7. State Space Analysis

7.1. Introduction

So far we have studied linear time-invariant systems based on their input-output relationships, which are known as the external descriptions of the systems. In this chapter we discuss the method of *state space* representations of systems, which are known as the internal descriptions of the systems. The representation of systems in this form has many advantages:

1. It provides an insight into the behavior of the system.
2. It allows us to handle systems with multiple inputs and outputs in a unified way.
3. It can be extended to nonlinear and time-varying systems.

Since the state space representation is given in terms of matrix equations, the reader should have some familiarity with matrix or linear algebra. A brief review is given in App. A.

7.2. The Concept of State

7.2.1. A. Definition:

The *state* of a system at time t_0 (or n_0) is defined as the minimal information that is sufficient to determine the state and the output of the system for all times $t \geq t_0$ (or $n \geq n_0$) when the input to the system is also known for all times $t \geq t_0$ (or $n \geq n_0$). The variables that contain this information are called the *state variables*. Note that this definition of the state of the system applies only to causal systems.

Consider a single-input single-output LTI electric network whose structure is known. Then the complete knowledge of the input $x(t)$ over the time interval $-\infty$ to t is sufficient to determine the output $y(t)$ over the same time interval. However, if the input $x(t)$ is known over only the time interval t_0 to t, then the current through the inductors and the voltage across the capacitors at some time t_0 must be known in order to determine the output $y(t)$ over the time interval t_0 to t. These currents and voltages constitute the "state" of the network at time t_0. In this sense, the state of the network is related to the memory of the network.

7.2.2. B. Selection of State Variables:

Since the state variables of a system can be interpreted as the "memory elements" of the system, for discrete-time systems which are formed by unit-delay elements, amplifiers, and adders, we choose the outputs of the unit-delay elements as the state variables of the system (Prob. 7.1). For continuous-time systems which are formed by integrators, amplifiers, and adders, we choose the outputs of the integrators as the state variables of the system (Prob. 7.3). For a continuous-time system containing physical energy-storing elements, the outputs of these memory elements can be chosen to be the state variables of the system (Probs. 7.4 and 7.5). If the system is described by the difference or differential equation, the state variables can be chosen as shown in the following sections.

Note that the choice of state variables of a system is not unique. There are infinitely many choices for any given system.

7.3. State Space Representation of Discrete-Time LTI Systems

7.3.1. A. Systems Described by Difference Equations:

Suppose that a single-input single-output discrete-time LTI system is described by an Nth-order difference equation

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = x[n]$$

(7.1)

We know from previous discussion that if $x[n]$ is given for $n \geq 0$, Eq. (7.1) requires N initial conditions $y[-1]$, $y[-2]$, ..., $y[-N]$ to uniquely determine the complete solution for $n > 0$. That is, N values are required to specify the state of the system at any time.

Let us define N state variables $q_1[n]$, $q_2[n]$, ..., $q_N[n]$ as

$$\begin{aligned} q_1[n] &= y[n-N] \\ q_2[n] &= y[n-(N-1)] = y[n-N+1] \\ &\vdots \\ q_N[n] &= y[n-1] \end{aligned}$$

(7.2)

Then from Eqs. (7.2) and (7.1) we have

$$\begin{aligned} q_1[n+1] &= q_2[n] \\ q_2[n+1] &= q_3[n] \\ &\vdots \\ q_N[n+1] &= -a_N q_1[n] - a_{N-1} q_2[n] - \cdots - a_1 q_N[n] + x[n] \end{aligned}$$

(7.3a)

and

$$y[n] = -a_N q_1[n] - a_{N-1} q_2[n] - \cdots - a_1 q_N[n] + x[n]$$

(7.3b)

In matrix form Eqs. (7.3a) and (7.3b) can be expressed as

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \\ \vdots \\ q_N[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} x[n]$$

(7.4a)

$$y[n] = \begin{bmatrix} -a_N & -a_{N-1} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix} + [1]x[n]$$

(7.4b)

Now we define an $N \times 1$ matrix (or N-dimensional vector) $\mathbf{q}[n]$, which we call the *statevector*:

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix}$$

(7.5)

Then Eqs. (7.4a) and (7.4b) can be rewritten compactly as

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

(7.6a)

$$y[n] = \mathbf{c}\mathbf{q}[n] + dx[n]$$

(7.6b)

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} -a_N & -a_{N-1} & \cdots & -a_1 \end{bmatrix} \qquad d = 1$$

Equations (7.6a) and (7.6b) are called an *N-dimensional state space representation* (or *state equations*) of the system, and the $N \times N$ matrix $\mathbf{A}$ is termed the *system matrix*. The solution of Eqs. (7.6a) and (7.6b) for a given initial state is discussed in Sec. 7.5.

7.3.2. B. Similarity Transformation:

As mentioned before, the choice of state variables is not unique and there are infinitely many choices of the state variables for any given system. Let $\mathbf{T}$ be any $N \times N$ *nonsingular* matrix (App. A) and define a new state vector

$$\mathbf{v}[n] = \mathbf{T}\mathbf{q}[n]$$

(7.7)

where $\mathbf{q}[n]$ is the old state vector which satisfies Eqs. (7.6a) and (7.6b). Since $\mathbf{T}$ is nonsingular; that is, $\mathbf{T}^{-1}$ exists, and we have

$$\mathbf{q}[n] = \mathbf{T}^{-1}\mathbf{v}[n]$$

(7.8)

Now

$$\begin{aligned}\mathbf{v}[n+1] &= \mathbf{T}\mathbf{q}[n+1] = \mathbf{T}(\mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]) \\ &= \mathbf{T}\mathbf{A}\mathbf{q}[n] + \mathbf{T}\mathbf{b}x[n] = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\,\mathbf{v}[n] + \mathbf{T}\mathbf{b}x[n]\end{aligned}$$

(7.9a)

$$y[n] = \mathbf{c}\mathbf{q}[n] + dx[n] = \mathbf{c}\mathbf{T}^{-1}\,\mathbf{v}[n] + dx[n]$$

(7.9b)

Thus, if we let

$$\hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$$

(7.10a)

$$\hat{\mathbf{b}} = \mathbf{T}\mathbf{b} \qquad \hat{\mathbf{c}} = \mathbf{c}\mathbf{T}^{-1} \qquad \hat{d} = d$$

(7.10b)

then Eqs. (7.9a) and (7.9b) become

$$\mathbf{v}[n+1] = \hat{\mathbf{A}}\mathbf{v}[n] + \hat{\mathbf{b}}x[n]$$

(7.11a)

$$y[n] = \hat{\mathbf{c}}\mathbf{v}[n] + \hat{d}x[n]$$

(7.11b)

Equations (7.11a) and (7.11b) yield the same output $y[n]$ for a given input $x[n]$ with different state equations. In matrix algebra, Eq. (7.10a) is known as the *similarity transformation* and matrices **A** and $\hat{\mathbf{A}}$ are called *similar matrices* (App. A).

7.3.3. C. Multiple-Input Multiple-Output Systems:

If a discrete-time LTI system has m inputs and p outputs and N state variables, then a state space representation of the system can be expressed as

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{B}\mathbf{x}[n]$$

(7.12a)

$$\mathbf{y}[n] = \mathbf{C}\mathbf{q}[n] + \mathbf{D}\mathbf{x}[n]$$

(7.12b)

where

$$\mathbf{q}[n]=\begin{bmatrix} q_1[n] \\ q_2[n] \\ \vdots \\ q_N[n] \end{bmatrix} \qquad \mathbf{x}[n]=\begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_m[n] \end{bmatrix} \qquad \mathbf{y}[n]=\begin{bmatrix} y_1[n] \\ y_2[n] \\ \vdots \\ y_p[n] \end{bmatrix}$$

and

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}_{N\times N} \qquad \mathbf{B}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{Nm} \end{bmatrix}_{N\times m}$$

$$\mathbf{C}=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & a_{p2} & \cdots & c_{pN} \end{bmatrix}_{p\times N} \qquad \mathbf{D}=\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}_{p\times m}$$

7.4. State Space Representation of Continuous-Time LTI Systems

7.4.1. A. Systems Described by Differential Equations:

Suppose that a single-input single-output continuous-time LTI system is described by an$N$ th-order differential equation

$$\frac{d^N y(t)}{dt^N}+a_1\frac{d^{N-1}y(t)}{dt^{N-1}}+\cdots+a_N y(t)=x(t)$$

(7.13)

One possible set of initial conditions is $y(0), y^{(1)}(0), \ldots, y^{(N-1)}(0)$, where $y^{(k)}(t) = d^k y(t)/dt^k$. Thus, let us define N state variables $q_1(t), q_2(t), \ldots, q_N(t)$ as

$$\begin{aligned} q_1(t) &= y(t) \\ q_2(t) &= y^{(1)}(t) \\ &\vdots \\ q_N(t) &= y^{(N-1)}(t) \end{aligned}$$

(7.14)

Then from Eqs. (7.14) and (7.13) we have

$$\begin{aligned}\dot{q}_1(t) &= q_2(t)\\ \dot{q}_2(t) &= q_3(t)\\ &\vdots\\ \dot{q}_N(t) &= -a_N q_1(t) - a_{N-1} q_2(t) - \cdots - a_1 q_N(t) + x(t)\end{aligned}$$

(7.15a)

and

$$y(t) = q_1(t)$$

(7.15b)

where $\dot{q}_k(t) = dq_k(t)/dt$.

In matrix form Eqs. (7.15a) and (7.15b) can be expressed as

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \vdots\\ \dot{q}_N(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ \vdots\\ q_N(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \vdots\\ 1\end{bmatrix} x(t)$$

(7.16a)

$$y(t) = [1 \quad 0 \quad \cdots \quad 0]\begin{bmatrix}q_1(t)\\ q_2(t)\\ \vdots\\ q_N(t)\end{bmatrix}$$

(7.16b)

Now we define an $N \times 1$ matrix (or N-dimensional vector) $\mathbf{q}(t)$ which we call the state vector:

$$\mathbf{q}(t) = \begin{bmatrix}q_1(t)\\ q_2(t)\\ \vdots\\ q_N(t)\end{bmatrix}$$

(7.17)

The derivative of a matrix is obtained by taking the derivative of each element of the matrix. Thus,

$$\frac{d\mathbf{q}(t)}{dt} = \dot{\mathbf{q}}(t) = \begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \vdots\\ \dot{q}_N(t)\end{bmatrix}$$

(7.18)

Then Eqs. (7.16a) and (7.16b) can be rewritten compactly as

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$

(7.19a)

$$y(t) = \mathbf{c}\mathbf{q}(t)$$

(7.19b)

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

As in the discrete-time case, Eqs. (7.19a) and (7.19b) are called an N-dimensional state space representation (or state equations) of the system, and the $N \times N$ matrix **A** is termed the system matrix. In general, state equations of a single-input single-output continuous time LTI system are given by

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$

(7.20a)

$$y(t) = \mathbf{c}\mathbf{q}(t) + dx(t)$$

(7.20b)

As in the discrete-time case, there are infinitely many choices of state variables for any given system. The solution ofEqs. (7.20a) and (7.20b) for a given initial state are discussed in Sec. 7.6.

7.4.2. B. Multiple-Input Multiple-Output Systems:

If a continuous-time LTI system has m inputs, p outputs, and N state variables, then a state space representation of the system can be expressed as

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{x}(t)$$

(7.21a)

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{x}(t)$$

(7.21b)

where

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{bmatrix} \qquad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{bmatrix} \qquad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

and

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}_{N \times N} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{Nm} \end{bmatrix}_{N \times m}
$$

$$
\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & a_{p2} & \cdots & c_{pN} \end{bmatrix}_{p \times N} \qquad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}_{p \times m}
$$

7.5. Solutions of State Equations for Discrete-Time LTI Systems

7.5.1. A. Solution in the Time Domain:

Consider an N-dimensional state representation

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

(7.22a)

$$y[n] = \mathbf{c}\mathbf{q}[n] + dx[n]$$

(7.22b)

where **A**, **b**, **c**, and d are $N \times N$, $N \times 1$, $1 \times N$, and 1×1 matrices, respectively. One method of finding **q**[n], given the initial state **q**[0], is to solve Eq. (7.22a) iteratively. Thus,

$$
\begin{aligned}
\mathbf{q}[1] &= \mathbf{A}\mathbf{q}[0] + \mathbf{b}x[0] \\
\mathbf{q}[2] &= \mathbf{A}\mathbf{q}[1] + \mathbf{b}x[1] = \mathbf{A}\{\mathbf{A}\mathbf{q}[0] + \mathbf{b}x[0]\} + \mathbf{b}x[1] \\
&= \mathbf{A}^2\mathbf{q}[0] + \mathbf{A}\mathbf{b}x[0] + \mathbf{b}x[1]
\end{aligned}
$$

By continuing this process, we obtain

$$
\begin{aligned}
\mathbf{q}[n] &= \mathbf{A}^n\mathbf{q}[0] + \mathbf{A}^{n-1}\mathbf{b}x[0] + \cdots + \mathbf{b}x[n-1] \\
&= \mathbf{A}^n\mathbf{q}[0] + \sum_{k=0}^{n-1} \mathbf{A}^{n-1-k}\mathbf{b}x[k] \qquad n > 0
\end{aligned}
$$

(7.23)

If the initial state is **q**[n_0] and $x[n]$ is defined for $n \geq n_0$, then, proceeding in a similar manner, we obtain

$$\mathbf{q}[n] = \mathbf{A}^{n-n_0}\mathbf{q}[n_0] + \sum_{k=0}^{n-1} \mathbf{A}^{n-1-k}\mathbf{b}x[n_0 + k] \qquad n > n_0$$

(7.24)

The matrix $\mathbf{A}^n$ is the *n*-fold product

$$\mathbf{A}^n = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{n}$$

and is known as the *state-transition* matrix of the discrete-time system. Substituting Eq. (7.23) into Eq. (7.22b), we obtain

$$y[n] = \mathbf{c}\mathbf{A}^n\mathbf{q}[0] + \sum_{k=0}^{n-1} \mathbf{c}\mathbf{A}^{n-1-k}\mathbf{b}x[k] + dx[n] \qquad n > 0$$

(7.25)

The first term $\mathbf{c}\mathbf{A}^n\mathbf{q}[0]$ is the zero-input response, and the second and third terms together form the zero-state response.

7.5.2. B. Determination of A^n:

Method 1: Let **A** be an $N \times N$ matrix. The *characteristic equation* of **A** is defined to be (App. A)

$$c(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = 0$$

(7.26)

where $|\lambda\mathbf{I} - \mathbf{A}|$ means the determinant of $\lambda\mathbf{I} - \mathbf{A}$ and **I** is the *identity matrix* (or *unit matrix*) of *N*th order. The roots of $c(\lambda) = 0$, λ_k ($k = 1, 2, \ldots, N$), are known as the *eigenvalues* of **A**. By the *Cayley-Hamilton theorem* $\mathbf{A}^n$ can be expressed as [App. A, Eq. (A.57)]

$$\mathbf{A}^n = b_0\mathbf{I} + b_1\mathbf{A} + \cdots + b_{N-1}\mathbf{A}^{N-1}$$

(7.27)

When the eigenvalues λ_k are all distinct, the coefficients $b_0, b_1, \ldots, b_{N-1}$ can be found from the conditions

$$b_0 + b_1\lambda_k + \cdots + b_{N-1}\lambda_k^{N-1} = \lambda_k^n \qquad k = 1, 2, \ldots, N$$

(7.28)

For the case of repeated eigenvalues, see Prob. 7.25.

Method 2: The second method of finding $\mathbf{A}^n$ is based on the *diagonalization* of a matrix **A**. If eigenvalues λ_k of **A** are all distinct, then $\mathbf{A}^n$ can be expressed as [App. A, Eq. (A.53)]

$$\mathbf{A}^n = \mathbf{P}\begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N^n \end{bmatrix}\mathbf{P}^{-1}$$

(7.29)

where matrix **P** is known as the *diagonalization matrix* and is given by [App. A, Eq. (A.36)]

$$\mathbf{P} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_N]$$

(7.30)

and $\mathbf{x}_k (k = 1, 2, \ldots, N)$ are the *eigenvectors* of **A** defined by

$$\mathbf{A}\mathbf{x}_k = \lambda_k \mathbf{x}_k \qquad k = 1, 2, \ldots, N$$

(7.31)

Method 3: The third method of finding $\mathbf{A}^n$ is based on the *spectral decomposition* of a matrix **A**. When all eigenvalues of **A** are distinct, then **A** can be expressed as

$$\mathbf{A} = \lambda_1 \mathbf{E}_1 + \lambda_2 \mathbf{E}_2 + \cdots + \lambda_N \mathbf{E}_N = \sum_{k=1}^{N} \lambda_k \mathbf{E}_k$$

(7.32)

where λ_k $(k = 1, 2, \ldots, N)$ are the distinct eigenvalues of **A** and $\mathbf{E}_k$ $(k = 1, 2, \ldots, N)$ are called *constituent matrices*, which can be evaluated as [App. A, Eq. (A.67)]

$$\mathbf{E}_k = \frac{\prod_{\substack{m=1 \\ m \neq k}}^{N} (\mathbf{A} - \lambda_m \mathbf{I})}{\prod_{\substack{m=1 \\ m \neq k}}^{N} (\lambda_k - \lambda_m)}$$

(7.33)

Then we have

$$\mathbf{A}^n = \lambda_1^n \mathbf{E}_1 + \lambda_2^n \mathbf{E}_2 + \cdots + \lambda_N^n \mathbf{E}_N$$

(7.34)

Method 4: The fourth method of finding $\mathbf{A}^n$ is based on the *z*-transform.

$$\mathbf{A}^n = \mathfrak{Z}_I^{-1}\left\{(z\mathbf{I} - \mathbf{A})^{-1} z\right\}$$

(7.35)

which is derived in the following section [Eq. (7.41)].

7.5.3. C. The *z*-Transform Solution:

Taking the unilateral *z*-transform of Eqs. (7.22a) and (7.22b) and using Eq. (4.51), we get

$$z\mathbf{Q}(z) - z\mathbf{q}(0) = \mathbf{A}\mathbf{Q}(z) + \mathbf{b}X(z)$$

(7.36a)

$$Y(z) = \mathbf{c}\mathbf{Q}(z) + dX(z)$$

(7.36b)

where $X(z) = \mathfrak{Z}_I\{x[n]\}$, $Y(z) = \mathfrak{Z}_I\{y[n]\}$, and

$$\mathbf{Q}(z) = \mathcal{Z}_I\{\mathbf{q}[n]\} = \begin{bmatrix} Q_1(z) \\ Q_2(z) \\ \vdots \\ Q_N(z) \end{bmatrix} \qquad \text{where } Q_k(z) = \mathcal{Z}_I\{q_k[n]\}$$

Rearranging Eq. (7.36a), we have

$$(z\mathbf{I} - \mathbf{A})\mathbf{Q}(z) = z\mathbf{q}(0) + \mathbf{b}X(z)$$

(7.37)

Premultiplying both sides of Eq. (7.37) by $(z\mathbf{I} - \mathbf{A})^{-1}$ yields

$$\mathbf{Q}(z) = (z\mathbf{I} - \mathbf{A})^{-1} z\mathbf{q}(0) + (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z)$$

(7.38)

Hence, taking the inverse unilateral z-transform of Eq. (7.38), we get

$$\mathbf{q}[n] = \mathcal{Z}_I^{-1}\left\{(z\mathbf{I} - \mathbf{A})^{-1} z\right\}\mathbf{q}(0) + \mathcal{Z}_I^{-1}\left\{(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z)\right\}$$

(7.39)

Substituting Eq. (7.39) into Eq. (7.22b), we get

$$y[n] = \mathbf{c}\mathcal{Z}_I^{-1}\left\{(z\mathbf{I} - \mathbf{A})^{-1} z\right\}\mathbf{q}(0) + \mathbf{c}\mathcal{Z}_I^{-1}\left\{(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z)\right\} + dx[n]$$

(7.40)

A comparison of Eq. (7.39) with Eq. (7.23) shows that

$$\mathbf{A}^n = \mathcal{Z}_I^{-1}\left\{(z\mathbf{I} - \mathbf{A})^{-1} z\right\}$$

(7.41)

7.5.4. D. System Function *H*(*z*):

In Sec. 4.6 the system function $H(z)$ of a discrete-time LTI system is defined by $H(z) = Y(z)/X(z)$ with zero initial conditions. Thus, setting $\mathbf{q}[0] = \mathbf{0}$ in Eq. (7.38), we have

$$\mathbf{Q}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z)$$

(7.42)

The substitution of Eq. (7.42) into Eq. (7.36b) yields

$$Y(z) = [\mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]\, X(z)$$

(7.43)

Thus,

$$H(z) = [\mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]$$

(7.44)

7.5.5. E. Stability:

From Eqs. (7.25) and (7.29) or (7.34) we see that if the magnitudes of all eigenvalues λ_k of the system matrix **A** are less than unity, that is,

$$|\lambda_k| < 1 \qquad \text{all } k$$

(7.45)

then the system is said to be *asymptotically stable*; that is, if, undriven, its state tends to zero from any finite initial state $\mathbf{q}_0$. It can be shown that if all eigenvalues of **A** are distinct and satisfy the condition (7.45), then the system is also BIBO stable.

7.6. Solutions of State Equations for Continuous-Time LTI Systems

7.6.1. A. Laplace Transform Method:

Consider an *N*-dimensional state space representation

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$

(7.46a)

$$y(t) = \mathbf{c}\mathbf{q}(t) + dx(t)$$

(7.46b)

where **A**, **b**, **c**, and *d* are $N \times N$, $N \times 1$, $1 \times N$, and 1×1 matrices, respectively. In the following we solve Eqs. (7.46a) and (7.46b) with some initial state $\mathbf{q}(0)$ by using the unilateral Laplace transform. Taking the unilateral Laplace transform of Eqs. (7.46a) and (7.46b) and using Eq. (3.44), we get

$$s\mathbf{Q}(s) - \mathbf{q}(0) = \mathbf{A}\mathbf{Q}(s) + \mathbf{b}X(s)$$

(7.47a)

$$Y(s) = \mathbf{c}\mathbf{Q}(s) + dX(s)$$

(7.47b)

where $X(s) = \mathcal{L}_I\{x(t)\}$, $Y(s) = \mathcal{L}_I\{y(t)\}$, and

$$\mathbf{Q}(s) = \mathcal{L}_I\{\mathbf{q}(t)\} = \begin{bmatrix} Q_1(s) \\ Q_2(s) \\ \vdots \\ Q_N(s) \end{bmatrix} \qquad \text{where } Q_k(s) = \mathcal{L}_I\{q_k(t)\}$$

Rearranging Eq. (7.47a), we have

$$(s\mathbf{I} - \mathbf{A})\mathbf{Q}(s) = \mathbf{q}(0) + \mathbf{b}X(s)$$

(7.48)

Premultiplying both sides of Eq. (7.48) by $(s\mathbf{I} - \mathbf{A})^{-1}$ yields

$$\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{q}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(s)$$

(7.49)

Substituting Eq. (7.49) into Eq. (7.47b), we get

$$Y(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{q}(0) + [\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]\,X(s)$$

(7.50)

Taking the inverse Laplace transform of Eq. (7.50), we obtain the output $y(t)$. Note that $\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{q}(0)$ corresponds to the zero-input response and that the second term corresponds to the zero-state response.

7.6.2. B. System Function *H*(*s*):

As in the discrete-time case, the system function $H(s)$ of a continuous-time LTI system is defined by $H(s) = Y(s)/X(s)$ with zero initial conditions. Thus, setting $\mathbf{q}(0) = \mathbf{0}$ in Eq. (7.50), we have

$$Y(s) = [\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]\,X(s)$$

(7.51)

Thus,

$$H(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d$$

(7.52)

7.6.3. C. Solution in the Time Domain:

Following

$$e^{at} = 1 + at + \frac{a^2}{2!}t^2 + \cdots + \frac{a^k}{k!}t^k + \cdots$$

we define

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}t^2 + \cdots + \frac{\mathbf{A}^k}{k!}t^k + \cdots$$

(7.53)

where $k! = k(k-1) \ldots 2 \,.\, 1$. If $t = 0$, then Eq. (7.53) reduces to

$$e^{\mathbf{0}} = \mathbf{I}$$

(7.54)

where **0** is an $N \times N$ zero matrix whose entries are all zeros. As in $e^{a(t-\tau)} = e^{at}e^{-a\tau} = e^{-a\tau}e^{at}$, we can show that

$$e^{\mathbf{A}(t-\tau)} = e^{\mathbf{A}t}e^{-\mathbf{A}\tau} = e^{-\mathbf{A}\tau}e^{\mathbf{A}t}$$

(7.55)

Setting $\tau = t$ in Eq. (7.55), we have

$$e^{\mathbf{A}t}e^{-\mathbf{A}t} = e^{-\mathbf{A}t}e^{\mathbf{A}t} = e^{\mathbf{0}} = \mathbf{I}$$

(7.56)

Thus,

$$e^{-\mathbf{A}t} = (e^{\mathbf{A}t})^{-1}$$

(7.57)

which indicates that $e^{-\mathbf{A}t}$ is the inverse of $e^{\mathbf{A}t}$.

The differentiation of Eq. (7.53) with respect to t yields

$$\begin{aligned}\frac{d}{dt}e^{\mathbf{A}t} &= \mathbf{0} + \mathbf{A} + \frac{\mathbf{A}^2}{2!}2t + \cdots + \frac{\mathbf{A}^k}{k!}kt^{k-1} + \cdots \\ &= \mathbf{A}\left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}t^2 + \cdots\right] \\ &= \left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}t^2 + \cdots\right]\mathbf{A}\end{aligned}$$

which implies

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

(7.58)

Now using the relationship [App. A, Eq. (A.70)]

$$\frac{d}{dt}(\mathbf{AB}) = \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt}$$

and Eq. (7.58), we have

$$\begin{aligned}\frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{q}(t)\right] &= \left[\frac{d}{dt}e^{-\mathbf{A}t}\right]\mathbf{q}(t) + e^{-\mathbf{A}t}\dot{\mathbf{q}}(t) \\ &= -e^{-\mathbf{A}t}\mathbf{A}\mathbf{q}(t) + e^{-\mathbf{A}t}\dot{\mathbf{q}}(t)\end{aligned}$$

(7.59)

Now premultiplying both sides of Eq. (7.46a) by $e^{-\mathbf{A}t}$, we obtain

$$e^{-\mathbf{A}t}\dot{\mathbf{q}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{q}(t) + e^{-\mathbf{A}t}\mathbf{b}x(t)$$

or

$$e^{-\mathbf{A}t}\dot{\mathbf{q}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{q}(t) = e^{-\mathbf{A}t}\mathbf{b}x(t)$$

(7.60)

From Eq. (7.59)Eq. (7.60) can be rewritten as

$$\frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{q}(t)\right] = e^{-\mathbf{A}t}\mathbf{b}x(t)$$

(7.61)

Integrating both sides of Eq. (7.61) from 0 to t, we get

$$e^{-\mathbf{A}t}\mathbf{q}(t)\Big|_0^t = \int_0^t e^{-\mathbf{A}\tau}\mathbf{b}x(\tau)\,d\tau$$

or

$$e^{-\mathbf{A}t}\mathbf{q}(t) - \mathbf{q}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{b}x(\tau)\,d\tau$$

Hence

$$e^{-\mathbf{A}t}\mathbf{q}(t) = \mathbf{q}(0) + \int_0^t e^{-\mathbf{A}\tau}\mathbf{b}x(\tau)\,d\tau$$

(7.62)

Premultiplying both sides of Eq. (7.62) by $e^{\mathbf{A}t}$ and using Eqs. (7.55) and (7.56), we obtain

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}x(\tau)\,d\tau$$

(7.63)

If the initial state is $\mathbf{q}(t_0)$ and we have $x(t)$ for $t \geq t_0$, then

$$\mathbf{q}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{q}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{b}x(\tau)\,d\tau$$

(7.64)

which is obtained easily by integrating both sides of Eq. (7.61) from t_0 to t. The matrix function $e^{\mathbf{A}t}$ is known as the state-transition matrix of the continuous-time system. Substituting Eq. (7.63) into Eq. (7.46b), we obtain

$$y(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{q}(0) + \int_0^t \mathbf{c}e^{\mathbf{A}(t-\tau)}\mathbf{b}x(\tau)\,d\tau + dx(t)$$

(7.65)

7.6.4. D. Evaluation of e^{At}:

Method 1: As in the evaluation of $\mathbf{A}^n$, by the *Cayley-Hamilton* theorem we have

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} + \cdots + b_{N-1}\mathbf{A}^{N-1}$$

(7.66)

When the eigenvalues λ_k of **A** are all distinct, the coefficients $b_0, b_1, \ldots, b_{N-1}$ can be found from the conditions

$$b_0 + b_1\lambda_k + \cdots + b_{N-1}\lambda_k^{N-1} = e^{\lambda_k t} \qquad k = 1, 2, \ldots, N$$

(7.67)

For the case of repeated eigenvalues see Prob. 7.45.

Method 2: Again, as in the evaluation of $\mathbf{A}^n$, we can also evaluate $e^{\mathbf{A}t}$ based on the diagonalization of **A**. If all eigenvalues λ_k of **A** are distinct, we have

$$e^{\mathbf{A}t} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_N t} \end{bmatrix}\mathbf{P}^{-1}$$

(7.68)

where **P** is given by Eq. (7.30).

Method 3: We could also evaluate $e^{\mathbf{A}t}$ using the spectral decomposition of **A**, that is, find constituent matrices $\mathbf{E}_k$ ($k = 1, 2, \ldots, N$) for which

$$\mathbf{A} = \lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2 + \cdots + \lambda_N\mathbf{E}_N$$

(7.69)

where λ_k ($k = 1, 2, \ldots, N$) are the distinct eigenvalues of **A**. Then, when eigenvalues λ_k of **A** are all distinct, we have

$$e^{\mathbf{A}t} = e^{\lambda_1 t}\mathbf{E}_1 + e^{\lambda_2 t}\mathbf{E}_2 + \cdots + e^{\lambda_N t}\mathbf{E}_N$$

(7.70)

Method 4: Using the Laplace transform, we can calculate $e^{\mathbf{A}t}$. Comparing Eqs. (7.63) and (7.49), we see that

$$e^{\mathbf{A}t} = \mathscr{L}_I^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}$$

(7.71)

7.6.5. E. Stability:

From Eqs. (7.63) and (7.68) or (7.70), we see that if all eigenvalues λ_k of the system matrix **A** have negative real parts, that is,

$$\mathrm{Re}\{\lambda_k\} < 0 \qquad \text{all } k$$

(7.72)

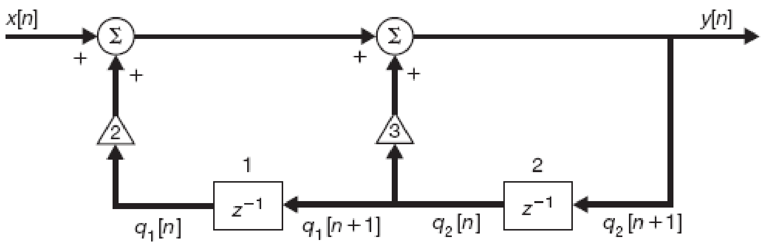
then the system is said to be *asymptotically stable*. As in the discrete-time case, if all eigenvalues of **A** are distinct and satisfy the condition (7.72), then the system is also BIBO stable.

7.7. SOLVED PROBLEMS

7.7.1. State Space Representation

7.1. Consider the discrete-time LTI system shown in Fig. 7-1. Find the state space representation of the system by choosing the outputs of unit-delay elements 1 and 2 as state variables $q_1[n]$ and $q_2[n]$, respectively.

Figure 7-1



From Fig. 7-1 we have

$$q_1[n+1] = q_2[n]$$
$$q_2[n+1] = 2q_1[n] + 3q_2[n] + x[n]$$
$$y[n] = 2q_1[n] + 3q_2[n] + x[n]$$

In matrix form

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + x[n]$$

(7.73a)

or

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$y[n] = \mathbf{c}\mathbf{q}[n] + dx[n]$$

(7.73b)

where

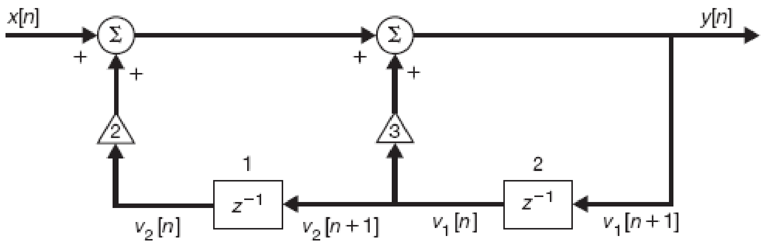
$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 2 & 3 \end{bmatrix} \qquad d = 1$$

7.2. Redo Prob. 7.1 by choosing the outputs of unit-delay elements 2 and 1 as state variables $v_1[n]$ and $v_2[n]$, respectively, and verify the relationships in Eqs. (7.10a) and (7.10b).

We redraw Fig. 7-1 with the new state variables as shown in Fig. 7-2. From Fig. 7-2 we have

$$v_1[n+1] = 3v_1[n] + 2v_2[n] + x[n]$$
$$v_2[n+1] = v_1[n]$$
$$y[n] = 3v_1[n] + 2v_2[n] + x[n]$$

Figure 7-2



In matrix form

$$\begin{bmatrix} v_1[n+1] \\ v_2[n+1] \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]$$
$$y[n] = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + x[n]$$

(7.74a)

or

$$\mathbf{v}[n+1] = \hat{\mathbf{A}}\mathbf{v}[n] + \hat{\mathbf{b}}x[n]$$
$$y[n] = \hat{\mathbf{c}}\mathbf{v}[n] + \hat{d}x[n]$$

(7.74b)

where

$$\mathbf{v}[n] = \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} \qquad \hat{\mathbf{A}} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \qquad \hat{\mathbf{b}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{\mathbf{c}} = \begin{bmatrix} 3 & 2 \end{bmatrix} \qquad \hat{d} = 1$$

Note that $v_1[n] = q_2[n]$ and $v_2[n] = q_1[n]$. Thus, we have

$$\mathbf{v}[n] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{q}[n] = \mathbf{T}\mathbf{q}[n]$$

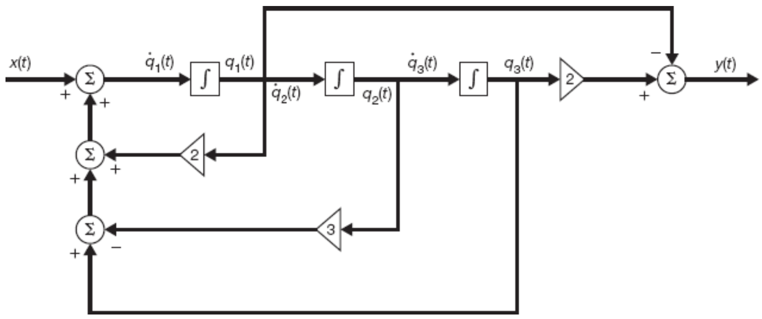
Now using the results from Prob. 7.1, we have

$$\mathbf{TAT}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \hat{\mathbf{A}}$$
$$\mathbf{Tb} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{\mathbf{b}}$$
$$\mathbf{cT}^{-1} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \end{bmatrix} = \hat{\mathbf{c}} \qquad d = 1 = \hat{d}$$

which are the relationships in Eqs. (7.10a) and (7.10b).

7.3. Consider the continuous-time LTI system shown in Fig. 7-3. Find a state space representation of the system.

Figure 7-3



We choose the outputs of integrators as the state variables $q_1(t)$, $q_2(t)$, and $q_3(t)$ as shown in Fig. 7-3. Then from Fig. 7-3 we obtain

$$\dot{q}_1(t) = 2q_1(t) - 3q_2(t) + q_3(t) + x(t)$$

$$\dot{q}_2(t) = q_1(t)$$

$$\dot{q}_3(t) = q_2(t)$$

$$y(t) = -q_1(t) + 2q_3(t)$$

In matrix form

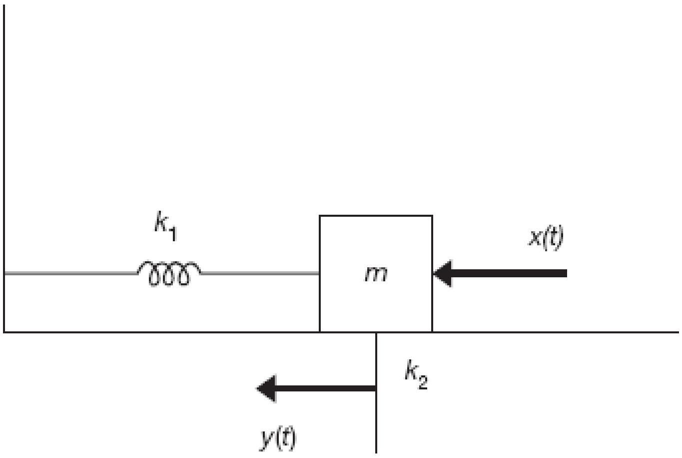
$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \mathbf{q}(t)$$

(7.75)

7.4. Consider the mechanical system shown in Fig. 7-4. It consists of a block with mass m connected to a wall by a spring. Let k_1 be the spring constant and k_2 be the viscous friction coefficient. Let the output $y(t)$ be the displacement of the block and the input $x(t)$ be the applied force. Find a state space representation of the system.

Figure 7-4 *Mechanical system.*



By Newton's law we have

$$m\ddot{y}(t) = -k_1 y(t) - k_2 \dot{y}(t) + x(t)$$

or

$$m\ddot{y}(t) + k_2 \dot{y}(t) + k_1 y(t) = x(t)$$

The potential energy and kinetic energy of a mass are stored in its position and velocity. Thus, we select the state variables $q_1(t)$ and $q_2(t)$ as

$$q_1(t) = y(t)$$
$$q_2(t) = \dot{y}(t)$$

Then we have

$$\begin{aligned}\dot{q}_1(t) &= q_2(t)\\ \dot{q}_2(t) &= -\frac{k_1}{m}q_1(t) - \frac{k_2}{m}q_2(t) + \frac{1}{m}x(t)\\ y(t) &= q_1(t)\end{aligned}$$

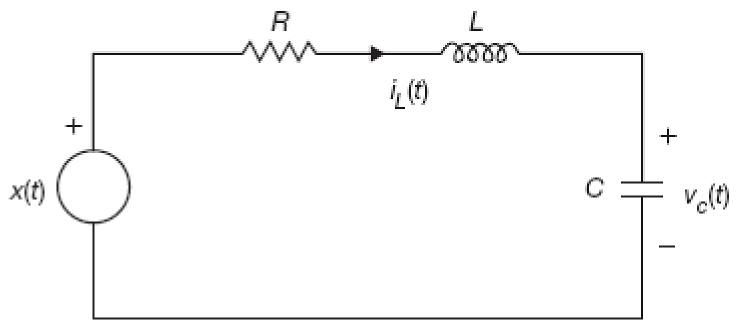
In matrix form

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 0 & 1 \\ -\dfrac{k_1}{m} & -\dfrac{k_2}{m} \end{bmatrix}\mathbf{q}(t) + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}x(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{q}(t)$$

(7.76)

7.5. Consider the *RLC* circuit shown in Fig. 7-5. Let the output $y(t)$ be the loop current. Find a state space representation of the circuit.

Figure 7-5 *RLC circuit.*



We choose the state variables $q_1(t) = i_L(t)$ and $q_2(t) = v_c(t)$. Then by Kirchhoff's law we get

$$L\dot{q}_1(t) + Rq_1(t) + q_2(t) = x(t)$$
$$C\dot{q}_2(t) = q_1(t)$$
$$y(t) = q_1(t)$$

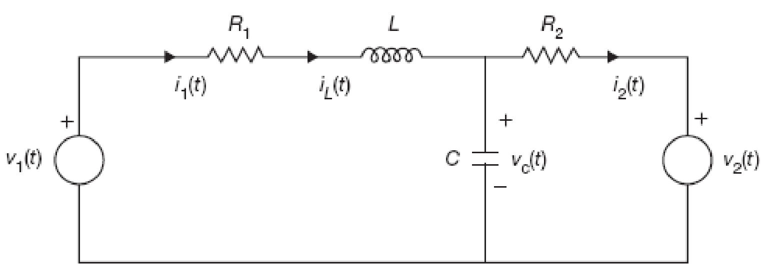
Rearranging and writing in matrix form, we get

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} x(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{q}(t)$$

(7.77)

7.6. Find a state space representation of the circuit shown in Fig. 7-6, assuming that the outputs are the currents flowing in R_1 and R_2.

Figure 7-6



We choose the state variables $q_1(t) = i_L(t)$ and $q_2(t) = v_c(t)$. There are two voltage sources and let $x_1(t) = v_1(t)$ and $x_2(t) = v_2(t)$. Let $y_1(t) = i_1(t)$ and $y_2(t) = i_2(t)$. Applying Kirchhoff's law to each loop, we obtain

$$L\dot{q}_1(t) + R_1q_1(t) + q_2(t) = x_1(t)$$
$$q_2(t) - [q_1(t) - C\dot{q}_2(t)]\,R_2 = x_2(t)$$
$$y_1(t) = q_1(t)$$
$$y_2(t) = \frac{1}{R_2}[q_2(t) - x_2(t)]$$

Rearranging and writing in matrix form, we get

$$\dot{\mathbf{q}}(t)=\begin{bmatrix}-\frac{R_1}{L} & -\frac{1}{L}\\ \frac{1}{C} & -\frac{1}{R_2C}\end{bmatrix}\mathbf{q}(t)+\begin{bmatrix}\frac{1}{L} & 0\\ 0 & \frac{1}{R_2C}\end{bmatrix}\mathbf{x}(t)$$
$$\mathbf{y}(t)=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{R_2}\end{bmatrix}\mathbf{q}(t)+\begin{bmatrix}0 & 0\\ 0 & -\frac{1}{R_2}\end{bmatrix}\mathbf{x}(t)$$

(7.78)

where

$$\mathbf{q}(t)=\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}\qquad \mathbf{x}(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\qquad \mathbf{y}(t)=\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}$$

7.7.2. State Equations of Discrete-Time LTI Systems Described by Difference Equations

7.7. Find state equations of a discrete-time system described by

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=x[n]$$

(7.79)

Choose the state variables $q_1[n]$ and $q_2[n]$ as

$$q_1[n]=y[n-2]$$
$$q_2[n]=y[n-1]$$

(7.80)

Then from Eqs. (7.79) and (7.80) we have

$$q_1[n+1]=q_2[n]$$
$$q_2[n+1]=-\frac{1}{8}q_1[n]+\frac{3}{4}q_2[n]+x[n]$$
$$y[n]=-\frac{1}{8}q_1[n]+\frac{3}{4}q_2[n]+x[n]$$

In matrix form

$$\mathbf{q}[n+1]=\begin{bmatrix}0 & 1\\ -\frac{1}{8} & \frac{3}{4}\end{bmatrix}\mathbf{q}[n]+\begin{bmatrix}0\\ 1\end{bmatrix}x[n]$$
$$y[n]=\begin{bmatrix}-\frac{1}{8} & \frac{3}{4}\end{bmatrix}\mathbf{q}[n]+x[n]$$

(7.81)

7.8. Find state equations of a discrete-time system described by

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] + \frac{1}{2}x[n-1]$$

(7.82)

Because of the existence of the term $\frac{1}{2}x[n-1]$ on the right-hand side of Eq. (7.82), the selection of $y[n-2]$ and $y[n-1]$ as state variables will not yield the desired state equations of the system. Thus, in order to find suitable state variables, we construct a simulation diagram of Eq. (7.82) using unit-delay elements, amplifiers, and adders. Taking the z-transforms of both sides of Eq. (7.82) and rearranging, we obtain

$$Y(z) = \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + X(z) + \frac{1}{2}z^{-1}X(z)$$

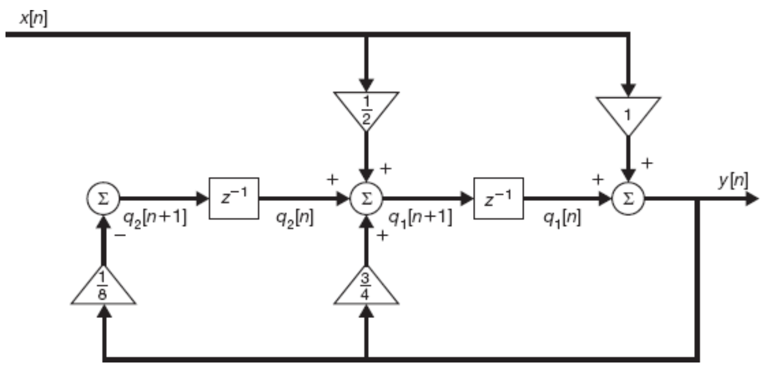
from which (noting that z^{-k} corresponds to k unit time delays) the simulation diagram in Fig. 7-7 can be drawn. Choosing the outputs of unit-delay elements as state variables as shown in Fig. 7-7, we get

$$y[n] = q_1[n] + x[n]$$

$$\begin{aligned} q_1[n+1] &= q_2[n] + \frac{3}{4}y[n] + \frac{1}{2}x[n] \\ &= \frac{3}{4}q_1[n] + q_2[n] + \frac{5}{4}x[n] \end{aligned}$$

$$q_2[n+1] = -\frac{1}{8}y[n] = -\frac{1}{8}q_1[n] - \frac{1}{8}x[n]$$

Figure 7-7



In matrix form

$$\mathbf{q}[n+1] = \begin{bmatrix} \frac{3}{4} & 1 \\ -\frac{1}{8} & 0 \end{bmatrix}\mathbf{q}[n] + \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{8} \end{bmatrix}x[n]$$

$$y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{q}[n] + x[n]$$

(7.83)

7.9. Find state equations of a discrete-time LTI system with system function

$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$$

(7.84)

From the definition of the system function [Eq. (4.41)]

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}}{1+a_1z^{-1}+a_2z^{-2}}$$

we have

$$(1+a_1z^{-1}+a_2z^{-2})Y(z)=(b_0+b_1z^{-1}+b_2z^{-2})X(z)$$

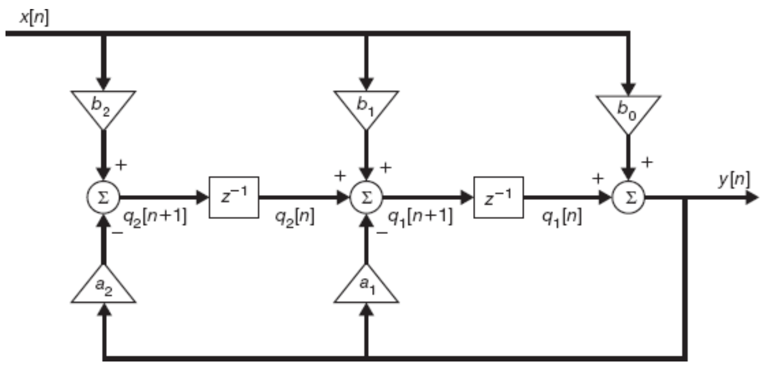
Rearranging the above equation, we get

$$Y(z)=-a_1z^{-1}Y(z)-a_2z^{-2}Y(z)+b_0X(z)+b_1z^{-1}X(z)+b_2z^{-2}X(z)$$

from which the simulation diagram in Fig. 7-8 can be drawn. Choosing the outputs of unit-delay elements as state variables as shown in Fig. 7-8, we get

$$\begin{aligned}
y[n]&=q_1[n]+b_0x[n]\\
q_1[n+1]&=-a_1y[n]+q_2[n]+b_1x[n]\\
&=-a_1q_1[n]+q_2[n]+(b_1-a_1b_0)x[n]\\
q_2[n+1]&=-a_2y[n]+b_2x[n]\\
&=-a_2q_1[n]+(b_2-a_2b_0)x[n]
\end{aligned}$$

Figure 7-8 Canonical simulation of the first form.



In matrix form

$$\mathbf{q}[n+1]=\begin{bmatrix}-a_1 & 1\\ -a_2 & 0\end{bmatrix}\mathbf{q}[n]+\begin{bmatrix}b_1-a_1b_0\\ b_2-a_2b_0\end{bmatrix}x[n]$$

$$y[n]=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{q}[n]+b_0x[n]$$

(7.85)

Note that in the simulation diagram in Fig. 7-8 the number of unit-delay elements is 2 (the order of the system) and is the minimum number required. Thus, Fig. 7-8 is known as the *canonical simulation of the first form* and Eq. (7.85) is known as the *canonical state representation of the first form*.

7.10. Redo Prob. 7.9 by expressing $H(z)$ as

$$H(z) = H_1(z)\, H_2(z)$$

where

$$H_1(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} \qquad H_2(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

Let

$$H_1(z) = \frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

(7.86)

$$H_2(z) = \frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

(7.87)

Then we have

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) = X(z)$$

(7.88)

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z)$$

(7.89)

Rearranging Eq. (7.88), we get

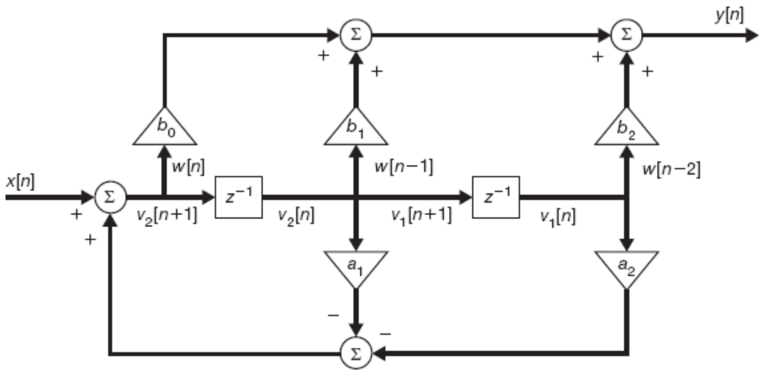
$$W(z) = -a_1 z^{-1}\, W(z) - a_2 z^{-2}\, W(z) + X(z)$$

(7.90)

From Eqs. (7.89) and (7.90) the simulation diagram in Fig. 7-9 can be drawn. Choosing the outputs of unit-delay elements as state variables as shown in Fig. 7-9, we have

$$\begin{aligned}
v_1[n+1] &= v_2[n] \\
v_2[n+1] &= -a_2 v_1[n] - a_1 v_2[n] + x[n] \\
y[n] &= b_2 v_1[n] + b_1 v_2[n] + b_0 v_2[n+1] \\
&= (b_2 - b_0 a_2) v_1[n] + (b_1 - b_0 a_1) v_2[n] + b_0 x[n]
\end{aligned}$$

Figure 7-9 Canonical simulation of the second form.



In matrix form

$$\mathbf{v}[n+1]=\begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\mathbf{v}[n]+\begin{bmatrix}0\\1\end{bmatrix}x[n]$$
$$y[n]=\begin{bmatrix}b_2-b_0a_2 & b_1-b_0a_1\end{bmatrix}\mathbf{v}[n]+b_0x[n]$$

(7.91)

The simulation in Fig. 7-9 is known as the *canonical simulation of the second form*, and Eq. (7.91) is known as the *canonical state representation of the second form*.

7.11. Consider a discrete-time LTI system with system function

$$H(z)=\frac{z}{2z^2-3z+1}$$

(7.92)

Find a state representation of the system.

Rewriting $H(z)$ as

$$H(z)=\frac{z}{2z^2\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)}=\frac{\frac{1}{2}z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

(7.93)

Comparing Eq. (7.93) with Eq. (7.84) in Prob. 7.9, we see that

$$a_1=-\frac{3}{2}\qquad a_2=\frac{1}{2}\qquad b_0=0\qquad b_1=\frac{1}{2}\qquad b_2=0$$

Substituting these values into Eq. (7.85) in Prob. 7.9, we get

$$\mathbf{q}[n+1]=\begin{bmatrix}\frac{3}{2} & 1\\ -\frac{1}{2} & 0\end{bmatrix}\mathbf{q}[n]+\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}x[n]$$
$$y[n]=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{q}[n]$$

(7.94)

7.12. Consider a discrete-time LTI system with system function

$$H(z)=\frac{z}{2z^2-3z+1}=\frac{z}{2(z-1)\left(z-\frac{1}{2}\right)}$$

(7.95)

Find a state representation of the system such that its system matrix**A** is diagonal.

First we expand $H(z)$ in partial fractions as

$$H(z)=\frac{z}{2(z-1)\left(z-\frac{1}{2}\right)}=\frac{z}{z-1}-\frac{z}{z-\frac{1}{2}}$$

$$=\frac{1}{1-z^{-1}}-\frac{1}{1-\frac{1}{2}z^{-1}}=H_1(z)+H_2(z)$$

where

$$H_1(z)=\frac{1}{1-z^{-1}} \qquad H_2(z)=\frac{-1}{1-\frac{1}{2}z^{-1}}$$

Let

$$H_k(z)=\frac{\alpha_k}{1-p_k z^{-1}}=\frac{Y_k(z)}{X(z)}$$

(7.96)

Then

$$(1-p_k z^{-1})Y_k(z)=\alpha_k X(z)$$

or

$$Y_k(z)=p_k z^{-1}Y_k(z)+\alpha_k X(z)$$

from which the simulation diagram in Fig. 7-10 can be drawn. Thus, $H(z) = H_1(z) + H_2(z)$ can be simulated by the diagram in Fig. 7-11 obtained by parallel connection of two systems. Choosing the outputs of unit-delay elements as state variables as shown in Fig. 7-11, we have

$$q_1[n+1]=q_1[n]+x[n]$$

$$q_2[n+1]=\frac{1}{2}q_2[n]-x[n]$$

$$y[n]=q_1[n+1]+q_2[n+1]=q_1[n]+\frac{1}{2}q_2[n]$$

Figure 7-10

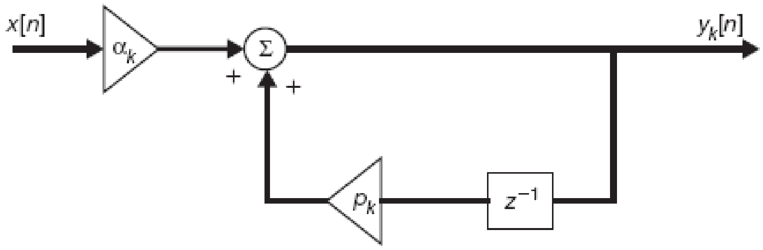
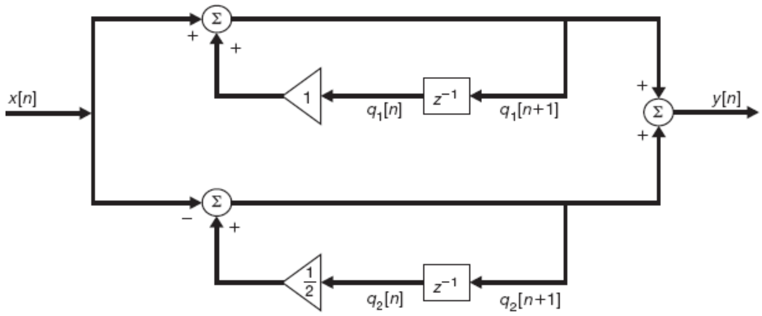


Figure 7-11



In matrix form

$$\mathbf{q}[n+1]=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}\mathbf{q}[n]+\begin{bmatrix}1\\ -1\end{bmatrix}x[n]$$

$$y[n]=\begin{bmatrix}1 & \frac{1}{2}\end{bmatrix}\mathbf{q}[n]$$

(7.97)

Note that the system matrix **A** is a diagonal matrix whose diagonal elements consist of the poles of $H(z)$.

7.13. Sketch a block diagram of a discrete-time system with the state representation

$$\mathbf{q}[n+1]=\begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{2}{3}\end{bmatrix}\mathbf{q}[n]+\begin{bmatrix}0\\ 1\end{bmatrix}x[n]$$

$$y[n]=\begin{bmatrix}3 & -2\end{bmatrix}\mathbf{q}[n]$$

(7.98)

We rewrite Eq. (7.98) as

$$q_1[n+1]=q_2[n]$$

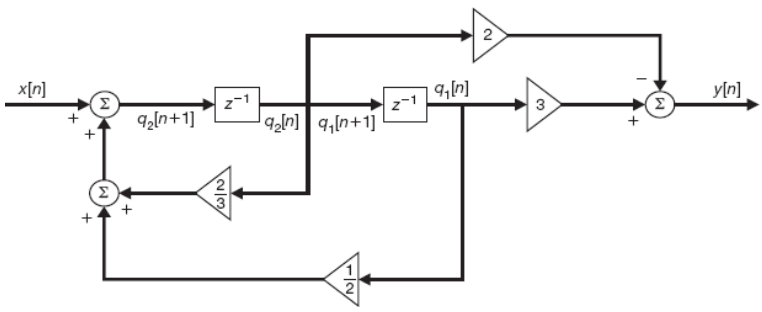
$$q_2[n+1]=\frac{1}{2}q_1[n]+\frac{2}{3}q_2[n]+x[n]$$

$$y[n]=3q_1[n]-2q_2[n]$$

(7.99)

from which we can draw the block diagram in Fig. 7-12.

Figure 7-12



7.7.3. State Equations of Continuous-Time LTI Systems Described by Differential Equations

7.14. Find state equations of a continuous-time LTI system described by

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$

(7.100)

Choose the state variables as

$$q_1(t) = y(t)$$
$$q_2(t) = \dot{y}(t)$$

(7.101)

Then from Eqs. (7.100) and (7.101) we have

$$\dot{q}_1(t) = q_2(t)$$
$$\dot{q}_2(t) = -2q_1(t) - 3q_2(t) + x(t)$$
$$y(t) = q_1(t)$$

In matrix form

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{q}(t)$$

(7.102)

7.15. Find state equations of a continuous-time LTI system described by

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 4\dot{x}(t) + x(t)$$

(7.103)

Because of the existence of the term $4\dot{x}(t)$ on the right-hand side of Eq. (7.103), the selection of $y(t)$ and $\dot{y}(t)$ as state variables will not yield the desired state equations of the system. Thus, in order to find suitable state variables, we construct a simulation diagram of Eq. (7.103) using integrators, amplifiers, and adders. Taking the Laplace transforms of both sides of Eq. (7.103), we obtain

$$s^2Y(s) + 3sY(s) + 2Y(s) = 4sX(s) + X(s)$$

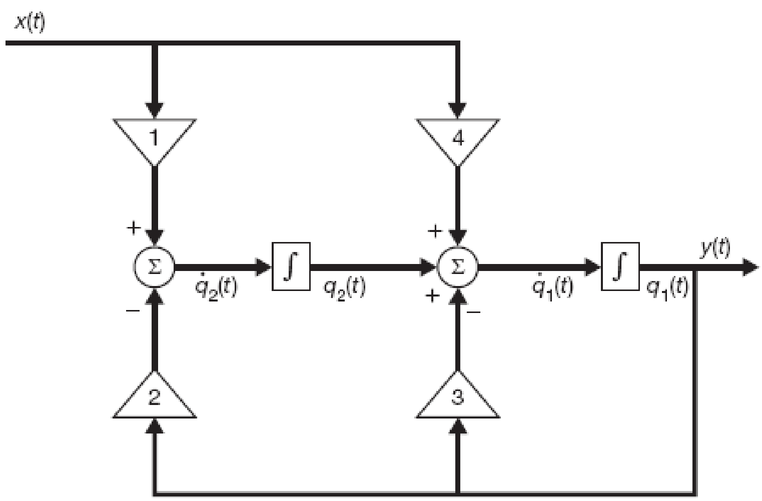
Dividing both sides of the above expression by s^2 and rearranging, we get

$$Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + 4s^{-1}X(s) + s^{-2}X(s)$$

from which (noting that s^{-k} corresponds to integration of k times) the simulation diagram in Fig. 7-13 can be drawn. Choosing the outputs of integrators as state variables as shown in Fig. 7-13, we get

$$\begin{aligned}\dot{q}_1(t) &= -3q_1(t) + q_2(t) + 4x(t)\\ \dot{q}_2(t) &= -2q_1(t) + x(t)\\ y(t) &= q_1(t)\end{aligned}$$

Figure 7-13



In matrix form

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{q}(t)$$

(7.104)

7.16. Find state equations of a continuous-time LTI system with system function

$$H(s) = \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{s_3 + a_1s^2 + a_2s + a_3}$$

(7.105)

From the definition of the system function [Eq.(3.37)]

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3}$$

we have

$$(s^3 + a_1s^2 + a_2s + a_3)Y(s) = (b_0s^3 + b_1s^2 + b_2s + b_3)X(s)$$

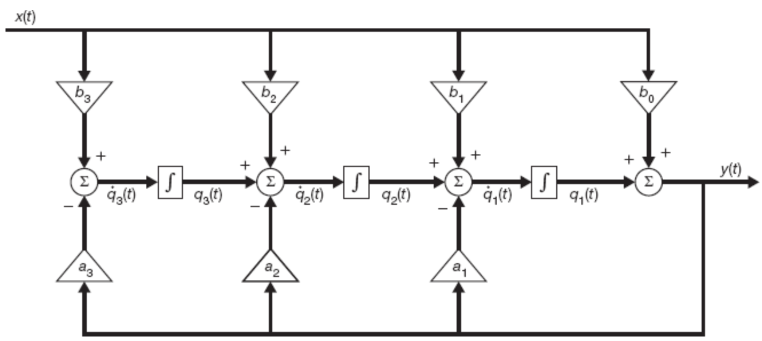
Dividing both sides of the above expression by s^3 and rearranging, we get

$$\begin{aligned} Y(s) = &-a_1s^{-1}Y(s) - a_2s^{-2}Y(s) - a_3s^{-3}Y(s) \\ &+ b_0X(s) + b_1s^{-1}X(s) + b_2s^{-2}X(s) + b_3s^{-3}X(s) \end{aligned}$$

from which (noting that s^{-k} corresponds to integration of k times) the simulation diagram in Fig. 7-14 can be drawn. Choosing the outputs of integrators as state variables as shown in Fig. 7-14, we get

$$\begin{aligned} y(t) &= q_1(t) + b_0x(t) \\ \dot{q}_1(t) &= -a_1y(t) + q_2(t) + b_1x(t) \\ &= -a_1q_1(t) + q_2(t) + (b_1 - a_1b_0)x(t) \\ \dot{q}_2(t) &= -a_2y(t) + q_3(t) + b_2x(t) \\ &= -a_2q_1(t) + q_3(t) + (b_2 - a_2b_0)x(t) \\ \dot{q}_3(t) &= -a_3y(t) + b_3x(t) \\ &= -a_3q_1(t) + (b_3 - a_3b_0)x(t) \end{aligned}$$

Figure 7-14 Canonical simulation of the first form.



In matrix form

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}\mathbf{q}(t) + \begin{bmatrix} b_1 - a_1b_0 \\ b_2 - a_2b_0 \\ b_3 - a_3b_0 \end{bmatrix}x(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{q}(t) + b_0x(t)$$

(7.106)

As in the discrete-time case, the simulation of $H(s)$ shown in Fig. 7-14 is known as the canonical simulation of the first form, and Eq. (7.106) is known as the canonical state representation of the first form.

7.17. Redo Prob. 7.16 by expressing $H(s)$ as

$$H(s) = H_1(s)H_2(s)$$

where

$$H_1(s) = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$$
$$H_2(s) = b_0 s^3 + b_1 s^2 + b_2 s + b_3$$

Let

$$H_1(s) = \frac{W(s)}{X(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$$
$$H_2(s) = \frac{Y(s)}{W(s)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$$

(7.107)

Then we have

$$(s^3 + a_1 s^2 + a_2 s + a_3)W(s) = X(s)$$
$$Y(s) = (b_0 s^3 + b_1 s^2 + b_2 s + b_3)W(s)$$

Rearranging the above equations, we get

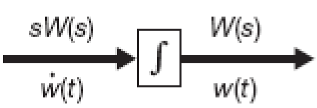
$$s^3 W(s) = -a_1 s^2 W(s) - a_2 s W(s) - a_3 W(s) + X(s)$$
$$Y(s) = b_0 s^3 W(s) + b_1 s^2 W(s) + b_2 s W(s) + b_3 W(s)$$

from which, noting the relation shown in Fig. 7-15, the simulation diagram in Fig. 7-16 can be drawn. Choosing the outputs of integrators as state variables as shown in Fig. 7-16, we have

$$\begin{aligned}
\dot{v}_1(t) &= v_2(t) \\
\dot{v}_2(t) &= v_3(t) \\
\dot{v}_3(t) &= -a_3 v_1(t) - a_2 v_2(t) - a_1 v_3(t) + x(t) \\
y(t) &= b_3 v_1(t) + b_2 v_2(t) + b_1 v_3(t) + b_0 \dot{v}_3(t) \\
&= (b_3 - a_3 b_0) v_1(t) + (b_2 - a_2 b_0) v_2(t) \\
&\quad + (b_1 - a_1 b_0) v_3(t) + b_0 x(t)
\end{aligned}$$

(7.108)

Figure 7-15



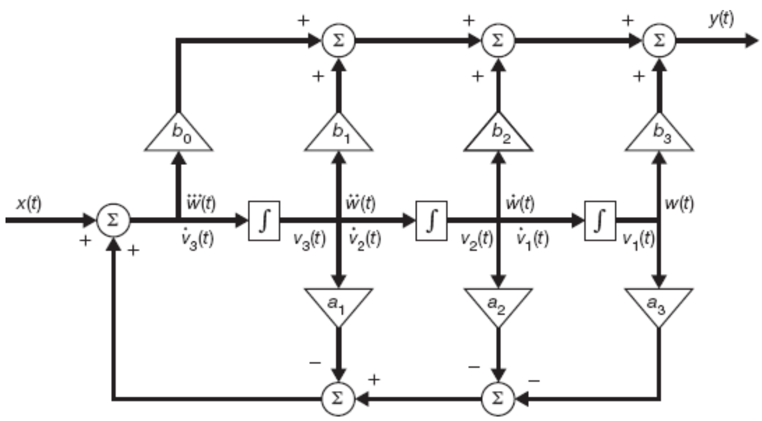
In matrix form

$$\dot{\mathbf{v}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x(t)$$
$$y(t) = \begin{bmatrix} b_3 - a_3 b_0 & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \mathbf{v}(t) + b_0 x(t)$$

(7.109)

As in the discrete-time case, the simulation of $H(s)$ shown in Fig. 7-16 is known as the canonical simulation of the second form, and Eq. (7.109) is known as the canonical state representation of the second form.

Figure 7-16 Canonical simulation of the second form.



7.18. Consider a continuous-time LTI system with system function

$$H(s)=\frac{3s+7}{(s+1)(s+2)(s+5)}$$

(7.110)

Find a state representation of the system.

Rewrite $H(s)$ as

$$H(s)=\frac{3s+7}{(s+1)(s+2)(s+5)}=\frac{3s+7}{s^3+8s^2+17s+10}$$

(7.111)

Comparing Eq. (7.111) with Eq. (7.105) in Prob. 7.16, we see that

$$a_1=8 \qquad a_2=17 \qquad a_3=10 \qquad b_0=b_1=0 \qquad b_2=3 \qquad b_3=7$$

Substituting these values into Eq. (7.106) in Prob. 7.16, we get

$$\dot{\mathbf{q}}(t)=\begin{bmatrix}-8 & 1 & 0\\ -17 & 0 & 1\\ -10 & 0 & 0\end{bmatrix}\mathbf{q}(t)+\begin{bmatrix}0\\3\\7\end{bmatrix}x(t)$$
$$y(t)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\mathbf{q}(t)$$

(7.112)

7.19. Consider a continuous-time LTI system with system function

$$H(s)=\frac{3s+7}{(s+1)(s+2)(s+5)}$$

(7.113)

Find a state representation of the system such that its system matrix **A** is diagonal.

First we expand $H(s)$ in partial fractions as

$$H(s)=\frac{3s+7}{(s+1)(s+2)(s+5)}=\frac{1}{s+1}-\frac{\frac{1}{3}}{s+2}-\frac{\frac{2}{3}}{s+5}$$
$$=H_1(s)+H_2(s)+H_3(s)$$

where

$$H_1(s)=\frac{1}{s+1}\qquad H_2(s)=-\frac{\frac{1}{3}}{s+2}\qquad H_3(s)=-\frac{\frac{2}{3}}{s+5}$$

Let

$$H_k(s)=\frac{\alpha_k}{s-p_k}=\frac{Y_k(s)}{X(s)}$$

(7.114)

Then

$$(s-p_k)Y_k(s)=\alpha_k X(s)$$

or

$$Y_k(s)=p_k s^{-1}Y_k(s)+\alpha_k s^{-1}X(s)$$

from which the simulation diagram in Fig. 7-17 can be drawn. Thus, $H(s) = H_1(s) + H_2(s) + H_3(s)$ can be simulated by the diagram in Fig. 7-18 obtained by parallel connection of three systems. Choosing the outputs of integrators as state variables as shown in Fig. 7-18, we get

$$\dot{q}_1(t)=-q_1(t)+x(t)$$
$$\dot{q}_2(t)=-2q_2(t)-\frac{1}{3}x(t)$$
$$\dot{q}_3(t)=-5q_3(t)-\frac{2}{3}x(t)$$
$$y(t)=q_1(t)+q_2(t)+q_3(t)$$

Figure 7-17

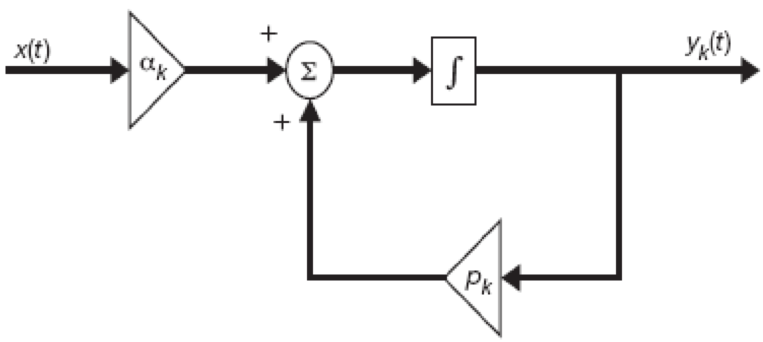
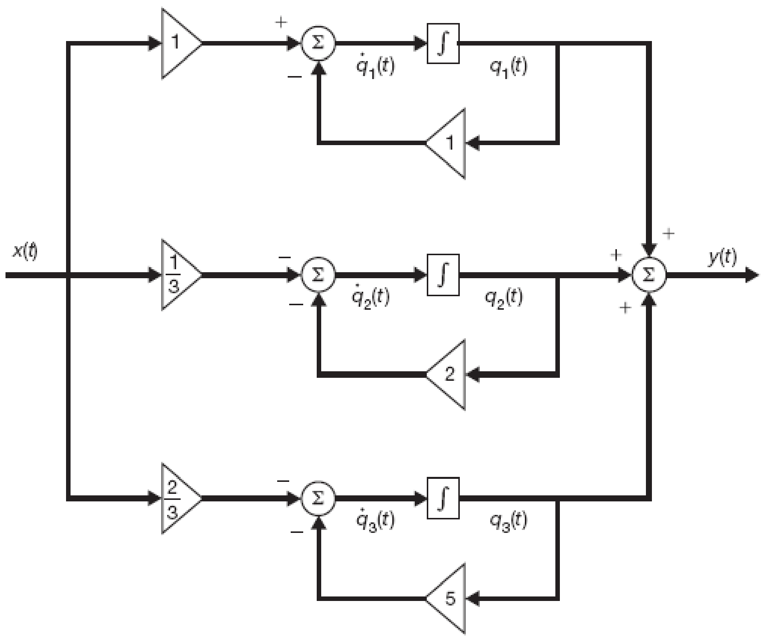


Figure 7-18



In matrix form

$$\dot{\mathbf{q}}(t)=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -5\end{bmatrix}\mathbf{q}(t)+\begin{bmatrix}1\\ -\frac{1}{3}\\ -\frac{2}{3}\end{bmatrix}x(t)$$

$$y(t)=\begin{bmatrix}1 & 1 & 1\end{bmatrix}\mathbf{q}(t)$$

(7.115)

Note that the system matrix **A** is a diagonal matrix whose diagonal elements consist of the poles of $H(s)$.

7.7.4. Solutions of State Equations for Discrete-Time LTI Systems

7.20. Find $\mathbf{A}^n$ for

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -\frac{1}{8} & \frac{3}{4}\end{bmatrix}$$

by the Cayley-Hamilton theorem method.

First, we find the characteristic polynomial $c(\lambda)$ of **A**.

$$c(\lambda)=|\lambda\mathbf{I}-\mathbf{A}|=\begin{bmatrix}\lambda & -1\\ \frac{1}{8} & \lambda-\frac{3}{4}\end{bmatrix}$$

$$=\lambda^2-\frac{3}{4}\lambda+\frac{1}{8}=\left(\lambda-\frac{1}{2}\right)\left(\lambda-\frac{1}{4}\right)$$

Thus, the eigenvalues of **A** are $\lambda_1=\frac{1}{2}$ and $\lambda_2=\frac{1}{4}$. Hence, by Eqs. (7.27) and (7.28) we have

$$\mathbf{A}^n=b_0\mathbf{I}+b_1\mathbf{A}=\begin{bmatrix}b_0 & b_1\\ -\frac{1}{8}b_1 & b_0+\frac{3}{4}b_1\end{bmatrix}$$

and b_0 and b_1 are the solutions of

$$b_0+b_1\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^n$$

$$b_0+b_1\left(\frac{1}{4}\right)=\left(\frac{1}{4}\right)^n$$

from which we get

$$b_0=-\left(\frac{1}{2}\right)^n+2\left(\frac{1}{4}\right)^n \qquad b_1=4\left(\frac{1}{2}\right)^n-4\left(\frac{1}{4}\right)^n$$

Hence,

$$\mathbf{A}^n=\begin{bmatrix}-\left(\frac{1}{2}\right)^n+2\left(\frac{1}{4}\right)^n & 4\left(\frac{1}{2}\right)^n-4\left(\frac{1}{4}\right)^n\\ -\frac{1}{2}\left(\frac{1}{2}\right)^n+\frac{1}{2}\left(\frac{1}{4}\right)^n & 2\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n\end{bmatrix}$$

$$=\left(\frac{1}{2}\right)^n\begin{bmatrix}-1 & 4\\ -\frac{1}{2} & 2\end{bmatrix}+\left(\frac{1}{4}\right)^n\begin{bmatrix}2 & -4\\ \frac{1}{2} & -1\end{bmatrix}$$

7.21. Repeat Prob. 7.20 using the diagonalization method.

Let **x** be an eigenvector of **A** associated with λ. Then

$$[\lambda\mathbf{I}-\mathbf{A}]\mathbf{x}=0$$

For $\lambda=\lambda_1=\frac{1}{2}$ we have

$$\begin{bmatrix} \frac{1}{2} & -1 \\ \frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions of this system are given by $x_1 = 2x_2$. Thus, the eigenvectors associated with λ_1 are those vectors of the form

$$\mathbf{x}_1 = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \alpha \neq 0$$

For $\lambda = \lambda_2 = \frac{1}{4}$ we have

$$\begin{bmatrix} \frac{1}{4} & -1 \\ \frac{1}{8} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions of this system are given by $x_1 = 4x_2$. Thus, the eigenvectors associated with λ_2 are those vectors of the form

$$\mathbf{x}_2 = \beta \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \beta \neq 0$$

Let $\alpha = \beta = 1$ in the above expressions and let

$$\mathbf{P} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

Then

$$\mathbf{P}^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 2 \\ \frac{1}{2} & -1 \end{bmatrix}$$

and by Eq. (7.29) we obtain

$$\mathbf{A}^n = \mathbf{P}\Lambda^n\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} \left(\frac{1}{2}\right)^n & 0 \\ 0 & \left(\frac{1}{4}\right)^n \end{bmatrix}\mathbf{P}^{-1} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \left(\frac{1}{2}\right)^n & 0 \\ 0 & \left(\frac{1}{4}\right)^n \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & 2 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{4}\right)^n & 4\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n \\ -\frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{2}\right)^n & 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \end{bmatrix}$$

$$= \left(\frac{1}{2}\right)^n\begin{bmatrix} -1 & 4 \\ -\frac{1}{2} & 2 \end{bmatrix} + \left(\frac{1}{4}\right)^n\begin{bmatrix} 2 & -4 \\ \frac{1}{2} & -1 \end{bmatrix}$$

7.22. Repeat Prob. 7.20 using the spectral decomposition method.

Since all eigenvalues of **A** are distinct, by Eq. (7.33) we have

$$\mathbf{E}_1 = \frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_2\mathbf{I}) = \frac{1}{\frac{1}{2} - \frac{1}{4}}\left(\mathbf{A} - \frac{1}{4}\mathbf{I}\right) = 4\begin{bmatrix} -\frac{1}{4} & 1 \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -\frac{1}{2} & 2 \end{bmatrix}$$

$$\mathbf{E}_2 = \frac{1}{\lambda_2 - \lambda_1}(\mathbf{A} - \lambda_1\mathbf{I}) = \frac{1}{\frac{1}{4} - \frac{1}{2}}\left(\mathbf{A} - \frac{1}{2}\mathbf{I}\right) = -4\begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{8} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ \frac{1}{2} & -1 \end{bmatrix}$$

Then, by Eq. (7.34) we obtain

$$\mathbf{A}^n = \left(\frac{1}{2}\right)^n\mathbf{E}_1 + \left(\frac{1}{4}\right)^n\mathbf{E}_2 = \left(\frac{1}{2}\right)^n\begin{bmatrix} -1 & 4 \\ -\frac{1}{2} & 2 \end{bmatrix} + \left(\frac{1}{4}\right)^n\begin{bmatrix} 2 & -4 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\left(\frac{1}{2}\right)^n + 2\left(\frac{1}{4}\right)^n & 4\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n \\ -\frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{4}\right)^n & 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \end{bmatrix}$$

7.23. Repeat Prob. 7.20 using the z-transform method.

First, we must find $(z\mathbf{I} - \mathbf{A})^{-1}$.

$$(z\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix} z & -1 \\ \frac{1}{8} & z-\frac{3}{4} \end{bmatrix}^{-1}=\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\begin{bmatrix} z-\frac{3}{4} & 1 \\ -\frac{1}{8} & z \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{z-\frac{3}{4}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} & \dfrac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \\ \dfrac{-\frac{1}{8}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} & \dfrac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \end{bmatrix}$$

$$=\begin{bmatrix} -\dfrac{1}{z-\frac{1}{2}}+2\dfrac{1}{z-\frac{1}{4}} & 4\dfrac{1}{z-\frac{1}{2}}-4\dfrac{1}{z-\frac{1}{4}} \\ -\dfrac{1}{2}\dfrac{1}{z-\frac{1}{2}}+\dfrac{1}{2}\dfrac{1}{z-\frac{1}{4}} & 2\dfrac{1}{z-\frac{1}{2}}-\dfrac{1}{z-\frac{1}{4}} \end{bmatrix}$$

Then by Eq. (7.35) we obtain

$$\mathbf{A}^n=\mathfrak{Z}_I^{-1}\{(z\mathbf{I}-\mathbf{A})^{-1}z\}$$

$$=\mathfrak{Z}_I^{-1}\begin{bmatrix} -\dfrac{z}{z-\frac{1}{2}}+2\dfrac{z}{z-\frac{1}{4}} & 4\dfrac{z}{z-\frac{1}{2}}-4\dfrac{z}{z-\frac{1}{4}} \\ -\dfrac{1}{2}\dfrac{z}{z-\frac{1}{2}}+\dfrac{1}{2}\dfrac{z}{z-\frac{1}{4}} & 2\dfrac{z}{z-\frac{1}{2}}-\dfrac{z}{z-\frac{1}{4}} \end{bmatrix}$$

$$=\begin{bmatrix} -\left(\frac{1}{2}\right)^n+2\left(\frac{1}{4}\right)^n & 4\left(\frac{1}{2}\right)^n-4\left(\frac{1}{4}\right)^n \\ -\frac{1}{2}\left(\frac{1}{2}\right)^n+\frac{1}{2}\left(\frac{1}{4}\right)^n & 2\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n \end{bmatrix}$$

$$=\left(\frac{1}{2}\right)^n\begin{bmatrix} -1 & 4 \\ -\frac{1}{2} & 2 \end{bmatrix}+\left(\frac{1}{4}\right)^n\begin{bmatrix} 2 & -4 \\ \frac{1}{2} & -1 \end{bmatrix}$$

From the above results we note that when the eigenvalues of **A** are all distinct, the spectral decomposition method is computationally the most efficient method of evaluating $\mathbf{A}^n$.

7.24. Find $\mathbf{A}^n$ for

$$\mathbf{A}=\begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{4}{3} \end{bmatrix}$$

The characteristic polynomial $c(\lambda)$ of **A** is

$$c(\lambda)=|\lambda\mathbf{I}-\mathbf{A}|=\begin{vmatrix} \lambda & -1 \\ \frac{1}{3} & \lambda-\frac{4}{3} \end{vmatrix}$$

$$=\lambda^2-\frac{4}{3}\lambda+\frac{1}{3}=(\lambda-1)\left(\lambda-\frac{1}{3}\right)$$

Thus, the eigenvalues of **A** are $\lambda_1 = 1$ and $\lambda_2=\frac{1}{3}$, and by Eq. (7.33) we have

$$\mathbf{E}_1=\frac{1}{\lambda_1-\lambda_2}(\mathbf{A}-\lambda_2\mathbf{I})=\frac{1}{1-\frac{1}{3}}\left(\mathbf{A}-\frac{1}{3}\mathbf{I}\right)=\frac{3}{2}\begin{bmatrix} -\frac{1}{3} & 1 \\ -\frac{1}{3} & 1 \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$\mathbf{E}_2=\frac{1}{\lambda_2-\lambda_1}(\mathbf{A}-\lambda_1\mathbf{I})=\frac{1}{\frac{1}{3}-1}(\mathbf{A}-\mathbf{I})=-\frac{3}{2}\begin{bmatrix} -1 & 1 \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}=\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Thus, by Eq. (7.34) we obtain

$$\mathbf{A}^n=(1)^n\mathbf{E}_1+\left(\frac{1}{3}\right)^n\mathbf{E}_2=\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}+\left(\frac{1}{3}\right)^n\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$=\begin{bmatrix} -\frac{1}{2}+\frac{3}{2}\left(\frac{1}{3}\right)^n & \frac{3}{2}-\frac{3}{2}\left(\frac{1}{3}\right)^n \\ -\frac{1}{2}+\frac{1}{2}\left(\frac{1}{3}\right)^n & \frac{3}{2}-\frac{1}{2}\left(\frac{1}{3}\right)^n \end{bmatrix}$$

7.25. Find $\mathbf{A}^n$ for

$$\mathbf{A}=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

The characteristic polynomial $c(\lambda)$ of **A** is

$$c(\lambda)=|\lambda\mathbf{I}-\mathbf{A}|=\begin{vmatrix} \lambda-2 & 1 \\ 0 & \lambda-2 \end{vmatrix}=(\lambda-2)^2$$

Thus, the eigenvalues of **A** are $\lambda_1 = \lambda_2 = 2$. We use the Cayley-Hamilton theorem to evaluate $\mathbf{A}^n$. By Eq. (7.27) we have

$$\mathbf{A}^n = b_0\mathbf{I} + b_1\mathbf{A} = \begin{bmatrix} b_0 + 2b_1 & b_1 \\ 0 & b_0 + 2b_1 \end{bmatrix}$$

where b_0 and b_1 are determined by setting λ = 2 in the following equations [App. A, Eqs. (A.59) and (A.60)]:

$$b_0 + b_1\lambda = \lambda^n$$
$$b_1 = n\lambda^{n-1}$$

Thus,

$$b_0 + 2b_1 = 2^n$$
$$b_1 = n2^{n-1}$$

from which we get

$$b_0 = (1-n)2^n \qquad b_1 = n2^{n-1}$$

and

$$\mathbf{A}^n = \begin{bmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{bmatrix}$$

7.26. Consider the matrix **A** in Prob. 7.25. Let **A** be decomposed as

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathbf{D} + \mathbf{N}$$

where

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

a. Show that $\mathbf{N}^2 = \mathbf{0}$.

b. Show that **D** and **N** commute, that is, **DN** = **ND**.

c. Using the results from parts (*a*) and (*b*), find $\mathbf{A}^n$.

a. By simple multiplication we see that

$$\mathbf{N}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

b. Since the diagonal matrix **D** can be expressed as **2I**, we have

$$\mathbf{DN} = 2\mathbf{IN} = 2\mathbf{N} = 2\mathbf{NI} = \mathbf{N}(2\mathbf{I}) = \mathbf{ND}$$

that is, **D** and **N** commute.

c. Using the binomial expansion and the result from part(*b*), we can write

$$(\mathbf{D}+\mathbf{N})^n = \mathbf{D}^n + n\mathbf{D}^{n-1}\mathbf{N} + \frac{n(n-1)}{2!}\mathbf{D}^{n-2}\mathbf{N}^2 + \cdots + \mathbf{N}^n$$

Since $\mathbf{N}^2 = 0$, then $\mathbf{N}^k = 0$ for $k \geq 2$, and we have

$$\mathbf{A}^n = (\mathbf{D} + \mathbf{N})^n = \mathbf{D}^n + n\mathbf{D}^{n-1}\mathbf{N}$$

Thus [see App. A, Eq. (A.43)],

$$\begin{aligned}\mathbf{A}^n &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^n + n\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{n-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix} + n\begin{bmatrix} 2^{n-1} & 0 \\ 0 & 2^{n-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix} + n\begin{bmatrix} 0 & 2^{n-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{bmatrix}\end{aligned}$$

which is the same result obtained in Prob. 7.25.

Note that a square matrix $\mathbf{N}$ is called *nilpotent of index r* if $\mathbf{N}^{r-1} \neq \mathbf{0}$ and $\mathbf{N}^r = \mathbf{0}$.

7.27. The *minimal polynomial* $m(\lambda)$ of $\mathbf{A}$ is the polynomial of lowest order having 1 as its leading coefficient such that $m(\mathbf{A}) = \mathbf{0}$. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

a. Find the minimal polynomial $m(\lambda)$ of $\mathbf{A}$.

b. Using the result from part (*a*), find $\mathbf{A}^n$.

a. The characteristic polynomial $c(\lambda)$ of $\mathbf{A}$ is

$$c(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda + 2 & -1 \\ 0 & -4 & \lambda - 1 \end{vmatrix} = (\lambda + 3)(\lambda - 2)^2$$

Thus, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -3$ and $\lambda_2 = \lambda_3 = 2$. Consider

$$m(\lambda) = (\lambda + 3)(\lambda - 2) = \lambda^2 + \lambda - 6$$

Now

$$m(\mathbf{A}) = \mathbf{A}^2 + \mathbf{A} - 6\mathbf{I} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}^2 + \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

Thus, the minimal polynomial of **A** is

$$m(\lambda) = (\lambda + 3)(\lambda - 2) = \lambda^2 + \lambda - 6$$

b. From the result from part (*a*) we see that $\mathbf{A}^n$ can be expressed as a linear combination of **I** and **A** only, even though the order of **A** is 3. Thus, similar to the result from the Cayley-Hamilton theorem, we have

$$\mathbf{A}^n = b_0\mathbf{I} + b_1\mathbf{A} = \begin{bmatrix} b_0 & b_1 \\ -\frac{1}{8}b_1 & b_0 + \frac{3}{4}b_1 \end{bmatrix}$$

where b_0 and b_1 are determined by setting $\lambda = -3$ and $\lambda = 2$ in the equation

$$b_0 + b_1\lambda = \lambda^n$$

Thus,

$$b_0 - 3b_1 = (-3)^n$$
$$b_0 + 2b_1 = 2^n$$

from which we get

$$b_0 = \frac{2}{5}(-3)^n + \frac{3}{5}(2)^n \qquad b_1 = -\frac{1}{5}(-3)^n + \frac{1}{5}(2)^n$$

and

$$\mathbf{A}^n = \begin{bmatrix} (2)^n & 0 & 0 \\ 0 & \frac{4}{5}(-3)^n + \frac{1}{5}(2)^n & -\frac{1}{5}(-3)^n + \frac{1}{5}(2)^n \\ 0 & -\frac{4}{5}(-3)^n + \frac{4}{5}(2)^n & \frac{1}{5}(-3)^n + \frac{4}{5}(2)^n \end{bmatrix}$$

$$= (-3)^n \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{1}{5} \\ 0 & -\frac{4}{5} & \frac{1}{5} \end{bmatrix} + (2)^n \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{4}{5} & \frac{4}{5} \end{bmatrix}$$

7.28. Using the spectral decomposition method, evaluate $\mathbf{A}^n$ for matrix **A** in Prob. 7.27.

Since the minimal polynomial of **A** is

$$m(\lambda) = (\lambda + 3)(\lambda - 2) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

which contains only simple factors, we can apply the spectral decomposition method to evaluate $\mathbf{A}^n$. Thus, by Eq. (7.33) we have

$$\mathbf{E}_1 = \frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_2 \mathbf{I}) = \frac{1}{-3-2}(\mathbf{A} - 2\mathbf{I})$$

$$= -\frac{1}{5}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{1}{5} \\ 0 & -\frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

$$\mathbf{E}_2 = \frac{1}{\lambda_2 - \lambda_1}(\mathbf{A} - \lambda_1 \mathbf{I}) = \frac{1}{2-(-3)}(\mathbf{A} + 3\mathbf{I})$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{4}{5} & \frac{4}{5} \end{bmatrix}$$

Thus, by Eq. (7.34) we get

$$\mathbf{A}^n = (-3)^n \mathbf{E}_1 + (2)^n \mathbf{E}_2$$

$$= (-3)^n \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{1}{5} \\ 0 & -\frac{4}{5} & \frac{1}{5} \end{bmatrix} + (2)^n \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{4}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} (2)^n & 0 & 0 \\ 0 & \frac{4}{5}(-3)^n + \frac{1}{5}(2)^n & -\frac{1}{5}(-3)^n + \frac{1}{5}(2)^n \\ 0 & -\frac{4}{5}(-3)^n + \frac{4}{5}(2)^n & \frac{1}{5}(-3)^n + \frac{4}{5}(2)^n \end{bmatrix}$$

which is the same result obtained in Prob. 7.27(*b*).

7.29. Consider the discrete-time system in Prob. 7.7. Assume that the system is initially relaxed.

a. Using the state space representation, find the unit step response of the system.

b. Find the system function $H(z)$.

a. From the result of Prob. 7.7 we have

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$y[n] = \mathbf{c}\mathbf{q}[n] + dx[n]$$

where

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -\frac{1}{8} & \frac{3}{4}\end{bmatrix} \qquad \mathbf{b}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \mathbf{c}=\begin{bmatrix}-\frac{1}{8} & \frac{3}{4}\end{bmatrix} \qquad d=1$$

Setting **q**[0] = **0** and $x[n] = u[n]$ in Eq. (7.25), the unit step response $s[n]$ is given by

$$s[n]=\sum_{k=0}^{n-1}\mathbf{c}\mathbf{A}^{n-1-k}\mathbf{b}u[k]+du[n]$$

(7.116)

Now, from Prob. 7.20 we have

$$\mathbf{A}^n=\left(\frac{1}{2}\right)^n\begin{bmatrix}-1 & 4\\ -\frac{1}{2} & 2\end{bmatrix}+\left(\frac{1}{4}\right)^n\begin{bmatrix}2 & -4\\ \frac{1}{2} & -1\end{bmatrix}$$

and

$$\begin{aligned}\mathbf{c}\mathbf{A}^{n-1-k}\mathbf{b}&=\begin{bmatrix}-\frac{1}{8} & \frac{3}{4}\end{bmatrix}\left\{\left(\frac{1}{2}\right)^{n-1-k}\begin{bmatrix}-1 & 4\\ -\frac{1}{2} & 2\end{bmatrix}+\left(\frac{1}{4}\right)^{n-1-k}\begin{bmatrix}2 & -4\\ \frac{1}{2} & -1\end{bmatrix}\right\}\begin{bmatrix}0\\1\end{bmatrix}\\ &=\left(\frac{1}{2}\right)^{n-1-k}\begin{bmatrix}-\frac{1}{8} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}-1 & 4\\ -\frac{1}{2} & 2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ &\quad+\left(\frac{1}{4}\right)^{n-1-k}\begin{bmatrix}-\frac{1}{8} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}2 & -4\\ \frac{1}{2} & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ &=\left(\frac{1}{2}\right)^{n-1-k}-\frac{1}{4}\left(\frac{1}{4}\right)^{n-1-k}=2\left(\frac{1}{2}\right)^{n-k}-\left(\frac{1}{4}\right)^{n-k}\end{aligned}$$

Thus,

$$s[n]=\sum_{k=0}^{n-1}\left[2\left(\frac{1}{2}\right)^{n-k}-\left(\frac{1}{4}\right)^{n-k}\right]+1$$

$$=2\left(\frac{1}{2}\right)^{n}\sum_{k=0}^{n-1}2^{k}-\left(\frac{1}{4}\right)^{n}\sum_{k=0}^{n-1}4^{k}+1$$

$$=2\left(\frac{1}{2}\right)^{n}\left(\frac{1-2^{n}}{1-2}\right)-\left(\frac{1}{4}\right)^{n}\left(\frac{1-4^{n}}{1-4}\right)+1$$

$$=-2\left(\frac{1}{2}\right)^{n}+2+\frac{1}{3}\left(\frac{1}{4}\right)^{n}-\frac{1}{3}+1$$

$$=\frac{8}{3}-2\left(\frac{1}{2}\right)^{n}+\frac{1}{3}\left(\frac{1}{4}\right)^{n}\qquad n\geq 0$$

which is the same result obtained in Prob. 4.32(*c*)

b. By Eq. (7.44) the system function $H(z)$ is given by

$$H(z)=\mathbf{c}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}+d$$

Now

$$(z\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix} z & -1 \\ \frac{1}{8} & z-\frac{3}{4}\end{bmatrix}^{-1}=\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\begin{bmatrix} z-\frac{3}{4} & 1 \\ -\frac{1}{8} & z\end{bmatrix}$$

Thus,

$$H(z)=\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\begin{bmatrix}-\frac{1}{8}, & \frac{3}{4}\end{bmatrix}\begin{bmatrix} z-\frac{3}{4} & 1 \\ -\frac{1}{8} & z\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+1$$

$$=\frac{-\frac{1}{8}+\frac{3}{4}z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}+1=\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

which is the same result obtained in Prob. 4.32(*a*).

7.30. Consider the discrete-time LTI system described by

$$\mathbf{q}[n+1]=\mathbf{A}\mathbf{q}[n]+\mathbf{b}x[n]$$
$$y[n]=\mathbf{c}\mathbf{q}[n]+dx[n]$$

a. Show that the unit impulse response $h[n]$ of the system is given by

$$h[n]=\begin{cases} d & n=0 \\ \mathbf{c}\mathbf{A}^{n-1}\mathbf{b} & n>0 \\ 0 & n<0 \end{cases}$$

(7.117)

b. Using Eq. (7.117), find the unit impulse response $h[n]$ of the system in Prob. 7.29.

a. By setting $\mathbf{q}[0] = \mathbf{0}$, $x[k]$, $= \delta[k]$, and $x[n] = \delta[n]$ in Eq. (7.25), we obtain

$$h[n]=\sum_{k=0}^{n-1}\mathbf{c}\mathbf{A}^{n-1-k}\mathbf{b}\delta[k]+d\delta[n]$$

(7.118)

Note that the sum in Eq. (7.118) has no terms for $n = 0$ and that the first term is $\mathbf{c}\mathbf{A}^{n-1}\mathbf{b}$ for $n > 0$. The second term on the right-hand side of Eq. (7.118) is equal to d for $n = 0$ and zero otherwise. Thus, we conclude that

$$h[n]=\begin{cases} d & n=0 \\ \mathbf{c}\mathbf{A}^{n-1}\mathbf{b} & n>0 \\ 0 & n<0 \end{cases}$$

b. From the result of Prob. 7.29 we have

$$\mathbf{A}=\begin{bmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{bmatrix} \qquad \mathbf{b}=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}=\begin{bmatrix} -\frac{1}{8} & \frac{3}{4} \end{bmatrix} \qquad d=1$$

and

$$\mathbf{c}\mathbf{A}^{n-1}\mathbf{b}=\left(\frac{1}{2}\right)^{n-1}-\frac{1}{4}\left(\frac{1}{4}\right)^{n-1} \qquad n\geq 1$$

Thus, by Eq. (7.117)$h[n]$ is

$$h[n]=\begin{cases} 1 & n=0 \\ \left(\frac{1}{2}\right)^{n-1}-\left(\frac{1}{4}\right)^{n-1} & n\geq 1 \\ 0 & n<0 \end{cases}$$

which is the same result obtained in Prob. 4.32(b).

7.31. Use the state space method to solve the difference equation [Prob. 4.38(b)]

$$3y[n]-4y[n-1]+y[n-2]=x[n]$$

(7.119)

with $x[n]=\left(\frac{1}{2}\right)^n u[n]$ and $y[-1] = 1, y[-2] = 2$.

Rewriting Eq. (7.119), we have

$$y[n] - \frac{4}{3} y[n-1] + \frac{1}{3} y[n-2] = \frac{1}{3} x[n]$$

Let $q_1[n] = y[n-2]$ and $q_2[n] = y[n-1]$. Then

$$q_1[n+1] = q_2[n]$$

$$q_2[n+1] = -\frac{1}{3} q_1[n] + \frac{4}{3} q_2[n] + \frac{1}{3} x[n]$$

$$y[n] = -\frac{1}{3} q_1[n] + \frac{4}{3} q_2[n] + \frac{1}{3} x[n]$$

In matrix form

$$\mathbf{q}_1[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$y[n] = \mathbf{c}\mathbf{q}[n] + dx[n]$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{4}{3} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \end{bmatrix} \qquad d = \frac{1}{3}$$

and

$$\mathbf{q}[0] = \begin{bmatrix} q_1[0] \\ q_2[0] \end{bmatrix} = \begin{bmatrix} y[-2] \\ y[-1] \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then, by Eq. (7.25)

$$y[n] = \mathbf{c}\mathbf{A}^n \mathbf{q}[0] + \sum_{k=0}^{n-1} \mathbf{c}\mathbf{A}^{n-1-k} \mathbf{b}x[k] + dx[n] \qquad n > 0$$

Now from the result of Prob. 7.24 we have

$$\mathbf{A}^n = \begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{4}{3} \end{bmatrix}^n = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} + \left(\frac{1}{3}\right)^n \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and

$$\mathbf{c}\mathbf{A}^n\mathbf{q}[0] = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \end{bmatrix} \left\{ \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} + \left(\frac{1}{3}\right)^n \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} + \frac{1}{6}\left(\frac{1}{3}\right)^n$$

$$\mathbf{c}\mathbf{A}^{n-1-k}\mathbf{b} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \end{bmatrix} \left\{ \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} + \left(\frac{1}{3}\right)^{n-1-k} \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \right\} \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}$$

$$= \frac{1}{2} - \frac{1}{18}\left(\frac{1}{3}\right)^{n-1-k} = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^{n+1-k}$$

Thus,

$$y[n] = \frac{1}{2} + \frac{1}{6}\left(\frac{1}{3}\right)^n + \sum_{k=0}^{n-1}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^{n+1-k}\right]\left(\frac{1}{2}\right)^k + \frac{1}{3}\left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2} + \frac{1}{6}\left(\frac{1}{3}\right)^n + \frac{1}{2}\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k - \frac{1}{2}\left(\frac{1}{3}\right)^{n+1}\sum_{k=0}^{n-1}\left(\frac{3}{2}\right)^k + \frac{1}{3}\left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2} + \frac{1}{6}\left(\frac{1}{3}\right)^n + \frac{1}{2}\left[\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right] - \frac{1}{2}\left(\frac{1}{3}\right)^{n+1}\left[\frac{1-\left(\frac{3}{2}\right)^n}{1-\frac{3}{2}}\right] + \frac{1}{3}\left(\frac{1}{2}\right)^n$$

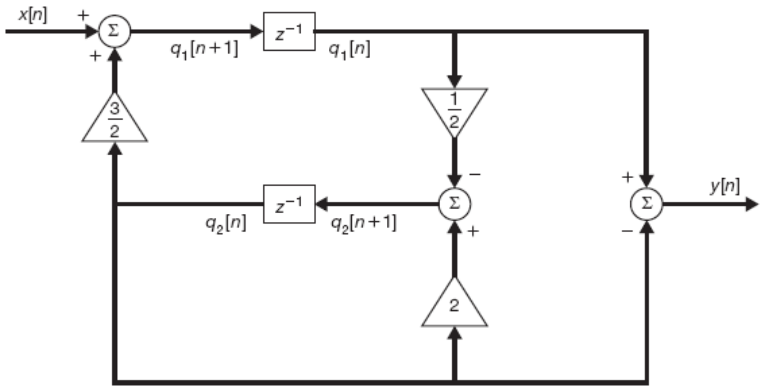
$$= \frac{1}{2} + \frac{1}{6}\left(\frac{1}{3}\right)^n + 1 - \left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{3}\right)^n - \frac{1}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n$$

$$= \frac{3}{2} - \left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{3}\right)^n \qquad n > 0$$

which is the same result obtained in Prob. 4.38(*b*).

7.32. Consider the discrete-time **LTI** system shown in Fig. 7-19.

Figure 7-19



a. Is the system asymptotically stable?

b. Find the system function $H(z)$.

c. Is the system **BIBO** stable?

a. From Fig. 7-19 and choosing the state variables $q_1[n]$ and $q_2[n]$ as shown, we obtain

$$q_1[n+1] = \frac{3}{2}q_2[n] + x[n]$$

$$q_2[n+1] = -\frac{1}{2}q_1[n] + 2q_2[n]$$

$$y[n] = q_1[n] - q_2[n]$$

In matrix form

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$y[n] = \mathbf{c}\mathbf{q}[n]$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{3}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Now

$$c(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -\frac{3}{2} \\ \frac{1}{2} & \lambda - 2 \end{vmatrix} = \lambda(\lambda - 2) + \frac{3}{4} = \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right)$$

Thus, the eigenvalues of **A** are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{3}{2}$. Since $|\lambda_2| > 1$, the system is not asymptotically stable.

b. By Eq. (7.44) the system function $H(z)$ is given by

$$H(z) = \mathbf{c}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} = [1 \quad -1]\begin{bmatrix} z & -\frac{3}{2} \\ \frac{1}{2} & z-2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)}[1 \quad -1]\begin{bmatrix} z-2 & \frac{3}{2} \\ -\frac{1}{2} & z \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{z-\frac{3}{2}}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)} = \frac{1}{z-\frac{1}{2}}$$

c. Note that there is pole-zero cancellation in $H(z)$ at $z = \frac{3}{2}$. Thus, the only pole of $H(z)$ is $\frac{1}{2}$, which lies inside the unit circle of the z-plane. Hence, the system is BIBO stable.

Note that even though the system is BIBO stable, it is essentially unstable if it is not initially relaxed.

7.33. Consider an Nth-order discrete-time LTI system with the state equation

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

The system is said to be *controllable* if it is possible to find a sequence of N input samples $x[n_0]$, $x[n_0+1]$, ..., $x[n_0+N-1]$ such that it will drive the system from $\mathbf{q}[n_0] = \mathbf{q}_0$ to $\mathbf{q}[n_0+N] = \mathbf{q}_1$ and $\mathbf{q}_0$ and $\mathbf{q}_1$ are any finite states. Show that the system is controllable if the *controllability matrix* defined by

$$\mathbf{M}_c = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \cdots \quad \mathbf{A}^{N-1}\mathbf{b}]$$

(7.120)

has rank N.

We assume that $n_0 = 0$ and $\mathbf{q}[0] = \mathbf{0}$. Then, by Eq. (7.23) we have

$$\mathbf{q}[N] = \sum_{k=0}^{N-1} \mathbf{A}^{N-1-k}\,\mathbf{b}x[k]$$

(7.121)

which can be rewritten as

$$\mathbf{q}[N] = [\mathbf{b} \ \ \mathbf{A}\mathbf{b} \ \ \cdots \ \ \mathbf{A}^{N-1}\mathbf{b}]\begin{bmatrix} x[N-1] \\ x[N-1] \\ \vdots \\ x[0] \end{bmatrix}$$

(7.122)

Thus, if $\mathbf{q}[N]$ is to be an arbitrary N-dimensional vector and also to have a nonzero input sequence, as required for controllability, the coefficient matrix in Eq. (7.122) must be nonsingular; that is, the matrix

$$\mathbf{M}_c = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \cdots \quad \mathbf{A}^{N-1}\mathbf{b}]$$

must have rank N.

7.34. Consider an Nth-order discrete-time LTI system with state space representation

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$
$$y[n] = \mathbf{c}\mathbf{q}[n]$$

The system is said to be *observable* if, starting at an arbitrary time index n_0, it is possible to determine the state $\mathbf{q}[n_0] = \mathbf{q}_0$ from the output sequence $y[n_0], y[n_0+1], \ldots, y[n_0+N-1]$. Show that the system is observable if the *observability matrix* defined by

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \vdots \\ \mathbf{c}\mathbf{A}^{N-1} \end{bmatrix}$$

(7.123)

has rank N.

We assume that $n_0 = 0$ and $x[n] = 0$. Then, by Eq. (7.25) the output $y[n]$ for $n = 0, 1, \ldots, N-1$, with $x[n] = 0$, is given by

$$y[n] = \mathbf{c}\mathbf{A}^n\mathbf{q}[0] \qquad n = 0, 1, \ldots, N-1$$

(7.124)

or

$$y[0] = \mathbf{c}\mathbf{q}[0]$$
$$y[1] = \mathbf{c}\mathbf{A}\mathbf{q}[0]$$
$$\vdots$$
$$y[N-1] = \mathbf{c}\mathbf{A}^{N-1}\mathbf{q}[0]$$

(7.125)

Rewriting Eq. (7.125) as a matrix equation, we get

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \vdots \\ \mathbf{c}\mathbf{A}^{N-1} \end{bmatrix} \mathbf{q}[0]$$

(7.126)

Thus, to find a unique solution for $\mathbf{q}[0]$, the coefficient matrix of Eq. (7.126) must be nonsingular; that is, the matrix

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \vdots \\ \mathbf{cA}^{N-1} \end{bmatrix}$$

must have rank *N*.

7.35. Consider the system in Prob. 7.7.

a. Is the system controllable?

b. Is the system observable?

c. Find the system function $H(z)$.

a. From the result of Prob. 7.7 we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{4} \end{bmatrix} \qquad d = 1$$

Now

$$\mathbf{Ab} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}$$

and by Eq. (7.120) the controllability matrix is

$$\mathbf{M}_c = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & \frac{3}{4} \end{bmatrix}$$

and $|\mathbf{M}_c| = -1 \neq 0$. Thus, its rank is 2, and hence the system is controllable.

b. Similarly,

$$\mathbf{cA} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} -\frac{3}{32} & \frac{7}{16} \end{bmatrix}$$

and by Eq. (7.123) the observability matrix is

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{4} \\ -\frac{3}{32} & \frac{7}{16} \end{bmatrix}$$

and $|\mathbf{M}_o| = -\frac{1}{64} \neq 0$. Thus, its rank is 2, and hence the system is observable.

c. By Eq. (7.44) the system function $H(z)$ is given by

$$H(z)=\mathbf{c}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}+d=\begin{bmatrix}-\frac{1}{8} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}z & -1\\ \frac{1}{8} & z-\frac{3}{4}\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}+1$$

$$=\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\begin{bmatrix}-\frac{1}{8} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}z-\frac{3}{4} & 1\\ -\frac{1}{8} & z\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+1$$

$$=\frac{\frac{3}{4}z-\frac{1}{8}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}+1=\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$=\frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

7.36. Consider the system in Prob. 7.7. Assume that

$$\mathbf{q}[0]=\begin{bmatrix}0\\1\end{bmatrix}$$

Find $x[0]$ and $x[1]$ such that $\mathbf{q}[2]=\mathbf{0}$.

From Eq. (7.23) we have

$$\mathbf{q}[2]=\mathbf{A}^2\mathbf{q}[0]+\mathbf{A}\mathbf{b}x[0]+\mathbf{b}x[1]=\mathbf{A}^2\mathbf{q}[0]+\begin{bmatrix}\mathbf{b} & \mathbf{A}\mathbf{b}\end{bmatrix}\begin{bmatrix}x[1]\\x[0]\end{bmatrix}$$

Thus,

$$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{1}{8} & \frac{3}{4}\end{bmatrix}^2\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}0 & 1\\ 1 & \frac{3}{4}\end{bmatrix}\begin{bmatrix}x[1]\\x[0]\end{bmatrix}$$

$$=\begin{bmatrix}\frac{3}{4}\\ \frac{7}{16}\end{bmatrix}+\begin{bmatrix}x[0]\\ x[1]+\frac{3}{4}x[0]\end{bmatrix}$$

from which we obtain $x[0]=-\frac{3}{4}$ and $x[1]=\frac{1}{8}$.

7.37. Consider the system in Prob. 7.7. We observe $y[0]=1$ and $y[1]=0$ with $x[0]=x[1]=0$.

Find the initial state $\mathbf{q}[0]$.

Using Eq. (7.125), we have

$$\begin{bmatrix}y[0]\\y[1]\end{bmatrix}=\begin{bmatrix}\mathbf{c}\\ \mathbf{c}\mathbf{A}\end{bmatrix}\mathbf{q}[0]$$

Thus,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{4} \\ -\frac{3}{32} & \frac{7}{16} \end{bmatrix} \begin{bmatrix} q_1[0] \\ q_2[0] \end{bmatrix}$$

Solving for $q_1[0]$ and $q_2[0]$, we obtain

$$\mathbf{q}[0] = \begin{bmatrix} q_1[0] \\ q_2[0] \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{3}{4} \\ -\frac{3}{32} & \frac{7}{16} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 28 \\ 6 \end{bmatrix}$$

7.38. Consider the system in Prob. 7.32.

a. Is the system controllable?

b. Is the system observable?

a. From the result of Prob. 7.32 we have

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{3}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Now

$$\mathbf{Ab} = \begin{bmatrix} 0 & \frac{3}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$$

and by Eq. (7.120) the controllability matrix is

$$\mathbf{M}_c = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

and $|\mathbf{M}_c| = -\frac{1}{2} \neq 0$. Thus, its rank is 2, and hence the system is controllable.

b. Similarly,

$$\mathbf{cA} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and by Eq. (7.123) the observability matrix is

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and $|\mathbf{M}_o| = 0$. Thus, its rank is less than 2, and hence the system is not observable.

Note from the result from Prob. 7.32(b) that the system function $H(z)$ has pole-zero cancellation. If $H(z)$ has pole-zero cancellation, then the system cannot be both controllable and observable.

7.7.5. Solutions of State Equations for Continuous-Time LTI Systems

7.39. Find $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

using the Cayley-Hamilton theorem method.

First, we find the characteristic polynomial $c(\lambda)$ of $\mathbf{A}$.

$$c(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ 6 & \lambda + 5 \end{vmatrix}$$
$$= \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3)$$

Thus, the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -2$ and $\lambda_2 = -3$. Hence, by Eqs. (7.66) and (7.67) we have

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} = \begin{bmatrix} b_0 & b_1 \\ -6b_1 & b_0 - 5b_1 \end{bmatrix}$$

and b_0 and b_1 are the solutions of

$$b_0 - 2b_1 = e^{-2t}$$
$$b_0 - 3b_1 = e^{-3t}$$

from which we get

$$b_0 = 3e^{-2t} - 2e^{-3t} \qquad b_1 = e^{-2t} - e^{-3t}$$

Hence,

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$
$$= e^{-2t}\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3t}\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$$

7.40. Repeat Prob. 7.39 using the diagonalization method.

Let $\mathbf{x}$ be an eigenvector of $\mathbf{A}$ associated with λ. Then

$$[\lambda\mathbf{I} - \mathbf{A}]\mathbf{x} = \mathbf{0}$$

For λ = λ_1 = −2 we have

$$\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions of this system are given by $x_2 = -2x_1$. Thus, the eigenvectors associated with λ_1 are those vectors of the form

$$\mathbf{x}_1 = \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{with } \alpha \neq 0$$

For λ = λ_2 = −3 we have

$$\begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions of this system are given by $x_2 = -3x_1$. Thus, the eigenvectors associated with λ_2 are those vectors of the form

$$\mathbf{x}_2 = \beta \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \text{with } \beta \neq 0$$

Let $\alpha = \beta = 1$ in the above expressions and let

$$\mathbf{P} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$$

Then

$$\mathbf{P}^{-1} = -\begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$$

and by Eq. (7.68) we obtain

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3t}\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$$

7.41. Repeat Prob. 7.39 using the spectral decomposition method.

Since all eigenvalues of **A** are distinct, by Eq. (7.33) we have

$$\mathbf{E}_1 = \frac{1}{\lambda_1 - \lambda_2}(\mathbf{A} - \lambda_2\mathbf{I}) = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$$

$$\mathbf{E}_2 = \frac{1}{\lambda_2 - \lambda_1}(\mathbf{A} - \lambda_1\mathbf{I}) = -(\mathbf{A} + 2\mathbf{I}) = \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$$

Then by Eq. (7.70) we obtain

$$e^{\mathbf{A}t} = e^{-2t}\mathbf{E}_1 + e^{-3t}\mathbf{E}_2 = e^{-2t}\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3t}\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

7.42. Repeat Prob. 7.39 using the Laplace transform method.

First, we must find $(s\mathbf{I} - \mathbf{A})^{-1}$.

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}^{-1} = \frac{1}{(s+2)(s+3)}\begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ -\dfrac{6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{3}{s+2} - \dfrac{2}{s+3} & \dfrac{1}{s+2} - \dfrac{1}{s+3} \\ -\dfrac{6}{s+2} + \dfrac{6}{s+3} & -\dfrac{2}{s+2} + \dfrac{3}{s+3} \end{bmatrix}$$

Then, by Eq. (7.71) we obtain

$$e^{\mathbf{A}t} = \mathscr{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

Again we note that when the eigenvalues of **A** are all distinct, the spectral decomposition method is computationally the most efficient method of evaluating $e^{\mathbf{A}t}$.

7.43. Find $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

The characteristic polynomial $c(\lambda)$ of **A** is

$$c(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda+2 & -1 \\ -1 & \lambda+2 \end{vmatrix}$$

$$= \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3)$$

Thus, the eigenvalues of **A** are $\lambda_1 = -1$ and $\lambda_2 = -3$. Since all eigenvalues of **A** are distinct, by Eq. (7.33) we have

$$\mathbf{E}_1 = -\frac{1}{2}(\mathbf{A}+3\mathbf{I}) = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\mathbf{E}_2 = -\frac{1}{2}(\mathbf{A}+\mathbf{I}) = -\frac{1}{2}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

Then, by Eq. (7.70) we obtain

$$e^{\mathbf{A}t} = e^{-t}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} + e^{-3t}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2}e^{-t}+\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t}\\ \frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t}+\frac{1}{2}e^{-3t}\end{bmatrix}$$

7.44. Given matrix

$$\mathbf{A} = \begin{bmatrix}0 & -2 & 1\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix}$$

a. Show that **A** is nilpotent of index 3.

b. Using the result from part (*a*) find $e^{\mathbf{A}t}$.

a. By direct multiplication we have

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{bmatrix}0 & -2 & 1\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & -2 & 1\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & -6\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix}0 & 0 & -6\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & -2 & 1\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Thus, **A** is nilpotent of index 3.

b. By definition (7.53) and the result from part (*a*)

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2}\mathbf{A}^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2t & t-3t^2 \\ 0 & 1 & 3t \\ 0 & 0 & 1 \end{bmatrix}$$

7.45. Find $e^{\mathbf{A}t}$ for matrix **A** in Prob. 7.44 using the Cayley-Hamilton theorem method.

First, we find the characteristic polynomial $c(\lambda)$ of **A**.

$$c(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & 2 & -1 \\ 0 & \lambda & -3 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3$$

Thus, $\lambda = 0$ is the eigenvalues of **A** with multiplicity 3. By Eq. (7.66) we have

$$e^{\mathbf{A}t} = b_0\mathbf{I} + b_1\mathbf{A} + b_2\mathbf{A}^2$$

where b_0, b_1, and b_2 are determined by setting $\lambda = 0$ in the following equations [App. A, Eqs. (A.59) and (A.60)]:

$$\begin{aligned} b_0 + b_1\lambda + b_2\lambda^2 &= e^{\lambda t} \\ b_1 + 2b_2\lambda &= te^{\lambda t} \\ 2b_2 &= t^2e^{\lambda t} \end{aligned}$$

Thus,

$$b_0 = 1 \qquad b_1 = t \qquad b_2 = \frac{t^2}{2}$$

Hence,

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2}\mathbf{A}^2$$

which is the same result obtained in Prob. 7.44(*b*).

7.46. Show that

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$$

provided **A** and **B** commute; that is, **AB** = **BA**.

By Eq. (7.53)

$$
\begin{aligned}
e^{\mathbf{A}}e^{\mathbf{B}} &= \left(\sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{A}^k\right)\left(\sum_{m=0}^{\infty}\frac{1}{m!}\mathbf{B}^m\right)\\
&= \left(\mathbf{I}+\mathbf{A}+\frac{1}{2!}\mathbf{A}^2+\cdots\right)\left(\mathbf{I}+\mathbf{B}+\frac{1}{2!}\mathbf{B}^2+\cdots\right)\\
&= \mathbf{I}+\mathbf{A}+\mathbf{B}+\frac{1}{2!}\mathbf{A}^2+\mathbf{AB}+\frac{1}{2!}\mathbf{B}^2+\cdots\\
e^{\mathbf{A}+\mathbf{B}} &= \mathbf{I}+(\mathbf{A}+\mathbf{B})+\frac{1}{2!}(\mathbf{A}+\mathbf{B})^2+\cdots\\
&= \mathbf{I}+\mathbf{A}+\mathbf{B}+\frac{1}{2!}\mathbf{A}^2+\frac{1}{2}\mathbf{AB}+\frac{1}{2}\mathbf{BA}+\frac{1}{2!}\mathbf{B}^2+\cdots
\end{aligned}
$$

and

$$e^{\mathbf{A}}e^{\mathbf{B}}-e^{\mathbf{A}+\mathbf{B}}=\frac{1}{2}(\mathbf{AB}-\mathbf{BA})+\cdots$$

Thus, if **AB** = **BA**, then

$$e^{\mathbf{A}+\mathbf{B}}=e^{\mathbf{A}}e^{\mathbf{B}}$$

7.47. Consider the matrix

$$\mathbf{A}=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$$

Now we decompose **A** as

$$\mathbf{A}=\mathbf{\Lambda}+\mathbf{N}$$

where

$$\mathbf{\Lambda}=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\qquad\text{and}\qquad\mathbf{N}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

a. Show that the matrix **N** is nilpotent of index 3.

b. Show that **Λ** and **N** commute; that is, **ΛN** = **NΛ**.

c. Using the results from parts (*a*) and (*b*), find $e^{\mathbf{A}t}$.

a. By direct multiplication we have

$$\mathbf{N}^2 = \mathbf{NN} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{N}^3 = \mathbf{N}^2\mathbf{N} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, **N** is nilpotent of index 3.

b. Since the diagonal matrix **Λ** can be expressed as 2**I**, we have

$$\mathbf{\Lambda N} = 2\mathbf{IN} = 2\mathbf{N} = 2\mathbf{NI} = \mathbf{N}(2\mathbf{I}) = \mathbf{N\Lambda}$$

that is, **Λ** and **N** commute.

c. Since **Λ** and **N** commute, then, by the result from Prob. 7.46

$$e^{\mathbf{A}t} = e^{(\mathbf{\Lambda} + \mathbf{N})t} = e^{\mathbf{\Lambda}t}e^{\mathbf{N}t}$$

Now [see App. A, Eq. (A.49)]

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{2t}\mathbf{I}$$

and using similar justification as in Prob. 7.44(*b*), we have

$$\begin{aligned} e^{\mathbf{N}t} &= \mathbf{I} + t\mathbf{N} + \frac{t^2}{2!}\mathbf{N}^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Thus,

$$e^{\mathbf{A}t} = e^{\mathbf{\Lambda}t}e^{\mathbf{N}t} = e^{2t}\mathbf{I}e^{\mathbf{N}t} = e^{2t}e^{\mathbf{N}t} = e^{2t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

7.48. Using the state variables method, solve the second-order linear differential equation

$$y''(t) + 5y'(t) + 6y(t) = x(t)$$

(7.127)

with the initial conditions $y(0) = 2, y'(0) = 1$, and $x(t) = e^{-t}u(t)$ (Prob. 3.38).

Let the state variables $q_1(t)$ and $q_2(t)$ be

$$q_1(t) = y(t) \qquad q_2(t) = y'(t)$$

Then the state space representation of Eq. (7.127) is given by [Eq. (7.19)]

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$
$$y(t) = \mathbf{c}\mathbf{q}(t)$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \mathbf{q}[0] = \begin{bmatrix} q_1[0] \\ q_2[0] \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus, by Eq. (7.65)

$$y(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{q}(0) + \int_0^t \mathbf{c}e^{\mathbf{A}(t-\tau)}\mathbf{b}x(\tau)\, d\tau$$

with $d = 0$. Now, from the result of Prob. 7.39,

$$e^{\mathbf{A}t} = e^{-2t}\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3t}\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{c}e^{\mathbf{A}t}\mathbf{q}(0) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\left\{ e^{-2t}\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3t}\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= 7e^{-2t} - 5e^{-3t} \end{aligned}$$

$$\begin{aligned} \mathbf{c}e^{\mathbf{A}(t-\tau)}\mathbf{b} &= \begin{bmatrix} 1 & 0 \end{bmatrix}\left\{ e^{-2(t-\tau)}\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3(t-\tau)}\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}\right\}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= e^{-2(t-\tau)} - e^{-3(t-\tau)} \end{aligned}$$

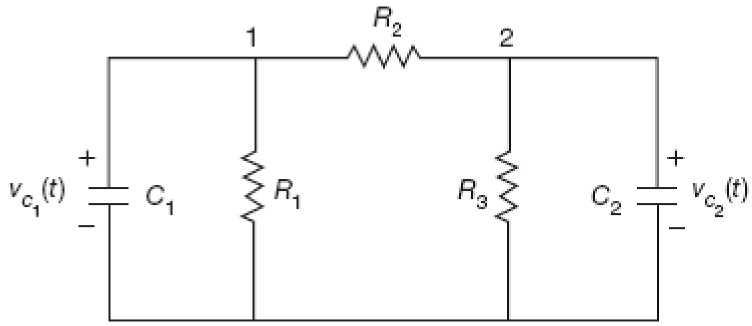
Thus,

$$\begin{aligned} y(t) &= 7e^{-2t} - 5e^{-3t} + \int_0^t (e^{-2(t-\tau)} - e^{-3(t-\tau)})e^{-\tau}\, d\tau \\ &= 7e^{-2t} - 5e^{-3t} + e^{-2t}\int_0^t e^{\tau}\, d\tau - e^{-3t}\int_0^t e^{2\tau} d\tau \\ &= \frac{1}{2}e^{-t} + 6e^{-2t} - \frac{9}{2}e^{-3t} \qquad t > 0 \end{aligned}$$

which is the same result obtained in Prob. 3.38.

7.49. Consider the network shown in Fig. 7-20. The initial voltages across the capacitors C_1 and C_2 are $\frac{1}{2}$ V and 1 V, respectively. Using the state variable method, find the voltages across these capacitors for $t > 0$. Assume that $R_1 = R_2 = R_3 = 1\ \Omega$ and $C_1 = C_2 = 1$ F.

Figure 7-20



Let the state variables $q_1(t)$ and $q_2(t)$ be

$$q_1(t) = v_{C_1}(t) \qquad q_2(t) = v_{C_2}(t)$$

Applying Kirchhoff's current law at nodes 1 and 2, we get

$$C_1\dot{q}_1(t) + \frac{q_1(t)}{R_1} + \frac{q_1(t) - q_2(t)}{R_2} = 0$$

$$C_2\dot{q}_2(t) + \frac{q_2(t)}{R_3} + \frac{q_2(t) - q_1(t)}{R_2} = 0$$

Substituting the values of R_1, R_2, R_3, C_1, and C_2 and rearranging, we obtain

$$\dot{q}_1(t) = -2q_1(t) + q_2(t)$$
$$\dot{q}_2(t) = q_1(t) - 2q_2(t)$$

In matrix form

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t)$$

with

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{q}(0) = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Then, by Eq. (7.63) with $x(t) = 0$ and using the result from Prob. 7.43, we get

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) = \left\{ e^{-t}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + e^{-3t}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

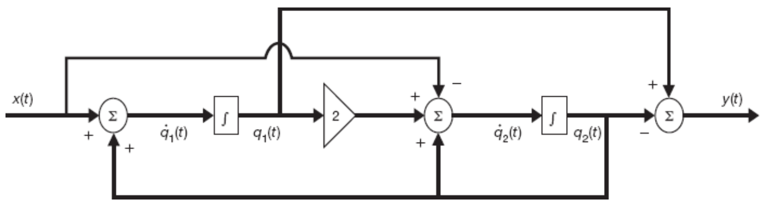
$$= \begin{bmatrix} \frac{3}{4}e^{-t} - \frac{1}{4}e^{-3t} \\ \frac{3}{4}e^{-t} + \frac{1}{4}e^{-3t} \end{bmatrix}$$

Thus,

$$v_{C_1}(t)=\frac{3}{4}e^{-t}-\frac{1}{4}e^{-3t} \qquad \text{and} \qquad v_{C_2}(t)=\frac{3}{4}e^{-t}+\frac{1}{4}e^{-3t}$$

7.50. Consider the continuous-time LTI system shown in Fig. 7-21.

Figure 7-21



a. Is the system asymptotically stable?

b. Find the system function $H(s)$.

c. Is the system BIBO stable?

a. From Fig. 7-21 and choosing the state variables $q_1(t)$ and $q_2(t)$ as shown, we obtain

$$\begin{aligned}\dot{q}_1(t) &= q_2(t) + x(t)\\ \dot{q}_2(t) &= 2q_1(t) + q_2(t) - x(t)\\ y(t) &= q_1(t) - q_2(t)\end{aligned}$$

In matrix form

$$\begin{aligned}\dot{\mathbf{q}}(t) &= \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)\\ y(t) &= \mathbf{c}\mathbf{q}(t)\end{aligned}$$

where

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix} \qquad \mathbf{b}=\begin{bmatrix}1\\ -1\end{bmatrix} \qquad \mathbf{c}=\begin{bmatrix}1 & -1\end{bmatrix}$$

Now

$$c(\lambda)=|\lambda\mathbf{I}-\mathbf{A}|=\begin{vmatrix}\lambda & -1\\ -2 & \lambda-1\end{vmatrix}=\lambda^2-\lambda-2=(\lambda+1)(\lambda-2)$$

Thus, the eigenvalues of **A** are $\lambda_1 = -1$ and $\lambda_2 = 2$. Since Re{ λ_2} > 0, the system is not asymptotically stable.

b. By Eq. (7.52) the system function $H(s)$ is given by

$$\begin{aligned}H(s)=\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}&=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}s & -1\\ -2 & s-1\end{bmatrix}^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}\\ &=\frac{1}{(s+1)(s-2)}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}s-1 & 1\\ 2 & s\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}\\ &=\frac{2(s-2)}{(s+1)(s-2)}=\frac{2}{s+1}\end{aligned}$$

c. Note that there is pole-zero cancellation in $H(s)$ at $s = 2$. Thus, the only pole of $H(s)$ is −1, which is located in the left-hand side of the s-plane. Hence, the system is BIBO stable.

Again, it is noted that the system is essentially unstable if the system is not initially relaxed.

7.51. Consider an Nth-order continuous-time LTI system with state equation

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$

The system is said to be *controllable* if it is possible to find an input $x(t)$ which will drive the system from $\mathbf{q}(t_0) = \mathbf{q}_0$ to $\mathbf{q}(t_1) = \mathbf{q}_1$ in a specified finite time and $\mathbf{q}_0$ and $\mathbf{q}_1$ are any finite state vectors. Show that the system is controllable if the *controllability matrix* defined by

$$\mathbf{M}_c = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \cdots \quad \mathbf{A}^{N-1}\mathbf{b}]$$

(7.128)

has rank N.

We assume that $t_0 = 0$ and $\mathbf{q}[0] = \mathbf{0}$. Then, by Eq. (7.63) we have

$$\mathbf{q}_1 = \mathbf{q}(t_1) = e^{\mathbf{A}t_1}\int_0^{t_1} e^{-\mathbf{A}\tau}\mathbf{b}x(\tau)\,d\tau$$

(7.129)

Now, by the Cayley-Hamilton theorem we can express $e^{-\mathbf{A}\tau}$ as

$$e^{-\mathbf{A}\tau} = \sum_{k=0}^{N-1}\alpha_k(\tau)\mathbf{A}^k$$

(7.130)

Substituting Eq. (7.130) into Eq. (7.129) and rearranging, we get

$$\mathbf{q}_1 = e^{\mathbf{A}t_1} = \left[\sum_{k=0}^{N-1}\mathbf{A}^k\mathbf{b}\int_0^{t_1}\alpha_k(\tau)x(\tau)\,d\tau\right]$$

(7.131)

Let

$$\int_0^{t_1}\alpha_k(\tau)x(\tau)\,d\tau = \beta_k$$

Then Eq. (7.131) can be rewritten as

$$e^{-\mathbf{A}t_1}\mathbf{q}_1 = \sum_{k=0}^{N-1}\mathbf{A}^k\mathbf{b}\beta_k$$

or

$$e^{-\mathbf{A}t_1}\mathbf{q}_1 = \begin{bmatrix}\mathbf{b} & \mathbf{A}\mathbf{b} & \cdots & \mathbf{A}^{N-1}\mathbf{b}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N-1}\end{bmatrix}$$

(7.132)

For any given state $\mathbf{q}_1$ we can determine from Eq. (7.132) unique β_k's (k = 0, 1, ..., N − 1), and hence $x(t)$, if the coefficients matrix of Eq. (7.132) is nonsingular, that is, the matrix

$$\mathbf{M}_c = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \cdots \quad \mathbf{A}^{N-1}\mathbf{b}]$$

has rank N.

7.52. Consider an Nth-order continuous-time LTI system with state space representation

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$
$$y(t) = \mathbf{c}\mathbf{q}(t)$$

The system is said to be *observable* if any initial state $\mathbf{q}(t_0)$ can be determined by examining the system output $y(t)$ over some finite period of time from t_0 to t_1. Show that the system is observable if the *observability matrix* defined by

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathbf{A} \\ \vdots \\ \mathbf{c}\mathbf{A}^{N-1} \end{bmatrix}$$

has rank N.

We prove this by contradiction. Suppose that the rank of $\mathbf{M}_o$ is less than N. Then there exists an initial state $\mathbf{q}[0] = \mathbf{q}_0 \neq \mathbf{0}$ such that

$$\mathbf{M}_o\mathbf{q}_0 = \mathbf{0}$$

or

$$\mathbf{c}\mathbf{q}_0 = \mathbf{c}\mathbf{A}\mathbf{q}_0 = \cdots = \mathbf{c}\mathbf{A}^{N-1}\mathbf{q}_0 = 0$$

(7.134)

Now from Eq. (7.65), for $x(t)$ = 0 and t_0 = 0,

$$y(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{q}_0$$

(7.135)

However, by the Cayley-Hamilton theorem, $e^{\mathbf{A}t}$ can be expressed as

$$e^{\mathbf{A}t} = \sum_{k=0}^{N-1} \alpha_k(t)\mathbf{A}^k$$

(7.136)

Substituting Eq. (7.136) into Eq. (7.135), we get

$$y(t) = \sum_{k=0}^{N-1} \alpha_k(t)\mathbf{c}\mathbf{A}^k\mathbf{q}_0 = 0$$

(7.137)

in view of Eq. (7.134). Thus, $\mathbf{q}_0$ is indistinguishable from the zero state, and hence, the system is not observable. Therefore, if the system is to be observable, then $\mathbf{M}_o$ must have rank N.

7.53. Consider the system in Prob. 7.50.

a. Is the system controllable?

b. Is the system observable?

a. From the result from Prob. 7.50 we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Now

$$\mathbf{Ab} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and by Eq. (7.128) the controllability matrix is

$$\mathbf{M}_c = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and $|\mathbf{M}_c| = 0$. Thus, it has a rank less than 2, and hence, the system is not controllable.

b. Similarly,

$$\mathbf{cA} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \end{bmatrix}$$

and by Eq. (7.133) the observability matrix is

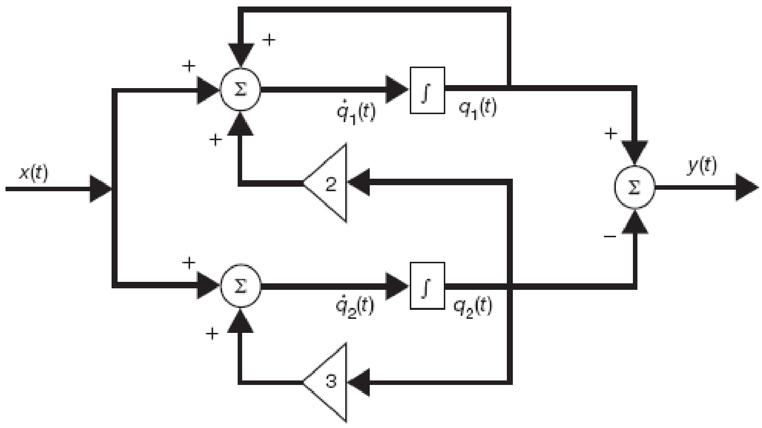
$$\mathbf{M}_o = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$$

and $|\mathbf{M}_0| = -2 \neq 0$. Thus, its rank is 2, and hence, the system is observable.

Note from the result from Prob. 7.50(b) that the system function $H(s)$ has pole-zero cancellation. As in the discrete-time case, if $H(s)$ has pole-zero cancellation, then the system cannot be both controllable and observable.

7.54. Consider the system shown in Fig. 7-22.

Figure 7-22



a. Is the system controllable?

b. Is the system observable?

c. Find the system function $H(s)$.

a. From Fig. 7-22 and choosing the state variables $q_1(t)$ and $q_2(t)$ as shown, we have

$$\dot{q}_1(t) = q_1(t) + 2q_2(t) + x(t)$$
$$\dot{q}_2(t) = 3q_2(t) + x(t)$$
$$y(t) = q_1(t) - q_2(t)$$

In matrix form

$$\dot{\mathbf{q}}_1(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$
$$y(t) = \mathbf{c}\mathbf{q}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Now

$$\mathbf{Ab} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

and by Eq. (7.128) the controllability matrix is

$$\mathbf{M}_c = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

and $|\mathbf{M}_c| = 0$. Thus, its rank is less than 2, and hence, the system is not controllable.

b. Similarly,

$$\mathbf{cA} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

and by Eq. (7.133) the observability matrix is

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

and $|\mathbf{M}_o| = 0$. Thus, its rank is less than 2, and hence, the system is not observable.

c. By Eq. (7.52) the system function $H(s)$ is given by

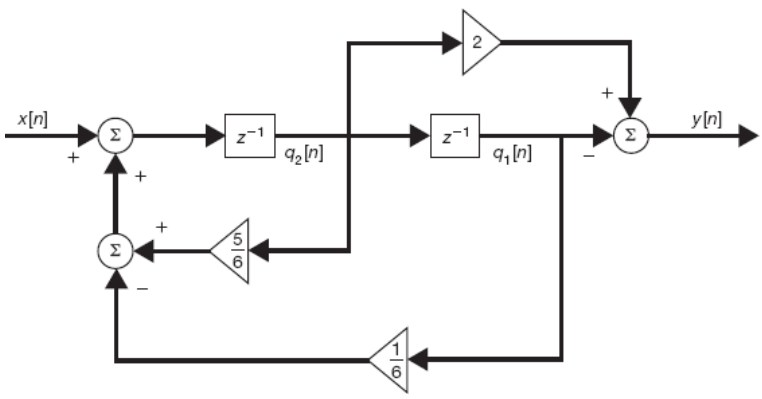
$$\begin{aligned} H(s) &= \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \\ &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s-1 & -2 \\ 0 & s-3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{(s-1)(s-3)} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s-3 & 2 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \end{aligned}$$

Note that the system is both uncontrollable and unobservable.

7.8. SUPPLEMENTARY PROBLEMS

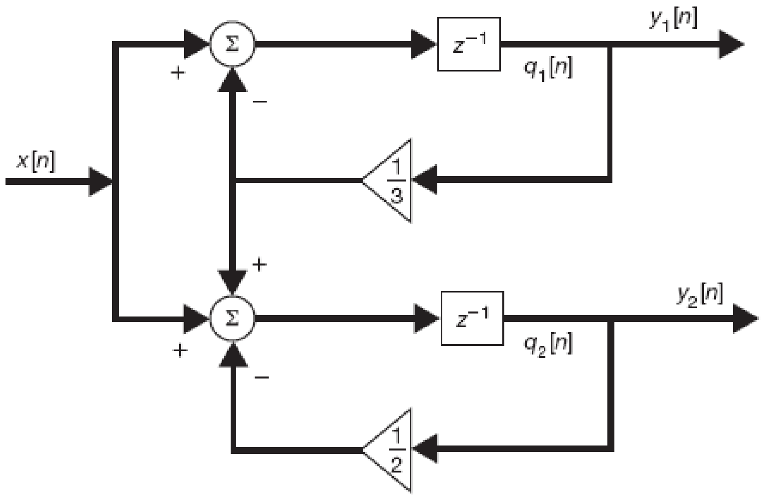
7.55. Consider the discrete-time LTI system shown in Fig. 7-23. Find the state space representation of the system with the state variables $q_1[n]$ and $q_2[n]$ as shown.

Figure 7-23



7.56. Consider the discrete-time LTI system shown in Fig. 7-24. Find the state space representation of the system with the state variables $q_1[n]$ and $q_2[n]$ as shown.

Figure 7-24



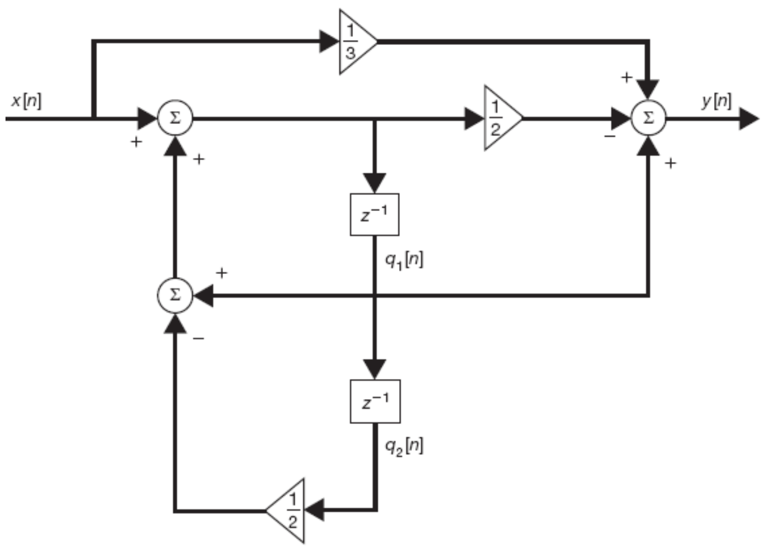
7.57. Consider the discrete-time LTI system shown in Fig. 7-25.

a. Find the state space representation of the system with the state variables $q_1[n]$ and $q_2[n]$ as shown.

b. Find the system function $H(z)$.

c. Find the difference equation relating $x[n]$ and $y[n]$.

Figure 7-25



7.58. A discrete-time LTI system is specified by the difference equation

$$y[n] + y[n-1] - 6y[n-2] = 2\,x[n-1] + x[n-2]$$

Write the two canonical forms of state representation for the system.

7.59. Find $\mathbf{A}^n$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

a. Using the Cayley-Hamilton theorem method.

b. Using the diagonalization method.

7.60. Find $\mathbf{A}^n$ for

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

a. Using the spectral decomposition method.

b. Using the *z*-transform method.

7.61. Given a matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

a. Find the minimal polynomial $m(\lambda)$ of **A**.

b. Using the result from part (*a*), find $\mathbf{A}^n$.

7.62. Consider the discrete-time LTI system with the following state space representation:

$$\mathbf{q}[n+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{q}[n] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x[n]$$
$$y[n] = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{q}[n]$$

a. Find the system function $H(z)$.

b. Is the system controllable?

c. Is the system observable?

7.63. Consider the discrete-time LTI system in Prob. 7.55.

a. Is the system asymptotically stable?

b. Is the system BIBO stable?

c. Is the system controllable?

d. Is the system observable?

7.64. The controllability and observability of an LTI system may be investigated by diagonalizing the system matrix**A**. A system with a state space representation

$$\mathbf{v}[n+1] = \mathbf{\Lambda}\mathbf{v}[n] + \hat{\mathbf{b}}x[n]$$

$$y[n] = \hat{\mathbf{c}}\mathbf{v}[n]$$

(where $\mathbf{\Lambda}$ is a diagonal matrix) is controllable if the vector $\hat{\mathbf{b}}$ has no zero elements, and it is observable if the vector $\hat{\mathbf{c}}$ has no zero elements. Consider the discrete-time LTI system in Prob. 7.55.

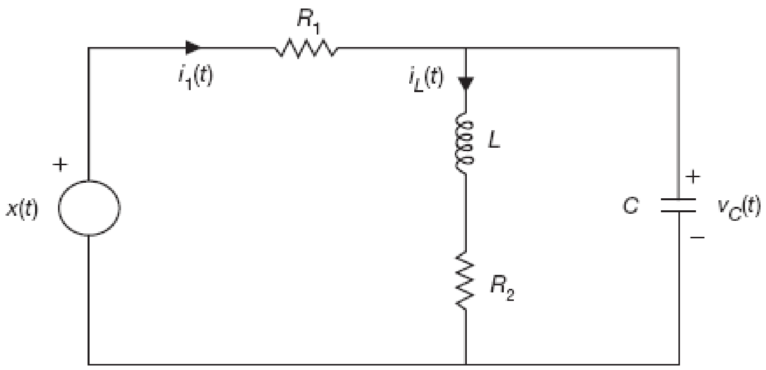
a. Let **v**[n] = **Tq**[n]. Find the matrix **T** such that the new state space representation will have a diagonal system matrix.

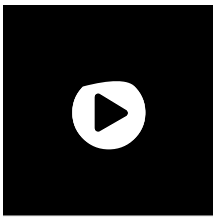
b. Write the new state space representation of the system.

c. Using the result from part (b), investigate the controllability and observability of the system.

7.65. Consider the network shown in Fig. 7-26. Find a state space representation for the network with the state variables $q_1(t) = i_L(t)$, $q_2(t) = v_C(t)$ and outputs $y_1(t)$, $= i_1(t)$, $y_2(t) = v_C(t)$, assuming $R_1 = R_2 = 1\ \Omega$, $L = 1\ H$, and $C = 1$ F.

Figure 7-26





Schaum's Signals and Systems Supplementary Problem 7.65: State Space Representation

This video demonstrates how to find the state space representation of a system.

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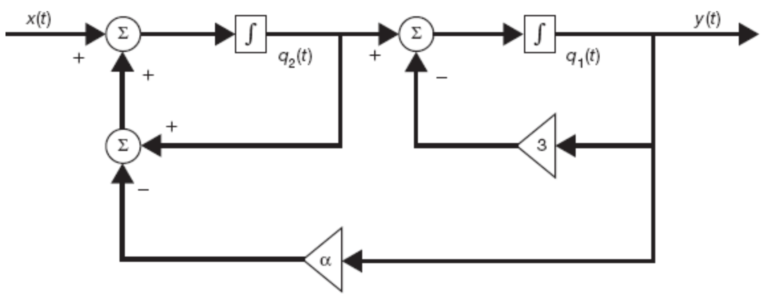
7.66. Consider the continuous-time LTI system shown in Fig. 7-27

Figure 7-27



a. Find the state space representation of the system with the state variables $q_1(t)$ and $q_2(t)$ as shown.

b. For what values of a will the system be asymptotically stable?

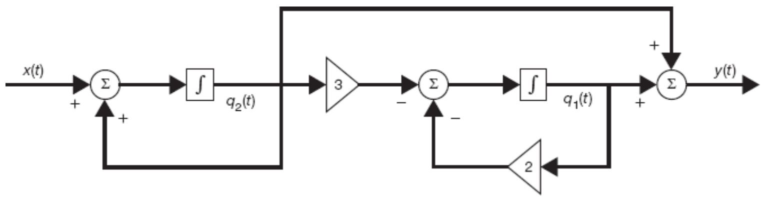
7.67. A continuous-time LTI system is described by

$$H(s) = \frac{3s^2 - 1}{s^3 + 3s^2 - s - 2}$$

Write the two canonical forms of state representation for the system.

7.68. Consider the continuous-time LTI system shown in Fig. 7-28.

Figure 7-28

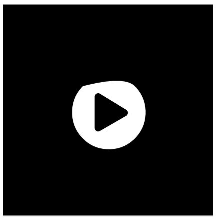


a. Find the state space representation of the system with the state variables $q_1(t)$ and $q_2(t)$ as shown.

b. Is the system asymptotically stable?

c. Find the system function $H(s)$.

d. Is the system BIBO stable?



Schaum's Signals and Systems Supplementary Problem 7.68: Asymptotically Stable System

This video demonstrates how to find the state space representation of a system and determine whether it is asymptotically and/or BIBO stable.

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2013

7.69. Find $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

a. Using the Cayley-Hamilton theorem method.

b. Using the spectral decomposition method.

7.70. Consider the matrix **A** in Prob. 7.69. Find $e^{-\mathbf{A}t}$ and show that $e^{-\mathbf{A}t} = [e^{\mathbf{A}t}]^{-1}$.

7.71. Find $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

a. Using the diagonalization method.

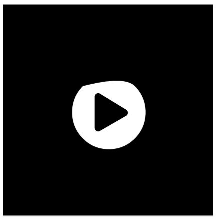
b. Using the Laplace transform method.

7.72. Consider the network in Prob. 7.65 (Fig. 7-26). Find $v_C(t)$ if $x(t) = u(t)$ under an initially relaxed condition.

7.73. Using the state space method, solve the linear differential equation

$$y''(t) + 3y'(t) + 2y(t) = 0$$

with the initial conditions $y(0) = 0, y'(0) = 1$.



Schaum's Signals and Systems Supplementary Problem 7.73: State Space Method

This video demonstrates how to use the state space method to solve a linear differential equation.

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2013

7.74. As in the discrete-time case, controllability and observability of a continuous-time LTI system may be investigated by diagonalizing the system matrix **A**. A system with state space representation

$$\dot{\mathbf{v}}(t) = \boldsymbol{\Lambda}\mathbf{v}(t) + \hat{\mathbf{b}}x(y)$$

$$y(t) = \hat{\mathbf{c}}\mathbf{v}(t)$$

where **Λ** is a diagonal matrix, is controllable if the vector $\hat{\mathbf{b}}$ has no zero elements, and is observable if the vector $\hat{\mathbf{c}}$ has no zero elements. Consider the continuous-time system in Prob. 7.50.

a. Find a new state space representation of the system by diagonalizing the system matrix **A**.

b. Is the system controllable?

c. Is the system observable?

7.9. ANSWERS TO SUPPLEMENTARY PROBLEMS

7.55.

$$\mathbf{q}[n+1] = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \mathbf{q}[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = \begin{bmatrix} -1 & 2 \end{bmatrix} \mathbf{q}[n]$$

7.56.

$$\mathbf{q}[n+1] = \begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \mathbf{q}[n] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{q}[n]$$

7.57.

$$(a)\quad \mathbf{q}[n+1]=\begin{bmatrix}1 & -\frac{1}{2}\\ 1 & 0\end{bmatrix}\mathbf{q}[n]+\begin{bmatrix}1\\0\end{bmatrix}x[n]$$

$$y[n]=\begin{bmatrix}\frac{1}{2} & \frac{1}{4}\end{bmatrix}\mathbf{q}[n]-\frac{1}{6}x[n]$$

$$(b)\quad H(z)=-\frac{1}{6}\frac{z^2-4z-1}{z^2-z+\frac{1}{2}}$$

$$(c)\quad y[n]-y[n-1]+\frac{1}{2}y[n-2]=-\frac{1}{6}x[n]+\frac{2}{3}x[n-1]+\frac{1}{6}x[n-2]$$

7.58.

$$(1)\quad \mathbf{q}[n+1]=\begin{bmatrix}-1 & 1\\ 6 & 0\end{bmatrix}\mathbf{q}[n]+\begin{bmatrix}2\\1\end{bmatrix}x[n]$$

$$y[n]=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{q}[n]$$

$$(2)\quad \mathbf{v}[n+1]=\begin{bmatrix}0 & 1\\ 6 & -1\end{bmatrix}\mathbf{v}[n]+\begin{bmatrix}0\\1\end{bmatrix}x[n]$$

$$y[n]=\begin{bmatrix}1 & 2\end{bmatrix}\mathbf{v}[n]$$

7.59. $\mathbf{A}^n=\begin{bmatrix}-2\left(\frac{1}{2}\right)^n+3\left(\frac{1}{3}\right)^n & 6\left(\frac{1}{2}\right)^n-6\left(\frac{1}{3}\right)^n\\ -\left(\frac{1}{2}\right)^n+\left(\frac{1}{3}\right)^n & 3\left(\frac{1}{2}\right)^n-2\left(\frac{1}{3}\right)^n\end{bmatrix}$

7.60. $\mathbf{A}^n=\begin{bmatrix}(3)^n & 0 & 0\\ 0 & \frac{1}{5}(2)^n+\frac{4}{5}(-3)^n & \frac{1}{5}(2)^n-\frac{1}{5}(-3)^n\\ 0 & \frac{4}{5}(2)^n-\frac{4}{5}(-3)^n & \frac{4}{5}(2)^n-\frac{1}{5}(-3)^n\end{bmatrix}$

7.61.

$$(a)\quad m(\lambda)=(\lambda-3)(\lambda+3)=\lambda^2-9$$

$$(b)\quad \mathbf{A}^n=\frac{1}{3}\begin{bmatrix}3^n+2(-3)^n & 3^n-(-3)^n & 3^n-(-3)^n\\ 3^n-(-3)^n & 3^n+2(-3)^n & 3^n-(-3)^n\\ 3^n-(-3)^n & 3^n-(-3)^n & 3^n+2(-3)^n\end{bmatrix}$$

7.62.

a. $H(z)=\dfrac{1}{(z-1)^2}$

b. The system is controllable.

c. The system is not observable.

7.63.

a. The system is asymptotically stable.

b. The system is BIBO stable.

c. The system is controllable.

d. The system is not observable.

7.64.

a. $\mathbf{T}=\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

b. $\mathbf{v}[n+1]=\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\mathbf{v}[n]+\begin{bmatrix} -2 \\ 3 \end{bmatrix}x[n]$

$y[n]=\begin{bmatrix} -1 & 0 \end{bmatrix}\mathbf{v}[n]$

c. The system is controllable but not observable.

7.65.

$$\dot{\mathbf{q}}(t)=\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\mathbf{q}(t)+\begin{bmatrix} 0 \\ 1 \end{bmatrix}x(t)$$

$$\mathbf{y}(t)=\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\mathbf{q}(t)+\begin{bmatrix} 1 \\ 0 \end{bmatrix}x(t)$$

7.66.

$$(a)\quad \dot{\mathbf{q}}(t)=\begin{bmatrix} -3 & 1 \\ -\alpha & 1 \end{bmatrix}\mathbf{q}(t)+\begin{bmatrix} 0 \\ 1 \end{bmatrix}x(t)$$

$$y(t)=\begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{q}(t)$$

$$(b)\quad \alpha \geq 4$$

7.67.

$$(1)\quad \dot{\mathbf{q}}(t)=\begin{bmatrix}-3 & 1 & 0\\ 1 & 0 & 1\\ 2 & 0 & 0\end{bmatrix}\mathbf{q}(t)+\begin{bmatrix}3\\ 0\\ -1\end{bmatrix}x(t)$$

$$y(t)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\mathbf{q}(t)$$

$$(2)\quad \dot{\mathbf{v}}(t)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 1 & -3\end{bmatrix}\mathbf{v}(t)+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}x(t)$$

$$y(t)=\begin{bmatrix}-1 & 0 & 3\end{bmatrix}\mathbf{v}(t)$$

7.68.

a. $$\dot{\mathbf{q}}(t)=\begin{bmatrix}-2 & -3\\ 0 & 1\end{bmatrix}\mathbf{q}(t)+\begin{bmatrix}0\\ 1\end{bmatrix}x(t)$$

$$y(t)=\begin{bmatrix}1 & 1\end{bmatrix}\mathbf{q}(t)$$

b. The system is not asymptotically stable.

c. $H(s)=\dfrac{1}{s+2}$

d. The system is BIBO stable.

7.69. $e^{\mathbf{A}t}=e^{-t}\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$

7.70. $e^{-\mathbf{A}t}=e^{t}\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$

7.71. $e^{\mathbf{A}t}=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$

7.72. $v_C(t)=\dfrac{1}{2}(1+e^{-t}\sin t-e^{-t}\cos t),\ t>0$

7.73. $y(t)=e^{-t}-e^{-2t}, t>0$

7.74.

a. $$\dot{\mathbf{v}}(t)=\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\mathbf{v}(t)+\begin{bmatrix}1\\ 0\end{bmatrix}x(t)$$

$$y(t)=\begin{bmatrix}2 & -1\end{bmatrix}\mathbf{v}(t)$$

b. The system is not controllable.

c. The system is observable