9. Power Spectral Density and Random Signals in Linear System

9.1. Introduction

In this chapter, the notion of power spectral density for a random signal is introduced. This concept enables us to study wide-sense stationary random signals in the frequency domain and define a white-noise process. The response of a linear system to random signal is then studied.

9.2. Correlations and Power Spectral Densities

In the following, we assume that all random processes are WSS.

9.2.1. A. Autocorrelation $R_{XX}(\tau)$:

The autocorrelation of $X(t)$ is [Eq. (8.35)]

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

(9.1)

9.2.1.1. Properties of $R_{XX}(\tau)$:

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

(9.2)

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

(9.3)

$$R_{XX}(0) = E[X^2(t)]$$

(9.4)

Property 3 [Eq. (9.4)] is easily obtained by setting $\tau = 0$ in Eq. (9.1). If we assume that $X(t)$ is a voltage waveform across a 1−Ω resistor, then $E[X^2(t)]$ is the average value of power delivered to the 1−Ω resistor by $X(t)$. Thus, $E[X^2(t)]$ is often called the average power of $X(t)$. Properties 1 and 2 are verified in Prob. (9.1).

In case of a discrete-time random process $X(n)$, the autocorrelation function of $X(n)$ is defined by [Eq. (8.39)]

$$R_{XX}(k) = E[X(n)X(n+k)]$$

(9.5)

Various properties of $R_{XX}(k)$ similar to those of $R_{XX}(\tau)$ can be obtained by replacing τ by k in Eqs. (9.2) to (9.4).

9.2.2. B. Cross-Correlation $R_{XY}(\tau)$:

9.2.2. B. Cross-Correlation $R_{XY}(\tau)$:

The cross-correlation of $X(t)$ and $Y(t)$ is [Eq. (8.42)]

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

(9.6)

9.2.2.1. Properties of $R_{XY}(\tau)$:

$$R_{XY}(-\tau) = R_{YX}(\tau)$$

(9.7)

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$$

(9.8)

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

(9.9)

These properties are verified in Prob. 9.2.

Similarly, the cross-correlation function of two discrete-time jointly WSS random sequences $X(n)$ and $Y(n)$ is defined by

$$R_{XY}(k) = E[X(n)Y(n+k)]$$

(9.10)

And various properties of $R_{XY}(k)$ similar to those of $R_{XY}(\tau)$ can be obtained by replacing τ by k in Eqs. (9.7) to (9.9).

9.2.3. C. Power Spectral Density or Power Spectrum:

Let $R_{XX}(\tau)$ be the autocorrelation of $X(t)$. Then the *power spectral density* (or *power spectrum*) of $X(t)$ is defined by the Fourier transform of $R_{XX}(\tau)$ as

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\,d\tau$$

(9.11)

Thus,

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}\,d\omega$$

(9.12)

Equations (9.11) and (9.12) are known as the *Wiener-Khinchin relations.*

9.2.3.1. Properties of $S_{XX}(\omega)$:

$$S_{XX}(\omega) \text{ is real and } S_{XX}(\omega) \geq 0$$

(9.13)

$$S_{XX}(-\omega) = S_{XX}(\omega)$$

(9.14)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega = R_{XX}(0) = E[X^2(t)]$$

(9.15)

Similarly, the power spectral density $S_{XX}(\Omega)$ of a discrete-time random process $X(n)$ is defined as the Fourier transform of $R_{XX}(k)$:

$$S_{XX}(\Omega) = \sum_{k=-\infty}^{\infty} R_{XX}(k)e^{-j\Omega k}$$

(9.16)

Thus, taking the inverse Fourier transform of $S_{XX}(\Omega)$, we obtain

$$R_{XX}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\Omega)e^{j\Omega k}\,d\Omega$$

(9.17)

9.2.3.2. Properties of $S_X(\Omega)$:

$$S_{XX}(\Omega + 2\pi) = S_{XX}(\Omega)$$

(9.18)

$$S_{XX}(\Omega) \text{ is real and } S_{XX}(\Omega) \geq 0.$$

(9.19)

$$S_{XX}(-\Omega) = S_{XX}(\Omega)$$

(9.20)

$$E[X^2(n)] = R_{XX}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\Omega)\,d\Omega$$

(9.21)

Note that property 1 [Eq. (9.18)] follows from the fact that $e^{-j\Omega k}$ is periodic with period 2π. Hence it is sufficient to define $S_{XX}(\Omega)$ only in the range $(-\pi, \pi)$.

9.2.4. D. Cross-Power Spectral Densities:

The *cross-power spectral density* (or *cross-power spectrum*) $S_{XY}(\omega)$ of two continuous-time random processes $X(t)$ and $Y(t)$ is defined as the Fourier transform of $R_{XY}(\tau)$:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}\,d\tau$$

(9.22)

Thus, taking the inverse Fourier transform of $S_{XY}(\omega)$, we get

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau}\, d\omega$$

(9.23)

9.2.4.1. Properties of $S_{XY}(\omega)$:

Unlike $S_{XX}(\omega)$, which is a real-valued function of ω, $S_{XY}(\omega)$, in general, is a complex-valued function.

$$S_{XY}(\omega) = S_{YX}(-\omega)$$

(9.24)

$$S_{XY}(-\omega) = S_{XY}^{*}(\omega)$$

(9.25)

Similarly, the cross-power spectral density $S_{XY}(\Omega)$ of two discrete-time random processes $X(n)$ and $Y(n)$ is defined as the Fourier transform of $R_{XY}(k)$:

$$S_{XY}(\Omega) = \sum_{k=-\infty}^{\infty} R_{XY}(k) e^{-j\Omega k}$$

(9.26)

Thus, taking the inverse Fourier transform of $S_{XY}(\Omega)$, we get

$$R_{XY}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XY}(\Omega) e^{j\Omega k}\, d\Omega$$

(9.27)

9.2.4.2. Properties of $S_{XY}(\Omega)$:

Unlike $S_{XX}(\Omega)$, which is a real-valued function of Ω, $S_{XY}(\Omega)$, in general, is a complex-valued function.

$$S_{XY}(\Omega + 2\pi) = S_{XY}(\Omega)$$

(9.28)

$$S_{XY}(\Omega) = S_{YX}(-\Omega)$$

(9.29)

$$S_{XY}(-\Omega) = S_{XY}^{*}(\Omega)$$

(9.30)

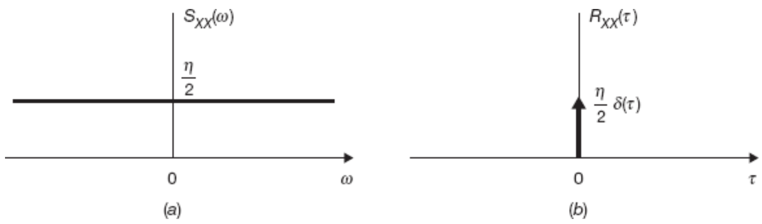
9.3. White Noise

A random process $X(t)$ is called *white noise* if [Fig. 9-1(*a*)]

$$S_{XX}(\omega) = \frac{\eta}{2}$$

(9.31)

Figure 9-1 *White noise.*



Taking the inverse Fourier transform of Eq. (9.31), we have

$$R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$$

(9.32)

which is illustrated in Fig. 9-1(*b*). It is usually assumed that the mean of white noise is zero.

Similarly, a zero-mean discrete-time random sequence $X(n)$ is called a discrete-time white noise if

$$S_{XX}(\Omega) = \sigma^2$$

(9.33)

Again the power spectral density of $X(n)$ is constant. Note that $S_{XX}(\Omega + 2\pi) = S_{XX}(\Omega)$ and the average power of $X(n)$ is $\sigma^2 = \text{Var}[X(n)]$, which is constant. Taking the discrete-time inverse Fourier transform of Eq. (9.33), we have

$$R_{XX}(k) = \sigma^2\,\delta(k)$$

(9.34)

9.3.1.1. Band-Limited White Noise:

A random process $X(t)$ is called *band-limited white noise* if

$$S_{XX}(\omega) = \begin{cases} \frac{\eta}{2} & |\omega| \le \omega_B \\ 0 & |\omega| > \omega_B \end{cases}$$

(9.35)

Then

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B} \frac{\eta}{2} e^{j\omega\tau}\,d\omega = \frac{\eta\omega_B}{2\pi}\frac{\sin \omega_B\tau}{\omega_B\tau}$$

(9.36)

And $S_{XX}(\omega)$ and $R_{XX}(\tau)$ of band-limited white noise are shown in Fig. 9-2.

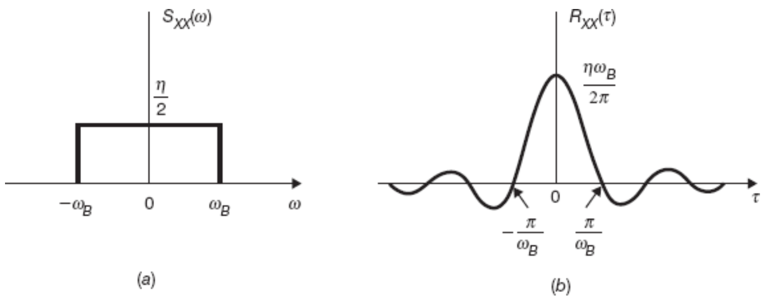


Figure 9-2 Band-limited white noise.

Note that the term *white* or *band-limited white* refers to the spectral shape of the process $X(t)$ only, and these terms do not imply that the distribution associated with $X(t)$ is Gaussian.

9.3.1.2. Narrowband Random Process:

Suppose that $X(t)$ is a WSS process with zero mean and its power spectral density $S_{XX}(\omega)$ is nonzero only in some narrow frequency band of width $2W$ that is very small compared to a center frequency ω_c. Then the process $X(t)$ is called a *narrowband random process*.

In many communication systems, a narrowband process (or noise) is produced when white noise (or broadband noise) is passed through a narrowband linear filter. When a sample function of the narrowband process is viewed on an oscilloscope, the observed waveform appears as a sinusoid of random amplitude and phase. For this reason, the narrowband noise $X(t)$ is conveniently represented by the expression

$$X(t) = V(t)\cos[\omega_c t + \phi(t)]$$

(9.37)

9.4. Response of Linear System to Random Input

9.4.1. A. Linear System:

As we discussed in Chap. 1 (Sec. 1.5), a system is a mathematical model for a physical process that relates the input (or excitation) signal x to the output (or response) y, and the system is viewed as a transformation (or mapping) of x into y. This transformation is represented by the operator **T** as (Eq. (1.60))

$$y = \mathbf{T}x$$

(9.38)

For a continuous-time linear time-invariant (LTI) system, Eq. (9.38) can be expressed as Eq. (2.60)

$$y(t) = \int_{-\infty}^{\infty} h(\alpha)x(t-\alpha)\,d\alpha = h(t) * x(t)$$

(9.39)

where $h(t)$ is the *impulse response* of a continuous-time LTI system. For a discrete-time LTI system, Eq. (9.38) can be expressed as (Eq. (2.45))

$$y(n)=\sum_{i=-\infty}^{\infty} h(i)x(n-i)=h(n)*x(n)$$

(9.40)

where $h(n)$ is the *impulse response* (or *unit sample response*) of a discrete-time LTI system.

9.4.2. B. Response of a Continuous-Time Linear System to Random Input:

When the input to a continuous-time linear system represented by Eq. (9.38) is a random process $\{X(t), t \in T_X\}$, then the output will also be a random process $\{Y(t), t \in T_Y\}$; that is,

$$\mathbf{T}\{X(t), t\in T_x\} = \{Y(t), t\in T_y\}$$

(9.41)

For any input sample function $x_i(t)$, the corresponding output sample function is

$$y_i(t) = \mathbf{T}\{x_i(t)\}$$

(9.42)

If the system is LTI, then by Eq. (9.39), we can write

$$Y(t)=\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)\,d\alpha = h(t)*X(t)$$

(9.43)

Note that Eq. (9.43) is a stochastic integral. Then

$$\begin{aligned}\mu_Y(t)=E[Y(t)]&=E\left[\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)\,d\alpha\right]\\&=\int_{-\infty}^{\infty} h(\alpha)E[X(t-\alpha)]\,d\alpha\\&=\int_{-\infty}^{\infty} h(\alpha)\mu_X(t-\alpha)\,d\alpha=h(t)*\mu_X(t)\end{aligned}$$

(9.44)

$$\begin{aligned}R_{YY}(t_1,t_2)&=E[Y(t_1)Y(t_2)]\\&=E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)X(t_1-\alpha)h(\beta)X(t_2-\beta)\,d\alpha d\beta\right]\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)E[X(t_1-\alpha)X(t_2-\beta)]\,d\alpha d\beta\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{XX}(t_1-\alpha,t_2-\beta)]\,d\alpha d\beta\end{aligned}$$

(9.45)

If the input $X(t)$ is WSS, then from Eq. (9.43) we have

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\alpha)\mu_X d\alpha = \mu_X \int_{-\infty}^{\infty} h(\alpha)d\alpha = \mu_X H(0)$$

(9.46)

where $H(0)$ is the frequency response of the linear system at $\omega = 0$. Thus, the mean of the output is a constant.

The autocorrelation of the output given in Eq. (9.45) becomes

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{XX}(t_2 - t_1 + \alpha - \beta)\, d\alpha d\beta$$

(9.47)

which indicates that $R_{YY}(t_1, t_2)$ is a function of the time difference $\tau = t_2 - t_1$. Hence,

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{XX}(\tau + \alpha - \beta)\, d\alpha d\beta$$

(9.48)

Thus, we conclude that if the input $X(t)$ is WSS, the output $Y(t)$ is also WSS.

The cross-correlation function between input $X(t)$ and $Y(t)$ is given by

$$\begin{aligned} R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\ &= E\left[X(t_1)\int_{-\infty}^{\infty} h(\alpha)X(t_2 - \alpha)\, d\alpha\right] \\ &= \int_{-\infty}^{\infty} h(\alpha)E[X(t_1)X(t_2 - \alpha)]\, d\alpha \\ &= \int_{-\infty}^{\infty} h(\alpha)R_{XX}(t_1, t_2 - \alpha)\, d\alpha \end{aligned}$$

(9.49)

When input $X(t)$ is WSS, Eq. (9.49) becomes

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\alpha)R_{XX}(t_2 - t_1 - \alpha)\, d\alpha$$

(9.50)

which indicates that $R_{XY}(t_1, t_2)$ is a function of the time difference $\tau = t_2 - t_1$. Hence

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(\alpha)R_{XX}(\tau - \alpha)\, d\alpha = h(\tau) * R_{XX}(\tau)$$

(9.51)

Thus, we conclude that if the input $X(t)$ to an LTI system is WSS, the output $Y(t)$ is also WSS. Moreover, the input $X(t)$ and output $Y(t)$ are jointly WSS.

In a similar manner, it can be shown that (Prob. 9.11)

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} h(-\alpha)R_{XY}(\tau - \alpha)\, d\alpha = h(-\tau) * R_{XY}(\tau)$$

(9.52)

Substituting Eq. (9.51) into Eq. (9.52), we have

$$R_{YY}(\tau) = h(-\tau) * h(\tau) * R_{XX}(\tau)$$

(9.53)

Now taking Fourier transforms of Eq. (9.51), (9.52), and (9.53) and using convolution property of Fourier transform [Eq. (5.58)], we obtain

$$S_{XY}(\omega) = H(\omega) S_{XX}(\omega)$$

(9.54)

$$S_{YY}(\omega) = H^*(\omega) S_{XY}(\omega)$$

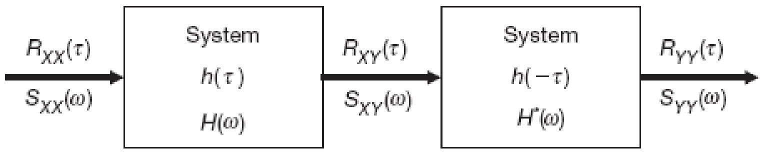
(9.55)

$$S_{YY}(\omega) = H^*(\omega) H(\omega) S_{XX}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

(9.56)

The schematic of these relations is shown in Fig. 9-3.

Figure 9-3



Equation (9.56) indicates the important result that the power spectral density of the output is the product of the power spectral density of the input and the magnitude squared of the frequency response of the system.

When the autocorrelation of the output $R_{YY}(\tau)$ is desired, it is easier to determine the power spectral density $S_{YY}(\omega)$ and then to evaluate the inverse Fourier transform (Prob. 9.13). Thus,

$$\begin{aligned} R_{YY}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(\omega) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{XX}(\omega) e^{j\omega\tau} d\omega \end{aligned}$$

(9.57)

By Eq. (9.4), the average power in the output $Y(t)$ is

$$E[Y^2(t)] = R_{YY}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{XX}(\omega) d\omega$$

(9.58)

9.4.3. C. Response of a Discrete-Time LTI System to Random Input:

When the input to a discrete-time LTI system is a discrete-time random sequence $X(n)$, then by Eq. (2.39), the output $Y(n)$ is

$$Y(n)=\sum_{i=-\infty}^{\infty} h(i)X(n-i)$$

(9.59)

The autocorrelation function of $Y(n)$ is given by (Prob. 9.22)

$$R_{YY}(n,m)=\sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(i)h(l)R_{XX}(n-i,m-l)$$

(9.60)

The cross-correlation function of $X(n)$ and $Y(n)$ is given by (Prob. 9.23)

$$R_{XY}(n,m)=E[X(n)Y(m)]=\sum_{l=-\infty}^{\infty} h(l)R_{XX}(n,m-l)$$

(9.61)

When $X(n)$ is WSS, then from Eq. (9.59)

$$\mu_Y(n)=E[Y(n)]=\mu_X\sum_{i=-\infty}^{\infty} h(i)=\mu_X H(0)$$

(9.62)

where $H(0)=H(\Omega)|_{\Omega=0}$ and $H(\Omega)$ is the frequency response of the system defined by the Fourier transform of $H(n)$.

The autocorrelation function of $Y(n)$ is, from Eq. (9.60)

$$R_{YY}(n,m)=\sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(i)h(l)R_{XX}(m-n+i-l)$$

(9.63)

Setting $m=n+k$, we get

$$R_{YY}(n,n+k)=\sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(i)h(l)R_{XX}(k+i-l)=R_{YY}(k)$$

(9.64)

Similarly, from Eq. (9.61), we obtain

$$R_{XY}(k)=\sum_{l=-\infty}^{\infty} h(l)R_{XX}(k-l)=h(k)*R_{XX}(k)$$

(9.65)

and (Prob. 9.24)

$$R_{YY}(k)=\sum_{l=-\infty}^{\infty} h(-l)R_{XY}(k-l)=h(-k)*R_{XY}(k)$$

(9.66)

Substituting Eq. (9.65) into Eq. (9.66), we obtain

$$R_{YY}(k) = h(-k) * h(k) * R_{XX}(k)$$

(9.67)

Now taking Fourier transforms of Eq. (9.65), (9.66) and (9.67), we obtain

$$S_{XY}(\Omega) = H(\Omega)\, S_{XX}(\Omega)$$

(9.68)

$$S_{YY}(\Omega) = H^*(\Omega)\, S_{XY}(\Omega)$$

(9.69)

$$S_{YY}(\Omega) = H^*(\Omega)H(\Omega)S_{XX}(\Omega) = |H(\Omega)|^2\, S_{XX}(\Omega)$$

(9.70)

Similarly, when the autocorrelation function of the output $R_{YY}(k)$ is desired, it is easier to determine the power spectral density $S_{YY}(\Omega)$ and then take the inverse Fourier transform. Thus,

$$R_{YY}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{YY}(\Omega)e^{j\Omega k}\, d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\Omega)|^2\, S_{XX}(\Omega)e^{j\Omega k}\, d\Omega$$

(9.71)

By Eq. (9.21), the average power in the output $Y(n)$ is

$$E\left[Y^2(n)\right] = R_{YY}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\Omega)|^2\, S_{XX}(\Omega)\, d\Omega$$

(9.72)

9.5. SOLVED PROBLEMS

9.5.1. Correlations and Power Spectral Densities

9.1. Let $X(t)$ be a WSS random process. Verify Eqs. (9.2) and (9.3); that is,

a. $R_{XX}(-\tau) = R_{XX}(\tau)$

b. $|R_{XX}(\tau)| \le R_{XX}(0)$

a. (*a*) From Eq. (8.35)

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

Setting $t + \tau = t'$, we have

$$\begin{aligned} R_{XX}(\tau) &= E[X(t' - \tau)X(t')] \\ &= E[X(t')X(t' - \tau)] = R_{XX}(-\tau) \end{aligned}$$

$$E[[X(t) \pm X(t+\tau)]^2] \geq 0$$

or
$$E[X^2(t) \pm 2X(t)X(t+\tau) + X^2(t+\tau)] \geq 0$$

or
$$E[X^2(t)] \pm 2E[X(t)X(t+\tau)] + E[X^2(t+\tau)] \geq 0$$

or
$$2R_{XX}(0) \pm 2R_{XX}(\tau) \geq 0$$

Hence,
$$R_{XX}(0) \geq |R_{XX}(\tau)|$$

9.2. Let $X(t)$ and $Y(t)$ be WSS random processes. Verify Eqs. (9.7) and (9.8); that is,

a. $R_{XY}(-\tau) = R_{YX}(\tau)$

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$$

a. By Eq. (8.42)

$$R_{XY}(-\tau) = E[X(t)Y(t-\tau)]$$

Setting $t - \tau = t'$, we obtain

$$R_{XY}(-\tau) = E[X(t'+\tau)Y(t')] = E[Y(t')X(t'+\tau)] = R_{YX}(\tau)$$

b. From the Cauchy-Schwarz inequality Eq. (B.129) (Appendix B), it follows that

$$\{E[X(t)Y(t+\tau)]\}^2 \leq E[X^2(t)]E[Y^2(t+\tau)]$$

or

$$[R_{XY}(\tau)]^2 \leq R_{XX}(0)R_{YY}(0)$$

Thus,

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$$

9.3. Show that the power spectrum of a (real) random process $X(t)$ is real, and verify Eq. (9.14); that is,

$$S_{XX}(-\omega) = S_{XX}(\omega)$$

From Eq. (9.11) and by expanding the exponential, we have

$$\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\,d\tau \\ &= \int_{-\infty}^{\infty} R_{XX}(\tau)(\cos\omega\tau - j\sin\omega\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} R_{XX}(\tau)\cos\omega\tau\,d\tau - j\int_{-\infty}^{\infty} R_{XX}(\tau)\sin\omega\tau\,d\tau \end{aligned}$$

(9.73)

Since $R_{XX}(-\tau) = R_{XX}(\tau)$ [Eq. (9.2)] imaginary term in Eq. (9.73) then vanishes and we obtain

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\cos\omega\tau\,d\tau$$

(9.74)

which indicates that $S_{XX}(\omega)$ is real.

Since the cosine is an even function of its arguments, that is, cos $(-\omega\tau)$ = cos $\omega\tau$, it follows that

$$S_{XX}(-\omega) = S_{XX}(\omega)$$

which indicates that the power spectrum of $X(t)$ is an even function of frequency.

9.4. Let $X(t)$ and $Y(t)$ be both zero-mean and WSS random processes. Consider the random process $Z(t)$ defined by

$$Z(t) = X(t) + Y(t)$$

(9.75)

a. Determine the autocorrelation and the power spectrum of $Z(t)$ if $X(t)$ and $Y(t)$ are jointly WSS.

b. Repeat part (*a*) if $X(t)$ and $Y(t)$ are orthogonal.

c. Show that if $X(t)$ and $Y(t)$ are orthogonal, then the mean square of $Z(t)$ is equal to the sum of the mean squares of $X(t)$ and $Y(t)$.

a. The autocorrelation of $Z(t)$ is given by

$$\begin{aligned} R_{ZZ}(t_1, t_2) &= E[Z(t_1)Z(t_2)] \\ &= E[[X(t_1) + Y(t_1)][X(t_2) + Y(t_2)]] \\ &= E[X(t_1)X(t_2)] + E[X(t_1)Y(t_2)] \\ &\quad + E[Y(t_1)X(t_2)] + E[Y(t_1)Y(t_2)] \\ &= R_{XX}(t_1, t_2) + R_{XY}(t_1, t_2) + R_{YX}(t_1, t_2) + R_{YY}(t_1, t_2) \end{aligned}$$

(9.76)

If $X(t)$ and $Y(t)$ are jointly WSS, then we have

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

(9.77)

where $\tau = t_2 - t_1$.

Taking the Fourier transform of both sides of Eq. (9.77), we obtain

$$S_{ZZ}(\omega) = S_{XX}(\omega) + S_{XY}(\omega) + S_{YX}(\omega) + S_{YY}(\omega)$$

(9.78)

b. If $X(t)$ and $Y(t)$ are orthogonal [Eq. (8.47)],

$$R_{XY}(\tau) = R_{YX}(\tau) = 0$$

Then Eqs. (9.77) and (9.78) become

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

(9.79)

and

$$S_{ZZ}(\omega) = S_{XX}(\omega) + S_{YY}(\omega)$$

(9.80)

c. From Eqs. (9.79) and (8.37)

$$R_{ZZ}(0) = R_{XX}(0) + R_{YY}(0)$$

or

$$E[Z^2(t)] = E[X^2(t)] + E[Y^2(t)]$$

(9.81)

which indicates that the mean square of $Z(t)$ is equal to the sum of the mean squares of $X(t)$ and $Y(t)$.

9.5. Two random processes $X(t)$ and $(Y(t)$ are given by

$$X(t) = A\cos(\omega t + \Theta)$$

(9.82)

$$Y(t) = A\sin(\omega t + \Theta)$$

(9.83)

where A and ω are constants and θ is a uniform random variable over $[0, 2\pi]$. Find the cross-correlation of $X(t)$ and $Y(t)$, and verify Eq. (9.7).

From Eq. (8.42), the cross-correlation of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(t, t+\tau) &= E[X(t)Y(t+\tau)] \\ &= E[A^2 \cos(\omega t + \Theta)\sin[\omega(t+\tau)+\Theta]] \\ &= \frac{A^2}{2}E[\sin(2\omega t + \omega\tau + 2\Theta) - \sin(-\omega\tau)] \\ &= \frac{A^2}{2}\sin\omega\tau = R_{XY}(\tau) \end{aligned}$$

(9.84)

Similarly,

$$\begin{aligned} R_{YX}(t, t+\tau) &= E[Y(t)X(t+\tau)] \\ &= E[A^2 \sin(\omega t + \Theta)\cos[\omega(t+\tau)+\Theta]] \\ &= \frac{A^2}{2}E[\sin(2\omega t + \omega\tau + 2\Theta) + \sin(-\omega\tau)] \\ &= -\frac{A^2}{2}\sin\omega\tau = R_{YX}(\tau) \end{aligned}$$

(9.85)

From Eqs. (9.84) and (9.85)

$$R_{XY}(-\tau) = \frac{A^2}{2}\sin\omega(-\tau) = -\frac{A^2}{2}\sin\omega\tau = R_{YX}(\tau)$$

which verifies Eq. (9.7).

9.6. A class of modulated random signal $Y(t)$ is defined by

$$Y(t) = AX(t) \cos(\omega_c t + \Theta)$$

(9.86)

where $X(t)$ is the random message signal and $A\cos(\omega_c t + \Theta)$ is the carrier. The random message signal $X(t)$ is a zero-mean stationary random process with autocorrelation $R_{XX}(\tau)$ and power spectrum $S_{XX}(\omega)$. The carrier amplitude A and the frequency ω_c are constants, and phase Θ is a random variable uniformly distributed over $[0, 2\pi]$. Assuming that $X(t)$ and Θ are independent, find the mean, autocorrelation, and power spectrum of $Y(t)$.

$$\begin{aligned}\mu_Y(t) &= E[Y(t)] = E[AX(t)\cos(\omega_c t + \Theta)] \\ &= AE[X(t)]E[\cos(\omega_c t + \Theta)] = 0\end{aligned}$$

since $X(t)$ and θ are independent and $E[X(t)] = 0$.

$$\begin{aligned}R_{YY}(t, t+\tau) &= E[Y(t)Y(t+\tau)] \\ &= E[A^2 X(t)X(t+\tau)\cos(\omega_c t + \Theta)\cos[\omega_c(t+\tau) + \Theta]] \\ &= \frac{A^2}{2}E[X(t)X(t+\tau)]E[\cos\omega_c\tau + \cos(2\omega_c t + \omega_c\tau + 2\Theta)] \\ &= \frac{A^2}{2}R_{XX}(\tau)\cos\omega_c\tau = R_{YY}(\tau)\end{aligned}$$

(9.87)

Since the mean of $Y(t)$ is a constant and the autocorrelation of $Y(t)$ depends only on the time difference τ, $Y(t)$ is WSS. Thus,

$$S_{YY}(\omega) = \mathscr{F}[R_{YY}(\tau)] = \frac{A^2}{2}\mathscr{F}[R_{XX}(\tau)\cos\omega_c\tau]$$

By Eqs. (9.11) and (5.144)

$$\begin{aligned}&\mathscr{F}[R_{XX}(\tau)] = S_{XX}(\omega) \\ &\mathscr{F}(\cos\omega_c\tau) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)\end{aligned}$$

Then, using the frequency convolution theorem (5.59) and Eq. (2.59), we obtain

$$\begin{aligned}S_{YY}(\omega) &= \frac{A^2}{4\pi}S_{XX}(\omega) * [\pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)] \\ &= \frac{A^2}{4}[S_{XX}(\omega - \omega_c) + S_{XX}(\omega + \omega_c)]\end{aligned}$$

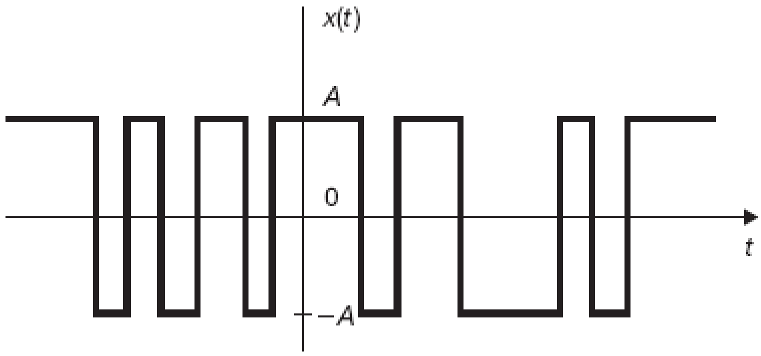
(9.88)

9.7. Consider a random process $X(t)$ that assumes the values $\pm A$ with equal probability. A typical sample func-tion of $X(t)$ is shown in Fig. 9-4. The average number of polarity switches (zero crossings) per unit time is α. The probability of having exactly k crossings in time τ is given by the Poisson distribution [Eq. (B.48)]

$$P(Z = k) = e^{-\alpha\tau}\frac{(\alpha\tau)^k}{k!}$$

(9.89)

Figure 9-4



where Z is the random variable representing the number of zero crossing. The process $X(t)$ is known as the *telegraph* signal. Find the autocorrelation and the power spectrum of $X(t)$.

If τ is any positive time interval, then

$$\begin{aligned}
R_{XX}(t, t+\tau) &= E[X(t)X(t+\tau)] \\
&= A^2 P[\,X(t) \text{ and } X(t+\tau) \text{ have same signs}] \\
&\quad + (-A^2) P[X(t) \text{ and } X(t+\tau) \text{ have different signs}] \\
&= A^2 P[Z \text{ even in } (t, t+\tau)] - A^2 P[Z \text{ odd in } (t, t+\tau)] \\
&= A^2 \sum_{k \text{ even}} e^{-\alpha\tau} \frac{(\alpha\tau)^k}{k!} - A^2 \sum_{k \text{ odd}} e^{-\alpha\tau} \frac{(\alpha\tau)^k}{k!} \\
&= A^2 e^{-\alpha\tau} \sum_{k=0}^{\infty} \frac{(\alpha\tau)^k}{k!} (-1)^k \\
&= A^2 e^{-\alpha\tau} \sum_{k=0}^{\infty} \frac{(-\alpha\tau)^k}{k!} = A^2 e^{-\alpha\tau} e^{-\alpha\tau} = A^2 e^{-2\alpha\tau}
\end{aligned}$$

(9.90)

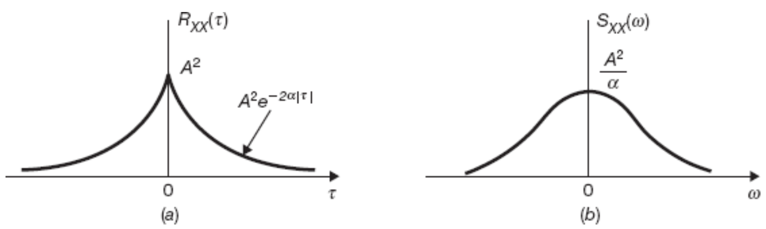
which indicates that the autocorrelation depends only on the time difference τ. By Eq. (9.2), the complete solution that includes $\tau < 0$ is given by

$$R_{XX}(\tau) = A^2 e^{-2\alpha|\tau|}$$

(9.91)

which is sketched in Fig. 9-5(*a*).

Figure 9-5



Taking the Fourier transform of both sides of Eq. (9.91), we see that the power spectrum of $X(t)$ is [Eq. (5.138)]

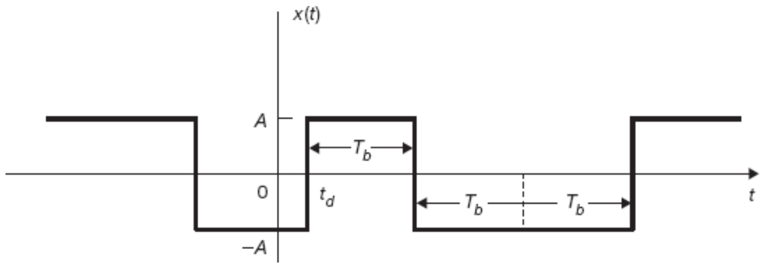
$$S_{XX}(\omega) = A^2 \frac{4\alpha}{\omega^2 + (2\alpha)^2}$$

(9.92)

which is sketched in Fig. 9-5(b).

9.8. Consider a random binary process $X(t)$ consisting of a random sequence of binary symbols 1 and 0. A typical sample function of $X(t)$ is shown in Fig. 9-6. It is assumed that

Figure 9-6 Random binary signal.



1. The symbols 1 and 0 are represented by pulses of amplitude +A and −A V, respectively, and duration T_bs.
2. The two symbols 1 and 0 are equally likely, and the presence of a 1 or 0 in any one interval is independent of the presence in all other intervals.
3. The pulse sequence is not synchronized, so that the starting time t_d of the first pulse after t = 0 is equally likely to be anywhere between 0 to T_b. That is, t_d is the sample value of a random variable T_d uniformly distributed over [0, T_b].

Find the autocorrelation and power spectrum of $X(t)$.

The random binary process $X(t)$ can be represented by

$$X(t) = \sum_{k=-\infty}^{\infty} A_k p(t - kT_b - T_d)$$

(9.93)

where {A_k} is a sequence of independent random variables with $P[A_k = A] = P[A_k = -A] = 1–2$, $p(t)$ is a unit amplitude pulse of duration T_b, and T_d is a random variable uniformly distributed over [0, T_b].

$$\mu_X(t) = E[X(t)] = E[A_k] = \frac{1}{2}A + \frac{1}{2}(-A) = 0$$

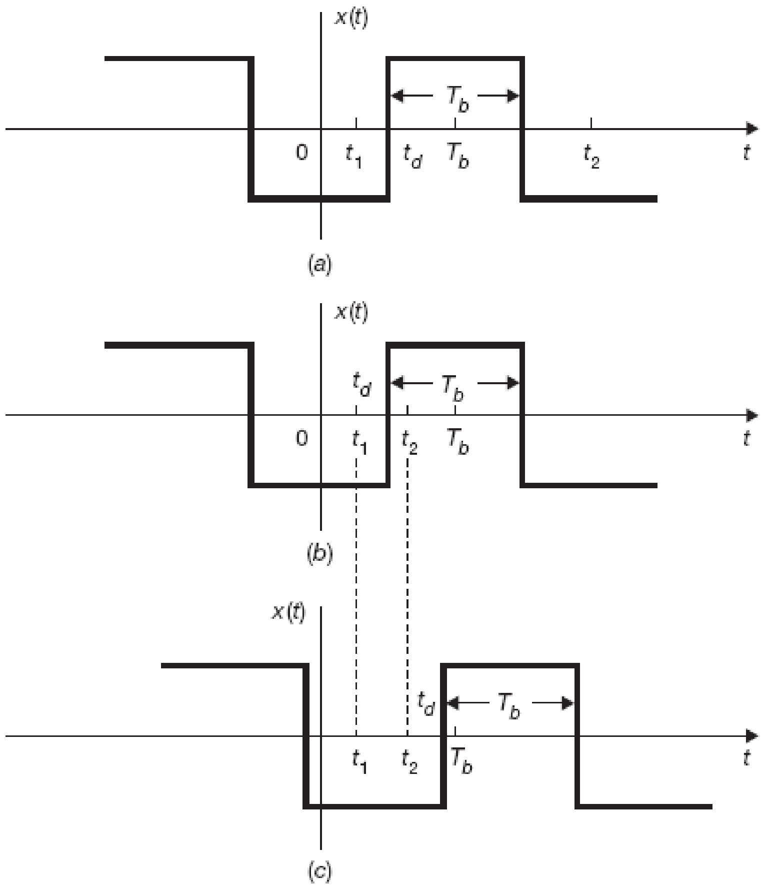
(9.94)

Let $t_2 > t_1$. When $t_2 - t_1 > T_b$, then t_1 and t_2 must fall in different pulse intervals [Fig. 9-7(a)] and the random variables $X(t_1)$ and $X(t_2)$ are therefore independent. We thus have

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = E[X(t_1)]E[X(t_2)] = 0$$

(9.95)

Figure 9-7



When $t_2 - t_1 < T_b$, then depending on the value of T_d, t_1 and t_2 may or may not be in the same pulse interval [Fig. 9-7(*b*) and (*c*)]. If we let *B* denote the random event "t_1 and t_2 are in adjacent pulse intervals," then we have

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)\,|\,B]\,P(B) + E[X(t_1)X(t_2)\,|\,\bar{B}]\,P(\bar{B})$$

Now

$$E[X(t_1)X(t_2)\,|\,B] = E[X(t_1)]E[X(t_2)] = 0$$

$$E[X(t_1)X(t_2)\,|\,\bar{B}] = A^2$$

Since $P(B)$ will be the same when t_1 and t_2 fall in any time range of length T_b, it suffices to consider the case $0 < t < T_b$, as shown in Fig. 9-7(*b*). From Fig. 9-7 (*b*);

$$\begin{aligned} P(B) &= P(t_1 < T_d < t_2) \\ &= \int_{t_1}^{t_2} f_{T_d}(t_d)\,dt_d = \int_{t_1}^{t_2} \frac{1}{T_b}\,dt_d = \frac{t_2 - t_1}{T_b} \end{aligned}$$

From Eq. (B.4), we have

$$P(\bar{B}) = 1 - P(B) = 1 - \frac{t_2 - t_1}{T_b}$$

Thus,

$$R_{XX}(t_1, t_2) = A^2\left(1 - \frac{t_2 - t_1}{T_b}\right) = R_{XX}(\tau)$$

(9.96)

where $\tau = t_2 - t_1$.

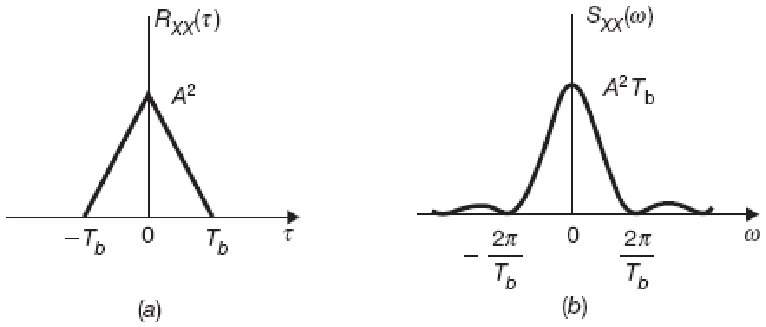
Since $R_{XX}(-\tau) = R_{XX}(\tau)$, we conclude that

$$R_{XX}(\tau) = \begin{cases} A^2\left(1 - \frac{|\tau|}{T_b}\right) & |\tau| \le T_b \\ 0 & |\tau| > T_b \end{cases}$$

(9.97)

which is plotted in Fig. 9-8(*a*).

Figure 9-8



From Eqs. (9.94) and (9.97), we see that $X(t)$ is WSS. Thus, from Eq. (9.11), the power spectrum of $X(t)$ is (Prob. 5.67)

$$S_{XX}(\omega) = A^2T_b\left[\frac{\sin(\omega T_b/2)}{\omega T_b/2}\right]^2$$

(9.98)

which is plotted in Fig. 9-8(*b*).

9.5.2. Response of Linear System to Random Input

9.9. A WSS random process $X(t)$ is applied to the input of an LTI system with impulse response $h(t) = 3e^{-2t}u(t)$. Find the mean value of the output $Y(t)$ of the system if $E[X(t)] = 2$.

By Eq. (5.45), the frequency response $H(\omega)$ of the system is

$$H(\omega) = \mathscr{F}[h(t)] = 3\frac{1}{j\omega + 2}$$

Then, by Eq. (9.46), the mean value of $Y(t)$ is

$$\mu_Y(t) = E[Y(t)] = \mu_X H(0) = 2\left(\frac{3}{2}\right) = 3$$

9.10. Let $Y(t)$ be the output of an LTI system with impulse response $h(t)$, when $X(t)$ is applied as input. Show that

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\beta) R_{XX}(t_1, t_2 - \beta)\, d\beta$$

(9.99)

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\alpha) R_{XY}(t_1, -\alpha, t_2)\, d\alpha$$

(9.100)

a. Using Eq. (9.43), we have

$$\begin{aligned} R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\ &= E\left[X(t_1)\int_{-\infty}^{\infty} h(\beta)X(t_2 - \beta)d\beta\right] \\ &= \int_{-\infty}^{\infty} h(\beta)E[X(t_1)X(t_2 - \beta)]\, d\beta \\ &= \int_{-\infty}^{\infty} h(\beta)R_{XX}(t_1, t_2 - \beta)\, d\beta \end{aligned}$$

b. Similarly,

$$\begin{aligned} R_{YY}(t_1, t_2) &= E[Y(t_1)Y(t_2)] \\ &= E\left[\int_{-\infty}^{\infty} h(\alpha)X(t_1 - \alpha)\, d\alpha Y(t_2)\right] \\ &= \int_{-\infty}^{\infty} h(\alpha)E[X(t_1 - \alpha)]Y(t_2)d\alpha \\ &= \int_{-\infty}^{\infty} h(\alpha)R_{XY}(t_1 - \alpha, t_2)\, d\alpha \end{aligned}$$

9.11. Let $X(t)$ be a WSS random input process to an LTI system with impulse response $h(t)$, and let $Y(t)$ be the corresponding output process. Show that

$$R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$$

(9.101)

$$R_{YY}(\tau) = h(-\tau) * R_{XY}(\tau)$$

(9.102)

$$S_{XY}(\omega) = H(\omega) S_{XX}(\omega)$$

(9.103)

$$S_{YY}(\omega) = H^*(\omega) S_{XY}(\omega)$$

(9.104)

where * denotes the convolution and $H^*(\omega)$ is the complex conjugate of $H(\omega)$.

a. If $X(t)$ is WSS, then Eq. (9.99) of Prob. 9.10 becomes

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\beta) R_{XX}(t_2 - t_1 - \beta)\, d\beta$$

(9.105)

which indicates that $R_{XY}(t_1, t_2)$ is a function of the time difference $\tau = t_2 - t_1$ only. Hence, Eq. (9.105) yields

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(\beta) R_{XX}(\tau - \beta)\, d\beta = h(\tau) * R_{XX}(\tau)$$

b. Similarly, if $X(t)$ is WSS, then Eq. (9.100) becomes

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\alpha) R_{XY}(t_2 - t_1 + \alpha)\, d\alpha$$

or

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} h(\alpha) R_{XY}(\tau + \alpha)\, d\alpha = h(-\tau) * R_{XY}(\tau)$$

c. Taking the Fourier transform of both sides of Eq. (9.101) and using Eqs. (9.22) and (5.58), we obtain

$$S_{XY}(\omega) = H(\omega)\, S_{XX}(\omega)$$

d. Similarly, taking the Fourier transform of both sides of Eq. (9.102) and using Eqs. (9.11), (5.58), and (5.53), we obtain

$$S_{YY}(\omega) = H^*(\omega) S_{XY}(\omega)$$

Note that by combining Eqs. (9.103) and (9.104), we obtain Eq. (9.56); that is,

$$S_{YY}(\omega) = H^*(\omega) H(\omega) S_{XX}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

9.12. Let $X(t)$ and $Y(t)$ be the wide-sense stationary random input process and random output process, respectively, of a quadrature phase-shifting filter ($-\pi/2$ rad phase shifter of Prob. 5.48). Show that

$$R_{XX}(\tau) = R_{YY}(\tau)$$

(9.106)

$$R_{XY}(\tau) = \hat{R}_{XX}(\tau)$$

(9.107)

where $\hat{R}_{XX}(\tau)$ is the Hilbert transform of $R_{XX}(\tau)$.

a. The Hilbert transform $\hat{X}(t)$ of $X(t)$ was defined in Prob. 5.48 as the output of a quadrature phase-shifting filter with

$$h(t) = \frac{1}{\pi t} \qquad H(\omega) = -j\,\mathrm{sgn}(\omega)$$

Since $|H(\omega)|^2 = 1$, we conclude that if $X(t)$ is a WSS random signal, then $Y(t) = \hat{X}(t)$ and by Eq. (9.56)

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = S_{XX}(\omega)$$

Hence,

$$R_{YY}(\tau) = \mathscr{F}^{-1}[S_{YY}(\omega)] = \mathscr{F}^{-1}[S_{XX}(\omega)] = R_{XX}(\tau)$$

b. Using Eqs. (9.101) and (5.174), we have

$$R_{XY}(\tau) = h(\tau) * R_{XX}(\tau) = \frac{1}{\pi t} * R_{XX}(\tau) = \hat{R}_{XX}(\tau)$$

9.13. A WSS random process $X(t)$ with autocorrelation

$$R_{XX}(\tau) = Ae^{-a|\tau|}$$

where A and a are real positive constants, is applied to the input of an LTI system with impulse response

$$h(t) = e^{-bt}u(t)$$

where b is a real positive constant. Find the autocorrelation of the output $Y(t)$ of the system.

Using Eq. (5.45), we see that the frequency response $H(\omega)$ of the system is

$$H(\omega) = \mathscr{F}[h(t)] = \frac{1}{j\omega + b}$$

So

$$|H(\omega)|^2 = \frac{1}{\omega^2 + b^2}$$

Using Eq. (5.138), we see that the power spectral density of $X(t)$ is

$$S_{XX}(\omega) = \mathscr{F}[R_{XX}(\tau)] = A\frac{2a}{\omega^2 + a^2}$$

By Eq. (9.56), the power spectral density of $Y(t)$ is

$$\begin{aligned}
S_{YY}(\omega) &= |H(\omega)|^2 S_{XX}(\omega) \\
&= \left(\frac{1}{\omega^2 + b^2}\right)\left(\frac{2aA}{\omega^2 + a^2}\right) \\
&= \frac{aA}{(a^2 - b^2)b}\left(\frac{2b}{\omega^2 + b^2}\right) - \frac{A}{a^2 - b^2}\left(\frac{2a}{\omega^2 + a^2}\right)
\end{aligned}$$

Taking the inverse Fourier transform of both sides of the above equation and using Eq. (5.139), we obtain

$$R_{YY}(\tau) = \frac{aA}{(a^2 - b^2)b}e^{-b|\tau|} - \frac{A}{a^2 - b^2}e^{-a|\tau|}$$

9.14. Verify Eq. (9.13); that is, the power spectrum of any WSS process $X(t)$ is real and

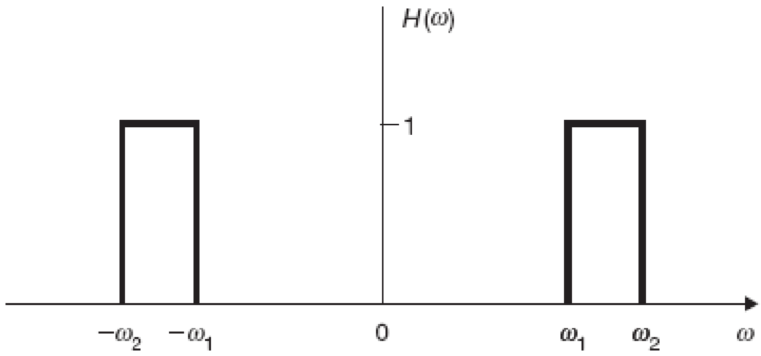
$$S_{XX}(\omega) \geq 0$$

for every ω.

The realness of the power spectrum of $X(t)$ was shown in Prob. 9.3. Consider an ideal bandpass filter with frequency response (Fig. 9-9)

$$H(\omega) = \begin{cases} 1 & \omega_1 \leq |\omega| \leq \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

Figure 9-9



with a random process $X(t)$ as its input. From Eq. (9.56) it follows that the power spectrum $S_{YY}(\omega)$ of the resulting output $Y(t)$ equals

$$S_{YY}(\omega) = \begin{cases} S_{XX}(\omega) & \omega_1 \le |\omega| \le \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

Hence, from Eq. (9.58), we have

$$E[Y^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega = 2\left(\frac{1}{2\pi}\right)\int_{\omega_1}^{\omega_2} S_{XX}(\omega)\, d\omega \ge 0$$

(9.108)

which indicates that the area of $S_{XX}(\omega)$ in any interval of ω is nonnegative. This is possible only if $S_{XX}(\omega) \ge 0$ for every ω.

9.15. Consider a WSS process $X(t)$ with autocorrelation $R_{XX}(\tau)$ and power spectrum $S_{XX}(\omega)$. Let $X'(t) = dX(t)/dt$. Show that

$$R_{XX'}(\tau) = \frac{dR_{XX}(\tau)}{d\tau}$$

(9.109)

$$R_{X'X'}(\tau) = -\frac{d^2 R_{XX}(\tau)}{d\tau^2}$$

(9.110)

$$S_{X'X'}(\omega) = \omega^2 S_{XX}(\omega)$$

(9.111)

A system with frequency response $H(\omega) = j\omega$ is a differentiator (Fig. 9-10). Thus, if $X(t)$ is its input, then its output is $Y(t) = X'(t)$ [see Eq. (5.55)].

Figure 9-10 Differentiator.

a. From Eq. (9.103)

$$S_{XX'}(\omega) = H(\omega)S_{XX}(\omega) = j\omega S_{XX}(\omega)$$

Taking the inverse Fourier transform of both sides, we obtain

$$R_{XX'}(\tau) = \frac{dR_{XX}(\tau)}{d\tau}$$

b. From Eq. (9.104)

$$S_{X'X'}(\omega) = H^*(\omega)S_{XX'}(\omega) = -j\omega S_{XX'}(\omega)$$

Again taking the inverse Fourier transform of both sides and using the result of part (*a*), we have

$$R_{X'X'}(\tau) = -\frac{dR_{XX'}(\tau)}{d\tau} = -\frac{d^2R_{XX}(\tau)}{d\tau^2}$$

c. From Eq. (9.56)

$$S_{X'X'}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = |j\omega|^2 S_{XX}(\omega) = \omega^2 S_{XX}(\omega)$$

9.16. Suppose that the input to the differentiator of Fig. 9-10 is the zero-mean random telegraph signal of Prob. 9.7.

a. Determine the power spectrum of the differentiator output and plot it.

b. Determine the mean-square value of the differentiator output.

a. From Eq. (9.92) of Prob. 9.7

$$S_{XX}(\omega) = A^2 \frac{4\alpha}{\omega^2 + (2\alpha)^2}$$

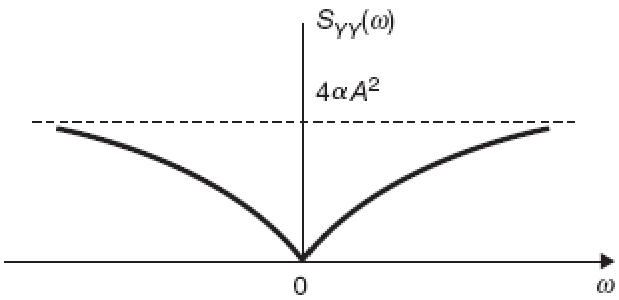
For the differentiator $H(\omega) = j\omega$, and from Eq. (9.56), we have

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = A^2 \frac{4\alpha\omega^2}{\omega^2 + (2\alpha)^2}$$

(9.112)

which is plotted in Fig. 9-11.

Figure 9-11



b. From Eq. (9.58) or Fig. 9-11

$$E[Y^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega = \infty$$

9.17. Suppose the random telegraph signal of Prob. 9.7 is the input to an ideal bandpass filter with unit gain and narrow bandwidth $W_B(= 2\pi B)(<<\omega_c)$ centered at $\omega_c = 2a$. Find the dc component and the average power of the output.

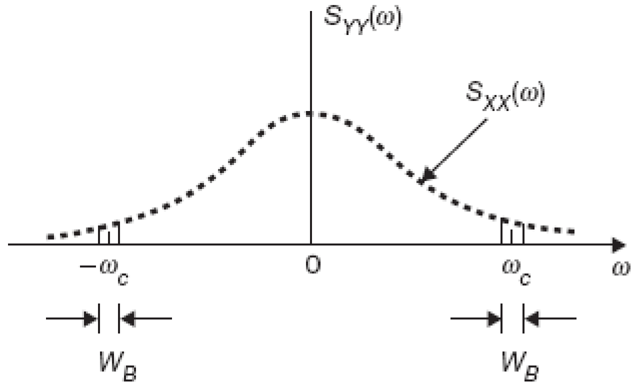
From Eqs. (9.56) and (9.92) and Fig. 9-5 (*b*), the resulting output power spectrum

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

is shown in Fig. 9-12. Since $H(0) = 0$, from Eq. (9.46) we see that

$$\mu_Y = H(0)\mu_X = 0$$

Figure 9-12



Hence, the dc component of the output is zero.

From Eq. (9.92) (Prob. 9.7)

$$S_{XX}(\pm\omega_c) = A^2 \frac{4\alpha}{(2\alpha)^2 + (2\alpha)^2} = \frac{A^2}{(2\alpha)}$$

Since $W_B << \omega_c$,

$$S_{YY}(\omega) \approx \begin{cases} \dfrac{A^2}{2\alpha} & |\omega - \omega_c| < \dfrac{W_B}{2} \\ 0 & \text{otherwise} \end{cases}$$

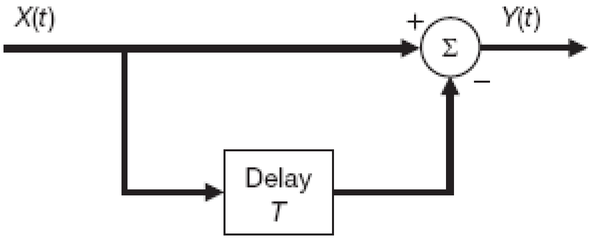
The average output power is

$$E[Y^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega$$
$$\approx \frac{1}{2\pi}(2W_B)\left(\frac{A^2}{2\alpha}\right) = \frac{A^2 W_B}{2\pi\alpha} = \frac{A^2 B}{\alpha}$$

(9.113)

9.18. Suppose that a WSS random process $X(t)$ with power spectrum $S_{XX}(\omega)$ is the input to the filter shown in Fig. 9-13. Find the power spectrum of the output process $Y(t)$.

Figure 9-13



From Fig. 9-13, $Y(t)$ can be expressed as

$$Y(t) = X(t) - X(t - T)$$

(9.114)

From Eq. (2.1) the impulse response of the filter is

$$h(t) = \delta(t) - \delta(t - T)$$

and by Eqs. (5.140) and (5.50) the frequency response of the filter is

$$H(\omega) = 1 - e^{-j\omega T}$$

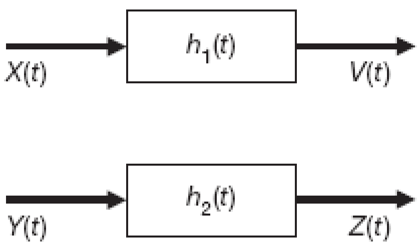
Thus, by Eq. (9.56) the output power spectrum is

$$\begin{aligned} S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) &= \left|1 - e^{-j\omega T}\right|^2 S_{XX}(\omega) \\ &= \left[(1 - \cos\omega T)^2 + \sin^2 \omega T\right] S_{XX}(\omega) \\ &= 2(1 - \cos\omega T) S_{XX}(\omega) \end{aligned}$$

(9.115)

9.19. Suppose that $X(t)$ is the input to an LTI system with impulse response $h_1(t)$ and that $Y(t)$ is the input to another LTI system with impulse response $h_2(t)$. It is assumed that $X(t)$ and $Y(t)$ are jointly wide-sense stationary. Let $V(t)$ and $Z(t)$ denote the random process at the respective system outputs (Fig. 9-14). Find the cross-correlation and cross spectral density of $V(t)$ and $Z(t)$ in terms of the cross-correlation and cross spectral density of $X(t)$ and $Y(t)$.

Figure 9-14



Using Eq. (9.43), we have

$$\begin{aligned}R_{VZ}(t_1,t_2) &= E[V(t_1)Z(t_2)] \\ &= E\left[\int_{-\infty}^{\infty} X(t_1-\alpha)h_1(\alpha)\,d\alpha \int_{-\infty}^{\infty} Y(t_2-\beta)h_2(\beta)\,d\beta\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2(\beta)E[X(t_1-\alpha)Y(t_2-\beta)]\,d\alpha d\beta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2(\beta)R_{XY}(t_1-\alpha,t_2-\beta)\,d\alpha d\beta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2(\beta)R_{XY}(t_2-t_1+\alpha-\beta)\,d\alpha d\beta\end{aligned}$$

(9.116)

since $X(t)$ and $Y(t)$ are jointly WSS.

Equation (9.116) indicates that $R_{VZ}(t_1, t_2)$ depends only on the time difference $\tau = t_2 - t_1$. Thus,

$$R_{VZ}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2(\beta)R_{XY}(\tau+\alpha-\beta)\,d\alpha\,d\beta$$

(9.117)

Taking the Fourier transform of both sides of Eq. (9.117), we obtain

$$\begin{aligned}S_{VZ}(\omega) &= \int_{-\infty}^{\infty} R_{VZ}(\tau)e^{-j\omega\tau}\,d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2(\beta)R_{XY}(\tau+\alpha-\beta)e^{-j\omega\tau}\,d\alpha\,d\beta\,d\tau\end{aligned}$$

Let $\tau + \alpha - \beta = \lambda$, or equivalently $\tau = \lambda - \alpha + \beta$. Then

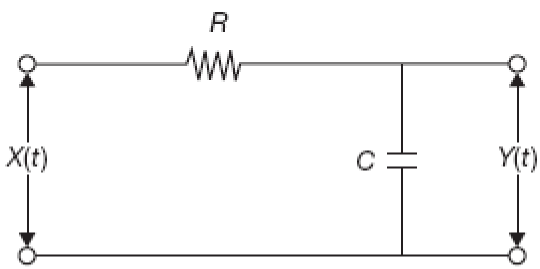
$$\begin{aligned}S_{VZ}(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\alpha)h_2(\beta)R_{XY}(\lambda)e^{-j\omega(\lambda-\alpha+\beta)}\,d\alpha\,d\beta\,d\lambda \\ &= \int_{-\infty}^{\infty} h_1(\alpha)e^{j\omega\alpha}\,d\alpha \int_{-\infty}^{\infty} h_2(\beta)e^{-j\omega\beta}\,d\beta \int_{-\infty}^{\infty} R_{XY}(\lambda)e^{-j\omega\lambda}\,d\lambda \\ &= H_1(-\omega)H_2(\omega)S_{XY}(\omega) \\ &= H_1^*(\omega)H_2(\omega)S_{XY}(\omega)\end{aligned}$$

(9.118)

where $H_1(\omega)$ and $H_2(\omega)$ are the frequency responses of the respective systems in Fig. 9-14.

9.20. The input $X(t)$ to the *RC* filter shown in Fig. 9-15 is a white-noise process.

Figure 9-15 RC filter.



a. Determine the power spectrum of the output process $Y(t)$.

b. Determine the autocorrelation and the mean-square value of $Y(t)$.

From Eq. (5.91) the frequency response of the *RC* filter is

$$H(\omega) = \frac{1}{1 + j\omega RC}$$

a. From Eqs. (9.31) and (9.56)

$$S_{XX}(\omega) = \frac{\eta}{2}$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \frac{1}{1 + (\omega RC)^2} \frac{\eta}{2}$$

(9.119)

b. Rewriting Eq. (9.119) as

$$S_{YY}(\omega) = \frac{\eta}{2} \frac{1}{2RC} \frac{2[1/(RC)]}{\omega^2 + [1/(RC)]^2}$$

and using the Fourier transform pair Eq. (5.138), we obtain

$$R_{YY}(\tau) = \frac{\eta}{2} \frac{1}{2RC} e^{-|\tau|/(RC)}$$

(9.120)

Finally, from Eq. (9.120)

$$E[Y^2(t)] = R_{YY}(0) = \frac{\eta}{4RC}$$

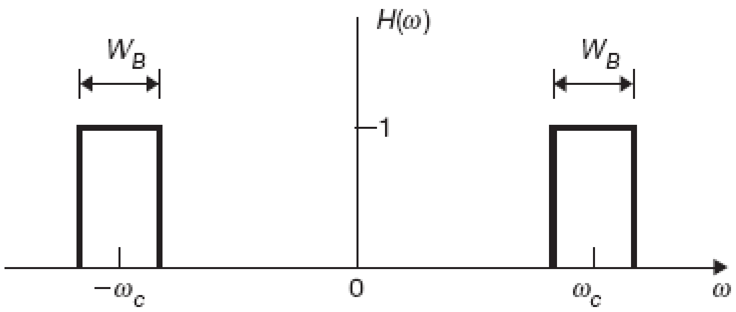
(9.121)

9.21. The input $X(t)$ to an ideal bandpass filter having the frequency response characteristic shown in Fig. 9-16 is a white-noise process. Determine the total noise power at the output of the filter.

$$S_{XX}(\omega) = \frac{\eta}{2}$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \frac{\eta}{2} |H(\omega)|^2$$

Figure 9-16



The total noise power at the output of the filter is

$$E[Y^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\,d\omega = \frac{1}{2\pi}\frac{\eta}{2}\int_{-\infty}^{\infty} |H(\omega)|^2\,d\omega$$
$$= \frac{\eta}{2}\frac{1}{2\pi}(2W_B) = \eta B$$

(9.122)

where $B = W_B/(2\pi)$ (in Hz).

9.22. Verify Eq. (9.60); that is

$$R_{YY}(n, m) = \sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(i)h(l)\,R_{XX}(n-i, m-l)$$

From Eq. (9.59) we have

$$R_{YY}(n, m) = E[Y(n)Y(m)] = E\left[\sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(i)h(l)X(n-i)X(m-l)\right]$$
$$= \sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(i)h(l)E[X(n-i)X(m-l)]$$
$$= \sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(i)h(l)\,R_{XX}(n-i, m-l)$$

9.23. Verify Eq. (9.61); that is

$$R_{XY}(n, m) = \sum_{l=-\infty}^{\infty} h(l)\,R_{XX}(n, m-l)$$

From Eq. (9.59), we have

$$R_{XY}(n, m) = E[X(n)Y(m)] = E\left[X(n)\sum_{l=-\infty}^{\infty} h(l)X(m-l)\right]$$
$$= \sum_{l=-\infty}^{\infty} h(l)E[X(n)X(m-l)]$$
$$= \sum_{l=-\infty}^{\infty} h(l)\,R_{XX}(n, m-l)$$

9.24. Verify Eq. (9.66); that is

$$R_{YY}(k) = h(-k) * R_{XY}(k)$$

From Eq. (9.64), and using Eq. (9.65), we obtain

$$R_{YY}(k)=\sum_{i=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}h(i)h(l)R_{XX}(k+i-l)$$
$$=\sum_{i=-\infty}^{\infty}h(i)R_{XY}(k+i)$$
$$=\sum_{l=-\infty}^{\infty}h(-l)R_{XY}(k-l)=h(-k)*R_{XY}(k)$$

9.25. The output $Y(n)$ of a discrete-time system is related to the input $X(n)$ by

$$Y(n) = X(n) - X(n-1)$$

(9.123)

If the input is a zero-mean discrete-time white noise with power spectral density σ^2, find $E[Y^2(t)]$.

The impulse response $h(n)$ of the system is given by

$$h(n) = \delta(n) - \delta(n-1)$$

Taking the discrete-time Fourier transform of $h(n)$, the frequency response $H(\Omega)$ of the system is given by

$$H(\Omega) = 1 - e^{-j\Omega}$$

Then

$$|H(\Omega)|^2 = |1 - e^{-j\Omega}|^2 = (1-\cos\Omega)^2 + \sin^2\Omega = 2(1-\cos\Omega)$$

(9.124)

Since $S_{XX}(\Omega) = \sigma^2$, and by Eq. (9.70), we have

$$S_{YY}(\Omega) = S_{XX}(\Omega)|H(\Omega)|^2 = 2\sigma^2(1-\cos\Omega)$$

(9.125)

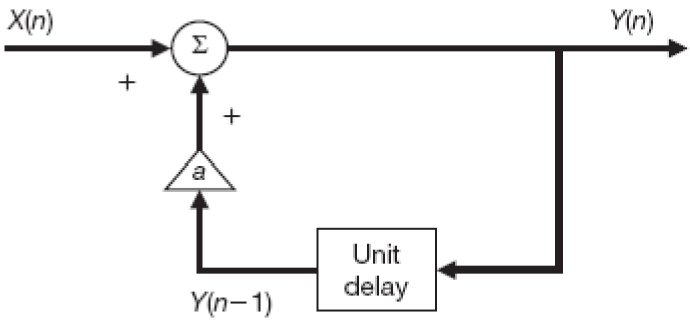
Thus, by Eq. (9.72) we obtain

$$E[Y^2(t)] = R_{YY}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{YY}(\Omega)\,d\Omega$$
$$=\frac{2\sigma^2}{2\pi}\int_{-\pi}^{\pi}(1-\cos\Omega)\,d\Omega = 2\sigma^2$$

(9.126)

9.26. The discrete-time system shown in Fig. 9-17 consists of one unit delay element and one scalar multiplier ($a < 1$). The input $X(n)$ is discrete-time white noise with average power σ^2. Find the spectral density and average power of the output $Y(n)$.

Figure 9-17



From Fig. 9-17, $Y(n)$ and $X(n)$ are related by

$$Y(n) = aY(n-1) + X(n)$$

(9.127)

The impulse response $h(n)$ of the system is defined by

$$h(n) = ah(n-1) + \delta(n)$$

(9.128)

Solving Eq. (9.128), we obtain

$$h(n) = a^n u(n)$$

(9.129)

where $u(n)$ is the unit step sequence defined by

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Taking the Fourier transform of Eq. (9.129), we obtain

$$H(\Omega) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \frac{1}{1 - ae^{-j\Omega}} \qquad a < 1, |\Omega| < \pi$$

Now, by Eq. (9.34),

$$S_{XX}(\Omega) = \sigma^2 \qquad |\Omega| < \pi$$

and by Eq. (9.70) the power spectral density of $Y(n)$ is

$$\begin{aligned} S_{YY}(\Omega) &= |H(\Omega)|^2 S_{XX}(\Omega) = H(\Omega)H(-\Omega)S_{XX}(\Omega) \\ &= \frac{\sigma^2}{(1 - ae^{-j\Omega})(1 - ae^{j\Omega})} \\ &= \frac{\sigma^2}{1 + a^2 - 2a\cos\Omega} \qquad |\Omega| < \pi \end{aligned}$$

(9.130)

Taking the inverse Fourier transform of Eq. (9.130), we obtain

$$R_{YY}(k) = \frac{\sigma^2}{1-a^2} a^{|k|}$$

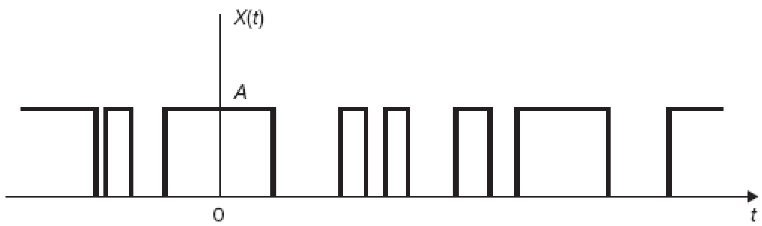
Thus, by Eq. (9.72) the average power of $Y(n)$ is

$$E[Y^2(n)] = R_{YY}(0) = \frac{\sigma^2}{1-a^2}$$

9.6. SUPPLEMENTARY PROBLEMS

9.27. A sample function of a random telegraph signal $X(t)$ is shown in Fig. 9-18. This signal makes independent random shifts between two equally likely values, A and 0. The number of shifts per unit time is governed by the Poisson distribution with parameter α.

Figure 9-18



a. Find the autocorrelation and the power spectrum of $X(t)$.

b. Find the rms value of $X(t)$.

9.28. Suppose that $X(t)$ is a Gaussian process with

$$\mu_X = 2 \qquad R_{XX}(\tau) = 5e^{-02|\tau|}$$

Find the probability that $X(4) \leq 1$.

9.29. The output of a filter is given by

$$Y(t) = X(t+T) - X(t-T)$$

where $X(t)$ is a WSS process with power spectrum $S_{XX}(\omega)$ and T is a constant. Find the power spectrum of $Y(t)$.

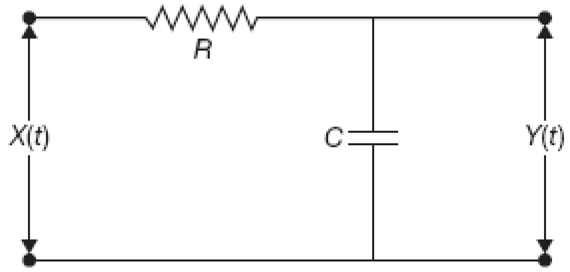
9.30. Let $\hat{X}(t)$ is the Hilbert transform of a WSS process $X(t)$. Show that

$$R_{X\hat{X}}(0) = E[X(t)\hat{X}(t)] = 0$$

9.31. A WSS random process $X(t)$ is applied to the input of an LTI system with impulse response $h(t) = 3e^{-2t}u(t)$. Find the mean value of $Y(t)$ of the system if $E[X(t)] = 2$.

9.32. The input $X(t)$ to the RC filter shown in Fig. 9-19 is a white noise specified by Eq. (9.31). Find the mean-square value of $Y(t)$.

Figure 9-19 *RC filter.*



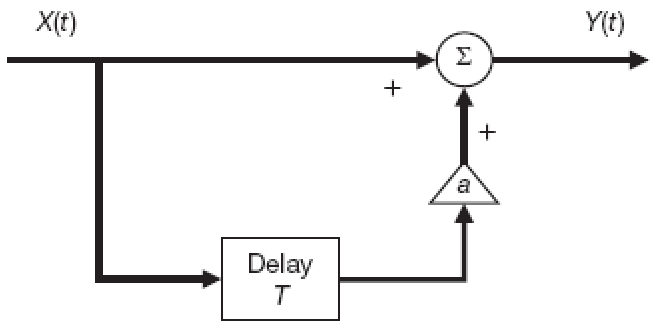
9.33. The input $X(t)$ to a differentiator is the random telegraph signal of Prob. 9.7.

a. Determine the power spectral density of the differentiator output.

b. Find the mean-square value of the differentiator output.

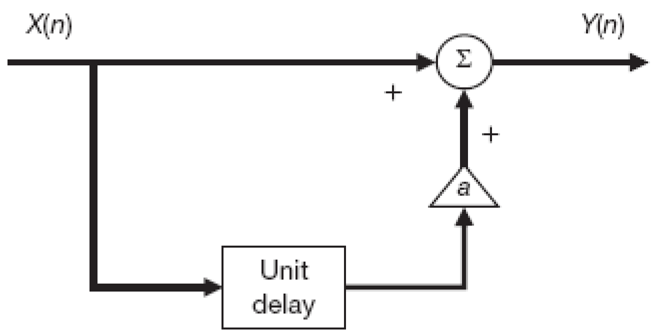
9.34. Suppose that the input to the filter shown in Fig. 9-20 is a white noise specified by Eq. (9.51). Find the power spectral density of $Y(t)$.

Figure 9-20



9.35. Suppose that the input to the discrete-time filter shown in Fig. 9-21 is a discrete-time white noise with average power σ^2. Find the power spectral density of $Y(n)$.

Figure 9-21



9.7. ANSWERS TO SUPPLEMENTARY PROBLEMS

9.27.

a. $R_{XX}(\tau)=\dfrac{A^2}{4}(1+e^{-2\alpha|\tau|});\ S_{XX}(\omega)=\dfrac{A^2}{2}\pi\delta(\omega)+A^2\dfrac{4\alpha}{\omega^2+(2\alpha)^2}$

b. $\frac{A}{2}$

9.28. 0.159

9.29. $S_{YY}(\omega) = 4\sin^2\omega T S_{XX}(\omega)$

9.30. *Hint:* Use relation (*b*) of Prob. 9.12 and definition (5.174).

9.31. *Hint:* Use Eq. (9.46).

3

9.32. *Hint:* Use Eqs. (9.57) and (9.58).

$\eta/(4RC)$

9.34. $(a)\quad S_Y(\omega) = \dfrac{4\alpha\omega^2}{\omega^2 + 4\alpha^2}$

$(b)\quad E[Y^2(t)] = \infty$

9.34. $S_Y(\omega) = \sigma^2 \to \eta{-}2\ (1 + a^2 + 2a\cos\omega T)$

9.35. $S_Y(\Omega) = \sigma^2(1 + a^2 + 2a\cos\Omega)$