D. Review of Complex Numbers

D.1. Representation of Complex Numbers

The complex number *z* can be expressed in several ways.

Cartesian or rectangular form:

$$z = a + jb$$

(D.1)

where $j = \sqrt{-1}$ and *a* and *b* are real numbers referred to the *real part* and the *imaginary part* of *z*. *a* and *b* are often expressed as

$$a = \mathrm{Re}\{z\} \qquad b = \mathrm{Im}\{z\}$$

(D.2)

where "Re" denotes the "real part of" and "Im" denotes the "imaginary part of."

Polar form:

$$z = re^{j\theta}$$

(D.3)

where $r > 0$ is the *magnitude* of *z* and θ is the *angle* or *phase* of *z*. These quantities are often written as

$$r = |z| \qquad \theta = \angle z$$

(D.4)

Fig. D-1 is the graphical representation of *z*. Using Euler's formula,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

(D.5)

or from Fig. D-1 the relationships between the Cartesian and polar representations of *z* are

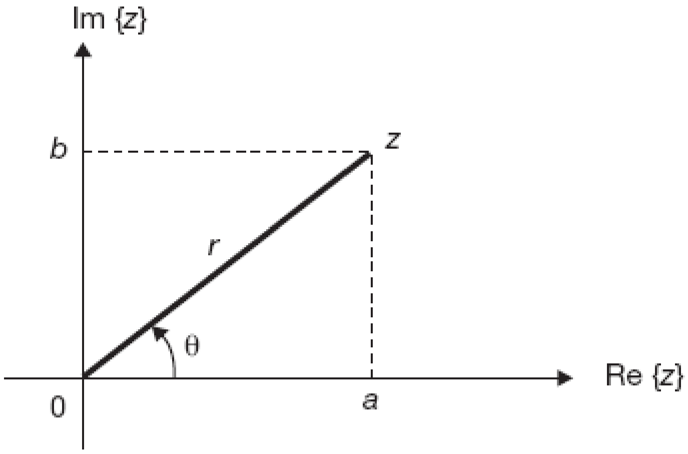
$$a = r\cos\theta \qquad b = r\sin\theta$$

(D.6a)

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

(D.6b)

Figure D-1



D.2. Addition, Multiplication, and Division

If $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, then

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$$

(D.7)

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2)$$

$$\frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)}$$

(D.8)

$$= \frac{(a_1 a_2 + b_1 b_2) + j(-a_1 b_2 + b_1 a_2)}{a_2^2 + b_2^2}$$

(D.9)

If $z_1 = r_1 e^{j\theta 1}$ and $z_2 = r_2 e^{j\theta 2}$, then

$$z_1 z_2 = (r_1 r_2) e^{j(\theta_1 + \theta_2)}$$

(D.10)

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{j(\theta_1 - \theta_2)}$$

(D.11)

D.3. The Complex Conjugate

The *complex conjugate* of z is denoted by z^* and is given by

$$z^* = a - jb = re^{-j\theta}$$

(D.12)

Useful relationships:

1. $zz^* = r^2$
2. $\dfrac{z}{z^*} = e^{j2\theta}$
3. $z + z^* = 2\,\mathrm{Re}\{z\}$
4. $z - z^* = j2\,\mathrm{Im}\{\mathrm{z}\}$
5. $(z_1 + z_2)^* = z_1^* + z_2^*$
6. $(z_1 z_2)^* = z_1^* z_2^*$
7. $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$

D.4. Powers and Roots of Complex Numbers

The *n*th power of the complex number $z = re^{j\theta}$ is

$$z^n = r^n e^{jn\theta} = r^n(\cos n\theta + j \sin n\theta)$$

(D.13)

from which we have De Moivre's relation

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

(D.14)

The *n*th root of a complex *z* is the number *w* such that

$$w^n = z = re^{j\theta}$$

(D.15)

Thus, to find the *n*th root of a complex number *z,* we must solve

$$w^n - re^{j\theta} = 0$$

(D.16)

which is an equation of degree *n* and hence has *n* roots. These roots are given by

$$w_k = r^{1/n} e^{j[\theta + 2(k-1)\pi]/n} \qquad k = 1, 2, \ldots, n$$

(D.17)