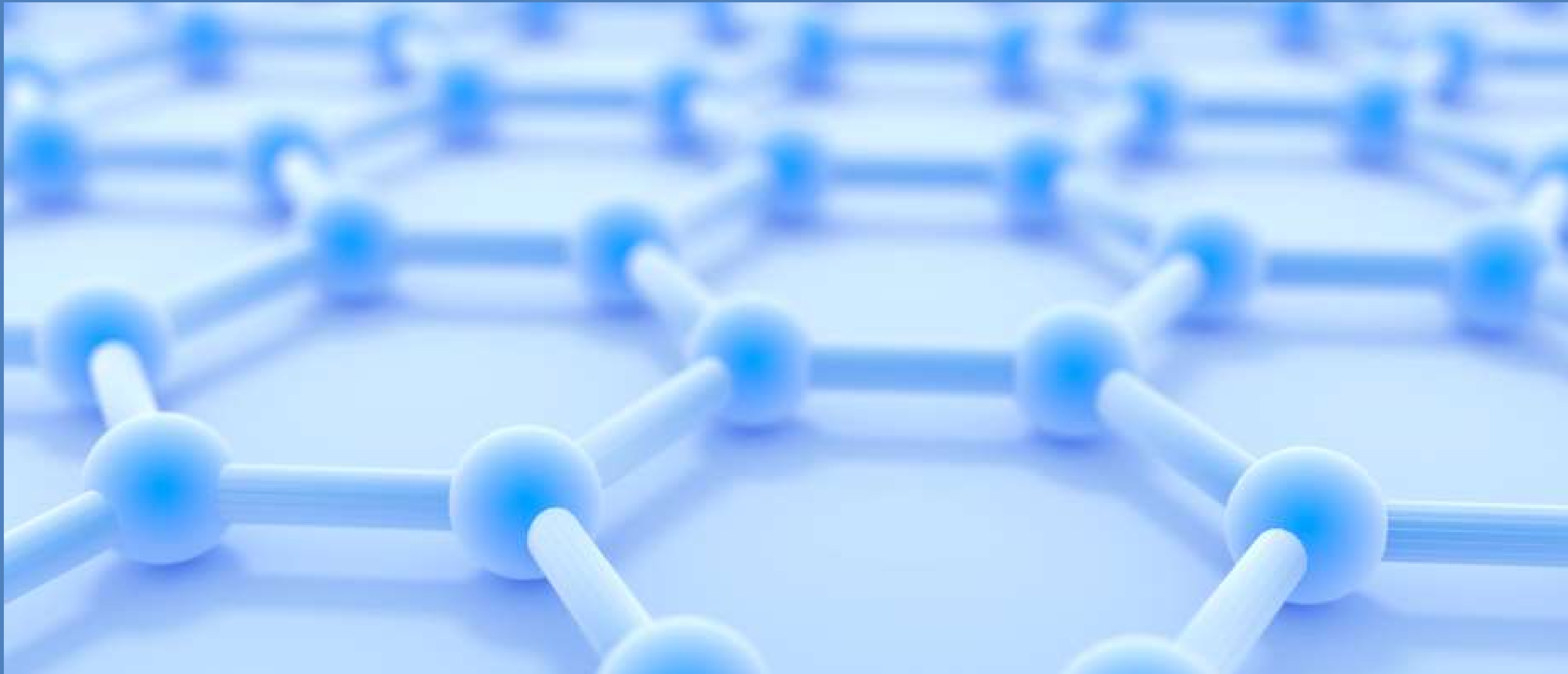


Photonics

aa 2020/2021

Prof. Maria Antonietta Vincenti
Università degli Studi di Brescia

Wave Optics:
Monochromatic Waves and Simple optical components

UNIVERSITY OF BRESCIA

Introduction

Wave optics includes all the Concepts we discussed in the **ray optics** section.
We could say that **ray optics is the limit of wave optics when the wavelength is infinitesimally short**. Put another way: if the objects under investigation are much bigger than the incident wavelength, ray theory is sufficient to describe most of the observable phenomena.

With the wave optics approach light is described as a **wavefunction** (which is a scalar function) that obeys to a second-order differential equation called the **wave equation.**

Limits of wave optics:
- Not capable of giving a complete picture or reflection and refraction at interfaces;
- Not able to describe polarization effects (they require a vectorial representation);

Postulates of wave optics

➢ **Wave Equation:**

Light propagates in free space with a speed **c_0= 3 x 10^8 m/s.** In a medium with refractive index **n**, light travels at the reduced speed **c = c_0/n.**

An optical wave is described by a real function u(**r**,t) known as **wavefunction**. The wavefunction satisfies the following wave equation:

$$\boxed{\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0}$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

NOTE: The <u>principle of superposition</u> is valid for the wave equation so any sum of two solution to the wave equation is still a solution of the wave equation.

NOTE 2: The wave equation is valid only for <u>locally homogeneous</u> media.

UNIVERSITY OF BRESCIA

Postulates of wave optics

- **Intensity:**

The **optical intensity** I(**r**,t), defined as the optical power per unit area (W/cm^2), is proportional to the average of the squared wavefunction:

$$\boxed{I(\boldsymbol{r},t) = 2\langle u^2(\boldsymbol{r},t)\rangle}$$

Where the average operator is applied over a time longer than an optical cycle but much shorter than the overall pulse length. This implies that the pulses under consideration must be sufficiently long to contain several optical cycles.

- **Power:**

the **optical power** P(t) - measured in Watts – that flows into an area A normal to the direction of propagation is:

$$\boxed{\mathrm{P(t)} = \int_A I(\boldsymbol{r},t)dA}$$

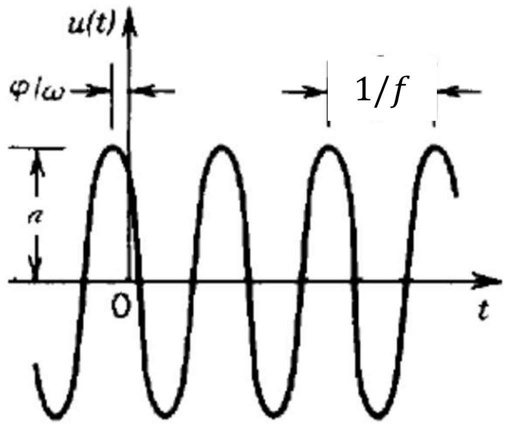
- **Energy:**

The **optical energy** – measured in Joules – in a given time interval is the integral of optical power over that time interval.

Monochromatic Waves

A monochromatic wave is represented by a wave function with harmonic time dependence:

$$u(\boldsymbol{r},t) = \underbrace{a(\boldsymbol{r})}_{\text{amplitude}}\cos[\underbrace{2\pi f}_{\text{angular frequency } \omega}t + \underbrace{\varphi(\boldsymbol{r})}_{\text{phase}}]$$



Period T = 1/ν = 2π/ω

Complex Wavefunction

It is convenient to represent the real wavefunction in terms of a **complex wavefunction** :

$$U(\boldsymbol{r},t) = a(r)\exp(j2\pi f t)\exp[j\varphi(\boldsymbol{r})]$$

So that:

$$u(\boldsymbol{r},t) = Re[U(\boldsymbol{r},t)] = \frac{1}{2}[U(\boldsymbol{r},t) + U(\boldsymbol{r},t)^*]$$

Like the real wavefunction, also the complex wavefunction must satisfy the wave equation:

$$\boxed{\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0}$$

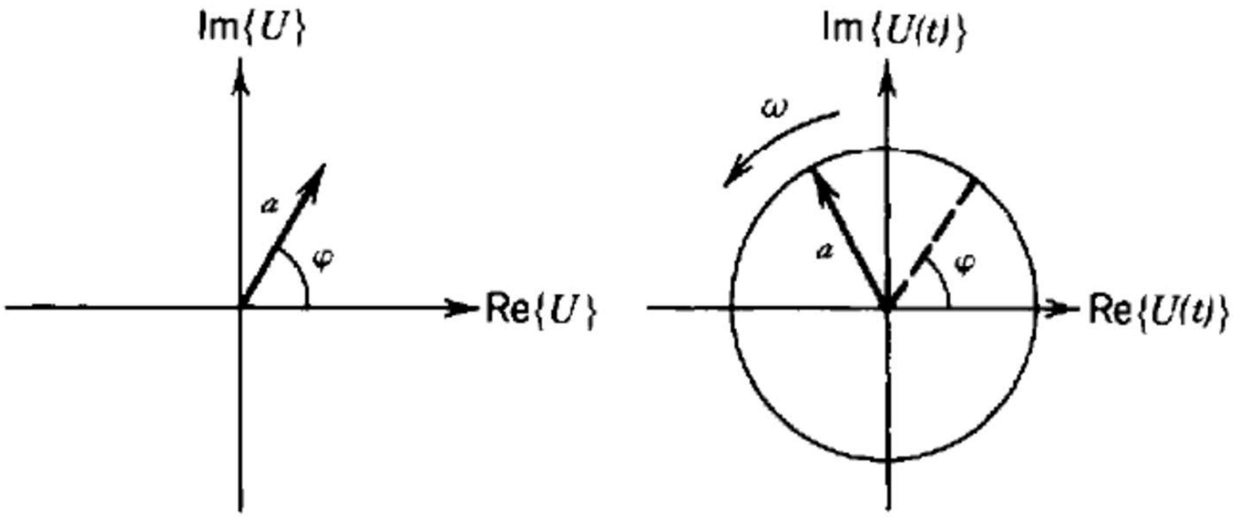
Complex Amplitude

We can re-write the complex wavefunction also in the following form:

$$U(\boldsymbol{r},t) = a(\boldsymbol{r})\exp(j2\pi f t)\exp[j\varphi(\boldsymbol{r})] = U(\boldsymbol{r})\exp(j2\pi f t)$$

Where U(**r**) is now the **complex amplitude** of the wave. Therefore, the real wavefunction can be written in the following form:

$$u(\boldsymbol{r},t) = Re[U(\boldsymbol{r})\exp[j2\pi f t]\,] = \frac{1}{2}[U(\boldsymbol{r})\exp(j2\pi f t) + U(\boldsymbol{r})^{*}\exp(-j2\pi f t)]$$



Helmholtz Equation

If we now replace $U(\boldsymbol{r},t) = U(\boldsymbol{r})\exp(j2\pi f t)$ in the wave equation we obtain the **Helmholtz equation**:

$$\boxed{\nabla^2 U + k^2 U = 0}$$

Where we can define the **wavenumber**:

$$\boxed{k = \frac{2\pi f}{c} = \frac{\omega}{c}}$$

The **optical intensity** of a monochromatic wave is:

$$\boxed{I(\boldsymbol{r}) = |U(\boldsymbol{r})|^2}$$

The **wavefronts** are defined as the surfaces where φ(**r**) = constant.

Solutions of the wave equation: Plane waves

The **plane wave** has a complex amplitude:

$$\boxed{U(\boldsymbol{r}) = A\exp(-jk\boldsymbol{r}) = A\exp[-\mathrm{j}(k_x x + k_y \mathrm{y} + k_z \mathrm{z})]}$$

Where A is a complex constant called **complex envelope** and $\mathbf{k} = (k_x, k_y, k_z)$ is the **wavevector.**

If we replace the expression of the plane wave amplitude in the Helmholtz equation we find that the magnitude of the wavevector **k** is the wavenumber ($k^2 = k_x^2 + k_y^2 + k_z^2$).

In a plane wave consecutive wavefronts are separated by a **wavelength**:

$$\boxed{\lambda = \frac{2\pi}{k} = \frac{c}{f}}$$

The plane wave has a constant **intensity** $I(\boldsymbol{r}) = |A|^2$ everywhere in space so it carries **infinite power.**

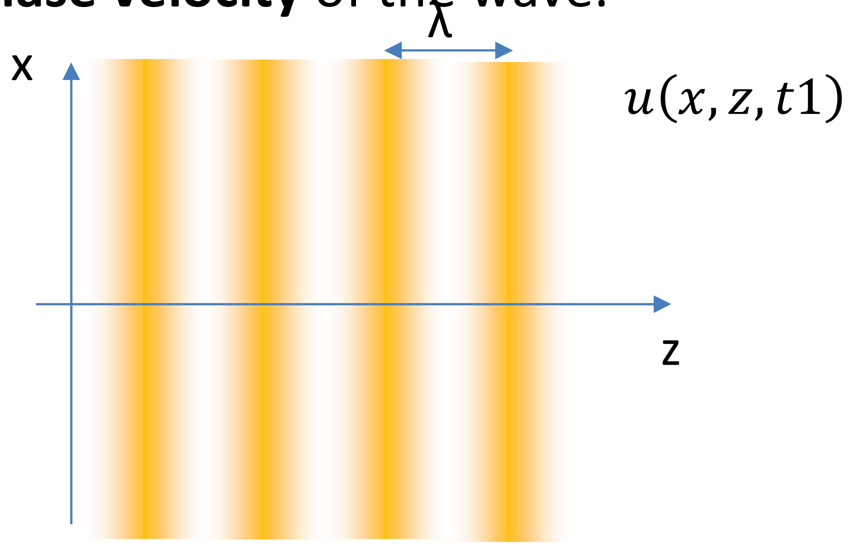
Solutions of the wave equation: Plane waves

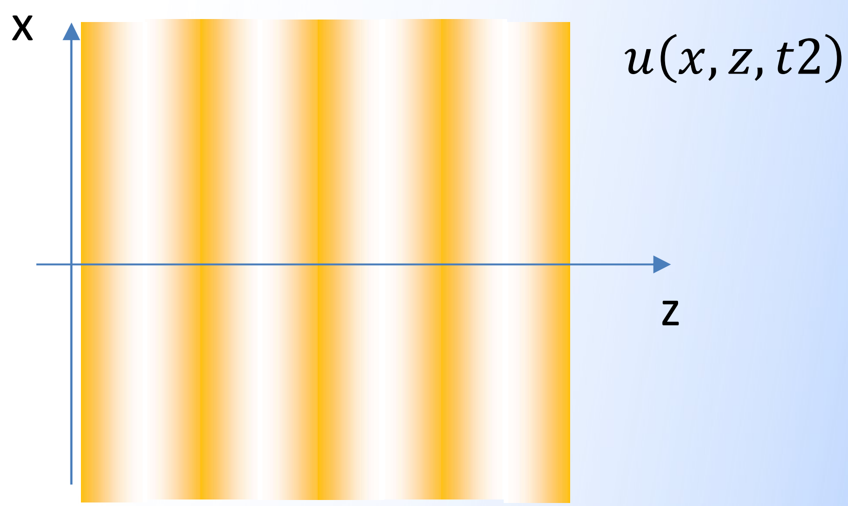
Assuming the plane wave propagates in the z direction then $k=k_z$ and $U(\boldsymbol{r}) = A\exp(-jkz)$. The corresponding wave function will be:

$$u(\boldsymbol{r},t) = |A|\cos[2\pi ft - kz + \arg(A)] = |A|\cos\left[2\pi f\left(t - \frac{z}{c}\right) + \arg(A)\right]$$

The wavefront is periodic in time with period T=1/f and in space with periodicity λ=2π/k (wavelength).

The phase varies in time and position as a function of $\left(t - \frac{z}{c}\right)$, therefore c is also the **phase velocity** of the wave.





Solutions of the wave equation: Plane waves

As a monochromatic plane wave propagates through media with different refractive indices n, its frequency remains the same, but wavelength, velocity and wavenumber change in the following way:

$$c = \frac{c_0}{n}$$

$$\lambda = \frac{\lambda_0}{n}$$

$$k = n\,\mathrm{k}_0$$

Solutions of the wave equation: Spherical waves

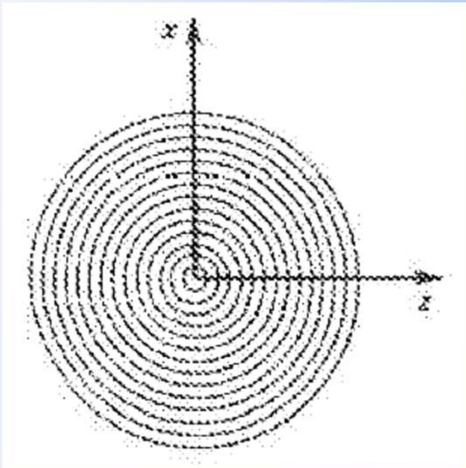
Another simple solution of the Helmholtz equation is the **spherical wave:**

$$\boxed{U(\boldsymbol{r}) = \frac{A_0}{r}\exp(-jkr)}$$

Where r is the distance from the origin and A_0 is a constant.
The **intensity** is inversely proportional to the square of the distance:

$$\boxed{I(\boldsymbol{r}) = \frac{|A_0|^2}{r^2}}$$

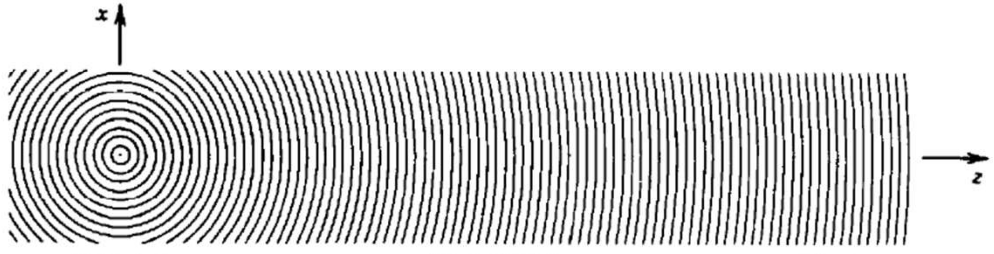
Assuming for simplicity $\arg(A_0)=0$ the **wavefronts** are the surfaces $kr=2\pi q$ or $r=q\lambda$, with q integer. The phase fronts are **concentric spheres** separated by a radial distance $\lambda=2\pi/k$. The wavefronts advance radially with **phase velocity** c.



Solutions of the wave equation: Paraboloidal waves

In the limit where we are examining a spherical wave in a position z very close to the z axis and far from the origin (for which $\sqrt{x^2+y^2} \ll z$) we can assume to work in the paraxial approximation. Under these circumstances we can write a **Fresnel approximation** of a spherical wave as follows:

$$\boxed{U(\boldsymbol{r}) \approx \frac{A_0}{r}\exp(-jkz)\exp\left(-jk\frac{x^2+y^2}{2z}\right)}$$



This approximation will be very useful for diffraction problems. In this regime the spherical wave is well approximated by a **paraboloidal wave**. When z becomes very large the paraboloidal phase factor tends to zero and the wave can be approximated with a plane wave.

NOTE: The Fresnel approximation is valid if $kz\theta^4/8 \ll \pi$, where $\theta = \sqrt{x^2+y^2}/z$. For points (x,y) in a circle of radius *a* centered about the z axis the validity condition is:

$$\boxed{\frac{N_F \theta_m^2}{4} \ll 1}$$

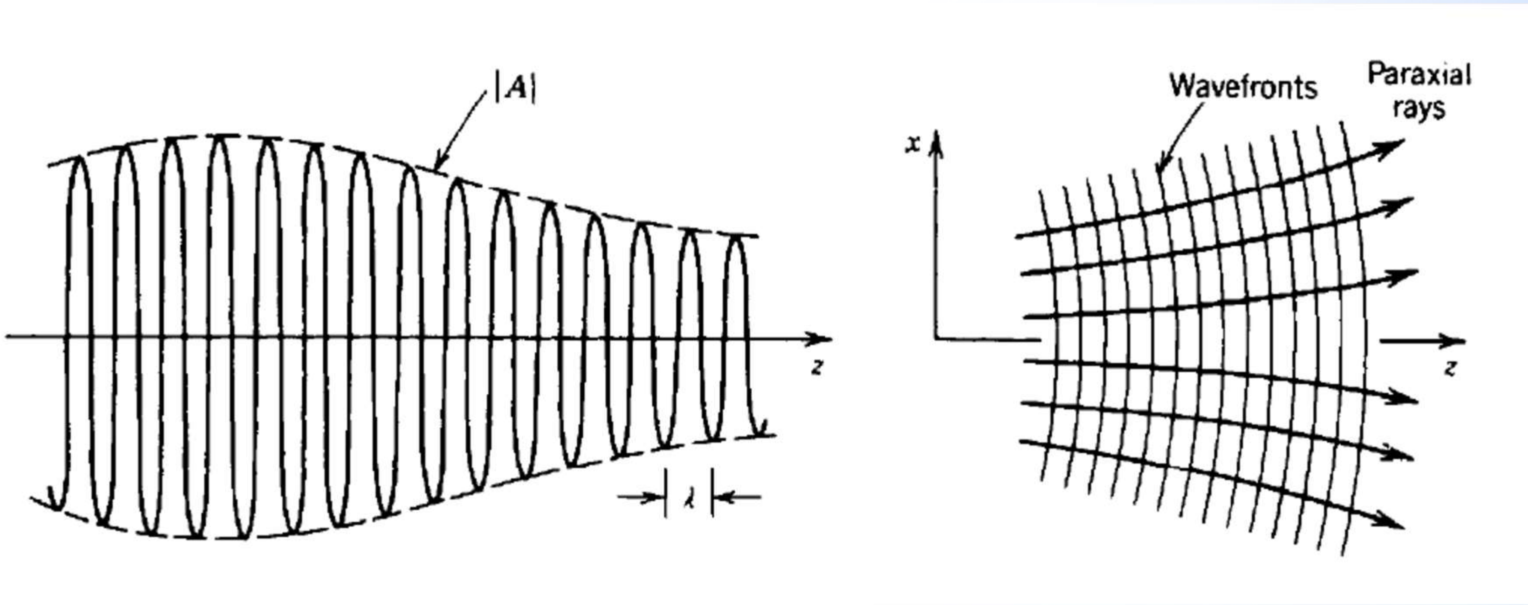
Where $\theta_m = \frac{a}{z}$ is the maximum angle and $N_F = \frac{a^2}{\lambda z}$ is the **Fresnel Number.**

Solutions of the wave equation: Paraxial waves

A paraxial wave is a wave with wavefront normals that are paraxial rays. An easy wave to make a paraxial wave is to start with a plane wave and **modify the complex envelope A making it a slowly varying function of position:**

$$\boxed{U(\boldsymbol{r}) = A(\mathbf{r})\exp(-jkz)}$$

The variation of the envelope A(**r**) must be slow within the distance of a wavelength.



Solutions of the wave equation: Paraxial waves

To satisfy the Helmholtz equation, the paraxial wave complex envelope A(**r**) must satisfy another partial differential equation.

The paraxial approximation means that in a distance Δz=λ the change in amplitude ΔA is much smaller than A.

Since $\Delta A = \left(\frac{\partial A}{\partial z}\right)\Delta z = \left(\frac{\partial A}{\partial z}\right)\lambda$ it follows that:

$$\frac{\partial A}{\partial z} \ll \frac{A}{\lambda} = \frac{Ak}{2\pi} \ll kA.$$

It is also true that the derivative of A over z must be slowly varying, therefore:

$$\frac{\partial^2 A}{\partial z^2} \ll k^2 A$$

Replacing these two expressions in the Helmholtz equation and neglecting the second derivative of A we get:

$$\boxed{\nabla_T^2 A - j2k\frac{\partial A}{\partial z} = 0}$$

Paraxial Helmholtz equation

Simple optical Components

- MIRRORS
- PLANAR BOUNDARIES
- PLATES AND LENSES
- GRADED INDEX COMPONENTS

Mirrors

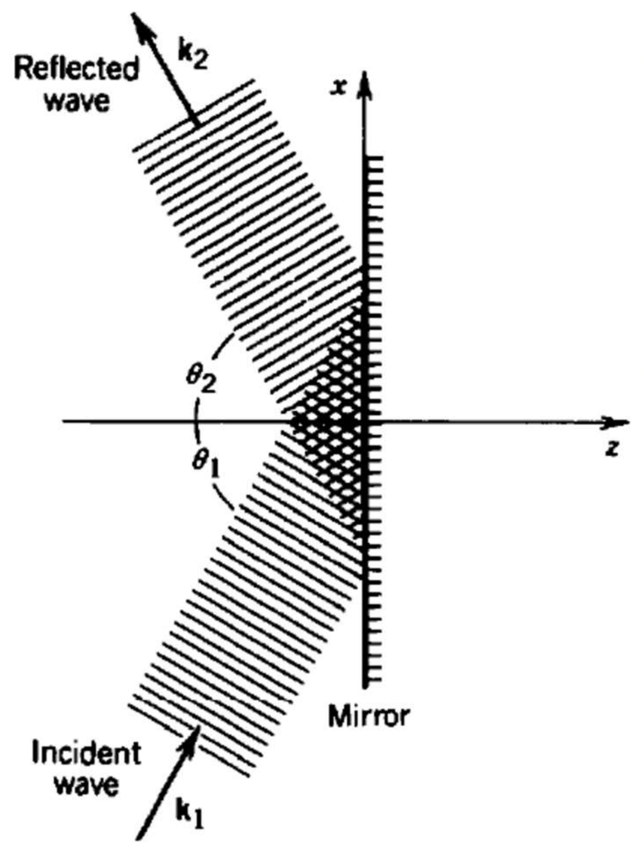
A plane wave of wavevector k_1 incident on a planar mirror located at z=0 plane is reflected into a plane wave of wavevector k_2. Angles of incidence and reflection are θ_1 and θ_2. The sum of the two waves satisfies the Helmholtz equation only if $k_1 = k_2 = k_0$.

Moreover, the boundary condition at the mirror Plane z=0 imposes that the two wavefronts must match for all **r**=(x,y,0). In other words:

$$k_1 \cdot \boldsymbol{r} = k_2 \cdot \boldsymbol{r},$$

Replacing **r**=(x,y,0) in $k_1 = (k_0 sin\theta_1, 0, k_0 cos\theta_1)$ and $k_2 = (k_0 sin\theta_2, 0, -k_0 cos\theta_2)$ we get:

$$k_0 x sin\theta_1 = k_0 x sin\theta_2 \Rightarrow \boxed{\theta_1 = \theta_2}$$



UNIVERSITY OF BRESCIA

Planar boundary

We now consider a plane wave of wavevector k_1 incident on a planar boundary between two homogeneous media with refractive indices n_1 and n_2, located at z=0. The plane is refracted into a plane wave of wavevector k_2. and reflected into a plane wave with wavevector k_3.
The sum of the three waves satisfies the Helmholtz equation if

$$k_1 = k_3 = n_1 k_0 \quad \text{and} \quad k_2 = n_2 k_0.$$

The boundary condition at z=0 imposes that the wavefronts must match for all **r**=(x,y,0).
In other words:

$$k_1 \cdot \boldsymbol{r} = k_2 \cdot \boldsymbol{r} = k_3 \cdot \boldsymbol{r},$$

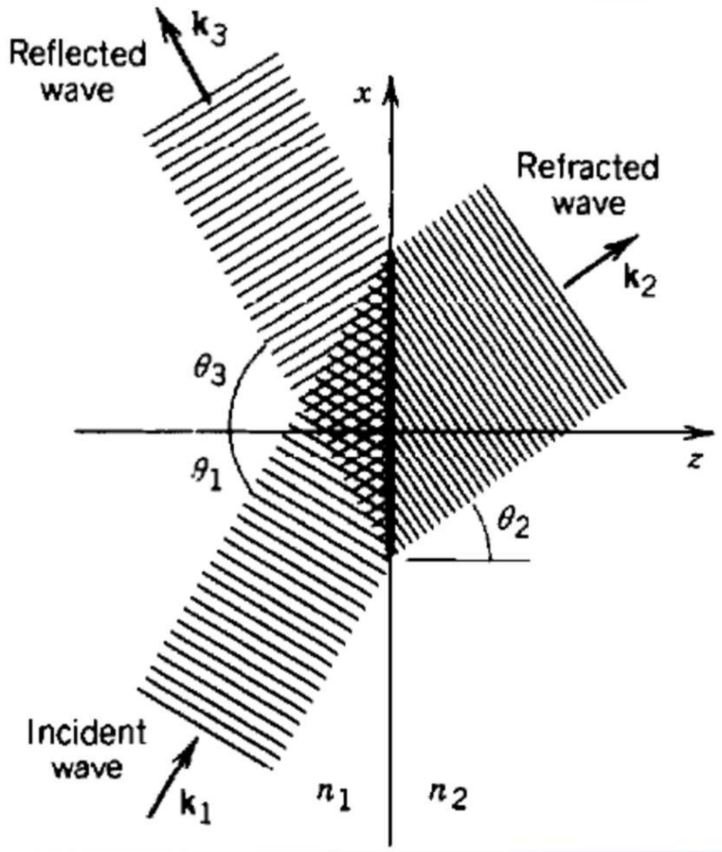
Replacing **r**=(x,y,0) in

$$k_1 = (k_0 n_1 sin\theta_1, 0, k_0 n_1 cos\theta_1)\ ,$$
$$k_2 = (k_0 n_2 sin\theta_2, 0, k_0 n_2 cos\theta_2)$$
$$k_3 = (k_0 n_1 sin\theta_3, 0, -k_0 n_1 cos\theta_3)$$

we get:

SNELL'S LAW

$$\boxed{n_1 sin\theta_1 = n_2 sin\,\theta_2}$$

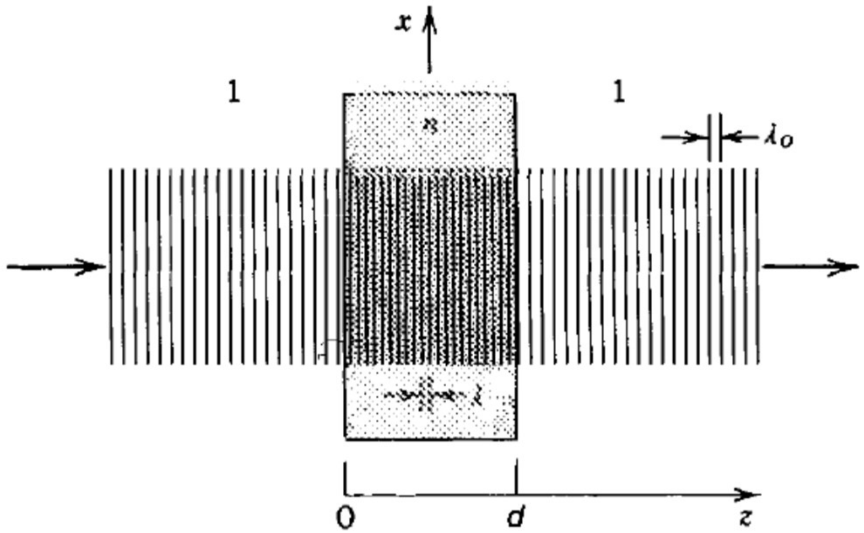


Transparent Plates

Let's consider the transmission of a plane wave through a transparent plate of refractive index n and thickness d. The incident wave travels in the z direction. U(x,y,z) is the complex amplitude of the wave. Since reflection at the boundaries are ignored U(x,y,z) is continuous at the boundaries.
The **complex amplitude transmittance** is defined as:

$$t(x,y) = \frac{U(x,y,d)}{U(x,y,0)}$$

Once inside the plate, the wave propagates as a plane wave with wavenumber nk_0, so that U(x,y,z) is proportional to $\exp(-jnk_0 d)$. It follows that:



$$\boxed{t(x,y) = \frac{U(x,y,d)}{U(x,y,0)} = \exp(-jnk_0 d)}$$

The plate introduces a phase shift:

$$nk_0 d = 2\pi\left(\frac{d}{\lambda}\right)$$

Transparent Plates

If the plane wave is incident at an angle θ with a wavevector **k** the refracted wave will have wavevector $\mathbf{k_1}$ and will refract with an angle θ_1 according to the Snell's law. At the second interface the wave will refract again and exit with wavevector **k** and same angle of the input wave θ.

Inside the plate the complex amplitude is proportional to

$$\exp(-jnk_0(z cos\theta_1 + x sin\theta_1))$$

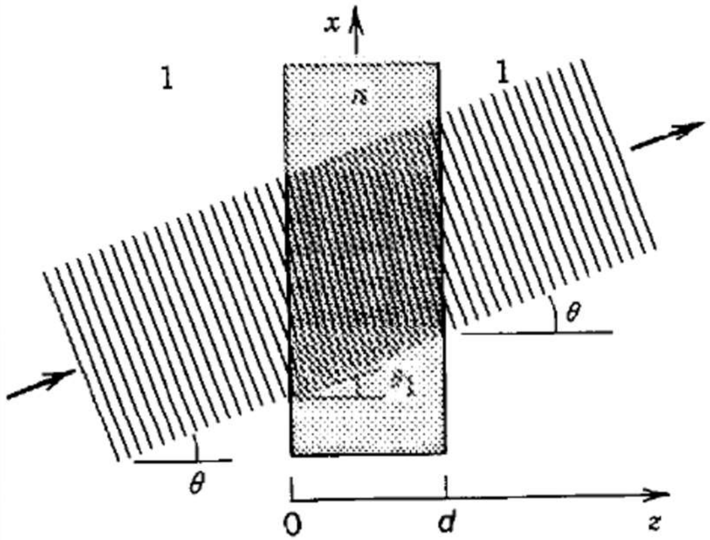
So that the complex amplitude transmittance will be:

$$\boxed{t(x,y) = \exp(-jnk_0 d cos\theta_1)}$$

If ***θ* is small** the wave is paraxial $\theta_1 \approx \frac{\theta}{n}$ is also small and Transmittance can be written as:

$$\boxed{t(x,y) \approx \exp(-jnk_0 d)\exp\left(\frac{jk_0\theta^2 d}{2n}\right)}$$

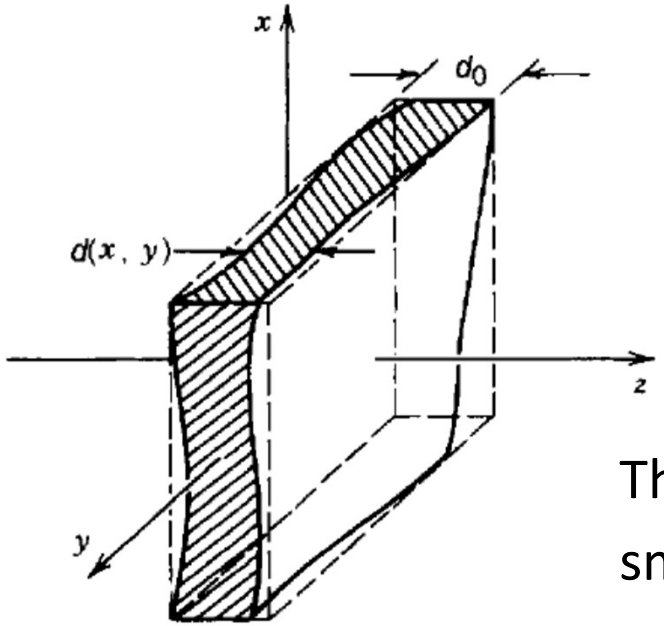
If the plate is sufficiently **thin** and incident **angle is small** Transmittance is identical to normal incidence.



Thin Transparent Plates of varying thickness

Let's now consider a **thin transparent plate** has a **thickness that varies smoothly** as a function of x and y, i.e., d(x,y), and we assume a **paraxial wave** is incident on that plate. If the overall coordinate along z varies between 0 and d_0 we can consider our plane wave that crosses a thin plate of material of thickness d(x,y) and a layer of air of total thicknes d_0 - d(x,y), so that the total transmittance would be the product of two optical components, the plate and the air layer and can be written as follows:

$$t(x,y) = \exp[-jnk_0 d(x,y)] \exp\left[-jk_0\left(d_0 - d(x,y)\right)\right]$$



$$\boxed{t(x,y) \approx h_0 \exp[-j(n-1)k_0 d(x,y)]}$$

where $h_0 = \exp[-jk_0 d_0]$ is a constant phase factor.
This expression is valid in the paraxial approximation and for small thicknesses d_0.

Thin lens

The transmittance expression for the thin plate of variable thickness can be applied in the case of a planoconvex thin lens. The lens can be considered a cap of a sphere with radius R. The thickness at the generic point (x,y) is d(x,y) = d_0-PQ=d_0-(R-QC), or

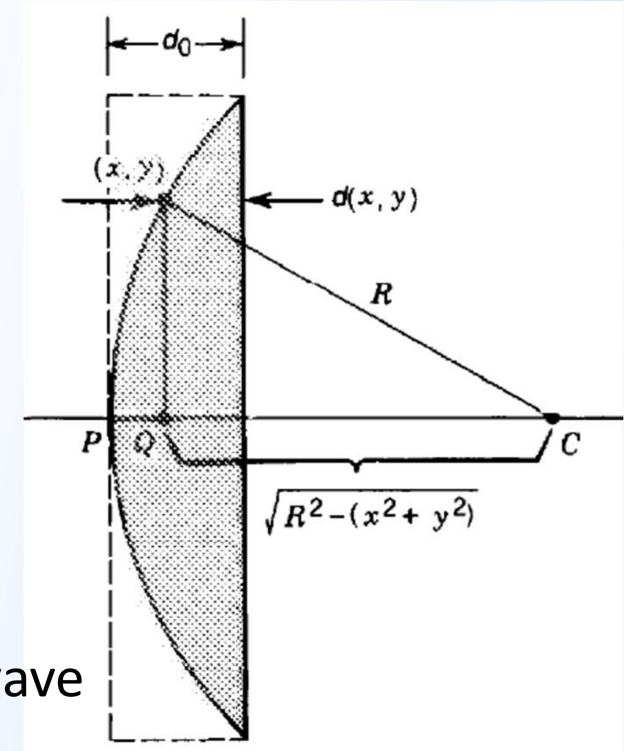
$$d(x,y) = d_0 - \left[\mathrm{R} - \sqrt{R^2 - (x^2 + y^2)}\right]$$

If we only consider points for which $x^2 + y^2 \ll \mathrm{R}^2$ we can use the same expansion in Taylor series that we used for the Fresnel approximation and write:

$$d(x,y) \approx d_0 - \frac{x^2 + y^2}{2R}$$

By replacing this expression in the transmittance for the Thin variable plate expression we get:

$$\boxed{t(x,y) \approx h_0 \exp\left[-jk_0 \frac{x^2+y^2}{2f}\right], \text{ with } f = \frac{R}{n-1} \text{ focal length.}}$$

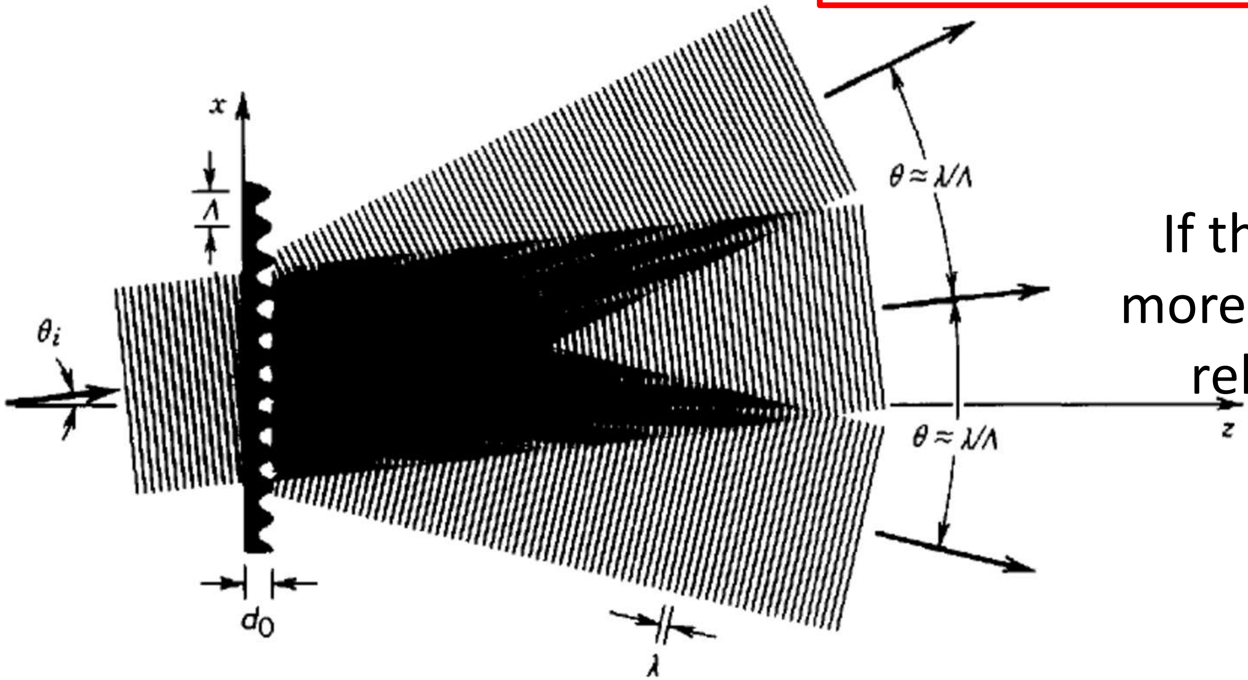


NOTE: this lens transforms a plane wave into a paraboloidal wave

Diffraction Gratings

A **diffraction grating** is an optical component that periodically modulates the phase and amplitude of an incident wave. If the grating is made of a thin transparent plate with thickness that varies periodically in x with period Λ, this grating transforms an incident plane wave with $\lambda \ll \Lambda$ and incident with a small angle θ_i into several plane waves exiting at angles:

$$\theta_q = \theta_i + q\frac{\lambda}{\Lambda}$$



Where q=0,+1,-1,+2,-2,… are the diffraction orders. If the incident angle is not small then a more general equation can be written to relate the incident and exiting angles:

$$\sin\theta_q = \sin\theta_i + q\frac{\lambda}{\Lambda}$$

Diffraction gratings are typically used as filters or spectrum analyzers.

Graded-index optical components

We have seen that the effect of an optical component of variable thickness is basically to impart a certain phase shift to the incoming wave.

The same phase shift can be introduced by a transparent planar plate of fixed thickness but with variable refractive index.

From this follows that the complex amplitude of a thin transparent plate of thickness d_0 with variable refractive index n(x,y) is:

$$\boxed{t(x,y) = \exp(-jn(x,y)k_0 d_0)}$$