CHAPTER 5
SIGNAL PROPAGATION IN OPTICAL FIBER WITH CHROMATIC DISPERSION

Philippe DI BIN
University of Limoges
philippe.dibin@unilim.fr

Introduction

Introduction

Optical fiber links are mainly designed for information and signal transmission

→ Need to control the quality of the transmitted signal
→ Understanding the propagation properties of the optical signal in the optical fiber

Aim of the chapter 5

→ Calculation of time domain modulated signals after propagation
→ Management of the chromatic dispersion
→ Application to the propagation of a Gaussian pulse

Conditions

→ Single mode propagation
→ Application to any mode propagation, any fiber, any propagation media
→ Linear propagation only – No non-linear effect (Kerr, Pockels, Raman, …)

Properties of a modulated signal – Optical pulse

❑Complex modulation envelope (1)

- **Optical carrier : monochromatic wave at frequency $\boldsymbol{\nu_0}$**

Spectral complex amplitude $\vec{E}(x,y,z,\nu_0) = \vec{e}(x,y,z)\,\Psi(x,y)\,S(z,\nu_0)$

Unitary vector Polarisation — *Transverse field repartition* — *Spectral complex amplitude @ν_0* $S(z,\nu_0) = |S(z,\nu_0)|e^{j\phi(\nu_0)}$

Time domain expression

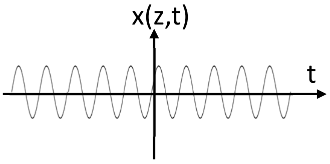
$$s(z,t) = S(z,\nu_0)e^{j2\pi\nu_0 t} = S(z,\nu_0)e^{j\omega_0 t} = |S(z,\nu_0)|\,e^{j(\omega_0 t+\phi(\nu_0))}$$

$$x(z,t) = \mathcal{Re}(s(z,t)) = |S(z,\nu_0)|\cos(\omega_0 t + \phi(\nu_0))$$

Properties

- Defined for $-\infty \le t \le \infty$
- Constant amplitude $|S(z,\nu_0)|$
- Constant frequency ν_0
- Constant temporal phase $\phi(\nu_0)$

Constant parameters of the wave
⇔ No information is carried



Properties of a modulated signal – Optical pulse

❑Complex modulation envelope (2)

- **Modulated optical carrier at a given z position**

Time domain expression

$$s(z,t) = a(z,t)e^{j\omega_0 t} = |a(z,t)|\,e^{j(\omega_0 t+\phi(t))}$$

$$x(z,t) = \mathcal{Re}(s(z,t)) = |a(z,t)|\cos(\omega_0 t + \phi(t))$$

$a(z,t)$ is the modulation complex envelop : $\boldsymbol{a(z,t) = |a(z,t)|\,e^{j\phi(t)}}$

Spectral domain

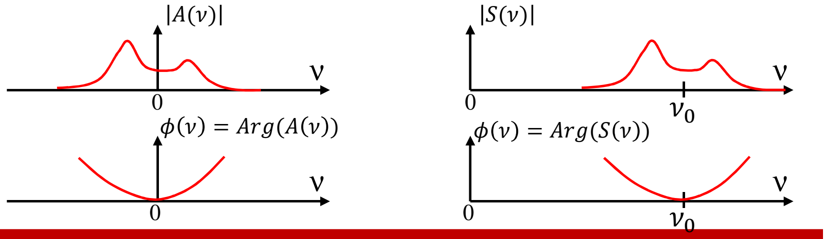
$$\mathbf{S}(\boldsymbol{z},\boldsymbol{\nu}) = Fourier\ Transform\big(s(z,t)\big) = FT\big(a(z,t)e^{j\omega_0 t}\big) = A(z,\nu) * \delta(\nu-\nu_0) = \boldsymbol{A(z,\nu-\nu_0)}$$

$A(z,\nu)$ modulation signal complex spectrum

$S(z,\nu)$ modulated signal complex spectrum

$$\boldsymbol{S(z,\nu) = A(z,\nu-\nu_0)}$$

at z position



Properties of a modulated signal – Optical pulse

❑Complex modulation envelope (3)

- **Temporal domain signal reconstruction**

The time domain signal linked to the signal spectrum or the modulation spectrum is the sum of all the individual spectral components $S(z,\nu)\,e^{j\omega t}$ defined by their :

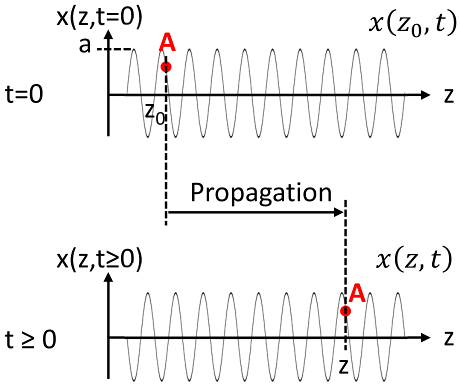
- Amplitude $|S(z,\nu)|$
- Pulsation $\omega = 2\pi\nu$
- Phase $\phi(z,\nu) = \arg(S(z,\nu))$

$$\boldsymbol{s(z,t) = \int_{-\infty}^{+\infty} S(z,\nu)\, e^{j\omega t}\, dt = \int_{-\infty}^{+\infty} A(z,\nu-\nu_0)\, e^{j\omega t}\, dt = FT^{-1}[S(z,\nu)]}$$

Phase velocities dispersion

❑Phase velocities – phase index (1)

- **Propagation of a continuous monochromatic wave**



$$x(z_0,t) = a\cos(\omega_0 t + \phi_{t0})$$

$$x(z,t) = a\cos\left(\omega_0\left(t - \frac{z-z_0}{\mathrm{v}_\phi}\right) + \phi_{t0}\right)$$

v_ϕ is the propagation velocity of a point A corresponding to a value of phase of the monochromatic signal.

- v_ϕ is the **phase velocity** defined only for a monochromatic signal

(one frequency – one phase)

- $n_e = \frac{c}{\mathrm{v}_\phi}$

is the **effective index** of the wave

Phase velocities dispersion

❑Phase velocities – phase index (2)

- Propagation of a continuous monochromatic wave

Propagation constant of the spectral component at ν_0

$$\beta(\nu_0) = \beta_0 = \frac{2\pi}{\lambda_0} n_e = k_0 n_e = \frac{2\pi\nu_0}{c} n_e = \frac{\omega_0}{c} n_e \quad \Leftrightarrow \quad \omega_0 = \beta_0 \frac{c}{n_e} = \beta_0 \mathrm{v}_\phi$$

$$\Leftrightarrow \mathrm{v}_\phi(\nu_0) = \frac{\omega_0}{\beta_0} \qquad \Leftrightarrow \mathrm{v}_\phi(\nu) = \frac{\omega}{\beta} = \frac{c}{n_e}$$

$$x(z,t) = a\cos\left(\omega_0\left(t - \frac{z - z_0}{v_\phi}\right) + \phi_{t0}\right) = a\cos(\omega_0 t - \beta_0(z - z_0) + \phi_{t0})$$

$$\boldsymbol{x(z,t) = a\cos(\overbrace{\omega_0 t - \beta_0 z - \underbrace{\beta_0 z_0 + \phi_{t0}}_{\textit{Origin phase}\ \phi_0}}^{\textit{Spatio-temporal phase}\ \phi(z,t)})}$$

Temporal phase $\boldsymbol{\phi(t)}$ ***Spatial phase*** $\boldsymbol{\phi(z)}$

Phase velocities dispersion

❑Phase velocities dispersion law

- The refractive index of a material is wavelength (frequency) dependent
- Guiding effect make the effective index of the wave dependent on wavelength (frequency)

$$n_e = n_e(\omega) \Leftrightarrow \beta = \frac{2\pi}{\lambda_0} n_e(\omega) = k_0 n_e(\omega) = \beta(\omega)$$

The propagation constant depends on material properties and guide structure

We can generalize from a Taylor development :

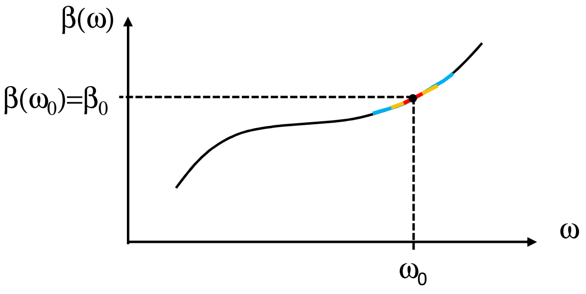
$$\beta(\omega) = \beta(\omega_0) + (\omega - \omega_0)\left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0} + \frac{1}{2}(\omega - \omega_0)^2 \left.\frac{\partial^2\beta}{\partial\omega^2}\right|_{\omega_0} + \ \ldots$$

$$\boldsymbol{\Delta\beta(\omega) = \beta(\omega) - \beta(\omega_0) = \sum_{n=1}^{\infty} \left.\frac{\partial^n \beta(\omega)}{\partial\omega^n}\right|_{\omega_0} \frac{(2\pi f)^n}{n!} = \sum_{n=1}^{\infty} \beta_n \frac{(2\pi)^n}{n!} f^n}$$

with $\boldsymbol{\beta_n = \left.\frac{\partial^n \beta(\omega)}{\partial\omega^n}\right|_{\omega_0}}$ **and** $\boldsymbol{f = \nu - \nu_0}$

Phase velocities dispersion

❑Phase velocities dispersion law



The greater is the number of known terms,
the wider bandwidth is described around ω_0

$\beta_0 = \beta(\omega_0)$ @ ω_0 (monochromatic wave)

+

$\beta_1 = \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0}$ @ narrow bandwidth around ω_0 (large optical pulses)

+

$\beta_2 = \left.\frac{\partial^2\beta}{\partial\omega^2}\right|_{\omega_0}$ @ larger bandwidth around ω_0 (short optical pulses)

+

$\beta_3 = \left.\frac{\partial^3\beta}{\partial\omega^3}\right|_{\omega_0}$ @ very larger bandwidth around ω_0 (ultra-short optical pulses)

Each spectral component with pulsation $\omega = 2\pi\nu$ propagates with $\beta(\omega) = \frac{2\pi}{\lambda}\ n_e(\omega)$

$$\boldsymbol{s(z,t) = a(\omega)e^{j(\omega_0 t-\beta(\omega)z)}}$$
$$\boldsymbol{x(z,t) = a(\omega)\cos(\omega_0 t-\beta(\omega)z)}$$

$$\mathrm{v}_\phi(\omega) = \frac{c}{n_e(\omega)} = \frac{2\pi c}{\lambda\,\beta(\omega)} = \frac{2\pi\nu}{\beta(\omega)}$$

$$\boxed{\boldsymbol{\mathrm{v}_\phi(\omega) = \frac{\omega}{\beta}}}$$ Phase velocities dispersion

---

Group velocities dispersion

❑Complex modulation envelop propagation (1)

- **Time domain signal general expression**

$$s(z,t) = a(z,t)\ e^{j(2\pi\nu_0 t-\beta_0 z)}$$

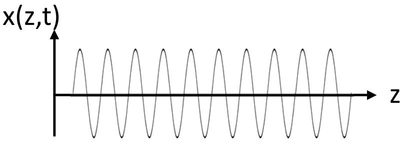
Complex modulation Envelop (amplitude and phase modulation)

ν_0 frequency optical carrier propagating with $\beta_0=\beta(\omega_0)=\beta(2\pi\nu_0)$

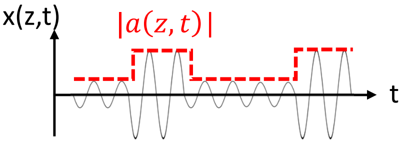
- **Information carrying signal** (useful signal is modulation signal)

$$a(z,t) = s(z,t)\ e^{j(\beta_0 z-2\pi\nu_0 t)}$$

Unmodulated optical carrier
→ no information



Modulated optical carrier
→ information transmission



Group velocities dispersion

❑Complex modulation envelop propagation (2)

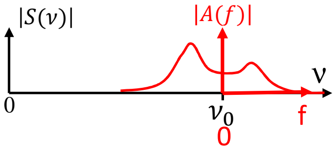
- **At emission position (z=0)**

$$s(z=0,t) = a(z=0,t)\, e^{j(2\pi\nu_0 t)}$$

$$S(z=0,\nu) = A(0,\nu) * \delta(\nu - \nu_0)$$
$$S(0,\nu) = A(0,\nu - \nu_0) = A(0,f)$$

f : low frequencies compared to ν

$$\boldsymbol{S(0,\nu) = A(0,f)}$$



- **At z>0**

$$s(z,t) = a(z,t)\, e^{j(2\pi\nu_0 t - \beta_0 z)}$$
$$S(z,\nu) = e^{j(-\beta_0 z)}\ A(z,\nu) * \delta(\nu - \nu_0)$$

$$A(z,f) = e^{j(\beta_0 z)}\ S(z,\nu)$$

Let's determinate $S(z,\nu)$ for each position z. Each spectral component propagates with $\beta(\omega)$, leading to :

$$A(z,f) = e^{j(\beta_0 z)}\ S(z=0,\nu)\, e^{-j\beta(\omega)z}$$
$$A(z,f) = A(z=0,f)\, e^{-j(\beta(\omega)-\beta_0)z}$$

$$\boldsymbol{A(z,f) = A(z=0,f)\quad e^{-j\Delta\beta(\omega)z}}$$

Spectrum of complex modulation at z	*Spectrum of complex modulation at z=0*	*Relative spectral phase shift induced by propagation*

Group velocities dispersion

❑Complex modulation envelop propagation (3)

- **Calculation of the time domain signal** from spectrum after propagation

$$a(z,t) = \int_{-\infty}^{+\infty} \left[A(o,f) e^{j\Delta\beta(\omega)z}\right]\ e^{j2\pi f t}\ df$$

which corresponds to an inverse-Fourier integral

$$\boldsymbol{a(z,t) = a(0,t) * FT^{-1}\left[e^{j\Delta\beta(\omega)z}\right]}$$

Giving the relation between the input signal and the output signal after propagation
$\Delta\beta(\omega)$ describing the propagation medium

Group velocities dispersion

❑Equation of the evolution of complex modulation envelope (1)

In a general case, optical power attenuation induced by the propagation must be described by its attenuation coefficient α (Neper/m)

$$A(z,f) = A(z=0,f)\, e^{-j\Delta\beta(\omega)z}\, e^{-\frac{\alpha z}{2}} \qquad with\ \alpha = \frac{1}{L}\ln\frac{P(z)}{P(z=0)}$$

$$a(z,t) = \int_{-\infty}^{+\infty} \underbrace{\left[A(o,f)e^{-\left(j\Delta\beta(\omega)+\frac{\alpha}{2}\right)z}\right]}_{A(z,f)}\ e^{j2\pi ft}\, df$$

$$\frac{\partial a(z,t)}{\partial z} = \int_{-\infty}^{+\infty} -\left(j\Delta\beta(\omega)+\frac{\alpha}{2}\right)A(z,f)\ \ e^{j2\pi ft}\, df = -\int_{-\infty}^{+\infty}\left(\frac{\alpha}{2}+j2\pi\beta_1 f + j4\pi^2\pi\frac{\beta_2}{2}f^2+\cdots\right)A(z,f)\ \ e^{j2\pi ft}\, df$$

Group velocities dispersion

❑Equation of the evolution of complex modulation envelope (2)

The time domain derivatives are

$$\left.\begin{aligned}\frac{\partial a(z,t)}{\partial t} &= j2\pi\int_{-\infty}^{+\infty} f\,A(z,f)\ \ e^{j2\pi ft}\, df\\ \frac{\partial^2 a(z,t)}{\partial t^2} &= -4\pi^2\int_{-\infty}^{+\infty} f\,A(z,f)\ \ e^{j2\pi ft}\, df\end{aligned}\right\}$$

we introduce these expressions in $\frac{\partial a(z,t)}{\partial z}$

$$\frac{\partial a(z,t)}{\partial z} = -\int_{-\infty}^{+\infty}\left(\frac{\alpha}{2}+j2\pi\beta_1 f + j4\pi^2\,\frac{\beta_2}{2}f^2+\cdots\right)A(z,f)\ \ e^{j2\pi ft}\, df$$

$$\frac{\partial a(z,t)}{\partial z} = -\int_{-\infty}^{+\infty}\left(\frac{\alpha}{2}A(z,f)\right)\, e^{j2\pi ft}\, df - \mathbf{j2\pi}\beta_1\int_{-\infty}^{+\infty}\left(\boldsymbol{fA(z,f)}\ \boldsymbol{e^{j2\pi ft}}\ \boldsymbol{df}\right. - \mathbf{j4\pi^2}\frac{\beta_2}{2}\int_{-\infty}^{+\infty}\left(\boldsymbol{f^2A(z,f)}\right)\ \boldsymbol{e^{j2\pi ft}}\ \boldsymbol{df}+\cdots$$

$$\frac{\partial a(z,t)}{\partial z} = -\frac{\alpha}{2}\,FT^{-1}[A(z,f)] - \beta_1\frac{\boldsymbol{\partial a(z,t)}}{\boldsymbol{\partial t}} + j\frac{\beta_2}{2}\,\frac{\boldsymbol{\partial^2 a(z,t)}}{\boldsymbol{\partial t^2}}$$

$$\boxed{\frac{\partial a(z,t)}{\partial z} + \beta_1\frac{\partial a(z,t)}{\partial t} + j\frac{\beta_2}{2}\frac{\partial^2 a(z,t)}{\partial t^2} = -\frac{\alpha}{2}a(z,t)}$$

Differential equation of the evolution of the complex envelop along the propagation including losses and dispersion up to the 2nd order.

Group velocities dispersion

❑Group velocity v_g (1)

- **Carrier modulation induces spectrum broadening**

  ⇔ a set of frequencies ν propagates in the fiber with their propagation constant β(ω)

- **First case** : optical spectrum is narrow around ν_0 ⇔ Development of β(ω) at first order

$$\beta(\omega)=\beta(\omega_0)+(\omega-\omega_0)\left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0} \quad\Leftrightarrow\quad \Delta\beta(\omega)=\beta(\omega)-\beta(\omega_0)=(\omega-\omega_0)\left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0} = 2\pi f\ \beta_1$$

$$a(z,t)=\int_{-\infty}^{+\infty}\left[A(0,f)e^{j2\pi f\ \beta_1 z}\right]\ e^{j2\pi f t}\ df \quad = FT^{-1}\left[A(0,f)e^{j2\pi f\ \beta_1 z}\right] \quad = \mathrm{a}(0,\mathrm{t})*\delta(t-\beta_1 z)$$

$$\boldsymbol{a(z,t)=a(0,t-\beta_1 z)=a\left(0,t-T_g\right)}$$ Time domain translation of the signal without deformation

$$\boxed{\boldsymbol{T_g=\frac{z}{\mathrm{v}_g}=\beta_1 z \quad\Leftrightarrow\quad \mathrm{v}_g=\frac{1}{\beta_1}=\frac{d\omega}{d\beta}}}$$

T_g and v_g : **group time** and **group velocity** (frequencies group around n_0), respectively, i.e. the time and the speed of propagation of the modulation

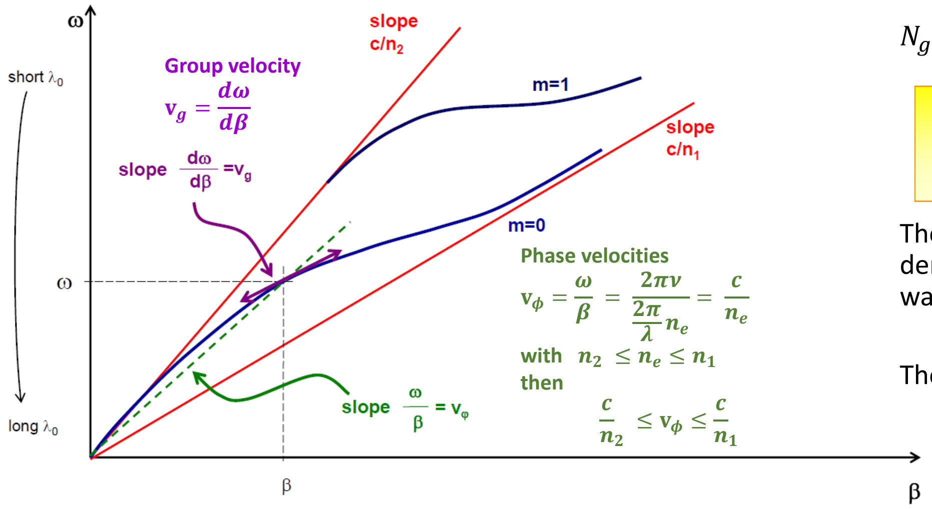
$$\boldsymbol{\phi(f)=2\pi f\ \beta_1 z}$$ Linear spectral phase term corresponding to a time domain translation of the signal

Module Title : Linear propagation in optical fibers 2021-2022 - 15 -
Copyright notice : This material can be freely used within the E.M.I.M.E.O. Erasmus Mundus consortium. Explicit authorisation of the authors is required for its use outside this E.M.I.M.E.O. consortium. This learning Programme has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Group velocities dispersion

❑Group velocity v_g (2)

Remember chap III of Dominique PAGNOUX course



❑Group index N_g

$$N_g=\frac{c}{\mathrm{v}_g}=c\frac{d\omega}{d\beta}=c\frac{d(kn_e)}{d(kc)}=\frac{d(kn_e)}{dk}$$

$$N_g=n_e+k\frac{dn_e}{dk} \qquad \left(\text{with } \mathrm{k}=\frac{2\pi}{\lambda}\right)$$

$$\boxed{\boldsymbol{N_g=n_e-\lambda\frac{dn_e(\lambda)}{d\lambda}}}$$

The group index is related to the first derivative of n_e with respect to the wavelength λ.

The group time $\mathbf{T_g}$ is

$$\boldsymbol{T_g=\frac{z}{\mathrm{v}_g}=\frac{z}{c}N_g}$$

Module Title : Linear propagation in optical fibers 2021-2022 - 16 -
Copyright notice : This material can be freely used within the E.M.I.M.E.O. Erasmus Mundus consortium. Explicit authorisation of the authors is required for its use outside this E.M.I.M.E.O. consortium. This learning Programme has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Group velocities dispersion

❑Mobile temporal axis (relative time) (1)

- **Second case :** broader spectrum around ν_0 ⇔ Development of $\beta(\omega)$ at second order

$$\beta(\omega) = \beta(\omega_0) + (\omega - \omega_0)\left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0} + \frac{1}{2}(\omega - \omega_0)^2\left.\frac{\partial^2\beta}{\partial\omega^2}\right|_{\omega_0} \Leftrightarrow$$

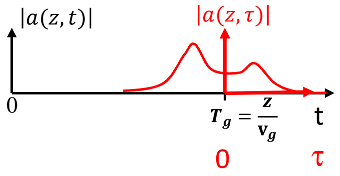
$$\boldsymbol{\Delta\beta(\omega) = 2\pi f\, \beta_1 + 2\pi^2 f^2\, \beta_2}$$

Related to time group T_g Time delay only No signal distortion (for $2\pi f\,\beta_1$)

Related to Signal distortion (for $2\pi^2 f^2\,\beta_2$)

Interest is on signal distortion, not on propagation time

⇒ Definition of a relative time axis



$$\boldsymbol{\tau} = \boldsymbol{t} - \boldsymbol{T_g} = t - \frac{z}{v_g} = t - \beta_1 z$$

$$\boldsymbol{t = \tau + T_g = \tau + \beta_1 z}$$

Group velocities dispersion

❑Mobile temporal axis (relative time) (2)

- **Time domain signal after propagation**

$$a(z,\tau) = \int_{-\infty}^{+\infty} \left[A(0,f)e^{-j(2\pi f\,\beta_1 + 2\pi^2 f^2\,\beta_2)z}\right]\, e^{j2\pi f(\tau+\beta_1 z\,)}\, df$$

$$\boldsymbol{a(z,\tau) = \int_{-\infty}^{+\infty} \left[A(0,f)e^{-j2\pi^2 f^2\,\beta_2 z}\right]\, e^{j2\pi f(\tau\,)}\, df}$$

✓ β_1, T_g, v_g disappeared from this equation : propagation time is not taken into account

✓ $e^{j2\pi^2 f^2\,\beta_2 z}$ is the distortion term :

At z=0 or if $\beta_2 = 0$: no distortion

⇒ we are like in the first case (1st order Taylor development), propagation without distortion

⇒ $a(z,\tau) = a(0,t)$

For simplification purpose, we change the notation : τ **becomes t but is still a relative time**

$$\boldsymbol{a(z,\tau) = a(z,t)}$$

Group velocities dispersion

❑Chromatic dispersion (1)

- **Time domain signal after propagation**

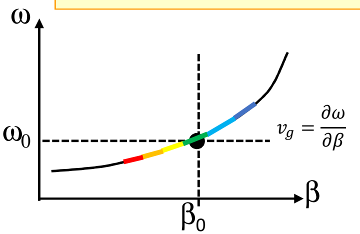
$$a(z,t)=\int_{-\infty}^{+\infty}\left[A(0,f)e^{-j2\pi^2 f^2\,\beta_2 z}\right]\,e^{j2\pi f(t)}\,df$$

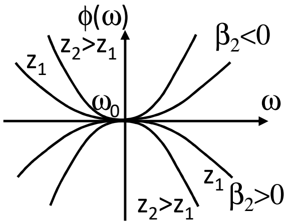
$$\boldsymbol{a(z,t)=a(0,t)*FT^{-1}\left[e^{-j2\pi^2 f^2\,\beta_2 z}\right]}$$

$$\boldsymbol{A(z,f)=A(0,f)\;.\;e^{-j2\pi^2 f^2\,\beta_2 z}=A(0,f)\,.\,e^{j\phi(f)}}$$

Initial spectrum (complex)

Spectral phase shift $\phi(f)=-2\pi^2 f^2\,\beta_2 z$





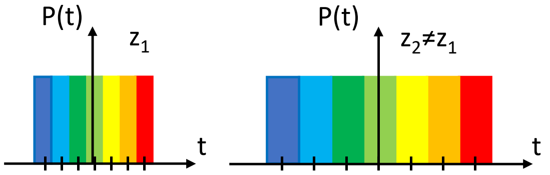
$\phi(f)=-2\pi^2 f^2\,\beta_2 z$ is a quadratic spectral phase shift which induces signal distortion

$$\boldsymbol{T_g(f)=-\frac{d\phi(f)}{d\omega}=-\frac{1}{2\pi}\frac{d\phi(f)}{df}}$$

$$\Leftrightarrow\; T_g(f)=2\pi\beta_2\mathrm{z}\,\mathrm{f}\;=2\pi\beta_2\mathrm{z}\,(\nu-\nu_0)$$
$$=\;\beta_2\mathrm{z}\,(\omega-\omega_0)=\;\beta_2\mathrm{z}\,\Delta\omega$$

$$\boldsymbol{T_g(f)=\beta_2\mathrm{z}\,\Delta\omega}$$

Group time of the different spectral components



Group velocities dispersion

❑Chromatic dispersion (2)

- **Chromatic dispersion coefficient D (1)**

$$\beta_2=\frac{\partial^2\beta}{\partial\omega^2}=\frac{\partial}{\partial\omega}\beta_1=\frac{\partial}{\partial\omega}\left(\frac{1}{\mathrm{v}_g}\right)=\frac{1}{c}\frac{d\mathrm{v}_g^{-1}}{d\lambda}\frac{d\lambda}{dk}=-\frac{\lambda^2}{2\pi c}\left(\frac{d\mathrm{v}_g^{-1}}{d\lambda}\right)$$

$$\beta_2=-\frac{\lambda^2}{2\pi c}\left(\frac{d\mathrm{v}_g^{-1}}{d\lambda}\right)$$

$$\boldsymbol{\beta_2=-\frac{\lambda^2}{2\pi c}\,\mathrm{D}=-\frac{\lambda}{\omega}\,D}$$

$$\boldsymbol{D=\frac{d\mathrm{v}_g^{-1}}{d\lambda}=-\frac{2\pi c}{\lambda^2}\beta_2=-\frac{\omega}{\lambda}\beta_2}$$

D is the chromatic dispersion coefficient of the fiber

Unit : s m^{-2} (usual unit is ps nm^{-1} km^{-1})

Group velocities dispersion

❑Chromatic dispersion (3)

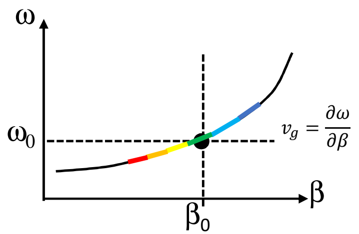
- **Chromatic dispersion coefficient D (2)**

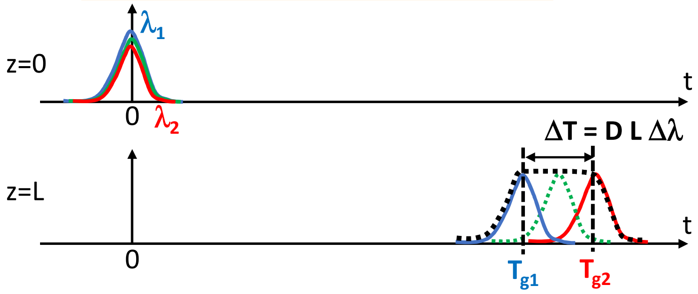
$$D = \frac{dv_g^{-1}}{d\lambda} = \frac{d\left(\frac{T_g}{L}\right)}{d\lambda} = \frac{1}{L}\frac{d\,T_g}{d\lambda}$$

D practical unit is ps nm^{-1} km^{-1}

$$\Leftrightarrow \quad \underbrace{\Delta T_g}_{\text{Relative delay (ps)}} = \underbrace{D}_{}\ \underbrace{L}_{\text{Propag. Length (km)}}\ \underbrace{\Delta\lambda}_{\text{Difference in wavlength (nm)}}$$

$$\Leftrightarrow \quad T_{g2} - T_{g1} = D\,L\,(\lambda_2 - \lambda_1)$$





For $\lambda_2 > \lambda_1$:
If D>0 → ΔT>0
Signal distortion

If D<0 → ΔT<0
Signal distortion

If D=0 → ΔT=0
No dispersion effect

Group velocities dispersion

❑Chromatic dispersion (4)

- **Chromatic dispersion compensation (1)**

At first fiber output $A(L,f) = A(0,f)\ .\ e^{-j2\pi^2\beta_2\,L\,f^2}$

At second fiber output $A'(L',f) = A'(0,f).\ e^{-j2\pi^2\beta_2'\,L'\,f^2} = A(L,f)\ .\ e^{-j2\pi^2\beta_2'\,L'\,f^2}$

$$A'(L',f) = A(0,f)\ e^{-j2\pi^2(\beta_2\,L+\beta_2'\,L')\,f^2}$$

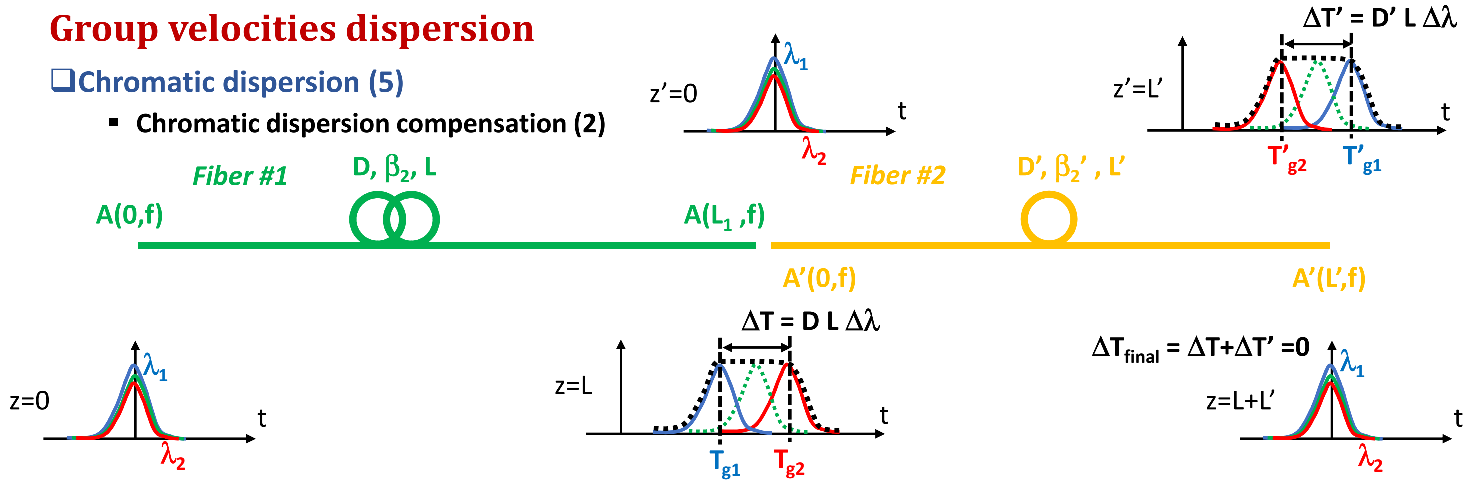
Dispersion effect is cancelled if following **compensation condition** is achieved

$$\beta_2\,L + \beta_2'\,L' = 0 \quad \Leftrightarrow \quad D\,L + D'\,L' = 0 \ \Leftrightarrow \ D' = -\frac{L}{L'}D \quad or \quad L' = -\frac{D}{D'}L$$

Group velocities dispersion

❑Chromatic dispersion (5)

- Chromatic dispersion compensation (2)



$\Delta T_{final} = \Delta T + \Delta T' = D L \Delta\lambda + D' L' \Delta\lambda = (D L + D' L') \Delta\lambda$

Assuming compensation condition is achieved, then :

$\Delta T_{final} = 0$

General case for N fibers

$$\Delta T_{final} = \sum_{i=1}^{N} D_i L_i \Delta\lambda$$

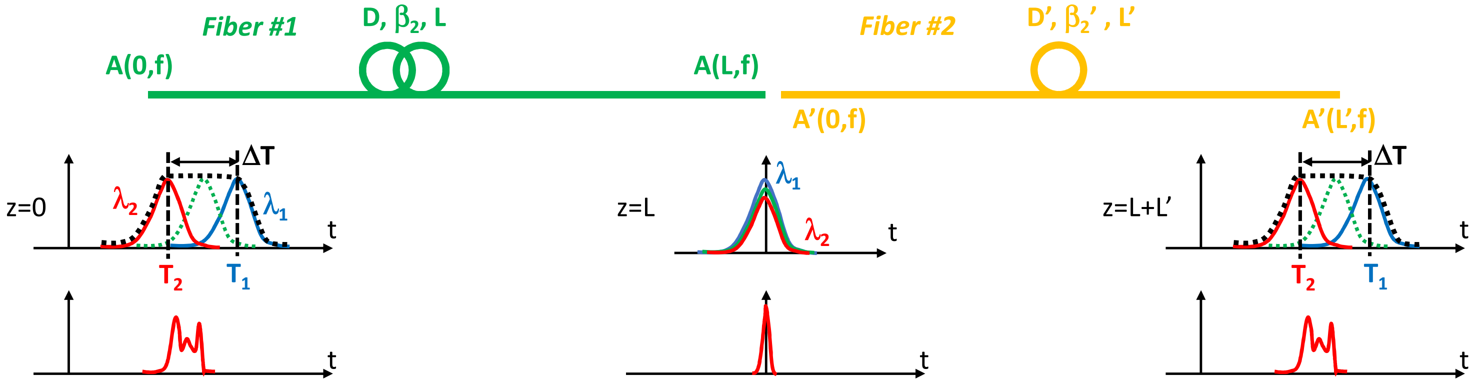
D value can be negative, null or positive depending on refractive index profile, material, and wavelength

---

Group velocities dispersion

❑Chromatic dispersion (6)

- Chromatic dispersion compensation (3)



With compensated dispersion, a set of fibers is a transparent “black box”. The bandwidth is virtually infinite. The new limits are now :

- The difficulty to get a dispersion compensation over a broad spectral band ($\beta_3 \neq 0$)
- The polarization mode dispersion PMD (intrinsic and extrinsic birefringency of fibers)

Propagation of a Gaussian pulse

❑Complex Gaussian pulse

- **Complex Gaussian function**

$$\boldsymbol{a(z,t) = A_0\, e^{-\frac{t^2}{2\sigma(z)^2}}}$$

with the z-dependent complex variance

$$\boldsymbol{\sigma(z)^2 = a(z) + j\, b(z)}$$

$$\Leftrightarrow a(z,t) = A_0\, e^{-\frac{t^2}{2(a(z)+j\,b(z))}} = A_0\, e^{-\frac{t^2}{2\left(\frac{a(z)^2+b(z)^2}{a(z)}\right)}}\;\; e^{j\frac{b\,t^2}{2(a(z)^2+b(z)^2)}}$$

$$\Leftrightarrow \boldsymbol{a(z,t) = A_0\, e^{-\frac{t^2}{2T(z)^2}}\;\; e^{j\phi(z,t)}}$$

Amplitude term (modulus) *Phase term*

with

$$\boldsymbol{T^2(z) = \frac{a(z)^2 + b(z)^2}{a(z)}}$$

$$\boldsymbol{\phi(z,t) = \frac{b\,t^2}{2(a(z)^2 + b(z)^2)}}$$

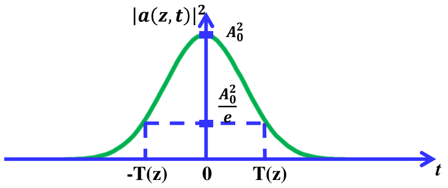
Propagation of a Gaussian pulse

❑Complex Gaussian pulse (2)

- **Pulsewidth**

$$\boldsymbol{|a(z,t)|^2 = A_0^2\;\; e^{-\frac{t^2}{T(z)^2}}}$$

T(z) is the Half-Width time duration at 1/e of the maximum intensity

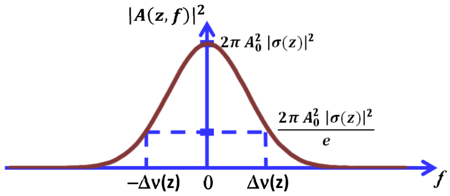


$$\boldsymbol{T^2(z) = \frac{a(z)^2 + b(z)^2}{a(z)} = \frac{\left|\sigma(z)^2\right|^2}{\mathcal{R}e(\sigma(z)^2)}}$$

- **Pulse spectrum**

$$A(z,f) = FT(a(z,t)) = A_0\,|\sigma(z)|\,\sqrt{2\pi}\;\; e^{-2\pi^2\sigma(z)^2 f^2}$$

$$|A(z,f)|^2 = 2\pi\, A_0^2\, |\sigma(z)|^2\;\; e^{-4\pi^2 a(z) f^2} = A_0^2\, |\sigma(z)|^2\, 2\pi\;\; e^{-\frac{f^2}{\Delta\nu(z)^2}}$$



Δν(z) is the Half-Width spectral width at 1/e of the maximum intensity

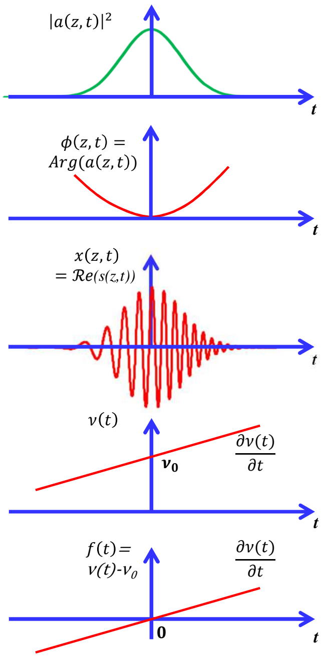
$$\boldsymbol{\Delta\nu(z)^2 = \frac{1}{4\pi^2 a(z)} = \frac{1}{4\pi^2 \mathcal{R}e(\sigma(z)^2)}}$$

$$\boldsymbol{\Delta\nu(z) = \frac{1}{2\pi\sqrt{a(z)}} = \frac{1}{2\pi\sqrt{\mathcal{R}e(\sigma(z)^2)}}}$$

Propagation of a Gaussian pulse

❑Complex Gaussian pulse (3)

- **Pulse chirp parameter (1)**



The phase of a(z,t) is the phase of the modulated signal

$$\phi(z,t) = Arg(a(z,t)) = \frac{b(z)t^2}{2(a(z)^2+b(z)^2)}$$

$$s(z,t) = a(z,t)e^{j(\omega_0 t-\beta z)} = |a(z,t)|\, e^{j(\omega_0 t-\beta z+\phi(z,t))}$$

Instantaneous frequency of the signal (carrier)

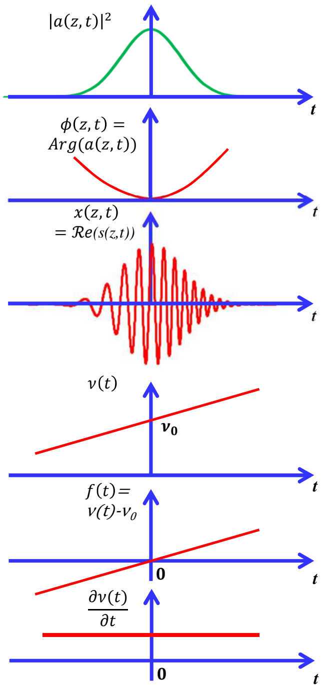
$$\boldsymbol{\nu(t) = \frac{1}{2\pi}\frac{\partial\phi(z,t)}{\partial t} = \frac{b(z)t}{2\pi\,(a(z)^2+b(z)^2)}}$$

Linear frequency drift vs time

Propagation of a Gaussian pulse

❑Complex Gaussian pulse (4)

- **Pulse chirp parameter (2)**



$$\frac{\partial\nu(t)}{\partial t} = \frac{\partial f(t)}{\partial t} = \frac{1}{2\pi}\frac{\partial^2\phi(z,t)}{\partial t^2} = \frac{b(z)}{2\pi\,(a(z)^2+b(z)^2)} \text{ is time independent}$$

$$\frac{\partial\nu(t)}{\partial t} = \frac{\frac{b(z)}{a(z)}}{2\pi\,\frac{a(z)^2+b(z)^2}{a(z)}} = \frac{C(z)}{2\pi\,T(z)^2}$$

C(z) is the chirp parameter of the pulse. Its value is z-dependent and is representative of the carrier frequency drift

$$\boldsymbol{C(z) = \frac{b(z)}{a(z)} = \frac{\mathcal{Im}(\sigma(z)^2)}{\mathcal{Re}(\sigma(z)^2)} = 2\pi\,T(z)^2\frac{\partial\nu(t)}{\partial t}}$$

$$\boldsymbol{\phi(z,t) = \frac{C(z)t^2}{2\,T(z)^2}}$$

Propagation of a Gaussian pulse

❑Complex Gaussian pulse (5)

- **Pulse chirp parameter (3)**

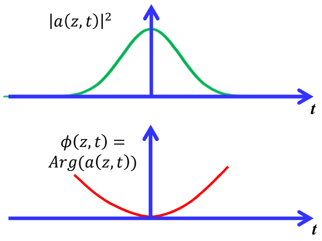
$$a(z,t) = A_0\, e^{-\frac{t^2}{2T(z)^2}}\; e^{j\phi(z,t)}$$

$$\Leftrightarrow \quad \boldsymbol{a(z,t) = A_0\, e^{-(1-jC(z))\frac{t^2}{2\,T(z)^2}}}$$

$$a(z,t) = A_0\, e^{-\frac{t^2}{2T(z)^2}}\; e^{j\frac{C(z)t^2}{2\,T(z)^2}}$$

This equation can describe the effect of the fiber dispersion on pulsewidth and its link with the frequency drift (chirp)

Amplitude, instantaneous frequency and phase of the complex Gaussian pulse is fully determinated by the parameters T(z) and C(z)



Propagation of a Gaussian pulse

❑Gaussian pulse propagation (1)

- **At z=0 :** the pulse is described at the origin by the initial values T_0, C_0, σ_0^2

$$C_0 = \frac{\mathcal{Im}(\sigma_0^2)}{\mathcal{Re}(\sigma_0^2)} \qquad T_0^2 = \frac{\left|\sigma_0^2\right|^2}{\mathcal{Re}(\sigma_0^2)} \qquad a(0,t) = A_0\, e^{-(1-jC_0)\frac{t^2}{2\,T_0{}^2}}$$

$$\boldsymbol{\Delta\nu_0^2 = \frac{1}{4\pi^2\mathcal{Re}(\sigma_0^2)}} \qquad A(0,f) = FT(a(0,t)) = A_0\,|\sigma_0|\,\sqrt{2\pi}\;\, e^{-2\pi^2\sigma_0^2 f^2}$$

- **At z>0 during propagation**

$$A(z,f) = A(0,f)\;\, e^{-j2\pi^2\beta_2\, z\, f^2} = A_0\,|\sigma_0|\,\sqrt{2\pi}\;\, e^{-2\pi^2\sigma_0^2 f^2} e^{-j2\pi^2\beta_2\, z\, f^2}$$

$$A(z,f) = A_0\,|\sigma_0|\,\sqrt{2\pi}\;\, e^{-2\pi^2\boldsymbol{(\sigma_0^2+j\beta_2\, z)}f^2} = A_0\,|\sigma_0|\,\sqrt{2\pi}\;\, e^{-2\pi^2\boldsymbol{\sigma(z)^2}f^2}$$

$$\Leftrightarrow \quad \boldsymbol{\sigma(z)^2 = \sigma_0^2 + j\beta_2\, z}$$

Complex variance of a Gaussian pulse along the propagation

Only the imaginary part of σ(z) evoluates along the propagation

Propagation of a Gaussian pulse

❑Gaussian pulse propagation (2)

- **Pulse spectral width :**

$$\sigma(z)^2 = \sigma_0^2 + j\beta_2\, z = \mathcal{R}e(\sigma_0^2) + j\,\mathcal{J}m(\sigma_0^2) + j\beta_2\, z$$

$$= \boldsymbol{\mathcal{R}e}\left(\boldsymbol{\sigma_0^2}\right) + j\,\left(\mathcal{J}m(\sigma_0^2) + \beta_2\, z\right) = \boldsymbol{\mathcal{R}e}\left(\boldsymbol{\sigma(z)^2}\right) + j\,\mathcal{J}m\left(\sigma(z)^2\right)$$

The real part of $\sigma(z)^2$ is constant along all the propagation leading to

$$\boxed{\boldsymbol{\Delta\nu(z)^2 = \frac{1}{4\pi^2\mathcal{R}e\left(\sigma(z)^2\right)} = \frac{1}{4\pi^2\mathcal{R}e\left(\sigma_0^2\right)} = \Delta\nu_0^2 = \text{cste}\ \forall z}}$$

The spectral width $\Delta\nu(z)$ of the pulse spectrum remains constant along the propagation
Dispersion has no effect of the pulse spectral width (spectrum modulus)
but only on spectral phase

Propagation of a Gaussian pulse

❑Gaussian pulse invariant (1)

$$\boldsymbol{T(z)^2 = \frac{\left|\sigma(z)^2\right|^2}{\mathcal{R}e\left(\sigma(z)^2\right)} = \frac{\mathcal{R}e^2\left(\sigma(z)^2\right) + \mathcal{J}m^2\left(\sigma(z)^2\right)}{\mathcal{R}e\left(\sigma(z)^2\right)} \cdot \frac{\mathcal{R}e\left(\sigma(z)^2\right)}{\mathcal{R}e\left(\sigma(z)^2\right)} = \left(1 + \frac{\mathcal{J}m^2\left(\sigma(z)^2\right)}{\mathcal{R}e^2\left(\sigma(z)^2\right)}\right)\mathcal{R}e\left(\sigma(z)^2\right)}$$

$$\boldsymbol{T(z)^2 = \left(1 + \frac{\mathcal{J}m^2\left(\sigma(z)^2\right)}{\mathcal{R}e^2\left(\sigma(z)^2\right)}\right)\mathcal{R}e\left(\sigma_0^2\right)}$$

$$\boldsymbol{T(z)^2 = \left(1 + C(z)^2\right)\frac{1}{4\pi^2\Delta\nu^2}}$$

$$\boxed{\boldsymbol{\frac{T(z)^2}{1 + C(z)^2} = \frac{1}{4\pi^2\Delta\nu^2} = \text{cste}}}$$

Gaussian pulse propagation invariant

$T^2(z)$ decreases
as
$C^2(z)$ decreases

Propagation of a Gaussian pulse

❑Gaussian pulse invariant (2)

- Case C(z=z_m) =0 at a position z_m (1)

⇒ Pulse is unchirped ⇒ T(z_m) is minimal : **T(z_m)=T_m**

$$\frac{T(z)^2}{1+C(z)^2} = \frac{1}{4\pi^2\Delta\nu^2} = \text{cste} \qquad T(z_m)^2 = \frac{1}{4\pi^2\Delta\nu^2} = \text{cste} = T_m^2$$

$$\boldsymbol{T(z_m) = T_m = \frac{1}{2\pi\,\Delta\nu}}$$

Final equation of the propagation invariant

$$\boldsymbol{\frac{T(z)^2}{1+C(z)^2} = \frac{T_0^2}{1+C_0^2} = \frac{1}{4\pi^2\Delta\nu^2} = T_m^2 = \text{cste}}$$

Propagation of a Gaussian pulse

❑Gaussian pulse invariant (3)

- Case of a null chirp C(z=z_m) =0 at a position z_m (2)

$$\phi(t) = C\,\frac{t^2}{2T^2} = 0$$

a(z_m) is real, i.e. no phase or frequency variation

$$a(z_m,t) = A_0\, e^{-(1-jC(z_m))\frac{t^2}{2\,T(z_m)^2}} = A_0\, e^{-\frac{t^2}{2T_m^2}} = |a(z_m,t)|$$

$$A(z_m,f) = A_0 T_m \sqrt{2\pi}\, e^{-2\pi^2 T_m^2 f^2} = |A(z_m,f)|$$

$$\boldsymbol{|A(z_m,f)| = \textbf{FT}\,(|a(z_m,t)|)}$$

The pulse is « Transform Limited »

i.e. its pulsewidth is limited by the Fourier Transform and the spectral width

Propagation of a Gaussian pulse

❑Gaussian pulse invariant (4)

- Case C(z=z_m) =0 at a position z_m (3)

Short duration needs large spectral width Δν

$$T(z_m) = T_m = \frac{1}{2\pi\,\Delta\nu}$$

From the invariant equation, for an unchirped pulse (z=z_m) :

$$\boxed{2\,T_m.\,2\,\Delta\nu = \frac{2}{\pi}}$$

2 T_m is the full pulsewidth
2 Δν is the spectrum full width

For a chirped pulse (z≠z_m) :

$$\boxed{2\,T(z).\,2\,\Delta\nu = \frac{2}{\pi}\sqrt{1 + C(z)^2} \quad \left(> \frac{2}{\pi}\right)}$$

Propagation of a Gaussian pulse

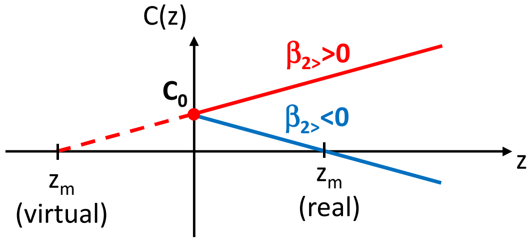
❑Chirp evolution along the propagation

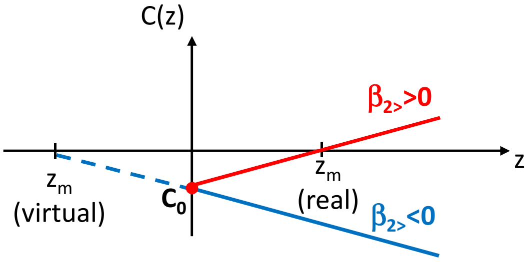
$$\sigma(z)^2 = \sigma_0^2 + j\beta_2\, z \quad \Leftrightarrow \quad C(z) = \frac{b(z)}{a(z)} = \frac{\mathcal{Im}(\sigma(z)^2)}{\mathcal{Re}(\sigma(z)^2)} = \frac{\mathcal{Im}(\sigma_0^2) + \beta_2\, z}{\mathcal{Re}(\sigma_0^2)} = C_0 + \frac{\beta_2\, z}{\mathcal{Re}(\sigma_0^2)} = C_0 + \frac{\beta_2\, z}{T_m^2}$$

$$\boxed{\Leftrightarrow \quad C(z) = C_0 + \frac{\beta_2\, z}{T_m^2}}$$

β_2 is a fiber property (dispersion)

C_0 and T_m are pulse properties





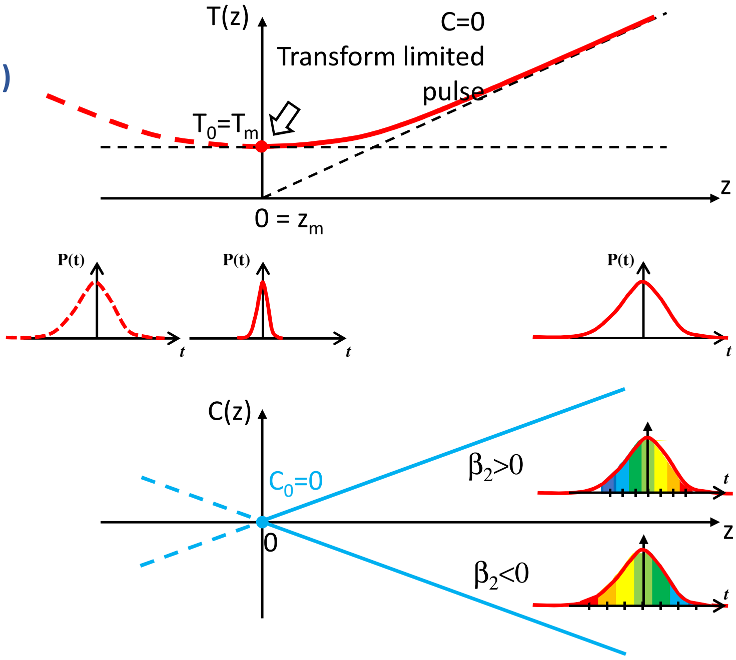
$$z_m = -\frac{C_0\, T_m^2}{\beta_2}$$

Propagation of a Gaussian pulse

❑Pulsewidth evolution along the propagation (1)

$$T(z)^2 = T_m^2[1 + C(z)^2]$$

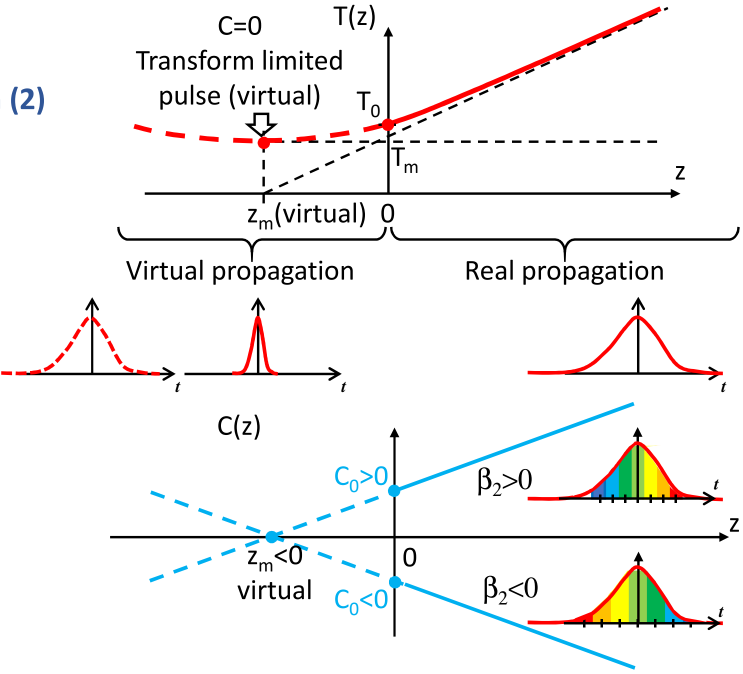
- **C_0=0** : chirped initial pulse
  - ✓ Only pulsewidth increase possible
  - ✓ No minimal pulsewidth T_m reaching possible



Propagation of a Gaussian pulse

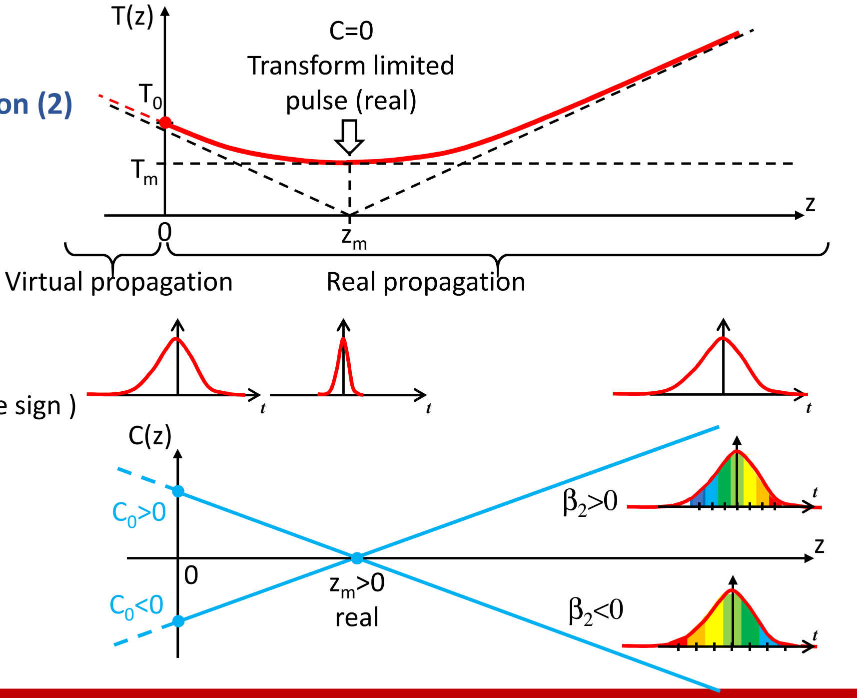
❑Pulse width evolution along the propagation (2)

- **C_0 and β_2** are of same sign : z_m < 0
  - ✓ Only pulsewidth increase possible
  - ✓ No minimal pulsewidth T_m reaching possible



Propagation of a Gaussian pulse

❑Pulse width evolution along the propagation (2)



- **C_0 and β_2** are of opposite sign : $z_m > 0$
  - ✓ Minimal pulsewidth T_m reached at $z=z_m$ ($C(z)=0$)
  - ✓ Linear pulse compression ratio (possible only if C_0 and β_2 are of opposite sign)

$$\frac{T_0}{T_m} = \sqrt{1 + C_0^2}$$