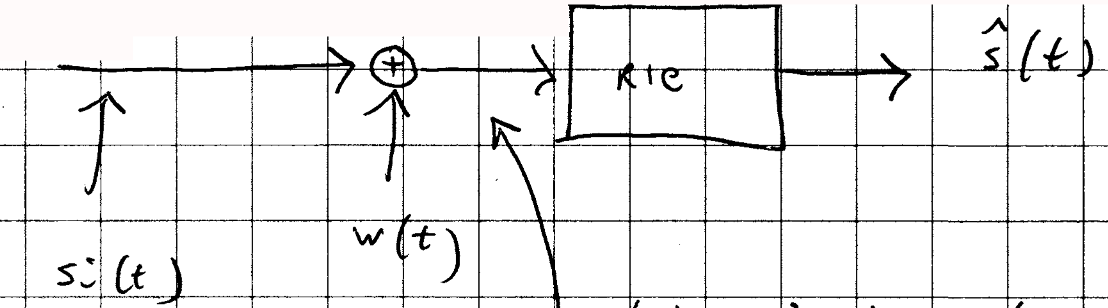
Principles of digital Communications in presence of AWGN



$r(t) = s_i(t) + w(t)$

$\{m_i\}_{i=1}^{M}$

The receiver choses the message which maximize the "a posteriori" probabilty

RIC: min della $P(\varepsilon)$ → SCELGO il messaggio PIÙ probabile A POSTERIORI !!!

BAYES: $P(A_i/B) = \dfrac{P(B/A_i)\,P(A_i)}{P(B)}$

$$P(A_i/X) = \frac{f_X(x/A_i)\,P(A_i)}{f_X(x)}$$

$f_X(x)$ → è indip. da "i"

It is independent from "i"

$A_i \to s_i(t)$; $B \to r(t) = s_i(t) + n(t)$

Infinit dimensions (n(t))

è ad ∞ dimensioni (Se AWGN ....)

Se $n(t)$ è AWGN, si ha:

$$E[r_k / s_i] = \begin{cases} s_{ik} ; & m \le N \\ 0 ; & m > N \end{cases}$$

DIM. dello spazio dei segnali

N is the signal space dimension

coordinate di $r(t)$, rispetto alla base che rappresenta segnali e rumore scegliendo n componenti ... 2

Considering n components ...

$$f(r_1, r_2, \ldots, r_n / s_i) =$$

$$= \prod_{k=1}^{M} \left(\frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \exp\left[-\frac{(r_k - s_{ik})^2}{2 \frac{N_0}{2}} \right] \right) \cdot$$

$$\cdot \prod_{k=M+1}^{n} \frac{1}{\sqrt{2\pi \sigma_k^2}} \exp\left[-\frac{r_k^2}{2\sigma_k^2} \right] =$$

Se devo cercare il max. al variare di i, posso non considerare i termini che NON DIPENDONO da i

→ (IRRILEVANTI ...)

We are looking for the maximum with respect to "i" ... therefore the terms that are independent of "i" are not relevant ... (irrelevant)

È come dire che le r_k; $k \leq M$ sono una STATISTICA SUFFICIENTE ... ai fini della dec. ottima

Le r_k; $k > M$ sono "INUTILI" ai fini della dec. OTTIMA (è anche INUTILE determinarle ...)

We can say that the component of the received signal in the signal space represent a "sufficient statistic" for the optimal detection ... therefore the component of the received signal out of the signal space are "irrelevant" (beeing orthogonal and therefore (beeing gaussian) statistically independent) ...

Se cambia la $f(r/s_i)$ è proporzionale a

$$f(r/s_i) \equiv \exp\left[-\frac{1}{N_0}\sum_{k=1}^{N}(r_k - s_{ik})^2\right] =$$

proportional to ...

$$= \exp\left[-\frac{1}{N_0}\underbrace{\|r - s_i\|^2}\right]$$

square distances between r(t) and si(t), in the signal vector space (N domensional)

distanza (al quadrato) tra $r(t)$ e $s_i(t)$ nello spazio vett. ad N dimensioni

$$\|r - s_i\|^2 = \int (r(t) - s_i(t))^2\, dt =$$

$$= \sum_{k=1}^{\infty}(r_k - s_{ik})^2 = \sum_{k=1}^{N}(r_k - s_{ik})^2 + \sum_{k=N+1}^{\infty} r_k^2$$

TORNANDO alla regola di Bayes, si ha:

$$f(r/s_i)\,P(s_i) \equiv \exp\left[-\frac{1}{N_0}\|r - s_i\|^2\right] P(s_i)$$

PROB. a PRIORI ...

Likelihood function ...

VEROSIMIGLIANZE ...

Se vogliamo il max (o il MIN), possiamo usare il log ...

Using log() ...

$$\|r - s_i\|^2 - N_0 \log P(s_i)$$

ML

MAP

MAP: richiede: $\frac{N_0}{2}$, $P(s_i)$!!!

ML: richiede solo la DISTANZA minima ...

I can predefine some "decision regions", associated to each possible transmitted signal ...

Posso precalcolare le "regioni di DECISIONE"

In generale: $\|r - s_i\|^2 = \|r\|^2 + \|s_i\|^2 - 2\langle r, s_i\rangle$

$\langle r, s_i\rangle - \frac{1}{2}\|s_i\|^2 \longrightarrow$ (devo valutare il MASSIMO)

we have ro evaluate the maximum

$$\langle r, s_i\rangle = \sum_{k=1}^{N} r_k s_{ik}$$

$$r_k = \int r(t)\, \phi_k(t)\, dt$$

$$\langle r, s_i\rangle = \int r(t)\, s_i(t)\, dt$$

in case of band-pass signals ...

Se è passa-banda $\longrightarrow$ $r_k = \frac{1}{2}\int z(t)\, z_k^*(t)\, dt$ ...

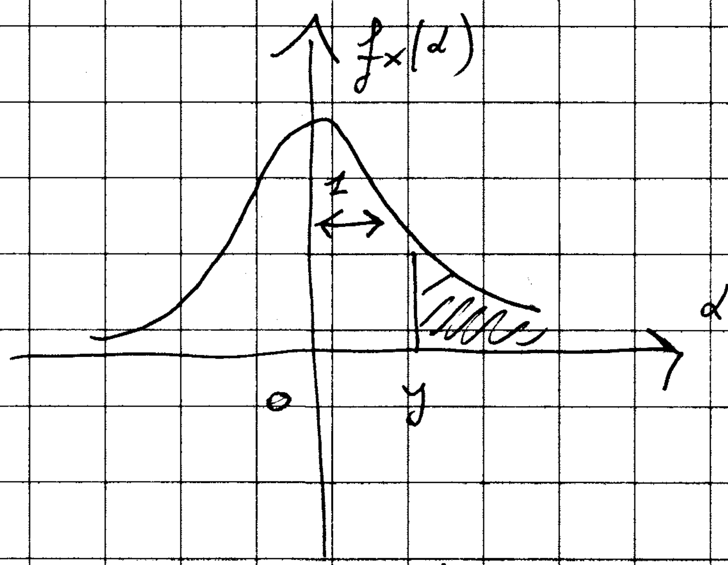
PROB. di ERRORE — Error probabilities

$$P(E) = \sum_{i=1}^{M} P(s_i)\, P(E/s_i) = \sum_{i=1}^{M} P(s_i) \sum_{j \neq i} P(s_j/s_i)$$

Tr. Binaria

$$P(s_2/s_1) = P(s_1/s_2) = P(E) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) = Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

$$Q(y) = \int_y^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$$



Se $s_2 = -s_1$

$$P(E) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$P(E)$	10^{-3}	10^{-5}	10^{-7}	10^{-10}	10^{-13}
E_b/N_0 [dB]	6,79	9,59	11,31	13,06	14,31

$$\left(Q(y) \approx \frac{1}{\sqrt{2\pi}\, y}\, e^{-y^2/2} \qquad y > 3 \right)$$

$$\log_{10} Q(\gamma) \approx -0{,}22\,\gamma^2 - 1{,}04$$

$$P_b(E) = \frac{1}{\log_2 M} \sum_{1}^{M} {}_i \, P(s_i) \sum_{j \neq i} n_{ij} \, P(s_j / s_i)$$

number of bits differing between the bit mapping associated to si and sj

(n° di bit in cui s_i ed s_j differiscono)

$$\frac{P(E)}{\log_2 M} \leq P_b(E) \leq P(E)$$

P errore sul bit

P(E) sul SIMBOLO

UNION BOUND

$$P(E) = \sum_{1}^{M} {}_i \, P(s_i) \sum_{j \neq i} P(s_j / s_i) \leq \sum_{i=1}^{M} P(s_i) \sum_{j \neq i} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

$$P_b(E) = \frac{1}{\log_2 M} \sum_{1}^{M} {}_i \, P(s_i) \sum_{j \neq i} n_{ij} \, P(s_j / s_i) \leq (\cdot)$$

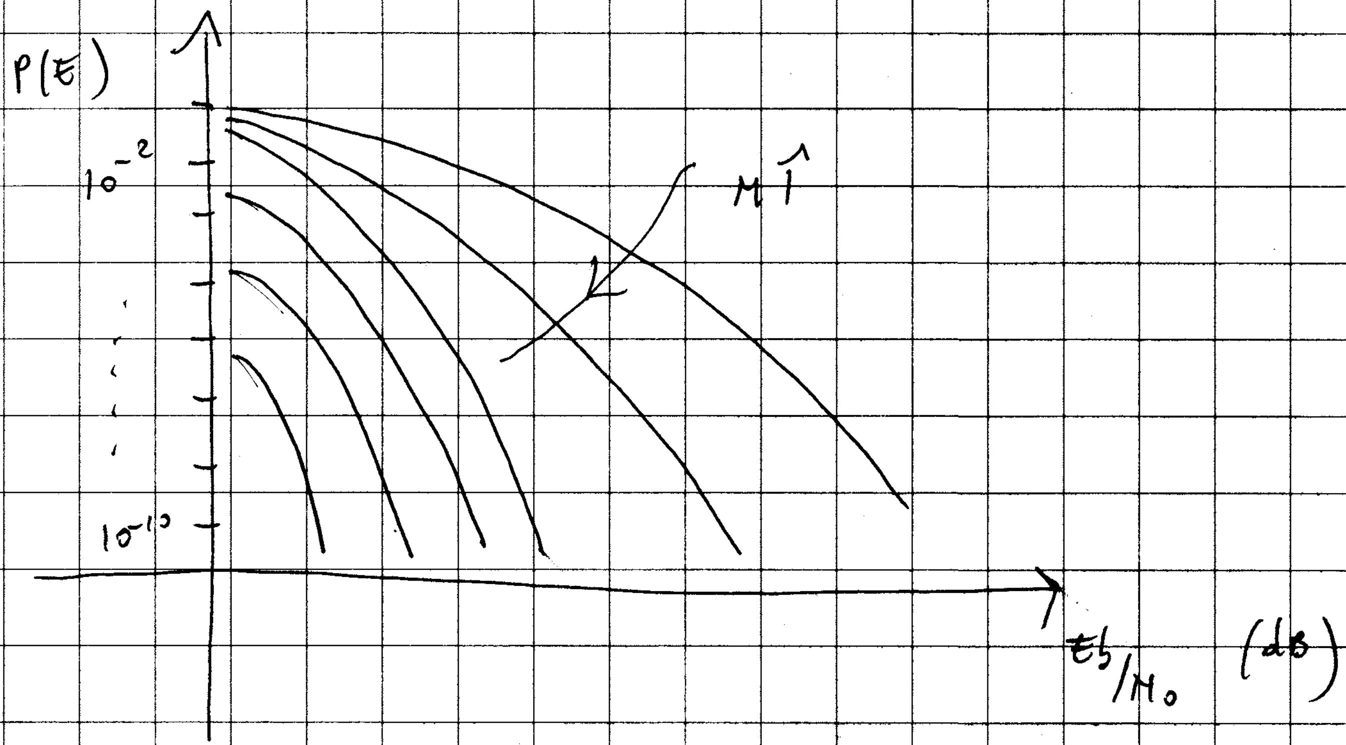
$$\leq \frac{1}{\log_2 M} \sum_{1}^{M} {}_i \, P(s_i) \sum_{j \neq i} n_{ij} \, Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

Segnali ORTOGONALI

Orthogonal signals

$$P(E) \leq (M-1)\, Q\left(\sqrt{\frac{E_s}{N_0}}\right) = (M-1)\, Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$

$$P_b(E) \leq \frac{M}{2}\, Q\left(\sqrt{\frac{E_b \log_2 M}{N_0}}\right)$$



$$Q(y) \leq \frac{1}{2} e^{-y^2/2} \quad \text{se } y \geq 0$$

$$P(E) < M \exp\left(-\frac{E_b \log_2 M}{2N_0}\right) =$$

$$= \exp\left(-\log_2 M \left(\frac{E_b}{2N_0} - \log 2\right)\right)$$

$$P(e) < \exp\left(-\log_2 M\,\overbrace{\left(\frac{E_b}{2N_0}-\log 2\right)}^{\geq 0}\right) \xrightarrow{M\to\infty} 0$$

Se $\dfrac{E_b}{N_0} > 2\log 2$ $\quad (1{,}42\ \mathrm{dB})$

$\dfrac{E_b}{N_0} = 9{,}59\,\mathrm{dB} \Rightarrow P(E) = 10^{-5}$ (bin. antipodale ---)

In realtà basta $\dfrac{E_b}{N_0} > \log 2$ $\quad (-1{,}59\ \mathrm{dB})$!!!!

Per avere $P(E) = 0$ se $M \to \infty$!!!!