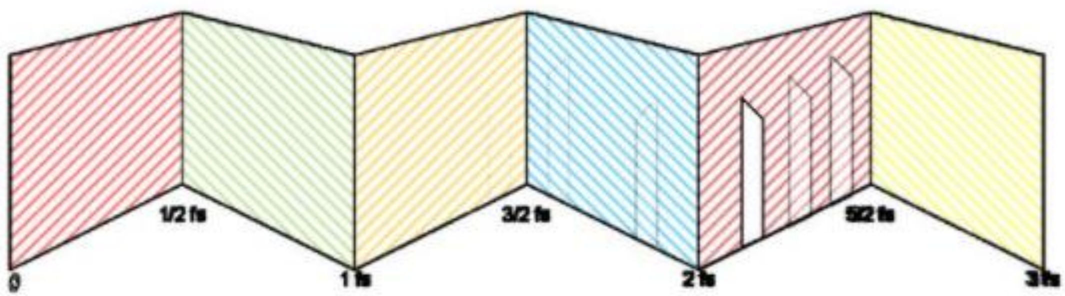
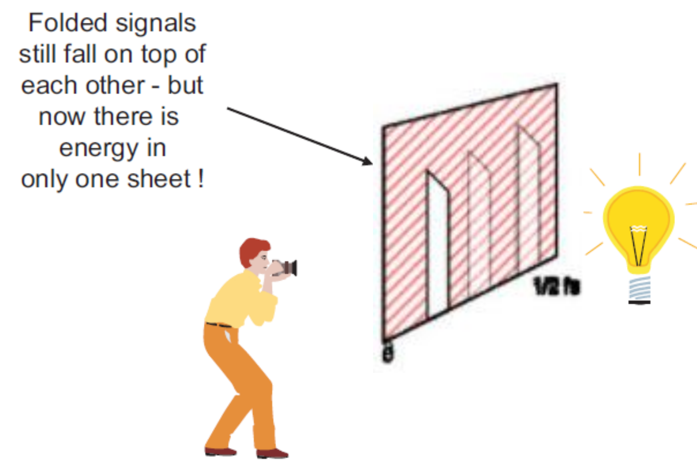
Digital Systems for Signal Processing

Interpolation and Decimation

Nyquist Zones

- The process of sampling a signal outside the first Nyquist zone is often referred to as undersampling, or harmonic sampling.
  - Note that the image, which falls in the first Nyquist zone, contains all the information in the original signal with the exception of its original location.

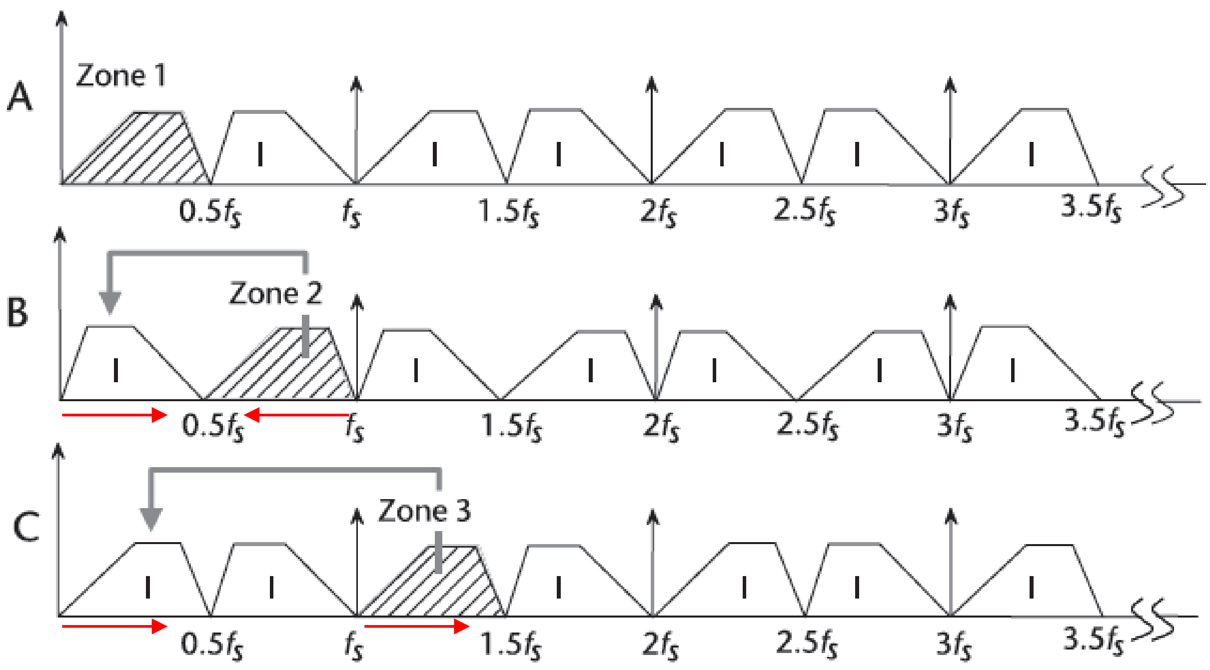




We can clearly restate the Nyquist criteria as: *“A signal must be sampled at a rate equal to or greater than twice its bandwidth in order to preserve all the signal information”*.

Nyquist Zones

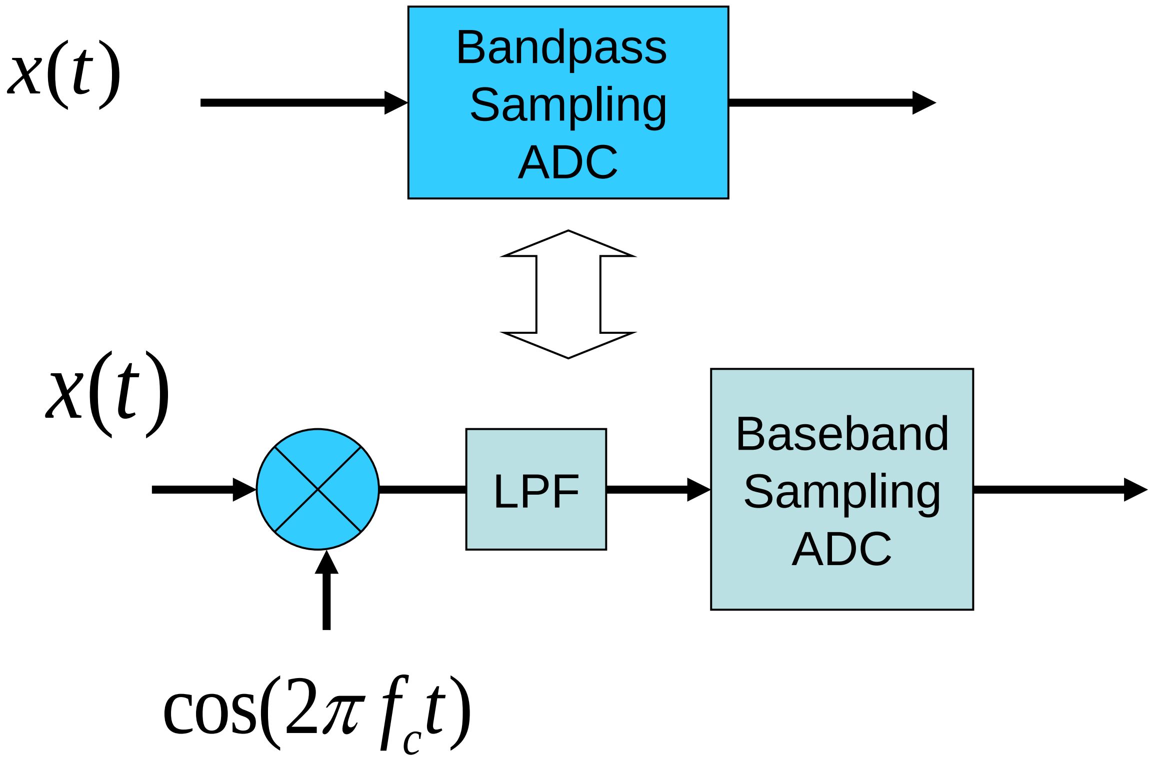
- Sampling signals above the first Nyquist zone has become popular in communications because the process is equivalent to analog mixing.
- An ADC suitable for undersampling applications must maintain dynamic performance into the higher order Nyquist zones.



Undersampling avoid problem of DC correction, i.e. HPF in order to remove the LO leakage down mixed near the DC component

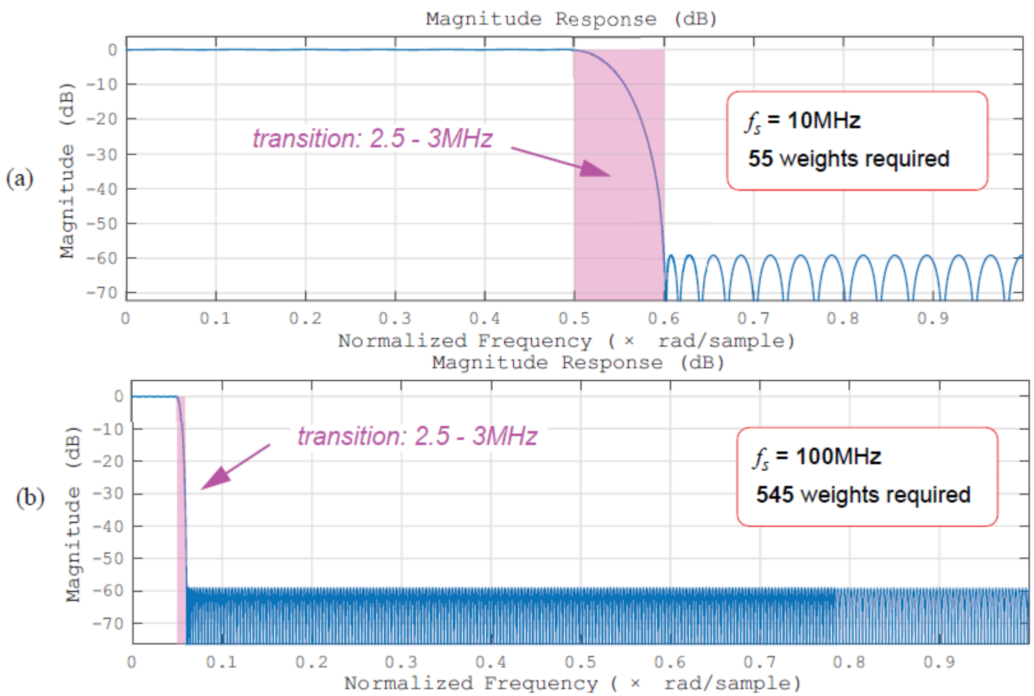
Undersampling

- A Undersampling ADC (aka Band Pass Sampling ADC) works like a Mixer and a Baseband-Sampling ADC



Sample Rate Changes: Decimation and Interpolation

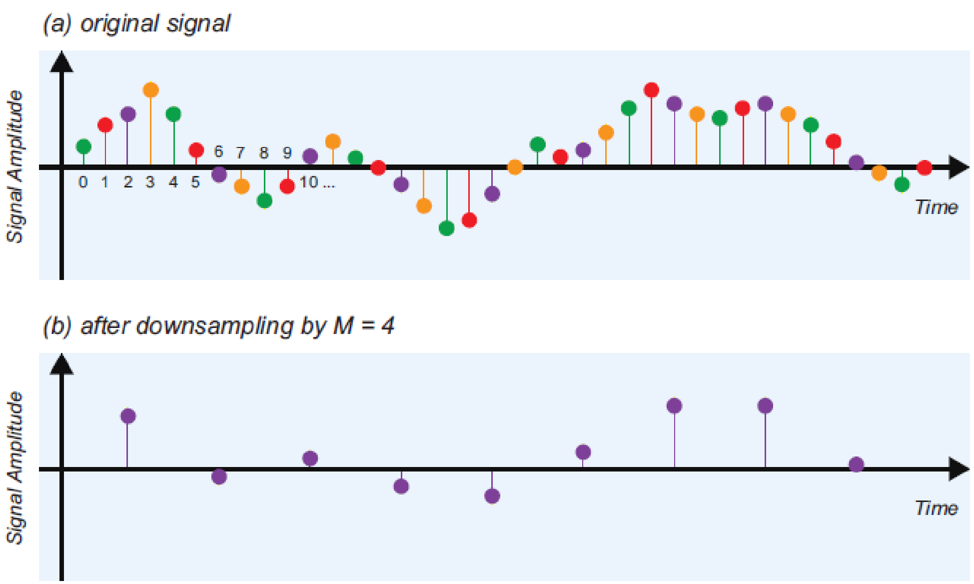
- Choosing a sampling rate in excess of that required by Nyquist (2fmax) will still of course be valid, but:
  - Processing samples at an unnecessarily high rate means the computational requirements are inflated.
  - Any given response over a defined bandwidth can be achieved with fewer weights, if the sampling rate is lower.



- The two LPF have a transition band extending from 2.5-3MHz, and identical specifications for passband ripple and stopband attenuation.
- The important point is that the transition bandwidth is relative to the sampling rate.

Sample Rate Changes: Decimation and Interpolation

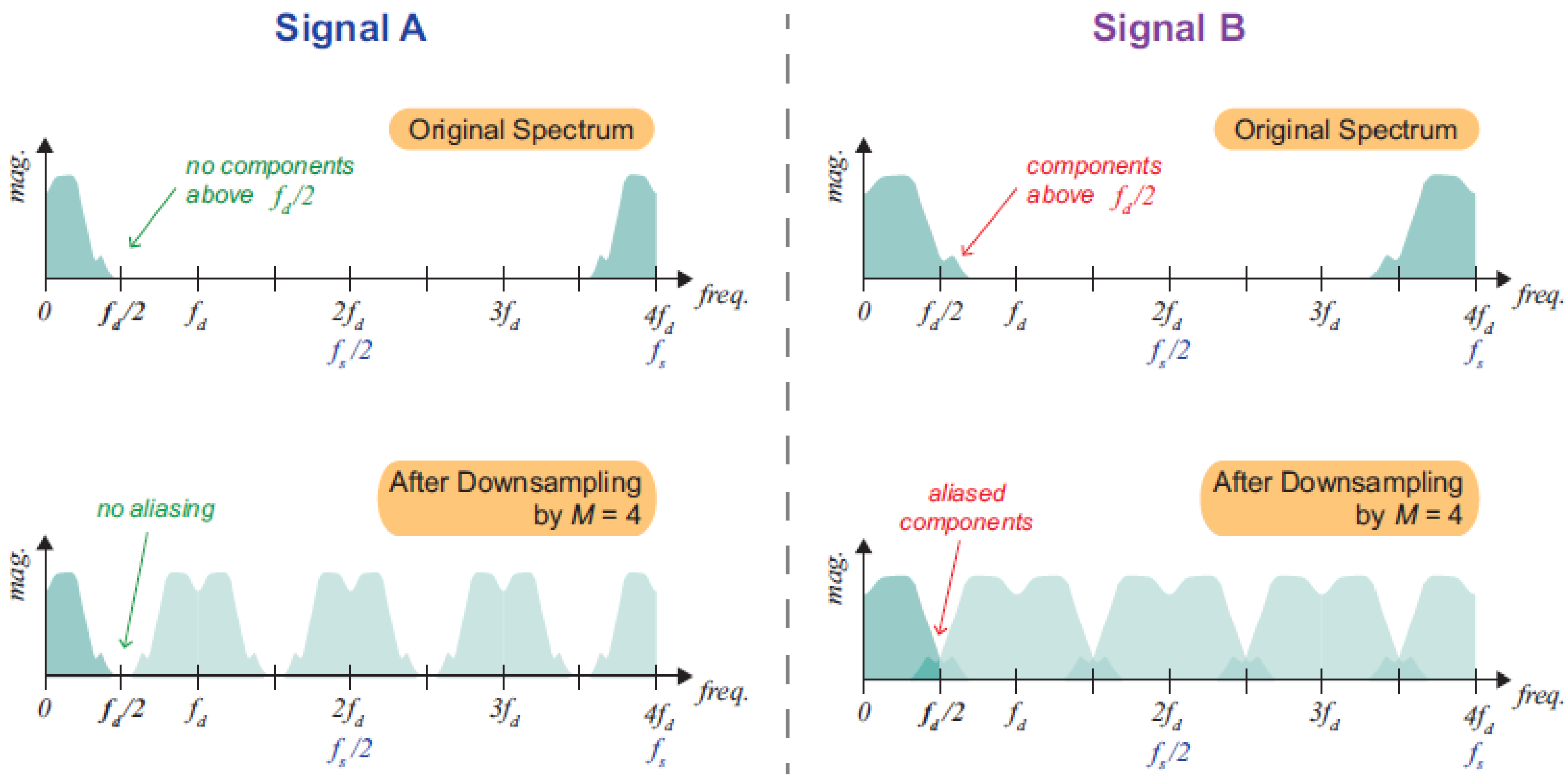
- Decimation is the process of reducing the sampling rate of a signal.
  - comprises two operations: (i) *downsampling* and (ii) lowpass filtering
- Downsampling by the factor *M* effectively means that only every *M*th sample from the original set is retained, while the intermediate samples are discarded.
  - The phase of downsampling is significant, because there are *M* possible phases of samples that could be retained.



In the frequency domain, we must be careful that the new rate continues to fulfil the requirements of Nyquist sampling; in other words, that the new sampling rate is still in excess of twice the signal bandwidth.

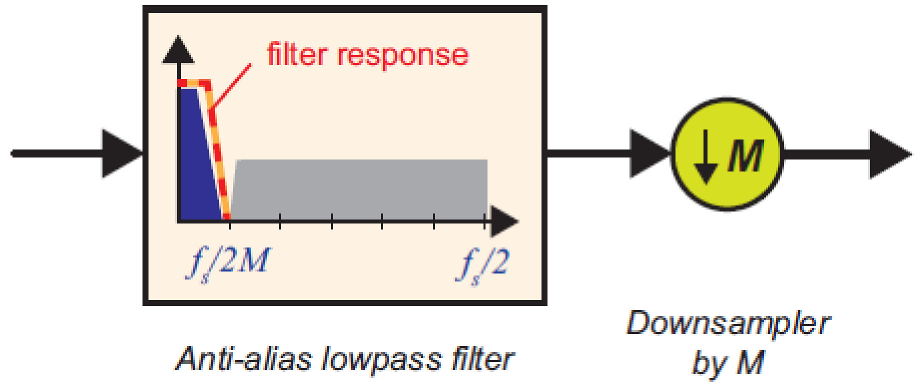
Sample Rate Changes: Decimation and Interpolation

- Let's consider two example signals (A and B), whose sampling rate has been reduced by a factor of 4 from *fs* to *fd*.
  - spectral repetitions occur at integer multiples of the sampling rate.



Sample Rate Changes: Decimation and Interpolation

- The signal must be appropriately bandlimited, prior to downsampling!
  - apply an ‘antialias’ LPF, cutting off at *fd/2* (equivalent to *fs/2M*).



- Downsampling: M-fold stretched version of the resampled spectrum

$$x_r(m) = x(m)\cdot p(m) = \sum_{k=-\infty}^{\infty} x(k)\delta(m-Mk) = \begin{cases} x(Mm), m = 0, \pm 1, \pm 2, \ldots \\ 0, otherwise \end{cases}$$

$$p(m) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-j2\pi km/M}$$

$$x_d(m) = x_r(mM)$$

$$X_d(f) = \sum_{m=-\infty}^{\infty} x_d(m) e^{-j2\pi fm}$$

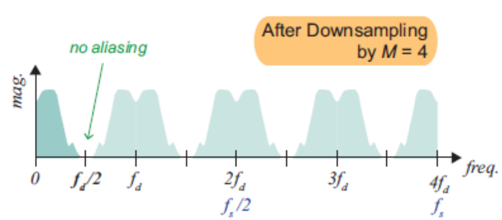
$$= \sum_{m=-\infty}^{\infty} x_r(mM) e^{-j2\pi \frac{f}{M} mM}$$

$$= \sum_{m=-\infty}^{\infty} x_r(m') e^{-j2\pi \frac{f}{M} m'},$$

$$with\ m' = mM$$

$$= X_r\left(\frac{f}{M}\right)$$

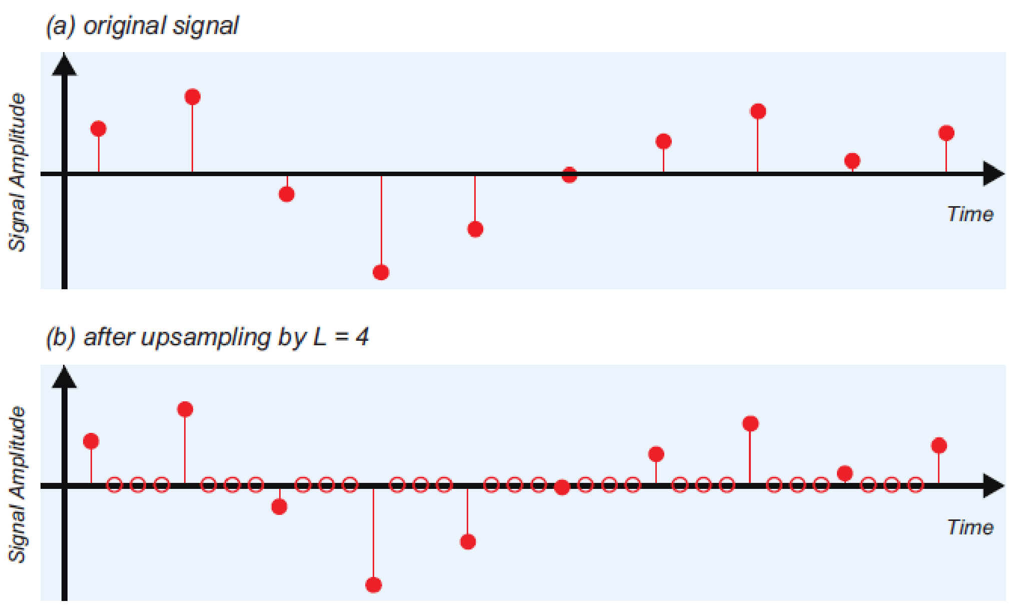
$$X_r(f) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(f + \frac{k}{M}\right)$$



π ← π

Sample Rate Changes: Decimation and Interpolation

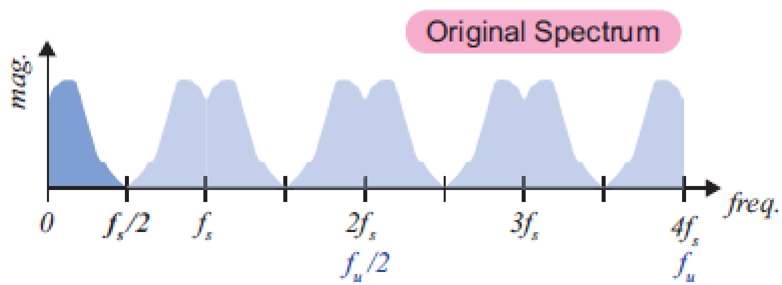
- Interpolation is the process of augmenting the sampling.
  - comprises two operations: (i) *upsampling* and (ii) lowpass filtering
- Upsampling by the factor *L* involves inserting *L-1* zeros between the original samples.

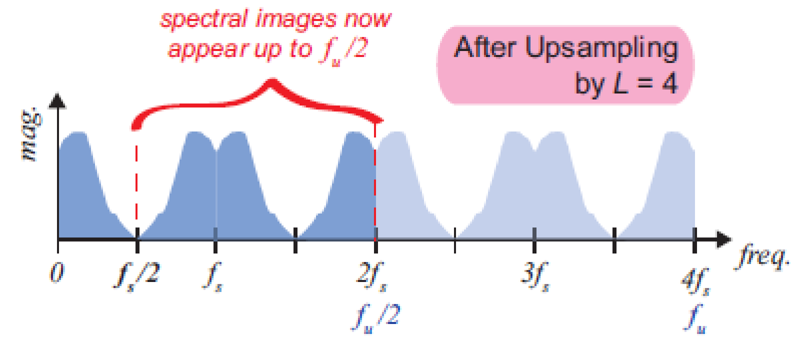


In the frequency domain, the effect of raising the sampling rate by upsampling is that the spectral images at integer multiples of the original sampling rate, *fs*, now fall within the range 'visible' with the new, higher sampling rate, *fu* .

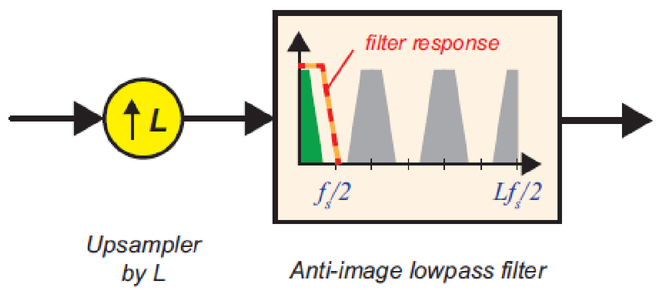
Sample Rate Changes: Decimation and Interpolation

- Let's consider two example signals (A and B), whose sampling rate has been increased by a factor of 4 from *fs* to *fu*.
  - spectral repetitions occur at integer multiples of the sampling rate.





- Spectral images must be removed, in order to achieve a signal with an equivalent spectrum to the original (in the time domain, this will be observed as a 'smoothing out' of the waveform).
  - A LPF will cut off at approximately *fs/2= fu/2L*



Sample Rate Changes: Decimation and Interpolation

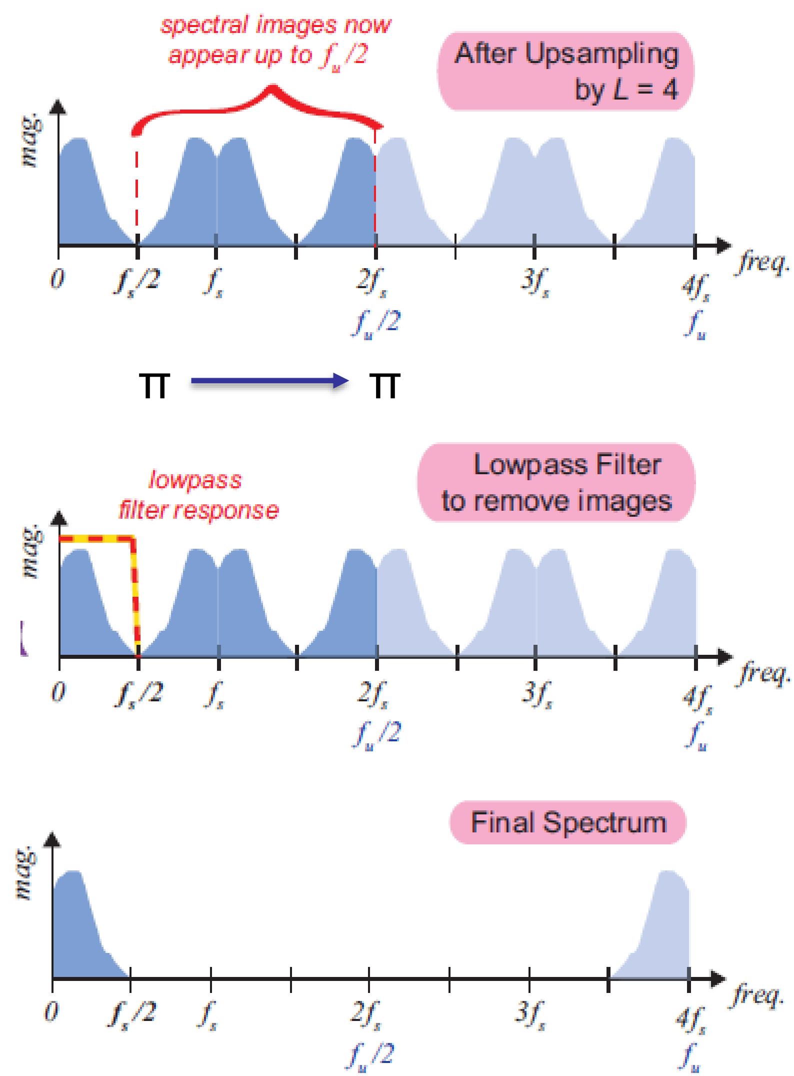
- Upsampling: I-fold compressed version of the spectrum

$$x_z(m)\begin{cases} x\left(\dfrac{m}{I}\right), m = 0, \pm I, \pm 2I, \dots \\ 0, otherwise \end{cases}$$

$$X_z(f) = \sum_{m=-\infty}^{\infty} x_z(m) e^{-j2\pi f m}$$

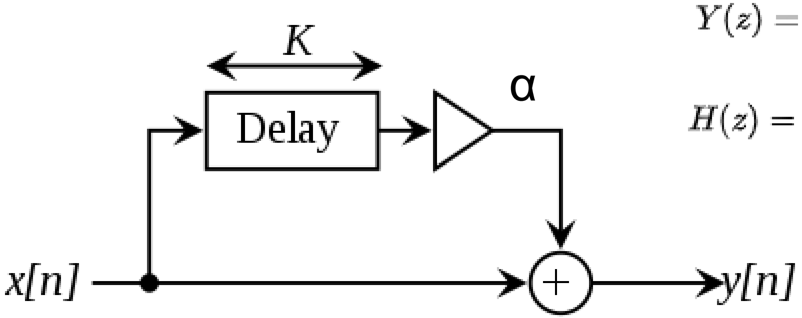
$$= \sum_{m=-\infty}^{\infty} x(m) e^{-j2\pi f I m}$$

$$= X(If)$$



COMB filter

- Comb filter is a filter implemented by adding a delayed version of a signal to itself, causing constructive and destructive interference.
- The frequency response of a comb filter consists of a series of regularly spaced notches in between regularly spaced peaks (sometimes called teeth) giving the appearance of a comb.



$$y[n] = x[n] + \alpha x[n-K]$$

$$Y(z) = \left(1 + \alpha z^{-K}\right) X(z)$$

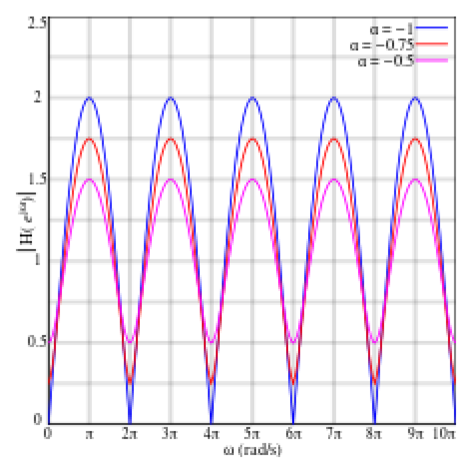
$$H(z) = \frac{Y(z)}{X(z)} = 1 + \alpha z^{-K}$$

$$H\left(e^{j\Omega}\right) = 1 + \alpha e^{-j\Omega K}$$

$$\left|H\left(e^{j\Omega}\right)\right| = \sqrt{\left(1+\alpha^2\right) + 2\alpha\cos(\Omega K)}$$

For $\alpha=-1$, the first maximum occurs at half the delay period and repeat at even multiples of the delay frequency and the minimum is null

$$f = \frac{1}{2K}, \frac{3}{2K}, \frac{5}{2K} \cdots$$

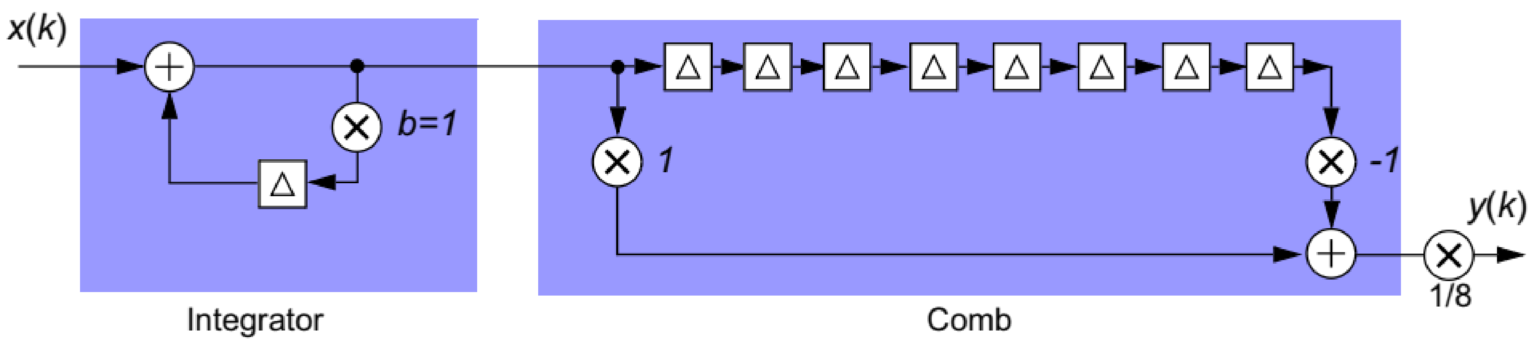


K = 1

CIC

- A cascaded integrator–comb (CIC filter) is an efficient implementation of a moving-average filter (normalization by a power of 2 is easy to carry out).
- If K = 8:

$$H(z) = \left(\frac{1}{1-z^{-1}}\right)(1-z^{-8})$$



- $$H(z) = \left(\frac{1}{1-z^{-1}}\right)(1-z^{-8}) = \frac{1-z^{-8}}{1-z^{-1}} = 1+z^{-1}+z^{-2}+z^{-3}+z^{-4}+z^{-5}+z^{-6}+z^{-7}$$

- Indeed:

$$1-z^{-8} = (1+z^{-1}+z^{-2}+z^{-3}+z^{-4}+z^{-5}+z^{-6}+z^{-7})(1-z^{-1})$$

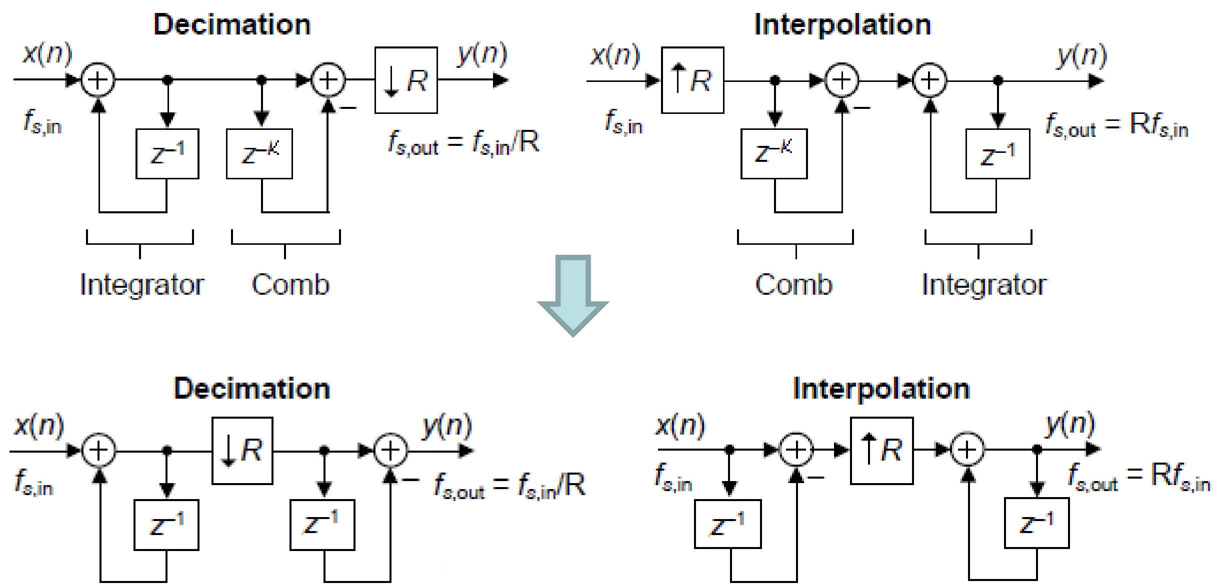
$$\begin{aligned}1-z^{-8} = 1 &+ \cancel{z^{-1}} + \cancel{z^{-2}} + \cancel{z^{-3}} + \cancel{z^{-4}} + \cancel{z^{-5}} + \cancel{z^{-6}} + \cancel{z^{-7}} \\ &- \cancel{z^{-1}} - \cancel{z^{-2}} - \cancel{z^{-3}} - \cancel{z^{-4}} - \cancel{z^{-5}} - \cancel{z^{-6}} - \cancel{z^{-7}} - z^{-8}\end{aligned}$$

CIC

- A CIC can be combined with an interpolator or decimator, where comb delay K=R=interpolation/decimation rate (and α = -1)
- The system function for the composite CIC filter referenced to the high sampling rate, fs is:

$$H(z) = \sum_{k=0}^{RM-1} z^{-k}$$
$$= \frac{1 - z^{-RM}}{1 - z^{-1}}$$

R = decimation or interpolation ratio
M = number of samples per stage
(usually 1 but sometimes 2)



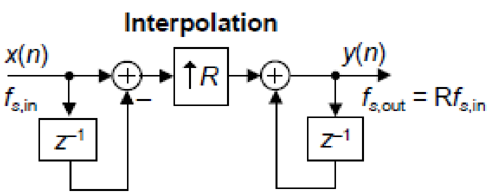
- Properties:
  - Linear phase response
  - Utilize only delay and addition and subtraction

CIC

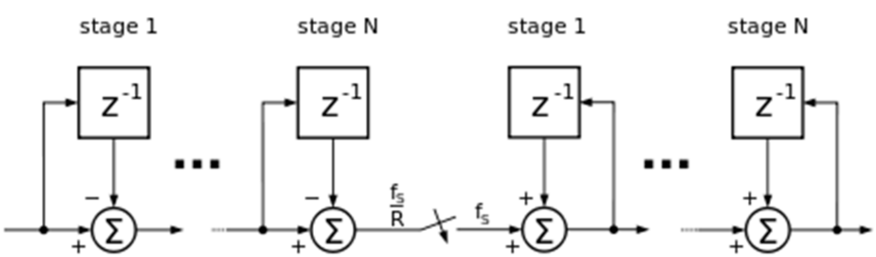
- CIC low-pass response can be improved using multiple stages
- But baseband is «worse»

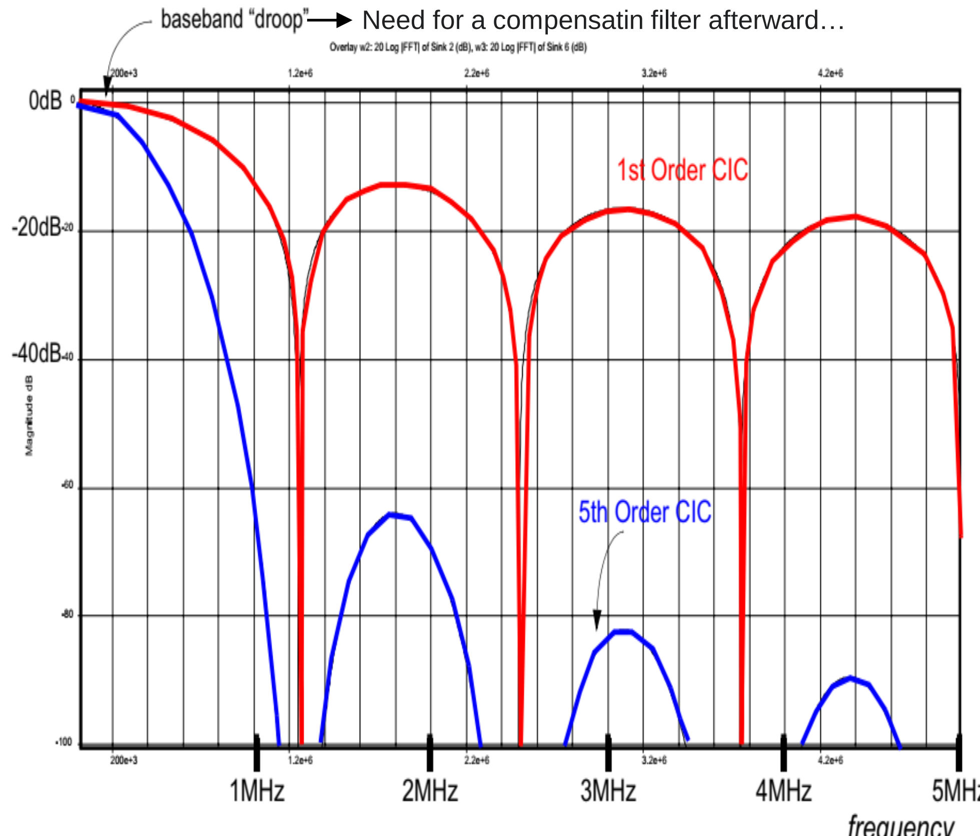
$$H(z) = \left[\sum_{k=0}^{RM-1} z^{-k}\right]^N = \left(\frac{1 - z^{-RM}}{1 - z^{-1}}\right)^N$$

R = decimation or interpolation ratio
M = number of samples per stage (usually 1 but sometimes 2)
N = number of stages in filter



Multistage (Interpolator)

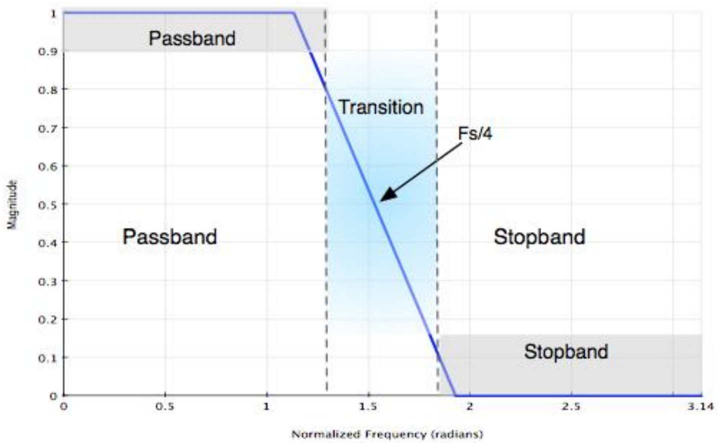




Plots of CIC and cascade of 5 CICs for 8th order moving average

Half-band filter

- Half-band filters are lowpass FIR filters with cut-off frequency of one-quarter of sampling frequency f_s and odd symmetry about $f_s/4$.
- And it so happens that almost half of the coefficients are zero.
- The passband and stopband bandwiths are equal, making these filters useful for decimation-by-2 and interpolation-by-2.



For a HBF with an odd filter length N, there are 3 non-zero coefficients at the temporal center. Thereafter, in both directions, the odd coefficients have zero values: only half multiplications are required!

10th order (N=11) Half-band FIR filter structure