ACCESS Engineering

1. Signals and Systems

1.1. Introduction

The concept and theory of signals and systems are needed in almost all electrical engineering fields and in many other engineering and scientific disciplines as well. In this chapter we introduce the mathematical description and representation of signals and systems and their classifications. We also define several important basic signals essential to our studies.

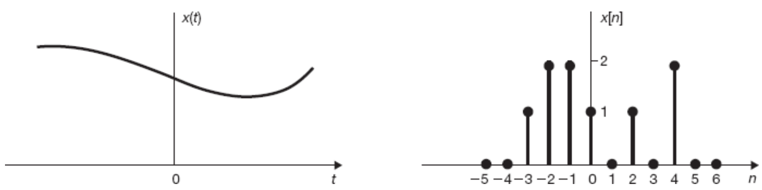
1.2. Signals and Classification of Signals

A *signal* is a function representing a physical quantity or variable, and typically it contains information about the behavior or nature of the phenomenon. For instance, in an *RC* circuit the signal may represent the voltage across the capacitor or the current flowing in the resistor. Mathematically, a signal is represented as a function of an independent variable t. Usually t represents time. Thus, a signal is denoted by $x(t)$.

1.2.1. A. Continuous-Time and Discrete-Time Signals:

A signal $x(t)$ is a *continuous-time* signal if t is a continuous variable. If t is a discrete variable—that is, $x(t)$ is defined at discrete times—then $x(t)$ is a *discrete-time* signal. Since a discrete-time signal is defined at discrete times, a discrete-time signal is often identified as a *sequence* of numbers, denoted by $\{x_n\}$ or $x[n]$, where n = integer. Illustrations of a continuous-time signal $x(t)$ and of a discrete-time signal $x[n]$ are shown in Fig. 1-1.

Figure 1-1 Graphical representation of (a) continuous-time and (b) discrete-time signals.



A discrete-time signal $x[n]$ may represent a phenomenon for which the independent variable is inherently discrete. For instance, the daily closing stock market average is by its nature a signal that evolves at discrete points in time (that is, at the close of each day). On the other hand a discrete-time signal $x[n]$ may be obtained by *sampling* a continuous-time signal $x(t)$ such as

$$x(t_0), x(t_1), \ldots, x(t_n), \ldots$$

or in a shorter form as

$$x[0], x[1], \ldots, x[n], \ldots$$

or

$$x_0, x_1, \ldots, x_n, \ldots$$

where we understand that

$$x_n = x[n] = x(t_n)$$

and x_n's are called *samples* and the time interval between them is called the *sampling interval*. When the sampling intervals are equal (uniform sampling), then

$$x_n = x[n] = x(nT_s)$$

where the constant T_s is the sampling interval.

A discrete-time signal $x[n]$ can be defined in two ways:

1. We can specify a rule for calculating the nth value of the sequence. For example,

$$x[n] = x_n = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

   or

$$\{x_n\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \left(\frac{1}{2}\right)^n, \ldots\right\}$$

2. We can also explicitly list the values of the sequence. For example, the sequence shown in Fig. 1-1(*b*) can be written as

$$\{x_n\} = \{\ldots, 0, 0, \underset{\uparrow}{1}, 2, 2, 1, 0, 1, 0, 2, 0, 0, \ldots\}$$

or

$$\{x_n\} = \{\underset{\uparrow}{1}, 2, 2, 1, 0, 1, 0, 2\}$$

   We use the arrow to denote the $n = 0$ term. We shall use the convention that if no arrow is indicated, then the first term corresponds to $n = 0$ and all the values of the sequence are zero for $n < 0$.

The sum and product of two sequences are defined as follows:

$$\begin{aligned} \{c_n\} &= \{a_n\} + \{b_n\} \rightarrow c_n = a_n + b_n \\ \{c_n\} &= \{a_n\}\{b_n\} \quad \rightarrow c_n = a_n b_n \\ \{c_n\} &= \alpha\{a_n\} \qquad \rightarrow c_n = \alpha a_n \qquad \alpha = \text{constant} \end{aligned}$$

1.2.2. B. Analog and Digital Signals:

If a continuous-time signal $x(t)$ can take on any value in the continuous interval (a, b), where a may be $-\infty$ and b may be $+\infty$, then the continuous-time signal $x(t)$ is called an *analog* signal. If a discrete-time signal $x[n]$ can take on only a finite number of distinct values, then we call this signal a *digital* signal.

1.2.3. C. Real and Complex Signals:

A signal $x(t)$ is a *real* signal if its value is a real number, and a signal $x(t)$ is a *complex* signal if its value is a complex number. A general complex signal $x(t)$ is a function of the form

$$x(t) = x_1(t) + jx_2(t)$$

(1.1)

where $x_1(t)$ and $x_2(t)$ are real signals and $j = \sqrt{-1}$.

Note that in Eq. (1.1) t represents either a continuous or a discrete variable.

1.2.4. D. Deterministic and Random Signals:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signal can be modeled by a known function of time t. *Random* signals are those signals that take random values at any given time and must be characterized statistically. Random signals will be discussed in Chaps. 8 and 9.

1.2.5. E. Even and Odd Signals:

A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if

$$x(-t) = x(t)$$
$$x[-n] = x[n]$$

(1.2)

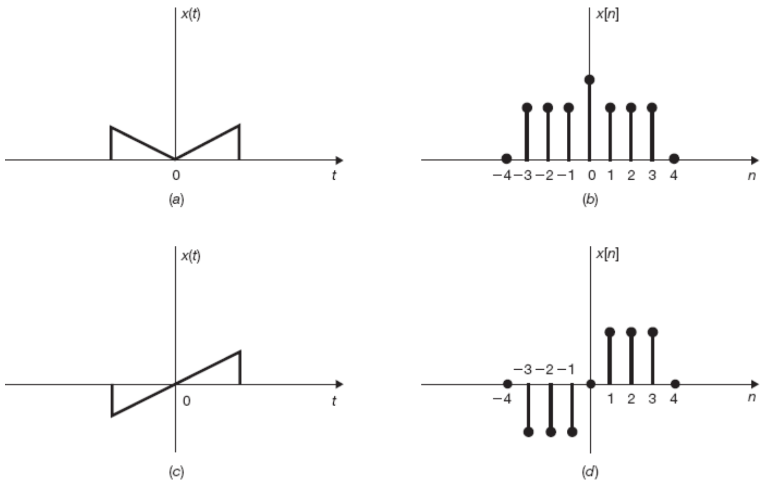
A signal $x(t)$ or $x[n]$ is referred to as an *odd* signal if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

(1.3)

Examples of even and odd signals are shown in Fig. 1-2.

Figure 1-2 Examples of even signals (a and b) and odd signals (c and d).



Any signal $x(t)$ or $x[n]$ can be expressed as a sum of two signals, one of which is even and one of which is odd. That is,

$$x(t) = x_e(t) + x_o(t)$$
$$x[n] = x_e[n] + x_o[n]$$

(1.4)

Where

$$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\} \qquad \text{even part of } x(t)$$
$$x_e[n] = \frac{1}{2}\{x[n] + x[-n]\} \qquad \text{even part of } x[n]$$

(1.5)

$$x_o(t) = \frac{1}{2}\{x(t) - x(-t)\} \qquad \text{odd part of } x(t)$$
$$x_o[n] = \frac{1}{2}\{x[n] - x[-n]\} \qquad \text{odd part of } x[n]$$

(1.6)

Note that the product of two even signals or of two odd signals is an even signal and that the product of an even signal and an odd signal is an odd signal (Prob. 1.7).

1.2.6. F. Periodic and Nonperiodic Signals:

A continuous-time signal $x(t)$ is said to be *periodic with period T* if there is a positive nonzero value of T for which

$$x(t + T) = x(t) \qquad \text{all } t$$

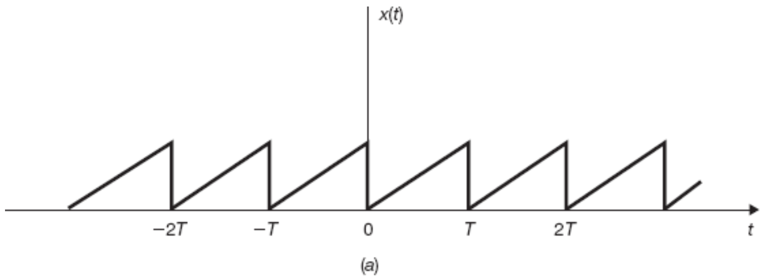
(1.7)

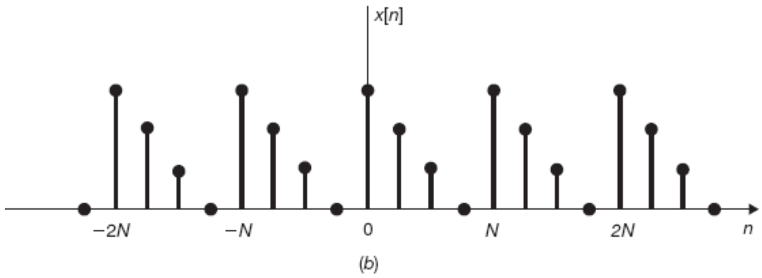
An example of such a signal is given in Fig. 1-3(*a*). From Eq. (1.7) or Fig. 1-3(*a*) it follows that

$$x(t + mT) = x(t)$$

(1.8)

Figure 1-3 *Examples of periodic signals.*





for all t and any integer m. The *fundamental period* T_0 of $x(t)$ is the smallest positive value of T for which Eq. (1.7) holds. Note that this definition does not work for a constant signal $x(t)$ (known as a dc signal). For a constant signal $x(t)$ the fundamental period is undefined since $x(t)$ is periodic for *any* choice of T (and so there is no smallest positive value). Any continuous-time signal which is not periodic is called a *nonperiodic* (or *aperiodic*) signal.

Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) $x[n]$ is *periodic with period N* if there is a positive integer N for which

$$x[n+N] = x[n] \qquad \text{all } n$$

(1.9)

An example of such a sequence is given in Fig. 1-3(b). From Eq. (1.9) and Fig. 1-3(b) it follows that

$$x[n+mN] = x[n]$$

(1.10)

for all n and any integer m. The fundamental period N_0 of $x[n]$ is the smallest positive integer N for which Eq. (1.9) holds. Any sequence which is not periodic is called a *nonperiodic* (or *aperiodic*) sequence.

Note that a sequence obtained by uniform sampling of a periodic continuous-time signal may not be periodic (Probs. 1.12 and 1.13). Note also that the sum of two continuous-time periodic signals may not be periodic but that the sum of two periodic sequences is always periodic (Probs. 1.14 and 1.15).

1.2.7. G. Energy and Power Signals:

Consider $v(t)$ to be the voltage across a resistor R producing a current $i(t)$. The instantaneous power $p(t)$ per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t)$$

(1.11)

Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i^2(t)\, dt \quad \text{joules}$$

(1.12)

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)\, dt \quad \text{watts}$$

(1.13)

For an arbitrary continuous-time signal $x(t)$, the *normalized energy content E* of $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

(1.14)

The *normalized average power P* of $x(t)$ is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

(1.15)

Similarly, for a discrete-time signal $x[n]$, the normalized energy content E of $x[n]$ is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

(1.16)

The normalized average power P of $x[n]$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

(1.17)

Based on definitions (1.14) to (1.17), the following classes of signals are defined:

1. $x(t)$ (or $x[n]$) is said to be an *energy* signal (or sequence) if and only if $0 < E < \infty$, and so $P = 0$.
2. $x(t)$ (or $x[n]$) is said to be a *power* signal (or sequence) if and only if $0 < P < \infty$, thus implying that $E = \infty$.
3. Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Note that a periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period (Prob. 1.18).

1.3. Basic Continuous-Time Signals

1.3.1. A. The Unit Step Function:

The *unit step* function $u(t)$, also known as the *Heaviside unit* function, is defined as

$$u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$

(1.18)

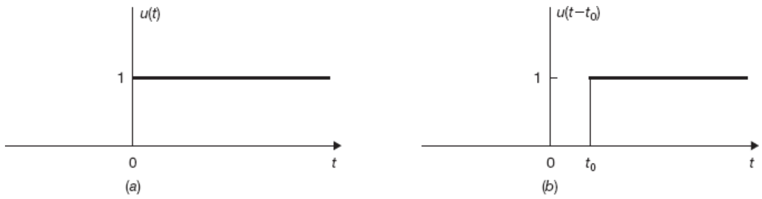
which is shown in Fig. 1-4(*a*). Note that it is discontinuous at $t = 0$ and that the value at $t = 0$ is undefined. Similarly, the shifted unit step function $u(t - t_0)$ is defined as

$$u(t-t_0)=\begin{cases}1 & t>t_0\\ 0 & t<t_0\end{cases}$$

(1.19)

which is shown in Fig. 1-4(*b*)

Figure 1-4 (a) Unit step function; (b) shifted unit step function.



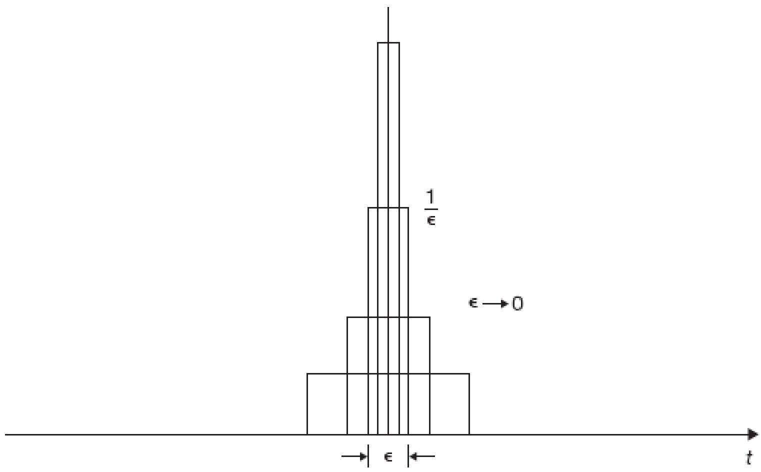
1.3.2. B. The Unit Impulse Function:

The *unit impulse* function $\delta(t)$, also known as the *Dirac delta* function, plays a central role in system analysis. Traditionally, $\delta(t)$ is often defined as the limit of a suitably chosen conventional function having unity area over an infinitesimal time interval as shown in Fig. 1-5 and possesses the following properties:

$$\delta(t)=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases}$$

$$\int_{-\varepsilon}^{\varepsilon}\delta(t)\,dt=1$$

Figure 1-5



But an ordinary function which is everywhere 0 except at a single point must have the integral 0 (in the Riemann integral sense). Thus, $\delta(t)$ cannot be an ordinary function and mathematically it is defined by

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt = \phi(0)$$

(1.20)

where $\phi(t)$ is any regular function continuous at $t = 0$.

An alternative definition of $\delta(t)$ is given by

$$\int_a^b \phi(t)\delta(t)\,dt = \begin{cases} \phi(0) & a < 0 < b \\ 0 & a < b < 0 \quad \text{or} \quad 0 < a < b \\ \text{undefined} & a = 0 \quad \text{or} \quad b = 0 \end{cases}$$

(1.21)

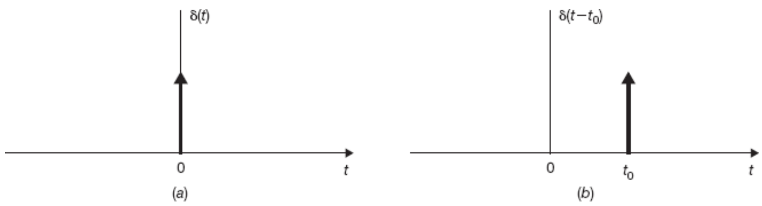
Note that Eq. (1.20) or (1.21) is a symbolic expression and should not be considered an ordinary Riemann integral. In this sense, $\delta(t)$ is often called a *generalized function* and $\phi(t)$ is known as a *testing function*. A different class of testing functions will define a different generalized function (Prob. 1.24). Similarly, the delayed delta function $\delta(t - t_0)$ is defined by

$$\int_{-\infty}^{\infty} \phi(t)\delta(t - t_0)\,dt = \phi(t_0)$$

(1.22)

where $\phi(t)$ is any regular function continuous at $t = t_0$. For convenience, $\delta(t)$ and $\delta(t - t_0)$ are depicted graphically as shown in Fig. 1-6.

Figure 1-6 (a) Unit impulse function; (b) shifted unit impulse function.



Some additional properties of $\delta(t)$ are

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

(1.23)

$$\delta(-t) = \delta(t)$$

(1.24)

$$x(t)\delta(t) = x(0)\delta(t)$$

(1.25)

if $x(t)$ is continuous at $t = 0$.

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

(1.26)

if $x(t)$ is continuous at $t = t_0$.

Using Eqs. (1.22) and (1.24), any continuous-time signal $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

(1.27)

1.3.2.1. Generalized Derivatives:

If $g(t)$ is a generalized function, its nth generalized derivative $g^{(n)}(t) = d^n g(t)/dt^n$ is defined by the following relation:

$$\int_{-\infty}^{\infty} \phi(t) g^{(n)}(t)\,dt = (-1)^n \int_{-\infty}^{\infty} \phi^{(n)}(t) g(t)\,dt$$

(1.28)

where $\phi(t)$ is a testing function which can be differentiated an arbitrary number of times and vanishes outside some fixed interval and $\phi^{(n)}(t)$ is the nth derivative of $\phi(t)$. Thus, by Eqs. (1.28) and (1.20) the derivative of $\delta(t)$ can be defined as

$$\int_{-\infty}^{\infty} \phi(t)\delta'(t)\,dt = -\phi'(0)$$

(1.29)

where $\phi(t)$ is a testing function which is continuous at $t = 0$ and vanishes outside some fixed interval and $\phi'(0) = d\phi(t)/dt|_{t=0}$. Using Eq. (1.28), the derivative of $u(t)$ can be shown to be $\delta(t)$ (Prob. 1.28); that is,

$$\delta(t) = u'(t) = \frac{du(t)}{dt}$$

(1.30)

Then the unit step function $u(t)$ can be expressed as

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

(1.31)

Note that the unit step function $u(t)$ is discontinuous at $t = 0$; therefore, the derivative of $u(t)$ as shown in Eq. (1.30) is not the derivative of a function in the ordinary sense and should be considered a generalized derivative in the sense of a generalized function. From Eq. (1.31) we see that $u(t)$ is undefined at $t = 0$ and

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

by Eq. (1.21) with $\phi(t) = 1$. This result is consistent with the definition (1.18) of $u(t)$.

Note that the properties (or identities) expressed by Eqs. (1.23) to (1.26) and Eq. (1.30) can not be verified by using the conventional approach of $\delta(t)$ as shown in Fig. 1-5.

1.3.3. C. Complex Exponential Signals:

The *Complex exponential* signal

$$x(t) = e^{j\omega_0 t}$$

(1.32)

is an important example of a complex signal. Using Euler's formula, this signal can be defined as

$$x(t) = e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

(1.33)

Thus, $x(t)$ is a complex signal whose real part is $\cos\omega_0 t$ and imaginary part is $\sin\omega_0 t$. An important property of the complex exponential signal $x(t)$ in Eq. (1.32) is that it is periodic. The fundamental period T_0 of $x(t)$ is given by (Prob. 1.9)

$$T_0 = \frac{2\pi}{\omega_0}$$

(1.34)

Note that $x(t)$ is periodic for any value of ω_0.

1.3.3.1. General Complex Exponential Signals:

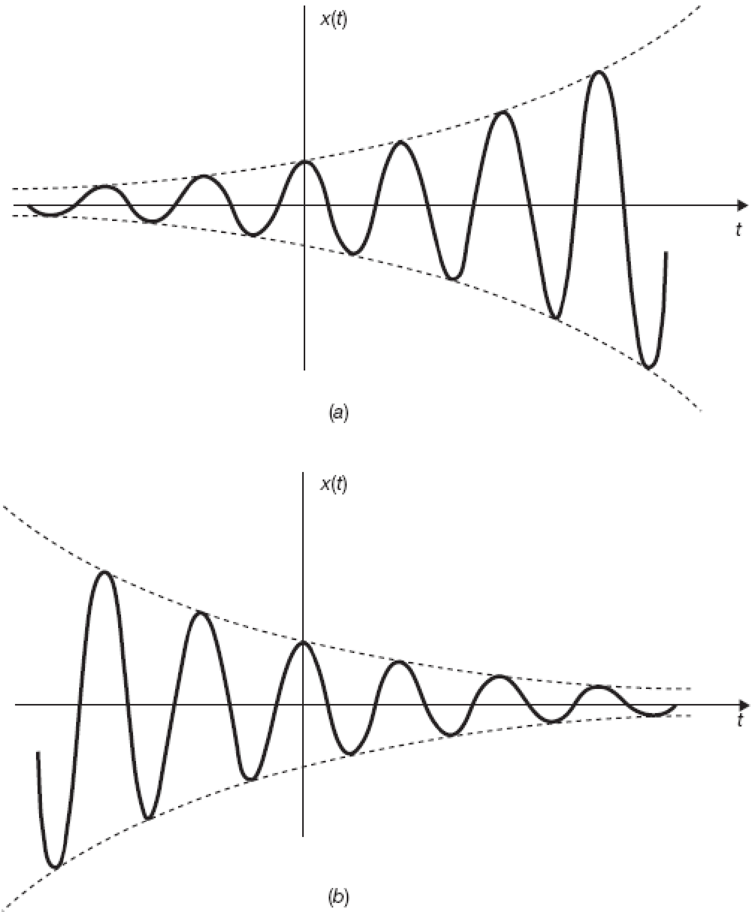
Let $s = \sigma + j\omega$ be a complex number. We define $x(t)$ as

$$x(t) = e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

(1.35)

Then signal $x(t)$ in Eq. (1.35) is known as a *general complex exponential* signal whose real part $e^{\sigma t}\cos\omega t$ and imaginary part $e^{\sigma t}\sin\omega t$ are exponentially increasing ($\sigma > 0$) or decreasing ($\sigma < 0$) sinusoidal signals (Fig. 1-7).

Figure 1-7 *(a) Exponentially increasing sinusoidal signal; (b) exponentially decreasing sinusoidal signal.*



1.3.3.2. Real Exponential Signals:

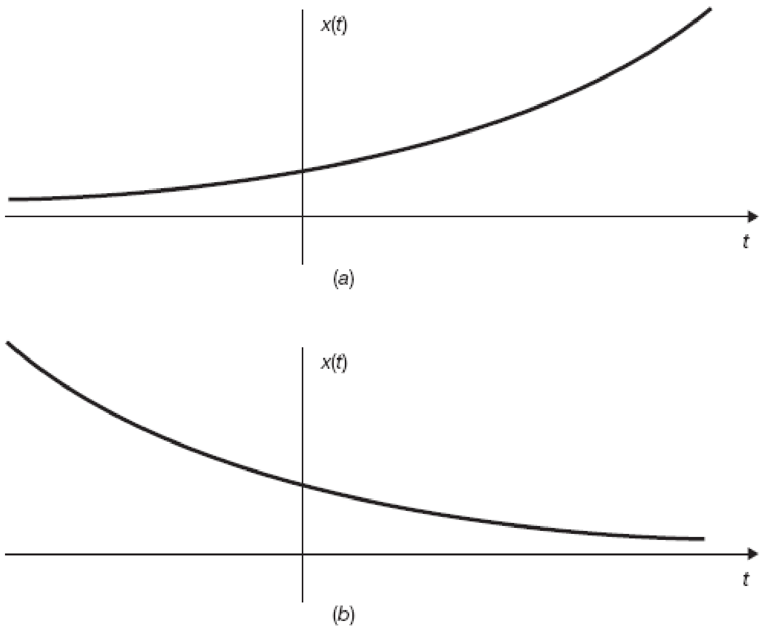
Note that if $s = \sigma$ (a real number), then Eq. (1.35) reduces to a *real exponential* signal

$$x(t) = e^{\sigma t}$$

(1.36)

As illustrated in Fig. 1-8, if $\sigma > 0$, then $x(t)$ is a growing exponential; and if $\sigma < 0$, then $x(t)$ is a decaying exponential.

Figure 1-8 *Continuous-time real exponential signals. (a) σ > 0; (b) σ < 0.*



1.3.4. D. Sinusoidal Signals:

A continuous-time *sinusoidal* signal can be expressed as

$$x(t) = A\cos(\omega_0 t + \theta)$$

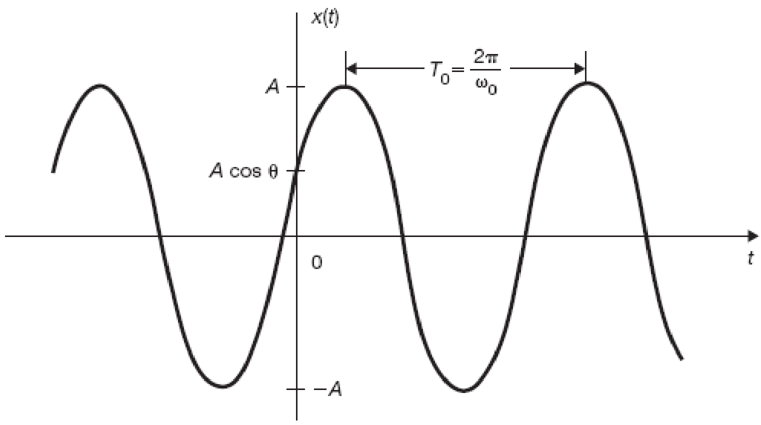
(1.37)

where A is the *amplitude* (real), ω_0 is the *radian frequency* in radians per second, and θ is the *phase angle* in radians. The sinusoidal signal $x(t)$ is shown in Fig. 1-9, and it is periodic with fundamental period

$$T_0 = \frac{2\pi}{\omega_0}$$

(1.38)

Figure 1-9 *Continuous-time sinusoidal signal.*



The reciprocal of the fundamental period T_0 is called the *fundamental frequency* f_0:

$$f_0 = \frac{1}{T_0} \quad \text{hertz (Hz)}$$

(1.39)

From Eqs. (1.38) and (1.39) we have

$$\omega_0 = 2\pi f_0$$

(1.40)

which is called the *fundamental angular frequency.* Using Euler's formula, the sinusoidal signal in Eq. (1.37) can be expressed as

$$A\cos(\omega_0 t + \theta) = A\,\mathrm{Re}\{e^{j(\omega_0 t+\theta)}\}$$

(1.41)

where "Re" denotes "real part of." We also use the notation "Im" to denote "imaginary part of." Then

$$A\,\mathrm{Im}\{e^{j(\omega_0 t+\theta)}\} = A\sin(\omega_0 t+\theta)$$

(1.42)

1.4. Basic Discrete-Time Signals

1.4.1. A. The Unit Step Sequence:

The *unit step* sequence $u[n]$ is defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(1.43)

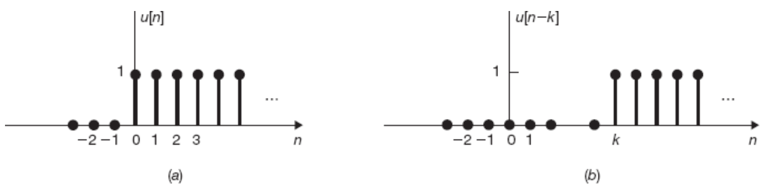
which is shown in Fig. 1-10(*a*). Note that the value of $u[n]$ at $n = 0$ is defined [unlike the continuous-time step function$u(t)$ at $t = 0$] and equals unity. Similarly, the shifted unit step sequence $u[n - k]$ is defined as

$$u[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}$$

(1.44)

which is shown in Fig. 1-10(*b*).

Figure 1-10 (a) Unit step sequence; (b) shifted unit step sequence.



1.4.2. B. The Unit Impulse Sequence:

The *unit impulse* (or *unit sample*) sequence $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

(1.45)

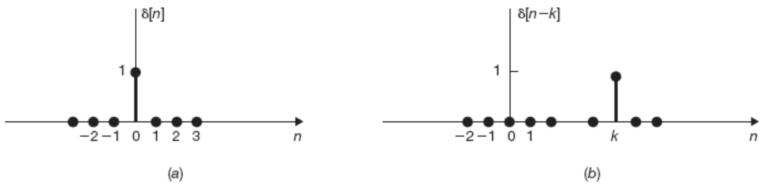
which is shown in Fig. 1-11(*a*). Similarly, the shifted unit impulse (or sample) sequence $\delta[n-k]$ is defined as

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

(1.46)

which is shown in Fig. 1-11(*b*).

Figure 1-11 (a) Unit impulse (sample) sequence; (b) shifted unit impulse sequence.



Unlike the continuous-time unit impulse function $\delta(t)$, $\delta[n]$ is defined without mathematical complication or difficulty. From definitions (1.45) and (1.46) it is readily seen that

$$x[n]\delta[n] = x[0]\delta[n]$$

(1.47)

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

(1.48)

which are the discrete-time counterparts of Eqs. (1.25) and (1.26), respectively. From definitions (1.43) to (1.46), $\delta[n]$ and $u[n]$ are related by

$$\delta[n] = u[n] - u[n-1]$$

(1.49)

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] = \sum_{k=0}^{\infty} \delta[n-k]$$

(1.50)

which are the discrete-time counterparts of Eqs. (1.30) and (1.31), respectively.

Using definition (1.46), any sequence $x[n]$ can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

(1.51)

which corresponds to Eq. (1.27) in the continuous-time signal case.

1.4.3. C. Complex Exponential Sequences:

The *complex exponential* sequence is of the form

$$x[n] = e^{j\Omega_0 n}$$

(1.52)

Again, using Euler's formula, $x[n]$ can be expressed as

$$x[n] = e^{j\Omega_0 n} = \cos \Omega_0 n + j \sin \Omega_0 n$$

(1.53)

Thus, $x[n]$ is a complex sequence whose real part is $\cos \Omega_0 n$ and imaginary part is $\sin \Omega_0 n$.

1.4.3.1. Periodicity of $e^{j\Omega_0 n}$:

In order for $e^{j\Omega_0 n}$ to be periodic with period N (> 0), Ω_0 must satisfy the following condition (Prob. 1.11):

$$\frac{\Omega_0}{2\pi} = \frac{m}{N} \qquad m = \text{positive integer}$$

(1.54)

Thus, the sequence $e^{j\Omega_0 n}$ is not periodic for any value of Ω_0. It is periodic only if $\Omega_0/2\pi$ is a rational number. Note that this property is quite different from the property that the continuous-time signal $e^{j\omega_0 t}$ is periodic for any value of ω_0. Thus, if Ω_0 satisfies the periodicity condition in Eq. (1.54), $\Omega_0 \neq 0$, and N and m have no factors in common, then the fundamental period of the sequence $x[n]$ in Eq. (1.52) is N_0 given by

$$N_0 = m\left(\frac{2\pi}{\Omega_0}\right)$$

(1.55)

Another very important distinction between the discrete-time and continuous-time complex exponentials is that the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0 but that this is not the case for the signals $e^{j\Omega_0 n}$.

Consider the complex exponential sequence with frequency $(\Omega_0 + 2\pi k)$, where k is an integer:

$$e^{j(\Omega_0 + 2\pi k)n} = e^{j\Omega_0 n} e^{j2\pi k n} = e^{j\Omega_0 n}$$

(1.56)

since $e^{j2\pi kn} = 1$. From Eq. (1.56) we see that the complex exponential sequence at frequency Ω_0 is the same as that at frequencies $(\Omega_0 \pm 2\pi)$, $(\Omega_0 \pm 4\pi)$, and so on. Therefore, in dealing with discrete-time exponentials, we need only consider an interval of length 2π in which to choose Ω_0. Usually, we will use the interval $0 \le \Omega_0 < 2\pi$ or the interval $-\pi \le \Omega_0 < \pi$.

1.4.3.2. General Complex Exponential Sequences:

The most general complex exponential sequence is often defined as

$$x[n] = C\alpha^n$$

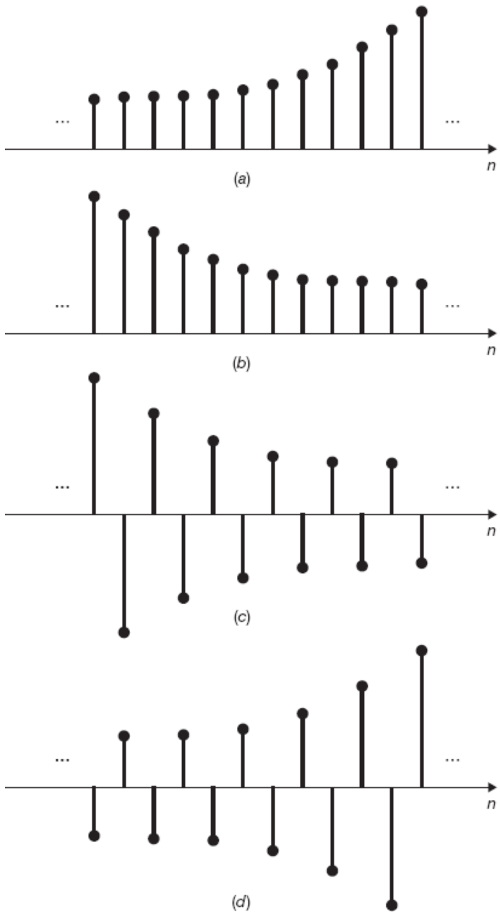
(1.57)

where C and α are, in general, complex numbers. Note that Eq. (1.52) is the special case of Eq. (1.57) with $C = 1$ and $\alpha = e^{j\Omega 0}$.

1.4.3.3. Real Exponential Sequences:

If C and α in Eq. (1.57) are both real, then $x[n]$ is a real exponential sequence. Four distinct cases can be identified: $\alpha > 1$, $0 < \alpha < 1$, $-1 < \alpha < 0$, and $\alpha < -1$. These four real exponential sequences are shown in Fig. 1-12. Note that if $\alpha = 1$, $x[n]$ is a constant sequence, whereas if $\alpha = -1$, $x[n]$ alternates in value between $+C$ and $-C$.

Figure 1-12 Real exponential sequences. (a) α > 1; (b) 1 > α > 0; (c) 0 > α > −1; (d) α < −1.



1.4.4. D. Sinusoidal Sequences:

A *sinusoidal* sequence can be expressed as

$$x[n] = A\cos(\Omega_0 n + \theta)$$

(1.58)

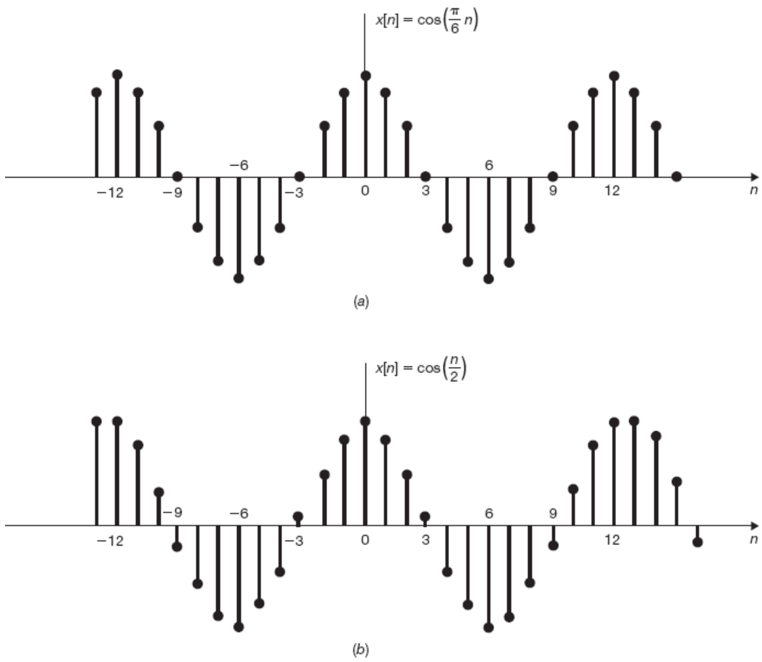
If n is dimensionless, then both Ω_0 and θ have units of radians. Two examples of sinusoidal sequences are shown in Fig. 1-13. As before, the sinusoidal sequence in Eq. (1.58) can be expressed as

$$A\cos(\Omega_0 n+\theta)=A\,\mathrm{Re}\{e^{j(\Omega_0 n+\theta)}\}$$

(1.59)

As we observed in the case of the complex exponential sequence in Eq. (1.52), the same observations [Eqs. (1.54) and (1.56)] also hold for sinusoidal sequences. For instance, the sequence in Fig. 1-13(*a*) is periodic with fundamental period 12, but the sequence in Fig. 1-13(*b*) is not periodic.

Figure 1-13 Sinusoidal sequences. (a) x[n] = cos(πn/6); (b) x[n] = cos(n/2).



1.5. Systems and Classification of Systems

1.5.1. A. System Representation:

A *system* is a mathematical model of a physical process that relates the *input* (or *excitation*) signal to the *output* (or *response*) signal.

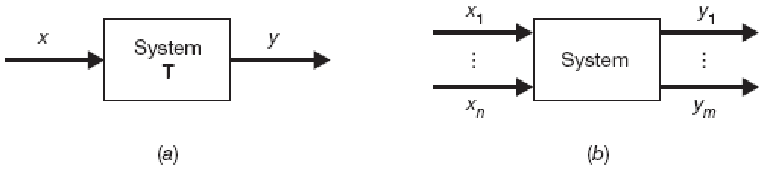
Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a *transformation* (or *mapping*) of x into y. This transformation is represented by the mathematical notation

$$y=\mathbf{T}x$$

(1.60)

where **T** is the *operator* representing some well-defined rule by which x is transformed into y. Relationship (1.60) is depicted as shown in Fig. 1-14(*a*). Multiple input and/or output signals are possible, as shown in Fig. 1-14(*b*). We will restrict our attention for the most part in this text to the single-input, single-output case.

Figure 1-14 *System with single or multiple input and output signals.*



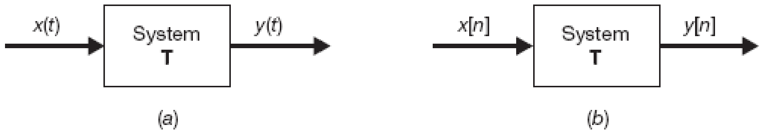
1.5.2. B. Deterministic and Stochastic Systems:

If the input and output signals *x* and *y* are deterministic signals, then the system is called a deterministic system. If the input and output signals *x* and *y* are random signals, then the system is called a *stochastic* system.

1.5.3. C. Continuous-Time and Discrete-Time Systems:

If the input and output signals *x* and *y* are continuous-time signals, then the system is called a *continuous-time system* [Fig. 1-15(*a*)]. If the input and output signals are discrete-time signals or sequences, then the system is called a *discrete-time system* [Fig. 1-15(*b*)].

Figure 1-15 *(a) Continuous-time system; (b) discrete-time system.*



Note that in a continuous-time system the input $x(t)$ and output $y(t)$ are often expressed by a differential equation (see Prob. 1.32) and in a discrete-time system the input $x[n]$ and output $y[n]$ are often expressed by a difference equation (see Prob. 1.37).

1.5.4. D. Systems with Memory and without Memory

A system is said to be *memoryless* if the output at any time depends on only the input at that same time. Otherwise, the system is said to have *memory*. An example of a memoryless system is a resistor R with the input $x(t)$ taken as the current and the voltage taken as the output $y(t)$. The input-output relationship (Ohm's law) of a resistor is

$$y(t) = Rx(t)$$

(1.61)

An example of a system with memory is a capacitor C with the current as the input $x(t)$ and the voltage as the output $y(t)$; then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

(1.62)

A second example of a system with memory is a discrete-time system whose input and output sequences are related by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

(1.63)

1.5.5. E. Causal and Noncausal Systems:

A system is called *causal* if its output at the present time depends on only the present and/or past values of the input. Thus, in a causal system, it is not possible to obtain an output before an input is applied to the system. A system is called *noncausal* (or *anticipative*) if its output at the present time depends on future values of the input. Example of noncausal systems are

$$y(t) = x(t+1)$$

(1.64)

$$y[n] = x[-n]$$

(1.65)

Note that all memoryless systems are causal, but not vice versa.

1.5.6. F. Linear Systems and Nonlinear Systems:

If the operator **T** in Eq. (1.60) satisfies the following two conditions, then **T** is called a *linear operator* and the system represented by a linear operator **T** is called a *linear system*:

1.5.6.1. 1. Additivity:

Given that $\mathbf{T}x_1 = y_1$ and $\mathbf{T}x_2 = y_2$, then

$$\mathbf{T}\{x_1 + x_2\} = y_1 + y_2$$

(1.66)

for any signals x_1 and x_2.

1.5.6.2. 2. Homogeneity (or *Scaling*):

$$\mathbf{T}\{\alpha x\} = \alpha y$$

(1.67)

for any signals x and any scalar α.

Any system that does not satisfy Eq. (1.66) and/or Eq. (1.67) is classified as a *nonlinear* system. Eqs. (1.66) and (1.67) can be combined into a single condition as

$$\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 y_1 + \alpha_2 y_2$$

(1.68)

where α_1 and α_2 are arbitrary scalars. Eq. (1.68) is known as the *superposition property*. Examples of linear systems are the resistor [Eq. (1.61)] and the capacitor [Eq. (1.62)]. Examples of nonlinear systems are

$$y = x^2$$

(1.69)

$$y = \cos x$$

(1.70)

Note that a consequence of the homogeneity (or scaling) property [Eq. (1.67)] of linear systems is that *a zero input yields a zero output*. This follows readily by setting $\alpha = 0$ in Eq. (1.67). This is another important property of linear systems.

1.5.7. G. Time-Invariant and Time-Varying Systems:

A system is called *time-invariant* if a time shift (delay or advance) in the input signal causes the same time shift in the output signal. Thus, for a continuous-time system, the system is time-invariant if

$$\mathbf{T}\{x(t-\tau)\} = y(t-\tau)$$

(1.71)

for any real value of τ. For a discrete-time system, the system is time-invariant (or *shift-invariant*) if

$$\mathbf{T}\{x[n-k]\} = y[n-k]$$

(1.72)

for any integer k. A system which does not satisfy Eq. (1.71) (continuous-time system) or Eq. (1.72) (discrete-time system) is called a *time-varying* system. To check a system for time-invariance, we can compare the shifted output with the output produced by the shifted input (Probs. 1.33 to 1.39).

1.5.8. H. Linear Time-Invariant Systems:

If the system is linear and also time-invariant, then it is called a *linear time-invariant* (LTI) system.

1.5.9. I. Stable Systems:

A system is *bounded-input/bounded-output* (BIBO) *stable* if for any bounded input x defined by

$$|x| \le k_1$$

(1.73)

the corresponding output y is also bounded defined by

$$|y| \le k_2$$

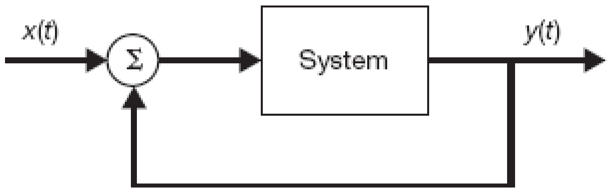
(1.74)

where k_1 and k_2 are finite real constants. An *unstable* system is one in which not all bounded inputs lead to bounded output. For example, consider the system where output $y[n]$ is given by $y[n] = (n + 1)u[n]$, and input $x[n] = u[n]$ is the unit step sequence. In this case the input $u[n] = 1$, but the output $y[n]$ increases without bound as n increases.

1.5.10. J. Feedback Systems:

A special class of systems of great importance consists of systems having *feedback*. In a *feedback system*, the output signal is fed back and added to the input to the system as shown in Fig. 1-16.

Figure 1-16 Feedback system.

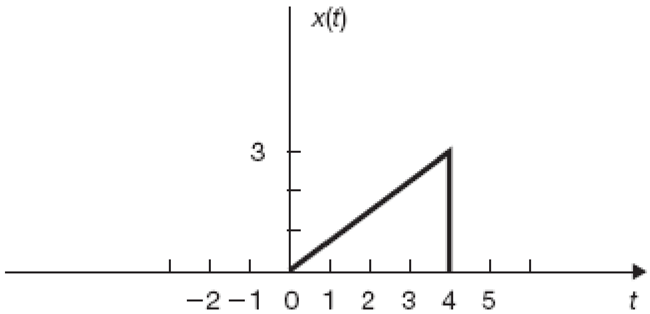


1.6. SOLVED PROBLEMS

1.6.1. Signals and Classification of Signals

1.1. A continuous-time signal $x(t)$ is shown in Fig. 1-17. Sketch and label each of the following signals.

Figure 1-17



(*a*) $x(t-2)$; (*b*) $x(2t)$; (*c*) $x(t/2)$; (*d*) $x(-t)$

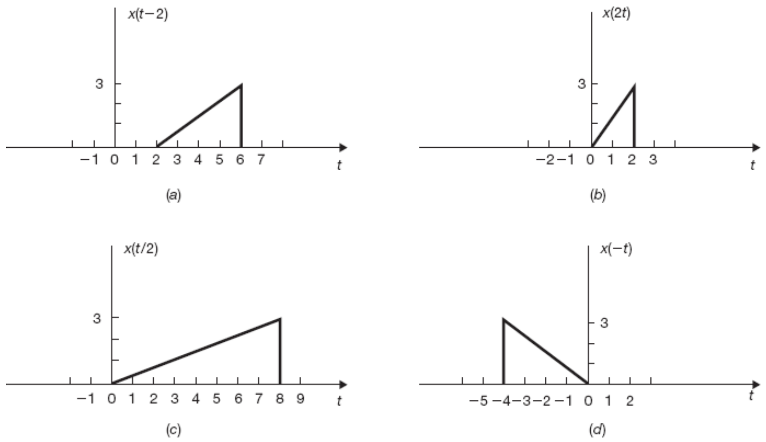
a. $x(t-2)$ is sketched in Fig. 1-18(*a*).

b. $x(2t)$ is sketched in Fig. 1-18(*b*).

c. $x(t/2)$ is sketched in Fig. 1-18(*c*).

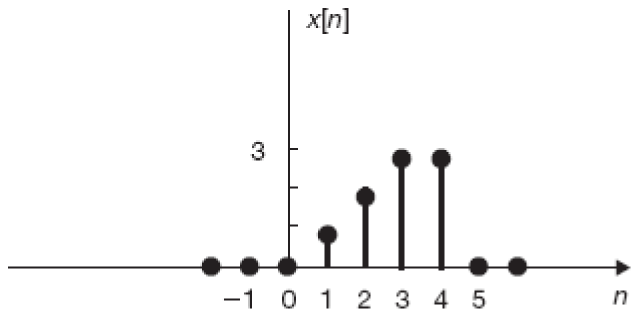
d. $x(-t)$ is sketched in Fig. 1-18(*d*).

Figure 1-18



1.2. A discrete-time signal $x[n]$ is shown in Fig. 1-19. Sketch and label each of the following signals.

Figure 1-19



(*a*) $x[n-2]$; (*b*) $x[2n]$; (*c*) $x[-n]$; (*d*) $x[-n+2]$

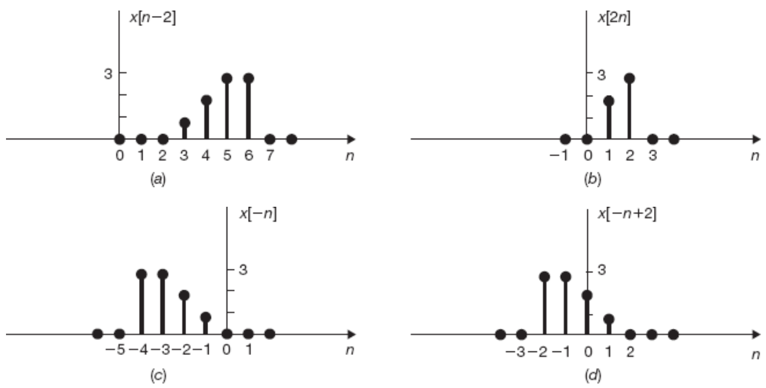
a. $x[n-2]$ is sketched in Fig. 1-20(*a*).

b. $x[2n]$ is sketched in Fig. 1-20(*b*).

c. $x[-n]$ is sketched in Fig. 1-20(*c*).

d. $x[-n+2]$ is sketched in Fig. 1-20(*d*).

Figure 1-20



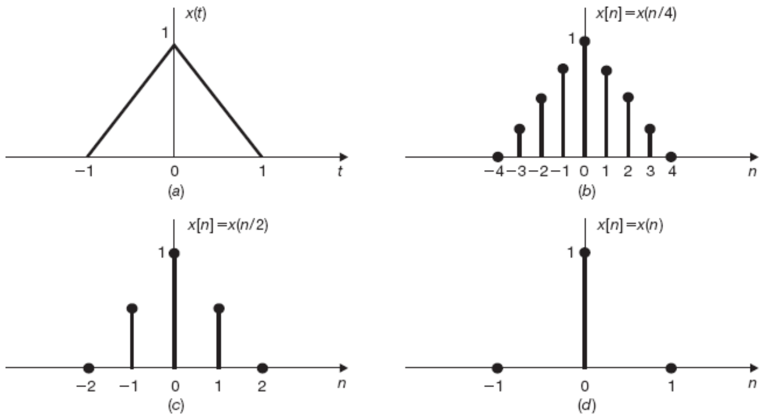
1.3. Given the continuous-time signal specified by

$$x(t)=\begin{cases}1-|t| & -1\le t\le 1\\ 0 & \text{otherwise}\end{cases}$$

determine the resultant discrete-time sequence obtained by uniform sampling of $x(t)$ with a sampling interval of (*a*) 0.25 s, (*b*) 0.5 s, and (*c*) 1.0 s.

It is easier to take the graphical approach for this problem. The signal $x(t)$ is plotted in Fig. 1-21(*a*). Figs. 1-21(*b*) to (*d*) give plots of the resultant sampled sequences obtained for the three specified sampling intervals.

Figure 1-21



a. T_s = 0.25 s. From Fig. 1-21(*b*) we obtain

$$x[n]=\{\ldots,0,0.25,0.5,0.75,\underset{\uparrow}{1},0.75,0.5,0.25,0,\ldots\}$$

b. T_s = 0.5 s. From Fig. 1-21(*c*) we obtain

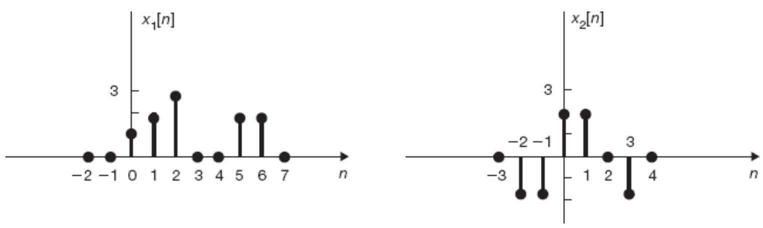
$$x[n]=\{\ldots,0,0.5,\underset{\uparrow}{1},0.5,0,\ldots\}$$

c. T_s = 1 s. From Fig. 1-21(*d*) we obtain

$$x[n]=\{\ldots,0,\underset{\uparrow}{1},0,\ldots\}=\delta[n]$$

1.4. Using the discrete-time signals $x_1[n]$ and $x_2[n]$ shown in Fig. 1-22, represent each of the following signals by a graph and by a sequence of numbers.

Figure 1-22

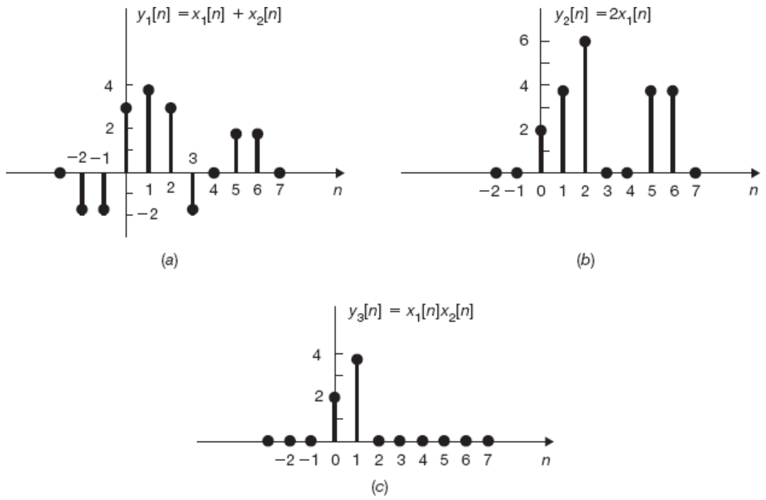


(*a*) $y_1[n] = x_1[n] + x_2[n]$; (*b*) $y_2[n] = 2x_1[n]$; (*c*) $y_3[n] = x_1[n]x_2[n]$

a. $y_1[n]$ is sketched in Fig. 1-23(a). From Fig. 1-23(a) we obtain

$$y_1[n] = \{\ldots, 0, -2, -2, \underset{\uparrow}{3}, 4, 3, -2, 0, 2, 2, 0, \ldots\}$$

Figure 1-23



b. $y_2[n]$ is sketched in Fig. 1-23(b). From Fig. 1-23(b) we obtain

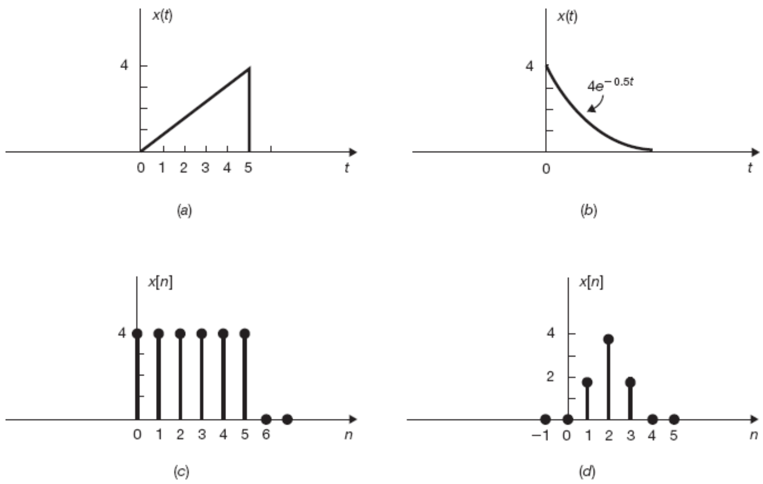
$$y_2[n] = \{\ldots, 0, \underset{\uparrow}{2}, 4, 6, 0, 0, 4, 4, 0, \ldots\}$$

c. $y_3[n]$ is sketched in Fig. 1-23(c). From Fig. 1-23(c) we obtain

$$y_3[n] = \{\ldots, 0, \underset{\uparrow}{2}, 4, 0, \ldots\}$$

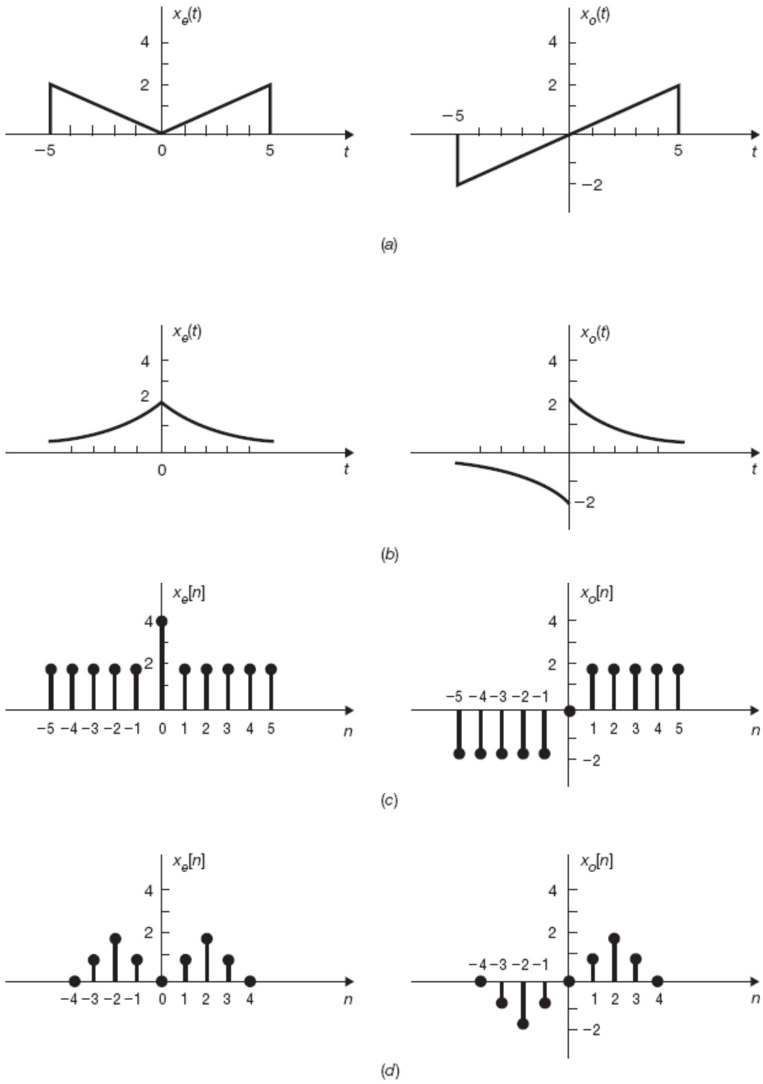
1.5. Sketch and label the even and odd components of the signals shown in Fig. 1-24.

Figure 1-24



Using Eqs. (1.5) and (1.6), the even and odd components of the signals shown in Fig. 1-24 are sketched in Fig. 1-25.

Figure 1-25



1.6. Find the even and odd components of $x(t) = e^{jt}$.

Let $x_e(t)$ and $x_o(t)$ be the even and odd components of e^{jt}, respectively.

$$e^{jt} = x_e(t) + x_o(t)$$

From Eqs. (1.5) and (1.6) and using Euler's formula, we obtain

$$x_e(t) = \frac{1}{2}(e^{jt} + e^{-jt}) = \cos t$$

$$x_o(t) = \frac{1}{2}(e^{jt} - e^{-jt}) = j \sin t$$

1.7. Show that the product of two even signals or of two odd signals is an even signal and that the product of an even and an odd signal is an odd signal.

Let $x(t) = x_1(t)x_2(t)$. If $x_1(t)$ and $x_2(t)$ are both even, then

$$x(-t) = x_1(-t)x_2(-t) = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ and $x_2(t)$ are both odd, then

$$x(-t) = x_1(-t)x_2(-t) = -x_1(t)\,[-x_2(t)] = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ is even and $x_2(t)$ is odd, then

$$x(-t) = x_1(-t)\,x_2(-t) = x_1(t)\,[-x_2(t)] = -x_1(t)x_2(t) = -x(t)$$

and $x(t)$ is odd. Note that in the above proof, variable t represents either a continuous or a discrete variable.

1.8. Show that

a. If $x(t)$ and $x[n]$ are even, then

$$\int_{-a}^{a} x(t)\,dt = 2\int_{0}^{a} x(t)\,dt$$

(1.75a)

$$\sum_{n=-k}^{k} x[n] = x[0] + 2\sum_{n=1}^{k} x[n]$$

(1.75b)

b. If $x(t)$ and $x[n]$ are odd, then

$$x(0) = 0 \qquad \text{and} \qquad x[0] = 0$$

(1.76)

$$\int_{-a}^{a} x(t)\,dt = 0 \qquad \text{and} \qquad \sum_{n=-k}^{k} x[n] = 0$$

(1.77)

a. We can write

$$\int_{-a}^{a} x(t)\,dt = \int_{-a}^{0} x(t)\,dt + \int_{0}^{a} x(t)\,dt$$

Letting $t = -\lambda$ in the first integral on the right-hand side, we get

$$\int_{-a}^{0} x(t)\,dt = \int_{a}^{0} x(-\lambda)\,(-d\lambda) = \int_{0}^{a} x(-\lambda)\,d\lambda$$

Since $x(t)$ is even, that is, $x(-\lambda) = x(\lambda)$, we have

$$\int_{0}^{a} x(-\lambda)\,d\lambda = \int_{0}^{a} x(\lambda)\,d\lambda = \int_{0}^{a} x(t)\,dt$$

Hence,

$$\int_{-a}^{a} x(t)\,dt = \int_{0}^{a} x(t)\,dt + \int_{0}^{a} x(t)\,dt = 2\int_{0}^{a} x(t)\,dt$$

Similarly,

$$\sum_{n=-k}^{k} x[n] = \sum_{n=-k}^{-1} x[n] + x[0] + \sum_{n=1}^{k} x[n]$$

Letting $n = -m$ in the first term on the right-hand side, we get

$$\sum_{n=-k}^{-1} x[n] = \sum_{m=1}^{k} x[-m]$$

Since $x[n]$ is even, that is, $x[-m] = x[m]$, we have

$$\sum_{m=1}^{k} x[-m] = \sum_{m=1}^{k} x[m] = \sum_{n=1}^{k} x[n]$$

Hence,

$$\sum_{n=-k}^{k} x[n] = \sum_{n=1}^{k} x[n] + x[0] + \sum_{n=1}^{k} x[n] = x[0] + 2\sum_{n=1}^{k} x[n]$$

b. Since $x(t)$ and $x[n]$ are odd, that is, $x(-t) = -x(t)$ and $x[-n] = -x[n]$, we have

$$x(-0) = -x(0) \qquad \text{and} \qquad x[-0] = -x[0]$$

Hence,

$$x(-0) = x(0) = -x(0) \Rightarrow x(0) = 0$$
$$x[-0] = x[0] = -x[0] \Rightarrow x[0] = 0$$

Similarly,

$$\int_{-a}^{a} x(t)\,dt = \int_{-a}^{0} x(t)\,dt + \int_{0}^{a} x(t)\,dt = \int_{0}^{a} x(-\lambda)\,d\lambda + \int_{0}^{a} x(t)\,dt$$
$$= -\int_{0}^{a} x(\lambda)\,d\lambda + \int_{0}^{a} x(t)\,dt = -\int_{0}^{a} x(t)\,dt + \int_{0}^{a} x(t)\,dt = 0$$

and

$$\sum_{n=-k}^{k} x[n] = \sum_{n=-k}^{-1} x[n] + x[0] + \sum_{n=1}^{k} x[n] = \sum_{m=1}^{k} x[-m] + x[0] + \sum_{n=1}^{k} x[n]$$
$$= -\sum_{m=1}^{k} x[m] + x[0] + \sum_{n=1}^{k} x[n] = -\sum_{n=1}^{k} x[n] + x[0] + \sum_{n=1}^{k} x[n]$$
$$= x[0] = 0$$

in view of Eq. (1.76).

1.9. Show that the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

is periodic and that its fundamental period is $2\pi/\omega_0$.

By Eq. (1.7), $x(t)$ will be periodic if

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t}$$

Since

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t} e^{j\omega_0 T}$$

we must have

$$e^{j\omega_0 T} = 1$$

(1.78)

If $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T. If $\omega_0 \neq 0$, Eq. (1.78) holds if

$$\omega_0 T = m2\pi \qquad \text{or} \qquad T = m\frac{2\pi}{\omega_0} \qquad m = \text{positive integer}$$

Thus, the fundamental period T_0, the smallest positive T, of $x(t)$ is given by $2\pi/\omega_0$.

1.10. Show that the sinusoidal signal

$$x(t) = \cos(\omega_0 t + \theta)$$

is periodic and that its fundamental period is $2\pi/\omega_0$.

The sinusoidal signal $x(t)$ will be periodic if

$$\cos[\omega_0(t + T) + \theta] = \cos(\omega_0 t + \theta)$$

We note that

$$\cos[\omega_0(t + T) + \theta] = \cos[\omega_0 t + \theta + \omega_0 T] = \cos(\omega_0 t + \theta)$$

if

$$\omega_0 T = m2\pi \qquad \text{or} \qquad T = m\frac{2\pi}{\omega_0} \qquad m = \text{positive integer}$$

Thus, the fundamental period T_0 of $x(t)$ is given by $2\pi/\omega_0$.

1.11. Show that the complex exponential sequence

$$x[n] = e^{j\Omega_0 n}$$

is periodic only if $\Omega_0/2\pi$ is a rational number.

By Eq. (1.9), $x[n]$ will be periodic if

$$e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} e^{j\Omega_0 N} = e^{j\Omega_0 n}$$

or

$$e^{j\Omega_0 N} = 1$$

(1.79)

Equation (1.79) holds only if

$$\Omega_0 N = m2\pi \qquad m = \text{positive integer}$$

or

$$\frac{\Omega_0}{2\pi} = \frac{m}{N} = \text{rational numbers}$$

(1.80)

Thus, $x[n]$ is periodic only if $\Omega_0/2\pi$ is a rational number.

1.12. Let $x(t)$ be the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

with radian frequency ω_0 and fundamental period $T_0 = 2\pi/\omega_0$. Consider the discrete-time sequence $x[n]$ obtained by uniform sampling of $x(t)$ with sampling interval T_s. That is,

$$x[n] = x(nT_s) = e^{j\omega_0 nT_s}$$

Find the condition on the value of T_s so that $x[n]$ is periodic.

If $x[n]$ is periodic with fundamental period N_0, then

$$e^{j\omega_0(n+N_0)T_s} = e^{j\omega_0 nT_s} e^{j\omega_0 N_0 T_s} = e^{j\omega_0 nT_s}$$

Thus, we must have

$$e^{j\omega_0 N_0 T_s} = 1 \Rightarrow \omega_0 N_0 T_s = \frac{2\pi}{T_0} N_0 T_s = m2\pi \qquad m = \text{positive integer}$$

or

$$\frac{T_s}{T_0} = \frac{m}{N_0} = \text{rational number}$$

(1.81)

Thus, $x[n]$ is periodic if the ratio T_s/T_0 of the sampling interval and the fundamental period of $x(t)$ is a rational number.

Note that the above condition is also true for sinusoidal signals $x(t) = \cos(\omega_0 t + \theta)$.

1.13. Consider the sinusoidal signal

$$x(t) = \cos 15t$$

a. Find the value of sampling interval T_s such that $x[n] = x(nT_s)$ is a periodic sequence.

b. Find the fundamental period of $x[n] = x(nT_s)$ if $T_s = 0.1\pi$ seconds.

a. The fundamental period of $x(t)$ is $T_0 = 2\pi/\omega_0 = 2\pi/15$. By Eq. (1.81), $x[n] = x(nT_s)$ is periodic if

$$\frac{T_s}{T_0} = \frac{T_s}{2\pi/15} = \frac{m}{N_0}$$

(1.82)

where m and N_0 are positive integers. Thus, the required value of T_s is given by

$$T_s = \frac{m}{N_0} T_0 = \frac{m}{N_0} \frac{2\pi}{15}$$

(1.83)

b. Substituting $T_s = 0.1\pi = \pi/10$ in Eq. (1.82), we have

$$\frac{T_s}{T_0} = \frac{\pi/10}{2\pi/15} = \frac{15}{20} = \frac{3}{4}$$

Thus, $x[n] = x(nT_s)$ is periodic. By Eq. (1.82)

$$N_0 = m\frac{T_0}{T_s} = m\frac{4}{3}$$

The smallest positive integer N_0 is obtained with $m = 3$. Thus, the fundamental period of $x[n] = x(0.1\pi n)$ is $N_0 = 4$.

1.14. Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2, respectively. Under what conditions is the sum $x(t) = x_1(t) + x_2(t)$ periodic, and what is the fundamental period of $x(t)$ if it is periodic?

Since $x_1(t)$ and $x_2(t)$ are periodic with fundamental periods T_1 and T_2, respectively, we have

$$x_1(t) = x_1(t + T_1) = x_1(t + mT_1) \qquad m = \text{positive integer}$$
$$x_2(t) = x_2(t + T_2) = x_2(t + kT_2) \qquad k = \text{positive integer}$$

Thus,

$$x(t) = x_1(t + mT_1) + x_2(t + kT_2)$$

In order for $x(t)$ to be periodic with period T, one needs

$$x(t + T) = x_1(t + T) + x_2(t + T) = x_1(t + mT_1) + x_2(t + kT_2)$$

Thus, we must have

$$mT_1 = kT_2 = T$$

(1.84)

or

$$\frac{T_1}{T_2} = \frac{k}{m} = \text{rational number}$$

(1.85)

In other words, the sum of two periodic signals is periodic only if the ratio of their respective periods can be expressed as a rational number. Then the fundamental period is the least common multiple of T_1 and T_2, and it is given by Eq. (1.84) if the integers m and k are relative prime. If the ratio T_1/T_2 is an irrational number, then the signals $x_1(t)$ and $x_2(t)$ do not have a common period and $x(t)$ cannot be periodic.

1.15. Let $x_1[n]$ and $x_2[n]$ be periodic sequences with fundamental periods N_1 and N_2, respectively. Under what conditions is the sum $x[n] = x_1[n] + x_2[n]$ periodic, and what is the fundamental period of $x[n]$ if it is periodic?

Since $x_1[n]$ and $x_2[n]$ are periodic with fundamental periods N_1 and N_2, respectively, we have

$$x_1[n] = x_1[n + N_1] = x_1[n + mN_1] \qquad m = \text{positive integer}$$
$$x_2[n] = x_2[n + N_2] = x_2[n + kN_2] \qquad k = \text{positive integer}$$

Thus,

$$x[n] = x_1[n + mN_1] + x_2[n + kN_2]$$

In order for $x[n]$ to be periodic with period N, one needs

$$x[n+N] = x_1[n+N] + x_2[n+N] = x_1[n+mN_1] + x_2[n+kN_2]$$

Thus, we must have

$$mN_1 = kN_2 = N$$

(1.86)

Since we can always find integers m and k to satisfy Eq. (1.86), it follows that the sum of two periodic sequences is also periodic and its fundamental period is the least common multiple of N_1 and N_2.

1.16. Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

$$(a)\ \ x(t) = \cos\left(t + \frac{\pi}{4}\right) \qquad (b)\ \ x(t) = \sin\frac{2\pi}{3}t$$

$$(c)\ \ x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t \qquad (d)\ \ x(t) = \cos t + \sin\sqrt{2}t$$

$$(e)\ \ x(t) = \sin^2 t \qquad (f)\ \ x(t) = e^{j[(\pi/2)t-1]}$$

$$(g)\ \ x[n] = e^{j(\pi/4)n} \qquad (h)\ \ x[n] = \cos\frac{1}{4}n$$

$$(i)\ \ x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n \qquad (j)\ \ x[n] = \cos^2\frac{\pi}{8}n$$

$$(a)\ \ x(t) = \cos\left(t + \frac{\pi}{4}\right) = \cos\left(\omega_0 t + \frac{\pi}{4}\right) \rightarrow \omega_0 = 1$$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 2\pi$.

$$(b)\ \ x(t) = \sin\frac{2\pi}{3}t \rightarrow \omega_0 = \frac{2\pi}{3}$$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 3$.

$$(c)\ \ x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t = x_1(t) + x_2(t)$$

where $x_1(t) = \cos(\pi/3)t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 6$ and $x_2(t) = \sin(\pi/4)t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = 8$. Since $T_1/T_2 = \frac{6}{8} = \frac{3}{4}$ is a rational number, $x(t)$ is periodic with fundamental period $T_0 = 4T_1 = 3T_2 = 24$.

$$(d)\ \ x(t) = \cos t + \sin\sqrt{2}t = x_1(t) + x_2(t)$$

where $x_1(t) = \cos t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 2\pi$ and $x_2(t) = \sin\sqrt{2}t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \sqrt{2}\pi$. Since $T_1/T_2 = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

(e) Using the trigonometric identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, we can write

$$x(t) = \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t = x_1(t) + x_2(t)$$

where $x_1(t) = \frac{1}{2}$ is a dc signal with an arbitrary period and $x_2(t) = -\frac{1}{2}\cos 2t = -\frac{1}{2}\cos\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \pi$. Thus, $x(t)$ is periodic with fundamental period $T_0 = \pi$.

(*f*) $x(t) = e^{j[(\pi/2)t-1]} = e^{-j}e^{j(\pi/2)t} = e^{-j}e^{j\omega_0 t} \rightarrow \omega_0 = \frac{\pi}{2}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 4$.

(*g*) $x[n] = e^{j(\pi/4)n} = e^{j\Omega_0 n} \rightarrow \Omega_0 = \frac{\pi}{4}$

Since $\Omega_0/2\pi = \frac{1}{8}$ is a rational number, $x[n]$ is periodic, and by Eq. (1.55) the fundamental period is $N_0 = 8$.

(*h*) $x[n] = \cos\frac{1}{4}n = \cos\Omega_0 n \rightarrow \Omega_0 = \frac{1}{4}$

Since $\Omega_0/2\pi = 1/8\pi$ is not a rational number, $x[n]$ is nonperiodic.

(*i*) $x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n = x_1[n] + x_2[n]$

where

$$x_1[n] = \cos\frac{\pi}{3}n = \cos\Omega_1 n \rightarrow \Omega_1 = \frac{\pi}{3}$$
$$x_2[n] = \sin\frac{\pi}{4}n = \cos\Omega_2 n \rightarrow \Omega_2 = \frac{\pi}{4}$$

Since $\Omega_1/2\pi = \frac{1}{6}$ (= rational number), $x_1[n]$ is periodic with fundamental period $N_1 = 6$, and since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, from the result of Prob. 1.15, $x[n]$ is periodic and its fundamental period is given by the least common multiple of 6 and 8, that is, $N_0 = 24$.

(*j*) Using the trigonometric identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we can write

$$x[n] = \cos^2\frac{\pi}{8}n = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{4}n = x_1[n] + x_2[n]$$

where $x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n$ is periodic with fundamental period $N_1 = 1$ and $x_2[n] = \frac{1}{2}\cos(\pi/4)n = \frac{1}{2}\cos\Omega_2 n \rightarrow \Omega_2 = \pi/4$. Since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, $x[n]$ is periodic with fundamental period $N_0 = 8$ (the least common multiple of N_1 and N_2).

1.17. Show that if $x(t + T) = x(t)$, then

$$\int_\alpha^\beta x(t)\,dt = \int_{\alpha+T}^{\beta+T} x(t)\,dt$$

(1.87)

$$\int_0^T x(t)\,dt = \int_a^{a+T} x(t)\,dt$$

(1.88)

for any real α, β, and a.

If $x(t + T) = x(t)$, then letting $t = \tau - T$, we have

$$x(\tau - T + T) = x(\tau) = x(\tau - T)$$

and

$$\int_{\alpha}^{\beta} x(t)\,dt = \int_{\alpha+T}^{\beta+T} x(\tau - T)\,d\tau = \int_{\alpha+T}^{\beta+T} x(\tau)\,d\tau = \int_{\alpha+T}^{\beta+T} x(t)\,dt$$

Next, the right-hand side of Eq. (1.88) can be written as

$$\int_{a}^{a+T} x(t)\,dt = \int_{a}^{0} x(t)\,dt + \int_{0}^{a+T} x(t)\,dt$$

By Eq. (1.87) we have

$$\int_{a}^{0} x(t)\,dt = \int_{a+T}^{T} x(t)\,dt$$

Thus,

$$\begin{aligned}\int_{a}^{a+T} x(t)\,dt &= \int_{a+T}^{T} x(t)\,dt + \int_{0}^{a+T} x(t)\,dt \\ &= \int_{0}^{a+T} x(t)\,dt + \int_{a+T}^{T} x(t)\,dt = \int_{0}^{T} x(t)\,dt\end{aligned}$$

1.18. Show that if $x(t)$ is periodic with fundamental period T_0, then the normalized average power P of $x(t)$ defined by Eq. (1.15) is the same as the average power of $x(t)$ over any interval of length T_0, that is,

$$P = \frac{1}{T_0}\int_{0}^{T_0} |x(t)|^2\,dt$$

(1.89)

By Eq. (1.15)

$$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt$$

Allowing the limit to be taken in a manner such that T is an integral multiple of the fundamental period, $T = kT_0$, the total normalized energy content of $x(t)$ over an interval of length T is k times the normalized energy content over one period. Then

$$P = \lim_{k\to\infty}\left[\frac{1}{kT_0}k\int_{0}^{T_0} |x(t)|^2\,dt\right] = \frac{1}{T_0}\int_{0}^{T_0} |x(t)|^2\,dt$$

1.19. The following equalities are used on many occasions in this text. Prove their validity.

$$\sum_{n=0}^{N-1}\alpha^n = \begin{cases}\dfrac{1-\alpha^N}{1-\alpha} & \alpha \neq 1 \\ N & \alpha = 1\end{cases}$$

(1.90)

$$\sum_{n=0}^{\infty}\alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

(1.91)

$$\sum_{n=k}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha} \qquad |\alpha|<1$$

(1.92)

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2} \qquad |\alpha|<1$$

(1.93)

a. Let

$$S = \sum_{n=0}^{N-1} \alpha^n = 1+\alpha+\alpha^2+\cdots+\alpha^{N-1}$$

(1.94)

Then

$$\alpha S = \alpha \sum_{n=0}^{N-1} \alpha^n = \alpha+\alpha^2+\alpha^3+\cdots+\alpha^N$$

(1.95)

Subtracting Eq. (1.95) from Eq. (1.94), we obtain

$$(1-\alpha)S = 1-\alpha^N$$

Hence if $\alpha \neq 1$, we have

$$S = \sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

(1.96)

If $\alpha = 1$, then by Eq. (1.94)

$$\sum_{n=0}^{N-1} \alpha^n = 1+1+1+\cdots+1 = N$$

b. For $|\alpha|<1$, $\lim_{N\to\infty} \alpha^N = 0$. Then by Eq. (1.96) we obtain

$$\sum_{n=0}^{\infty} \alpha^n = \lim_{N\to\infty} \sum_{n=0}^{N-1} \alpha^n = \lim_{N\to\infty} \frac{1-\alpha^N}{1-\alpha} = \frac{1}{1-\alpha}$$

c. Using Eq. (1.91), we obtain

$$\sum_{n=k}^{\infty} \alpha^n = \alpha^k+\alpha^{k+1}+\alpha^{k+2}+\cdots$$
$$= \alpha^k(1+\alpha+\alpha^2+\cdots) = \alpha^k \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^k}{1-\alpha}$$

d. Taking the derivative of both sides of Eq. (1.91) with respect to α, we have

$$\frac{d}{d\alpha}\left(\sum_{n=0}^{\infty}\alpha^n\right)=\frac{d}{d\alpha}\left(\frac{1}{1-\alpha}\right)=\frac{1}{(1-\alpha)^2}$$

and

$$\frac{d}{d\alpha}\left(\sum_{n=0}^{\infty}\alpha^n\right)=\sum_{n=0}^{\infty}\frac{d}{d\alpha}\alpha^n=\sum_{n=0}^{\infty}n\alpha^{n-1}=\frac{1}{\alpha}\sum_{n=0}^{\infty}n\alpha^n$$

Hence,

$$\frac{1}{\alpha}\sum_{n=0}^{\infty}n\alpha^n=\frac{1}{(1-\alpha)^2} \qquad \text{or} \qquad \sum_{n=0}^{\infty}n\alpha^n=\frac{\alpha}{(1-\alpha)^2}$$

1.20. Determine whether the following signals are energy signals, power signals, or neither.

a. $x(t) = e^{-at}u(t)$, $a > 0$

b. $x(t) = A\cos(\omega_0 t + \theta)$

c. $x(t) = tu(t)$

d. $x[n] = (-0.5)^n u[n]$

e. $x[n] = u[n]$

f. $x[n] = 2e^{j3n}$

a. $E=\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_0^{\infty}e^{-2at}\,dt=\dfrac{1}{2a}<\infty$

Thus, $x(t)$ is an energy signal.

b. The sinusoidal signal $x(t)$ is periodic with $T_0 = 2\pi/\omega_0$. Then by the result from Prob. 1.18, the average power of $x(t)$ is

$$\begin{aligned}P&=\frac{1}{T_0}\int_0^{T_0}[x(t)]^2\,dt=\frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0}A^2\cos^2(\omega_0 t+\theta)\,dt\\&=\frac{A^2\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\frac{1}{2}[1+\cos(2\omega_0 t+2\theta)]\,dt=\frac{A^2}{2}<\infty\end{aligned}$$

Thus, $x(t)$ is a power signal. Note that periodic signals are, in general, power signals.

$$E=\lim_{T\to\infty}\int_{-T/2}^{T/2}|x(t)|^2\,dt=\lim_{T\to\infty}\int_0^{T/2}t^2\,dt=\lim_{T\to\infty}\frac{(T/2)^3}{3}=\infty$$

$$P=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt=\lim_{T\to\infty}\frac{1}{T}\int_0^{T/2}t^2\,dt=\lim_{T\to\infty}\frac{1}{T}\frac{(T/2)^3}{3}=\infty$$

Thus, $x(t)$ is neither an energy signal nor a power signal.

d. By definition (1.16) and using Eq. (1.91), we obtain

$$E=\sum_{n=-\infty}^{\infty}|x[n]|^2=\sum_{n=0}^{\infty}0.25^n=\frac{1}{1-0.25}=\frac{4}{3}<\infty$$

Thus, $x[n]$ is an energy signal.

e. By definition (1.17)

$$P=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2$$

$$=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=0}^{N}1^2=\lim_{N\to\infty}\frac{1}{2N+1}(N+1)=\frac{1}{2}<\infty$$

Thus, $x[n]$ is a power signal.

f. Since $|x[n]| = |2e^{j3n}| = 2|e^{j3n}| = 2$,

$$P=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}2^2$$

$$=\lim_{N\to\infty}\frac{1}{2N+1}4(2N+1)=4<\infty$$

Thus, $x[n]$ is a power signal.

1.6.2. Basic Signals

1.21. Show that

$$u(-t)=\begin{cases}0 & t>0\\ 1 & t<0\end{cases}$$

(1.97)

Let $\tau = -t$. Then by definition (1.18)

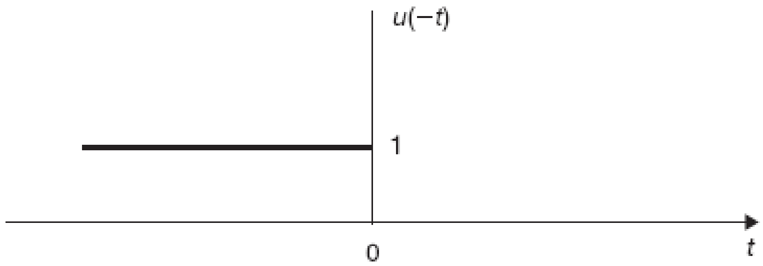
$$u(-t)=u(\tau)=\begin{cases}1 & \tau>0\\ 0 & \tau<0\end{cases}$$

Since $\tau > 0$ and $\tau < 0$ imply, respectively, that $t < 0$ and $t > 0$, we obtain

$$u(-t)=\begin{cases}0 & t>0\\ 1 & t<0\end{cases}$$

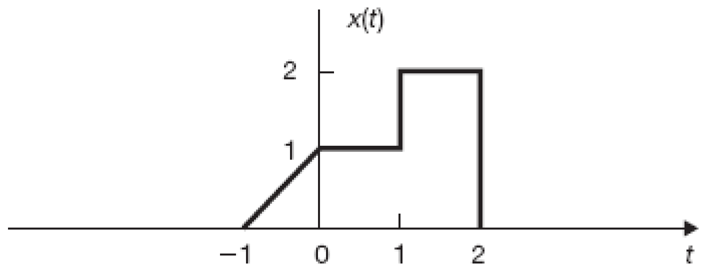
which is shown in Fig. 1-26.

Figure 1-26



1.22. A continuous-time signal $x(t)$ is shown in Fig. 1-27. Sketch and label each of the following signals.

Figure 1-27



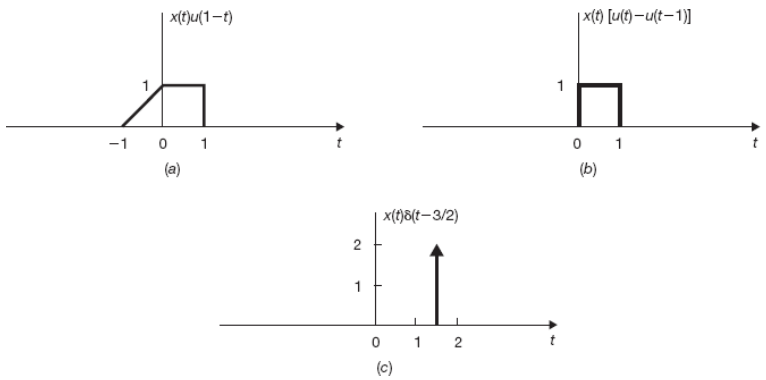
$x(t)u(1-t)$; (*b*) $x(t)[u(t) - u(t-1)]$; (*c*) $x(t)\delta(t - \frac{3}{2})$

a. By definition (1.19)

$$u(1-t) = \begin{cases} 1 & t < 1 \\ 0 & t > 1 \end{cases}$$

and $x(t)u(1-t)$ is sketched in Fig. 1-28(*a*).

Figure 1-28



b. By definitions (1.18) and (1.19)

$$u(t) - u(t-1) = \begin{cases} 1 & 0 < t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and $x(t)[u(t) - u(t-1)]$ is sketched in Fig. 1-28(*b*).

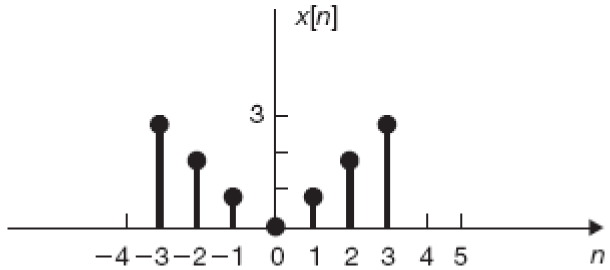
c. By Eq. (1.26)

$$x(t)\delta\left(t-\frac{3}{2}\right)=x\left(\frac{3}{2}\right)\delta\left(t-\frac{3}{2}\right)=2\delta\left(t-\frac{3}{2}\right)$$

which is sketched in Fig. 1-28(*c*).

1.23. A discrete-time signal $x[n]$ is shown in Fig. 1-29. Sketch and label each of the following signals.

Figure 1-29



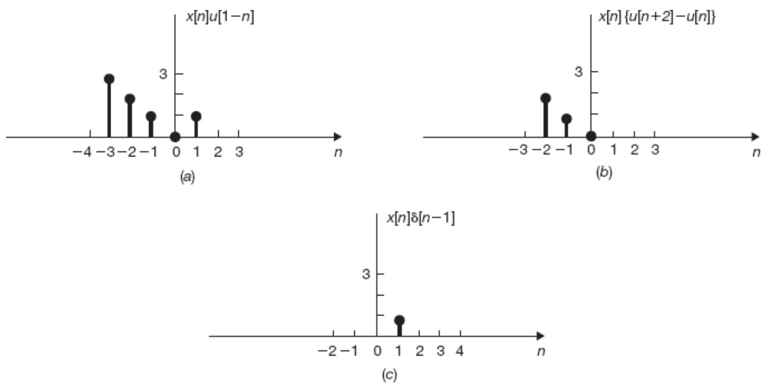
(*a*) $x[n]u[1-n]$; (*b*) $x[n]\{u[n+2]-u[n]\}$; (*c*) $x[n]\delta[n-1]$

a. By definition (1.44)

$$u[1-n]=\begin{cases}1 & n\le 1\\ 0 & n>1\end{cases}$$

and $x[n]u[1-n]$ is sketched in Fig. 1-30(*a*).

Figure 1-30



b. By definitions (1.43) and (1.44)

$$u[n+2]-u[n]=\begin{cases}1 & -2\le n<0\\ 0 & \text{otherwise}\end{cases}$$

and $x[n]\{u[n+2]-u[n]\}$ is sketched in Fig. 1-30(*b*).

c. By definition (1.48)

$$x[n]\delta[n-1]=x[1]\delta[n-1]=\delta[n-1]=\begin{cases}1 & n=1\\ 0 & n\ne 1\end{cases}$$

which is sketched in Fig. 1-30(*c*).

1.24. The unit step function $u(t)$ can be defined as a generalized function by the following relation:

$$\int_{-\infty}^{\infty} \phi(t)u(t)\,dt = \int_{0}^{\infty} \phi(t)\,dt$$

(1.98)

where $\phi(t)$ is a testing function which is integrable over $0 < t < \infty$. Using this definition, show that

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Rewriting Eq. (1.98) as

$$\int_{-\infty}^{\infty} \phi(t)u(t)\,dt = \int_{-\infty}^{0} \phi(t)u(t)\,dt + \int_{0}^{\infty} \phi(t)u(t)\,dt = \int_{0}^{\infty} \phi(t)\,dt$$

we obtain

$$\int_{-\infty}^{0} \phi(t)u(t)\,dt = \int_{0}^{\infty} \phi(t)[1 - u(t)]\,dt$$

This can be true only if

$$\int_{-\infty}^{0} \phi(t)u(t)\,dt = 0 \quad \text{and} \quad \int_{0}^{\infty} \phi(t)[1 - u(t)]\,dt = 0$$

These conditions imply that

$$\phi(t)u(t) = 0,\ t < 0 \quad \text{and} \quad \phi(t)[1 - u(t)] = 0,\ t > 0$$

Since $\phi(t)$ is arbitrary, we have

$$u(t) = 0,\ t < 0 \quad \text{and} \quad 1 - u(t) = 0,\ t > 0$$

that is,

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

1.25. Verify Eqs. (1.23) and (1.24); that is,

(a) $\delta(at) = \dfrac{1}{|a|}\delta(t)$; $\quad (b) \quad \delta(-t) = \delta(t)$

The proof will be based on the following *equivalence* property:

Let $g_1(t)$ and $g_2(t)$ be generalized functions. Then the equivalence property states that $g_1(t) = g_2(t)$ if and only if

$$\int_{-\infty}^{\infty} \phi(t)\,g_1(t)\,dt = \int_{-\infty}^{\infty} \phi(t)\,g_2(t)\,dt$$

(1.99)

for all suitably defined testing functions $\phi(t)$.

a. With a change of variable, $at = \tau$, and hence $t = \tau/a$, $dt = (1/a)\,d\tau$, we obtain the following equations:

   If $a > 0$,

$$\int_{-\infty}^{\infty}\phi(t)\delta(at)\,dt=\frac{1}{a}\int_{-\infty}^{\infty}\phi\left(\frac{\tau}{a}\right)\delta(\tau)\,d\tau=\frac{1}{a}\phi\left(\frac{\tau}{a}\right)\bigg|_{\tau=0}=\frac{1}{|a|}\phi(0)$$

If $a<0$,

$$\int_{-\infty}^{\infty}\phi(t)\delta(at)\,dt=\frac{1}{a}\int_{\infty}^{-\infty}\phi\left(\frac{\tau}{a}\right)\delta(\tau)\,d\tau=-\frac{1}{a}\int_{-\infty}^{\infty}\phi\left(\frac{\tau}{a}\right)\delta(\tau)\,d\tau$$
$$=-\frac{1}{a}\phi\left(\frac{\tau}{a}\right)\bigg|_{\tau=0}=\frac{1}{|a|}\phi(0)$$

Thus, for any a

$$\int_{-\infty}^{\infty}\phi(t)\delta(at)\,dt=\frac{1}{|a|}\phi(0)$$

Now, using Eq. (1.20) for $\phi(0)$, we obtain

$$\int_{-\infty}^{\infty}\phi(t)\delta(at)\,dt=\frac{1}{|a|}\phi(0)=\frac{1}{|a|}\int_{-\infty}^{\infty}\phi(t)\delta(t)\,dt$$
$$=\int_{-\infty}^{\infty}\phi(t)\frac{1}{|a|}\delta(t)\,dt$$

for any $\phi(t)$. Then, by the equivalence property (1.99), we obtain

$$\delta(at)=\frac{1}{|a|}\delta(t)$$

b. Setting $a=-1$ in the above equation, we obtain

$$\delta(-t)=\frac{1}{|-1|}\delta(t)=\delta(t)$$

which shows that $\delta(t)$ is an even function.

1.26.

a. Verify Eq. (1.26):

$$x(t)\,\delta(t-t_0)=x(t_0)\,\delta(t-t_0)$$

if $x(t)$ is continuous at $t=t_0$.

b. Verify Eq. (1.25):

$$x(t)\,\delta(t)=x(0)\,\delta(t)$$

if $x(t)$ is continuous at $t=0$.

a. If $x(t)$ is continuous at $t=t_0$, then by definition (1.22) we have

$$\int_{-\infty}^{\infty} \phi(t)[x(t)\delta(t-t_0)]\,dt = \int_{-\infty}^{\infty} [\phi(t)x(t)]\delta(t-t_0)\,dt = \phi(t_0)x(t_0)$$
$$= x(t_0)\int_{-\infty}^{\infty} \phi(t)\delta(t-t_0)\,dt$$
$$= \int_{-\infty}^{\infty} \phi(t)[x(t_0)\delta(t-t_0)]\,dt$$

for all $\phi(t)$ which are continuous at $t = t_0$. Hence, by the equivalence property (1.99) we conclude that

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

b. Setting $t_0 = 0$ in the above expression, we obtain

$$x(t)\delta(t) = x(0)\delta(t)$$

1.27. Show that

a. $t\delta(t) = 0$

b. $\sin t\delta(t) = 0$

c. $\cos t\delta(t-\pi) = -\delta(t-\pi)$

Using Eqs. (1.25) and (1.26), we obtain

a. $t\delta(t) = (0)\delta(t) = 0$

b. $\sin t\delta(t) = (\sin 0)\delta(t) = (0)\delta(t) = 0$

c. $\cos t\delta(t-\pi) = (\cos\pi)\,\delta(t-\pi) = (-1)\delta(t-\pi) = -\delta(t-\pi)$

1.28. Verify Eq. (1.30):

$$\delta(t) = u'(t) = \frac{du(t)}{dt}$$

From Eq. (1.28) we have

$$\int_{-\infty}^{\infty} \phi(t)u'(t)\,dt = -\int_{-\infty}^{\infty} \phi'(t)u(t)\,dt$$

(1.100)

where $\phi(t)$ is a testing function which is continuous at $t = 0$ and vanishes outside some fixed interval. Thus, $\phi'(t)$ exists and is integrable over $0 < t < \infty$ and $\phi(\infty) = 0$. Then using Eq. (1.98) or definition (1.18), we have

$$\int_{-\infty}^{\infty} \phi(t)u'(t)\,dt = -\int_{0}^{\infty} \phi'(t)\,dt = -\phi(t)\Big|_0^{\infty} = -[\phi(\infty)-\phi(0)]$$
$$= \phi(0) = \int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt$$

Since $\phi(t)$ is arbitrary and by equivalence property (1.99), we conclude that

$$\delta(t) = u'(t) = \frac{du(t)}{dt}$$

1.29. Show that the following properties hold for the derivative of $\delta(t)$:

$$\int_{-\infty}^{\infty} \phi(t)\delta'(t)\,dt = -\phi'(0) \qquad \text{where } \phi'(0) = \left.\frac{d\phi(t)}{dt}\right|_{t=0}$$

(1.101)

$$t\delta'(t) = -\delta(t)$$

(1.102)

a. Using Eqs. (1.28) and (1.20), we have

$$\int_{-\infty}^{\infty} \phi(t)\delta'(t)\,dt = -\int_{-\infty}^{\infty} \phi'(t)\delta(t)\,dt = -\phi'(0)$$

b. Using Eqs. (1.101) and (1.20), we have

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(t)[t\delta'(t)]\,dt &= \int_{-\infty}^{\infty} [t\phi(t)]\delta'(t)\,dt = -\frac{d}{dt}[t\phi(t)]\Big|_{t=0} \\ &= -[\phi(t) + t\phi'(t)]\Big|_{t=0} = -\phi(0) \\ &= -\int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt = \int_{-\infty}^{\infty} \phi(t)[-\delta(t)]\,dt\end{aligned}$$

Thus, by the equivalence property (1.99) we conclude that

$$t\delta'(t) = -\delta(t)$$

1.30. Evaluate the following integrals:

$$(a) \quad \int_{-1}^{1} (3t^2+1)\delta(t)\,dt$$

$$(b) \quad \int_{1}^{2} (3t^2+1)\delta(t)\,dt$$

$$(c) \quad \int_{-\infty}^{\infty} (t^2+\cos \pi t)\delta(t-1)\,dt$$

$$(d) \quad \int_{-\infty}^{\infty} e^{-t}\delta(2t-2)\,dt$$

$$(e) \quad \int_{-\infty}^{\infty} e^{-t}\delta'(t)\,dt$$

a. By Eq. (1.21), with $a = -1$ and $b = 1$, we have

$$\int_{-1}^{1} (3t^2+1)\delta(t)\,dt = (3t^2+1)\Big|_{t=0} = 1$$

b. By Eq. (1.21), with $a = 1$ and $b = 2$, we have

$$\int_{1}^{2} (3t^2+1)\delta(t)\,dt = 0$$

c. By Eq. (1.22)

$$\begin{aligned}\int_{-\infty}^{\infty} (t^2+\cos \pi t)\delta(t-1)\,dt &= (t^2+\cos \pi t)\Big|_{t=1} \\ &= 1+\cos \pi = 1-1 = 0\end{aligned}$$

d. Using Eqs. (1.22) and (1.23), we have

$$\int_{-\infty}^{\infty} e^{-t}\delta(2t-2)\,dt = \int_{-\infty}^{\infty} e^{-t}\delta[2(t-1)]\,dt$$

$$= \int_{-\infty}^{\infty} e^{-t}\frac{1}{|2|}\delta(t-1)\,dt = \frac{1}{2}e^{-t}\Big|_{t=1} = \frac{1}{2e}$$

e. By Eq. (1.29)

$$\int_{-\infty}^{\infty} e^{-t}\delta'(t)\,dt = -\frac{d}{dt}(e^{-t})\Big|_{t=0} = e^{-t}\Big|_{t=0} = 1$$

1.31. Find and sketch the first derivatives of the following signals:

a. $x(t) = u(t) - u(t-a),\ a > 0$

b. $x(t) = t[u(t) - u(t-a)],\ a > 0$

c. $x(t) = \operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$

a. Using Eq. (1.30), we have

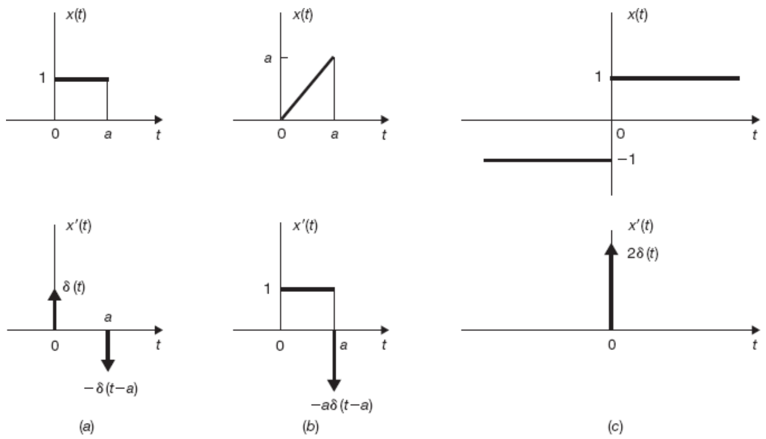
$$u'(t) = \delta(t) \qquad \text{and} \qquad u'(t-a) = \delta(t-a)$$

Then

$$x'(t) = u'(t) - u'(t-a) = \delta(t) - \delta(t-a)$$

Signals $x(t)$ and $x'(t)$ are sketched in Fig. 1-31(a).

Figure 1-31



b. Using the rule for differentiation of the product of two functions and the result from part (a), we have

$$x'(t) = [u(t) - u(t-a)] + t[\delta(t) - \delta(t-a)]$$

But by Eqs. (1.25) and (1.26)

$$t\delta(t) = (0)\delta(t) = 0 \qquad \text{and} \qquad t\delta(t-a) = a\delta(t-a)$$

Thus,

$$x'(t) = u(t) - u(t-a) - a\delta(t-a)$$

Signals $x(t)$ and $x'(t)$ are sketched in Fig. 1-31(b).

c. $x(t) = \operatorname{sgn} t$ can be rewritten as

$$x(t) = \operatorname{sgn} t = u(t) - u(-t)$$

Then using Eq. (1.30), we obtain

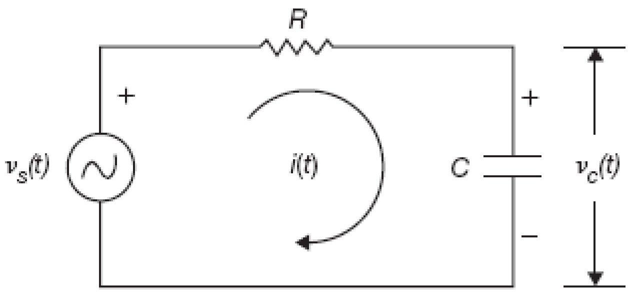
$$x'(t) = u'(t) - u'(-t) = \delta(t) - [-\delta(t)] = 2\delta(t)$$

Signals $x(t)$ and $x'(t)$ are sketched in Fig. 1-31(c).

1.6.3. Systems and Classification of Systems

1.32. Consider the *RC* circuit shown in Fig. 1-32. Find the relationship between the input $x(t)$ and the output $y(t)$

Figure 1-32 RC circuit.



a. If $x(t) = v_s(t)$ and $y(t) = v_c(t)$.

b. If $x(t) = v_s(t)$ and $y(t) = i(t)$.

a. Applying Kirchhoff's voltage law to the *RC* circuit in Fig. 1-32, we obtain

$$v_s(t) = Ri(t) + v_c(t)$$

(1.103)

The current $i(t)$ and voltage $v_c(t)$ are related by

$$i(t) = C\frac{dv_c(t)}{dt}$$

(1.104)

Letting $v_s(t) = x(t)$ and $v_c(t) = y(t)$ and substituting Eq. (1.04) into Eq. (1.103), we obtain

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

or

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

(1.105)

Thus, the input-output relationship of the *RC* circuit is described by a first-order linear differential equation with constant coefficients.

b. Integrating Eq. (1.104), we have

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau$$

(1.106)

Substituting Eq. (1.106) into Eq. (1.103) and letting $v_s(t) = x(t)$ and $i(t) = y(t)$, we obtain

$$Ry(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\, d\tau = x(t)$$

or

$$y(t) + \frac{1}{RC}\int_{-\infty}^{t} y(\tau)\, d\tau = \frac{1}{R}x(t)$$

Differentiating both sides of the above equation with respect to *t*, we obtain

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{R}\frac{dx(t)}{dt}$$

(1.107)

Thus, the input-output relationship is described by another first-order linear differential equation with constant coefficients.

1.33. Consider the capacitor shown in Fig. 1-33. Let input $x(t) = i(t)$ and output $y(t) = v_c(t)$.

Figure 1-33

a. Find the input-output relationship.

b. Determine whether the system is (*i*) memoryless, (*ii*) causal, (*iii*) linear, (*iv*) time-invariant, or (*v*) stable.

a. Assume the capacitance *C* is constant. The output voltage $y(t)$ across the capacitor and the input current $x(t)$ are related by [Eq. (1.106)]

$$y(t) = \mathbf{T}\{x(t)\} = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$$

(1.108)

b. i. From Eq. (1.108) it is seen that the output $y(t)$ depends on the past and the present values of the input. Thus, the system is not memoryless.

ii. Since the output $y(t)$ does not depend on the future values of the input, the system is causal.

iii. Let $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$. Then

$$\begin{aligned} y(t) = \mathbf{T}\{x(t)\} &= \frac{1}{C}\int_{-\infty}^{t}[\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)]\,d\tau \\ &= \alpha_1\left[\frac{1}{C}\int_{-\infty}^{t} x_1(\tau)\,d\tau\right] + \alpha_2\left[\frac{1}{C}\int_{-\infty}^{t} x_2(\tau)\,d\tau\right] \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

Thus, the superposition property (1.68) is satisfied and the system is linear.

iv. Let $y_1(t)$ be the output produced by the shifted input current $x_1(t) = x(t - t_0)$.

Then

$$\begin{aligned} y_1(t) = \mathbf{T}\{x(t - t_0)\} &= \frac{1}{C}\int_{-\infty}^{t} x(\tau - t_0)\,d\tau \\ &= \frac{1}{C}\int_{-\infty}^{t-t_0} x(\lambda)\,d\lambda = y(t - t_0) \end{aligned}$$

Hence, the system is time-invariant.

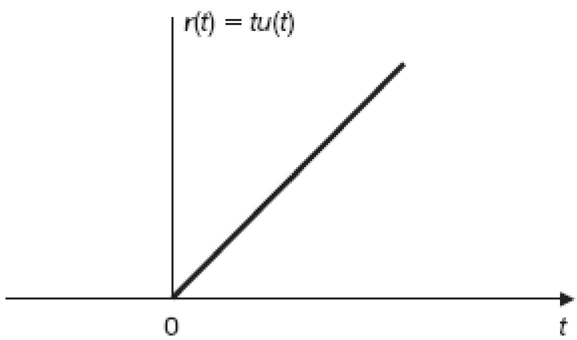
v. Let $x(t) = k_1 u(t)$, with $k_1 \neq 0$. Then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} k_1 u(\tau)\,d\tau = \frac{k_1}{C}\int_{0}^{t} d\tau = \frac{k_1}{C} t u(t) = \frac{k_1}{C} r(t)$$

(1.109)

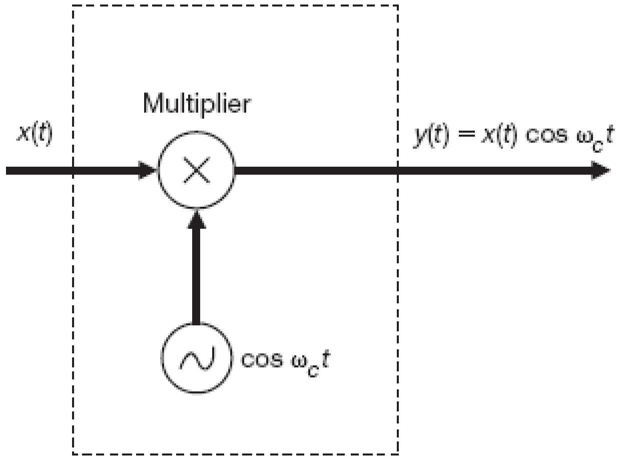
where $r(t) = tu(t)$ is known as the *unit ramp* function (Fig. 1-34). Since $y(t)$ grows linearly in time without bound, the system is not BIBO stable.

Figure 1-34 Unit ramp function.



1.34. Consider the system shown in Fig. 1-35. Determine whether it is (*a*) memoryless, (*b*) causal, (*c*) linear, (*d*) time-invariant, or (*e*) stable.

Figure 1-35



a. From Fig. 1-35 we have

$$y(t) = \mathbf{T}\{x(t)\} = x(t)\cos\omega_c t$$

Since the value of the output $y(t)$ depends on only the present values of the input $x(t)$, the system is memoryless.

b. Since the output $y(t)$ does not depend on the future values of the input $x(t)$, the system is causal.

c. Let $x(t) = \alpha_1 x(t) + \alpha_2 x(t)$. Then

$$\begin{aligned} y(t) = \mathbf{T}\{x(t)\} &= [\alpha_1 x_1(t) + \alpha_2 x_2(t)]\cos\omega_c t \\ &= \alpha_1 x_1(t)\cos\omega_c t + \alpha_2 x_2(t)\cos\omega_c t \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

Thus, the superposition property (1.68) is satisfied and the system is linear.

d. Let $y_1(t)$ be the output produced by the shifted input $x_1(t) = x(t - t_0)$. Then

$$y_1(t) = \mathbf{T}\{x(t - t_0)\} = x(t - t_0)\cos\omega_c t$$

But

$$y(t - t_0) = x(t - t_0)\cos\omega_c(t - t_0) \neq y_1(t)$$

Hence, the system is not time-invariant.

e. Since $|\cos\omega_c t| \leq 1$, we have

$$|y(t)| = |x(t)\cos\omega_c t| \leq |x(t)|$$

Thus, if the input $x(t)$ is bounded, then the output $y(t)$ is also bounded and the system is BIBO stable.

1.35. A system has the input-output relation given by

$$y = \mathbf{T}\{x\} = x^2$$

(1.110)

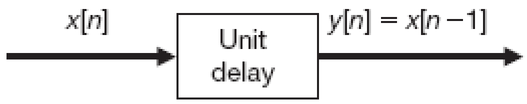
Show that this system is nonlinear.

$$\mathbf{T}\{x_1 + x_2\} = (x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2$$
$$\neq \mathbf{T}\{x_1\} + \mathbf{T}\{x_2\} = x_1^2 + x_2^2$$

Thus, the system is nonlinear.

1.36. The discrete-time system shown in Fig. 1-36 is known as the *unit delay* element. Determine whether the system is (*a*) memoryless, (*b*) causal, (*c*) linear, (*d*) time-invariant, or (*e*) stable.

Figure 1-36 Unit delay element



a. The system input-output relation is given by

$$y[n] = \mathbf{T}\{x[n]\} = x[n-1]$$

(1.111)

Since the output value at n depends on the input values at $n - 1$, the system is not memoryless.

b. Since the output does not depend on the future input values, the system is causal.

c. Let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. Then

$$y[n] = \mathbf{T}\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]$$
$$= \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

Thus, the superposition property (1.68) is satisfied and the system is linear.

d. Let $y_1[n]$ be the response to $x_1[n] = x[n - n_0]$. Then

$$y_1[n] = \mathbf{T}\{x_1[n]\} = x_1[n-1] = x[n-1-n_0]$$

and

$$y[n-n_0] = x[n-n_0-1] = x[n-1-n_0] = y_1[n]$$

Hence, the system is time-invariant.

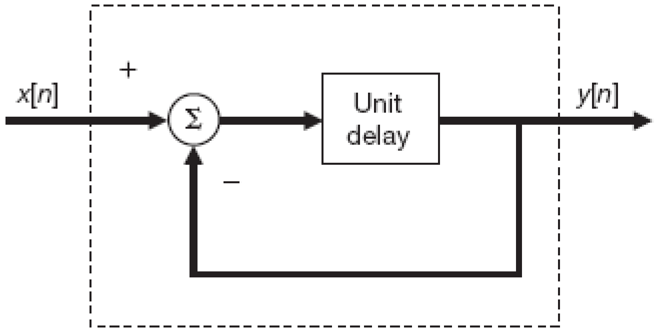
e. Since

$$|y[n]| = |x[n-1]| \le k \qquad \text{if } |x[n]| \le k \text{ for all } n$$

the system is BIBO stable.

1.37. Find the input-output relation of the feedback system shown in Fig. 1-37.

Figure 1-37



From Fig. 1-37 the input to the unit delay element is $x[n] - y[n]$. Thus, the output $y[n]$ of the unit delay element is [Eq. (1.111)]

$$y[n] = x[n-1] - y[n-1]$$

Rearranging, we obtain

$$y[n] + y[n-1] = x[n-1]$$

(1.112)

Thus, the input-output relation of the system is described by a first-order difference equation with constant coefficients.

1.38. A system has the input-output relation given by

$$y[n] = \mathbf{T}\{x[n]\} = nx[n]$$

(1.113)

Determine whether the system is (*a*) memoryless, (*b*) causal, (*c*) linear, (*d*) time-invariant, or (*e*) stable.

a. Since the output value at n depends on only the input value at n, the system is memoryless.
b. Since the output does not depend on the future input values, the system is causal.
c. Let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. Then

$$y[n] = \mathbf{T}\{x[n]\} = n\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\}$$
$$= \alpha_1 n x_1[n] + \alpha_2 n x_2[n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

Thus, the superposition property (1.68) is satisfied and the system is linear.

d. Let $y_1[n]$ be the response to $x_1[n] = x[n - n_0]$. Then

$$y_1[n] = \mathbf{T}\{x[n-n_0]\} = nx[n-n_0]$$

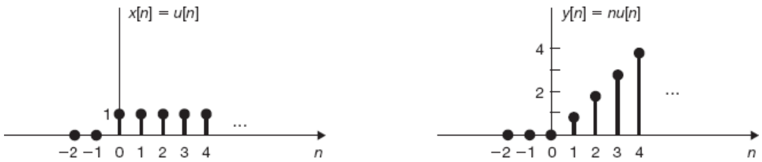
But

$$y[n-n_0] = (n-n_0)\, x[n-n_0] \neq y_1[n]$$

Hence, the system is not time-invariant.

e. Let $x[n] = u[n]$. Then $y[n] = nu[n]$. Thus, the bounded unit step sequence produces an output sequence that grows without bound (Fig. 1-38) and the system is not BIBO stable.

Figure 1-38



1.39. A system has the input-output relation given by

$$y[n] = \mathbf{T}\{x[n]\} = x[k_0 n]$$

(1.114)

where k_0 is a positive integer. Is the system time-invariant?

Let $y_1[n]$ be the response to $x_1[n] = x[n - n_0]$. Then

$$y_1[n] = \mathbf{T}\{x_1[n]\} = x_1[k_0 n] = x[k_0 n - n_0]$$

But

$$y[n - n_0] = x[k_0(n - n_0)] \neq y_1[n]$$

Hence, the system is not time-invariant unless $k_0 = 1$. Note that the system described by Eq. (1.114) is called a *compressor*. It creates the output sequence by selecting every k_0th sample of the input sequence. Thus, it is obvious that this system is time-varying.

1.40. Consider the system whose input-output relation is given by the linear equation

$$y = ax + b$$

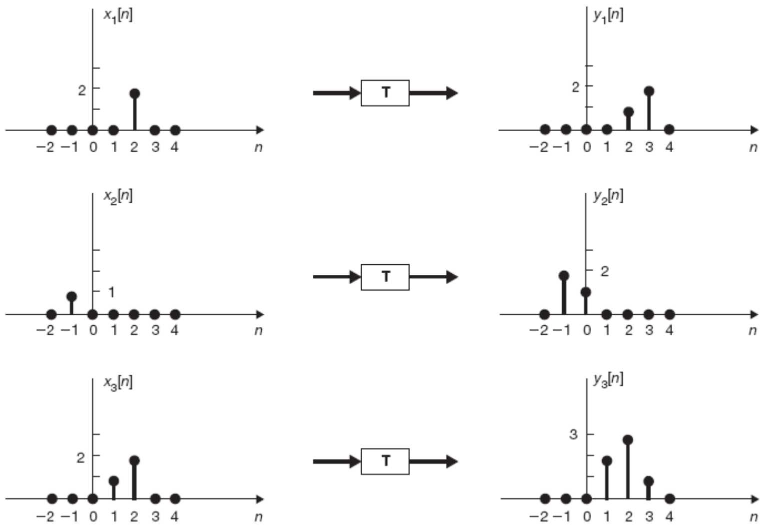
(1.115)

where x and y are the input and output of the system, respectively, and a and b are constants. Is this system linear?

If $b \neq 0$, then the system is not linear because $x = 0$ implies $y = b \neq 0$. If $b = 0$, then the system is linear.

1.41. The system represented by **T** in Fig. 1-39 is known to be time-invariant. When the inputs to the system are $x_1[n]$, $x_2[n]$, and $x_3[n]$, the outputs of the system are $y_1[n]$, $y_2[n]$, and $y_3[n]$ as shown. Determine whether the system is linear.

Figure 1-39



From Fig. 1-39 it is seen that

$$x_3[n] = x_1[n] + x_2[n-2]$$

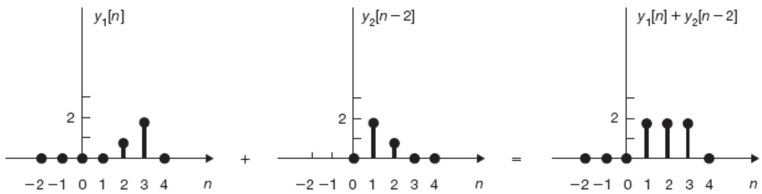
Thus, if **T** is linear, then

$$\mathbf{T}\{x_3[n]\} = \mathbf{T}\{x_1[n]\} + \mathbf{T}\{x_2[n-2]\} = y_1[n] + y_2[n-2]$$

which is shown in Fig. 1-40. From Figs. 1-39 and 1-40 we see that

$$y_3[n] \neq y_1[n] + y_2[n-2]$$

Figure 1-40



Hence, the system is not linear.

1.42. Give an example of a system that satisfies the condition of additivity(1.66) but not the condition of homogeneity (1.67).

Consider a discrete-time system represented by an operator **T** such that

$$y[n] = \mathbf{T}\{x[n]\} = x^*[n]$$

(1.116)

where $x^*[n]$ is the complex conjugate of $x[n]$. Then

$$\mathbf{T}\{x_1[n] + x_2[n]\} = \{x_1[n] + x_2[n]\}^* = x_1^*[n] + x_2^*[n] = y_1[n] + y_2[n]$$

Next, if α is any arbitrary complex-valued constant, then

$$\mathbf{T}\{\alpha\, x[n]\} = \{\alpha\, x[n]\}^* = \alpha^* x^*[n] = \alpha^* y[n] \neq \alpha\, y[n]$$

Thus, the system is additive but not homogeneous.

1.43.

a. Show that the causality for a continuous-time linear system is equivalent to the following statement: For any time t_0 and any input $x(t)$ with $x(t) = 0$ for $t \le t_0$, the output $y(t)$ is zero for $t \le t_0$.

b. Find a nonlinear system that is causal but does not satisfy this condition.

c. Find a nonlinear system that satisfies this condition but is not causal.

a. Since the system is linear, if $x(t) = 0$ for all t, then $y(t) = 0$ for all t. Thus, if the system is causal, then $x(t) = 0$ for $t \le t_0$ implies that $y(t) = 0$ for $t \le t_0$. This is the necessary condition. That this condition is also sufficient is shown as follows: let $x_1(t)$ and $x_2(t)$ be two inputs of the system and let $y_1(t)$ and $y_2(t)$ be the corresponding outputs. If $x_1(t) = x_2(t)$ for $t \le t_0$, or $x(t) = x_1(t) - x_2(t) = 0$ for $t \le t_0$, then $y_1(t) = y_2(t)$ for $t \le t_0$, or $y(t) = y_1(t) - y_2(t) = 0$ for $t \le t_0$.

b. Consider the system with the input-output relation

$$y(t) = x(t) + 1$$

This system is nonlinear (Prob. 1.40) and causal since the value of $y(t)$ depends on only the present value of $x(t)$. But with $x(t) = 0$ for $t \le t_0$, $y(t) = 1$ for $t \le t_0$.

c. Consider the system with the input-output relation

$$y(t) = x(t)x(t+1)$$

It is obvious that this system is nonlinear (see Prob. 1.35) and noncausal since the value of $y(t)$ at time t depends on the value of $x(t + 1)$ of the input at time $t + 1$. Yet $x(t) = 0$ for $t \le t_0$ implies that $y(t) = 0$ for $t \le t_0$.

1.44. Let **T** represent a continuous-time LTI system. Then show that

$$\mathbf{T}\{e^{st}\} = \lambda e^{st}$$

(1.117)

where s is a complex variable and λ is a complex constant.

Let $y(t)$ be the output of the system with input $x(t) = e^{st}$. Then

$$\mathbf{T}\{e^{st}\} = y(t)$$

Since the system is time-invariant, we have

$$\mathbf{T}\{e^{s(t+t_0)}\} = y(t + t_0)$$

for arbitrary real t_0. Since the system is linear, we have

$$\mathbf{T}\{e^{s(t+t_0)}\} = \mathbf{T}\{e^{st}\, e^{st_0}\} = e^{st_0}\mathbf{T}\{e^{st}\} = e^{st_0}y(t)$$

Hence,

$$y(t + t_0) = e^{st_0}y(t)$$

Setting $t = 0$, we obtain

$$y(t_0) = y(0)e^{st_0}$$

(1.118)

Since t_0 is arbitrary, by changing t_0 to t, we can rewrite Eq. (1.118) as

$$y(t) = y(0)\, e^{st} = \lambda e^{st}$$

or

$$\mathbf{T}\{e^{st}\} = \lambda\, e^{st}$$

where $\lambda = y(0)$.

1.45. Let **T** represent a discrete-time LTI system. Then show that

$$\mathbf{T}\{z^n\} = \lambda z^n$$

(1.119)

where z is a complex variable and λ is a complex constant.

Let $y[n]$ be the output of the system with input $x[n] = z^n$. Then

$$\mathbf{T}\{z^n\} = y[n]$$

Since the system is time-invariant, we have

$$\mathbf{T}\{z^{n+n_0}\} = y[n + n_0]$$

for arbitrary integer n_0. Since the system is linear, we have

$$\mathbf{T}\{z^{n+n_0}\} = \mathbf{T}\{z^n z^{n_0}\} = z^{n_0}\mathbf{T}\{z^n\} = z^{n_0} y[n]$$

Hence,

$$y[n + n_0] = z^{n_0} y[n]$$

Setting $n = 0$, we obtain

$$y[n_0] = y[0] z^{n_0}$$

(1.120)

Since n_0 is arbitrary, by changing n_0 to n, we can rewrite Eq. (1.120) as

$$y[n] = y[0] z^n = \lambda\, z^n$$

or

$$\mathbf{T}\{z^n\} = \lambda\, z^n$$

where $\lambda = y[0]$.

In mathematical language, a function $x(\cdot)$ satisfying the equation

$$\mathbf{T}\{x(\cdot)\} = \lambda x(\cdot)$$

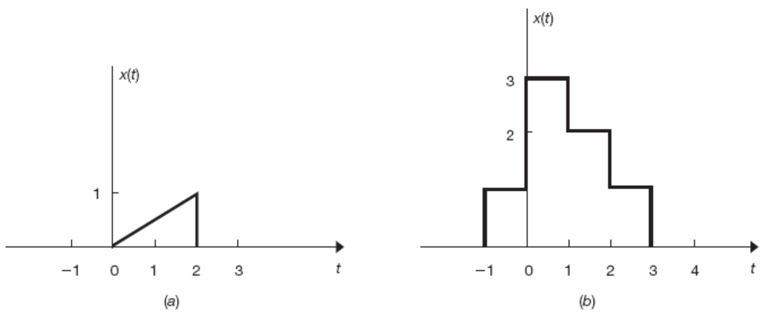
(1.121)

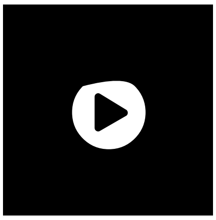
is called an *eigenfunction* (or *characteristic function*) of the operator **T**, and the constant λ is called the *eigenvalue* (or *characteristic value*) corresponding to the eigenfunction $x(\cdot)$. Thus, Eqs. (1.117) and (1.119) indicate that the complex exponential functions are eigenfunctions of any LTI system.

1.7. SUPPLEMENTARY PROBLEMS

1.46. Express the signals shown in Fig. 1-41 in terms of unit step functions.

Figure 1-41





Schaum's Signals and Systems Supplementary Problem 1.46: Unit Step Functions Example

This video shows how to express signal in terms of unit step functions.

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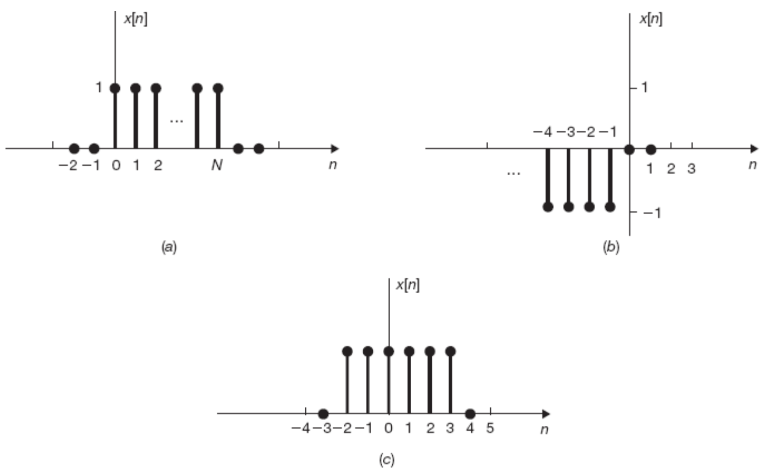
1.47. Express the sequences shown in Fig. 1-42 in terms of unit step sequences.

Figure 1-42



1.48. Determine the even and odd components of the following signals:

$$(a)\quad x(t) = u(t)$$

$$(b)\quad x(t) = \sin\left(\omega_0 t + \frac{\pi}{4}\right)$$

$$(c)\quad x[n] = e^{j(\Omega_0 n + \pi/2)}$$

$$(d)\quad x[n] = \delta[n]$$

1.49. Let $x(t)$ be an arbitrary signal with even and odd parts denoted by $x_e(t)$ and $x_o(t)$, respectively. Show that

$$\int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} x_e^2(t)\, dt + \int_{-\infty}^{\infty} x_o^2(t)\, dt$$

1.50. Let $x[n]$ be an arbitrary sequence with even and odd parts denoted by $x_e[n]$ and $x_o[n]$, respectively. Show that

$$\sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$$

1.51. Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

$$(a)\quad x(t) = \cos\left(2t + \frac{\pi}{4}\right)$$

$$(b)\quad x(t) = \cos^2 t$$

$$(c)\quad x(t) = (\cos 2\pi t)u(t)$$

$$(d)\quad x(t) = e^{j\pi t}$$

$$(e)\quad x[n] = e^{j[(n/4)-\pi]}$$

$$(f)\quad x[n] = \cos\left(\frac{\pi n^2}{8}\right)$$

$$(g)\quad x[n] = \cos\left(\frac{n}{2}\right)\cos\left(\frac{\pi n}{4}\right)$$

$$(h)\quad x[n] = \cos\left(\frac{\pi n}{4}\right) + \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{2}\right)$$

1.52. Show that if $x[n]$ is periodic with period N, then

$$(a)\quad \sum_{k=n_0}^{n} x[k] = \sum_{k=n_0+N}^{n+N} x[k]; \qquad (b)\quad \sum_{k=0}^{N} x[k] = \sum_{k=n_0}^{n_0+N} x[k]$$

1.53.

a. What is $\delta(2t)$?

b. What is $\delta[2n]$?

1.54. Show that

$$\delta'(-t) = -\delta'(t)$$

1.55. Evaluate the following integrals:

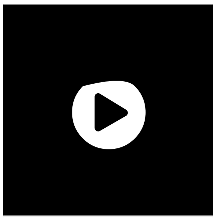
(a) $\int_{-\infty}^{t} (\cos \tau) u(\tau)\, d\tau$ (b) $\int_{-\infty}^{t} (\cos \tau) \delta(\tau)\, d\tau$

(c) $\int_{-\infty}^{\infty} (\cos t) u(t-1) \delta(t)\, dt$ (d) $\int_{0}^{2\pi} t \sin \frac{t}{2} \delta(\pi - t)\, dt$

1.56. Consider a continuous-time system with the input-output relation

$$y(t) = \mathbf{T}\{x(t)\} = \frac{1}{T} \int_{t-T/2}^{t+T/2} x(\tau)\, d\tau$$

Determine whether this system is (*a*) linear, (*b*) time-invariant, (*c*) causal.



Schaum's Signals and Systems Supplementary Problem 1.56: System Properties Example 1

This video illustrates how to determine if a system is linear, time-invariant, and causal.

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1.57. Consider a continuous-time system with the input-output relation

$$y(t) = \mathbf{T}\{x(t)\} = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT_s)$$

Determine whether this system is (*a*) linear, (*b*) time-invariant.

1.58. Consider a discrete-time system with the input-output relation

$$y[n] = \mathbf{T}\{x[n]\} = x^2[n]$$

Determine whether this system is (*a*) linear, (*b*) time-invariant.

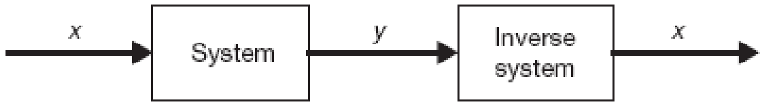
1.59. Give an example of a system that satisfies the condition of homogeneity(1.67) but not the condition of additivity (1.66).

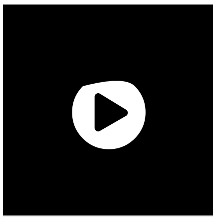
1.60. Give an example of a linear time-varying system such that with a periodic input the corresponding output is not periodic.

1.61. A system is called *invertible* if we can determine its input signal x uniquely by observing its output signal y. This is illustrated in Fig. 1-43. Determine if each of the following systems is invertible. If the system is invertible, give the inverse system.

$$(a)\quad y(t) = 2x(t)$$
$$(b)\quad y(t) = x^2(t)$$
$$(c)\quad y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$
$$(d)\quad y[n] = \sum_{k=-\infty}^{n} x[k]$$
$$(e)\quad y[n] = nx[n]$$

Figure 1-43





Schaum's Signals and Systems Supplementary Problem 1.61: System Properties Example 2

This video illustrates how to determine if a system is invertible.

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1.8. ANSWERS TO SUPPLEMENTARY PROBLEMS

1.46.

a. $x(t) = \frac{t}{2}[u(t) - u(t-2)]$

b. $x(t) = u(t+1) + 2u(t) - u(t-1) - u(t-2) - u(t-3)$

1.47.

a. $x[n] = u[n] - u[n-(N+1)]$

b. $x[n] = -u[-n-1]$

c. $x[n] = u[n+2] - u[n-4]$

1.48.

$$(a)\quad x_e(t) = \frac{1}{2},\ x_o(t) = \frac{1}{2}\operatorname{sgn} t$$
$$(b)\quad x_e(t) = \frac{1}{\sqrt{2}}\cos\omega_0 t,\ x_o(t) = \frac{1}{\sqrt{2}}\sin\omega_0 t$$
$$(c)\quad x_e[n] = j\cos\Omega_0 n,\ x_o[n] = -\sin\Omega_0 \boldsymbol{n}$$
$$(d)\quad x_e[n] = \delta[n],\ x_o[n] = 0$$

1.49. *Hint:* Use the results from Prob. 1.7 and Eq. (1.77).

1.50. *Hint:* Use the results from Prob. 1.7 and Eq. (1.77).

1.51.

a. Periodic, period = π

b. Periodic, period = π

c. Nonperiodic

d. Periodic, period = 2

e. Nonperiodic

f. Periodic, period = 8

g. Nonperiodic

h. Periodic, period = 16

1.52. *Hint:* See Prob. 1.17.

1.53.

a. $\delta(2t) = \frac{1}{2}\delta(t)$

b. $\delta[2n] = \delta[n]$

1.54. *Hint:* Use Eqs. (1.101) and (1.99).

1.55.

a. $\sin t$

b. 1 for $t > 0$ and 0 for $t < 0$; not defined for $t = 0$

c. 0

d. π

1.56. (*a*) Linear; (*b*) Time-invariant; (*c*) Noncausal

1.57. (*a*) Linear; (*b*) Time-varying

1.58. (*a*) Nonlinear; (*b*) Time-invariant

1.59. Consider the system described by

$$y(t) = \mathbf{T}\{x(t)\} = \left[\int_a^b [x(\tau)]^2\, d\tau\right]^{1/2}$$

1.60. $y[n] = \mathbf{T}\{x[n]\} = nx[n]$

1.61.

(*a*) Invertible; $x(t) = \frac{1}{2}y(t)$

(*b*) Not invertible

(*c*) Invertible; $x(t) = \frac{dy(t)}{dt}$

(*d*) Invertible; $x[n] = y[n] - y[n-1]$

(*e*) Not invertible