Photonics / Nanophotonics

Lecture 1:
Problems on
Electromagnetic and
Polarization optics

Problem 1: Show that for a medium described by the Drude model the product of the phase velocity and Group velocity equals c^2. Assume the medium to be lossless ($\gamma = 0$).

A material modeled with Drude oscillator has permittivity

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 - \cancel{j\gamma\omega}_{\,0}} = 1 - \frac{\omega_p^3}{\omega^2}$$

Phase velocity:

$$V_p = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon}}$$

Group velocity:

$$V_g = \frac{\partial \omega}{\partial k} \qquad k = \frac{2\pi}{\lambda} = \frac{\omega n}{c}$$

$$V_g = \frac{\partial \omega}{\partial k} = \frac{\partial \omega}{\partial \left(\frac{\omega n}{c}\right)} = c\,\frac{\partial \omega}{\partial n\omega}$$

The product of phase and group velocity is:

$$v_p v_g = \frac{c}{n} \cdot c \cdot \frac{\partial \omega}{\partial n\omega} =$$

$$n = \sqrt{\varepsilon} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}$$

$$\frac{\partial \omega}{\partial n\omega} = \frac{1}{\frac{\partial n\omega}{\partial \omega}}$$

$$\frac{\partial n\omega}{\partial \omega} = \frac{\partial}{\partial \omega}\, \omega \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}} =$$

$$= \omega \underbrace{\frac{\partial}{\partial \omega}\left[1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}}_{} + \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{\frac{1}{2}} =$$

$$= \frac{\omega_p^2}{\omega^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-\frac{1}{2}} + \left[1 - \frac{\omega_p^2}{\omega^2}\right]^{\frac{1}{2}} =$$

$$= \frac{\omega_p^2}{\omega^2\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} + \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{\omega_p^2 + \omega^2\left(1 - \frac{\omega_p^2}{\omega^2}\right)}{\omega^2\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} =$$

$$= \frac{\cancel{\omega_p^2} + \omega^2 - \cancel{\omega_p^2}}{\omega^2 \sqrt{1 - \frac{\omega_p^2}{\omega^2}}} = \frac{1}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$

$$V_p V_g = \frac{c^2}{n} \frac{\partial \omega}{\partial n \omega} = \frac{c^2}{\cancel{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}} \cdot \cancel{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} = c^2$$

Problem 2: Wave Retarders

Write the Jones Matrices for:

a) a pi/2 wave retarder with fast axis along the x direction;

b) a pi wave retarder with fast axis at 45° with respect to the x direction;

c) a pi/2 wave retarder with fast axis along the y direction;

If these three elements are placed together with element c following b and following a, calculate the rotation introduced by the whole system.

$$M_a = \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\Gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix}$$

$\Gamma = \frac{\pi}{2}$

$$M_c = \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\Gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}$$

$\Gamma = -\frac{\pi}{2}$

$$M_b = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\Gamma} \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} =$$

$\Gamma = \pi$

$\Gamma = \pi$

$\theta = 45^\circ$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{In} \rightarrow \boxed{M_a}\,\boxed{M_b}\,\boxed{M_c} \rightarrow \text{Out}$$

$$A = M_C M_B M_A = j \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow +90^\circ \text{ rotation}$$

$$B = M_A M_B M_C = j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow -90^\circ \text{ rotation}$$

You can also define

$$M_b' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e^{j\frac{\pi}{2}} & 0 \\ 0 & e^{-j\frac{\pi}{2}} \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

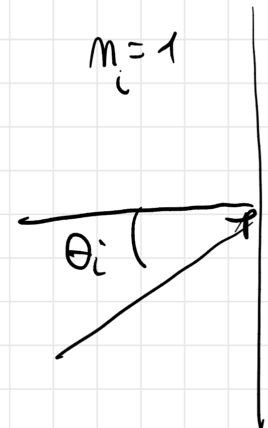
$$= j \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A' = M_C M_B' M_A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B' = M_A M_B' M_C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\underline{A' = e^{j\frac{\pi}{2}} A}$$

Problem 3: A plane wave is incident from air (n=1) onto a glass plate (n=1.5) at an angle of 45°. Determine the reflectance R for TE and TM polarization.



For TE polarization

$$r_{TE} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} =$$

$$= \frac{\frac{\sqrt{2}}{2} - 1.5\cos(28.12°)}{\frac{\sqrt{2}}{2} + 1.5\cos(28.12°)} = -0.3033$$

$$\underline{\theta_i = 45°} \qquad \boxed{n_i \sin\theta_i = n_t \sin\theta_t}$$

$$\theta_t = \operatorname{asin}\left(\frac{n_i}{n_t}\sin\theta_i\right) = \underline{28.12°}$$

$$R_{TE} = |r_{TE}|^2 = 0.092 \rightarrow R_{TE} = 9.2\ \%$$

For TM polarization

$$r_{TM} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t} = \frac{1.5 \cdot \frac{\sqrt{2}}{2} - \cos(28.12°)}{\cos(28.12°) + 1.5\frac{\sqrt{2}}{2}} =$$

$$= 0.09$$

$$R_{TM} = |r_{TM}|^2 = 0.0085 \rightarrow \underline{R_{TM} = 0.85\%}$$

The Average reflectance is $R = \frac{R_{TE} + R_{TM}}{2} = \underline{5.02\%}$

Problem 4: An unpolarized plane wave is incident from free space onto a quartz crystal. Quartz is a positive uniaxial crystal with ne=1.553 and no=1.544. The angle of incidence is 30°.
The optic axis lies in the plane of incidence and is perpendicular to the direction of the incident wave before it enters the crystal.
Determine the direction of the wavevectors of the two refracted components.

(a) • The ordinary wave refracts accordingly to n_o

(b) • The extraordinary wave refracts accordingly to $n(\theta)$ that satisfies the relation

$$\boxed{\frac{1}{n^2(\theta)} = \frac{\cos^2(\theta)}{n_o^2} + \frac{\sin^2\theta}{n_e^2}}$$

$\theta = 30°$

(a)

$$\theta_o = \operatorname{asin}\left(\frac{\overset{1}{\overset{\shortparallel}{n_i}}}{\underset{1.544}{\underset{\shortparallel}{n_o}}} \sin \overset{30°}{\overset{\shortparallel}{\theta_i}}\right) = 19.42°$$

(b)

$$n(\theta) = \sqrt{\frac{1}{\frac{\cos^2(\theta)}{n_o^2} + \frac{\sin^2\theta}{n_e^2}}} = \underline{\underline{1.5462}}$$

$$\theta_e = \arcsin\left(\frac{\overset{1}{n_i}}{\underset{1.5462}{n(\theta)}} \sin(\underset{30^\circ}{\theta_i}) \right) = \underline{\underline{19.40^\circ}}$$

Problem 5: Quartz is a positive uniaxial crystal with ne=1.553 and no=1.544. Calculate:

a) the retardation per mm at 633nm;
b) at what thickness/thicknesses the crystal acts as a quarter wave retarder.

a) For a generically polarized incident wave the phase retardation is a function of d and λ:

$$\Gamma = \frac{2\pi (n_o - n_e) d}{\lambda_0}$$

Retardation/mm is therefore

$$\frac{\Gamma}{d\,(mm)} = \frac{2\pi (n_o - n_e)}{\lambda_0} = \frac{2\pi (1.544 - 1.553)}{633 \cdot 10^{-9}} =$$

$$= (-8.93 \cdot 10^{-5})(10^{9})(10^{-3}) = -89.3 \, \frac{rad}{mm}$$

b) The crystal will act as a quarter-wave retarder when

$$\Gamma = \frac{2\pi (n_o - n_e) d}{\lambda_0} = \frac{\pi}{2} + 2\pi m$$

setting $m=0$

$$d = \frac{\pi}{2} \frac{\lambda_0}{2\pi (n_o - n_e)} = \frac{\lambda_0}{4|(n_o - n_e)|} = \frac{633 \cdot 10^{-9}}{4(1.553 - 1.544)} =$$

$$= 17.58\ \mu m$$

Adding 2π we get

$m = 1$ $\quad d = \frac{5}{2}\pi \frac{\lambda_0}{2\pi |(n_e - n_o)|} = 87.92\ \mu m$

$m = 2 \ldots$