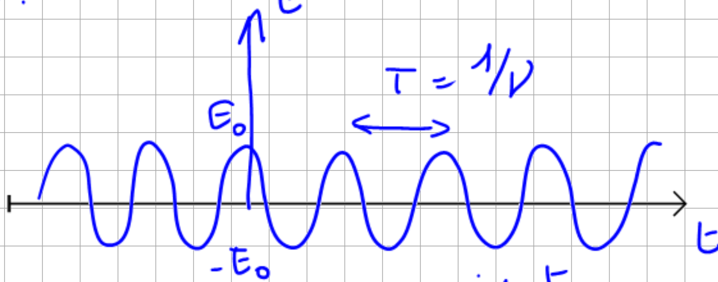
Tutorial 1 : FOURIER OPTICS 15/09/21

Reminder :



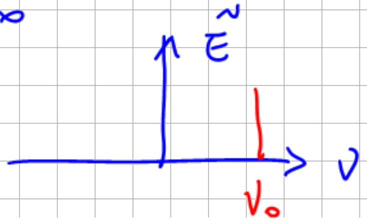
$E = E_0 \cos(\omega t)$

$\omega = 2\pi . \nu$ ↳ temporal frequency

Complex form : $E = E_0 e^{j\omega t} = E_0 e^{j 2\pi \nu_0 t}$

↓ FT ⇒ $\int_{-\infty}^{+\infty} E . e^{-j 2\pi \nu t} dt$

$\tilde{E} = E_0 \delta(\nu - \nu_0)$



Exercise 1: Temporal signal versus spatial beam

A - Decomposition of a temporal signal into a sum of monochromatic signals

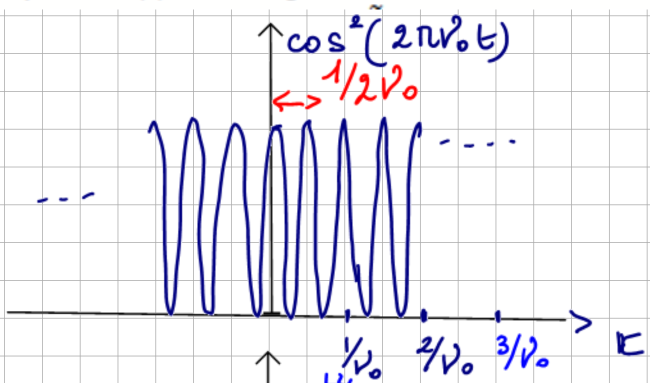
Note: A temporal signal is monochromatic when it is described by a sinusoidal function (from $t \rightarrow -\infty$ to $t \rightarrow +\infty$.

At any point M in space, the electric field E(t) of a light radiation is given by:

$$E(t) = a.\cos^2(2\pi\nu_0 t).e^{-\pi t^2/\tau^2}$$

with a = constant (V/m), ν_0 = 0.3 10^{15}Hz, $\tau = 3/\nu_0$ (time constant (s)).

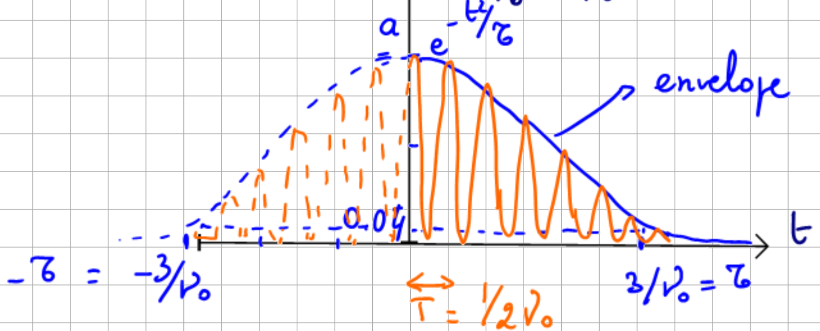
1) Plot E(t). Is this signal monochromatic?



$\cos^2\left(\frac{\pi t}{T}\right) = \cos^2(2\pi\nu_0 t)$

⇓

$T = \frac{1}{2\nu_0}$



$e^{-\pi t^2/\tau^2} = ?$ when $t = \tau = 3/\nu_0$

$e^{-\pi} = 0.04$

Envelope of finite size (τ) → It is not a monochromatic wave

2) $\tilde{E}(\nu) = FT(E)$

$$\boxed{\begin{array}{l} FT\left(e^{-\pi t^2}\right) = e^{-\pi\nu^2} \\ FT\left(f(at)\right) = \frac{1}{|a|}\tilde{f}\left(\frac{\nu}{a}\right) \\ a = cst \end{array}}$$

$$E(t) = a\,\overbrace{\cos^2(2\pi\nu_0 t)}^{①}\,\underbrace{e^{-\pi t^2/\tau^2}}_{②}$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

① $FT\left(\frac{\cos 4\pi\nu_0 t + 1}{2}\right) = FT\left(\frac{\cos 4\pi\nu_0 t}{2}\right) + FT\left(\frac{1}{2}\right)$

$FT\left(\frac{1}{2}\right) = \frac{1}{2}\delta(\nu)$

$\frac{1}{2}FT\left(\cos 4\pi\nu_0 t\right) = \frac{1}{\cancel{2}\,4} FT\left(e^{j4\pi\nu_0 t} + e^{-j4\pi\nu_0 t}\right)$

$FT\left(e^{j4\pi\nu_0 t}\right) = \int_{-\infty}^{+\infty} e^{j4\pi\nu_0 t} \cdot e^{-j2\pi\nu t}\,dt$

$= \int_{-\infty}^{+\infty} 1 \times e^{-j2\pi(\underline{\nu - 2\nu_0})t}\,dt = \delta(\nu - 2\nu_0)$

$FT\left(e^{-j4\pi\nu_0 t}\right) = \delta(\nu + 2\nu_0)$

① $= \frac{1}{2}\left(\frac{1}{2}\delta(\nu - 2\nu_0) + \frac{1}{2}\delta(\nu + 2\nu_0) + \delta(\nu)\right)$

② $= e^{-\pi(t/\tau)^2}$

$a = 1/\tau$

$FT\left(e^{-\pi(t/\tau)^2}\right) = \tau e^{-\underbrace{\pi(\nu\tau)^2}}$ unitless!!

$f(at)$

$e^{-\pi(t/\tau)^2}$

$e^{-\pi t^2} \xrightarrow{FT} e^{-\pi\nu^2}$

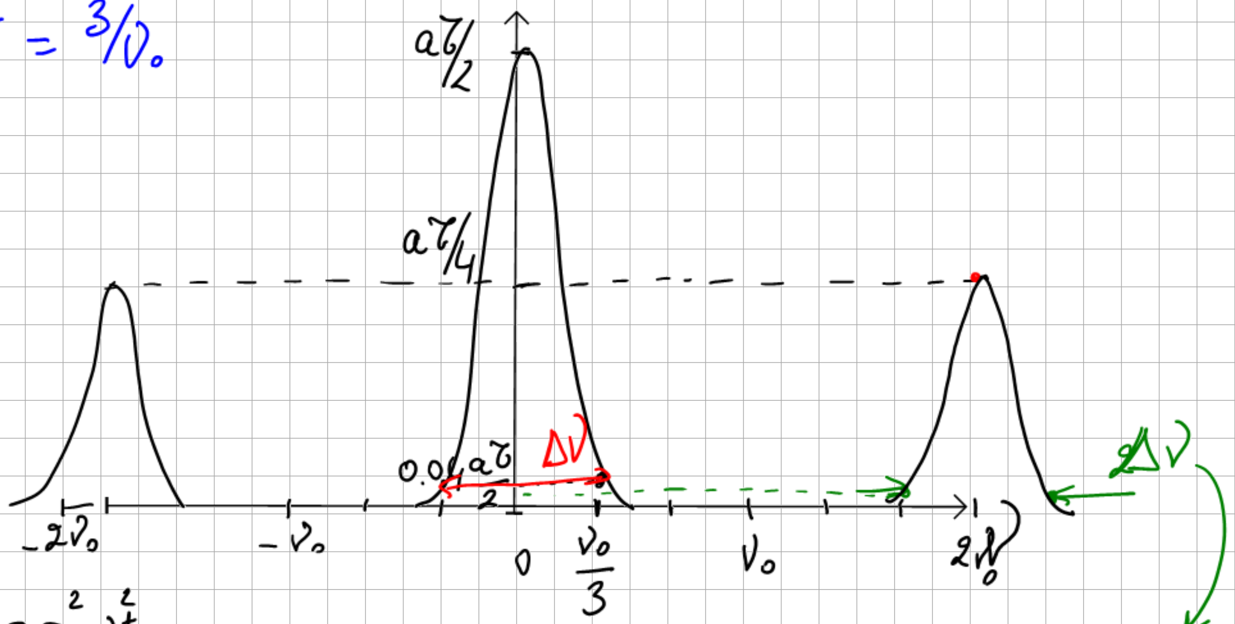
$f(at) \xrightarrow{FT} \frac{1}{|a|}\tilde{f}\left(\frac{\nu}{a}\right)$

$\tilde{E} = a\ ① * ②$

$\tilde{E} = \frac{a\tau}{2}\left(\frac{1}{2}\delta(\nu - 2\nu_0) + \frac{1}{2}\delta(\nu + 2\nu_0) + \delta(\nu)\right) * e^{-\pi\tau^2\nu^2}$

$$\boxed{\tilde{E}(\nu) = \frac{a\tau}{2}\left(\frac{1}{2}\, e^{-\pi\tau^2(\nu-2\nu_0)^2} + \frac{1}{2}\, e^{-\pi\tau^2(\nu+2\nu_0)^2} + e^{-\pi\tau^2\nu^2}\right)}$$

3) $\tau = 3/\nu_0$



$$e^{-\pi\tau^2\nu^{+2}} = e^{-\pi} \simeq 0.04 \Rightarrow \nu^+ = \frac{\Delta\nu}{2} = 1/\tau = \frac{\nu_0}{3} \Rightarrow \boxed{\Delta\nu = \frac{2\nu_0}{3}}$$

$$\Delta\nu = 2/\tau$$

Monochromatic components $\Delta\nu \to 0$

$\Rightarrow \tau \to +\infty$

$\Rightarrow$ Superposition of 3 monochromatic components

B)

B - Decomposition of a monochromatic wave into a sum of monochromatic plane waves

At a given time t, consider a monochromatic wave ($\lambda_0 = 1\mu m$) which propagates in the plane (x,z). Its spatial profile is described in the plane of abscissa z = 0 by the electric field:

$$E(x, z=0) = a.cos^2(2\pi N_0 x).e^{-\pi x^2/L^2},$$

1) $N_0 \longleftrightarrow \nu_0$

$x \longleftrightarrow t$

$L \longleftrightarrow \tau$

2) Temporal domain

$$\tilde{E}(\nu) = \frac{a\tau}{2}\left(\frac{1}{2}e^{-\pi\tau^2(\nu-2\nu_0)^2} + \frac{1}{2}e^{-\pi\tau^2(\nu+2\nu_0)^2} + e^{-\pi\tau^2\nu^2}\right)$$

Spatial domain

$$\tilde{E}(N_x) = \frac{aL}{2}\left(\frac{1}{2}e^{-\pi L^2(N_x-2N_0)^2} + \frac{1}{2}e^{-\pi L^2(N_x+2N_0)^2} + e^{-\pi L^2 N_x^2}\right)$$

3) Same drawing as A 3)

3 components in the <u>spatial</u> frequency spectrum.

centered on $\left.\begin{array}{l}(m^{-1})\ N_x = 0 \\ (m^{-1})\ N_x = 2N_0 \\ (m^{-1})\ N_x = -2N_0\end{array}\right)$ of bandwidth $\Delta N = 2/L$

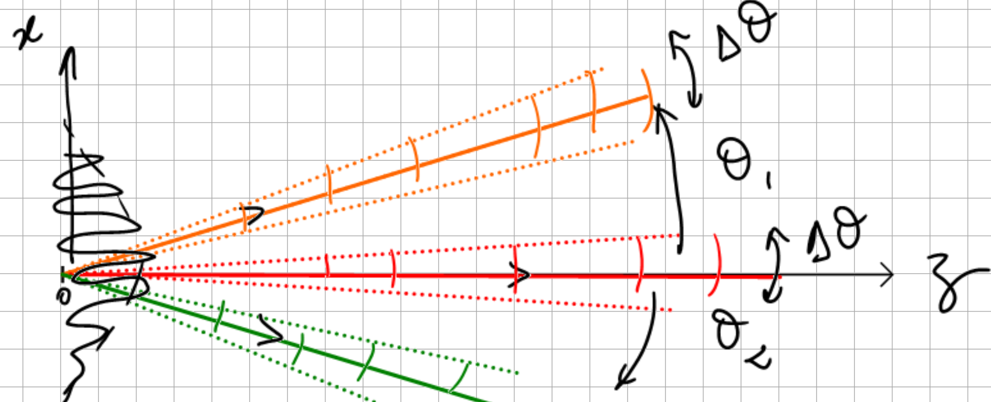
4) $N_x = \dfrac{\sin\theta}{\lambda}$

$N_x = 0 \Rightarrow \theta_0 = 0$

$N_x = 2N_0 \Rightarrow \dfrac{\sin\theta_1}{\lambda} = 2N_0 \Rightarrow \sin\theta_1 = 2N_0\lambda = 2.10^{3}\,10^{-6}$
$= 2.10^{-3}$

$\Rightarrow \boxed{\theta_1 \simeq 2.10^{-3}\ \text{rad}}$ $\qquad \sin\theta \simeq \theta$

$N_3 = -2N_0 \Rightarrow \boxed{\theta_2 = -2\,10^{-3}\ \text{rad}}$



$$\Delta N = 2/L \Rightarrow \Delta\theta = \,? \qquad N_x = \frac{\sin\theta}{\lambda}$$

$$\Delta N = \frac{\cos\theta}{\lambda}\,\Delta\theta \qquad \cos\theta \simeq 1 \text{ (paraxial approximation)}$$

$$\Delta\theta = \lambda\,\Delta N = \frac{2\lambda}{L} = \frac{2}{3}N_0.\lambda$$

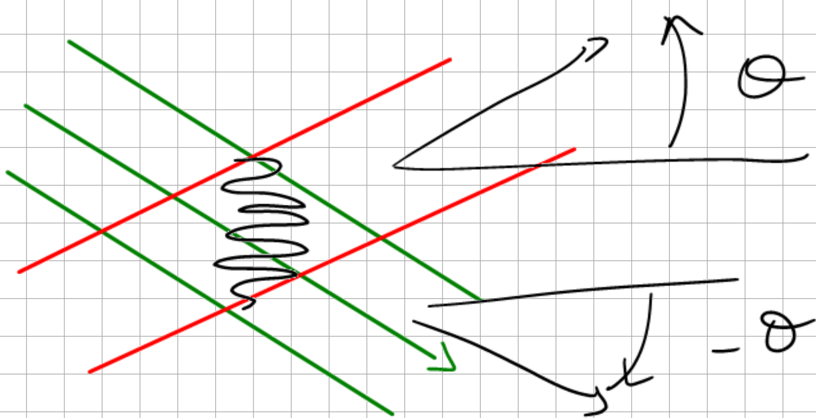
$$\Delta\theta = \frac{2}{3}10^{3}.10^{-6} = \frac{2}{3}10^{-3}\text{ rad} = 0.66\text{ mrad}$$

5) Each component is not a plane wave because

$\Delta\theta \neq 0 \iff \Delta N \neq 0$

To have plane waves components $\Rightarrow \Delta N \to 0$

$\iff L \to +\infty$

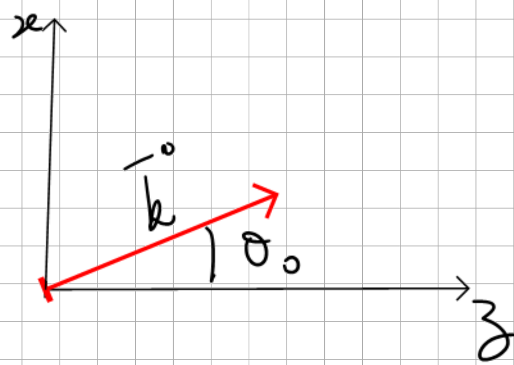


Exercise 2 :

1) $$E_0(x,z,t) = a_0 e^{j(\omega_0 t - \vec{k}.\vec{OM})}$$

$$= a_0 \cancel{e^{j\omega_0 t}} \cdot e^{-j\vec{k}.\vec{OM}}$$

$$E_0(x,z) =$$



$$\vec{OM} : \begin{vmatrix} x \\ z \end{vmatrix} \qquad \vec{k} = \frac{2\pi}{\lambda_0}\begin{vmatrix} \sin\theta_0 \\ \cos\theta_0 \end{vmatrix}$$

$$\vec{OM}.\vec{k} = x\sin\theta_0 + z\cos\theta_0$$

$$\boxed{E_0(x,z) = a_0 \, e^{-j\frac{2\pi}{\lambda_0}(x \sin\theta_0 + z\cos\theta_0)}}$$

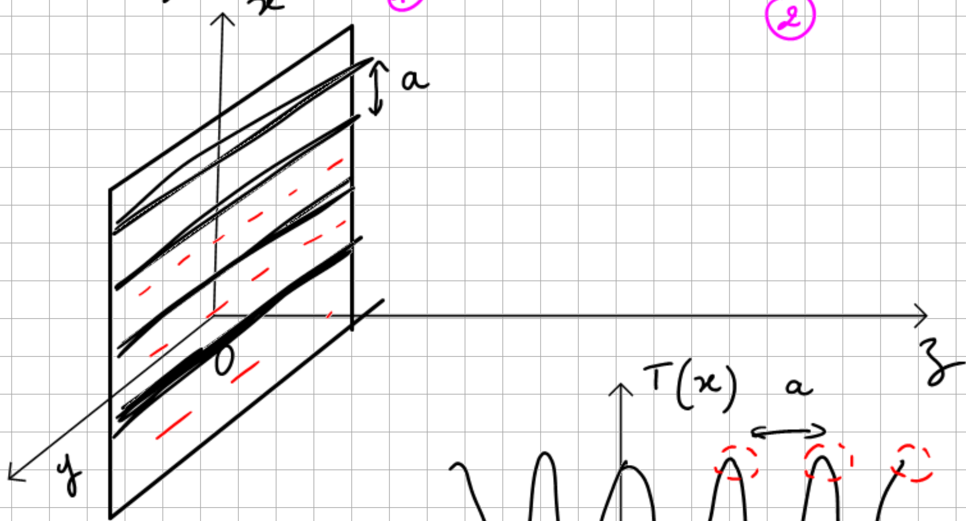
2) $$E_0(x,0) = a_0 \, e^{-j\left(\frac{2\pi}{\lambda_0}\sin\theta_0 \cdot x\right)}$$

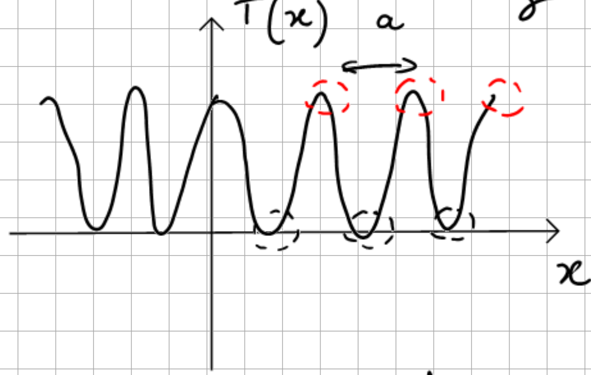
$$\boxed{E_0(x,0) = a_0 \, e^{-j 2\pi N_{x_0} x} \quad \text{with} \quad N_{x_0} = \frac{\sin\theta_0}{\lambda_0}}$$

3) $$\mathcal{T}(x) = \frac{1}{2}\left(1 + \cos 2\pi \frac{x}{a}\right)$$

$$E(x,0^+) = E_0(x,0^-) \cdot \mathcal{T}(x)$$

$$= \frac{a_0}{2} \underbrace{e^{-j 2\pi N_{x_0} x}}_{1} \cdot \left(\underbrace{1 + \cos\left(\frac{2\pi x}{a}\right)}_{2}\right)$$



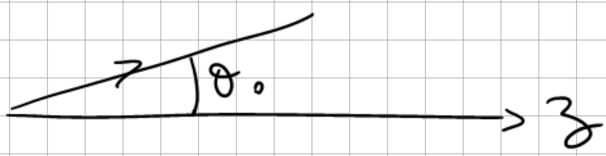


4) ⚠ $$FT(f(x)) = \int_{-\infty}^{+\infty} f(x) \, e^{+j 2\pi N_x x} \, dx$$

$$\tilde{1} = \int_{-\infty}^{+\infty} e^{-j 2\pi N_{x_0} x} \, e^{j 2\pi N_x x} \, dx = \int_{-\infty}^{+\infty} 1 \times e^{j 2\pi (N_x - N_{x_0}) x} \, dx$$

$$= \delta(N_x - N_{x_0})$$

$$\delta(N_x - N_{x_0}) \qquad N_{x_0} = \frac{\sin\theta_0}{\lambda}$$



$$\tilde{②} = FT\left(1 + \cos\frac{2\pi x}{a}\right) = \delta(N_x) + \frac{1}{2}\left(\delta\left(N_x - \frac{1}{a}\right) + \delta\left(N_x + \frac{1}{a}\right)\right)$$

$$\tilde{E}(N_x, z=0) = \frac{a_0}{2}\,\delta(N_x - N_{x_0}) * \left(\delta(N_x) + \frac{1}{2}\delta\left(N_x - \frac{1}{a}\right) + \frac{1}{2}\delta\left(N_x + \frac{1}{a}\right)\right)$$

$$\tilde{E}(N_x, z=0) = \frac{a_0}{2}\left(\delta(N_x - N_{x_0}) + \frac{1}{2}\delta\left(N_x - N_{x_0} - \frac{1}{a}\right) + \frac{1}{2}\delta\left(N_x - N_{x_0} + \frac{1}{a}\right)\right)$$

$$\begin{bmatrix} N_x = N_{x_0} \\ N_x = N_{x_0} + \frac{1}{a} \\ N_x = N_{x_0} - \frac{1}{a} \end{bmatrix}$$ Location of 3 Dirac

5) $\boxed{\sin\theta = \sin\theta_0 + \frac{k\lambda}{a}}$ $k \in \mathbb{Z}$

$N_x = \frac{\sin\theta}{\lambda}$ → Direction of observation after the grating

$N_{x_0} = \frac{\sin\theta_0}{\lambda}$ → Direction of propagation of the incident PW

$$\left.\begin{aligned} \frac{\sin\theta}{\lambda} &= \frac{\sin\theta_0}{\lambda} \iff \sin\theta = \sin\theta_0 && k=0 \\ \frac{\sin\theta}{\lambda} &= \frac{\sin\theta_0}{\lambda} + \frac{1}{a} \iff \sin\theta = \sin\theta_0 + \frac{\lambda}{a} && k=+1 \\ \frac{\sin\theta}{\lambda} &= \frac{\sin\theta_0}{\lambda} - \frac{1}{a} \iff \sin\theta = \sin\theta_0 - \frac{\lambda}{a} && k=-1 \end{aligned}\right\}$$