Lecture 5: CPM: (Continous Phase Modulation)

* It is an evolution of the frequency modulation

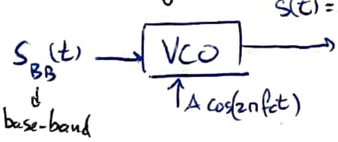
Basic Characteristics

1. Constant envelope: We can push the amplifier in saturation $\Rightarrow$ Efficient use of power
2. Phase Continuity:
   * One of the main problems of freq. mod is the BW. To reduce the BW Phase Continuity was introduced
   * To guarantee the phase continuity we have to introduce a correlation between different symbols $\Rightarrow$ It introduces a memory in the system
   * Because of this memory $\Rightarrow$ Optimal Receiver is very complicated to build (in the real application optimal receiver is never used)
3. Basic Principle: Frequency Modulation of a PAM signal.
   The starting point is a PAM $\Rightarrow$ the I apply FM

Modulation Basics

* Block diagram

$$s(t) = A\cos[\overbrace{2\pi f_c t + \varphi(t)}^{\theta_i(t)}]$$



$\Rightarrow S_{BB}(t)$ is a PAM signal: $S_{BB}(t) = \sum_k a_k\, g(t-kT)$
(it is the same signal we have used in the previous modulations)
$\hookrightarrow a_k$: useful information (i.e. ± 1)
$\hookrightarrow g(t-kT)$: basic pulse (i.e. rectangular signal)

$\Rightarrow$ VCO: The freq. modulator

$\Rightarrow s(t)$: Is the signal obtained applying a FM of a PAM signal

$\Rightarrow \varphi(t) \rightarrow$ The shape of CPM signal (phase of the carrier)

$$\varphi(t) = 2\pi h \int_{-\infty}^{t} \sum_k a_k\, g(t'-kT)\, dt' \xrightarrow{q(t) = \int_{-\infty}^{t} g(t')dt'} \varphi(t) = 2\pi h \sum_k a_k\, q(t-kT)$$

$\hookrightarrow$ Two degree of freedom
* Select the values of h
* Select the shape of $g(t)$

Parameters:
* $\rightarrow g(t)$: shape of the pulse
* $\rightarrow a_k$: alphabet of the source ($\pm 1 \ldots$)

} Decide the shape of $S_{BB}(t)$

* $\rightarrow h$: modulation index

Conventions:
* $\rightarrow a_k = \pm 1$ (Binary Mod)
* $\rightarrow$ Area of $g(t) = \int g(t)dt = \frac{1}{2}$

* Could be useful to work with the complex envelope

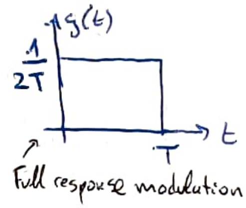
$$s(t) = \mathrm{Re}\left\{ A e^{j\varphi} e^{j2\pi f_c t} \right\} \Longrightarrow \hat{s}(t) = A e^{j\varphi(t)}$$

* Two type of situations
1. Total Response CPM $\Rightarrow$ if $g(t) \leq T_s$
   the contribution of a_k is given inside one symbol time
2. Partial Response CPM $\Rightarrow$ $g(t) > T_s$

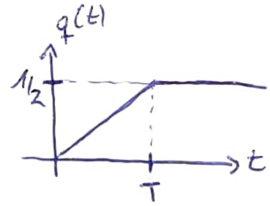
MSK Modulation (Minimum Shift-Keying)

* Specific case related to CPM

* **Parameters**
  - ↳ Pulse shape: $g(t)$: { - rectangular shape; - area: $1/2$ }



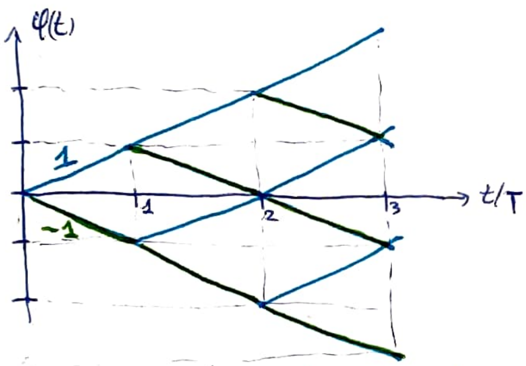
$q(t)$ is the integral of $g(t)$ ⟹



  - ↳ Modulation index: $h = \frac{1}{2}$

* The phase we get is a ramp multiplied by $\frac{\pi}{2} a_k$

* **MSK Phase three**



⟹ This diagram is useful to see what happens to the phase depending on the transmitted signal.

⟹ The shape of the Phase three is decide by the parameters of the modulation:
- ↳ pulse shape $g(t)$
- ↳ modulation index h
- ↳ by the possible a_k ($\pm1, \pm3, ..$)

⟹ Given one path we know unequivocally the original signal. The task of the receiver is observe $\varphi(t)$ and look at the phase three to determine the most similar path

↳ It is a system with memory because the position of the phase three depends also on the past.

↳ Due to the perturbation of the noise the receiver will see a phase which is not a path on the three

⟹ The phase is continuos but the first derivative is not continous ⟶ not smooth transition ⟹ Very large BW
⟶ MSK is not used because of that

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* **Mathematical expression of the phase in MSK**

↳ The instantaneous phase θ_i is composed by 3 parts: the carrier + the past + current symbol

$$\theta_i(t) = \underbrace{2\pi f_c t}_{\text{carrier}} + \underbrace{\frac{\pi}{2}\sum_{n=-\infty}^{k-1} a_n}_{\text{past}} + \underbrace{\frac{\pi}{2} a_k \cdot \frac{(t-kT)}{T}}_{\text{present symbol}}$$

⟹ The position on the phase three is determine by the past and current symbol.

↳ Rearranging ⟹ $\theta_i(t) = \underbrace{2\pi\left(f_c + \frac{a_k}{4T}\right)t}_{\text{present}} + \underbrace{\frac{\pi}{2}\sum_{n=-\infty}^{k-1} a_n}_{\text{past}} - \underbrace{\frac{\pi}{2} k a_k}_{\text{residual term}}$

↳ The signal will be: $S(t) = A\cos\left[2\pi f_c t + \frac{2\pi}{4} a_k t + \frac{\pi}{2}\sum_{n=-\infty}^{k-1} a_n - \frac{\pi}{2} k a_k\right]$ ⟹

$s(t) = A\cos\theta_i$

The interesting thing is that depending on a_k we will have:

$a_k = +1 \Rightarrow f_1 = f_c + \frac{1}{4T}$
$a_k = -1 \Rightarrow f_2 = f_c - \frac{1}{4T}$
} I am shifting the freq. of the carrier

Therefore MSK can be also considered as a special case of FSK

✱ Interpretation as amplitude modulation: O-PSK

↳ It is demonstrated that $s(t)$ can be seen as amplitude modulation (instead of a FM)

$$s(t) = \sqrt{\frac{2E_b}{T}} \left[c_k h_a(t-k2T)\cos(2\pi f_c t) - d_k h_a(t-k2T-T)\sin(2\pi f_c t) \right]$$

- c_k and d_k are related to a_k
- $\Delta f = \frac{1}{2T}$

Amplitude mod. of cosine carrier and amplitude mod. of sine carrier

↳ It is sensible because with the complex envelope any modulated signal can be interpreted in different way

⇒ Polar notation: $Ae^{j\varphi}$

⇒ Cartesian notation $a+jb$ ⇒ so, amplitud mod. of cosine carrier and sine carrier

↳ • There is an equation that relates c_k and d_k to a_k

• In this case Δf is: $\Delta f = \frac{1}{2T}$ → the time symbol is $T_s = 2T$ but in $2T$ we are sending 2 symbols

• There is an off-set between cosine carrier and sine carrier $\begin{cases} h_a(t-k2T) \\ h_a(t-k2T-T) \end{cases}$

(they are also shifted)

↳ Shape of $h_a(t)$:

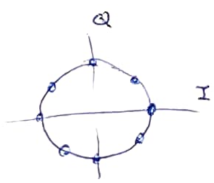
$h_a(t) = \cos\left(\frac{\pi t}{2T}\right)$, $-T \le t \le T$

$h_a(t-T) = \sin\left(\frac{\pi t}{2T}\right)$, $0 \le t \le 2T$

$h_a(t)$ is not longer the rectangular shape ⇒ O-PSK (offset)

MSK can be seen as Freq. mod., as Amplitude mod and as a CPM.

⇒ This result is important because if it's true that this is an amplitude modulation of two carrier one in-quadrature and the other in-phase, we can use the receiver of QAM ⇒ The receiver is not complicated



MSK Optimal Receiver

* Rember : Optimal receiver is a receiver that maximizes the cross-correlation corrected by the energy (it is just one possible implementation) O.R. $\Rightarrow \max\left[(r, s_i) - \frac{1}{2}\|s_i\|^2\right]$

* Observations: (
  - ↳ The energy is the same for any bit (due to constant envelope)
  - ↳ S_i is not the signal associated to time symbol i because we have the memory. Instead i is an entire transmitted sequence (a path on phase three)
    One symbol $i \Rightarrow$ entire path
  - ↳ Number of possible signals is very big ($2^{\text{possible paths}}$) $\rightarrow$ Optimal receiver have to check all

* Scalar product (r, s_i)
  - ↳ We use the complex envelope: $\xrightarrow[\text{product}]{\text{scalar}} \int s_1(t) s_2(t)\,dt = \frac{1}{2}\,\mathrm{Re}\left\{\int z_1(t) z_2^*(t)\,dt\right\}$ (scalar product ⇔ correlation)
  - ↳ In our case:
    $z_1(t) = z(t) \rightarrow$ complex envelope of received signal
    $z_2^*(t) = \tilde{S}_i^*(t) \rightarrow$ complex envelope of CPM signal $\rightarrow$

$$\begin{cases} \varphi(t) = 2\pi \cdot \underbrace{\tfrac{1}{2}}_{h} \sum_k a_k\, q(t-kT) \Rightarrow \tilde{S}(t) = A e^{j\varphi(t)} \Rightarrow \\ \Rightarrow \tilde{S}^*(t) = A e^{-j\varphi(t)} \end{cases}$$

$$\int z(t)\,\tilde{S}_i^*(t)\,dt = \frac{1}{2}\,\mathrm{Re}\left\{A \int_{-\infty}^{+\infty} z(t) \exp\left[-j\pi \sum_k a_k\, q(t-kT)\right] dt\right\}$$

Evaluating we get P(E)

$P(E)_{MSK} = Q\sqrt{\frac{2E_b}{N_0}} \rightarrow$ Same of the binary antipolar

↳ The task of the opt. receiver is the evaluation of this equation changing the sequence a_k
$\Rightarrow$ Very difficult implementation

The difficulty comes from the fact that one signal i is a sequence of a_k (not only one a_i) so if I transmit 100 symbols we have 2^{100} different possibilities
$\Rightarrow$ Impossible to make.

* Case of Partial response: $g(t)$ duration $> T \Rightarrow g(t) \neq 0, \forall t \in (0, LT)$
  - ↳ when we give a symbol the contribution to the phase expires in more than one symbol time $\Rightarrow$ The equation is more complicated

$$\varphi(t) = \underbrace{\pi h \sum_{n=-\infty}^{k-L} a_n}_{\text{the past until } L} + \underbrace{2\pi h \sum_{n=k-L+1}^{k-1} a_n\, q(t-kT)}_{\substack{\text{intermediate situation, the transtion} \\ \text{between one symbol and another}}} + \underbrace{2\pi h\, a_k\, q(t-kT)}_{\text{present}}$$

  - ↳ Complexity: Exponential with L
    ↳ how many symbols are involved in the transition of the phase
  - ↳ Very complicated, never implemented

Simplified Receivers

In the real situation we don't use optimal receivers, but simplified ones.
There are 3 strategies

① First strategy works on reducing L, that is the number of symbols involved in the transition.
The shape of $q(t)$ is changed and we use an approximation that is a shorter Version ⟹ reducing L ⟹ reducing the complexity and memory
We will reduce the performance of the system

② Second strategy tries to approximate the CPM signal as a sum of sinusoids.
I make the receiver tailored for a limited number of sinusoids instead of the real signal.

③ Third strategy ⟹ Real used strategy

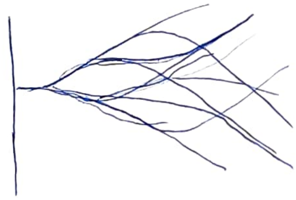
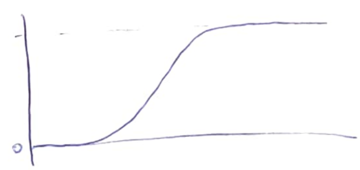
* It can be mathematically demonstrated that CPM signal can be interpreted as a sum of PAM signals.
* The idea used in MSK (that MSK is composed by one sinusoid modulated in-phase and in-quadrature) can be extended to any type of CPM
* The number of PAM signals we need is 2^{L-1} to get the optimal solution. It can be difficult to manage so many signals.
* In the practical application ⟹ We take only the more important components ⟹ ⟹ It is not longer the optimal receiver but the implementation is easier

TFM (Tamed Frequency Modulation)

* It was introduced to try to reduce the bandwidth.

* Characteristics of TFM:
  - ↳ Modulation index: $h = \frac{1}{2}$
  - ↳ g(t) duration: $L = 3$ or 4
  - ↳ Pulse shape $g(t) \xrightarrow[\text{fourier transf}]{} G(f) = \frac{1}{2}\cos^2 \pi f T \frac{\pi f T}{\sin \pi f T}$

    It was introduced as an evolution of MSK. Starting from MSK try to arrange g(t) to reduce the bandwidth.
    The phase transitions are very soft $\Rightarrow$ reduce BW (compact spectrum)

  - ↳ Phase three 
    ↳ q(t) 

  - ↳ Prob. of error.
    TFM → $d^2 = 1{,}59$
    MSK → $d^2 = 2$
    } worse P(E)

  - ↳ Energy on the BW for one
    TFM → $h_0(t) = 98\%$
    MSK → $h_0(t) = 100\%$

  - ↳ Relatively simple transmitter and receiver implementation

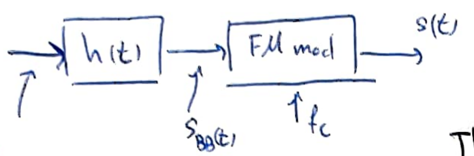
GMSK (Gaussian Minimum Shift Keying)

*** Basic Idea**

To obtain a CPM we start from a base-band signal and apply a freq. modulator

↳ Using a rectangular pulse shape ⇒ MSK mod.

The idea is to smooth the rectangular signal using a filter with gaussian impulse response



$\sum_k a_k \, \frac{rect}{2T}\left(\frac{t-T/2}{T}\right)$

We have the convolution of a rect and a gaussian

The fourier transform of a gaussian pulse is a gaussian:

$$H(f) = \exp\left(-\frac{f^2}{B^2}\,\frac{\ln 2}{2}\right)$$

* B is the BW when the attenuation is 3dB

↳ increasing B ⇒ increase BW

*** Parameters**

↳ Modulation index: $h = \frac{1}{2}$

↳ Pulse shape: A rect filtered by a gaussian signal

↳ Parameter

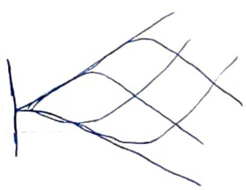
Parameter B

$B = 0{,}3/T$ ⇒ GSM

$B = 0{,}2/T$ ⇒ quasi TFM

$B = \infty$ ⇒ MSK

↳ Phase three : smooth



Less compact spectrum density
(trade-off between spectrum and P(E))

↳ Prob. of error

$$\left.\begin{array}{l} \text{GMSK} \rightarrow d^2 = 1.79 \\ \text{TFM} \rightarrow d^2 = 1.59 \\ \text{MSK} \rightarrow d^2 = 2 \end{array}\right\} \text{Improves } P(E) \text{ respect TFM}$$

↳ Energy on the BW

GMSK → 99,6%

TFM → 98%

MSK → 100%

↳ Receiver similar to MSK so quite simple.