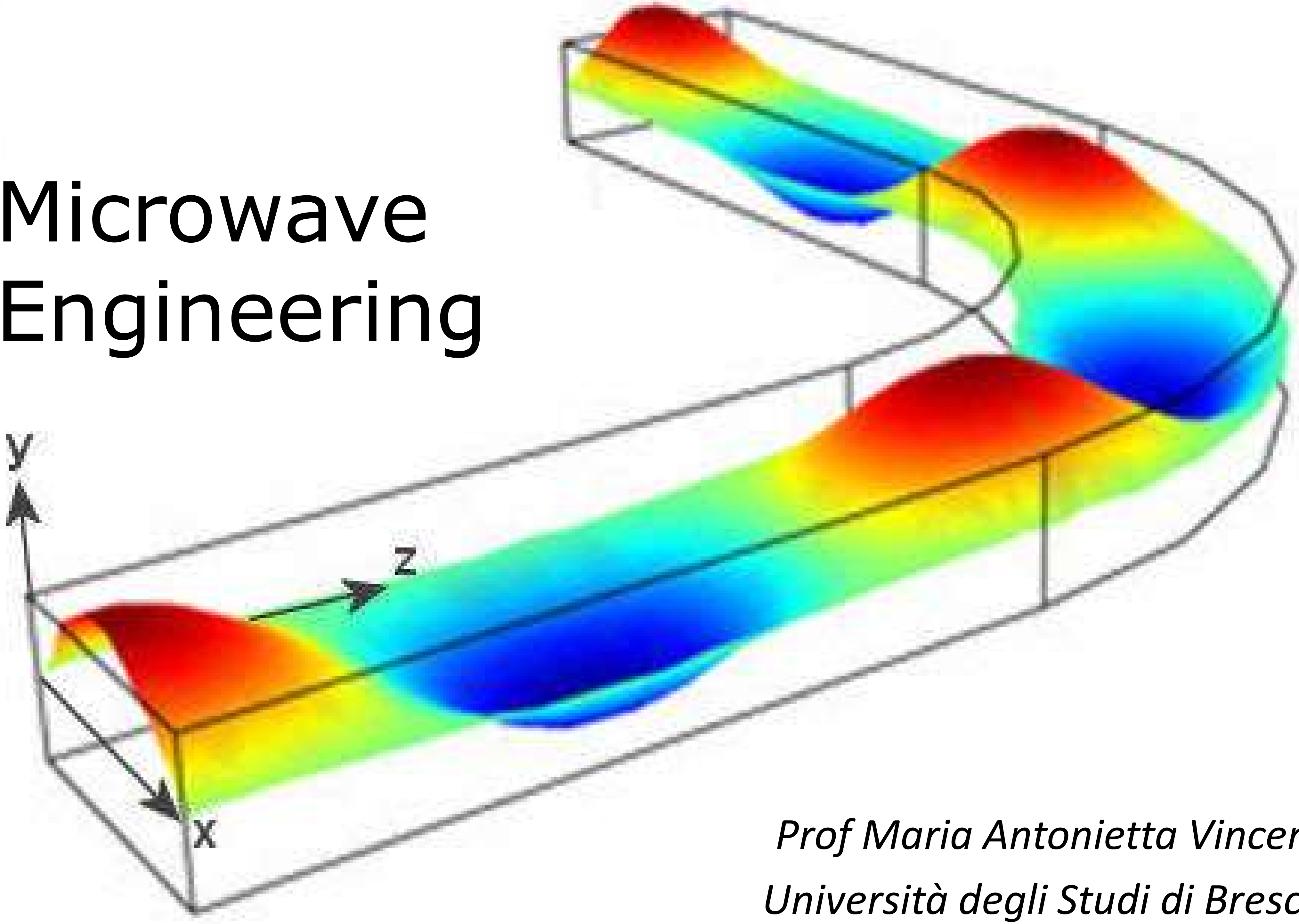
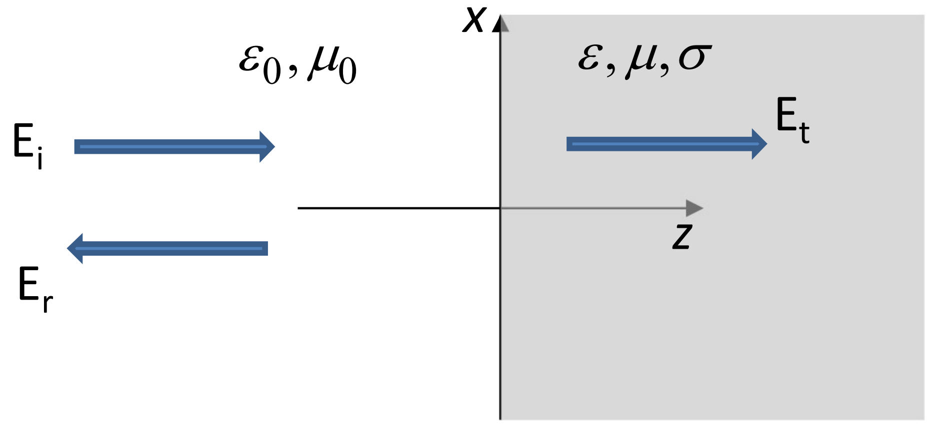
Microwave Engineering



Prof Maria Antonietta Vincenti
Università degli Studi di Brescia

Transmission, Reflection at Normal and Oblique Incidence
Reciprocity theorem, Image theory

Interface
with a generic lossy medium



Without loss of generality we can assume the fields propagating in the z direction and therefore for z<0 we can write:

$$\mathbf{E}_i = \hat{x} E_0 e^{-jk_0 z} \qquad \mathbf{E}_r = \hat{x} \Gamma E_0 e^{jk_0 z}$$

$$\mathbf{H}_i = \hat{y} \frac{E_0}{\eta_0} e^{-jk_0 z} \qquad \mathbf{H}_r = -\hat{y} \Gamma \frac{E_0}{\eta_0} e^{jk_0 z}$$

For z>0 we can write:

$$\mathbf{E}_t = \hat{x} T E_0 e^{-\gamma z} \qquad \eta = \frac{j\omega\mu}{\gamma}$$

$$\mathbf{H}_t = \hat{y} T \frac{E_0}{\eta} e^{-\gamma z} \qquad \gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\sigma/\omega\varepsilon}$$

UNIVERSITY OF BRESCIA

Interface with a generic lossy medium

Reflection and Transmission coefficients can be found by imposing the continuity of the tangential components of the electric and magnetic fields at the interface z=0:

$$\mathbf{E}_i + \mathbf{E}_r = \mathbf{E}_t$$

$$\mathbf{H}_i + \mathbf{H}_r = \mathbf{H}_t$$

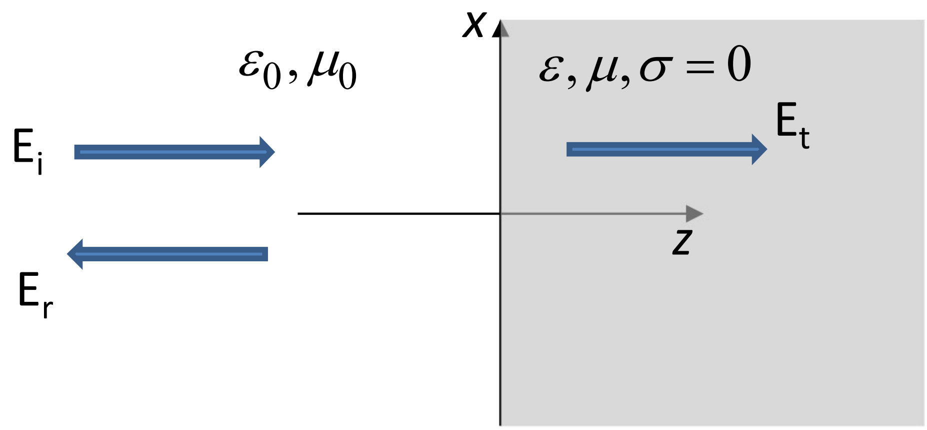
$$\Downarrow$$

$$\Gamma = \frac{\eta - \eta_0}{\eta + \eta_0}$$

$$\mathrm{T} = 1 + \Gamma = \frac{2\eta}{\eta + \eta_0}$$

Where Γ and T are complex quantities.

Interface with a lossless medium



Without loss of generality we can assume the fields propagating in the z direction and therefore for z<0 we can write:

$$\mathbf{E}_i = \hat{x} E_0 e^{-jk_0 z} \qquad \mathbf{E}_r = \hat{x} \Gamma E_0 e^{jk_0 z}$$

$$\mathbf{H}_i = \hat{y} \frac{E_0}{\eta_0} e^{-jk_0 z} \qquad \mathbf{H}_r = -\hat{y} \Gamma \frac{E_0}{\eta_0} e^{jk_0 z}$$

For z>0 we can write:

$$\mathbf{E}_t = \hat{x} T E_0 e^{-\gamma z} \qquad \eta = \frac{j\omega\mu}{\gamma}$$

$$\mathbf{H}_t = \hat{y} T \frac{E_0}{\eta} e^{-\gamma z} \qquad \gamma = j\beta = j\omega\sqrt{\mu\varepsilon} = jk_0\sqrt{\mu_r \varepsilon_r}$$

Γ and T have the same expressions than lossy case and they are real quantities.

Interface with a lossless medium

Some meaningful quantities we can define in the dielectric are:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}}, v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}, \eta = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Conservation of energy can be demonstrated by computing the Poynting vectors in the two regions. For z<0 we have:

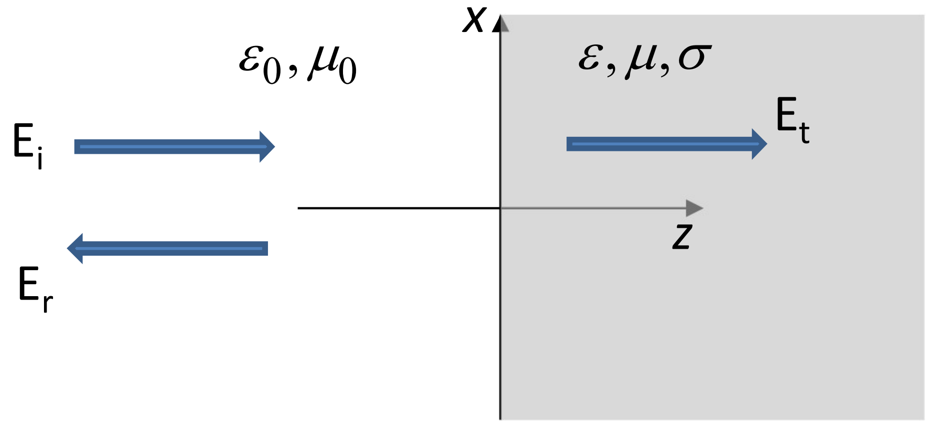
$$\mathbf{S}^- = \mathbf{E}\times\mathbf{H}^* = \left(\mathbf{E}_i + \mathbf{E}_r\right)\times\left(\mathbf{H}_i + \mathbf{H}_r\right)^* = \hat{z}\frac{\left|E_0\right|^2}{\eta_0}\left(1-\left|\Gamma\right|^2 + 2j\Gamma\sin 2k_0 z\right)$$

For z>0 we have:

$$\mathbf{S}^+ = \mathbf{E}\times\mathbf{H}^* = \mathbf{E}_t\times\mathbf{H}_t^* = \hat{z}\frac{\left|E_0\right|^2\left|\mathrm{T}\right|^2}{\eta} = \hat{z}\frac{\left|E_0\right|^2}{\eta_0}\left(1-\left|\Gamma\right|^2\right)$$

For z=0 we find $\mathbf{S}^+ = \mathbf{S}^-$

Interface with a good conductor



Without loss of generality, we can assume the fields propagating in the z direction and therefore for z<0 we can write:

$$\mathbf{E}_i = \hat{x} E_0 e^{-jk_0 z} \qquad \mathbf{E}_r = \hat{x} \Gamma E_0 e^{jk_0 z}$$

$$\mathbf{H}_i = \hat{y} \frac{E_0}{\eta_0} e^{-jk_0 z} \qquad \mathbf{H}_r = -\hat{y} \Gamma \frac{E_0}{\eta_0} e^{jk_0 z}$$

For z>0 we can write:

$$\mathbf{E}_t = \hat{x} T E_0 e^{-\gamma z}$$

$$\mathbf{H}_t = \hat{y} T \frac{E_0}{\eta} e^{-\gamma z}$$

$$\gamma = \alpha + j\beta = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = (1+j)\frac{1}{\delta_s}$$

$$\eta = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\frac{1}{\sigma\delta_s}$$

Where Γ and T are complex quantities.

Interface with a good conductor

UNIVERSITY OF BRESCIA

For z<0 the complex Poynting vector is:

$$\mathbf{S}^- = \mathbf{E}\times\mathbf{H}^* = \left(\mathbf{E}_i + \mathbf{E}_r\right)\times\left(\mathbf{H}_i + \mathbf{H}_r\right)^* = \hat{z}\frac{\left|E_0\right|^2}{\eta_0}\left(1-\left|\Gamma\right|^2 + \Gamma - \Gamma^*\right)$$

For z>0 we have:

$$\mathbf{S}^+ = \mathbf{E}\times\mathbf{H}^* = \mathbf{E}_t \times \mathbf{H}_t{}^* = \hat{z}\frac{\left|E_0\right|^2\left|\mathrm{T}\right|^2}{\eta^*}e^{-2\alpha z} = \hat{z}\frac{\left|E_0\right|^2}{\eta_0}\left(1-\left|\Gamma\right|^2 + \Gamma - \Gamma^*\right)e^{-2\alpha z}$$

For z=0 we find **S**$^+$ = **S**$^-$

NOTE: The power balance is not obtained if we were to separate incident and reflected Poynting vectors. The only way we can still recover the balance is to use the time averaged quantities of the power flows P=1/2 Re(ExH*)

NOTE 2: The power in the lossy conductor decays exponentially as expected according to the attenuation factor $e^{-2\alpha z}$

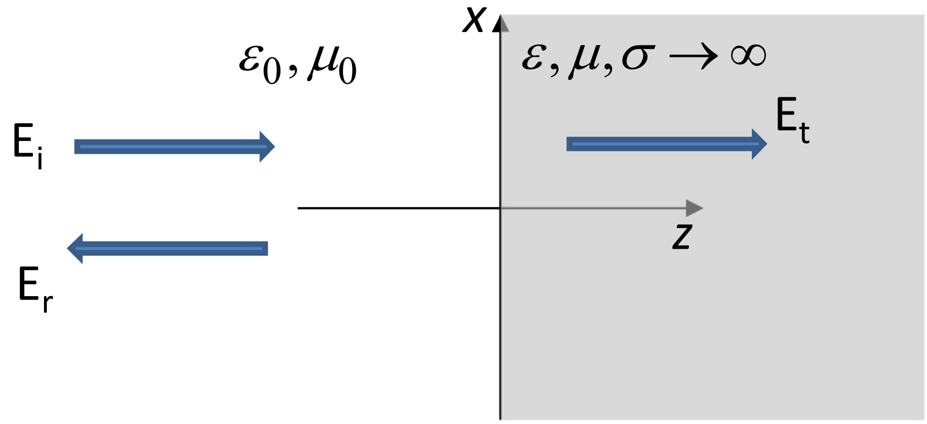
For a good conductor the volume current in the conducting region is:

$$\mathbf{J}_t = \sigma\mathbf{E}_t = \hat{x}\sigma E_0 \mathrm{T} e^{-\gamma z}\quad \mathrm{A/m}^2$$

While the average power dissipated can be calculated from Joule's law as:

$$\mathbf{P}_t = \frac{1}{2}\int_V \mathbf{E}_t\cdot\mathbf{J}_t dv = \int_0^\infty\int_0^\infty\int_0^\infty \sigma\mathbf{E}_t\cdot\mathbf{E}_t dxdydz = \frac{\sigma\left|E_0\right|^2\left|\mathrm{T}\right|^2}{4\alpha}$$

Interface with a perfect conductor



In this specific scenario we observe that if:

$$\sigma \to \infty$$

$$\alpha \to \infty$$

$$\eta = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} \to 0$$

Then:

$$\delta_s = \sqrt{\frac{\omega\mu\sigma}{2}} \to 0$$

$$\mathrm{T} \to 0$$

$$\Gamma \to -1$$

In other words there are no fields that propagate into the perfect conductor.

Interface with a perfect conductor

Therefore, we can write the total fields as:

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_r = \hat{x}E_0(e^{-jk_0z} - e^{jk_0z}) = -\hat{x}2jE_0 \sin k_0 z$$

$$\mathbf{H} = \mathbf{H}_i + \mathbf{H}_r = \hat{y}\frac{E_0}{\eta_0}(e^{-jk_0z} + e^{jk_0z}) = \hat{y}\frac{2}{\eta_0}E_0 \cos k_0 z$$

At z=0:

$$\mathbf{E} = 0$$

$$\mathbf{H} = \hat{y}\frac{2}{\eta_0}E_0$$

While for z<0 the Poynting vector is:

$$\mathbf{S}^- = \mathbf{E} \times \mathbf{H}^* = -\hat{z}\frac{4j}{\eta_0}\left|E_0\right|^2 \sin k_0 z \cos k_0 z$$

For a perfect conductor a volume current density reduces to a surface current:

$$\mathbf{J}_s = \hat{n} \times \mathbf{H} = -\hat{z} \times \left(\hat{y}\frac{2}{\eta_0}E_0 \cos k_0 z \right)\Bigg|_{z=0} = \hat{x}\frac{2}{\eta_0}E_0 \quad \text{A/m}$$

Surface Impedance

If we have a good conductor in the z>0 region, most of the power that is transmitted is rapidly dissipated into heat and can be quantified as:

$$\mathbf{P}_t = \frac{1}{2}\int_V \mathbf{E}_t \cdot \mathbf{J}_t dv = \frac{\sigma |E_0|^2 |\mathrm{T}|^2}{4\alpha}$$

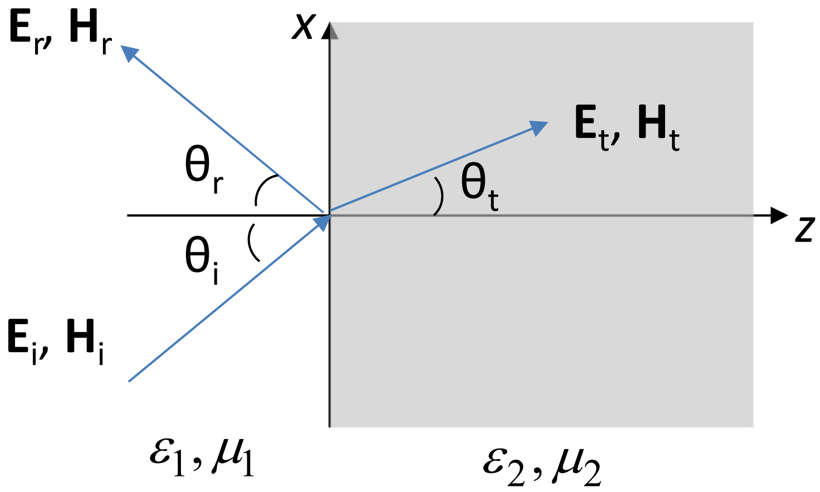
Since:

$$\left.\begin{aligned} \mathrm{T} &= \frac{2\eta}{\eta+\eta_0} \\ \eta &= (1+j)\frac{1}{\sigma\delta_s} \\ \alpha &= \frac{1}{\delta_s} \end{aligned}\right\} \Rightarrow \frac{\sigma|\mathrm{T}|^2}{\alpha} = \frac{\sigma\delta_s 4|\eta|^2}{|\eta+\eta_0|^2} \simeq \frac{8}{\sigma\delta_s \eta_0{}^2} = \frac{8R_s}{\eta_0{}^2} \Rightarrow \mathbf{P}_t = \frac{2|E_0|^2 R_s}{\eta_0}$$

Where the surface resistance of the metal is:

$$R_s = \mathrm{Re}(\eta) = \mathrm{Re}\left(\frac{1+j}{\sigma\delta_s}\right) = \frac{1}{\sigma\delta_s} = \sqrt{\frac{\omega\mu}{2\sigma}}$$

Oblique incidence at a dielectric interface



Parallel Polarization (TM Wave): If the E-field of the wave is in the plane of incidence then the wave is called a TM-wave;

Perpendicular Polarization (TE Wave): If the E-field of the wave is perpendicular to the plane of incidence then the wave is called a TE-wave.

UNIVERSITY
OF BRESCIA

Oblique incidence at a dielectric interface

Parallel Polarization (TM)

(electric field vector in the xz plane)

$$\mathbf{E}_i = E_0\left(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i\right)e^{-jk_1\left(x\sin\theta_i + z\cos\theta_i\right)}$$

$$\mathbf{H}_i = \frac{E_0}{\eta_1}\hat{y}e^{-jk_1\left(x\sin\theta_i + z\cos\theta_i\right)}$$

$$\mathbf{E}_r = E_0\Gamma\left(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r\right)e^{-jk_1\left(x\sin\theta_r - z\cos\theta_r\right)}$$

$$\mathbf{H}_r = -\frac{E_0\Gamma}{\eta_1}\hat{y}e^{-jk_1\left(x\sin\theta_r - z\cos\theta_r\right)}$$

$$k_1 = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_1}$$

$$\eta_1 = \sqrt{\mu_0 / \varepsilon_0\varepsilon_1}$$

$$\mathbf{E}_t = E_0\mathrm{T}\left(\hat{x}\cos\theta_t - \hat{z}\sin\theta_t\right)e^{-jk_2\left(x\sin\theta_t + z\cos\theta_t\right)}$$

$$\mathbf{H}_t = \frac{E_0}{\eta_2}\mathrm{T}\hat{y}e^{-jk_2\left(x\sin\theta_t + z\cos\theta_t\right)}$$

$$k_2 = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_2}$$

$$\eta_2 = \sqrt{\mu_0 / \varepsilon_0\varepsilon_2}$$

Oblique incidence at a dielectric interface

UNIVERSITY OF BRESCIA

Parallel Polarization – Boundary Conditions

At $z = 0$ the H_y and E_x fields must be continuous across the interface

$$\cos\theta_i e^{-jk_1(x\sin\theta_i)} + \Gamma\cos\theta_r e^{-jk_1(x\sin\theta_r)} = \mathrm{T}\cos\theta_t e^{-jk_2(x\sin\theta_t)}$$

$$\frac{1}{\eta_1} e^{-jk_1(x\sin\theta_i)} - \frac{\Gamma}{\eta_1} e^{-jk_1(x\sin\theta_r)} = \frac{\mathrm{T}}{\eta_2} e^{-jk_2(x\sin\theta_t)}$$

These boundary conditions have to be valid for all values of x, therefore we have:

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

$$\Downarrow$$

$$\theta_i = \theta_r$$

$$\boxed{k_1 \sin\theta_i = k_2 \sin\theta_t}$$ Snell's law

These arguments ensure that the phase terms vary with x at the same rate at the interface (phase matching condition)

UNIVERSITY OF BRESCIA

Oblique incidence at a dielectric interface

Imposing such phase-matching condition allows to find the reflection and transmission coefficients:

$$\Gamma = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$\mathrm{T} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

For this polarization we can find an angle of incidence that gives Γ=0, which is called Brewster's angle θ_b:

$$\Gamma = 0 \Rightarrow \eta_2 \cos\theta_t = \eta_1 \cos\theta_b$$

$$\Downarrow$$

$$\theta_b = a\tan\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

UNIVERSITY
OF BRESCIA

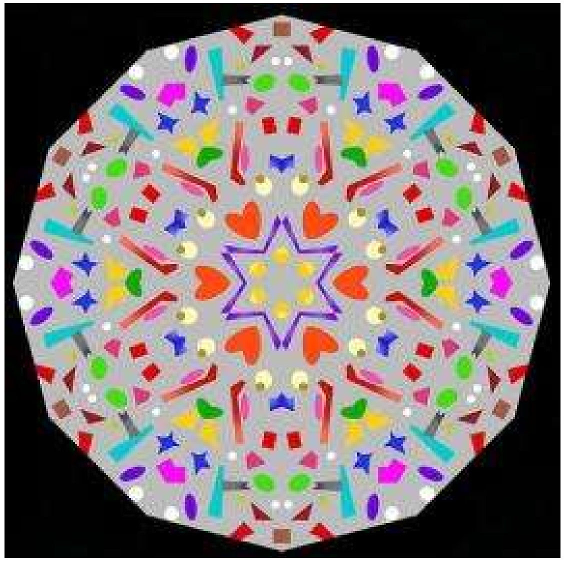
Brewster’s Angle

Today’s Culture Moment

Sir David Brewster

1781 –1868

- Scottish scientist
- Studied at University of Edinburgh at age 12
- Independently discovered Fresnel lens
- Editor of *Edinburgh Encyclopedia and contributor to Encyclopedia Britannica (7th and 8th editions)*
- Inventor of the Kaleidoscope
- Nominated (1849) to the National Institute of France.



Kaleidoscope

Oblique incidence at a dielectric interface

UNIVERSITY OF BRESCIA

Perpendicular Polarization (TE)

(electric field vector orthogonal to the xz plane)

$$\mathbf{E}_i = E_0 \hat{y} e^{-jk_1\left(x\sin\theta_i + z\cos\theta_i\right)}$$

$$\mathbf{H}_i = \frac{E_0}{\eta_1}\left(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i\right)e^{-jk_1\left(x\sin\theta_i + z\cos\theta_i\right)}$$

$$\mathbf{E}_r = E_0 \Gamma \hat{y} e^{-jk_1\left(x\sin\theta_r - z\cos\theta_r\right)}$$

$$\mathbf{H}_r = \frac{E_0\Gamma}{\eta_1}\left(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r\right)e^{-jk_1\left(x\sin\theta_r - z\cos\theta_r\right)}$$

$$k_1 = \omega\sqrt{\mu_0\varepsilon_1}$$

$$\eta_1 = \sqrt{\mu_0 / \varepsilon_1}$$

$$\mathbf{E}_t = E_0 \mathrm{T} \hat{y} e^{-jk_2\left(x\sin\theta_t + z\cos\theta_t\right)}$$

$$\mathbf{H}_t = \frac{E_0\mathrm{T}}{\eta_2}\left(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t\right)e^{-jk_2\left(x\sin\theta_t + z\cos\theta_t\right)}$$

$$k_2 = \omega\sqrt{\mu_0\varepsilon_2}$$

$$\eta_2 = \sqrt{\mu_0 / \varepsilon_2}$$

Oblique incidence at a dielectric interface

Perpendicular Polarization – Boundary Conditions

At z = 0 the E_y and H_x fields must be continuous across the interface

$$e^{-jk_1(x\sin\theta_i)} + \Gamma e^{-jk_1(x\sin\theta_r)} = \mathrm{T} e^{-jk_2(x\sin\theta_t)}$$

$$-\frac{1}{\eta_1}\cos\theta_i e^{-jk_1(x\sin\theta_i)} + \frac{\Gamma}{\eta_1}\cos\theta_r e^{-jk_1(x\sin\theta_r)} = -\frac{\mathrm{T}}{\eta_2}\cos\theta_t e^{-jk_2(x\sin\theta_t)}$$

These boundary conditions have to be valid for all values of x, therefore we have the same Snell's law of the parallel polarization:

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

$$\Downarrow$$

$$\theta_i = \theta_r$$

$$k_1 \sin\theta_i = k_2 \sin\theta_t$$

These arguments ensure that the phase terms vary with x at the same rate at the interface (phase matching condition)

Oblique incidence at a dielectric interface

UNIVERSITY OF BRESCIA

Imposing such phase-matching condition allows to find the reflection and transmission coefficients:

$$\Gamma = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$\mathrm{T} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

For this polarization we cannot find the Brewster angle θ_b :

$$\Gamma = 0 \Rightarrow \eta_2 \cos\theta_i = \eta_1 \cos\theta_t$$

$$\Downarrow$$

If we apply Snell's law we get:

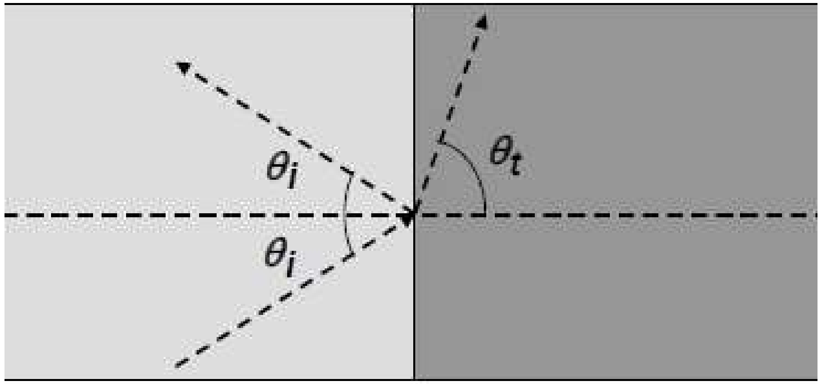
$$k_2^2\left(\eta_2^2 - \eta_1^2\right) = \left(k_2^2\eta_2^2 - k_1^2\eta_1^2\right)\sin\theta_t \quad \xrightarrow{\boxed{\text{Not feasible}}} \quad \left(k_2^2\eta_2^2 - k_1^2\eta_1^2\right) = 0$$

Total Internal Reflection & Surface waves

UNIVERSITY OF BRESCIA

$\varepsilon_1 > \varepsilon_2$

If θ_i is increased, then θ_t will eventually become 90°. The value of θ_i for which θ_t is 90° is called the critical angle θ_c



$$\sin(\theta_t) = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin(\theta_i)$$

$$\Downarrow$$

$$\theta_c = a\sin\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

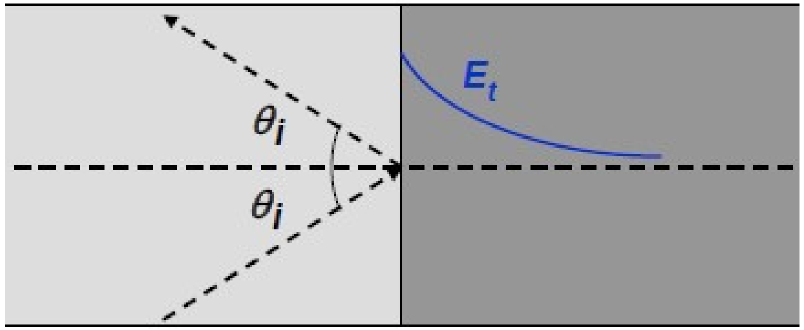
If θ_i is increased beyond θ_c the wave is not transmitted but is completely (100%) reflected at the interface back into the medium of incidence.

This phenomenon is called TOTAL INTERNAL REFLECTION and it happens for both parallel and perpendicular polarization.

Total Internal Reflection & Surface waves

UNIVERSITY OF BRESCIA

$\varepsilon_1 > \varepsilon_t$ and $\theta_i > \theta_c$



$$\theta_c = a\sin\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

If $\theta_i > \theta_c$ then $\sin(\theta_t)>1$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t}$$ is imaginary (transmission angle loses significance)

We can replace the expression of the transmitted fields in medium 2 as follows:

$$\mathbf{E}_t = E_0\mathrm{T}\left(\frac{j\alpha}{k_2}\hat{x} - \frac{\beta}{k_2}\hat{z}\right)e^{-j\beta x}e^{-\alpha z}$$

$$\mathbf{H}_t = \frac{E_0\mathrm{T}}{\eta_2}\hat{y}e^{-j\beta x}e^{-\alpha z}$$

Propagation in the x direction
Decays along z
NOTE: no energy flows in the z direction which can be verified calculating the Poynting vector

These expressions are obtained observing that $-jk_2\sin\theta_t$ is imaginary when $\sin\theta_t>1$, while $-jk_2\cos\theta_t$ is real so that we can assume $\sin\theta_t=\beta/k_2$ and $\cos\theta_t=-j\alpha/k_2$. Using these field expressions for the Helmoltz equation we get:

$$-\beta^2 + \alpha^2 + k_2^2 = 0$$

Total Internal Reflection & Surface waves

At z = 0 the H_y and E_x fields have to be equal to those that are generally assumed for the parallel polarization, therefore we have:

$$\cos\theta_i e^{-jk_1 x \sin\theta_i} + \Gamma \cos\theta_r e^{-jk_1 x \sin\theta_r} = \mathrm{T}\frac{j\alpha}{k_2} e^{-j\beta x}$$

$$\frac{1}{\eta_1} e^{-jk_1 x \sin\theta_i} - \frac{\Gamma}{\eta_1} e^{-jk_1 x \sin\theta_r} = \frac{\mathrm{T}}{\eta_2} e^{-j\beta x}$$

To satisfy the phase-matching condition we must have:

$$k_1 \sin\theta_i = k_1 \sin\theta_r = \beta$$

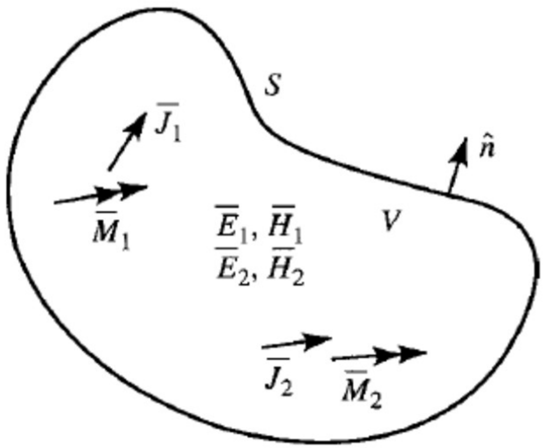
$$\theta_i = \theta_r \qquad\qquad k_1 \sin\theta_i = \beta$$

$$\alpha = \sqrt{\beta^2 - k_2^2} = \sqrt{k_1^2 \sin\theta_i - k_2^2}$$

Positive real number

Lorentz Reciprocity Theorem

Consider two set of sources J_1, M_1 and J_2, M_2, that generate two sets of fields E_1, H_1 and E_2, H_2, located in a volume V enclosed by the surface S. Maxwell-s equation have to be satisfied so that:



$$\nabla \times \mathbf{E}_1 = -j\omega\mu\mathbf{H}_1 - \mathbf{M}_1 \qquad \nabla \times \mathbf{E}_2 = -j\omega\mu\mathbf{H}_2 - \mathbf{M}_2$$

$$\nabla \times \mathbf{H}_1 = j\omega\varepsilon\mathbf{E}_1 + \mathbf{J}_1 \qquad \nabla \times \mathbf{H}_2 = j\omega\varepsilon\mathbf{E}_2 + \mathbf{J}_2$$

Recalling that $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}$

the quantity $\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1)$ can be expanded as:

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = \mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{J}_2 \cdot \mathbf{E}_1 + \mathbf{M}_2 \cdot \mathbf{H}_1 - \mathbf{M}_1 \cdot \mathbf{H}_2$$

Lorentz Reciprocity Theorem

UNIVERSITY OF BRESCIA

Integrating over the volume V and applying the divergence theorem we get:

$$\int_V \nabla \cdot \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) dv = \int_S \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) dS = \int_V \left(\mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{J}_2 \cdot \mathbf{E}_1 + \mathbf{M}_2 \cdot \mathbf{H}_1 - \mathbf{M}_1 \cdot \mathbf{H}_2\right) dv$$

<u>Case 1 – S encloses no sources</u> (J_1=J_2=M_1=M_2=0)

$$\int_S \left(\mathbf{E}_1 \times \mathbf{H}_2\right) dS = \int_S \left(\mathbf{E}_2 \times \mathbf{H}_1\right) dS$$

<u>Case 2 – S bounds a perfect conductor</u> (surface integral is 0)

$$\int_V \left(\mathbf{J}_2 \cdot \mathbf{E}_1 - \mathbf{M}_2 \cdot \mathbf{H}_1\right) dv = \int_V \left(\mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{M}_1 \cdot \mathbf{H}_2\right) dv$$

<u>Case 3 – S is a sphere at infinity</u> (fields can be considered plane waves therefore $\mathbf{H} = \hat{n} \times \frac{\mathbf{E}}{\eta}$)

$$\left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) \cdot \hat{n} = \left(\hat{n} \times \mathbf{E}_1\right) \cdot \mathbf{H}_2 - \left(\hat{n} \times \mathbf{E}_2\right) \cdot \mathbf{H}_1 = \frac{\mathbf{H}_1}{\eta} \cdot \mathbf{H}_2 - \frac{\mathbf{H}_2}{\eta} \cdot \mathbf{H}_1 = 0$$

$$\Downarrow$$

$$\int_V \left(\mathbf{J}_2 \cdot \mathbf{E}_1 - \mathbf{M}_2 \cdot \mathbf{H}_1\right) dv = \int_V \left(\mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{M}_1 \cdot \mathbf{H}_2\right) dv$$

Image Theory

In many problems a current source (electric or magnetic) is located in the vicinity of a conducting ground plane. Image theory permits the removal of the ground plane by placing a virtual image source on the other side of the ground plane.

The presence of a current density in z=d generates a plane wave traveling in the negative direction. Such plane wave hits the ground plane and is reflected, so that a plane wave traveling in the positive direction is formed. The fields in the two regions on the left and right of the ground plane can be written as:

$$\mathbf{E}_{x,s} = A\left(e^{jk_0 z} - e^{-jk_0 z}\right), \quad 0 < z < d$$

$$\mathbf{E}_{x,+} = Be^{-jk_0 z}, \quad z > d$$

$$\mathbf{H}_{y,s} = -\frac{A}{\eta_0}\left(e^{jk_0 z} - e^{-jk_0 z}\right), \quad 0 < z < d$$

$$\mathbf{H}_{y,+} = \frac{B}{\eta_0}e^{-jk_0 z}, \quad z>d$$

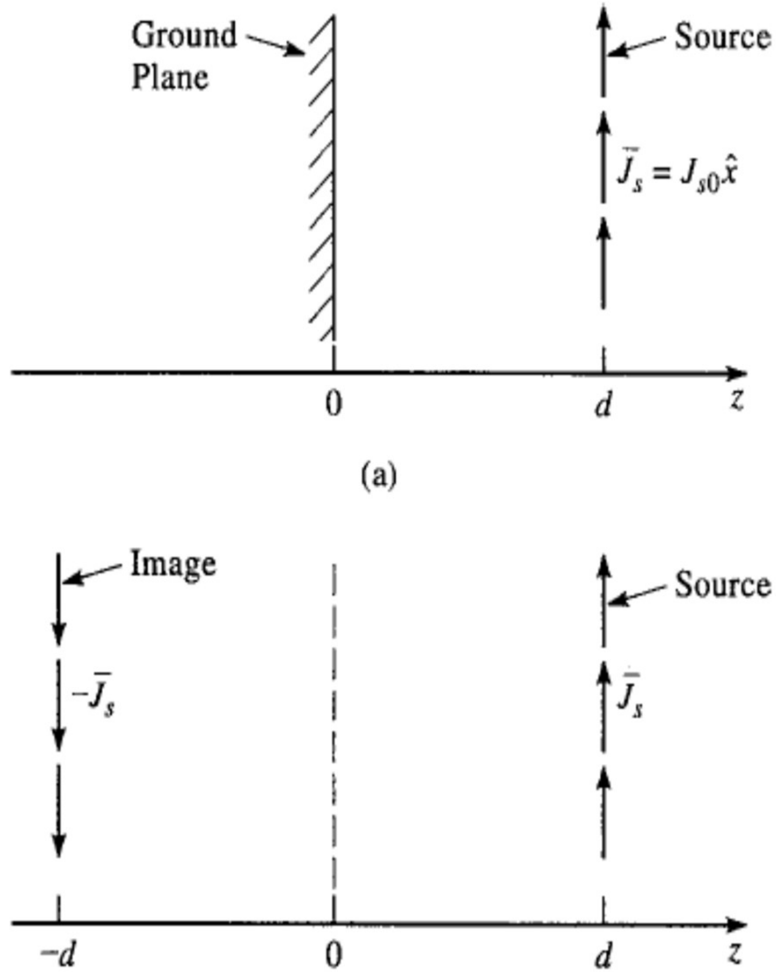


Image Theory

UNIVERSITY OF BRESCIA

Two boundary conditions have to be satisfied to evaluate A and B of the fields. Boundary conditions can be imposed for *z=0* and *z=d*. Fields have been constructed so that E_x=0 in z=0 while the tangential components of E and H have to be continuous at z=d and include the presence of the surface current:

$$E_x^s = E_x^+\Big|_{z=d}$$

$$\mathbf{J}_s = \hat{z} \times \hat{y}\left(H_y^+ + H_y^s\right)\Big|_{z=d}$$

$$\Downarrow$$

$$2jA\sin k_0 d = Be^{-jk_0 d}$$

$$J_{s0} = -\frac{B}{\eta_0} - e^{-jk_0 d} - \frac{2A}{\eta_0}\cos k_0 d$$

$$\Downarrow$$

$$A = -\frac{J_{s0}\eta_0}{2}e^{-jk_0 d}$$

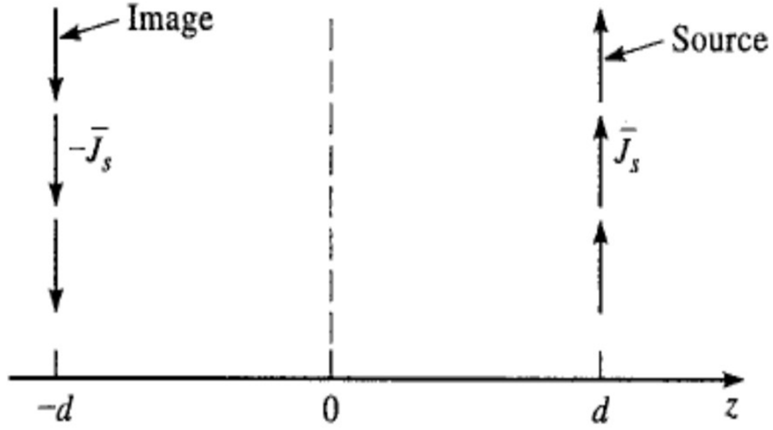
$$B = -jJ_{s0}\eta_0 \sin k_0 d$$

So fields can be re-written by replacing the explicit expressions for A and B

Image Theory

UNIVERSITY OF BRESCIA

The fields due to a source placed in z=-d can be calculated using the same procedure. In this case we will consider a current –$\boldsymbol{J}_s$.



Other image theory equivalences: