5. Fourier Analysis of Continuous-Time Signals and Systems

5.1. Introduction

In previous chapters we introduced the Laplace transform and the *z*-transform to convert time-domain signals into the complex *s*-domain and *z*-domain representations that are, for many purposes, more convenient to analyze and process. In addition, greater insights into the nature and properties of many signals and systems are provided by these transformations. In this chapter and the following one, we shall introduce other transformations known as Fourier series and Fourier transform which convert time-domain signals into frequency-domain (or *spectral*) representations. In addition to providing spectral representations of signals, Fourier analysis is also essential for describing certain types of systems and their properties in the frequency domain. In this chapter we shall introduce Fourier analysis in the context of continuous-time signals and systems.

5.2. Fourier Series Representation of Periodic Signals

5.2.1. A. Periodic Signals:

In Chap. 1 we defined a continuous-time signal $x(t)$ to be periodic if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t$$

(5.1)

The fundamental period T_0 of $x(t)$ is the smallest positive value of T for which Eq. (5.1) is satisfied, and $1/T_0 = f_0$ is referred to as the *fundamental frequency*.

Two basic examples of periodic signals are the real sinusoidal signal

$$x(t) = \cos(\omega_0 t + \phi)$$

(5.2)

and the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

(5.3)

where $\omega_0 = 2\pi/T_0 = 2\pi f_0$ is called the *fundamental angular frequency.*

5.2.2. B. Complex Exponential Fourier Series Representation:

The complex exponential Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.4)

where c_k are known as the *complex Fourier coefficients* and are given by

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

(5.5)

where $\int_{T_0}$ denotes the integral over any one period and 0 to T_0 or $-T_0/2$ to $T_0/2$ is commonly used for the integration. Setting $k = 0$ in Eq. (5.5), we have

$$c_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$

(5.6)

which indicates that c_0 equals the average value of $x(t)$ over a period.

When $x(t)$ is real, then from Eq. (5.5) it follows that

$$c_{-k} = c_k^*$$

(5.7)

where the asterisk indicates the complex conjugate.

5.2.3. C. Trigonometric Fourier Series:

The trigonometric Fourier series representation of a periodic signal$x(t)$ with fundamental period T_0 is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.8)

where a_k and b_k are the Fourier coefficients given by

$$a_k = \frac{2}{T_0}\int_{T_0} x(t) \cos k\omega_0 t\, dt$$

(5.9a)

$$b_k = \frac{2}{T_0}\int_{T_0} x(t) \sin k\omega_0 t\, dt$$

(5.9b)

The coefficients a_k and b_k and the complex Fourier coefficients c_k are related by (Prob. 5.3)

$$\frac{a_0}{2} = c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k})$$

(5.10)

From Eq. (5.10) we obtain

$$c_k = \frac{1}{2}(a_k - jb_k) \qquad c_{-k} = \frac{1}{2}(a_k + jb_k)$$

(5.11)

When $x(t)$ is real, then a_k and b_k are real and by Eq. (5.10) we have

$$a_k = 2\,\mathrm{Re}[c_k] \qquad b_k = -2\,\mathrm{Im}[c_k]$$

(5.12)

5.2.3.1. Even and Odd Signals:

If a periodic signal $x(t)$ is even, then $b_k = 0$ and its Fourier series (5.8) contains only cosine terms:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.13)

If $x(t)$ is odd, then $a_k = 0$ and its Fourier series contains only sine terms:

$$x(t) = \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.14)

5.2.4. D. Harmonic Form Fourier Series:

Another form of the Fourier series representation of a real periodic signal $x(t)$ with fundamental period T_0 is

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t - \theta_k) \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.15)

Equation (5.15) can be derived from Eq. (5.8) and is known as the *harmonic form* Fourier series of $x(t)$. The term C_0 is known as the *dc component*, and the term $C_k \cos(k\omega_0 t - \theta_k)$ is referred to as the *kth harmonic component* of $x(t)$. The first harmonic component $C_1 \cos(\omega_0 t - \theta_1)$ is commonly called the *fundamental component* because it has the same fundamental period as $x(t)$. The coefficients C_k and the angles θ_k are called the *harmonic amplitudes* and *phase angles*, respectively, and they are related to the Fourier coefficients a_k and b_k by

$$C_0 = \frac{a_0}{2} \qquad C_k = \sqrt{a_k^2 + b_k^2} \qquad \theta_k = \tan^{-1}\frac{b_k}{a_k}$$

(5.16)

For a real periodic signal $x(t)$, the Fourier series in terms of complex exponentials as given in Eq. (5.4) is mathematically equivalent to either of the two forms in Eqs. (5.8) and (5.15). Although the latter two are common forms for Fourier series, the complex form in Eq. (5.4) is more general and usually more convenient, and we will use that form almost exclusively.

5.2.5. E. Convergence of Fourier Series:

It is known that a periodic signal $x(t)$ has a Fourier series representation if it satisfies the following Dirichlet conditions:

1. $x(t)$ is absolutely integrable over any period; that is,

$$\int_{T_0} |x(t)|\, dt < \infty$$

(5.17)

2. $x(t)$ has a finite number of maxima and minima within any finite interval of t.
3. $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

Note that the Dirichlet conditions are sufficient but not necessary conditions for the Fourier series representation (Prob. 5.8).

5.2.6. F. Amplitude and Phase Spectra of a Periodic Signal:

Let the complex Fourier coefficients c_k in Eq. (5.4) be expressed as

$$c_k = |c_k|\, e^{j\phi_k}$$

(5.18)

A plot of $|c_k|$ versus the angular frequency ω is called the *amplitude spectrum* of the periodic signal $x(t)$, and a plot of ϕ_k versus ω is called the *phase spectrum* of $x(t)$. Since the index k assumes only integers, the amplitude and phase spectra are not continuous curves but appear only at the discrete frequencies $k\omega_0$. They are therefore referred to as *discrete frequency spectra* or *line spectra.*

For a real periodic signal $x(t)$ we have $c_{-k} = c^*_k$. Thus,

$$|c_{-k}| = |c_k| \qquad \phi_{-k} = -\phi_k$$

(5.19)

Hence, the amplitude spectrum is an even function of ω, and the phase spectrum is an odd function of ω for a real periodic signal.

5.2.7. G. Power Content of a Periodic Signal:

In Chap. 1 (Prob. 1.18) we introduced the average power of a periodic signal $x(t)$ over any period as

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2\, dt$$

(5.20)

If $x(t)$ is represented by the complex exponential Fourier series in Eq. (5.4), then it can be shown that (Prob. 5.14)

$$\frac{1}{T_0}\int_{T_0}|x(t)|^2\,dt=\sum_{k=-\infty}^{\infty}|c_k|^2$$

(5.21)

Equation (5.21) is called *Parseval's identity* (or *Parseval's theorem*) for the Fourier series.

5.3. The Fourier Transform

5.3.1. A. From Fourier Series to Fourier Transform:

Let $x(t)$ be a nonperiodic signal of finite duration; that is,

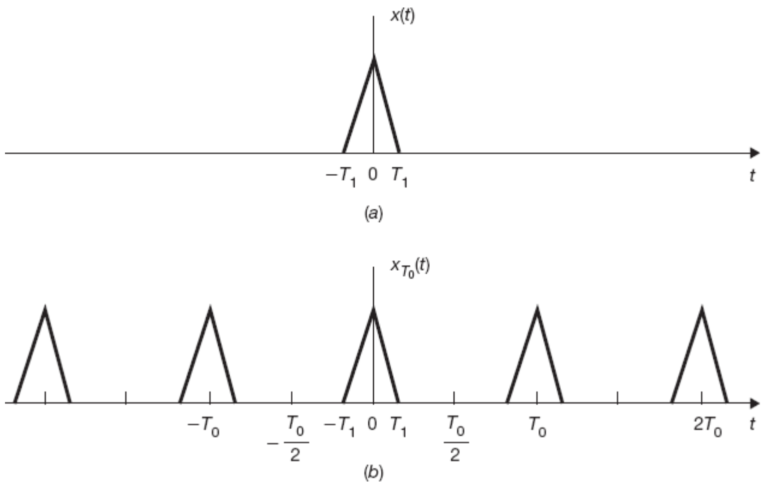
$$x(t)=0 \qquad |t|>T_1$$

Such a signal is shown in Fig. 5-1(*a*). Let $x_{T_0}(t)$ be a periodic signal formed by repeating $x(t)$ with fundamental period T_0 as shown in Fig. 5-1(*b*). If we let $T_0 \twoheadrightarrow \infty$, we have

$$\lim_{T_0\to\infty} x_{T_0}(t)=x(t)$$

(5.22)

Figure 5-1 (a) Nonperiodic signal x(t); (b) periodic signal formed by periodic extension of x(t).



The complex exponential Fourier series of $x_{T_0}(t)$ is given by

$$x_{T_0}(t)=\sum_{k=-\infty}^{\infty} C_k\, e^{jk\omega_0 t} \qquad \omega_0=\frac{2\pi}{T_0}$$

(5.23)

where

$$c_k=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\,dt$$

(5.24a)

Since $x_{T_0}(t) = x(t)$ for $|t| < T_0/2$ and also since $x(t) = 0$ outside this interval, Eq. (5.24a) can be rewritten as

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

(5.24b)

Let us define $X(\omega)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

(5.25)

Then from Eq. (5.24b) the complex Fourier coefficients c_k can be expressed as

$$c_k = \frac{1}{T_0} X(k\omega_0)$$

(5.26)

Substituting Eq. (5.26) into Eq. (5.23), we have

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0)\, e^{jk\omega_0 t}$$

or

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0)\, e^{jk\omega_0 t} \omega_0$$

(5.27)

As $T_0 \twoheadrightarrow \infty$, $\omega_0 = 2\pi/T_0$ becomes infinitesimal ($\omega_0 \twoheadrightarrow 0$). Thus, let $\omega_0 = \Delta\omega$. Then Eq. (5.27) becomes

$$x_{T_0}(t)\Big|_{T_0 \to \infty} \to \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega)\, e^{jk\Delta\omega t} \Delta\omega$$

(5.28)

Therefore,

$$x(t) = \lim_{T_0 \to \infty} x_{T_0}(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega)\, e^{jk\Delta\omega t} \Delta\omega$$

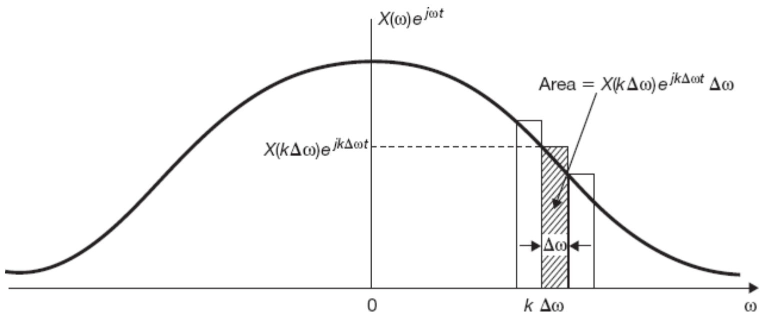
(5.29)

The sum on the right-hand side of Eq. (5.29) can be viewed as the area under the function $X(\omega)\, e^{j\omega t}$, as shown in Fig. 5-2. Therefore, we obtain

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}\,d\omega$$

(5.30)

Figure 5-2 Graphical interpretation of Eq. (5.29).



which is the Fourier representation of a nonperiodic $x(t)$.

5.3.2. B. Fourier Transform Pair:

The function $X(\omega)$ defined by Eq. (5.25) is called the *Fourier transform* of $x(t)$, and Eq. (5.30) defines the *inverse Fourier transform of* $X(\omega)$. Symbolically they are denoted by

$$X(\omega)=\mathscr{F}\{x(t)\}=\int_{-\infty}^{\infty}x(t)\,e^{j\omega t}\,dt$$

(5.31)

$$x(t)=\mathscr{F}^{-1}\{X(\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\,e^{j\omega t}\,d\omega$$

(5.32)

and we say that $x(t)$ and $X(\omega)$ form a Fourier transform pair denoted by

$$x(t)\leftrightarrow X(\omega)$$

(5.33)

5.3.3. C. Fourier Spectra:

The Fourier transform $X(\omega)$ of $x(t)$ is, in general, complex, and it can be expressed as

$$X(\omega)=|X(\omega)|\;e^{j\phi(\omega)}$$

(5.34)

By analogy with the terminology used for the complex Fourier coefficients of a periodic signal $x(t)$, the Fourier transform $X(\omega)$ of a nonperiodic signal $x(t)$ is the frequency-domain specification of $x(t)$ and is referred to as the *spectrum* (or *Fourier spectrum*) of $x(t)$. The quantity $|X(\omega)|$ is called the *magnitude spectrum* of $x(t)$, and $\phi(\omega)$ is called the *phase spectrum* of $x(t)$.

If $x(t)$ is a real signal, then from Eq. (5.31) we get

$$X(-\omega)=\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt$$

(5.35)

Then it follows that

$$X(-\omega) = X^*(\omega)$$

(5.36a)

and

$$|X(-\omega)| = |X(\omega)| \qquad \phi(-\omega) = -\phi(\omega)$$

(5.36b)

Hence, as in the case of periodic signals, the amplitude spectrum $|X(\omega)|$ is an even function and the phase spectrum $\phi(\omega)$ is an odd function of ω.

5.3.4. D. Convergence of Fourier Transforms:

Just as in the case of periodic signals, the sufficient conditions for the convergence of $X(\omega)$ are the following (again referred to as the Dirichlet conditions):

1. $x(t)$ is absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

   (5.37)

2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval, and each of these discontinuities is finite.

Although the above Dirichlet conditions guarantee the existence of the Fourier transform for a signal, if impulse functions are permitted in the transform, signals which do not satisfy these conditions can have Fourier transforms (Prob. 5.23).

5.3.5. E. Connection between the Fourier Transform and the Laplace Transform:

Equation (5.31) defines the Fourier transform of $x(t)$ as

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

(5.38)

The bilateral Laplace transform of $x(t)$, as defined in Eq. (4.3), is given by

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

(5.39)

Comparing Eqs. (5.38) and (5.39), we see that the Fourier transform is a special case of the Laplace transform in which $s = j\omega$; that is,

$$X(s)\big|_{s=j\omega} = \mathscr{F}\{x(t)\}$$

(5.40)

Setting $s = \sigma + j\omega$ in Eq. (5.39), we have

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t}dt = \int_{-\infty}^{\infty}[x(t)e^{-\sigma t}]\, e^{-j\omega t}\, dt$$

or

$$X(\sigma + j\omega) = \mathscr{F}\{x(t)e^{-\sigma t}\}$$

(5.41)

which indicates that the bilateral Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)\, e^{-\sigma t}$.

Since the Laplace transform may be considered a generalization of the Fourier transform in which the frequency is generalized from $j\omega$ to $s = \sigma + j\omega$, the complex variable s is often referred to as the *complex frequency*.

Note that since the integral in Eq. (5.39) is denoted by $X(s)$, the integral in Eq. (5.38) may be denoted as $X(j\omega)$. Thus, in the remainder of this book both $X(\omega)$ and $X(j\omega)$ mean the same thing whenever we connect the Fourier transform with the Laplace transform. Because the Fourier transform is the Laplace transform with $s = j\omega$, it should not be assumed automatically that the Fourier transform of a signal $x(t)$ is the Laplace transform with s replaced by $j\omega$. If $x(t)$ is absolutely integrable, that is, if $x(t)$ satisfies condition (5.37), the Fourier transform of $x(t)$ can be obtained from the Laplace transform of $x(t)$ with $s = j\omega$. This is not generally true of signals which are not absolutely integrable. The following examples illustrate the above statements.

EXAMPLE 5.1 Consider the unit impulse function $\delta(t)$.

From Eq. (3.13) the Laplace transform of $\delta(t)$ is

$$\mathscr{L}\{\delta(t)\} = 1 \qquad \text{all } s$$

(5.42)

By definitions (5.31) and (1.20) the Fourier transform of $\delta(t)$ is

$$\mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$$

(5.43)

Thus, the Laplace transform and the Fourier transform of $\delta(t)$ are the same.

EXAMPLE 5.2 Consider the exponential signal

$$x(t) = e^{-at}u(t) \qquad a > 0$$

From Eq. (3.8) the Laplace transform of $x(t)$ is given by

$$\mathscr{L}\{x(t)\} = X(s) = \frac{1}{s+a} \qquad \mathrm{Re}(s) > -a$$

(5.44)

By definition (5.31) the Fourier transform of $x(t)$ is

$$\mathscr{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-j\omega t}\, dt$$
$$= \int_{0^+}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{1}{a+j\omega}$$

(5.45)

Thus, comparing Eqs. (5.44) and (5.45), we have

$$X(\omega) = X(s)\big|_{s=j\omega}$$

(5.46)

Note that $x(t)$ is absolutely integrable.

EXAMPLE 5.3 Consider the unit step function $u(t)$.

From Eq. (3.14) the Laplace transform of $u(t)$ is

$$\mathscr{L}\{u(t)\} = \frac{1}{s} \qquad \operatorname{Re}(s) > 0$$

(5.47)

The Fourier transform of $u(t)$ is given by (Prob. 5.30)

$$\mathscr{F}\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

(5.48)

Thus, the Fourier transform of $u(t)$ cannot be obtained from its Laplace transform. Note that the unit step function $u(t)$ is not absolutely integrable.

5.4. Properties of the Continuous-Time Fourier Transform

Basic properties of the Fourier transform are presented in the following. Many of these properties are similar to those of the Laplace transform (see Sec. 3.4).

5.4.1. A. Linearity:

$$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$$

(5.49)

5.4.2. B. Time Shifting:

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$$

(5.50)

Equation (5.50) shows that the effect of a shift in the time domain is simply to add a linear term $-\omega t_0$ to the original phase spectrum $\theta(\omega)$. This is known as a *linear phase shift* of the Fourier transform $X(\omega)$.

5.4.3. C. Frequency Shifting:

$$e^{j\omega_0 t}x(t) \leftrightarrow X(\omega-\omega_0)$$

(5.51)

The multiplication of $x(t)$ by a complex exponential signal $e^{j\omega_0 t}$ is sometimes called *complex modulation*. Thus, Eq. (5.51) shows that complex modulation in the time domain corresponds to a shift of $X(\omega)$ in the frequency domain. Note that the frequency-shifting property Eq. (5.51) is the dual of the time-shifting property Eq. (5.50).

5.4.4. D. Time Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

(5.52)

where *a* is a real constant. This property follows directly from the definition of the Fourier transform. Equation (5.52) indicates that scaling the time variable *t* by the factor *a* causes an inverse scaling of the frequency variable ω by $1/a$, as well as an amplitude scaling of $X(\omega/a)$ by $1/|a|$. Thus, the scaling property (5.52) implies that time compression of a signal ($a > 1$) results in its spectral expansion and that time expansion of the signal ($a < 1$) results in its spectral compression.

5.4.5. E. Time Reversal:

$$x(-t) \leftrightarrow X(-\omega)$$

(5.53)

Thus, time reversal of $x(t)$ produces a like reversal of the frequency axis for $X(\omega)$. Equation (5.53) is readily obtained by setting $a = -1$ in Eq. (5.52).

5.4.6. F. Duality (or Symmetry):

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

(5.54)

The duality property of the Fourier transform has significant implications. This property allows us to obtain both of these dual Fourier transform pairs from one evaluation of Eq. (5.31) (Probs. 5.20 and 5.22).

5.4.7. G. Differentiation in the Time Domain:

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

(5.55)

Equation (5.55) shows that the effect of differentiation in the time domain is the multiplication of $X(\omega)$ by $j\omega$ in the frequency domain (Prob. 5.28).

5.4.8. H. Differentiation in the Frequency Domain:

$$(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega}$$

(5.56)

Equation (5.56) is the dual property of Eq. (5.55).

5.4.9. I. Integration in the Time Domain:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \pi X(0)\,\delta(\omega) + \frac{1}{j\omega} X(\omega)$$

(5.57)

Since integration is the inverse of differentiation, Eq. (5.57) shows that the frequency domain operation corresponding to time-domain integration is multiplication by $1/j\omega$, but an additional term is needed to account for a possible dc component in the integrator output. Hence, unless $X(0) = 0$, a dc component is produced by the integrator (Prob. 5.33).

5.4.10. J. Convolution:

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega)\, X_2(\omega)$$

(5.58)

Equation (5.58) is referred to as the *time convolution theorem*, and it states that convolution in the time domain becomes multiplication in the frequency domain (Prob. 5.31). As in the case of the Laplace transform, this convolution property plays an important role in the study of continuous-time LTI systems (Sec. 5.5) and also forms the basis for our discussion of filtering (Sec. 5.6).

5.4.11. K. Multiplication:

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

(5.59)

The multiplication property (5.59) is the dual property of Eq. (5.58) and is often referred to as the *frequency convolution theorem*. Thus, multiplication in the time domain becomes convolution in the frequency domain (Prob. 5.35).

5.4.12. L. Additional Properties:

If $x(t)$ is real, let

$$x(t) = x_e(t) + x_o(t)$$

(5.60)

where $x_e(t)$ and $x_o(t)$ are the even and odd components of $x(t)$, respectively. Let

$$x(t) \leftrightarrow X(\omega) = A(\omega) + jB(\omega)$$

Then

$$X(-\omega) = X^*(\omega)$$

(5.61a)

$$x_e(t) \leftrightarrow \mathrm{Re}\{X(\omega)\} = A(\omega)$$

(5.61b)

$$x_o(t) \leftrightarrow j\,\mathrm{Im}\{X(\omega)\} = jB(\omega)$$

(5.61c)

Equation (5.61a) is the necessary and sufficient condition for $x(t)$ to be real (Prob. 5.39). Equations (5.61b) and (5.61c) show that the Fourier transform of an even signal is a real function of ω and that the Fourier transform of an odd signal is a pure imaginary function of ω.

5.4.13. M. Parseval's Relations:

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda)\,d\lambda = \int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda)\,d\lambda$$

(5.62)

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\,d\omega$$

(5.63)

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

(5.64)

Equation (5.64) is called *Parseval's identity* (or *Parseval's theorem*) for the Fourier transform. Note that the quantity on the left-hand side of Eq. (5.64) is the normalized energy content E of $x(t)$ [Eq. (1.14)]. Parseval's identity says that this energy content E can be computed by integrating $|X(\omega)|^2$ over all frequencies ω. For this reason, $|X(\omega)|^2$ is often referred to as the *energy-density spectrum* of $x(t)$, and Eq. (5.64) is also known as the *energy theorem*.

Table 5-1 contains a summary of the properties of the Fourier transform presented in this section. Some common signals and their Fourier transforms are given in Table 5-2.

Table 5-1 Properties of the Fourier Transform

PROPERTY	SIGNAL	FOURIER TRANSFORM
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\pi X(0)\,\delta(\omega) + \frac{1}{j\omega} X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Real signal	$x(t) = x_e(t) + x_o(t)$	$X(\omega) = A(\omega) + jB(\omega)$ $X(-\omega) = X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\} = A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\} = jB(\omega)$
Parseval's relations		

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda)\, d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$$

Table 5-2 Common Fourier Transforms Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\dfrac{1}{j\omega}$
$e^{-at}u(t), a>0$	$\dfrac{1}{j\omega+a}$
$te^{-at}u(t), a>0$	$\dfrac{1}{(j\omega+a)^2}$
$e^{-a\|t\|}, a>0$	$\dfrac{2a}{a^2+\omega^2}$
$\dfrac{1}{a^2+t^2}$	$e^{-a\|\omega\|}$
$e^{-at^2}, a>0$	$\sqrt{\dfrac{\pi}{a}}\,e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \|t\|<a\\0 & \|t\|>a\end{cases}$	$2a\dfrac{\sin\omega a}{\omega a}$
$\dfrac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \|\omega\|<a\\0 & \|\omega\|>a\end{cases}$
$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0), \omega_0=\dfrac{2\pi}{T}$

5.5. The Frequency Response of Continuous-Time LTI Systems

5.5.1. A. Frequency Response:

In Sec. 2.2 we showed that the output $y(t)$ of a continuous-time LTI system equals the convolution of the input $x(t)$ with the impulse response $h(t)$; that is,

$$y(t) = x(t) * h(t)$$

(5.65)

Applying the convolution property (5.58), we obtain

$$Y(\omega) = X(\omega)H(\omega)$$

(5.66)

where $Y(\omega)$, $X(\omega)$, and $H(\omega)$ are the Fourier transforms of $y(t)$, $x(t)$, and $h(t)$, respectively. From Eq. (5.66) we have

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

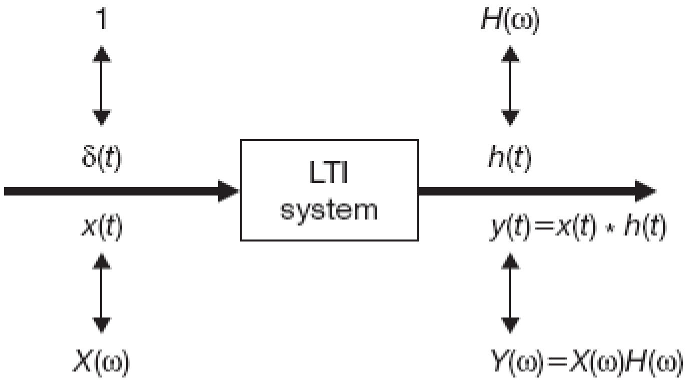
(5.67)

The function $H(\omega)$ is called the *frequency response* of the system. Relationships represented by Eqs. (5.65) and (5.66) are depicted in Fig. 5-3. Let

$$H(\omega) = |H(\omega)|e^{j\theta_H(\omega)}$$

(5.68)

Figure 5-3 Relationships between inputs and outputs in an LTI system.



Then $|H(\omega)|$ is called the *magnitude response* of the system, and $\theta_H(\omega)$ the *phase response* of the system.

Consider the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

(5.69)

with Fourier transform (Prob. 5.23)

$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

(5.70)

Then from Eqs. (5.66) and (1.26) we have

$$Y(\omega) = 2\pi H(\omega_0)\,\delta(\omega - \omega_0)$$

(5.71)

Taking the inverse Fourier transform of $Y(\omega)$, we obtain

$$y(t) = H(\omega_0)\,e^{j\omega_0 t}$$

(5.72)

which indicates that the complex exponential signal $e^{j\omega_0 t}$ is an eigenfunction of the LTI system with corresponding eigenvalue $H(\omega_0)$, as previously observed in Chap. 2 (Sec. 2.4 and Prob. 2.17]. Furthermore, by the linearity property (5.49), if the input $x(t)$ is periodic with the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\,e^{jk\omega_0 t}$$

(5.73)

then the corresponding output $y(t)$ is also periodic with the Fourier series

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0)\,e^{jk\omega_0 t}$$

(5.74)

If $x(t)$ is not periodic, then from Eq. (5.30)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega$$

(5.75)

and using Eq. (5.66), the corresponding output $y(t)$ can be expressed as

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)\,e^{j\omega t}\,d\omega$$

(5.76)

Thus, the behavior of a continuous-time LTI system in the frequency domain is completely characterized by its frequency response $H(\omega)$. Let

$$X(\omega) = |X(\omega)|\,e^{j\theta_X(\omega)} \qquad Y(\omega) = |Y(\omega)|\,e^{j\theta_Y(\omega)}$$

(5.77)

Then from Eq. (5.66) we have

$$|Y(\omega)| = |X(\omega)||H(\omega)|$$

(5.78a)

$$\theta_Y(\omega) = \theta_X(\omega) + \theta_H(\omega)$$

(5.78b)

Hence, the magnitude spectrum $|X(\omega)|$ of the input is multiplied by the magnitude response $|H(\omega)|$ of the system to determine the magnitude spectrum $|Y(\omega)|$ of the output, and the phase response $\theta_H(\omega)$ is added to the phase spectrum $\theta_X(\omega)$ of the input to produce the phase spectrum $\theta_Y(\omega)$ of the output. The magnitude response $|H(\omega)|$ is sometimes referred to as the *gain* of the system.

5.5.2. B. Distortionless Transmission:

For distortionless transmission through an LTI system we require that the exact input signal shape be reproduced at the output, although its amplitude may be different and it may be delayed in time. Therefore, if $x(t)$ is the input signal, the required output is

$$y(t) = Kx(t - t_d)$$

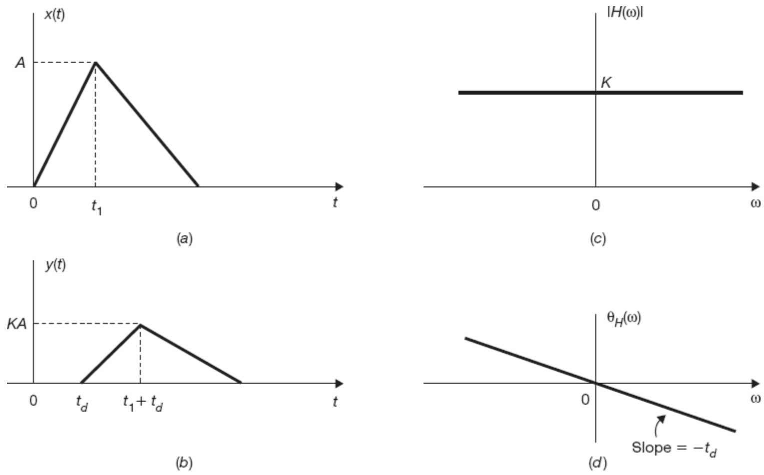
(5.79)

where t_d is the *time delay* and K (> 0) is a *gain constant.* This is illustrated in Figs. 5-4(*a*) and (*b*). Taking the Fourier transform of both sides of Eq. (5.79), we get

$$Y(\omega) = Ke^{-j\omega t_d} X(\omega)$$

(5.80)

Figure 5-4 Distortionless transmission.



Thus, from Eq. (5.66) we see that for distortionless transmission, the system must have

$$H(\omega) = |H(\omega)|e^{j\theta_H(\omega)} = Ke^{-j\omega t_d}$$

(5.81)

Thus,

$$|H(\omega)| = K$$

(5.82a)

$$\theta_H(\omega) = -j\omega t_d$$

(5.82b)

That is, the amplitude of $H(\omega)$ must be constant over the entire frequency range, and the phase of $H(\omega)$ must be linear with the frequency. This is illustrated in Figs. 5-4(c) and (d).

5.5.2.1. Amplitude Distortion and Phase Distortion:

When the amplitude spectrum $|H(\omega)|$ of the system is not constant within the frequency band of interest, the frequency components of the input signal are transmitted with a different amount of gain or attenuation. This effect is called *amplitude distortion*. When the phase spectrum $\theta_H(\omega)$ of the system is not linear with the frequency, the output signal has a different waveform than the input signal because of different delays in passing through the system for different frequency components of the input signal. This form of distortion is called *phase distortion*.

5.5.3. C. LTI Systems Characterized by Differential Equations:

As discussed in Sec. 2.5, many continuous-time LTI systems of practical interest are described by linear constant-coefficient differential equations of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

(5.83)

with $M \le N$. Taking the Fourier transform of both sides of Eq. (5.83) and using the linearity property (5.49) and the time-differentiation property (5.55), we have

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

or

$$Y(\omega) \sum_{k=0}^{N} a_k (j\omega)^k = X(\omega) \sum_{k=0}^{M} b_k (j\omega)^k$$

(5.84)

Thus, from Eq. (5.67)

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

(5.85)

which is a rational function of ω. The result (5.85) is the same as the Laplace transform counterpart $H(s) = Y(s)/X(s)$ with $s = j\omega$ [Eq. (3.40)]; that is,

$$H(\omega) = H(s)\big|_{s=j\omega} = H(j\omega)$$

5.6. Filtering

One of the most basic operations in any signal processing system is *filtering*. Filtering is the process by which the relative amplitudes of the frequency components in a signal are changed or perhaps some frequency components are suppressed. As we saw in the preceding section, for continuous-time LTI systems, the spectrum of the output is that of the input multiplied by the frequency response of the system. Therefore, an LTI system acts as a filter on the input signal. Here the word "filter" is used to denote a system that exhibits some sort of frequency-selective behavior.

5.6.1. A. Ideal Frequency-Selective Filters:

An *ideal* frequency-selective filter is one that exactly passes signals at one set of frequencies and completely rejects the rest. The band of frequencies passed by the filter is referred to as the *pass band*, and the band of frequencies rejected by the filter is called the *stop band*.

The most common types of ideal frequency-selective filters are the following.

5.6.1.1. 1. Ideal Low-Pass Filter:

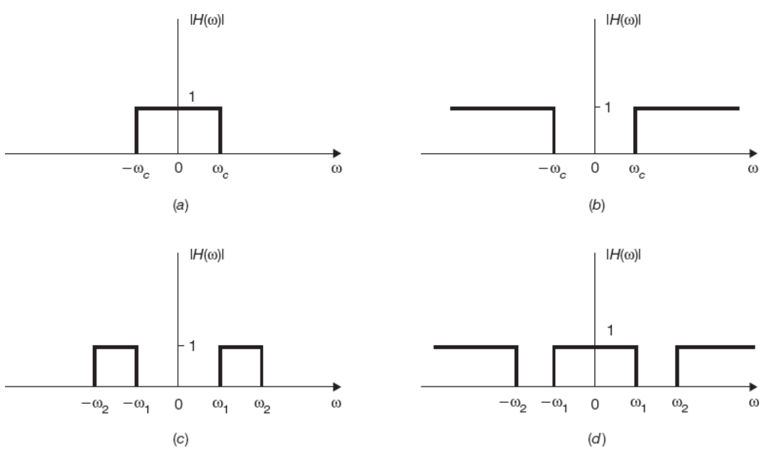
An ideal low-pass filter (LPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

(5.86)

which is shown in Fig. 5-5(*a*). The frequency ω_c is called the *cutoff* frequency.

Figure 5-5 Magnitude responses of ideal frequency-selective filters.



5.6.1.2. 2. Ideal High-Pass Filter:

An ideal high-pass filter (HPF) is specified by

$$|H(\omega)| = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| > \omega_c \end{cases}$$

(5.87)

which is shown in Fig. 5-5(*b*).

5.6.1.3. 3. Ideal Bandpass Filter:

An ideal bandpass filter (BPF) is specified by

$$|H(\omega)| = \begin{cases} 1 & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

(5.88)

which is shown in Fig. 5-5(*c*).

5.6.1.4. 4. Ideal Bandstop Filter:

An ideal bandstop filter (BSF) is specified by

$$|H(\omega)| = \begin{cases} 0 & \omega_1 < |\omega| < \omega_2 \\ 1 & \text{otherwise} \end{cases}$$

(5.89)

which is shown in Fig. 5-5(*d*).

In the above discussion, we said nothing regarding the phase response of the filters. To avoid phase distortion in the filtering process, a filter should have a linear phase characteristic over the pass band of the filter; that is [Eq. (5.82b)],

$$\theta_H(\omega) = -\omega t_d$$

(5.90)

where t_d is a constant.

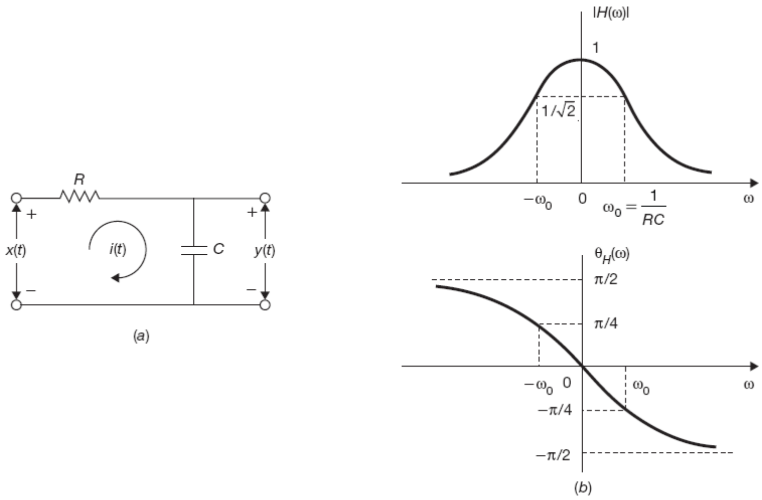
Note that all ideal frequency-selective filters are noncausal systems.

5.6.2. B. Nonideal Frequency-Selective Filters:

As an example of a simple continuous-time causal frequency-selective filter, we consider the *RC* filter shown in Fig. 5-6(*a*). The output $y(t)$ and the input $x(t)$ are related by (Prob. 1.32)

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Figure 5-6 RC filter and its frequency response.



Taking the Fourier transforms of both sides of the above equation, the frequency response $H(\omega)$ of the *RC* filter is given by

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{1}{1+j\omega RC}=\frac{1}{1+j\omega/\omega_0}$$

(5.91)

where $\omega_0 = 1/RC$. Thus, the amplitude response $|H(\omega)|$ and phase response $\theta_H(\omega)$ are given by

$$|H(\omega)|=\frac{1}{|1+j\omega/\omega_0|}=\frac{1}{\left[1+(\omega/\omega_0)^2\right]^{1/2}}$$

(5.92)

$$\theta_H(\omega)=-\tan^{-1}\frac{\omega}{\omega_0}$$

(5.93)

which are plotted in Fig. 5-6(*b*). From Fig. 5-6(*b*) we see that the *RC* network in Fig. 5-6(*a*) performs as a low-pass filter.

5.7. Bandwidth

5.7.1. A. Filter (or System) Bandwidth:

One important concept in system analysis is the *bandwidth* of an LTI system. There are many different definitions of system bandwidth.

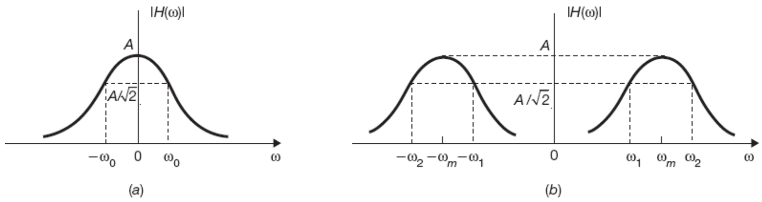
5.7.1.1. 1. Absolute Bandwidth:

The bandwidth W_B of an ideal low-pass filter equals its cutoff frequency; that is, $W_B = \omega_c$ [Fig. 5-5(*a*)]. In this case W_B is called the *absolute bandwidth*. The absolute bandwidth of an ideal bandpass filter is given by $W_B = \omega_2 - \omega_1$ [Fig. 5-5(*c*)]. A bandpass filter is called *narrowband* if $W_B << \omega_0$, where $\omega_0 = \frac{1}{2}(\omega_1 + \omega_2)$ is the center frequency of the filter. No bandwidth is defined for a high-pass or a bandstop filter.

5.7.1.2. 2. 3-dB (or Half-Power) Bandwidth:

For causal or practical filters, a common definition of filter (or system) bandwidth is the 3-dB bandwidth $W_{3\text{ dB}}$. In the case of a low-pass filter, such as the *RC* filter described by Eq. (5.92) or in Fig. 5-6(*b*), $W_{3\text{ dB}}$ is defined as the positive frequency at which the amplitude spectrum $|H(\omega)|$ drops to a value equal to $|H(0)|/\sqrt{2}$, as illustrated in Fig. 5-7(*a*). Note that $|H(0)|$ is the peak value of $H(\omega)$ for the low-pass *RC* filter. The 3-dB bandwidth is also known as the *half-power* bandwidth because a voltage or current attenuation of 3 dB is equivalent to a power attenuation by a factor of 2. In the case of a bandpass filter, $W_{3\text{ dB}}$ is defined as the difference between the frequencies at which $|H(\omega)|$ drops to a value equal to $1/\sqrt{2}$ times the peak value $|H(\omega_m)|$ as illustrated in Fig. 5-7(*b*). This definition of $W_{3\text{ dB}}$ is useful for systems with unimodal amplitude response (in the positive frequency range) and is a widely accepted criterion for measuring a system's bandwidth, but it may become ambiguous and nonunique with systems having multiple peak amplitude responses.

Figure 5-7 Filter bandwidth.



Note that each of the preceding bandwidth definitions is defined along the positive frequency axis only and always defines positive frequency, or one-sided, bandwidth only.

5.7.2. B. Signal Bandwidth:

The *bandwidth* of a signal can be defined as the range of positive frequencies in which "most" of the energy or power lies. This definition is rather ambiguous and is subject to various conventions (Probs. 5.57 and 5.76).

5.7.2.1. 3-dB Bandwidth:

The bandwidth of a signal $x(t)$ can also be defined on a similar basis as a filter bandwidth such as the 3-dB bandwidth, using the magnitude spectrum $|X(\omega)|$ of the signal. Indeed, if we replace $|H(\omega)|$ by $|X(\omega)|$ in Figs. 5-5(*a*) to (*c*), we have frequency-domain plots of *low-pass*, *high-pass*, and *bandpass* signals.

5.7.2.2. Band-Limited Signal:

A signal $x(t)$ is called a *band-limited* signal if

$$|X(\omega)| = 0 \qquad |\omega| > \omega_M$$

(5.94)

Thus, for a band-limited signal, it is natural to define ω_M as the bandwidth.

5.8. SOLVED PROBLEMS

5.8.1. Fourier Series

5.1. We call a set of signals $\{\Phi_n(t)\}$ *orthogonal* on an interval (a, b) if any two signals $\Phi_m(t)$ and $\Phi_k(t)$ in the set satisfy the condition

$$\int_a^b \Psi_m(t)\,\Psi_k^*(t)\,dt = \begin{cases} 0 & m \neq k \\ \alpha & m = k \end{cases}$$

(5.95)

where * denotes the complex conjugate and $\alpha \neq 0$. Show that the set of complex exponentials $\{e^{jk\omega_0 t}: k = 0, \pm 1, \pm 2, \ldots\}$ is orthogonal on any interval over a period T_0, where $T_0 = 2\pi/\omega_0$.

For any t_0 we have

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}\,dt = \frac{1}{jm\omega_0} e^{jm\omega_0 t}\Big|_{t_0}^{t_0+T_0} = \frac{1}{jm\omega_0}\left(e^{jm\omega_0(t_0+T_0)} - e^{jm\omega_0 t_0}\right)$$
$$= \frac{1}{jm\omega_0} e^{jm\omega_0 t_0}\left(e^{jm2\pi} - 1\right) = 0 \qquad m \neq 0$$

(5.96)

since $e^{jm2\pi} = 1$. When $m = 0$, we have $e^{jm\omega_0 t}\big|_{m=0} = 1$ and

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}\,dt = \int_{t_0}^{t_0+T_0} dt = T_0$$

(5.97)

Thus, from Eqs. (5.96) and (5.97) we conclude that

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t}\left(e^{jk\omega_0 t}\right)^* dt = \int_{t_0}^{t_0+T_0} e^{j(m-k)\omega_0 t}\,dt = \begin{cases} 0 & m \neq k \\ T_0 & m = k \end{cases}$$

(5.98)

which shows that the set $\{e^{jk\omega_0 t}: k = 0, \pm 1, \pm 2, \ldots\}$ is orthogonal on any interval over a period T_0.

5.2. Using the orthogonality condition (5.98), derive Eq. (5.5) for the complex Fourier coefficients.

From Eq. (5.4)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Multiplying both sides of this equation by $e^{-jm\omega_0 t}$ and integrating the result from t_0 to $(t_0 + T_0)$, we obtain

$$\int_{t_0}^{t_0+T_0} x(t)e^{-jm\omega_0 t}\,dt = \int_{t_0}^{t_0+T_0}\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)e^{-jm\omega_0 t}\,dt$$
$$= \sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_0} e^{j(k-m)\omega_0 t}\,dt$$

(5.99)

Then by Eq. (5.98), Eq. (5.99) reduces to

$$\int_{t_0}^{t_0+T_0} x(t)e^{-jm\omega_0 t}\,dt = c_m T_0$$

(5.100)

Changing index m to k, we obtain Eq. (5.5); that is,

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,e^{-jk\omega_0 t}\,dt$$

(5.101)

We shall mostly use the following two special cases for Eq. (5.101): $t_0 = 0$ and $t_0 = -T_0/2$, respectively. That is,

$$c_k = \frac{1}{T_0}\int_{0}^{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

(5.102a)

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\omega_0 t}\,dt$$

(5.102b)

5.3. Derive the trigonometric Fourier series Eq. (5.8) from the complex exponential Fourier series Eq. (5.4).

Rearranging the summation in Eq. (5.4) as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = c_0 + \sum_{k=1}^{\infty}(c_k e^{jk\omega_0 t} + c_{-k}e^{-jk\omega_0 t})$$

and using Euler's formulas

$$e^{\pm jk\omega_0 t} = \cos k\omega_0 t \pm j\sin k\omega_0 t$$

we have

$$x(t) = c_0 + \sum_{k=1}^{\infty}[(c_k + c_{-k})\cos k\omega_0 t + j(c_k - c_{-k})\sin k\omega_0 t]$$

(5.103)

Setting

$$c_0 = \frac{a_0}{2} \qquad c_k + c_{-k} = a_k \qquad j(c_k - c_{-k}) = b_k$$

(5.104)

Eq. (5.103) becomes

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

5.4. Determine the complex exponential Fourier series representation for each of the following signals:

a. $x(t) = \cos \omega_0 t$

b. $x(t) = \sin \omega_0 t$

c. $x(t) = \cos\left(2t + \frac{\pi}{4}\right)$

d. $x(t) = \cos 4t + \sin 6t$

e. $x(t) = \sin^2 t$

a. Rather than using Eq. (5.5) to evaluate the complex Fourier coefficients c_k using Euler's formula, we get

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) = \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\cos \omega_0 t$ are

$$c_1 = \frac{1}{2} \qquad c_{-1} = \frac{1}{2} \qquad c_k = 0, |k| \neq 1$$

b. In a similar fashion we have

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}) = -\frac{1}{2j}e^{-j\omega_0 t} + \frac{1}{2j}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier coefficients for $\sin \omega_0 t$ are

$$c_1 = \frac{1}{2j} \qquad c_{-1} = -\frac{1}{2j} \qquad c_k = 0, |k| \neq 1$$

c. The fundamental angular frequency ω_0 of $x(t)$ is 2. Thus,

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Now

$$x(t) = \cos\left(2t + \frac{\pi}{4}\right) = \frac{1}{2}(e^{j(2t+\pi/4)} + e^{-j(2t+\pi/4)})$$

$$= \frac{1}{2}e^{-j\pi/4}e^{-j2t} + \frac{1}{2}e^{j\pi/4}e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\cos(2t + \pi/4)$ are

$$c_1 = \frac{1}{2}e^{j\pi/4} = \frac{1}{2}\frac{1+j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1+j)$$
$$c_{-1} = \frac{1}{2}e^{-j\pi/4} = \frac{1}{2}\frac{1-j}{\sqrt{2}} = \frac{\sqrt{2}}{4}(1-j)$$
$$c_k = 0 \qquad |k| \neq 1$$

d. By the result from Prob. 1.14 the fundamental period T_0 of $x(t)$ is π and $\omega_0 = 2\pi/T_0 = 2$. Thus,

$$x(t) = \cos 4t + \sin 6t = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Again using Euler's formula, we have

$$x(t) = \cos 4t + \sin 6t = \frac{1}{2}(e^{j4t} + e^{-j4t}) + \frac{1}{2j}(e^{j6t} - e^{-j6t})$$
$$= -\frac{1}{2j}e^{-j6t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j4t} + \frac{1}{2j}e^{j6t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Thus, the complex Fourier coefficients for $\cos 4t + \sin 6t$ are

$$c_{-3} = -\frac{1}{2j} \qquad c_{-2} = \frac{1}{2} \qquad c_2 = \frac{1}{2} \qquad c_3 = \frac{1}{2j}$$

and all other $c_k = 0$.

e. From Prob. 1.16(e) the fundamental period T_0 of $x(t)$ is π and $\omega_0 = 2\pi/T_0 = 2$. Thus,

$$x(t) = \sin^2 t = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

Again using Euler's formula, we get

$$x(t) = \sin^2 t = \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^2 = -\frac{1}{4}(e^{j2t} - 2 + e^{-j2t})$$
$$= -\frac{1}{4}e^{-j2t} + \frac{1}{2} - \frac{1}{4}e^{j2t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

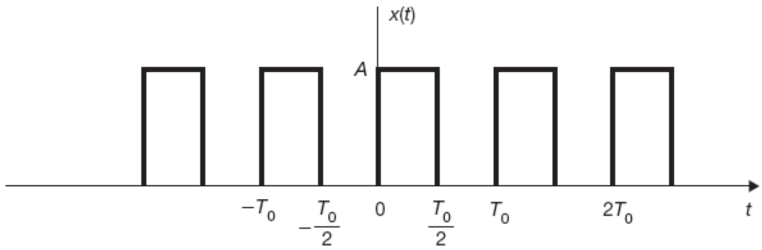
Thus, the complex Fourier coefficients for $\sin^2 t$ are

$$c_{-1} = -\frac{1}{4} \qquad c_0 = \frac{1}{2} \qquad c_1 = -\frac{1}{4}$$

and all other $c_k = 0$.

5.5. Consider the periodic square wave $x(t)$ shown in Fig. 5-8.

Figure 5-8



a. Determine the complex exponential Fourier series of $x(t)$.

b. Determine the trigonometric Fourier series of $x(t)$.

a. Let

$$x(t)=\sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_0 t} \qquad \omega_0=\frac{2\pi}{T_0}$$

Using Eq. (5.102a), we have

$$c_k=\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt=\frac{1}{T_0}\int_0^{T_0/2} A\, e^{-jk\omega_0 t}\, dt$$

$$=\frac{A}{-jk\omega_0 T_0} e^{-jk\omega_0 t}\Big|_0^{T_0/2}=\frac{A}{-jk\omega_0 T_0}\left(e^{-jk\omega_0 T_0/2}-1\right)$$

$$=\frac{A}{jk2\pi}\left(1-e^{-jk\pi}\right)=\frac{A}{jk2\pi}\left[1-(-1)^k\right]$$

since $\omega_0 T_0=2\pi$ and $e^{-jk\pi}=(-1)^k$. Thus,

$$c_k=0 \qquad k=2m\neq 0$$

$$c_k=\frac{A}{jk\pi} \qquad k=2m+1$$

$$c_0=\frac{1}{T_0}\int_0^{T_0} x(t)\, dt=\frac{1}{T_0}\int_0^{T_0/2} A\, dt=\frac{A}{2}$$

Hence,

$$c_0=\frac{A}{2} \qquad c_{2m}=0 \qquad c_{2m+1}=\frac{A}{j(2m+1)\pi}$$

(5.105)

and we obtain

$$x(t)=\frac{A}{2}+\frac{A}{j\pi}\sum_{m=-\infty}^{\infty}\frac{1}{2m+1}e^{j(2m+1)\omega_0 t}$$

(5.106)

b. From Eqs. (5.105), (5.10), and (5.12) we have

$$\frac{a_0}{2} = c_0 = \frac{A}{2} \qquad a_{2m} = b_{2m} = 0,\ m \neq 0$$

$$a_{2m+1} = 2\ \mathrm{Re}[c_{2m+1}] = 0 \qquad b_{2m+1} = -2\ \mathrm{Im}[c_{2m+1}] = \frac{2A}{(2m+1)\pi}$$

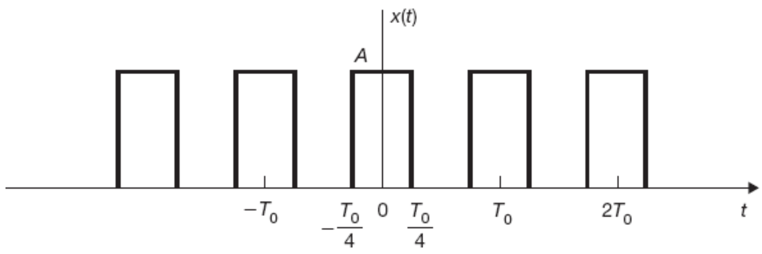
Substituting these values in Eq. (5.8), we get

$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)\omega_0 t$$
$$= \frac{A}{2} + \frac{2A}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right)$$

(5.107)

5.6. Consider the periodic square wave $x(t)$ shown in Fig. 5-9.

Figure 5-9



a. Determine the complex exponential Fourier series of $x(t)$.

b. Determine the trigonometric Fourier series of $x(t)$.

a. Let

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Using Eq. (5.102b), we have

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} A\, e^{-jk\omega_0 t}\, dt$$
$$= \frac{A}{-jk\omega_0 T_0}\left(e^{-jk\omega_0 T_0/4} - e^{jk\omega_0 T_0/4}\right)$$
$$= \frac{A}{-jk2\pi}\left(e^{-jk\pi/2} - e^{jk\pi/2}\right) = \frac{A}{k\pi}\sin\left(\frac{k\pi}{2}\right)$$

Thus,

$$c_k = 0 \qquad k = 2m \neq 0$$

$$c_k = (-1)^m \frac{A}{k\pi} \qquad k = 2m+1$$

$$c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0/2} A\,dt = \frac{A}{2}$$

Hence,

$$c_0 = \frac{A}{2} \qquad c_{2m} = 0,\ m \neq 0 \qquad c_{2m+1} = (-1)^m \frac{A}{(2m+1)\pi}$$

(5.108)

and we obtain

$$x(t) = \frac{A}{2} + \frac{A}{\pi}\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{2m+1} e^{j(2m+1)\omega_0 t}$$

(5.109)

b. From Eqs. (5.108), (5.10), and (5.12) we have

$$\frac{a_0}{2} = c_0 = \frac{A}{2} \qquad a_{2m} = 2\,\mathrm{Re}[c_{2m}] = 0,\ m \neq 0$$

$$a_{2m+1} = 2\,\mathrm{Re}[c_{2m+1}] = (-1)^m \frac{2A}{(2m+1)\pi} \qquad b_k = -2\,\mathrm{Im}[c_k] = 0$$

Substituting these values into Eq. (5.8), we obtain

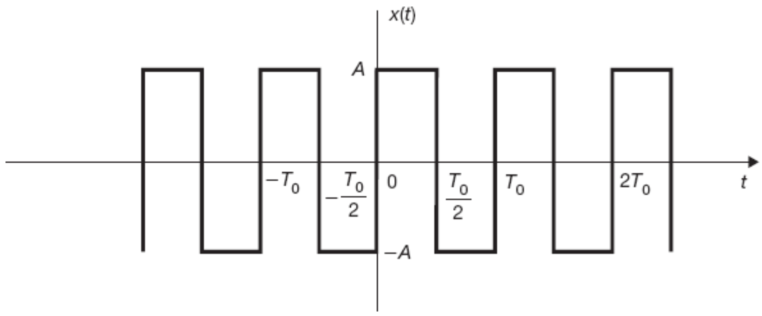
$$\begin{aligned} x(t) &= \frac{A}{2} + \frac{2A}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}\cos(2m+1)\omega_0 t \\ &= \frac{A}{2} + \frac{2A}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right) \end{aligned}$$

(5.110)

Note that $x(t)$ is even; thus, $x(t)$ contains only a dc term and cosine terms. Note also that $x(t)$ in Fig. 5-9 can be obtained by shifting $x(t)$ in Fig. 5-8 to the left by $T_0/4$.

5.7. Consider the periodic square wave $x(t)$ shown in Fig. 5-10.

Figure 5-10



a. Determine the complex exponential Fourier series of $x(t)$.

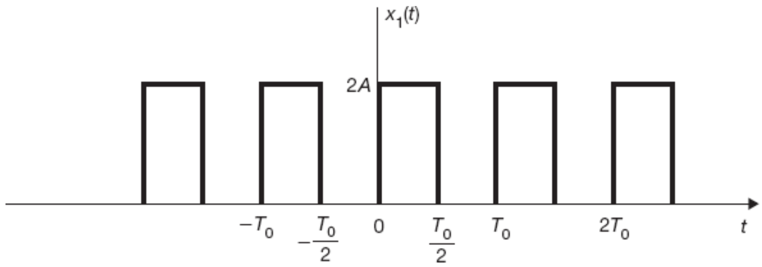
b. Determine the trigonometric Fourier series of $x(t)$.

Note that $x(t)$ can be expressed as

$$x(t) = x_1(t) - A$$

where $x_1(t)$ is shown in Fig. 5-11. Now comparing Fig. 5-11 and Fig. 5-8 in Prob. 5.5, we see that $x_1(t)$ is the same square wave of $x(t)$ in Fig. 5-8 except that A becomes $2A$.

Figure 5-11



a. Replacing A by $2A$ in Eq. (5.106), we have

$$x_1(t) = A + \frac{2A}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

Thus,

$$x(t) = x_1(t) - A = \frac{2A}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

(5.111)

b. Similarly, replacing A by $2A$ in Eq. (5.107), we have

$$x_1(t) = A + \frac{4A}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)\omega_0 t$$

Thus,

$$x(t) = \frac{4A}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)\omega_0 t$$
$$= \frac{4A}{\pi} \left(\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \cdots \right)$$

(5.112)

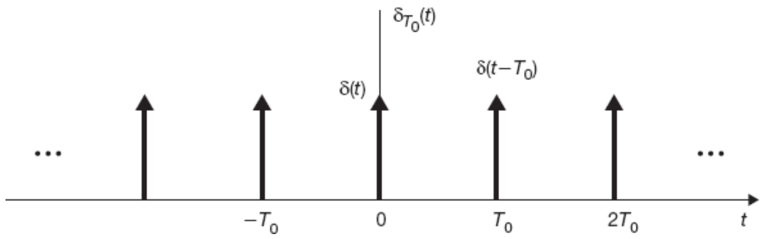
Note that $x(t)$ is odd; thus, $x(t)$ contains only sine terms.

5.8. Consider the periodic impulse train $\delta_{T0}(t)$ shown in Fig. 5-12 and defined by

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

(5.113)

Figure 5-12



a. Determine the complex exponential Fourier series of $\delta_{T0}(t)$.

b. Determine the trigonometric Fourier series of $\delta_{T0}(t)$.

a. Let

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Since $\delta(t)$ is involved, we use Eq. (5.102b) to determine the Fourier coefficients and we obtain

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}$$

(5.114)

Hence, we get

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.115)

b. Let

$$\delta_{T_0}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

Since $\delta_{T_0}(t)$ is even, $b_k = 0$, and by Eq. (5.9a), a_k are given by

$$a_k = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) \cos k\omega_0 t\, dt = \frac{2}{T_0}$$

(5.116)

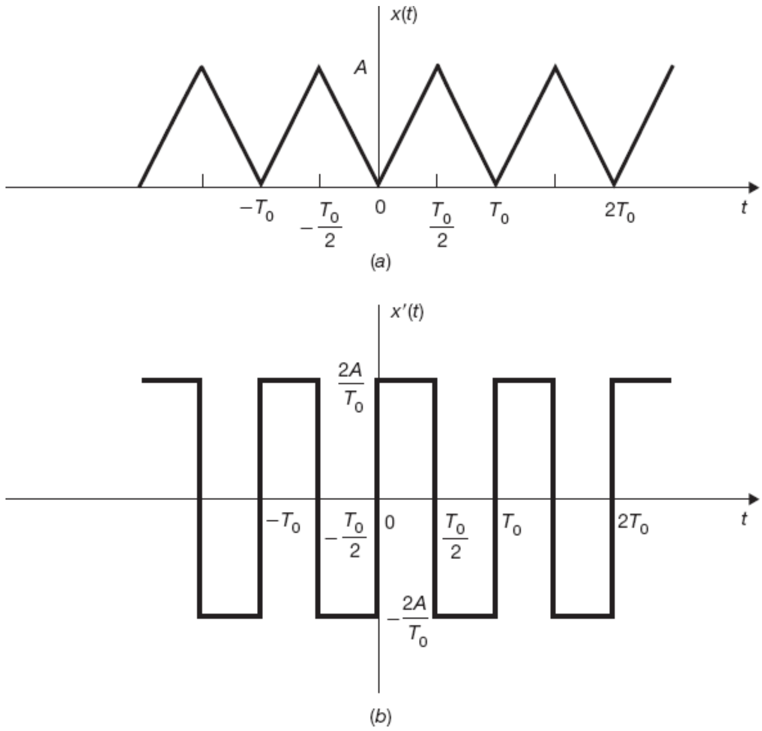
Thus, we get

$$\delta_{T_0}(t) = \frac{1}{T_0} + \frac{2}{T_0}\sum_{k=1}^{\infty} \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.117)

5.9. Consider the triangular wave $x(t)$ shown in Fig. 5-13(*a*). Using the differentiation technique, find (*a*) the complex exponential Fourier series of $x(t)$, and (*b*) the trigonometric Fourier series of $x(t)$.

Figure 5-13



The derivative $x'(t)$ of the triangular wave $x(t)$ is a square wave as shown in Fig. 5-13(*b*).

a. Let

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.118)

Differentiating Eq. (5.118), we obtain

$$x'(t) = \sum_{k=-\infty}^{\infty} jk\omega_0 c_k e^{jk\omega_0 t}$$

(5.119)

Equation (5.119) shows that the complex Fourier coefficients of $x'(t)$ equal $jk\omega_0 c_k$. Thus, we can find c_k ($k \neq 0$) if the Fourier coefficients of $x'(t)$ are known. The term c_0 cannot be determined by Eq. (5.119) and must be evaluated directly in terms of $x(t)$ with Eq. (5.6). Comparing Fig. 5-13(*b*) and Fig. 5-10, we see that $x'(t)$ in Fig. 5-13(*b*) is the same as $x(t)$ in Fig. 5-10 with A replaced by $2A/T_0$. Hence, from Eq. (5.111), replacing A by $2A/T_0$, we have

$$x'(t) = \frac{4A}{j\pi T_0} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

(5.120)

Equating Eqs. (5.119) and (5.120), we have

$$c_k = 0 \qquad\qquad k = 2m \neq 0$$

$$jk\omega_0 c_k = \frac{4A}{j\pi k T_0} \quad \text{or} \quad c_k = -\frac{2A}{\pi^2 k^2} \qquad k = 2m+1$$

From Fig. 5-13(*a*) and Eq. (5.6) we have

$$c_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt = \frac{A}{2}$$

Substituting these values into Eq. (5.118), we obtain

$$x(t) = \frac{A}{2} - \frac{2A}{\pi^2} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)^2} e^{j(2m+1)\omega_0 t}$$

(5.121)

b. (*b*) In a similar fashion, differentiating Eq. (5.8), we obtain

$$x'(t) = \sum_{k=1}^{\infty} k\omega_0 (b_k \cos k\omega_0 t - a_k \sin k\omega_0 t)$$

(5.122)

Equation (5.122) shows that the Fourier cosine coefficients of $x'(t)$ equal to $k\omega_0 b_k$ and that the sine coefficients equal to $-k\omega_0 a_k$. Hence, from Eq. (5.112), replacing A by $2A/T_0$, we have

$$x'(t) = \frac{8A}{\pi T_0} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)\omega_0 t$$

(5.123)

Equating Eqs. (5.122) and (5.123), we have

$$b_k = 0 \qquad a_k = 0 \qquad k = 2m \neq 0$$

$$-k\omega_0 a_k = \frac{8A}{\pi k T_0} \quad \text{or} \quad a_k = -\frac{4A}{\pi^2 k^2} \qquad k = 2m+1$$

From Eqs. (5.6) and (5.10) and Fig. 5-13(*a*) we have

$$\frac{a_0}{2} = c_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt = \frac{A}{2}$$

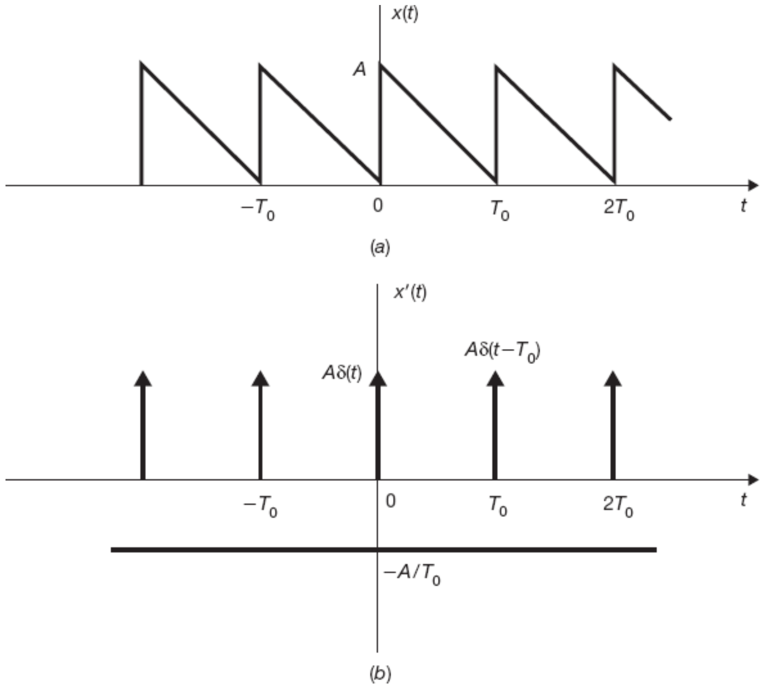
Substituting these values into Eq. (5.8), we get

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos(2m+1)\,\omega_0 t$$

(5.124)

5.10. Consider the triangular wave $x(t)$ shown in Fig. 5-14(a). Using the differentiation technique, find the triangular Fourier series of $x(t)$.

Figure 5-14



From Fig. 5-14(a) the derivative $x'(t)$ of the triangular wave $x(t)$ is, as shown in Fig. 5-14(b),

$$x'(t) = -\frac{A}{T_0} + A \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

(5.125)

Using Eq. (5.117), Eq. (5.125) becomes

$$x'(t) = \sum_{k=1}^{\infty} \frac{2A}{T_0} \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

(5.126)

Equating Eqs. (5.126) and (5.122), we have

$$a_k = 0, \ \ k \neq 0 \qquad k\omega_0 b_k = \frac{2A}{T_0} \qquad \text{or} \qquad b_k = \frac{A}{k\pi}$$

From Fig. 5-14(a) and Eq. (5.9a), we have

$$\frac{a_0}{2} = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt = \frac{A}{2}$$

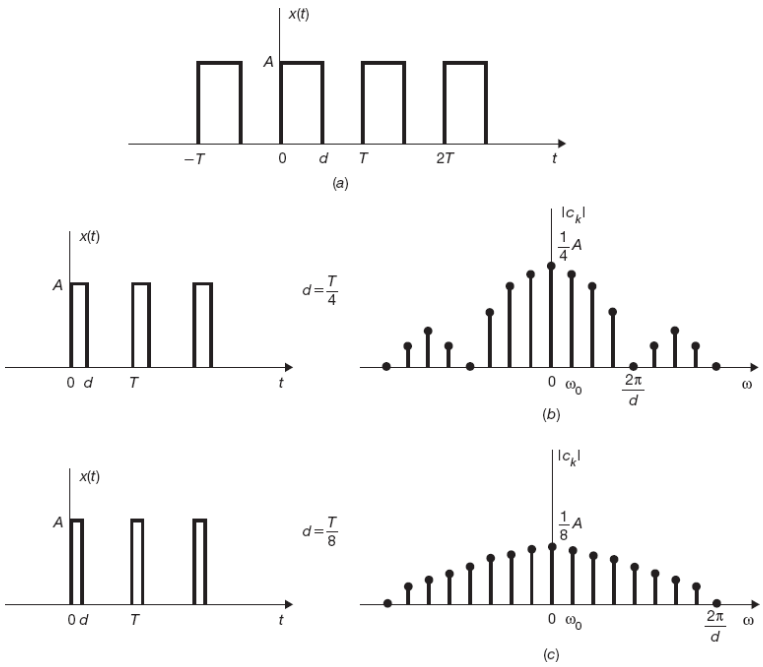
Thus, substituting these values into Eq. (5.8), we get

$$x(t)=\frac{A}{2}+\frac{A}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin k\omega_0 t \qquad \omega_0=\frac{2\pi}{T_0}$$

(5.127)

5.11. Find and sketch the magnitude spectra for the periodic square pulse train signal $x(t)$ shown in Fig. 5-15(a) for (a) $d = T_0/4$, and (b) $d = T_0/8$.

Figure 5-15



Using Eq. (5.102a), we have

$$\begin{aligned}
c_k &= \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{A}{T_0}\int_0^{d} e^{-jk\omega_0 t}\, dt \\
&= \frac{A}{T_0}\frac{1}{-jk\omega_0} e^{-jk\omega_0 t}\Big|_0^d = \frac{A}{T_0}\frac{1}{jk\omega_0}\left(1-e^{-jk\omega_0 d}\right) \\
&= \frac{A}{-jk\omega_0 T_0} e^{-jk\omega_0 d/2}\left(e^{jk\omega_0 d/2}-e^{-jk\omega_0 d/2}\right) \\
&= A\frac{d}{T_0}\frac{\sin(k\omega_0 d/2)}{k\omega_0 d/2} e^{-jk\omega_0 d/2}
\end{aligned}$$

(5.128)

Note that $c_k = 0$ whenever $k\omega_0 d/2 = m\pi$; that is,

$$n\omega_0=\frac{m2\pi}{d} \qquad m=0, \pm 1, \pm 2, \dots$$

a. $d = T_0/4$, $k\omega_0 d/2 = k\pi d/T_0 = k\pi/4$,

$$|c_k| = \frac{A}{4}\left|\frac{\sin(k\pi/4)}{k\pi/4}\right|$$

The magnitude spectrum for this case is shown in Fig. 5-15(*b*).

b. $d = T_0/8, k\omega_0 d/2 = k\pi d/T_0 = k\pi/8$,

$$|c_k| = \frac{A}{8}\left|\frac{\sin(k\pi/8)}{k\pi/8}\right|$$

The magnitude spectrum for this case is shown in Fig. 5-15(*c*).

5.12. If $x_1(t)$ and $x_2(t)$ are periodic signals with fundamental period T_0 and their complex Fourier series expressions are

$$x_1(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t} \qquad x_2(t) = \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

show that the signal $x(t) = x_1(t)x_2(t)$ is periodic with the same fundamental period T_0 and can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{\omega_0}$$

where c_k is given by

$$c_k = \sum_{m=-\infty}^{\infty} d_m e_{k-m}$$

(5.129)

Now

$$x(t+T_0) = x_1(t+T_0)x_2(t+T_0) = x_1(t)x_2(t) = x(t)$$

Thus, $x(t)$ is periodic with fundamental period T_0. Let

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Then

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_1(t)x_2(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left(\sum_{m=-\infty}^{\infty} d_m e^{jm\omega_0 t}\right) x_2(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \sum_{m=-\infty}^{\infty} d_m\left[\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_2(t)\, e^{-j(k-m)\omega_0 t}\, dt\right] = \sum_{m=-\infty}^{\infty} d_m e_{k-m}$$

since

$$e_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_2(t)\, e^{-jk\omega_0 t}\, dt$$

and the term in brackets is equal to e_{k-m}.

5.13. Let $x_1(t)$ and $x_2(t)$ be the two periodic signals in Prob. 5.12. Show that

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_1(t)x_2(t)\, dt = \sum_{k=-\infty}^{\infty} d_k e_{-k}$$

(5.130)

Equation (5.130) is known as *Parseval's relation* for periodic signals.

From Prob. 5.12 and Eq. (5.129) we have

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_1(t)x_2(t)\, e^{-jk\omega_0 t}\, dt = \sum_{m=-\infty}^{\infty} d_m e_{k-m}$$

Setting $k = 0$ in the above expression, we obtain

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_1(t)x_2(t)\, dt = \sum_{m=-\infty}^{\infty} d_m e_{-m} = \sum_{k=-\infty}^{\infty} d_k e_{-k}$$

5.14. Verify Parseval's identity (5.21) for the Fourier series; that is,

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

then

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} c_k^*\, e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_{-k}^*\, e^{jk\omega_0 t}$$

(5.131)

where * denotes the complex conjugate. Equation (5.131) indicates that if the Fourier coefficients of $x(t)$ are c_k, then the Fourier coefficients of $x^*(t)$ are c^*_{-k}. Setting $x_1(t) = x(t)$ and $x_2(t) = x^*(t)$ in Eq. (5.130), we have $d_k = c_k$ and $e_k = c^*_{-k}$ or ($e_{-k} = c^*_k$), and we obtain

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t)\, dt = \sum_{k=-\infty}^{\infty} c_k c_k^*$$

(5.132)

or

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left|x(t)\right|^2 dt = \sum_{k=-\infty}^{\infty}\left|c_k\right|^2$$

5.15.

a. The periodic convolution $f(t) = x_1(t) \otimes x_2(t)$ was defined in Prob. 2.8. If d_n and e_n are the complex Fourier coefficients of $x_1(t)$ and $x_2(t)$, respectively, then show that the complex Fourier coefficients c_k of $f(t)$ are given by

$$c_k = T_0 d_k e_k$$

(5.133)

where T_0 is the fundamental period common to $x_1(t)$, $x_2(t)$, and $f(t)$.

b. Find the complex exponential Fourier series of $f(t)$ defined in Prob. 2.8(*c*).

a. From Eq. (2.70) (Prob. 2.8)

$$f(t) = x_1(t) \otimes x_2(t) = \int_0^{T_0} x_1(\tau)x_2(t-\tau)\,d\tau$$

Let

$$x_1(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t} \qquad x_2(t) = \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 t}$$

Then

$$f(t) = \int_0^{T_0} x(\tau)\left(\sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0(t-\tau)}\right)d\tau$$

$$= \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 \tau}\int_0^{T_0} x(\tau)e^{-jk\omega_0\tau}\,d\tau$$

Since

$$d_k = \frac{1}{T_0}\int_0^{T_0} x(\tau)e^{-jk\omega_0\tau}\,d\tau$$

we get

$$f(t) = \sum_{k=-\infty}^{\infty} T_0 d_k e_k e^{jk\omega_0 t}$$

(5.134)

which shows that the complex Fourier coefficients c_k of $f(t)$ equal $T_0 d_k e_k$.

b. In Prob. 2.8(*c*), $x_1(t) = x_2(t) = x(t)$, as shown in Fig. 2-12, which is the same as Fig. 5-8 (Prob. 5.5). From Eq. (5.105) we have

$$d_0 = e_0 = \frac{A}{2} \qquad d_k = e_k = \begin{cases} 0 & k = 2m,\ m \neq 0 \\ A/jk\pi & k = 2m+1 \end{cases}$$

Thus, by Eq. (5.133) the complex Fourier coefficients c_k of $f(t)$ are

$$c_0 = T_0 d_0 e_0 = T_0 \frac{A^2}{4}$$

$$c_k = T_0 d_k e_k = \begin{cases} 0 & k = 2m, m \neq 0 \\ -T_0 A^2 / k^2 \pi^2 & k = 2m+1 \end{cases}$$

Note that in Prob. 2.8(c), $f(t) = x_1(t) \otimes x_2(t)$, shown in Fig. 2-13($b$), is proportional to $x(t)$, shown in Fig. 5-13(a). Thus, replacing A by $A^2 T_0/2$ in the result from Prob. 5.9, we get

$$c_0 = T_0 \frac{A^2}{4} \qquad c_k = \begin{cases} 0 & k = 2m, m \neq 0 \\ -T_0 A^2 / k^2 \pi^2 & k = 2m+1 \end{cases}$$

which are the same results obtained by using Eq. (5.133).

5.8.2. Fourier Transform

5.16. (a) Verify the time-shifting property (5.50); that is,

$$x(t - t_0) \leftrightarrow e^{j\omega t_0} X(\omega)$$

By definition (5.31)

$$\mathscr{F}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t}\, dt$$

By the change of variable $\tau = t - t_0$, we obtain

$$\mathscr{F}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_0)}\, d\tau$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau}\, d\tau = e^{-j\omega t_0} X(\omega)$$

Hence,

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$$

5.17. Verify the frequency-shifting property (5.51); that is,

$$x(t) e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

By definition (5.31)

$$\mathscr{F}\{x(t)\, e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} x(t) e^{j\omega_0 t} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t}\, dt = X(\omega - \omega_0)$$

Hence,

$$x(t) e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

5.18. Verify the duality property (5.54); that is,

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

From the inverse Fourier transform definition (5.32), we have

$$\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega = 2\pi x(t)$$

Changing *t* to −*t*, we obtain

$$\int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}\, d\omega = 2\pi x(-t)$$

Now interchanging *t* and *ω*, we get

$$\int_{-\infty}^{\infty} X(t)e^{-j\omega t}\, dt = 2\pi x(-\omega)$$

Since

$$\mathscr{F}\{X(t)\} = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\, dt$$

we conclude that

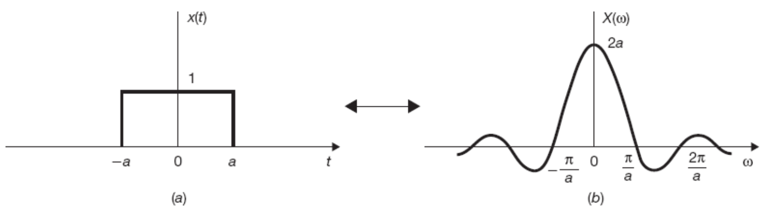
$$X(t) \leftrightarrow 2\pi x(-\omega)$$

5.19. Find the Fourier transform of the rectangular pulse signal $x(t)$ [Fig. 5-16(*a*)] defined by

$$x(t) = p_a(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$

(5.135)

Figure 5-16 Rectangular pulse and its Fourier transform.



By definition (5.31)

$$X(\omega) = \int_{-\infty}^{\infty} p_a(t)e^{-j\omega t}\, dt = \int_{-a}^{a} e^{-j\omega t}\, dt$$

$$= \frac{1}{j\omega}(e^{j\omega a} - e^{-j\omega a}) = 2\frac{\sin \omega a}{\omega} = 2a\frac{\sin \omega a}{\omega a}$$

Hence, we obtain

$$p_a(t) \leftrightarrow 2\frac{\sin \omega a}{\omega} = 2a\frac{\sin \omega a}{\omega a}$$

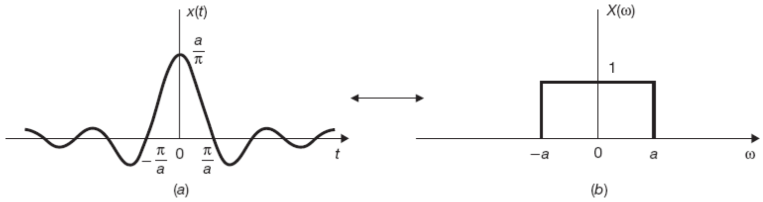
(5.136)

The Fourier transform $X(\omega)$ of $x(t)$ is sketched in Fig. 5-16(*b*).

5.20. Find the Fourier transform of the signal [Fig. 5-17(*a*)]

$$x(t)=\frac{\sin at}{\pi t}$$

Figure 5-17 sin at/πt and its Fourier transform.



From Eq. (5.136) we have

$$p_a(t)\leftrightarrow 2\frac{\sin \omega a}{\omega}$$

Now by the duality property (5.54), we have

$$2\frac{\sin at}{t}\leftrightarrow 2\pi p_a(-\omega)$$

Dividing both sides by 2π (and by the linearity property), we obtain

$$\frac{\sin at}{\pi t}\leftrightarrow p_a(-\omega)=p_a(\omega)$$

(5.137)

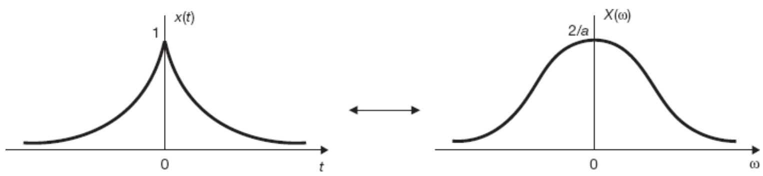
where $p_a(\omega)$ is defined by [see Eq. (5.135) and Fig. 5-17(*b*)]

$$p_a(\omega)=\begin{cases}1 & |\omega|<a\\ 0 & |\omega|>a\end{cases}$$

5.21. Find the Fourier transform of the signal [Fig. 5-18(*a*)]

$$x(t)=e^{-a|t|}\qquad a>0$$

Figure 5-18 $e^{-|a|t}$ and its Fourier transform.



Signal $x(t)$ can be rewritten as

$$x(t)=e^{-a|t|}=\begin{cases}e^{-at} & t>0\\ e^{at} & t<0\end{cases}$$

Then

$$X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$
$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

Hence, we get

$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2+\omega^2}$$

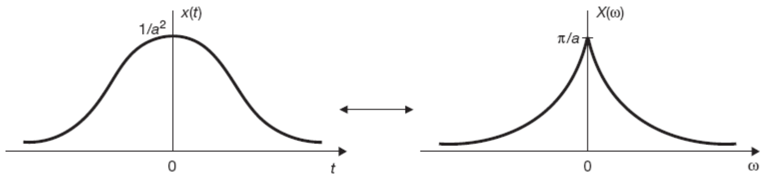
(5.138)

The Fourier transform $X(\omega)$ of $x(t)$ is shown in Fig. 5-18(b).

5.22. Find the Fourier transform of the signal [Fig. 5-19(a)]

$$x(t) = \frac{1}{a^2+t^2}$$

Figure 5-19 *$1/(a^2 + t^2)$ and its Fourier transform.*



From Eq. (5.138) we have

$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2+\omega^2}$$

Now by the duality property (5.54) we have

$$\frac{2a}{a^2+t^2} \leftrightarrow 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$$

Dividing both sides by $2a$, we obtain

$$\frac{1}{a^2+t^2} \leftrightarrow \frac{\pi}{a} e^{-a|\omega|}$$

(5.139)

The Fourier transform $X(\omega)$ of $x(t)$ is shown in Fig. 5-19(b).

5.23. Find the Fourier transforms of the following signals:

a. $x(t) = 1$

b. $x(t) = e^{j\omega_0 t}$

c. $x(t) = e^{-j\omega_0 t}$

d. $x(t) = \cos \omega_0 t$

e. $x(t) = \sin \omega_0 t$

a. By Eq. (5.43) we have

$$\delta(t) \leftrightarrow 1$$

(5.140)

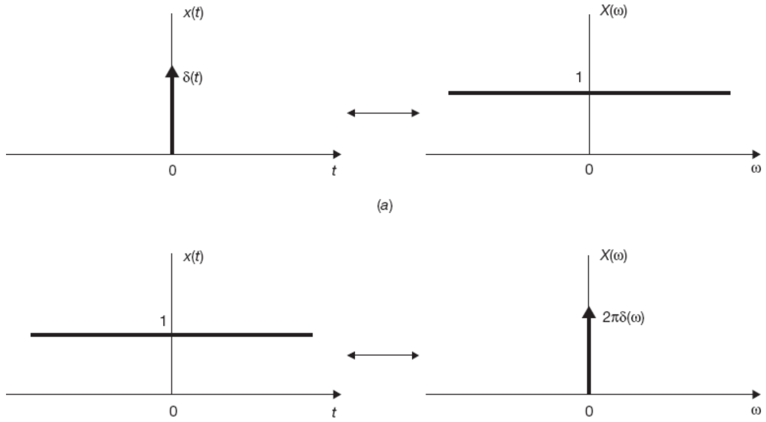
Thus, by the duality property (5.54) we get

$$1 \leftrightarrow 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

(5.141)

Figs. 5-20(*a*) and (*b*) illustrate the relationships in Eqs. (5.140) and (5.141), respectively.

Figure 5-20 (a) Unit impulse and its Fourier transform; (b) constant (dc) signal and its Fourier transform.



b. Applying the frequency-shifting property (5.51) to Eq. (5.141), we get

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

(5.142)

c. From Eq. (5.142), it follows that

$$e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega + \omega_0)$$

(5.143)

d. From Euler's formula we have

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{j\omega_0 t})$$

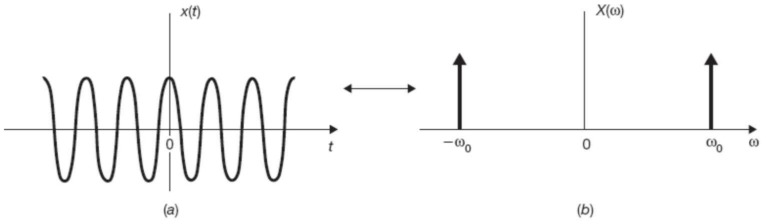
Thus, using Eqs. (5.142) and (5.143) and the linearity property (5.49), we get

$$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

(5.144)

Fig. 5-21 illustrates the relationship in Eq. (5.144).

Figure 5-21 Cosine signal and its Fourier transform.



e. Similarly, we have

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

and again using Eqs. (5.142) and (5.143), we get

$$\sin \omega_0 t \leftrightarrow -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

(5.145)

5.24. Find the Fourier transform of a periodic signal $x(t)$ with period T_0.

We express $x(t)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Taking the Fourier transform of both sides and using Eq. (5.142) and the linearity property (5.49), we get

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)$$

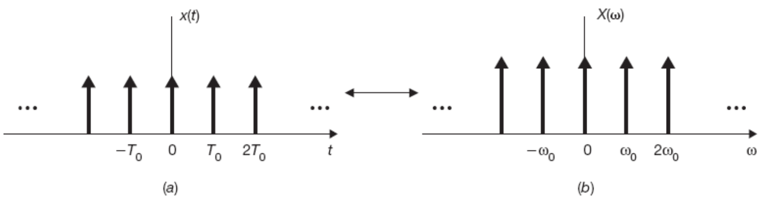
(5.146)

which indicates that the Fourier transform of a periodic signal consists of a sequence of equidistant impulses located at the harmonic frequencies of the signal.

5.25. Find the Fourier transform of the periodic impulse train [Fig. 5-22(*a*)]

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

Figure 5-22 Unit impulse train and its Fourier transform.



From Eq. (5.115) in Prob. 5.8, the complex exponential Fourier series of $\delta_{T0}(t)$ is given by

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Using Eq. (5.146), we get

$$\mathscr{F}[\delta_{T_0}(t)] = \frac{2\pi}{T_0} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

$$= \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \omega_0 \delta_{\omega_0}(\omega)$$

or

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_0) \leftrightarrow \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

(5.147)

Thus, the Fourier transform of a unit impulse train is also a similar impulse train [Fig. 5-22(*b*)].

5.26. Show that

$$x(t) \cos \omega_0 t \leftrightarrow \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)$$

(5.148)

and

$$x(t) \sin \omega_0 t \leftrightarrow -j\left[\frac{1}{2} X(\omega - \omega_0) - \frac{1}{2} X(\omega + \omega_0)\right]$$

(5.149)

Equation (5.148) is known as the *modulation theorem*.

From Euler's formula we have

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Then by the frequency-shifting property (5.51) and the linearity property (5.49), we obtain

$$\mathscr{F}[x(t) \cos \omega_0 \mathrm{t}] = \mathscr{F}\left[\frac{1}{2} x(t)\, e^{j\omega_0 t} + \frac{1}{2} x(t)\, e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)$$

Hence,

$$x(t) \cos \omega_0 t \leftrightarrow \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)$$

In a similar manner we have

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

and

$$\mathscr{F}[x(t)\sin \omega_0 t] = \mathscr{F}\left[\frac{1}{2j}x(t)\,e^{j\omega_0 t} - \frac{1}{2j}x(t)\,e^{-j\omega_0 t}\right]$$
$$= \frac{1}{2j}X(\omega - \omega_0) - \frac{1}{2j}X(\omega + \omega_0)$$

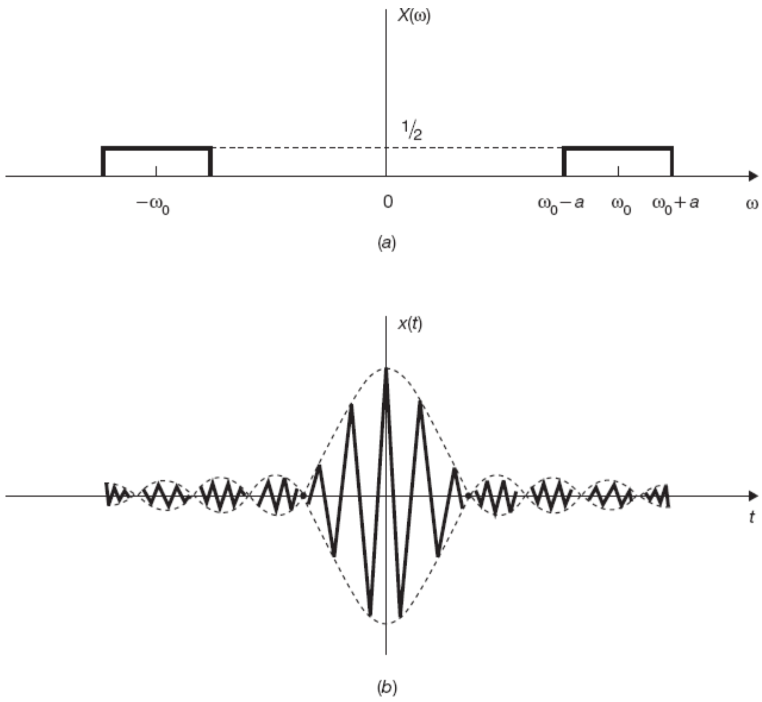
Hence,

$$x(t)\sin \omega_0 t \leftrightarrow -j\left[\frac{1}{2}X(\omega - \omega_0) - \frac{1}{2}X(\omega + \omega_0)\right]$$

5.27. The Fourier transform of a signal $x(t)$ is given by [Fig. 5-23(*a*)]

$$X(\omega) = \frac{1}{2}p_a(\omega - \omega_0) + \frac{1}{2}p_a(\omega + \omega_0)$$

Figure 5-23



Find and sketch $x(t)$.

From Eq. (5.137) and the modulation theorem (5.148), it follows that

$$x(t) = \frac{\sin at}{\pi t}\cos \omega_0 t$$

which is sketched in Fig. 5-23(*b*).

5.28. Verify the differentiation property (5.55); that is,

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

From Eq. (5.32) the inverse Fourier transform of $X(\omega)$ is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

(5.150)

Then

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\right]\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{\partial}{\partial t}(e^{j\omega t})\, d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)\, e^{j\omega t}\, d\omega\end{aligned}$$

(5.151)

Comparing Eq. (5.151) with Eq. (5.150), we conclude that $dx(t)/dt$ is the inverse Fourier transform of $j\omega X(\omega)$. Thus,

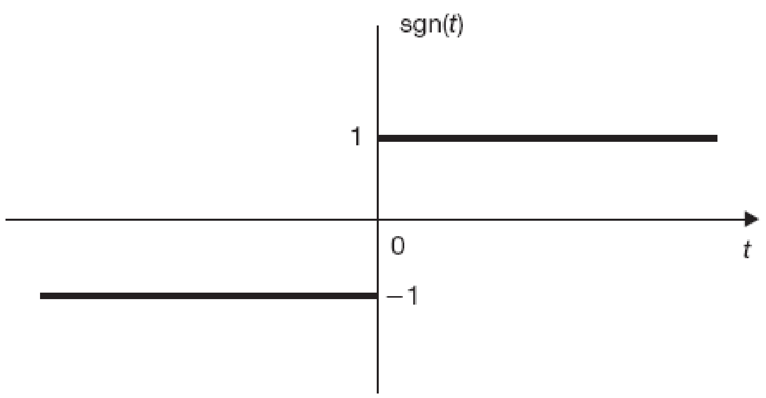
$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

5.29. Find the Fourier transform of the *signum* function, sgn(t) (Fig. 5-24), which is defined as

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

(5.152)

Figure 5-24 Signum function.



The signum function, sgn(t), can be expressed as

$$\operatorname{sgn}(t) = 2u(t) - 1$$

Using Eq. (1.30), we have

$$\frac{d}{dt}\text{sgn}(t) = 2\delta(t)$$

Let

$$\text{sgn}(t) \leftrightarrow X(\omega)$$

Then applying the differentiation property (5.55), we have

$$j\omega X(\omega) = \mathscr{F}[2\delta(t)] = 2 \rightarrow X(\omega) = \frac{2}{j\omega}$$

Hence,

$$\text{sgn}(t) \leftrightarrow \frac{2}{j\omega}$$

(5.153)

Note that sgn(t) is an odd function, and therefore its Fourier transform is a pure imaginary function ofω (Prob. 5.41).

5.30. Verify Eq. (5.48); that is,

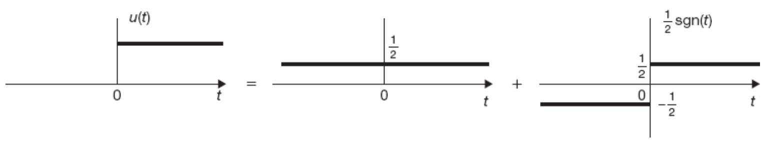
$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

(5.154)

As shown in Fig. 5-25, $u(t)$ can be expressed as

$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

Figure 5-25 Unit step function and its even and odd components.



Note that $\frac{1}{2}$ is the even component of $u(t)$ and $\frac{1}{2}$ sgn(t) is the odd component of $u(t)$. Thus, by Eqs. (5.141) and (5.153) and the linearity property (5.49), we obtain

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

5.31. Prove the time convolution theorem (5.58); that is,

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega)\, X_2(\omega)$$

By definitions (2.6) and (5.31), we have

$$\mathscr{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau\right]e^{-j\omega t}\,dt$$

Changing the order of integration gives

$$\mathscr{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t} dt\right] d\tau$$

By the time-shifting property (5.50)

$$\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t} dt = X_2(\omega)\, e^{-j\omega\tau}$$

Thus, we have

$$\begin{aligned}\mathscr{F}[x_1(t) * x_2(t)] &= \int_{-\infty}^{\infty} x_1(\tau) X_2(\omega)\, e^{-j\omega\tau}\, d\tau \\ &= \left[\int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau} d\tau\right] X_2(\omega) = X_1(\omega) X_2(\omega)\end{aligned}$$

Hence,

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega)\, X_2(\omega)$$

5.32. Using the time convolution theorem (5.58), find the inverse Fourier transform of $X(\omega) = 1/(a + j\omega)^2$.

From Eq. (5.45) we have

$$e^{-at} u(t) \leftrightarrow \frac{1}{a + j\omega}$$

(5.155)

Now

$$X(\omega) = \frac{1}{(a + j\omega)^2} = \left(\frac{1}{a + j\omega}\right)\left(\frac{1}{a + j\omega}\right)$$

Thus, by the time convolution theorem (5.58) we have

$$\begin{aligned}x(t) &= e^{-at} u(t) * e^{-at} u(t) \\ &= \int_{-\infty}^{\infty} e^{-a\tau} u(\tau)\, e^{-a(t-\tau)} u(t-\tau)\, d\tau \\ &= e^{-at} \int_0^t d\tau = te^{-at} u(t)\end{aligned}$$

Hence,

$$te^{-at} u(t) \leftrightarrow \frac{1}{(a + j\omega)^2}$$

(5.156)

5.33. Verify the integration property (5.57); that is,

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \pi X(0)\delta(\omega) + \frac{1}{j\omega} X(\omega)$$

From Eq. (2.60) we have

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

Thus, by the time convolution theorem (5.58) and Eq. (5.154), we obtain

$$\mathscr{F}[x(t) * u(t)] = X(\omega)\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = \pi X(\omega)\,\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

$$= \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

since $X(\omega)\delta(\omega) = X(0)\delta(\omega)$ by Eq. (1.25). Thus,

$$\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] \leftrightarrow \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

5.34. Using the integration property (5.57) and Eq. (1.31), find the Fourier transform of $u(t)$.

From Eq. (1.31) we have

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

Now from Eq. (5.140) we have

$$\delta(t) \leftrightarrow 1$$

Setting $x(\tau) = \delta(\tau)$ in Eq. (5.57), we have

$$x(t) = \delta(t) \leftrightarrow X(\omega) = 1 \qquad \text{and} \qquad X(0) = 1$$

and

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

5.35. Prove the frequency convolution theorem (5.59); that is,

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

By definitions (5.31) and (5.32) we have

$$\mathscr{F}[x_1(t)x_2(t)] = \int_{-\infty}^{\infty} x_1(t)x_2(t)\,e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\,e^{j\lambda t}d\lambda\right]x_2(t)e^{-j\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left[\int_{-\infty}^{\infty} x_2(t)\,e^{-j(\omega-\lambda)t}dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)X_2(\omega-\lambda)\,d\lambda = \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$$

Hence,

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$$

5.36. Using the frequency convolution theorem (5.59), derive the modulation theorem (5.148).

From Eq. (5.144) we have

$$\cos \omega_0 t \leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

By the frequency convolution theorem (5.59) we have

$$\begin{aligned} x(t)\cos\omega_0 t &\leftrightarrow \frac{1}{2\pi} X(\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)] \\ &= \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0) \end{aligned}$$

The last equality follows from Eq. (2.59).

5.37. Verify Parseval's relation (5.63); that is,

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega$$

From the frequency convolution theorem (5.59) we have

$$\mathscr{F}[x_1(t)x_2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda)\, d\lambda$$

that is,

$$\int_{-\infty}^{\infty} [x_1(t)\, x_2(t)]\, e^{-j\omega t}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda)\, d\lambda$$

Setting $\omega = 0$, we get

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) X_2(-\lambda)\, d\lambda$$

By changing the dummy variable of integration, we obtain

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega)\, d\omega$$

5.38. Prove Parseval's identity [Eq. (5.64)] or Parseval's theorem for the Fourier transform; that is,

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

By definition (5.31) we have

$$\begin{aligned} \mathscr{F}\{x^*(t)\} &= \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t}\, dt \\ &= \left[\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt\right]^* = X^*(-\omega) \end{aligned}$$

where * denotes the complex conjugate. Thus,

$$x^*(t) \leftrightarrow X^*(-\omega)$$

(5.157)

Setting $x_1(t) = x(t)$ and $x_2(t) = x^*(t)$ in Parseval's relation (5.63), we get

$$\int_{-\infty}^{\infty} x(t)x^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)\,d\omega$$

or

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

5.39. Show that Eq. (5.61a); that is,

$$X^*(\omega) = X(-\omega)$$

is the necessary and sufficient condition for $x(t)$ to be real.

By definition (5.31)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$$

If $x(t)$ is real, then $x^*(t) = x(t)$ and

$$\begin{aligned} X^*(\omega) &= \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right]^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt = X(-\omega) \end{aligned}$$

Thus, $X^*(\omega) = X(-\omega)$ is the necessary condition for $x(t)$ to be real. Next assume that $X^*(\omega) = X(-\omega)$. From the inverse Fourier transform definition (5.32)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Then

$$\begin{aligned} x^*(t) &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega\right]^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\,e^{-j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(-\omega)\,e^{-j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,e^{j\lambda t}d\lambda = x(t) \end{aligned}$$

which indicates that $x(t)$ is real. Thus, we conclude that

$$X^*(\omega) = X(-\omega)$$

is the necessary and sufficient condition for $x(t)$ to be real.

5.40. Find the Fourier transforms of the following signals:

a. $x(t) = u(-t)$

b. $x(t) = e^{at}u(-t)$, $a > 0$

From Eq. (5.53) we have

$$x(-t) \leftrightarrow X(-\omega)$$

Thus, if $x(t)$ is real, then by Eq. (5.61a) we have

$$x(-t) \leftrightarrow X(-\omega) = X^*(\omega)$$

(5.158)

a. From Eq. (5.154)

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

Thus, by Eq. (5.158) we obtain

$$u(-t) \leftrightarrow \pi\delta(\omega) - \frac{1}{j\omega}$$

(5.159)

b. From Eq. (5.155)

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega}$$

Thus, by Eq. (5.158) we get

$$e^{at}u(-t) \leftrightarrow \frac{1}{a - j\omega}$$

(5.160)

5.41. Consider a real signal $x(t)$ and let

$$X(\omega) = \mathscr{F}[x(t)] = A(\omega) + jB(\omega)$$

and

$$x(t) = x_e(t) + x_o(t)$$

where $x_e(t)$ and $x_o(t)$ are the even and odd components of $x(t)$, respectively. Show that

$$x_e(t) \leftrightarrow A(\omega)$$

(5.161a)

$$x_o(t) \leftrightarrow jB(\omega)$$

(5.161b)

From Eqs. (1.5) and (1.6) we have

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Now if $x(t)$ is real, then by Eq. (5.158) we have

$$x(t) \leftrightarrow X(\omega) = A(\omega) + jB(\omega)$$

$$x(-t) \leftrightarrow X(-\omega) = X^*(\omega) = A(\omega) - jB(\omega)$$

Thus, we conclude that

$$x_e(t) \leftrightarrow \frac{1}{2}X(\omega) + \frac{1}{2}X^*(\omega) = A(\omega)$$

$$x_o(t) \leftrightarrow \frac{1}{2}X(\omega) - \frac{1}{2}X^*(\omega) = jB(\omega)$$

Equations (5.161a) and (5.161b) show that the Fourier transform of a real even signal is a real function of ω, and that of a real odd signal is an imaginary function of ω, respectively.

5.42. Using Eqs. (5.161a) and (5.155), find the Fourier transform of $e^{-a|t|}$ ($a > 0$).

From Eq. (5.155) we have

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$

By Eq. (1.5) the even component of $e^{-at}u(t)$ is given by

$$\frac{1}{2}e^{-at}u(t) + \frac{1}{2}e^{at}u(-t) = \frac{1}{2}e^{-a|t|}$$

Thus, by Eq. (5.161a) we have

$$\frac{1}{2}e^{-a|t|} \leftrightarrow \mathrm{Re}\left(\frac{1}{a+j\omega}\right) = \frac{a}{a^2+\omega^2}$$

or

$$e^{-a|t|} \leftrightarrow \frac{2a}{a^2+\omega^2}$$

which is the same result obtained in Prob. 5.21 [Eq. (5.138)].

5.43. Find the Fourier transform of a Gaussian pulse signal

$$x(t) = e^{-at^2} \qquad a > 0$$

By definition (5.31)

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

(5.162)

Taking the derivative of both sides of Eq. (5.162) with respect to ω, we have

$$\frac{dX(\omega)}{d\omega} = -j\int_{-\infty}^{\infty} te^{-at^2} e^{-j\omega t}\, dt$$

Now, using the integration by parts formula

$$\int_{\alpha}^{\beta} u\, dv = uv\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} v\, du$$

and letting

$$u = e^{-j\omega t} \qquad \text{and} \qquad dv = te^{-at^2}\, dt$$

we have

$$du = -\,j\omega e^{-j\omega t}\, dt \qquad \text{and} \qquad v = -\frac{1}{2a}e^{-at^2}$$

and

$$\int_{-\infty}^{\infty} te^{-at^2} e^{-j\omega t}\, dt = -\frac{1}{2a}e^{-at^2} e^{-j\omega t}\Big|_{-\infty}^{\infty} - j\frac{\omega}{2a}\int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$
$$= -j\frac{\omega}{2a}\int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

since $a > 0$. Thus, we get

$$\frac{dX(\omega)}{d\omega} = -\frac{\omega}{2a}X(\omega)$$

Solving the above separable differential equation for $X(\omega)$, we obtain

$$X(\omega) = Ae^{-\omega^2/4a}$$

(5.163)

where A is an arbitrary constant. To evaluate A, we proceed as follows. Setting $\omega = 0$ in Eq. (5.162) and by a change of variable, we have

$$X(0) = A = \int_{-\infty}^{\infty} e^{-at^2}\, dt = 2\int_{0}^{\infty} e^{-at^2}\, dt = \frac{2}{\sqrt{a}}\int_{0}^{\infty} e^{-\lambda^2}\, d\lambda = \sqrt{\frac{\pi}{a}}$$

Substituting this value of A into Eq. (5.163), we get

$$X(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

(5.164)

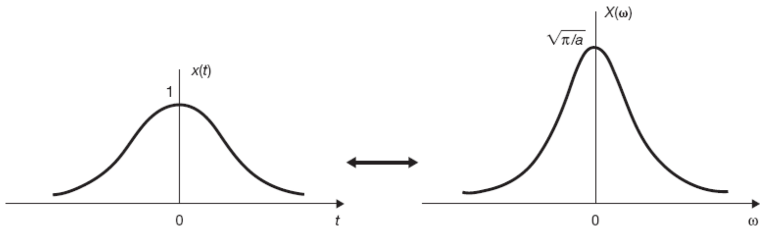
Hence, we have

$$e^{-at^2},\, a > 0 \leftrightarrow \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

(5.165)

Note that the Fourier transform of a Gaussian pulse signal is also a Gaussian pulse in the frequency domain. Fig. 5-26 shows the relationship in Eq. (5.165).

Figure 5-26 *Gaussian pulse and its Fourier transform.*



5.8.3. Frequency Response

5.44. Using the Fourier transform, redo Prob. 2.25.

The system is described by

$$y'(t) + 2y(t) = x(t) + x'(t)$$

Taking the Fourier transforms of the above equation, we get

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega) + j\omega X(\omega)$$

or

$$(j\omega + 2)\, Y(\omega) = (1 + j\omega)\, X(\omega)$$

Hence, by Eq. (5.67) the frequency response $H(\omega)$ is

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 + j\omega}{2 + j\omega} = \frac{2 + j\omega - 1}{2 + j\omega} = 1 - \frac{1}{2 + j\omega}$$

Taking the inverse Fourier transform of $H(\omega)$, the impulse response $h(t)$ is

$$h(t) = \delta(t) - e^{-2t}u(t)$$

Note that the procedure is identical to that of the Laplace transform method with s replaced by $j\omega$ (Prob. 3.29).

5.45. Consider a continuous-time LTI system described by

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

(5.166)

Using the Fourier transform, find the output $y(t)$ to each of the following input signals:

a. $x(t) = e^{-t}u(t)$
b. $x(t) = u(t)$

a. (*a*) Taking the Fourier transforms of Eq. (5.166), we have

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

Hence,

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{1}{2+j\omega}$$

From Eq. (5.155)

$$X(\omega)=\frac{1}{1+j\omega}$$

and

$$Y(\omega)=X(\omega)H(\omega)=\frac{1}{(1+j\omega)(2+j\omega)}=\frac{1}{1+j\omega}-\frac{1}{2+j\omega}$$

Therefore,

$$y(t)=(e^{-t}-e^{-2t})\,u(t)$$

b. From Eq. (5.154)

$$X(\omega)=\pi\delta(\omega)+\frac{1}{j\omega}$$

Thus, by Eq. (5.66) and using the partial-fraction expansion technique, we have

$$\begin{aligned}Y(\omega)=X(\omega)H(\omega)&=\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]\frac{1}{2+j\omega}\\&=\pi\delta(\omega)\frac{1}{2+j\omega}+\frac{1}{j\omega(2+j\omega)}\\&=\frac{\pi}{2}\delta(\omega)+\frac{1}{2}\frac{1}{j\omega}-\frac{1}{2}\frac{1}{2+j\omega}\\&=\frac{1}{2}\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]-\frac{1}{2}\frac{1}{2+j\omega}\end{aligned}$$

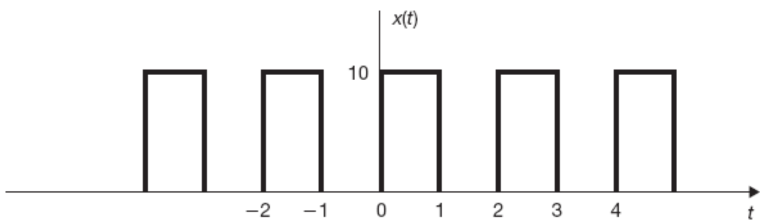
where we used the fact that $f(\omega)\delta(\omega)=f(0)\delta(\omega)$ [Eq. (1.25)]. Thus,

$$y(t)=\frac{1}{2}u(t)-\frac{1}{2}e^{-2t}u(t)=\frac{1}{2}(1-e^{-2t})u(t)$$

We observe that the Laplace transform method is easier in this case because of the Fourier transform of $u(t)$.

5.46. Consider the LTI system in Prob. 5.45. If the input $x(t)$ is the periodic square waveform shown in Fig. 5-27, find the amplitude of the first and third harmonics in the output $y(t)$.

Figure 5-27



Note that $x(t)$ is the same $x(t)$ shown in Fig. 5-8 [Prob. 5.5]. Thus, setting $A = 10$, $T_0 = 2$, and $\omega_0 = 2\pi/T_0 = \pi$ in Eq. (5.106), we have

$$x(t) = 5 + \frac{10}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\pi t}$$

Next, from Prob. 5.45

$$H(\omega) = \frac{1}{2+j\omega} \to H(k\omega_0) = H(k\pi) = \frac{1}{2+jk\pi}$$

Thus, by Eq. (5.74) we obtain

$$y(t) = 5H(0) + \frac{10}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} H[(2m+1)\pi] e^{j(2m+1)\pi t}$$

$$= \frac{5}{2} + \frac{10}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)[2+j(2m+1)\pi]} e^{j(2m+1)\pi t}$$

(5.167)

Let

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

The harmonic form of $y(t)$ is given by [Eq. (5.15)]

$$y(t) = D_0 + \sum_{k=1}^{\infty} D_k \cos(k\omega_0 t - \phi_k)$$

where D_k is the amplitude of the kth harmonic component of $y(t)$. By Eqs. (5.11) and (5.16), D_k and d_k are related by

$$D_k = 2|d_k|$$

(5.168)

Thus, from Eq. (5.167), with $m = 0$, we obtain

$$D_1 = 2|d_1| = 2\left|\frac{10}{j\pi(2+j\pi)}\right| = 1.71$$

With $m = 1$, we obtain

$$D_3 = 2|d_3| = 2\left|\frac{10}{j\pi(3)(2+j3\pi)}\right| = 0.22$$

5.47. The most widely used graphical representation of the frequency response $H(\omega)$ is the *Bode plot* in which the quantities $20 \log_{10}|H(\omega)|$ and $\theta_H(\omega)$ are plotted versus ω, with ω plotted on a logarithmic scale. The quantity $20 \log_{10}|H(\omega)|$ is referred to as the magnitude expressed in *decibels* (dB), denoted by $|H(\omega)|_{dB}$. Sketch the Bode plots for the following frequency responses:

$$(a)\quad H(\omega) = 1 + \frac{j\omega}{10}$$

$$(b)\quad H(\omega) = \frac{1}{1 + j\omega/100}$$

$$(c)\quad H(\omega) = \frac{10^4(1 + j\omega)}{(10 + j\omega)(100 + j\omega)}$$

a. $|H(\omega)|_{\text{dB}} = 20 \log_{10}|H(\omega)| = 20 \log_{10}\left|1 + j\dfrac{\omega}{10}\right|$

For ω << 10,

$$|H(\omega)|_{\text{dB}} = 20 \log_{10}\left|1 + j\frac{\omega}{10}\right| \to 20 \log_{10} 1 = 0 \qquad \text{as } \omega \to 0$$

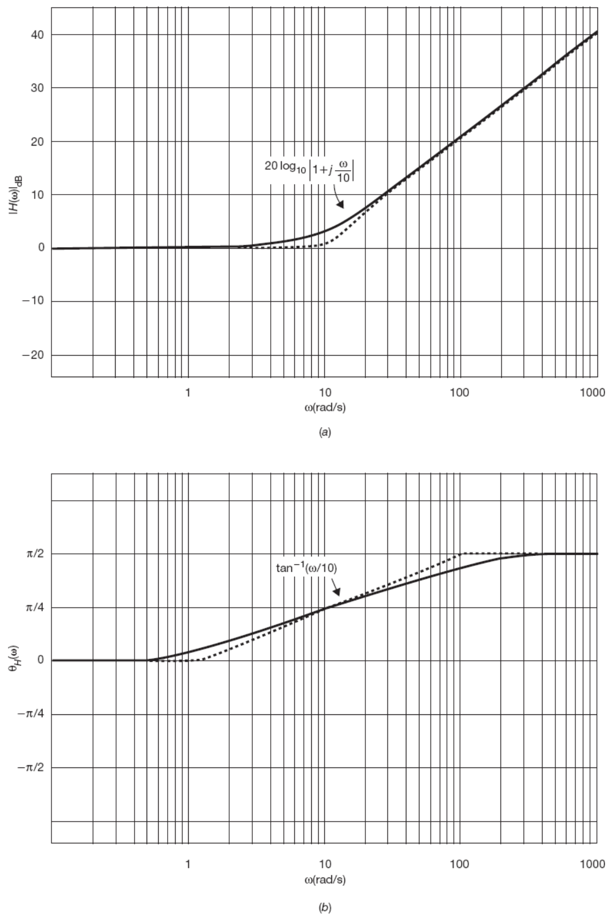
For ω >> 10,

$$|H(\omega)|_{\text{dB}} = 20 \log_{10}\left|1 + j\frac{\omega}{10}\right| \to 20 \log_{10}\left(\frac{\omega}{10}\right) \qquad \text{as } \omega \to 0$$

On a log frequency scale, 20 $\log_{10}(\omega/10)$ is a straight line with a slope of 20 dB/decade (a decade is a 10-to-1 change in frequency). This straight line intersects the 0-dB axis at ω = 10 [Fig. 5-28(*a*)]. (This value of ω is called the *corner frequency*.) At the corner frequency ω = 10

$$H(10)|_{\text{dB}} = 20 \log_{10}|1 + j1| = 20 \log_{10}\sqrt{2} \approx 3 \text{ dB}$$

Figure 5-28



The plot of $|H(\omega)|_{dB}$ is sketched in Fig. 5-28(*a*). Next,

$$\theta_H(\omega) = \tan^{-1}\frac{\omega}{10}$$

Then

$$\theta_H(\omega) = \tan^{-1}\frac{\omega}{10} \to 0 \qquad \text{as } \omega \to 0$$

$$\theta_H(\omega) = \tan^{-1}\frac{\omega}{10} \to \frac{\pi}{2} \qquad \text{as } \omega \to \infty$$

At $\omega = 10$, $\theta_H(10) = \tan^{-1} 1 = \pi/4$ radian (rad). The plot of $\theta_H(\omega)$ is sketched in Fig. 5-28(*b*). Note that the dotted lines represent the straight-line approximation of the Bode plots.

b. $$|H(\omega)|_{dB} = 20\log_{10}\left|\frac{1}{1+j\omega/100}\right| = -20\log_{10}\left|1+j\frac{\omega}{100}\right|$$

For $\omega \ll 100$,

$$|H(\omega)|_{dB} = -20\log_{10}\left|1+j\frac{\omega}{100}\right| \to -20\log_{10}1 = 0 \qquad \text{as } \omega \to 0$$

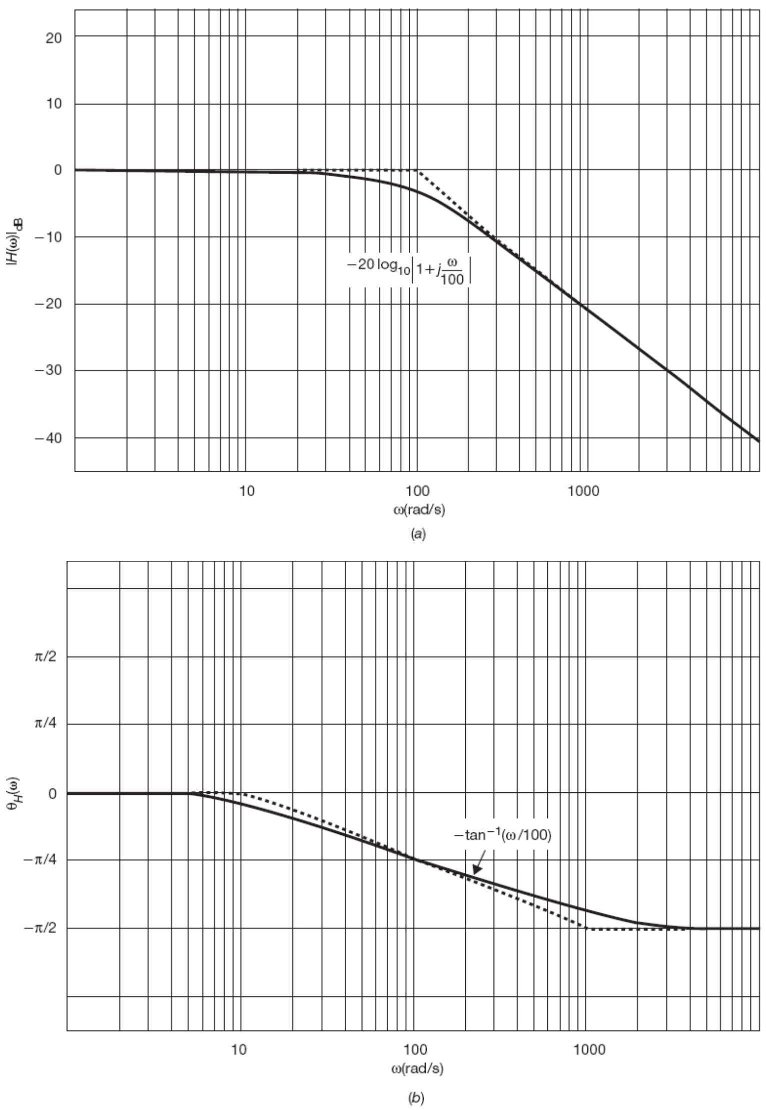
For $\omega \gg 100$,

$$\left|H(\omega)\right|_{\text{dB}} = -20\log_{10}\left|1 - j\frac{\omega}{100}\right| \to -20\log_{10}\left(\frac{\omega}{100}\right) \qquad \text{as } \omega \to \infty$$

On a log frequency scale $-20\log_{10}(\omega/100)$ is a straight line with a slope of -20 dB/decade. This straight line intersects the 0-dB axis at the corner frequency $\omega = 100$ [Fig. 5-29(a)]. At the corner frequency $\omega = 100$

$$H(100)\big|_{\text{dB}} = -20\log_{10}\sqrt{2} \approx -3\text{ dB}$$

Figure 5-29 ***Bode plots.***



The plot of $|H(\omega)|_{\text{dB}}$ is sketched in Fig. 5-29(a). Next,

$$\theta_H(\omega) = -\tan^{-1}\frac{\omega}{100}$$

Then

$$\theta_H(\omega) = -\tan^{-1}\frac{\omega}{100} \to 0 \qquad \text{as } \omega \to 0$$

$$\theta_H(\omega) = -\tan^{-1}\frac{\omega}{100} \to -\frac{\pi}{2} \qquad \text{as } \omega \to \infty$$

At ω = 100, $\theta_H(100) = -\tan^{-1} 1 = -\pi/4$ rad. The plot of $\theta_H(\omega)$ is sketched in Fig. 5-29(*b*).

c. First, we rewrite $H(\omega)$ in standard form as

$$H(\omega) = \frac{10(1+j\omega)}{(1+j\omega/10)(1+j\omega/100)}$$

Then

$$\begin{aligned}|H(\omega)|_{\text{dB}} &= 20 \log_{10} 10 + 20 \log_{10} |1+j\omega| \\ &\quad - 20 \log_{10} \left|1+j\frac{\omega}{10}\right| - 20 \log_{10} \left|1+j\frac{\omega}{100}\right|\end{aligned}$$

Note that there are three corner frequencies, $\omega = 1$, $\omega = 10$, and $\omega = 100$. At corner frequency $\omega = 1$

$$H(1)|_{\text{dB}} = 20 + 20 \log_{10} \sqrt{2} - 20 \log_{10} \sqrt{1.01} - 20 \log_{10} \sqrt{1.0001} \approx 23 \text{ dB}$$

At corner frequency $\omega = 10$

$$H(10)|_{\text{dB}} = 20 + 20 \log_{10} \sqrt{101} - 20 \log_{10} \sqrt{2} - 20 \log_{10} \sqrt{1.01} \approx 37 \text{ dB}$$

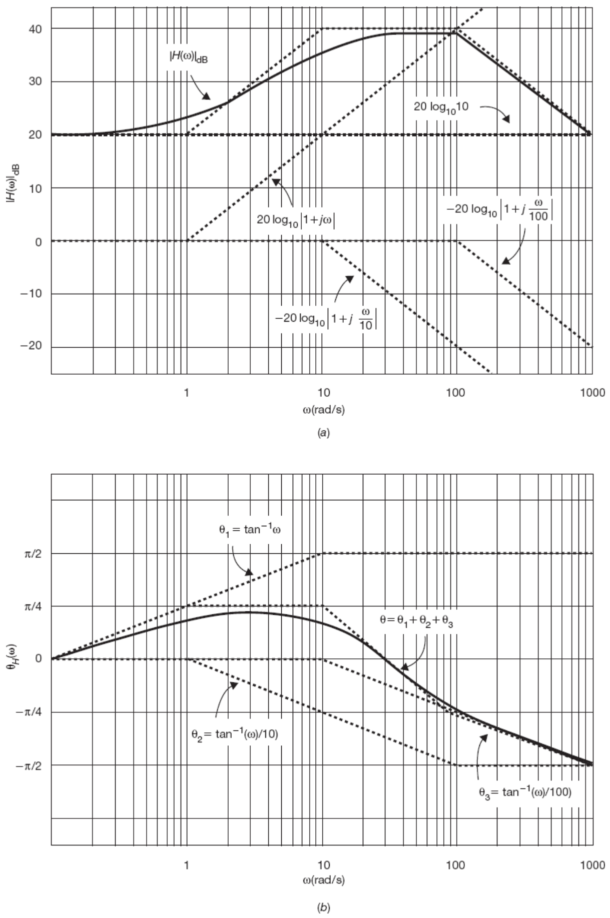
At corner frequency $\omega = 100$

$$H(100)|_{\text{dB}} = 20 + 20 \log_{10} \sqrt{10{,}001} - 20 \log_{10} \sqrt{101} - 20 \log_{10} \sqrt{2} \approx 37 \text{ dB}$$

The Bode amplitude plot is sketched in Fig. 5-30(*a*). Each term contributing to the overall amplitude is also indicated. Next,

$$\theta_H(\omega) = \tan^{-1} \omega - \tan^{-1} \frac{\omega}{10} - \tan^{-1} \frac{\omega}{100}$$

Figure 5-30 **Bode plots.**



Then

$$\theta_H(\omega) = \to 0 - 0 - 0 = 0 \qquad \text{as } \omega \to 0$$

$$\theta_H(\omega) = \to \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2} \qquad \text{as } \omega \to \infty$$

and

$$\theta_H(1) = \tan^{-1}(1) - \tan^{-1}(0.1) - \tan^{-1}(0.01) = 0.676 \text{ rad}$$
$$\theta_H(10) = \tan^{-1}(10) - \tan^{-1}(1) - \tan^{-1}(0.1) = 0.586 \text{ rad}$$
$$\theta_H(100) = \tan^{-1}(100) - \tan^{-1}(10) - \tan^{-1}(1) = -0.696 \text{ rad}$$

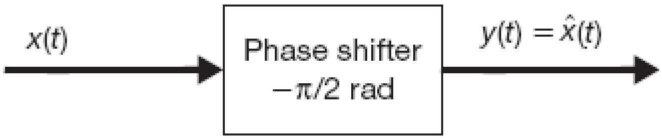
The plot of $\theta_H(\omega)$ is sketched in Fig. 5-30(*b*).

5.48. An ideal $(-\pi/2)$ radian (or – 90°) phase shifter (Fig. 5-31) is defined by the frequency response

$$H(\omega) = \begin{cases} e^{-j(\pi/2)} & \omega > 0 \\ e^{j(\pi/2)} & \omega < 0 \end{cases}$$

(5.169)

Figure 5-31 *−π/2 rad phase shifter.*



a. Find the impulse response $h(t)$ of this phase shifter.

b. Find the output $y(t)$ of this phase shifter due to an arbitrary input $x(t)$.

c. (*c*) Find the output $y(t)$ when $x(t) = \cos \omega_0 t$.

a. (*a*) Since $e^{-j\pi/2} = -j$ and $e^{j\pi/2} = j$, $H(\omega)$ can be rewritten as

$$H(\omega) = -j \operatorname{sgn}(\omega)$$

(5.170)

where

$$\operatorname{sgn}(\omega) = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases}$$

(5.171)

Now from Eq. (5.153)

$$\operatorname{sgn}(t) \leftrightarrow \frac{2}{j\omega}$$

and by the duality property (5.54) we have

$$\frac{2}{jt} \leftrightarrow 2\pi \operatorname{sgn}(-\omega) = -2\pi \operatorname{sgn}(\omega)$$

or

$$\frac{1}{\pi t} \leftrightarrow -j \operatorname{sgn}(\omega)$$

(5.172)

since $\operatorname{sgn}(\omega)$ is an odd function of ω. Thus, the impulse response $h(t)$ is given by

$$h(t) = \mathscr{F}^{-1}[H(\omega)] = \mathscr{F}^{-1}[-j \operatorname{sgn}(\omega)] = \frac{1}{\pi t}$$

(5.173)

b. By Eq. (2.6)

$$y(t) = x(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} \, d\tau$$

(5.174)

The signal $y(t)$ defined by Eq. (5.174) is called the *Hilbert transform* of $x(t)$ and is usually denoted by $\hat{x}(t)$.

c. From Eq. (5.144)

$$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Then

$$\begin{aligned} Y(\omega) = X(\omega)H(\omega) &= \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)][-j\,\mathrm{sgn}(\omega)] \\ &= -j\pi\,\mathrm{sgn}(\omega_0)\delta(\omega-\omega_0) - j\pi\,\mathrm{sgn}(-\omega_0)\delta(\omega+\omega_0) \\ &= -j\pi\delta(\omega-\omega_0) + j\pi\delta(\omega+\omega_0) \end{aligned}$$

since $\mathrm{sgn}(\omega_0) = 1$ and $\mathrm{sgn}(-\omega_0) = -1$. Thus, from Eq. (5.145) we get

$$y(t) = \sin \omega_0 t$$

Note that $\cos(\omega_0 t - \pi/2) = \sin \omega_0 t$.

5.49. Consider a causal continuous-time LTI system with frequency response

$$H(\omega) = A(\omega) + jB(\omega)$$

Show that the impulse response $h(t)$ of the system can be obtained in terms of $A(\omega)$ or $B(\omega)$ alone.

Since the system is causal, by definition

$$h(t) = 0 \qquad t < 0$$

Accordingly,

$$h(-t) = 0 \qquad t > 0$$

Let

$$h(t) = h_e(t) + h_o(t)$$

where $h_e(t)$ and $h_o(t)$ are the even and odd components of $h(t)$, respectively. Then from Eqs. (1.5) and (1.6) we can write

$$h(t) = 2h_e(t) = 2h_o(t)$$

(5.175)

From Eqs. (5.61b) and (5.61c) we have

$$h_e(t) \leftrightarrow A(\omega) \qquad \text{and} \qquad h_o(t) \leftrightarrow jB(\omega)$$

Thus, by Eq. (5.175)

$$h(t) = 2h_e(t) = 2\mathscr{F}^{-1}[A(\omega)] \qquad t > 0$$

(5.176a)

$$h(t) = 2h_o(t) = 2\mathscr{F}^{-1}[jB(\omega)] \qquad t > 0$$

(5.176b)

Equation (5.176a) and (5.176b) indicate that $h(t)$ can be obtained in terms of $A(\omega)$ or $B(\omega)$ alone.

5.50. Consider a causal continuous-time LTI system with frequency response

$$H(\omega) = A(\omega) + jB(\omega)$$

If the impulse response $h(t)$ of the system contains no impulses at the origin, then show that $A(\omega)$ and $B(\omega)$ satisfy the following equation:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{B(\lambda)}{\omega - \lambda}\, d\lambda$$

(5.177a)

$$B(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A(\lambda)}{\omega - \lambda}\, d\lambda$$

(5.177b)

As in Prob. 5.49, let

$$h(t) = h_e(t) + h_o(t)$$

Since $h(t)$ is causal, that is, $h(t) = 0$ for $t < 0$, we have

$$h_e(t) = -h_o(t) \qquad t < 0$$

Also from Eq. (5.175) we have

$$h_e(t) = h_o(t) \qquad t > 0$$

Thus, using Eq. (5.152), we can write

$$h_e(t) = h_o(t)\ \mathrm{sgn}(t)$$

(5.178a)

$$h_o(t) = h_e(t)\ \mathrm{sgn}(t)$$

(5.178b)

Now, from Eqs. (5.61b), (5.61c), and (5.153) we have

$$h_e(t) \leftrightarrow A(\omega) \qquad h_o(t) \leftrightarrow jB(\omega) \qquad \mathrm{sgn}(t) \leftrightarrow \frac{2}{j\omega}$$

Thus, by the frequency convolution theorem (5.59) we obtain

$$A(\omega) = \frac{1}{2\pi} jB(\omega) * \frac{2}{j\omega} = \frac{1}{\pi} B(\omega) * \frac{1}{\omega} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{B(\lambda)}{\omega - \lambda}\, d\lambda$$

and

$$jB(\omega) = \frac{1}{2\pi} A(\omega) * \frac{2}{j\omega} = -j\frac{1}{\pi} A(\omega) * \frac{1}{\omega}$$

or

$$B(\omega) = -\frac{1}{\pi}A(\omega) * \frac{1}{\omega} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A(\lambda)}{\omega-\lambda}d\lambda$$

Note that $A(\omega)$ is the Hilbert transform of $B(\omega)$ [Eq. (5.174)] and that $B(\omega)$ is the negative of the Hilbert transform of $A(\omega)$.

5.51. The real part of the frequency response $H(\omega)$ of a causal LTI system is known to be $\pi\delta(\omega)$. Find the frequency response $H(\omega)$ and the impulse function $h(t)$ of the system.

Let

$$H(\omega) = A(\omega) + jB(\omega)$$

Using Eq. (5.177b), with $A(\omega) = \pi\delta(\omega)$, we obtain

$$B(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\pi\delta(\lambda)}{\omega-\lambda}d\lambda = -\int_{-\infty}^{\infty}\delta(\lambda)\frac{1}{\omega-\lambda}d\lambda = -\frac{1}{\omega}$$

Hence,

$$H(\omega) = \pi\delta(\omega) - j\frac{1}{\omega} = \pi\delta(\omega) + \frac{1}{j\omega}$$

and by Eq. (5.154)

$$h(t) = u(t)$$

5.8.4. Filtering

5.52. Consider an ideal low-pass filter with frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The input to this filter is

$$x(t) = \frac{\sin at}{\pi t}$$

a. Find the output $y(t)$ for $a < \omega_c$.

b. Find the output $y(t)$ for $a > \omega_c$.

c. In which case does the output suffer distortion?

a. From Eq. (5.137) (Prob. 5.20) we have

$$x(t) = \frac{\sin at}{\pi t} \leftrightarrow X(\omega) = p_a(\omega) = \begin{cases} 1 & |\omega| < a \\ 0 & |\omega| > a \end{cases}$$

Then when $a < \omega_c$, we have

$$Y(\omega) = X(\omega)H(\omega) = X(\omega)$$

Thus,

$$y(t)=x(t)=\frac{\sin at}{\pi t}$$

b. When $a > \omega_c$, we have

$$Y(\omega) = X(\omega)H(\omega) = H(\omega)$$

Thus,

$$y(t)=h(t)=\frac{\sin \omega_c t}{\pi t}$$

c. In case (*a*), that is, when $\omega_c > a$, $y(t) = x(t)$ and the filter does not produce any distortion. In case (*b*), that is, when $\omega_c < a$, $y(t) = h(t)$ and the filter produces distortion.

5.53. Consider an ideal low-pass filter with frequency response

$$H(\omega)=\begin{cases}1 & |\omega|<4\pi \\ 0 & |\omega|>4\pi\end{cases}$$

The input to this filter is the periodic square wave shown in Fig. 5-27. Find the output $y(t)$.

Setting $A = 10$, $T_0 = 2$, and $\omega_0 = 2\pi/T_0 = \pi$ in Eq. (5.107) (Prob. 5.5), we get

$$x(t)=5+\frac{20}{\pi}\left(\sin \pi t+\frac{1}{3}\sin 3\pi t+\frac{1}{5}\sin 5\pi t+\cdots\right)$$

Since the cutoff frequency ω_c of the filter is 4π rad, the filter passes all harmonic components of $x(t)$ whose angular frequencies are less than 4π rad and rejects all harmonic components of $x(t)$ whose angular frequencies are greater than 4π rad. Therefore,

$$y(t)=5+\frac{20}{\pi}\sin \pi t+\frac{20}{3\pi}\sin 3\pi t$$

5.54. Consider an ideal low-pass filter with frequency response

$$H(\omega)=\begin{cases}1 & |\omega|<\omega_c \\ 0 & |\omega|>\omega_c\end{cases}$$

The input to this filter is

$$x(t) = e^{-2t}u(t)$$

Find the value of ω_c such that this filter passes exactly one-half of the normalized energy of the input signal $x(t)$.

From Eq. (5.155)

$$X(\omega)=\frac{1}{2+j\omega}$$

Then

$$Y(\omega) = X(\omega)H(\omega) = \begin{cases} \dfrac{1}{2 + j\omega} & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The normalized energy of $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_0^{\infty} e^{-4t} \, dt = \frac{1}{4}$$

Using Parseval's identity (5.64), the normalized energy of $y(t)$ is

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{d\omega}{4 + \omega^2}$$
$$= \frac{1}{\pi} \int_0^{\omega_c} \frac{d\omega}{4 + \omega^2} = \frac{1}{2\pi} \tan^{-1} \frac{\omega_c}{2} = \frac{1}{2} E_x = \frac{1}{8}$$

from which we obtain

$$\frac{\omega_c}{2} = \tan \frac{\pi}{4} = 1 \qquad \text{and} \qquad \omega_c = 2 \text{ rad/s}$$

5.55. The *equivalent bandwidth* of a filter with frequency response $H(\omega)$ is defined by

$$W_{eq} = \frac{1}{|H(\omega)|^2_{\max}} \int_0^{\infty} |H(\omega)|^2 \, d\omega$$

(5.179)

where $|H(\omega)|_{max}$ denotes the maximum value of the magnitude spectrum. Consider the low-pass *RC* filter shown in Fig. 5-6(*a*).

a. Find its 3-dB bandwidth $W_{3\text{ dB}}$.
b. Find its equivalent bandwidth W_{eq}.

a. From Eq. (5.91) the frequency response $H(\omega)$ of the *RC* filter is given by

$$H(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j(\omega/\omega_0)}$$

where $\omega_0 = 1/RC$. Now

$$|H(\omega)| = \frac{1}{[1 + (\omega/\omega_0)^2]^{1/2}}$$

The amplitude spectrum $|H(\omega)|$ is plotted in Fig. 5-6(*b*). When $\omega = \omega_0 = 1/RC$, $|H(\omega_0)| = 1/\sqrt{2}$. Thus, the 3-dB bandwidth of the *RC* filter is given by

$$W_{3\text{dB}} = \omega_0 = \frac{1}{RC}$$

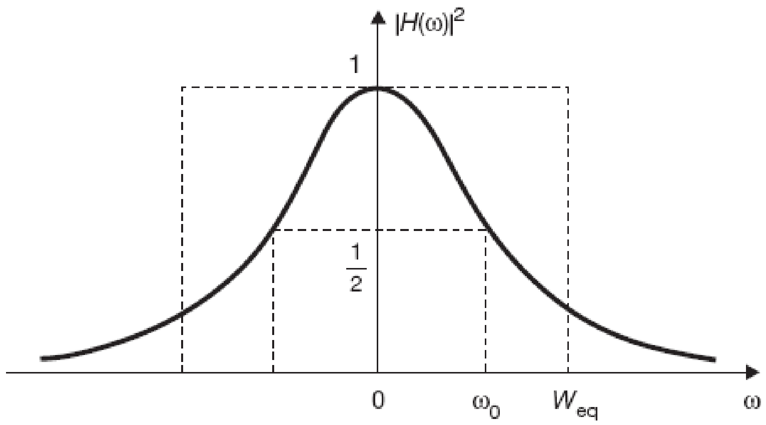
b. From Fig. 5-6(*b*) we see that $|H(0)| = 1$ is the maximum of the magnitude spectrum. Rewriting $H(\omega)$ as

$$H(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{RC}\frac{1}{1/RC + j\omega}$$

and using Eq. (5.179), the equivalent bandwidth of the *RC* filter is given by (Fig. 5-32)

$$W_{\text{eq}} = \frac{1}{(RC)^2}\int_0^\infty \frac{d\omega}{(1/RC)^2 + \omega^2} = \frac{1}{(RC)^2}\frac{\pi}{2/RC} = \frac{\pi}{2RC}$$

Figure 5-32 Filter bandwidth.



5.56. The risetime t_r of the low-pass *RC* filter in Fig. 5-6(*a*) is defined as the time required for a unit step response to go from 10 to 90 percent of its final value. Show that

$$t_r = \frac{0.35}{f_{3\,\text{dB}}}$$

where $f_{3\text{ dB}} = W_{3\text{ dB}}/2\pi = 1/2\pi RC$ is the 3-dB bandwidth (in hertz) of the filter.

From the frequency response $H(\omega)$ of the *RC* filter, the impulse response is

$$h(t) = \frac{1}{RC}e^{-t/RC}\,u(t)$$

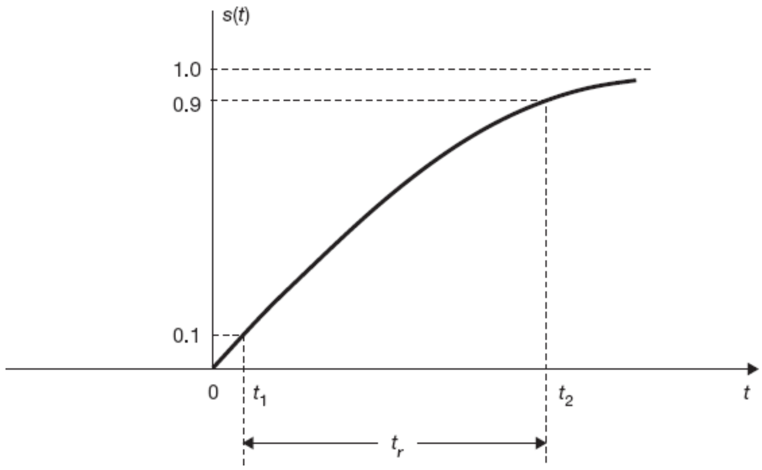
Then, from Eq. (2.12) the unit step response $s(t)$ is found to be

$$s(t) = \int_0^t h(\tau)\,d\tau = \int_0^t \frac{1}{RC}e^{-\tau/RC}\,d\tau = (1 - e^{-t/RC})\,u(t)$$

which is sketched in Fig. 5-33. By definition of the risetime

$$t_r = t_2 - t_1$$

Figure 5-33



where

$$s(t_1) = 1 - e^{-t_1/RC} = 0.1 \rightarrow e^{-t_1/RC} = 0.9$$
$$s(t_2) = 1 - e^{-t_2/RC} = 0.9 \rightarrow e^{-t_2/RC} = 0.1$$

Dividing the first equation by the second equation on the right-hand side, we obtain

$$e^{(t_2 - t_1)/RC} = 9$$

and

$$t_r = t_2 - t_1 = RC\ln(9) = 2.197RC = \frac{2.197}{2\pi f_{3\,\mathrm{dB}}} = \frac{0.35}{f_{3\,\mathrm{dB}}}$$

which indicates the inverse relationship between bandwidth and risetime.

5.57. Another definition of bandwidth for a signal $x(t)$ is the 90 percent *energy containment bandwidth* W_{90}, defined by

$$\frac{1}{2\pi}\int_{-W_{90}}^{W_{90}} |X(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{W_{90}} |X(\omega)|^2 d\omega = 0.9E_x$$

(5.180)

where E_x is the normalized energy content of signal $x(t)$. Find the W_{90} for the following signals:

$$(a)\quad x(t) = e^{-at}u(t),\ a > 0$$
$$(b)\quad x(t) = \frac{\sin at}{\pi t}$$

a. From Eq. (5.155)

$$x(t) = e^{-at}u(t) \leftrightarrow X(\omega) = \frac{1}{a + j\omega}$$

From Eq. (1.14)

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2at} dt = \frac{1}{2a}$$

Now, by Eq. (5.180)

$$\frac{1}{\pi}\int_0^{W_{90}}|X(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{W_{90}}\frac{d\omega}{a^2+\omega^2} = \frac{1}{a\pi}\tan^{-1}\left(\frac{W_{90}}{a}\right) = 0.9\frac{1}{2a}$$

from which we get

$$\tan^{-1}\left(\frac{W_{90}}{a}\right) = 0.45\pi$$

Thus,

$$W_{90} = a\tan(0.45\pi) = 6.31a \qquad \text{rad/s}$$

b. From Eq. (5.137)

$$x(t) = \frac{\sin at}{\pi t} \leftrightarrow X(\omega) = p_a(\omega) = \begin{cases} 1 & |\omega| < a \\ 0 & |\omega| > a \end{cases}$$

Using Parseval's identity (5.64), we have

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 dt = \frac{1}{\pi}\int_0^{\infty}|X(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^a d\omega = \frac{a}{\pi}$$

Then, by Eq. (5.180)

$$\frac{1}{\pi}\int_0^{W_{90}}|X(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{W_{90}} d\omega = \frac{W_{90}}{\pi} = 0.9\frac{a}{\pi}$$

from which we get

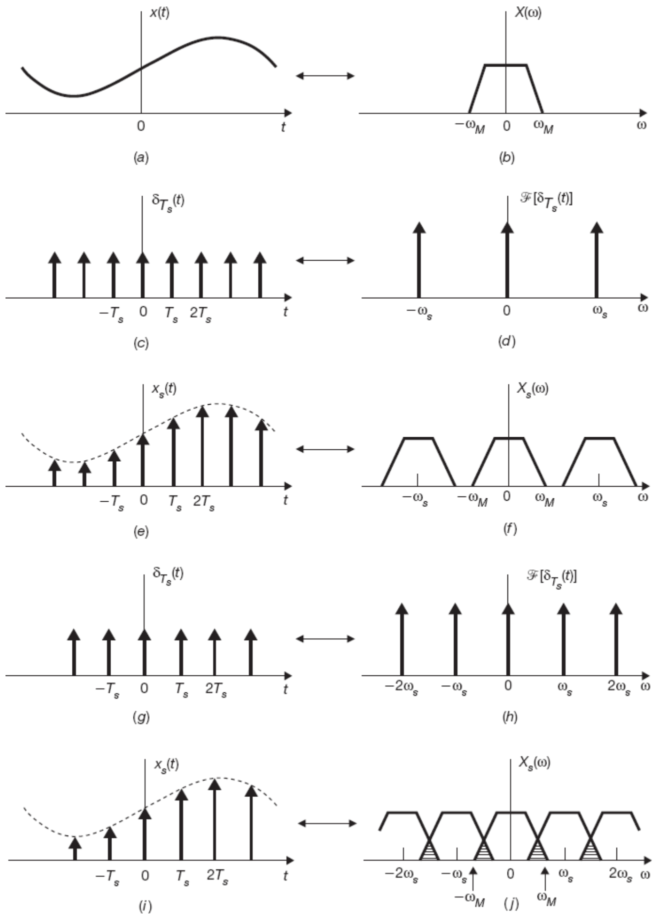
$$W_{90} = 0.9a \qquad \text{rad/s}$$

Note that the absolute bandwidth of $x(t)$ is a (radians/second).

5.58. Let $x(t)$ be a real-valued band-limited signal specified by [Fig. 5-34(b)]

$$X(\omega) = 0 \qquad |\omega| > \omega_M$$

Figure 5-34 *Ideal sampling.*



Let $x_s(t)$ be defined by

$$x_s(t) = x(t)\delta_{T_s}(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

(5.181)

a. Sketch $x_s(t)$ for $T_s < \pi/\omega_M$ and for $T_s > \pi/\omega_M$.
b. Find and sketch the Fourier spectrum $X_s(\omega)$ of $x_s(t)$ for $T_s < \pi/\omega_M$ and for $T_s > \pi/\omega_M$.

a. (*a*) Using Eq. (1.26), we have

$$x_s(t) = x(t)\,\delta_{T_s}(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$= \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT_s) = \sum_{k=-\infty}^{\infty} x(kT_s)\,\delta\,(t - kT_s)$$

(5.182)

The sampled signal $x_s(t)$ is sketched in Fig. 5-34(*c*) for $T_s < \pi/\omega_M$, and in Fig. 5-34(*i*) for $T_s > \pi/\omega_M$.

The signal $x_s(t)$ is called the *ideal sampled signal*, T_s is referred to as the *sampling interval* (or *period*), and $f_s = 1/T_s$ is referred to as the *sampling rate* (or *frequency*).

b. From Eq. (5.147) (Prob. 5.25) we have

$$\delta_{T_s}(t) \leftrightarrow \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \qquad \omega_s = \frac{2\pi}{T_s}$$

Let

$$x_s(t) \leftrightarrow X_s(\omega)$$

Then, according to the frequency convolution theorem (5.59), we have

$$X_s(\omega) = \mathscr{F}[x(t)\delta_{T_s}(t)] = \frac{1}{2\pi}\left[X(\omega) * \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\right]$$
$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_s)$$

Using Eq. (1.26), we obtain

$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

(5.183)

which shows that $X_s(\omega)$ consists of periodically repeated replicas of $X(\omega)$ centered about $k\omega_s$ for all k. The Fourier spectrum $X_s(\omega)$ is shown in Fig. 5-34(*f*) for $T_s < \pi/\omega_M$ (or $\omega_s > 2\omega_M$), and in Fig. 5-34(*j*) for $T_s > \pi/\omega_M$ (or $\omega_s < 2\omega_M$), where $\omega_s = 2\pi/T_s$. It is seen that no overlap of the replicas $X(\omega - k\omega_s)$ occurs in $X_s(\omega)$ for $\omega_s \geq 2\omega_M$ and that overlap of the spectral replicas is produced for $\omega_s < 2\omega_M$. This effect is known as *aliasing*.

5.59. Let $x(t)$ be a real-valued band-limited signal specified by

$$X(\omega) = 0 \qquad |\omega| > \omega_M$$

Show that $x(t)$ can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \frac{\sin \omega_M (t - kT_s)}{\omega_M (t - kT_s)}$$

(5.184)

where $T_s = \pi/\omega_M$.

Let

$$x(t) \leftrightarrow X(\omega)$$
$$x_s(t) = x(t)\delta_{T_s}(t) \leftrightarrow X_s(\omega)$$

From Eq. (5.183) we have

$$T_s X_s(\omega) = \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

(5.185)

Then, under the following two conditions,

$$(1)\quad X(\omega)=0, |\omega|>\omega_M \qquad \text{and} \qquad (2)\quad T_s=\frac{\pi}{\omega_M}$$

we see from Eq. (5.185) that

$$X(\omega)=\frac{\pi}{\omega_M}X_s(\omega) \qquad |\omega|<\omega_M$$

(5.186)

Next, taking the Fourier transform of Eq. (5.182), we have

$$X_s(\omega)=\sum_{k=-\infty}^{\infty} x(kT_s)\, e^{-jkT_s\omega}$$

(5.187)

Substituting Eq. (5.187) into Eq. (5.186), we obtain

$$X(\omega)=\frac{\pi}{\omega_M}\sum_{k=-\infty}^{\infty} x(kT_s)e^{-jkT_s\omega} \qquad |\omega|<\omega_M$$

(5.188)

Taking the inverse Fourier transform of Eq. (5.188), we get

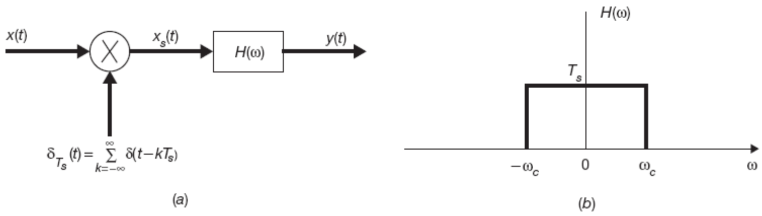
$$\begin{aligned}
x(t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega \\
&=\frac{1}{2\omega_M}\int_{-\omega_M}^{\omega_M}\sum_{k=-\infty}^{\infty} x(kT_s)e^{j\omega(t-kT_s)}\, d\omega \\
&=\sum_{k=-\infty}^{\infty} x(kT_s)\frac{1}{2\omega_M}\int_{-\omega_M}^{\omega_M} e^{j\omega(t-kT_s)}\, d\omega \\
&=\sum_{k=-\infty}^{\infty} x(kT_s)\frac{\sin \omega_M(t-kT_s)}{\omega_M(t-kT_s)}
\end{aligned}$$

From Probs. 5.58 and 5.59 we conclude that a band-limited signal which has no frequency components higher than f_M hertz can be recovered completely from a set of samples taken at the rate of f_s ($\geq 2f_M$) samples per second. This is known as the *uniform sampling theorem* for low-pass signals. We refer to $T_s = \pi/\omega_M = 1/2f_M$ ($\omega_M = 2\pi f_M$) as the *Nyquist sampling interval* and $f_s = 1/T_s = 2f_M$ as the *Nyquist sampling rate.*

5.60. Consider the system shown in Fig. 5-35(*a*). The frequency response $H(\omega)$ of the ideal low-pass filter is given by [Fig. 5-35(*b*)]

$$H(\omega)=T_s p_{\omega_c}(\omega)=\begin{cases} T_s & |\omega|<\omega_c \\ 0 & |\omega|>\omega_c \end{cases}$$

Figure 5-35



Show that if $\omega_c = \omega_s/2$, then for any choice of T_s,

$$y(mT_s) = x(mT_s) \qquad m = 0, \pm 1, \pm 2, \ldots$$

From Eq. (5.137) the impulse response $h(t)$ of the ideal low-pass filter is given by

$$h(t) = T_s \frac{\sin \omega_c t}{\pi t} = \frac{T_s \omega_c}{\pi} \frac{\sin \omega_c t}{\omega_c t}$$

(5.189)

From Eq. (5.182) we have

$$x_s(t) = x(t)\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t-kT_s)$$

By Eq. (2.6) and using Eqs. (2.7) and (1.26), the output $y(t)$ is given by

$$\begin{aligned} y(t) = x_s(t) * h(t) &= \left[\sum_{k=-\infty}^{\infty} x(kT_s)\delta(t-kT_s)\right] * h(t) \\ &= \sum_{k=-\infty}^{\infty} x(kT_s)[h(t) * \delta(t-kT_s)] \\ &= \sum_{k=-\infty}^{\infty} x(kT_s) h(t-kT_s) \end{aligned}$$

Using Eq. (5.189), we get

$$y(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \frac{T_s \omega_c}{\pi} \frac{\sin \omega_c (t-kT_s)}{\omega_c (t-kT_s)}$$

If $\omega_c = \omega_s/2$, then $T_s\omega_c/\pi = 1$ and we have

$$y(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \frac{\sin[\omega_s(t-kT_s)/2]}{\omega_s(t-kT_s)/2}$$

Setting $t = mT_s$ (m = integer) and using the fact that $\omega_s T_s = 2\pi$, we get

$$y(mT_s) = \sum_{k=-\infty}^{\infty} x(kT_s) \frac{\sin \pi(m-k)}{\pi(m-k)}$$

Since

$$\frac{\sin \pi(m-k)}{\pi(m-k)} = \begin{cases} 0 & m \neq k \\ 0 & m = k \end{cases}$$

we have

$$y(mT_s) = x(mT_s) \qquad m = 0, \pm 1, \pm 2, \ldots$$

which shows that without any restriction on $x(t)$, $y(mT_s) = x(mT_s)$ for any integer value of m.

Note from the sampling theorem (Probs. 5.58 and 5.59) that if $\omega_s = 2\pi/T_s$ is greater than twice the highest frequency present in $x(t)$ and $\omega_c = \omega_s/2$, then $y(t) = x(t)$. If this condition on the bandwidth of $x(t)$ is not satisfied, then $y(t) \neq x(t)$. However, if $\omega_c = \omega_s/2$, then $y(mT_s) = x(mT_s)$ for any integer value of m.

5.9. SUPPLEMENTARY PROBLEMS

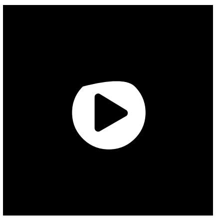
5.61. Consider a rectified sine wave signal $x(t)$ defined by

$$x(t) = |A \sin \pi t|$$

a. Sketch $x(t)$ and find its fundamental period.

b. Find the complex exponential Fourier series of $x(t)$.

c. Find the trigonometric Fourier series of $x(t)$.



Schaum's Signals and Systems Supplementary Problem 5.61: Fourier Series

This video illustrates how to find the trigonometric and complex exponential Fourier Series of a continuous time signal.

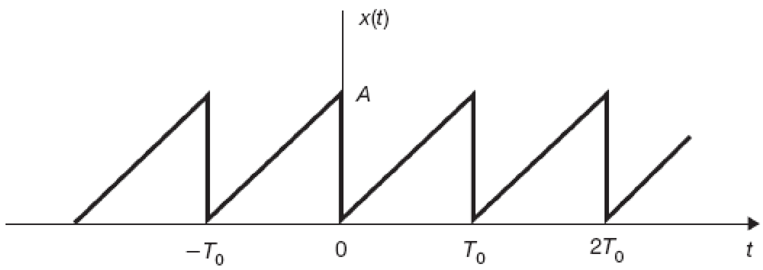
Carlotta A. Berry, Associate Professor, Electrical and Computer Engineering, Rose-Hulman Institute of Technology
2013

5.62. Find the trigonometric Fourier series of a periodic signal $x(t)$ defined by

$$x(t) = t^2, -\pi < t < \pi \qquad \text{and} \qquad x(t + 2\pi) = x(t)$$

5.63. Using the result from Prob. 5.10, find the trigonometric Fourier series of the signal $x(t)$ shown in Fig. 5-36.

Figure 5-36



5.64. Derive the harmonic form Fourier series representation (5.15) from the trigonometric Fourier series representation (5.8).

5.65. Show that the mean-square value of a real periodic signal $x(t)$ is the sum of the mean-square values of its harmonics.

5.66. Show that if

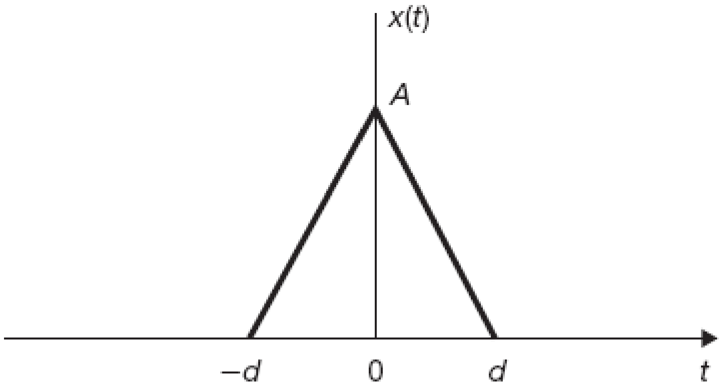
$$x(t) \leftrightarrow X(\omega)$$

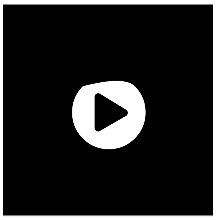
then

$$x^{(n)}(t) = \frac{d^n x(t)}{dt^n} \leftrightarrow (j\omega)^n X(\omega)$$

5.67. Using the differentiation technique, find the Fourier transform of the triangular pulse signal shown in Fig. 5-37.

Figure 5-37





Schaum's Signals and Systems Supplementary Problem 5.67/5.69: Fourier Transforms

This video illustrated how to find the Fourier transform and inverse Fourier transform of a continuous time signal.

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5.68. Find the inverse Fourier transform of

$$X(\omega) = \frac{1}{(a + j\omega)^N}$$

5.69. Find the inverse Fourier transform of

$$X(\omega) = \frac{1}{2 - \omega^2 + j3\omega}$$

5.70. Verify the frequency differentiation property (5.56); that is,

$$(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega}$$

5.71. Find the Fourier transform of each of the following signals:

a. $x(t) = \cos \omega_0 t u(t)$

b. $x(t) = \sin \omega_0 t u(t)$

c. $x(t) = e^{-at} \cos \omega_0 t u(t)$, $a > 0$

d. $x(t) = e^{-at} \sin \omega_0 t u(t)$, $a > 0$

5.72. Let $x(t)$ be a signal with Fourier transform $X(\omega)$ given by

$$X(\omega) = \begin{cases} 1 & |\omega| < 1 \\ 0 & |\omega| > 1 \end{cases}$$

Consider the signal

$$y(t) = \frac{d^2 x(t)}{dt^2}$$

Find the value of

$$\int_{-\infty}^{\infty} |y(t)|^2 \, dt$$

5.73. Let $x(t)$ be a real signal with the Fourier transform $X(\omega)$. The *analytical signal* $x_+(t)$ associated with $x(t)$ is a complex signal defined by

$$x_+(t) = x(t) + j\hat{x}(t)$$

where $\hat{x}(t)$ is the Hilbert transform of $x(t)$.

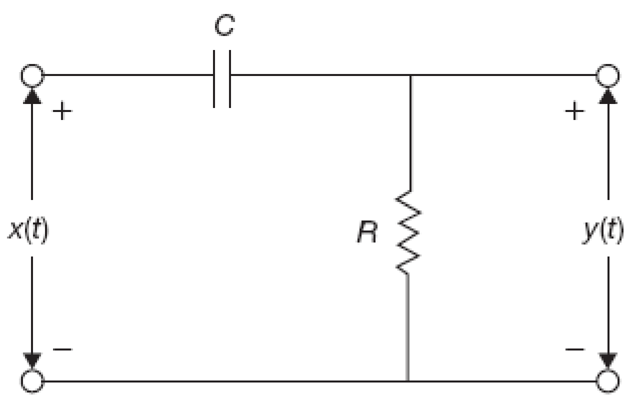
a. Find the Fourier transform $X_+(\omega)$ of $x_+(t)$.

b. Find the analytical signal $x_+(t)$ associated with $\cos \omega_0 t$ and its Fourier transform $X_+(\omega)$.

5.74. Consider a continuous-time LTI system with frequency response $H(\omega)$. Find the Fourier transform $S(\omega)$ of the unit step response $s(t)$ of the system.

5.75. Consider the *RC* filter shown in Fig. 5-38. Find the frequency response $H(\omega)$ of this filter and discuss the type of filter.

Figure 5-38



Schaum's Signals and Systems Supplementary Problem 5.75: RC Filters

This video demonstrates how to use Fourier transforms to find the frequency response, H(w) and the type of filter.

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5.76. Determine the 99 percent energy containment bandwidth for the signal

$$x(t)=\frac{1}{t^2+a^2}$$

5.77. The *sampling theorem in the frequency domain* states that if a real signal $x(t)$ is a duration-limited signal, that is,

$$x(t)=0 \qquad |t|>t_M$$

then its Fourier transform $X(\omega)$ can be uniquely determined from its values $X(n\pi/t_M)$ at a series of equidistant points spaced π/t_M apart. In fact, $X(\omega)$ is given by

$$X(\omega)=\sum_{n=-\infty}^{\infty} X\left(\frac{n\pi}{t_M}\right)\frac{\sin(\omega t_M-n\pi)}{\omega t_M-n\pi}$$

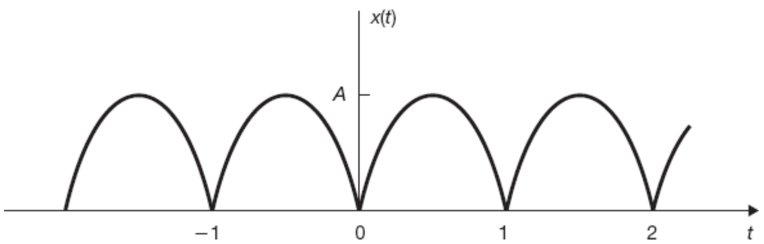
Verify the above sampling theorem in the frequency domain.

5.10. ANSWERS TO SUPPLEMENTARY PROBLEMS

5.61.

a. $X(t)$ is sketched in Fig. 5-39 and $T_0 = 1$.

Figure 5-39



b. $x(t)=-\frac{2A}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{4k^2-1}e^{jk2\pi t}$

c. $x(t)=\frac{2A}{\pi}-\frac{4A}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos k2\pi t$

5.62. $x(t)=\frac{\pi^2}{3}+4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kt$

5.63. $x(t)=\frac{A}{2}-\frac{A}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin k\omega_0 t \qquad \omega_0=\frac{2\pi}{T_0}$

5.64. *Hint:* Rewrite $a_k \cos k\omega_0 t + b_k \sin k\omega_0 t$ as

$$\sqrt{a_k^2+b_k^2}\left[\frac{a_k}{(a_k^2+b_k^2)^{1/2}}\cos k\omega_0 t+\frac{b_k}{(a_k^2+b_k^2)^{1/2}}\sin k\omega_0 t\right]$$

and use the trigonometric formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

5.65. *Hint:* Use Parseval's identity (5.21) for the Fourier series and Eq. (5.168).

5.66. *Hint:* Repeat the time-differentiation property (5.55).

5.67. $Ad\left[\dfrac{\sin(\omega d/2)}{\omega d/2}\right]^2$

5.68. *Hint:* Differentiate Eq. (5.155) N times with respect to (a).

$$\frac{t^{N-1}}{(N-1)!}e^{-at}u(t)$$

5.69. *Hint:* Note that

$$2-\omega^2+j3\omega=2+(j\omega)^2+j3\omega=(1+j\omega)(2+j\omega)$$

and apply the technique of partial-fraction expansion.

$x(t) = (e^{-t} - e^{-2t})u(t)$

5.70. *Hint:* Use definition (5.31) and proceed in a manner similar to Prob. 5.28.

5.71. *Hint:* Use multiplication property (5.59).

$$(a)\quad X(\omega)=\frac{\pi}{2}\delta(\omega-\omega_0)+\frac{\pi}{2}\delta(\omega+\omega_0)+\frac{j\omega}{(j\omega)^2+\omega_0^2}$$

$$(b)\quad X(\omega)=\frac{\pi}{2j}\delta(\omega-\omega_0)-\frac{\pi}{2j}\delta(\omega+\omega_0)+\frac{\omega_0}{(j\omega)^2+\omega_0^2}$$

$$(c)\quad X(\omega)=\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$$

$$(d)\quad X(\omega)=\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$$

5.72. *Hint:* Use Parseval's identity (5.64) for the Fourier transform.

$1/3\pi$

5.73. $(a)\quad X_+(\omega)=2X(\omega)u(\omega)=\begin{cases}2X(\omega) & \omega>0\\ 0 & \omega<0\end{cases}$

$$(b)\quad x_+(t)=e^{j\omega_0 t},\quad X_+(\omega)=2\pi\,\delta(\omega-\omega_0)$$

5.74. *Hint:* Use Eq. (2.12) and the integration property (5.57).

$S(\omega) = \pi H(0)\delta(\omega) + (1/j\omega)\,H(\omega)$

5.75. $H(\omega) = \dfrac{j\omega}{(1/RC) + j\omega}$, high-pass filter

5.76. $W_{99} = 2.3/a$ radians/second or $f_{99} = 0.366/a$ hertz

5.77. *Hint:* Expand $x(t)$ in a complex Fourier series and proceed in a manner similar to that for Prob. 5.59.