Spatial Optics

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CH3 Fourier optics

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Free propagations and lenses

⟹ Data processing applied to spatial optics

Free propagation

usual for any data processing problem

(Linearity
space invariance)

$$in(x,y) \xrightarrow{*\,PSF} out(x,y) = in * PSF = FT^{-1}(\tilde{in}.TF)$$

FT ↓ ↑ FT^{-1}

L~~T~~SI system Σ

$$\tilde{in}(N_x,N_y) \xrightarrow{.TF} \tilde{out}(N_x,N_y)=\tilde{in}(N_x,N_y).TF$$

transfert function

Previous chapter
(gaussian beams)

$$\boxed{\boldsymbol{TF_z(N_x,N_y) = e^{-j\frac{2\pi}{\lambda}z}.e^{+j\pi\lambda z(N_x{}^2+N_y{}^2)}}}$$

Point Spread function ?

in (x, y) = point.
(pin hole) ⟹ Point spread function
$PSF_d(x, y)$.

Point Spread function of a free propagation over a distance z

$$PSF_z(x,y) = spheric\ wave$$

$$PSF_d(x,y) = Cte\,.\,e^{-j\pi(\frac{x^2+y^2}{\lambda d})} \quad \text{with } z = d$$

Convolution

$$f_d(x,y) = f_0(x,y) * e^{-j\pi(\frac{x^2+y^2}{\lambda d})}$$

$$f_d(x,y) = \int_{-\infty}^{+\infty} f_0(x_0,y_0))\ e^{-j\pi(\frac{(x-x_0)^2+(y-y_0)^2}{\lambda d})} dx_0 dy_0$$

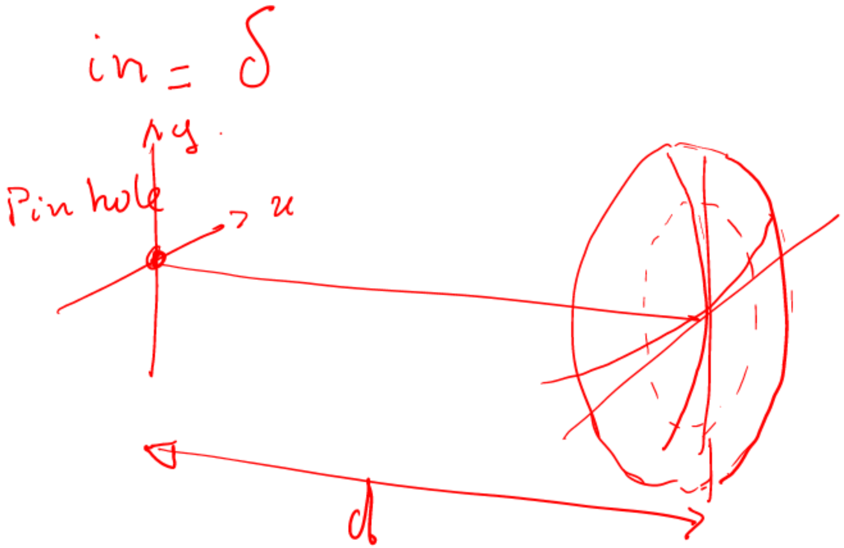
$$f_d(x,y) = e^{-j\pi(\frac{x^2+y^2}{\lambda d})} \int_{-\infty}^{+\infty} f_0(x_0,y_0)\, e^{-j\pi(\frac{x_0^2+y_0^2}{\lambda d})} e^{j2\pi(\frac{xx_0+yy_0}{\lambda d})} dx_0 dy_0$$

3

Point Spread function of a free propagation over a distance z

$$PSF_z(x, y) = spheric\ wave$$

$$PSF_d(x, y) = Cte\ .\ e^{-j\pi\left(\frac{x^2+y^2}{\lambda d}\right)} \quad \text{with } z = d$$

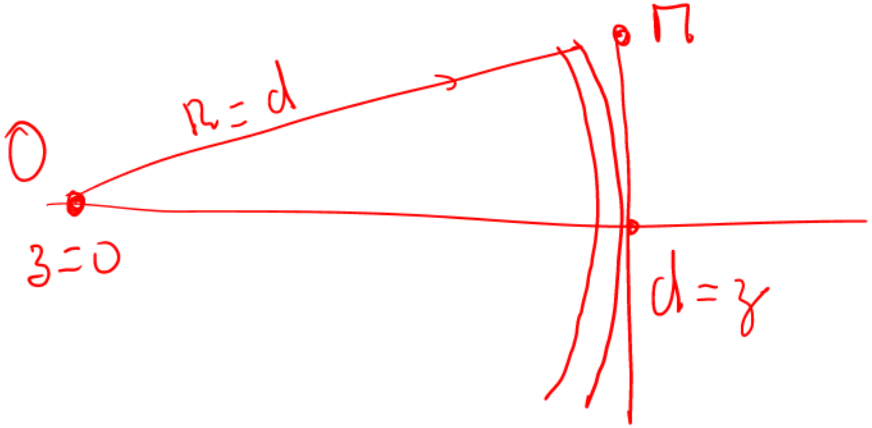


$E_d = PSF_d(x, y)$

$|PSF_d| \ \# \ c^{te} \quad \forall x, y$

$PSF_d = SW(x, y)$

$= e^{-j\frac{2\pi}{\lambda}|\vec{OM}|}$



$\vec{OM} \begin{vmatrix} x \\ y \\ d \end{vmatrix}$

$|\vec{OM}| = \sqrt{x^2+y^2+d^2}$

$= d\sqrt{1 + \frac{x^2+y^2}{d^2}}$

$\# \ d\left(1 + \frac{x^2+y^2}{2d^2}\right)$

$$OM = d + \frac{x^2+y^2}{2d}$$

$$\boxed{PSF_d(x,y) = e^{-j\frac{2\pi d}{\lambda}} \cdot e^{-j\pi\frac{x^2+y^2}{\lambda d}}}$$

Convolution

$$f_d(x,y) = f_0(x,y) * e^{-j\pi(\frac{x^2+y^2}{\lambda d})}$$

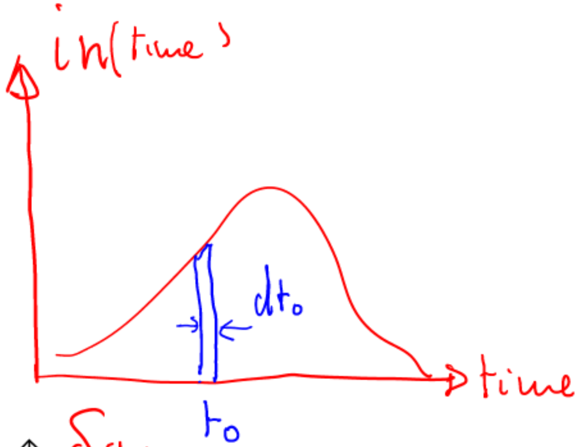
$$f_d(x,y) = \int_{-\infty}^{+\infty} f_0(x_0,y_0))\ e^{-j\pi(\frac{(x-x_0)^2+(y-y_0)^2}{\lambda d})} dx_0 dy_0$$

$$f_d(x,y) = e^{-j\pi(\frac{x^2+y^2}{\lambda d})} \int_{-\infty}^{+\infty} f_0(x_0,y_0)\, e^{-j\pi(\frac{x_0^2\ +y_0^2}{\lambda d})} e^{j2\pi(\frac{xx_0\ +yy_0}{\lambda d})} dx_0 dy_0$$

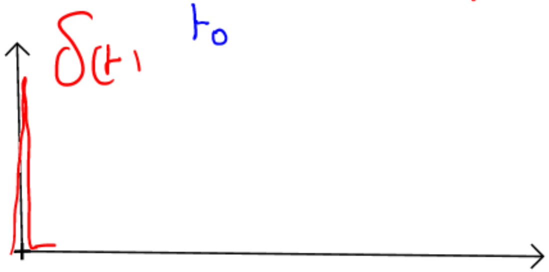
what is a convolution

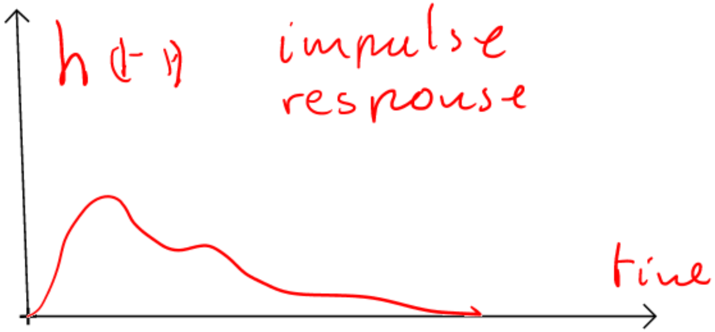
why linearity and time invariance
space

are
Mandatory!



out ?

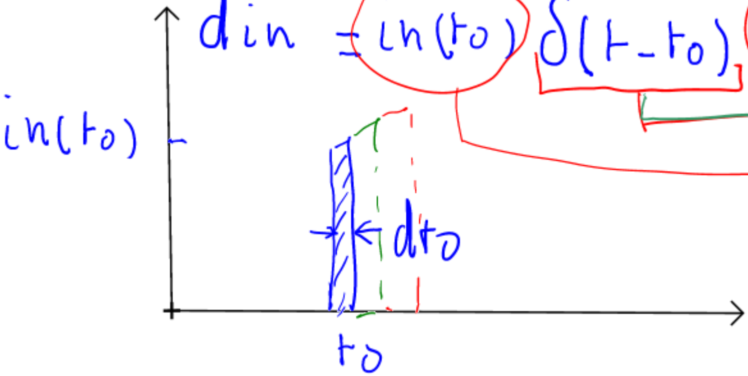




$$d\,in = in(t_0)\,\delta(t-t_0)\,dt_0 \qquad dout = h(t-t_0)\,in(t_0)\,dt_0$$

Linearity

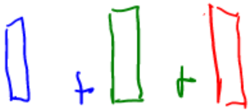
time invariance



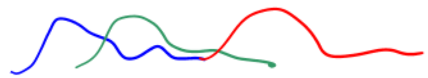
$\Downarrow$

Superposition of all the contributions,

superposition of all the output contributions

in = 

= superposition of all the contributions

out = 

Linearity

$$in(t) = \sum \square$$

$$= \int_{-\infty}^{+\infty} d\,in$$

$$= \int_{-\infty}^{+\infty} in(t_0)\,\delta(t-t_0)\,dt_0$$

$$= in * \delta = in$$

$$out = \int d\,out$$

$$= \int_{-\infty}^{+\infty} in(t_0)\,h(t-t_0)\,dt_0$$

$$= in * h$$

For spatial domain:

$$h(t) \longrightarrow PSF(x, y).$$

$$d\,in = in(t_0)\,dt_0 \longrightarrow d\,in = in(x_0; y_0)\,dx_0\,dy_0$$

$$\delta(t-t_0) \downarrow h(t-t_0) \qquad \delta(x-x_0; y-y_0) \Downarrow PSF(x-x_0; y-y_0)$$

$$out = \int_{-\infty}^{+\infty} in(t_0) \cdot h(t-t_0)\,dt_0$$

$$out = \iint_{-\infty}^{+\infty} in(x_0; y_0)\,PSF(x-x_0; y-y_0)\,dx_0\,dy_0$$

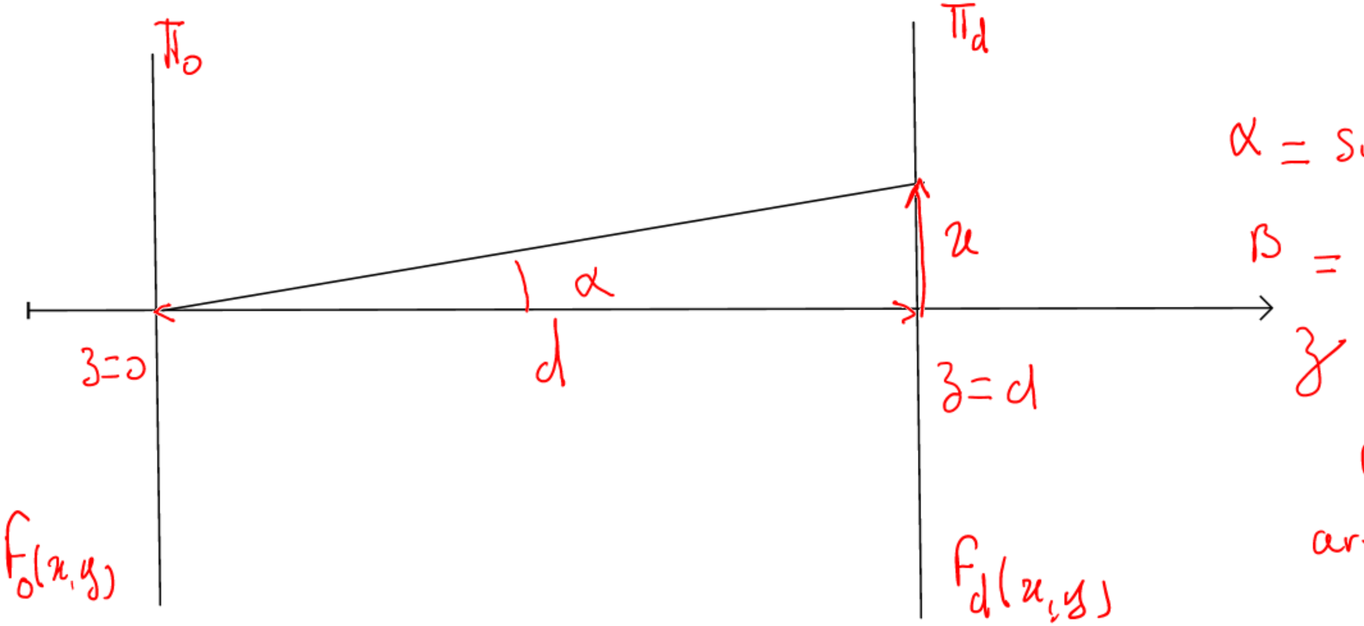
$$= in * PSF$$

$$PSF = e^{-j\pi \frac{x^2+y^2}{\lambda d}}$$

$$out(x,y) = \iint in(x_0, y_0)\, e^{-j\pi \frac{(x-x_0)^2+(y-y_0)^2}{\lambda d}}\, dx_0\, dy_0$$

$$= \iint in(x_0,y_0)\, \underbrace{e^{-j\pi \frac{x^2+y^2}{\lambda d}}} \cdot e^{-j\pi \frac{x_0^2+y_0^2}{\lambda d}} \cdot \underbrace{e^{+j2\pi \frac{(x x_0 + y y_0)}{\lambda d}}}_{\text{Look like a FT}}\, dx_0\, dy_0$$

$$out(x,y) = e^{-j\pi \frac{x^2+y^2}{\lambda d}} \iint in(x_0,y_0)\, e^{-j\pi \frac{x_0^2+y_0^2}{\lambda d}}\, e^{+j2\pi \left(x_0 \underbrace{\left(\frac{x}{\lambda d}\right)}_{N_x} + y_0 \underbrace{\left(\frac{y}{\lambda d}\right)}_{N_y}\right)}\, dx_0\, dy_0$$



$$\alpha = \sin\alpha = \frac{x}{d}$$

$$\beta = \sin\beta = \frac{y}{d}$$

α and β are very small.

In CH1 and CH2 definition of the spatial frequency as

$$N_x = \frac{\sin\alpha}{\lambda} \# \frac{x}{\lambda d}$$

$$N_y = \frac{\sin\alpha}{\lambda} \# \frac{y}{\lambda d}$$

$$\boxed{Out(x,y) = F_d(x,y) = e^{-j\pi\frac{(x^2+y^2)}{\lambda d}} \iint F_o(x_o;y_o)\, e^{-j\pi\frac{x_o^2+y_o^2}{\lambda d}} \cdot e^{j2\pi(x_o N_x + y_o N_y)}\, dx_o\, dy_o}$$

Transmission of a lens (focal length *f*)

Linearity yes

space invariance !!!no!!!

$$f_{\Pi}(x,y) = f_{\Pi 0}(x,y).\tau(x,y) = f_{\Pi 0}(x,y)\, e^{j\pi(\frac{x^2+y^2}{\lambda f})}$$

Lens

Free propagation distance *d*

$$f_{\Pi L-}(x,y) \xrightarrow{.\tau(x,y)} f_{\Pi L+}(x,y) \xrightarrow{*\,PSF} f_d(x,y)$$

$$f_{d=f}(x,y) = e^{-j\pi(\frac{x^2+y^2}{\lambda d})} \int_{-\infty}^{+\infty} f_{\Pi L-}(x_0,y_0)\, \underbrace{e^{j\pi(\frac{x_0^2\ +y_0^2}{\lambda f})}\, e^{-j\pi(\frac{x_0^2\ +y_0^2}{\lambda d})}}\, e^{j2\pi(\frac{xx_0\ +yy_0}{\lambda d})} dx_0 dy_0$$

If *d* = *f* >>> simplification

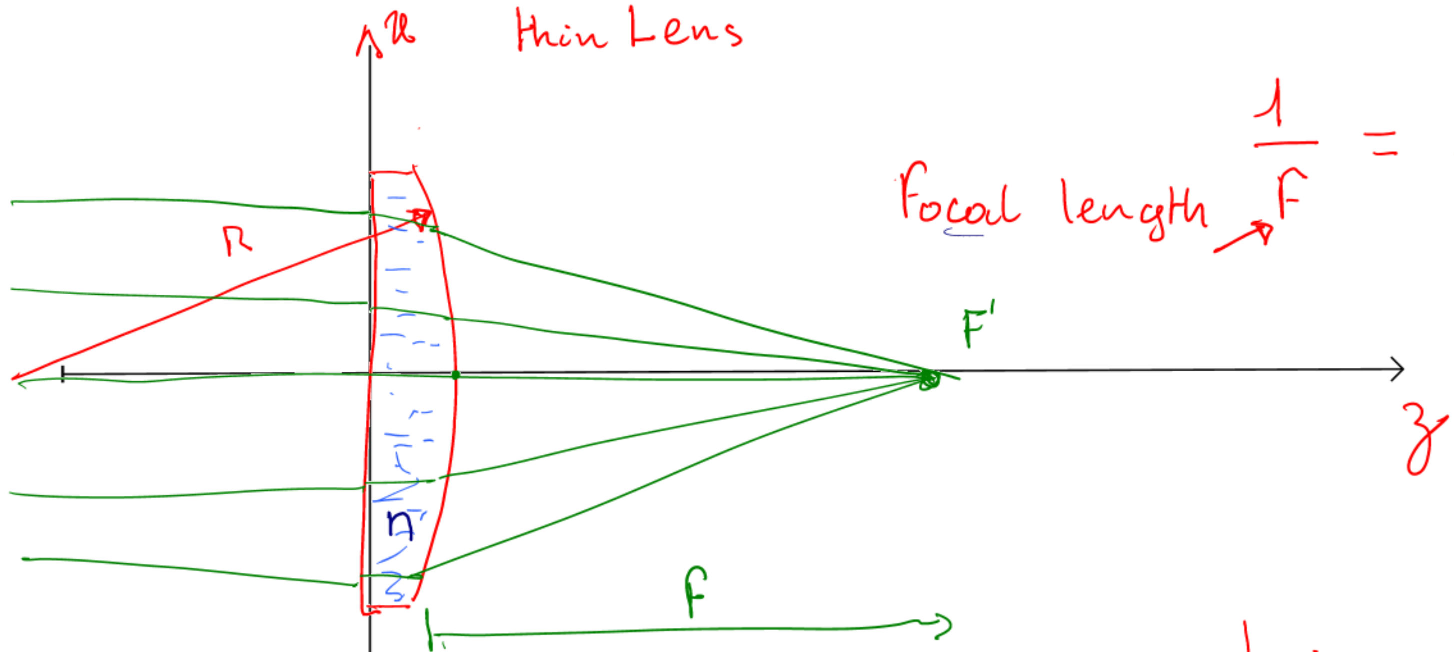
$$f_{d=f}(x,y) = e^{-j\pi(\frac{x^2+y^2}{\lambda d})} \int_{-\infty}^{+\infty} f_{\Pi L-}(x_0,y_0)\, e^{j2\pi(\frac{xx_0\ +yy_0}{\lambda d})} dx_0 dy_0$$

Transmission of a lens (focal length f)

Linearity yes

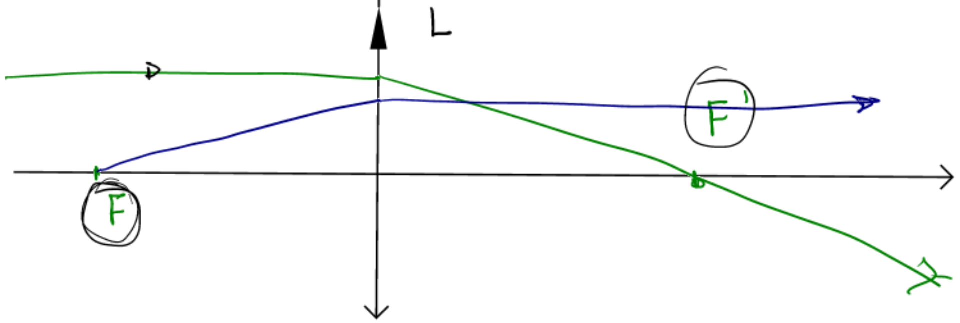
space invariance !!!no!!!

$$f_{\Pi}(x,y) = f_{\Pi 0}(x,y).\,\tau(x,y) = f_{\Pi 0}(x,y)\; e^{j\pi\left(\frac{x^2+y^2}{\lambda f}\right)}$$



refractive index

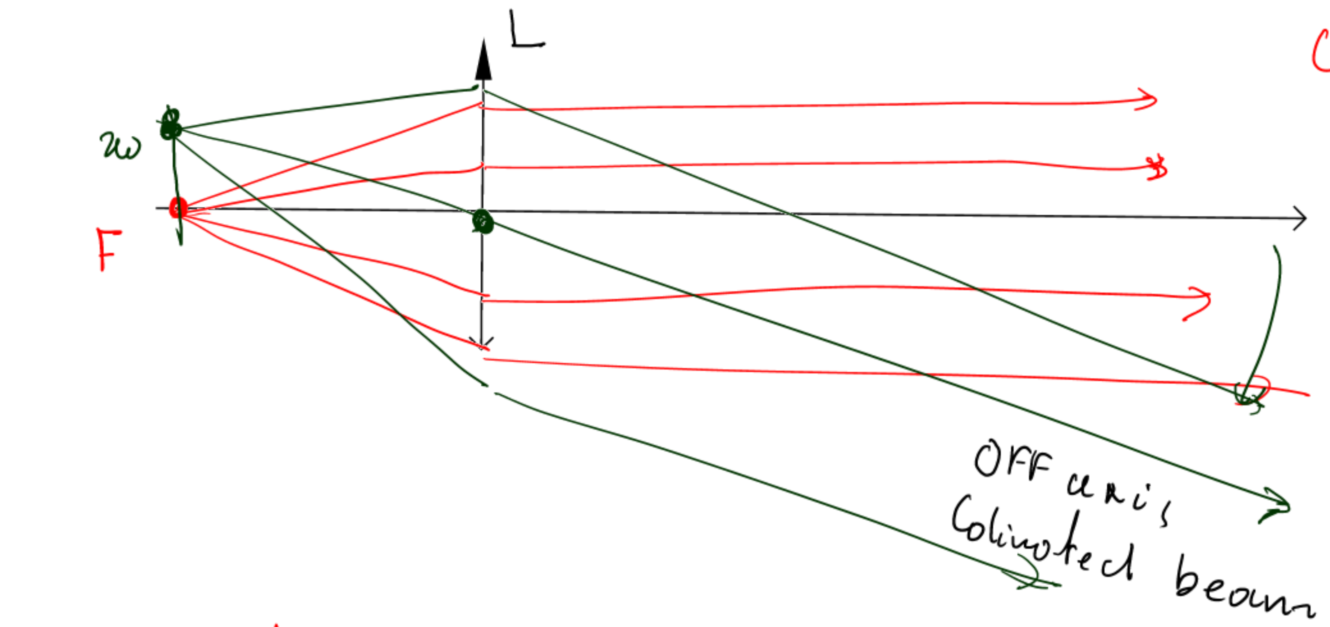
focal length $\frac{1}{F} = \frac{(n-1)}{R}$ ← curvature of the spherical surface



Linearity ✓ ok

Space invariance ✗

No use of data processing such as PSF



Colimated beam.

opt. axis.

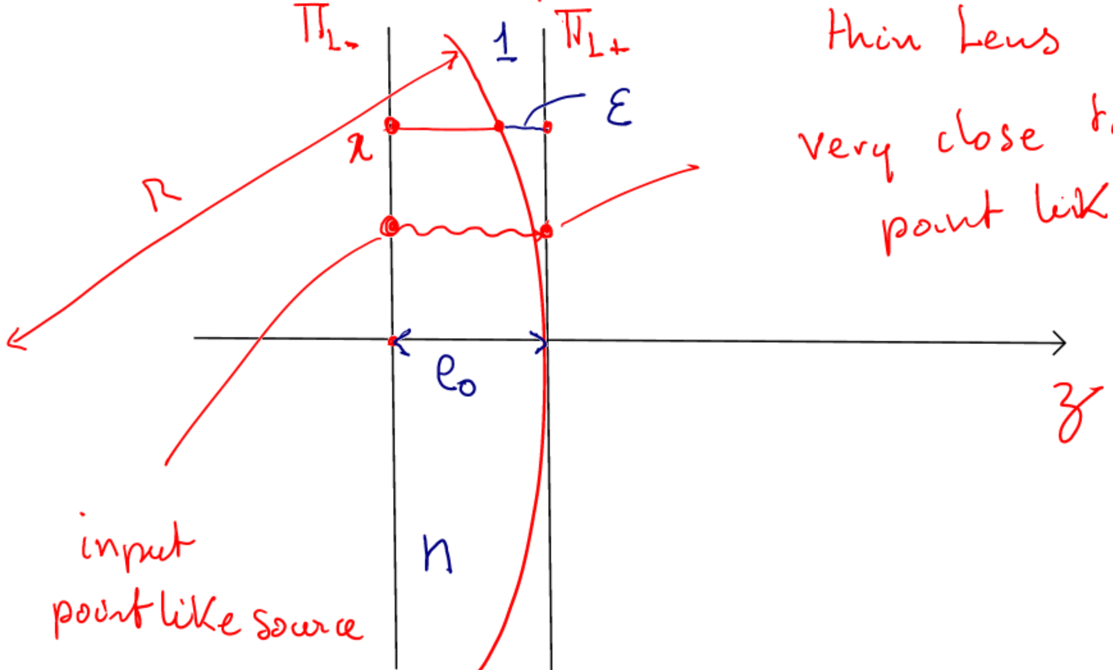
no shift by x_0

change of output direction

colimated beam.

Linearity means $\longrightarrow$ Transmission through this Lens

$\tau_L(x,y)$.



thin Lens

very close to a point like source.

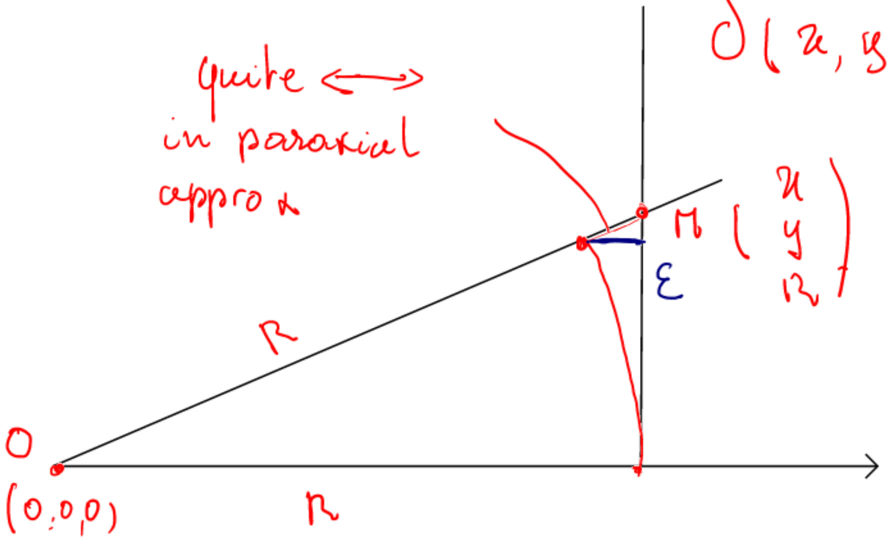
propagation

= optical path

$\Downarrow$

phase shift.

quite ⟷
in paraxial
approx



$$\delta(x,y) = n(e_0 - \varepsilon) + \varepsilon$$

what is the value of ε

$$\varepsilon = \overline{OM} - R \qquad \overline{OM} = \begin{vmatrix} x \\ y \\ R \end{vmatrix}$$

$$= \sqrt{R^2 + x^2 + y^2} - R$$

$$= R\sqrt{1 + \frac{x^2+y^2}{R^2}} - R$$

$$\simeq R\left(\cancel{1} + \frac{x^2+y^2}{2R^2}\right) - \cancel{R} = \frac{x^2+y^2}{2R}.$$

$$T_L(x,y) = e^{-j\frac{2\pi\delta}{\lambda}} =$$

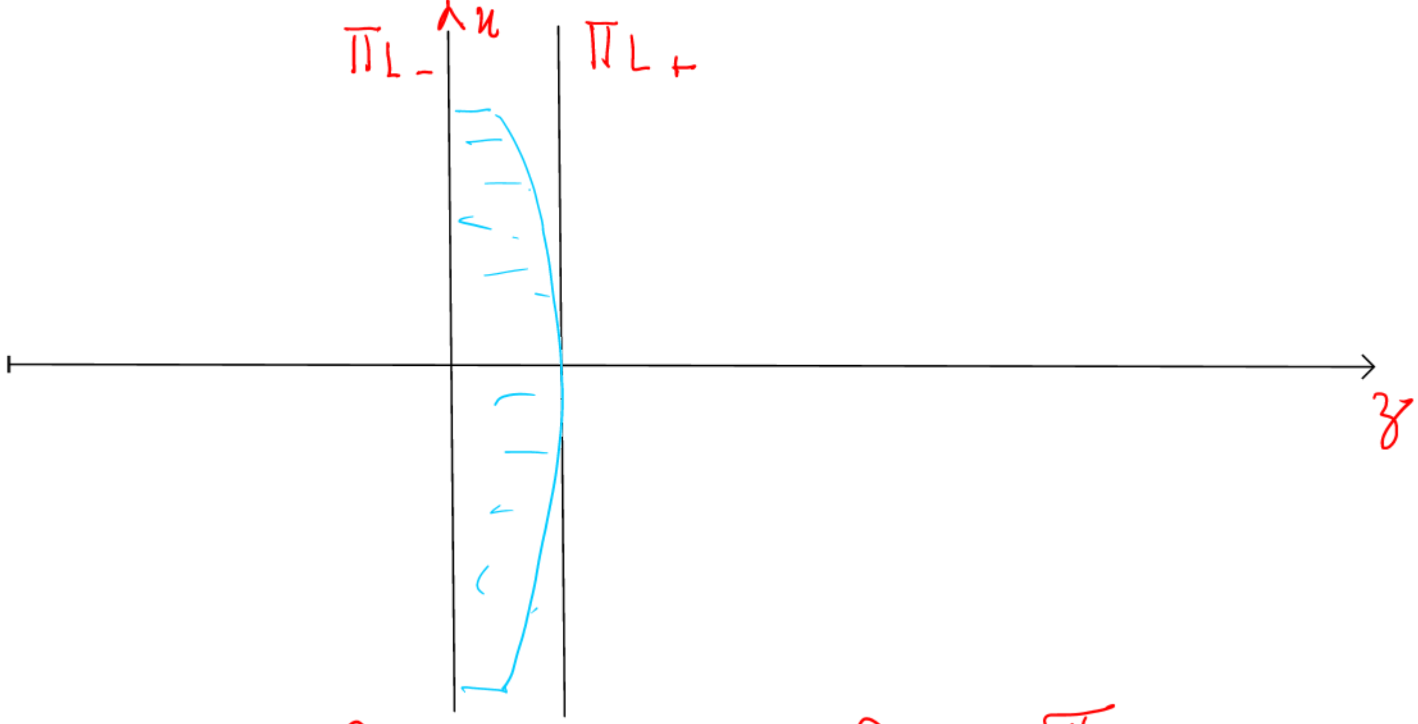
$$= e^{-j\frac{2\pi}{\lambda}\left(ne_0 - (n-1)\varepsilon\right)}$$

$$= e^{-j\frac{2\pi n e_0}{\lambda}}\, e^{+j\frac{2\pi}{\lambda}(n-1)\frac{x^2+y^2}{2R}}$$

$\frac{1}{f}$

$$= e^{-j\frac{2\pi n e_0}{\lambda}}\, e^{+j\pi\frac{x^2+y^2}{f\lambda}}$$

$$\boxed{T_L(x,y) = e^{+j\pi\frac{x^2+y^2}{\lambda f}}}$$



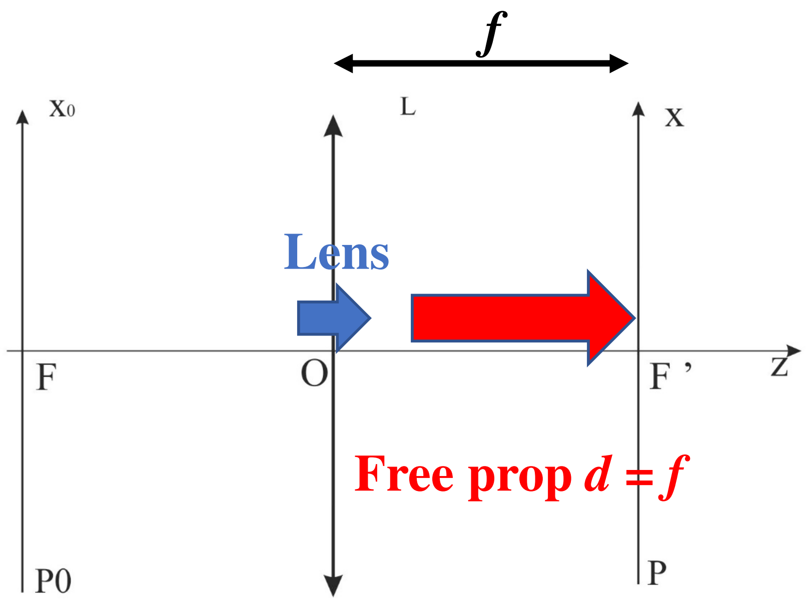
f_{L-} $\qquad$ $f_{L+} = f_{L-} \cdot \tau_L$

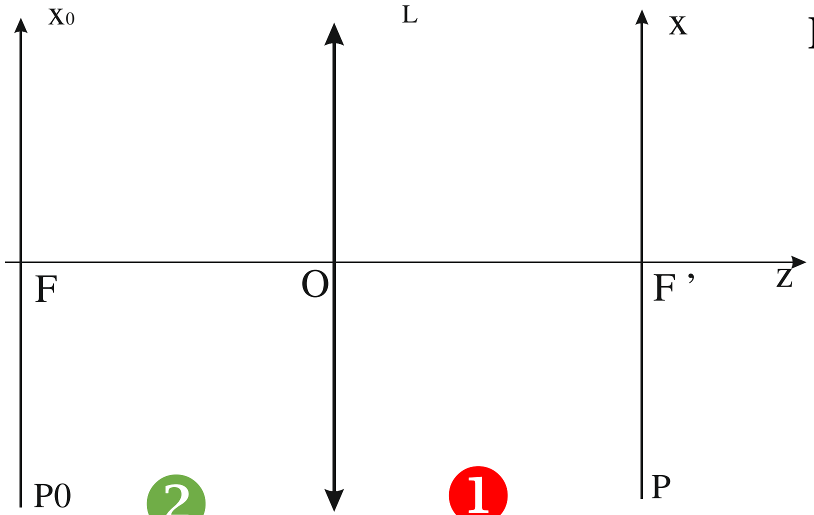
$$f_{d=f}(x,y) = e^{-j\pi\left(\frac{x^2+y^2}{\lambda f}\right)} \int_{-\infty}^{+\infty} f_{\Pi L-}(x_0,y_0)\, e^{j2\pi\left(\frac{xx_0+yy_0}{\lambda f}\right)} dx_0 dy_0$$

And denoting

$$N_x = \frac{x}{\lambda f} \text{ and } N_y = \frac{y}{\lambda f}$$

$$\boxed{f_{d=f}(x,y) = e^{-j\pi\lambda f(N_x^2+N_y^2)} FT(f_{\Pi L-})}$$





For a global scheme

propagation > lens > propagations

$$f_0 \xrightarrow{2} f_{\Pi 0} \xrightarrow{1} f_{d=f}$$

1 $$\boxed{f_{d=f}(x,y) = e^{-j\pi\lambda f(N_x^2+N_y^2)} FT(f_{\Pi L-})}$$

$$\tilde{f}_0 \xrightarrow{2} \boxed{FT(f_{\Pi L-}) = TF_{d_0}.FT(f_0)} \quad TF_{d_0}(N_x, N_y) = e^{-j\frac{2\pi}{\lambda}d_0}.e^{+j\pi\lambda d_0({N_x}^2+{N_y}^2)}$$

as

$$f_{d=f}(x,y) = e^{-j\pi\lambda f(N_x^2+N_y^2)} e^{+j\pi\lambda d_0(N_x^2+N_y^2)} FT(f_0)$$

$$f_{d=f}(x,y) = e^{-j\pi(\frac{x^2+y^2}{\lambda f})} \int_{-\infty}^{+\infty} f_{\Pi L-}(x_0, y_0)\, e^{j2\pi(\frac{xx_0\ +yy_0}{\lambda f})} dx_0 dy_0$$

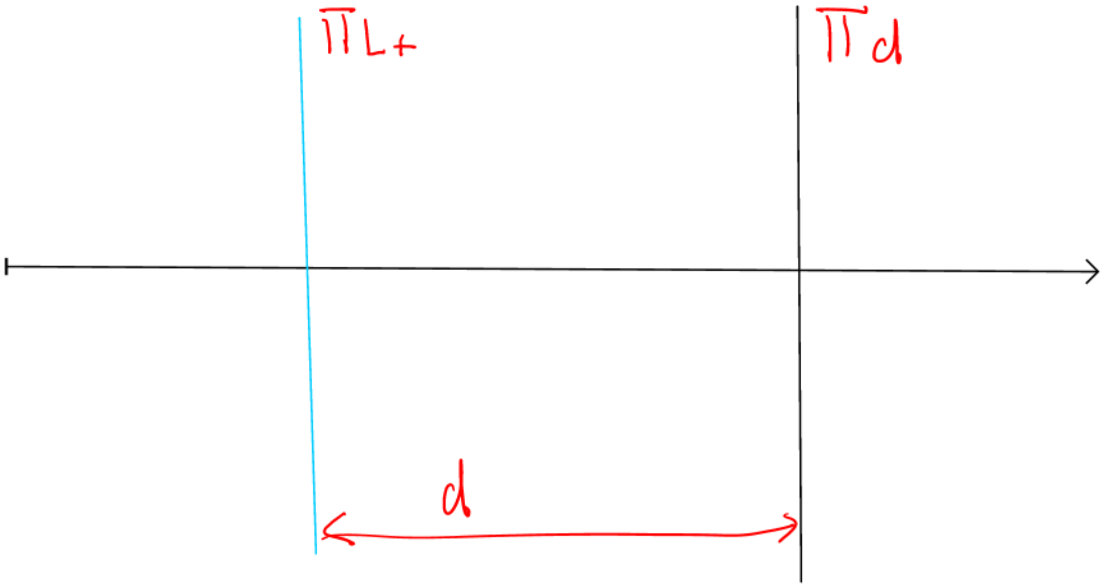
And denoting

$$N_x = \frac{x}{\lambda f} \ and \ N_y = \frac{y}{\lambda f}$$

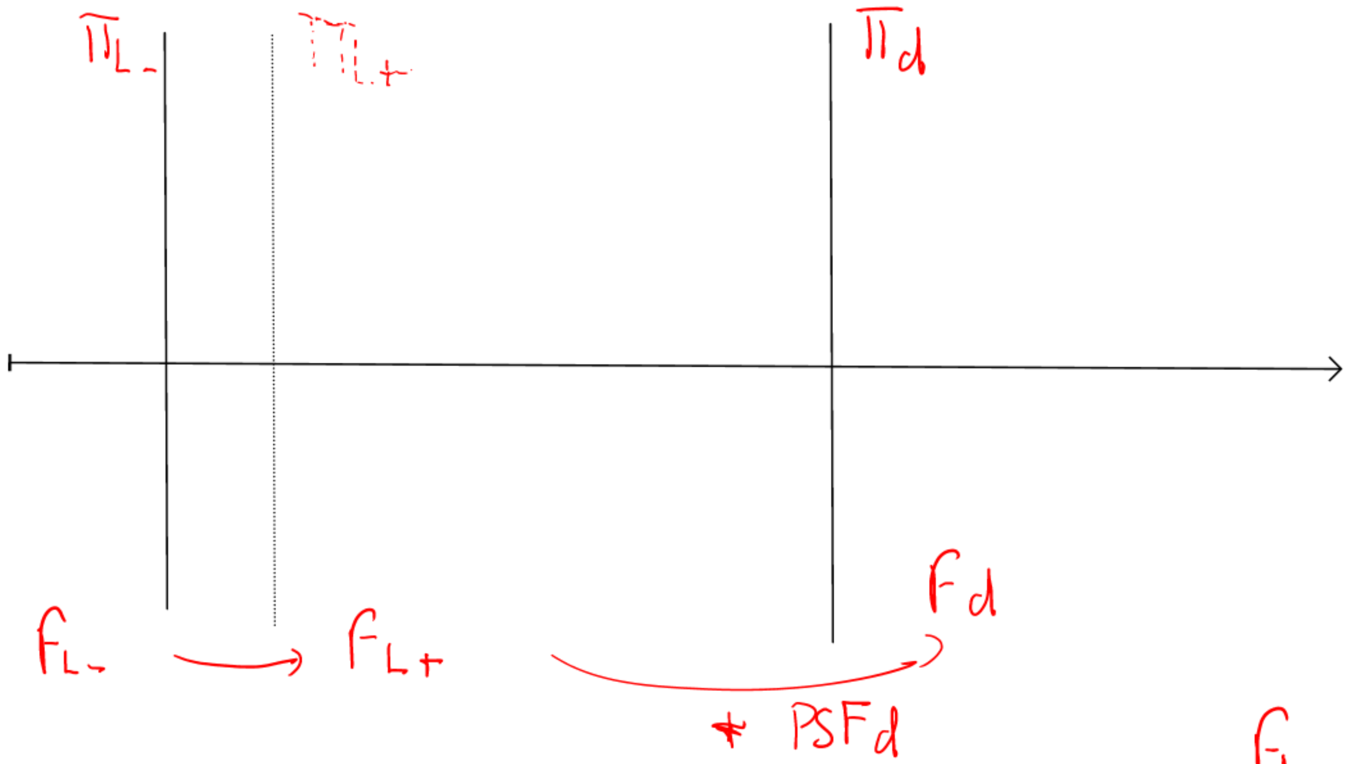
$$\boxed{f_{d=f}(x,y) = e^{-j\pi\lambda f(N_x^2+N_y^2)} FT(f_{\Pi L-})}$$

f

$$f_d(x,y) = e^{-j\pi\frac{x^2+y^2}{\lambda d}} \iint f_{L+}(x_0,y_0)\, e^{-j\pi\frac{x_0^2+y_0^2}{\lambda d}}\, e^{+j2\pi\, x_0 N_x + y_0 N_y}\, dx_0 dy_0$$



but $F_{L+} = T_L(x,y)\, F_{L-}$



$$F_{L+} = F_{L-}\, e^{+j\pi \frac{x^2+y^2}{\lambda f}}$$

$$F_d(x,y) = e^{-j\pi \frac{x^2+y^2}{\lambda d}} \iint \underbrace{F_{L-}(x_0,y_0)\, e^{+j\pi \frac{x_0^2+y_0^2}{\lambda f}}}_{F_{L+}} \cdot e^{-j\pi \frac{x_0^2+y_0^2}{\lambda d}}\, e^{+j2\pi (x_0 N_x + y_0 N_y)}\, dx_0\, dy_0$$

Assumption

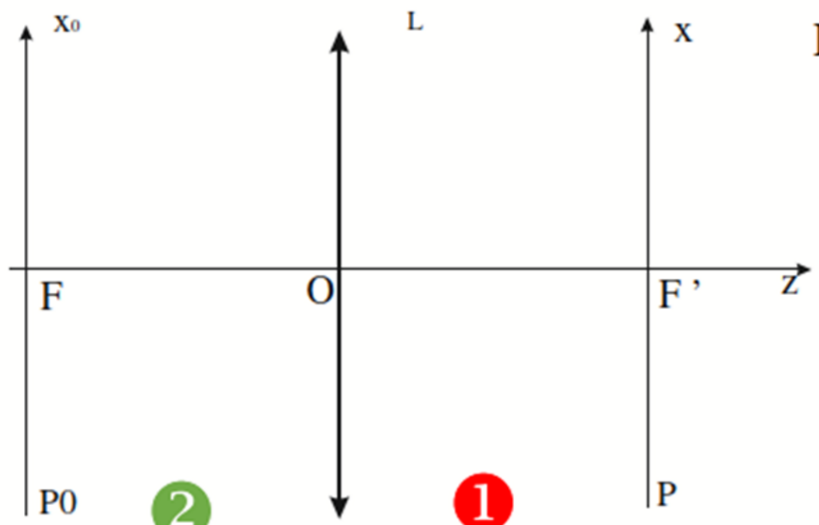
$d = f$

$$F_d(x,y) = e^{-j\pi \frac{x^2+y^2}{\lambda f}} \underbrace{\iint F_{L-}(x_0,y_0)\, e^{+j2\pi(x_0 N_x + y_0 N_y)}\, dx_0\, dy_0}_{FT(F_{L-}) = \tilde{F}_{L-}}$$

$$N_x = \frac{\sin\alpha}{\lambda} \# \frac{x}{\lambda d} \Leftarrow \quad d = f$$

$$N_y = \frac{\sin\beta}{\lambda} \# \frac{y}{\lambda d}$$

$$\boxed{F_d(N_x, N_y) = e^{-j\pi\lambda f (N_x^2 + N_y^2)}\, \tilde{F}_{L-}}$$



For a global scheme

propagation > lens > propagations

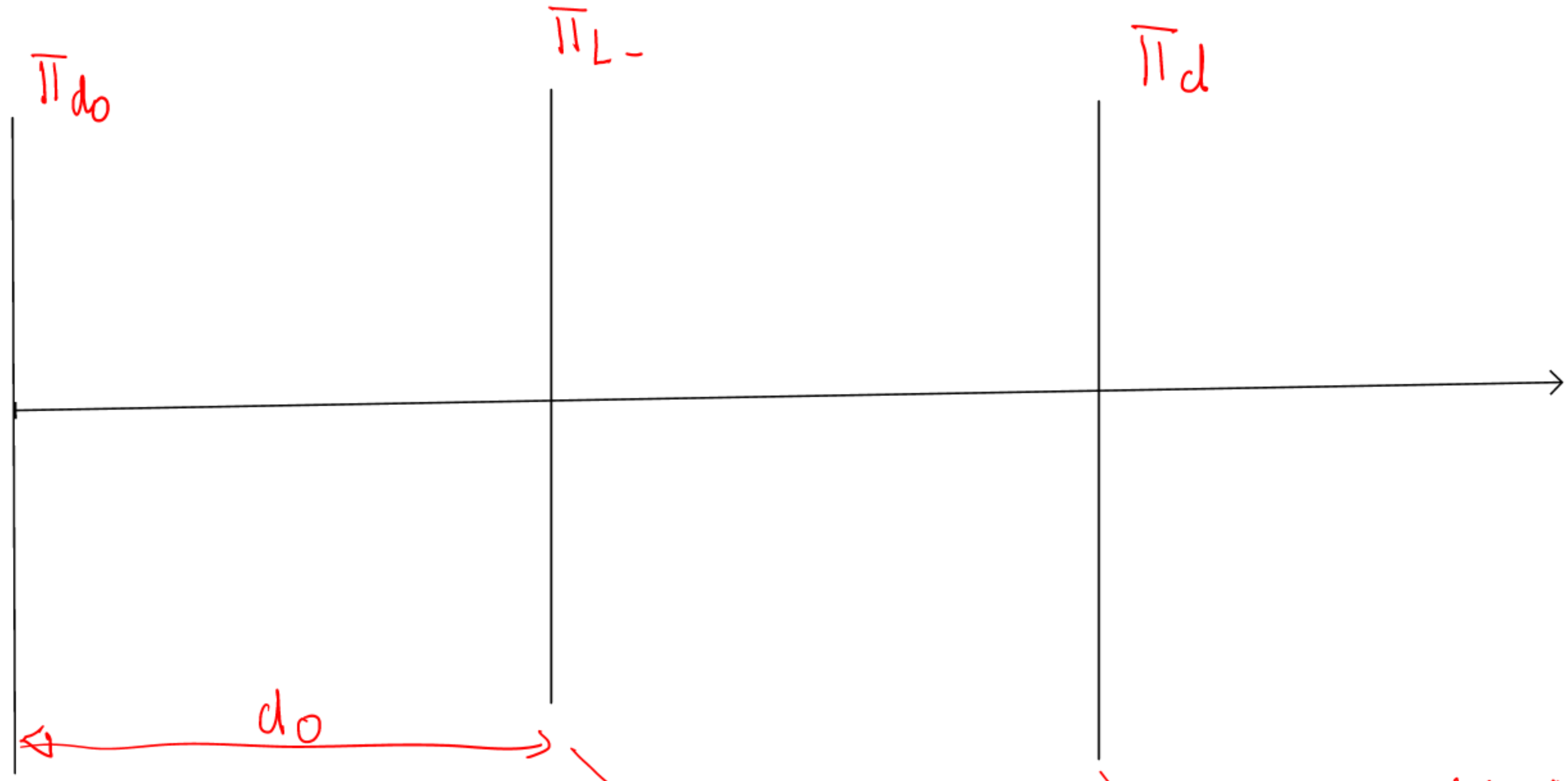
$$f_0 \xrightarrow{2} f_{\Pi L0} \xrightarrow{1} f_{d=f}$$

1 $$\boxed{f_{d=f}(x,y) = e^{-j\pi\lambda f(N_x^2+N_y^2)} FT(f_{\Pi L-})}$$

$$\widetilde{f_0} \xrightarrow{2} \boxed{FT(f_{\Pi L-}) = TF_{d_0}.FT(f_0)} \quad TF_{d_0}(N_x,N_y) = e^{-j\frac{2\pi}{\lambda}d_0}.\, e^{+j\pi\lambda d_0({N_x}^2+{N_y}^2)}$$

as

$$f_{d=f}(x,y) = e^{-j\pi\lambda f(N_x^2+N_y^2)} e^{+j\pi\lambda d_0(N_x^2+N_y^2)} FT(f_0)$$



$$F_0 \xrightarrow{*\, PSF_{d_0}} F_{L-} \qquad F_d = e^{-j\pi\lambda f(N_x^2+N_y^2)}\ \tilde{F}_{L-}$$

$$\tilde{F}_0 \xrightarrow{\times\, TF_{d_0}} \tilde{F}_{L-} = \tilde{F}_0 \cdot TF_{d_0}$$

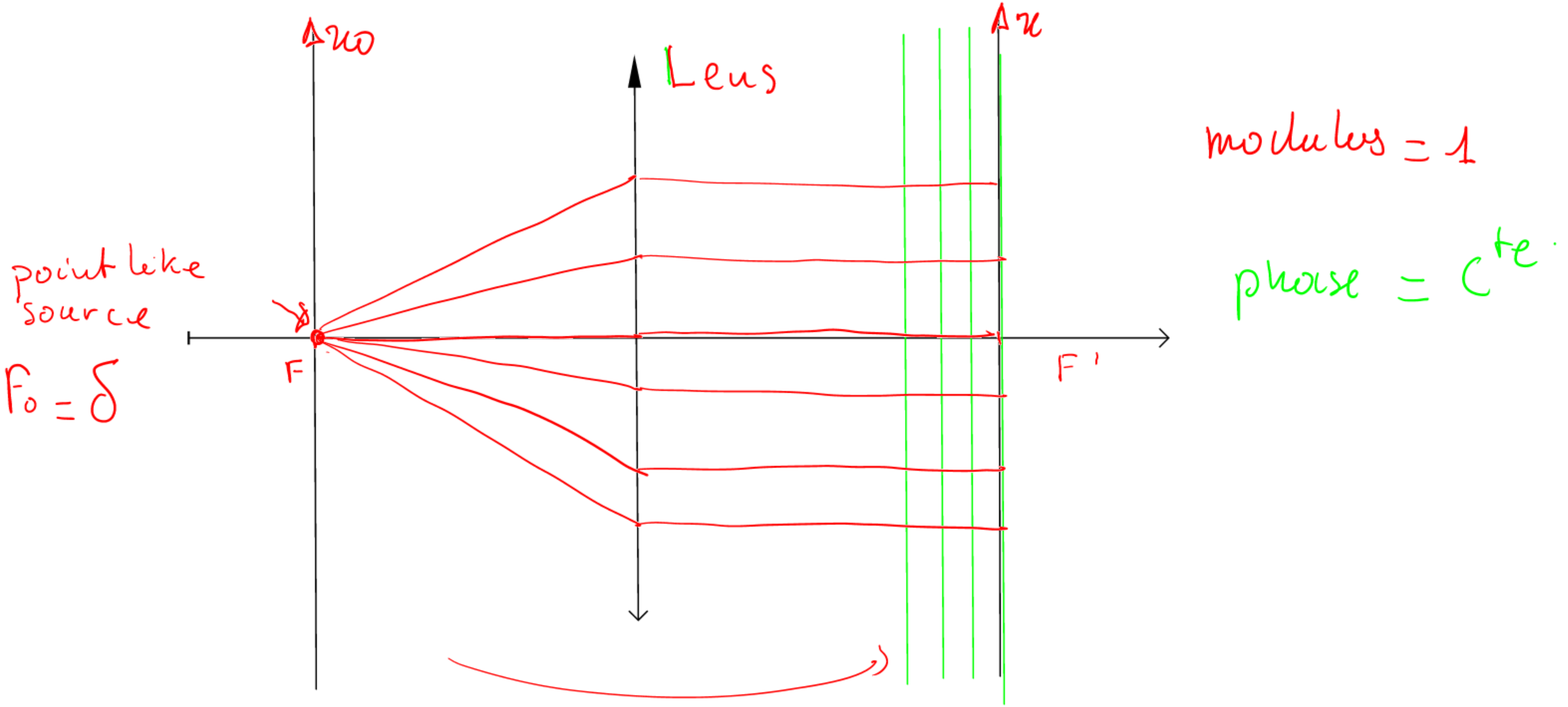
with $TF_{d_0} = e^{+j\pi\lambda d_0(N_x^2+N_y^2)}$

$$F_d = e^{-j\pi\lambda f(N_x^2+N_y^2)} \cdot \tilde{F}_0\, e^{+j\pi\lambda d_0(N_x^2+N_y^2)}$$

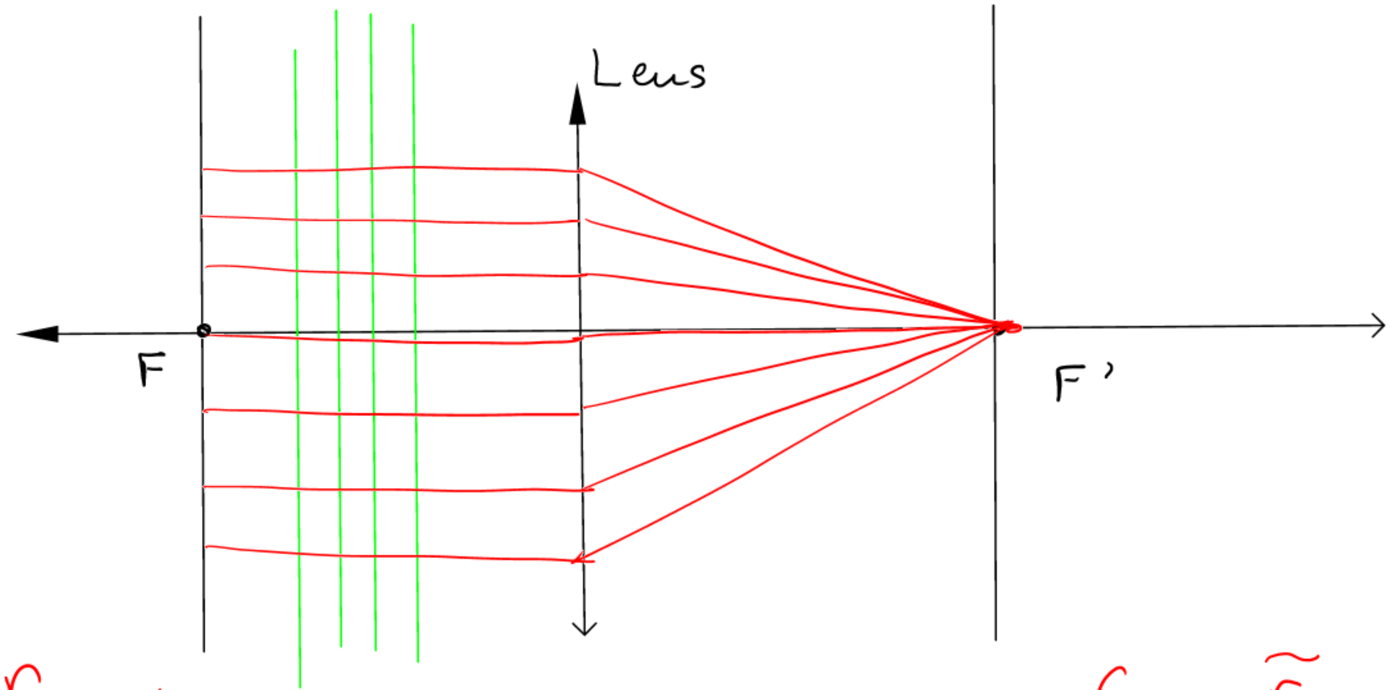
Assumption $d_0 = f$

$$\boxed{F_d = \tilde{F}_0}$$

$$F_d = FT(F_0)$$

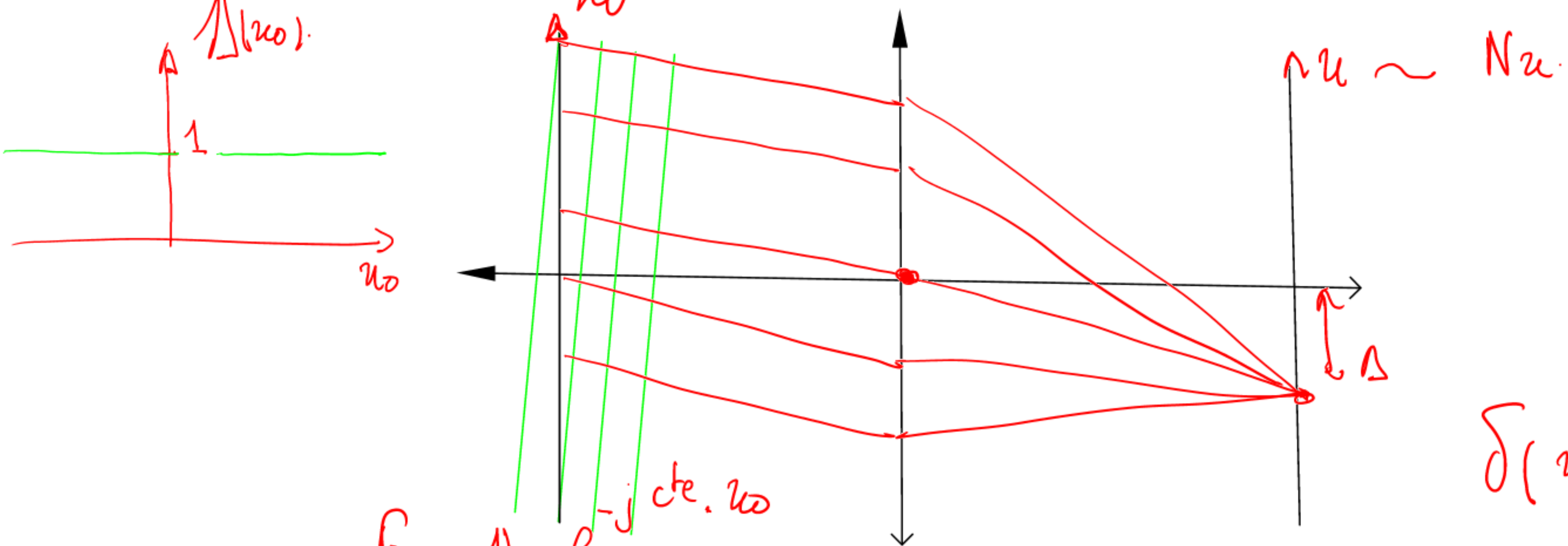


$$FT(\delta) = 1$$



$F_o = \mathbb{1}$

$F_d = \tilde{F_o} = \delta$

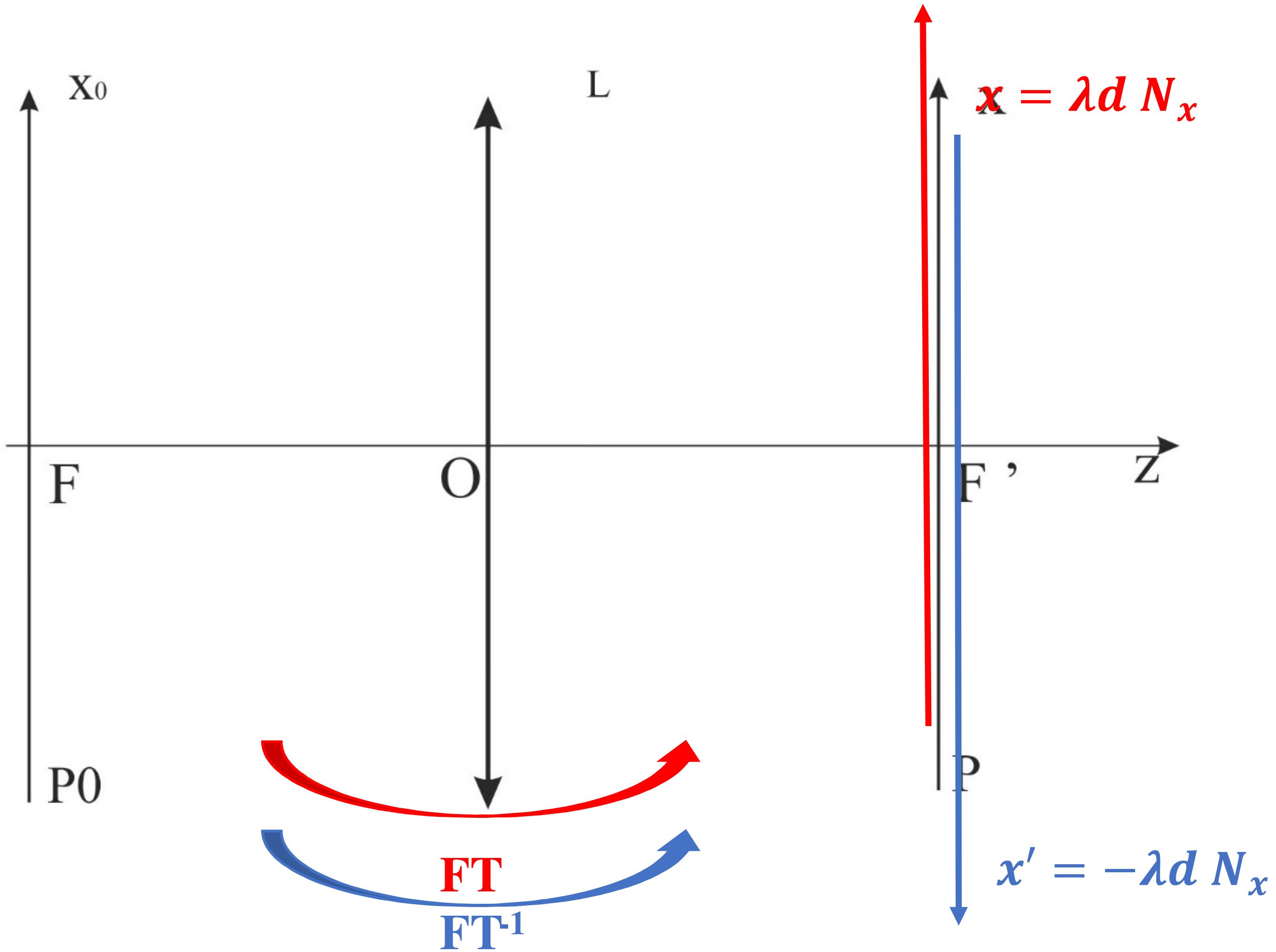


$\delta(u - \Delta)$

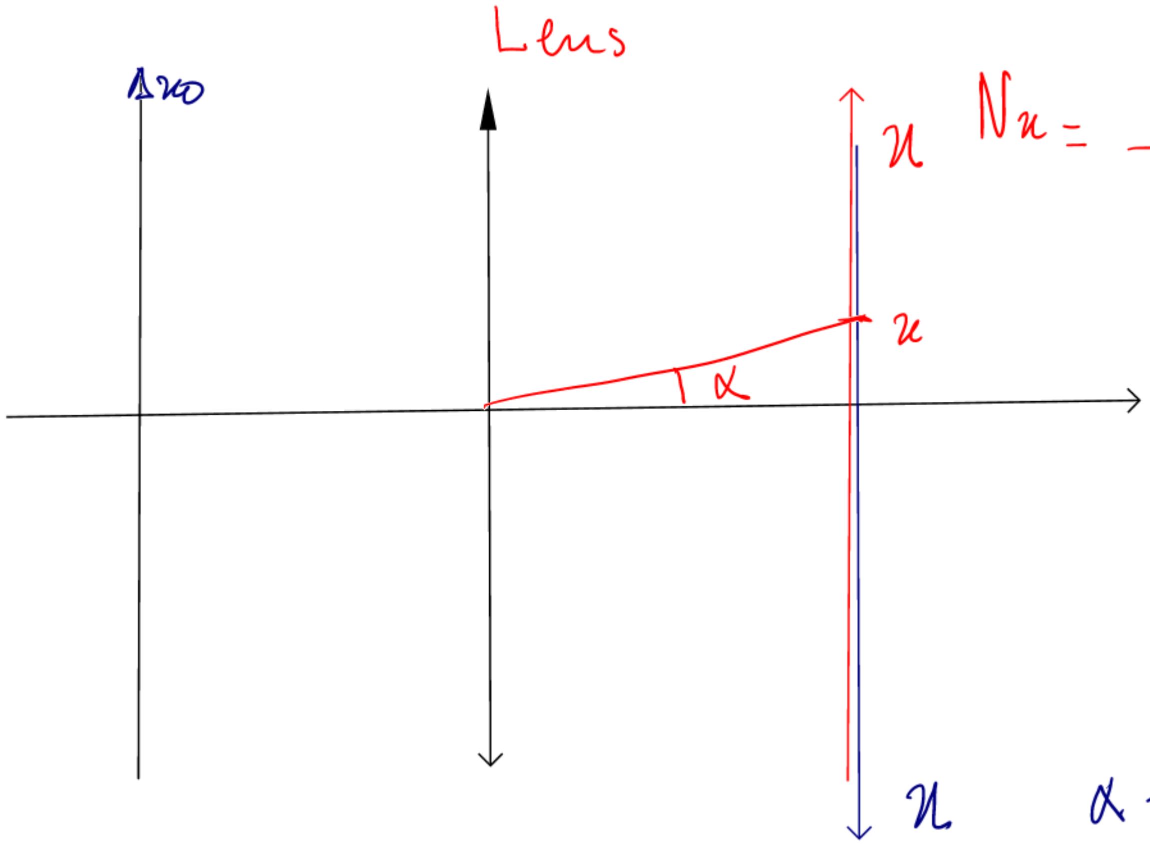
$F_o = \mathbb{1} \cdot e^{-j\, c^{te} \cdot u_o}$

$$f_{d=f}(x,y) = \underbrace{e^{-j\pi\lambda f(N_x^2+N_y^2)}e^{+j\pi\lambda f(N_x^2+N_y^2)}}_{\text{If } d_0 = f}FT(f_0)$$

$$\boxed{f_{d=f}(x,y) = FT(f_0)}$$



$$\tilde{F}_0 = \iint F_0(x_0, y_0)\, e^{+j 2\pi (x_0 N_x + y_0 N_y)}\, dx_0 dy_0$$

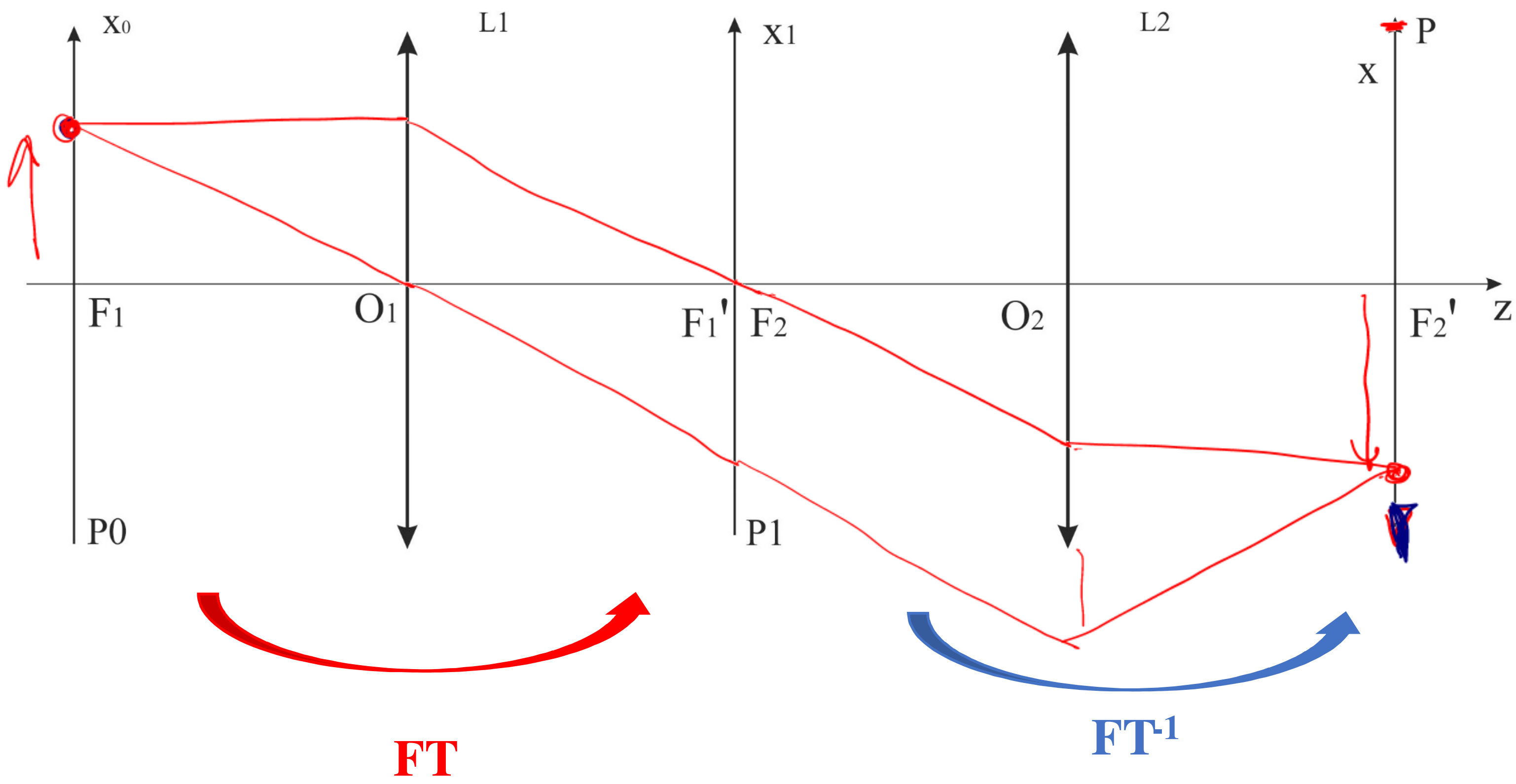


$$N_x = \frac{x}{\lambda F} \qquad N_y = \frac{y}{\lambda F}$$

$$\alpha = -\frac{x}{F} \qquad N_x = -\frac{x}{\lambda F}$$

$$FT^{-1}(F_0) = \iint F_0(x_0, y_0)\, e^{-j 2\pi (x_0 N_x + y_0 N_y)}\, dx_0 dy_0$$

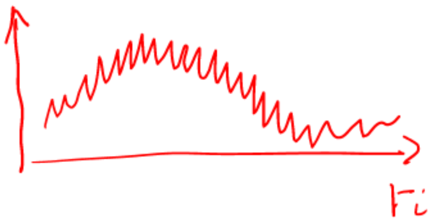
Spatial Filtering

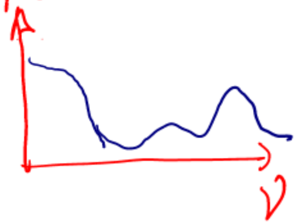


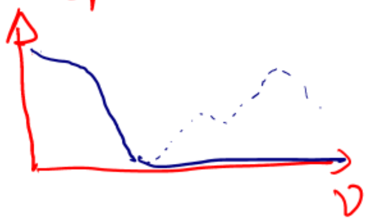
Filtering process

$$in(t) \longrightarrow \widetilde{in}(\nu) \xrightarrow[\text{Filter}]{\times H(\nu)} \widetilde{out}(\nu) = \widetilde{in}\cdot H \xrightarrow{FT^{-1}} out$$

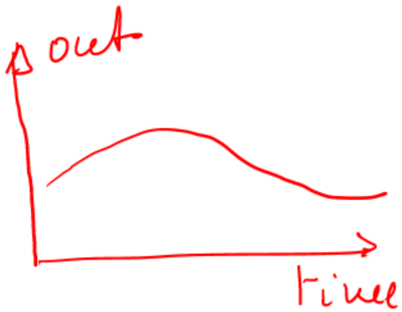
input function — spectrum — output spectrum



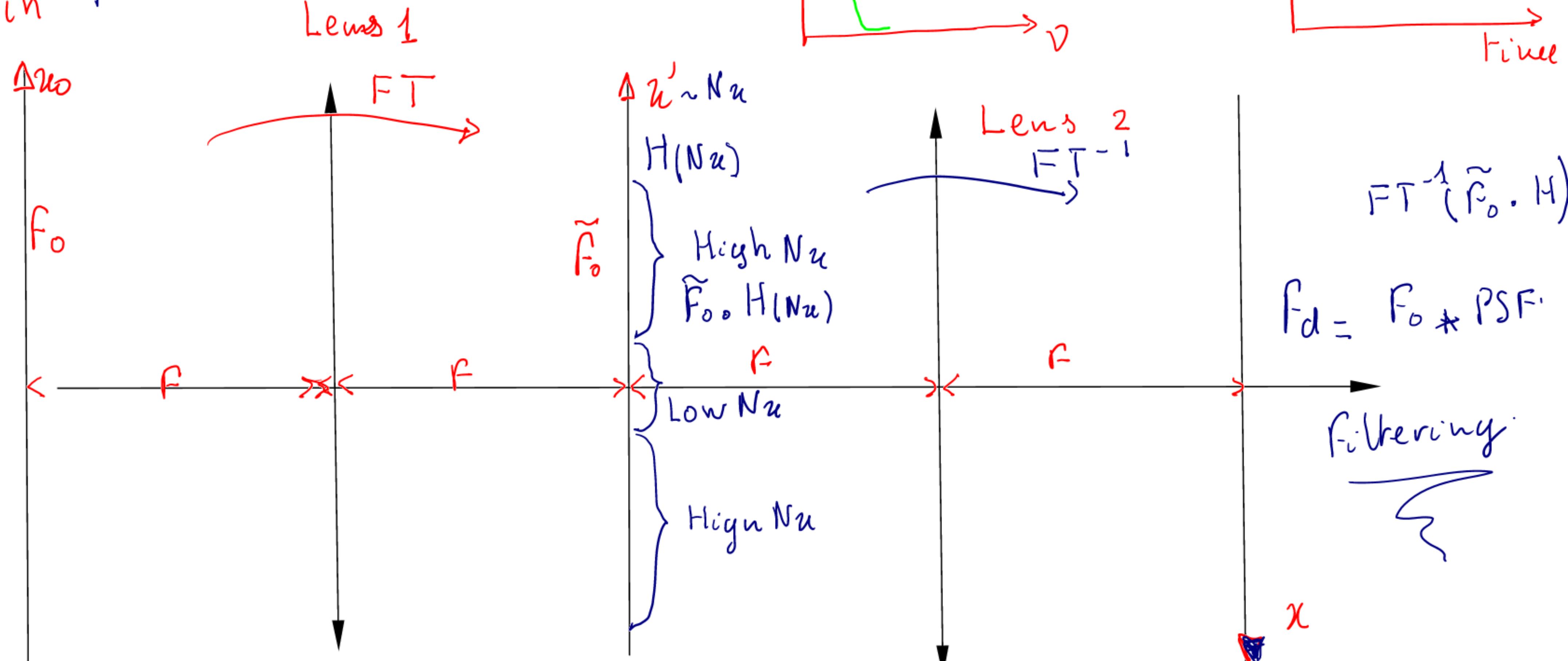




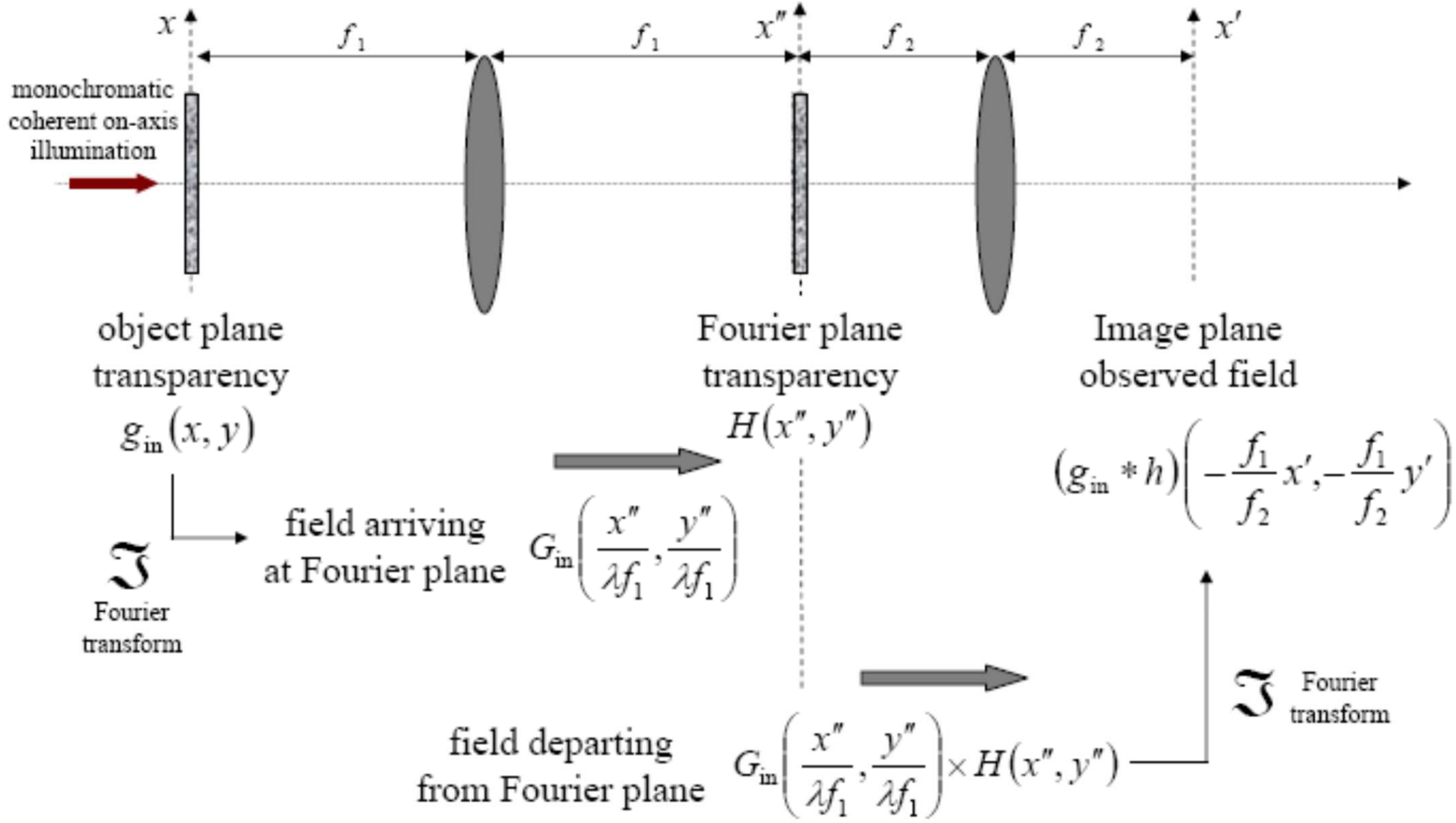
Low pass filter.



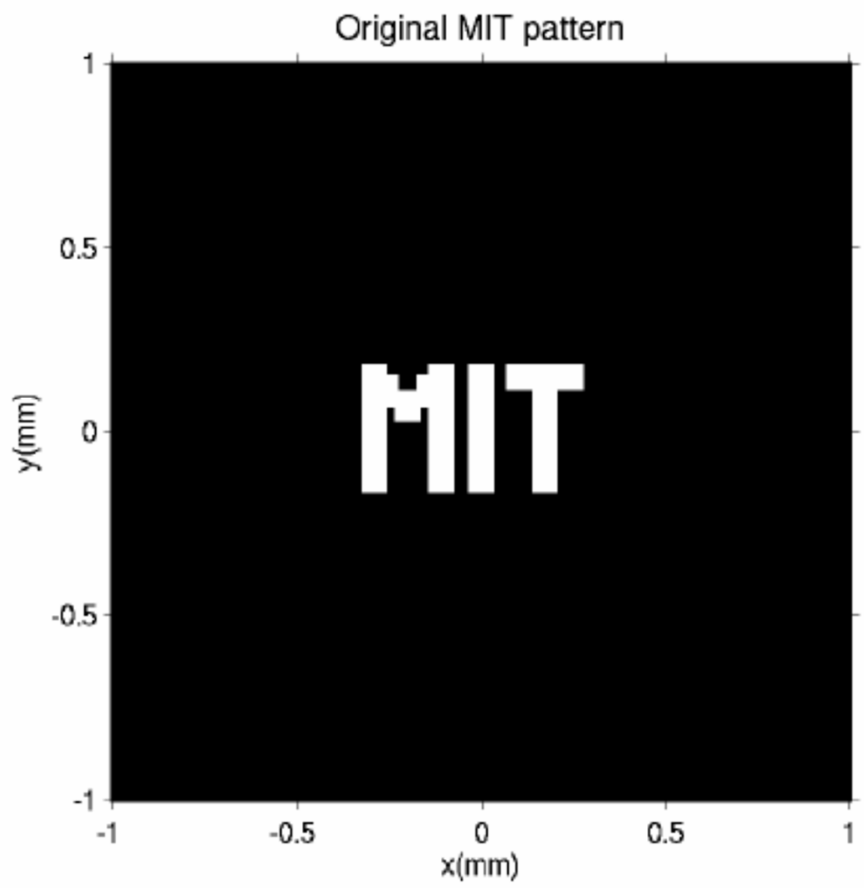
$\boxed{\text{4F setup}}$



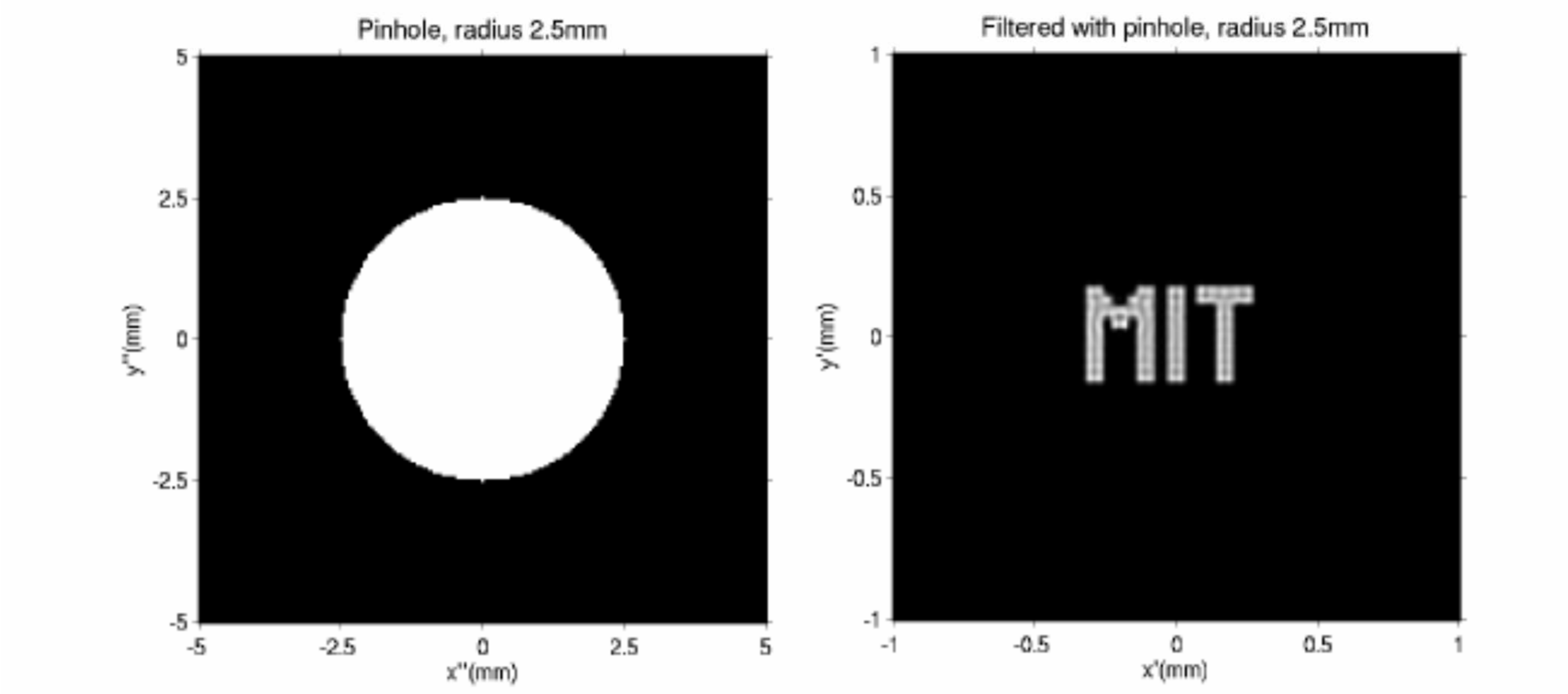
Spatial filtering with the 4F system



Examples: the amplitude MIT pattern



Weak low–pass filtering



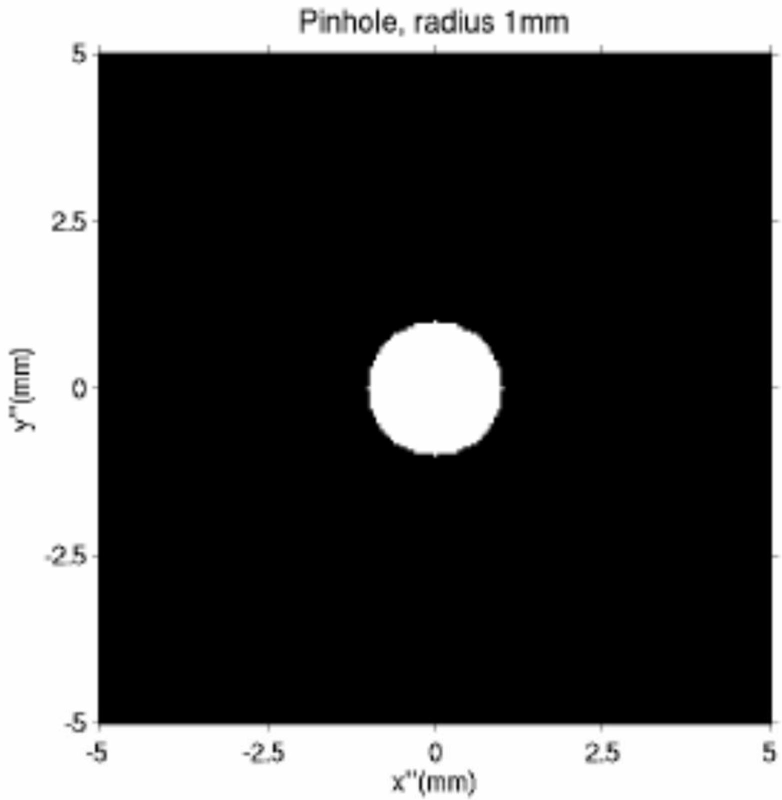
f_1=20cm
λ=0.5μm

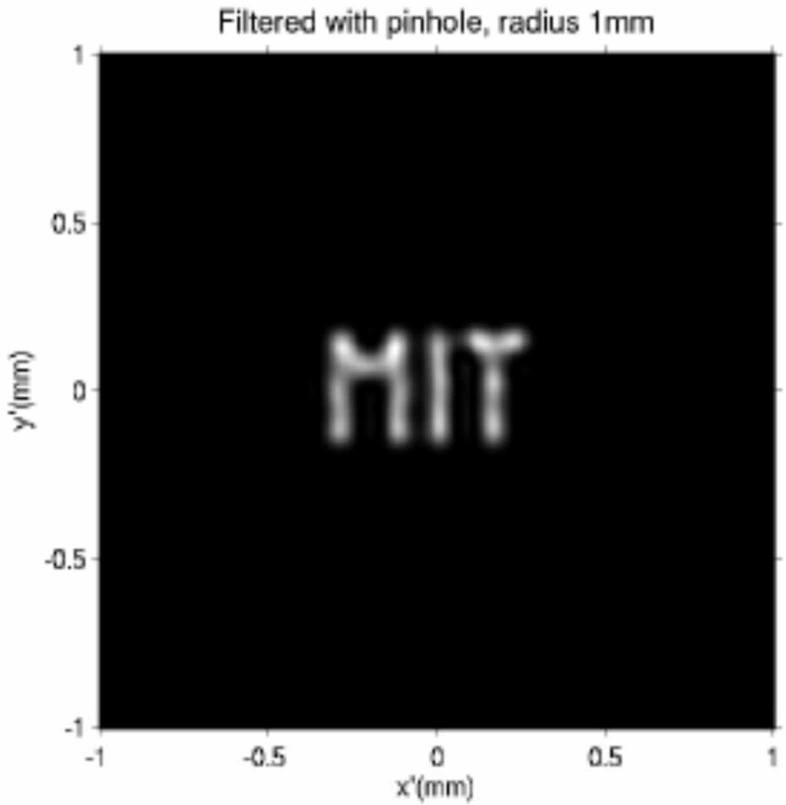
Fourier filter

Intensity @ image plane

Moderate low–pass filtering

(*aka* blurring)



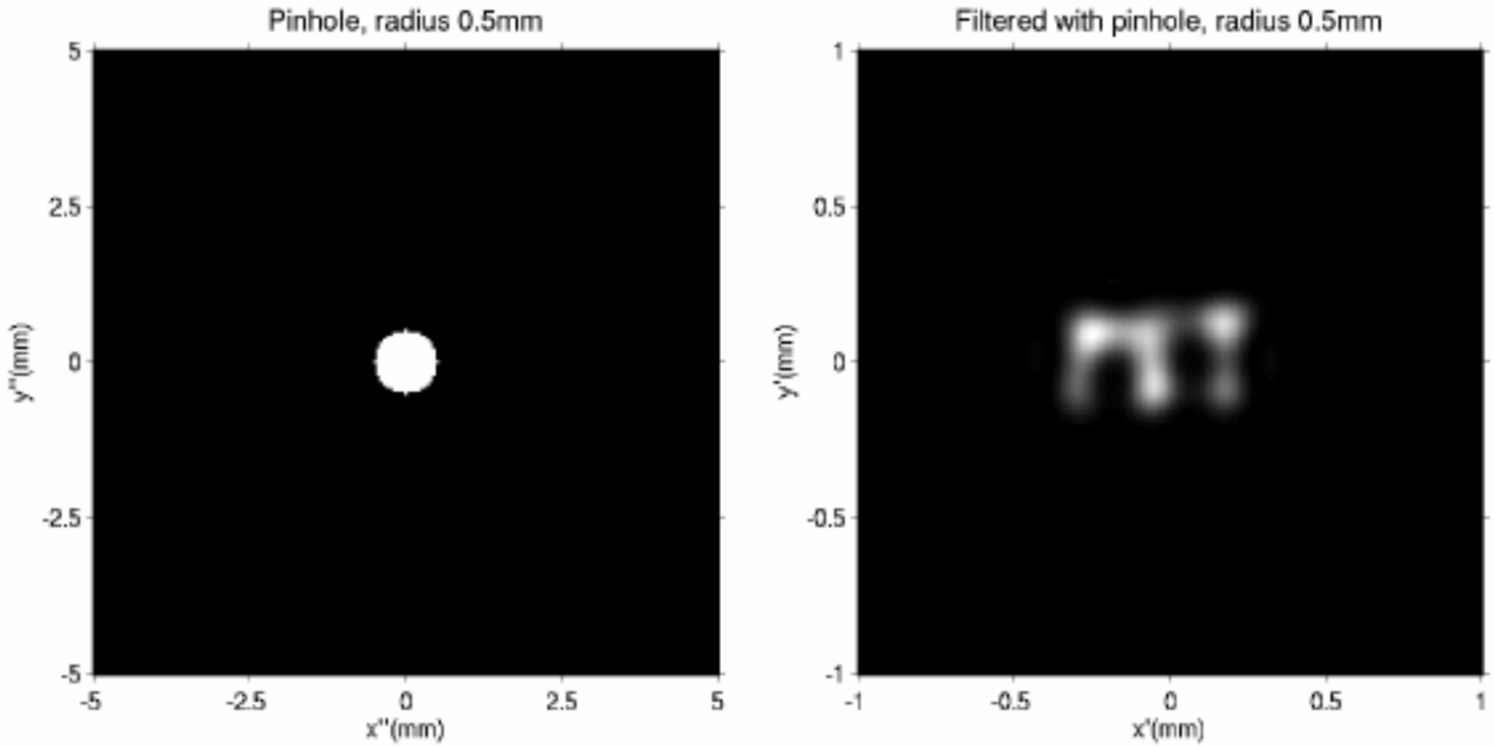


f_1=20cm
λ=0.5µm

Fourier filter

Intensity @ image plane

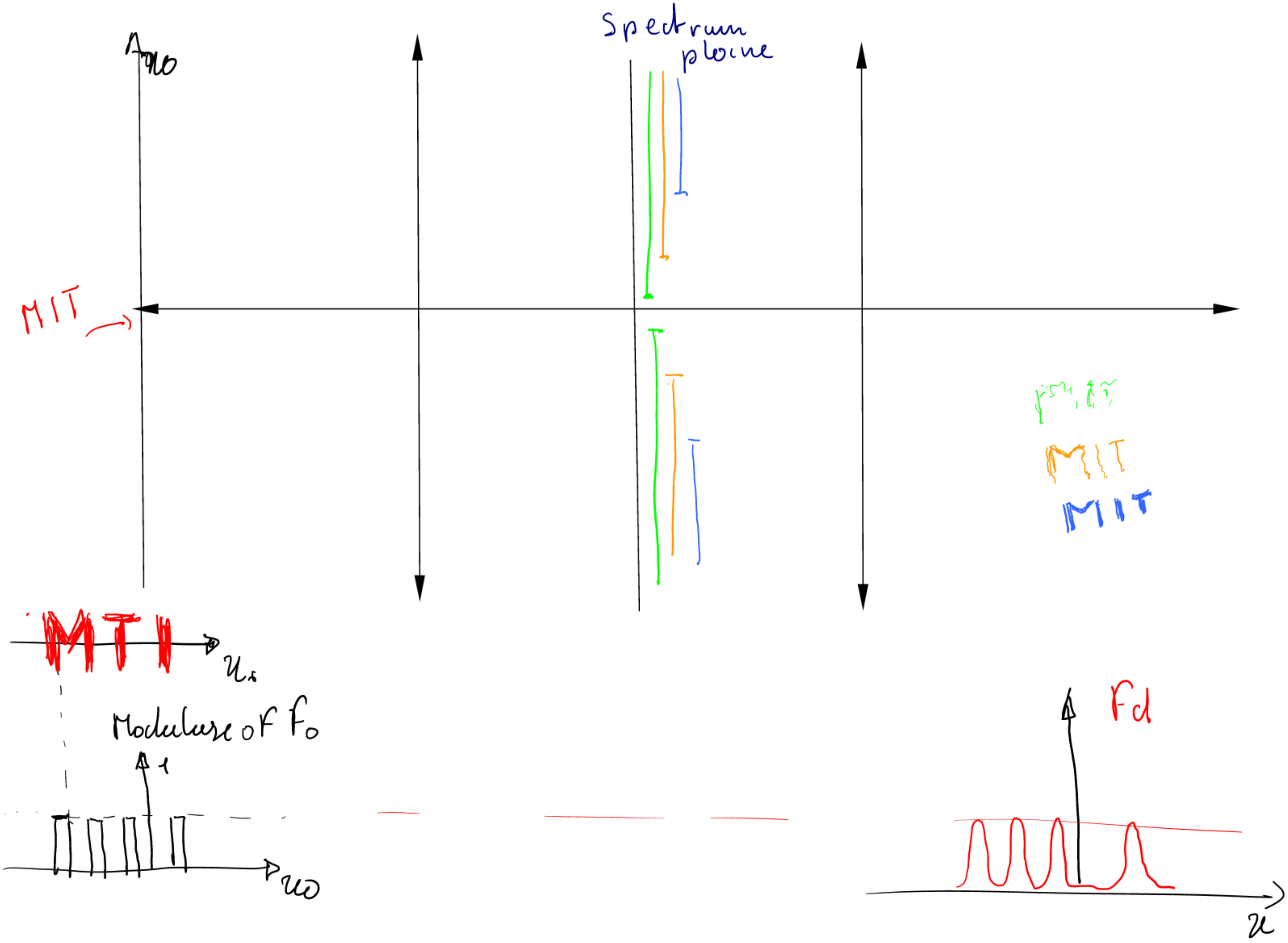
Strong low–pass filtering



f_1=20cm
λ=0.5μm

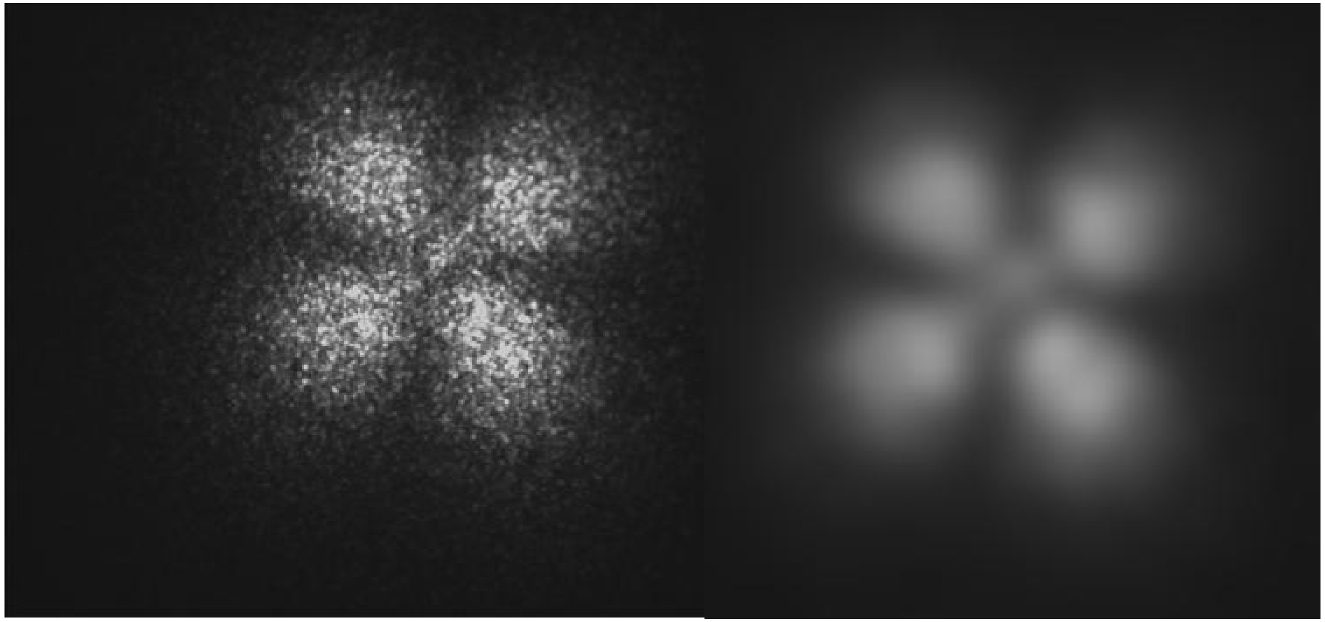
Fourier filter

Intensity @ image plane


Spectrum plane
MIT
MIT
MIT
Fd

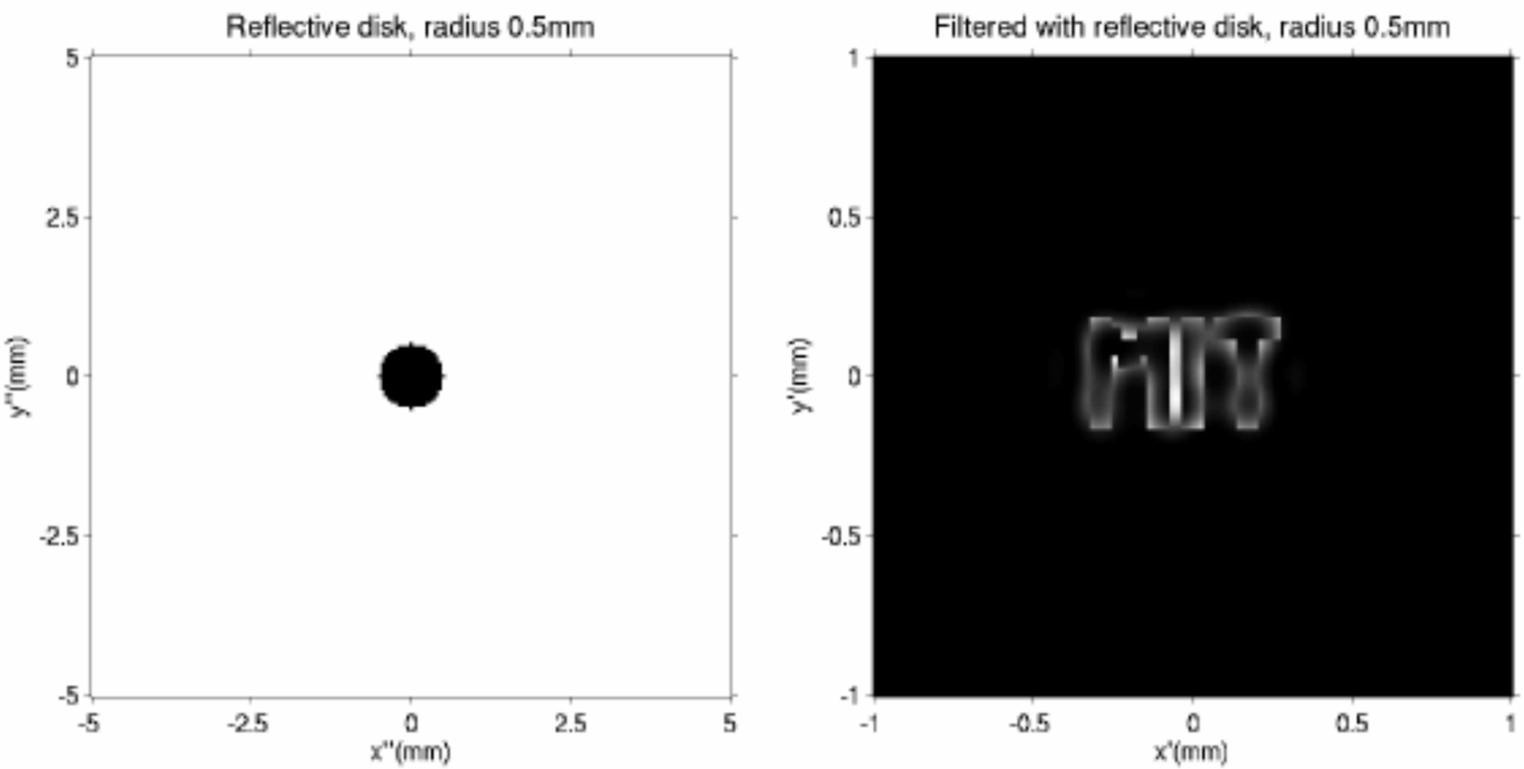
Low pass filter

Filtrage des bruits (par exemple lorsque le signal intéressant est dans les basses fréquences)



Low pass filter.

Moderate high–pass filtering



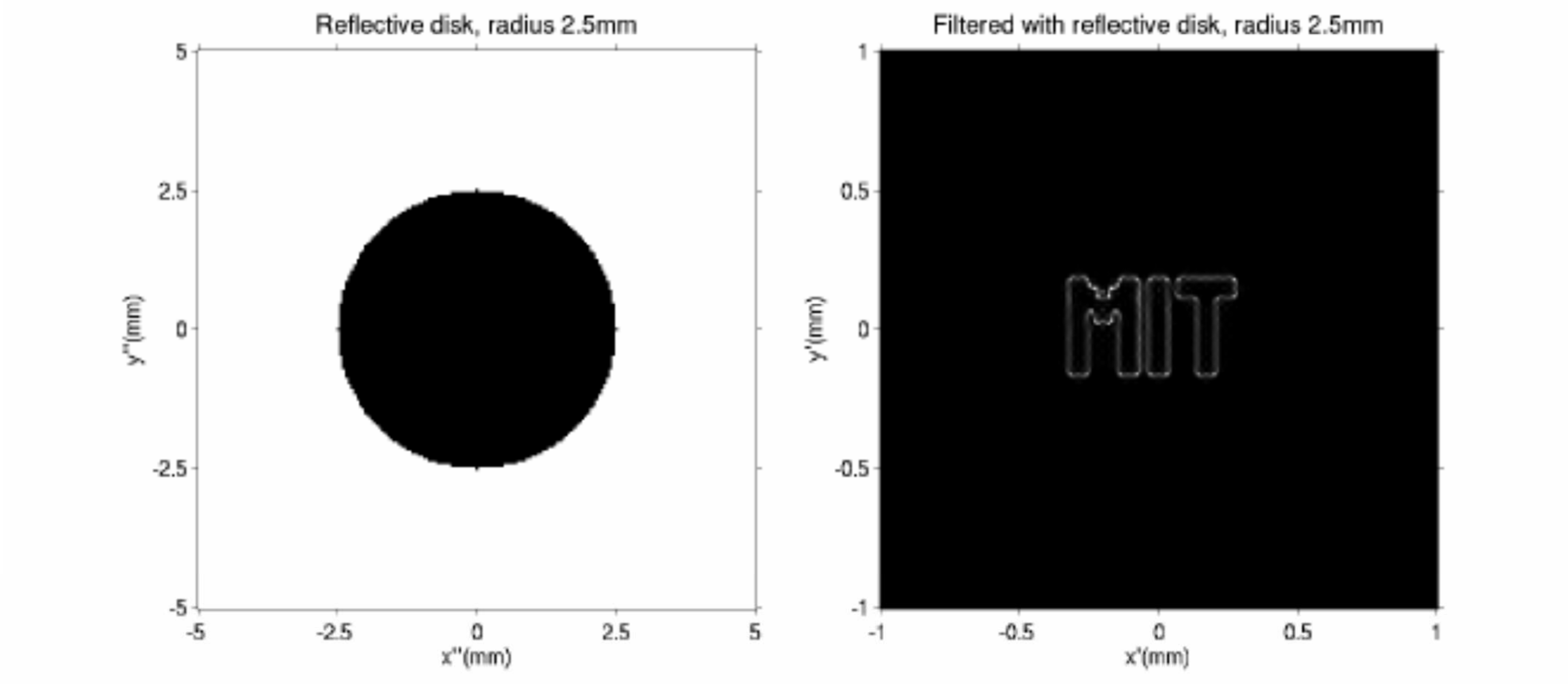
f_1=20cm
λ=0.5μm

Fourier filter

Intensity @ image plane

Strong high–pass filtering

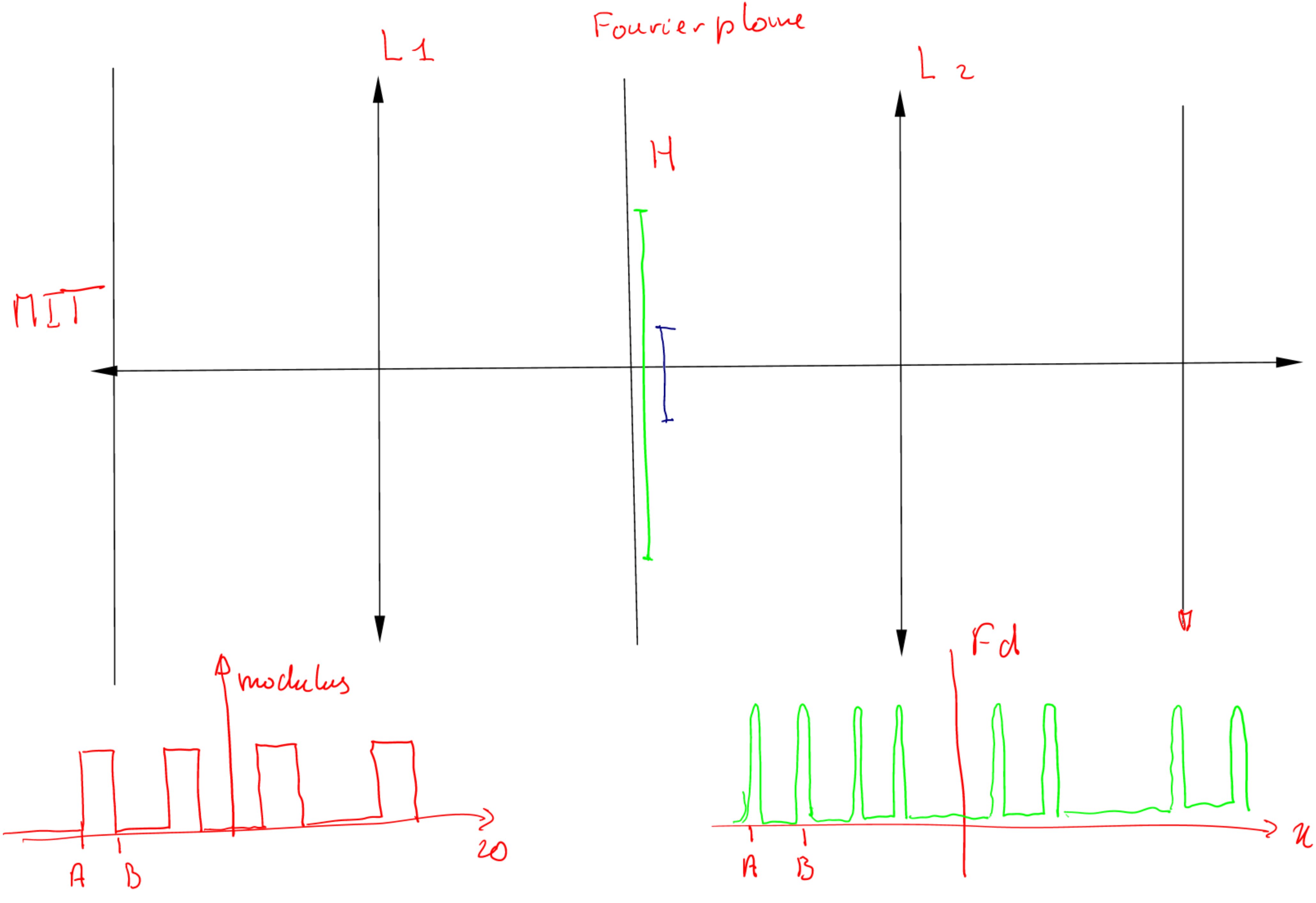
(*aka* edge enhancement)



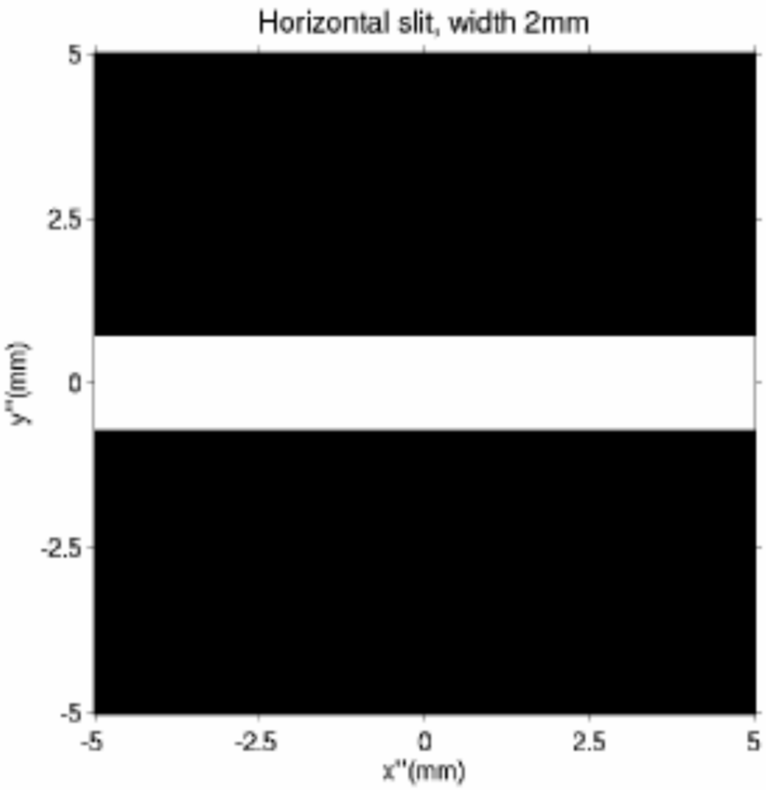
f_1=20cm
λ=0.5μm

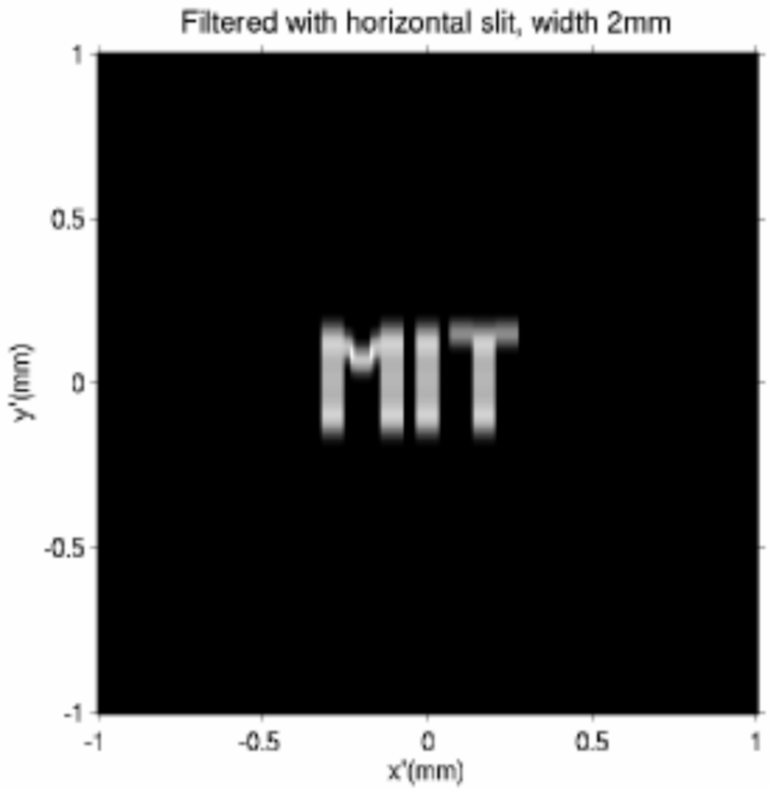
Fourier filter

Intensity @ image plane


Fourier plane
L 1
L 2
H
MIT
Fd
modulus
A
B
20
A
B
u

1-dimensional blurring



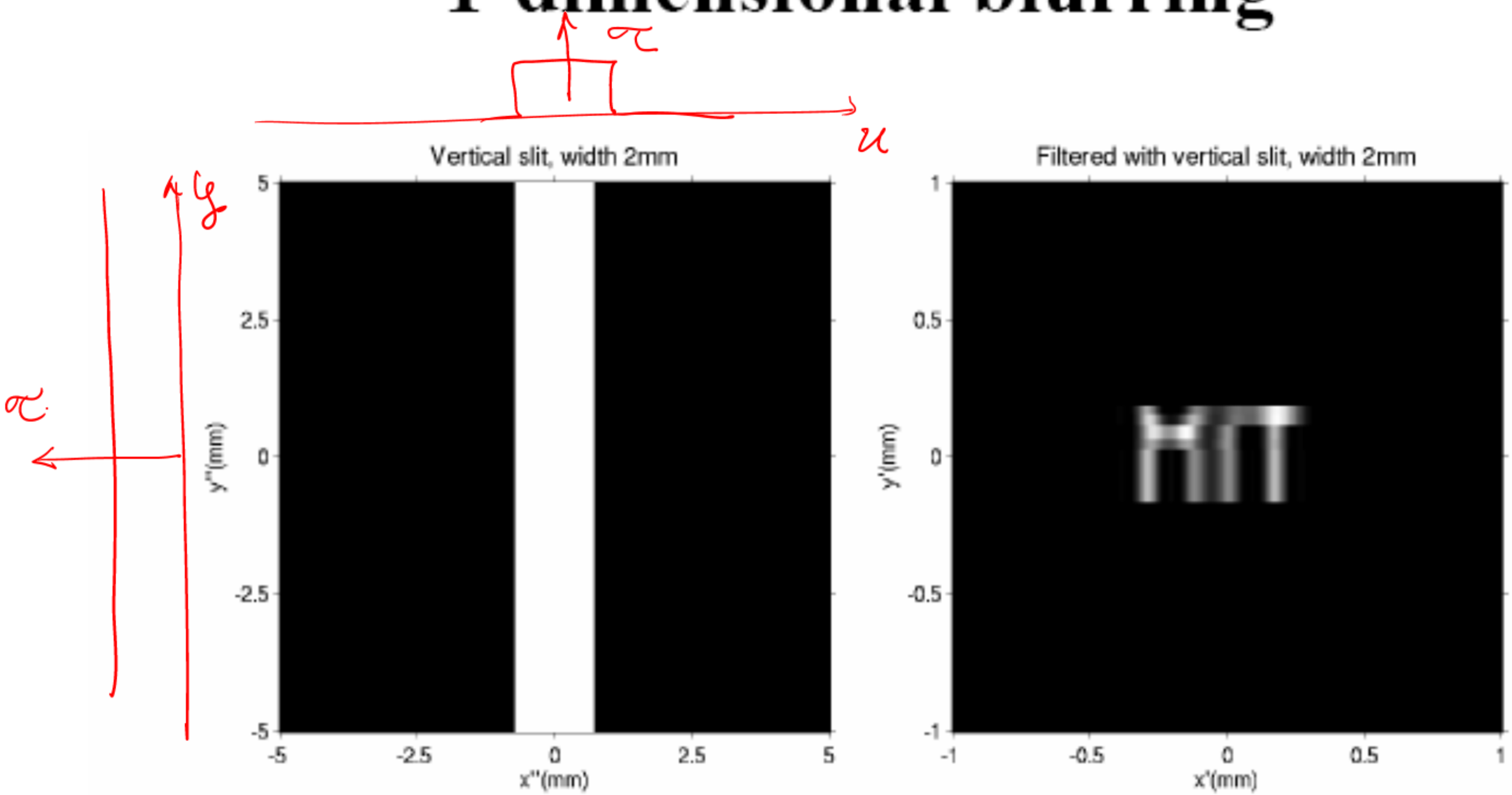


f_1=20cm
λ=0.5μm

Fourier filter

Intensity @ image plane

1-dimensional blurring

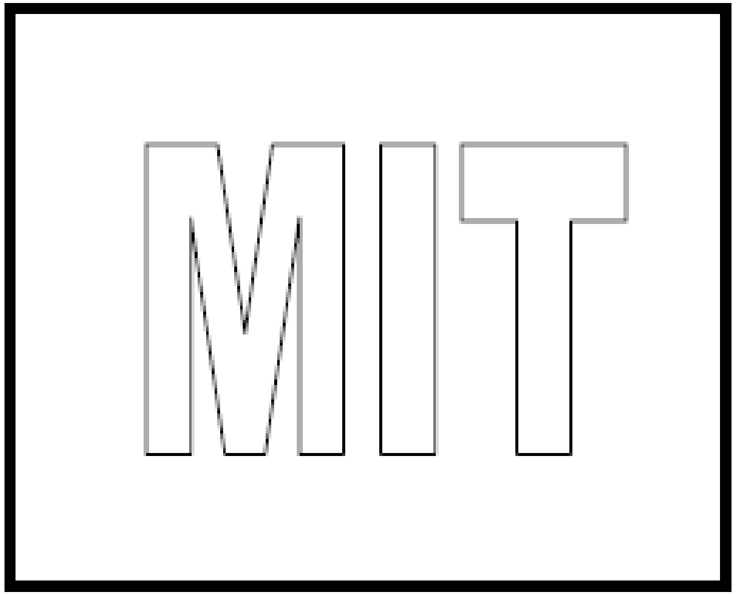


f_1=20cm
λ=0.5μm

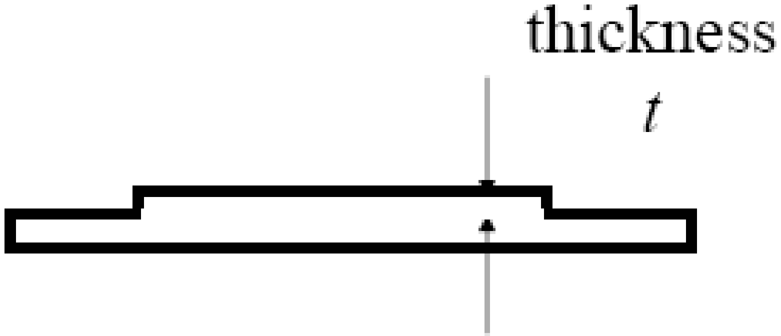
Fourier filter

Intensity @ image plane

Phase objects



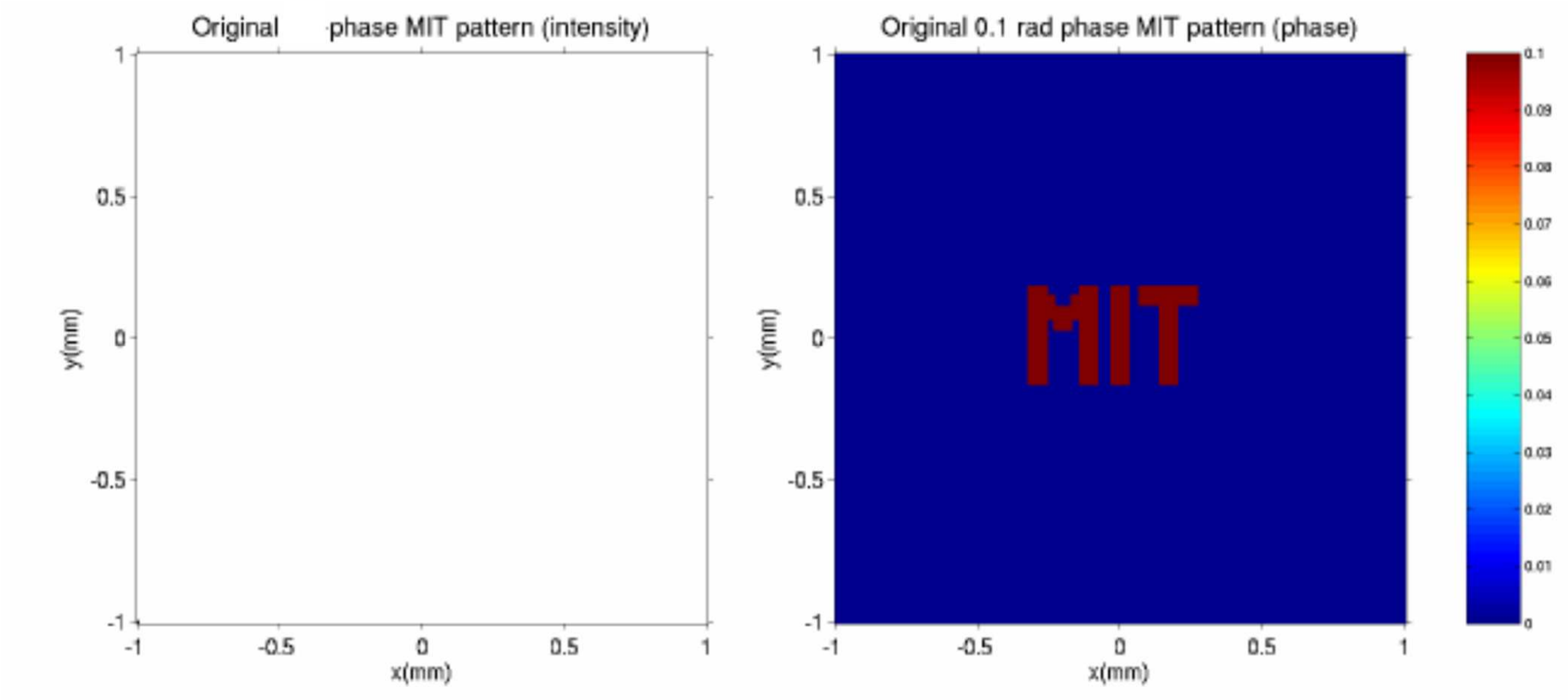
glass plate
(transparent)



protruding part
phase-shifts
coherent illumination
by amount $\varphi=2\pi(n-1)t/\lambda$

Often useful in imaging biological objects (cells, etc.)

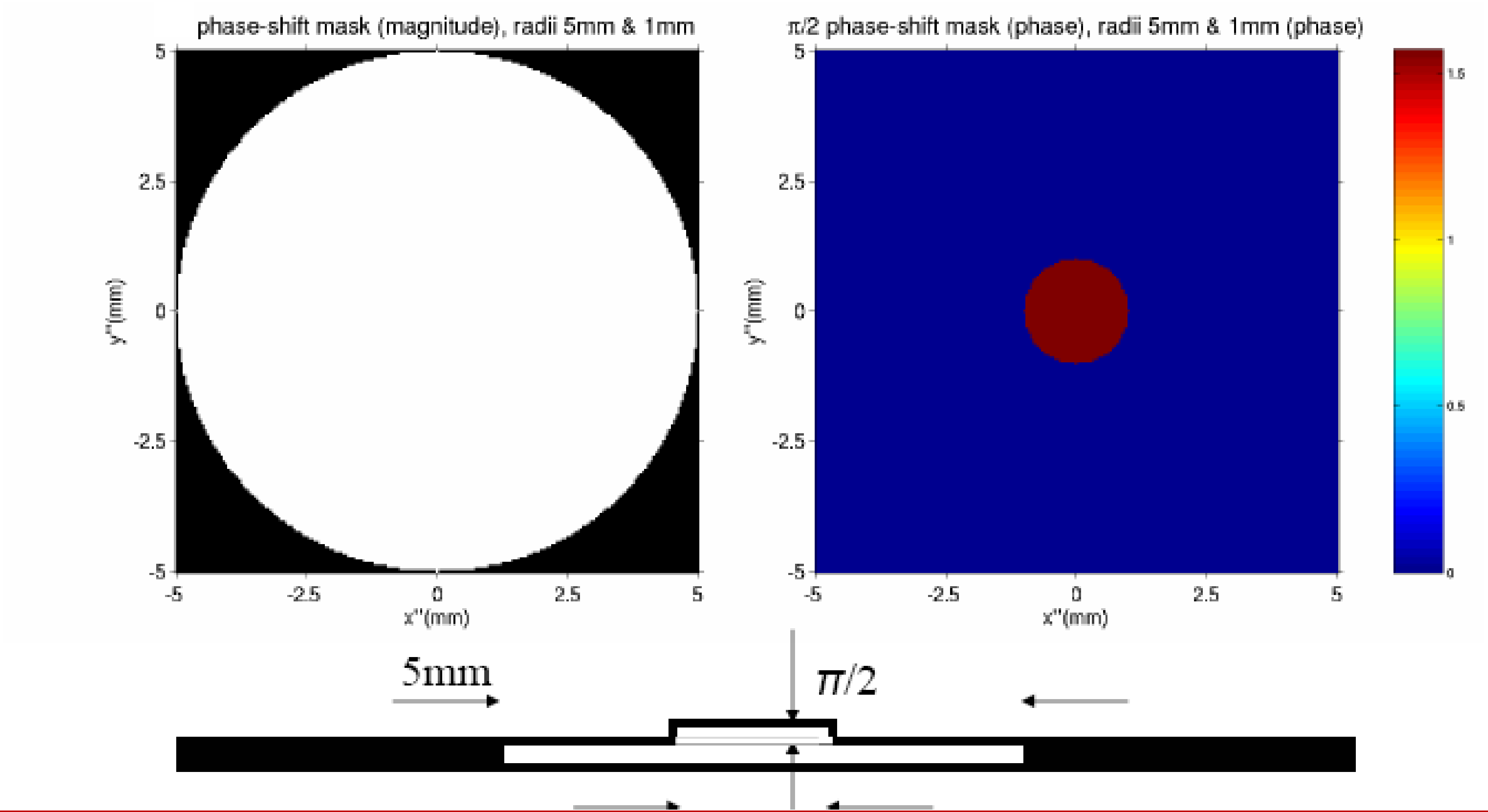
Viewing phase objects



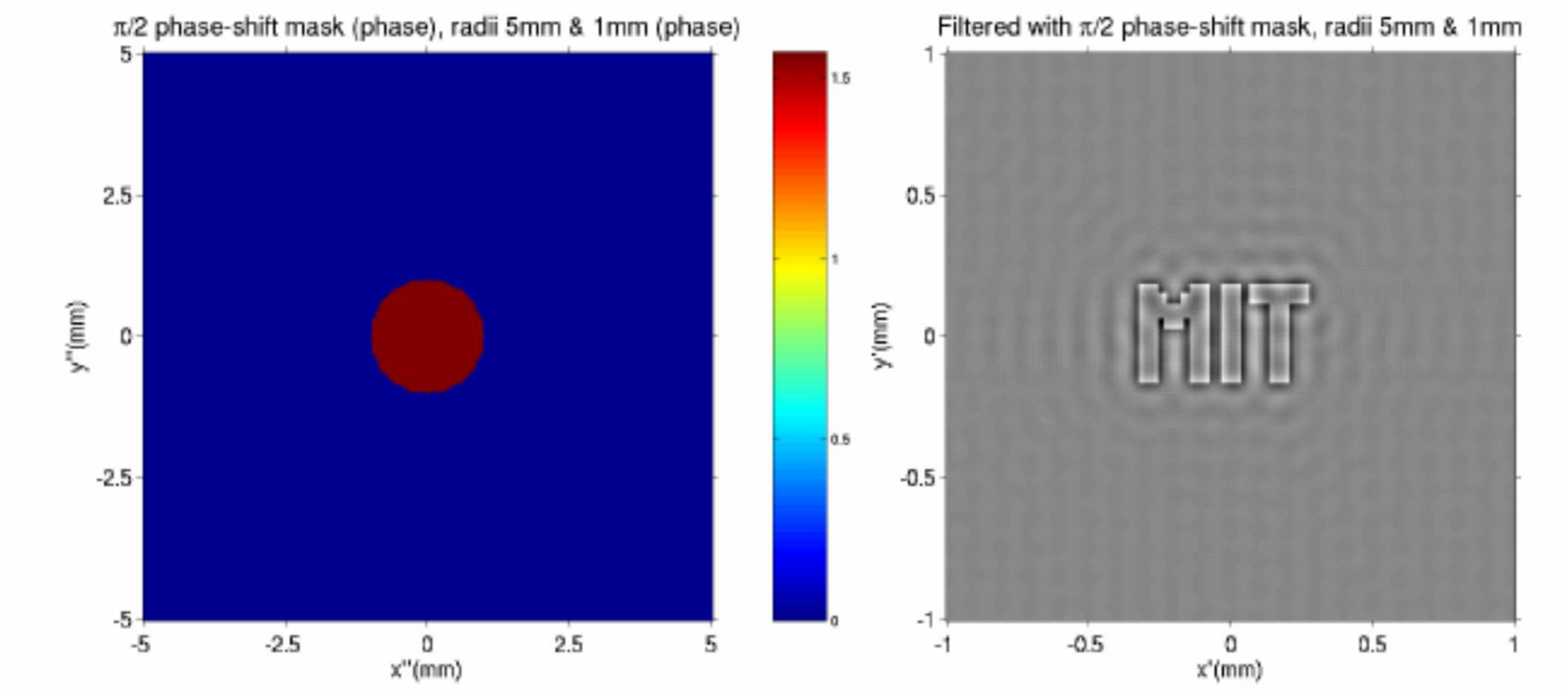
Intensity
(object is invisible)

Amplitude
(need interferometer)

Zernicke phase-shift mask



Imaging with Zernicke mask

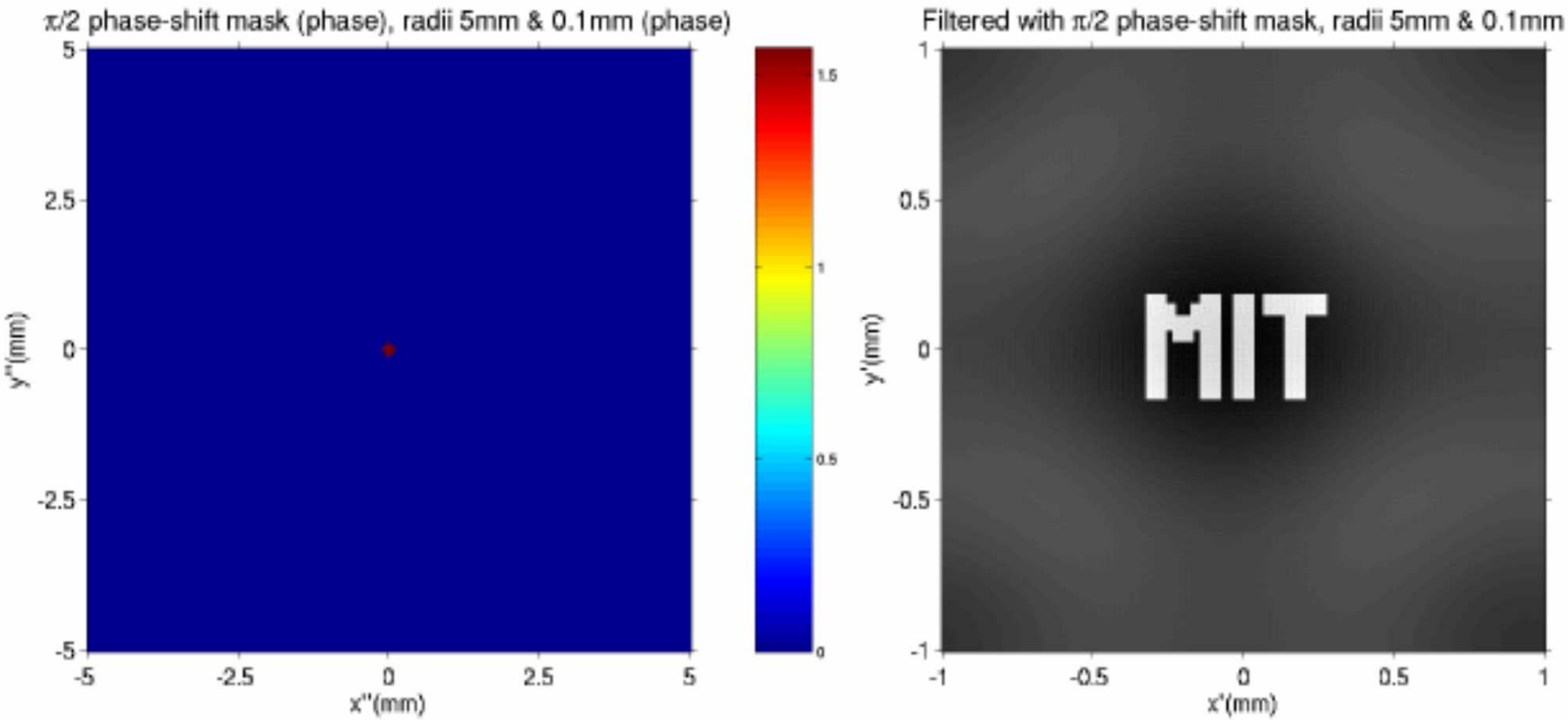


f_1=20cm
λ=0.5μm

Fourier filter

Intensity @ image plane

Imaging with Zernicke mask

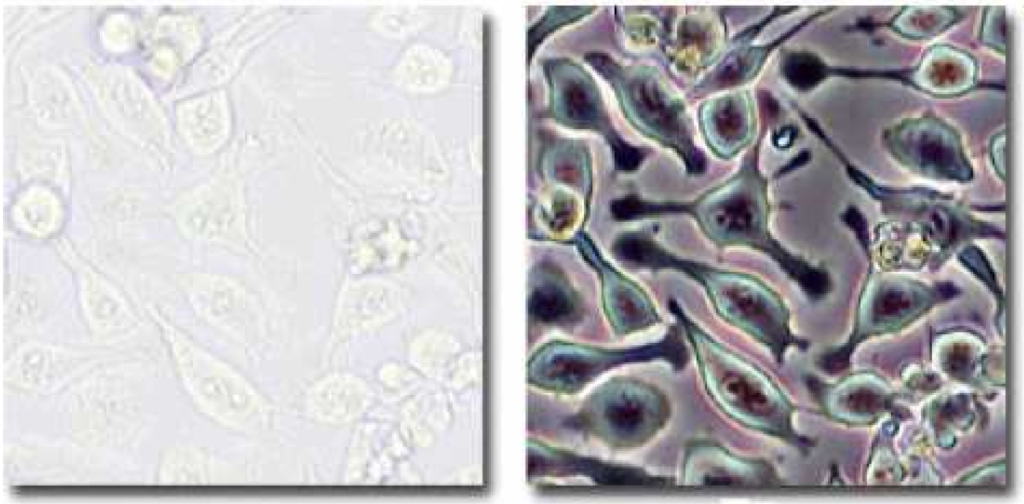


f_1=20cm
λ=0.5μm

Fourier filter

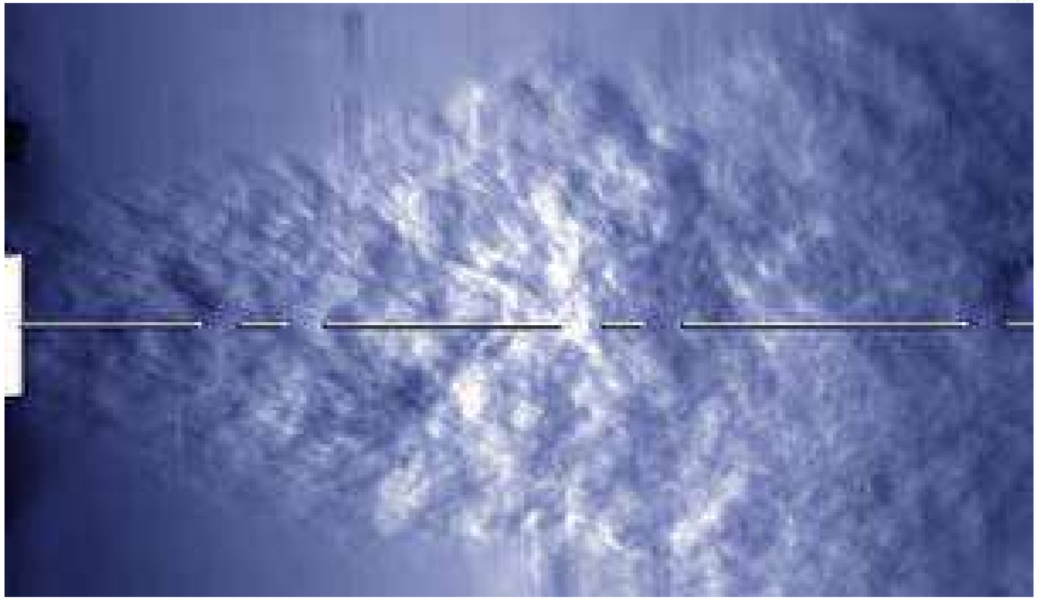
Intensity @ image plane

microscopy



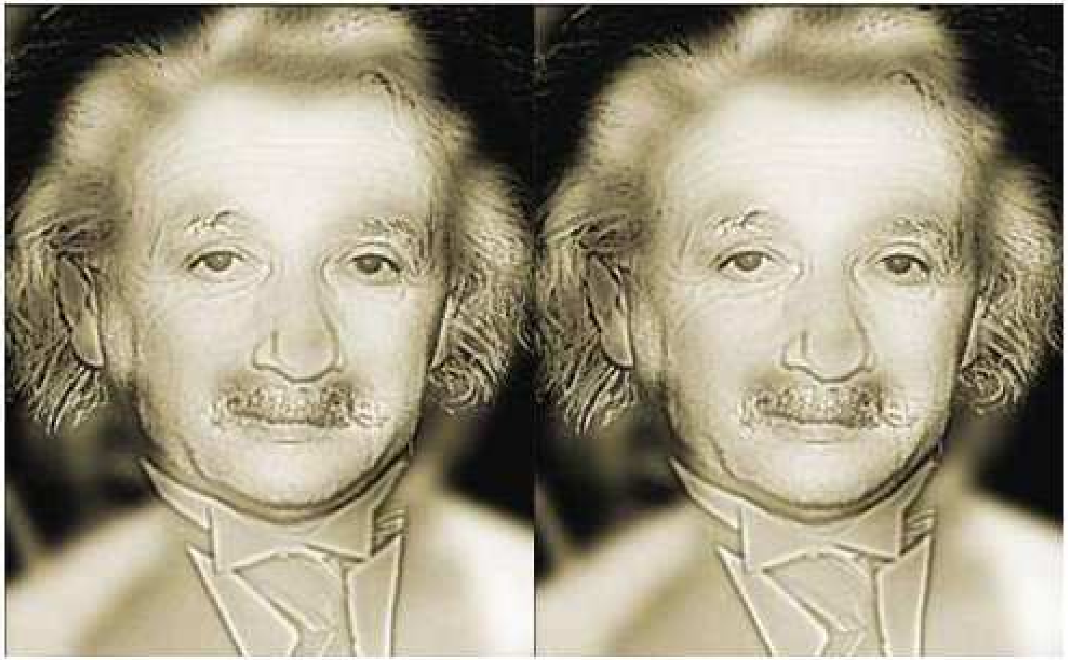
Fond clair   Contraste de phase

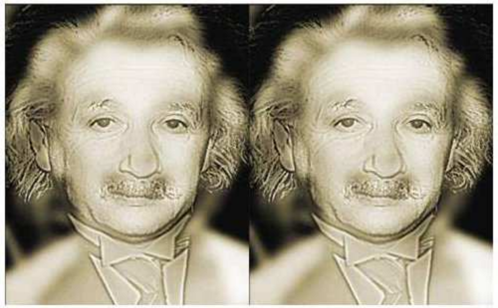
Fluid mechanics



Application to incoherent beams

Abraham Lincoln, par Salvador Dali





he/she « Marylin Einstein »