

Photonics

aa 2021/2022

Prof. Maria Antonietta Vincenti
Università degli Studi di Brescia



Electromagnetic Optics: Poynting Theorem and material properties



Poynting Theorem

“The power flowing out of a given volume V is equal to the time rate of the decrease in the energy stored within V minus the ohmic losses”

Let's start with time-harmonic
Maxwell's equation for a linear,
stationary, isotropic medium:

$$\begin{aligned}\mathbf{B} &= \mu \mathbf{H} = (\mu' - j\mu'') \mathbf{H} \\ \mathbf{D} &= \varepsilon \mathbf{E} = (\varepsilon' - j\varepsilon'') \mathbf{E}\end{aligned}$$

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} = -j\omega \mu \mathbf{H} \longrightarrow 1. \text{ Scalar product of this equation by } \mathbf{H}^*$$

$$\nabla \times \mathbf{H}^* = -j\omega \varepsilon^* \mathbf{E}^* + \mathbf{J}_S^* \longrightarrow 2. \text{ Scalar product of this equation by } \mathbf{E}$$

Trick: c.c. of
the Ampere's
law

3. Subtract

$$\mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^* = -j\omega(\mu' - j\mu'') |\mathbf{H}|^2 + j\omega(\varepsilon' + j\varepsilon'') |\mathbf{E}|^2 - \mathbf{E} \cdot \mathbf{J}_S^*$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \nabla \cdot \mathbf{S}$$



Poynting Theorem

$$\mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^* = -j\omega(\mu' - j\mu'')|\mathbf{H}|^2 + j\omega(\varepsilon' + j\varepsilon'')|\mathbf{E}|^2 - \mathbf{E} \cdot \mathbf{J}_S^*$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega\mu'|\mathbf{H}|^2 - \omega\mu''|\mathbf{H}|^2 + j\omega\varepsilon'|\mathbf{E}|^2 - \omega\varepsilon''|\mathbf{E}|^2 - \mathbf{E} \cdot \mathbf{J}_S^*$$

Integrate over the volume and divide by 2.

Then apply the divergence theorem

$$\iiint_V \nabla \cdot \mathbf{F} dV = \oiint_S (\mathbf{F} \cdot \hat{n}) dS$$

$$-\frac{1}{2} \iiint_V \mathbf{E} \cdot \mathbf{J}_S^* dV =$$

$$\oiint_S \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \cdot \hat{n} dS + \frac{1}{2} \omega \iiint_V (\varepsilon''|\mathbf{E}|^2 + \mu''|\mathbf{H}|^2) dV - j\omega \iiint_V \frac{1}{2} (\varepsilon'|\mathbf{E}|^2 - \mu'|\mathbf{H}|^2) dV$$

The phasor of the **Poynting vector** is defined as
flowing in the direction of \mathbf{k}

$$\mathbf{S} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \text{ [W/m}^2\text{]} = \text{Power per unit surface}$$



Interpretation of the Poynting Theorem

$$-\frac{1}{2} \iiint_V \mathbf{E} \cdot \mathbf{J}_S^* dV = \oiint_S \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \cdot \hat{n} dS + \frac{1}{2} \omega \iiint_V \left(\varepsilon'' |\mathbf{E}|^2 + \mu'' |\mathbf{H}|^2 \right) dV - j\omega \iiint_V \frac{1}{2} \left(\varepsilon' |\mathbf{E}|^2 - \mu' |\mathbf{H}|^2 \right) dV$$

Power provided
by the source

Power flowing
through

Power loss
through Joule
heating

Reactive power
stored by the
e.m. field

$$P_S = P_F + P_L + P_R$$



Light Intensity

$$P_F = \oiint_S \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \cdot \hat{n} dS = \oiint_S \mathbf{S} \cdot \hat{n} dS$$

Power flow through surface S

The light intensity at any point is the real part of the Poynting vector

$$I = \text{Re}(\mathbf{S}) = \frac{1}{2} \text{Re}(\mathbf{E} \times \mathbf{H}^*)$$

For a plane wave propagating in a homogeneous isotropic medium, the Poynting vector has the same direction of the wavevector \mathbf{k} . In addition, since the relation between the electric field and magnetic field amplitudes is

$$E / H = Z = Z_0 \sqrt{\mu_r / \epsilon_r}$$

then the light intensity can be written as

$$I = |\mathbf{S}| = \frac{1}{2} |\mathbf{E} \times \mathbf{H}^*| = \frac{1}{2} \frac{E^2}{Z} = \frac{1}{2} \frac{E^2}{Z_0} \sqrt{\frac{\epsilon_r}{\mu_r}}$$

Non-magnetic material

$$\mu_r = 1$$

LIGHT INTENSITY

$$I = \frac{1}{2} \epsilon_0 c n E^2$$



Constitutive Relations

In a generic medium we can write the constitutive relations as:

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

Where:

$\varepsilon = \varepsilon_r \varepsilon_0$ [F/m] is the permittivity

$\mu = \mu_r \mu_0$ [H/m] is the permeability

σ [S/m] is the electric conductivity



Polarization and Susceptibility

The constitutive relations can be written as a function of the electric and magnetic polarization densities \mathbf{P} and \mathbf{M} as follows:

$$\begin{aligned}\mathbf{D} &= \varepsilon \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}_e \\ \mathbf{B} &= \mu \mathbf{H} = \mu_0 \mu_r \mathbf{H} = \mu_0 \mathbf{H} + \mathbf{P}_m\end{aligned}$$

Polarization densities are related to the electric and magnetic fields by the electric and magnetic susceptibilities as follows:

$$\mathbf{P}_e = \varepsilon_0 \chi \mathbf{E} \quad \longrightarrow \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}_e = \varepsilon_0 (1 + \chi) \mathbf{E} \quad \text{therefore} \quad \varepsilon_r = 1 + \chi$$

For symmetry:

$$\mathbf{P}_m = \mu_0 \chi_m \mathbf{H} \quad \longrightarrow \quad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{P}_m = \mu_0 (1 + \chi_m) \mathbf{H} \quad \text{therefore} \quad \mu_r = 1 + \chi_m$$

Note that in general we will deal with materials with no magnetic response ($\mathbf{P}_m=0$)



Materials Properties

Materials can be:

- ***linear/nonlinear***
response is independent/dependent on field strength
- ***homogeneous/inhomogeneous***
response is independent/dependent on position within the medium
- ***isotropic/anisotropic***
response is independent/dependent on direction of propagation
- ***stationary/non-stationary***
response is independent/dependent on time
- ***dispersive/non-dispersive***
response is dependent/independent on time and space frequency



linear/nonlinear

medium response as a function of field strength

Any medium is nonlinear

The **general constitutive relation for a nonlinear medium** derives from the following expression of the polarization density:

$$P = \varepsilon_0 \chi^{(1)} E + \varepsilon_0 \chi^{(2)} E^2 + \varepsilon_0 \chi^{(3)} E^3 + \dots + \varepsilon_0 \chi^{(n)} E^n$$

Where $\chi^{(1)}$ is the linear susceptibility tensor, and $\chi^{(2)}$, $\chi^{(3)}$, $\chi^{(n)}$ are the second, third and n-th order nonlinear susceptibilities of the medium.

The nonlinear susceptibilities are usually very small, so in many cases we work in the so called “linear” regime where:

$$\chi^{(2)} E \ll \chi^{(1)}, \quad \chi^{(3)} E^2 \ll \chi^{(1)}, \dots, \quad \chi^{(n)} E^{n-1} \ll \chi^{(1)}$$

Therefore, we can assume:

$$P \approx \varepsilon_0 \chi^{(1)} E$$



Materials Properties

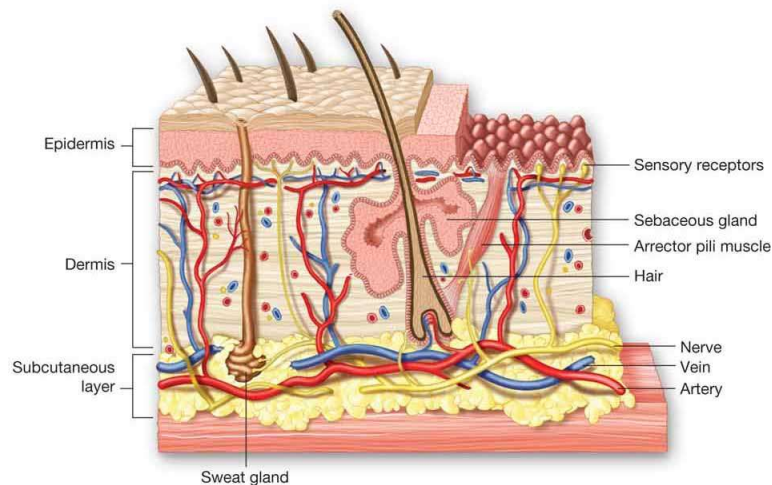
homogeneous/inhomogeneous

response as a function of position within the medium

There is no material in nature that is in fact homogeneous, because any structure is finite in space.

However, we can consider a complicated structure made by many different materials that are distributed in space (hence inhomogeneous) as a collection of homogeneous media and apply a piecewise constant approximation. Moreover, when the features composing the materials are sufficiently small, we can use some simple techniques to simplify this condition and make the material virtually homogeneous.

Human skin



Model of human skin





Materials Properties

isotropic/anisotropic

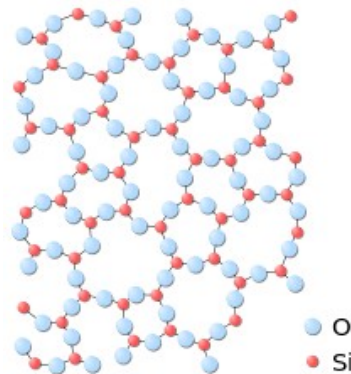
response as a function of direction of propagation

Isotropic

$$\mathbf{D} = \epsilon \mathbf{E}$$

ϵ is a scalar

Glass



Anisotropic

$$\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E}$$

ϵ is a tensor

Liquid crystal



$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- Gas, liquids and amorphous solids are isotropic because molecules are oriented randomly in space, so macroscopically, the material behaves isotropically.
- On the other hand, if the molecules/particles have a preferred orientation, then the material shows anisotropy.



Materials Properties

stationary/non-stationary

response as a function of time

- In a stationary medium, the optical properties (permittivity, permeability, conductivity) do not change in time.
- A stationary medium is homogeneous in time

Non-stationary

$$\mathbf{D}(t) = \varepsilon(t)\mathbf{E}(t)$$

Stationary

$$\mathbf{D}(t) = \varepsilon\mathbf{E}(t)$$

Examples of non-stationary media may be fluids, objects in movement, optically/electrically/thermally excited materials



Materials Properties

dispersive/non-dispersive

response is dependent/independent on time and space frequency

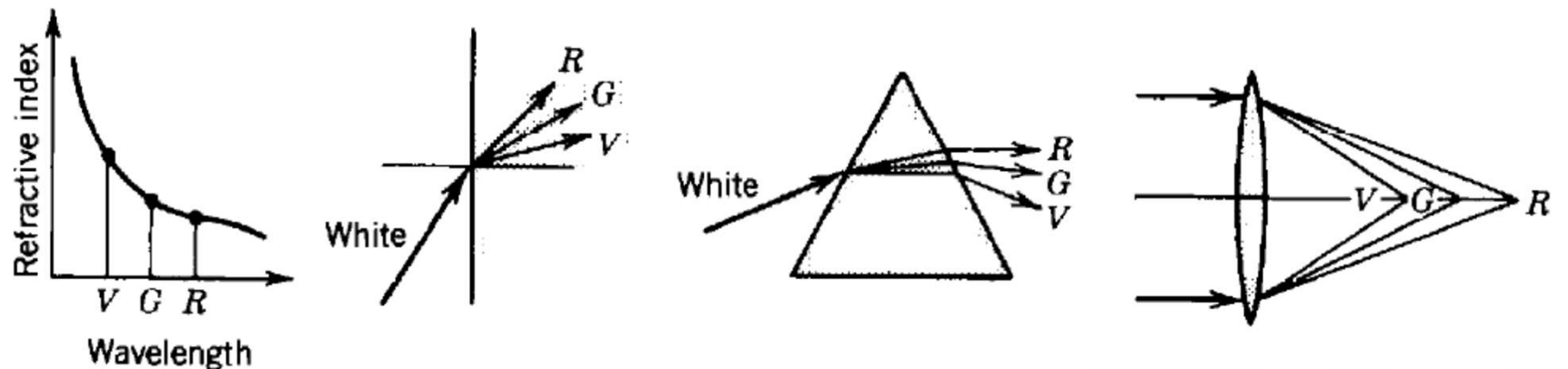
$$\mathbf{D}(\mathbf{r}, \omega) = \varepsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)$$

- Complex, frequency-domain permittivity is $\varepsilon = \varepsilon' - j\varepsilon''$. Similar expression may be written for μ and σ . The optical properties change with frequency. This is a manifestation of temporal dispersion.
- Why the permittivity $\varepsilon(\mathbf{r}, \omega)$, as well as the permeability and the conductivity, are functions of frequency? Matter is made of molecules, atoms and electrons. These entities interact with electromagnetic waves (light or RF waves) and may be resonant at certain frequencies. Near these resonances the complex (frequency-domain) permittivity $\varepsilon = \varepsilon' - j\varepsilon''$ undergoes a strong modulation with a peak of ε'' .

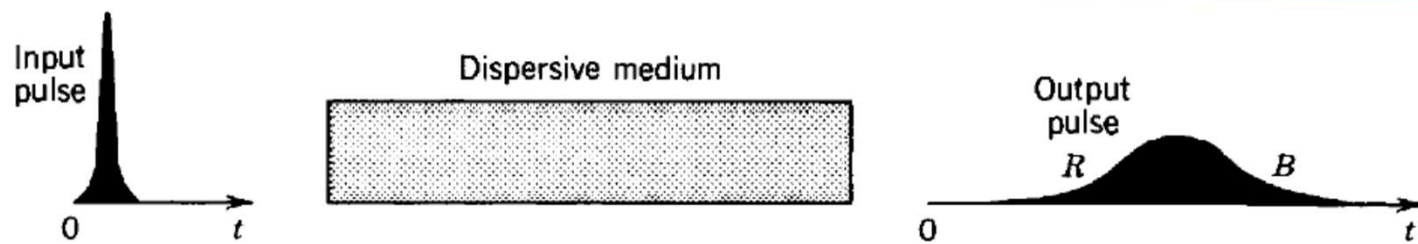


Materials Properties

Dispersion is extremely important in optical components since **light refracts at different angles depending on the incident wavelength**. Prisms and lenses will therefore refract light into a range of directions.



Moreover, since speed of light in a medium also depends on the refractive index, each of the **frequency components in a short pulse will experience a different time delay**. This phenomenon is more evident the longer is the propagation distance in the medium.





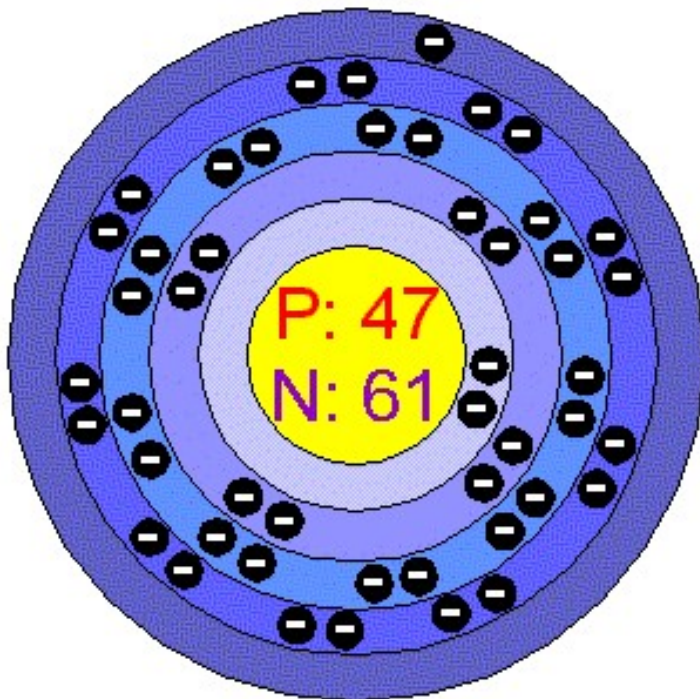
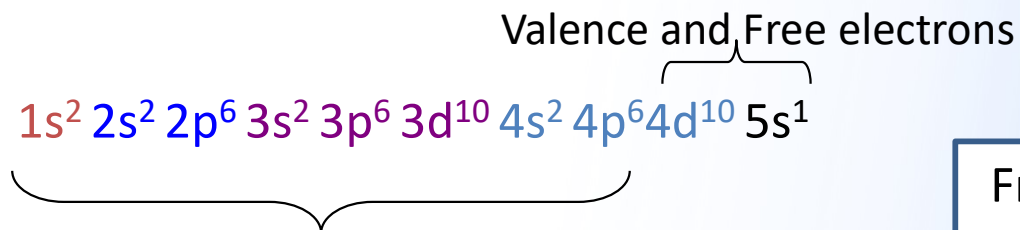
Materials Properties

But what is the physical origin of dispersion in materials?

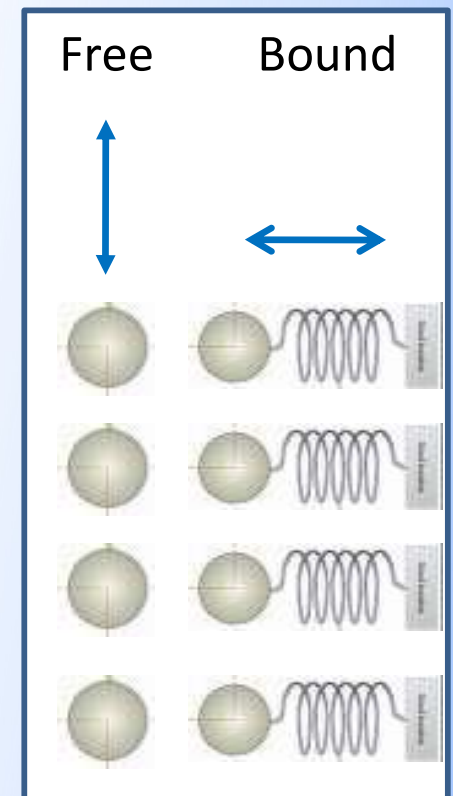
There are two kind of electrons in atoms:

FREE ELECTRONS and **BOUND ELECTRONS**

Let's take a look for example at the electronic configuration of a silver atom:



- ~1 free electron per atom
- up to 10 valence band electrons available. 4d orbital contributes to conductivity/dielectric constant at wavelengths in the visible range





Materials Dispersion: Non-conducting media

In an isotropic non-conducting medium, such as dielectrics or semiconductors, the electrons are **BOUND** to the atoms.

Like every other mass, electrons move under the action of forces. Those forces are responsible for their movement as described in Newton's second law:

$$\mathbf{F} = m\mathbf{a}$$

In particular the forces that an electron feels are:

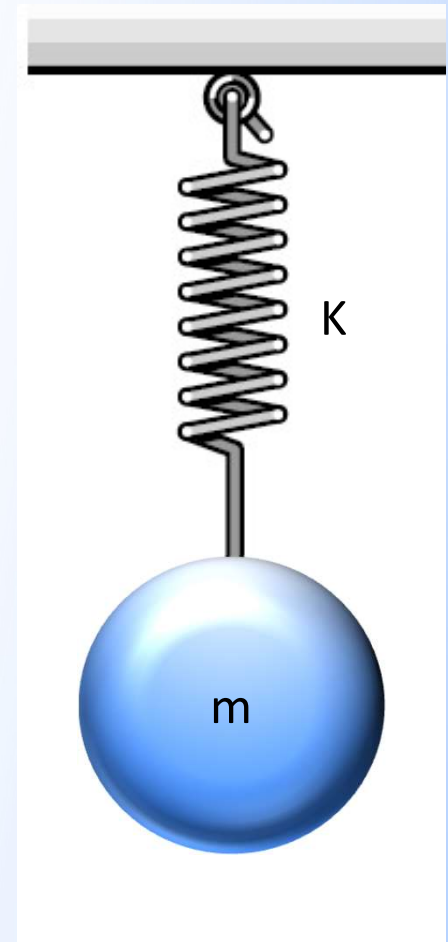
- The Coulomb force $-e\mathbf{E}$
- The Lorentz Force $-e\mathbf{v} \times \mathbf{B}$

Therefore we can write:

$$m\mathbf{a} = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B}$$

To describe the motion of the electron properly we should consider the equation of a damped harmonic oscillator, that reads:

$$m \frac{d^2 \mathbf{r}}{dt^2} + \underbrace{m\gamma \frac{d\mathbf{r}}{dt}}_{\text{damping}} + \underbrace{K\mathbf{r}}_{\text{Hooke's law}} = -e\mathbf{E} - \cancel{e\mathbf{v} \times \mathbf{B}}$$





Materials Dispersion: Non-conducting media

$$m \frac{d^2 \mathbf{r}}{dt^2} + m\gamma \frac{d\mathbf{r}}{dt} + K\mathbf{r} = -e\mathbf{E}$$

Dividing by m we get:

$$\frac{d^2 \mathbf{r}}{dt^2} + \gamma \frac{d\mathbf{r}}{dt} + \omega_0^2 \mathbf{r} = -\frac{e}{m} \mathbf{E}, \quad \omega_0 = \sqrt{\frac{K}{m}} \quad \text{Effective resonance frequency}$$

The dipole moment is: $\mathbf{p} = -e\mathbf{r}$

Since the macroscopic polarization of a medium can be written as $\mathbf{P} = -N e \mathbf{r}$, where N is the total number of electrons per unit volume. If we multiply the equation by Ne we can write:

$$\frac{d^2 \mathbf{P}}{dt^2} + \gamma \frac{d\mathbf{P}}{dt} + \omega_0^2 \mathbf{P} = \varepsilon_0 \omega_p^2 \mathbf{E}, \quad \varepsilon_0 \omega_p^2 = \frac{Ne^2}{m}$$



Materials Dispersion: Non-conducting media

Let's now assume our electric field varies harmonically with time with the usual factor $e^{j\omega t}$. Assuming the electron has the same harmonic dependence we can write:

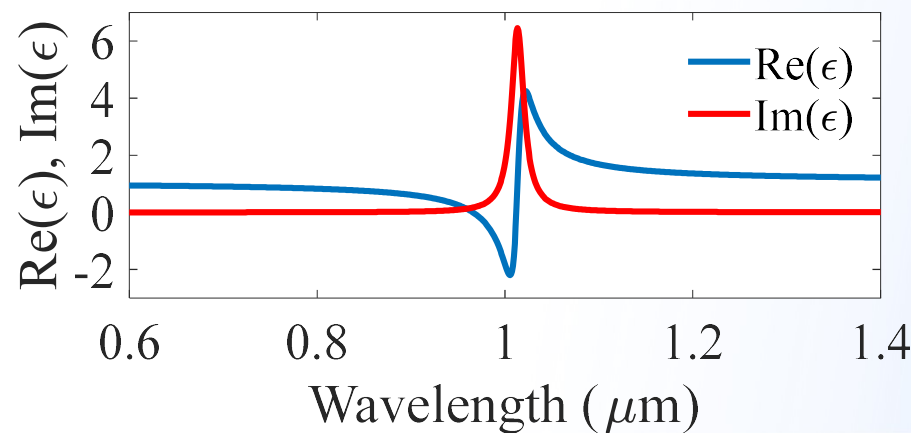
$$-\omega^2 \mathbf{P} + j\gamma\omega \mathbf{P} + \omega_0^2 \mathbf{P} = \varepsilon_0 \omega_p^2 \mathbf{E} \qquad \mathbf{P} = -\varepsilon_0 \frac{\omega_p^2}{\omega^2 - \omega_0^2 - j\gamma\omega} \mathbf{E}$$



$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - j\gamma\omega}$$

Lorentz oscillator

This equation implies that we should find a resonance in proximity of the frequency ω_0 . This resonance phenomenon translates into a large change in the refractive index of the medium and strong light absorption at or near the resonance frequency.



NOTE: In general the permittivity of a medium can be described with a superposition of Lorentz oscillators



Materials Dispersion: Conducting media

A similar analysis can be conducted for **FREE** electrons, or conduction electrons. Those electrons populate the outer shells of the atom and they also feel the action of the applied electric field. However, FREE electrons do not undergo the elastic restoring force we saw for the BOUND electrons. Therefore, the equation of motion for these electrons can be written as:

$$m \frac{d^2 \mathbf{r}}{dt^2} + \frac{m}{\tau} \frac{d\mathbf{r}}{dt} = -e\mathbf{E}$$

The friction constant τ ($1/\gamma$) can be related to the static conductivity if we write:

$$\mathbf{J} = -Ne\mathbf{v}$$

So that the equation of motion becomes:

$$\frac{d\mathbf{J}}{dt} + \frac{1}{\tau} \mathbf{J} = \frac{Ne^2}{m} \mathbf{E}$$



Materials Dispersion: Conducting media

The decay of a transient current is governed by the associated homogeneous equation:

$$\frac{d\mathbf{J}}{dt} + \frac{1}{\tau} \mathbf{J} = 0$$

Whose solution is $\mathbf{J} = \mathbf{J}_0 e^{-t/\tau}$. Thus a transient current will decay to 1/e of its initial value in a time τ (relaxation time). For a static electric field we have:

$$\frac{1}{\tau} \mathbf{J} = -\frac{Ne^2}{m} \mathbf{E}$$



$$\sigma = \frac{Ne^2}{m} \tau$$



Materials Dispersion: Conducting media

Let's now assume our electric field varies harmonically with time with the usual factor $e^{j\omega t}$. For simplicity we can also go back to the polarization equation and write:

$$\frac{d^2 \mathbf{P}}{dt^2} + \gamma \frac{d\mathbf{P}}{dt} = \varepsilon_0 \omega_p^2 \mathbf{E}, \quad \varepsilon_0 \omega_p^2 = \frac{Ne^2}{m}, \quad \gamma = \frac{1}{\tau}$$



$$-\omega^2 \mathbf{P} + j\gamma\omega \mathbf{P} = \varepsilon_0 \omega_p^2 \mathbf{E}$$

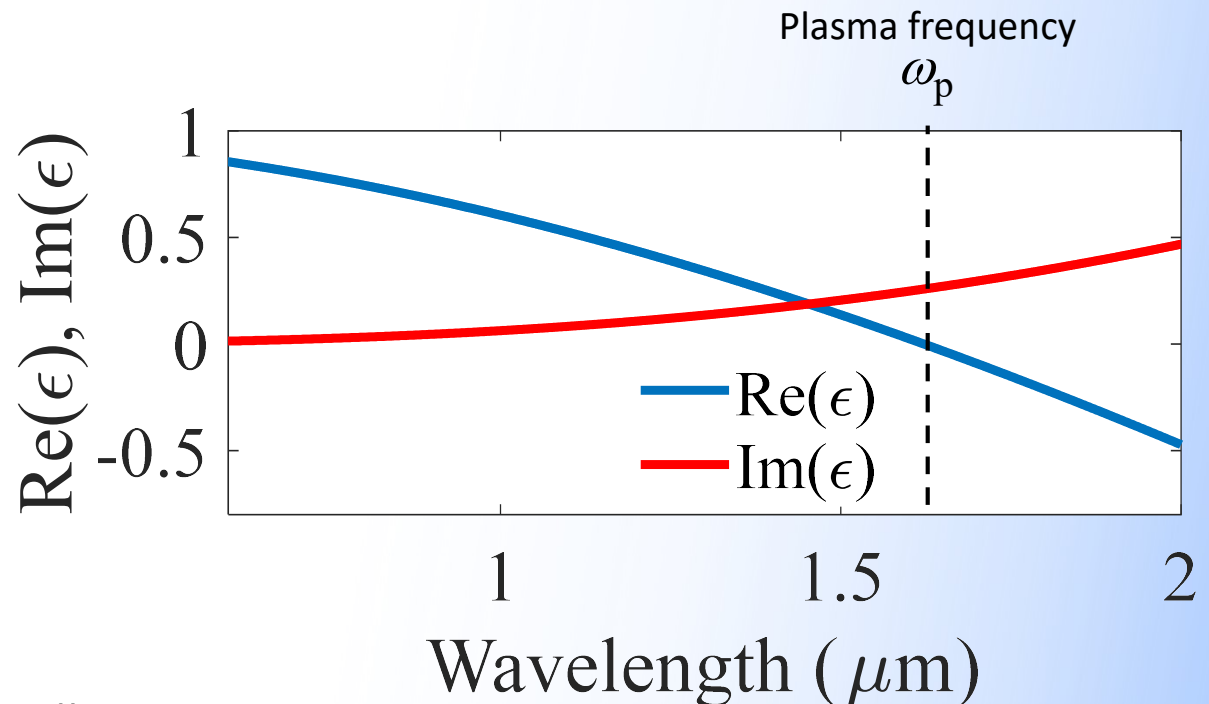


$$\mathbf{P} = -\varepsilon_0 \frac{\omega_p^2}{\omega^2 - j\gamma\omega} \mathbf{E}$$



$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 - j\gamma\omega}$$

Drude oscillator



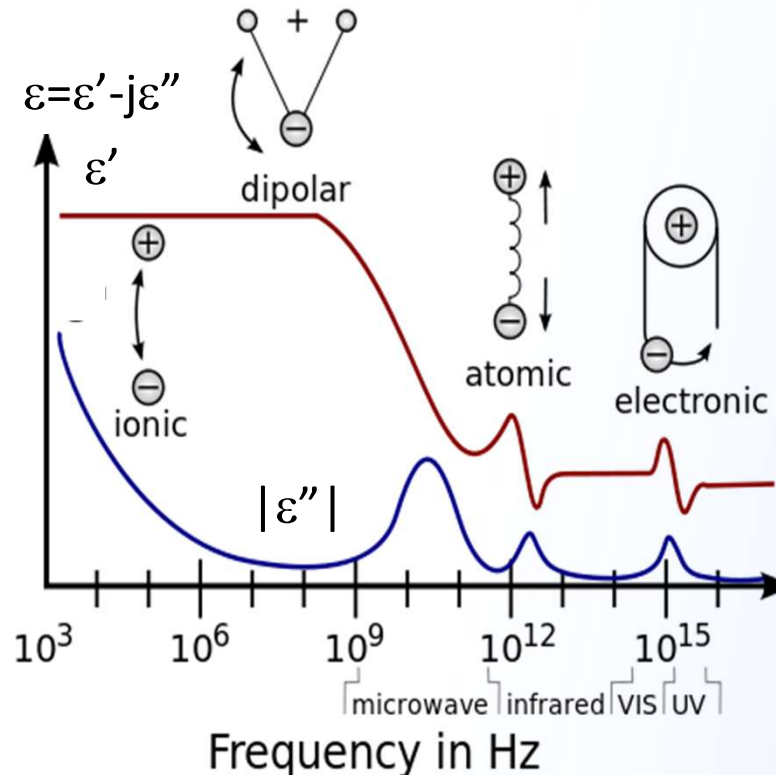


Materials Dispersion

IMPORTANT: The majority of material are not perfect conductors or perfect insulators, therefore they should be modeled as a superposition of oscillators that take into account the contributions of both FREE and BOUND electrons. In general the permittivity of a medium can be written as:

$$\varepsilon = 1 - \frac{\omega_{pD}^2}{\omega^2 - j\gamma_D\omega} - \sum_i \frac{\omega_{pi}^2}{\omega^2 - \omega_{0i}^2 - j\gamma_i\omega}$$

Dispersion of water permittivity
 $\varepsilon = \varepsilon' - j\varepsilon''$





The Sellmeier Equation

In a medium with multiple resonances, permittivity and, therefore, susceptibility, is the sum of a number of terms. We know that the relation between the permittivity, refractive index and susceptibility is:

$$\varepsilon_r = n^2 = 1 + \chi$$

For most optically transparent materials working far from their absorption resonances the relation between refractive index and wavelength assumes the following form:

$$n^2 \approx 1 + \sum_i \chi_{0i} \frac{\lambda^2}{\lambda^2 - \lambda_i^2}$$



Wave velocities: non-dispersive media

In a non-dispersive media wave velocity corresponds to the phase velocity.

The **phase velocity** is the velocity at which the phase of any frequency component of the wave travels. In other words is the ratio between the space travelled by a plane of the wave over the time it takes to travel that space. If we consider the temporal solution of the wave equation assuming the wave propagates only in the z direction:

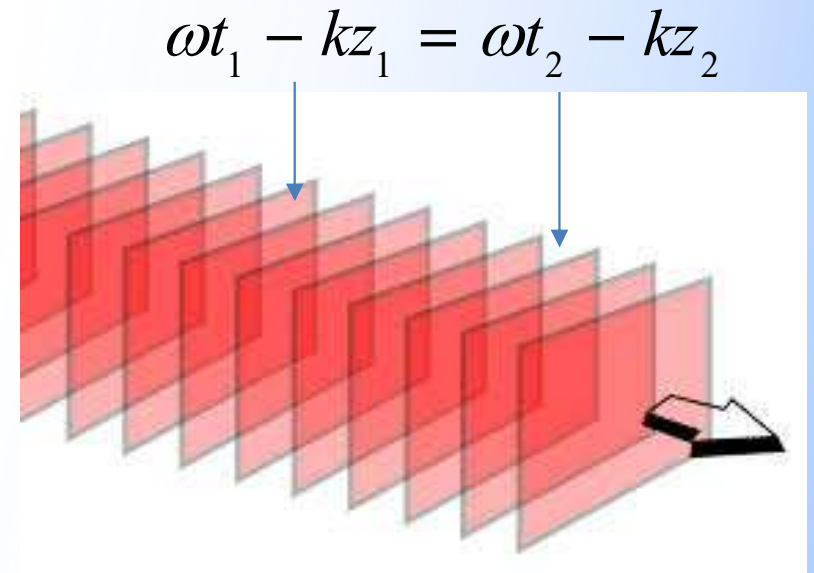
$$E(t) = E_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r})$$

The surface of constant phase are planes:

$$\omega t - \mathbf{k} \cdot \mathbf{r} = \text{constant}$$

Let's assume \mathbf{k} is in the z direction, then we can write the velocity as

$$v_p = \Delta z / \Delta t = \omega / k = 1 / \sqrt{\mu \epsilon} = c / n$$





Wave velocities: Dispersive media

What happens if a wave packet, i.e., a collection of waves with different frequencies travels through a dispersive medium?

We have already seen that each component of the pulse will travel at different speed. If the dispersion is relatively weak, we can define the group velocity as:

$$v_g = \Delta\omega / \Delta k$$

For a medium with dispersive index $n(\omega)$, the group velocity is:

$$v_g = \partial(ck/n) / \partial k = c/n - ck/n^2 \partial n / \partial k = v_p (1 - \frac{k}{n} \frac{\partial n}{\partial k})$$

This expression can be also expressed as a function of wavelength as:

$$v_g = \frac{c}{N} \quad N = n - \lambda_0 \frac{\partial n}{\partial \lambda_0}$$

Where N is referred to as **group index**.



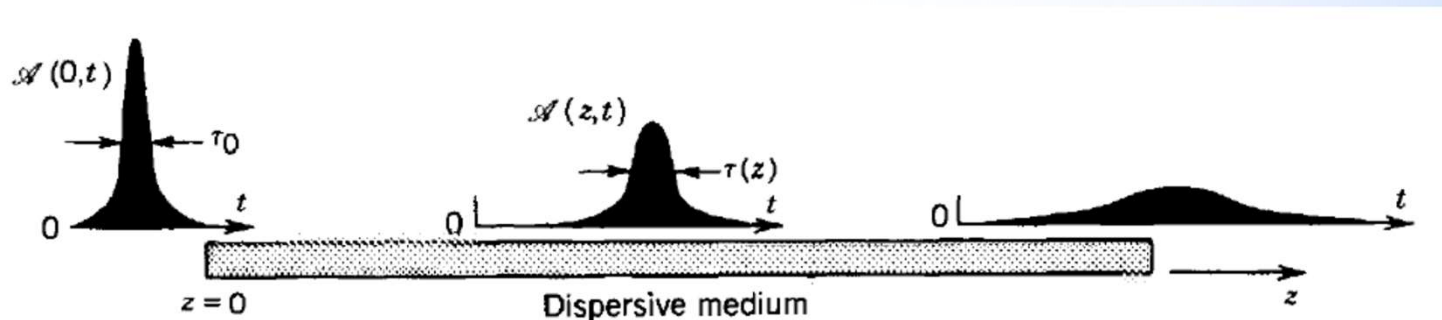
Wave velocities: Dispersive media

Since the group velocity is itself often a frequency dependent quantity, each frequency component will undergo a different time delay, effectively spreading the pulse over time. This phenomenon is called **group velocity dispersion (GVD)**.

To estimate the spread of the pulse we can evaluate the **dispersion coefficient**:

$$D_v = \frac{\partial}{\partial \omega} \left(\frac{1}{v_g(\omega)} \right) = \frac{\partial^2 k}{\partial \omega^2}$$

We can schematically see the pulse broadening phenomenon as the pulse travels along a dispersive medium:



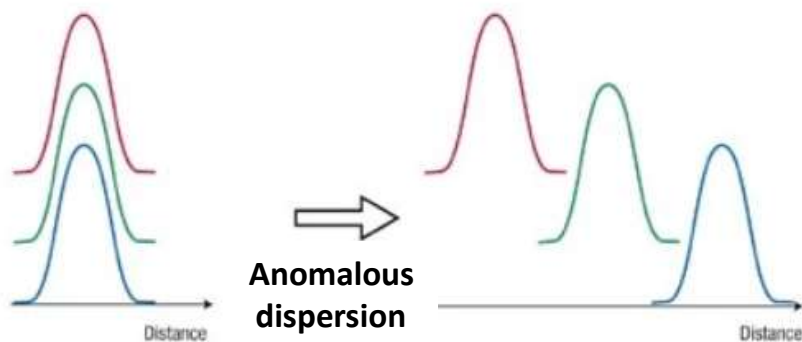
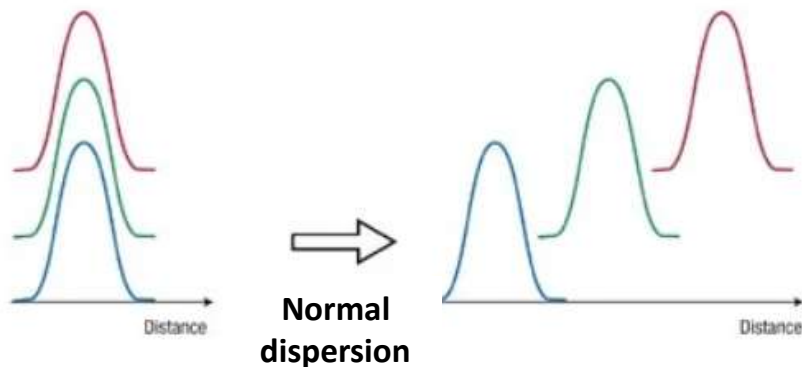


Wave velocities: Normal and anomalous dispersion

The sign of the dispersion coefficient D_v does not affect how much the pulse is broadened but changes the phase of the complex envelope of the optical pulse.

In media with **normal dispersion** ($D_v > 0$), higher frequencies have higher travel time than lower frequencies. Put another way: shorter wavelengths arrive later than longer wavelengths.

In media with **anomalous dispersion** ($D_v < 0$), higher frequencies have lower travel time than lower frequencies. Put another way: shorter wavelengths arrive earlier than longer wavelengths.



NOTE: Artificial/Engineered nanostructures can be designed to alternate normal and anomalous dispersions to achieve peculiar functionalities that cannot be realized in natural materials.