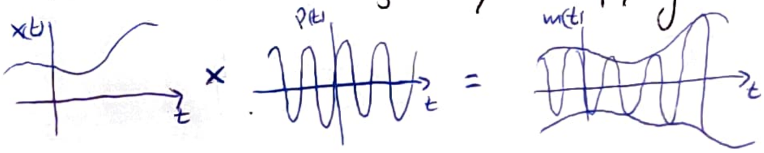
Lecture 1: Basic Theory

Amplitude Modulation

Basic idea: I have the modulating signal $x(t)$ (the signal I want to send) and it is associated to a carrier (sinusoidal signal) $p(t) = A_0 \cos(2\pi f_0 t)$

We obtain the modulated signal by multiplying $\rightarrow$ $m(t) = x(t)\, A_0 \cos(2\pi f_0 t)$



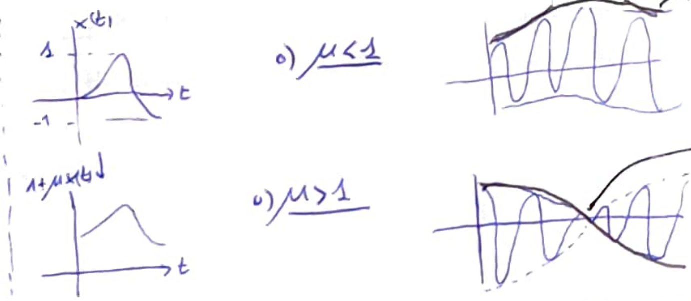
- If something change the amplitude we destroy information (distortion) (Noise, non-linearities)
- Modulated signal <u>is not</u> a sinusoidal signal

↳ AM with Transmitted Carrier

We use it to manage the situation when $x(t)$ crosses the t axis.

$$m(t) = [1 + \mu \cdot x(t)]\, A_0 \cos(2\pi f_0 t)$$

- We are transmitting also the carrier so we waste power because the carrier doesn't transmit information. Not efficient in terms of power



signal envelope never crosses zero line

signal envelope crosse zero. Phase inversion

circuit will be more complicated

- The spectrum is obtained by Fourier transform. Multiplication becomes a convolution. We also will have the carrier in the spectrum
- the BW is twice the original BW $B_T = 2W$

- If I know the negative freqs. I can obtain positive freqs and viceversa. Negative freqs are the complex conjugate of the positive (Hermitian property)
- The useful information is just in one of the side bands

↳ AM with Suppressed Carrier

Basic idea: Remove the carrier, we only have the product $m(t) = x(t) \cdot p(t)$

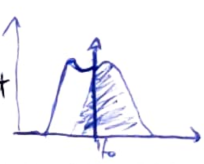
- The problem: Suppressing the carrier I can cross the axis $\Rightarrow$ Phase inversion
- Demodulation is more difficult. I have to understand the phase inversion
- Require less power $\Rightarrow$ More efficient. Reduce the power $\Rightarrow$ Transmit longer distances
- BW is twice the original: $B_T = 2W$

↳ Single Side Band (SSB)

Basic idea: I remove carrier (suppressed carrier) and also one of the two side bands.

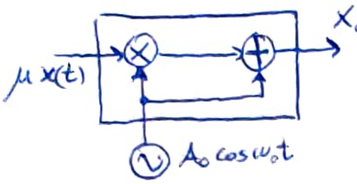
- Very efficient in BW: $B_T = W$ (the minimum reachable, less than W will destroy information)
- Very efficient in Power

- Problems:
  - → To remove the carrier f_0 we need a band-pass filter at high freq. It is difficult to implement a sharp filter.
- Solution: Vestigial Side Band (VSB)
  - → We don't remove complety the side band just in a soft way $\Rightarrow$ Easier filter to implement

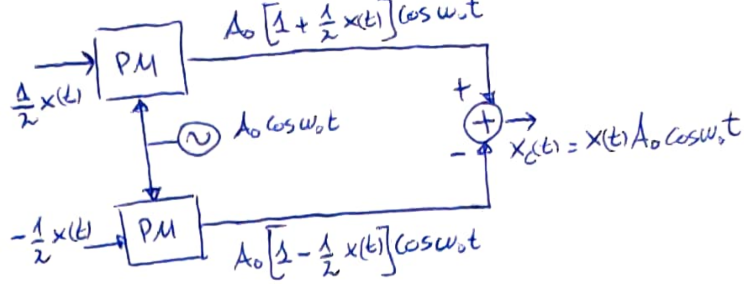


Basic Modulators Circuits

- Product Modulator (PM)



- Balanced Modulator



Demodulation

We received the modulated signal $m(t)$ $\Longrightarrow$ We want to obtain the original signal $x(t)$

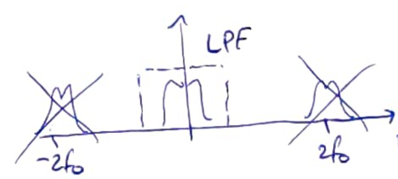
Synchronous (Coherent) Demodulation

- It's the demodulation of the AM with suppressed carrier
- Very easy idea:
  - I multiply the $m(t)$ by the same sinusoidal signal, so I shift again the signal.
  - The signal will go back to the freq. around ϕHz and I apply LPF
- Problems:
  - We need the knowledge of the carrier at the receiver ⇒ Very accurate info of Freq. and Phase.
  - Since the carrier is not transmitted we need circuits that are able to recover the carrier from the signal. Complicated.

① modulated signal
$m(t) = A_0\, x(t)\cos(2\pi f_0 t)$

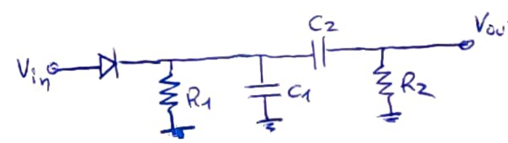
② multiply
$d(t) = m(t)\cdot\cos(2\pi f_0 t) = A_0\, x(t)\cos^2(\cdot)$
$= \frac{A_0\, x(t)}{2}\left[1+\cos(2\pi f_0 t)\right]$

③ LPF
$\underbrace{\cancel{\cos(2\pi f_0 t)}}_{\to 0} \Rightarrow d(t) = \frac{A_0\, x(t)}{2}$



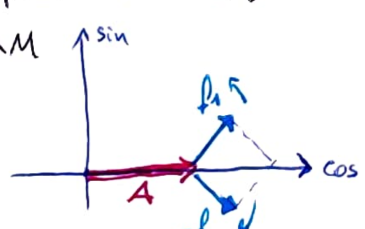
Envelope Demodulation

- It works if we have AM with transmitted carrier
- The envelope never crosses the zero axis ⇒ No phase inversion
- The circuit to make the envelope detector is very cheap and very easy to build



Phasors

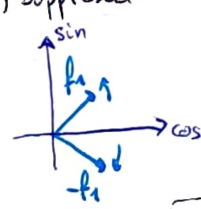
- We can represent signals with phasors
$x(t) = A\cos(2\pi f_1 t + \varphi) = Ae^{j\varphi}$
- If I use axis: cos and sin I can represent the carrier just with cos
- AM



* The component in sin is compensated, we only have component in cos. The modulated signal doesn't take into acc. sin axis. Is always in cos axis

Amplitude carrier
Modulated signal

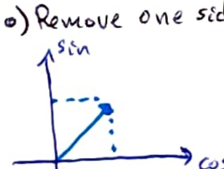
- Suppresed Carrier Mod. (DSB-SC)



* We don't have the vector of the carrier
* The sin axis is not influenced by modulation

→ This approach allow us to use the sin carrier on the same channel because sin and cos won't interfere between them

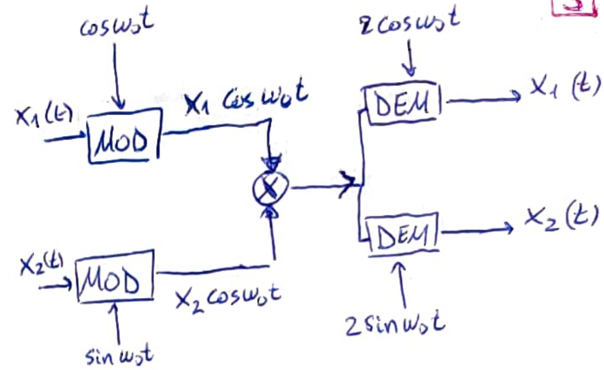
- Remove one side band (SSB-SC)



* Now the vector influences the cos and the sin. I have both components

↳ Quadrature Amplitude Modulation

- We can transmit two signals around f_0: one modulated by cosine, other one modulated by sine } Transmitting 2 signals instead of one is reducing BW by a factor 2



- The receiver has to be coherent to demodulate the two signals. We need cos and sine at receiver
  - ⇒ If we have the phase wrong, we have a rotation and the components do not compensate each other and we mix the two signals.

  Carrier synchronization very important.

- QAM cannot be used in SSB because we will mix the signals

Angle Modulation (Freq. and Phase Modulation)

- Basic idea: Put the information not in the amplitude but in the phase of the carrier.

  $m(t) = A_0 \cos[2\pi f_0 t + \varphi(t)]$ ⇒ $\varphi(t) = K\,x(t)$ → Phase modulation

  ⇒ $\varphi(t) = K \int_{-\infty}^{t} x(t)\,dt$ → Freq. modulation (The info stays in the derivative of phase → FM
  Phase ⇒ Integral of freq.
  Freq. ⇒ Derivative of phase)

- Instantaneous Frequency: Take the derivative of the argument of cosine
  - ↳ We use FM → $f_{inst}(t) = f_0 + K\,x(t)$ → The information is in the instantaneous freq.

- The envelope (and amplitude) is constant so it is robust against non-linearities. Reduce effect of noise. Also this system is non-linear.

- Problem: We cannot do the fourier transform ⇒ We cannot calculate the BW

  Approximation given by Carson: $B_T = 2W + \Delta f_{pp}$ * Δf_{pp} = freq. deviation depends on the characteristics of the modulation.

- Important parameter: K → Modulation index.
  - ↳ If I increase K ⇒ increase the oscillation in instantaneous freq. ⇒ The BW increases
  - ↳ Increasing K I increase BW but I reduce the noise

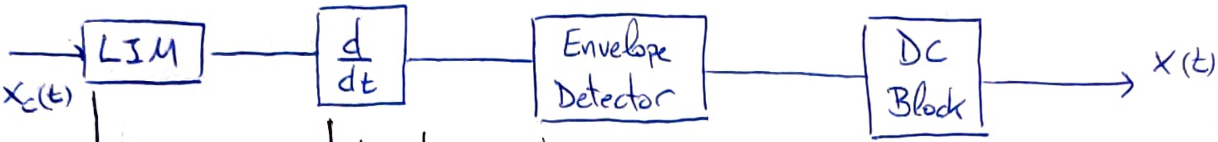
Demodulation : Discriminator

$$X_c(t) = A_c \cos \theta_c(t)$$

$$\dot{X}_c(t) = -A_c\,\dot{\theta}_c(t) \cdot \sin \theta_c(t)$$

$$= -2\pi A_c \left[f_c + f_\Delta x(t)\right] \sin\left(\theta_c(t) \pm 180^\circ\right)$$

- The information is in the derivative $\dot{\theta}_c$
  After the derivative we have the info in the amplitude of the signal ⇒ I can use an envelope detector
- The modulation is not coherent



LIM ↳ Removes spurious amplitude oscillations due to amplifier. It limits the amplitude

$\frac{d}{dt}$ ↳ Derivative

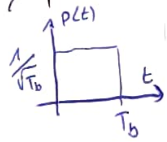
Envelope Detector ↳

DC Block ↳ Removes the frequency of the carrier

- **Use of PLL for Demodulation**
  - ↳ Non-linear feedbak system
  - ↳ It obtains the phase and make the derivative, then we get the information
  - ↳ Higher cost but works better than the discriminator

Digital Communication techniques

- Basic idea: We sample the signal, we do a quantization until the signal is transformed in a sequence of "1" and "0". Transmission of sequence of "1" and "0"
- The task of the receiver is to stimate if the signal was "1" or "0".
- Probability of Error is crucial: If the decision of "1" and "0" was incorrect we have one error. i.e $P(E) = 10^{-10}$ ⇝ In average from 10^{10} bits transmitted there is one bit wrong
- First idea: Pulse Amplitude Modulation (PAM) (base-band): I take one pulse and I multiply by positive or negative number to say "1" or "0"



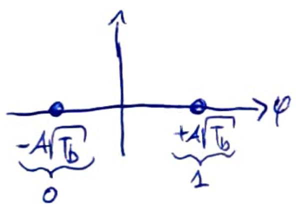
$1 \to +A\sqrt{T_b}$
$0 \to -A\sqrt{T_b}$

↳ T_b: Duration of the pulse, how much time is needed to transmit one bit

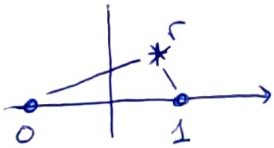
↳ Noise: It will corrupt the shape and will make more difficult to know if it is "1" or "0"
To work against the noise ⇒ Increase T_b (it reduces the transmission velocity)

↳ Constellation: Vector space representation

- Without Noise



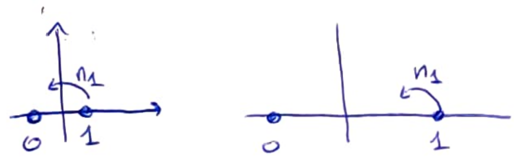
- With Noise



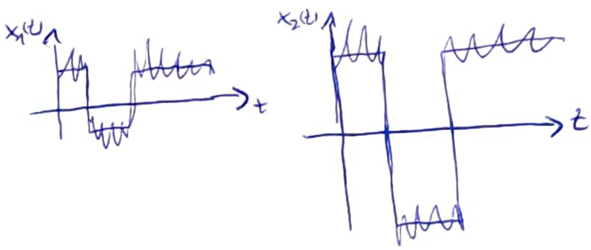
→ This is binary system ("0", "1")
→ At the receiver we decide if transmitted signal was "1" or "0"
→ received signal: $r(t) = \underset{\uparrow \text{ info}}{x(t)} + \underset{\uparrow \text{ noise}}{n(t)}$

↳ Probability of Errors: $P(E)$

- It depends on "distance" between the possible signals
Increasing the distance ⇒ Reduce noise effect



- Increase distance ⇒ Increase the amplitude
Increase the amplitude ⇒ I need more power



Increase power (amplitude) ⇒ Reduce Noise

↳ **R_b:** Bit-rate → It is related to the time spent for a bit.

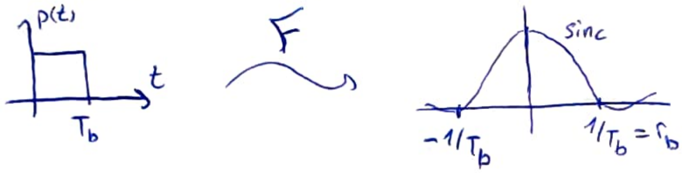
$$r_b = \frac{1}{T_b} \quad [bit/sec]$$

•) Increase r_b ⇒ decrease T_b ⇒ Less energy to the signal ⇒ distance decrease

•) $r_b \uparrow \Rightarrow T_b \downarrow \Rightarrow E = \frac{P}{r_b} = P T_b \downarrow \Rightarrow P(E) \uparrow$

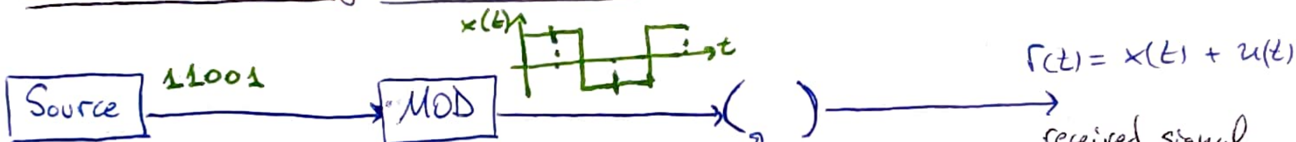
↳ **BW:** Bandwidth

•) Bandwidth is related to the Fourier transform and therefore to r_b



$B_T \simeq r_b$

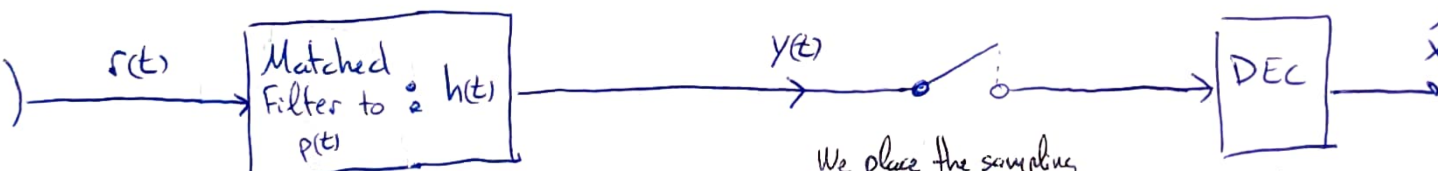
Receiver in Digital Modulation (base-band PAM)



Produces a sequence of "1" and "0" at certain bit-rate: r_b

The sequence will be modulated and will produce PAM signal

channel and noise



We try to clean our signal: we use a filter
Among all posible filters the best is the matched filter
Match filter: The filter has the shape of the transmitted signal p(t)
The shape of the filter: Impulse response h(t)

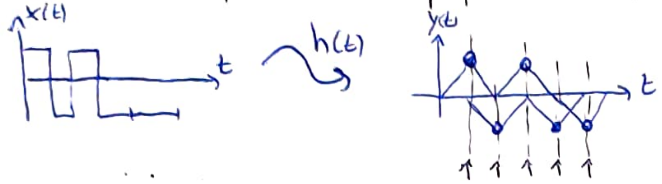
We place the sampling here to have the best amplitude for the useful signal.
We will sample the noise that's why is better to do after filtering

Decide if "1" or "0"

Inter Symbol Interference (ISI)

It can happen that our symbols could interfere and it would lead to make errors.
We use **equalizers** to reduce the posible inter symbol intereference

Sometimes even if there is superposition, after sampling we can have ISI = 0



We see here that it doesn't affect

- Important to control ISI: ISI ↑ ⇒ P(E) ↑
- Strategy to reduce ISI: Increase T_b ↑ but reduce r_b ↓

Multi-Level Transmission

With more levels we can send more bits. For example: 4 levels ⇒ 2 bits
Now we also use symbol time T_s ⟶ In the example $T_s = 2T_b$
⟶ Symbol rate r_s ⟶ " " " $r_s = \frac{r_b}{2}$
⟶ used BW $B_T \sim r_s$ ⟶ " " " $B_T \simeq r_s \simeq \frac{r_b}{2}$ reduced by number of 2

In general:

M levels ⇒ $\log_2(M)$ bits

$\Rightarrow T_s = \log_2(M)\, T_b$,

$\Rightarrow r_s = \frac{r_b}{\log_2(M)}$

$\Rightarrow B_T \simeq \frac{r_b}{\log_2(M)}$

* Pay attention: The distance between two possibilities is reduce.
We have more signal with the same power
* Good in terms of BW, but worse in terms of P(E)
* If I want to maintain P(E) ⇒ I need increase power

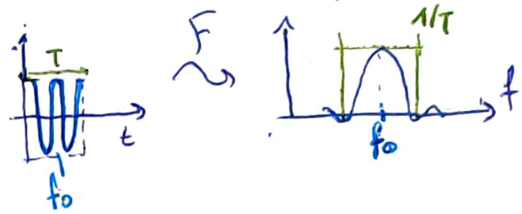
* Increase M ↑ ⇒ Reduce BW ↓ (with log progression) ⇒ Increase P(E) ↑
- Maintain P(E) ≃ ⇒ Increase Power transmitted ↑

Band-Pass Modulation

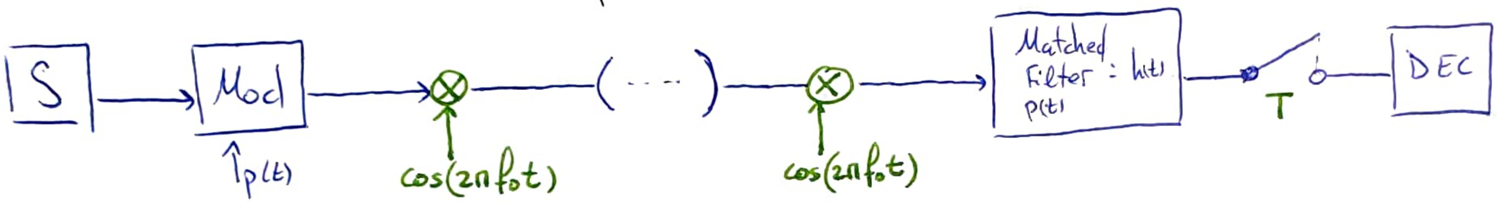
In the PAM we had base-band mod. so we don't shift the spectrum.

- In this case we want to shift the spectrum

Basic idea: I multiply the pulse by a sinusoidal carrier so the spectrum will be shifted around the central freq. f_0



- Block Diagram: We add the sinusoidal carrier in the modulation part and also in the demodulation part.



↳ **Very important**: We need to know the carrier (freq f_0 and phase φ) at receiver (like QAM) and also I need to know the sampling period (T: time synchronization) and also the filter shape (p(t)) → take into account that the filter should be adapted to the signal that arrives to the receiver, not the one that is in the transmitter ⇒ I have to take into account the channel and stimate the characteristic of the channel (it changes in time also)

Coherent system
I know
carrier f_0 and φ
Pulse shape p(t)
sampling time T

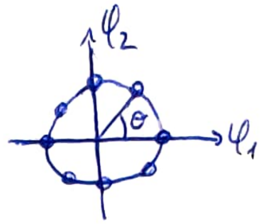
↳ We have 4 parameter
⇒ 2 related to Analog mod.
⇒ 2 related to PAM mod.

↳ Because we are using an analog mod with sinusoidal carrier ⇒ We can use QAM
→ Use cosine carrier and sine carrier ⇒ Both are orthogo. so at receiver can be demod. with no errors.

- QAM
- Very efficient in BW
- Weak point: Amplitude mod. is linear mod ⇒ require a linear system ⇒ not good against non-linearities
- Requiere power to maintain P(E) → therefore to maintain BW
- Requiere a coherent receiver

Non linear
shrink!!

Phase Shift-Keying: PSK



$\varphi_1 \to$ carrier 1
$\varphi_2 \to$ orthogonal carrier

- The information is related to the angle of the point therefore is the phase of the vector

  The constellation is placed on a circle.

- If I have non-linear distortion $\Rightarrow$ the radius reduces $\Rightarrow$ the distance reduces $\Rightarrow$ more errors
  BUT $\Rightarrow$ the receiver is working because the constellation shape remains the same (a circle)

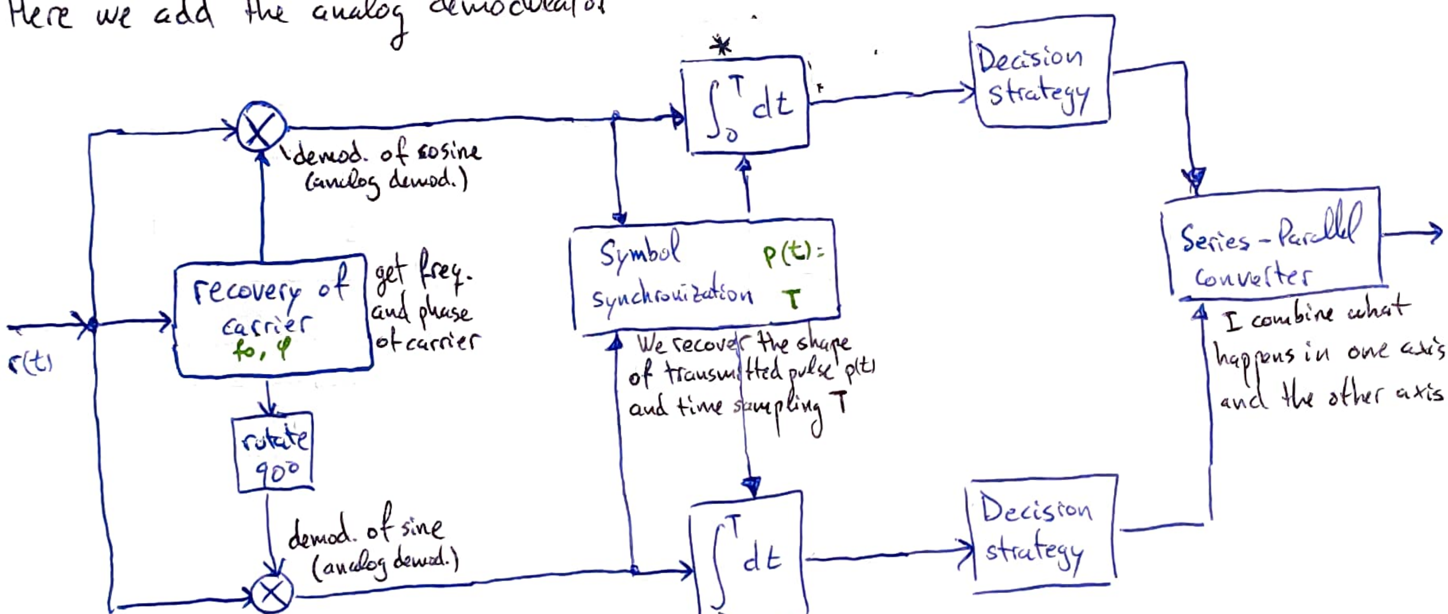
- The area inside the circle is not used $\Rightarrow$ waste of power

- the receiver is coherent

Summary:
- $\to$ Stronger against non-linearities
- $\to$ Inefficient in terms of power
- $\to$ Needs a coherent receiver

Receiver (for QAM and PSK)

- It is an evolution of receiver of base-band PAM, there is the matched filter and the sampling.
  Here we add the analog demodulator



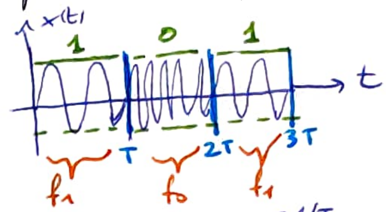
(*) $\to$ Convolution with matched filter
It stops at T so it is also the sampling

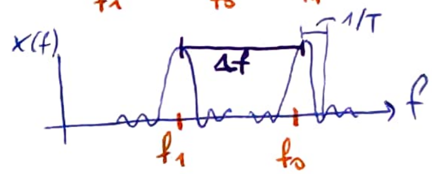
- If I increase the number of levels the complexity of the receiver <u>does not</u> increase

Frequency Shift Keying: FSK

- It is an evolution of Freq. Mod.
- Basic idea: Two distinguish between "1" and "0" instead of using positive amplitude and negative → I use two frequencies
  - → The envelope is constant so it is robust against non-linearities
  - → Depending of the distance between f_1 and f_0 we will have better P(E) or not.

    Increase distance ⇒ better P(E) ⇒ worse BW





- If the distance Δf is a multiple of $1/T$ the two symbols are orthogonal: $\Delta f = \frac{K}{T} \Rightarrow \perp$ symbols, $K \in$ integer

  The best BW: K=1

- For multilevel transmission M-FSK: If I increase the number of levels I increase the BW.

  Increase M ↑ ⇒ Increase BW ↑

  It is because I need more band ($f_1, f_0 \rightarrow$)

  $$B_T = (M+1)\frac{1}{T_s} = \frac{(M+1)\, r_b}{\log_2 M}$$

- Number of dimensions is the number of levels (in QAM if number of levels increases, number of dimensions stay ~~b~~being 2)

  ↳ Adding levels ⇒ does not increase the power (effective)

Summary
- → Power effective
- → Robust against non-linearities
- → Not needed use of coherent receiver
- → Needs to much BW

Signals Spaces

Signals as Vectors

↳ Linear space (or vector space) is a set of elements called vectors
With two operators: + addition
* multiplication by scalar

↳ Example 1 ⇒ The set of all complex-valued discrete-time signals of the form $\{y_k\}$ that have finite energy $\sum_k |y_k|^2 < \infty$ is a linear space

↳ Example 2 ⇒ The set of complex-valued continous-time signals $y(t)$ that have finite energy $\int_{-\infty}^{\infty} |y(t)|^2 dt < \infty$ is a linear space.

↳ **Span** ⇒ For instance if I take two signals $s_1(t)$, $s_2(t)$ and I multiply each of them by a different real number I will obtain many possible linear combinations

In general we have M possible signals $\{s_1(t), \dots, s_M(t)\}$, the set of vectors obtained by all linear comb. of $s_1(t), \dots, s_M(t)$ is the span

$S = \text{span}\{s_1(t), \dots, s_M(t)\}$

↳ **Inner product**: To deduce the length of a vector, distance between two vectors and angle between two vectors.

Discrete → $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i^* = y^* x$
(y_i^*: complex conjug.; y^*: ↳ transpose conjug.)

↳ Norm: $\langle x, x\rangle = x^* x = \sum_{i=1}^{n} |x_i|^2 = \|x\|^2$

* The length: $\|x\|$ * The energy: $\|x\|^2$

Continous → $\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$ → $\|x(t)\|^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt$

↳ **Swartz Inequality** ⇒ $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

↳ **Correlation**: In context of continous signals sometimes the inner product is the correlation and it is a sort of measure of the similarity between two signals

If $\langle x, y\rangle = 0$ means that $x(t)$ and $y(t)$ are orthogonal → $x \perp y$

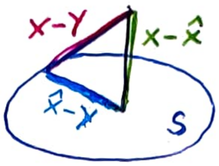
◦) Completeness: There are no "missing" vectors that are arbitrarily close to vectors in the space but are not themselves in the space.

◦) Inner product + Completeness ⇒ Hilbert Space

↳ Projection onto a Subspace

H a Hilbert space, S a subspace of H. Definition of a projection on S of any $X \in H$ as the unique element $\hat{X} \in S$ that is closest to X

$$\|X - \hat{X}\| = \min_{Y \in S} \|X - Y\|$$



* The error is minimum!

↳ Basis

Basis: In the context of digital telecom. the transmitter sends one signal of the possible M signals in the set $\{s_1, .., s_M\}$. Then, the optimal receiver will make a projection of the received signal $r(t)$ spanned by the M signals (will make a projection onto the vector space generated by the signals)

- Definition: a set $\{\phi_1(t); \phi_2(t), ...\}$ is orthonormal when each signal has unit energy (unit length) and $\int_{-\infty}^{\infty} \phi_i(t)\, \phi_k^*(t)\, dt = \delta_{ik} = \begin{cases} 1 \rightarrow i = k \\ 0 \rightarrow i \neq k \end{cases}$

- An orthonormal basis for signal space $S = \text{span}\{s_1(t), .., s_M(t)\}$ is the minimal set of N orthonormal functions $\{\phi_1(t), .., \phi_N(t)\}$ which every element $s(t) \in S$ can be expressed as a linear comb. of the basis $s(t) = \sum_{i=1}^{N} s_i\, \phi_i(t)$

- Any received signal can be expressed as: $r(t) = \hat{r}(t) + e(t)$

  $\hat{r}(t) = \sum_{i=1}^{N} r_i \phi_i \rightarrow$ projection of $r(t)$ onto S

  $e(t) = r(t) - \hat{r}(t) \rightarrow$ projection error, orthogonal to the space S

↳ Gram-Schmidt

Gram-Schmidt: Orthonormalization procedure, given a set of signals it is always possible to determine the base of the vector space

There are infinite number of possible bases of the vector space

↳

↳ Geometry of Signal Space

We have orthonormal basis → $\{\phi_1(t), \dots, \phi_N(t)\}$ for signal space S spanned by $\{s_1(t), \dots, s_M(t)\}$

We can have the k-th signal $s_k(t)$ as linear. comb. of the basis and the coefficients are obtained by: $s_{k,i} = \int_{-\infty}^{\infty} s_k(t)\,\phi_i^*(t)\,dt$ → projection on axis i

So k-th signal is uniquely specified by $s_k = [s_{k,1}, s_{k,2}, \dots, s_{k,N}]^T$

Instead of a function of time $s_k(t)$ can be represented by a vector in euclidean space
From function of time $\xrightarrow{\text{to}}$ set of numbers

∘) The scalar product of two signals can be obtained by the scalar product between the coefficients of the two signals

$$\langle s_j(t), s_k(t)\rangle = \langle s_j, s_k\rangle \quad \text{(From integral to a sum)}$$

$$\int_{-\infty}^{\infty} s_j(t)\, s_k^*(t)\,dt = \sum_{i=1}^{N} s_{j,i}\, s_{k,i}^*$$

∘) Energy of the signal: $\|s_k(t)\|^2$

$$\|s_k(t)\|^2 = \int_{-\infty}^{\infty} |s_k(t)|^2 dt = \sum_{i=1}^{N} |s_{k,i}|^2 = \|s_k\|^2$$

∘) Distance

$$\|s_j(t) - s_k(t)\|^2 = \int_{-\infty}^{\infty} |s_j(t) - s_k(t)|^2 dt =$$

$$= \sum_{j=1}^{N} |s_j - s_k|^2 = \|s_j - s_k\|^2$$

Pass-Band Signals

A pass-band signal is a signal that the spectrum is centered around f_0 (not zero).

Complex Envelope

If is true that for a signal $s(t)$ we can write:

$$S(t) = \mathrm{Re}\left\{ z(t)\, e^{j2\pi f_0 t} \right\} = \frac{z(t)}{2} e^{j2\pi f_0 t} + \frac{z^*(t)}{2} e^{-j2\pi f_0 t}$$

Then $z(t)$ is the complex envelope of $s(t)$ $\quad z(t) = |z(t)| \cos\left[2\pi f_0 t + \arg(z(t))\right]$

↳ The fourier transform of z → is relate to the fourier transform of s shifted by f_0 and multiplied by unit step function $U = \begin{cases} 0 & \text{if } f<0 \\ 1 & \text{if } f>0 \end{cases}$

$$Z(f) = 2\, S(f+f_0)\, U(f+f_0)$$

↳ If we have $s(t)$ like:

$$S(t) = A(t)\cos\left[2\pi f_0 t + \varphi(t)\right] \Rightarrow z(t) = A(t)\, e^{j\varphi(t)} \rightsquigarrow z(t) = x(t) + j\,y(t)$$

another representation quadrature and phase → $s(t) = x(t)\cos(2\pi f_0 t) - y(t)\sin(2\pi f_0 t)$

↳ The product:

$$s_1(t)\cdot s_2(t) = \frac{1}{2}\mathrm{Re}\left\{ z_1(t)\, z_2^*(t) \right\} + \frac{1}{2}\mathrm{Re}\left\{ z_1(t)\, z_2(t)\, e^{j4\pi f_0 t} \right\}$$

and the correlation: $\int s_1(t)\, s_2(t)\, dt = \frac{1}{2}\mathrm{Re}\left\{ \int z_1(t)\, z_2^*(t)\, dt \right\}$

and the energy: $\int s^2(t)\, dt = \frac{1}{2}\int |z(t)|^2 dt$

↳ If we have a base function that have complex envelope

$$\phi_k(t) = \mathrm{Re}\left\{ z_k(t)\, e^{j2\pi f_0 t} \right\} = A_k(t)\cos\left[2\pi f_0 t + \varphi_k(t)\right]$$

- Multiplying by j → Rotation of 90°

$$j z_k(t) \Rightarrow \phi_{k'}(t) = -A_k(t)\sin\left[2\pi f_0 t + \varphi_k(t)\right]$$

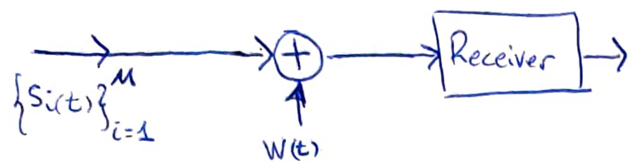
$\phi_k + \phi_{k'}$ → Useful in QAM where we have two signals in two axis we can use the notation

$$s_{ik}\,\phi_k(t) + s_{ik'}\,\phi_{k'}(t) = \mathrm{Re}\left\{ (s_{ik} + j s_{ik'})\, z_k(t)\, e^{j2\pi f_0 t} \right\}$$

* In the real part the cosine aspect
* In the imaginary part the sine aspect

Optimal Receiver

o) Basic model



o) Maths: Signal $S_i(t)$

→ S_i is a vector in the vector space

→ M is the number of possible signals

→ N is the dimension of vector space

$$S_i(t) = \sum_{k=1}^{N} S_{i,k}\,\phi_k \quad ; \quad S_{ik} = \int s_i(t)\,\phi_k(t)\,dt$$

→ ϕ_k is a orthonormal base

$$\langle \phi_i(t), \phi_k(t) \rangle = \delta_{ik} = \begin{cases} 1 & i=k \\ 0 & i \neq 0 \end{cases}$$

→ Property: Inner product ⇒ Correlation

$$\langle S_i(t), S_j(t) \rangle = \int_0^T S_i(t)\,S_j(t)\,dt = \sum_{k=1}^{N} S_{ik}\,S_{jk}$$

* The transmitter sends one signal out of M different signals → i.e. $S_3(t)$
  (we understand as transmitted what reaches the receiver)
* The receiver ⇒ Coherent receiver: know exactly the shape $S_i(t)$
* Decision of transmitter ⇒ Modelled as random variable with M different possible values and we know the probability of transmit
* After the channel ⇒ Addition of white noise $W(t)$
  ↳ Additive, white, gaussian noise
* Receiver ⇒ Make the decision about what was the transmitted symbol.
  It tries to minimize the errors (good decision)

o) Maths: AWGN (noise)

→ Noise is a random process ⇒ is composed by infinite possibilities

→ Noise can be represented in a vector space ⇒ infinite dimension vector space

↳ We take the consideration of the noise in a time interval → $[0, T]$

$$n(t) = \sum_{k=-\infty}^{\infty} n_k\,\phi_k(t)$$ → Noise can be represented as linear comb. of elements of a base ϕ_k; the base is infinite dimension

$$n_k = \int_0^T n(t)\,\phi_k(t)\,dt$$ → Component of the noise in the axis k. Since $n(t)$ is a random process ⇒ the n_k is a random variable (continuos one)

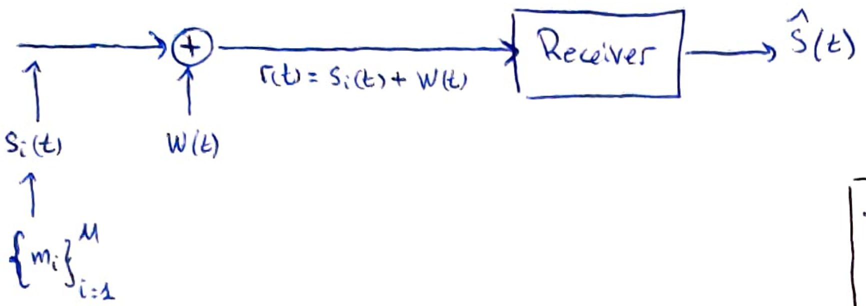
→ Being $n(t)$ a gaussian process ⇒ n_k gaussian random variable: characterize by 2 parameters.
→ Average
→ Variance

* Average (expected) value: $E[n_k] = 0$ (centered in zero)
* Expected value of product ⇒ Cross-correlation: $E[n_k n_j] = \begin{cases} 0 & k \neq j \\ N_0/2 & k = j \end{cases}$ > orthogonal
  (it's how two random variables are related)
  ↳ power spectrum of the noise (variance of the coordinate of the noise)

* Two random variables are "equal" not because are the same point by point but because on average the energy of the difference is zero.

↘

•) Bak to the model



① The transmitter decide one symbol out of a set of M symbols : $\{m_i\}_{i=1}^{M}$

Strong connection between m_i and s_i. Because we are working with coherent receiver, so if I transmit m_3 I know the shape of s_3

From math POV this set is a discrete random variable, only can assume M different values.

Any symbol m_i ⇒ Characterize by its probability

② The received signal $r(t)$ is the addition of transmitted signal and the noise. Also we have coherent receiver

③ The task of receiver: Observe the received signal and select the most probable transmitted signal.

④ Maximize the probability of good decision ⇒
⇒ The receiver have to maximize the posteriori probability ⇒ Minimize the pob. of error $P(\varepsilon)$

⑤ The output of the receiver is $\hat{s}(t)$. It is the stimation of the transmitted signal.

→ A priori probability : Prob. related to the transmitter. The prob. of one symbol to be transmitted

→ A posteriori probability : Prob of saying one symbol or another after the observation of received signal.

→ We use Bayes theorem (general)

$$P(A/B) = \frac{P(B/A) \cdot P(A)}{P(B)}$$

$P(A/B) \rightarrow$ prob of event A when B is true
$P(B/A) \rightarrow$ prob of event B when A is true
$P(A), P(B) \rightarrow$ prob events A or B

→ Bayes Theorem (our case) ⇒ An evolution
* transmitted signal A_i → discrete random variable
* received signal X → continous random variable due to the noise.

↳ Because cont. rand. var. we work with prob. density function $f_x(\,)$

$$P(A_i/X) = \frac{f_x(X/A_i)\, P(A_i)}{f_x(X)}$$

$f_x(X)$ → independent of i so no need to calculate

This is the prob. of A_i being transmitted when X is received
We evaluate all possible $P(A_i/X)$ and take the one that is maximum → M evaluations

→ The noise is modelled as seen AWGN
Gaussian random process

•) Evaluation of posteriori probability: $P(A_i/X)$

*We relate: $\begin{cases} A_i \longrightarrow s_i(t) : \text{transmitted signal} \\ X \longrightarrow r(t) = s_i(t) + n(t) : \text{received signal} \end{cases}$ > are represented as vectors, we work in the vector space

* Expected value of r_K given s_i: I'm transmitting s_i what is the component of received signal along axis ϕ_K

$r_K \rightarrow$ The coordinate of received signal with respect to the axis ϕ_K

$s_i \rightarrow$ The vector associated to the transmitted signal in position i.

$E[r_K/s_i] = \begin{cases} s_{iK}, & K \leq N \rightarrow \text{We work in the vector space of the signal} \\ 0, & K > N \rightarrow \text{We work outside of vector space (only due to the noise)} \end{cases}$

↳ *The component of received signal will be influenced by the transmitted signal (received = transmitted + noise)

* r_K is a random variable $\Rightarrow$ r_K conditioned to s_i is a gaussian random variable

↳ It is because $r(t) = s_i(t) + n(t)$ → and if I fix s_i (not a variable anymore) the only thing I left is the noise, which is a gaussian random variable and we know the variance.

* The components $(r_1, r_2, \ldots)$ along different axis are orthogonal, so the cross-correlation is equal to zero so what happen in ϕ_1 and what happen in $\phi_2, \ldots \phi_3 \ldots$ etc $\Rightarrow$ What happen is independent from one axis to the other.

So because the correlation is equal to zero $\Rightarrow$ the variables are independent (incorrelated)

↳ our component of noise are independent

* Now we evaluate the probability density of the received signal given the transmitted signal

↳ We evaluate the so-called: <u>Likehood function</u> $\Rightarrow f_X(r/s_i) = f_X(X/A_i)$

↳ I have n components of the received signal $(r_1, r_2, \ldots, r_n)$

<u>Only some</u> components are inside the vector space of the signal

n greater than $N \rightarrow n \gg N$. n should be infinite but we have to approximate

In reality we don't calculate complete likehood function (because $n \neq \infty$) we calculate an <u>approximate likehood func.</u>

$f(r_1, r_2, \ldots, r_n / s_i)$ <

↳ So we have:

- Components are gaussian random variables
- Statistical independence
- Different average and variance values if we are inside or outside the vector space

$$f(r_1, r_2, \ldots, r_n / s_i) = \prod_{K=1}^{N} \left(\frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \exp\left[-\frac{(r_K - s_i)^2}{2 \cdot \frac{N_0}{2}} \right] \right) \prod_{K=N+1}^{n} \left(\frac{1}{\sqrt{2\pi \sigma_K^2}} \exp\left[-\frac{(r_K - 0)^2}{2\sigma_K^2} \right] \right)$$

* Now we consider some simplifications
  - We don't need to calculate $f(r_1, \dots, r_n / s_i)$, we just need to know for which s_i it is maximum, so when it is greatest.
  - We don't need to consider what is happening outside the vector space $\left(\prod_{k=N+1}^{n}\right)$
  - We only consider what is inside the vector space. Also, we name: $r = r_1, r_2, \dots, r_n$

$$f(r/s_i) = \exp\left[-\frac{1}{N_0}\sum_{k=1}^{N}(r_k - s_{ik})^2\right] = \exp\left[-\frac{1}{N_0}\|r - s_i\|^2\right]$$

↳ This is the distance to power of 2 ⇒ is the energy of difference

  - So finally, the optimal receiver evaluates the distances and the decide the signal that is closer to the received signal

* To evaluate the a posteriori probability we multiply the likehood function by a priori probability

$$f(r/s_i)\cdot P(s_i) = \exp\left[-\frac{1}{N_0}\|r-s_i\|^2\right]P(s_i) \xrightarrow{\log} -\frac{1}{N_0}\left[\|r-s_i\|^2 - N_0\log(P(s_i))\right]$$

taking the max of negative quantity ⟶ is the same as take the minim. of the same quantity with positive value

$$\Rightarrow \|r-s_i\|^2 - N_0\log(P(s_i)) \rightarrow$$

To get the maximum a posteriori probability I have to minimize this term

* Two strategies
  - Maximum likehood receiver: Only minimize the distance $\min\left[\|r-s_i\|^2\right]$, very simple to implement
  - Maximum a posteriori receiver: Minimizes the full term $\min\left[\|r-s_i\|^2 - N_0\log(P(s_i))\right]$, the best but more complicated

* Using Maximum Likehood Receiver: Two strategies
  - Minimize the distance $\|r-s_i\|^2 = \|r\|^2 + \|s_i\|^2 - 2\langle r, s_i\rangle$

① Minimize the distance $\|r-s_i\|^2$

② Maximize the correlation $\langle r, s_i\rangle$

↳ Calculate using vector space: $\langle r, s_i\rangle = \sum_{k=1}^{N} r_k s_{ik}$, $r_k = \int r(t)\phi_k(t)\,dt$

↳ Calculate in analog domain: $\langle r, s_i\rangle = \int r(t) s_i(t)\,dt$

⇒ For band-pass signals use complex envelope: $r_k = \frac{1}{2}\int z(t)\, z_k^*(t)\,dt$

Error Probabilities

* We will consider the binary modulation because is simple. In the case of more complicated modulations the evaluation of error prob. may be not possible and we would have to use the stimation theory or computer simulations.

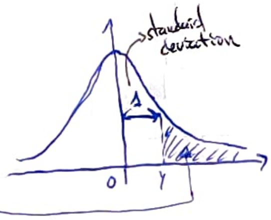
* when I am having an error? P(E)
  ↳ { → when transmitting s_1 I decided s_2 → $P(s_1/s_2)$; → when transmitting s_2 I decided s_1 → $P(s_2/s_1)$ } ⟶ The same prob $P(s_1/s_2) = P(s_2/s_1) = P(E)$

↳ In binary mod. we have two regions. When the component of the noise moves the signal from one region to the other region

Being d the distance between the two signals, if the noise component is greater than d/2, we will have an error.

$$Q(y) = \int_y^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

normalized gaussian v_0



So in our case, the prob. greater than d/2 will make an error and we pass from normalized gaussian to our gaussian with $\sigma^2 = N_0/2$

$$P(E) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

↳ We see that: Prob. of error P(E) is related to the distance $P(E) \sim d$

* The distance is related to the energy given to one bit ⇒ (E_b)

  ↳ Prob. of error $\xrightarrow[\text{relate to}]{}$ distances $\xrightarrow[\text{related to}]{}$ energy ⇒ power × time

  ↳ Given same power: Increase bit rate ⇒ decrease energy ⇒ increase prob. error

  $r_b \uparrow \Rightarrow E_b \downarrow \Rightarrow P(E) \uparrow$

* Remember: In binary antipolar modulation

$$P(E) = Q\left(\frac{2E_b}{N_0}\right)$$

↳ It is the reference point