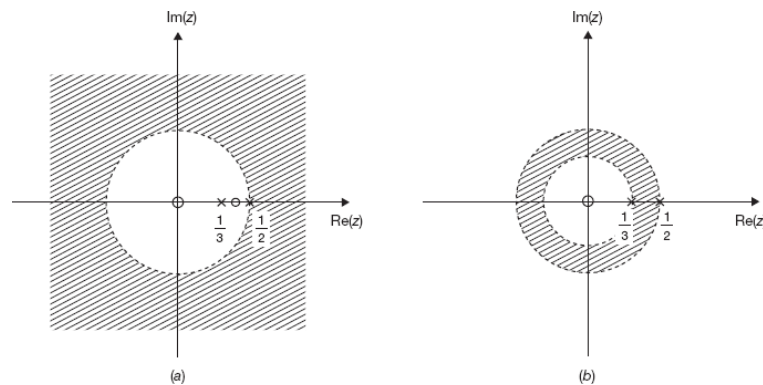
We see that the ROCs in Eqs. (4.58) and (4.59) overlap, and thus,

$$X(z) = \frac{z}{z - \frac{1}{2}} + \frac{z}{z - \frac{1}{3}} = \frac{2z\left(z - \frac{5}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} \quad |z| > \frac{1}{2}$$

(4.60)

From Eq. (4.60) we see that  $X(z)$  has two zeros at  $z = 0$  and  $z = \frac{5}{12}$  and two poles at  $z = \frac{1}{2}$  and  $z = \frac{1}{3}$  and that the ROC is  $|z| > \frac{1}{2}$ , as sketched in Fig. 4-5(a).

Figure 4-5



b. From (Table 4-1)

$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{z}{z - \frac{1}{3}} \quad |z| > \frac{1}{3}$$

(4.61)

$$\left(\frac{1}{2}\right)^n u[-n-1] \leftrightarrow -\frac{z}{z - \frac{1}{2}} \quad |z| < \frac{1}{2}$$

(4.62)

We see that the ROCs in Eqs. (4.61) and (4.62) overlap, and thus

$$X(z) = \frac{z}{z - \frac{1}{3}} - \frac{z}{z - \frac{1}{2}} = -\frac{1}{6} \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} \quad \frac{1}{3} < |z| < \frac{1}{2}$$

(4.63)

From Eq. (4.63) we see that  $X(z)$  has one zero at  $z = 0$  and two poles at  $z = \frac{1}{2}$  and  $z = \frac{1}{3}$  and that the ROC is  $\frac{1}{3} < |z| < \frac{1}{2}$ , as sketched in Fig. 4-5(b).

c. From (Table 4-1)

$$\left(\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{z}{z - \frac{1}{2}} \quad |z| > \frac{1}{2}$$

(4.64)

$$\left(\frac{1}{3}\right)^n u[-n-1] \leftrightarrow -\frac{z}{z - \frac{1}{3}} \quad |z| < \frac{1}{3}$$

(4.65)

We see that the ROCs in Eqs. (4.64) and (4.65) do not overlap and that there is no common ROC, and thus  $x[n]$  will not have  $X(z)$ .

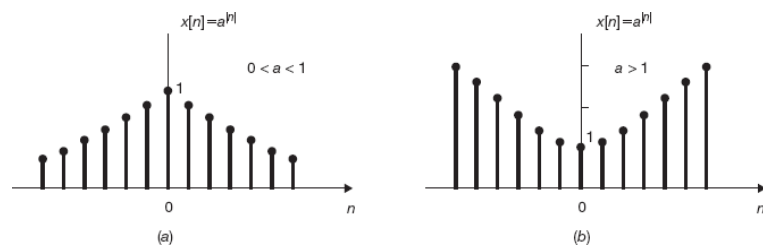
4.7. Let

$$x[n] = a^{|n|} \quad a > 0$$

(4.66)

- Sketch  $x[n]$  for  $a < 1$  and  $a > 1$ .
- Find  $X(z)$  and sketch the zero-pole plot and the ROC for  $a < 1$  and  $a > 1$ .
- The sequence  $x[n]$  is sketched in Figs. 4-6(a) and (b) for both  $a < 1$  and  $a > 1$ .

Figure 4-6



b. Since  $x[n]$  is a two-sided sequence, we can express it as

$$x[n] = a^n u[n] + a^{-n} u[-n-1]$$

(4.67)

From (Table 4-1)

$$a^n u[n] \leftrightarrow \frac{z}{z - a} \quad |z| > a$$

(4.68)

$$a^{-n} u[-n-1] \leftrightarrow -\frac{z}{z - 1/a} \quad |z| < \frac{1}{a}$$

(4.69)

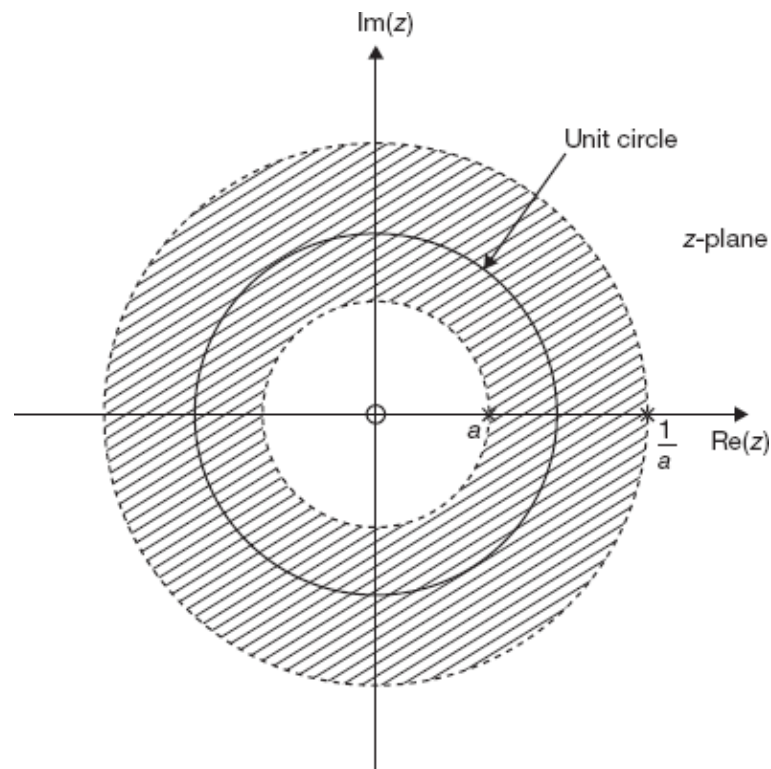
If  $a < 1$ , we see that the ROCs in Eqs. (4.68) and (4.69) overlap, and thus,

$$X(z) = \frac{z}{z-a} - \frac{z}{z-1/a} = \frac{a^2-1}{a} \frac{z}{(z-a)(z-1/a)} \quad a < |z| < \frac{1}{a}$$

(4.70)

From Eq. (4.70) we see that  $X(z)$  has one zero at the origin and two poles at  $z = a$  and  $z = 1/a$  and that the ROC is  $a < |z| < 1/a$ , as sketched in Fig. 4-7. If  $a > 1$ , we see that the ROCs in Eqs. (4.68) and (4.69) do not overlap and that there is no common ROC, and thus  $x[n]$  will not have  $X(z)$ .

Figure 4-7



## 4.8.2. Properties of the z-Transform

4.8. Verify the time-shifting property (4.18); that is,

$$x[n - n_0] \Leftrightarrow z^{-n_0} X(z) \quad R' \supset R \cap \{0 < |z| < \infty\}$$

By definition (4.3)

$$\mathfrak{Z}\{x[n - n_0]\} = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n}$$

By the change of variables  $m = n - n_0$ , we obtain

$$\begin{aligned}\mathcal{Z}\{x[n - n_0]\} &= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)} \\ &= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-n_0} X(z)\end{aligned}$$

Because of the multiplication by  $z^{-n_0}$ , for  $n_0 > 0$ , additional poles are introduced at  $z = 0$  and will be deleted at  $z = \infty$ . Similarly, if  $n_0 < 0$ , additional zeros are introduced at  $z = 0$  and will be deleted at  $z = \infty$ . Therefore, the points  $z = 0$  and  $z = \infty$  can be either added to or deleted from the ROC by time shifting. Thus, we have

$$x[n - n_0] \leftrightarrow z^{-n_0} X(z) \quad R' \supset R \cap \{0 < |z| < \infty\}$$

where  $R$  and  $R'$  are the ROCs before and after the time-shift operation.

**4.9.** Verify Eq. (4.21); that is,

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \quad R' = |z_0| R$$

By definition (4.3)

$$\mathcal{Z}\{z_0^n x[n]\} = \sum_{n=-\infty}^{\infty} (z_0^n x[n])z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} = X\left(\frac{z}{z_0}\right)$$

A pole (or zero) at  $z = z_k$  in  $X(z)$  moves to  $z = z_0 z_k$ , and the ROC expands or contracts by the factor  $|z_0|$ . Thus, we have

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \quad R' = |z_0| R$$

**4.10.** Find the z-transform and the associated ROC for each of the following sequences:

- $x[n] = \delta[n - n_0]$
  - $x[n] = u[n - n_0]$
  - $x[n] = a^{n+1}u[n + 1]$
  - $x[n] = u[-n]$
  - $x[n] = a^{-n}u[-n]$
- a. From Eq. (4.15)

$$\delta[n] \leftrightarrow 1 \quad \text{all } z$$

Applying the time-shifting property (4.18), we obtain

$$\begin{aligned}\delta[n - n_0] &\leftrightarrow z^{-n_0} & 0 < |z|, n_0 > 0 \\ & & |z| < \infty, n_0 < 0\end{aligned}$$

(4.71)

b. From Eq. (4.16)

$$u[n] \leftrightarrow \frac{z}{z-1} \quad |z| > 1$$

Again by the time-shifting property (4.18) we obtain

$$u[n - n_0] \leftrightarrow z^{-n_0} \frac{z}{z-1} = \frac{z^{-(n_0-1)}}{z-1} \quad 1 < |z| < \infty$$

(4.72)

c. From Eqs. (4.8) and (4.10)

$$a^n u[n] \leftrightarrow \frac{z}{z-a} \quad |z| > |a|$$

By Eq. (4.20) we obtain

$$a^{n+1} u[n+1] \leftrightarrow z \frac{z}{z-a} = \frac{z^2}{z-a} \quad |a| < |z| < \infty$$

(4.73)

d. From Eq. (4.16)

$$u[n] \leftrightarrow \frac{z}{z-1} \quad |z| > 1$$

By the time-reversal property (4.23) we obtain

$$u[-n] \leftrightarrow \frac{1/z}{1/z-1} = \frac{1}{1-z} \quad |z| < 1$$

(4.74)

e. From Eqs. (4.8) and (4.10)

$$a^n u[n] \leftrightarrow \frac{z}{z-a} \quad |z| > |a|$$

Again by the time-reversal property (4.23) we obtain

$$a^{-n} u[-n] \leftrightarrow \frac{1/z}{1/z-a} = \frac{1}{1-az} \quad |z| < \frac{1}{|a|}$$

(4.75)

4.11. Verify the multiplication by  $n$  (or differentiation in  $z$ ) property (4.24); that is,

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz} \quad R' = R$$

From definition (4.3)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Differentiating both sides with respect to  $z$ , we have

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} -nx[n] z^{-n-1}$$

and



$$-z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} \{nx[n]\} z^{-n} = \mathfrak{Z}\{nx[n]\}$$

Thus, we conclude that

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz} \quad R' = R$$

**4.12.** Find the z-transform of each of the following sequences:

a.  $x[n] = na^n u[n]$

b.  $x[n] = na^{n-1} u[n]$

a. From Eqs. (4.8) and (4.10)

$$a^n u[n] \leftrightarrow \frac{z}{z-a} \quad |z| > |a|$$

(4.76)

Using the multiplication by  $n$  property (4.24), we get

$$na^n u[n] \leftrightarrow -z \frac{d}{dz} \left( \frac{z}{z-a} \right) = \frac{az}{(z-a)^2} \quad |z| > |a|$$

(4.77)

b. Differentiating Eq. (4.76) with respect to  $a$ , we have

$$na^{n-1} u[n] \leftrightarrow \frac{d}{da} \left( \frac{z}{z-a} \right) = \frac{z}{(z-a)^2} \quad |z| > |a|$$

(4.78)

Note that dividing both sides of Eq. (4.77) by  $a$ , we obtain Eq. (4.78).

**4.13.** Verify the convolution property (4.26); that is,

$$x_1[n] * x_2[n] \leftrightarrow X_1(z) X_2(z) \quad R' \supset R_1 \cap R_2$$

By definition (2.35)

$$y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k]$$

Thus, by definition (4.3)

$$Y[z] = \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \right) z^{-n} = \sum_{k=-\infty}^{\infty} x_1[k] \left( \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n} \right)$$

Noting that the term in parentheses in the last expression is the z-transform of the shifted signal  $x_2[n-k]$ , then by the time-shifting property (4.18), we have

$$Y[z] = \sum_{k=-\infty}^{\infty} x_1[k] [z^{-k} X_2(z)] = \left( \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \right) X_2(z) = X_1(z) X_2(z)$$

with an ROC that contains the intersection of the ROC of  $X_1(z)$  and  $X_2(z)$ . If a zero of one transform cancels a pole of the other, the ROC of  $Y(z)$  may be larger. Thus, we conclude that

$$x_1[n] * x_2[n] \leftrightarrow X_1(z) X_2(z) \quad R' \supset R_1 \cap R_2$$

**4.14.** Verify the accumulation property (4.25); that is,

$$\sum_{k=-\infty}^n x[k] \leftrightarrow \frac{1}{1-z^{-1}} X(z) = \frac{z}{z-1} X(z) \quad R' \supset R \cap \{|z| > 1\}$$

From Eq. (2.40) we have

$$y[n] = \sum_{k=-\infty}^n x[k] = x[n] * u[n]$$

Thus, using Eq. (4.16) and the convolution property (4.26), we obtain

$$Y(z) = X(z) \left( \frac{1}{1-z^{-1}} \right) = X(z) \left( \frac{z}{z-1} \right)$$

with the ROC that includes the intersection of the ROC of  $X(z)$  and the ROC of the  $z$ -transform of  $u[n]$ . Thus,

$$\sum_{k=-\infty}^n x[k] \leftrightarrow \frac{1}{1-z^{-1}} X(z) = \frac{z}{z-1} X(z) \quad R' \supset R \cap \{|z| > 1\}$$

### 4.8.3. Inverse $z$ -Transform

**4.15.** Find the inverse  $z$ -transform of

$$X(z) = z^2 \left( 1 - \frac{1}{2} z^{-1} \right) (1 - z^{-1}) (1 + 2z^{-1}) \quad 0 < |z| < \infty$$

(4.79)

Multiplying out the factors of Eq. (4.79), we can express  $X(z)$  as

$$X(z) = z^2 + \frac{1}{2} z - \frac{5}{2} + z^{-1}$$

Then, by definition (4.3),

$$X(z) = x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1}$$

and we get

$$X[n] = \left\{ \dots, 0, 1, \frac{1}{2}, -\frac{5}{2}, 1, 0, \dots \right\}$$

↑

**4.16.** Using the power series expansion technique, find the inverse  $z$ -transform of the following  $X(z)$ :

$$(a) \quad X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$(b) \quad X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

- a. Since the ROC is  $|z| > |a|$ , that is, the exterior of a circle,  $x[n]$  is a right-sided sequence. Thus, we must divide to obtain a series in the power of  $z^{-1}$ . Carrying out the long division, we obtain

$$\begin{array}{r} 1 + az^{-1} + a^2z^{-2} + \dots \\ 1 - az^{-1} \overline{) 1} \\ \underline{1 - az^{-1}} \phantom{+ \dots} \\ az^{-1} \\ \underline{az^{-1} - a^2z^{-2}} \phantom{+ \dots} \\ a^2z^{-2} \\ \vdots \end{array}$$

Thus,

$$X(z) = \frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \dots + a^kz^{-k} + \dots$$

and so by definition (4.3) we have

$$\begin{array}{llllll} x[n] = 0 & n < 0 & & & & \\ x[0] = 1 & x[1] = a & x[2] = a^2 & \dots & x[k] = a^k & \dots \end{array}$$

Thus, we obtain

$$x[n] = a^n u[n]$$

- b. Since the ROC is  $|z| < |a|$ , that is, the interior of a circle,  $x[n]$  is a left-sided sequence. Thus, we must divide so as to obtain a series in the power of  $z$  as follows. Multiplying both the numerator and denominator of  $X(z)$  by  $z$ , we have

$$X(z) = \frac{z}{z - a}$$

and carrying out the long division, we obtain

$$\begin{array}{r} -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \dots \\ -a + z \overline{) z} \\ \underline{z - a^{-1}z^2} \phantom{+ \dots} \\ a^{-1}z^2 \\ \underline{a^{-1}z^2 - a^{-2}z^3} \phantom{+ \dots} \\ a^{-2}z^3 \\ \vdots \end{array}$$

Thus,

$$X(z) = \frac{1}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \dots - a^{-k}z^k - \dots$$

and so by definition (4.3) we have

$$x[n] = 0 \quad n \geq 0$$

$$x[-1] = -a^{-1} \quad x[-2] = -a^{-2} \quad x[-3] = -a^{-3} \quad \dots \quad x[-k] = -a^{-k} \quad \dots$$

Thus, we get

$$x[n] = -a^n u[-n - 1]$$

**4.17.** Find the inverse z-transform of the following  $X(z)$ :

$$(a) \quad X(z) = \log\left(\frac{1}{1 - az^{-1}}\right), |z| > |a|$$

$$(b) \quad X(z) = \log\left(\frac{1}{1 - a^{-1}z}\right), |z| < |a|$$

a. The power series expansion for  $\log(1 - r)$  is given by

$$\log(1 - r) = - \sum_{n=1}^{\infty} \frac{1}{n} r^n \quad |r| < 1$$

(4.80)

Now

$$X(z) = \log\left(\frac{1}{1 - az^{-1}}\right) = -\log(1 - az^{-1}) \quad |z| > |a|$$

Since the ROC is  $|z| > |a|$ , that is,  $|az^{-1}| < 1$ , by Eq. (4.80),  $X(z)$  has the power series expansion

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n} (az^{-1})^n = \sum_{n=1}^{\infty} \frac{1}{n} a^n z^{-n}$$

from which we can identify  $x[n]$  as

$$x[n] = \begin{cases} (1/n)a^n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

or

$$x[n] = \frac{1}{n} a^n u[n - 1]$$

(4.81)

$$X(z) = \log\left(\frac{1}{1 - a^{-1}z}\right) = -\log(1 - a^{-1}z) \quad |z| < |a|$$

Since the ROC is  $|z| < |a|$ , that is,  $|a^{-1}z| < 1$ , by Eq. (4.80),  $X(z)$  has the power series expansion

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n} (a^{-1}z)^n = \sum_{n=-1}^{-\infty} -\frac{1}{n} (a^{-1}z)^{-n} = \sum_{n=-1}^{-\infty} -\frac{1}{n} a^n z^{-n}$$

from which we can identify  $x[n]$  as

$$x[n] = \begin{cases} 0 & n \geq 0 \\ -(1/n)a^n & n \leq -1 \end{cases}$$

or

$$x[n] = -\frac{1}{n}a^n u[-n-1]$$

(4.82)

**4.18.** Using the power series expansion technique, find the inversez-transform of the following  $X(z)$ :

$$(a) \quad X(z) = \frac{z}{2z^2 - 3z + 1} \quad |z| < \frac{1}{2}$$

$$(b) \quad X(z) = \frac{z}{2z^2 - 3z + 1} \quad |z| > 1$$

- a. Since the ROC is  $|z| < \frac{1}{2}$ ,  $x[n]$  is a left-sided sequence. Thus, we must divide to obtain a series in power of  $z$ . Carrying out the long division, we obtain

$$\begin{array}{r} z + 3z^2 + 7z^3 + 15z^4 + \dots \\ 1 - 3z + 2z^2 \overline{)z} \\ \underline{z - 3z^2 + 2z^3} \phantom{+ \dots} \\ 3z^2 - 2z^3 \phantom{+ \dots} \\ \underline{3z^2 - 9z^3 + 6z^4} \phantom{+ \dots} \\ 7z^3 - 6z^4 \phantom{+ \dots} \\ \underline{7z^3 - 21z^4 + 14z^5} \phantom{+ \dots} \\ 15z^4 \dots \end{array}$$

Thus,

$$X(z) = \dots + 15z^4 + 7z^3 + 3z^2 + z$$

and so by definition (4.3) we obtain

$$x[n] = \{\dots, 15, 7, 3, 1, 0\}$$

↑

- b. Since the ROC is  $|z| > 1$ ,  $x[n]$  is a right-sided sequence. Thus, we must divide so as to obtain a series in power of  $z^{-1}$  as follows:

$$\begin{array}{r} \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \dots \\ 2z^2 - 3z + 1 \overline{)z} \\ \underline{z - \frac{3}{2} - \frac{1}{2}z^{-1}} \\ \frac{3}{2} - \frac{1}{2}z^{-1} \\ \underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}} \\ \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\ \vdots \end{array}$$

Thus,

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \dots$$

and so by definition (4.3) we obtain

$$x[n] = \left\{ 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots \right\}$$

**4.19.** Using partial-fraction expansion, redo Prob. 4.18.

$$X(z) = \frac{z}{2z^2 - 3z + 1} = \frac{z}{2(z-1)\left(z - \frac{1}{2}\right)} \quad |z| < \frac{1}{2}$$

Using partial-fraction expansion, we have

$$\frac{X(z)}{z} = \frac{1}{2z^2 - 3z + 1} = \frac{1}{2(z-1)\left(z - \frac{1}{2}\right)} = \frac{c_1}{z-1} + \frac{c_2}{z - \frac{1}{2}}$$

where

$$c_1 = \frac{1}{2\left(z - \frac{1}{2}\right)} \Big|_{z=1} = 1 \quad c_2 = \frac{1}{2(z-1)} \Big|_{z=1/2} = -1$$

and we get

$$X(z) = \frac{z}{z-1} - \frac{z}{z - \frac{1}{2}} \quad |z| < \frac{1}{2}$$

Since the ROC of  $X(z)$  is  $|z| < \frac{1}{2}$ ,  $x[n]$  is a left-sided sequence, and from (Table 4-1) we get

$$x[n] = -u[-n-1] + \left(\frac{1}{2}\right)^n u[-n-1] = \left[\left(\frac{1}{2}\right)^n - 1\right] u[-n-1]$$

which gives

$$x[n] = \{\dots, 15, 7, 3, 1, 0\}$$

$$\uparrow$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z - \frac{1}{2}} \quad |z| > 1$$

Since the ROC of  $X(z)$  is  $|z| > 1$ ,  $x[n]$  is a right-sided sequence, and from (Table 4-1) we get

$$x[n] = u[n] - \left(\frac{1}{2}\right)^n u[n] = \left[1 - \left(\frac{1}{2}\right)^n\right] u[n]$$

which gives

$$x[n] = \left\{ 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots \right\}$$

**4.20.** Find the inverse z-transform of

$$X(z) = \frac{z}{z(z-1)(z-2)^2} \quad |z| > 2$$

Using partial-fraction expansion, we have

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)^2} = \frac{c_1}{z-1} + \frac{\lambda_1}{z-2} + \frac{\lambda_2}{(z-2)^2}$$

(4.83)

where

$$c_1 = \frac{1}{(z-2)^2} \Big|_{z=1} = 1 \quad \lambda_2 = \frac{1}{z-1} \Big|_{z=2} = 1$$

Substituting these values into Eq. (4.83), we have

$$\frac{1}{(z-1)(z-2)^2} = \frac{1}{z-1} + \frac{\lambda_1}{z-2} + \frac{1}{(z-2)^2}$$

Setting  $z = 0$  in the above expression, we have

$$-\frac{1}{4} = -1 - \frac{\lambda_1}{2} + \frac{1}{4} \rightarrow \lambda_1 = -1$$

Thus,

$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{(z-2)^2} \quad |z| > 2$$

Since the ROC is  $|z| > 2$ ,  $x[n]$  is a right-sided sequence, and from (Table 4-1) we get

$$x[n] = (1 - 2^n + n2^{n-1})u[n]$$

**4.21.** Find the inverse z-transform of

$$X(z) = \frac{2z^3 - 5z^2 + z + 3}{(z-1)(z-2)} \quad |z| < 1$$

$$X(z) = \frac{2z^3 - 5z^2 + z + 3}{(z-1)(z-2)} = \frac{2z^3 - 5z^2 + z + 3}{z^2 - 3z + 2}$$

Note that  $X(z)$  is an improper rational function; thus, by long division, we have

$$X(z) = 2z + 1 + \frac{1}{z^2 - 3z + 2} = 2z + 1 + \frac{1}{(z-1)(z-2)}$$

Let

$$X_1(z) = \frac{1}{(z-1)(z-2)}$$

Then

$$\frac{X_1(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{c_1}{z} + \frac{c_2}{z-1} + \frac{c_3}{z-2}$$

where

$$c_1 = \left. \frac{1}{(z-1)(z-2)} \right|_{z=0} = \frac{1}{2} \quad c_2 = \left. \frac{1}{z(z-2)} \right|_{z=1} = -1$$

$$c_3 = \left. \frac{1}{z(z-1)} \right|_{z=2} = \frac{1}{2}$$

Thus,

$$X_1(z) = \frac{1}{2} - \frac{z}{z-1} + \frac{1}{2} \frac{z}{z-2}$$

and

$$X(z) = 2z + \frac{3}{2} - \frac{z}{z-1} + \frac{1}{2} \frac{z}{z-2} \quad |z| < 1$$

Since the ROC of  $X(z)$  is  $|z| < 1$ ,  $x[n]$  is a left-sided sequence, and from (Table 4-1) we get

$$x[n] = 2\delta[n+1] + \frac{3}{2}\delta[n] + u[-n-1] - \frac{1}{2}2^n u[-n-1]$$

$$= 2\delta[n+1] + \frac{3}{2}\delta[n] + (1 - 2^{n-1})u[-n-1]$$

**4.22.** Find the inverse z-transform of

$$X(z) = \frac{3}{z-2} \quad |z| > 2$$

$X(z)$  can be rewritten as

$$X(z) = \frac{3}{z-2} = 3z^{-1} \left( \frac{z}{z-2} \right) \quad |z| > 2$$

Since the ROC is  $|z| > 2$ ,  $x[n]$  is a right-sided sequence, and from (Table 4-1) we have

$$2^n u[n] \leftrightarrow \frac{z}{z-2}$$

Using the time-shifting property (4.18), we have

$$2^{n-1} u[n-1] \leftrightarrow z^{-1} \left( \frac{z}{z-2} \right) = \frac{1}{z-2}$$

Thus, we conclude that

$$x[n] = 3(2)^{n-1} u[n-1]$$



**4.23.** Find the inverse z-transform of

$$X(z) = \frac{2 + z^{-2} + 3z^{-4}}{z^2 + 4z + 3} \quad |z| > 0$$

We see that  $X(z)$  can be written as

$$X(z) = (2z^{-1} + z^{-3} + 3z^{-5}) X_1(z)$$

where

$$X_1(z) = \frac{z}{z^2 + 4z + 3}$$

Thus, if

$$x_1[n] \leftrightarrow X_1(z)$$

then by the linearity property (4.17) and the time-shifting property (4.18), we get

$$x[n] = 2x_1[n-1] + x_1[n-3] + 3x_1[n-5]$$

(4.84)

Now

$$\frac{X_1(z)}{z} = \frac{1}{z^2 + 4z + 3} = \frac{1}{(z+1)(z+3)} = \frac{c_1}{z+1} + \frac{c_2}{z+3}$$

Where

$$c_1 = \frac{1}{z+3} \Big|_{z=-1} = \frac{1}{2} \quad c_2 = \frac{1}{z+1} \Big|_{z=-3} = -\frac{1}{2}$$

Then

$$X_1(z) = \frac{1}{2} \frac{z}{z+1} - \frac{1}{2} \frac{z}{z+3} \quad |z| > 0$$

Since the ROC of  $X_1(z)$  is  $|z| > 0$ ,  $x_1[n]$  is a right-sided sequence, and from (Table 4-1) we get

$$x_1[n] = \frac{1}{2} [(-1)^n - (-3)^n] u[n]$$

Thus, from Eq. (4.84) we get

$$\begin{aligned} x[n] &= [(-1)^{n-1} - (-3)^{n-1}] u[n-1] + \frac{1}{2} [(-1)^{n-3} - (-3)^{n-3}] u[n-3] \\ &\quad + \frac{3}{2} [(-1)^{n-5} - (-3)^{n-5}] u[n-5] \end{aligned}$$

**4.24.** Find the inverse z-transform of

$$X(z) = \frac{1}{(1 - az^{-1})^2} \quad |z| > |a|$$

$$X(z) = \frac{1}{(1 - az^{-1})^2} = \frac{z^2}{(z - a)^2} \quad |z| > |a|$$

(4.85)

From Eq. (4.78) (Prob. 4.12)

$$na^{n-1}u[n] \leftrightarrow \frac{z}{(z - a)^2} \quad |z| > |a|$$

(4.86)

Now, from Eq. (4.85)

$$X(z) = z \left[ \frac{z}{(z - a)^2} \right] \quad |z| > |a|$$

and applying the time-shifting property (4.20) to Eq. (4.86), we get

$$x[n] = (n + 1) a^n u[n + 1] = (n + 1) a^n u[n]$$

(4.87)

since  $x[-1] = 0$  at  $n = -1$ .

## 4.8.4. System Function

**4.25.** Using the z-transform, redo Prob. 2.28.

From Prob. 2.28,  $x[n]$  and  $h[n]$  are given by

$$x[n] = u[n] \quad h[n] = \alpha^n u[n] \quad 0 < \alpha < 1$$

From (Table 4-1)

$$x[n] = u[n] \leftrightarrow X(z) = \frac{z}{z - 1} \quad |z| > 1$$

$$h[n] = \alpha^n u[n] \leftrightarrow H(z) = \frac{z}{z - \alpha} \quad |z| > |\alpha|$$

Then, by Eq. (4.40)

$$Y(z) = X(z)H(z) = \frac{z^2}{(z - 1)(z - \alpha)} \quad |z| > 1$$

Using partial-fraction expansion, we have

$$\frac{Y(z)}{z} = \frac{z}{(z - 1)(z - \alpha)} = \frac{c_1}{z - 1} + \frac{c_2}{z - \alpha}$$

where

$$c_1 = \left. \frac{z}{z-\alpha} \right|_{z=1} = \frac{1}{1-\alpha} \quad c_2 = \left. \frac{z}{z-1} \right|_{z=\alpha} = -\frac{\alpha}{1-\alpha}$$

Thus,

$$Y(z) = \frac{1}{1-\alpha} \frac{z}{z-1} - \frac{\alpha}{1-\alpha} \frac{z}{z-\alpha} \quad |z| > 1$$

Taking the inverse z-transform of  $Y(z)$ , we get

$$y[n] = \frac{1}{1-\alpha} u[n] - \frac{\alpha}{1-\alpha} \alpha^n u[n] = \left( \frac{1-\alpha^{n+1}}{1-\alpha} \right) u[n]$$

which is the same as [Eq. \(2.134\)](#).

**4.26.** Using the z-transform, redo [Prob. 2.29](#).

a. From [Prob. 2.29\(a\)](#),  $x[n]$  and  $h[n]$  are given by

$$x[n] = \alpha^n u[n] \quad h[n] = \beta^n u[n]$$

From [Table 4-1](#)

$$\begin{aligned} x[n] = \alpha^n u[n] &\leftrightarrow X(z) = \frac{z}{z-\alpha} \quad |z| > |\alpha| \\ h[n] = \beta^n u[n] &\leftrightarrow H(z) = \frac{z}{z-\beta} \quad |z| > |\beta| \end{aligned}$$

Then

$$Y(z) = X(z)H(z) = \frac{z^2}{(z-\alpha)(z-\beta)} \quad |z| > \max(\alpha, \beta)$$

Using partial-fraction expansion, we have

$$\frac{Y(z)}{z} = \frac{z}{(z-\alpha)(z-\beta)} = \frac{c_1}{z-\alpha} + \frac{c_2}{z-\beta}$$

where

$$c_1 = \left. \frac{z}{z-\beta} \right|_{z=\alpha} = \frac{\alpha}{\alpha-\beta} \quad c_2 = \left. \frac{z}{z-\alpha} \right|_{z=\beta} = -\frac{\beta}{\alpha-\beta}$$

Thus,

$$Y(z) = \frac{\alpha}{\alpha-\beta} \frac{z}{z-\alpha} - \frac{\beta}{\alpha-\beta} \frac{z}{z-\beta} \quad |z| > \max(\alpha, \beta)$$

and

$$y[n] = \left[ \frac{\alpha}{\alpha-\beta} \alpha^n - \frac{\beta}{\alpha-\beta} \beta^n \right] u[n] = \left( \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta} \right) u[n]$$

which is the same as [Eq. \(2.135\)](#). When  $\alpha = \beta$ ,

$$Y(z) = \frac{z^2}{(z - \alpha)^2} \quad |z| > \alpha$$

Using partial-fraction expansion, we have

$$\frac{Y(z)}{z} = \frac{z}{(z - \alpha)^2} = \frac{\lambda_1}{z - \alpha} + \frac{\lambda_2}{(z - \alpha)^2}$$

where

$$\lambda_2 = z \Big|_{z=\alpha} = \alpha$$

and

$$\frac{z}{(z - \alpha)^2} = \frac{\lambda_1}{z - \alpha} + \frac{\alpha}{(z - \alpha)^2}$$

Setting  $z = 0$  in the above expression, we have

$$0 = -\frac{\lambda_1}{\alpha} + \frac{1}{\alpha} \rightarrow \lambda_1 = 1$$

Thus,

$$Y(z) = \frac{z}{z - \alpha} + \frac{\alpha z}{(z - \alpha)^2} \quad |z| > \alpha$$

and from (Table 4-1) we get

$$y[n] = (\alpha^n + n\alpha^n) u[n] = \alpha^n (1 + n) u[n]$$

Thus, we obtain the same results as Eq. (2.135).

b. From Prob. 2.29(b),  $x[n]$  and  $h[n]$  are given by

$$x[n] = \alpha^n u[n] \quad h[n] = \alpha^{-n} u[-n] \quad 0 < \alpha < 1$$

From (Table 4-1) and Eq. (4.75)

$$\begin{aligned} x[n] = \alpha^n u[n] &\leftrightarrow X(z) = \frac{z}{z - \alpha} \quad |z| > |\alpha| \\ h[n] = \alpha^{-n} u[-n] &\leftrightarrow H(z) = \frac{1}{1 - \alpha z} = -\frac{1}{\alpha(z - 1/\alpha)} \quad |z| < \frac{1}{|\alpha|} \end{aligned}$$

Then

$$Y(z) = X(z)H(z) = -\frac{1}{\alpha} \frac{z}{(z - \alpha)(z - 1/\alpha)} \quad \alpha < |z| < \frac{1}{\alpha}$$

Using partial-fraction expansion, we have

$$\frac{Y(z)}{z} = -\frac{1}{\alpha} \frac{1}{(z - \alpha)(z - 1/\alpha)} = -\frac{1}{\alpha} \left( \frac{c_1}{z - \alpha} + \frac{c_2}{z - 1/\alpha} \right)$$

where

$$c_1 = \frac{1}{z - 1/\alpha} \Big|_{z=\alpha} = -\frac{\alpha}{1 - \alpha^2} \quad c_2 = \frac{1}{z - \alpha} \Big|_{z=1/\alpha} = \frac{\alpha}{1 - \alpha^2}$$

Thus,

$$Y(z) = \frac{1}{1 - \alpha^2} \frac{z}{z - \alpha} - \frac{1}{1 - \alpha^2} \frac{z}{z - 1/\alpha} \quad \alpha < |z| < \frac{1}{\alpha}$$

and from (Table 4-1) we obtain

$$\begin{aligned} y[n] &= \frac{1}{1 - \alpha^2} \alpha^n u[n] - \frac{1}{1 - \alpha^2} \left\{ -\left(\frac{1}{\alpha}\right)^n u[-n - 1] \right\} \\ &= \frac{1}{1 - \alpha^2} \alpha^n u[n] + \frac{1}{1 - \alpha^2} \alpha^{-n} u[-n - 1] = \frac{1}{1 - \alpha^2} \alpha^{|n|} \end{aligned}$$

which is the same as Eq. (2.137).

**4.27.** Using the z-transform, redo Prob. 2.30.

From Fig. 2-23 and definition (4.3)

$$\begin{aligned} x[n] &= \{1, 1, 1, 1\} \leftrightarrow X(z) = 1 + z^{-1} + z^{-2} + z^{-3} \\ h[n] &= \{1, 1, 1\} \leftrightarrow H(z) = 1 + z^{-1} + z^{-2} \end{aligned}$$

Thus, by the convolution property (4.26)

$$\begin{aligned} Y(z) &= X(z)H(z) = (1 + z^{-1} + z^{-2} + z^{-3})(1 + z^{-1} + z^{-2}) \\ &= 1 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5} \end{aligned}$$

Hence,

$$h[n] = \{1, 2, 3, 3, 2, 1\}$$

which is the same result obtained in Prob. 2.30.

**4.28.** Using the z-transform, redo Prob. 2.32.

Let  $x[n]$  and  $y[n]$  be the input and output of the system. Then

$$\begin{aligned} x[n] &= u[n] \quad \leftrightarrow \quad X(z) = \frac{z}{z - 1} \quad |z| > 1 \\ y[n] &= \alpha^n u[n] \quad \leftrightarrow \quad Y(z) = \frac{z}{z - \alpha} \quad |z| > |\alpha| \end{aligned}$$

Then, by Eq. (4.41)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z - 1}{z - \alpha} \quad |z| > \alpha$$

Using partial-fraction expansion, we have

$$\frac{H(z)}{z} = \frac{z - 1}{z(z - \alpha)} = \frac{c_1}{z} + \frac{c_2}{z - \alpha}$$

where

$$c_1 = \left. \frac{z-1}{z-\alpha} \right|_{z=0} = \frac{1}{\alpha} \quad c_2 = \left. \frac{z-1}{z} \right|_{z=\alpha} = \frac{\alpha-1}{\alpha} = -\frac{1-\alpha}{\alpha}$$

Thus,

$$H(z) = \frac{1}{\alpha} - \frac{1-\alpha}{\alpha} \frac{z}{z-\alpha} \quad |z| > \alpha$$

Taking the inverse z-transform of  $H(z)$ , we obtain

$$h[n] = \frac{1}{\alpha} \delta[n] - \frac{1-\alpha}{\alpha} \alpha^n u[n]$$

When  $n = 0$ ,

$$h[0] = \frac{1}{\alpha} - \frac{1-\alpha}{\alpha} = 1$$

Then

$$h[n] = \begin{cases} 1 & n = 0 \\ -(1-\alpha)\alpha^{n-1} & n \geq 1 \end{cases}$$

Thus,  $h[n]$  can be rewritten as

$$h[n] = \delta[n] - (1-\alpha) \alpha^{n-1} u[n-1]$$

which is the same result obtained in [Prob. 2.32](#).

**4.29.** The output  $y[n]$  of a discrete-time LTI system is found to be  $2\left(\frac{1}{3}\right)^n u[n]$  when the input  $x[n]$  is  $u[n]$ .

- Find the impulse response  $h[n]$  of the system.
- Find the output  $y[n]$  when the input  $x[n]$  is  $\left(\frac{1}{2}\right)^n u[n]$ .

$$x[n] = u[n] \leftrightarrow X(z) = \frac{z}{z-1} \quad |z| > 1$$

$$y[n] = 2\left(\frac{1}{3}\right)^n u[n] \leftrightarrow Y(z) = \frac{2z}{z-\frac{1}{3}} \quad |z| > \frac{1}{3}$$

Hence, the system function  $H(z)$  is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(z-1)}{z-\frac{1}{3}} \quad |z| > \frac{1}{3}$$

Using partial-fraction expansion, we have

$$\frac{H(z)}{z} = \frac{2(z-1)}{z\left(z-\frac{1}{3}\right)} = \frac{c_1}{z} + \frac{c_2}{z-\frac{1}{3}}$$

where

$$c_1 = \left. \frac{2(z-1)}{z - \frac{1}{3}} \right|_{z=0} = 6 \quad c_2 = \left. \frac{2(z-1)}{z} \right|_{z=1/3} = -4$$

Thus,

$$H(z) = 6 - 4 \frac{z}{z - \frac{1}{3}} \quad |z| > \frac{1}{3}$$

Taking the inverse z-transform of  $H(z)$ , we obtain

$$h[n] = 6\delta[n] - 4\left(\frac{1}{3}\right)^n u[n]$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \leftrightarrow X(z) = \frac{z}{z - \frac{1}{2}} \quad |z| > \frac{1}{2}$$

Then,

$$Y(z) = X(z)H(z) = \frac{2z(z-1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} \quad |z| > \frac{1}{2}$$

Again by partial-fraction expansion we have

$$\frac{Y(z)}{z} = \frac{2(z-1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} = \frac{c_1}{z - \frac{1}{2}} + \frac{c_2}{z - \frac{1}{3}}$$

where

$$c_1 = \left. \frac{2(z-1)}{z - \frac{1}{3}} \right|_{z=1/2} = -6 \quad c_2 = \left. \frac{2(z-1)}{z - \frac{1}{2}} \right|_{z=1/3} = 8$$

Thus,

$$Y(z) = -6 \frac{z}{z - \frac{1}{2}} + 8 \frac{z}{z - \frac{1}{3}} \quad |z| > \frac{1}{2}$$

Taking the inverse z-transform of  $Y(z)$ , we obtain

$$y[n] = \left[ -6\left(\frac{1}{2}\right)^n + 8\left(\frac{1}{3}\right)^n \right] u[n]$$

**4.30.** If a discrete-time LTI system is BIBO stable, show that the ROC of its system function  $H(z)$  must contain the unit circle; that is,  $|z| = 1$ .

A discrete-time LTI system is BIBO stable if and only if its impulse response  $h[n]$  is absolutely summable, that is [Eq. (2.49)],

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Now

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Let  $z = e^{j\Omega}$  so that  $|z| = |e^{j\Omega}| = 1$ . Then

$$\begin{aligned} |H(e^{j\Omega})| &= \left| \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |h[n]e^{-j\Omega n}| = \sum_{n=-\infty}^{\infty} |h[n]| < \infty \end{aligned}$$

Therefore, we see that if the system is stable, then  $H(z)$  converges for  $z = e^{j\Omega}$ . That is, for a stable discrete-time LTI system, the ROC of  $H(z)$  must contain the unit circle  $|z| = 1$ .

**4.31.** Using the z-transform, redo [Prob. 2.38](#).

a. From [Prob. 2.38](#) the impulse response of the system is

$$h[n] = \alpha^n u[n]$$

Then

$$H(z) = \frac{z}{z - \alpha} \quad |z| > |\alpha|$$

Since the ROC of  $H(z)$  is  $|z| > |\alpha|$ ,  $z = \infty$  is included. Thus, by the result from [Prob. 4.5](#) we conclude that  $h[n]$  is a causal sequence. Thus, the system is causal.

b. If  $|\alpha| > 1$ , the ROC of  $H(z)$  does not contain the unit circle  $|z| = 1$ , and hence the system will not be stable. If  $|\alpha| < 1$ , the ROC of  $H(z)$  contains the unit circle  $|z| = 1$ , and hence the system will be stable.

**4.32.** A causal discrete-time LTI system is described by

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

(4.88)

where  $x[n]$  and  $y[n]$  are the input and output of the system, respectively.

- Determine the system function  $H(z)$ .
- Find the impulse response  $h[n]$  of the system.
- Find the step response  $s[n]$  of the system.
- Taking the z-transform of [Eq. \(4.88\)](#), we obtain

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z)$$

or



$$\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right)Y(z) = X(z)$$

Thus,

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}} \\ &= \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} \quad |z| > \frac{1}{2} \end{aligned}$$

b. Using partial-fraction expansion, we have

$$\frac{H(z)}{z} = \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} = \frac{c_1}{z - \frac{1}{2}} + \frac{c_2}{z - \frac{1}{4}}$$

where

$$c_1 = \left. \frac{z}{z - \frac{1}{4}} \right|_{z=1/2} = 2 \quad c_2 = \left. \frac{z}{z - \frac{1}{2}} \right|_{z=1/4} = -1$$

Thus,

$$H(z) = 2 \frac{z}{z - \frac{1}{2}} - \frac{z}{z - \frac{1}{4}} \quad |z| > \frac{1}{2}$$

Taking the inverse z-transform of  $H(z)$ , we get

$$\begin{aligned} h[n] &= \left[ 2 \left( \frac{1}{2} \right)^n - \left( \frac{1}{4} \right)^n \right] u[n] \\ x[n] = u[n] &\leftrightarrow X(z) = \frac{z}{z-1} \quad |z| > 1 \end{aligned}$$

Then

$$Y(z) = X(z)H(z) = \frac{z^3}{(z-1)\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} \quad |z| > 1$$

Again using partial-fraction expansion, we have

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} = \frac{c_1}{z-1} + \frac{c_2}{z - \frac{1}{2}} + \frac{c_3}{z - \frac{1}{4}}$$

where

$$c_1 = \left. \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} \right|_{z=1} = \frac{8}{3} \quad c_2 = \left. \frac{z^2}{(z-1)\left(z - \frac{1}{4}\right)} \right|_{z=1/2} = -2$$

$$c_3 = \left. \frac{z^2}{(z-1)\left(z - \frac{1}{2}\right)} \right|_{z=1/4} = \frac{1}{3}$$

Thus,

$$Y(z) = \frac{8}{3} \frac{z}{z-1} - 2 \frac{z}{z - \frac{1}{2}} + \frac{1}{3} \frac{z}{z - \frac{1}{4}} \quad |z| > 1$$

Taking the inverse z-transformation of  $Y(z)$ , we obtain

$$y[n] = s[n] = \left[ \frac{8}{3} - 2 \left( \frac{1}{2} \right)^n + \frac{1}{3} \left( \frac{1}{4} \right)^n \right] u[n]$$

**4.33.** Using the z-transform, redo [Prob. 2.41](#).

As in [Prob. 2.41](#), from [Fig. 2-30](#) we see that

$$q[n] = 2q[n-1] + x[n]$$

$$y[n] = q[n] + 3q[n-1]$$

Taking the z-transform of the above equations, we get

$$Q(z) = 2z^{-1} Q(z) + X(z)$$

$$Y(z) = Q(z) + 3z^{-1} Q(z)$$

Rearranging, we get

$$(1 - 2z^{-1}) Q(z) = X(z)$$

$$(1 + 3z^{-1}) Q(z) = Y(z)$$

from which we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1}}{1 - 2z^{-1}}$$

(4.89)

Rewriting [Eq. \(4.89\)](#), we have

$$(1 - 2z^{-1}) Y(z) = (1 + 3z^{-1}) X(z)$$

or

$$Y(z) - 2z^{-1} Y(z) = X(z) + 3z^{-1} X(z)$$

(4.90)

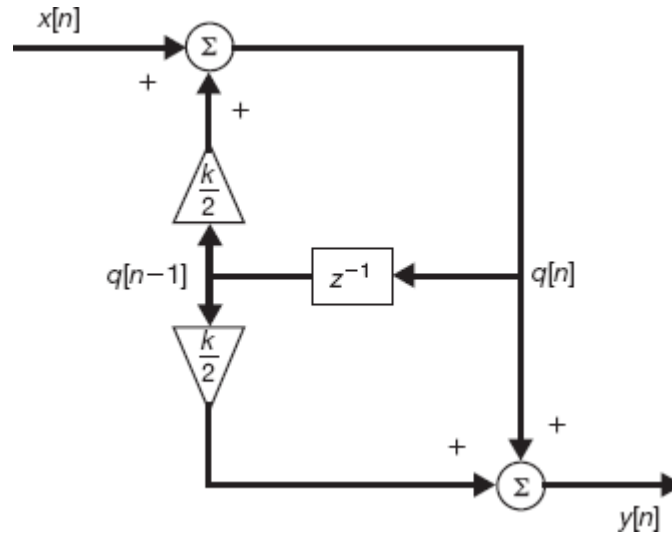
Taking the inverse z-transform of [Eq. \(4.90\)](#) and using the time-shifting property [\(4.18\)](#), we obtain

$$y[n] - 2y[n-1] = x[n] + 3x[n-1]$$

which is the same as Eq. (2.148).

**4.34.** Consider the discrete-time system shown in Fig. 4-8. For what values of  $k$  is the system BIBO stable?

Figure 4-8



From Fig. 4-8 we see that

$$q[n] = x[n] + \frac{k}{2}q[n-1]$$

$$y[n] = q[n] + \frac{k}{3}q[n-1]$$

Taking the z-transform of the above equations, we obtain

$$Q(z) = X(z) + \frac{k}{2}z^{-1}Q(z)$$

$$Y(z) = Q(z) + \frac{k}{3}z^{-1}Q(z)$$

Rearranging, we have

$$\left(1 - \frac{k}{2}z^{-1}\right)Q(z) = X(z)$$

$$\left(1 + \frac{k}{3}z^{-1}\right)Q(z) = Y(z)$$

from which we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + (k/3)z^{-1}}{1 - (k/2)z^{-1}} = \frac{z + k/3}{z - k/2} \quad |z| > \left|\frac{k}{2}\right|$$

which shows that the system has one zero at  $z = -k/3$  and one pole at  $z = k/2$  and that the ROC is  $|z| > |k/2|$ . Thus, as shown in Prob. 4.30, the system will be BIBO stable if the ROC contains the unit circle,  $|z| = 1$ . Hence, the system is stable only if  $|k| < 2$ .

## 4.8.5. Unilateral z-Transform

**4.35.** Find the unilateral z-transform of the following  $x[n]$ :

a.  $x[n] = a^n u[n]$

b.  $x[n] = a^{n+1} u[n+1]$

a. Since  $x[n] = 0$  for  $n < 0$ ,  $X_I(z) = X(z)$ , and from Example 4.1 we have

$$X_I(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad |z| > |a|$$

(4.91)

b. By definition (4.49) we have

$$\begin{aligned} X_I(z) &= \sum_{n=0}^{\infty} a^{n+1} u[n+1] z^{-n} = \sum_{n=0}^{\infty} a^{n+1} z^{-n} = a \sum_{n=0}^{\infty} (az^{-1})^n \\ &= a \frac{1}{1 - az^{-1}} = \frac{az}{z - a} \quad |z| > |a| \end{aligned}$$

(4.92)

Note that in this case  $x[n]$  is not a causal sequence; hence,  $X_I(z) \neq X(z)$  [see Eq. (4.73) in Prob. 4.10].

**4.36.** Verify Eqs. (4.50) and (4.51); that is, for  $m \geq 0$ ,

a.  $x[n - m] \Leftrightarrow z^{-m} X_I(z) + z^{-m+1} x[-1] + z^{-m+2} x[-2] + \dots + x[-m]$

b.  $x[n + m] \Leftrightarrow z^m X_I(z) - z^m x[0] - z^{m-1} x[1] - \dots - z x[m - 1]$

a. By definition (4.49) with  $m \geq 0$  and using the change in variable  $k = n - m$ , we have

$$\begin{aligned} \mathcal{Z}_I\{x[n - m]\} &= \sum_{n=0}^{\infty} x[n - m] z^{-n} = \sum_{k=-m}^{\infty} x[k] z^{-(m+k)} \\ &= z^{-m} \left\{ \sum_{k=0}^{\infty} x[k] z^{-k} + \sum_{k=-1}^{-m} x[k] z^{-k} \right\} \\ &= z^{-m} \{X_I(z) + x[-1]z + x[-2]z^2 + \dots + x[-m]z^m\} \\ &= z^{-m} X_I(z) + z^{-m+1} x[-1] + z^{-m+2} x[-2] + \dots + x[-m] \end{aligned}$$

b. With  $m \geq 0$

$$\begin{aligned} \mathcal{Z}_I\{x[n + m]\} &= \sum_{n=0}^{\infty} x[n + m] z^{-n} = \sum_{k=m}^{\infty} x[k] z^{-(k-m)} \\ &= z^m \left\{ \sum_{k=0}^{\infty} x[k] z^{-k} - \sum_{k=0}^{m-1} x[k] z^{-k} \right\} \\ &= z^m \{X_I(z) - (x[0] + x[1]z^{-1} + \dots + x[m-1]z^{-(m-1)})\} \\ &= z^m X_I(z) - z^m x[0] - z^{m-1} x[1] - \dots - z x[m-1] \end{aligned}$$

**4.37.** Using the unilateral z-transform, redo Prob. 2.42.

The system is described by

$$y[n] - ay[n-1] = x[n]$$

(4.93)

with  $y[-1] = y_{-1}$  and  $x[n] = Kb^n u[n]$ . Let

$$y[n] \leftrightarrow Y_I(z)$$

Then from Eq. (4.50)

$$y[n-1] \leftrightarrow z^{-1}Y_I(z) + y[-1] = z^{-1}Y_I(z) + y_{-1}$$

From (Table 4-1) we have

$$x[n] \leftrightarrow X_I(z) = K \frac{z}{z-b} \quad |z| > |b|$$

Taking the unilateral z-transform of Eq. (4.93), we obtain

$$Y_I(z) - a\{z^{-1}Y_I(z) + y_{-1}\} = K \frac{z}{z-b}$$

or

$$(1 - az^{-1})Y_I(z) = ay_{-1} + K \frac{z}{z-b}$$

or

$$\left(\frac{z-a}{z}\right)Y_I(z) = ay_{-1} + K \frac{z}{z-b}$$

Thus,

$$Y_I(z) = ay_{-1} \frac{z}{z-a} + K \frac{z^2}{(z-a)(z-b)}$$

Using partial-fraction expansion, we obtain

$$Y_I(z) = ay_{-1} \frac{z}{z-a} + \frac{K}{b-a} \left( b \frac{z}{z-b} - a \frac{z}{z-a} \right)$$

Taking the inverse z-transform of  $Y_I(z)$ , we get

$$\begin{aligned} y[n] &= ay_{-1} a^n u[n] + K \frac{b}{b-a} b^n u[n] - K \frac{a}{b-a} a^n u[n] \\ &= \left( y_{-1} a^{n+1} + K \frac{b^{n+1} - a^{n+1}}{b-a} \right) u[n] \end{aligned}$$

which is the same as Eq. (2.158).

**4.38.** For each of the following difference equations and associated input and initial conditions, determine the output  $y[n]$ :

- $y[n] - \frac{1}{2}y[n-1] = x[n]$ , with  $x[n] = (\frac{1}{3})^n$ ,  $y[-1] = 1$
- $3y[n] - 4y[n-1] + y[n-2] = x[n]$ , with  $x[n] = (\frac{1}{2})^n$ ,  $y[-1] = 1$ ,  $y[-2] = 2$

$$x[n] \leftrightarrow X_I(z) = \frac{z}{z - \frac{1}{3}} \quad |z| > \left| \frac{1}{3} \right|$$

Taking the unilateral z-transform of the given difference equation, we get

$$Y_I(z) - \frac{1}{2}\{z^{-1}Y_I(z) + y[-1]\} = X_I(z)$$

Substituting  $y[-1] = 1$  and  $X_I(z)$  into the above expression, we get

$$\left(1 - \frac{1}{2}z^{-1}\right)Y_I(z) = \frac{1}{2} + \frac{z}{z - \frac{1}{3}}$$

or

$$\left(\frac{z - \frac{1}{2}}{z}\right)Y_I(z) = \frac{1}{2} + \frac{z}{z - \frac{1}{3}}$$

Thus,

$$Y_I(z) = \frac{1}{2} \frac{z}{z - \frac{1}{2}} + \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} = \frac{7}{2} \frac{z}{z - \frac{1}{2}} - 2 \frac{z}{z - \frac{1}{3}}$$

Hence,

$$y[n] = 7\left(\frac{1}{2}\right)^{n+1} - 2\left(\frac{1}{3}\right)^n \quad n \geq -1$$

$$x[n] \leftrightarrow X_I(z) = \frac{z}{z - \frac{1}{2}} \quad |z| > \left| \frac{1}{2} \right|$$

Taking the unilateral z-transform of the given difference equation, we obtain

$$3Y_I(z) - 4\{z^{-1}Y_I(z) + y[-1]\} + \{z^{-2}Y_I(z) + z^{-1}y[-1] + y[-2]\} = X_I(z)$$

Substituting  $y[-1] = 1$ ,  $y[-2] = 2$ , and  $X_I(z)$  into the above expression, we get

$$(3 - 4z^{-1} + z^{-2})Y_I(z) = 2 - z^{-1} + \frac{z}{z - \frac{1}{2}}$$

or

$$\frac{3(z-1)\left(z - \frac{1}{3}\right)}{z^2}Y_I(z) = \frac{3z^2 - 2z + \frac{1}{2}}{z\left(z - \frac{1}{2}\right)}$$

Thus,

$$Y_I(z) = \frac{z \left( 3z^2 - 2z + \frac{1}{2} \right)}{3(z-1) \left( z - \frac{1}{2} \right) \left( z - \frac{1}{3} \right)}$$

$$= \frac{3}{2} \frac{z}{z-1} - \frac{z}{z - \frac{1}{2}} + \frac{1}{2} \frac{z}{z - \frac{1}{3}}$$

Hence,

$$y[n] = \frac{3}{2} - \left( \frac{1}{2} \right)^n + \frac{1}{2} \left( \frac{1}{3} \right)^n \quad n \geq -2$$

**4.39.** Let  $x[n]$  be a causal sequence and

$$x[n] \leftrightarrow X(z)$$

Show that

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

(4.94)

Equation (4.94) is called the *initial value theorem* for the  $z$ -transform.

Since  $x[n] = 0$  for  $n < 0$ , we have

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots$$

As  $z \rightarrow \infty$ ,  $z^{-n} \rightarrow 0$  for  $n > 0$ . Thus, we get

$$\lim_{z \rightarrow \infty} X(z) = x[0]$$

**4.40.** Let  $x[n]$  be a causal sequence and

$$x[n] \leftrightarrow X(z)$$

Show that if  $X(z)$  is a rational function with all its poles strictly inside the unit circle except possibly for a first-order pole at  $z = 1$ , then

$$\lim_{N \rightarrow \infty} x[N] = \lim_{z \rightarrow 1} (1 - z^{-1})X(z)$$

(4.95)

Equation (4.95) is called the *final value theorem* for the  $z$ -transform.

From the time-shifting property (4.19) we have

$$\mathcal{Z}\{x[n] - x[n-1]\} = (1 - z^{-1})X(z)$$

(4.96)

The left-hand side of Eq. (4.96) can be written as

$$\sum_{n=0}^{\infty} \{x[n] - x[n-1]\} z^{-n} = \lim_{N \rightarrow \infty} \sum_{n=0}^N \{x[n] - x[n-1]\} z^{-n}$$

If we now let  $z \rightarrow 1$ , then from Eq. (4.96) we have

$$\lim_{z \rightarrow 1} (1 - z^{-1})X(z) = \lim_{N \rightarrow \infty} \sum_{n=0}^N \{x[n] - x[n-1]\} = \lim_{N \rightarrow \infty} x[N]$$

## 4.9. SUPPLEMENTARY PROBLEMS

4.41. Find the z-transform of the following  $x[n]$ :

$$(a) \quad x[n] = \left\{ \frac{1}{2}, 1, -\frac{1}{3} \right\}$$

$$(b) \quad x[n] = 2\delta[n+2] - 3\delta[n-2]$$

$$(c) \quad x[n] = 3 \left( -\frac{1}{2} \right)^n u[n] - 2(3)^n u[-n-1]$$

$$(d) \quad x[n] = 3 \left( \frac{1}{2} \right)^n u[n] - 2 \left( \frac{1}{4} \right)^n u[-n-1]$$

4.42. Show that if  $x[n]$  is a left-sided sequence and  $X(z)$  converges from some value of  $z$ , then the ROC of  $X(z)$  is of the form

$$|z| < r_{\min} \quad \text{or} \quad 0 < |z| < r_{\min}$$

where  $r_{\min}$  is the smallest magnitude of any of the poles of  $X(z)$ .

4.43. Given

$$X(z) = \frac{z(z-4)}{(z-1)(z-2)(z-3)}$$

- State all the possible regions of convergence.
- For which ROC is  $X(z)$  the z-transform of a causal sequence?

4.44. Verify the time-reversal property (4.23); that is,

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \quad R' = \frac{1}{R}$$

4.45. Show the following properties for the z-transform.

- If  $x[n]$  is even, then  $X(z^{-1}) = X(z)$ .
- If  $x[n]$  is odd, then  $X(z^{-1}) = -X(z)$ .
- If  $x[n]$  is odd, then there is a zero in  $X(z)$  at  $z = 1$ .

4.46. Consider the continuous-time signal

$$x(t) = e^{-at} \quad t \geq 0$$



Let the sequence  $x[n]$  be obtained by uniform sampling of  $x(t)$  such that  $x[n] = x(nT_s)$ , where  $T_s$  is the sampling interval. Find the z-transform of  $x[n]$ .

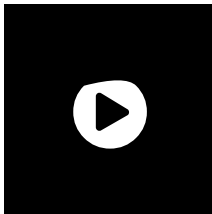
**4.47.** Derive the following transform pairs:

$$(\cos \Omega_0 n) u[n] \leftrightarrow \frac{z^2 - (\cos \Omega_0)z}{z^2 - (2 \cos \Omega_0)z + 1} \quad |z| > 1$$

$$(\sin \Omega_0 n) u[n] \leftrightarrow \frac{(\sin \Omega_0)z}{z^2 - (2 \cos \Omega_0)z + 1} \quad |z| > 1$$

**4.48.** Find the z-transforms of the following  $x[n]$ :

- $x[n] = (n - 3)u[n - 3]$
- $x[n] = (n - 3)u[n]$
- $x[n] = u[n] - u[n - 3]$
- $x[n] = n\{u[n] - u[n - 3]\}$



*Schaum's Signals and Systems Supplementary Problem 4.48: z-Transforms*

This video demonstrates how to find the z-transform of a discrete-time system.

Carlotta A. Berry, Associate Professor, Electrical and Computer Engineering, Rose-Hulman Institute of Technology  
2013

**4.49.** Using the relation

$$a^n u[n] \leftrightarrow \frac{z}{z - a} \quad |z| > |a|$$

find the z-transform of the following  $x[n]$ :

- $x[n] = na^{n-1}u[n]$
- $x[n] = n(n - 1)a^{n-2}u[n]$
- $x[n] = n(n - 1) \dots (n - k + 1) a^{n-k}u[n]$

**4.50.** Using the z-transform, verify [Eqs. \(2.130\)](#) and [\(2.131\)](#) in [Prob. 2.27](#); that is,

- $x[n] * \delta[n] = x[n]$
- $x[n] * \delta[n - n_0] = x[n - n_0]$

**4.51.** Using the z-transform, redo [Prob. 2.47](#).

**4.52.** Find the inverse z-transform of

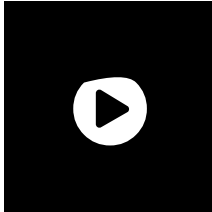
$$X(z) = e^{a/z} \quad |z| > 0$$

**4.53.** Using the method of long division, find the inverse z-transform of the following  $X(z)$ :

$$(a) \quad X(z) = \frac{z}{(z - 1)(z - 2)}, \quad |z| < 1$$

$$(b) \quad X(z) = \frac{z}{(z-1)(z-2)}, \quad 1 < |z| < 2$$

$$(c) \quad X(z) = \frac{z}{(z-1)(z-2)}, \quad |z| > 2$$



*Schaum's Signals and Systems Supplementary Problem 4.53/4.54: Inverse z-Transforms*

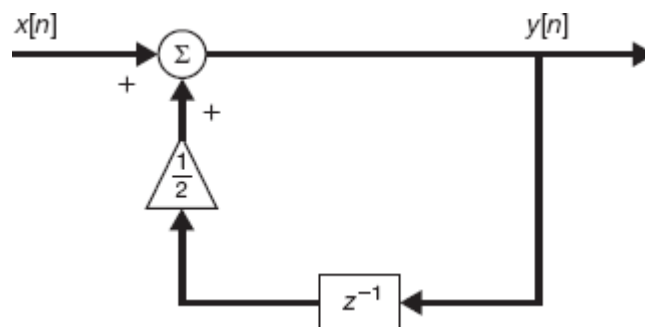
This video demonstrates how to find the inverse z-transform of several discrete-time signals.

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2013

4.54. Using the method of partial-fraction expansion, redo Prob. 4.53.

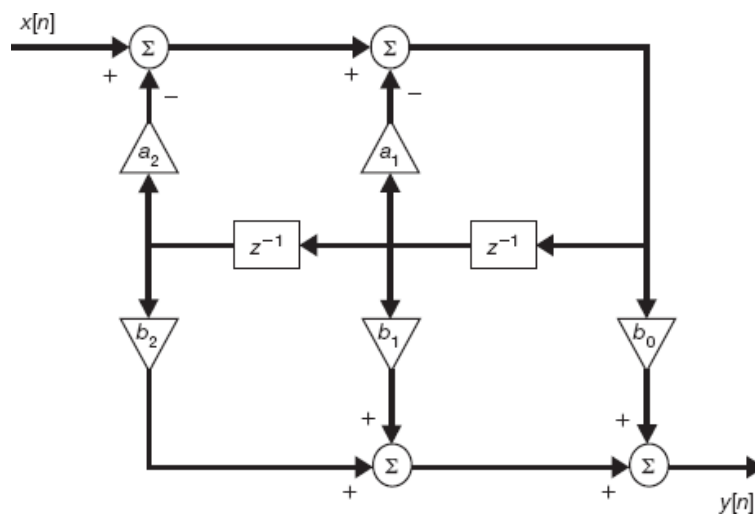
4.55. Consider the system shown in Fig. 4-9. Find the system function  $H(z)$  and its impulse response  $h[n]$ .

Figure 4-9

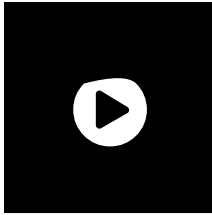


4.56. Consider the system shown in Fig. 4-10.

Figure 4-10



- Find the system function  $H(z)$ .
- Find the difference equation relating the output  $y[n]$  and input  $x[n]$ .



*Schaum's Signals and Systems Supplementary Problem 4.56: Transfer Functions Example 2*

This video demonstrates how to use z-transforms to find the transfer function,  $H(z)$  and the difference equation for the system.

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**4.57.** Consider a discrete-time LTI system whose system function  $H(z)$  is given by

$$H(z) = \frac{z}{z - \frac{1}{2}} \quad |z| > \frac{1}{2}$$

- Find the step response  $s[n]$ .
- Find the output  $y[n]$  to the input  $x[n] = nu[n]$ .

**4.58.** Consider a causal discrete-time system whose output  $y[n]$  and input  $x[n]$  are related by

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n]$$

- Find its system function  $H(z)$ .
- Find its impulse response  $h[n]$ .

**4.59.** Using the unilateral z-transform, solve the following difference equations with the given initial conditions:

- $y[n] - 3y[n-1] = x[n]$ , with  $x[n] = 4u[n]$ ,  $y[-1] = 1$
- $y[n] - 5y[n-1] + 6y[n-2] = x[n]$ , with  $x[n] = u[n]$ ,  $y[-1] = 3$ ,  $y[-2] = 2$

**4.60.** Determine the initial and final values of  $x[n]$  for each of the following  $X(z)$ :

$$(a) \quad X(z) = \frac{2z \left( z - \frac{5}{12} \right)}{\left( z - \frac{1}{2} \right) \left( z - \frac{1}{3} \right)}, \quad |z| > \frac{1}{2}$$

$$(b) \quad X(z) = \frac{z}{2z^2 - 3z + 1}, \quad |z| > 1$$

## 4.10. ANSWERS TO SUPPLEMENTARY PROBLEMS

**4.41.**

$$(a) \quad X(z) = \frac{1}{2} + z^{-1} - \frac{1}{3}z^{-2}, \quad 0 < |z|$$

$$(b) \quad X(z) = 2z^3 - 3z^{-3}, \quad 0 < |z| < \infty$$

$$(c) \quad X(z) = \frac{z(5z-8)}{\left(z + \frac{1}{2}\right)(z-3)}, \quad \frac{1}{2} < |z| < 3$$

$$(d) \quad X(z) \text{ does not exist.}$$

4.42. *Hint:* Proceed in a manner similar to Prob. 4.5.

4.43.

$$a. \quad 0 < |z| < 1, 1 < |z| < 2, 2 < |z| < 3, |z| > 3$$

$$b. \quad |z| > 3$$

4.44. *Hint:* Change  $n$  to  $-n$  in definition (4.3).

4.45. *Hint:*

a. Use Eqs. (1.2) and (4.23).

b. Use Eqs. (1.3) and (4.23).

c. Use the result from part (b).

$$4.46. \quad X(z) = \frac{1}{1 - e^{-\alpha T_s} z^{-1}}$$

4.47. *Hint:* Use Euler's formulas.

$$\cos \Omega_0 n = \frac{1}{2}(e^{j\Omega_0 n} + e^{-j\Omega_0 n}) \quad \sin \Omega_0 n = \frac{1}{2j}(e^{j\Omega_0 n} - e^{-j\Omega_0 n})$$

and use Eqs. (4.8) and (4.10) with  $a = e^{\pm j\Omega_0}$ .

4.48.

$$(a) \quad \frac{z^{-2}}{(z-1)^2}, \quad |z| > 1$$

$$(b) \quad \frac{-3z^2 + 4z}{(z-1)^2}, \quad |z| > 1$$

$$(c) \quad \frac{z - z^{-2}}{z-1}, \quad |z| > 1$$

$$(d) \quad \frac{z - 4z^{-2} + 3z^{-3}}{(z-1)^2}, \quad |z| > 1$$

4.49. *Hint:* Differentiate both sides of the given relation consecutively with respect to  $a$ .

$$(a) \frac{z}{(z-a)^2}, |z| > |a|$$

$$(b) \frac{2z}{(z-a)^3}, |z| > |a|$$

$$(c) \frac{k!z}{(z-a)^{k+1}}, |z| > |a|$$

4.50. *Hint:* Use Eq. (4.26) of the z-transform and transform pairs 1 and 4 from (Table 4-1).

4.51. *Hint:* Use Eq. (4.26) and (Table 4-1).

4.52. *Hint:* Use the power series expansion of the exponential function  $e^r$ .

$$x[n] = \frac{a^n}{n!} u[n]$$

4.53.

$$(a) \ x[n] = \{\dots, \frac{7}{8}, \frac{3}{4}, \frac{1}{2}, 0\}$$

↑

$$(b) \ x[n] = \{\dots, -\frac{1}{8}, -\frac{1}{4}, -\frac{1}{2}, -1, -1, -1, \dots\}$$

↑

$$(c) \ x[n] = \{0, 1, 3, 7, 15, \dots\}$$

4.54.

a.  $x[n] = (1 - 2^n)u[-n - 1]$

b.  $x[n] = -u[n] - 2^n u[-n - 1]$

c.  $x[n] = (-1 + 2^n)u[n]$

4.55.  $H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \ h[n] = \left(\frac{1}{2}\right)^n u[n]$

4.56.

$$(a) \ H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$(b) \ y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

4.57.

$$(a) \ s[n] = \left[2 - \left(\frac{1}{2}\right)^n\right] u[n]$$

$$(b) \ y[n] = 2 \left[ \left(\frac{1}{2}\right)^n + n - 1 \right] u[n]$$

4.58.

$$(a) \quad H(z) = \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}, \quad |z| > \frac{1}{2}$$

$$(b) \quad h[n] = \left[ 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \right] u[n]$$

4.59.

$$(a) \quad y[n] = -2 + 9(3)^n, \quad n \geq -1$$

$$(b) \quad y[n] = \frac{1}{2} + 8(2)^n - \frac{9}{2}(3)^n, \quad n \geq -2$$

4.60.

a.  $x[0] = 2, x[\infty] = 0$

b.  $x[0] = 0, x[\infty] = 1$