2. Linear Time-Invariant Systems

2.1. Introduction

Two most important attributes of systems are linearity and time-invariance. In this chapter we develop the fundamental input-output relationship for systems having these attributes. It will be shown that the input-output relationship for LTI systems is described in terms of a convolution operation. The importance of the convolution operation in LTI systems stems from the fact that knowledge of the response of an LTI system to the unit impulse input allows us to find its output to any input signals. Specifying the input-output relationships for LTI systems by differential and difference equations will also be discussed.

2.2. Response of a Continuous-Time LTI System and the Convolution Integral

2.2.1. A. Impulse Response:

The *impulse response* $h(t)$ of a continuous-time LTI system (represented by $\mathbf{T}$) is defined to be the response of the system when the input is $\delta(t)$, that is,

$$h(t) = \mathbf{T}\{\delta(t)\}$$

(2.1)

2.2.2. B. Response to an Arbitrary Input:

From Eq. (1.27) the input $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

(2.2)

Since the system is linear, the response $y(t)$ of the system to an arbitrary input $x(t)$ can be expressed as

$$\begin{aligned} y(t) &= \mathbf{T}\{x(t)\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\, d\tau \end{aligned}$$

(2.3)

Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\}$$

(2.4)

Substituting Eq. (2.4) into Eq. (2.3), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

(2.5)

Equation (2.5) indicates that a continuous-time LTI system is completely characterized by its impulse response $h(t)$.

2.2.3. C. Convolution Integral:

Equation (2.5) defines the *convolution* of two continuous-time signals $x(t)$ and $h(t)$ denoted by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

(2.6)

Equation (2.6) is commonly called the *convolution integral*. Thus, we have the fundamental result that *the output of any continuous-time LTI system is the convolution of the input* $x(t)$ *with the impulse response* $h(t)$ *of the system*. Fig. 2-1 illustrates the definition of the impulse response $h(t)$ and the relationship of Eq. (2.6).

Figure 2-1 Continuous-time LTI system.

2.2.4. D. Properties of the Convolution Integral:

The convolution integral has the following properties.

2.2.4.1. 1. Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

(2.7)

2.2.4.2. 2. Associative:

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

(2.8)

2.2.4.3. 3. Distributive:

$$x(t) * \{h_1(t)\} + h_2(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

(2.9)

2.2.5. E. Convolution Integral Operation:

Applying the commutative property (2.7) of convolution to Eq. (2.6), we obtain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

(2.10)

which may at times be easier to evaluate than Eq. (2.6). From Eq. (2.6) we observe that the convolution integral operation involves the following four steps:

1. The impulse response $h(\tau)$ is time-reversed (that is, reflected about the origin) to obtain $h(-\tau)$ and then shifted by t to form $h(t-\tau) = h[-(\tau-t)]$, which is a function of τ with parameter t.
2. The signal $x(\tau)$ and $h(t-\tau)$ are multiplied together for all values of τ with t fixed at some value.
3. The product $x(\tau)h(t-\tau)$ is integrated over all τ to produce a single output value $y(t)$.
4. Steps 1 to 3 are repeated as t varies over $-\infty$ to ∞ to produce the entire output $y(t)$.

Examples of the above convolution integral operation are given in Probs. 2.4 to 2.6.

2.2.6. F. Step Response:

The *step response* $s(t)$ of a continuous-time LTI system (represented by **T**) is defined to be the response of the system when the input is $u(t)$; that is,

$$s(t) = \mathbf{T}\{u(t)\}$$

(2.11)

In many applications, the step response $s(t)$ is also a useful characterization of the system. The step response $s(t)$ can be easily determined by Eq. (2.10); that is,

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

(2.12)

Thus, the step response $s(t)$ can be obtained by integrating the impulse response $h(t)$. Differentiating Eq. (2.12) with respect to t, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

(2.13)

Thus, the impulse response $h(t)$ can be determined by differentiating the step response $s(t)$.

2.3. Properties of Continuous-Time LTI Systems

2.3.1. A. Systems with or without Memory:

Since the output $y(t)$ of a memoryless system depends on only the present input $x(t)$, then, if the system is also linear and time-invariant, this relationship can only be of the form

$$y(t) = Kx(t)$$

(2.14)

where K is a (gain) constant. Thus, the corresponding impulse response $h(t)$ is simply

$$h(t) = K\delta(t)$$

(2.15)

Therefore, if $h(t_0) \neq 0$ for $t_0 \neq 0$, then continuous-time LTI system has memory.

2.3.2. B. Causality:

As discussed in Sec. 1.5D, a causal system does not respond to an input event until that event actually occurs. Therefore, for a causal continuous-time LTI system, we have

$$h(t) = 0 \qquad t < 0$$

(2.16)

Applying the causality condition (2.16) to Eq. (2.10), the output of a causal continuous-time LTI system is expressed as

$$y(t) = \int_0^{\infty} h(\tau)x(t-\tau)\, d\tau$$

(2.17)

Alternatively, applying the causality condition (2.16) to Eq. (2.6), we have

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\, d\tau$$

(2.18)

Equation (2.18) shows that the only values of the input $x(t)$ used to evaluate the output $y(t)$ are those for $\tau \leq t$.

Based on the causality condition (2.16), any signal $x(t)$ is called *causal* if

$$x(t) = 0 \qquad t < 0$$

(2.19a)

and is called *anticausal* if

$$x(t) = 0 \qquad t > 0$$

(2.19b)

Then, from Eqs. (2.17), (2.18), and (2.19a), when the input $x(t)$ is causal, the output $y(t)$ of a causal continuous-time LTI system is given by

$$y(t) = \int_0^{t} h(\tau)x(t-\tau)\, d\tau = \int_0^{t} x(\tau)h(t-\tau)\, d\tau$$

(2.20)

2.3.3. C. Stability:

The BIBO (bounded-input/bounded-output) stability of an LTI system (Sec. 1.5H) is readily ascertained from its impulse response. It can be shown (Prob. 2.13) that a continuous-time LTI system is BIBO stable if its impulse response is absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

(2.21)

2.4. Eigenfunctions of Continuous-Time LTI Systems

In Chap. 1 (Prob. 1.44) we saw that the eigenfunctions of continuous-time LTI systems represented by **T** are the complex exponentials e^{st}, with s a complex variable. That is,

$$\mathbf{T}\{e^{st}\} = \lambda e^{st}$$

(2.22)

where λ is the eigenvalue of **T** associated with e^{st}. Setting $x(t) = e^{st}$ in Eq. (2.10), we have

$$\begin{aligned} y(t) = \mathbf{T}\{e^{st}\} &= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau = \left[\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau\right] e^{st} \\ &= H(s)\, e^{st} = \lambda e^{st} \end{aligned}$$

(2.23)

where

$$\lambda = H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

(2.24)

Thus, the eigenvalue of a continuous-time LTI system associated with the eigenfunction e^{st} is given by $H(s)$, which is a complex constant whose value is determined by the value of s via Eq. (2.24). Note from Eq. (2.23) that $y(0) = H(s)$ (see Prob. 1.44).

The above results underlie the definitions of the Laplace transform and Fourier transform, which will be discussed in Chaps. 3 and 5.

2.5. Systems Described by Differential Equations

2.5.1. A. Linear Constant-Coefficient Differential Equations:

A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

(2.25)

where coefficients a_k and b_k are real constants. The order N refers to the highest derivative of $y(t)$ in Eq. (2.25). Such differential equations play a central role in describing the input-output relationships of a wide variety of electrical, mechanical, chemical, and biological systems. For instance, in the *RC* circuit considered in Prob. 1.32, the input $x(t) = v_s(t)$ and the output $y(t) = v_c(t)$ are related by a first-order constant-coefficient differential equation [Eq. (1.105)]

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

The general solution of Eq. (2.25) for a particular input $x(t)$ is given by

$$y(t) = y_p(t) + y_h(t)$$

(2.26)

where $y_p(t)$ is a *particular solution* satisfying Eq. (2.25) and $y_h(t)$ is a *homogeneous solution* (or *complementary solution*) satisfying the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0$$

(2.27)

The exact form of $y_h(t)$ is determined by N auxiliary conditions. Note that Eq. (2.25) does not completely specify the output $y(t)$ in terms of the input $x(t)$ unless auxiliary conditions are specified. In general, a set of auxiliary conditions are the values of

$$y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1}y(t)}{dt^{N-1}}$$

at some point in time.

2.5.2. B. Linearity:

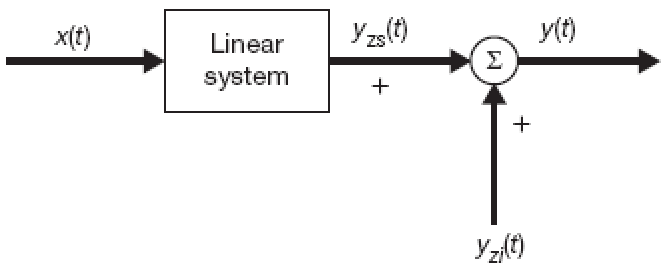
The system specified by Eq. (2.25) will be linear only if all of the auxiliary conditions are zero (seeProb. 2.21). If the auxiliary conditions are not zero, then the response $y(t)$ of a system can be expressed as

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

(2.28)

where $y_{zi}(t)$, called the *zero-input response*, is the response to auxiliary conditions, and $y_{zs}(t)$, called the *zero-state response*, is the response of a linear system with zero auxiliary conditions. This is illustrated in Fig. 2-2.

Figure 2-2 Zero-state and zero-input responses.



Note that $y_{zi}(t) \neq y_h(t)$ and $y_{zs}(t) \neq y_p(t)$ and that in general $y_{zi}(t)$ contains $y_h(t)$ and $y_{zs}(t)$ contains both $y_h(t)$ and $y_p(t)$ (see Prob. 2.20).

2.5.3. C. Causality:

In order for the linear system described by Eq. (2.25) to be causal we must assume the condition of *initial rest* (or an *initially relaxed condition*). That is, if $x(t) = 0$ for $t \leq t_0$, then assume $y(t) = 0$ for $t \leq t_0$ (see Prob. 1.43). Thus, the response for $t > t_0$ can be calculated from Eq. (2.25) with the initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

where

$$\frac{d^k y(t_0)}{dt^k} = \left.\frac{d^k y(t)}{dt^k}\right|_{t=t_0}.$$

Clearly, at initial rest $y_{zi}(t) = 0$.

2.5.4. D. Time-Invariance:

For a linear causal system, initial rest also implies time-invariance (Prob. 2.22).

2.5.5. E. Impulse Response:

The impulse response $h(t)$ of the continuous-time LTI system described by Eq. (2.25) satisfies the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k h(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k \delta(t)}{dt^k}$$

(2.29)

with the initial rest condition. Examples of finding impulse responses are given in Probs. 2.23 to 2.25. In later chapters, we will find the impulse response by using transform techniques.

2.6. Response of a Discrete-Time LTI System and Convolution Sum

2.6.1. A. Impulse Response:

The *impulse response* (or *unit sample response*) $h[n]$ of a discrete-time LTI system (represented by **T**) is defined to be the response of the system when the input is $\delta[n]$; that is,

$$h[n] = \mathbf{T}\{\delta[n]\}$$

(2.30)

2.6.2. B. Response to an Arbitrary Input:

From Eq. (1.51) the input $x[n]$ can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\ \delta[n-k]$$

(2.31)

Since the system is linear, the response $y[n]$ of the system to an arbitrary input $x[n]$ can be expressed as

$$\begin{aligned} y[n] &= \mathbf{T}\{x[n]\} = \mathbf{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\ \delta[n-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} x[k]\,\mathbf{T}\{\delta[n-k]\} \end{aligned}$$

(2.32)

Since the system is time-invariant, we have

$$h[n-k] = \mathbf{T}\{\delta[n-k]\}$$

(2.33)

Substituting Eq. (2.33) into Eq. (2.32), we obtain

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

(2.34)

Equation (2.34) indicates that a discrete-time LTI system is completely characterized by its impulse response $h[n]$.

2.6.3. C. Convolution Sum:

Equation (2.34) defines the *convolution* of two sequences $x[n]$ and $h[n]$ denoted by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

(2.35)

Equation (2.35) is commonly called the *convolution sum*. Thus, again, we have the fundamental result that *the output of any discrete-time LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$ of the system*.

Fig. 2-3 illustrates the definition of the impulse response $h[n]$ and the relationship of Eq. (2.35).

Figure 2-3 Discrete-time LTI system.

2.6.4. D. Properties of the Convolution Sum:

The following properties of the convolution sum are analogous to the convolution integral properties shown in Sec. 2.3.

2.6.4.1. 1. Commutative:

$$x[n] * h[n] = h[n\,] * x[n]$$

(2.36)

2.6.4.2. 2. Associative:

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

(2.37)

2.6.4.3. 3. Distributive:

$$x[n] * \{h_1[n]\} + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

(2.38)

2.6.5. E. Convolution Sum Operation:

Again, applying the commutative property (2.36) of the convolution sum to Eq. (2.35), we obtain

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

(2.39)

which may at times be easier to evaluate than Eq. (2.35). Similar to the continuous-time case, the convolution sum [Eq. (2.35)] operation involves the following four steps:

1. The impulse response $h[k]$ is time-reversed (that is, reflected about the origin) to obtain $h[-k]$ and then shifted by n to form $h[n-k] = h[-(k-n)]$, which is a function of k with parameter n.
2. Two sequences $x[k]$ and $h[n-k]$ are multiplied together for all values of k with n fixed at some value.
3. The product $x[k]h[n-k]$ is summed over all k to produce a single output sample $y[n]$.
4. Steps 1 to 3 are repeated as n varies over $-\infty$ to ∞ to produce the entire output $y[n]$.

Examples of the above convolution sum operation are given in Probs. 2.28 and 2.30.

2.6.6. F. Step Response:

The *step response* $s[n]$ of a discrete-time LTI system with the impulse response $h[n]$ is readily obtained from Eq. (2.39) as

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

(2.40)

From Eq. (2.40) we have

$$h[n] = s[n] - s[n-1]$$

(2.41)

Equations (2.40) and (2.41) are the discrete-time counterparts of Eqs. (2.12) and (2.13), respectively.

2.7. Properties of Discrete-Time LTI Systems

2.7.1. A. Systems with or without Memory:

Since the output $y[n]$ of a memoryless system depends on only the present input $x[n]$, then, if the system is also linear and time-invariant, this relationship can only be of the form

$$y[n] = Kx[n]$$

(2.42)

where K is a (gain) constant. Thus, the corresponding impulse response is simply

$$h[n] = K\delta[n]$$

(2.43)

Therefore, if $h[n_0] \neq 0$ for $n_0 \neq 0$, the discrete-time LTI system has memory.

2.7.2. B. Causality:

Similar to the continuous-time case, the causality condition for a discrete-time LTI system is

$$h[n] = 0 \qquad n < 0$$

(2.44)

Applying the causality condition (2.44) to Eq. (2.39), the output of a causal discrete-time LTI system is expressed as

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

(2.45)

Alternatively, applying the causality condition (2.44) to Eq. (2.35), we have

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

(2.46)

Equation (2.46) shows that the only values of the input $x[n]$ used to evaluate the output $y[n]$ are those for $k \le n$.

As in the continuous-time case, we say that any sequence $x[n]$ is called *causal* if

$$x[n] = 0 \qquad n < 0$$

(2.47a)

and is called *anticausal* if

$$x[n] = 0 \qquad n \geq 0$$

(2.47b)

Then, when the input $x[n]$ is causal, the output $y[n]$ of a causal discrete-time LTI system is given by

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k] = \sum_{k=0}^{n} x[k]h[n-k]$$

(2.48)

2.7.3. C. Stability:

It can be shown (Prob. 2.37) that a discrete-time LTI system is BIBO stable if its impulse response is absolutely summable; that is,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

(2.49)

2.8. Eigenfunctions of Discrete-Time LTI Systems

In Chap. 1 (Prob. 1.45) we saw that the eigenfunctions of discrete-time LTI systems represented by **T** are the complex exponentials z^n, with z a complex variable. That is,

$$\mathbf{T}\{z^n\} = \lambda z^n$$

(2.50)

where λ is the eigenvalue of **T** associated with z^n. Setting $x[n] = z^n$ in Eq. (2.39), we have

$$\begin{aligned} y[n] = \mathbf{T}\{z^n\} &= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} = \left[\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}\right] z^n \\ &= H(z)\, z^n = \lambda z^n \end{aligned}$$

(2.51)

where

$$\lambda = H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

(2.52)

Thus, the eigenvalue of a discrete-time LTI system associated with the eigenfunction z^n is given by $H(z)$, which is a complex constant whose value is determined by the value of z via Eq. (2.52). Note from Eq. (2.51) that $y[0] = H(z)$ (see Prob. 1.45).

The above results underlie the definitions of the z-transform and discrete-time Fourier transform, which will be discussed in Chaps. 4 and 6.

2.9. Systems Described by Difference Equations

The role of differential equations in describing continuous-time systems is played by *difference equations* for discrete-time systems.

2.9.1. A. Linear Constant-Coefficient Difference Equations:

The discrete-time counterpart of the general differential equation (2.25) is the Nth-order linear constant-coefficient difference equation given by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

(2.53)

where coefficients a_k and b_k are real constants. The order N refers to the largest delay of $y[n]$ in Eq. (2.53). An example of the class of linear constant-coefficient difference equations is given in Chap. 1 (Prob. 1.37). Analogous to the continuous-time case, the solution of Eq. (2.53) and all properties of systems, such as linearity, causality, and time-invariance, can be developed following an approach that directly parallels the discussion for differential equations. Again we emphasize that the system described by Eq. (2.53) will be causal and LTI if the system is initially at rest.

2.9.2. B. Recursive Formulation:

An alternate and simpler approach is available for the solution of Eq. (2.53). Rearranging Eq. (2.53) in the form

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

(2.54)

we obtain a formula to compute the output at time n in terms of the present input and the previous values of the input and output. From Eq. (2.54) we see that the need for auxiliary conditions is obvious and that to calculate $y[n]$ starting at $n = n_0$, we must be given the values of $y[n_0 - 1], y[n_0 - 2], \ldots, y[n_0 - N]$ as well as the input $x[n]$ for $n \geq n_0 - M$. The general form of Eq. (2.54) is called a *recursive equation*, since it specifies a recursive procedure for determining the output in terms of the input and previous outputs. In the special case when $N = 0$, from Eq. (2.53) we have

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

(2.55)

which is a *nonrecursive equation*, since previous output values are not required to compute the present output. Thus, in this case, auxiliary conditions are not needed to determine $y[n]$.

2.9.3. C. Impulse Response:

Unlike the continuous-time case, the impulse response $h[n]$ of a discrete-time LTI system described by Eq. (2.53) or, equivalently, by Eq. (2.54) can be determined easily as

$$h[n]=\frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k\,\delta[n-k]-\sum_{k=1}^{N} a_k\,h[n-k]\right\}$$

(2.56)

For the system described by Eq. (2.55) the impulse response $h[n]$ is given by

$$h[n]=\frac{1}{a_0}\sum_{k=0}^{M} b_k\,\delta[n-k]=\begin{cases} b_n/a_0 & 0\le n\le M \\ 0 & \text{otherwise}\end{cases}$$

(2.57)

Note that the impulse response for this system has finite terms; that is, it is nonzero for only a finite time duration. Because of this property, the system specified by Eq. (2.55) is known as a *finite impulse response* (FIR) system. On the other hand, a system whose impulse response is nonzero for an infinite time duration is said to be an *infinite impulse response* (IIR) system. Examples of finding impulse responses are given in Probs. 2.44 and 2.45. In Chap. 4, we will find the impulse response by using transform techniques.

2.10. SOLVED PROBLEMS

2.10.1. Responses of a Continuous-Time LTI System and Convolution

2.1. Verify Eqs. (2.7) and (2.8); that is,

a. $x(t) * h(t) = h(t) * x(t)$

b. $\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$

a. By definition (2.6)

$$x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

By changing the variable $t - \tau = \lambda$, we have

$$x(t)*h(t)=\int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)\,d\lambda=\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\,d\lambda=h(t)*x(t)$$

b. Let $x(t) * h_1(t) = f_1(t)$ and $h_1(t) * h_2(t) = f_2(t)$. Then

$$f_1(t)=\int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)\,d\tau$$

and

$$\{x(t) * h_1(t)\} * h_2(t) = f_1(t) * h_2(t) = \int_{-\infty}^{\infty} f_1(\sigma) h_2(t-\sigma)\, d\sigma$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) h_1(\sigma - \tau)\, d\tau \right] h_2(t-\sigma)\, d\sigma$$

Substituting $\lambda = \sigma - \tau$ and interchanging the order of integration, we have

$$\{x(t) * \mathrm{h}_1(t)\} * h_2(t) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h_1(\lambda) h_2(t-\tau-\lambda)\, d\lambda \right] d\tau$$

Now, since

$$f_2(t) = \int_{-\infty}^{\infty} h_1(\lambda) h_2(t-\lambda)\, d\lambda$$

we have

$$f_2(t-\tau) = \int_{-\infty}^{\infty} h_1(\lambda) h_2(t-\tau-\lambda)\, d\lambda$$

Thus,

$$\{x(t) * h_1(t)\} * h_2(t) = \int_{-\infty}^{\infty} x(\tau) f_2(t-\tau)\, d\tau$$

$$= x(t) * f_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

2.2. Show that

$$x(t) * \delta(t) = x(t)$$

(2.58)

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

(2.59)

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

(2.60)

$$x(t) * u(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$$

(2.61)

a. By definition (2.6) and Eq. (1.22) we have

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau = x(\tau)\big|_{\tau=t} = x(t)$$

b. By Eqs. (2.7) and (1.22) we have

$$x(t) * \delta(t-t_0) = \delta(t-t_0) * x(t) = \int_{-\infty}^{\infty} \delta(\tau - t_0) x(t-\tau)\, d\tau$$

$$= x(t-\tau)\big|_{\tau=t_0} = x(t-t_0)$$

c. By Eqs. (2.6) and (1.19) we have

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$\text{since } u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases}$$

d. In a similar manner, we have

$$x(t) * u(t-t_0) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau-t_0)\,d\tau = \int_{-\infty}^{t-t_0} x(\tau)\,d\tau$$

$$\text{since } u(t-\tau-t_0) = \begin{cases} 1 & \tau < t-t_0 \\ 0 & \tau > t-t_0 \end{cases}.$$

2.3. Let $y(t) = x(t) * h(t)$. Then show that

$$x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$$

(2.62)

By Eq. (2.6) we have

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

(2.63a)

and

$$x(t-t_1) * h(t-t_2) = \int_{-\infty}^{\infty} x(\tau-t_1)h(t-\tau-t_2)\,d\tau$$

(2.63b)

Let $\tau - t_1 = \lambda$. Then $\tau = \lambda + t_1$ and Eq. (2.63b) becomes

$$x(t-t_1) * h(t-t_2) = \int_{-\infty}^{\infty} x(\lambda)h(t-t_1-t_2-\lambda)\,d\lambda$$

(2.63c)

Comparing Eqs. (2.63a) and (2.63c), we see that replacing t in Eq. (2.63a) by $t - t_1 - t_2$, we obtain Eq. (2.63c). Thus, we conclude that

$$x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$$

2.4. The input $x(t)$ and the impulse response $h(t)$ of a continuous time LTI system are given by

$$x(t) = u(t) \qquad h(t) = e^{-\alpha t}u(t),\ \alpha > 0$$

a. Compute the output $y(t)$ by Eq. (2.6).

b. Compute the output $y(t)$ by Eq. (2.10).

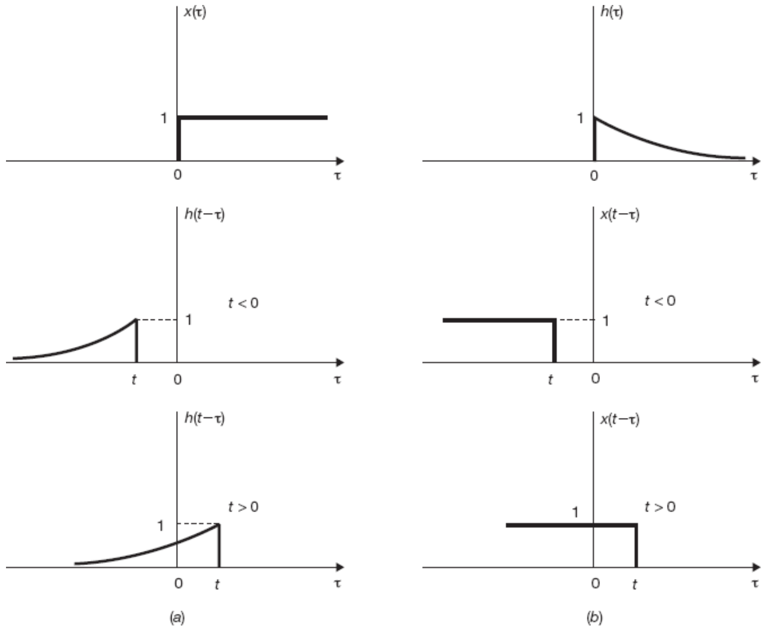
a. By Eq. (2.6)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Functions $x(\tau)$ and $h(t - \tau)$ are shown in Fig. 2-4(a) for $t < 0$ and $t > 0$. From Fig. 2-4(a) we see that for $t < 0$, $x(\tau)$ and $h(t - \tau)$ do not overlap, while for $t > 0$, they overlap from $\tau = 0$ to $\tau = t$. Hence, for $t < 0$, $y(t) = 0$. For $t > 0$, we have

$$y(t) = \int_0^t e^{-\alpha(t-\tau)}\, d\tau = e^{-\alpha t}\int_0^t e^{\alpha\tau}\, d\tau$$
$$= e^{-\alpha t}\frac{1}{\alpha}(e^{\alpha t} - 1) = \frac{1}{\alpha}(1 - e^{-\alpha t})$$

Figure 2-4



Thus, we can write the output $y(t)$ as

$$y(t) = \frac{1}{\alpha}(1 - e^{-\alpha t})u(t)$$

(2.64)

b. By Eq. (2.10)

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

Functions $h(\tau)$ and $x(t - \tau)$ are shown in Fig. 2-4(b) for $t < 0$ and $t > 0$. Again from Fig. 2-4(b) we see that for $t < 0$, $h(\tau)$ and $x(t - \tau)$ do not overlap, while for $t > 0$, they overlap from $\tau = 0$ to $\tau = t$. Hence, for $t < 0$, $y(t) = 0$. For $t > 0$, we have

$$y(t) = \int_0^t e^{-\alpha\tau}\, d\tau = \frac{1}{\alpha}(1 - e^{-\alpha t})$$

Thus, we can write the output $y(t)$ as

$$y(t) = \frac{1}{\alpha}(1 - e^{-\alpha t})u(t)$$

(2.65)

which is the same as Eq. (2.64).

2.5. Compute the output $y(t)$ for a continuous-time LTI system whose impulse response $h(t)$ and the input $x(t)$ are given by

$$h(t) = e^{-\alpha t} u(t) \qquad x(t) = e^{\alpha t} u(-t) \quad \alpha > 0$$

By Eq. (2.6)

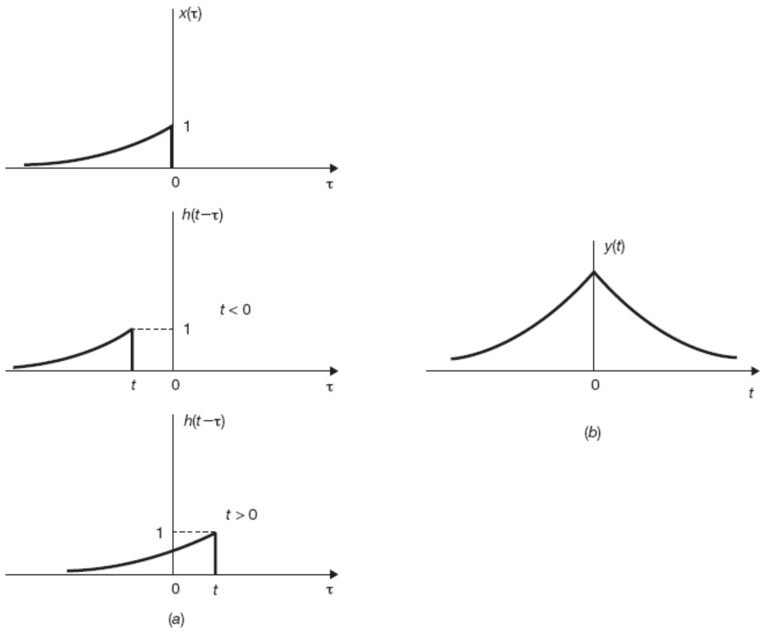
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

Functions $x(\tau)$ and $h(t - \tau)$ are shown in Fig. 2-5 (a) for $t < 0$ and $t > 0$. From Fig. 2-5 (a) we see that for $t < 0$, $x(\tau)$ and $h(t - \tau)$ overlap from $\tau = -\infty$ to $\tau = t$, while for $t > 0$, they overlap from $\tau = -\infty$ to $\tau = 0$. Hence, for $t < 0$, we have

$$y(t) = \int_{-\infty}^{t} e^{\alpha\tau}\, e^{-\alpha(t-\tau)} d\tau = e^{-\alpha t} \int_{-\infty}^{t} e^{2\alpha\tau} d\tau = \frac{1}{2\alpha} e^{\alpha t}$$

(2.66a)

Figure 2-5



For $t > 0$, we have

$$y(t) = \int_{-\infty}^{0} e^{\alpha\tau}\, e^{-\alpha(t-\tau)} d\tau = e^{-\alpha t} \int_{-\infty}^{0} e^{2\alpha\tau} d\tau = \frac{1}{2\alpha} e^{-\alpha t}$$

(2.66b)

Combining Eqs. (2.66a) and (2.66b), we can write $y(t)$ as

$$y(t) = \frac{1}{2\alpha} e^{-\alpha|t|} \qquad \alpha > 0$$

(2.67)

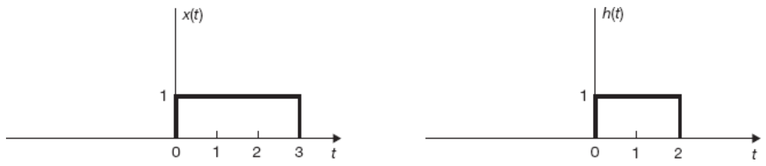
which is shown in Fig. 2-5(b).

2.6. Evaluate $y(t) = x(t) * h(t)$, where $x(t)$ and $h(t)$ are shown in Fig. 2-6, (a) by an analytical technique, and (b) by a graphical method.

a. We first express $x(t)$ and $h(t)$ in functional form:

$$x(t) = u(t) - u(t-3) \qquad h(t) = u(t) - u(t-2)$$

Figure 2-6



Then, by Eq. (2.6) we have

$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} [u(\tau)-u(\tau-3)]\,[u(t-\tau)-u(t-\tau-2)]\,d\tau \\ &= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)\,d\tau - \int_{-\infty}^{\infty} u(\tau)u(t-2-\tau)\,d\tau \\ &\quad - \int_{-\infty}^{\infty} u(\tau-3)u(t-\tau)\,d\tau + \int_{-\infty}^{\infty} u(\tau-3)u(t-2-\tau)\,d\tau \end{aligned}$$

Since

$$u(\tau)u(t-\tau) = \begin{cases} 1 & 0<\tau<t,\, t>0 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau)u(t-2-\tau) = \begin{cases} 1 & 0<\tau<t-2,\, t>2 \\ 0 & \text{otherwise} \end{cases}$$

$$u(\tau-3)u(t-\tau) = \begin{cases} 1 & 3<\tau<t,\, t>3 \\ 0 & \text{otherwise} \end{cases}$$

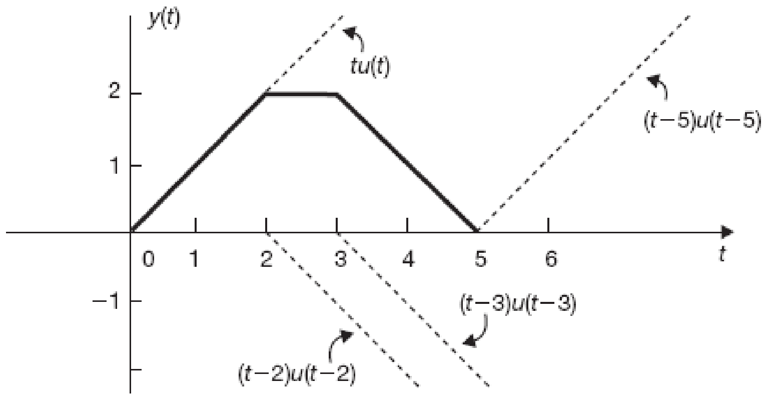
$$u(\tau-3)u(t-2-\tau) = \begin{cases} 1 & 3<\tau<t-2,\, t>5 \\ 0 & \text{otherwise} \end{cases}$$

we can express $y(t)$ as

$$\begin{aligned} y(t) &= \left(\int_0^t d\tau\right)u(t) - \left(\int_0^{t-2} d\tau\right)u(t-2) \\ &\quad - \left(\int_3^t d\tau\right)u(t-3) + \left(\int_3^{t-2} d\tau\right)u(t-5) \\ &= tu(t) - (t-2)u(t-2) - (t-3)u(t-3) + (t-5)u(t-5) \end{aligned}$$

which is plotted in Fig. 2-7.

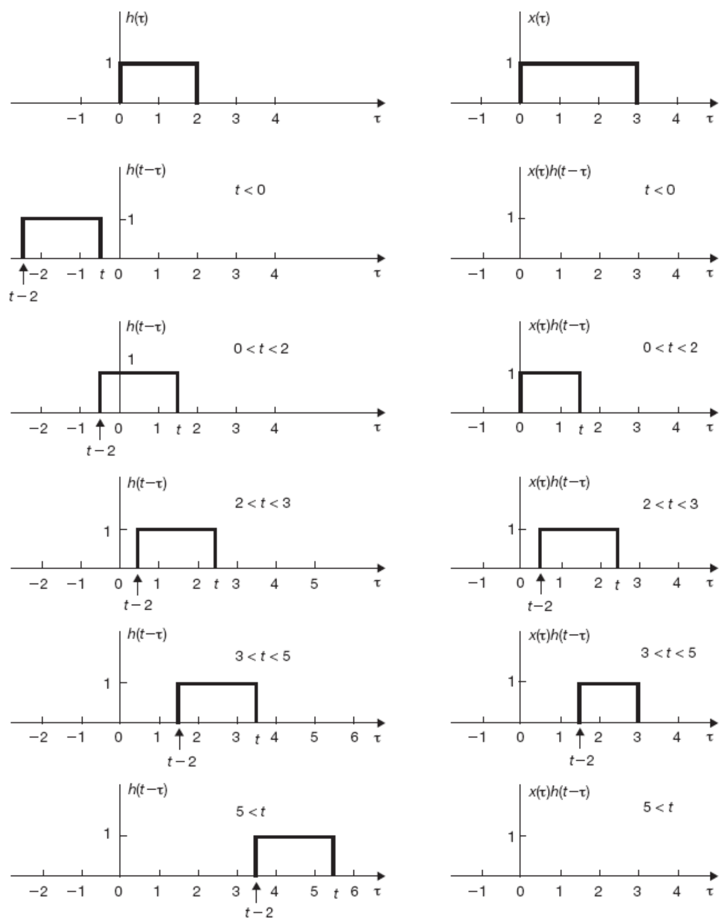
Figure 2-7



b. Functions $h(\tau)$, $x(\tau)$ and $h(t-\tau)$, $x(\tau)h(t-\tau)$ for different values of t are sketched in Fig. 2-8. From Fig. 2-8 we see that $x(\tau)$ and $h(t-\tau)$ do not overlap for $t < 0$ and $t > 5$, and hence, $y(t) = 0$ for $t < 0$ and $t > 5$. For the other intervals, $x(\tau)$ and $h(t-\tau)$ overlap. Thus, computing the area under the rectangular pulses for these intervals, we obtain

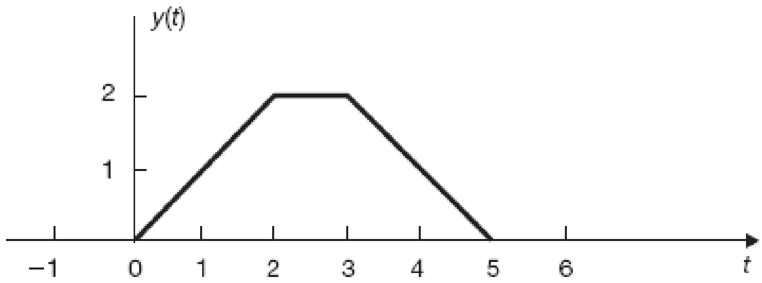
$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t \le 2 \\ 2 & 2 < t \le 3 \\ 5 - t & 3 < t \le 5 \\ 0 & 5 < t \end{cases}$$

Figure 2-8



which is plotted in Fig. 2-9.

Figure 2-9

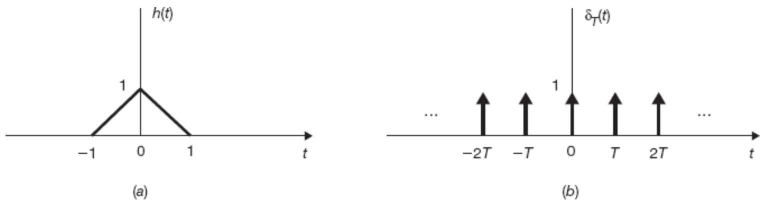


2.7. Let $h(t)$ be the triangular pulse shown in Fig. 2-10(a) and let $x(t)$ be the unit impulse train [Fig. 2-10(b)] expressed as

$$x(t) = \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

(2.68)

Figure 2-10



Determine and sketch $y(t) = h(t) * x(t)$ for the following values of T: (a) $T = 3$, (b) $T = 2$, (c) $T = 1.5$.

Using Eqs. (2.59) and (2.9), we obtain

$$y(t) = h(t) * \delta_T(t) = h(t) * \left[\sum_{n=-\infty}^{\infty} \delta(t - nT) \right]$$
$$= \sum_{n=-\infty}^{\infty} h(t) * \delta(t - nT) = \sum_{n=-\infty}^{\infty} h(t - nT)$$

(2.69)

a. For $T = 3$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 3n)$$

which is sketched in Fig. 2-11(a).

b. For $T = 2$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 2n)$$

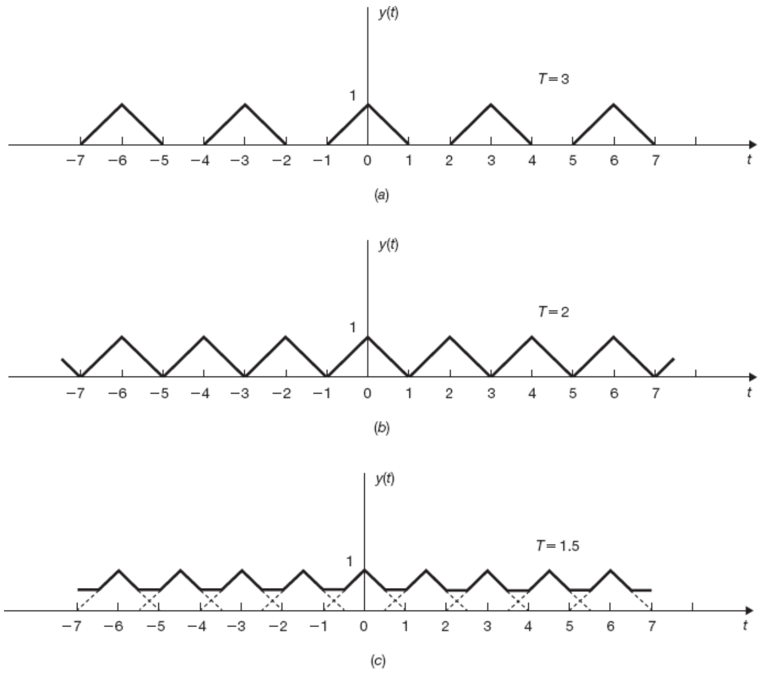
which is sketched in Fig. 2-11(b).

c. For $T = 1.5$, Eq. (2.69) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - 1.5n)$$

which is sketched in Fig. 2-11(*c*). Note that when $T < 2$, the triangular pulses are no longer separated and they overlap.

Figure 2-11



2.8. If $x_1(t)$ and $x_2(t)$ are both periodic signals with a common period T_0, the convolution of $x_1(t)$ and $x_2(t)$ does not converge. In this case, we define the *periodic convolution* of $x_1(t)$ and $x_2(t)$ as

$$f(t) = x_1(t) \otimes x_2(t) = \int_0^{T_0} x_1(\tau) x_2(t - \tau)\, d\tau$$

(2.70)

a. Show that $f(t)$ is periodic with period T_0.

b. Show that

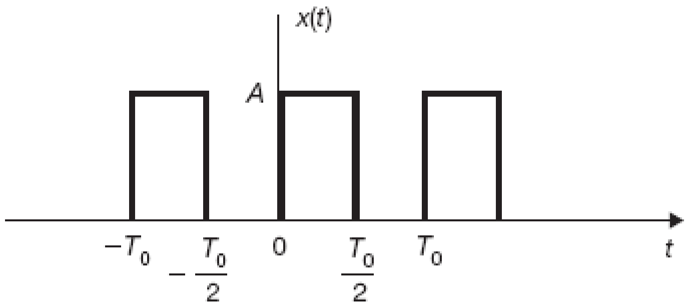
$$f(t) = \int_a^{a+T_0} x_1(\tau) x_2(t - \tau)\, d\tau$$

(2.71)

for any a.

c. Compute and sketch the periodic convolution of the square-wave signal $x(t)$ shown in Fig. 2-12 with itself.

Figure 2-12



a. Since $x_2(t)$ is periodic with period T_0, we have

$$x_2(t + T_0 - \tau) = x_2(t - \tau)$$

Then from Eq. (2.70) we have

$$\begin{aligned} f(t+T_0) &= \int_0^{T_0} x_1(\tau)x_2(t+T_0-\tau)\,d\tau \\ &= \int_0^{T_0} x_1(\tau)x_2(t-\tau)\,d\tau = f(t) \end{aligned}$$

Thus, $f(t)$ is periodic with period T_0.

b. Since both $x_1(\tau)$ and $x_2(\tau)$ are periodic with the same period T_0, $x_1(\tau)x_2(t - \tau)$ is also periodic with period T_0. Then using property (1.88) (Prob. 1.17), we obtain

$$f(t) = \int_0^{T_0} x_1(\tau)x_2(t-\tau)\,d\tau = \int_a^{a+T_0} x_1(\tau)x_2(t-\tau)\,d\tau$$

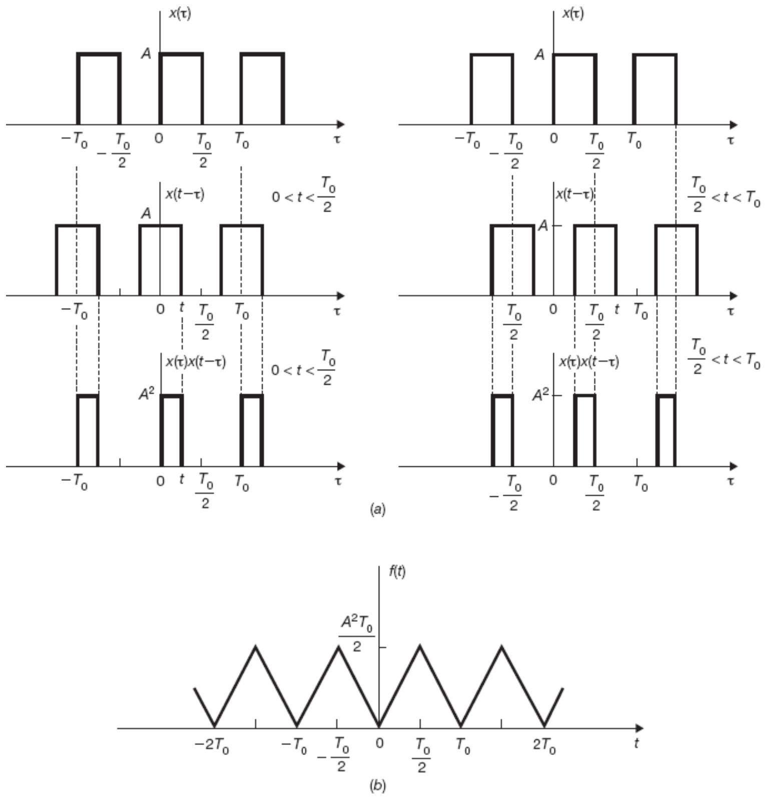
for an arbitrary a.

c. We evaluate the periodic convolution graphically. Signals $x(\tau)$, $x(t - \tau)$, and $x(\tau)x(t - \tau)$ are sketched in Fig. 2-13(*a*), from which we obtain

$$f(t) = \begin{cases} A^2 t & 0 < t \le T_0/2 \\ -A^2(t - T_0) & T_0/2 < t \le T_0 \end{cases} \quad \text{and} \quad f(t+T_0) = f(t)$$

which is plotted in Fig. 2-13(*b*).

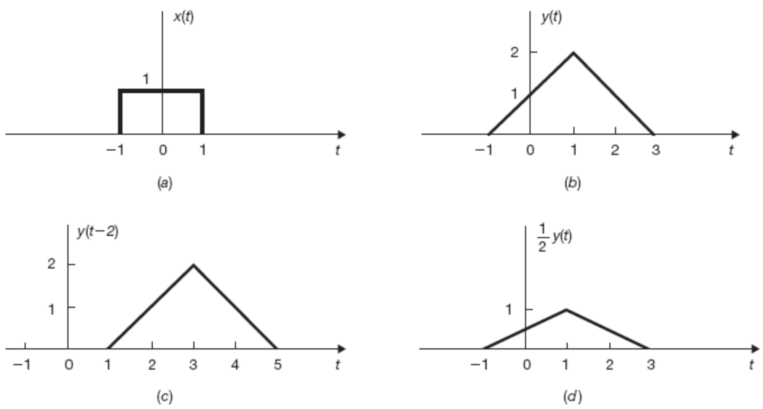
Figure 2-13



2.10.2. Properties of Continuous-Time LTI Systems

2.9. The signals in Figs. 2-14(*a*) and (*b*) are the input $x(t)$ and the output $y(t)$, respectively, of a certain continuous-time LTI system. Sketch the output to the following inputs: (*a*) $x(t-2)$; (*b*) $\frac{1}{2}x(t)$.

Figure 2-14



a. Since the system is time-invariant, the output will be $y(t-2)$, which is sketched in Fig. 2-14(*c*).

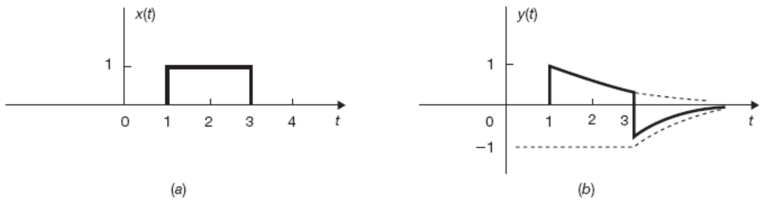
b. Since the system is linear, the output will be 1–2$y(t)$, which is sketched in Fig. 2-14(*d*).

2.10. Consider a continuous-time LTI system whose step response is given by

$$s(t) = e^{-t}u(t)$$

Determine and sketch the output of this system to the input $x(t)$ shown in Fig. 2-15(a).

Figure 2-15



From Fig. 2-15(a) the input $x(t)$ can be expressed as

$$x(t) = u(t-1) - u(t-3)$$

Since the system is linear and time-invariant, the output $y(t)$ is given by

$$\begin{aligned} y(t) &= s(t-1) - s(t-3) \\ &= e^{-(t-1)}u(t-1) - e^{-(t-3)}u(t-3) \end{aligned}$$

which is sketched in Fig. 2-15(b).

2.11. Consider a continuous-time LTI system described by (see Prob. 1.56)

$$y(t) = \mathbf{T}\{x(t)\} = \frac{1}{T}\int_{t-T/2}^{t+T/2} x(\tau)\,d\tau$$

(2.72)

a. Find and sketch the impulse response $h(t)$ of the system.

b. Is this system causal?

a. Equation (2.72) can be rewritten as

$$y(t) = \frac{1}{T}\int_{-\infty}^{t+T/2} x(\tau)\,d\tau - \frac{1}{T}\int_{-\infty}^{t-T/2} x(\tau)\,d\tau$$

(2.73)

Using Eqs. (2.61) and (2.9), Eq. (2.73) can be expressed as

$$\begin{aligned} y(t) &= \frac{1}{T}x(t) * u\left(t+\frac{T}{2}\right) - \frac{1}{T}x(t) * u\left(t-\frac{T}{2}\right) \\ &= x(t) * \frac{1}{T}\left[u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)\right] = x(t) * h(t) \end{aligned}$$

(2.74)

Thus, we obtain

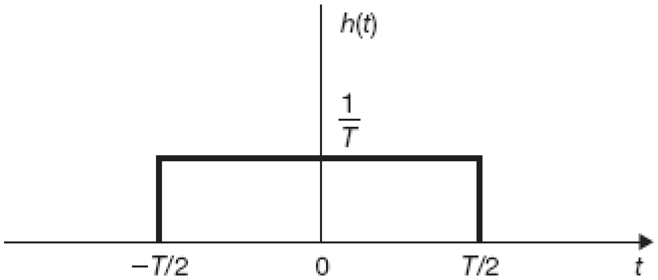
$$h(t) = \frac{1}{T}\left[u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)\right] = \begin{cases} 1/T & -T/2 < t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

(2.75)

which is sketched in Fig. 2-16.

b. From Fig. 2-16 or Eq. (2.75) we see that $h(t) \neq 0$ for $t < 0$. Hence, the system is not causal.

Figure 2-16



2.12. Let $y(t)$ be the output of a continuous-time LTI system with input $x(t)$. Find the output of the system if the input is $x'(t)$, where $x'(t)$ is the first derivative of $x(t)$.

From Eq. (2.10)

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

Differentiating both sides of the above convolution integral with respect to t, we obtain

$$\begin{aligned} y'(t) &= \frac{d}{dt}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau\right] = \int_{-\infty}^{\infty} \frac{d}{dt}[h(\tau)x(t-\tau)\,d\tau] \\ &= \int_{-\infty}^{\infty} h(\tau)x'(t-\tau)\,d\tau = h(t) * x'(t) \end{aligned}$$

(2.76)

which indicates that $y'(t)$ is the output of the system when the input is $x'(t)$.

2.13. Verify the BIBO stability condition [Eq. (2.21)] for continuous-time LTI systems.

Assume that the input $x(t)$ of a continuous-time LTI system is bounded, that is,

$$|x(t)| \leq k_1 \qquad \text{all } t$$

(2.77)

Then, using Eq. (2.10), we have

$$\begin{aligned} |y(t)| &= \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau\right| \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\,d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\,d\tau \leq k_1 \int_{-\infty}^{\infty} |h(\tau)|\,d\tau \end{aligned}$$

since $|x(t-\tau)| \leq k_1$ from Eq. (2.77). Therefore, if the impulse response is absolutely integrable, that is,

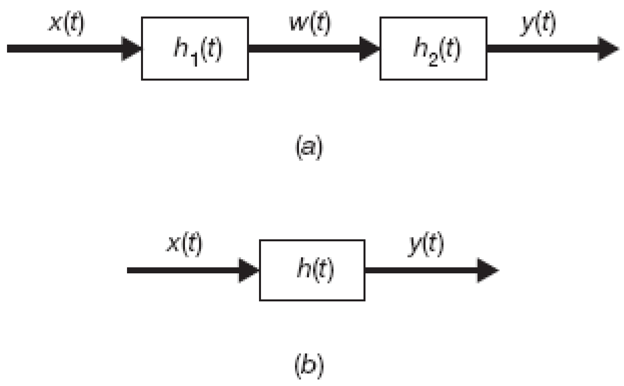
$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau = K < \infty$$

then $|y(t)| \leq k_1 K = k_2$ and the system is BIBO stable.

2.14. The system shown in Fig. 2-17(*a*) is formed by connecting two systems *in cascade*. The impulse responses of the systems are given by $h_1(t)$ and $h_2(t)$, respectively, and

$$h_1(t) = e^{-2t}u(t) \qquad h_2(t) = 2e^{-t}u(t)$$

Figure 2-17



i. Find the impulse response $h(t)$ of the overall system shown in Fig. 2-17(b).

ii. Determine if the overall system is BIBO stable.

a. Let $w(t)$ be the output of the first system. By Eq. (2.6)

$$w(t) = x(t) * h_1(t)$$

(2.78)

Then we have

$$y(t) = w(t) * h_2(t) = [x(t) * h_1(t)] * h_2(t)$$

(2.79)

But by the associativity property of convolution (2.8), Eq. (2.79) can be rewritten as

$$y(t) = x(t) * [h_1(t) * h_2(t)] = x(t) * h(t)$$

(2.80)

Therefore, the impulse response of the overall system is given by

$$h(t) = h_1(t) * h_2(t)$$

(2.81)

Thus, with the given $h_1(t)$ and $h_2(t)$, we have

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)\,2e^{-(t-\tau)}u(t-\tau)\,d\tau \\ &= 2e^{-t}\int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)\,d\tau = 2e^{-t}\left[\int_0^t e^{-\tau}\,d\tau\right]u(t) \\ &= 2(e^{-t} - e^{-2t})u(t) \end{aligned}$$

b. Using the above $h(t)$, we have

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = 2\int_0^{\infty} (e^{-\tau} - e^{-2\tau})\, d\tau = 2\left[\int_0^{\infty} e^{-\tau}\, d\tau - \int_0^{\infty} e^{-2\tau}\, d\tau\right]$$
$$= 2\left(1 - \frac{1}{2}\right) = 1 < \infty$$

Thus, the system is BIBO stable.

2.10.3. Eigenfunctions of Continuous-Time LTI Systems

2.15. Consider a continuous-time LTI system with the input-output relation given by

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\, d\tau$$

(2.82)

a. Find the impulse response $h(t)$ of this system.

b. Show that the complex exponential function e^{st} is an eigenfunction of the system.

c. Find the eigenvalue of the system corresponding to e^{st} by using the impulse response $h(t)$ obtained in part (*a*).

a. From Eq. (2.82), definition (2.1), and Eq. (1.21) we get

$$h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau)\, d\tau = e^{-(t-\tau)}\Big|_{\tau=0} = e^{-t} \qquad t > 0$$

Thus,

$$h(t) = e^{-t} u(t)$$

(2.83)

b. Let $x(t) = e^{st}$. Then

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}\, e^{s\tau}\, d\tau = e^{-t} \int_{-\infty}^{t} e^{(s+1)\tau}\, d\tau$$
$$= \frac{1}{s+1} e^{st} = \lambda e^{st} \qquad \text{if } \operatorname{Re} s > -1$$

(2.84)

Thus, by definition (2.22)e^{st} is the eigenfunction of the system and the associated eigenvalue is

$$\lambda = \frac{1}{s+1}$$

(2.85)

c. Using Eqs. (2.24) and (2.83), the eigenvalue associated with e^{st} is given by

$$\lambda = H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau = \int_{-\infty}^{\infty} e^{-\tau} u(\tau)\, e^{-s\tau}\, d\tau$$
$$= \int_0^{\infty} e^{-(s+1)\tau}\, d\tau = \frac{1}{s+1} \qquad \text{if } \operatorname{Re} s > -1$$

which is the same as Eq. (2.85).

2.16. Consider the continuous-time LTI system described by

$$y(t)=\frac{1}{T}\int_{t-T/2}^{t+T/2} x(\tau)\,d\tau$$

(2.86)

a. Find the eigenvalue of the system corresponding to the eigenfunction e^{st}.

b. Repeat part (*a*) by using the impulse function $h(t)$ of the system.

a. Substituting $x(\tau) = e^{s\tau}$ in Eq. (2.86), we obtain

$$\begin{aligned} y(t) &= \frac{1}{T}\int_{t-T/2}^{t+T/2} e^{s\tau}\,d\tau \\ &= \frac{1}{sT}(e^{sT/2}-e^{-sT/2})\,e^{st} = \lambda\, e^{st} \end{aligned}$$

Thus, the eigenvalue of the system corresponding to e^{st} is

$$\lambda = \frac{1}{sT}(e^{sT/2}-e^{-sT/2})$$

(2.87)

b. From Eq. (2.75) in Prob. 2.11 we have

$$h(t)=\frac{1}{T}\left[u\left(t+\frac{T}{2}\right)-u\left(t-\frac{T}{2}\right)\right]=\begin{cases}1/T & -T/2<t\le T/2\\ 0 & \text{otherwise}\end{cases}$$

Using Eq. (2.24), the eigenvalue $H(s)$ corresponding to e^{st} is given by

$$H(s)=\int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau=\frac{1}{T}\int_{-T/2}^{T/2} e^{-s\tau}\,d\tau=\frac{1}{sT}(e^{sT/2}-e^{-sT/2})$$

which is the same as Eq. (2.87).

2.17. Consider a stable continuous-time LTI system with impulse response $h(t)$ that is real and even. Show that $\cos \omega t$ and $\sin \omega t$ are eigenfunctions of this system with the same real eigenvalue.

By setting $s = j\omega$ in Eqs. (2.23) and (2.24), we see that $e^{j\omega t}$ is an eigenfunction of a continuous-time LTI system and the corresponding eigenvalue is

$$\lambda = H(j\omega)=\int_{-\infty}^{\infty} h(\tau)\,e^{-j\omega\tau}\,d\tau$$

(2.88)

Since the system is stable, that is,

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau<\infty$$

then

$$\int_{-\infty}^{\infty}\left|h(\tau)\,e^{-j\omega\tau}\right|d\tau=\int_{-\infty}^{\infty}|h(\tau)|\left|e^{-j\omega\tau}\right|d\tau=\int_{-\infty}^{\infty}|h(\tau)|\,d\tau<\infty$$

since $|e^{-j\omega\tau}| = 1$. Thus, $H(j\omega)$ converges for any ω. Using Euler's formula, we have

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} h(\tau)(\cos\omega\tau - j\sin\omega\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)\cos\omega\tau\, d\tau - j\int_{-\infty}^{\infty} h(\tau)\sin\omega\tau\, d\tau$$

(2.89)

Since $\cos\omega\tau$ is an even function of τ and $\sin\omega\tau$ is an odd function of τ, and if $h(t)$ is real and even, then $h(\tau)\cos\omega\tau$ is even and $h(\tau)\sin\omega\tau$ is odd. Then by Eqs. (1.75a) and (1.77), Eq. (2.89) becomes

$$H(j\omega) = 2\int_{0}^{\infty} h(\tau)\cos\omega\tau\, d\tau$$

(2.90)

Since $\cos\omega\tau$ is an even function of ω, changing ω to $-\omega$ in Eq. (2.90) and changing j to $-j$ in Eq. (2.89), we have

$$H(-j\omega) = H(j\omega)^* = 2\int_{0}^{\infty} h(\tau)\cos(-\omega\tau)\, d\tau$$
$$= 2\int_{0}^{\infty} h(\tau)\cos\omega\tau\, d\tau = H(j\omega)$$

(2.91)

Thus, we see that the eigenvalue $H(j\omega)$ corresponding to the eigenfunction $e^{j\omega t}$ is real. Let the system be represented by **T**. Then by Eqs. (2.23), (2.24), and (2.91) we have

$$\mathbf{T}\{e^{j\omega t}\} = H(j\omega)\, e^{j\omega t}$$

(2.92a)

$$\mathbf{T}\{e^{-j\omega t}\} = H(-j\omega)\, e^{-j\omega t} = H(j\omega)\, e^{-j\omega t}$$

(2.92b)

Now, since **T** is linear, we get

$$\mathbf{T}\{\cos\omega t\} = \mathbf{T}\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \frac{1}{2}\mathbf{T}\{e^{j\omega t}\} + \frac{1}{2}\mathbf{T}\{e^{-j\omega t}\}$$
$$= H(j\omega)\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = H(j\omega)\cos\omega t$$

(2.93a)

and

$$\mathbf{T}\{\sin\omega t\} = \mathbf{T}\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\} = \frac{1}{2j}\mathbf{T}\{e^{j\omega t}\} - \frac{1}{2j}\mathbf{T}\{e^{-j\omega t}\}$$
$$= H(j\omega)\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\} = H(j\omega)\sin\omega t$$

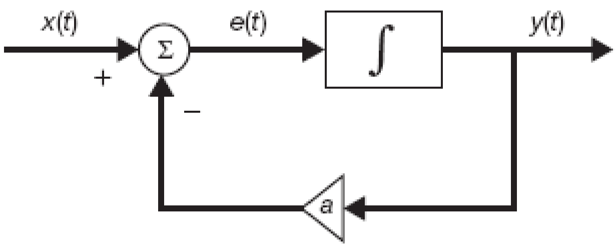
(2.93b)

Thus, from Eqs. (2.93a) and (2.93b) we see that $\cos\omega t$ and $\sin\omega t$ are the eigenfunctions of the system with the same real eigenvalue $H(j\omega)$ given by Eq. (2.88) or (2.90).

2.10.4. Systems Described by Differential Equations

2.18. The continuous-time system shown in Fig. 2-18 consists of one integrator and one scalar multiplier. Write a differential equation that relates the output $y(t)$ and the input $x(t)$.

Figure 2-18



Let the input of the integrator shown in Fig. 2-18 be denoted by $e(t)$. Then the input-output relation of the integrator is given by

$$y(t) = \int_{-\infty}^{t} e(\tau)\, d\tau$$

(2.94)

Differentiating both sides of Eq. (2.94) with respect to t, we obtain

$$\frac{dy(t)}{dt} = e(t)$$

(2.95)

Next, from Fig. 2-18 the input $e(t)$ to the integrator is given by

$$e(t) = x(t) - ay(t)$$

(2.96)

Substituting Eq. (2.96) into Eq. (2.95), we get

$$\frac{dy(t)}{dt} = x(t) - ay(t)$$

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

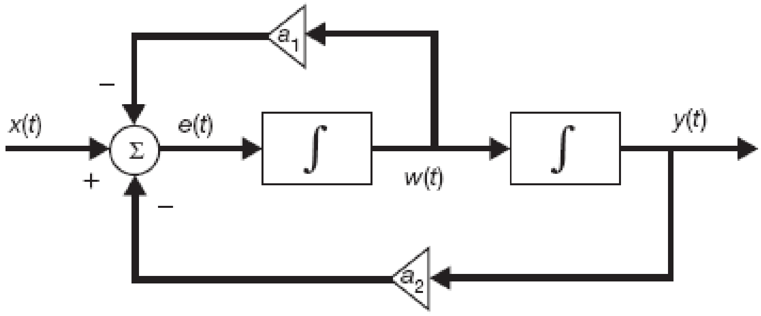
(2.97)

or

which is the required first-order linear differential equation.

2.19. The continuous-time system shown in Fig. 2-19 consists of two integrators and two scalar multipliers. Write a differential equation that relates the output $y(t)$ and the input $x(t)$.

Figure 2-19



Let $e(t)$ and $w(t)$ be the input and the output of the first integrator in Fig. 2-19, respectively. Using Eq. (2.95), the input to the first integrator is given by

$$e(t) = \frac{dw(t)}{dt} = -a_1 w(t) - a_2 y(t) + x(t)$$

(2.98)

Since $w(t)$ is the input to the second integrator in Fig. 2-19, we have

$$w(t) = \frac{dy(t)}{dt}$$

(2.99)

Substituting Eq. (2.99) into Eq. (2.98), we get

$$\frac{d^2 y(t)}{dt^2} = -a_1 \frac{dy(t)}{dt} - a_2 y(t) + x(t)$$

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = x(t)$$

(2.100)

or

which is the required second-order linear differential equation.

Note that, in general, the order of a continuous-time LTI system consisting of the interconnection of integrators and scalar multipliers is equal to the number of integrators in the system.

2.20. Consider a continuous-time system whose input $x(t)$ and output $y(t)$ are related by

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

(2.101)

where a is a constant.

a. Find $y(t)$ with the auxiliary condition $y(0) = y_0$ and

$$x(t) = Ke^{-bt}u(t)$$

(2.102)

b. Express $y(t)$ in terms of the zero-input and zero-state responses.

a. Let

$$y(t) = y_p(t) + y_h(t)$$

where $y_p(t)$ is the particular solution satisfying Eq. (2.101) and $y_h(t)$ is the homogeneous solution which satisfies

$$\frac{dy_h(t)}{dt} + ay_h(t) = 0$$

(2.103)

Assume that

$$y_p(t) = Ae^{-bt} \qquad t > 0$$

(2.104)

Substituting Eq. (2.104) into Eq. (2.101), we obtain

$$-bAe^{-bt} + aAe^{-bt} = Ke^{-bt}$$

from which we obtain $A = K/(a - b)$, and

$$y_p(t) = \frac{K}{a-b}e^{-bt} \qquad t > 0$$

(2.105)

To obtain $y_h(t)$, we assume

$$y_h(t) = Be^{st}$$

Substituting this into Eq. (2.103) gives

$$sBe^{st} + aBe^{st} = (s + a)\, Be^{st} = 0$$

from which we have $s = -a$ and

$$y_h(t) = Be^{-at}$$

Combining $y_p(t)$ and $y_h(t)$, we get

$$y(t) = Be^{-at} + \frac{K}{a-b}e^{-bt} \qquad t > 0$$

(2.106)

From Eq. (2.106) and the auxiliary condition $y(0) = y_0$, we obtain

$$B = y_0 - \frac{K}{a-b}$$

Thus, Eq. (2.106) becomes

$$y(t)=\left(y_0-\frac{K}{a-b}\right)e^{-at}+\frac{K}{a-b}e^{-bt} \qquad t>0$$

(2.107)

For $t < 0$, we have $x(t) = 0$, and Eq. (2.101) becomes Eq. (2.103). Hence,

$$y(t) = Be^{-at} \qquad t<0$$

From the auxiliary condition $y(0) = y_0$ we obtain

$$y(t) = y_0e^{-at} \qquad t<0$$

(2.108)

b. Combining Eqs. (2.107) and (2.108), $y(t)$ can be expressed in terms of $y_{zi}(t)$ (zero-input response) and $y_{zs}(t)$ (zero-state response) as

$$\begin{aligned} y(t) &= y_0e^{-at}+\frac{K}{a-b}(e^{-bt}-e^{-at})u(t) \\ &= y_{zi}(t)+y_{zs}(t) \end{aligned}$$

(2.109)

where

$$y_{zi}(t)=y_0e^{-at}$$

(2.110a)

$$y_{zs}(t)=\frac{K}{a-b}(e^{-bt}-e^{-at})u(t)$$

(2.110b)

2.21. Consider the system in Prob. 2.20.

a. Show that the system is not linear if $y(0) = y_0 \neq 0$.

b. Show that the system is linear if $y(0) = 0$.

a. Recall that a linear system has the property that zero input produces zero output (Sec. 1.5E). However, if we let $K = 0$ in Eq. (2.102), we have $x(t) = 0$, but from Eq. (2.109) we see that

$$y(t) = y_0e^{-at} \neq 0 \qquad y_0 \neq 0$$

Thus, this system is nonlinear if $y(0) = y_0 \neq 0$.

b. If $y(0) = 0$, the system is linear. This is shown as follows. Let $x_1(t)$ and $x_2(t)$ be two input signals, and let $y_1(t)$ and $y_2(t)$ be the corresponding outputs. That is,

$$\frac{dy_1(t)}{dt}+ay_1(t)=x_1(t)$$

(2.111)

$$\frac{dy_2(t)}{dt}+ay_2(t)=x_2(t)$$

(2.112)

with the auxiliary conditions

$$y_1(0) = y_2(0) = 0$$

(2.113)

Consider

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

where α_1 and α_2 are any complex numbers. Multiplying Eq. (2.111) by α_1 and Eq. (2.112) by α_2 and adding, we see that

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

satisfies the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

and also, from Eq. (2.113),

$$y(0) = \alpha_1 y_1(0) + \alpha_2 y_2(0) = 0$$

Therefore, $y(t)$ is the output corresponding to $x(t)$, and thus the system is linear.

2.22. Consider the system in Prob. 2.20. Show that the initial rest condition $y(0) = 0$ also implies that the system is time-invariant.

Let $y_1(t)$ be the response to an input $x_1(t)$ and

$$x_1(t) = 0 \qquad t \leq 0$$

(2.114)

Then

$$\frac{dy_1(t)}{dt} + ay_1(t) = x_1(t)$$

(2.115)

and

$$y_1(0) = 0$$

(2.116)

Now, let $y_2(t)$ be the response to the shifted input $x_2(t) = x_1(t - \tau)$. From Eq. (2.114) we have

$$x_2(t) = 0 \qquad t \leq \tau$$

(2.117)

Then $y_2(t)$ must satisfy

$$\frac{dy_2(t)}{dt} + ay_2(t) = x_2(t)$$

(2.118)

and

$$y_2(\tau) = 0$$

(2.119)

Now, from Eq. (2.115) we have

$$\frac{dy_1(t-\tau)}{dt} + ay_1(t-\tau) = x_1(t-\tau) = x_2(t)$$

If we let $y_2(t) = y_1(t - \tau)$, then by Eq. (2.116) we have

$$y_2(\tau) = y_1(\tau - \tau) = y_1(0) = 0$$

Thus, Eqs. (2.118) and (2.119) are satisfied and we conclude that the system is time-invariant.

2.23. Consider the system in Prob. 2.20. Find the impulse response $h(t)$ of the system.

The impulse response $h(t)$ should satisfy the differential equation

$$\frac{dh(t)}{dt} + ah(t) = \delta(t)$$

(2.120)

The homogeneous solution $h_h(t)$ to Eq. (2.120) satisfies

$$\frac{dh_h(t)}{dt} + ah_h(t) = 0$$

(2.121)

To obtain $h_h(t)$, we assume

$$h_h(t) = ce^{st}$$

Substituting this into Eq. (2.121) gives

$$sce^{st} + ace^{st} = (s + a)\, ce^{st} = 0$$

from which we have $s = -a$ and

$$h_h(t) = ce^{-at}u(t)$$

(2.122)

We predict that the particular solution $h_p(t)$ is zero since $h_p(t)$ cannot contain $\delta(t)$. Otherwise, $h(t)$ would have a derivative of $\delta(t)$ that is not part of the right-hand side of Eq. (2.120). Thus,

$$h(t) = ce^{-at}u(t)$$

(2.123)

To find the constant c, substituting Eq. (2.123) into Eq. (2.120), we obtain

$$\frac{d}{dt}[ce^{-at}u(t)] + ace^{-at}u(t) = \delta(t)$$

or

$$-ace^{-at}u(t) + ce^{-at}\frac{du(t)}{dt} + ace^{-at}u(t) = \delta(t)$$

Using Eqs. (1.25) and (1.30), the above equation becomes

$$ce^{-at}\frac{du(t)}{dt} = ce^{-at}\delta(t) = c\delta(t) = \delta(t)$$

so that $c = 1$. Thus, the impulse response is given by

$$h(t) = e^{-at}u(t)$$

(2.124)

2.24. Consider the system in Prob. 2.20 with $y(0) = 0$.

a. Find the step response $s(t)$ of the system without using the impulse response $h(t)$.

b. Find the step response $s(t)$ with the impulse response $h(t)$ obtained in Prob. 2.23.

c. Find the impulse response $h(t)$ from $s(t)$.

a. In Prob. 2.20

$$x(t) = Ke^{-bt}u(t)$$

Setting $K = 1, b = 0$, we obtain $x(t) = u(t)$ and then $y(t) = s(t)$. Thus, setting $K = 1, b = 0$, and $y(0) = y_0 = 0$ in Eq. (2.109), we obtain the step response

$$s(t) = \frac{1}{a}(1 - e^{-at})u(t)$$

(2.125)

b. Using Eqs. (2.12) and (2.124) in Prob. 2.23, the step response $s(t)$ is given by

$$\begin{aligned} s(t) &= \int_{-\infty}^{t} h(\tau)d\tau = \int_{-\infty}^{t} e^{-a\tau}u(\tau)d\tau \\ &= \left[\int_{-\infty}^{t} e^{-a\tau}d\tau\right]u(t) = \frac{1}{a}(1 - e^{-at})u(t) \end{aligned}$$

which is the same as Eq. (2.125).

c. Using Eqs. (2.13) and (2.125), the impulse response $h(t)$ is given by

$$\begin{aligned} h(t) = s'(t) &= \frac{d}{dt}\left[\frac{1}{a}(1 - e^{-at})u(t)\right] \\ &= e^{-at}u(t) + \frac{1}{a}(1 - e^{-at})u'(t) \end{aligned}$$

Using Eqs. (1.25) and (1.30), we have

$$\frac{1}{a}(1-e^{-at})u'(t)=\frac{1}{a}(1-e^{-at})\delta(t)=\frac{1}{a}(1-1)\delta(t)=0$$

Thus,

$$h(t)=e^{-at}u(t)$$

which is the same as Eq. (1.124).

2.25. Consider the system described by

$$y'(t) + 2y(t) = x(t) + x'(t)$$

(2.126)

Find the impulse response $h(t)$ of the system.

The impulse response $h(t)$ should satisfy the differential equation

$$h'(t) + 2h(t) = \delta(t) + \delta'(t)$$

(2.127)

The homogeneous solution $h_h(t)$ to Eq. (2.127) is [see Prob. 2.23 and Eq. (2.122)]

$$h_h(t) = c_1 e^{-2t} u(t)$$

Assuming the particular solution $h_p(t)$ of the form

$$h_p(t) = c_2 \delta(t)$$

the general solution is

$$h(t) = c_1 e^{-2t} u(t) + c_2 \delta(t)$$

(2.128)

The delta function $\delta(t)$ must be present so that $h'(t)$ contributes $\delta'(t)$ to the left-hand side of Eq. (1.127). Substituting Eq. (2.128) into Eq. (2.127), we obtain

$$\begin{aligned}- 2c_1 e^{-2t} u(t) + c_1 e^{-2t} u'(t) + c_2 \delta'(t) + 2c_1 e^{-2t} u(t) + 2c_2 \delta(t)\\ = \delta(t) + \delta'(t)\end{aligned}$$

Again, using Eqs. (1.25) and (1.30), we have

$$(c_1 + 2c_2)\,\delta(t) + c_2\,\delta'(t) = \delta(t) + \delta'(t)$$

Equating coefficients of $\delta(t)$ and $\delta'(t)$, we obtain

$$c_1 + 2c_2 = 1 \qquad c_2 = 1$$

from which we have $c_1 = -1$ and $c_2 = 1$. Substituting these values in Eq. (2.128), we obtain

$$h(t) = -e^{-2t} u(t) + \delta(t)$$

(2.129)

2.10.5 Responses of a Discrete-Time LTI System and Convolution

2.26. Verify Eqs. (2.36) and (2.37); that is,

a. $x[n] * h[n] = h[n] * x[n]$

b. $\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$

a. By definition (2.35)

$$x[n]*h[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

By changing the variable $n - k = m$, we have

$$x[n]*h[n]=\sum_{m=-\infty}^{\infty} x[n-m]h[m]=\sum_{m=-\infty}^{\infty} h[m]x[n-m]=h[n]*x[n]$$

b. Let $x[n] * h_1[n] = f_1[n]$ and $h_1[n] * h_2[n] = f_2[n]$. Then

$$f_1[n]=\sum_{k=-\infty}^{\infty} x[k]h_1[n-k]$$

and

$$\{x[n]*h_1[n]\}*h_2[n]=f_1[n]*h_2[n]=\sum_{m=-\infty}^{\infty} f_1[m]h_2[n-m]$$

$$=\sum_{m=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x[k]h_1[m-k]\right]h_2[n-m]$$

Substituting $r = m - k$ and interchanging the order of summation, we have

$$\{x[n]*h_1[n]\}*h_2[n]=\sum_{k=-\infty}^{\infty} x[k]\left(\sum_{r=-\infty}^{\infty} h_1[r]h_2[n-k-r]\right)$$

Now, since

$$f_2[n]=\sum_{r=-\infty}^{\infty} h_1[r]h_2[n-r]$$

we have

$$f_2[n-k]=\sum_{r=-\infty}^{\infty} h_1[r]h_2[n-k-r]$$

Thus,

$$\{x[n]*h_1[n]\}*h_2[n]=\sum_{k=-\infty}^{\infty} x[k]\,f_2[n-k]$$

$$=x[n]*f_2[n]=x[n]*\{h_1[n]*h_2[n]\}$$

2.27. Show that

$$x[n] * \delta[n] = x[n]$$

(2.130)

$$x[n] * \delta[n - n_0] = x[n - n_0]$$

(2.131)

$$x[n] * u[n] = \sum_{k=-\infty}^{\infty} x[k]$$

(2.132)

$$x[n] * u[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

(2.133)

a. By Eq. (2.35) and property (1.46) of $\delta[n - k]$ we have

$$x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

b. Similarly, we have

$$x[n] * \delta[n - n_0] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n - k - n_0] = x[n - n_0]$$

c. By Eq. (2.35) and definition (1.44) of $u[n - k]$ we have

$$x[n] * u[n] = \sum_{k=-\infty}^{\infty} x[k]\,u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

d. In a similar manner, we have

$$x[n] * u[n - n_0] = \sum_{k=-\infty}^{\infty} x[k]\,u[n - k - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

2.28. The input $x[n]$ and the impulse response $h[n]$ of a discrete-time LTI system are given by

$$x[n] = u[n] \qquad h[n] = \alpha^n u[n] \qquad 0 < \alpha < 1$$

a. Compute the output $y[n]$ by Eq. (2.35).

b. Compute the output $y[n]$ by Eq. (2.39).

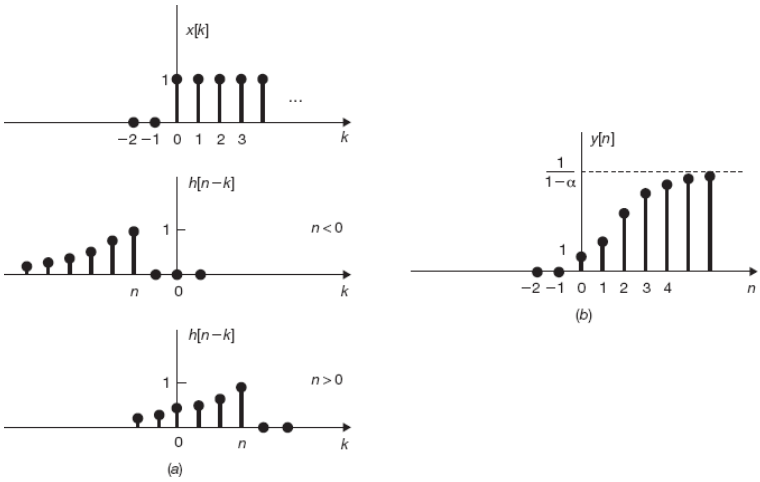
a. By Eq. (2.35) we have

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Sequences $x[k]$ and $h[n - k]$ are shown in Fig. 2-20(a) for $n < 0$ and $n > 0$. From Fig. 2-20(a) we see that for $n < 0$, $x[k]$ and $h[n - k]$ do not overlap, while for $n \geq 0$, they overlap from $k = 0$ to $k = n$. Hence, for $n < 0$, $y[n] = 0$. For $n \geq 0$, we have

$$y[n]=\sum_{k=0}^{n}\alpha^{n-k}$$

Figure 2-20



Changing the variable of summation k to $m = n - k$ and using Eq. (1.90), we have

$$y[n]=\sum_{m=n}^{0}\alpha^{m}=\sum_{m=0}^{n}\alpha^{m}=\frac{1-\alpha^{n+1}}{1-\alpha}$$

Thus, we can write the output $y[n]$ as

$$y[n]=\left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u[n]$$

(2.134)

which is sketched in Fig. 2-20(*b*).

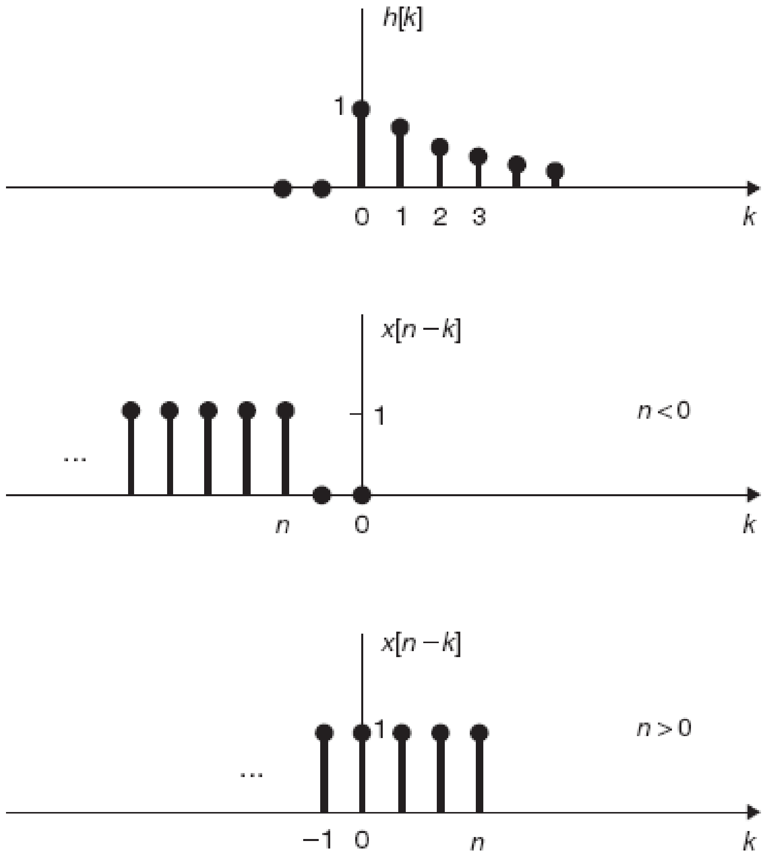
b. By Eq. (2.39)

$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

Sequences $h[k]$ and $x[n - k]$ are shown in Fig. 2-21 for $n < 0$ and $n > 0$. Again from Fig. 2-21 we see that for $n < 0$, $h[k]$ and $x[n - k]$ do not overlap, while for $n \geq 0$, they overlap from $k = 0$ to $k = n$. Hence, for $n < 0$, $y[n] = 0$. For $n \geq 0$, we have

$$y[n]=\sum_{k=0}^{n}\alpha^{k}=\frac{1-\alpha^{n+1}}{1-\alpha}$$

Figure 2-21



Thus, we obtain the same result as shown in Eq. (2.134).

2.29. Compute $y[n] = x[n] * h[n]$, where

a. $x[n] = \alpha^n u[n]$, $h[n] = \beta^n u[n]$

b. $x[n] = \alpha^n u[n]$, $h[n] = \alpha^{-n} u[-n]$, $0 < \alpha < 1$

a. From Eq. (2.35) we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k u[k]\beta^{n-k}u[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k \beta^{n-k} u[k]u[n-k]$$

since

$$u[k]u[n-k] = \begin{cases} 1 & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

we have

$$y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \qquad n \ge 0$$

Using Eq. (1.90), we obtain

$$y[n] = \begin{cases} \beta^n \dfrac{1-(\alpha/\beta)^{n+1}}{1-(\alpha/\beta)} u[n] & \alpha \neq \beta \\ \beta^n (n+1) u[n] & \alpha = \beta \end{cases}$$

(2.135a)

or

$$y[n] = \begin{cases} \dfrac{1}{\beta-\alpha}(\beta^{n+1}-\alpha^{n+1}) u[n] & \alpha \neq \beta \\ \beta^n (n+1) u[n] & \alpha = \beta \end{cases}$$

(2.135b)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k u[k] \alpha^{-(n-k)} u[-(n-k)]$$
$$= \sum_{k=-\infty}^{\infty} \alpha^{-n} \alpha^{2k} u[k] u[k-n]$$

For $n \leq 0$, we have

$$u[k]u[k-n] = \begin{cases} 1 & 0 \leq k \\ 0 & \text{otherwise} \end{cases}$$

Thus, using Eq. (1.91), we have

$$y[n] = \alpha^{-n} \sum_{k=0}^{\infty} \alpha^{2k} = \alpha^{-n} \sum_{k=0}^{\infty} (\alpha^2)^k = \frac{\alpha^{-n}}{1-\alpha^2} \qquad n \leq 0$$

(2.136a)

For $n \geq 0$, we have

$$u[k]u[k-n] = \begin{cases} 1 & n \leq k \\ 0 & \text{otherwise} \end{cases}$$

Thus, using Eq. (1.92), we have

$$y[n] = \alpha^{-n} \sum_{k=n}^{\infty} (\alpha^2)^k = \alpha^{-n} \frac{\alpha^{2n}}{1-\alpha^2} = \frac{\alpha^n}{1-\alpha^2} \qquad n \geq 0$$

(2.136b)

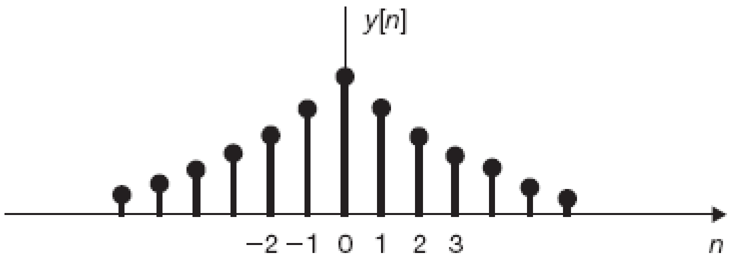
Combining Eqs. (2.136a) and (2.136b), we obtain

$$y[n] = \frac{\alpha^{|n|}}{1-\alpha^2} \qquad \text{all } n$$

(2.137)

which is sketched in Fig. 2-22.

Figure 2-22



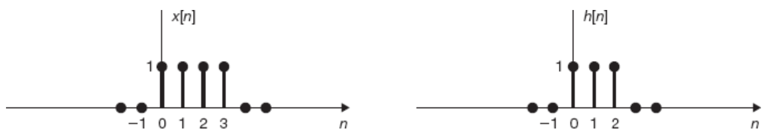
2.30. Evaluate $y[n] = x[n] * h[n]$, where $x[n]$ and $h[n]$ are shown in Fig. 2-23, (*a*) by an analytical technique, and (*b*) by a graphical method.

a. Note that $x[n]$ and $h[n]$ can be expressed as

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$$

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Figure 2-23



Now, using Eqs. (2.38), (2.130), and (2.131), we have

$$\begin{aligned} x[n] * h[n] &= x[n] * \{\delta[n] + \delta[n-1] + \delta[n-2]\} \\ &= x[n] * \delta[n] + x[n] * \delta[n-1] + x[n] * \delta[n-2] \\ &= x[n] + x[n-1] + x[n-2] \end{aligned}$$

Thus,

$$\begin{aligned} y[n] = {} & \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] \\ & + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] \\ & + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] \end{aligned}$$

or

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3\delta[n-3] + 2\delta[n-4] + \delta[n-5]$$

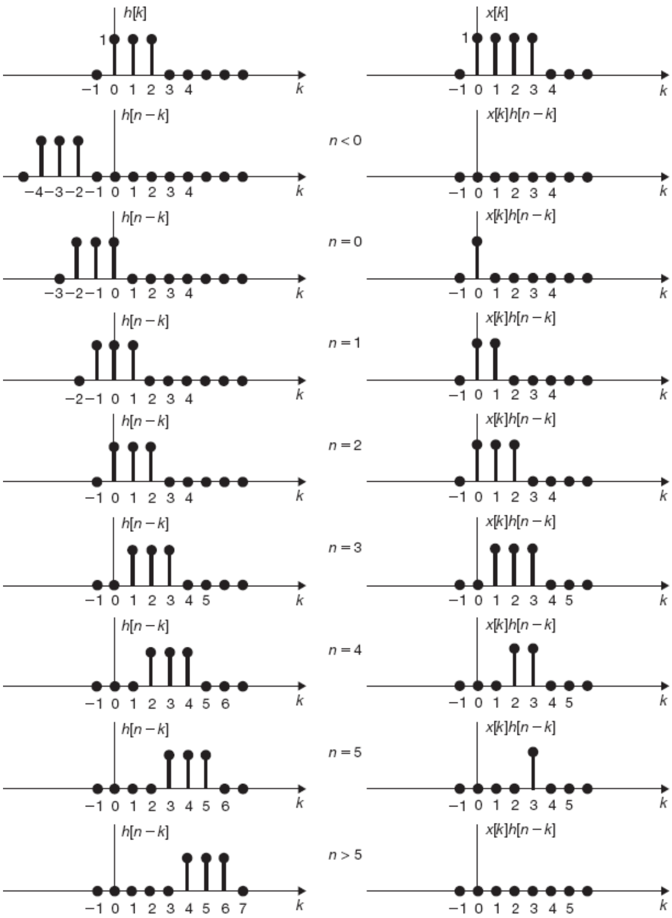
or

$$y[n] = \{1, 2, 3, 3, 2, 1\}$$

b. Sequences $h[k]$, $x[k]$ and $h[n-k]$, $x[k]h[n-k]$ for different values of n are sketched in Fig. 2-24. From Fig. 2-24 we see that $x[k]$ and $h[n-k]$ do not overlap for $n < 0$ and $n > 5$, and hence, $y[n] = 0$ for $n < 0$ and $n > 5$. For $0 \leq n \leq 5$, $x[k]$ and $h[n-k]$ overlap. Thus, summing $x[k]h[n-k]$ for $0 \leq n \leq 5$, we obtain

$$y[0] = 1 \qquad y[1] = 2 \qquad y[2] = 3 \qquad y[3] = 3 \qquad y[4] = 2 \qquad y[5] = 1$$

Figure 2-24

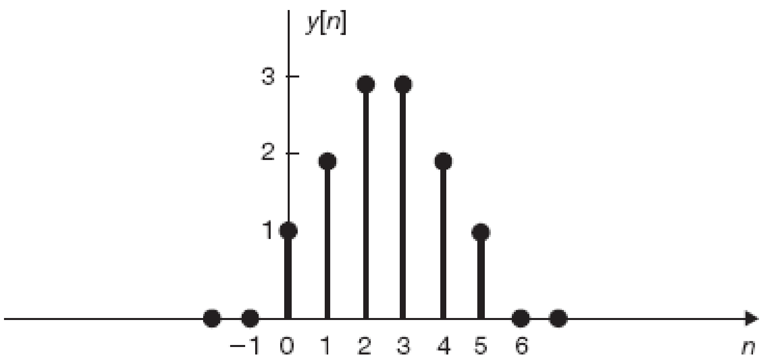


or

$$y[n] = \{1, 2, 3, 3, 2, 1\}$$

which is plotted in Fig. 2-25.

Figure 2-25



2.31. If $x_1[n]$ and $x_2[n]$ are both periodic sequences with common period N, the convolution of $x_1[n]$ and $x_2[n]$ does not converge. In this case, we define the *periodic convolution* of $x_1[n]$ and $x_2[n]$ as

$$f[n] = x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k]\, x_2[n-k]$$

(2.138)

Show that $f[n]$ is periodic with period N.

Since $x_2[n]$ is periodic with period N, we have

$$x_2[(n-k)+N] = x_2[n-k]$$

Then from Eq. (2.138) we have

$$\begin{aligned} f[n+N] &= \sum_{k=0}^{N-1} x_1[k]x_2[n+N-k] = \sum_{k=0}^{N-1} x_1[k]x_2[(n-k)+N] \\ &= \sum_{k=0}^{N-1} x_1[k]x_2[(n-k)] = f[n] \end{aligned}$$

Thus, $f[n]$ is periodic with period N.

2.32. The step response $s[n]$ of a discrete-time LTI system is given by

$$s[n] = \alpha^n u[n] \qquad 0 < \alpha < 1$$

Find the impulse response $h[n]$ of the system.

From Eq. (2.41) the impulse response $h[n]$ is given by

$$\begin{aligned} h[n] &= s[n] - s[n-1] = \alpha^n u[n] - \alpha^{n-1} u[n-1] \\ &= \{\delta[n] + \alpha^n u[n-1]\} - \alpha^{n-1} u[n-1] \\ &= \delta[n] - (1-\alpha)\alpha^{n-1} u[n-1] \end{aligned}$$

2.10.6. Properties of Discrete-Time LTI Systems

2.33. Show that if the input $x[n]$ to a discrete-time LTI system is periodic with period N, then the output $y[n]$ is also periodic with period N.

Let $h[n]$ be the impulse response of the system. Then by Eq. (2.39) we have

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Let $n = m + N$. Then

$$y[m+N] = \sum_{k=-\infty}^{\infty} h[k]x[m+N-k] = \sum_{k=-\infty}^{\infty} h[k]x[(m-k)+N]$$

Since $x[n]$ is periodic with period N, we have

$$x[(m-k)+N] = x[m-k]$$

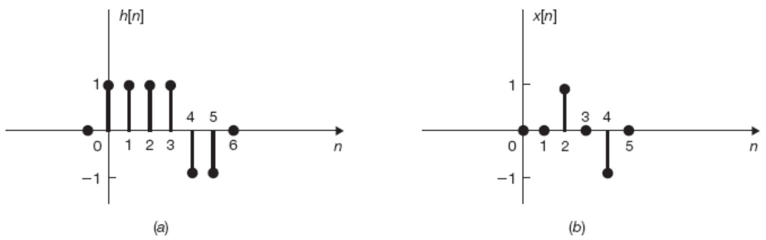
Thus,

$$y[m+N] = \sum_{k=-\infty}^{\infty} h[k]x[m-k] = y[m]$$

which indicates that the output $y[n]$ is periodic with period N.

2.34. The impulse response $h[n]$ of a discrete-time LTI system is shown in Fig. 2-26(a). Determine and sketch the output $y[n]$ of this system to the input $x[n]$ shown in Fig. 2-26(b) without using the convolution technique.

Figure 2-26



From Fig. 2-26(b) we can express $x[n]$ as

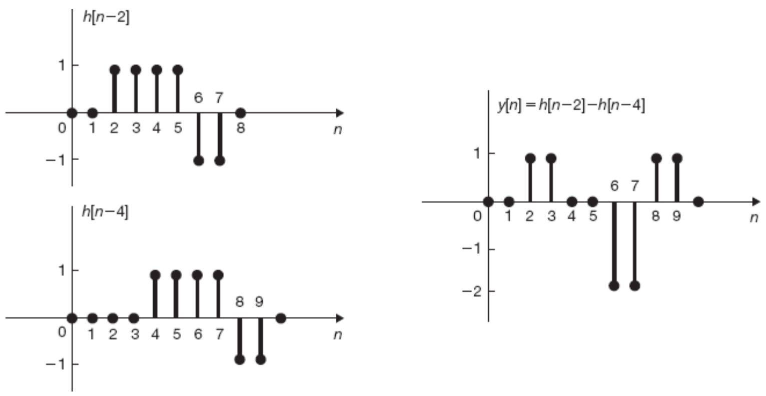
$$x[n] = \delta[n-2] - \delta[n-4]$$

Since the system is linear and time-invariant and by the definition of the impulse response, we see that the output $y[n]$ is given by

$$y[n] = h[n-2] - h[n-4]$$

which is sketched in Fig. 2-27.

Figure 2-27



2.35. A discrete-time system is causal if for every choice of n_0 the value of the output sequence $y[n]$ at $n = n_0$ depends on only the values of the input sequence $x[n]$ for $n \le n_0$ (see Sec. 1.5D). From this definition derive the causality condition (2.44) for a discrete-time LTI system; that is,

$$h[n] = 0 \qquad n < 0$$

From Eq. (2.39) we have

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{-1} h[k]x[n-k] + \sum_{k=0}^{\infty} h[k]x[n-k] \end{aligned}$$

(2.139)

Note that the first summation represents a weighted sum of future values of $x[n]$. Thus, if the system is causal, then

$$\sum_{k=-\infty}^{-1} h[k]x[n-k]=0$$

This can be true only if

$$h[n]=0 \qquad n<0$$

Now if $h[n] = 0$ for $n < 0$, then Eq. (2.139) becomes

$$y[n]=\sum_{k=0}^{\infty} h[k]x[n-k]$$

which indicates that the value of the output $y[n]$ depends on only the past and the present input values.

2.36. Consider a discrete-time LTI system whose input $x[n]$ and output $y[n]$ are related by

$$y[n]=\sum_{k=-\infty}^{n} 2^{k-n}x[k+1]$$

Is the system causal?

By definition (2.30) and Eq. (1.48) the impulse response $h[n]$ of the system is given by

$$h[n]=\sum_{k=-\infty}^{n} 2^{k-n}\delta[k+1]=\sum_{k=-\infty}^{n} 2^{-(n+1)}\delta[k+1]=2^{-(n+1)}\sum_{k=-\infty}^{n}\delta[k+1]$$

By changing the variable $k + 1 = m$ and by Eq. (1.50), we obtain

$$h[n]=2^{-(n+1)}\sum_{m=-\infty}^{n+1}\delta[m]=2^{-(n+1)}u[n+1]$$

(2.140)

From Eq. (2.140) we have $h[-1] = u[0] = 1 \neq 0$. Thus, the system is not causal.

2.37. Verify the BIBO stability condition [Eq. (2.49)] for discrete-time LTI systems.

Assume that the input $x[n]$ of a discrete-time LTI system is bounded, that is,

$$|x[n]| \le k_1 \qquad \text{all } n$$

(2.141)

Then, using Eq. (2.35), we have

$$|y[n]|=\left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty}|h[k]||x[n-k]| \le k_1\sum_{k=-\infty}^{\infty}|h[k]|$$

Since $|x[n-k]| \le k_1$ from Eq. (2.141). Therefore, if the impulse response is absolutely summable, that is,

$$\sum_{k=-\infty}^{\infty}|h[k]|=K<\infty$$

we have

$$|y[n]| \leq k_1 K = k_2 < \infty$$

and the system is BIBO stable.

2.38. Consider a discrete-time LTI system with impulse response $h[n]$ given by

$$h[n] = \alpha^n u[n]$$

a. Is this system causal?

b. Is this system BIBO stable?

a. Since $h[n] = 0$ for $n < 0$, the system is causal.

b. Using Eq. (1.91) (Prob. 1.19), we have

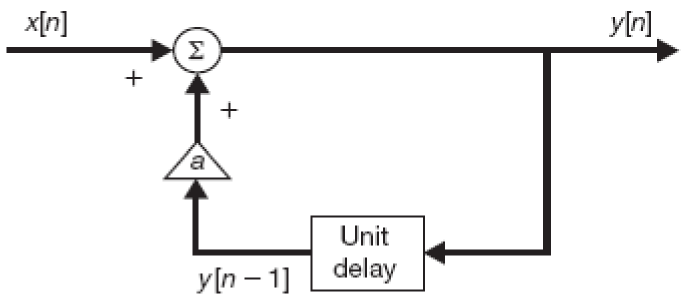
$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} \left|\alpha^k u[n]\right| = \sum_{k=0}^{\infty} |\alpha|^k = \frac{1}{1-|\alpha|} \qquad |\alpha| < 1$$

Therefore, the system is BIBO stable if $|\alpha| < 1$ and unstable if $|\alpha| \geq 1$.

2.10.7. Systems Described by Difference Equations

2.39. The discrete-time system shown in Fig. 2-28 consists of one unit delay element and one scalar multiplier. Write a difference equation that relates the output $y[n]$ and the input $x[n]$.

Figure 2-28



In Fig. 2-28 the output of the unit delay element is $y[n-1]$. Thus, from Fig. 2-28 we see that

$$y[n] = ay[n-1] + x[n]$$

(2.142)

or

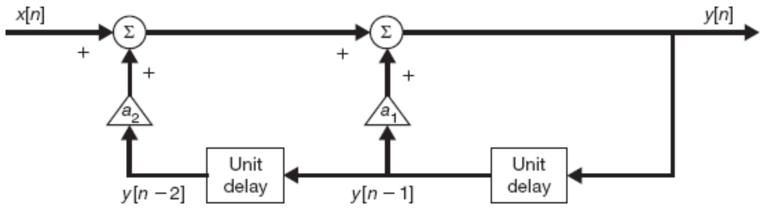
$$y[n] - ay[n-1] = x[n]$$

(2.143)

which is the required first-order linear difference equation.

2.40. The discrete-time system shown in Fig. 2-29 consists of two unit delay elements and two scalar multipliers. Write a difference equation that relates the output $y[n]$ and the input $x[n]$.

Figure 2-29



In Fig. 2-29 the output of the first (from the right) unit delay element is $y[n - 1]$ and the output of the second (from the right) unit delay element is $y[n - 2]$. Thus, from Fig. 2-29 we see that

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + x[n]$$

(2.144)

or

$$y[n] - a_1 y[n-1] - a_2 y[n-2] = x[n]$$

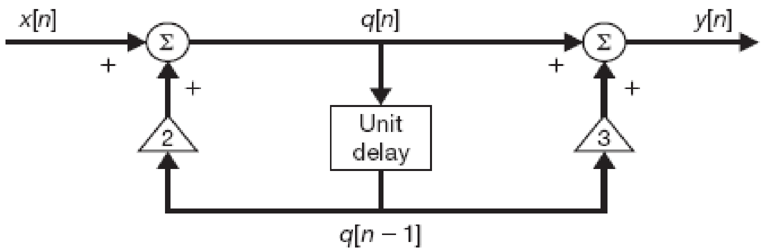
(2.145)

which is the required second-order linear difference equation.

Note that, in general, the order of a discrete-time LTI system consisting of the interconnection of unit delay elements and scalar multipliers is equal to the number of unit delay elements in the system.

2.41. Consider the discrete-time system in Fig. 2-30. Write a difference equation that relates the output $y[n]$ and the input $x[n]$.

Figure 2-30



Let the input to the unit delay element be $q[n]$. Then from Fig. 2-30 we see that

$$q[n] = 2q[n-1] + x[n]$$

(2.146a)

$$y[n] = q[n] + 3q[n-1]$$

(2.146b)

Solving Eqs. (2.146a) and (2.146b) for $q[n]$ and $q[n - 1]$ in terms of $x[n]$ and $y[n]$, we obtain

$$q[n] = \frac{2}{5} y[n] + \frac{3}{5} x[n]$$

(2.147a)

$$q[n-1]=\frac{1}{5}y[n]-\frac{1}{5}x[n]$$

(2.147b)

Changing n to $(n - 1)$ in Eq. (2.147a), we have

$$q[n-1]=\frac{2}{5}y[n-1]+\frac{3}{5}x[n-1]$$

(2.147c)

Thus, equating Eq. (2.147b) and Eq. (2.147c), we have

$$\frac{1}{5}y[n]-\frac{1}{5}x[n]=\frac{2}{5}y[n-1]+\frac{3}{5}x[n-1]$$

Multiplying both sides of the above equation by 5 and rearranging terms, we obtain

$$y[n] - 2y[n-1] = x[n] + 3x[n-1]$$

(2.148)

which is the required difference equation.

2.42. Consider a discrete-time system whose input $x[n]$ and output $y[n]$ are related by

$$y[n] - ay[n-1] = x[n]$$

(2.149)

where a is a constant. Find $y[n]$ with the auxiliary condition $y[-1] = y_{-1}$ and

$$x[n] = Kb^n u[n]$$

(2.150)

Let

$$y[n] = y_p[n] + y_h[n]$$

where $y_p[n]$ is the particular solution satisfying Eq. (2.149) and $y_h[n]$ is the homogeneous solution which satisfies

$$y[n] - ay[n-1] = 0$$

(2.151)

Assume that

$$y_p[n] = Ab^n \qquad n \geq 0$$

(2.152)

Substituting Eq. (2.152) into Eq. (2.149), we obtain

$$Ab^n - aAb^{n-1} = Kb^n$$

from which we obtain $A = Kb/(b - a)$, and

$$y_p[n] = \frac{K}{b-a} b^{n+1} \qquad n \geq 0$$

(2.153)

To obtain $y_h[n]$, we assume

$$y_h[n] = Bz^n$$

Substituting this into Eq. (2.151) gives

$$Bz^n - aBz^{n-1} = (z-a)\,Bz^{n-1} = 0$$

from which we have $z = a$ and

$$y_h[n] = Ba^n$$

(2.154)

Combining $y_p[n]$ and $y_h[n]$, we get

$$y[n] = Ba^n + \frac{K}{b-a} b^{n+1} \qquad n \geq 0$$

(2.155)

In order to determine B in Eq. (2.155) we need the value of $y[0]$. Setting $n = 0$ in Eqs. (2.149) and (2.150), we have

$$y[0] - ay[-1] = y[0] - ay_{-1} = x[0] = K$$

or

$$y[0] = K + ay_{-1}$$

(2.156)

Setting $n = 0$ in Eq. (2.155), we obtain

$$y[0] = B + K\frac{b}{b-a}$$

(2.157)

Therefore, equating Eqs. (2.156) and (2.157), we have

$$K + ay_{-1} = B + K\frac{b}{b-a}$$

from which we obtain

$$B = ay_{-1} - K\frac{a}{b-a}$$

Hence, Eq. (2.155) becomes

$$y[n] = y_{-1}a^{n+1} + K\frac{b^{n+1} - a^{n+1}}{b-a} \qquad n \geq 0$$

(2.158)

For $n < 0$, we have $x[n] = 0$, and Eq. (2.149) becomes Eq. (2.151). Hence,

$$y[n] = Ba^n$$

(2.159)

From the auxiliary condition $y[-1] = y_{-1}$, we have

$$y[-1] = y_{-1} = Ba^{-1}$$

from which we obtain $B = y_{-1}a$. Thus,

$$y[n] = y_{-1}a^{n+1} \qquad n < 0$$

(2.160)

Combining Eqs. (2.158) and (2.160), $y[n]$ can be expressed as

$$y[n] = y_{-1}a^{n+1} + K\frac{b^{n+1} - a^{n+1}}{b-a}u[n]$$

(2.161)

Note that as in the continuous-time case (Probs. 2.21 and 2.22), the system described by Eq. (2.149) is not linear if $y[-1] \neq 0$. The system is causal and time-invariant if it is initially at rest; that is, $y[-1] = 0$. Note also that Eq. (2.149) can be solved recursively (see Prob. 2.43).

2.43. Consider the discrete-time system in Prob. 2.42. Find the output $y[n]$ when $x[n] = K\delta[n]$ and $y[-1] = y_{-1} = \alpha$.

We can solve Eq. (2.149) for successive values of $y[n]$ for $n \geq 0$ as follows: rearrange Eq. (2.149) as

$$y[n] = ay[n-1] + x[n]$$

(2.162)

Then

$$\begin{aligned}
y[0] &= ay[-1] + x[0] = a\alpha + K \\
y[1] &= ay[0] + x[1] = a(a\alpha + K) \\
y[2] &= ay[1] + x[2] = a^2(a\alpha + K) \\
&\vdots \\
y[n] &= ay[n-1] + x[n] = a^n(a\alpha + K) = a^{n+1}\alpha + a^nK
\end{aligned}$$

(2.163)

Similarly, we can also determine $y[n]$ for $n < 0$ by rearranging Eq. (2.149) as

$$y[n-1] = \frac{1}{a}\{y[n] - x[n]\}$$

(2.164)

Then

$$
\begin{aligned}
y[-1] &= \alpha \\
y[-2] &= \frac{1}{a}\{y[-1] - x[-1]\} = \frac{1}{a}\alpha = a^{-1}\alpha \\
y[-3] &= \frac{1}{a}\{y[-2] - x[-2]\} = a^{-2}\alpha \\
&\vdots \\
y[-n] &= \frac{1}{a}\{y[-n+1] - x[-n+1]\} = a^{-n+1}\alpha
\end{aligned}
$$

(2.165)

Combining Eqs. (2.163) and (2.165), we obtain

$$y[n] = a^{n+1}\,\alpha + Ka^n u[n]$$

(2.166)

2.44. Consider the discrete-time system in Prob. 2.43 for an initially at rest condition.

a. Find in impulse response $h[n]$ of the system.
b. Find the step response $s[n]$ of the system.
c. Find the impulse response $h[n]$ from the result of part (b).

a. Setting $K = 1$ and $y[-1] = \alpha = 0$ in Eq. (2.166), we obtain

$$h[n] = a^n u[n]$$

(2.167)

b. Setting $K = 1$, $b = 1$, and $y[-1] = y_{-1} = 0$ in Eq. (2.161), we obtain

$$s[n] = \left(\frac{1 - a^{n+1}}{1 - a}\right) u[n]$$

(2.168)

c. From Eqs. (2.41) and (2.168) the impulse response $h[n]$ is given by

$$h[n] = s[n] - s[n-1] = \left(\frac{1 - a^{n+1}}{1 - a}\right) u[n] - \left(\frac{1 - a^n}{1 - a}\right) u[n-1]$$

When $n = 0$,

$$h[0] = \left(\frac{1 - a}{1 - a}\right) u[0] = 1$$

When $n \geq 1$,

$$h[n] = \frac{1}{1 - a}[1 - a^{n+1} - (1 - a^n)] = \frac{a^n(1 - a)}{1 - a} = a^n$$

Thus,

$$h[n] = a^n u[n]$$

which is the same as Eq. (2.167).

2.45. Find the impulse response $h[n]$ for each of the causal LTI discrete-time systems satisfying the following difference equations and indicate whether each system is a FIR or an IIR system.

a. $y[n] = x[n] - 2x[n-2] + x[n-3]$

b. $y[n] + 2y[n-1] = x[n] + x[n-1]$

c. $y[n] - \frac{1}{2}y[n-2] = 2x[n] - x[n-2]$

a. By definition (2.56)

$$h[n] = \delta[n] - 2\delta[n-2] + \delta[n-3]$$

or

$$h[n] = \{1, 0, -2, 1\}$$

Since $h[n]$ has only four terms, the system is a FIR system.

b. $h[n] = -2h[n-1] + \delta[n] + \delta[n-1]$

Since the system is causal, $h[-1] = 0$. Then

$$\begin{aligned}
h[0] &= -2h[-1] + \delta[0] + \delta[-1] = \delta[0] = 1 \\
h[1] &= -2h[0] + \delta[1] + \delta[0] = -2 + 1 = -1 \\
h[2] &= -2h[1] + \delta[2] + \delta[1] = -2(-1) = 2 \\
h[3] &= -2h[2] + \delta[3] + \delta[2] = -2(2) = -2^2 \\
&\vdots \\
h[n] &= -2h[n-1] + \delta[n] + \delta[n-1] = (-1)^n 2^{n-1}
\end{aligned}$$

Hence,

$$h[n] = \delta[n] + (-1)^n 2^{n-1} u[n-1]$$

Since $h[n]$ has infinite terms, the system is an IIR system.

c. $h[n] = \frac{1}{2}h[n-2] + 2\delta[n] - \delta[n-2]$

Since the system is casual, $h[-2] = h[-1] = 0$. Then

$$\begin{aligned}
h[0] &= \frac{1}{2}h[-2] + 2\delta[0] - \delta[-2] = 2\delta[0] = 2 \\
h[1] &= \frac{1}{2}h[-1] + 2\delta[1] - \delta[-1] = 0 \\
h[2] &= \frac{1}{2}h[0] + 2\delta[2] - \delta[0] = \frac{1}{2}(2) = -1 = 0 \\
h[3] &= \frac{1}{2}h[1] + 2\delta[3] - \delta[1] = 0 \\
&\vdots
\end{aligned}$$

Hence,

$$h[n] = 2\delta[n]$$

Since $h[n]$ has only one term, the system is a FIR system.

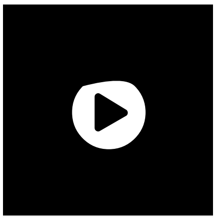
2.11. SUPPLEMENTARY PROBLEMS

2.46. Compute the convolution $y(t) = x(t) * h(t)$ of the following pair of signals:

$$(a)\quad x(t) = \begin{cases} 1 & -a < t \le a \\ 0 & \text{otherwise} \end{cases},\quad h(t) = \begin{cases} 1 & -a < t \le a \\ 0 & \text{otherwise} \end{cases}$$

$$(b)\quad x(t) = \begin{cases} t & 0 < t \le T \\ 0 & \text{otherwise} \end{cases},\quad h(t) = \begin{cases} 1 & 0 < t \le 2T \\ 0 & \text{otherwise} \end{cases}$$

$$(c)\quad x(t) = u(t-1),\ h(t) = e^{-3t}u(t)$$



Schaum's Signals and Systems Supplementary Problem 2.46: Convolution Example

This video shows how to find the convolution of a pair of signals.

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2.47. Compute the convolution sum $y[n] = x[n] * h[n]$ of the following pairs of sequences:

a. $x[n] = u[n]$, $h[n] = 2^n u[-n]$

b. $x[n] = u[n] - u[n - N]$, $h[n] = a^n u[n]$, $0 < a < 1$

c. $x[n] = (1–2)^n u[n]$, $h[n] = \delta[n] - 1–2\ \delta[n - 1]$

2.48. Show that if $y(t) = x(t) * h(t)$, then

$$y'(t) = x'(t) * h(t) = x(t) * h'(t)$$

2.49. Show that

$$x(t) * \delta'(t) = x'(t)$$

2.50. Let $y[n] = x[n] * h[n]$. Then show that

$$x[n - n_1] * h[n - n_2] = y[n - n_1 - n_2]$$

2.51. Show that

$$x_1[n] \otimes x_2[n] = \sum_{k=n_0}^{n_0+N-1} x_1[k]\, x_2[n-k]$$

for an arbitrary starting point n_0.

2.52. The step response $s(t)$ of a continuous-time LTI system is given by

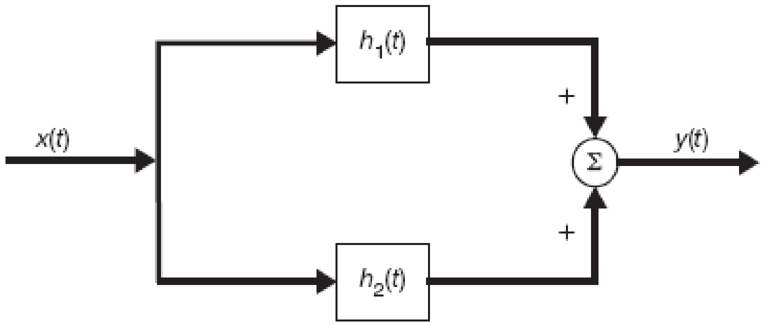
$$s(t) = [\cos \omega_0 t]u(t)$$

Find the impulse response $h(t)$ of the system.

2.53. The system shown in Fig. 2-31 is formed by connection two systems *in parallel*. The impulse responses of the systems are given by

$$h_1(t) = e^{-2t}u(t) \qquad \text{and} \qquad h_2(t) = 2e^{-t}u(t)$$

Figure 2-31



a. Find the impulse response $h(t)$ of the overall system.

b. Is the overall system stable?

2.54. Consider an integrator whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

a. Find the impulse response $h(t)$ of the integrator.

b. Is the integrator stable?

2.55. Consider a discrete-time LTI system with impulse response $h[n]$ given by

$$h[n] = \delta[n-1]$$

Is this system memoryless?

2.56. The impulse response of a discrete-time LTI system is given by

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Let $y[n]$ be the output of the system with the input

$$x[n] = 2\delta[n] + \delta[n-3]$$

Find $y[1]$ and $y[4]$.

2.57. Consider a discrete-time LTI system with impulse response $h[n]$ given by

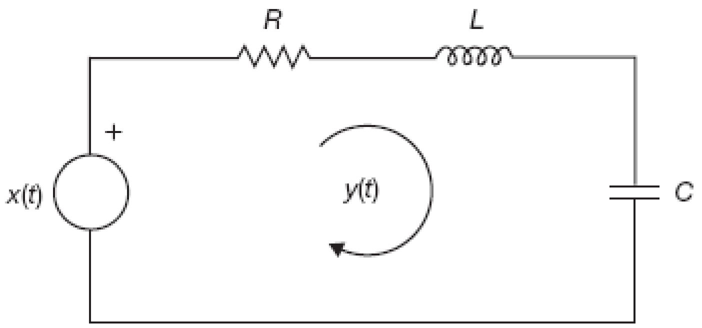
$$h[n] = \left(-\frac{1}{2}\right)^n u[n-1]$$

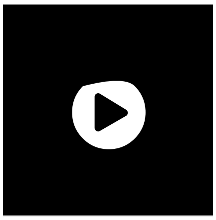
a. Is the system causal?

b. Is the system stable?

2.58. Consider the *RLC* circuit shown in Fig. 2-32. Find the differential equation relating the output current $y(t)$ and the input voltage $x(t)$.

Figure 2-32





Schaum's Signals and Systems Supplementary Problem 2.58: RLC Circuit Example

This video illustrates how to derive the differential equation for a series RLC circuit.

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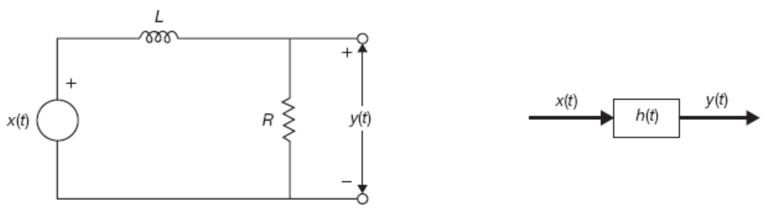
2.59. Consider the *RL* circuit shown in Fig. 2-33.

a. Find the differential equation relating the output voltage $y(t)$ across R and the the input voltage $x(t)$.

b. Find the impulse response $h(t)$ of the circuit.

c. Find the step response $s(t)$ of the circuit.

Figure 2-33



2.60. Consider the system in Prob. 2.20. Find the output $y(t)$ if $x(t) = e^{-at}u(t)$ and $y(0) = 0$.

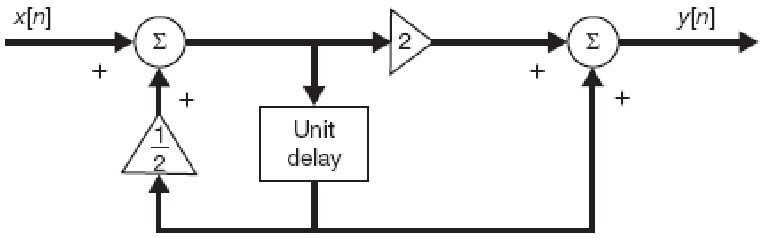
2.61. Is the system described by the differential equation

$$\frac{dy(t)}{dt} + 5y(t) + 2 = x(t)$$

linear?

2.62. Write the input-output equation for the system shown in Fig. 2-34.

Figure 2-34



2.63. Consider a discrete-time LTI system with impulse response

$$h[n] = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the input-output relationship of the system.

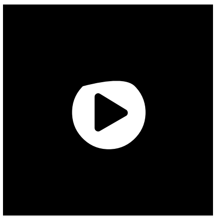
2.64. Consider a discrete-time system whose input $x[n]$ and output $y[n]$ are related by

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

with $y[-1] = 0$. Find the output $y[n]$ for the following inputs:

$$(a) \quad x[n] = \left(\frac{1}{3}\right)^n u[n];$$

$$(b) \quad x[n] = \left(\frac{1}{2}\right)^n u[n]$$



Schaum's Signals and Systems Supplementary Problem 2.64: Discrete-time system Example

This video shows how to find the output of a discrete-time system.

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2.65. Consider the system in Prob. 2.42. Find the eigenfunction and the corresponding eigenvalue of the system.

2.12. ANSWERS TO SUPPLEMENTARY PROBLEMS

2.46.

$$(a)\quad y(t)=\begin{cases}2a-|t| & |t|<2a\\ 0 & |t|\geq 2a\end{cases}$$

$$(b)\quad y(t)=\begin{cases}0 & t<0\\ \frac{1}{2}t^2 & 0<t\leq T\\ \frac{1}{2}T^2 & T<t\leq 2T\\ -\frac{1}{2}t^2+2T-\frac{5}{2}T^2 & 2T<t\leq 3T\\ 0 & 3T<t\end{cases}$$

$$(c)\quad \frac{1}{3}(1-e^{-3(t-1)})u(t-1)$$

2.47.

$$(a)\quad y[n]=\begin{cases}2^{1-n} & n\leq 0\\ 2 & n>0\end{cases}$$

$$(b)\quad y[n]=\begin{cases}0 & n<0\\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & n\leq 0\leq N-1\\ \alpha^{n-N+1}\left(\dfrac{1-\alpha^N}{1-\alpha}\right) & N-1<n\end{cases}$$

$$(c)\quad y[n]=\delta[n]$$

2.48. *Hint:* Differentiate Eqs. (2.6) and (2.10) with respect to t.

2.49. *Hint:* Use the result from Prob. 2.48 and Eq. (2.58).

2.50. *Hint:* See Prob. 2.3.

2.51. *Hint:* See Probs. 2.31 and 2.8.

2.52. $h(t) = \delta(t) - \omega_0[\sin \omega_0 t]u(t)$

2.53.

a. $h(t) = (e^{-2t} + 2e^{-t})\, u(t)$

b. Yes

2.54.

a. $h(t) = u(t)$

b. No

2.55. No, the system has memory.

2.56. $y[1] = 1$ and $y[4] = \frac{5}{8}$

2.57. (*a*) Yes; (*b*) Yes

2.58. $\dfrac{d^2y(t)}{dt^2}+\dfrac{R}{L}\dfrac{dy(t)}{dt}+\dfrac{1}{LC}y(t)=\dfrac{1}{L}\dfrac{dx(t)}{dt}$

2.59.

$$(a)\quad \frac{dy(t)}{dt}+\frac{R}{L}y(t)=\frac{R}{L}x(t)$$

$$(b)\quad h(t)=\frac{R}{L}e^{-(R/L)t}u(t)$$

$$(c)\quad s(t)=[1-e^{-(R/L)t}]u(t)$$

2.60. $te^{-at}u(t)$

2.61. No, it is nonlinear.

2.62. $2y[n]-y[n-1]=4x[n]+2x[n-1]$

2.63. $y[n]=x[n]+x[n-1]$

2.64.

$$(a)\quad y[n]=6\left[\left(\frac{1}{2}\right)^{n+1}-\left(\frac{1}{3}\right)^{n+1}\right]u[n]$$

$$(b)\quad y[n]=(n+1)\left(\frac{1}{2}\right)^{n}u[n]$$

2.65. $z^n,\ \lambda=\dfrac{z}{z-a}$