8. Random Signals

8.1. Introduction

Random signals, as mentioned in Chap. 1, are those signals that take random values at any given time and must be characterized statistically. However, when observed over a long period, a random signal may exhibit certain regularities that can be described in terms of probabilities and statistical averages. The probabilistic model used to describe random signals is called a random (or stochastic) process.

8.2. Random Processes

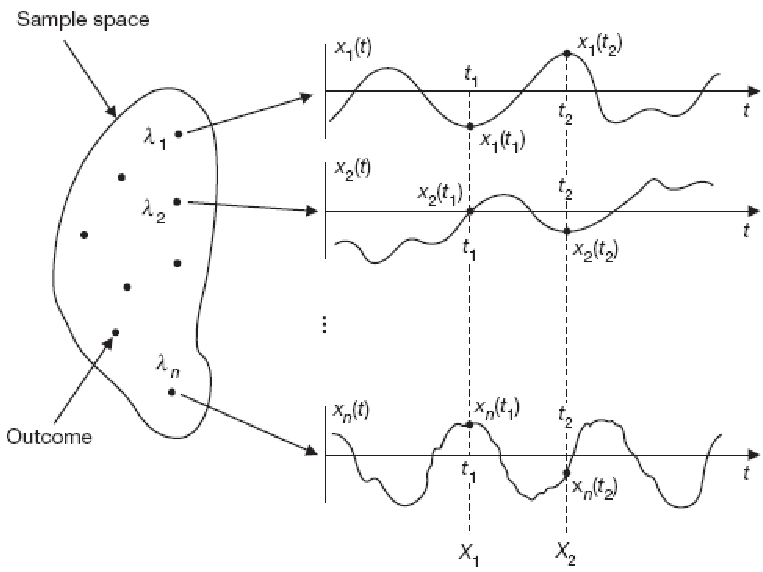
8.2.1. A. Definition:

Consider a random experiment with outcomes λ and a sample space S. If to every outcome $\lambda \in S$ we assign a real-valued time function $X(t, \lambda)$, we create a *random* (or *stochastic*) *process*. A random process $X(t, \lambda)$ is therefore a function of two parameters, the time t and the outcome λ. For a specific λ, say, λ_i, we have a single time function $X(t, \lambda_i) = x_i(t)$. This time function is called a *sample function* or a *realization of the process*. The totality of all sample functions is called an *ensemble*. For a specific time t_j, $X(t_j, \lambda) = X_j$ denotes a random variable. For fixed $t(= t_j)$ and fixed $\lambda(= \lambda_i)$, $X(t_j, \lambda_i) = x_i(t_j)$ is a number.

Thus, a random process is sometimes defined as a family of random variables indexed by the parameter $t \in T$, where T is called the *index set*.

Fig. 8-1 illustrates the concepts of the sample space of the random experiment, outcomes of the experiment, associated sample functions, and random variables resulting from taking two measurements of the sample functions.

Figure 8-1 Random process.



In the following we use the notation $X(t)$ to represent $X(t, \lambda)$.

EXAMPLE 8.1 Consider a random experiment of flipping a coin. The sample space is $S = \{H, T\}$ where H denotes the outcome that "head" appears and T denotes the outcome that "tail" appears. Let

$$X(t, H) = x_1(t) = \sin \omega_1 t$$
$$X(t, T) = x_2(t) = \sin \omega_2 t$$

where ω_1 and ω_2 are some fixed numbers. Then $X(t)$ is a random signal with $x_1(t)$ and $x_2(t)$ as sample functions. Note that $x_1(t)$ and $x_2(t)$ are deterministic signals. Randomness of $X(t)$ comes from the outcomes of flipping a coin.

EXAMPLE 8.2 Consider a random experiment of flipping a coin repeatedly and observing the sequence of outcomes. Then $S = \{\lambda_i, i = 1, 2, \ldots\}$, where $\lambda_i = H$ or T.

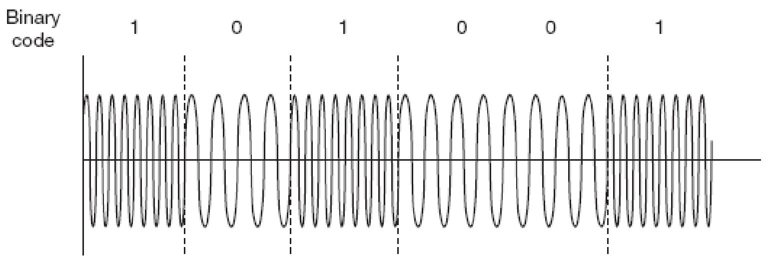
Let

$$X(t, \lambda_i) = \sin(\Omega_i t), \quad (i-1)T \le t \le iT$$

where $\Omega_i = \omega_1$ if $\lambda_i = H$ and $\Omega_i = \omega_2$ if $\lambda_i = T$.

One realization (or sample function) of the random signal $X(t)$ is shown in Fig. 8-2. This kind of random signal is the sort of signal that might be produced by a frequency shift keying (FSK) modem where the frequencies are determined by random sequence of data bits 1 or 0 (by replacing $H = 1$ and $T = 0$).

Figure 8-2



EXAMPLE 8.3 Often a random signal $X(t)$ is specified in terms of random variables.

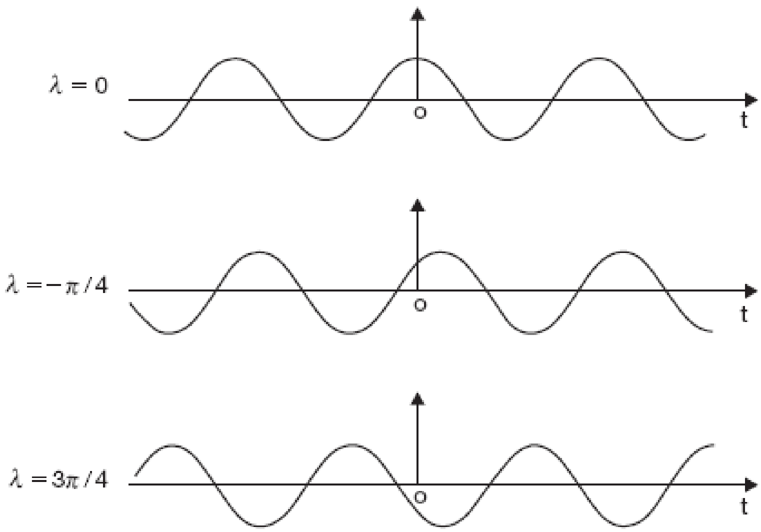
$$X(t) = a\cos(\omega_0 t + \Theta)$$

where a and ω_0 are fixed amplitude and frequency and Θ is a random variable (r.v.) uniformly distributed over $[0, 2\pi]$; that is, r.v. Θ is defined by $\Theta(\lambda) = \lambda$ for each λ in $S = [0, 2\pi]$. That is,

$$X(t, \lambda) = a\cos(\omega_0 t + \lambda) \text{ for } 0 \le \lambda \le 2\pi$$

The ensemble of $X(t, \lambda)$ is the set of cosine functions that have the same amplitude and frequency, but whose phase angle are functions of uniform r.v. over $S = [0, 2\pi]$. Some sampling functions of $X(t, \lambda)$ are plotted in Fig. 8-3.

Figure 8-3

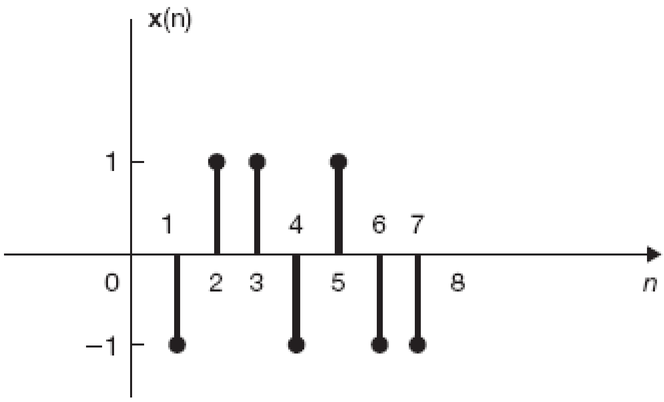


EXAMPLE 8.4 Let X_1, X_2, … be independent r.v. with

$$P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2} \text{ for each } n. \text{Let } X(n) = \{X_n, n \geq 0\} \text{ with } X_0 = 0.$$

Then $X(n)$ is a discrete-time random sequence. A sample sequence of $X(n)$ is shown in Fig. 8-4.

Figure 8-4



8.2.2. B. Description of a Random Process:

In a random process $\{X(t), t \in T\}$, the index set T is called the *parameter set* of the random process. The values assumed by $X(t)$ are called *states*, and the set of all possible values forms the *state spaceE* of the random process. If the index set T of a random process is discrete, then the process is called a *discrete-parameter* (or *discrete-time*) process. A discrete-parameter process is also called a *random sequence* and is denoted by $\{X_n, n = 1, 2, \ldots\}$. If T is continuous, then we have a *continuous-parameter* (or *continuous-time*) process. If the state space E of a random process is discrete, then the process is called a *discrete-state* process, often referred to as a *chain*. In this case, the state space E is often assumed to be $\{0, 1, 2, \ldots\}$. If the state space E is continuous, then we have a *continuous-state* process.

A complex random process $X(t)$ is defined by

$$X(t) = X_1(t) + jX_2(t)$$

where $X_1(t)$ and $X_2(t)$ are (real) random processes and $j = \sqrt{-1}$. Throughout this book, all random processes are real random processes unless specified otherwise.

8.3. Statistics of Random Processes

8.3.1. A. Probabilistic Expressions:

Consider a random process $X(t)$. For a particular time t_1, $X(t_1) = X_1$ is a random variable, and its distribution function $F_X(x_1; t_1)$ is defined as

$$F_X(x_1; t_1) = P\{X(t_1) \le x_1\}$$

(8.1)

where x_1 is any real number.

And $F_X(x_1; t_1)$ is called the *first-order distribution* of $X(t)$. The corresponding first-order density function is obtained by

$$f_X(x_1; t_1) = \frac{\partial F_X(x_1; t_1)}{\partial x_1}$$

(8.2)

Similarly, given t_1 and t_2, $X(t_1) = X_1$ and $X(t_2) = X_2$ represent two random variables. Their joint distribution is called the *second-order distribution* and is given by

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

(8.3)

where x_1 and x_2 are any real numbers.

The corresponding second-order density function is obtained by

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

(8.4)

In a similar manner, for n random variables $X(t_i) = X_i (i = 1, \ldots, n)$, the *nth-order distribution* is

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\}$$

(8.5)

The corresponding nth-order density function is

$$f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{\partial^n F_X(x_1, \ldots, x_n; t_1, \ldots, t_n)}{\partial x_1 \ldots \partial x_n}$$

(8.6)

In a similar manner, we can define a joint distribution between two random processes $X(t)$ and $Y(t)$. The joint distribution for $X(t_1)$ and $Y(t_2)$ is defined by

$$F_{XY}(x_1, y_2; t_1, t_2) = P\{X(t_1) \le x_1, Y(t_2) \le y_2\}$$

(8.7)

and corresponding joint density function by

$$f_{XY}(x_1, y_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial y_2} F_{XY}(x_1, y_2; t_1, t_2)$$

(8.8)

The joint nth-order distribution for $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} &F_{XY}(x_1,\ldots,x_n; y_1,\ldots,y_n; t_1,\ldots,t_n) \\ &\quad = P\{X(t_1) \le x_1,\ldots,X(t_{n1}) \le x_n; Y(t_1) \le y_1,\ldots,Y(t_n) \le y_n\} \end{aligned}$$

(8.9)

and the corresponding nth-order density function by

$$\begin{aligned} &f_{XY}(x_1,\ldots,x_n; y_1,\ldots,y_n; t_1,\ldots,t_n) \\ &\quad = \frac{\partial^{2n}}{\partial x_1 \cdots \partial x_n \partial y_1 \cdots \partial y_n} F_{XY}(x_1,\ldots,x_n; y_1,\ldots,y_n; t_1,\ldots,t_n) \end{aligned}$$

(8.10)

8.3.2. B. Statistical Averages:

As in the case of random variables, random processes are often described by using *statistical averages* (or *ensemble averages*).

The *mean* of $X(t)$ is defined by

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_X(x; t)\, dx$$

(8.11)

where $X(t)$ is treated as a random variable for a fixed value of t.

For discrete time processes, we use the following notation:

$$\mu_X(n) = E[X(n)] = \sum_n x_n p_X(x_n)$$

(8.12)

where $p_X(x_n) = P(X = x_n)$.

The *autocorrelation* of $X(t)$ is defined by

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)\,dx_1\,dx_2$$

(8.13)

The autocorrelation describes the relationship (correlation) between two samples of $X(t)$. In order to see how the correlation between two samples depends on how far apart the samples are spaced, the autocorrelation function is often expressed as

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

(8.1)

Note that

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = E[X(t_2)X(t_1)] = R_{XX}(t_2, t_1)$$

(8.15)

and

$$R_{XX}(t, t) = E[X^2(t)]$$

(8.16)

The *autocovariance* of $X(t)$ is defined by

$$C_{XX}(t_1, t_2) = E\{[X(t_1) - \mu_X(t_1)][X(t_2) - \mu_X(t_2)]\}$$
$$= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

(8.17)

It is clear that if $\mu_X(t) = 0$, then $C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2)$.

Note that $C_{XX}(t_1, t_2)$ and $R_{XX}(t_1, t_2)$ are deterministic functions of t_1 and t_2.

The variance of $X(t)$ is given by

$$\sigma_X^2(t) = \mathrm{Var}[X(t)] = E\{[X(t) - \mu_X(t)]^2\} = C_{XX}(t, t)$$

(8.18)

If $X(t)$ is a complex random process, then the autocorrelation and autocovariance of $X(t)$ are defined by

$$R_{XX}(t_1, t_2) = E[\,X(t_1)\,X^*(t_2)]$$

(8.19)

$$C_{XX}(t_1, t_2) = E\{[X(t_1) - \mu_X(t_1)]\,[X(t_2) - \mu_X(t_2)]^*\}$$

(8.20)

where $*$ denotes the complex conjugate.

In a similar manner, for discrete-time random processes (or random sequences), $X(n)$, the autocorrelation and autocovariance of $X(n)$ are defined by

$$R_{XX}(n_1, n_2) = E[\,X(n_1)\,X(n_2)]$$

(8.21)

$$C_{XX}(n_1, n_2) = E\{[X(n_1) - \mu_X(n_1)][X(n_2) - \mu_X(n_2)]\}$$

(8.22)

and

$$R_{XX}(n_1, n_2) = R_{XX}(n_2, n_1)$$

(8.23)

$$R_{XX}(n, n) = E[X^2(n)]$$

(8.24)

If $\mu_X(n) = 0$, then $C_{XX}(n_1, n_2) = R_{XX}(n_1, n_2)$.

For two different random signals $X(t)$ and $Y(t)$, we have the following definitions. The *cross-correlation* of $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 f_{XY}(x_1, y_2; t_1, t_2)\,dx_1\,dy_2 \end{aligned}$$

(8.25)

The *cross-covariance* of $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} C_{XY}(t_1, t_2) &= E\{[X(t_1) - \mu_X(t_1)][Y(t_2) - \mu_Y(t_2)]\} \\ &= R_{XY}(t_1, t_2) - \mu_X(t_1)\,\mu_Y(t_2) \end{aligned}$$

(8.26)

8.3.2.1. Some Properties of *X*(*t*) and *Y*(*t*):

Two random processes $X(t)$ and $Y(t)$ are *independent* if for all t_1 and t_2,

$$F_{XY}(x, y; t_1, t_2) = F_X(x; t_1)F_Y(y; t_2)$$

(8.27)

They are *uncorrelated* if for all t_1 and t_2

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2) = 0$$

(8.28)

or

$$R_{XY}(t_1, t_2) = \mu_X(t_1)\mu_Y(t_2)$$

(8.29)

They are *orthogonal* if for all t_1 and t_2

$$R_{XY}(t_1, t_2) = 0$$

(8.30)

By changing t_1 and t_2 by n_1 and n_2, respectively, similar definitions can be obtained for two different random sequences $X(n)$ and $Y(n)$.

8.3.3. C. Stationarity:

8.3.3.1. 1. Strict-Sense Stationary:

A random process $X(t)$ is called *strict-sense stationary* (SSS) if its statistics are invariant to a shift of origin. In other words, the process $X(t)$ is SSS if

$$f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = f_X(x_1, \ldots, x_n; t_1 + c, \ldots, t_n + c)$$

(8.31)

for any c.

From Eq. (8.31) it follows that $f_X(x_1; t_1) = f_X(x_1; t_1 + c)$ for any c. Hence, the first-order density of a stationary $X(t)$ is independent of t:

$$f_X(x_1; t) = f_X(x_1)$$

(8.32)

Similarly, $f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + c, t_2 + c)$ for any c. Setting $c = -t_1$, we obtain

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1)$$

(8.33)

which indicates that if $X(t)$ is SSS, the joint density of the random variables $X(t)$ and $X(t + \tau)$ is independent of t and depends only on the time difference τ.

8.3.3.2. 2. Wide-Sense Stationary:

A random process $X(t)$ is called *wide-sense stationary* (WSS) if its mean is constant

$$E[X(t)] = \mu_X$$

(8.34)

and its autocorrelation depends only on the time difference τ

$$E[X(t)X(t + \tau)] = R_{XX}(\tau)$$

(8.35)

From Eqs. (8.17) and (8.35) it follows that the autocovariance of a WSS process also depends only on the time difference τ:

$$C_{XX}(\tau) = R_{XX}(\tau) - \mu_X^2$$

(8.36)

Setting τ = 0 in Eq. (8.35), we obtain

$$E[X^2(t)] = R_{XX}(0)$$

(8.37)

Thus, the average power of a WSS process is independent of t and equals $R_{XX}(0)$.

Similarly, a discrete-time random process $X(n)$ is WSS if

$$E[X(n)] = \mu_X = \text{constant}$$

(8.38)

and

$$E[X(n)X(n+k)] = R_{XX}(k)$$

(8.39)

Then

$$C_{XX}(k) = R_{XX}(k) - \mu_X^2$$

(8.40)

Setting k = 0 in Eq. (8.39) we have

$$E[X^2(n)] = R_{XX}(0)$$

(8.41)

Note that an SSS process is WSS but a WSS process is not necessarily SSS.

Two processes $X(t)$ and $Y(t)$ are called *jointly wide-sense stationary* (jointly WSS) if each is WSS and their cross-correlation depends only on the time difference τ:

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$$

(8.42)

From Eq. (8.42) it follows that the cross-covariance of jointly WSS $X(t)$ and $Y(t)$ also depends only on the time difference τ:

$$C_{XY}(\tau) = R_{XY}(\tau)\mu_X\mu_Y$$

(8.43)

Similar to Eqs. (8.27) to (8.30), two jointly WSS random process $X(t)$ and $Y(t)$ are independent if for all x and y

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

(8.44)

They are uncorrelated if for all τ

$$C_{XY}(\tau) = R_{XY}(\tau) - \mu_X\mu_Y = 0$$

(8.45)

or

$$R_{XY}(\tau) = \mu_X \mu_Y$$

(8.46)

They are orthogonal if for all τ

$$R_{XY}(\tau) = 0$$

(8.47)

Similarly, two random sequences $X(n)$ and $Y(n)$ are jointly WSS if each is WSS and their cross-correlation depends only on the time difference k:

$$R_{XY}(n, n + k) = E[X(n)Y(n + k)] = R_{XY}(k)$$

(8.48)

Then the cross-covariance of jointly WSS $X(n)$ and $Y(n)$ is

$$C_{XY}(k) = R_{XY}(k) - \mu_X \mu_Y$$

(8.49)

They are uncorrelated if for all k

$$C_{XY}(k) = 0$$

(8.50)

or

$$R_{XY}(k) = \mu_X \mu_Y$$

(8.51)

They are orthogonal if for all k

$$R_{XY}(k) = 0$$

(8.52)

8.3.4. D. Time Averages and Ergodicity:

The *time-averaged mean* of a sample function $x(t)$ of a random process $X(t)$ is defined as

$$\bar{x} = \langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

(8.53)

where the symbol $\langle \cdot \rangle$ denotes *time-averaging*.

Similarly, the *time-averaged autocorrelation* of the sample function $x(t)$ is defined as

$$\overline{R}_{XX}(\tau) = \langle x(t)x(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt$$

(8.54)

Note that $\overline{x}$ and $\overline{R}_{XX}(\tau)$ are random variables; their values depend on which sample function of *X*(*t*) is used in the time-averaging evaluations.

If *X*(*t*) is stationary, then by taking the expected value on both sides of Eqs. (7.20) and (7.21), we obtain

$$E[\overline{x}] = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} E[x(t)]\,dt = \mu_X$$

(8.55)

which indicates that the expected value of the time-averaged mean is equal to the ensemble mean, and

$$E[\overline{R}_{XX}(\tau)] = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} E[x(t)x(t+\tau)]\,dt = R_{XX}(\tau)$$

(8.56)

which also indicates that the expected value of the time-averaged autocorrelation is equal to the ensemble autocorrelation.

A random process *X*(*t*) is said to be *ergodic* if time averages are the same for all sample functions and equal to the corresponding ensemble averages. Thus, in an ergodic process, all its statistics can be obtained by observing a single sample function *x*(*t*) = *X*(*t*, *λ*) (*λ* fixed) of the process.

A stationary process *X*(*t*) is called *ergodic* in the *mean* if

$$\overline{x} = \langle x(t)\rangle = E[X(t)] = \mu_X$$

(8.57)

Similarly, a stationary process *X*(*t*) is called *ergodic in the autocorrelation* if

$$\overline{R}_{XX}(\tau) = \langle x(t)x(t+\tau)\rangle = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

(8.58)

The time-averaged mean of a sample sequence *x*(*n*) of a random sequence *X*(*n*) is defined as

$$\overline{x} = \langle x(n)\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} x(n)$$

(8.59)

Similarly, the time-average autocorrelation of the sample sequence *x*(*n*) is defined as

$$\overline{R}_{XX}(k) = \langle x(n)x(n+k)\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} x(n)x(n+k)$$

(8.60)

If *X*(*n*) is stationary, then

$$E\left[\bar{x}\right] = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} E[x(n)] = \mu_X$$

(8.61)

and

$$E\left[\bar{R}_{XX}(k)\right] = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} E[x(n)\,x(n+k)] = R_{XX}(k)$$

(8.62)

Thus, $X(n)$ is also ergodic in the mean and autocorrelation if

$$\bar{x} = \langle x(n)\rangle = E[X(n)] = \mu_X$$

(8.63)

$$\bar{R}_{XX}(k) = \langle x(n)x(n+k)\rangle = E[x(n)\,x(n+k)] = R_{XX}(k)$$

(8.64)

Testing for the ergodicity of a random process is usually very difficult. A reasonable assumption in the random analysis of most random signals is that the random waveforms are ergodic in the mean and in the autocorrelation. Fundamental electrical engineering parameters, such as dc value, root-mean-square (rms) value, and average power can be related to the statistical averages of an ergodic random process. They are summarized in the following:

1. $\bar{x} = \langle x(t)\rangle$ is equal to the dc level of the signal.
2. $[\bar{x}]^2 = \langle x(t)\rangle^2$ is equal to the normalized power in the dc component.
3. $\bar{R}_{XX}(0) = \langle x^2(t)\rangle$ is equal to the total average normalized power.
4. $\bar{\sigma}_X^2 = \langle x^2(t)\rangle - \langle x(t)\rangle^2$ is equal to the average normalized power in the time-varying or ac component of the signal.
5. $\bar{\sigma}_X$ is equal to the rms value of the ac component of the signal.

8.4. Gaussian Random Process:

Consider a random process $X(t)$, and define n random variables $X(t_1), \ldots, X(t_n)$ corresponding to n time instants $t_1, \ldots, t_n$. Let **X** be a *random vector* ($n \times 1$ matrix) defined by

$$\mathbf{X} = \begin{bmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{bmatrix}$$

(8.65)

Let **x** be an n-dimensional vector ($n \times 1$ matrix) defined by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(8.66)

so that the event $\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\}$ is written $\{\mathbf{X} \le \mathbf{x}\}$. Then $X(t)$ is called a *Gaussian* (or normal) process if **X** has a jointly multivariate Gaussian density function for every finite set of $\{t_i\}$ and every n.

The multivariate Gaussian density function is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\,|\det \mathbf{C}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

(8.67)

where T denotes the "transpose," **μ** is the *vector means*, and **C** is the *covariance matrix*, given by

$$\boldsymbol{\mu} = E[\mathbf{X}] = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_n \end{bmatrix} = \begin{bmatrix} E[X(t_1)] \\ \vdots \\ E[X(t_n)] \end{bmatrix}$$

(8.68)

$$\mathbf{C} = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \cdots & \cdots & \cdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}$$

(8.69)

where

$$C_{ij} = C_{XX}(t_i, t_j) = R_{XX}(t_i, t_j) - \mu_i \mu_j$$

(8.70)

which is the covariance of $X(t_i)$ and $X(t_j)$, and det **C** is the determinant of the matrix **C**.

8.4.1.1. Alternate Definition:

A random process $X(t)$ is a Gaussian process if for any integers n and any subset $\{t_1, \ldots, t_n\}$ of T, and any real coefficients $a_k (1 \le k \le n)$, the r.v.

$$\sum_{k=1}^{n} a_k X(t_k) = a_1 X(t_1) + a_2 X(t_2) + \cdots a_n X(t_n)$$

(8.71)

is a Gaussian r.v..

Some of the important properties of a Gaussian process are as follows:

1. A Gaussian process $X(t)$ is completely specified by the set of means

$$\mu_i = E[X(t_i)] \qquad i = 1, \ldots, n$$

and the set of autocorrelations

$$R_{XX}(t_i, t_j) = E[X(t_i)X(t_j)] \qquad i, j = 1, \ldots, n$$

2. If the set of random variables $X(t_i)$, $i = 1, \ldots, n$, is uncorrelated, that is,

$$C_{ij} = 0 \qquad i \neq j$$

then $X(t_i)$ are independent.

3. If a Gaussian process $X(t)$ is WSS, then $X(t)$ is *SSS*.
4. If the input process $X(t)$ of a linear system is Gaussian, then the output process $Y(t)$ is also Gaussian.

8.5. SOLVED PROBLEMS

8.1. Consider a random process $X(t)$ defined by

$$X(t) = Y \cos \omega t \qquad t \geq 0$$

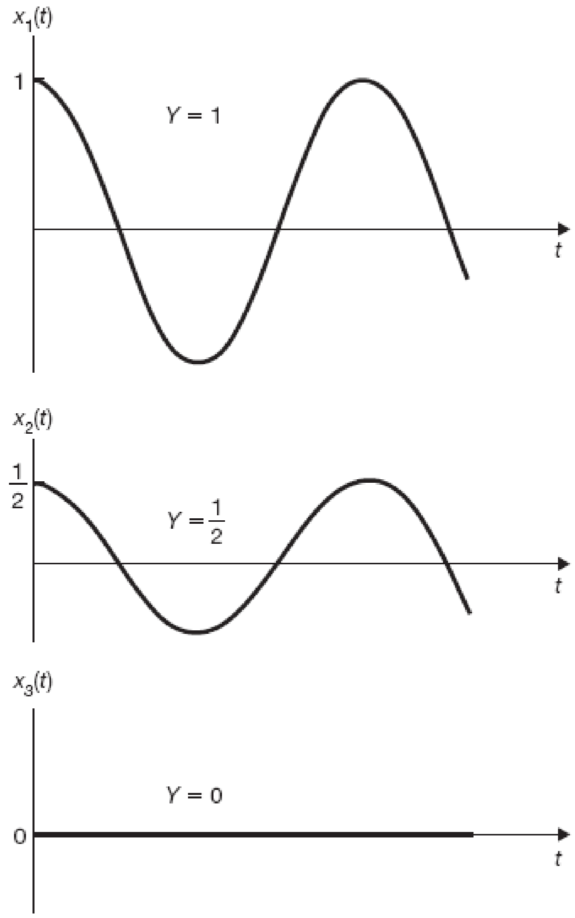
(8.72)

where ω is a constant and Y is a uniform r.v. over (0, 1).

a. Describe $X(t)$.
b. Sketch a few typical sample functions of $X(t)$.

a. The random process $X(t)$ is a continuous-parameter (or time), continuous-state random process. The state space is $E = \{x: -1 < x < 1\}$ and the index parameter set is $T = \{t: t \geq 0\}$.
b. Three sample functions of $X(t)$ are sketched in Fig. 8-5.

Figure 8-5



8.2. Consider a random signal $X(t)$ given by

$$X(t) = \sum_{k=-\infty}^{\infty} A_k p(t - kT_b - T_d)$$

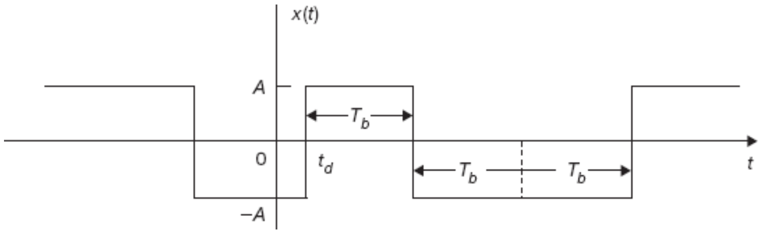
(8.73)

where $\{A_k\}$ is a sequence of independent r.v.'s with $P[A_k = A] = P[A_k = -A] = \frac{1}{2}$, $p(t)$ is a unit amplitude pulse of duration T_b, and T_d is a r.v. uniformly distributed over $[0, T_b]$.

a. Describe $X(t)$.
b. Sketch a sample function of $X(t)$.

a. The random signal $X(t)$ is a continuous time, discrete-state random process. The state space is $(A, -A)$, and the index parameter set is $T = (t; -\infty < t < \infty)$. $X(t)$ is known as a *random binary signal*.
b. A sample function of $X(t)$ is sketched in Fig. 8-6.

Figure 8-6



8.3. Consider a random process $\{X(t); t \geq 0\}$, where $X(t)$ represents the total number of "events" that have occurred in the interval $(0, t)$.

a. Describe $X(t)$.

b. Sketch a sample function of $X(t)$.

a. (*a*) From definition, $X(t)$ must satisfy the following conditions:

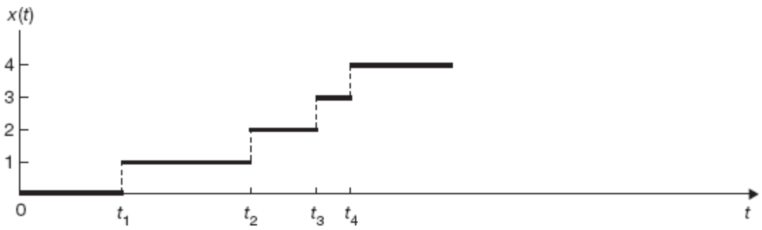
1. $X(t) \geq 0$ and $X(0) = 0$
2. $X(t)$ is integer valued.
3. $X(t_1) \leq X(t_2)$ if $t_1 < t_2$
4. $X(t_2) - X(t_1)$ equals the number of events that have occurred on the interval (t_1, t_2).

Thus, $X(t)$ is a continuous-time discrete state random process.

Note that $X(t)$ is known as a *counting* process. A counting process $X(t)$ is said to possess *independent increments* if the number of events which occur in disjoint intervals are independent.

b. A sample function of $X(t)$ is sketched in Fig. 8-7.

Figure 8-7 *A sampling function of a counting process.*



8.4. Let $W_1, W_2, \ldots$ be independent identically distributed (i.i.d.) zero-mean Gaussian r.v.'s. Let

$$X_n = \sum_{k=1}^{n} W_k = W_1 + W_2 + \cdots + W_n \qquad n = 1, 2, \ldots$$

(8.74)

with $X_0 = 0$. The collection of r.v.'s $X(n) = \{X_n, n \geq 0\}$ is a random process.

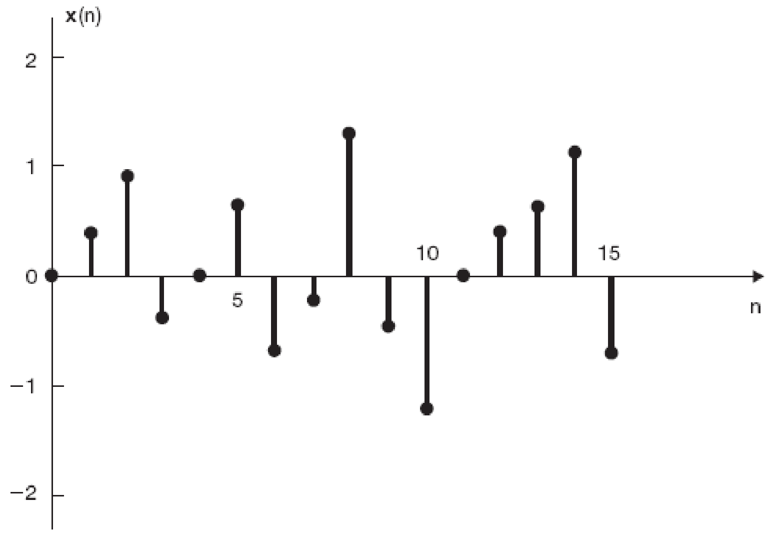
a. Describe $X(n)$.

b. Sketch a sample function of $X(n)$.

a. The random signal $X(n)$ is a discrete time, continuous-state random process. The state space is $E = (-\infty, \infty)$ and the index parameter set is $T = \{0, 1, 2, \ldots\}$.

b. A sample function of $X(t)$ is sketched in Fig. 8-8.

Figure 8-8



8.5. Let $Z_1, Z_2, \ldots$ be independent identically distributed r.v.'s with $P(Z_n = 1) = p$ and $P(Z_n = -1) = q = 1 - p$ for all n. Let

$$X_n = \sum_{i=1}^{n} Z_i \qquad n = 1, 2, \ldots$$

(8.75)

and $X_0 = 0$. The collection of r.v.'s $\{X_n, n \geq 0\}$ is a random process, and it is called the *simple random walk* X(n) in one dimension.

a. Describe the simple random walk $X(n)$.

b. Construct a typical sample sequence (or realization) of $X(n)$.

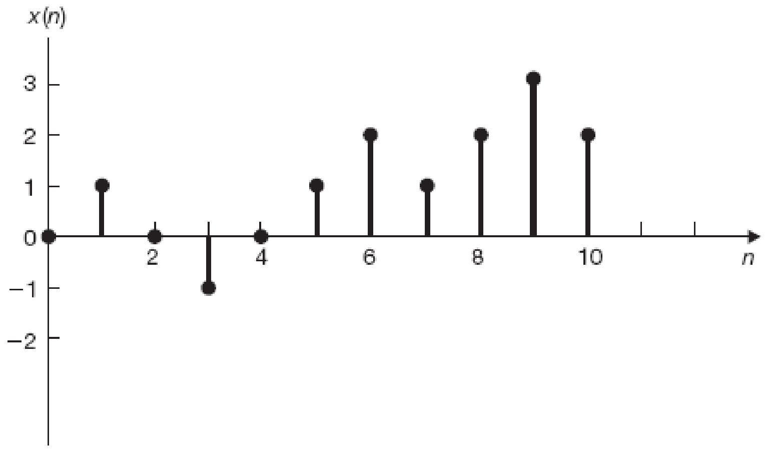
a. The simple random walk $X(n)$ is a discrete-parameter (or time), discrete-state random process. The state space is $E = \{ \ldots, -2, -1, 0, 1, 2, \ldots\}$, and the index parameter set is $T = \{ 0, 1, 2, \ldots\}$.

b. A sample sequence $x(n)$ of a simple random walk $X(n)$ can be produced by tossing a coin every second and letting $x(n)$ increase by unity if a head appears and decrease by unity if a tail appears. Thus, for instance,

n	0	1	2	3	4	5	6	7	8	9	10	...
Coin tossing		H	T	T	H	H	H	T	H	H	T	...
x(*n*)	0	1	0	−1	0	1	2	1	2	3	2	...

The sample sequence $x(n)$ obtained above is plotted in Fig. 8-9. The simple random walk $X(n)$ specified in this problem is said to be *unrestricted* because there are no bounds on the possible values of X_n.

Figure 8-9



8.6. Give an example of a complex random signal.

Consider a random signal $X(t)$ given by

$$X(t) = A(t)\cos[\omega t + \Theta(t)]$$

(8.76)

where ω is a constant, and $A(t)$ and $\Theta(t)$ are real random signals. Now $X(t)$ can be rewritten as

$$X(t) = \text{Re}\{A(t)\, e^{j\Theta(t)}\, e^{j\omega t}\} = \text{Re}\{Y(t)\, e^{j\omega t}\}$$

(8.77)

where Re denotes "take real part of." Then

$$Y(t) = A(t)e^{j\Theta(t)} = A(t)\cos\Theta(t) + j\,A(t)\sin\Theta(t)$$

(8.78)

is a complex random signal.

8.5.1. Statistics of Random Processes

8.7. Let a random signal $X(t)$ be specified by

$$X(t) = t - Y$$

(8.79)

where Y is an exponential r.v. with pdf

$$f_Y(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0\ , & y < 0 \end{cases}$$

Find the first-order cdf of $X(t)$, $F_X(x; t)$.

$$F_X(x;t) = P\{X(t) \le x\} = P\{t - Y \le x\}$$
$$= P\{Y \le t - X\}$$
$$= \int_{t-x}^{\infty} f_Y(y)\,dy = \int_{t-x}^{\infty} e^{-y} dy = e^{-(t-x)} \qquad t \ge x$$

Next, if $x > t$, then $t - x < 0$ and $Y \ge 0$, and

$$F_X(x;t) = P\{Y \le t - X\} = P\{Y \ge 0\}$$
$$= \int_0^{\infty} f_Y(y) dy = \int_0^{\infty} e^{-y} dy = 1 \qquad t < x$$

Thus,

$$F_X(x;t) = \begin{cases} e^{-(t-x)}, & t \ge x \\ 1 \quad , & t < x \end{cases}$$

(8.80)

8.8. A discrete-time random sequence $X(n)$ is defined by $X(n) = A^n (n \ge 0)$, where A is a uniform r.v. over (0, 1). Find the mean $\mu_X(n)$ and autocorrelation $R_{XX}(n, m)$ of $X(n)$.

The pdf of A is given by

$$f_A(a) = \begin{cases} 1, & 0 < a < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\mu_X(n) = E[X(n)] = E[A^n] = \int_0^1 a^n \, da = \frac{1}{n+1}$$

(8.81)

and

$$R_{XX}(n, m) = E[X(n)X(m)] = E[A^{n+m}] = \int_0^1 a^{n+m} da = \frac{1}{n+m+1}$$

(8.82)

8.9. Show that

$$R_{XX}(t, t) \ge 0$$

(8.83)

From definition (8.13)

$$R_{XX}(t, t) = E[X(t)X(t)] = E[X^2(t)] \ge 0$$

since $E[Y^2] \ge 0$, for any r.v. Y.

8.10. Show that

$$\left| R_{XX}(t_1, t_2) \right| \le \frac{R_{XX}(t_1, t_1) + R_{XX}(t_2, t_2)}{2}$$

(8.84)

Since $E[Y^2] \geq 0$, for any r.v. Y, we have

$$0 \leq E[(X(t_1) + X(t_2))^2] = R_{XX}(t_1, t_1) + 2R_{XX}(t_1, t_2) + R_{XX}(t_2, t_2)$$

(8.85)

$$0 \leq E[(X(t_1) - X(t_2))^2] = R_{XX}(t_1, t_1) - 2R_{XX}(t_1, t_2) + R_{XX}(t_2, t_2)$$

(8.86)

From Eqs. (8.85) and (8.86) we have

$$-R_{XX}(t_1, t_2) \leq \frac{R_{XX}(t_1, t_1) + R_{XX}(t_2, t_2)}{2}$$
$$R_{XX}(t_1, t_2) \leq \frac{R_{XX}(t_1, t_1) + R_{XX}(t_2, t_2)}{2}$$

which imply that

$$\left|R_{XX}(t_1, t_2)\right| \leq \frac{R_{XX}(t_1, t_1) + R_{XX}(t_2, t_2)}{2}$$

8.11. Consider a random signal given by

$$X(t) = A \cos \omega_0 t$$

(8.87)

where ω_0 is a constant and A is an uniform r.v. over [0,1]. Find the mean$\mu_X(t)$ and autocorrelation $R_{XX}(t_1, t_2)$ of $X(t)$.

The pdf of A is given by

$$f_A(a) = \begin{cases} 1, & 0 < a < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_X(t) = E[X(t)] = E[A \cos \omega_0 t] = E[A] \cos \omega_0 t = \frac{1}{2} \cos \omega_0 t$$

(8.88)

since

$$E[A] = \int_0^1 a\, f_A(a)\, da = \int_0^1 a\, da = \frac{1}{2}.$$

$$E[A] = \int_0^1 a\, f_A(a)\, da = \int_0^1 a\, da = \frac{1}{2}.$$

$$R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)] = E[A^2 \cos \omega_0 t_1 \cos \omega_0 t_2]$$

(8.89)

since

$$E[A^2] = \int_0^1 a^2 f_A(a)\,da = \int_0^1 a^2\,da = \frac{1}{3}.$$

8.12. A random sequence $X(n)$ is defined as

$$X(n) = An + B$$

(8.90)

where A and B are independent zero mean Gaussian r.v.'s of variance σ_A^2 and σ_B^2, respectively.

a. Find the mean $\mu_X(n)$ and autocorrelation $R_{XX}(n, m)$ of $X(n)$.

b. Find $E[X^2(n)]$.

$$\mu_X(n) = E[X(n)] = E[An + B] = E[A]n + E[B] = 0$$

(8.91)

since $E[A] = E[B] = 0$.

$$\begin{aligned} R_{XX}(n, m) &= E[X(n)X(m)] = E[(An + B)(Am + B)] \\ &= E[A^2]nm + E[AB](n + m) + E[B^2] \\ &= \sigma_A^2 nm + \sigma_B^2 \end{aligned}$$

(8.92)

since $E[AB] = E[BA] = E[A]E[B] = 0$.

b. Setting $m = n$ in Eq. (8.92), we obtain

$$E[X^2(n)] = \sigma_A^2 n^2 + \sigma_B^2$$

(8.93)

8.13. A counting process $X(t)$ of Prob. 8.3 is said to be a Poisson process with rate (or intensity) λ (> 0) if

1. $X(0) = 0$.
2. $X(t)$ has independent increments.
3. The number of events in any interval of length t is Poisson distributed with mean λt; that is, for all $s, t > 0$,

$$P[X(t + s) - X(s) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

(8.94)

a. Find the mean $\mu_X(t)$ and $E[X^2(t)]$.

b. Find the autocorrelation $R_{XX}(t_1, t_2)$ of $X(t)$.

a. Setting $s = 0$ in Eq. (8.94) and using condition 1, we have

$$P[X(t) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

(8.95)

Thus,

$$\mu_X(t) = E[X(t)] = \sum_{n=1}^{\infty} n\,P[X(t)=n] = \sum_{n=1}^{\infty} n e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

(8.96)

Now, the Taylor expansion of $e^{\lambda t}$ is given by

$$e^{\lambda t} = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}$$

Differentiating twice with respect to λt, we obtain

$$e^{\lambda t} = \sum_{n=0}^{\infty} n\frac{(\lambda t)^{n-1}}{n!} = \frac{1}{\lambda t}\sum_{n=1}^{\infty} n\frac{(\lambda t)^n}{n!}$$

(8.97)

$$e^{\lambda t} = \sum_{n=0}^{\infty} n(n-1)\frac{(\lambda t)^{n-2}}{n!} = \frac{1}{(\lambda t)^2}\sum_{n=1}^{\infty} n^2\frac{(\lambda t)^n}{n!} - \frac{1}{(\lambda t)^2}\sum_{n=1}^{\infty} n\frac{(\lambda t)^n}{n!}$$

(8.98)

Using Eqs. (9.97) and (9.98), we obtain

$$\mu_X(t) = E[X(t)] = e^{-\lambda t}\sum_{n=1}^{\infty} n\frac{(\lambda t)^n}{n!} = e^{-\lambda t}e^{\lambda t}(\lambda t) = \lambda t$$

(8.99)

and

$$\begin{aligned} E\left[X^2(t)\right] &= \sum_{n=1}^{\infty} n^2 P[X(t)=n] = e^{-\lambda t}\sum_{n=1}^{\infty} n^2\frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda t}\left[(\lambda t)^2 e^{\lambda t} + (\lambda t)e^{\lambda t}\right] = (\lambda t)^2 + (\lambda t) \end{aligned}$$

(8.100)

b. Next, let $t_1 < t_2$, the r.v.'s $X(t_1)$ and $X(t_2 - t_1)$ are independent since the intervals $(0, t_1)$ and (t_1, t_2) are non-overlapping, and they are Poisson distributed with mean λt_1 and $\lambda(t_2 - t_1)$, respectively. Thus,

$$E\{X(t_1)[X(t_2) - X(t_1)]\} = E[X(t_1)]E[X(t_2) - X(t_1)] = \lambda t_1\,\lambda(t_2 - t_1))$$

(8.101)

Now using identity

$$X(t_1)X(t_2) = X(t_1)[X(t_1) + X(t_2) - X(t_1)] = [X^2(t_1)] + X(t_1)[X(t_2) - X(t_1)]$$

we have

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] = E[X^2(t_1)] + E\{X(t_1)[X(t_2) - X(t_1)]\} \\ &= \lambda t_1 + \lambda^2 t_1^2 + \lambda t_1\,\lambda(t_2 - t_1) = \lambda t_1 + \lambda^2 t_1 t_2 \qquad t_1 \le t_2 \end{aligned}$$

(8.102)

Interchanging t_1 and t_2, we have

$$R_{XX}(t_1, t_2) = \lambda t_2 + \lambda^2 t_1 t_2 \qquad t_1 \geq t_2$$

(8.103)

Thus, combining Eqs. (102) and (8.103), we obtain

$$R_{XX}(t_1, t_2) = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2$$

(8.104)

8.14. Let $X(t)$ and $Y(t)$ be defined by

$$X(t) = A\cos \omega t + B\sin \omega t$$

(8.105)

$$Y(t) = B\cos \omega t - A\sin \omega t$$

(8.106)

where ω is constant and A and B are independent random variables both having zero mean and variance σ^2. Find the cross-correlation of $X(t)$ and $Y(t)$.

The cross-correlation of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\ &= E[(A\cos \omega t_1 + B\sin \omega t_1)(B\cos \omega t_2 - A\sin \omega t_2)] \\ &= E[AB](\cos \omega t_1 \cos \omega t_2 - \sin \omega t_1 \sin \omega t_2) \\ &\quad - E[A^2]\cos \omega t_1 \sin \omega t_2 + E[B^2]\sin \omega t_1 \cos \omega t_2 \end{aligned}$$

Since

$$E[AB] = E[A]E[B] = 0 \qquad E[A^2] = E[B^2] = \sigma^2$$

we have

$$\begin{aligned} R_{XY}(t_1, t_2) &= \sigma^2(\sin \omega t_1 \cos \omega t_2 - \cos \omega t_1 \sin \omega t_2) \\ &= \sigma^2 \sin \omega(t_1 - t_2) \end{aligned}$$

or

$$R_{XY}(\tau) = -\sigma^2 \sin \omega\tau$$

where

$$\tau = t_2 - t_1.$$

(8.107)

8.15. Consider a random process $X(t)$ given by

$$X(t) = A\cos(\omega t + \Theta)$$

(8.108)

where A and ω are constants and Θ is a uniform random variable over $[-\pi, \pi]$. Show that $X(t)$ is WSS.

From Eq. (B.57) (Appendix B), we have

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & -\pi \le \theta \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\begin{aligned} \mu_X(t) = E[X(t)] &= \int_{-\infty}^{\infty} A\cos(\omega t + \theta) f_\Theta(\theta)\, d\theta \\ &= \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta = 0 \end{aligned}$$

(8.109)

$$\begin{aligned} R_{XX}(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\cos[\omega(t+\tau)+\theta]\, d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}[\cos\omega t + \cos(2\omega t + 2\theta + \omega\tau)]\, d\theta \\ &= \frac{A^2}{2}\cos\omega\tau \end{aligned}$$

(8.110)

Since the mean of $X(t)$ is a constant and the autocorrelation of $X(t)$ is a function of time difference only, we conclude that $X(t)$ is WSS.

Note that $R_{XX}(\tau)$ is periodic with the period $T_0 = 2\pi/\omega$. A WSS random process is called *periodic* if its autocorrelation is periodic.

8.16. Consider a random process $X(t)$ given by

$$X(t) = A\cos(\omega t + \theta)$$

(8.111)

where ω and θ are constants and A is a random variable. Determine whether $X(t)$ is WSS.

$$\begin{aligned} \mu_X(t) &= E[X(t)] = E[A\cos(\omega t + \theta)] \\ &= \cos(\omega t + \theta)E[A] \end{aligned}$$

(8.112)

which indicates that the mean of $X(t)$ is not constant unless $E[A] = 0$.

$$\begin{aligned} R_{XX}(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E[(A^2\cos(\omega t + \theta)\cos[\omega(t+\tau)+\theta)]] \\ &= \frac{1}{2}[\cos\omega\tau + \cos(2\omega t + 2\theta + \omega\tau)]E[A^2] \end{aligned}$$

(8.113)

Thus, we see that the autocorrelation of $X(t)$ is not a function of the time difference τ only, and the process $X(t)$ is not WSS.

8.17. Consider a random process $X(t)$ given by

$$X(t) = A\cos \omega t + B\sin \omega t$$

(8.114)

where ω is constant and A and B are random variables.

a. Show that the condition

$$E[A] = E[B] = 0$$

(8.115)

is necessary for $X(t)$ to be stationary.

b. Show that $X(t)$ is WSS if and only if the random variables A and B are uncorrelated with equal variance; that is,

$$E[AB] = 0$$

(8.116)

and

$$E[A^2] = E[B^2] = \sigma^2$$

(8.117)

a. $\mu_X(t) = E[X(t)] = E[A]\cos \omega t + E[B]\sin \omega t$ must be independent of t for $X(t)$ to be stationary. This is possible only if $\mu_X(t) = 0$; that is,

$$E[A] = E[B] = 0$$

b. If $X(t)$ is WSS, then from Eq. (8.37)

$$E[X^2(0)] = E\left[X^2\left(\frac{\pi}{2\omega}\right)\right] = R_{XX}(0) = \sigma_X^2$$

But

$$X(0) = A \quad \text{and} \quad X\left(\frac{\pi}{2\omega}\right) = B$$

Thus,

$$E[A^2] = E[B^2] = \sigma_X^2 = \sigma^2$$

Using the preceding result, we obtain

$$\begin{aligned} R_{XX}(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E[(A\cos \omega t + B\sin \omega t)[A\cos \omega(t+\tau) + B\sin \omega(t+\tau)]] \\ &= \sigma^2 \cos \omega t + E[AB]\sin(2\omega t + \omega\tau) \end{aligned}$$

(8.118)

which will be a function of τ only if $E[AB] = 0$.

Conversely, if $E[AB] = 0$ and $E[A^2] = E[B^2] = \sigma^2$, then from the result of part (*a*) and Eq. (8.118), we have

$$\mu_X(t) = 0$$
$$R_{XX}(t, t+\tau) = \sigma^2 \cos \omega\tau = R_{XX}(\tau)$$

Hence, $X(t)$ is WSS.

8.18. A random process $X(t)$ is said to be *covariance-stationary* if the covariance of $X(t)$ depends only on the time difference $\tau = t_2 - t_1$; that is,

$$C_{XX}(t, t+\tau) = C_{XX}(\tau)$$

(8.119)

Let $X(t)$ be given by

$$X(t) = (A+1)\cos t + B \sin t$$

where A and B are independent random variables for which

$$E[A] = E[B] = 0 \qquad \text{and} \qquad E[A^2] = E[B^2] = 1$$

Show that $X(t)$ is not WSS, but it is covariance-stationary.

$$\begin{aligned} \mu_X(t) &= E[X(t)] = E[(A+1)\cos t + B\sin t)] \\ &= E[A+1]\cos t + E[B]\sin t \\ &= \cos t \end{aligned}$$

which depends on t. Thus, $X(t)$ cannot be WSS.

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= E[[(A+1)\cos t_1 + B\sin t_1][(A+1)\cos t_2 + B\sin t_2]] \\ &= E[(A+1)^2]\cos t_1 \cos t_2 + E[B^2]\sin t_1 \sin t_2 \\ &\quad + E[(A+1)B](\cos t_1 \sin t_2 + \sin t_1 \cos t_2) \end{aligned}$$

Now

$$\begin{aligned} E[(A+1)^2] &= E[A^2 + 2A + 1] = E[A^2] + 2E[A] + 1 = 2 \\ E[(A+1)B] &= E[AB] + E[B] = E[A]\,E[B] + E[B] = 0 \\ E[B^2] &= 1 \end{aligned}$$

Substituting these values into the expression of $R_{XX}(t_1, t_2)$, we obtain

$$\begin{aligned} R_{XX}(t_1, t_2) &= 2\cos t_1 \cos t_2 + \sin t_1 \sin t_2 \\ &= \cos(t_2 - t_1) + \cos t_1 \cos t_2 \end{aligned}$$

From Eq. (8.17), we have

$$\begin{aligned}C_{XX}(t_1, t_2) &= R_{XX}(t_1, t_2) - \mu_X(t_1)\,\mu_X(t_2)\\ &= \cos(t_2 - t_1) + \cos t_1 \cos t_2 - \cos t_1 \cos t_2\\ &= \cos(t_2 - t_1)\end{aligned}$$

Thus, $X(t)$ is covariance-stationary.

8.19. Show that if a random process $X(t)$ is WSS, then it must also be covariance stationary.

If $X(t)$ is WSS, then

$$\begin{aligned}E[X(t)] &= \mu\ (\text{constant}) \qquad \text{for all } t\\ R_{XX}(t, t+\tau) &= R_{XX}(\tau) \qquad \text{for all } t\end{aligned}$$

Now

$$\begin{aligned}C_{XX}(t, t+\tau) &= \text{Cov}[X(t)X(t+\tau)] = R_{XX}(t, t+\tau) - E[X(t)]\,E[X(t+\tau)]\\ &= R_{XX}(\tau) - \mu^2\end{aligned}$$

which indicates that $C_{XX}(t, t+\tau)$ depends only on τ. Thus, $X(t)$ is covariance stationary.

8.20. Show that if $X(t)$ is WSS, then

$$E[[X(t+\tau) - X(t)]^2] = 2[R_{XX}(0) - R_{XX}(\tau)]$$

(8.120)

where $R_{XX}(\tau)$ is the autocorrelation of $X(t)$.

Using the linearity of E (the expectation operator) and Eqs. (8.35) and (8.37), we have

$$\begin{aligned}E\left[[X(t+\tau) - X(t)]^2\right] &= E\left[X^2(t+\tau) - 2X(t+\tau)X(t) + X^2(t)\right]\\ &= E\left[X^2(t+\tau)\right] - 2E[X(t+\tau)X(t)] + E\left[X^2(t)\right]\\ &= R_{XX}(0) - 2R_{XX}(\tau) + R_{XX}(0)\\ &= 2[R_{XX}(0) - R_{XX}(\tau)]\end{aligned}$$

8.21. Let $X(t) = A\cos(\omega t + \Theta)$, where ω is constant and both A and Θ are r.v.'s with pdf $f_A(a)$ and $f_\Theta(\theta)$, respectively. Find the conditions that $X(t)$ is WSS.

$$\mu_X(t) = E[X(t)] = E[A\cos(\omega t + \Theta)] = \iint a\cos(\omega t + \theta)\, f_{A\Theta}(a, \theta)\, d\theta\, da$$

(8.121)

The first condition for the double integral to be independent of t is for A and Θ to be statistically independent. Then

$$\mu_X(t) = E[A\cos(\omega t + \Theta)] = \iint a\cos(\omega t + \theta)\, f_A(a) f_\Theta(\theta)\, d\theta\, da$$

(8.122)

The second condition is for Θ to be uniformly distributed over $[0, 2\pi]$. Then we have $\mu_X(t) = 0$ since 1—2π $\int^{2\pi}_0 \cos(\omega t + \theta)\, d\theta = 0$.

Next,

$$R_{XX}(t_1, t_2) = E[X(t_1\ X(t_2)] = E[\,A^2\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)]$$

(8.123)

Since A and Θ are independent, we have

$$R_{XX}(t_1, t_2) = \frac{1}{2} E[A^2]\; E\{\cos \omega (t_2 - t_1) + \cos[\omega (t_2 + t_1) + 2\Theta]\}$$

(8.124)

and $E[\cos \omega(t_2 + t_1) + 2\Theta] = 0$ since Θ is uniformly distributed over $[0, 2\pi]$.

Thus,

$$R_{XX}(t_1, t_2) = \frac{1}{2} E[A^2] \cos\omega (t_2 - t_1) = \frac{1}{2} E[A^2] \cos \omega\tau$$

(8.125)

So, we conclude that $X(t)$ is WSS if A and Θ are independent, and Θ is uniformly distributed over $[0, 2\pi]$.

8.22. Let $Z(t) = X(t) + Y(t)$, where random processes $X(t)$ and $Y(t)$ are independent and WSS. Is $Z(t)$ WSS?

$$\mu_Z(t) = E[Z(t)] = E[X(t) + Y(t)] = \mu_X + \mu_Y = \text{constant}$$

(8.126)

$$\begin{aligned} R_{ZZ}(t, t+\tau) &= E[Z(t)\, Z(t+\tau)] = E\{[X(t) + Y(t)][X(t+\tau) + Y(t+\tau)]\} \\ &= E[X(t)X(t+\tau)] + E[Y(t)Y(t+\tau)] + E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] \\ &= R_{XX}(\tau) + R_{YY}(\tau) + E[X(t)]E[Y(t+\tau)] + E[Y(t)]E[X(t+\tau)] \\ &= R_{XX}(\tau) + R_{YY}(\tau) + 2\mu_X \mu_Y \end{aligned}$$

(8.127)

Since the mean of $Z(t)$ is constant and its autocorrelation depends only on τ, $Z(t)$ is WSS.

8.23. Let $Z(t) = X(t) + Y(t)$, where random processes $X(t)$ and $Y(t)$ are jointly WSS. Show that if $X(t)$ and $Y(t)$ are orthogonal, then

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

(8.128)

$$\begin{aligned} R_{ZZ}(t, t+\tau) &= E[Z(t)\, Z(t+\tau)] = E\{[X(t) + Y(t)][X(t+\tau) + Y(t+\tau)]\} \\ &= E[X(t)X(t+\tau)] + E[Y(t)Y(t+\tau)] + E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] \\ &= R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) \end{aligned}$$

(8.129)

Since $X(t)$ and $Y(t)$ are orthogonal, then $R_{XY}(\tau) = 0$, and we have

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

8.24. A random signal $X(t)$ is defined as $X(t) = At + B$, where A and B are independent r.v.'s with both zero mean and unit variance. Is $X(t)$ WSS?

$$\mu_X(t) = E[X(t)] = E[At + B] = E[A]\, t + E[B] = 0$$

since $E[A] = E[B] = 0$.

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] = E[(At_1 + B)(At_2 + B)] \\ &= E[A^2 t_1 t_2 + A Bt_1 + B At_2 + B^2] \\ &= E[A^2]\, t_1 t_2 + E[AB]\, t_1 + E[BA]\, t_2 + E[B^2] \\ &= 1 + t_1 t_2 \end{aligned}$$

(8.130)

since $E[A^2] = E[B^2] = 1$ and $E[AB] = E[BA] = E[A]\,E[B] = 0$.

Since $R_{XX}(t_1, t_2)$ is not the function of $|t_2 - t_1|$, $X(t)$ is not WSS.

8.25. Let $X(n) = \{X_n, n \geq 0\}$ be a random sequence of iid r.v.'s with mean 0 and variance 1. Show that $X(n)$ is WSS.

$$\mu_X(n) = E[X(n)] = E[X_n] = 0 \text{ constant}$$

$$R_{XX}(n, n+k) = E[X(n)X(n+k)] = E[X_n X_{n+k}] = \begin{cases} E(X_n)E(X_{n+k}) = 0, & k \neq 0 \\ E(X_n^2) = 1\ , & k = 0 \end{cases}$$

(8.131)

which depends only on k. Thus $X(n)$ is WSS.

8.26. A random signal $X(t)$ is defined as $X(t) = A$, where A is a r.v. uniformly distributed over [0, 1]. Is $X(t)$ ergodic in the mean?

$$\mu_X(t) = E[X(t)] = E[A] = \int_0^1 a\, da = \frac{1}{2}$$

$$\overline{x} = \langle x(t) \rangle = \frac{1}{2T}\int_{-T}^{T} x(t)\, dt = A \text{ as } T \to \infty$$

Since $\overline{x} \neq \mu_X(t)$, $X(t)$ is not ergodic in the mean.

8.27. Show that the process $X(t)$ defined in Eq. (8.108) (Prob. 8.15) is ergodic in both the mean and the autocorrelation.

From Eq. (8.53), we have

$$\begin{aligned} \overline{x} = \langle x(t) \rangle &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} A\cos(\omega t + \theta)\, dt \\ &= \frac{A}{T_0}\int_{-T_0/2}^{T_0/2}\cos(\omega t + \theta)\, dt = 0 \end{aligned}$$

(8.132)

where $T_0 = 2\pi/\omega$.

From Eq. (8.54), we have

$$\begin{aligned} \overline{R}_{XX}(\tau) &= \langle x(t)x(t+\tau) \rangle \\ &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} A^2\cos(\omega t + \theta)\cos[\omega(t+\tau) + \theta]\, dt \\ &= \frac{A^2}{T_0}\int_{-T_0/2}^{T_0/2}\frac{1}{2}[\cos\omega\tau + \cos(2\omega t + 2\theta + \omega\tau)]\, dt \\ &= \frac{A^2}{2}\cos\omega\tau \end{aligned}$$

(8.133)

Thus, we have

$$\mu_X(t) = E[X(t)] = \langle x(t) \rangle = \bar{x}$$
$$R_{XX}(\tau) = E[X(t)X(t+\tau)] = \langle x(t)X(t+\tau) \rangle = \bar{R}_{XX}(\tau)$$

Hence, by definitions (8.57) and (8.58), we conclude that $X(t)$ is ergodic in both the mean and the autocorrelation.

8.28. Consider a random process $Y(t)$ defined by

$$Y(t) = \int_0^t X(\tau)d\tau$$

(8.134)

where $X(t)$ is given by

$$X(t) = A\cos\omega t$$

(8.135)

where ω is constant and $A = N[0; \sigma^2]$.

a. Determine the pdf of $Y(t)$ at $t = t_k$.
b. Is $Y(t)$ WSS?

$$Y(t_k) = \int_0^{t_k} A\cos\omega\tau\, d\tau = \frac{\sin\omega t_k}{\omega}A$$

(8.136)

Then from the result of Example B.10 (Appendix B) we see that $Y(t_k)$ is a Gaussian random variable with

$$E[Y(t_k)] = \frac{\sin\omega t_k}{\omega}E[A] = 0$$

(8.137)

and

$$\sigma_Y^2 = \mathrm{var}[Y(t_k)] = \left(\frac{\sin\omega t_k}{\omega}\right)^2\sigma^2$$

(8.138)

Hence, by Eq. (B.53), the pdf of $Y(t_k)$ is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}e^{-y^2/(2\sigma_Y^2)}$$

(8.139)

b. From Eqs. (8.137) and (8.138), the mean and variance of $Y(t)$ depend on time $t(t_k)$, so $Y(t)$ is not WSS.

8.29. Show that if a Gaussian random process is WSS, then it is SSS.

If the Gaussian process $X(t)$ is WSS, then

$$\mu_i = E[X(t_i)] = \mu(= \text{constant}) \qquad \text{for all } t_i$$

and

$$R_{XX}(t_i, t_j) = R_{XX}(t_j - t_i)$$

Therefore, in the expression for the joint probability density of Eq. (8.67) and Eqs. (8.68), (8.69), and (8.70).

$$\begin{aligned} \mu_1 &= \mu_2 = \cdots = \mu_n = \mu \rightarrow E[X(t_i)] = E[X(t_i + c)] \\ C_{ij} &= C_{XX}(t_i, t_j) = R_{XX}(t_i, t_j) - \mu_i\mu_j \\ &= R_{XX}(t_j - t_i) - \mu^2 = C_{XX}(t_i + c, t_j + c) \end{aligned}$$

for any c. It then follows that

$$f_{\mathbf{X}(t_i)}(\mathbf{x}) = f_{\mathbf{X}(t_i + c)}(\mathbf{x})$$

for any c. Therefore, $X(t)$ is SSS by Eq. (8.31)

8.30. Let **X** be an n-dimensional Gaussian random vector [Eq. (8.65)] with independent components. Show that the multivariate Gaussian joint density function is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\prod_{i=1}^{n}\sigma_i} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2\right]$$

(8.140)

where $\mu_i = E[X_i]$ and $\sigma^2{}_i = \text{var}(X_i)$.

The multivariate Gaussian density function is given by Eq. (8.67). Since $X_i = X(t_i)$ are independent, we have

$$C_{ij} = \begin{cases} \sigma_i^2 & i = j \\ 0 & i \neq j \end{cases}$$

(8.141)

Thus, from Eq. (8.69) the covariance matrix **C** becomes

$$\mathbf{C} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

(8.142)

It therefore follows that

$$|\det \mathbf{C}|^{1/2} = \sigma_1\sigma_2\cdots\sigma_n = \prod_{i=1}^{n}\sigma_1$$

(8.143)

and

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_n^2} \end{bmatrix}$$

(8.144)

Then we can write

$$(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \sum_{i=1}^{n}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2$$

(8.145)

Substituting Eqs. (8.143) and (8.145) into Eq. (8.67), we obtain Eq. (8.140).

8.6. SUPPLEMENTARY PROBLEMS

8.31. Consider a random process $X(t)$ defined by

$$X(t) = \cos \Omega t$$

where Ω is a random variable uniformly distributed over $[0,\omega_0]$. Determine whether $X(t)$ is stationary.

8.32. Consider the random process $X(t)$ defined by

$$X(t) = A\cos \omega t$$

where ω is a constant and A is a random variable uniformly distributed over [0, 1]. Find the autocorrelation and autocovariance of $X(t)$.

8.33. Let $X(t)$ be a WSS random process with autocorrelation

$$R_{XX}(\tau) = Ae^{-\alpha|\tau|}$$

Find the second moment of the random variable $Y = X(5) - X(2)$.

8.34. A random signal $X(t)$ is given by $X(t) = A(t) \cos(\omega t + \Theta)$, where $A(t)$ is a zero mean WSS random signal with autocorrelation $R_{AA}(\tau)$, and Θ is a r.v. uniformly distributed over $[0, 2\pi]$ and independent of $A(t)$. The total average power of $A(t)$ is 1 watt.

a. Show that $X(t)$ is WSS.

b. Find the total average power of $X(t)$.

8.35. A random signal $X(t)$ is given by $X(t) = A + B\cos(\omega t + \Theta)$, where A, B, and Θ are independent r.v.'s uniformly distributed over [0, 1], [0, 2] and $[0, 2\pi]$, respectively. Find the mean and the autocorrelation of $X(t)$.

8.36. Let $X(t)$ be a WSS random process with mean μ_X. Let $Y(t) = a + X(t)$. Is $Y(t)$ WSS?

8.37. Let $X(t)$ and $Y(t)$ be defined by

$$X(t) = A + Bt,\ Y(t) = B + At$$

where A and B are independent r.v.'s with zero means and variance σ^2_A and σ^2_B, respectively. Find the autocorrelations and cross-correlation of $X(t)$ and $Y(t)$.

8.38. Let $Z(t) = X(t)Y(t)$, where $X(t)$ and $Y(t)$ are independent and WSS. Is $Z(t)$ WSS?

8.39. Two random signals $X(t)$ and $Y(t)$ are given by

$$X(t) = A\cos\omega t + B\sin\omega t,\ Y(t) = B\cos\omega t - A\sin\omega t$$

where ω is a constant, and A and B are independent r.v.'s with zero mean and same variance σ^2. Find the cross-correlation function of $X(t)$ and $Y(t)$.

8.7. ANSWERS TO SUPPLEMENTARY PROBLEMS

8.31. Nonstationary.

Hint: Examine specific sample functions of $X(t)$ for different frequencies, say, $\Omega = \pi/2, \pi$, and 2π.

8.32.

$$R_{XX}(t_1, t_2) = \frac{1}{3}\cos t_1 \cos t_2$$

$$C_{XX}(t_1, t_2) = \frac{1}{12}\cos t_1 \cos t_2$$

8.33. $2A(1 - e^{-3a})$

8.34. (a) Yes. (b) $^1/_2$ watts.

8.35. $\mu_X(t) = \frac{1}{2}, \quad R_{XX}(t, t+\tau) = \frac{1}{3} + \frac{1}{3}\cos\omega\tau$

8.36. Yes.

8.37.

$$R_{XX}(t_1, t_2) = \sigma_A^2 + \sigma_B^2 t_1 t_2,\ R_{YY}(t_1, t_2) = \sigma_B^2 + \sigma_A^2 t_1 t_2$$

$$R_{XY}(t_1, t_2) = \sigma_A^2 t_2 + \sigma_B^2 t_1$$

8.38. Yes.

8.39. $R_{XY}(t_1, t_2) = \sigma^2 \sin\omega(t_1 - t_2)$