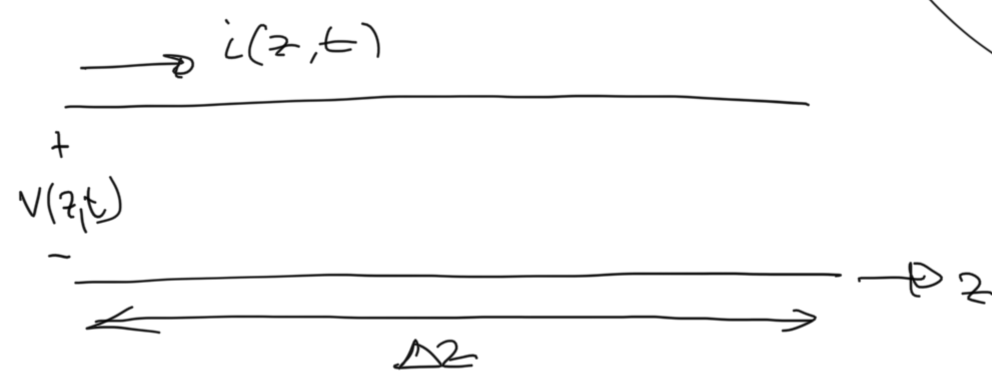
TRANSMISSION LINE THEORY

FIELD ANALYSIS

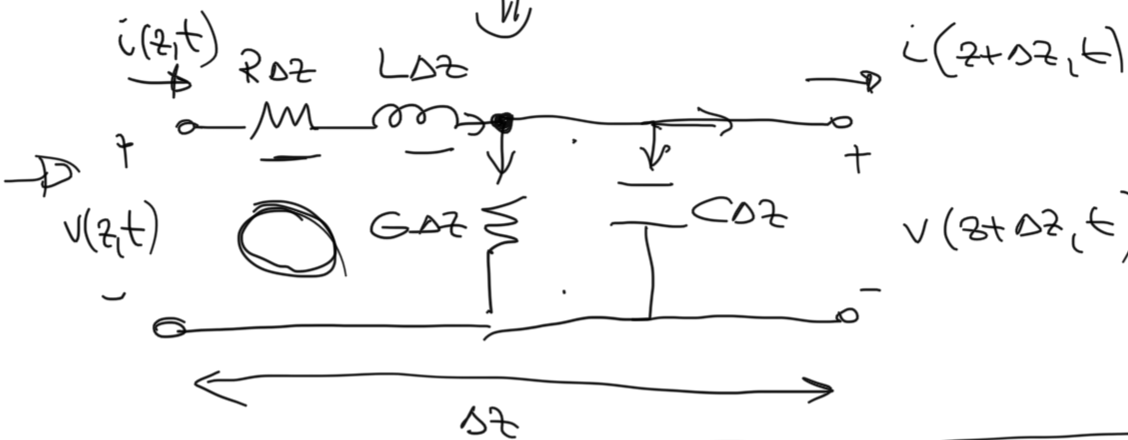
SIZES FRACTION OF WAVELENGTH

CIRCUIT MODEL

sizes << electrical wavelength



$\Updownarrow$



$R = \frac{\Omega}{m}$

$L = \frac{H}{m}$

$G = \frac{S}{m}$

$C = \frac{F}{m}$

$$\begin{cases} \dfrac{v(z,t)}{\Delta z} - R\cancel{\Delta z}\, i(z,t) - L\cancel{\Delta z}\dfrac{di(z,t)}{dt} - \dfrac{v(z+\Delta z,t)}{\Delta z} = 0 \\ \dfrac{i(z,t)}{\Delta z} - G\cancel{\Delta z}\, v(z+\Delta z,t) - C\cancel{\Delta z}\dfrac{\partial v(z+\Delta z,t)}{\partial t} - \dfrac{i(z+\Delta z,t)}{\Delta z} = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\dfrac{di(z,t)}{dt} \\ \dfrac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\dfrac{\partial v(z,t)}{\partial t} \end{cases}$$

TELEGRAPHER EQUATION

$$\frac{\partial}{\partial t} = j\omega$$

$$\begin{cases} \dfrac{dV(z)}{dz} = -(R + j\omega L)\, I(z) \\[2ex] \dfrac{dI(z)}{dz} = -(G + j\omega C)\, V(z) \end{cases}$$

PHASOR FORM

WAVE PROPAGATION:
APPY DERIVATIVE OVER Z

$$\frac{d^2V(z)}{dz^2} = -(R+j\omega L)\frac{dI(z)}{dz} \Rightarrow$$

$$\frac{d^2V(z)}{dz^2} = (R+j\omega L)(G+j\omega C)\,V(z)$$

$$\frac{d^2I(z)}{dz^2} = \underbrace{(R+j\omega L)(G+j\omega C)}\,I(z)$$

PROPAGATION CONSTANT

$$\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\begin{cases} \dfrac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \\[2ex] \dfrac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \end{cases}$$

$$\begin{cases} V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\ \boxed{I(z) = I_0^+ e^{\gamma z} + I_0^- e^{\gamma z}} \end{cases}$$ 1*

$$\frac{dV(z)}{dz} = -(R+j\omega L)\, I(z)$$

$$\frac{d}{dz}\left(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}\right) = -(R+j\omega L)\, I(z)$$

$$-\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} = -(R+j\omega L)\, I(z)$$

$$\boxed{I(z) = \frac{\gamma}{R+j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right)} \quad (1)$$

$$Z_0 = \frac{V}{I} = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R+j\omega L}{\gamma} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$\gamma \Rightarrow \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\rightarrow I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

$\gamma = \alpha + j\beta \implies$ WAVELENGTH : $\lambda = \frac{2\pi}{\beta}$

PHASE VELOCITY : $V_p = \frac{\omega}{\beta}$

LOSSLESS LINE $\quad G = R = 0$

$$\gamma = \sqrt{(\cancel{R}+j\omega L)(\cancel{G}+j\omega C)} = j\omega\sqrt{LC}$$

$$\underset{0}{} \qquad \underset{0}{}$$

$$\alpha = 0 \qquad \beta = \omega\sqrt{LC}$$

REAL NUMBER

$$Z_0 = \sqrt{\frac{\cancel{R}+j\omega L}{\ldots}} = \sqrt{\frac{L}{C}}$$

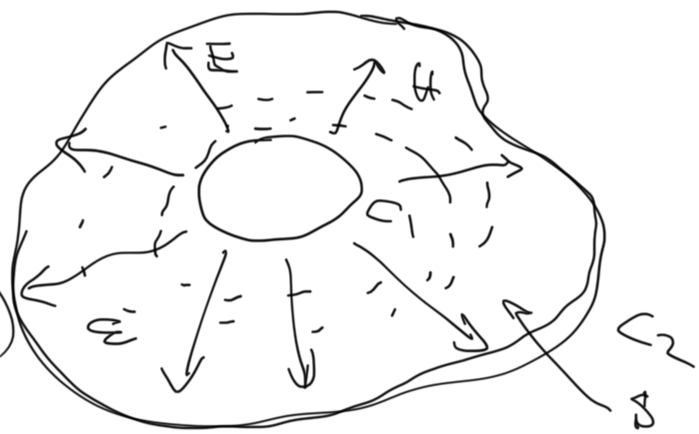
$$Z_0 = \sqrt{\frac{\quad}{\cancel{G}_{0} + j\omega C}} \qquad \sqrt{C}$$

WAVELENGTH $\Rightarrow$ $$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}}$$

PHASE VELOCITY $\Rightarrow$ $$V_p = \frac{\omega}{\beta} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{LC}} = \frac{1}{\sqrt{LC}}$$

FIELD ANALYSIS

$$\varepsilon = \varepsilon' - j\varepsilon''$$
$$= \varepsilon'(1 - j\tan\delta)$$



STORED MAGNETIC ENERGY

$$W_m = \frac{\mu}{4}\int_S H\cdot H^* \, dS \quad = \quad W_m = \frac{L|I_0|^2}{4}$$

$$L = \frac{\mu}{|I_0|^2}\int_S H\cdot H^* \, dS \quad \left[\frac{H}{m}\right]$$

STORED ELECTRIC ENERGY

$$W_e = \frac{\varepsilon}{4}\int_S E\cdot E^* \, dS$$

$$W_e = \frac{C|V_0|^2}{4}$$

$$\Downarrow$$

$$C = \frac{\varepsilon}{|V_0|^2} \int_S E \cdot E^* \, dS \quad \left[\frac{F}{m}\right]$$

POWER LOSS

$$P_C = \frac{R_S}{2} \int_{C_1 + C_2} H \cdot H^* \, dl$$

$$P_C = R \frac{|I_0|^2}{2}$$

$$\Downarrow$$

$$R = \frac{R_S}{|I_0|^2} \int_{C_1 + C_2} H \cdot H^* \, dl \quad \left[\frac{\Omega}{m}\right]$$

$$R_S = \frac{1}{\sigma \delta_S}$$ SURFACE RESISTANCE

POWER DISSIPATED IN DIELECTRIC FILLING THE LINE

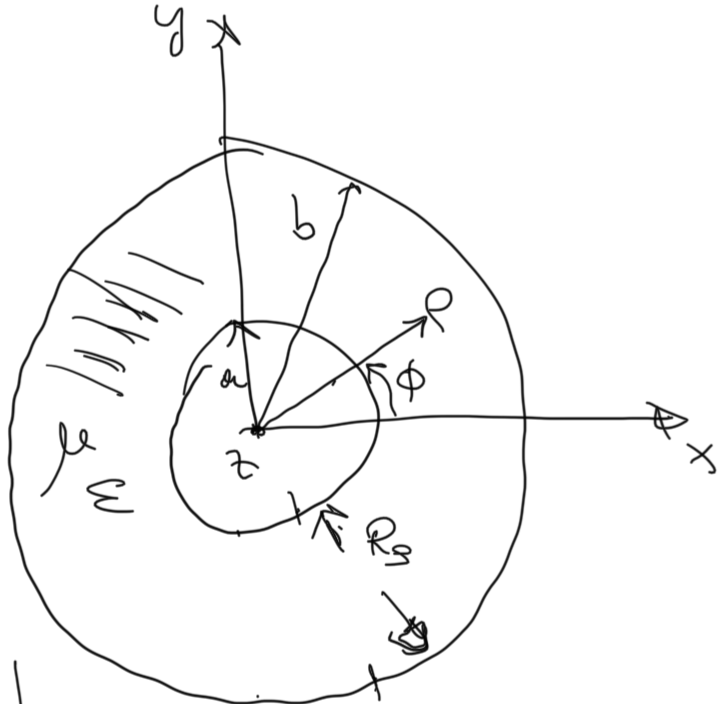
$$P_d = \frac{\omega \varepsilon''}{2} \int_S E \cdot E^* \, dS$$

$$P_d = G \frac{|V_0|^2}{2}$$

$$\Downarrow$$

$$G = \frac{\omega \varepsilon''}{|V_0|^2} \int_S E \cdot E^* \, dS \quad \left[\frac{S}{m}\right]$$

COAXIAL LINE



$$E_z = H_z = 0$$

$$\frac{\partial}{\partial \phi} = 0$$

$$\begin{cases} \nabla \times E = -j\omega\mu H \\ \nabla \times H = j\omega\varepsilon E \end{cases}$$

$$\varepsilon = \varepsilon' - j\varepsilon''$$

$$\nabla \times E \, / \, \cancel{\nabla \times H} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \cancel{\frac{\partial}{\partial \phi}}^{0} & \frac{\partial}{\partial z} \\ E_\rho / H_\rho & E_\phi / H_\phi & 0 \end{vmatrix}$$

$$\begin{cases} \cancel{-\hat{\rho}\frac{\partial E_\phi}{\partial z}} + \hat{\phi}\frac{\partial E_\rho}{\partial z} + \cancel{\hat{z}\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\phi)} = -j\omega\mu\left(\cancel{\hat{\rho}H_\rho} + \hat{\phi}H_\phi\right) \\ -\hat{\rho}\frac{\partial H_\phi}{\partial z} + \cancel{\hat{\phi}\frac{\partial H_\rho}{\partial z}} + \cancel{\hat{z}\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)} = j\omega\varepsilon\left(\hat{\rho}E_\rho + \cancel{\hat{\phi}E_\phi}\right) \end{cases}$$

$$\rho = a, b \Rightarrow E_\phi = 0$$

$$H_\rho = 0$$

$$\begin{cases} \dfrac{\partial E_\rho}{\partial z} = -j\omega\mu H_\phi \\ \dfrac{\partial H_\phi}{\partial z} = -j\omega\varepsilon E_\rho \end{cases} \qquad E_\rho = \frac{h(z)}{\rho} \qquad H_\phi = \frac{g(z)}{\rho}$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial z} h(z) = -j\omega\mu g(z) \\ \dfrac{\partial g(z)}{\partial z} = -j\omega\varepsilon h(z) \end{cases}$$

VOLTAGE

$$V(z) = \int_{\rho=a}^{b} E_\rho(\rho,z)\,d\rho = h(z)\int_{\rho=a}^{b}\frac{d\rho}{\rho} = \underline{h(z)}\ln\frac{b}{a}$$

CURRENT

$$I(z) = \int_{\phi=0}^{2\pi} H_\phi(a,z)\,a\,d\phi = \int_{\phi=0}^{2\pi}\frac{g(z)}{\cancel{a}}\cdot\cancel{a}\,d\phi = \underline{g(z)}\,2\pi$$

Teleg. eq. for coaxial line

$$\begin{cases} \dfrac{\partial V(z)}{\partial z} = -j\,\dfrac{\omega\mu\ln\frac{b}{a}}{2\pi}\,I(z) \\ \dfrac{\partial I(z)}{\partial z} = -j\omega\,(\underbrace{\varepsilon'-j\varepsilon''})\,2\pi\dfrac{V(z)}{\ln\frac{b}{a}} \end{cases}$$

ε a

$$\gamma^2 = \omega^2 \mu \varepsilon$$

Field Analysis

$$\begin{cases} E = \dfrac{V_0 \hat{\rho}}{\rho \ln \frac{b}{a}} e^{-\gamma z} \\ H = \dfrac{I_0 \hat{\phi}}{2\pi\rho} e^{-\gamma z} \end{cases}$$

① $$L = \frac{\mu}{|I_0|^2}\int_S H \cdot H^* = \frac{\mu}{(2\pi)^2}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b} \frac{1}{\rho^{\cancel{2}}}\cancel{\rho}\, d\rho\, d\phi =$$

$$= \frac{\mu}{2\pi} \ln\frac{b}{a} \left[\frac{H}{m}\right]$$

② $$C = \frac{\varepsilon}{|V_0|^2}\int_S E \cdot E^* \, dS = \frac{\varepsilon'}{\ln\left[\frac{b}{a}\right]^2}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b} \frac{1}{\rho^{\cancel{2}}}\cancel{\rho}\, d\rho\, d\phi =$$

$$= \frac{2\pi\varepsilon'}{\ln\frac{b}{a}} \left[\frac{F}{m}\right]$$

③ $$R = \frac{R_S}{|I_0|^2}\int_{C_1+C_2} H \cdot H^* \, dl = \frac{R_S}{(2\pi)^2}\left\{ \underbrace{\int_{\phi=0}^{2\pi} \frac{1}{a^2}\, a\, d\phi}_{C_1} + \underbrace{\int_{\phi=0}^{2\pi} \frac{1}{b^2}\, b\, d\phi}_{C_2} \right\}$$

$$= \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\left[\frac{\Omega}{m}\right]$$

$$④\ G = \frac{\omega\varepsilon''}{|V_0|^2}\int_S E \cdot E^* \, dS = \frac{\omega\varepsilon''}{\ln\left(\frac{b}{a}\right)^2}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b}\frac{1}{\rho^2}\rho \, d\rho \, d\phi =$$

$$= \frac{2\pi\omega\varepsilon''}{\ln\frac{b}{a}}\left[\frac{S}{m}\right]$$

$$\begin{cases} \dfrac{\partial E_\rho}{\partial z} = -j\omega\mu H_\phi \\ \dfrac{\partial H_\phi}{\partial z} = -j\omega\varepsilon E_\rho \end{cases}$$

$$\frac{\partial^2 E_\rho}{\partial z^2} = -\omega^2\mu\varepsilon E_\rho$$

$$\hookrightarrow \gamma^2 = -\omega^2\mu\varepsilon$$

$$\gamma = \alpha + j\beta$$

LOSSLESS MEDIA $\quad \beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{LC}$

TEM TRANSMISSION LINES

WAVE IMPEDANCE

$$Z = \frac{E_\rho}{\ldots} = \frac{\omega\mu}{\ldots} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$ MEDIUM IMPEDANCE

$$Z_w = \frac{\quad}{H_\phi} \quad \beta$$

LINE IMPEDANCE

$$Z_0 = \frac{V_0}{I_0} = \frac{E_\rho \ln\frac{b}{a}}{2\pi H_\phi} = \frac{\eta \ln\frac{b}{a}}{2\pi} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\ln\frac{b}{a}}{2\pi}$$

geometry dependent

$$\begin{cases} E_\rho = \dfrac{V_0}{\rho \ln\frac{b}{a}}\,\hat{\rho}\, e^{-j\beta z} \\ H_\phi = \dfrac{I_0}{2\pi\rho}\,\hat{\phi}\, e^{-j\beta z} \end{cases}$$

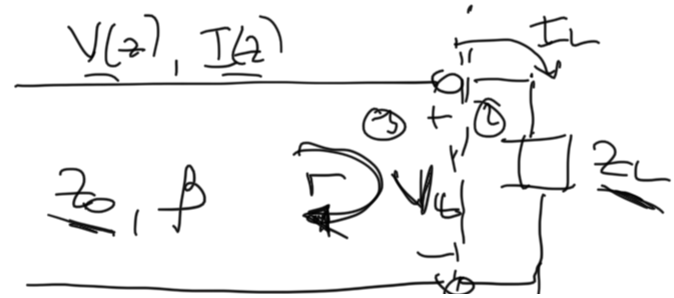
POWER FLOW

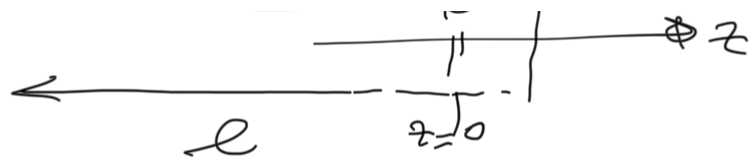
$$P = \frac{1}{2}\int_S E \times H^* \, dS =$$

$$= \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b} \frac{V_0 I_0^*}{2\pi\rho^2 \ln\frac{b}{a}}\,\rho\, d\rho\, d\phi =$$

$$= \frac{1}{2} V_0 I_0^* \frac{1}{\cancel{2\pi} \ln\frac{b}{a}} \cdot \cancel{2\pi} \cdot \cancel{\ln\frac{b}{a}} = \frac{1}{2} V_0 I_0^*$$

TERMINATED LOSSLESS TRANSMISSION LINE





$Z_L \neq Z_0$

$$\begin{cases} V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ I(z) = \dfrac{V_0^+}{Z_0} e^{-j\beta z} - \dfrac{V_0^-}{Z_0} e^{j\beta z} \end{cases}$$

At the load $(z=0)$: $\dfrac{V(0)}{I(0)} = Z_L$

$$Z_L = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0 \Rightarrow$$

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

Voltage reflection coefficient

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Rightarrow \begin{cases} V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma e^{j\beta z} \right] \\ I(z) = \dfrac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma e^{j\beta z} \right] \end{cases}$$

$$P_{av} = \frac{1}{2}\,\mathrm{Re}\left[V(z)\,I(z)^{*}\right] =$$

$$= \frac{1}{2}\,\frac{|V_0^+|^2}{Z_0}\,\mathrm{Re}\left\{1 \underbrace{- \Gamma^{*} e^{-2j\beta z} + \Gamma e^{2j\beta z}}_{\text{Purely imaginary}} - |\Gamma|^2\right\}$$

$$\underline{P_{av}} = \frac{1}{2}\,\frac{|V_0^+|^2}{Z_0}\left[1 - |\Gamma|^2\right]$$

Pav does not depend on z!

Maximum power transfer occurs $\Gamma = 0$

RETURN LOSS (RL) $= -20 \log|\Gamma|$ dB

$\Gamma = 0 \Rightarrow RL = \infty$

$\Gamma = 1 \Rightarrow RL = 0$

If $\Gamma = 0 \Rightarrow |V(z)| = V_0^+ \Rightarrow$ CONSTANT ALONG THE LINE

If $\Gamma \neq 0 \Rightarrow$ STANDING WAVE

$$|V(z)| = |V_0^+|\,\left|1 + \Gamma e^{2j\beta z}\right| = \qquad \ell = -z$$

$$= |V_0^+|\,\left|1 + \Gamma e^{-2j\beta \ell}\right| =$$

$$= |V_0^+|\,\left|1 + |\Gamma| e^{j(\theta - 2\beta\ell)}\right| \qquad \Gamma = |\Gamma| e^{j\theta}$$

Phase of the reflection coefficient

$$\begin{cases} V_{max} = |V_0^+|\,(1+|\Gamma|) \\ V_{min} = |V_0^+|\,(1-|\Gamma|) \end{cases}$$

$$\text{STANDING WAVE RATIO} = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$$\underbrace{1}_{\text{matched}} < SWR < \underbrace{\infty}_{\text{mismatched}}$$

DISTANCE BETWEEN TWO VOLTAGE MAXIMA

$$2\beta\ell = 2\pi \implies \ell = \frac{\cancel{2}\pi}{\cancel{2}\beta} = \frac{\pi\lambda}{2\pi} = \frac{\lambda}{2}$$

DISTANCE BETWEEN MAX and MIN.

$$2\beta\ell = \pi \implies \ell = \frac{\lambda}{4}$$

$$\Gamma(-\ell) = \frac{V_0^- e^{-j\beta\ell}}{V_0^+ e^{j\beta\ell}} = \Gamma(0)\, e^{-2j\beta\ell}$$

$$\Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_{IN} = Z(-\ell) = \frac{V(-\ell)}{I(-\ell)} = \frac{\cancel{V_0^+}\left[e^{j\beta\ell} + \Gamma e^{-j\beta\ell}\right]}{\cancel{V_0^+}\left[e^{j\beta\ell} - \Gamma e^{-j\beta\ell}\right]} Z_0 =$$

$$= Z_0 \frac{1 + \Gamma e^{-2j\beta\ell}}{\ldots}$$

$$1 - \Gamma e^{-2j\beta\ell}$$

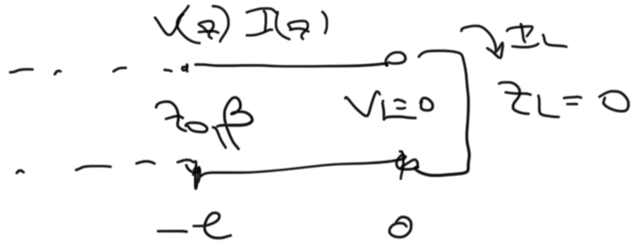
$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\boxed{Z_{IN} = Z_0 \frac{Z_L + jZ_0 \tan \beta\ell}{Z_0 + jZ_L \tan \beta\ell}}$$

TRANSMISSION LINE IMPEDANCE

SPECIAL CASES

(1) $\boxed{Z_L = 0}$ SHORT CIRCUIT TERMINATION

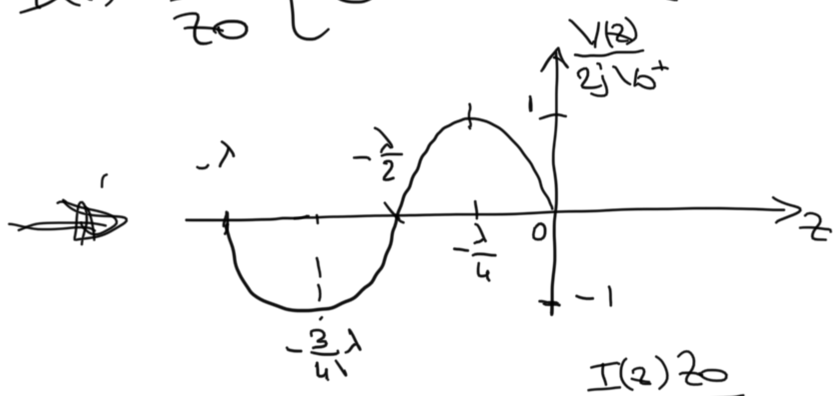


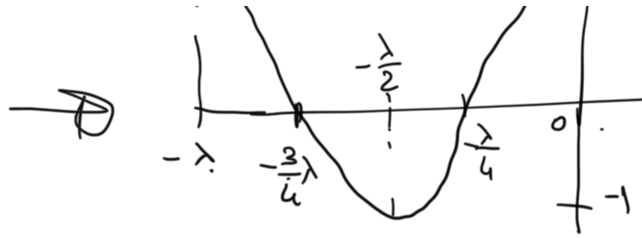
$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -1$$

$$SWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \infty$$

$$V(z) = V_0^+ \left[e^{-j\beta z} - e^{j\beta z}\right] = -2j\, V_0^+ \sin \beta z$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} + e^{j\beta z}\right] = 2\frac{V_0^+}{Z_0} \cos \beta z$$

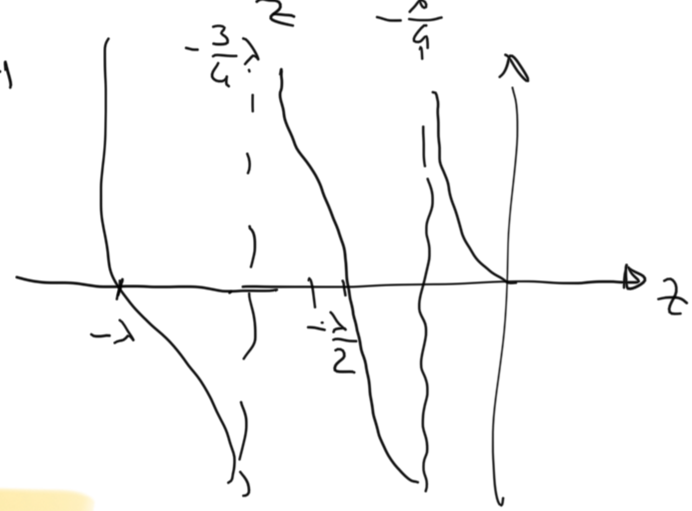




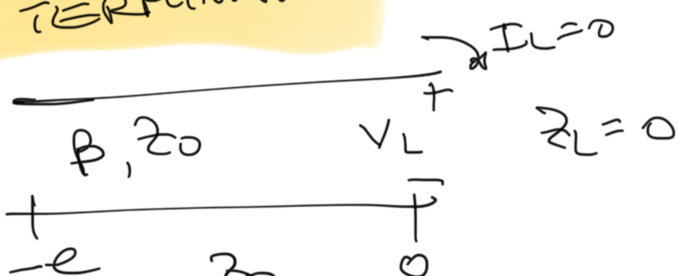
$Z_{IN} = jZ_0 \tan\beta\ell$

PURELY IMAGINARY

$-\infty < Z_{IN} \leq \infty$



② $Z_L = \infty$ OPEN CIRCUIT TERMINATION



$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1 - \frac{Z_0}{Z_L}}{1 + \frac{Z_0}{Z_L}} = 1$$

$$SWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \infty$$

$$V(z) = 2V_0^+ \cos\beta z$$

$$I(z) = -\frac{2jV_0^+}{Z_0} \sin\beta z$$

$$Z_{IN} = -jZ_0 \cot\beta\ell$$ purely imaginary

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③ $\ell = \lambda$

$\frac{\quad}{2}$

$$\beta l = \frac{\cancel{2\pi}}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{\cancel{2}} = \pi \qquad Z_{IN} = Z_L$$

$\tan \pi = 0$

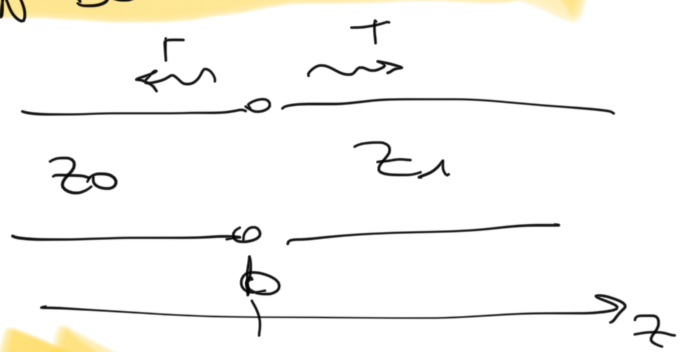
(4) $l = \frac{\lambda}{4} + n\frac{\lambda}{2}$

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2} \qquad \tan \beta l = \infty$$

$$Z_{IN} = \frac{Z_0^2}{Z_L}$$

QUARTER WAVE TRANSFORMER

(5) JUNCTION BETWEEN TWO LINES



$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

$$T = 1 + \Gamma = 1 + \frac{Z_1 - Z_0}{Z_1 + Z_0} = \frac{2Z_1}{Z_1 + Z_0}$$

INSERTION LOSS $\quad IL = -20 \log |T| \quad$ dB