Modal theory in step index fibers

Expressions of the fields – dispersion relationship – Mode cutoff – Modal phenomena

(lecture inspired from the book by Takanori Okoski "Optical fibers",

Academic Press, ISBN 012525260-9)

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The approach followed to determine the expression of the fields of the modes in optical fibers as well as the dispersion relationship for each mode is the same as that used for the slab waveguide or the rectangular dielectric waveguide. We just have to take under consideration that the section of the guide is now cylindrical.

I. Calculation of the components of the fields

General expression of the electric field (time-harmonic fields) : $\vec{\mathcal{E}}(x,y,z) = \Re e\left[\vec{E}(x,y).e^{j(\omega t-\beta z)}\right]$

Mode : electric field $\vec{E}(x,y) = \begin{vmatrix} E_x(x,y).\overrightarrow{ex} \\ E_y(x,y).\overrightarrow{ey} \\ E_z(x,y).\overrightarrow{ez} \end{vmatrix}$ magnetic field $\vec{H}(x,y) = \begin{vmatrix} H_x(x,y).\overrightarrow{ex} \\ H_y(x,y).\overrightarrow{ey} \\ H_z(x,y).\overrightarrow{ez} \end{vmatrix}$

Process followed :

→ *We develop the usual Maxwell equations in a dielectric medium with no electric charge and no current in it*

$$curl\vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t} \quad (1) \quad \text{with} \quad \vec{\mathcal{B}} = \mu\vec{\mathcal{H}} \quad ; \quad curl\vec{\mathcal{H}} = \varepsilon\frac{\partial \vec{\mathcal{E}}}{\partial t} \quad (2) \quad ; \quad div(\vec{\mathcal{D}}) = div(\varepsilon\vec{\mathcal{E}}) = \rho = 0$$

- We use the following relationships :

$$\frac{\partial}{\partial z}(\text{component}) = -j\beta.(\text{component}) \quad \text{and} \quad \frac{\partial}{\partial t}(\text{component}) = j\omega.(\text{component})$$

Warning: contrarily to the case of the slab waveguide studied in the previous chapter, there is no reason which could justify that $\frac{\partial}{\partial y}(\text{comp.})$ is zero. As well, no component is zero, *a priori*.

By developing (1) and (2), we obtain six differential equations relating the components E_x, E_y, E_z, H_x, H_y and H_z to each other.

$$\left| \begin{array}{ll} \dfrac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x & (3) \\ -j\beta E_x - \dfrac{\partial E_z}{\partial x} = -j\omega\mu H_y & (4) \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = -j\omega\mu H_z & (5) \end{array} \right.$$

and

$$\left| \begin{array}{ll} \dfrac{\partial H_z}{\partial y} + j\beta H_y = +j\omega\varepsilon E_x & (6) \\ -j\beta H_x - \dfrac{\partial H_z}{\partial x} = +j\omega\varepsilon E_y & (7) \\ \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = +j\omega\varepsilon E_z & (8) \end{array} \right.$$

→ *We write the transverse components E_x, E_y, H_x, and H_y as a function of the axial components E_z and H_z*

For example, in (6) we replace H_y by its value deduced from (4) → we'll find $E_x = f(E_z, H_z)$:

$$\frac{\partial H_z}{\partial y} + j\beta\left[\frac{\beta}{\omega\mu}E_x - \frac{j}{\omega\mu}\frac{\partial E_z}{\partial x}\right] = j\omega\varepsilon E_x$$

$$\Leftrightarrow E_x\left[j\frac{\beta^2}{\omega\mu} - j\omega\varepsilon\right] = -\frac{\partial H_z}{\partial y} - \frac{\beta}{\omega\mu}\frac{\partial E_z}{\partial x}$$

We multiply both sides of this equality by $\omega\mu$:

$$jE_x\left[\omega^2\varepsilon\mu - \beta^2\right] = \omega\mu\frac{\partial H_z}{\partial y} + \beta\frac{\partial E_z}{\partial x} \qquad (9)$$

We know that $\omega = k_0 c = \dfrac{k}{n_I}c = \dfrac{k}{\sqrt{\varepsilon_r}}\dfrac{1}{\sqrt{\varepsilon_0\mu_0}} = \dfrac{k}{\sqrt{\varepsilon\mu}}$ $\Leftrightarrow \omega^2\varepsilon\mu = k^2$ ($n_i = n_1$ or n_2)

And with $k^2 = \beta_t^2 + \beta^2$ we obtain $\omega^2\varepsilon\mu - \beta^2 = \beta_t^2$

Relation (9) becomes: $$E_x = \frac{-j}{\beta_t^2}\left[\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right] \qquad (10)$$

As well, for E_y, H_x et H_y we find:

$$E_y = \frac{-j}{\beta_t^2}\left[\beta\frac{\partial E_z}{\partial y} - \omega\mu\frac{\partial H_z}{\partial x}\right] \qquad (11)$$

$$H_x = \frac{-j}{\beta_t^2}\left[\beta\frac{\partial H_z}{\partial x} - \omega\varepsilon\frac{\partial E_z}{\partial y}\right] \qquad (12)$$

and $$H_y = \frac{-j}{\beta_t^2}\left[\beta\frac{\partial H_z}{\partial y} + \omega\varepsilon\frac{\partial E_z}{\partial x}\right] \qquad (13)$$

→ *We find the propagation equation expressed as a function of the component E_z by injecting (12) and (13) in (8).*

==> Demonstration:

$$(8) \Leftrightarrow \frac{\partial}{\partial x}\left[\frac{-j}{\beta_t^2}\left(\beta\frac{\partial H_z}{\partial y}+\omega\varepsilon\frac{\partial E_z}{\partial x}\right)\right]-\frac{\partial}{\partial y}\left[\frac{-j}{\beta_t^2}\left(\beta\frac{\partial H_z}{\partial x}-\omega\varepsilon\frac{\partial E_z}{\partial y}\right)\right]=j\omega\varepsilon E_z \qquad (14)$$

Then, we develop (14) which simplifies thanks to vanishing of terms containing H_z.

The remaining expression is : $\frac{-j\omega\varepsilon}{\beta_t^2}\left(\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}\right)=j\omega\varepsilon E_z$ $\boxed{\Leftrightarrow \frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\beta_t^2 E_z=0}$ (15)

(15) is the Helmoltz equation (or propagation equation) for the component E_z. We find a similar expression with H_z.

→ *In order to find the expressions of the components of the fields solutions of (10) to (13) and (15), in the case of an optical fiber which is a cylindrical waveguide, we must change both the coordinates system and the axis system.*

==> change from cartesian coodinates to cylindrical coordinates and change of axis system $((\vec{e_x},\vec{e_y},\vec{e_z})\rightarrow(\vec{e_r},\vec{e_\theta},\vec{e_z})$. (see Figure 1)

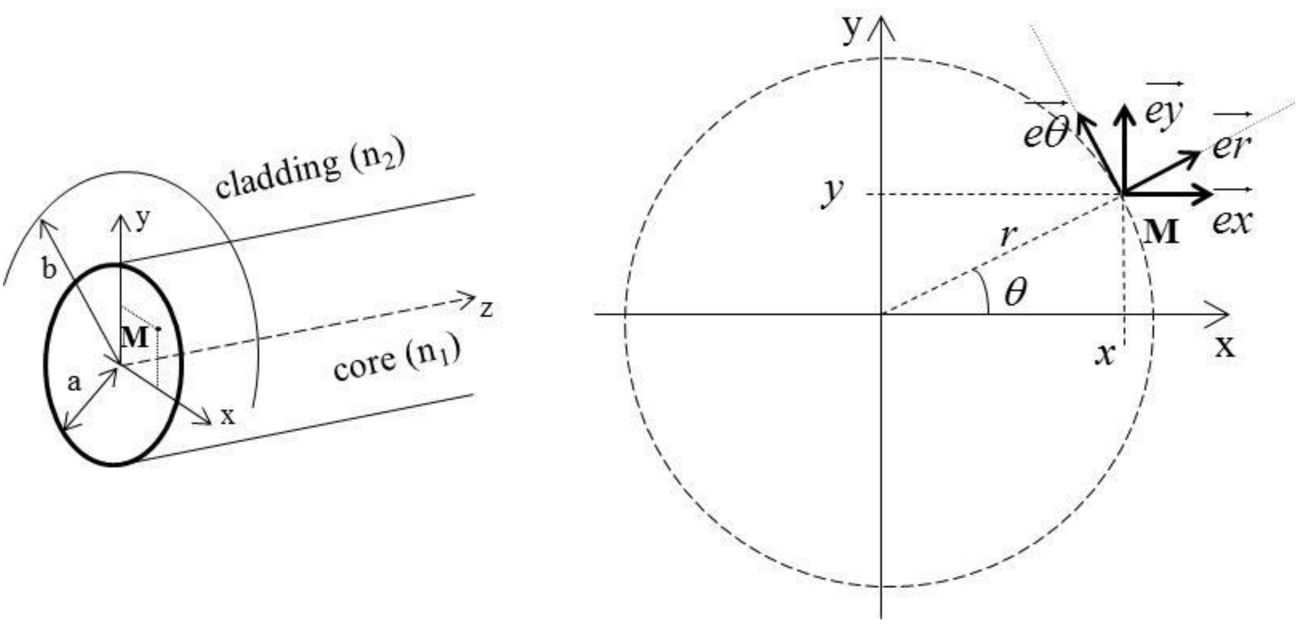

Figure 1 : change of axis system and change of coordinates system for the point M located in a cross section of a step index fiber

The mode $\vec{E}(x,y)=\left|\begin{array}{l}E_x(x,y).\overrightarrow{ex}\\E_y(x,y).\overrightarrow{ey}\\E_z(x,y).\overrightarrow{ez}\end{array}\right.$ becomes $\vec{E}(r,\theta)=\left|\begin{array}{l}E_r(r,\theta).\overrightarrow{er}\\E_\theta(r,\theta).\overrightarrow{e\theta}\\E_z(r,\theta).\overrightarrow{ez}\end{array}\right.$

We use the following conversion relationships :

* $x=r\cos\theta$ * $y=r\sin\theta$ * $r=\sqrt{x^2+y^2}$ *and* * $\theta=\tan^{-1}(y/x)$

* $\dfrac{dr}{dx}=\dfrac{1}{2}\left(x^2+y^2\right)^{-1/2}.2x\ =\dfrac{x}{r}$ * $\dfrac{d\theta}{dx}=\dfrac{1}{1+\left(\dfrac{y}{x}\right)^2}.\dfrac{-y}{x^2}\ =\dfrac{-y}{r^2}$

* $\dfrac{dr}{dy}=\dfrac{y}{r}$ * $\dfrac{d\theta}{dy}=\dfrac{x}{r^2}$

* et $\begin{pmatrix}E_r\\E_\theta\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}E_x\\E_y\end{pmatrix}=\begin{pmatrix}E_x\cos\theta+E_y\sin\theta\\-E_x\sin\theta+E_y\cos\theta\end{pmatrix}$

Calculation of $E_r=E_x\cos\theta+E_y\sin\theta$ with E_x and E_y given by (10) and (11) :

$$E_r=\underbrace{\frac{-j}{\beta_t^2}\left[\beta\frac{\partial E_z}{\partial x}+\omega\mu\frac{\partial H_z}{\partial y}\right]}_{E_x\uparrow}.\underbrace{\frac{x}{r}}_{\cos\theta\uparrow}\quad+\quad\underbrace{\frac{-j}{\beta_t^2}\left[\beta\frac{\partial E_z}{\partial y}-\omega\mu\frac{\partial H_z}{\partial x}\right]}_{E_y\uparrow}.\underbrace{\frac{y}{r}}_{\sin\theta\uparrow}\qquad(16)$$

with

$$\frac{\partial E_z}{\partial x}=\frac{\partial E_z}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial E_z}{\partial\theta}\frac{\partial\theta}{\partial x}=\frac{x}{r}\frac{\partial E_z}{\partial r}-\frac{y}{r^2}\frac{\partial E_z}{\partial\theta}\qquad(17)$$

$$\frac{\partial E_z}{\partial y}=\frac{\partial E_z}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial E_z}{\partial\theta}\frac{\partial\theta}{\partial y}=\frac{y}{r}\frac{\partial E_z}{\partial r}+\frac{x}{r^2}\frac{\partial E_z}{\partial\theta}\qquad(18)$$

We inject (17) and (18) in (16). We find, after simplification:

$$E_r=-\frac{j}{\beta_t^2}(\beta\frac{\partial E_z}{\partial r}+\frac{\omega\mu}{r}\frac{\partial H_z}{\partial\theta})\qquad(19)$$

Proceeding in the same way with E_θ, H_r, H_θ , we obtain:

$$E_\theta=-\frac{j}{\beta_t^2}(\beta\frac{1}{r}\frac{\partial E_z}{\partial r}-\omega\mu\frac{\partial H_z}{\partial r})\qquad(20)$$

$$H_r=-\frac{j}{\beta_t^2}(\beta\frac{\partial H_z}{\partial r}-\frac{\omega\varepsilon}{r}\frac{\partial E_z}{\partial\theta})\qquad(21)$$

$$H_\theta=-\frac{j}{\beta_t^2}(\beta\frac{1}{r}\frac{\partial H_z}{\partial\theta}+\omega\varepsilon\frac{\partial E_z}{\partial r})\qquad(22)$$

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→ *We also express the Helmoltz equation (15) in cylindrical coordinates:*

Starting from (15) we achieve the calculation of :

$$\frac{\partial^2 E_z}{\partial x^2} = \frac{\partial}{\partial r}\left[\frac{\partial E_z}{\partial x}\right].\underbrace{\frac{\partial r}{\partial x}}_{(17)^{\uparrow}} + \frac{\partial}{\partial \theta}\left[\frac{\partial E_z}{\partial x}\right].\underbrace{\frac{\partial \theta}{\partial x}}_{(17)^{\uparrow}} \quad \text{and} \quad \frac{\partial^2 E_z}{\partial y^2} = \frac{\partial}{\partial r}\left[\frac{\partial E_z}{\partial y}\right].\underbrace{\frac{\partial r}{\partial y}}_{(18)^{\uparrow}} + \frac{\partial}{\partial \theta}\left[\frac{\partial E_z}{\partial y}\right].\underbrace{\frac{\partial \theta}{\partial y}}_{(18)^{\uparrow}}$$

So, we find the following Helmoltz equation : $\boxed{\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2} + \beta_t^2 E_z = 0}$ (23)

Note: we can obtain a similar equation with H_z : ($\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2} + \beta_t^2 H_z = 0$)

→ *We solve the equation (23) by the method of separation of variables. Indeed, because the fiber exhibits a cylindrical symmetry around the direction z, the variables r and θ are independant ➔ we'll find $E_z(r,\theta)$.*

Therefore we write: $E_z(r,\theta) = R_z(r).T_z(\theta)$ In the following, we'll note: $E_z = R.\,T$ (24)

(24) injected in (23) leads to :

$$T\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}T\frac{\partial R}{\partial r} + \frac{1}{r^2}R\frac{\partial^2 T}{\partial \theta^2} + \beta_t^2 RT = 0$$

Multiplying both sides of this relation by $\frac{r^2}{RT}$ leads to :

$$\underbrace{\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + r^2\beta_t^2}_{\text{f(r)}} + \underbrace{\frac{1}{T}\frac{\partial^2 T}{\partial \theta^2}}_{\text{g}(\theta)} = 0 \qquad (25)$$

$$\left.\begin{aligned} \text{g}(\theta) &= -\nu^2 \quad \text{(independent of r)} \\ \text{f(r)} &= +\nu^2 \quad \text{(independent of } \theta) \end{aligned}\right\} \quad \text{where } \nu \text{ is a constant}$$

* **$\underline{g(\theta) = -\nu^2}$** $\Leftrightarrow \frac{1}{\mathrm{T}}\frac{\partial^2 \mathrm{T}}{\partial \theta^2} = -\nu^2 \Leftrightarrow \frac{\partial^2 \mathrm{T}}{\partial \theta^2} + \nu^2 \mathrm{T} = 0$ (26)

T is a periodic function of θ which takes the same value at each turn (i.e. when θ is increased by 2π).

$$\boxed{\text{if } \nu \neq 0 \qquad T(\theta) = \begin{cases} \cos(\nu\theta + \varphi_0) \\ \sin(\nu\theta + \varphi_0) \end{cases}}$$ to fullfill the previous condition, **ν must be INTEGER**

(27)

$\boxed{\text{if } \nu = 0 \qquad T(\theta) = \text{constant}}$;

Indeed, if ν=0, (26) becomes $\frac{\partial^2 T}{\partial \theta^2} = 0 \Leftrightarrow \frac{\partial T}{\partial \theta} = G \;\; (G\;\; cte) \Leftrightarrow \;\; T = G\theta + G'$

Because T must be periodic, it is necessary that G = 0 et T = G' = cte

* **<u>f(r) = +ν²</u>** $\Leftrightarrow \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\frac{\partial R}{\partial r} + r^2\beta_t^2 \;=\; \nu^2$ By multiplying both left and right terms by $\frac{R}{r^2}$, we find : $\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \left(\beta_t^2 - \frac{\nu^2}{r^2}\right) R = 0$ (28) —–> Bessel equation

The solution of (28) takes the form :

$$R(r) = \begin{cases} AJ_\nu(\beta_t r) + A'N_\nu(\beta_t r) & (\text{if } \beta_t \text{ is real}) \quad --> \text{because of the shape of the functions : in the core} \\ CK_\nu(|\beta_t| r) + C'I_\nu(|\beta_t| r) & (\text{if } \beta_t \text{ is imaginary}) --> \text{because of the shape of the functions : in the cladding} \end{cases}$$

where

A, A', C and C' are constants,

J_ν and N_ν are respectively Bessel functions of 1st 2nd kind, of ν order,

K_ν and I_ν are respectively modified Bessel functions of 1st 2nd kind, of ν order,

(see books on mathematics * G.N. Watson : "A treatise of the theory of Bessel functions" ; * Gouder : Les fonctions de Bessel (Masson (in French)) ; * Abramowitz and Steigun "Handbook of mathematical functions").

But $I_\nu \to \infty$ when $r \to \infty$ $\Rightarrow C' = 0$ (otherwise the field should be infinite far from the core : not physical)

but $N_\nu \to \infty$ when $r \to 0$ $\Rightarrow A' = 0$ (otherwise the field should tend towards infinite near the fiber axis: not physical)

It remains $$\boxed{R(r) = \begin{cases} AJ_\nu(\beta_t r) & (\text{if } \beta_t \text{ is real}) \quad --> \text{in the core} \\ CK_\nu(|\beta_t| r) & (\text{if } \beta_t \text{ is imaginary}) --> \text{in the cladding} \end{cases}}$$ (29)

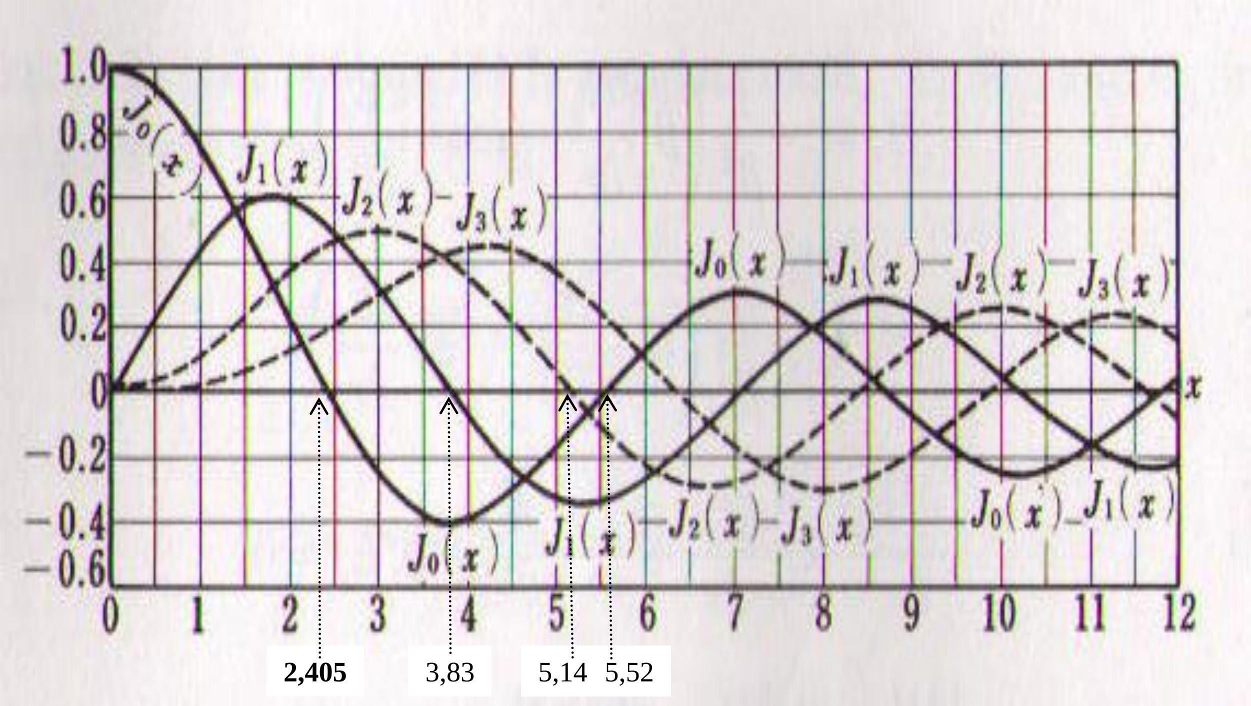


Figure 2 : First orders ν of the Bessel functions of the first kind J_ν (ν=0, 1, 2, 3)

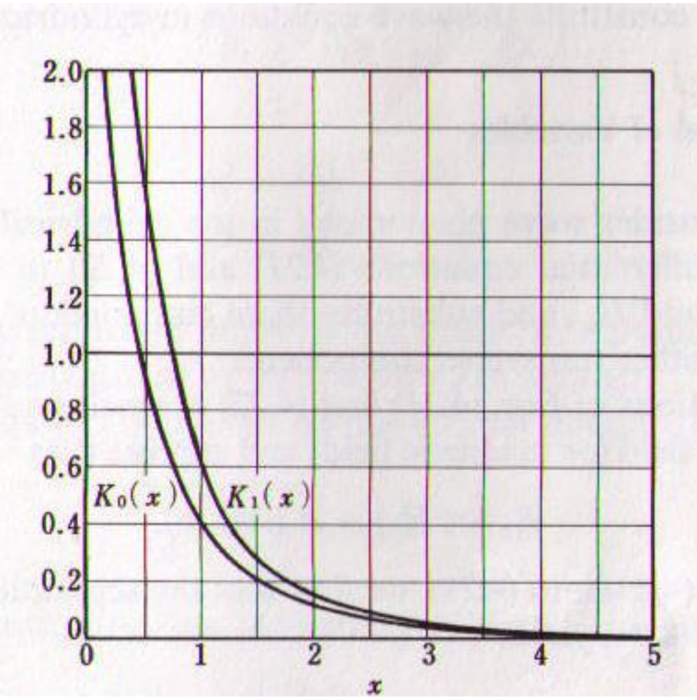


Figure 3 : First orders ν of the modified Bessel functions of the first kind K_ν (ν=0, 1)

→ *We find the propagation conditions for one mode, concerning its propagation constant*

In a the step index fiber with a uniform core that we consider, the refraction indices in the core (n_1) and in the cladding (n_2) are constant versus r, and the core radius is noted a (see Figure 1).

As already reminded in the case of a slab waveguide, the field will remain confined in the core if:

– β_t in the cladding ($=\beta_{t2}$) is imaginary (exponential decay when moving away from the axis)

– β_t in the core ($=\beta_{t1}$) is real.

We know that $k^2 = \beta_t^2 + \beta^2$ with $k = k_0 n_i$ (i = 1,2 in the core and in the cladding, respectively)

$$\Rightarrow \beta_{t2}^2 = k_0^2 n_2^2 - \beta^2 \le 0 \text{ in the cladding } (r \ge a) \text{ and}$$

$$\Rightarrow \beta_{t1}^2 = k_0^2 n_1^2 - \beta^2 > 0 \text{ in the core } (r < a).$$

β being real (no absorption), the two above inequalities result in the well known condition for guided propagation (see the chapter on "slab waveguide"): $\boxed{k_0 n_2 \le \beta \le k_0 n_1}$

→ *We specify the expression of the axial components of the fields, Ez and H_z*

* **<u>if ν ≠ 0</u>** From (24), (27) and (29), the general expression of E_z is then :

$$\begin{cases} E_z = AJ_\nu(\beta_{t1} r).\sin(\nu\theta + \varphi_0) \quad --> \text{in the core} \\ \quad = CK_\nu(|\beta_{t2}| r).\sin(\nu\theta + \varphi_0) --> \text{in the cladding} \end{cases} \qquad (30)$$

<u>Remarks</u> :

* we can choose φ_0=0 (the orientation of the axis Ox is then assumed such that E_z is zero on this axis (for θ=0))

* the general expression of H_z takes the same form, but it azimuthally varies like $\cos\nu\theta$. The is necessary in order to guarantee that the variations of the other components E_r, E_θ, H_r, *et* H_θ take the form f(r).$\cos\nu\theta$ or f(r) $\sin\nu\theta$ (see the relations (19) to (22)) . Such a form is required for allowing the continuity of the tangential components of $\vec{E}$ and $\vec{H}$ at the core-cladding boundary (r=a) whatever the value of θ.

Therefore, taking into account the above remarks, the general expression of H_z is :

$$\begin{cases} H_z = BJ_\nu(\beta_{t1} r).\cos\nu\theta \quad --> \text{in the core} \\ \quad = DK_\nu(|\beta_{t2}| r).\cos\nu\theta --> \text{in the cladding} \end{cases} \qquad (31)$$

Remarks :

1) Of course, if we have chosen to write E_z in the form $f_E(r).\cos \nu\theta$, H_z should be written in the form $f_H(r).\sin\nu\theta$

2) when $\nu\neq 0$, we never find $E_z = 0$ everywhere, or $H_z = 0$ everywhere. In this case, there are no TE or TM modes in the fiber. The modes are called "hybrid modes". They can be described as a linear combination of TE and TM modes. They are called :

3) EH modes if $H_z > E_z$ and HE modes if $E_z > H_z$.

This designation is arbitrary. It is the one which is used most often (see books by N.S. Kapany "optical waveguides" or by D. Marcuse "theory of dielectric optical waveguides") but sometimes we can find the opposite designation !

→ *We now introduce the normalized transverse propagation constants in the core (u) and in the cladding (w), and the normalized spatial frequency V.*

In the core (with radius a): $k = k_0 n_1$ thus $\beta_t = \beta_{t1} = \sqrt{k_0^2 n_1^2 - \beta^2}$ et $\boxed{u = a\beta_{t1} = a\sqrt{k_0^2 n_1^2 - \beta^2}}$ (32)

In the cladding : $k = k_0 n_2$ thus $|\beta_t| = |\beta_{t2}| = \sqrt{\beta^2 - k_0^2 n_2^2}$ et $\boxed{w = a|\beta_{t2}| = a\sqrt{\beta^2 - k_0^2 n_2^2}}$ (33)

Important : $\boxed{u^2 + w^2 = a^2 k_0^2 (n_1^2 - n_2^2) = \left(\frac{2\pi}{\lambda_0} a\sqrt{n_1^2 - n_2^2}\right)^2 = V^2}$ (34)

$\boxed{V = \frac{2\pi}{\lambda_0} a\sqrt{n_1^2 - n_2^2}}$ normalized spatial frequency of the fiber, at the operating wavelength

→ *We finally determine the expression of the transverse components of the fields (E_r, E_θ, H_r, and H_θ)*

Injecting the expressions (30) and (31) in (19) [→ E_r], (20) [→ E_θ], (21) [→ H_r] and (22) [→ H_θ], we obtain the expressions of the components of the fields in the core and in the cladding.
(see the two following pages)

$\nu \neq 0$

a. In the core

$$E_z = AJ_\nu\left(\frac{ur}{a}\right)\sin(\nu\theta) \qquad (C.1)$$

$$E_r = \left[-A\frac{j\beta}{(u/a)}J_\nu^{'}\left(\frac{ur}{a}\right) + B\frac{j\omega\mu_0}{(u/a)^2}\frac{\nu}{r}J_\nu\left(\frac{ur}{a}\right)\right]\sin(\nu\theta) \qquad (C.2)$$

$$E_\theta = \left[-A\frac{j\beta}{(u/a)^2}\frac{\nu}{r}J_\nu\left(\frac{ur}{a}\right) + B\frac{j\omega\mu_0}{(u/a)}J_\nu^{'}\left(\frac{ur}{a}\right)\right]\cos(\nu\theta) \qquad (C.3)$$

$$H_z = BJ_\nu\left(\frac{ur}{a}\right)\cos(\nu\theta) \qquad (C.4)$$

$$H_r = \left[A\frac{j\omega\varepsilon_1}{(u/a)^2}\frac{\nu}{r}J_\nu\left(\frac{ur}{a}\right) - B\frac{j\beta}{(u/a)}J_\nu^{'}\left(\frac{ur}{a}\right)\right]\cos(\nu\theta) \qquad (C.5)$$

$$H_\theta = \left[-A\frac{j\omega\varepsilon_1}{(u/a)}J_\nu^{'}\left(\frac{ur}{a}\right) + B\frac{j\beta}{(u/a)^2}\frac{\nu}{r}J_\nu\left(\frac{ur}{a}\right)\right]\sin(\nu\theta) \qquad (C.6)$$

b. In the cladding

$$E_z = CK_\nu\left(\frac{wr}{a}\right)\sin(\nu\theta) \qquad (C.7)$$

$$E_r = \left[C\frac{j\beta}{(w/a)}K_\nu^{'}\left(\frac{wr}{a}\right) - D\frac{j\omega\mu_0}{(w/a)^2}\frac{\nu}{r}K_\nu\left(\frac{wr}{a}\right)\right]\sin(\nu\theta) \qquad (C.8)$$

$$E_\theta = \left[C\frac{j\beta}{(w/a)^2}\frac{\nu}{r}K_\nu\left(\frac{wr}{a}\right) - D\frac{j\omega\mu_0}{(w/a)}K_\nu^{'}\left(\frac{wr}{a}\right)\right]\cos(\nu\theta) \qquad (C.9)$$

$$H_z = DK_\nu\left(\frac{wr}{a}\right)\cos(\nu\theta) \qquad (C.10)$$

$$H_r = \left[-C\frac{j\omega\varepsilon_2}{(w/a)^2}\frac{\nu}{r}K_\nu\left(\frac{wr}{a}\right) + D\frac{j\beta}{(w/a)}K_\nu^{'}\left(\frac{wr}{a}\right)\right]\cos(\nu\theta) \qquad (C.11)$$

$$H_\theta = \left[C\frac{j\varepsilon_2}{(w/a)}K_\nu^{'}\left(\frac{wr}{a}\right) - D\frac{j\beta}{(w/a)^2}\frac{\nu}{r}K_\nu\left(\frac{wr}{a}\right)\right]\sin(\nu\theta) \qquad (C.12)$$

ν = 0

In this case, the relations (C.1) to (C.12) are preserved, but without the factors $\sin(\nu\theta)$ and $\cos(\nu\theta)$, since T(θ) = cte.

We can now consider the case of TE modes by imposing $E_z = 0$. This implies that A = C = 0 (see (C.1) and (C.7)). Furthermore, $\dfrac{\partial(\text{component})}{\partial\theta} = 0$ since T(θ) = cte.
Thus, from the relationships (19) and (22), we can respectively deduce that $E_r = 0$ and $H_\theta = 0$. For the other components, it remains:

in the core :

$$E_\theta = B\frac{j\omega\mu_0}{(u/a)}J_0'\left(\frac{ur}{a}\right)$$
$$H_z = BJ_0\left(\frac{ur}{a}\right)$$
$$H_r = -B\frac{j\beta}{(u/a)}J_0'\left(\frac{ur}{a}\right)$$

in the cladding :

$$E_\theta = -D\frac{j\omega\mu_0}{(w/a)}K_0'\left(\frac{wr}{a}\right)$$
$$H_z = DK_0\left(\frac{wr}{a}\right)$$
$$H_r = D\frac{j\beta}{(w/a)}K_0'\left(\frac{wr}{a}\right)$$

As well, one can consider the case of TM modes by imposing $H_z = 0$. It is thus necessary that B = D = 0 (see (C.4) and (C.10)). Furthermore, we still have $\dfrac{\partial(\text{component})}{\partial\theta} = 0$ since T(θ) = cte.
Thus, from the relationships (20) and (21) we can respectively deduce that $E_\theta = 0$ and $H_r = 0$. For the other components it remains:

in the core

$$E_r = -A\frac{j\beta}{(u/a)}J_0'\left(\frac{ur}{a}\right)$$
$$E_z = AJ_0\left(\frac{ur}{a}\right)$$
$$H_\theta = -A\frac{j\omega\varepsilon_1}{(u/a)}J_0'\left(\frac{ur}{a}\right)$$

in the cladding

$$E_r = C\frac{j\beta}{(w/a)}K_0'\left(\frac{wr}{a}\right)$$
$$E_z = CK_0\left(\frac{wr}{a}\right)$$
$$H_\theta = C\frac{j\varepsilon_2}{(w/a)}K_0'\left(\frac{wr}{a}\right)$$

II. Classification of the modes – Dispersion equation

II. 1 General form of the dispersion equation

We write the continuity conditions of the normal components of $\vec{B}=\mu\vec{H}$ and $\vec{D}=\varepsilon\vec{E}$ (namely B_r and D_r) for r=a : $\mu_1 H_r^{coeur} = \mu_2 H_r^{gaine}$ at the core cladding interface, that is to say for r=a (with $\mu_1 = \mu_2 = \mu_0$) and $\varepsilon_1 E_r^{coeur} = \varepsilon_2 E_r^{gaine}$ for r = a (with $\varepsilon_1 \neq \varepsilon_2$)

We also write the continuity conditions of the tangential components of the fields $\vec{E}$ and $\vec{H}$ (namely E_z, H_z, E_θ, and H_θ) for r=a. That is to say :

$$\left|\begin{array}{l} E_z^{coeur}(\mathrm{r=a}) = E_z^{gaine}(\mathrm{r=a}) \Leftrightarrow (\mathrm{C.1})=(\mathrm{C.7}) \text{ avec } \frac{\mathrm{r}}{\mathrm{a}}=1 \\ H_z^{coeur}(\mathrm{r=a}) = H_z^{gaine}(\mathrm{r=a}) \Leftrightarrow (\mathrm{C.4})=(\mathrm{C.10}) \text{ avec } \frac{\mathrm{r}}{\mathrm{a}}=1 \\ E_\theta^{coeur}(\mathrm{r=a}) = E_\theta^{gaine}(\mathrm{r=a}) \Leftrightarrow (\mathrm{C.3})=(\mathrm{C.9}) \text{ avec } \frac{\mathrm{r}}{\mathrm{a}}=1 \\ H_\theta^{coeur}(\mathrm{r=a}) = H_\theta^{gaine}(\mathrm{r=a}) \Leftrightarrow (\mathrm{C.6})=(\mathrm{C.12}) \text{ avec } \frac{\mathrm{r}}{\mathrm{a}}=1 \end{array}\right. \qquad (35)$$

if *u* and *w* are known, the system (34) is a system of 4 equations with 4 unknowns which are A, B, C and D.

Under a matrix form, this system can be written: $[M].\begin{vmatrix} A \\ B \\ C \\ D \end{vmatrix} = 0$

In order to obtain non trivial solutions (A=B=C=D=0), these equations must be related => det [M] = 0. The calculation of this determinant (rather long and tedeous !) results in the following dispersion equation:

$$\left|\begin{array}{l} \underbrace{\left[\frac{J_\nu'(u)}{uJ_\nu(u)} + \frac{K_\nu'(w)}{wK_\nu(w)}\right]}_{F1} \underbrace{\left[\frac{\varepsilon_1 J_\nu'(u)}{\varepsilon_2 uJ_\nu(u)} + \frac{K_\nu'(w)}{wK_\nu(w)}\right]}_{F2} = \nu^2 \underbrace{\left(\frac{1}{u^2}+\frac{1}{\omega^2}\right)}_{F3} \underbrace{\left(\frac{\varepsilon_1}{\varepsilon_2}\frac{1}{u^2}+\frac{1}{w^2}\right)}_{F4} \qquad (36) \\ \text{with } u^2 + w^2 = V^2 = k_0^2 a^2 \left(n_1^2 - n_2^2\right) = \text{cte (set by the fiber and the operating wavelength)} \qquad (37) \end{array}\right.$$

For fixed ν and V, the resolution of the system (36)-(37) (by numerical computation !) allows to find a pair or several pairs of solutions (u,w). As we will see further, each pair of solution is associated to a transverse mode able to propagate in the fiber. With (32) and (33), we can deduce for each pair (fixed ν and V) the corresponding value of β.

If V varies, β varies → β=f(V) : dispersion curve of the considered mode.

For TE and TM modes, we have ν=0 ➔ 2ème side of (36) =0. One can show that :

- the dispersion equation for TE modes is F1=0
- the dispersion equation for TM modes is F2=0

Pour EH and HE modes, we have ν≠0, thus:

- the dispersion equation EH and HE modes is the entire relation (36).

II. 2 Designation of electromagnetic modes of optical fibers

The electromagnetic modes TE, TM, HE and EH are characterised by the index ν and by an index l which is an integer ≥1 → $TE_{0,l}$, $TM_{0,l}$, $EH_{\nu,l}$, $HE_{\nu,l}$.

The integer l indicates that the corresponding mode is that for which the propagation constant β is deduced from the lth pair of solutions (u,w) of the system (36)-(37), by means of (32) or (33).

Note: the first solution is that corresponding to the highest value of β (the closest to $k_0 n_1$) and the last one corresponds to the smallest value of β (the closest to $k_0 n_2$). In other words, in the calculations we consider u starting from 0 and growing up to V (see relation (32)).

II. 3 Weak guidance approximation (WGA)

We set $\boxed{\Delta = \left(\dfrac{n_1^2 - n_2^2}{2n_1^2}\right) = \dfrac{ON^2}{2n_1^2}}$ Δ is called "relative index difference".

If n_1 is close to n_2, we can write : $\Delta = \left(\dfrac{n_1^2 - n_2^2}{2n_1^2}\right) = \dfrac{(n_1+n_2)(n_1-n_2)}{2n_1^2} \simeq \dfrac{2n_1(n_1-n_2)}{2n_1^2} \simeq \dfrac{n_1-n_2}{n_1}$

In a standard fiber, at λ = 1.55µm, $n_1 \approx 1{,}45$ and $n_1 - n_2 \approx 0{,}01$

If $\Delta < 10^{-2}$, we say that the fiber guides light in the weak guidance conditions.

These are the more usual working conditions. Indeed, in order to limit the attenuation (loss), the dopant concentration in the core (Ge, Al, P…) is limited → the index of the core is thus just slightly higher than that of the cladding.

In the weak guidance approximation, $n_1 \approx n_2$, and thus $\varepsilon_1 \approx \varepsilon_2$. thus $F_2 \approx F_1$

The dispersion equation (36) becomes :

$$F_1^2 = \nu^2 F_3^2 \Leftrightarrow F_1 = \pm\nu\ F_3 \qquad (38)$$

- For TE and TM modes (ν=0), we obtain the same dispersion equation : $F_1 = 0$ (38')
  → exact equation for the case of TE modes ($F_1 = 0$)
  → approximate equation for the case of TM modes ($F_2 \approx F_1 = 0$)

- For hybrid EH and HE modes ($\nu \neq 0$), the equation (38) is associated to :
  → EH $F_1 = +\nu\ F_3$ (38")
  → HE $F_1 = -\nu\ F_3$ (38"')

To treat these equations, we use:

- the links between the different orders of Bessel functions: $J_{\nu+1}(x) = -J_{\nu-1}(x) + \dfrac{2\nu}{x} J_\nu(x)$ and $K_{\nu+1}(x) = K_{\nu-1}(x) + \dfrac{2\nu}{x} K_\nu(x)$.

- the relationships between the derivatives and the different orders (with $J_\nu'(x) = \dfrac{dJ_\nu}{dx}(x)$ and $K_\nu'(x) = \dfrac{dK_\nu}{dx}(x)$) : $J_\nu'(x) = \dfrac{1}{2}\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right)$ and $K_\nu'(x) = -\dfrac{1}{2}\left(K_{\nu-1}(x) + K_{\nu+1}(x)\right)$

and also : $J_{-\nu} = (-1)^\nu J_\nu$ et $K_{-\nu} = K_\nu$.

(in particular, one easily show that : $J_0' = \dfrac{1}{2}\left(J_{-1} - J_1\right) = -2J_1$ and $K_0' = \dfrac{-1}{2}\left(K_{-1} + K_1\right) = -2K_1$)

By means of these relations, we can write the dispersion relations (38'), (38") and (38"') under the form:

for (38') → TE or TM modes $$u\frac{J_0(u)}{J_1(u)}=\frac{-wK_0(w)}{K_1(w)}$$

for (38") → EH modes $$u\frac{J_\nu(u)}{J_{\nu+1}(u)}=\frac{-wK_\nu(w)}{K_{\nu+1}(w)}$$

for (38"') → HE modes $$u\frac{J_{\nu-2}(u)}{J_{\nu-1}(u)}=\frac{-wK_{\nu-2}(w)}{K_{\nu-1}(w)}$$

Now, if we introduce the parameter m such that : *m*=1 for TE or TM modes (ν=0), and

$$\left.\begin{array}{l} m=\nu+1 \text{ for EH modes} \\ m=\nu-1 \text{ for HE modes}\end{array}\right\} \nu\geq 1$$

we can unify the 3 expressions (38'), (38") and (38"') under the form:

$$\begin{cases} u\dfrac{J_{m-1}(u)}{J_m(u)}=\dfrac{-wK_{m-1}(w)}{K_m(w)} & (39) \\ \text{and we still have } \quad u^2+w^2=V^2 & (40)\end{cases}$$

Thus, for a given value of *m*, and a fixed value of *V*, we will find pairs (*u*, *w*), solutions of the system (39)-(40). For each pair, we can deduce the corresponding propagation constant *β*, with (32) or (33). Therefore, for the same *m* (and the same *β*) we can associate different EH, HE (and perhaps TE or TM) modes → these modes are degenerated modes.

Examples : * if m=3, a EH mode with ν+1=3 (ν=2) and a HE mode with ν-1=3 (ν=4) are degenerated. The first (= highest) value of *β* giving a solution for (39)-(40) will be the propagation constant of both the $EH_{2,1}$ mode and the $HE_{4,1}$ mode. The second value of *β* also corresponding to a solution will be the propagation constant of both the $EH_{2,2}$ mode and the $HE_{4,2}$ mode. And so on…

* if m=1, a TE (ν=0), a TM mode (ν=0) and a HE mode with ν-1=1 (ν=2) are degenerated. (there is no solution for EH modes with ν+1=1 as this would imply that ν=0, which is not allowed for EH modes). As in the previous example, the first (= highest) value of *β* giving a solution for (39)-(40) will be the propagation constant of the TE_{01}, TM_{01} and HE_{21} modes which are degenerated. qui sont dégénérés. The second value of *β* corresponding to a solution will be the propagation constant of the TE_{02}, TM_{02} and HE_{22} modes. And so on…

III. LP modes (Linearly Polarized) – Cutoff frequencies

III. 1 LP modes

Degenerated modes (which have the same dispersion equation and thus the same prpagation constant) are gathered in a "family" called " $LP_{m,l}$ family" or " $LP_{m,l}$ mode".

* for m=1 : $TE_{0,\ l}$, $TM_{0,\ l}$, $HE_{2,\ l}$ modes are degenerated. They are gathered in the $LP_{1,\ l}$ family or $LP_{1,\ l}$ mode;

* for m>1 : $EH_{m-1,\ l}$ and $HE_{m+1,\ l}$ are degenerated. They are gathered in the $LP_{m,\ l}$ family or $LP_{m,\ l}$ mode;

* for m=0 : the $HE_{1,\ l}$ mode constitutes the $LP_{0,\ l}$ mode.

One can show that degenerated modes have the same intensity distribution of the electric field, in a given linear polarization.

$$\left(\text{i.e. same } |E_x(r,\theta)|^2 \text{ or same } |E_y(r,\theta)|^2 \text{ or same } |E_{\text{any linear direction}}(r,\theta)|^2\right).$$

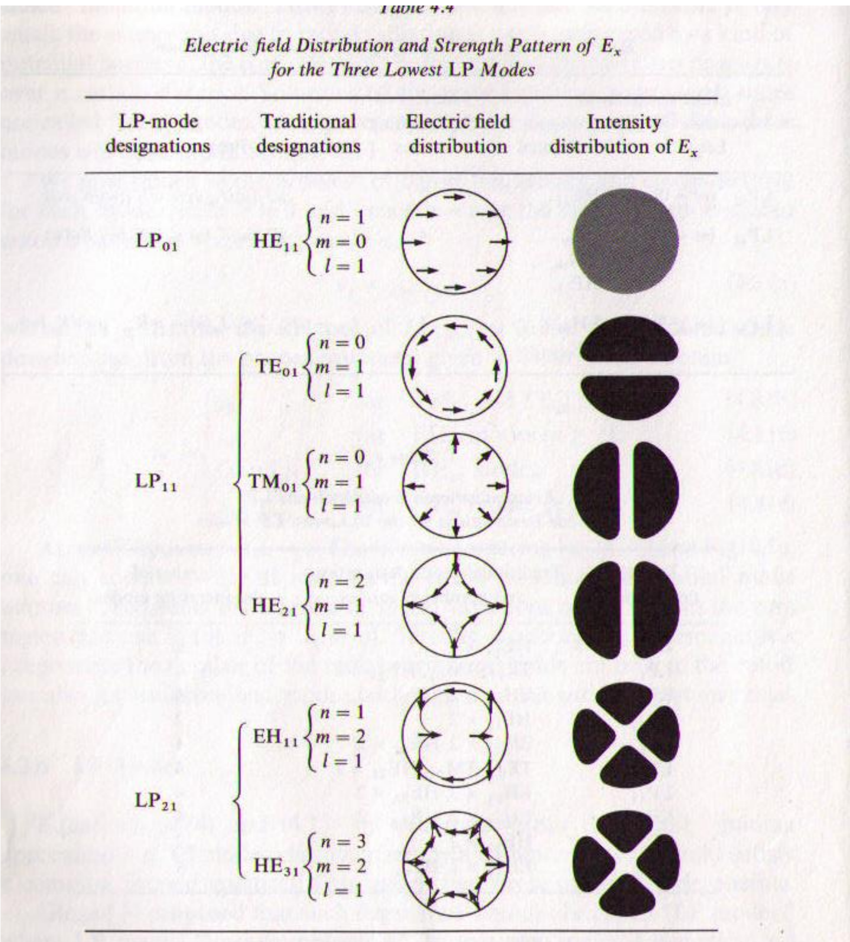


Table 4.4

Electric field Distribution and Strength Pattern of E_x for the Three Lowest LP Modes

LP-mode designations	Traditional designations	Electric field distribution	Intensity distribution of E_x
LP_{01}	HE_{11} $\{n=1,\ m=0,\ l=1$		
LP_{11}	TE_{01} $\{n=0,\ m=1,\ l=1$		
	TM_{01} $\{n=0,\ m=1,\ l=1$		
	HE_{21} $\{n=2,\ m=1,\ l=1$		
LP_{21}	EH_{11} $\{n=1,\ m=2,\ l=1$		
	HE_{31} $\{n=3,\ m=2,\ l=1$		

Figure 4 : First degenerated EM modes and intensity distribution in a given polarization direction x (corresponding LP modes)

In a LP$_{m,l}$ mode (weighted sum of "true" EM modes constituting the LP$_{m,l}$ family), the polarization is rectilinear. The field oriented along this polarization direction varies as follows:

- azimutally (i.e. when r=cte) → like the sinusoidal function sin $(m\theta)$ (the orientation of the coordinate system (Oxy) is chosen such that θ = 0 in the direction where the field is cancelled)
- radially (i.e. when θ=cte) → like the function $J_m\left(\frac{ur}{a}\right)$ in the core

and like the function $K_m\left(\frac{wr}{a}\right)$ in the cladding. The field has *l* cancellations along a radius r : (*l*-1) cancellations in the core and one in the cladding, from few microns beyond the core.

In other words, if *m*=0 (LP$_{0,l}$ mode), there are no azimuthal oscillations of the field and the mode is like a central circular spot surrounded by (*m*-1) rings. And if *m*≠0, the intensity distribution in a LP$_{m,l}$ mode is like a rosace with *l* rings of 2*m* lobes each.

Examples :

* if *m* = 0 : no azimuthal oscillations (→ rings) and radial variations following $J_0\left(\frac{ur}{a}\right)$

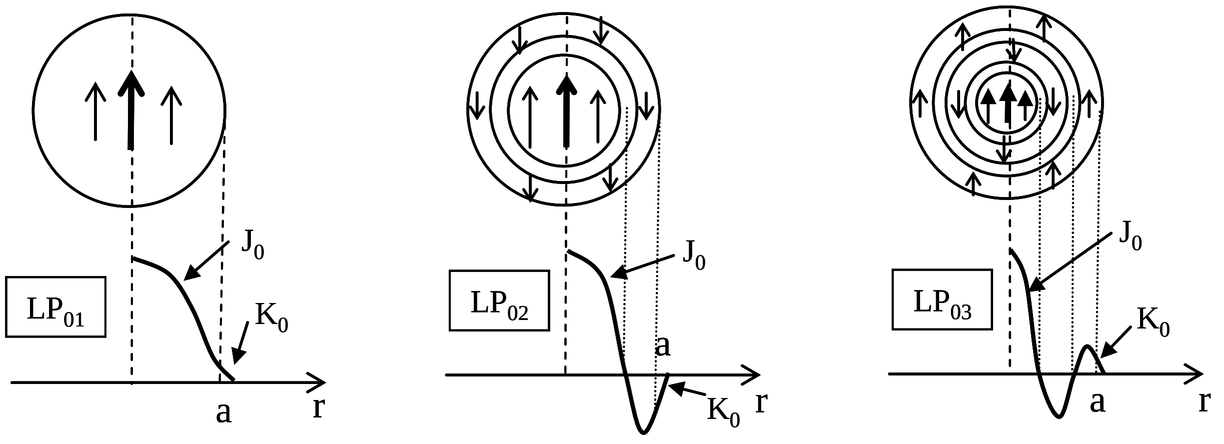
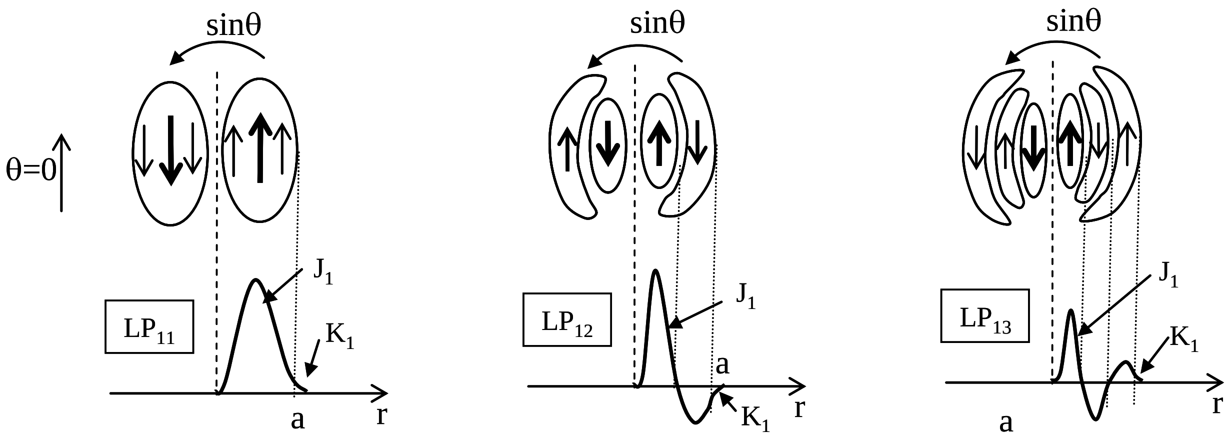


Figure 5 : distribution of the electric field in the first three LP$_{0l}$ modes

* <u>if *m*≠0</u> : azimutally, field variations following sin(mθ) → there are 2 m cancellations per revolution → 2m lobes.

radially, variations following $J_m\left(\frac{ur}{a}\right)$ → no energy in the center (as $J_m(0)=0$) and *l* rings (with 2m lobes each).

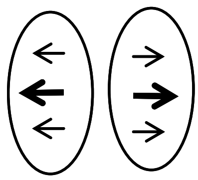


<u>Figure 6</u> : distribution of the electric field in the first three LP$_{1l}$ modes

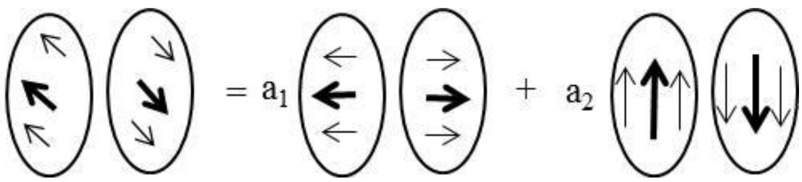
Note :

1) the orientation of the polarization of the field is independant of the direction of the cancellation

For example, for the LP$_{11}$ mode, we can meet the following situation:



or , more generally :



2) in the examples shown in Fig. 6, we have chosen θ=0 (cancellation of the field) in the direction Oy. But this direction, which depends on the excitation conditions of the modes (generally not controlled), can obviously be different. It is the case for the modes depicted in Fig. 7.

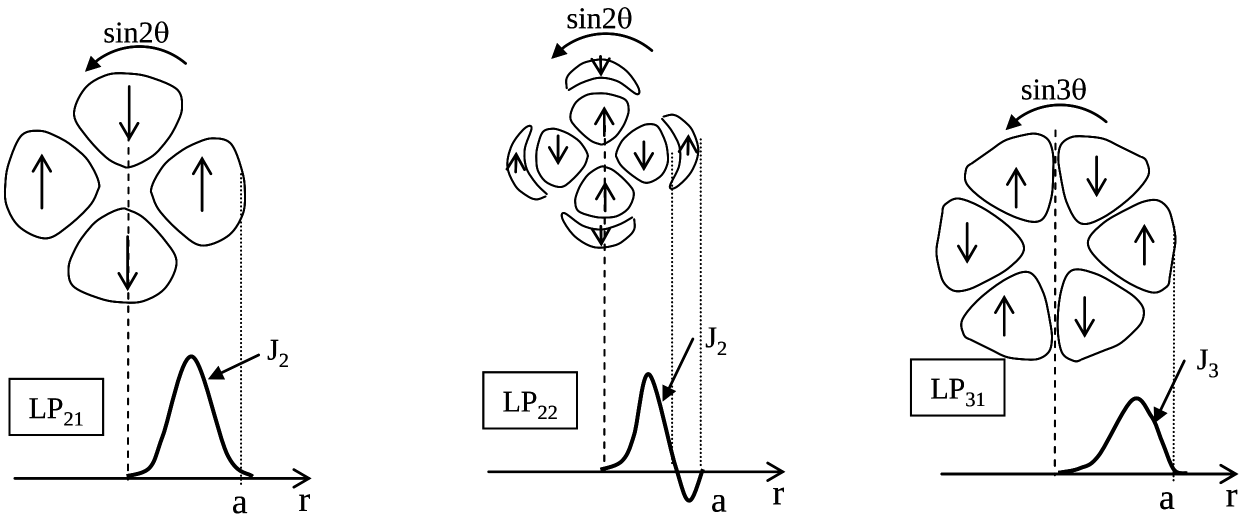


Figure 7 : distribution of the electric field in the LP_{21}, LP_{22}, and LP_{31} modes.

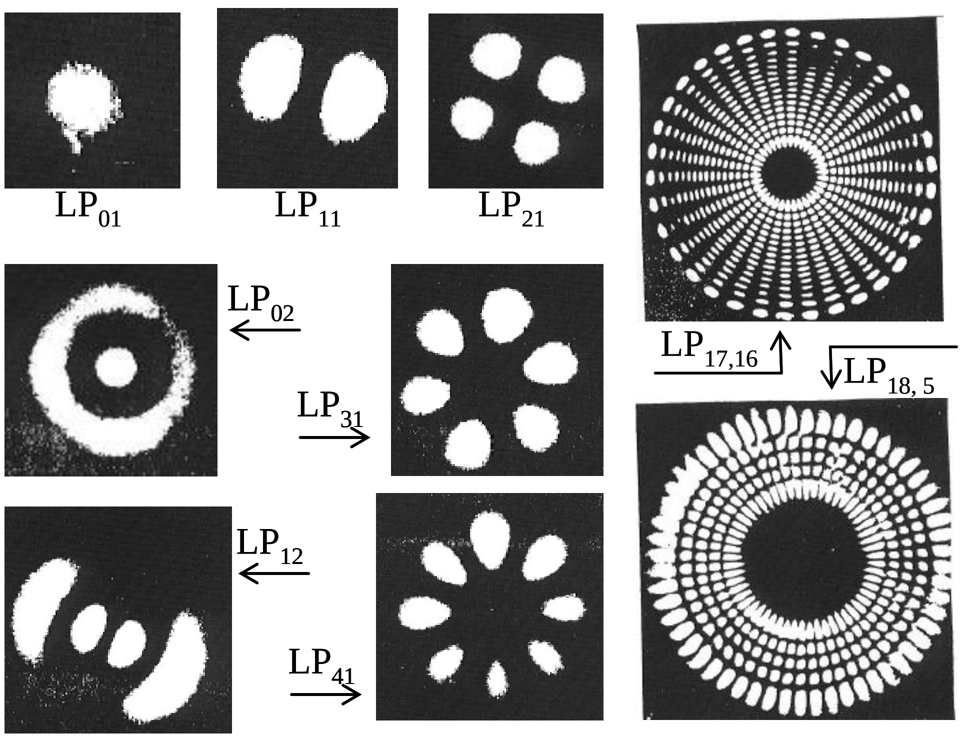


Figure 8 : photographies of the intensity distributions in various transverse modes of a step index fiber.

III. 2 Cutoff normalized spatial frequency of LP modes, cutoff wavelength

III.2.1 Cutoff normalized spatial frequency

For each "true" EM mode, or for each LP mode in the weak guidance approximation, one can plot the dispersion curve under the form β=f(V),

or under the form $n_{eff} = \dfrac{\beta}{k_0} = \mathrm{f}(V)$

or under the form $B = \dfrac{\beta^2 - k_0^2 n_2^2}{k_0^2\left(n_1^2 - n_2^2\right)} = \mathrm{f}(V)$ (41).

B is a normalized expression of β, so-called normalized propagation constant of the mode. Let us remind that the condition on the propagation constant required for the mode to be guided is: $k_0 n_2 \le \beta < k_0 n_1$. As B=0 when $\beta = \beta_{\min} = k_0 n_2$, and B=1 when $\beta = \beta_{\max} = k_0 n_1$, the condition for a mode to be guided is $0 \le B < 1$.

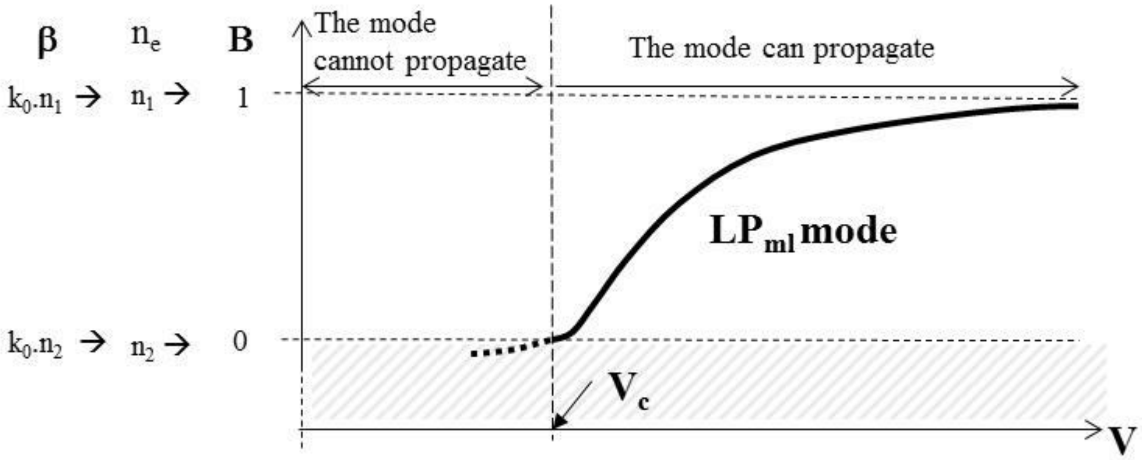


Figure 9 : domain of V where the dispersion curve of a $LP_{m,l}$ mode exists

Let us search the value of V corresponding to the limit of the propagation condition of a given mode ($\beta = k_0 n_2$ or $n_e = n_2$ or $B = 0$). This value will be called " cutoff normalized spatial frequency" and it will be noted V_c.

At the limit of the propagation conditions, we have $\beta = k_0 n_2$

$\Rightarrow w=0$ from (33)

$\Rightarrow V^2 = u^2$ from (34)

thus, the dispersion equation of the LP mode (eq. (39)) becomes $u\dfrac{J_{m-1}(u)}{J_m(u)} = 0$ with $u = V_c$ (42).

For this mode $\Rightarrow V = u = V_c$

For the LP$_{m,l}$ mode, V_c is the lth solution of (42), with V_c increasing from 0 to V.

Notes : - amonh the J_m functions, only the function J_0 does not cancel at the origin

- the functions J_m and J_{m-1} do not cancel simultaneously (for the same argument). Thus :

* <u>if m=0</u>, (42) becomes $u\dfrac{J_{-1}(u)}{J_0(u)} = u\dfrac{J_1(u)}{J_0(u)} = 0$ (43)

The first solution of (43) is $u=V_c=0$ (V_c for the LP$_{01}$ mode),

The second solution of (43) is the first zero (after u=0) of the function J_1 noted $j_{1,1}$ (V_c for the LP$_{02}$ mode),

The lth solution of (43) is the (l-1)th zero (after u=0) of the function J_1 noted $j_{1,l-1}$ (V_c for the LP$_{0,l}$ mode), etc

* <u>if m≠0</u>, one can show that $\lim\limits_{u\to 0} u\dfrac{J_{m-1}(u)}{J_m(u)} \neq 0$ therefore u=0 is not a solution of (42).

$\Rightarrow$ the lth solution of (42) is the lth zero (after u=0) of the function J_{m-1} noted $j_{m-1,l}$ (V_c for the mode LP$_{m,l}$).

<u>Summary :</u>

$$\begin{cases} m \neq 0 & l \geq 1 \quad V_c(LP_{m,l}) = j_{m-1,\,l} \\ \\ m = 0 & \begin{array}{ll} l = 1 & V_c(LP_{0,1}) = 0 \\ l > 1 & V_c\left(LP_{0,l}\right) = j_{1,\,l-1} \end{array} \end{cases}$$

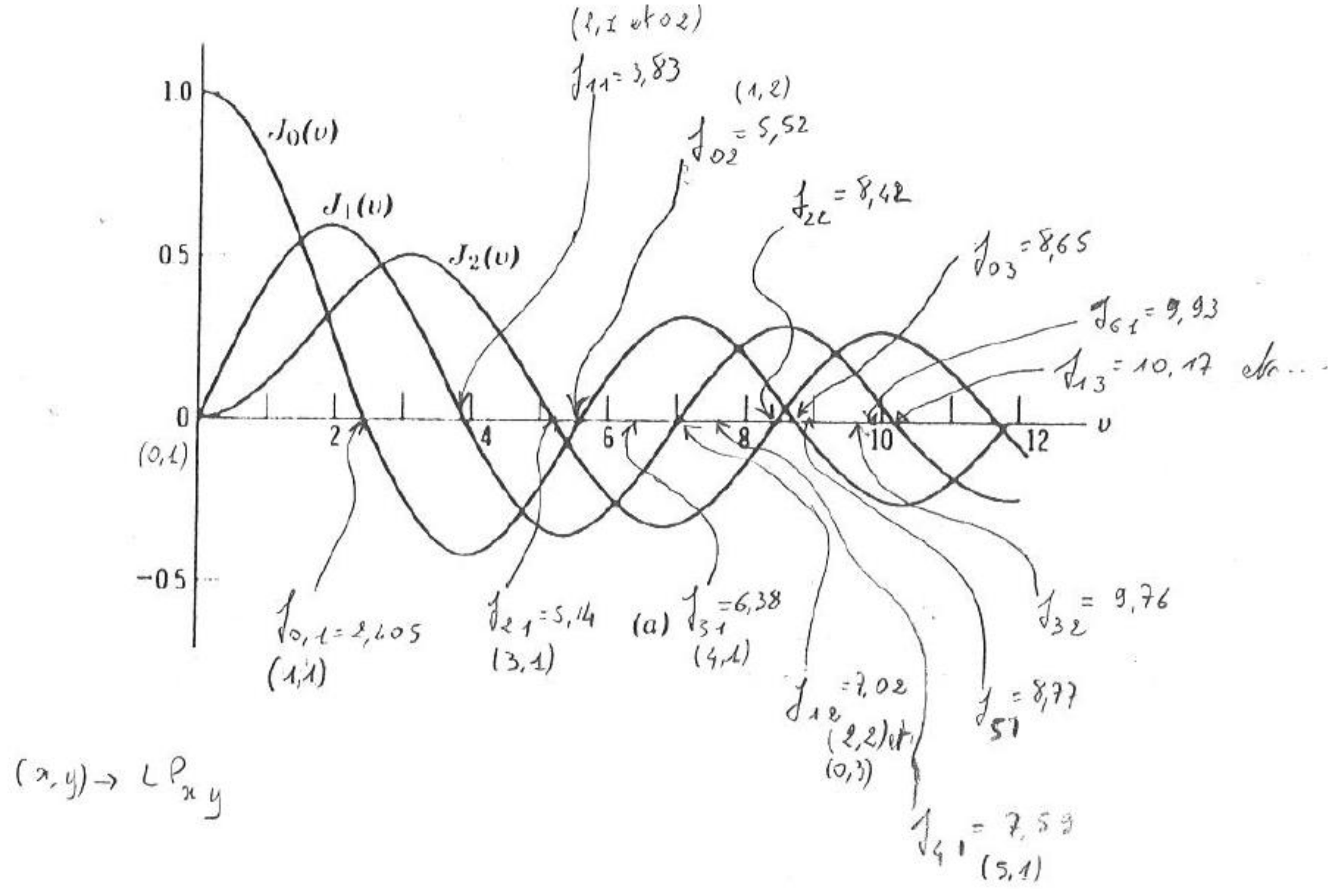


Figure 10 : the first zeros of the Bessel functions of the first kind (J_ν)

Back to the "true" electromagnetic modes, this leads to:

$$V_c = \begin{cases} j_{0,l} \quad \text{for } \mathrm{TE}_{0,l} \text{ and } \mathrm{TM}_{0,l} \text{ modes} \\ 0 \text{ and then } j_{1,l} \quad \text{for the } \mathrm{HE}_{1,1} \text{ and then } \mathrm{HE}_{1,l} \quad (l \geq 2) \text{ modes} \\ j_{\nu-2,l} \quad \text{for the } \mathrm{HE}_{\nu,l} \text{ modes with } \nu \geq 2 \\ j_{\nu,l} \quad \text{for the } \mathrm{EH}_{\nu,l} \text{ modes with } \nu \geq 1 \end{cases}$$

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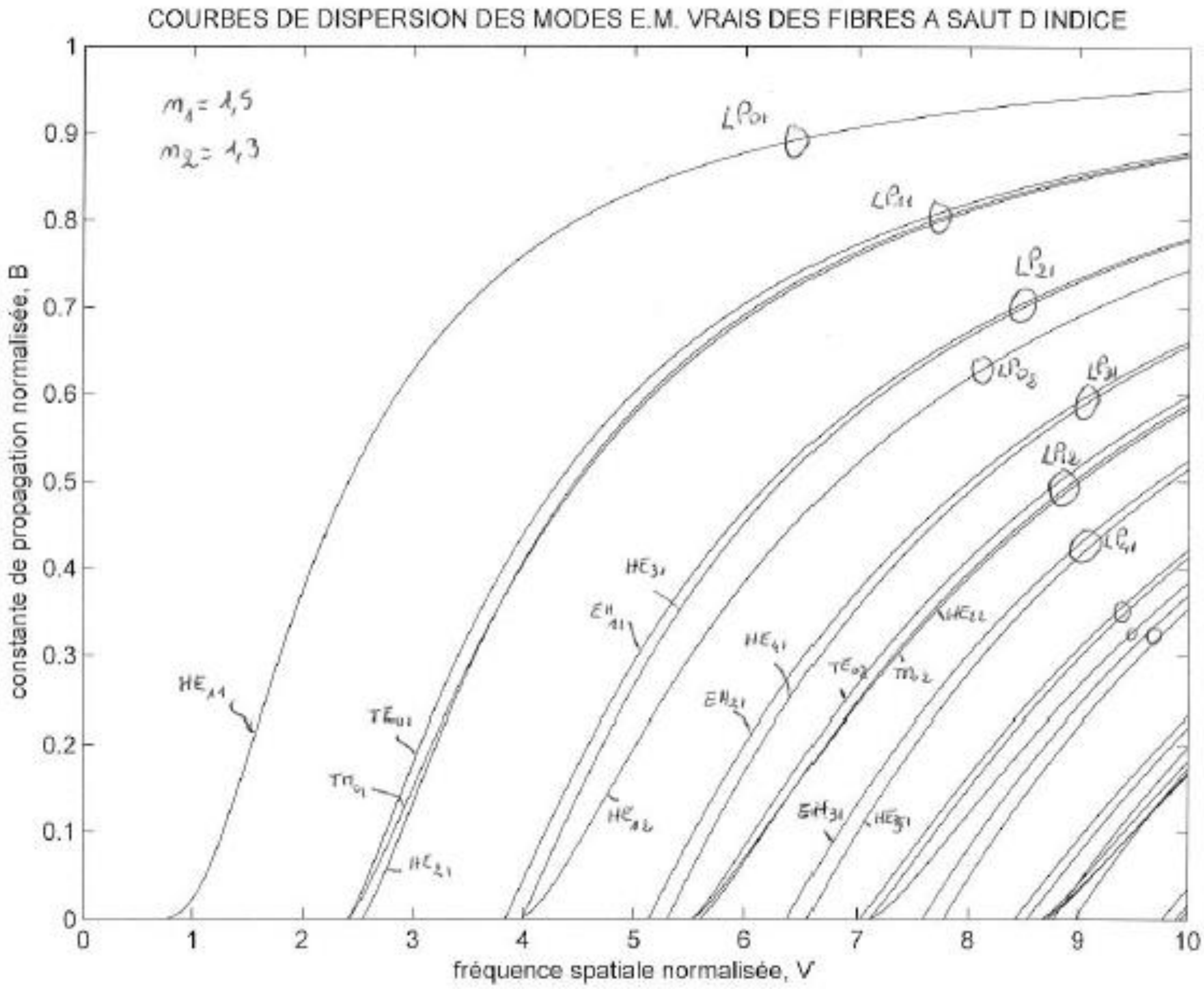


Figure 11 : dispersion curves of the first "true" electromagnetic modes of a step index fiber

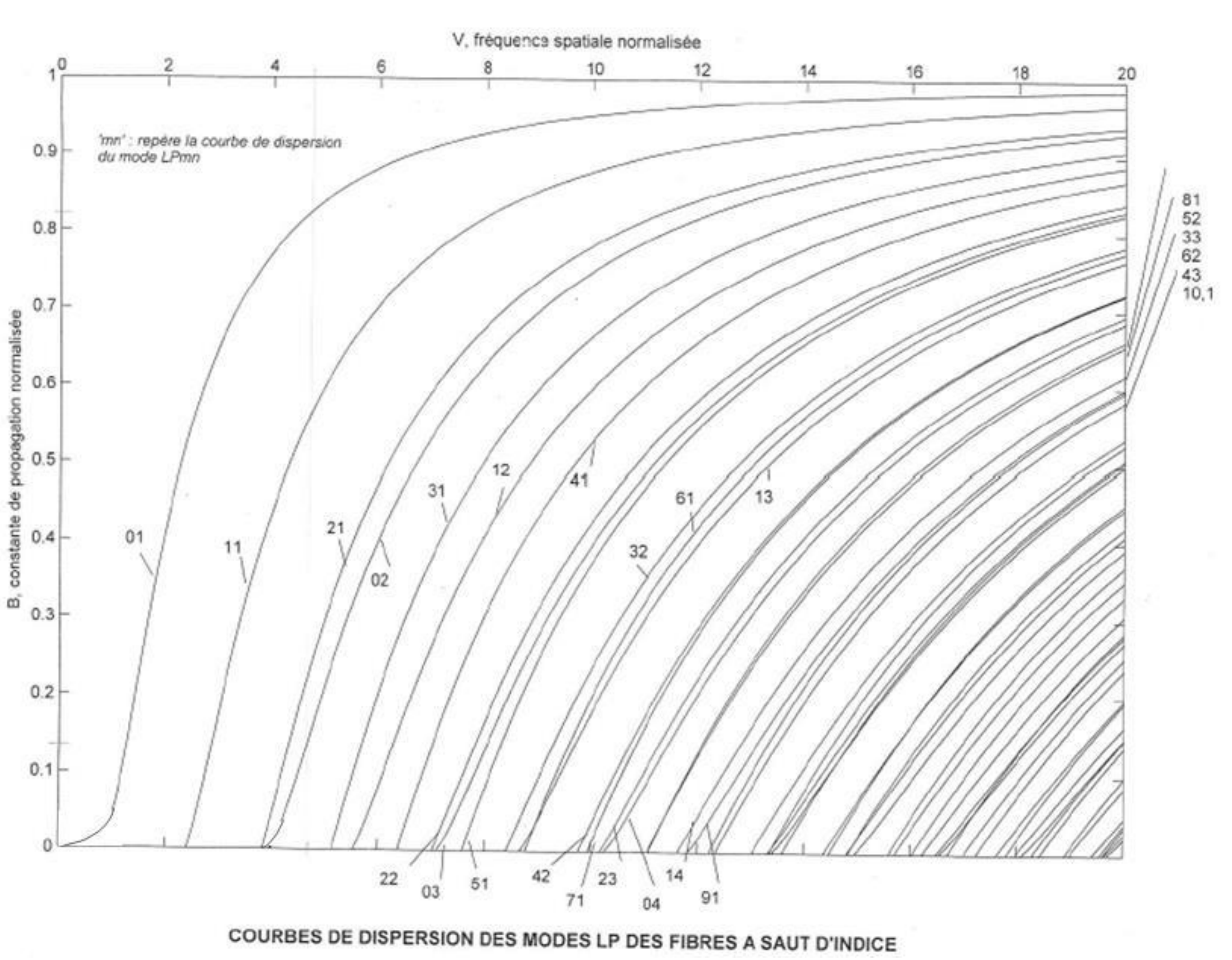
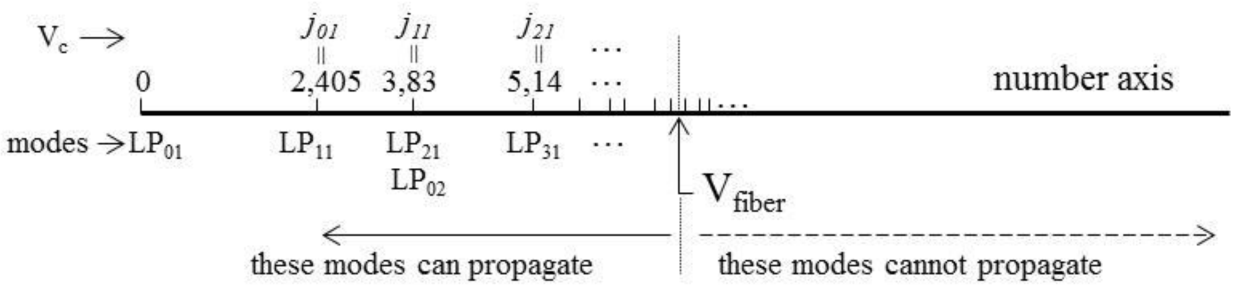


Figure 12 : dispersion curves of the first LP modes of a step index fiber

We remind again that a mode can propagate if $\beta \geq k_0 n_2$, that is to say if B ≥ 0.

On the dispersion curves, we can see that this condition results in: $\underbrace{V}_{\text{dépendent on the fiber}} > \underbrace{V_c}_{\text{dependent on the mode}}$

In other words, the propagation condtion of the LP$_{m,l}$ mode is $\underline{V > V_c(LP_{m,l})}$



<u>Figure 13</u> : cutoff conditions of LP modes in a step index fiber (the V_c values are different if the index profile is different)

Finally, we note that :

- the LP_{01} mode (<u>fundamental mode</u>) can alwyas propagate, theoretically ($V > V_c(LP_{0,1}) = 0$
- for 0 < V< 2,405 : the fundamental mode only can propagate → the fiber works in <u>single mode regime</u>
- for 2,405 < V : several modes can propagate → the fiber works in <u>multimode regime</u>

For example, for 2,405 < V < 3,83 : 2 modes can propagate (LP_{01} and LP_{11}) → "two mode regime" …

III.2.2 <u>Cutoff wavelength</u>

The cutoff wavelength of a mode X, noted λ_c, is such that $V(\lambda_c) = \frac{2\pi}{\lambda_c} aON = V_c(\text{X})$. In other words :

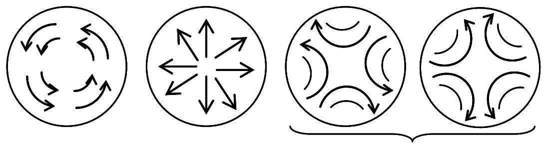
$\boxed{\lambda_c(X) = \frac{2\pi}{V_c(\text{X})} a.ON}$ where X is the considered LP mode. For example, for the 2[nd] mode LP_{11}, we have $\lambda_c(LP_{11}) = \frac{2\pi}{2.405} a.ON$

A given mode X can propagate if : $V_c(X) < V$ => $\frac{2\pi}{\lambda_c} aON < \frac{2\pi}{\lambda} aON$ => $\lambda < \lambda_c$

At the wavelength λ, a mode can propagate if $\lambda < \lambda_c$

A fiber is said "single mode fiber" if the fundamental mode only is able to propagate, thus if $\lambda > \frac{2\pi}{\lambda_c(LP_{11})} a.ON > \frac{2\pi}{2.405} a.ON$

Classification of the first LP modes versus their cutoff normalized spatial frequency:

LP Mode	V_c	Degenerated modes ((x) = number of polarizations)	Number of degenerated modes
LP_{01}	0	HE_{11} (2) = HE_{11x} et HE_{11y}	2
LP_{11}	2,405	TE_{01} (1), TM_{01} (1), et HE_{21} (2) $E_r = 0$ $E_\theta=0$  (field lines of the electric field)	4
LP_{21}	3,83	EH_{11} (2) et HE_{31} (2)	4
LP_{02}	3,83	HE_{12} (2)	2
LP_{31}	5,14	EH_{21} (2) et HE_{41} (2)	4
LP_{12}	5,52	TE_{02} (1), TM_{02} (1), et HE_{22} (2)	4
LP_{41}	6,38	EH_{31} (2) et HE_{51} (2)	4
LP_{22}	……	……….	….
etc…..		……….	….

III. 3 Number of guided modes in a step index or in a graded index fiber

One can show that a step index fiber can guide about $\mathcal{N} = \dfrac{V^2}{2}$ "true" electromagnetic modes (TE, TM, EH or HE), which corresponds to $\mathcal{N}' = \dfrac{1}{4}\left(\dfrac{V^2}{2}\right) = \dfrac{V^2}{8}$ LP modes (because there are ~ 4 degenerated polarizations for each LP mode).

In a graded index fiber, the Helmoltz equation is different (($n_1(r)$ instead of n_1 = cte) ➔ the expressions of the radial components of the fields are different (for example, they are Laguerre Gauss polynomials in the case of parabolic graded index fibers, instead of Bessel functions) ➔ the dispersion equations are different ➔ the cutoff spatial frequencies are different ➔ for a given value of V, the number of guided modes is different.

One can show that, for a gradient index of the form:

$$\begin{cases} n_{coeur} = n_1(r) = n_1\left[1-2\Delta\left(\dfrac{r}{a}\right)^g\right]^{1/2} & r \le a \\ n_{gaine} = n_2 & r \ge a \end{cases} \quad \text{with} \quad \Delta = \left(\frac{n_1^2 - n_2^2}{2n_1^2}\right) \ll 1$$

the number of electromagnetic modes able to be guided is $\mathcal{N} = \dfrac{V^2}{2}\dfrac{g}{g+2}$

* for g = ∞ , → $n_{core} = n_1$ (step index profile) ➔ $\mathcal{N} = \dfrac{V^2}{2}$ (already seen above) and the number of LP modes able to be guided is $\mathcal{N}' = \dfrac{\mathcal{N}}{4} = \dfrac{V^2}{8}$

* for g = 2 , → $n_{core} \approx n_1\left[1-\Delta\left(\dfrac{r}{a}\right)^2\right]$ (parabolic profile) ➔ $\mathcal{N} = \dfrac{V^2}{4}$ and the number of LP modes able to be guided is $\mathcal{N}' = \dfrac{\mathcal{N}}{4} = \dfrac{V^2}{16}$.

Thus, a step index fiber can guide twice as much modes than a parabolic index fiber having the same numerical aperture and the same core radius (→ same value of V).

IV Modal effects in multimode fibers

IV 1 Orders of the modes

The "order of a mode" is given by the number M =2*l*+m-1. The more the structure is complex (large number of lobes and/or large number of rings), the larger M is. The behaviour and propagation properties of the modes along their propagation depends on their order (see the following table).

M small	M large
→ mode with a simple structure → low order mode → energy preferentially close to the center → β close to $k_0 n_1$ → mode associated to paraxial rays → v_φ is small and v_g is high	→ mode with a "complex" structure → high order mode → energy close to the periphery (tubular modes) → β close to $k_0 n_2$ → mode associated to rays which are less inclined → v_φ is high and v_g is small

IV 2 Superimposition of modes (speckle), coupling, modal effects

In a multimode fiber, the part of the incident power which is coupled in a given mode by the incident beam corresponds the normalized integral overlap between the field of this incident beam ("exciter") and the field of the mode → see the chapter 5 (part of the course presented by P. DiBin). For example, a centered gaussian beam (even) can only excite even modes (LP_{0l}) because the integral overlap with all the other modes (odd) is zero. If the exciting beam is decentred (shifted or tilted), thus a part of its energy will be also coupled in odd modes.

In a multimode fiber, the superimposition of the modes excited and guided results in any section of the fiber in a complex distribution of the field so-called a "speckle".

If the fiber is uniform along z, and if it is not perturbated → there is no energy exchange (coupling) between the modes. However, the speckle changes along the propagation because the relative phase shifts between the modes change proportionally to z.

If the speckle is created by rather high order modes, it is composed numerous and fine grains. On the contrary, if it is created by low order modes (case of weakly multimode fibers), the speckle grains are few and large.

If the fiber is not uniform along z (changes of the optogeometrical profile) or/and if it is perturbated (bent, twisted, …), the modes can exchange energy (mode coupling). Mode coupling also occurs at imperfect splices. One can show that mode coupling preferentially occur between modes with close orders. Therefore, if only the fundamental mode is excited (rare case !), it will fisrt couple in low order modes (LP_{11}, LP_{21}, LP_{02}…), then these modes will couple further in modes having slightly higher orders. So, if the fiber is enough long and if the perturbations are enough strong, all the modes will be excited after a threshold length of propagation.

The characteristics of certain passive components (loss at splices and connectors, splitting ratio of certain types of couplers, …) depend on the modal population present in the fiber upstream of the component. For example, the loss is higher at misaligned splices for high order modes. → The components must be characterized with a well-known ("normalized") modal population upstream of this component. More precisely, the components must be characterized at the <u>modal equilibrium</u> (modes of any orders are excited and, statistically, the modal population is not changed in case of mode coupling). In order to generate the modal equilibrium, one can use a "modal equilibrium simulator" so-called "mode scrambler".

The detailed characterization of multimode passive components can require <u>selective modal excitation.</u> This means that modal excitation at the input is controlled in order to excite only one mode at a time, assuming no mode coupling along the fiber. Such pure modal excitation has been used for several years in novel very high bit rate communication systems based on the <u>mode division (= spatial) multiplexing technique</u>.

The knowledge of the fields distributions in the mode is of great importance in number of applications, in particular for the design of power amplifiers based on active double clad fibers (single mode doped core and multimode inner cladding).