3. Laplace Transform and Continuous-Time LTI Systems

3.1. Introduction

A basic result from Chap. 2 is that the response of an LTI system is given by convolution of the input and the impulse response of the system. In this chapter and the following one we present an alternative representation for signals and LTI systems. In this chapter, the Laplace transform is introduced to represent continuous-time signals in the s-domain (s is a complex variable), and the concept of the system function for a continuous-time LTI system is described. Many useful insights into the properties of continuous-time LTI systems, as well as the study of many problems involving LTI systems, can be provided by application of the Laplace transform technique.

3.2. The Laplace Transform

In Sec. 2.4 we saw that for a continuous-time LTI system with impulse response $h(t)$, the output $y(t)$ of the system to the complex exponential input of the form e^{st} is

$$y(t) = T\{e^{st}\} = H(s)e^{st}$$

(3.1)

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

(3.2)

3.2.1. A. Definition:

The function $H(s)$ in Eq. (3.2) is referred to as the Laplace transform of $h(t)$. For a general continuous-time signal $x(t)$, the Laplace transform $X(s)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

(3.3)

The variable s is generally complex-valued and is expressed as

$$s = \sigma + j\omega$$

(3.4)

The Laplace transform defined in Eq. (3.3) is often called the *bilateral* (or *two-sided*) Laplace transform in contrast to the *unilateral* (or *one-sided*) Laplace transform, which is defined as

$$X_I(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

(3.5)

where $0^- = \lim_{\varepsilon \to 0}(0 - \varepsilon)$. Clearly the bilateral and unilateral transforms are equivalent only if $x(t) = 0$ for $t < 0$. The unilateral Laplace transform is discussed in Sec. 3.8. We will omit the word "bilateral" except where it is needed to avoid ambiguity.

Equation (3.3) is sometimes considered an operator that transforms a signal $x(t)$ into a function $X(s)$ symbolically represented by

$$X(s) = \mathscr{L}\{x(t)\}$$

(3.6)

and the signal $x(t)$ and its Laplace transform $X(s)$ are said to form a Laplace transform pair denoted as

$$x(t) \leftrightarrow X(s)$$

(3.7)

3.2.2. B. The Region of Convergence:

The range of values of the complex variables s for which the Laplace transform converges is called the *region of convergence* (ROC). To illustrate the Laplace transform and the associated ROC, let us consider some examples.

EXAMPLE 3.1 Consider the signal

$$x(t) = e^{-at}u(t) \qquad a \text{ real}$$

(3.8)

Then by Eq. (3.3) the Laplace transform of $x(t)$ is

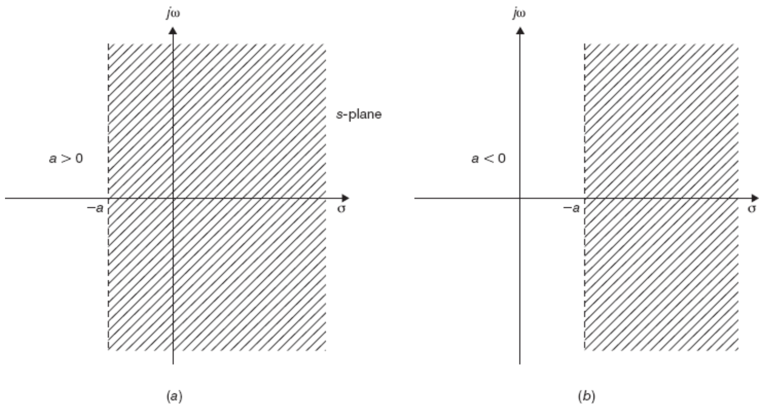
$$X(s) = \int_{-\infty}^{\infty} e^{-at}\, u(t) e^{-st}\, dt = \int_{0^+}^{\infty} e^{-(s+a)t}\, dt$$
$$= -\frac{1}{s+a} e^{-(s+a)t}\Big|_{0^+}^{\infty} = \frac{1}{s+a} \qquad \operatorname{Re}(s) > -a$$

(3.9)

because $\lim_{t \to \infty} e^{-(s+a)t} = 0$ only if $\operatorname{Re}(s + a) > 0$ or $\operatorname{Re}(s) > -a$.

Thus, the ROC for this example is specified in Eq. (3.9) as $\operatorname{Re}(s) > -a$ and is displayed in the complex plane as shown in Fig. 3-1 by the shaded area to the right of the line $\operatorname{Re}(s) = -a$. In Laplace transform applications, the complex plane is commonly referred to as the s-plane. The horizontal and vertical axes are sometimes referred to as the σ-axis and the $j\omega$-axis, respectively.

Figure 3-1 *ROC for Example 3.1.*



EXAMPLE 3.2 Consider the signal

$$x(t) = -e^{-at}u(-t) \qquad a \text{ real}$$

(3.10)

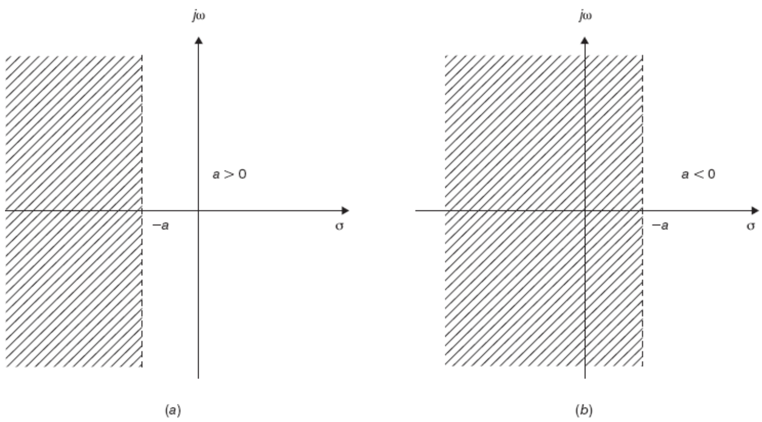
Its Laplace transform $X(s)$ is given by (Prob. 3.1)

$$X(s) = \frac{1}{s+a} \qquad \operatorname{Re}(s) < -a$$

(3.11)

Thus, the ROC for this example is specified in Eq. (3.11) as Re(s) < −a and is displayed in the complex plane as shown in Fig. 3-2 by the shaded area to the left of the line Re(s) = −a. Comparing Eqs. (3.9) and (3.11), we see that the algebraic expressions for $X(s)$ for these two different signals are identical except for the ROCs. Therefore, in order for the Laplace transform to be unique for each signal $x(t)$, *the ROC must be specified as part of the transform.*

Figure 3-2 *ROC for Example 3.2.*



3.2.3. C. Poles and Zeros of *X*(*s*):

Usually, $X(s)$ will be a rational function in s; that is,

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{a_0}{b_0}\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

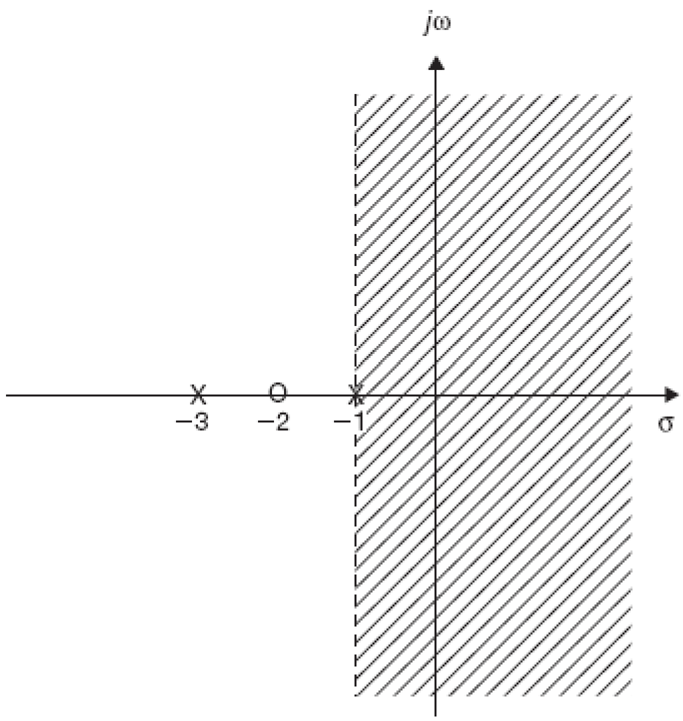
(3.12)

The coefficients a_k and b_k are real constants, and m and n are positive integers. The $X(s)$ is called a *proper* rational function if $n > m$, and an *improper* rational function if $n \le m$. The roots of the numerator polynomial, z_k, are called the *zeros* of $X(s)$ because $X(s) = 0$ for those values of s. Similarly, the roots of the denominator polynomial, p_k, are called the *poles* of $X(s)$ because $X(s)$ is infinite for those values of s. Therefore, the poles of $X(s)$ lie outside the ROC since $X(s)$ does not converge at the poles, by definition. The zeros, on the other hand, may lie inside or outside the ROC. Except for a scale factor a_0/b_0, $X(s)$ can be completely specified by its zeros and poles. Thus, a very compact representation of $X(s)$ in the s-plane is to show the locations of poles and zeros in addition to the ROC.

Traditionally, an "×" is used to indicate each pole location and an "o" is used to indicate each zero. This is illustrated in Fig. 3-3 for $X(s)$ given by

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{s+2}{(s+1)(s+3)} \qquad \text{Re}(s) > -1$$

Figure 3-3 s-plane representation of $X(s) = (2s + 4)/(s^2 + 4s + 3)$.



Note that $X(s)$ has one zero at $s = -2$ and two poles at $s = -1$ and $s = -3$ with scale factor 2.

3.2.4. D. Properties of the ROC:

As we saw in Examples 3.1 and 3.2, the ROC of $X(s)$ depends on the nature of $x(t)$. The properties of the ROC are summarized below. We assume that $X(s)$ is a rational function of s.

Property 1: The ROC does not contain any poles.

Property 2: If $x(t)$ is a *finite-duration* signal, that is, $x(t) = 0$ except in a finite interval $t_1 \le t \le t_2$ ($-\infty < t_1$ and $t_2 < \infty$), then the ROC is the entire s-plane except possibly $s = 0$ or $s = \infty$.

Property 3: If $x(t)$ is a *right-sided* signal, that is, $x(t) = 0$ for $t < t_1 < \infty$, then the ROC is of the form

$$\operatorname{Re}(s) > \sigma_{\max}$$

where σ_{max} equals the maximum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the right of the vertical line $\operatorname{Re}(s) = \sigma_{max}$ in the s-plane and thus to the right of all of the poles of $X(s)$.

Property 4: If $x(t)$ is a *left-sided* signal, that is, $x(t) = 0$ for $t > t_2 > -\infty$, then the ROC is of the form

$$\operatorname{Re}(s) < \sigma_{\min}$$

where σ_{min} equals the minimum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the left of the vertical line $\operatorname{Re}(s) = \sigma_{min}$ in the s-plane and thus to the left of all of the poles of $X(s)$.

Property 5: If $x(t)$ is a *two-sided* signal, that is, $x(t)$ is an infinite-duration signal that is neither right-sided nor left-sided, then the ROC is of the form

$$\sigma_1 < \operatorname{Re}(s) < \sigma_2$$

where σ_1 and σ_2 are the real parts of the two poles of $X(s)$. Thus, the ROC is a vertical strip in the s-plane between the vertical lines $\operatorname{Re}(s) = \sigma_1$ and $\operatorname{Re}(s) = \sigma_2$.

Note that Property 1 follows immediately from the definition of poles; that is, $X(s)$ is infinite at a pole. For verification of the other properties see Probs. 3.2 to 3.7.

3.3. Laplace Transforms of Some Common Signals

3.3.1. A. Unit Impulse Function $\delta(t)$:

Using Eqs. (3.3) and (1.20), we obtain

$$\mathscr{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = 1 \qquad \text{all } s$$

(3.13)

3.3.2. B. Unit Step Function $u(t)$:

$$\mathscr{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st}\, dt = \int_{0^+}^{\infty} e^{-st}\, dt$$
$$= -\frac{1}{s} e^{-st} \Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \operatorname{Re}(s) > 0$$

(3.14)

where $0^+ = \lim_{\varepsilon \to 0}(0 + \varepsilon)$.

3.3.3. C. Laplace Transform Pairs for Common Signals:

The Laplace transforms of some common signals are tabulated in Table 3-1. Instead of having to reevaluate the transform of a given signal, we can simply refer to such a table and read out the desired transform.

Table 3-1 Some Laplace Transforms Pairs

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\dfrac{1}{s}$	$\text{Re}(s) > 0$
$-u(-t)$	$\dfrac{1}{s}$	$\text{Re}(s) < 0$
$tu(t)$	$\dfrac{1}{s^2}$	$\text{Re}(s) > 0$
$t^k u(t)$	$\dfrac{k!}{s^{k+1}}$	$\text{Re}(s) > 0$
$e^{-at} u(t)$	$\dfrac{1}{s+a}$	$\text{Re}(s) > -\text{Re}(a)$
$-e^{-at} u(-t)$	$\dfrac{1}{s+a}$	$\text{Re}(s) < -\text{Re}(a)$
$te^{-at} u(t)$	$\dfrac{1}{(s+a)^2}$	$\text{Re}(s) > -\text{Re}(a)$
$-te^{-at}u(-t)$	$\dfrac{1}{(s+a)^2}$	$\text{Re}(s) < -\text{Re}(a)$
$\cos \omega_0 tu(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$\text{Re}(s) > 0$
$\sin \omega_0 tu(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}(s) > 0$
$e^{-at} \cos \omega_0 tu(t)$	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\text{Re}(s) > -\text{Re}(a)$
$e^{-at} \sin \omega_0 tu(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{Re}(s) > -\text{Re}(a)$

3.4. Properties of the Laplace Transform

Basic properties of the Laplace transform are presented in the following. Verification of these properties is given inProbs. 3.8 to 3.16.

3.4.1. A. Linearity:

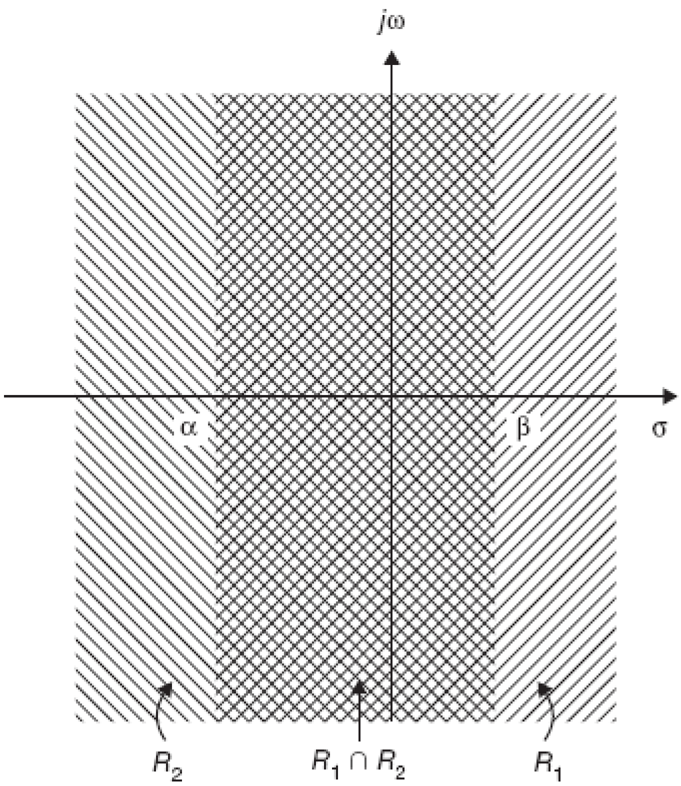
If

$$x_1(t) \leftrightarrow X_1(s) \qquad \text{ROC} = R_1$$
$$x_2(t) \leftrightarrow X_2(s) \qquad \text{ROC} = R_2$$
$$\text{Then} \qquad a_1x_1(t) + a_2x_2(t) \leftrightarrow a_1X_1(s) + a_2X_2(s) \qquad R' \supset R_1 \cap R_2$$

(3.15)

The set notation $A \supset B$ means that set A contains set B, while $A \cap B$ denotes the intersection of sets A and B, that is, the set containing all elements in both A and B. Thus, Eq. (3.15) indicates that the ROC of the resultant Laplace transform is at least as large as the region in common between R_1 and R_2. Usually we have simply $R' = R_1 \cap R_2$. This is illustrated in Fig. 3-4.

Figure 3-4 ROC of $a_1X_1(s) + a_2X_2(s)$.



3.4.2. B. Time Shifting:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$
$$\text{then} \qquad x(t - t_0) \leftrightarrow e^{-st_0}X(s) \qquad R' = R$$

(3.16)

Equation (3.16) indicates that the ROCs before and after the time-shift operation are the same.

3.4.3. C. Shifting in the *s*-Domain:

If

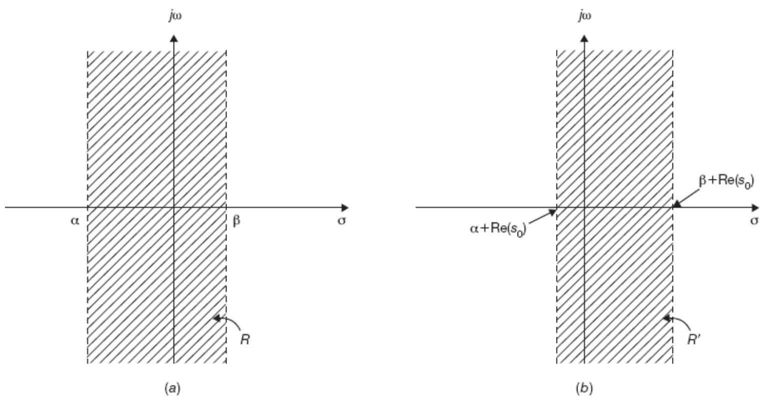
$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$e^{s_0 t} x(t) \leftrightarrow X(s - s_0) \qquad R' = R + \text{Re}(s_0)$$

(3.17)

Equation (3.17) indicates that the ROC associated with $X(s-s_0)$ is that of $X(s)$ shifted by $\text{Re}(s_0)$. This is illustrated in Fig. 3-5.

Figure 3-5 Effect on the ROC of shifting in the s-domain. (a) ROC of $X(s)$; (b) ROC of $X(s - s_0)$.



3.4.4. D. Time Scaling:

If

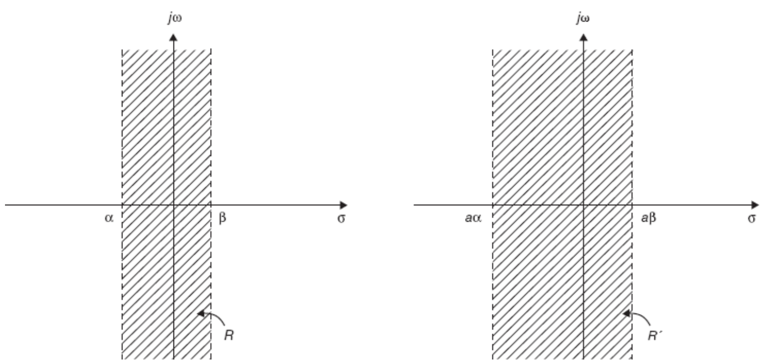
$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad R' = aR$$

(3.18)

Equation (3.18) indicates that scaling the time variable t by the factor a causes an inverse scaling of the variable s by $1/a$ as well as an amplitude scaling of $X(s/a)$ by $1/|a|$. The corresponding effect on the ROC is illustrated in Fig. 3-6.

Figure 3-6 Effect on the ROC of time scaling. (a) ROC of $X(s)$; (b) ROC of $X(s/a)$.



3.4.5. E. Time Reversal:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

$$\text{then} \qquad x(-t) \leftrightarrow X(-s) \qquad R' = -R$$

(3.19)

Thus, time reversal of $x(t)$ produces a reversal of both the σ- and $j\omega$-axes in the s-plane. Equation (3.19) is readily obtained by setting a =−1 in Eq. (3.18).

3.4.6. F. Differentiation in the Time Domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

$$\text{then} \qquad \frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R$$

(3.20)

Equation (3.20) shows that the effect of differentiation in the time domain is multiplication of the corresponding Laplace transform by s. The associated ROC is unchanged unless there is a pole-zero cancellation at $s = 0$.

3.4.7. G. Differentiation in the *s*-Domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

$$\text{then} \qquad -tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$$

(3.21)

3.4.8. H. Integration in the Time Domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

$$\text{then} \qquad \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) \qquad R' = R \cap \{\text{Re}(s) > 0\}$$

(3.22)

Equation (3.22) shows that the Laplace transform operation corresponding to time-domain integration is multiplication by $1/s$, and this is expected since integration is the inverse operation of differentiation. The form of R' follows from the possible introduction of an additional pole at $s = 0$ by the multiplication by $1/s$.

3.4.9. I. Convolution:

If

$$\begin{aligned} & x_1(t) \leftrightarrow X_1(s) && \text{ROC} = R_1 \\ & x_2(t) \leftrightarrow X_2(s) && \text{ROC} = R_2 \\ \text{then} \quad & x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s) && R' \supset R_1 \cap R_2 \end{aligned}$$

(3.23)

This convolution property plays a central role in the analysis and design of continuous-time LTI systems.

Table 3-2 summarizes the properties of the Laplace transform presented in this section.

Table 3-2 Properties of the Laplace Transform

PROPERTY	SIGNAL	TRANSFORM	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	$R'=R$
Shifting in s	$e^{s_0t}x(t)$	$X(s-s_0)$	$R'=R+\text{Re}(s_0)$
Time scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert}X(a)$	$R'=aR$
Time reversal	$x(-t)$	$X(-s)$	$R'=-R$
Differentiation in t	$\dfrac{dx(t)}{dt}$	$sX(s)$	$R' \supset R$
Differentiation in s	$-tx(t)$	$\dfrac{dX(s)}{ds}$	$R'=R$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\dfrac{1}{s}X(s)$	$R' \supset R \cap \{\text{Re}(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	$R' \supset R_1 \cap R_2$

3.5. The Inverse Laplace Transform

Inversion of the Laplace transform to find the signal $x(t)$ from its Laplace transform $X(s)$ is called the *inverse* Laplace transform, symbolically denoted as

$$x(t) = \mathscr{L}^{-1}\{X(s)\}$$

(3.24)

3.5.1. A. Inversion Formula:

There is a procedure that is applicable to all classes of transform functions that involves the evaluation of a line integral in complex s-plane; that is,

$$x(t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

(3.25)

In this integral, the real c is to be selected such that if the ROC of $X(s)$ is $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, then $\sigma_1 < c < \sigma_2$. The evaluation of this inverse Laplace transform integral requires understanding of complex variable theory.

3.5.2. B. Use of Tables of Laplace Transform Pairs:

In the second method for the inversion of $X(s)$, we attempt to express $X(s)$ as a sum

$$X(s) = X_1(s) + \cdots + X_n(s)$$

(3.26)

where $X_1(s), \ldots, X_n(s)$ are functions with known inverse transforms $x_1(t), \ldots, x_n(t)$. From the linearity property (3.15) it follows that

$$x(t) = x_1(t) + \cdots + x_n(t)$$

(3.27)

3.5.3. C. Partial-Fraction Expansion:

If $X(s)$ is a rational function, that is, of the form

$$X(s)=\frac{N(s)}{D(s)}=k\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

(3.28)

a simple technique based on partial-fraction expansion can be used for the inversion of $X(s)$.

(a) When $X(s)$ is a proper rational function, that is, when $m < n$:

3.5.3.1. 1. Simple Pole Case:

If all poles of $X(s)$, that is, all zeros of $D(s)$, are simple (or distinct), then $X(s)$ can be written as

$$X(s)=\frac{c_1}{s-p_1}+\cdots+\frac{c_n}{s-p_n}$$

(3.29)

where coefficients c_k are given by

$$c_k=(s-p_k)X(s)\big|_{s=p_k}$$

(3.30)

3.5.3.2. 2. Multiple Pole Case:

If $D(s)$ has multiple roots, that is, if it contains factors of the form $(s - p_i)^r$, we say that p_i is the *multiple pole of* $X(s)$ *with multiplicity* r. Then the expansion of $X(s)$ will consist of terms of the form

$$\frac{\lambda_1}{s - p_i} + \frac{\lambda_2}{(s - p_i)^2} + \cdots + \frac{\lambda_r}{(s - p_i)^r}$$

(3.31)

$$\text{where} \qquad \lambda_{r-k} = \frac{1}{k!}\frac{d^k}{ds^k}[(s - p_i)^r X(s)]\Big|_{s=p_i}$$

(3.32)

(*b*) When $X(s)$ is an improper rational function, that is, when $m \geq n$:

If $m \geq n$, by long division we can write $X(s)$ in the form

$$X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

(3.33)

where $N(s)$ and $D(s)$ are the numerator and denominator polynomials in s, respectively, of $X(s)$, the quotient $Q(s)$ is a polynomial in s with degree $m - n$, and the remainder $R(s)$ is a polynomial in s with degree strictly less than n. The inverse Laplace transform of $X(s)$ can then be computed by determining the inverse Laplace transform of $Q(s)$ and the inverse Laplace transform of $R(s)/D(s)$. Since $R(s)/D(s)$ is proper, the inverse Laplace transform of $R(s)/D(s)$ can be computed by first expanding into partial fractions as given above. The inverse Laplace transform of $Q(s)$ can be computed by using the transform pair

$$\frac{d^k\delta(t)}{dt^k} \leftrightarrow s^k \qquad k = 1, 2, 3, \ldots$$

(3.34)

3.6. The System Function

3.6.1. A. The System Function:

In Sec. 2.2 we showed that the output $y(t)$ of a continuous-time LTI system equals the convolution of the input $x(t)$ with the impulse response $h(t)$; that is,

$$y(t) = x(t) * h(t)$$

(3.35)

Applying the convolution property (3.23), we obtain

$$Y(s) = X(s)H(s)$$

(3.36)

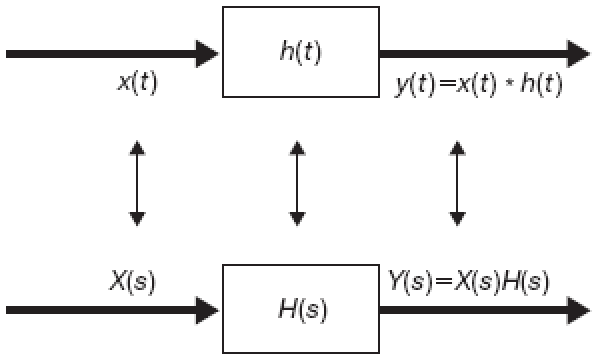
where $Y(s)$, $X(s)$, and $H(s)$ are the Laplace transforms of $y(t)$, $x(t)$, and $h(t)$, respectively. Equation (3.36) can be expressed as

$$H(s) = \frac{Y(s)}{X(s)}$$

(3.37)

The Laplace transform $H(s)$ of $h(t)$ is referred to as the *system function* (or the *transfer function*) of the system. By Eq. (3.37), the system function $H(s)$ can also be defined as the ratio of the Laplace transforms of the output $y(t)$ and the input $x(t)$. The system function $H(s)$ completely characterizes the system because the impulse response $h(t)$ completely characterizes the system. Fig. 3-7 illustrates the relationship of Eqs. (3.35) and (3.36).

Figure 3-7 Impulse response and system function.



3.6.2. B. Characterization of LTI Systems:

Many properties of continuous-time LTI systems can be closely associated with the characteristics of $H(s)$ in the s-plane and in particular with the pole locations and the ROC.

3.6.2.1. 1. Causality:

For a causal continuous-time LTI system, we have

$$h(t) = 0 \qquad t < 0$$

Since $h(t)$ is a right-sided signal, the corresponding requirement on $H(s)$ is that the ROC of $H(s)$ must be of the form

$$\text{Re}(s) > \sigma_{\max}$$

That is, the ROC is the region in the s-plane to the right of all of the system poles. Similarly, if the system is anticausal, then

$$h(t) = 0 \qquad t > 0$$

and $h(t)$ is left-sided. Thus, the ROC of $H(s)$ must be of the form

$$\text{Re}(s) < \sigma_{\min}$$

That is, the ROC is the region in the s-plane to the left of all of the system poles.

3.6.2.2. 2. Stability:

In Sec. 2.3 we stated that a continuous-time LTI system is BIBO stable if and only if [Eq. (2.21)]

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

The corresponding requirement on $H(s)$ is that the ROC of $H(s)$ contains the $j\omega$-axis (that is, $s = j\omega$) (Prob. 3.26).

3.6.2.3. 3. Causal and Stable Systems:

If the system is both causal and stable, then all the poles of $H(s)$ must lie in the left half of the s-plane; that is, they all have negative real parts because the ROC is of the form $\text{Re}(s) > \sigma_{max}$, and since the $j\omega$ axis is included in the ROC, we must have $\sigma_{max} < 0$.

3.6.3. C. System Function for LTI Systems Described by Linear Constant-Coefficient Differential Equations:

In Sec. 2.5 we considered a continuous-time LTI system for which input $x(t)$ and output $y(t)$ satisfy the general linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

(3.38)

Applying the Laplace transform and using the differentiation property (3.20) of the Laplace transform, we obtain

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

or

$$Y(s) \sum_{k=0}^{N} a_k s^k = X(s) \sum_{k=0}^{M} b_k s^k$$

(3.39)

Thus,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

(3.40)

Hence, $H(s)$ is always rational. Note that the ROC of $H(s)$ is not specified by Eq. (3.40) but must be inferred with additional requirements on the system such as the causality or the stability.

3.6.4. D. Systems Interconnection:

For two LTI systems [with $h_1(t)$ and $h_2(t)$, respectively] in cascade [Fig. 3-8(a)], the overall impulse response $h(t)$ is given by [Eq. (2.81), Prob. 2.14]

$$h(t) = h_1(t) * h_2(t)$$

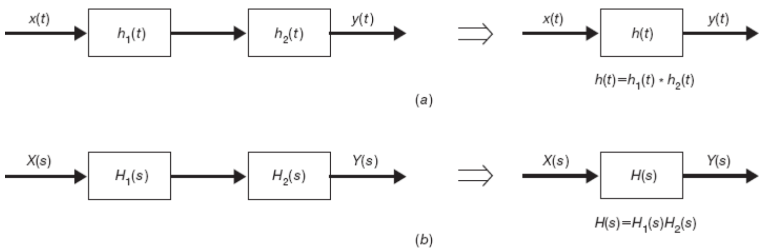


Figure 3-8 Two systems in cascade. (a) Time-domain representation; (b) s-domain representation.

Thus, the corresponding system functions are related by the product

$$H(s) = H_1(s)H_2(s)$$

(3.41)

This relationship is illustrated in Fig. 3-8(*b*).

Similarly, the impulse response of a parallel combination of two LTI systems [Fig. 3-9(*a*)] is given by (Prob. 2.53)

$$h(t) = h_1(t) + h_2(t)$$

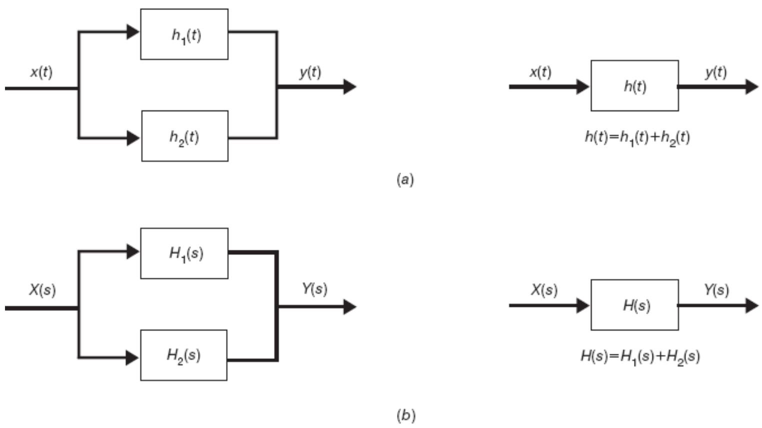


Figure 3-9 Two systems in parallel. (a) Time-domain representation; (b) s-domain representation.

Thus,

$$H(s) = H_1(s) + H_2(s)$$

(3.42)

This relationship is illustrated in Fig. 3-9(*b*).

3.7. The Unilateral Laplace Transform

3.7.1. A. Definitions:

The *unilateral* (or *one-sided*) Laplace transform $X_I(s)$ of a signal $x(t)$ is defined as [Eq. (3.5)]

$$X_I(s) = \int_{0^-}^{\infty} x(t) s^{-st}\, dt$$

(3.43)

The lower limit of integration is chosen to be 0^- (rather than 0 or 0^+) to permit $x(t)$ to include $\delta(t)$ or its derivatives. Thus, we note immediately that the integration from 0^- to 0^+ is zero except when there is an impulse function or its derivative at the origin. The unilateral Laplace transform ignores $x(t)$ for $t < 0$. Since $x(t)$ in Eq. (3.43) is a right-sided signal, the ROC of $X_I(s)$ is always of the form $\text{Re}(s) > \sigma_{max}$, that is, a right half-plane in the s-plane.

3.7.2. B. Basic Properties:

Most of the properties of the unilateral Laplace transform are the same as for the bilateral transform. The unilateral Laplace transform is useful for calculating the response of a causal system to a causal input when the system is described by a linear constant-coefficient differential equation with nonzero initial conditions. The basic properties of the unilateral Laplace transform that are useful in this application are the time-differentiation and time-integration properties which are different from those of the bilateral transform. They are presented in the following.

3.7.2.1. 1. Differentiation in the Time Domain:

$$\frac{dx(t)}{dt} \leftrightarrow sX_I(s) - x(0^-)$$

(3.44)

provided that $\lim_{t \to \infty} x(t)e^{-st} = 0$. Repeated application of this property yields

$$\frac{d^2x(t)}{dt^2} \leftrightarrow s^2 X_I(s) - sx(0^-) - x'(0^-)$$

(3.45)

$$\frac{d^n x(t)}{dt^n} \leftrightarrow s^n X_I(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) - \cdots - x^{(n-1)}(0^-)$$

(3.46)

where

$$x^{(r)}(0^-) = \left.\frac{d^r x(t)}{dt^r}\right|_{t=0^-}$$

3.7.2.2. 2. Integration in the Time Domain:

$$\int_{0^-}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s} X_I(s)$$

(3.47)

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s} X_I(s) + \frac{1}{s} \int_{-\infty}^{0^-} x(\tau)\, d\tau$$

(3.48)

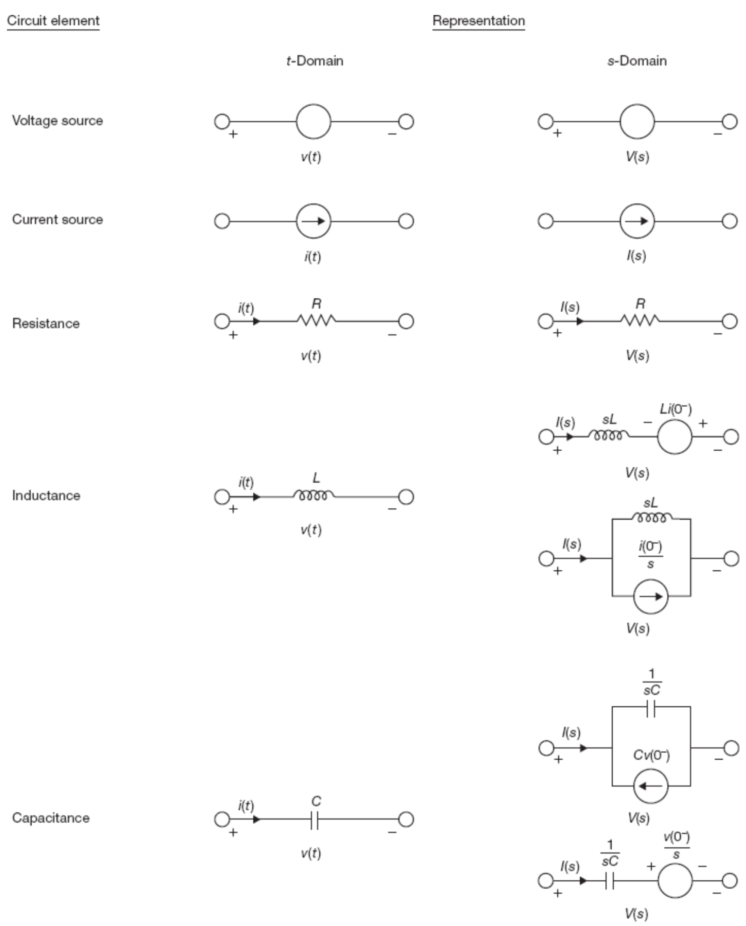
3.7.3. C. System Function:

Note that with the unilateral Laplace transform, the system function $H(s) = Y(s)/X(s)$ is defined under the condition that the LTI system is relaxed, that is, all initial conditions are zero.

3.7.4. D. Transform Circuits:

The solution for signals in an electric circuit can be found without writing integrodifferential equations if the circuit operations and signals are represented with their Laplace transform equivalents. [In this subsection the Laplace transform means the unilateral Laplace transform and we drop the subscript *I* in $X_I(s)$.] We refer to a circuit produced from these equivalents as a *transform circuit*. In order to use this technique, we require the Laplace transform models for individual circuit elements. These models are developed in the following discussion and are shown in Fig. 3-10. Applications of this transform model technique to electric circuits problems are illustrated in Probs. 3.40 to 3.42.

Figure 3-10 Representation of Laplace transform circuit-element models.



3.7.4.1. 1. Signal Sources:

$$v(t) \leftrightarrow V(s) \qquad i(t) \leftrightarrow I(s)$$

where $v(t)$ and $i(t)$ are the voltage and current source signals, respectively.

3.7.4.2. 2. Resistance *R*:

$$v(t) = Ri(t) \leftrightarrow V(s) = RI(s)$$

(3.49)

3.7.4.3. 3. Inductance *L*:

$$v(t) = L\frac{di(t)}{dt} \leftrightarrow V(s) = sLI(s) - Li(0^-)$$

(3.50)

The second model of the inductance *L* in Fig. 3-10 is obtained by rewriting Eq. (3.50) as

$$i(t) \leftrightarrow I(s) = \frac{1}{sL}V(s) + \frac{1}{s}i(0^-)$$

(3.51)

3.7.4.4. 4. Capacitance *C*:

$$i(t) = C\frac{dv(t)}{dt} \leftrightarrow I(s) = sCV(s) - Cv(0^-)$$

(3.52)

The second model of the capacitance *C* in Fig. 3-10 is obtained by rewriting Eq. (3.52) as

$$v(t) \leftrightarrow V(s) = \frac{1}{sC}I(s) + \frac{1}{s}v(0^-)$$

(3.53)

3.8. SOLVED PROBLEMS

3.8.1. Laplace Transform

3.1. Find the Laplace transform of

a. $x(t) = -e^{-at}u(-t)$

b. $x(t) = e^{at}u(-t)$

a. From Eq. (3.3)

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0^-} e^{-(s+a)t}dt$$
$$= \frac{1}{s+a}e^{-(s+a)t}\Big|_{-\infty}^{0-} = \frac{1}{s+a} \qquad \text{Re}(s) < -a$$

Thus, we obtain

$$-e^{-at}u(-t) \leftrightarrow \frac{1}{s+a} \qquad \text{Re}(s) < -a$$

(3.54)

b. Similarly,

$$X(s)=\int_{-\infty}^{\infty} e^{at}u(-t)\,e^{-st}\,dt=\int_{-\infty}^{0^-} e^{-(s-a)t}\,dt$$
$$=-\frac{1}{s-a}e^{-(s-a)t}\Big|_{-\infty}^{0-}=-\frac{1}{s-a}\qquad \mathrm{Re}(s)<a$$

Thus, we obtain

$$e^{at}u(-t)\leftrightarrow-\frac{1}{s-a}\qquad \mathrm{Re}(s)<a$$

(3.55)

3.2. A finite-duration signal $x(t)$ is defined as

$$x(t)\begin{cases}\neq 0 & t_1\le t\le t_2\\ =0 & \text{otherwise}\end{cases}$$

where t_1 and t_2 are finite values. Show that if $X(s)$ converges for at least one value of s, then the ROC of $X(s)$ is the entire s-plane.

Assume that $X(s)$ converges at $s = \sigma_0$; then by Eq. (3.3)

$$|X(s)|\le\int_{-\infty}^{\infty}\left|x(t)e^{-st}\right|dt=\int_{t_1}^{t_2}|x(t)|\,e^{-\sigma_0 t}dt<\infty$$

Let Re(s) = $\sigma_1 > \sigma_0$. Then

$$\int_{-\infty}^{\infty}\left|x(t)\,e^{-(\sigma_1+j\omega)t}\right|dt=\int_{t_1}^{t_2}|x(t)|\,e^{-\sigma_1 t}\,dt$$
$$=\int_{t_1}^{t_2}|x(t)|e^{-\sigma_0 t}e^{-(\sigma_1-\sigma_0)t}\,dt$$

Since $(\sigma_1 - \sigma_0) > 0$, $e{-}(\sigma_1 - \sigma_0)^t$ is a decaying exponential. Then over the interval where$x(t) \neq 0$, the maximum value of this exponential is $e{-}(\sigma_1 - \sigma_0)t_1$, and we can write

$$\int_{t_1}^{t_2}|x(t)|e^{-\sigma_1 t}\,dt<e^{-(\sigma_1-\sigma_0)t_1}\int_{t_1}^{t_2}|x(t)|\,e^{-\sigma_0 t}dt<\infty$$

(3.56)

Thus, $X(s)$ converges for Re(s) = $\sigma_1 > \sigma_0$. By a similar argument, if $\sigma_1 < \sigma_0$, then

$$\int_{t_1}^{t_2}|x(t)|e^{-\sigma_1 t}dt<e^{-(\sigma_1-\sigma_0)t_2}\int_{t_1}^{t_2}|x(t)|\,e^{-\sigma_0 t}dt<\infty$$

(3.57)

and again $X(s)$ converges for Re(s) = $\sigma_1 < \sigma_0$. Thus, the ROC of $X(s)$ includes the entire s-plane.

3.3. Let

$$x(t)=\begin{cases}e^{-at} & 0\le t\le T\\ 0 & \text{otherwise}\end{cases}$$

Find the Laplace transform of $x(t)$.

By Eq. (3.3)

$$X(s) = \int_0^T e^{-at} e^{-st}\,dt = \int_0^T e^{-(s+a)t}\,dt$$
$$= -\frac{1}{s+a} e^{-(s+a)t}\Big|_0^T = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

(3.58)

Since $x(t)$ is a finite-duration signal, the ROC of $X(s)$ is the entire s-plane. Note that from Eq. (3.58) it appears that $X(s)$ does not converge at $s = -a$. But this is not the case. Setting $s = -a$ in the integral in Eq. (3.58), we have

$$X(-a) = \int_0^T e^{-(a+a)t}\,dt = \int_0^T dt = T$$

The same result can be obtained by applying L'Hospital's rule to Eq. (3.58).

3.4. Show that if $x(t)$ is a right-sided signal and $X(s)$ converges for some value of s, then the ROC of $X(s)$ is of the form

$$\operatorname{Re}(s) > \sigma_{max}$$

where σ_{max} equals the maximum real part of any of the poles of $X(s)$.

Consider a right-sided signal $x(t)$ so that

$$x(t) = 0 \qquad t < t_1$$

and $X(s)$ converges for $\operatorname{Re}(s) = \sigma_0$. Then

$$|X(s)| \le \int_{-\infty}^{\infty} |x(t) e^{-st}|\,dt = \int_{-\infty}^{\infty} |x(t)| e^{-\sigma_0 t}\,dt$$
$$= \int_{t_1}^{\infty} |x(t)| e^{-\sigma_0 t}\,dt < \infty$$

Let $\operatorname{Re}(s) = \sigma_1 > \sigma_0$. Then

$$\int_{t_1}^{\infty} |x(t)| e^{-\sigma_1 t}\,dt = \int_{t_1}^{\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t}\,dt$$
$$< e^{-(\sigma_1 - \sigma_0)t_1} \int_{t_1}^{\infty} |x(t)| e^{-\sigma_0 t}\,dt < \infty$$

Thus, $X(s)$ converges for $\operatorname{Re}(s) = \sigma_1$ and the ROC of $X(s)$ is of the form $\operatorname{Re}(s) > \sigma_0$. Since the ROC of $X(s)$ cannot include any poles of $X(s)$, we conclude that it is of the form

$$\operatorname{Re}(s) > \sigma_{max}$$

where σ_{max} equals the maximum real part of any of the poles of $X(s)$.

3.5. Find the Laplace transform $X(s)$ and sketch the pole-zero plot with the ROC for the following signals $x(t)$:

a. $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$

b. $x(t) = e^{-3t}u(t) + e^{2t}u(-t)$

c. $x(t) = e^{2t}u(t) + e^{-3t}u(-t)$

a. From Table 3-1

$$e^{-2t}u(t) \leftrightarrow \frac{1}{s+2} \qquad \mathrm{Re}(s) > -2$$

(3.59)

$$e^{-3t}u(t) \leftrightarrow \frac{1}{s+3} \qquad \mathrm{Re}(s) > -3$$

(3.60)

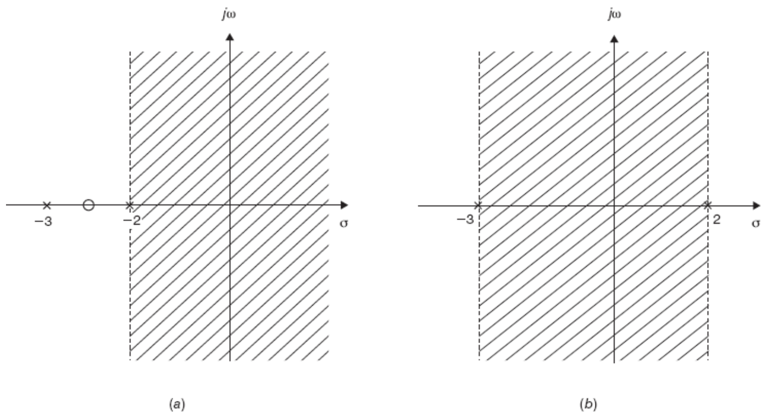
We see that the ROCs in Eqs. (3.59) and (3.60) overlap, and thus,

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3} = \frac{2\left(s+\frac{5}{2}\right)}{(s+2)(s+3)} \qquad \mathrm{Re}(s) > -2$$

(3.61)

From Eq. (3.61) we see that $X(s)$ has one zero at s = −5−2 and two poles at s = −2 and s = −3 and that the ROC is Re(s) > −2, as sketched in Fig. 3-11(a).

Figure 3-11



b. From Table 3-1

$$e^{-3t}u(t) \leftrightarrow \frac{1}{s+3} \qquad \mathrm{Re}(s) > -3$$

(3.62)

$$e^{2t}u(-t) \leftrightarrow -\frac{1}{s-2} \qquad \mathrm{Re}(s) < 2$$

(3.63)

We see that the ROCs in Eqs. (3.62) and (3.63) overlap, and thus,

$$X(s) = \frac{1}{s+3} - \frac{1}{s-2} = \frac{-5}{(s-2)(s+3)} \qquad -3 < \mathrm{Re}(s) < 2$$

(3.64)

From Eq. (3.64) we see that $X(s)$ has no zeros and two poles at s = 2 and s = −3 and that the ROC is −3 < Re(s) < 2, as sketched in Fig. 3-11(b).

c. From Table 3-1

$$e^{2t}u(t) \leftrightarrow \frac{1}{s-2} \qquad \mathrm{Re}(s) > 2$$

(3.65)

$$e^{-3t}u(-t) \leftrightarrow -\frac{1}{s+3} \qquad \mathrm{Re}(s) < -3$$

(3.66)

We see that the ROCs in Eqs. (3.65) and (3.66) do not overlap and that there is no common ROC; thus, $x(t)$ has no transform $X(s)$.

3.6. Let

$$x(t) = e^{-a|t|}$$

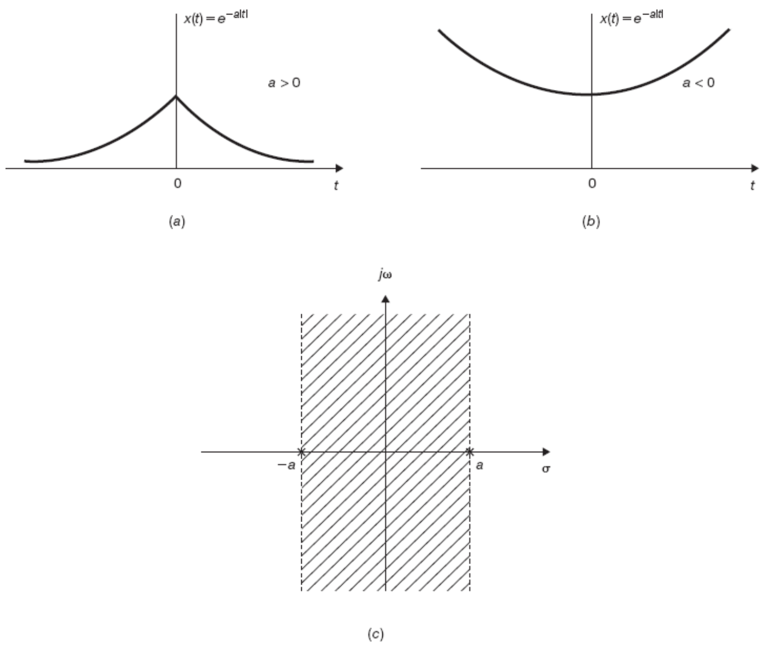
Find $X(s)$ and sketch the zero-pole plot and the ROC for $a > 0$ and $a < 0$.

The signal $x(t)$ is sketched in Figs. 3-12 (*a*) and (*b*) for both $a > 0$ and $a < 0$. Since $x(t)$ is a two-sided signal, we can express it as

$$x(t) = e^{-at}u(t) + e^{at}u(-t)$$

(3.67)

Figure 3-12



Note that $x(t)$ is continuous at $t = 0$ and $x(0^-) = x(0) = x(0^+) = 1$. From Table 3-1

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a} \qquad \mathrm{Re}(s) > -a$$

(3.68)

$$e^{at}u(-t) \leftrightarrow -\frac{1}{s-a} \qquad \mathrm{Re}(s) < a$$

(3.69)

If $a > 0$, we see that the ROCs in Eqs. (3.68) and (3.69) overlap, and thus,

$$X(s) = \frac{1}{s+a} - \frac{1}{s-a} = \frac{-2a}{s^2-a^2} \qquad -a < \mathrm{Re}(s) < a$$

(3.70)

From Eq. (3.70) we see that $X(s)$ has no zeros and two poles at $s = a$ and $s = -a$ and that the ROC is $-a < \mathrm{Re}(s) < a$, as sketched in Fig. 3-12(c). If $a < 0$, we see that the ROCs in Eqs. (3.68) and (3.69) do not overlap and that there is no common ROC; thus, $x(t)$ has no transform $X(s)$.

3.8.2. Properties of the Laplace Transform

3.7. Verify the time-shifting property (3.16); that is,

$$x(t-t_0) \leftrightarrow e^{-st_0}X(s) \qquad R' = R$$

By definition (3.3)

$$\mathscr{L}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)e^{-st}dt$$

By the change of variables $\tau = t - t_0$ we obtain

$$\mathscr{L}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau$$

$$= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau = e^{-st_0}X(s)$$

with the same ROC as for $X(s)$ itself. Hence,

$$x(t-t_0) \leftrightarrow e^{-st_0}X(s) \qquad R' = R$$

where R and R' are the ROCs before and after the time-shift operation.

3.8. Verify the time-scaling property (3.18); that is,

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right) \qquad R' = aR$$

By definition (3.3)

$$\mathscr{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt$$

By the change of variables $\tau = at$ with $a > 0$, we have

$$\mathscr{L}\{x(at)\} = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-(s/a)\tau}d\tau = \frac{1}{a}X\left(\frac{s}{a}\right) \qquad R' = aR$$

Note that because of the scaling s/a in the transform, the ROC of $X(s/a)$ is aR. With $a < 0$, we have

$$\mathscr{L}\{x(at)\} = \frac{1}{a}\int_{\infty}^{-\infty} x(\tau)e^{-(s/a)\tau}\,d\tau$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-(s/a)\tau}\,d\tau = -\frac{1}{a}X\left(\frac{s}{a}\right) \qquad R' = aR$$

Thus, combining the two results for $a > 0$ and $a < 0$, we can write these relationships as

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right) \qquad R' = aR$$

3.9. Find the Laplace transform and the associated ROC for each of the following signals:

a. $x(t) = \delta(t - t_0)$

b. $x(t) = u(t - t_0)$

c. $x(t) = e^{-2t}[u(t) - u(t - 5)]$

d. $x(t) = \sum_{k=0}^{\infty} \delta(t - kT)$

e. $x(t) = \delta(at + b)$, a, b real constants

a. Using Eqs. (3.13) and (3.16), we obtain

$$\delta(t - t_0) \leftrightarrow e^{-st_0} \qquad \text{all } s$$

(3.71)

b. Using Eqs. (3.14) and (3.16), we obtain

$$u(t - t_0) \leftrightarrow \frac{e^{-st_0}}{s} \qquad \mathrm{Re}(s) > 0$$

(3.72)

c. Rewriting $x(t)$ as

$$x(t) = e^{-2t}[u(t) - u(t - 5)] = e^{-2t}u(t) - e^{-2t}u(t - 5)$$
$$= e^{-2t}u(t) - e^{-10}e^{-2(t-5)}u(t - 5)$$

Then, from Table 3-1 and using Eq. (3.16), we obtain

$$X(s) = \frac{1}{s+2} - e^{-10}e^{-5s}\frac{1}{s+2} = \frac{1}{s+2}(1 - e^{-5(s+2)}) \qquad \mathrm{Re}(s) > -2$$

d. Using Eqs. (3.71) and (1.99), we obtain

$$X(s) = \sum_{k=0}^{\infty} e^{-skT} = \sum_{k=0}^{\infty} (e^{-sT})^k = \frac{1}{1 - e^{-sT}} \qquad \mathrm{Re}(s) > 0$$

(3.73)

e. Let

$$f(t) = \delta(at)$$

Then from Eqs. (3.13) and (3.18) we have

$$f(t)=\delta(at) \leftrightarrow F(s)=\frac{1}{|a|} \qquad \text{all } s$$

(3.74)

Now

$$\text{Now} \qquad x(t)=\delta(at+b)=\delta\left[a\left(t+\frac{b}{a}\right)\right]=f\left(t+\frac{b}{a}\right)$$

Using Eqs. (3.16) and (3.74), we obtain

$$X(s)=e^{sb/a}F(s)=\frac{1}{|a|}e^{sb/a} \qquad \text{all } s$$

(3.75)

3.10. Verify the time differentiation property (3.20); that is,

$$\frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R$$

From Eq. (3.24) the inverse Laplace transform is given by

$$x(t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

(3.76)

Differentiating both sides of the above expression with respect to t, we obtain

$$\frac{dx(t)}{dt}=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} sX(s)e^{st}ds$$

(3.77)

Comparing Eq. (3.77) with Eq. (3.76), we conclude that $dx(t)/dt$ is the inverse Laplace transform of $sX(s)$. Thus,

$$\frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R$$

Note that the associated ROC is unchanged unless a pole-zero cancellation exists at $s = 0$.

3.11. Verify the differentiation in s property (3.21); that is,

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$$

From definition (3.3)

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Differentiating both sides of the above expression with respect to s, we have

$$\frac{dX(s)}{ds}=\int_{-\infty}^{\infty}(-t)x(t)e^{-st}dt=\int_{-\infty}^{\infty}[-tx(t)]e^{-st}dt$$

Thus, we conclude that

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$$

3.12. Verify the integration property (3.22); that is,

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X(s) \qquad R' = R \cap \{\mathrm{Re}(s) > 0\}$$

Let

$$f(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow F(s)$$

Then

$$x(t) = \frac{df(t)}{dt}$$

Applying the differentiation property (3.20), we obtain

$$X(s) = sF(s)$$

Thus,

$$F(s) = \frac{1}{s}X(s) \qquad R' = R \cap \{\mathrm{Re}(s) > 0\}$$

The form of the ROC R' follows from the possible introduction of an additional pole at $s = 0$ by the multiplying by $1/s$.

3.13. Using the various Laplace transform properties, derive the Laplace transforms of the following signals from the Laplace transform of $u(t)$.

a. $\delta(t)$

b. $\delta'(t)$

c. $tu(t)$

d. $e^{-at}u(t)$

e. $te^{-at}u(t)$

f. $\cos \omega_0 t u(t)$

g. $e^{-at} \cos \omega_0 t u(t)$

a. From Eq. (3.14) we have

$$u(t) \leftrightarrow \frac{1}{s} \qquad \text{for } \mathrm{Re}(s) > 0$$

From Eq. (1.30) we have

$$\delta(t) = \frac{du(t)}{dt}$$

Thus, using the time-differentiation property (3.20), we obtain

$$\delta(t) \leftrightarrow s\frac{1}{s} = 1 \qquad \text{all } s$$

b. Again applying the time-differentiation property (3.20) to the result from part (*a*), we obtain

$$\delta'(t) \leftrightarrow s \qquad \text{all } s$$

(3.78)

c. Using the differentiation in *s* property (3.21), we obtain

$$tu(t) \leftrightarrow -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2} \qquad \text{Re}(s) > 0$$

(3.79)

d. Using the shifting in the *s*-domain property (3.17), we have

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a} \qquad \text{Re}(s) > -a$$

e. From the result from part (*c*) and using the differentiation in *s* property (3.21), we obtain

$$te^{-at}u(t) \leftrightarrow -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2} \qquad \text{Re}(s) > -a$$

(3.80)

f. From Euler's formula we can write

$$\cos \omega_0 tu(t) = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})u(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$$

Using the linearity property (3.15) and the shifting in the *s*-domain property (3.17), we obtain

$$\cos \omega_0 tu(t) \leftrightarrow \frac{1}{2}\frac{1}{s-j\omega_0} + \frac{1}{2}\frac{1}{s+j\omega_0} = \frac{s}{s^2+\omega_0^2} \qquad \text{Re}(s) > 0$$

(3.81)

g. Applying the shifting in the *s*-domain property (3.17) to the result from part (*f*), we obtain

$$e^{-at}\cos \omega_0 tu(t) \leftrightarrow \frac{s+a}{(s+a)^2+\omega_0^2} \qquad \text{Re}(s) > -a$$

(3.82)

3.14. Verify the convolution property (3.23); that is,

$$x_1(t) * x_2(t) \leftrightarrow X_1(s)\, X_2(s) \qquad R' \supset R_1 \cap R_2$$

Let

$$y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau$$

Then, by definition (3.3)

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau\right]e^{-st}dt \\ &= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-st}\, dt\right]d\tau \end{aligned}$$

Noting that the bracketed term in the last expression is the Laplace transform of the shifted signal $x_2(t - \tau)$, by Eq. (3.16) we have

$$Y(s)=\int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau}X_2(s)\,d\tau$$
$$=\left[\int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau}d\tau\right]X_2(s)=X_1(s)\,X_2(s)$$

with an ROC that contains the intersection of the ROC of $X_1(s)$ and $X_2(s)$. If a zero of one transform cancels a pole of the other, the ROC of $Y(s)$ may be larger. Thus, we conclude that

$$x_1(t)*x_2(t) \leftrightarrow X_1(s)\,X_2(s) \qquad R' \supset R_1 \cap R_2$$

3.15. Using the convolution property (3.23), verify Eq. (3.22); that is,

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X(s) \qquad R' = R \cap \{\mathrm{Re}(s)>0\}$$

We can write [Eq. (2.60), Prob. 2.2]

$$\int_{-\infty}^{t} x(\tau)\,d\tau = x(t)*u(t)$$

(3.83)

From Eq. (3.14)

$$u(t)\leftrightarrow\frac{1}{s} \qquad \mathrm{Re}(s)>0$$

and thus, from the convolution property (3.23) we obtain

$$x(t)*u(t)\leftrightarrow\frac{1}{s}X(s)$$

with the ROC that includes the intersection of the ROC of $X(s)$ and the ROC of the Laplace transform of $u(t)$. Thus,

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) \qquad R' = R\cap\{\mathrm{Re}(s)>0\}$$

3.8.3. Inverse Laplace Transform

3.16. Find the inverse Laplace transform of the following $X(s)$:

$$(a)\quad X(s)=\frac{1}{s+1},\ \mathrm{Re}(s)>-1$$
$$(b)\quad X(s)=\frac{1}{s+1},\ \mathrm{Re}(s)<-1$$
$$(c)\quad X(s)=\frac{s}{s^2+4},\ \mathrm{Re}(s)>0$$
$$(d)\quad X(s)=\frac{s+1}{(s+1)^2+4},\ \mathrm{Re}(s)>-1$$

a. From Table 3-1 we obtain

$$x(t)=e^{-t}u(t)$$

b. From Table 3-1 we obtain

$$x(t) = -e^{-t}u(-t)$$

c. From Table 3-1 we obtain

$$x(t) = \cos 2tu(t)$$

d. From Table 3-1 we obtain

$$s(t) = e^{-t}\cos 2tu(t)$$

3.17. Find the inverse Laplace transform of the following $X(s)$:

$$(a)\quad X(s) = \frac{2s+4}{s^2+4s+3}, \ \mathrm{Re}(s) > -1$$

$$(b)\quad X(s) = \frac{2s+4}{s^2+4s+3}, \ \mathrm{Re}(s) < -3$$

$$(c)\quad X(s) = \frac{2s+4}{s^2+4s+3}, \ -3 < \mathrm{Re}(s) < -1$$

Expanding by partial fractions, we have

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{s+2}{(s+1)(s+3)} = \frac{c_1}{s+1} + \frac{c_2}{s+3}$$

Using Eq. (3.30), we obtain

$$c_1 = (s+1)X(s)\big|_{s=-1} = 2\left.\frac{s+2}{s+3}\right|_{s=-1} = 1$$

$$c_2 = (s+3)X(s)\big|_{s=-3} = 2\left.\frac{s+2}{s+1}\right|_{s=-3} = 1$$

Hence,

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

a. The ROC of $X(s)$ is $\mathrm{Re}(s) > -1$. Thus, $x(t)$ is a right-sided signal and from Table 3-1 we obtain

$$x(t) = e^{-t}u(t) + e^{-3t}u(t) = (e^{-t} + e^{-3t})u(t)$$

b. The ROC of $X(s)$ is $\mathrm{Re}(s) < -3$. Thus, $x(t)$ is a left-sided signal and from Table 3-1 we obtain

$$x(t) = -e^{-t}u(-t) - e^{-3t}u(-t) = -(e^{-t} + e^{-3t})u(-t)$$

c. The ROC of $X(s)$ is $-3 < \mathrm{Re}(s) < -1$. Thus, $x(t)$ is a double-sided signal and from Table 3-1 we obtain

$$x(t) = -e^{-t}u(-t) + e^{-3t}u(t)$$

3.18. Find the inverse Laplace transform of

$$X(s) = \frac{5s+13}{s(s^2+4s+13)} \qquad \mathrm{Re}(s) > 0$$

We can write

$$s^2 + 4s + 13 = (s+2)^2 + 9 = (s+2-j3)(s+2+j3)$$

Then

$$X(s)=\frac{5s+13}{s(s^2+4s+13)}=\frac{5s+13}{s(s+2-j3)(s+2+j3)}$$
$$=\frac{c_1}{s}+\frac{c_2}{s-(-2+j3)}+\frac{c_3}{s-(-2-j3)}$$

where

$$c_1=sX(s)\big|_{s=0}=\frac{5s+13}{s^2+4s+13}\bigg|_{s=0}=1$$
$$c_2=(s+2-j3)X(s)\big|_{s=-2+j3}=\frac{5s+13}{s(s+2+j3)}\bigg|_{s=-2+j3}=-\frac{1}{2}(1+j)$$
$$c_3=(s+2+j3)X(s)\big|_{s=-2-j3}=\frac{5s+13}{s(s+2-j3)}\bigg|_{s=-2-j3}=-\frac{1}{2}(1-j)$$

Thus,

$$X(s)=\frac{1}{s}-\frac{1}{2}(1+j)\frac{1}{s-(-2+j3)}-\frac{1}{2}(1-j)\frac{1}{s-(-2-j3)}$$

The ROC of $X(s)$ is $\mathrm{Re}(s) > 0$. Thus, $x(t)$ is a right-sided signal and from Table 3-1 we obtain

$$x(t)=u(t)-\frac{1}{2}(1+j)e^{(-2+j3)t}u(t)-\frac{1}{2}(1-j)e^{(-2-j3)t}u(t)$$

Inserting the identity

$$e^{(-2\pm j3)t}=e^{-2t}e^{\pm j3t}=e^{-2t}(\cos 3t\pm j\sin 3t)$$

into the above expression, after simple computations we obtain

$$x(t)=u(t)-e^{-2t}(\cos 3t-\sin 3t)\,u(t)$$
$$=[1-e^{-2t}(\cos 3t-\sin 3t)]\,u(t)$$

Alternate Solution:

We can write $X(s)$ as

$$X(s)=\frac{5s+13}{s(s^2+4s+13)}=\frac{c_1}{s}+\frac{c_2 s+c_3}{s^2+4s+13}$$

As before, by Eq. (3.30) we obtain

$$c_1=sX(s)\big|_{s=0}=\frac{5s+13}{s^2+4s+13}\bigg|_{s=0}=1$$

Then we have

$$\frac{c_2 s+c_3}{s^2+4s+13}=\frac{5s+13}{s(s^2+4s+13)}-\frac{1}{s}=\frac{-s+1}{s^2+4s+13}$$

Thus,

$$X(s)=\frac{1}{s}-\frac{s-1}{s^2+4s+13}=\frac{1}{s}-\frac{s+2-3}{(s+2)^2+9}$$
$$=\frac{1}{s}-\frac{s+2}{(s+2)^2+3^2}+\frac{3}{(s+2)^2+3^2}$$

Then from Table 3-1 we obtain

$$x(t)=u(t)-e^{-2t}\cos 3tu(t)+e^{-2t}\sin 3tu(t)$$
$$=[1-e^{-2t}(\cos 3t-\sin 3t)]u(t)$$

3.19. Find the inverse Laplace transform of

$$X(s)=\frac{s^2+2s+5}{(s+3)(s+5)^2} \qquad \operatorname{Re}(s)>-3$$

We see that $X(s)$ has one simple pole at $s = -3$ and one multiple pole at $s = -5$ with multiplicity 2. Then by Eqs. (3.29) and (3.31) we have

$$X(s)=\frac{c_1}{s+3}+\frac{\lambda_1}{s+5}+\frac{\lambda_2}{(s+5)^2}$$

(3.84)

By Eqs. (3.30) and (3.32) we have

$$c_1=(s+3)X(s)\big|_{s=-3}=\left.\frac{s^2+2s+5}{(s+5)^2}\right|_{s=-3}=2$$
$$\lambda_2=(s+5)^2X(s)\big|_{s=-5}=\left.\frac{s^2+2s+5}{s+3}\right|_{s=-5}=-10$$
$$\lambda_1=\frac{d}{ds}\left[(s+5)^2X(s)\right]\bigg|_{s=-5}=\frac{d}{ds}\left[\frac{s^2+2s+5}{s+3}\right]\bigg|_{s=-5}$$
$$=\left.\frac{s^2+6s+1}{(s+3)^2}\right|_{s=-5}=-1$$

Hence,

$$X(s)=\frac{2}{s+3}-\frac{1}{s+5}-\frac{10}{(s+5)^2}$$

The ROC of $X(s)$ is $\operatorname{Re}(s) > -3$. Thus, $x(t)$ is a right-sided signal and from Table 3-1 we obtain

$$x(t)=2e^{-3t}u(t)-e^{-5t}u(t)-10te^{-5t}u(t)$$
$$=[2e^{-3t}-e^{-5t}-10te^{-5t}]\,u(t)$$

Note that there is a simpler way of finding λ_1 without resorting to differentiation. This is shown as follows: First find c_1 and λ_2 according to the regular procedure. Then substituting the values of c_1 and λ_2 into Eq. (3.84), we obtain

$$\frac{s^2+2s+5}{(s+3)(s+5)^2}=\frac{2}{s+3}+\frac{\lambda_1}{s+5}-\frac{10}{(s+5)^2}$$

Setting $s = 0$ on both sides of the above expression, we have

$$\frac{5}{75}=\frac{2}{3}+\frac{\lambda_1}{5}-\frac{10}{25}$$

from which we obtain $\lambda_1 = -1$.

3.20. Find the inverse Laplace transform of the following $X(s)$:

$$(a)\quad X(s)=\frac{2s+1}{s+2},\ \mathrm{Re}(s) > -2$$

$$(b)\quad X(s)=\frac{s^2+6s+7}{s^2+3s+2},\ \mathrm{Re}(s) > -1$$

$$(c)\quad X(s)=\frac{s^3+2s^2+6}{s^2+3s},\ \mathrm{Re}(s) > 0$$

a. $X(s)=\dfrac{2s+1}{s+2}=\dfrac{2(s+2)-3}{s+2}=2-\dfrac{3}{s+2}$

Since the ROC of $X(s)$ is $\mathrm{Re}(s) > -2$, $x(t)$ is a right-sided signal and from Table 3-1 we obtain

$$x(t) = 2\delta(t) - 3e^{-2t}u(t)$$

b. Performing long division, we have

$$X(s)=\frac{s^2+6s+7}{s^2+3s+2}=1+\frac{3s+5}{s^2+3s+2}=1+\frac{3s+5}{(s+1)(s+2)}$$

Let

$$X_1(s)=\frac{3s+5}{(s+1)(s+2)}=\frac{c_1}{s+1}+\frac{c_2}{s+2}$$

where

$$c_1=(s+1)X_1(s)\big|_{s=-1}=\left.\frac{3s+5}{s+2}\right|_{s=-1}=2$$

$$c_2=(s+2)X_1(s)\big|_{s=-2}=\left.\frac{3s+5}{s+1}\right|_{s=-2}=1$$

Hence,

$$X(s)=1+\frac{2}{s+1}+\frac{1}{s+2}$$

The ROC of $X(s)$ is $\mathrm{Re}(s) > -1$. Thus, $x(t)$ is a right-sided signal and from Table 3-1 we obtain

$$x(t) = \delta(t) + (2e^{-t} + e^{-2t})u(t)$$

c. Proceeding similarly, we obtain

$$X(s)=\frac{s^3+2s^2+6}{s^2+3s}=s-1+\frac{3s+6}{s(s+3)}$$

Let

$$X_1(s)=\frac{3s+6}{s(s+3)}=\frac{c_1}{s}+\frac{c_2}{s+3}$$

where

$$c_1=sX_1(s)\big|_{s=0}=\frac{3s+6}{s+3}\bigg|_{s=0}=2$$

$$c_2=(s+3)X_1(s)\big|_{s=-3}=\frac{3s+6}{s}\bigg|_{s=-3}=1$$

Hence,

$$X(s)=s-1+\frac{2}{s}+\frac{1}{s+3}$$

The ROC of $X(s)$ is $\mathrm{Re}(s) > 0$. Thus, $x(t)$ is a right-sided signal and from Table 3-1 and Eq. (3.78) we obtain

$$x(t) = \delta'(t) - \delta(t) + (2 + e^{-3t})u(t)$$

Note that all $X(s)$ in this problem are improper fractions and that $x(t)$ contains $\delta(t)$ or its derivatives.

3.21. Find the inverse Laplace transform of

$$X(s)=\frac{2+2se^{-2s}+4e^{-4s}}{s^2+4s+3} \qquad \mathrm{Re}(s) > -1$$

We see that $X(s)$ is a sum

$$X(s) = X_1(s) + X_2(s)\, e^{-2s} + X_3(s)e^{-4s}$$

where

$$X_1(s)=\frac{2}{s^2+4s+3} \qquad X_2(s)=\frac{2s}{s^2+4s+3} \qquad X_3(s)=\frac{4}{s^2+4s+3}$$

If

$$x_1(t) \leftrightarrow X_1(s) \qquad x_2(t) \leftrightarrow X_2(s) \qquad x_3(t) \leftrightarrow X_3(s)$$

then by the linearity property (3.15) and the time-shifting property (3.16) we obtain

$$x(t) = x_1(t) + x_2(t-2) + x_3(t-4)$$

(3.85)

Next, using partial-fraction expansions and from Table 3-1, we obtain

$$X_1(s)=\frac{1}{s+1}-\frac{1}{s+3} \leftrightarrow x_1(t)=(e^{-t}-e^{-3t})u(t)$$

$$X_2(s)=\frac{-1}{s+1}+\frac{3}{s+3} \leftrightarrow x_2(t)=(-e^{-t}+3e^{-3t})u(t)$$

$$X_3(s)=\frac{2}{s+1}-\frac{2}{s+3} \leftrightarrow x_3(t)=2(e^{-t}-e^{-3t})u(t)$$

Thus, by Eq. (3.85) we have

$$x(t) = (e^{-t} - e^{-3t})u(t) + [-e^{-(t-2)} + 3e^{-3(t-2)}]u(t-2)$$
$$+ 2[e^{-(t-4)} - e^{-3(t-4)}]u(t-4)$$

3.22. Using the differentiation in s property (3.21), find the inverse Laplace transform of

$$X(s) = \frac{1}{(s+a)^2} \qquad \operatorname{Re}(s) > -a$$

We have

$$-\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}$$

and from Eq. (3.9) we have

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a} \qquad \operatorname{Re}(s) > -a$$

Thus, using the differentiation in s property (3.21), we obtain

$$x(t) = te^{-at}u(t)$$

3.8.4. System Function

3.23. Find the system function $H(s)$ and the impulse response $h(t)$ of the RC circuit in Fig. 1-32 (Prob. 1.32).

a. Let

$$x(t) = v_s(t) \qquad y(t) = v_c(t)$$

In this case, the RC circuit is described by [Eq. (1.105)]

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

Taking the Laplace transform of the above equation, we obtain

$$sY(s) + \frac{1}{RC}Y(s) = \frac{1}{RC}X(s)$$

or

$$\left(s + \frac{1}{RC}\right)Y(s) = \frac{1}{RC}X(s)$$

Hence, by Eq. (3.37) the system function $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/RC}{s + 1/RC} = \frac{1}{RC}\,\frac{1}{s + 1/RC}$$

Since the system is causal, taking the inverse Laplace transform of $H(s)$, the impulse response $h(t)$ is

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \frac{1}{RC}e^{-t/RC}u(t)$$

b. Let

$$x(t) = v_s(t) \qquad y(t) = i(t)$$

In this case, the *RC* circuit is described by [Eq. (1.107)]

$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{R}\frac{dx(t)}{dt}$$

Taking the Laplace transform of the above equation, we have

$$sY(s)+\frac{1}{RC}Y(s)=\frac{1}{R}sX(s)$$

or

$$\left(s+\frac{1}{RC}\right)Y(s)=\frac{1}{R}sX(s)$$

Hence, the system function $H(s)$ is

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s/R}{s+1/RC}=\frac{1}{R}\frac{s}{s+1/RC}$$

In this case, the system function $H(s)$ is an improper fraction and can be rewritten as

$$H(s)=\frac{1}{R}\frac{s+1/RC-1/RC}{s+1/RC}=\frac{1}{R}-\frac{1}{R^2C}\frac{1}{s+1/RC}$$

Since the system is causal, taking the inverse Laplace transform of $H(s)$, the impulse response $h(t)$ is

$$h(t)=\mathscr{L}^{-1}\{H(s)\}=\frac{1}{R}\delta(t)-\frac{1}{R^2C}e^{-t/RC}u(t)$$

Note that we obtained different system functions depending on the different sets of input and output.

3.24. Using the Laplace transform, redo Prob. 2.5.

From Prob. 2.5 we have

$$h(t)=e^{-\alpha t}u(t) \qquad x(t)=e^{\alpha t}u(-t) \qquad \alpha>0$$

Using Table 3-1, we have

$$H(s)=\frac{1}{s+\alpha} \qquad \mathrm{Re}(s)>-\alpha$$

$$X(s)=-\frac{1}{s-\alpha} \qquad \mathrm{Re}(s)<\alpha$$

Thus,

$$Y(s)=X(s)H(s)=-\frac{1}{(s+\alpha)(s-\alpha)}=-\frac{1}{s^2-\alpha^2} \qquad -\alpha<\mathrm{Re}(s)<\alpha$$

and from Table 3-1 (or Prob. 3.6) the output is

$$y(t)=\frac{1}{2\alpha}e^{-\alpha|t|}$$

which is the same as Eq. (2.67).

3.25. The output $y(t)$ of a continuous-time LTI system is found to be $2e^{-3t}u(t)$ when the input $x(t)$ is $u(t)$.

a. Find the impulse response $h(t)$ of the system.

b. Find the output $y(t)$ when the input $x(t)$ is $e^{-t}u(t)$.

$$(a) \qquad x(t) = u(t),\ y(t) = 2e^{-3t}u(t)$$

Taking the Laplace transforms of $x(t)$ and $y(t)$, we obtain

$$X(s) = \frac{1}{s} \qquad \operatorname{Re}(s) > 0$$

$$Y(s) = \frac{2}{s+3} \qquad \operatorname{Re}(s) > -3$$

Hence, the system function $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s+3} \qquad \operatorname{Re}(s) > -3$$

Rewriting $H(s)$ as

$$H(s) = \frac{2s}{s+3} = \frac{2(s+3)-6}{s+3} = 2 - \frac{6}{s+3} \qquad \operatorname{Re}(s) > -3$$

and taking the inverse Laplace transform of $H(s)$, we have

$$h(t) = 2\delta(t) - 6e^{-3t}u(t)$$

Note that $h(t)$ is equal to the derivative of $2e^{-3t}u(t)$, which is the step response $s(t)$ of the system [see Eq. (2.13)].

$$(b) \qquad x(t) = e^{-t}u(t) \leftrightarrow \frac{1}{s+1} \qquad \operatorname{Re}(s) > -1$$

Thus,

$$Y(s) = X(s)\,H(s) = \frac{2s}{(s+1)(s+3)} \qquad \operatorname{Re}(s) > -1$$

Using partial-fraction expansions, we get

$$Y(s) = -\frac{1}{s+1} + \frac{3}{s+3}$$

Taking the inverse Laplace transform of $Y(s)$, we obtain

$$y(t) = (-e^{-t} + 3e^{-3t})\,u(t)$$

3.26. If a continuous-time LTI system is BIBO stable, then show that the ROC of its system function $H(s)$ must contain the imaginary axis; that is, $s = j\omega$.

A continuous-time LTI system is BIBO stable if and only if its impulse response $h(t)$ is absolutely integrable, that is [Eq. (2.21)],

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

By Eq. (3.3)

$$H(s) = \int_{-\infty}^{\infty} h(t)\,e^{-st}\,dt$$

Let $s = j\omega$. Then

$$|H(j\omega)| = \left|\int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt\right| \le \int_{-\infty}^{\infty} \left|h(t)e^{-j\omega t}\right| dt = \int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Therefore, we see that if the system is stable, then $H(s)$ converges for $s = j\omega$. That is, for a stable continuous-time LTI system, the ROC of $H(s)$ must contain the imaginary axis $s = j\omega$.

3.27. Using the Laplace transfer, redo Prob. 2.14

a. Using Eqs. (3.36) and (3.41), we have

$$Y(s) = X(s)H_1(s)H_2(s) = X(s)H(s)$$

where $H(s) = H_1(s)H_2(s)$ is the system function of the overall system. Now from Table 3-1 we have

$$h_1(t) = e^{-2t}u(t) \leftrightarrow H_1(s) = \frac{1}{s+2} \qquad \mathrm{Re}(s) > -2$$

$$h_2(t) = 2\,e^{-t}u(t) \leftrightarrow H_2(s) = \frac{2}{s+1} \qquad \mathrm{Re}(s) > -1$$

Hence,

$$H(s) = H_1(s)H_2(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2} \qquad \mathrm{Re}(s) > -1$$

Taking the inverse Laplace transfer of $H(s)$, we get

$$h(t) = 2(e^{-t} - e^{-2t})u(t)$$

b. Since the ROC of $H(s)$, $\mathrm{Re}(s) > -1$, contains the $j\omega$-axis, the overall system is stable.

3.28. Using the Laplace transform, redo Prob. 2.23.

The system is described by

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Taking the Laplace transform of the above equation, we obtain

$$sY(s) + aY(s) = X(s) \qquad \text{or} \qquad (s+a)Y(s) = X(s)$$

Hence, the system function $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+a}$$

Assuming the system is causal and taking the inverse Laplace transform of H(s), the impulse response $h(t)$ is

$$h(t) = e^{-at}u(t)$$

which is the same as Eq. (2.124).

3.29. Using the Laplace transform, redo Prob. 2.25.

The system is described by

$$y'(t) + 2y(t) = x(t) + x'(t)$$

Taking the Laplace transform of the above equation, we get

$$sY(s) + 2Y(s) = X(s) + sX(s)$$

or

$$(s + 2)Y(s) = (s + 1)X(s)$$

Hence, the system function $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s+2} = \frac{s+2-1}{s+2} = 1 - \frac{1}{s+2}$$

Assuming the system is causal and taking the inverse Laplace transform of $H(s)$, the impulse response $h(t)$ is

$$h(t) = \delta(t) - e^{-2t}u(t)$$

3.30. Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by

$$y''(t) + y'(t) - 2y(t) = x(t)$$

(3.86)

a. Find the system function $H(s)$.

b. Determine the impulse response $h(t)$ for each of the following three cases: (*i*) the system is causal, (*ii*) the system is stable, (*iii*) the system is neither causal nor stable.

a. Taking the Laplace transform of Eq. (3.86), we have

$$s^2Y(s) + sY(s) - 2Y(s) = X(s)$$

or

$$(s^2 + s - 2)Y(s) = X(s)$$

Hence, the system function $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)}$$

b. Using partial-fraction expansions, we get

$$H(s) = \frac{1}{(s+2)(s-2)} = -\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}$$

i. If the system is causal, then $h(t)$ is causal (that is, a right-sided signal) and the ROC of $H(s)$ is $\text{Re}(s) > 1$. Then from Table 3-1 we get

$$h(t) = -\frac{1}{3}(e^{-2t} - e^{t})u(t)$$

ii. If the system is stable, then the ROC of $H(s)$ must contain the $j\omega$-axis. Consequently the ROC of $H(s)$ is $-2 < \text{Re}(s) < 1$. Thus, $h(t)$ is two-sided and from Table 3-1 we get

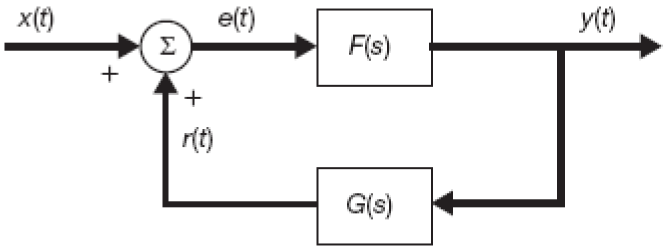
$$h(t) = -\frac{1}{3}e^{-2t}u(t) - \frac{1}{3}e^{t}u(-t)$$

iii. If the system is neither causal nor stable, then the ROC of $H(s)$ is $\text{Re}(s) < -2$. Then $h(t)$ is noncausal (that is, a left-sided signal) and from Table 3-1 we get

$$h(t) = \frac{1}{3}e^{-2t}u(-t) - \frac{1}{3}e^{t}u(-t)$$

3.31. The feedback interconnection of two causal subsystems with system functions $F(s)$ and $G(s)$ is depicted in Fig. 3-13. Find the overall system function $H(s)$ for this feedback system.

Figure 3-13 Feedback system.



Let

$$x(t) \leftrightarrow X(s) \qquad y(t) \leftrightarrow Y(s) \qquad r(t) \leftrightarrow R(s) \qquad e(t) \leftrightarrow E(s)$$

Then,

$$Y(s) = E(s)F(s)$$

(3.87)

$$R(s) = Y(s)G(s)$$

(3.88)

Since

$$e(t) = x(t) + r(t)$$

we have

$$E(s) = X(s) + R(s)$$

(3.89)

Substituting Eq. (3.88) into Eq. (3.89) and then substituting the result into Eq. (3.87), we obtain

$$Y(s) = [X(s) + Y(s)G(s)]F(s)$$

or

$$[1 - F(s)G(s)]\,Y(s) = F(s)X(s)$$

Thus, the overall system function is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{F(s)}{1 - F(s)G(s)}$$

(3.90)

3.8.5. Unilateral Laplace Transform

3.32. Verify Eqs. (3.44) and (3.45); that is,

$$(a) \quad \frac{dx(t)}{dt} \leftrightarrow sX_I(s) - x(0^-)$$

$$(b) \quad \frac{d^2x(t)}{dt^2} \leftrightarrow s^2X_I(s) - sx(0^-) - x'(0^-)$$

a. Using Eq. (3.43) and integrating by parts, we obtain

$$\mathscr{L}_I\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt$$
$$= x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}\,dt$$
$$= -x(0^-) + sX_I(s) \qquad \mathrm{Re}(s) > 0$$

Thus, we have

$$\frac{dx(t)}{dt} \leftrightarrow sX_I(s) - x(0^-)$$

b. Applying the above property to signal $x'(t) = dx(t)/dt$, we obtain

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\frac{dx(t)}{dt} \leftrightarrow s[sX_I(s) - x(0^-)] - x'(0^-)$$
$$= s^2X_I(s) - sx(0^-) - x'(0^-)$$

Note that Eq. (3.46) can be obtained by continued application of the above procedure.

3.33. Verify Eqs. (3.47) and (3.48); that is,

$$(a) \quad \int_{0^-}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X_I(s)$$

$$(b) \quad \int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X_I(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\,d\tau$$

(a) Let $$g(t) = \int_{0^-}^{t} x(\tau)\,d\tau$$

Then $$\frac{dg(t)}{dt} = x(t) \quad \text{and} \quad g(0^-) = 0$$

Now if

$$g(t) \leftrightarrow G_I(s)$$

then by Eq. (3.44)

$$X_I(s) = sG_I(s) - g(0^-) = sG_I(s)$$

Thus,

$$G_I(s) = \frac{1}{s}X_I(s)$$

or $$\int_{0^-}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X_I(s)$$

(b) We can write

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{0^-} x(\tau)\,d\tau + \int_{0^-}^{t} x(\tau)\,d\tau$$

Note that the first term on the right-hand side is a constant. Thus, taking the unilateral Laplace transform of the above equation and using Eq. (3.47), we get

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{s}X_I(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\,d\tau$$

3.34.

a. Show that the bilateral Laplace transform of $x(t)$ can be computed from two unilateral Laplace transforms.

b. Using the result obtained in part (*a*), find the bilateral Laplace transform of $e^{-2|t|}$.

a. The bilateral Laplace transform of $x(t)$ defined in Eq. (3.3) can be expressed as

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt=\int_{-\infty}^{0^-} x(t)e^{-st}dt+\int_{0^-}^{\infty} x(t)e^{-st}dt$$
$$=\int_{0^-}^{\infty} x(-t)e^{st}dt+\int_{0^-}^{\infty} x(t)e^{-st}dt$$

(3.91)

$$\text{Now} \qquad \int_{0^-}^{\infty} x(t)e^{-st}dt=X_I(s) \qquad \mathrm{Re}(s)>\sigma^+$$

(3.92)

Next, let

$$\mathscr{L}_I\{x(-t)\}=X_I^-(s)=\int_{0^-}^{\infty} x(-t)e^{-st}dt \qquad \mathrm{Re}(s)>\sigma^-$$

(3.93)

$$\text{Then} \qquad \int_{0^-}^{\infty} x(-t)e^{st}dt=\int_{0^-}^{\infty} x(-t)e^{-(-s)t}dt=X_I^-(-s) \qquad \mathrm{Re}(s)<\sigma^-$$

(3.94)

Thus, substituting Eqs. (3.92) and (3.94) into Eq. (3.91), we obtain

$$X(s)=X_I(s)+X_I^-(-s) \qquad \sigma^+<\mathrm{Re}(s)<\sigma^-$$

(3.95)

b. $x(t)=e^{-2|t|}$

1. $x(t) = e^{-2t}$ for $t > 0$, which gives

$$\mathscr{L}_I\{x(t)\}=X_I(s)=\frac{1}{s+2} \qquad \mathrm{Re}(s)>-2$$

2. $x(t) = e^{2t}$ for $t < 0$. Then $x(-t) = e^{-2t}$ for $t > 0$, which gives

$$\mathscr{L}_I\{x(-t)\}=X_I^-(s)=\frac{1}{s+2} \qquad \mathrm{Re}(s)>-2$$

Thus,

$$X_I^-(-s)=\frac{1}{-s+2}=-\frac{1}{s-2} \qquad \mathrm{Re}(s)<2$$

3. According to Eq. (3.95), we have

$$X(s)=X_I(s)+X_I^-(-s)=\frac{1}{s+2}-\frac{1}{s-2}$$
$$=-\frac{4}{s^2-4} \qquad -2<\mathrm{Re}(s)<2$$

(3.96)

which is equal to Eq. (3.70), with a = 2, in Prob. 3.6.

3.35. Show that

$$(a) \quad x(0^+) = \lim_{s\to\infty} sX_I(s)$$

(3.97)

$$(b) \quad \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX_I(s)$$

(3.98)

Equation (3.97) is called the *initial value theorem*, while Eq. (3.98) is called the *final value theorem* for the unilateral Laplace transform.

a. Using Eq. (3.44), we have

$$\begin{aligned} sX_I(s) - x(0^-) &= \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \\ &= \int_{0^-}^{0^+} \frac{dx(t)}{dt} e^{-st}\, dt + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \\ &= x(t)\Big|_{0^-}^{0^+} + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt \\ &= x(0^+) - x(0^-) + \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \end{aligned}$$

Thus,

$$sX_I(s) = x(0^+) = \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

and

$$\begin{aligned} \lim_{s\to\infty} sX_I(s) &= x(0^+) + \lim_{s\to\infty} \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \\ &= x(0^+) + \int_{0^+}^{\infty} \frac{dx(t)}{dt} \left(\lim_{s\to\infty} e^{-st}\right) dt = x(0^+) \end{aligned}$$

since $\lim_{s\to\infty} e^{-st} = 0$.

b. Again using Eq. (3.44), we have

$$\begin{aligned} \lim_{s\to 0} [sX_I(s) - x(0^-)] &= \lim_{s\to 0} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \\ &= \int_{0^-}^{\infty} \frac{dx(t)}{dt} \left(\lim_{s\to 0} e^{-st}\right) dt \\ &= \int_{0^-}^{\infty} \frac{dx(t)}{dt} dt = x(t)\Big|_{0^-}^{\infty} \\ &= \lim_{t\to\infty} x(t) - x(0^-) \end{aligned}$$

Since

$$\lim_{s\to 0} [sX_I(s) - x(0^-)] = \lim_{s\to 0} [sX_I(s)] - x(0^-)$$

we conclude that

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX_I(s)$$

3.36. The unilateral Laplace transform is sometimes defined as

$$\mathscr{L}_+\{x(t)\} = X_I^+(s) = \int_{0^+}^{\infty} x(t)e^{-st}dt$$

(3.99)

with 0^+ as the lower limit. (This definition is sometimes referred to as the 0^+ definition.)

a. Show that

$$\mathscr{L}_+\left\{\frac{dx(t)}{dt}\right\} = sX_I^+(s) - x(0^+) \qquad \mathrm{Re}(s) > 0$$

(3.100)

b. Show that

$$\mathscr{L}_+\{u(t)\} = \frac{1}{s}$$

(3.101)

$$\mathscr{L}_+\{\delta(t)\} = 0$$

(3.102)

a. Let $x(t)$ have unilateral Laplace transform $X_I^+(s)$. Using Eq. (3.99) and integrating by parts, we obtain

$$\begin{aligned}\mathscr{L}_+\left\{\frac{dx(t)}{dt}\right\} &= \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}dt \\ &= x(t)e^{-st}\Big|_{0^+}^{\infty} + s\int_{0^+}^{\infty} x(t)e^{-st}dt \\ &= -x(0^+) + sX_I^+(s) \qquad \mathrm{Re}(s) > 0\end{aligned}$$

Thus, we have

$$\frac{dx(t)}{dt} \leftrightarrow sX_I^+(s) - x(0^+)$$

b. By definition (3.99)

$$\begin{aligned}\mathscr{L}_+\{u(t)\} &= \int_{0^+}^{\infty} u(t)\,e^{-st}dt = \int_{0^+}^{\infty} e^{-st}dt \\ &= -\frac{1}{s}e^{-st}\Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \mathrm{Re}(s) > 0\end{aligned}$$

From Eq. (1.30) we have

$$\delta(t) = \frac{du(t)}{dt}$$

(3.103)

Taking the 0^+ unilateral Laplace transform of Eq. (3.103) and using Eq. (3.100), we obtain

$$\mathscr{L}_+\{\delta(t)\} = s\frac{1}{s} - u(0^+) = 1 - 1 = 0$$

This is consistent with Eq. (1.21); that is,

$$\mathscr{L}_+\{\delta(t)\} = \int_{0^+}^{\infty} \delta(t) e^{-st}\, dt = 0$$

Note that taking the 0^- unilateral Laplace transform of Eq. (3.103) and using Eq. (3.44), we obtain

$$\mathscr{L}_-\{\delta(t)\} = s\frac{1}{s} - u(0^-) = 1 - 0 = 1$$

3.8.6. Application of Unilateral Laplace Transform

3.37. Using the unilateral Laplace transform, redo Prob. 2.20.

The system is described by

$$y'(t) + ay(t) = x(t)$$

with $y(0) = y_0$ and $x(t) = Ke^{-bt} u(t)$.

Assume that $y(0) = y(0^-)$. Let

(3.104)

with $y(0) = y_0$ and $x(t) = Ke^{-bt} u(t)$.

Assume that $y(0) = y(0^-)$. Let

$$y(t) \leftrightarrow Y_I(s)$$

Then from Eq. (3.44)

$$y'(t) \leftrightarrow sY_I(s) - y(0^-) = sY_I(s) - y_0$$

From Table 3-1 we have

$$x(t) \leftrightarrow X_I(s) = \frac{K}{s+b} \qquad \mathrm{Re}(s) > -b$$

Taking the unilateral Laplace transform of Eq. (3.104), we obtain

$$[sY_I(s) - y_0] + aY_I(s) = \frac{K}{s+b}$$

or

$$(s+a)Y_I(s) = y_0 + \frac{K}{s+b}$$

Thus,

$$Y_I(s) = \frac{y_0}{s+a} + \frac{K}{(s+a)(s+b)}$$

Using partial-fraction expansions, we obtain

$$Y_I(s) = \frac{y_0}{s+a} + \frac{K}{a-b}\left(\frac{1}{s+b} - \frac{1}{s+a}\right)$$

Taking the inverse Laplace transform of $Y_I(s)$, we obtain

$$y(t)=\left[y_0e^{-at}+\frac{K}{a-b}(e^{-bt}-e^{-at})\right]u(t)$$

which is the same as Eq. (2.107). Noting that $y(0^+) = y(0) = y(0^-) = y_0$, we write $y(t)$ as

$$y(t)=y_0e^{-at}+\frac{K}{a-b}(e^{-bt}-e^{-at}) \qquad t\geq 0$$

3.38. Solve the second-order linear differential equation

$$y''(t)+5y'(t)+6y(t)=x(t)$$

(3.105)

with the initial conditions $y(0) = 2, y'(0) = 1$, and $x(t) = e^{-t}u(t)$.

Assume that $y(0)=y(0^-)$ and $y'(0)=y'(0^-)$. Let

$$y(t) \leftrightarrow Y_I(s)$$

Then from Eqs. (3.44) and (3.45)

$$y'(t) \leftrightarrow sY_I(s)-y(0^-)=sY_I(s)-2$$
$$y''(t) \leftrightarrow s^2Y_I(s)-sy(0^-)-y'(0^-)=s^2Y_I(s)-2s-1$$

From Table 3-1 we have

$$x(t) \leftrightarrow X_I(s)=\frac{1}{s+1}$$

Taking the unilateral Laplace transform of Eq. (3.105), we obtain

$$[s^2Y_I(s)-2s-1]+5[sY_I(s)-2]+6Y_I(s)=\frac{1}{s+1}$$

or

$$(s^2+5s+6)Y_I(s)=\frac{1}{s+1}+2s+11=\frac{2s^2+13s+12}{s+1}$$

Thus,

$$Y_I(s)=\frac{2s^2+13s+12}{(s+1)(s^2+5s+6)}=\frac{2s^2+13s+12}{(s+1)(s+2)(s+3)}$$

Using partial-fraction expansions, we obtain

$$Y_I(s)=\frac{1}{2}\frac{1}{s+1}+6\frac{1}{s+2}-\frac{9}{2}\frac{1}{s+3}$$

Taking the inverse Laplace transform of $Y_I(s)$, we have

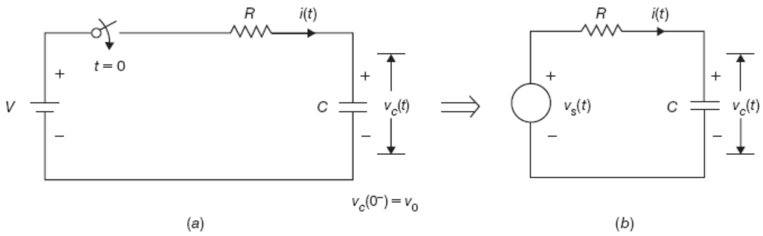
$$y(t)=\left(\frac{1}{2}e^{-t}+6e^{-2t}-\frac{9}{2}e^{-3t}\right)u(t)$$

Notice that $y(0^+) = 2 = y(0)$ and $y'(0^+) = 1 = y'(0)$; and we can write $y(t)$ as

$$y(t) = \frac{1}{2}e^{-t} + 6e^{-2t} - \frac{9}{2}e^{-3t} \qquad t \geq 0$$

3.39. Consider the *RC* circuit shown in Fig. 3-14(*a*). The switch is closed at $t = 0$. Assume that there is an initial voltage on the capacitor and $v_c(0^-) = v_0$.

Figure 3-14 RC circuit.



a. Find the current $i(t)$.

b. Find the voltage across the capacitor $v_C(t)$.

a. With the switching action, the circuit shown in Fig. 3-14(*a*) can be represented by the circuit shown in Fig. 3-14(*b*) with $v_s(t) = Vu(t)$. When the current $i(t)$ is the output and the input is $v_s(t)$, the differential equation governing the circuit is

$$Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau = v_s(t)$$

(3.106)

Taking the unilateral Laplace transform of Eq. (3.106) and using Eq. (3.48), we obtain

$$RI(s) + \frac{1}{C}\left[\frac{1}{s}I(s) + \frac{1}{s}\int_{-\infty}^{0^-} i(\tau)\,d\tau\right] = \frac{V}{s}$$

(3.107)

where $\quad I(s) = \mathscr{L}_I\{i(t)\}$

Now $\quad v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau$

and $\quad v_c(0^-) = \frac{1}{C}\int_{-\infty}^{0^-} i(\tau)\,d\tau = v_0$

Hence, Eq. (3.107) reduces to

$$\left(R + \frac{1}{Cs}\right)I(s) + \frac{v_0}{s} = \frac{V}{s}$$

Solving for $I(s)$, we obtain

$$I(s) = \frac{V - v_0}{s}\,\frac{1}{R + 1/Cs} = \frac{V - v_0}{R}\,\frac{1}{s + 1/RC}$$

Taking the inverse Laplace transform of $I(s)$, we get

$$i(t) = \frac{V - v_0}{R}e^{-t/RC}u(t)$$

b. When $v_C(t)$ is the output and the input is $v_s(t)$, the differential equation governing the circuit is

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

(3.108)

Taking the unilateral Laplace transform of Eq. (3.108) and using Eq. (3.44), we obtain

$$sV_c(s) - v_c(0^-) + \frac{1}{RC}V_c(s) = \frac{1}{RC}\frac{V}{s}$$

or

$$\left(s + \frac{1}{RC}\right)V_c(s) = \frac{1}{RC}\frac{V}{s} + v_0$$

Solving for $V_c(s)$, we have

$$V_c(s) = \frac{V}{RC}\frac{1}{s(s+1/RC)} + \frac{v_0}{s+1/RC}$$

$$= V\left(\frac{1}{s} - \frac{1}{s+1/RC}\right) + \frac{v_0}{s+1/RC}$$

Taking the inverse Laplace transform of $V_c(s)$, we obtain

$$v_c(t) = V[1 - e^{-t/RC}]u(t) + v_0 e^{-t/RC}u(t)$$

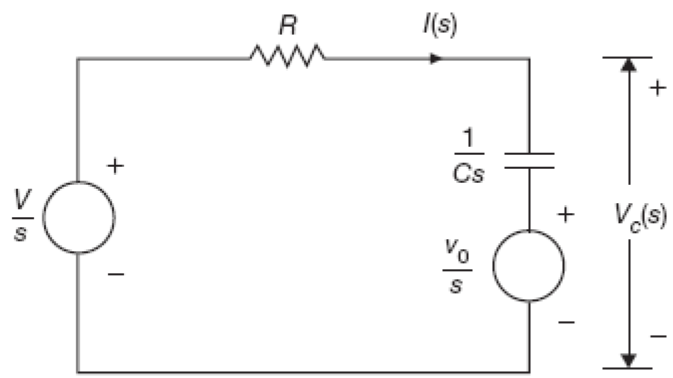
Note that $v_c(0^+) = v_0 = v_c(0^-)$. Thus, we write $v_c(t)$ as

$$v_c(t) = V(1 - e^{-t/RC}) + v_0 e^{-t/RC} \qquad t \geq 0$$

3.40. Using the transform network technique, redo Prob. 3.39.

a. Using Fig. 3-10, the transform network corresponding to Fig. 3-14 is constructed as shown in Fig. 3-15.

Figure 3-15 Transform circuit.



Writing the voltage law for the loop, we get

$$\left(R + \frac{1}{Cs}\right)I(s) + \frac{v_0}{s} = \frac{V}{s}$$

Solving for $I(s)$, we have

$$I(s) = \frac{V - v_0}{s}\frac{1}{R + 1/Cs} = \frac{V - v_0}{R}\frac{1}{s + 1/RC}$$

Taking the inverse Laplace transform of $I(s)$, we obtain

$$i(t)=\frac{V-v_0}{R}e^{-t/RC}u(t)$$

b. From Fig. 3.15 we have

$$V_c(s)=\frac{1}{Cs}I(s)+\frac{v_0}{s}$$

Substituting $I(s)$ obtained in part (a) into the above equation, we get

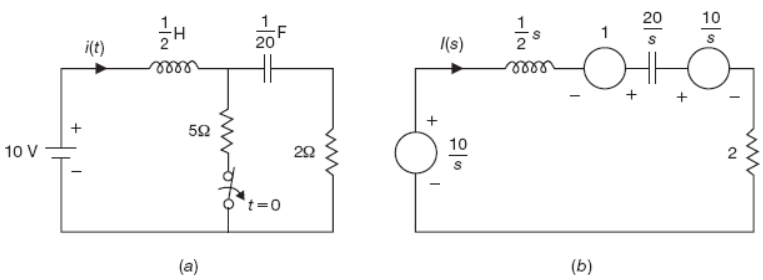
$$\begin{aligned}V_c(s)&=\frac{V-v_0}{RC}\frac{1}{s(s+1/RC)}+\frac{v_0}{s}\\&=(V-v_0)\left(\frac{1}{s}-\frac{1}{s+1/RC}\right)+\frac{v_0}{s}\\&=V\left(\frac{1}{s}-\frac{1}{s+1/RC}\right)+\frac{v_0}{s+1/RC}\end{aligned}$$

Taking the inverse Laplace transform of $V_c(s)$, we have

$$v_c(t)=V(1-e^{-t/RC})u(t)+v_0e^{-t/RC}u(t)$$

3.41. In the circuit in Fig. 3-16(a) the switch is in the closed position for a long time before it is opened at $t = 0$. Find the inductor current $i(t)$ for $t \geq 0$.

Figure 3-16



When the switch is in the closed position for a long time, the capacitor voltage is charged to 10 V and there is no current flowing in the capacitor. The inductor behaves as a short circuit, and the inductor current is $\frac{10}{5}=2\text{ A}$.

Thus, when the switch is open, we have $i(0^-)=2$ and $v_c(0^-)=10$; the input voltage is 10 V, and therefore it can be represented as $10u(t)$. Next, using Fig. 3-10, we construct the transform circuit as shown in Fig. 3-16(b).

From Fig. 3-16(b) the loop equation can be written as

$$\frac{1}{2}sI(s)-1+2I(s)+\frac{20}{s}I(s)+\frac{10}{s}=\frac{10}{s}$$

or

$$\left(\frac{1}{2}s+2+\frac{20}{s}\right)I(s)=1$$

Hence,

$$I(s)=\frac{1}{\frac{1}{2}s+2+20/s}=\frac{2s}{s^2+4s+40}$$
$$=\frac{2(s+2)-4}{(s+2)^2+6^2}=2\frac{(s+2)}{(s+2)^2+6^2}-\frac{2}{3}\frac{6}{(s+2)^2+6^2}$$

Taking the inverse Laplace transform of $I(s)$, we obtain

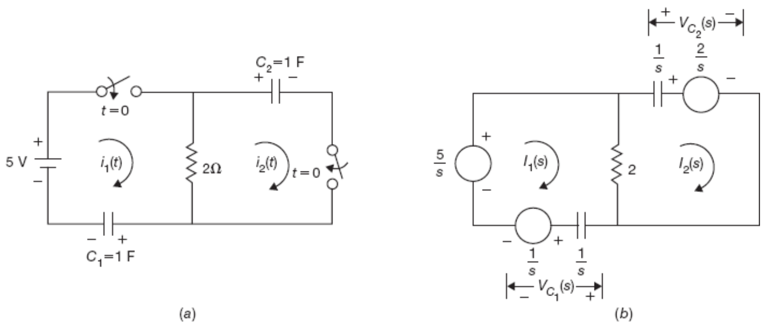
$$i(t)=e^{-2t}\left(2\cos 6t-\frac{2}{3}\sin 6t\right)u(t)$$

Note that $i(0^+)=2=i(0^-)$; that is, there is no discontinuity in the inductor current before and after the switch is opened. Thus, we have

$$i(t)=e^{-2t}(2\cos 6t-\frac{2}{3}\sin 6t)\qquad t\geq 0$$

3.42. Consider the circuit shown in Fig. 3-17(a). The two switches are closed simultaneously at $t = 0$. The voltages on capacitors C_1 and C_2 before the switches are closed are 1 and 2 V, respectively.

Figure 3-17



a. Find the currents $i_1(t)$ and $i_2(t)$.

b. Find the voltages across the capacitors at $t = 0^+$.

a. From the given initial conditions, we have

$$v_{C_1}(0^-)=1\text{ V}\qquad\text{and}\qquad v_{C_2}(0^-)=2\text{ V}$$

Thus, using Fig. 3-10, we construct a transform circuit as shown in Fig. 3-17(b). From

Fig. 3-17(b) the loop equations can be written directly as

$$\left(2+\frac{1}{s}\right)I_1(s)-2I_2(s)=\frac{4}{s}$$
$$-2I_1(s)+\left(2+\frac{1}{s}\right)I_2(s)=-\frac{2}{s}$$

Solving for $I_1(s)$ and $I_2(s)$ yields

$$I_1(s)=\frac{s+1}{s+\frac{1}{4}}=\frac{s+\frac{1}{4}+\frac{3}{4}}{s+\frac{1}{4}}=1+\frac{3}{4}\frac{1}{s+\frac{1}{4}}$$

$$I_2(s)=\frac{s-\frac{1}{2}}{s+\frac{1}{4}}=\frac{s+\frac{1}{4}-\frac{3}{4}}{s+\frac{1}{4}}=1-\frac{3}{4}\frac{1}{s+\frac{1}{4}}$$

Taking the inverse Laplace transforms of $I_1(s)$ and $I_2(s)$, we get

$$i_1(t)=\delta(t)+\frac{3}{4}e^{-t/4}u(t)$$

$$i_2(t)=\delta(t)-\frac{3}{4}e^{-t/4}u(t)$$

b. From Fig. 3-17(b) we have

$$V_{C_1}(s)=\frac{1}{s}I_1(s)+\frac{1}{s}$$

$$V_{C_2}(s)=\frac{1}{s}I_2(s)+\frac{2}{s}$$

Substituting $I_1(s)$ and $I_2(s)$ obtained in part (a) into the above expressions, we get

$$V_{C_1}(s)=\frac{1}{s}\frac{s+1}{s+\frac{1}{4}}+\frac{1}{s}$$

$$V_{C_2}(s)=\frac{1}{s}\frac{s-\frac{1}{2}}{s+\frac{1}{4}}+\frac{2}{s}$$

Then, using the initial value theorem (3.97), we have

$$v_{C_1}(0^+)=\lim_{s\to\infty} sV_{C_1}(s)=\lim_{s\to\infty}\frac{s+1}{s+\frac{1}{4}}+1=1+1=2\text{ V}$$

$$v_{C_2}(0^+)=\lim_{s\to\infty} sV_{C_2}(s)=\lim_{s\to\infty}\frac{s-\frac{1}{2}}{s+\frac{1}{4}}+2=1+2=3\text{ V}$$

Note that $v_{C_1}(0^+)\neq v_{C_1}(0^-)$ and $v_{C_2}(0^+)\neq v_{C_2}(0^-)$. This is due to the existence of a capacitor loop in the circuit resulting in a sudden change in voltage across the capacitors. This step change in voltages will result in impulses in $i_1(t)$ and $i_2(t)$. Circuits having a capacitor loop or an inductor star connection are known as *degenerative circuits*.

3.9. SUPPLEMENTARY PROBLEMS

3.43. Find the Laplace transform of the following $x(t)$:

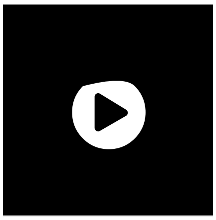
a. $x(t) = \sin \omega_0 t u(t)$

b. $x(t) = \cos(\omega_0 t + \phi)\, u(t)$

c. $x(t) = e^{-at}u(t) - e^{at}u(-t)$

d. $x(t) = 1$

e. $x(t) = \operatorname{sgn} t$



Schaum's Signals and Systems Supplementary Problem 3.43: Laplace Transforms

This video illustrates how to find the Laplace transform of a signal.

Carlotta A. Berry, Associate Professor, Electrical and Computer Engineering, Rose-Hulman Institute of Technology
2013

3.44. Find the Laplace transform of $x(t)$ given by

$$x(t) = \begin{cases} 1 & t_1 \le t \le t_2 \\ 0 & \text{otherwise} \end{cases}$$

3.45. Show that if $x(t)$ is a left-sided signal and $X(s)$ converges for some value of s, then the ROC of $X(s)$ is of the form

$$\operatorname{Re}(s) < \sigma_{\min}$$

where $\sigma_{\min}$ equals the minimum real part of any of the poles of $X(s)$.

3.46. Verify Eq. (3.21); that is,

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$$

3.47. Show the following properties for the Laplace transform:

a. If $x(t)$ is even, then $X(-s) = X(s)$; that is, $X(s)$ is also even.

b. If $x(t)$ is odd, then $X(-s) = -X(s)$; that is, $X(s)$ is also odd.

c. If $x(t)$ is odd, then there is a zero in $X(s)$ at $s = 0$.

3.48. Find the Laplace transform of

$$x(t) = (e^{-t}\cos 2t - 5e^{-2t})u(t) + \frac{1}{2}e^{2t}u(-t)$$

3.49. Find the inverse Laplace transform of the following $X(s)$:

$$(a)\quad X(s)=\frac{1}{s(s+1)^2},\ \mathrm{Re}(s) > -1$$

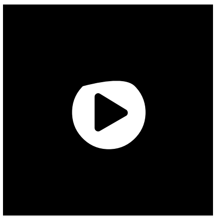
$$(b)\quad X(s)=\frac{1}{s(s+1)^2},\ -1<\mathrm{Re}(s)<0$$

$$(c)\quad X(s)=\frac{1}{s(s+1)^2},\ \mathrm{Re}(s) < -1$$

$$(d)\quad X(s)=\frac{s+1}{s^2+4s+13},\ \mathrm{Re}(s) > -2$$

$$(e)\quad X(s)=\frac{s}{(s^2+4)^2},\ \mathrm{Re}(s)>0$$

$$(f)\quad X(s)=\frac{s}{s^3+2s^2+9s+18},\ \mathrm{Re}(s) > -2$$



Schaum's Signals and Systems Supplementary Problem 3.49: Inverse Laplace Transforms

This video demonstrates how to find the inverse Laplace transform of a signal.

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3.50. Using the Laplace transform, redo Prob. 2.46.

3.51. Using the Laplace transform, show that

a. $x(t) * \delta(t) = x(t)$

b. $x(t) * \delta'(t) = x'(t)$

3.52. Using the Laplace transform, redo Prob. 2.54.

3.53. Find the output $y(t)$ of the continuous-time LTI system with

$$h(t) = e^{-2t}u(t)$$

for the each of the following inputs:

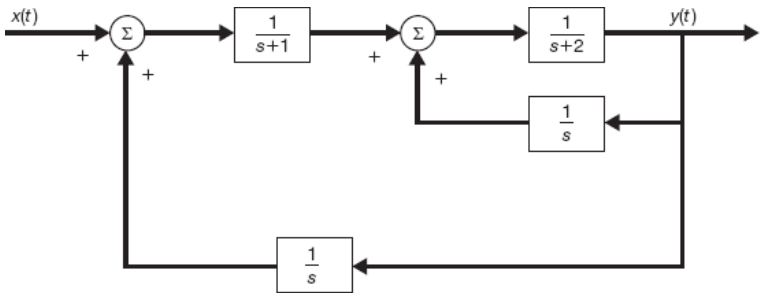
a. $x(t) = e^{-t}u(t)$

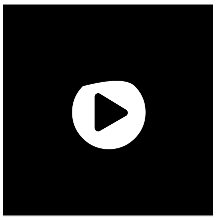
b. $x(t) = e^{-t}u(-t)$

3.54. The step response of an continuous-time LTI system is given by $(1 - e^{-t})\, u(t)$. For a certain unknown input $x(t)$, the output $y(t)$ is observed to be $(2 - 3e^{-t} + e^{-3t})u(t)$. Find the input $x(t)$.

3.55. Determine the overall system function $H(s)$ for the system shown in Fig. 3-18.

Figure 3-18





Schaum's Signals and Systems Supplementary Problem 3.55: Transfer Functions Example 1

This video demonstrates how to find the transfer function of a system described by a block diagram.

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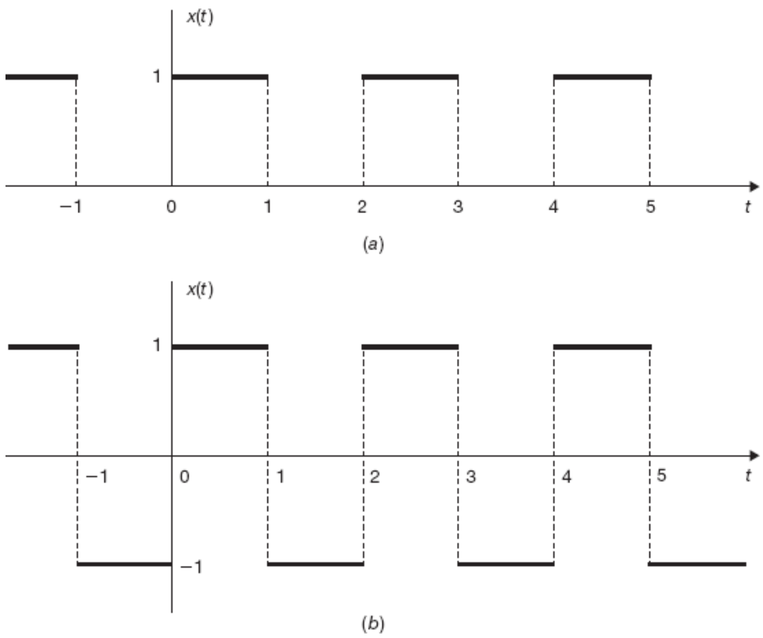
3.56. If $x(t)$ is a periodic function with fundamental period T, find the unilateral Laplace transform of $x(t)$.

3.57. Find the unilateral Laplace transforms of the periodic signals shown in Fig. 3-19.

Figure 3-19



3.58. Using the unilateral Laplace transform, find the solution of

$$y''(t) - y'(t) - 6y(t) = e^t$$

with the initial conditions $y(0) = 1$ and $y'(0) = 0$ for $t \geq 0$.

3.59. Using the unilateral Laplace transform, solve the following simultaneous differential equations:

$$y'(t) + y(t) + x'(t) + x(t) = 1$$

$$y'(t) - y(t) - 2x(t) = 0$$

with $x(0) = 0$ and $y(0) = 1$ for $t \geq 0$.

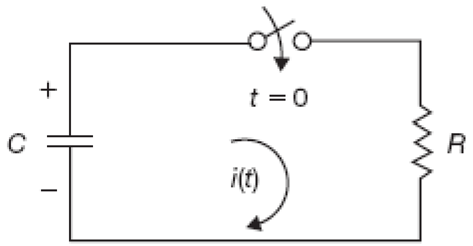
3.60. Using the unilateral Laplace transform, solve the following integral equations:

$$(a)\quad y(t) = 1 + a\int_0^t y(\tau)\,d\tau, t \geq 0$$

$$(b)\quad y(t) = e^t\left[1 + \int_0^t e^{-\tau} y(\tau)\,d\tau\right], t \geq 0$$

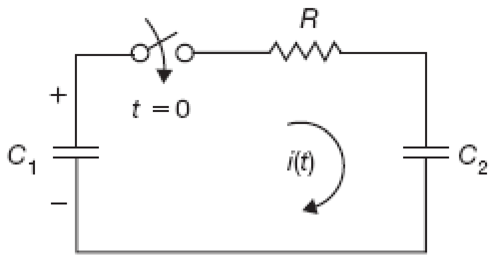
3.61. Consider the *RC* circuit in Fig. 3-20. The switch is closed at $t = 0$. The capacitor voltage before the switch closing is v_0. Find the capacitor voltage for $t \geq 0$.

Figure 3-20 RC circuit.



3.62. Consider the *RC* circuit in Fig. 3-21. The switch is closed at $t = 0$. Before the switch closing, the capacitor C_1 is charged to v_0 V and the capacitor C_2 is not charged.

Figure 3-21 RC circuit.



a. Assuming $C_1 = C_2 = C$, find the current $i(t)$ for $t \geq 0$.

b. Find the total energy E dissipated by the resistor R, and show that E is independent of R and is equal to half of the initial energy stored in C_1.

c. Assume that $R = 0$ and $C_1 = C_2 = C$. Find the current $i(t)$ for $t \geq 0$ and voltages $v_{C_1}(0^+)$ and $v_{C_2}(0^+)$.

3.10. ANSWERS TO SUPPLEMENTARY PROBLEMS

3.43.

a. $X(s) = \dfrac{\omega_0}{s^2 + \omega_0^2}, \mathrm{Re}(s) > 0$

b. $X(s) = \dfrac{s\cos\phi - \omega_0 \sin\phi}{s^2 + \omega_0^2}, \mathrm{Re}(s) > 0$

c. If $a > 0$, $X(s) = \dfrac{2s}{s^2 - a^2}$, $-a < \mathrm{Re}(s) < a$. If $a < 0$, $X(s)$ does not exist since $X(s)$ does not have an ROC.

d. *Hint:* $x(t) = u(t) + u(-t)$

$X(s)$ does not exist since $X(s)$ does not have an ROC.

e. *Hint:* $x(t) = u(t) - u(-t)$

$X(s)$ does not exist since $X(s)$ does not have an ROC.

3.44. $X(s) = \dfrac{1}{s}[e^{-st_1} - e^{-st_2}]$, all s

3.45. *Hint:* Proceed in a manner similar to Prob. 3.4.

3.46. *Hint:* Differentiate both sides of Eq. (3.3) with respect to s.

3.47. *Hint:*

a. Use Eqs. (1.2) and (3.17).

b. Use Eqs. (1.3) and (3.17).

c. Use the result from part (*b*) and Eq. (1.83a).

3.48. $X(s) = \dfrac{s+1}{(s+1)^2 + 4} - \dfrac{5}{s+2} - \dfrac{1}{2}\dfrac{1}{s-2}, -1 < \mathrm{Re}(s) < 2$

3.49.

$$
\begin{aligned}
&(a)\quad x(t) = (1 - e^{-t} - te^{-t})u(t)\\
&(b)\quad x(t) = -u(-t) - (1+t)e^{-t}u(t)\\
&(c)\quad x(t) = (-1 + e^{-t} + te^{-t})u(-t)\\
&(d)\quad x(t) = e^{-2t}\left(\cos 3t - \frac{1}{3}\sin 3t\right)u(t)\\
&(e)\quad x(t) = \frac{1}{4}t\sin 2\mathrm{tu}(t)\\
&(f)\quad x(t) = \left(-\frac{2}{13}e^{-2t} + \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t\right)u(t)
\end{aligned}
$$

3.50. *Hint:* Use Eq. (3.21) and Table 3-1.

3.51. *Hint:*

a. Use Eq. (3.21) and Table 3-1.

b. Use Eqs. (3.18) and (3.21) and Table 3-1.

3.52. *Hint:*

a. Find the system function $H(s)$ by Eq. (3.32) and take the inverse Laplace transform of $H(s)$.

b. Find the ROC of $H(s)$ and show that it does not contain the $j\omega$-axis.

3.53.

a. $y(t) = (e^{-t} - e^{-2t})\, u(t)$

b. $y(t) = e^{-t}u(-t) + e^{-2t}u(t)$

3.54. $x(t) = 2(1 - e^{-3t})u(t)$

3.55. *Hint:* Use the result from Prob. 3.31 to simplify the block diagram.

$$H(s) = \frac{2}{s^3 + 3s^2 + s - 2}$$

3.56. $X(s) = \dfrac{1}{1 - e^{-sT}} \displaystyle\int_{0^-}^{T} x(t)e^{-st}dt, \text{Re}(s) > 0$

3.57. $(a) \quad \dfrac{1}{s(1 + e^{-s})}, \text{Re}(s) > 0; \qquad (b) \quad \dfrac{1 - e^{-s}}{s(1 + e^{-s})}, \text{Re}(s) > 0$

3.58. $y(t) = -\dfrac{1}{6}e^{t} + \dfrac{2}{3}e^{-2t} + \dfrac{1}{2}e^{3t}, \; t \geq 0$

3.59. $x(t) = e^{-t} - 1, y(t) = 2 - e^{-t}, t \geq 0$

3.60.

a. $y(t) = e^{at}, t \geq 0;$

b. $y(t) = e^{2t}, t \geq 0$

3.61. $v_c(t) = v_0 e^{-t/RC}, t \geq 0$

3.62.

$$(a) \quad i(t) = (v_0 / R)e^{-2t/RC}, t \geq 0$$

$$(b) \quad E = \frac{1}{4}v_0^2 C$$

$$(c) \quad i(t) = \frac{1}{2}v_0 C\,\delta(t), v_{C_1}(0^+) = v_0 / 2 \neq v_{C_1}(0^-) = v_0, v_{C_2}(0^+) = v_0 / 2 \neq v_{C_2}(0^-) = 0$$