PROBLEM P3

THE ELECTRIC FIELD GENERATED BY AN ANTENNA IS

$$\vec{E} = E_\theta \hat{\theta} \qquad E_\theta = I_0 \, C \, \frac{1}{R} \, e^{-jkR} \sin^2\theta$$

WHERE I_0 IS THE CURRENT FEEDING THE ANTENNA, R IS THE DISTANCE BETWEEN THE OBSERVATION POINT AND THE ANTENNA (PLACED ON THE ORIGIN) AND C IS A CONSTANT

CALCULATE:

A) DIRECTIVITY

B) RADIATION RESISTANCE

SOLUTION

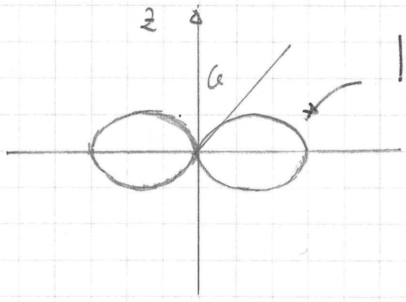
THE POWER DENSITY IS GIVEN BY $S = \frac{1}{2\eta} |E_\theta|^2$

$$S = \frac{1}{2\eta} \frac{1}{R^2} |I_0|^2 \sin^4\theta$$

BY DEFINITION THE POWER PATTERN IS $P(\theta,\varphi) = |F(\theta,\varphi)|^2 = \frac{S}{\max\{S\}} = \frac{|E|^2}{|E_{MAX}|^2}$

$$|F(\theta,\varphi)|^2 = \frac{|E|^2}{|E|^2_{MAX}} = \frac{|I_0|^2 C^2 \frac{1}{R^2} \sin^4\theta}{|I_0|^2 C^2 \frac{1}{R^2}} = \sin^4\theta$$

THE DIRECTIVITY D IS COMPUTED STARTING FROM THE POWER PATTERN $|F(\theta,\varphi)|^2$



$|F(\theta,\varphi)|^2 = \sin^4\theta$

$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{\iint_{4\pi} |F(\theta,\varphi)|^2 \, d\Omega} = \frac{4\pi}{\int_0^{2\pi} \int_0^{\pi} |F(\theta,\varphi)|^2 \sin\theta \, d\theta \, d\varphi}$$

$$D = \frac{4\pi}{\int_0^{2\pi}\int_0^{\pi} \sin^4\theta \sin\theta \, d\theta \, d\varphi}$$

THE DOUBLE INTEGRAL CAN BE WRITTEN AS THE PRODUCT OF TWO INTEGRALS OF A SINGLE VARIABLE

$$\int_0^{2\pi}\int_0^{\pi} \sin^4\theta \sin\theta \, d\theta \, d\varphi = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^5\theta \, d\theta$$

$$\int_0^{2\pi} d\varphi = 2\pi$$

$$\int_0^{\pi} \sin^5\theta \, d\theta = \int_0^{\pi} (1-\cos^2\theta)^2 \sin\theta \, d\theta$$

WE CHANGE THE INTEGRATION VARIABLE

$\cos\theta = x \qquad d\cos\theta = -\sin\theta \, d\theta = dx$

$$= \int_1^{-1} (1-x^2)^2 (-dx) = \int_{-1}^{1} (1 + x^4 - 2x^2) \, dx = \left(x + \frac{x^5}{5} - 2\frac{x^3}{3} \right\rvert_{-1}^{1} =$$

$$= 2 + \frac{2}{5} - \frac{4}{3} = \frac{30 + 6 - 20}{15} = \frac{16}{15}$$

AND THE DIRECTIVITY BECOMES

$$D = \frac{4\pi}{2\pi \frac{16}{15}} = \frac{30}{16} = \frac{15}{8} = 1{,}875$$

B) BY DEFINITION THE RADIATION RESISTANCE R_R FULFILLS THE RELATION

$$P_R = \frac{1}{2} R_R |I|^2$$

WHERE I IS THE CURRENT FEEDING THE ANTENNA AND P_R IS THE RADIATED POWER

THE RADIATED POWER IS OBTAINED BY INTEGRATING THE POYNTING VECTOR (POWER DENSITY) OVER THE SURFACE OF A SPHERE HAVING RADIUS R

$$P_R = \int_{\Sigma} \frac{1}{2\eta} |E|^2 \, d\Sigma = \int_0^{2\pi} \int_0^{\pi} \frac{1}{2\eta} |E|^2 R^2 \sin\theta \, d\theta \, d\varphi$$

$$P_R = \frac{1}{2\eta} \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{|I_0|^2}{R^2} C^2 R^2 \sin^4\theta \sin\theta \, d\theta$$

$$P_R = \frac{1}{2} \frac{C^2}{\eta} |I_0|^2 \, 2\pi \int_0^{\pi} \sin^5\theta \, d\theta$$

FROM A) WE KNOW THAT $\int_0^{\pi} \sin^5\theta \, d\theta = \frac{16}{15}$

$$P_R = \frac{1}{2} |I_0|^2 \frac{C^2}{\eta} \frac{32\pi}{15}$$

AND BY COMPARING THIS FORMULA WITH THE DEFINITION OF RADIATION RESISTANCE

$$P_R = \frac{1}{2} R_R |I_0|^2$$

WE IMMEDIATELY WRITE

$$R_R = \frac{C^2}{\eta} \frac{32\pi}{15} \simeq \frac{C^2}{120\pi} \frac{32\pi}{15} = C^2 \frac{4}{225} \simeq C^2 \, 0{,}0178 \, \Omega$$