MICROWAVE ENGINEERING

Lecture 9:

Smith Chart,

Quarter Wave

Transformer

Lossy Transmission

Lines

The Smith's Chart (1939 by P. Smith @ Bells lab)

↳ POLAR PLOT of the reflection coefficient Γ

$$\Gamma = |\Gamma| e^{j\theta}$$

$|\Gamma| \leq 1$
is plotted
as a radius

$-180^\circ \leq \theta \leq 180^\circ$
Measured from the right hand side of the horizontal diameter on the graph.

Reflection coeff. can be converted → Impedance
→ Admittance

On the chart we always plot NORMALIZED IMPEDANCES

$$z = \frac{Z}{Z_0}$$

If we have a lossless line with char. imp. Z_0 and terminated with a load Z_L:

$$\underbrace{\Gamma = \frac{z_L - 1}{z_L + 1}} = |\Gamma| e^{j\theta} \qquad z_L = \frac{Z_L}{Z_0}$$

Solving for z_L $\Longrightarrow$ $\Gamma(z_L + 1) = (z_L - 1) \Rightarrow \Gamma z_L + \Gamma = z_L - 1$

$$\boxed{z_L = \frac{1 + |\Gamma| e^{j\theta}}{1 - |\Gamma| e^{j\theta}}}$$

$\Gamma = \Gamma_r + j\Gamma_i$

$z_L = r_L + jx_L$

$$r_e + j\,x_L = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i} = \frac{\left[(1+\Gamma_r)+j\Gamma_i\right]\left[1-\Gamma_r+j\Gamma_i\right]}{\left[1-\Gamma_r-j\Gamma_i\right]\left[1-\Gamma_r+j\Gamma_i\right]} =$$

$$= \frac{1 \cancel{-\Gamma_r} + j\Gamma_i \cancel{+\Gamma_r} - \Gamma_r^2 \cancel{+j\Gamma_r\Gamma_i} + j\Gamma_i \cancel{-j\Gamma_i\Gamma_r} - \Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} =$$

$$= \frac{1-\Gamma_r^2-\Gamma_i^2+2j\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}$$

LOAD RESISTANCE

$$\boxed{r_l = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2}}$$

LOAD REACTANCE

$$\boxed{x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}}$$

We can rearrange in eq. of circles :

Resistance circles

$$\left(\Gamma_r - \frac{\Gamma_i}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2$$

Reactance circles

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$

For ex: if $r_L = 1$ $\longrightarrow$ Circle is centered in $\Gamma_r = 0.5$
$\Gamma_i = 0$

$$\left(\Gamma_r - \frac{1}{2}\right)^2 + \Gamma_i^2 = \left(\frac{1}{2}\right)^2$$

offset with respect to 0

radius

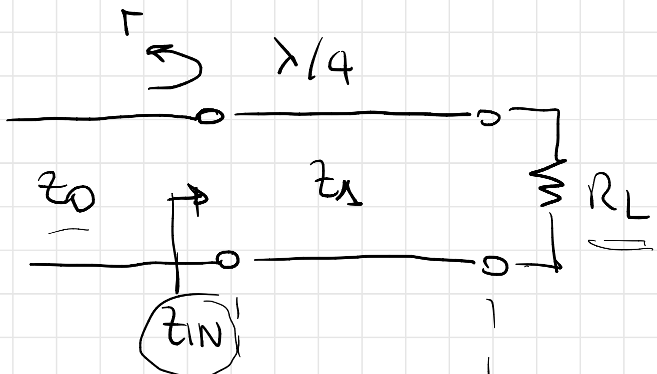
Smith chart can be used to graphycally solve transmission line eq.

$$z_{IN} = z_0 \frac{1+\Gamma e^{-2j\beta l}}{1-\Gamma e^{-2j\beta l}}$$

By plotting Γ on the chart the normalize z_{in} can be found by rotating clockwise an amount $2\beta l$ (subtract $2\beta l$ from θ) around the center.

Max rotation on chart $0.5\lambda \rightarrow 2\pi$

THE QUARTER WAVE TRANSFORMER



QUARTER WAVE TRANSFORMER

↑

loss less TL with Z_1 charact. impedance.

IMPORTANT:

R_L and Z_0 are given parameters and assumed to be REAL

To match a load to a line $\Rightarrow$ $\Gamma = 0$ at the input of the transformer

$$Z_{IN} = Z_1 \frac{R_L + j Z_1 \tan\beta\ell}{Z_1 + j R_L \tan\beta\ell}$$

$$\beta\ell = \frac{2\pi}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{4} = \frac{\pi}{2}$$

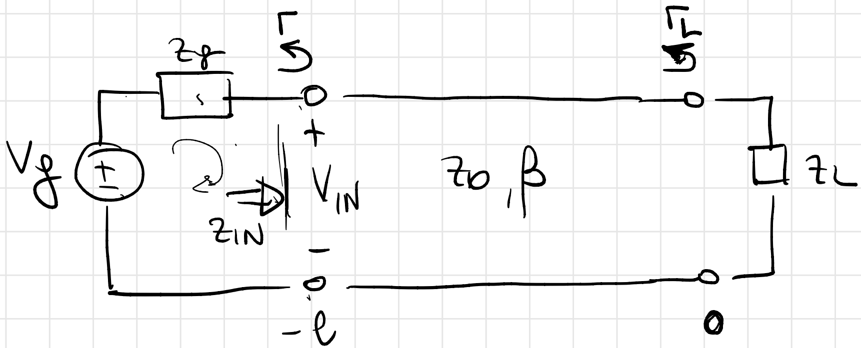
$$Z_{IN} = Z_1 \frac{\cancel{\frac{R_L}{\tan\beta\ell}} + \cancel{j} Z_1}{\cancel{\frac{Z_1}{\tan\beta\ell}} + \cancel{j} R_L} = \frac{Z_1^2}{R_L}$$

to get $\Gamma = 0$ $\Rightarrow$ $Z_{IN} = Z_0 = \dfrac{Z_1^2}{R_L}$ $\Rightarrow$ $\boxed{Z_1 = \sqrt{R_L Z_0}}$

SWR=1 for this condition

IMPORTANT: Matching is valid at 1 frequency

GENERATOR AND LOAD MISMATCHES



- Z_g, Z_L can be complex
- TL is lossless

$$Z_{IN} = Z_0 \frac{1 + \Gamma_L e^{-2j\beta\ell}}{1 - \Gamma_L e^{-2j\beta\ell}} = Z_0 \frac{Z_L + j Z_0 \tan\beta\ell}{Z_0 + j Z_L \tan\beta\ell}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

The voltage on the line is:

$$V(z) = V_0^+ \left(e^{-j\beta z} + \Gamma_L e^{j\beta z} \right)$$

In $z = -\ell$

$$V(-\ell) = \overbrace{V_g \frac{Z_{IN}}{Z_{IN} + Z_g}}^{V_{IN}} = V_0^+ \left[e^{j\beta \ell} + \Gamma_L e^{-j\beta \ell} \right]$$

$$V_0^+ = V_g \frac{Z_{IN}}{Z_{IN} + Z_g} \frac{1}{e^{j\beta \ell} + \Gamma_L e^{-j\beta \ell}}$$

$$Z_{IN} = Z_0 \frac{1 + \Gamma_L e^{-2j\beta \ell}}{1 - \Gamma_L e^{-2j\beta \ell}}$$

Replacing Z_{IN} in the expression of V_0^+

$$V_0^+ = V_g Z_0 \frac{1+\Gamma_L e^{-2j\beta \ell}}{1-\Gamma_L e^{-2j\beta \ell}} \cdot \frac{1}{Z_g + Z_0 \frac{1+\Gamma_L e^{-2j\beta \ell}}{1-\Gamma_L e^{-2j\beta \ell}}} \cdot \frac{1}{e^{j\beta \ell}+\Gamma_L e^{-j\beta \ell}}$$

$$= V_g Z_0 \frac{1+\Gamma_L e^{-2j\beta \ell}}{\cancel{1-\Gamma_L e^{-2j\beta \ell}}} \cdot \frac{\cancel{1-\Gamma_L e^{-2j\beta \ell}}}{Z_0\left(1+\Gamma_L e^{-2j\beta \ell}\right)+Z_g\left(1-\Gamma_L e^{-2j\beta \ell}\right)} \cdot \frac{1}{e^{j\beta \ell}+\Gamma_L e^{-j\beta \ell}}$$

multiply by $e^{-j\beta \ell}$

$$= V_g Z_0 \frac{\cancel{\left(1+\Gamma_L e^{-2j\beta \ell}\right)}}{Z_0\left(1+\Gamma_L e^{-2j\beta \ell}\right)+Z_g\left(1-\Gamma_L e^{-2j\beta \ell}\right)} \cdot \frac{e^{-j\beta \ell}}{\cancel{1+\Gamma_L e^{-2j\beta \ell}}} =$$

$$= V_g Z_0 \frac{e^{-j\beta \ell}}{Z_0+Z_g+(Z_0-Z_g)\Gamma_L e^{-2j\beta \ell}}$$

divide by Z_0+Z_g

$$= V_g \frac{Z_0}{Z_0+Z_g} \frac{e^{-j\beta \ell}}{1-\Gamma_L \Gamma_g e^{-j\beta \ell}}$$

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}$$

On the line $\quad SWR = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|}$

The power delivered at the load:

$$P = \frac{1}{2}\,\mathrm{Re}\left\{V_{IN} I_{IN}^{*}\right\} = \frac{1}{2}|V_{IN}|^2\,\mathrm{Re}\left\{\frac{1}{Z_{IN}}\right\} = \frac{1}{2}|V_g|^2\left|\frac{Z_{IN}}{Z_{IN}+Z_g}\right|^2 \underbrace{\mathrm{Re}\left\{\frac{1}{Z_{IN}}\right\}}_{\frac{R_{IN}-jX_{IN}}{R_{IN}^2+X_{IN}^2}}$$

$$\Uparrow$$

$$I_{IN} = \frac{V_{IN}}{Z_{IN}}$$

$$\left.\begin{aligned} \text{If } Z_{IN} &= R_{IN} + jX_{IN} \\ Z_g &= R_g + jX_g \end{aligned}\right\} \Rightarrow \boxed{P = \frac{1}{2}|V_g|^2\,\frac{R_{IN}}{(R_{IN}+R_g)^2+(X_{IN}+X_g)^2}}$$

Three special cases are:

1. LOAD IS MATCHED TO THE LINE $\quad Z_L = Z_0$

In this case $\Gamma_L = 0$, SWR $= 1 \implies Z_{IN} = Z_0$

Therefore

$$P = \frac{1}{2} |V_g|^2 \frac{Z_0}{(Z_0 + R_g)^2 + X_g^2}$$

2. GENERATOR IS MATCHED TO THE LOADED LINE: $\quad Z_{IN} = Z_g$

$$\Gamma = 0 \implies \frac{Z_{IN} - Z_g}{Z_{IN} + Z_g} = 0$$

We know that $\Gamma_L \neq 0 \implies$ there will be a standing wave on the line

$$P = \frac{1}{2} |V_g|^2 \frac{R_g}{R_g^2 + X_g^2}$$

(3) CONJUGATE MATCHING

If we want to max power at the load we can assume Z_g is fixed and we vary Z_{IN} until max power is delivered:

$$\boxed{\frac{\partial P}{\partial R_{IN}} = 0} \Rightarrow \frac{1}{(R_{IN}+R_g)^2 + (X_{IN}+X_g)^2} + \frac{-2R_{IN}(R_{IN}+R_g)}{(R_{IN}+R_g)^2 + (X_{IN}+X_g)^2} = 0$$

$$R_{IN}^2 + R_g^2 + 2\cancel{R_{IN} R_g} - 2R_{IN}^2 - \cancel{2R_{IN} R_g} + (X_{IN}+X_g)^2 = 0$$

$$\boxed{R_g^2 - R_{IN}^2 + (X_{IN}+X_g)^2 = 0}$$

or

$$\frac{\partial P}{\partial X_{IN}} = 0 \longrightarrow \frac{-2 R_{IN}(X_{IN}+X_g)}{\left[(R_{IN}+R_g)^2+(X_{IN}+X_g)^2\right]^2} = 0$$

$$\boxed{R_{IN}(X_{IN}+X_g) = 0}$$

If we solve simultaneously for R_{IN} and X_{IN} :

$$\boxed{R_{IN} = R_g} \qquad \boxed{X_{IN} = -X_g}$$

$$\boxed{Z_{IN} = Z_g^*} \quad \text{CONJUGATE MATCHING CONDITION}$$

In this case

$$\boxed{P = \frac{1}{2}|V_g|^2 \frac{1}{4R_g}}$$

MAX AVAILABLE POWER FOR A SPECIFIC GENERATOR

NOTE: Γ_1, Γ_g and Γ_e can be non-zero

NOTE 2: if X_g is Real $\Rightarrow$ Max power transfer $\Rightarrow$ $R_{IN} = R_g$

NOTE 3: Even if we achieve max power transfer we still have max 50% efficiency → Z_g needs to ~~be~~ be as small as possible!

LOSSY TRANSMISSION LINES

• LOW-LOSS LINE

$$\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)} = \ldots$$

$$= \sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) - \frac{RG}{\omega^2 LC}}$$

If losses are small:

$$\left.\begin{array}{l} R \ll \omega L \\ G \ll \omega C \end{array}\right\} \Rightarrow \quad RG \ll \omega^2 LC$$

$$\gamma \simeq j\omega\sqrt{LC}\sqrt{1 - \underbrace{j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)}} =$$

expanding in Taylor series

$$= j\omega\sqrt{LC}\left[1 - \frac{j}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)\right]$$

$$\alpha \simeq \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) = \frac{1}{2}\left(\frac{R}{Z_0} + G Z_0\right)$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

because

$$\beta \simeq \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \simeq \sqrt{\frac{L}{C}}$$

for small losses

THE DISTORTIONLESS LINE

unless line is lossless β is NOT a linear function of ω

$$\gamma = j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) - \frac{RG}{\omega^2 LC}}$$

$\Downarrow$

the PHASE VELOCITY $V_p = \frac{\omega}{\beta}$
will be different for different frequencies

$\Downarrow$

Different signal wavelength traveling on the line will arrive at different times!

$\Downarrow$

$\boxed{\text{DISPERSION!}}$

A distortionless line has : $\boxed{\frac{R}{L} = \frac{G}{C}}$

So the propagation constant is:

$$\gamma = j\omega\sqrt{LC}\,\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{R}{\omega L}\right) - \frac{R^2}{\omega^2 L^2}} =$$

$$= j\omega\sqrt{LC}\,\sqrt{\underbrace{1 - 2j\frac{R}{\omega L} - \frac{R^2}{\omega^2 L^2}}_{\left(1 - j\frac{R}{\omega L}\right)^2}} = j\omega\sqrt{LC}\left(1 - j\frac{R}{\omega L}\right) =$$

$$= \underbrace{j\omega\sqrt{LC}}_{\beta} + \underbrace{R\sqrt{\frac{C}{L}}}_{\alpha} = \alpha + j\beta$$

$$\boxed{\beta = \omega\sqrt{LC}}$$

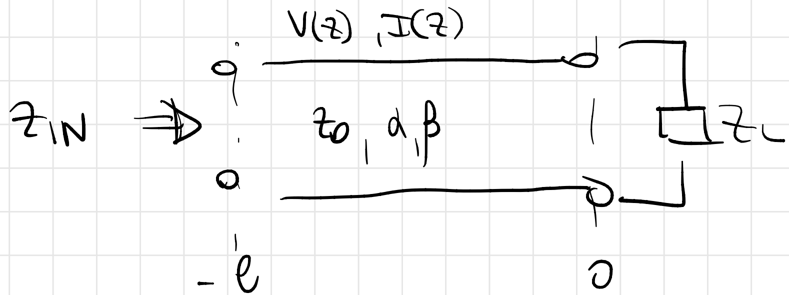
LINEAR FUNCTION OF W

$$\boxed{\alpha = R\sqrt{\frac{C}{L}}}$$

IT DOES NOT DEPEND OF FREQUENCY

NO DISTORTION

THE TERMINATED LOSSY LINE



Let's assume LOW LOSSESS $\rightarrow$ $Z_0 \simeq \sqrt{\frac{L}{C}}$

$$V(z) = V_0^+ \left[e^{-\gamma z} + \Gamma e^{\gamma z} \right]$$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-\gamma z} - \Gamma e^{\gamma z} \right]$$

At a distance l from the load:

$$\Gamma(l) = \Gamma e^{-2j\beta l} e^{-2\alpha l} = \Gamma e^{-2\gamma l}$$

The input impedance

$$Z_{IN} = \frac{V(-\ell)}{I(-\ell)} = Z_0 \frac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell}$$

The power delivered at $z = -\ell$ is

$$P_{IN} = \frac{1}{2} \operatorname{Re}\left\{ V(-\ell)\, I(-\ell)^* \right\} = \frac{|V_0^+|^2}{2Z_0} \left[e^{2\alpha\ell} - |\Gamma(\ell)|^2 e^{-2\alpha\ell} \right]$$

The power at the load is

$$P_L = \frac{1}{2} \operatorname{Re}\left\{ V(0) + I(0)^* \right\} = \frac{|V_0^+|^2}{2Z_0} \left(1 - |\Gamma|^2 \right)$$

The power loss:

$$\boxed{P_{LOSS} = P_{IN} - P_L}$$