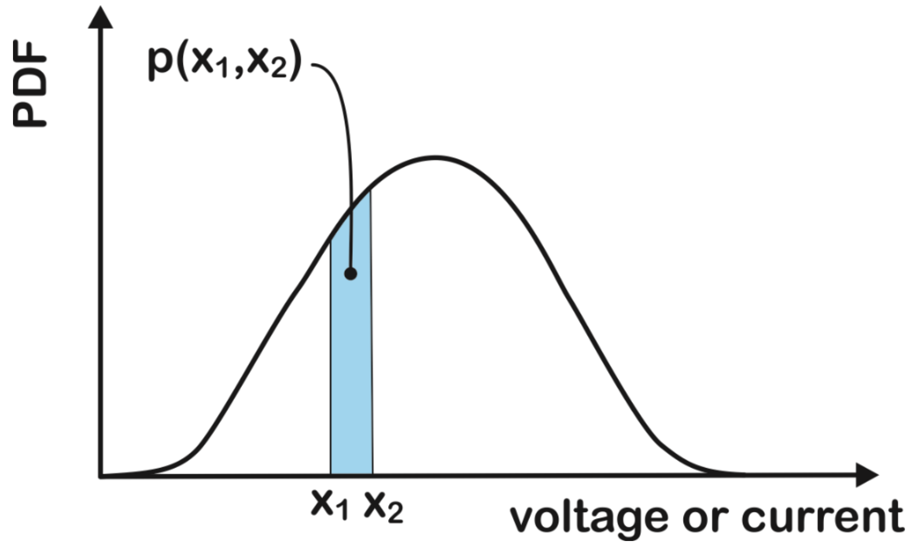
Probability density function



- a random signal (the noise) can be described just statistically: the PDF is a good descriptor
- by definition PDF $p(x_1, x_2)$ is the probability that the actual value of x is included in the interval $x_1 \div x_2$

- therefore the expected value, the average, is: $\bar{x} = E(x) = \int_{-\infty}^{+\infty} x \cdot PDF(x) \cdot dx$ in case of electronic noise the average can be assumed equal to zero

- the variance is $\sigma^2 = E(x - \bar{x})^2 = \int_{-\infty}^{+\infty} (x - \bar{x})^2 \cdot PDF(x) \cdot dx = \overline{x^2} - (\bar{x})^2$

Noise power

- a voltage V over a resistive load of resistance R dissipates o power equal to V^2/R, in the same way a current I flowing in a resistor of resistance R dissipates a power equal to RI^2
- if $R = 1\ \Omega$ the dissipated powers becomes V^2 and I^2
- by convention, the noise power is implicitly defined as the power dissipated by the noise when $R = 1\ \Omega$, therefore we assume V^2 or I^2 as the power of a voltage noise or current noise respectively

- $E(x-\bar{x})^2 = \int_{-\infty}^{+\infty}(x-\bar{x})^2 \cdot PDF(x) \cdot dx = \overline{x^2} - (\bar{x})^2$ **(1)**

  (x represent either a voltage or a current)

- assuming $\bar{x} = 0$ (offset = 0) we can say that $\overline{x^2}$ represents the average power of the noise statistically described by the variable x of eq. (1)

Power spectral density

- a random signal (the noise) can be described in the frequency-domain, however, since the time-domain waveform is not defined, also the Fourier transform of the random signal can not be defined
- we can define the function *power spectral density* $\boldsymbol{S}_x(f)$ (one sided) of a signal x such that the signal power P_{12} associated to the frequency band $\mathrm{f}_1 \div \mathrm{f}_2$ is given as:

$$P_{12} = \int_{f_1}^{f_2} \boldsymbol{S}_x(f) \cdot df$$

- the total power is: $P_{tot} = \int_0^{+\infty} \boldsymbol{S}_x(f) \cdot df$
- the total power can also be evaluated in the time-domain as

$$P_{tot} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt = \overline{x^2(t)}$$ **(offset = 0)**

- therefore $\int_0^{+\infty} \boldsymbol{S}_x(f) \cdot df = \overline{x^2(t)}$

Power spectral density

- the Wiener–Khinchin theorem (wide-sense stationary process)
  - given the autocorrelation function

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

  - the power spectral density (two-sided) coincides with the Fourier transform of that function:

$$S_x(f) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j2\pi f\tau} d\tau$$

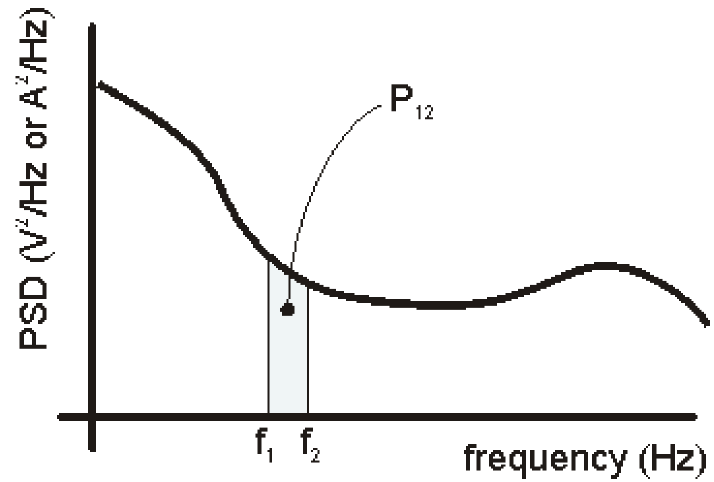
- total noise power:

$$\overline{x^2(t)} = \int_{-\infty}^{+\infty} S_x(f)df = 2\int_0^{+\infty} S_x(f)df = \int_0^{\infty} \boldsymbol{S}_x(f)df$$

- noise power in a given bandwidth: BW: $\overline{x_{BW}{}^2(t)} = \int_{f_1}^{f_2} \boldsymbol{S}_x(f)\,df$

Frequency distribution of the noise

- the power spectral density can be seen as the power dissipated on a resistive load with a resistance of 1Ω per unit band, that is in a band of $1\ Hz$



- the unit of measurement of the power spectral density is $\frac{[V]^2}{[Hz]}$ for a voltage noise and $\frac{[A]^2}{[Hz]}$ for a current noise
- in case we want to represent the noise with amplitude spectra the unit of measurement will be $\frac{[V]}{[\sqrt{Hz}]}$ for voltage noise and $\frac{[A]}{[\sqrt{Hz}]}$ for current noise

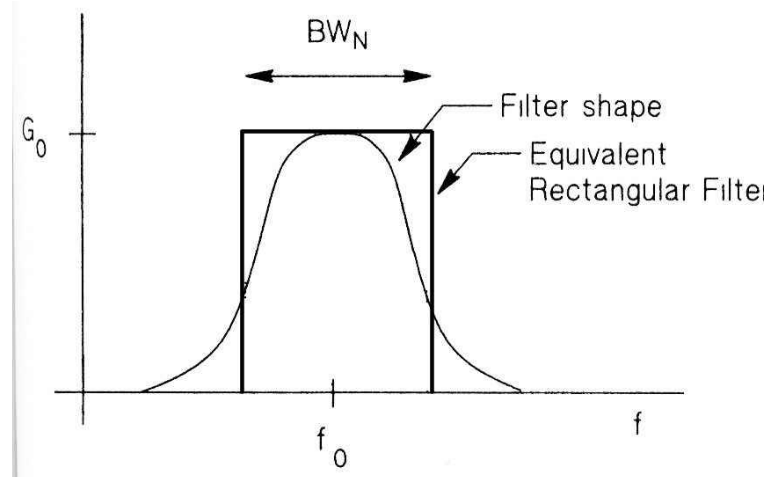
Noise equivalent bandwidth

- How much noise will be present at the output of a filter ?

$$P_n = \int_0^\infty G(f) \cdot N_0 \cdot df = N_0 \cdot \int_0^\infty G(f) \cdot df$$

N_0 = input noise PSD

$G(f)$ = power gain of the filter



- at the output of an ideal (rectangular response) filter having a power gain G_0 and a bandwidth BW_N the noise power should be:

$$P_n = N_0 \cdot G_0 \cdot BW_N$$

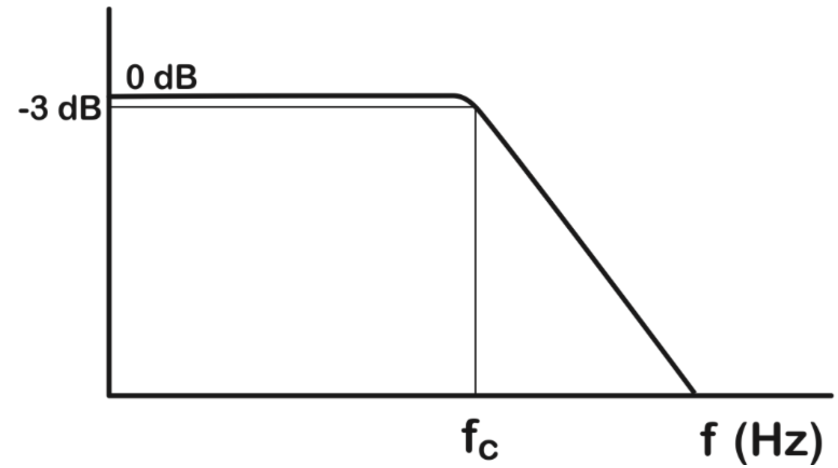
- the noise equivalent bandwidth is: $BW_N = \frac{1}{G_0}\int_0^\infty G(f) \cdot df$

Example

- **one pole low-pass filter:**

$$T(s) = \frac{1}{1+s\tau}$$

$$T(f) = \frac{f_c}{f_c+jf}$$



- **power gain:**

$$G(f) = |T(f)|^2 = \frac{{f_c}^2}{{f_c}^2+f^2}$$

- **noise equivalent bandwidth:**

$$BW_n = \frac{1}{G_0}\int_0^\infty G(f)df = \int_0^\infty \frac{{f_c}^2}{{f_c}^2+f^2}df = \frac{\pi}{2}\cdot f_c$$

Noise and decibel

- for a noise bandwidth of $1\ Hz$ we can write:

$$P_n(dBm, 1\ Hz) = 10\log\left(\frac{\overbrace{N_0 \cdot 1}^{noisepower(W)}}{\underbrace{0.001}_{reference(1mW)}}\right)$$

- for a generic noise bandwidth BW_N we can write:

$$P_n(dBm, BW_N) = 10\log\left(\frac{N_0 \cdot BW_N}{0.001}\right) = 10\log(BW_N) + P_n(dBm, 1Hz)$$

- for a given constant $PSD\ N_0$ we can switch from a bandwidth BW_1 to another bandwidth BW_2:

$$P_{\boldsymbol{n}}(dBm, BW_2) = 10\log\left(\frac{BW_2}{BW_1}\right) + P_{\boldsymbol{n}}(dBm, BW_1)$$

Noise measurement with a spectrum analyzer

- by construction, a spectrum analyzer provides a trace representing the power content of the input signal included in the resolution bandwidth as a function of frequency
- the output of the spectrum analyzer measuring chain $P_{out}(f)$ is always transformed into a PSD reading by normalizing the measurement chain output power $P_{mc}(f)$ by using the measuring chain signal bandwidth BW_S:

$$P_{readout}(f) = {}^{P_{out}(f)}/_{BW_S} \qquad (1)$$

- for noise measurements the normalization is based on the knowledge of the equivalent noise bandwidth of the measuring chain, so in (1) we use BW_N instead of BW_S
- typically, the equivalent noise bandwidth of the spectrum analyzer filter is wider than the signal bandwidth (-3 dB) of about 15-20%

Noise measurement with a spectrum analyzer

- for a given frequency, the equivalent noise power density at the input of the analyzer must be significantly lower than the noise power density we want to measure
- most spectrum analyzer has dedicated programs for noise measurements
- the noise level of the analyzer measuring chain depends on the frequency resolution of the measurement
- the rms meter of the analyzer is optimized to work with sinusoids, if the input is noise (random signal):
  - the noise level measurement must be corrected
  - the best analyzers are equipped with a calibration and auto-compensation mechanism

cont.

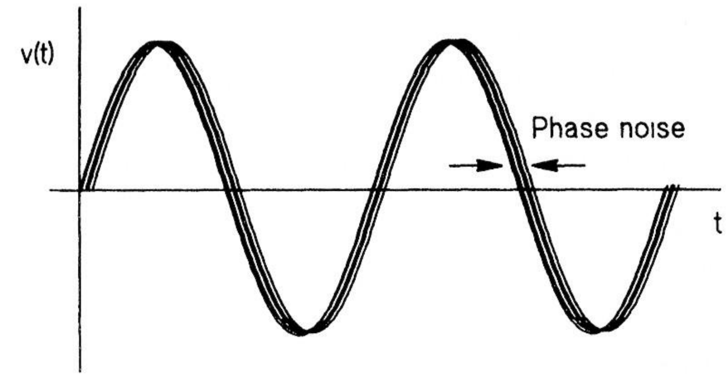
- since the noise is random, it would be necessary to take an infinite number of measurements to obtain a perfect measurement
- important note: during the noise measurements it is assumed that the noise is white inside the noise equivalent bandwidth of the measuring chain
- the displayed measurement result is always normalized to a noise equivalent bandwidth of $1\ Hz$

Phase noise

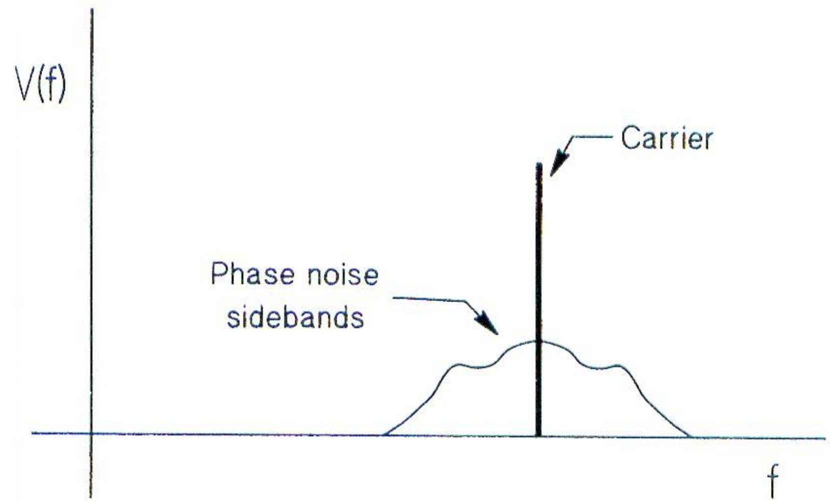
- a sinusoid which is noisy in phase is represented as:

$$v(t) = V_0 \sin[2\pi f_0 t + \Phi_N(t)]$$

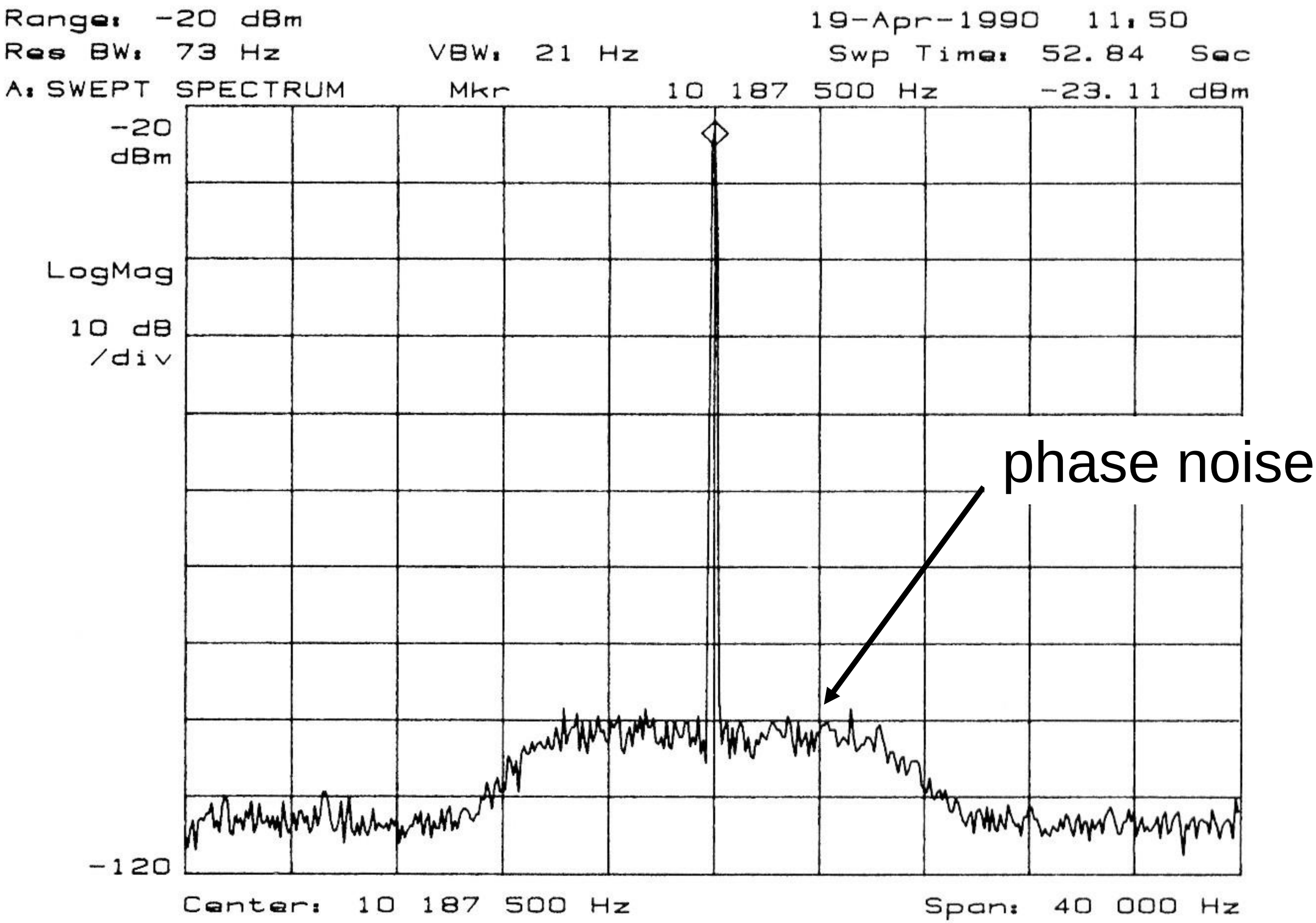
where $\Phi_N(t)$ is the phase noise



- the signal is angle modulated by the noise
- the result is the appearance of the symmetrical sidebands around the carrier



Phase noise: spectrum



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Phase noise

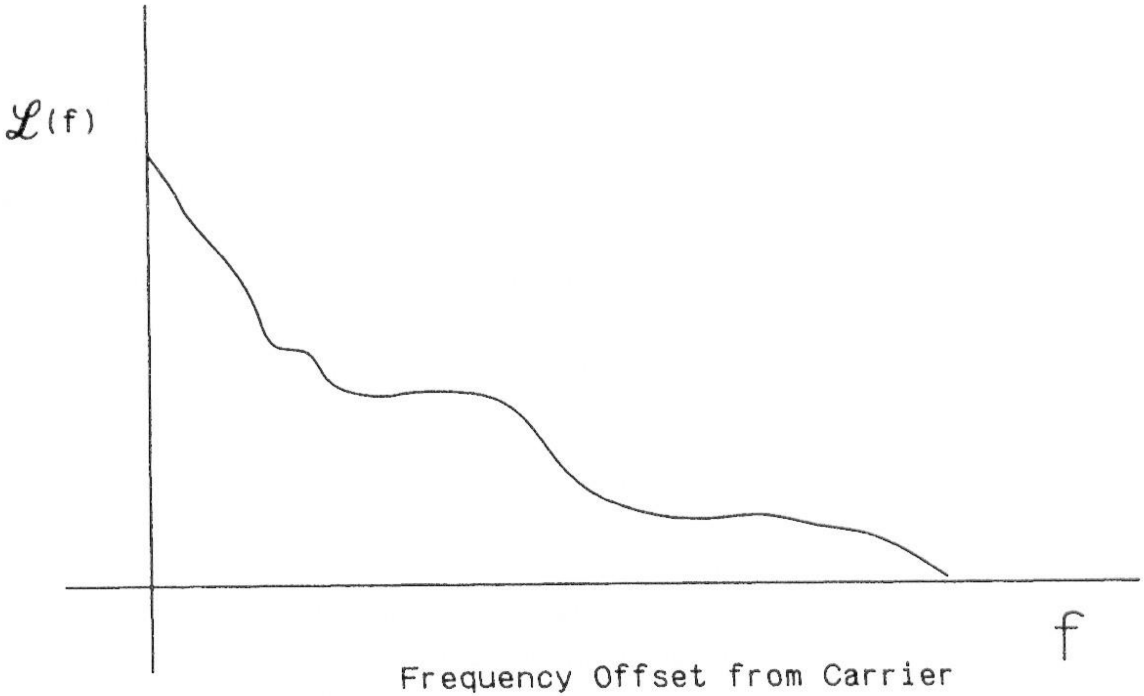
- the phase noise in the frequency domain can be expressed as single-sideband (SSB) phase-noise:

$$L(f) = \frac{V_N(1\ Hz\ BW)}{V_0}$$

where $V_N(1HzBW)$ is the rms noise level in a bandwidth of $1\ Hz$ at $f\ Hz$ away from the carrier, V_0 is the rms amplitude of the carrier

- in decibel $L(f)_{dB} = 20\log\frac{V_N(1HzBW)}{V_0}$

Phase noise: example



$$L(f)_{dB} = 20\log\frac{V_N(1HzBW)}{V_0}$$