ELEN E4810: Digital Signal Processing
Topic 3: Fourier domain

1. The Fourier domain
2. Discrete-Time Fourier Transform (DTFT)
3. Discrete Fourier Transform (DFT)
4. Convolution with the DFT

1. The Fourier Transform

- Basic observation (continuous time):
  A periodic signal can be decomposed into sinusoids at integer multiples of the fundamental frequency
- i.e. if $\tilde{x}(t) = \tilde{x}(t+T)$
  we can approach $\tilde{x}$ with

$$\tilde{x}(t) \approx \sum_{k=0}^{M} a_k \boxed{\cos\left(\frac{2\pi k}{T} t\right.} \left. + \phi_k \right)$$

Harmonics *of the fundamental*

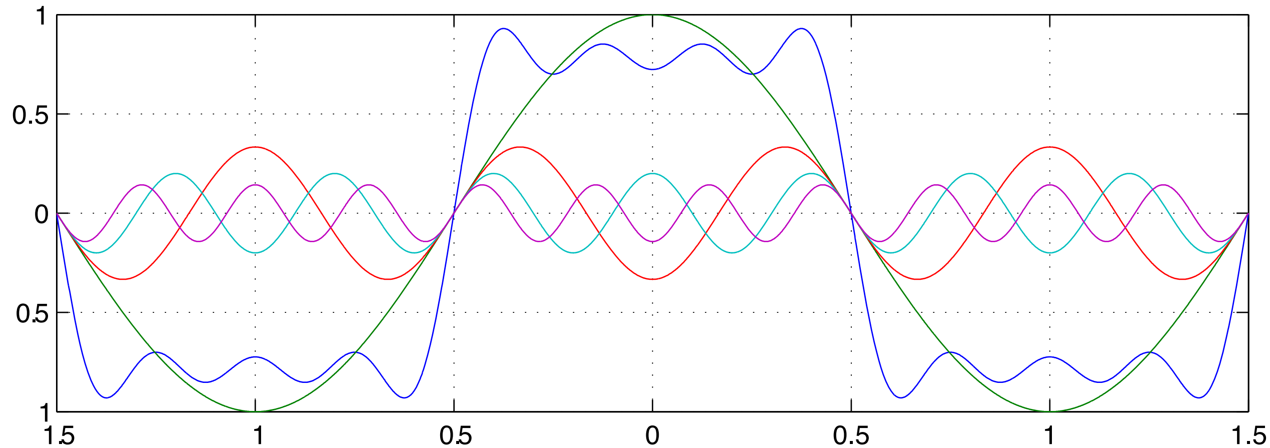
Fourier Series

$$\sum_{k=0}^{M} a_k \cos\left(\frac{2\pi k}{T} t + \phi_k\right)$$

- For a square wave,

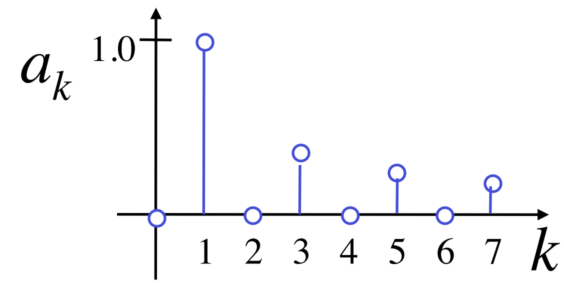
$$\phi_k = 0; \quad a_k = \begin{cases} (-1)^{\frac{k-1}{2}} \frac{1}{k} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

i.e. $x(t) = \cos\left(\frac{2\pi}{T} t\right) - \frac{1}{3} \cos\left(\frac{2\pi}{T} 3t\right) + \frac{1}{5} \cos\left(\frac{2\pi}{T} 5t\right) - \ldots$

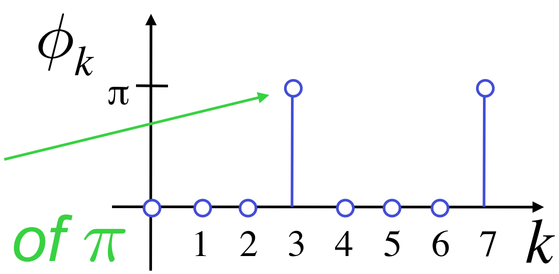


Fourier domain

- x is equivalently described by its Fourier Series parameters:



$$a_k = (-1)^{\frac{k-1}{2}} \frac{1}{k} \quad k = 1, 3, 5, \ldots$$



Negative a_k is equivalent to phase of π

- Complex form: $\tilde{x}(t) \approx \sum_{k=\boxed{-M}}^{M} c_k e^{j \frac{2\pi k}{T} t}$

Fourier analysis

- How to find $\{|c_k|\}, \{\arg\{c_k\}\}$?
  Inner product with complex sinusoids:

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi k}{T}t} dt$$

but
$e^{j\theta} = \cos\theta + j\sin\theta$

$$= \frac{1}{T}\left(\int x(t)\cos\left(\frac{2\pi k}{T}t\right)dt - j\int x(t)\sin\left(\frac{2\pi k}{T}t\right)dt\right)$$

Fourier analysis

- Consider $x(t) = cos\left(l\frac{2\pi}{T}t\right)$

  .. so c_k should $= 0$ except $k = \pm l$

- Then

$$c_k = \frac{1}{T}\left(\int x(t)\cos\tfrac{2\pi kt}{T}\,dt - j\int x(t)\sin\tfrac{2\pi kt}{T}\,dt\right)$$

$0 \because$ $even \cdot odd$

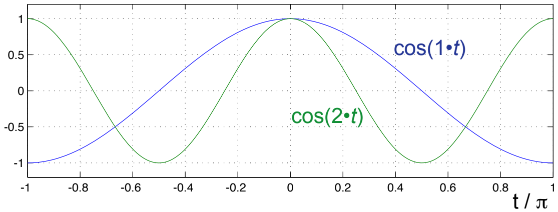
$$= \frac{1}{T}\left(\int \cos\tfrac{2\pi lt}{T}\cos\tfrac{2\pi kt}{T}\,dt - j\int \cos\tfrac{2\pi lt}{T}\sin\tfrac{2\pi kt}{T}\,dt\right)$$

Fourier analysis

- Works $\because$ if k, l are positive integers,

(say $T=2\pi$)

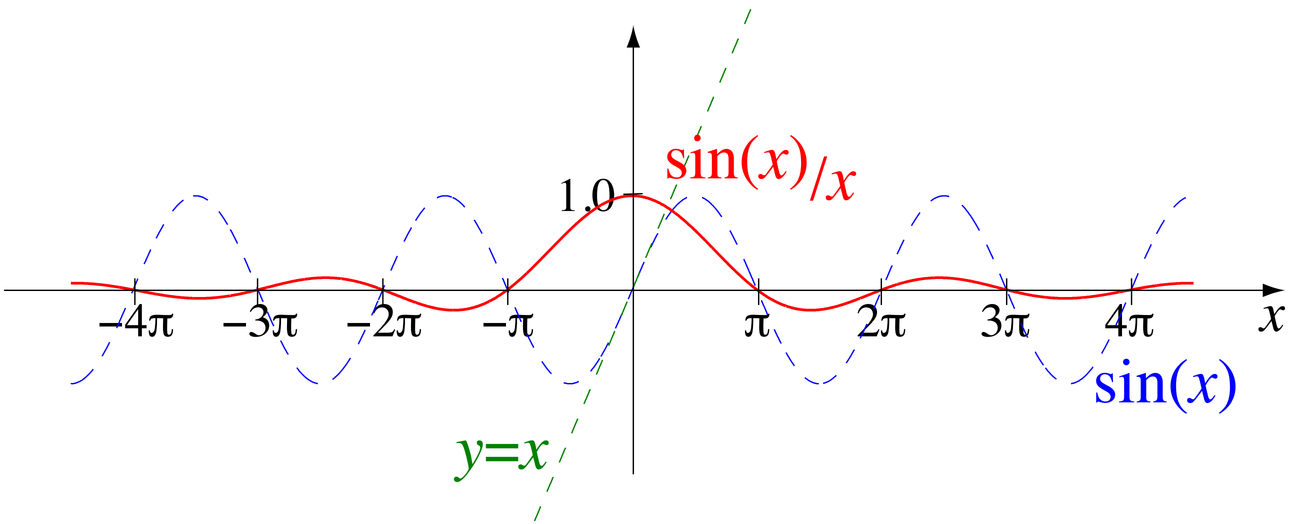
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(kt)\cdot\cos(lt)dt = \begin{cases} 1 & k=\pm l \\ 0 & \text{otherwise}\end{cases}$$



$$\begin{aligned}
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}\cos(k+l)t + \cos(k-l)t dt \\
&= \frac{1}{4\pi}\left[\frac{\sin(k+l)t}{k+l} + \frac{\sin(k-l)t}{k-l}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\left(\mathrm{sinc}\pi(k+l) + \mathrm{sinc}\pi(k-l)\right)
\end{aligned}$$

sinc

- $\operatorname{sinc} x \triangleq \dfrac{\sin x}{x}$



- $= 1$ when $x = 0$
  $= 0$ when $x = r\cdot\pi,\ r \neq 0,\ r = \pm 1, \pm 2, \pm 3, \ldots$

Fourier Analysis

- Thus, $c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt$

  because real & imag sinusoids in $e^{-j\frac{2\pi k}{T}t}$ pick out the corresponding sinusoidal components linearly combined

  in $x(t) = \sum_{k=-M}^{M} c_k e^{j\frac{2\pi k}{T}t}$

Fourier Transform

- Fourier series for periodic signals extends naturally to Fourier Transform for any (CT) signal (not just periodic):

$$X(j\Omega) = \int_{\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Fourier Transform (FT)

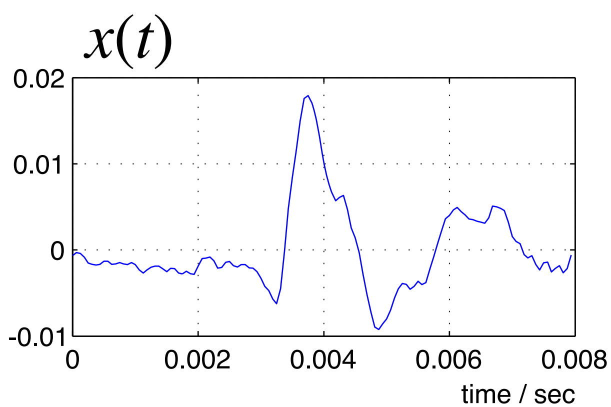
$$x(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

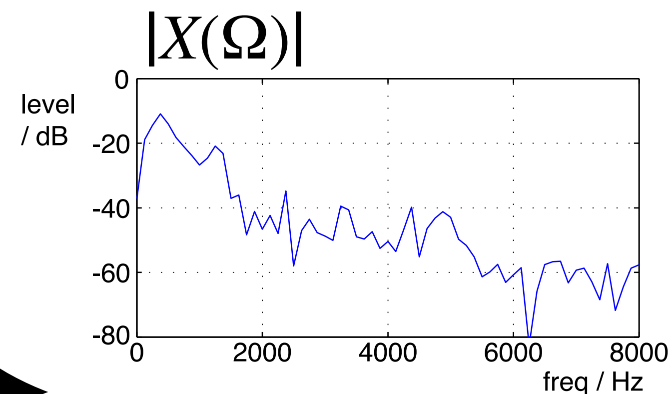
Inverse Fourier Transform (IFT)

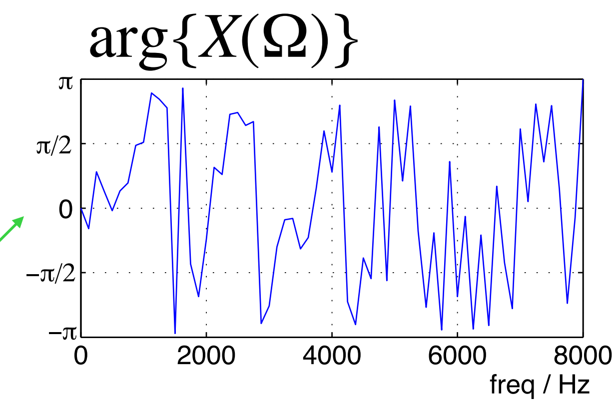
- Discrete index k → continuous freq. Ω

Fourier Transform

- Mapping between two continuous functions:







2π ambiguity

Fourier Transform of a sine

- Assume $x(t) = e^{j\Omega_0 t}$

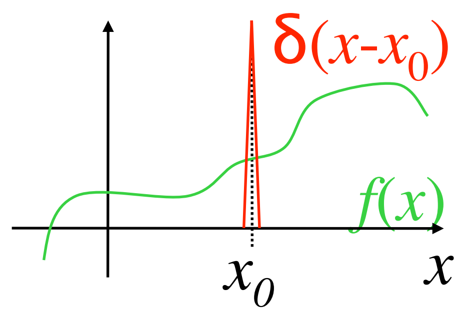
  Now, since $x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$

  ...we know $X(\Omega) = 2\pi\delta(\Omega - \Omega_0)$

  ...where $\delta(x)$ is the Dirac delta function

  (continuous time) i.e.

  $$\int \delta(x - x_0) f(x) dx = f(x_0)$$



- $\rightarrow$ $x(t) = Ae^{j\Omega_0 t} \leftrightarrow X(\Omega) = A\delta(\Omega - \Omega_0)$

Fourier Transforms

	Time	*Frequency*
Fourier Series (FS)	Continuous periodic $\tilde{x}(t)$	Discrete infinite c_k
Fourier Transform (FT)	Continuous infinite $x(t)$	Continuous infinite $X(\Omega)$
Discrete-Time FT (DTFT)	Discrete infinite $x[n]$	Continuous periodic $X(e^{j\omega})$
Discrete FT (DFT)	Discrete finite/pdc $\tilde{x}[n]$	Discrete finite/pdc $X[k]$

2. Discrete Time FT (DTFT)

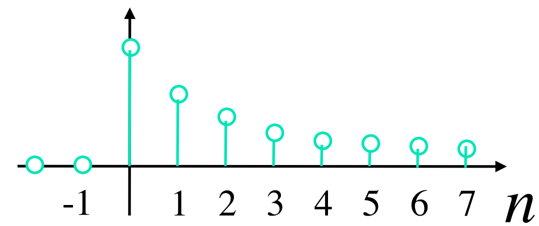
- FT defined for discrete sequences:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{DTFT}$$

- Summation (not integral)
- Discrete (normalized) frequency variable ω
- Argument is $e^{j\omega}$, not $j\omega$

DTFT example

- e.g. $x[n] = \alpha^n \cdot \mu[n]$, $|\alpha| < 1$



$$\Rightarrow \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n}$$

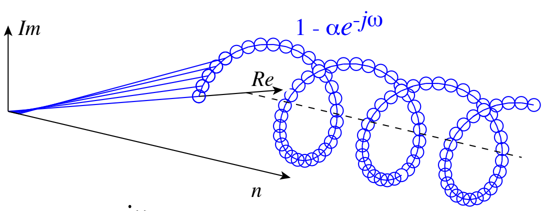
$$= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n$$

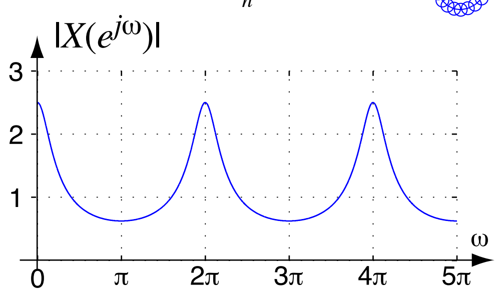
$$= \frac{1}{1 - \alpha e^{-j\omega}}$$

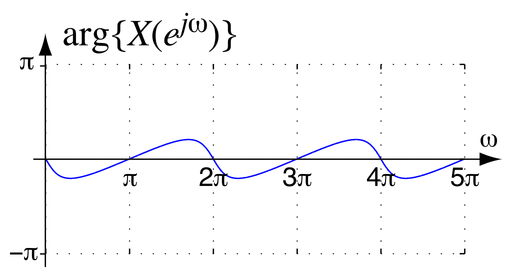
$$S = \sum_{n=0}^{\infty} c^n \Rightarrow cS = \sum_{n=1}^{\infty} c^n$$

$$\Rightarrow S - cS = c^0 = 1$$

$$\Rightarrow S = \frac{1}{1-c} \qquad (\,|c| < 1\,)$$





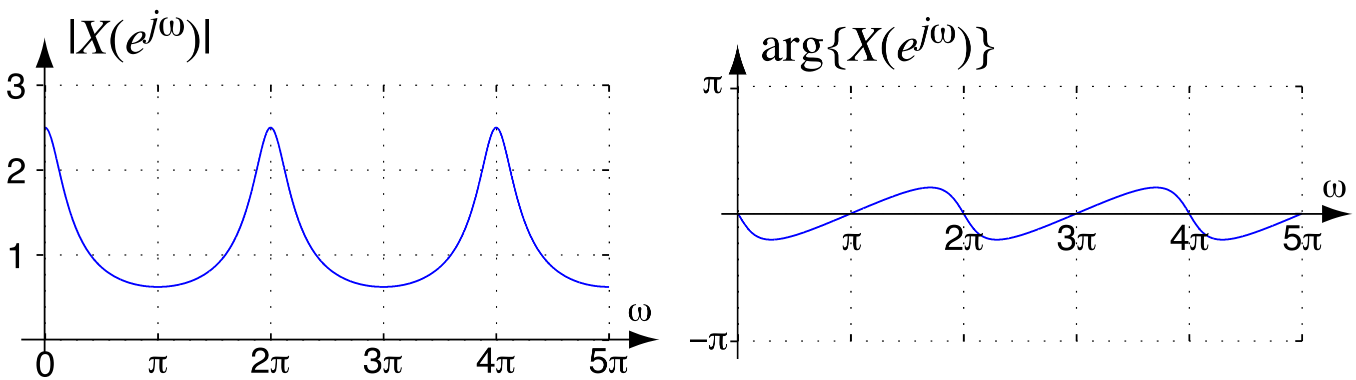


Periodicity of $X(e^{j\omega})$

- $X(e^{j\omega})$ has periodicity 2π in ω :

$$X(e^{j(\omega+2\pi)}) = \sum x[n] e^{-j(\omega+2\pi)n}$$

$$= \sum x[n] e^{-j\omega n} e^{-j2\pi n} = X(e^{j\omega})$$



- Phase ambiguity of $e^{j\omega}$ makes it implicit

Inverse DTFT (IDTFT)

- Same basic form as other IFTs:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \textit{IDTFT}$$

- Note: continuous, periodic $X(e^{j\omega})$

  discrete, infinite $x[n]$ ...

- IDTFT is actually forward Fourier Series (except for sign of ω)

IDTFT

- Verify by substituting in DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_l x[l]e^{-j\omega l}\right)e^{j\omega n}d\omega$$

$$= \sum_l x[l]\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-l)}d\omega$$

$= 0$ *unless* $n = l$

i.e. $= \delta[n\text{-}l]$

$$= \sum_l x[l]\,\mathrm{sinc}\pi(n-l) = x[n]$$

✓

sinc again

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-l)}dw = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-l)}}{j(n-l)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\pi(n-l)} - e^{-j\pi(n-l)}}{j(n-l)}\right)$$

$$= \frac{1}{2\pi}\left(\frac{2j\sin\pi(n-l)}{j(n-l)}\right) = \operatorname{sinc}\pi(n-l)$$

- Same as $\int\cos$ $\because$ imag $j\sin$ part cancels

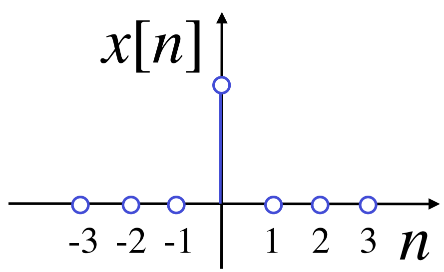
DTFTs of simple sequences

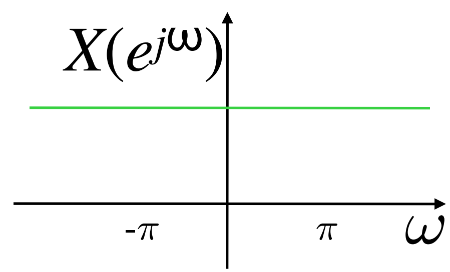
- $\underline{x[n] = \delta[n]} \Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$= e^{-j\omega 0} = 1 \quad \text{(for all } \omega\text{)}$$

- i.e.

$x[n]$		$X(e^{j\omega})$
$\delta[n]$	$\leftrightarrow$	1





DTFTs of simple sequences

IDTFT

- $x[n] = e^{j\omega_0 n}$: $\quad x[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$

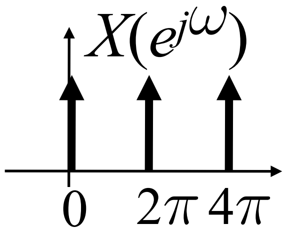
  $\Rightarrow X(e^{j\omega}) = 2\pi \cdot \delta(\omega - \omega_0)$ over $-\pi < \omega < \pi$

  but $X(e^{j\omega})$ must be periodic in ω $\Rightarrow$

  $$e^{j\omega_0 n} \quad \leftrightarrow \quad \sum_k 2\pi \cdot \delta(\omega - \omega_0 - 2\pi k)$$

- If $\omega_0 = 0$ then $x[n] = 1\ \forall\ n$

  so $$1 \quad \leftrightarrow \quad \sum_k 2\pi \cdot \delta(\omega - 2\pi k)$$



DTFTs of simple sequences

- From before:

$$\alpha^n \mu[n] \leftrightarrow \frac{1}{1-\alpha e^{-j\omega}} \qquad (\,|\alpha| < 1)$$

- $\mu[n]$ tricky - not finite

$$\mu[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \underbrace{\sum_k \pi\delta(\omega + 2\pi k)}_{\text{DTFT of 1/2}}$$

DTFT properties

- Linear:

$$\alpha g[n] + \beta h[n] \quad \leftrightarrow \quad \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$$

- Time shift:

$$g[n - n_0] \quad \leftrightarrow \quad e^{-j\omega n_0} G(e^{j\omega})$$

- Frequency shift:

$$e^{j\omega_0 n} g[n] \quad \leftrightarrow \quad G\left(e^{j(\omega-\omega_0)}\right)$$

'delay' in frequency

DTFT example

- $x[n] = \delta[n] + \alpha^n \mu[n\text{-}1] \quad \leftrightarrow \quad ?$

$$= \delta[n] + \alpha(\alpha^{n-1}\mu[n\text{-}1])$$

$$\Rightarrow X(e^{j\omega}) = 1 + \alpha\left(e^{-j\omega\cdot 1}\cdot\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$= 1 + \frac{\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}} = \frac{1-\alpha e^{-j\omega}+\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}}$$

$$= \frac{1}{1-\alpha e^{-j\omega}} \qquad \Rightarrow x[n] = \alpha^n \mu[n] \checkmark$$

DTFT symmetry

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- If $x[n] \leftrightarrow X(e^{j\omega})$ then...

$x[-n] \leftrightarrow X(e^{-j\omega})$ *from summation*

$x^*[n] \leftrightarrow X^*(e^{-j\omega})$ $(e^{-j\omega})^* = e^{j\omega}$

$$\mathrm{Re}\{x[n]\} \leftrightarrow X_{CS}(e^{j\omega}) = \frac{1}{2}\left[X\left(e^{j\omega}\right) + X^*\left(e^{-j\omega}\right)\right]$$

conjugate symmetry cancels Im parts on IDTFT

$$j\mathrm{Im}\{x[n]\} \leftrightarrow X_{CA}(e^{j\omega}) = \frac{1}{2}\left[X\left(e^{j\omega}\right) - X^*\left(e^{-j\omega}\right)\right]$$

$x_{cs}[n] \leftrightarrow \mathrm{Re}\{X(e^{j\omega})\}$

$x_{ca}[n] \leftrightarrow j\mathrm{Im}\{X(e^{j\omega})\}$

DTFT of real $x[n]$

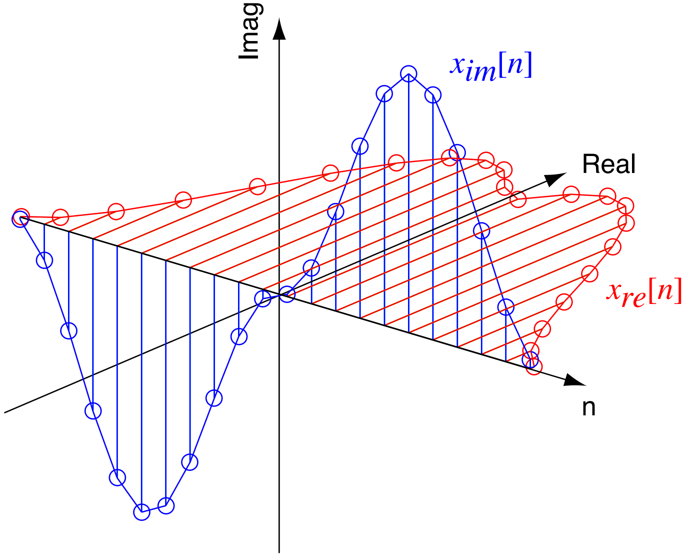
- When $x[n]$ is pure real, $\Rightarrow \underbrace{X(e^{j\omega}) = X^*(e^{-j\omega})}_{X_{CS}}$

$$x_{cs}[n] \equiv x_{ev}[n] = x_{ev}[-n] \quad \leftrightarrow \quad X_R(e^{j\omega}) = X_R(e^{-j\omega})$$

$$x_{ca}[n] \equiv x_{od}[n] = -x_{od}[-n] \quad \leftrightarrow \quad X_I(e^{j\omega}) = -X_I(e^{-j\omega})$$

$x[n]$ *real, even*

$\leftrightarrow X(e^{j\omega})$ *even, real*



DTFT and convolution

- Convolution: $x[n] = g[n] \circledast h[n]$

$$
\begin{aligned}
\Rightarrow X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left(g[n] \circledast h[n]\right) e^{-j\omega n} \\
&= \sum_{n} \left(\sum_{k} g[k] h[n-k] \right) e^{-j\omega n} \\
&= \sum_{k} \left(g[k] e^{-j\omega k} \sum_{n} h[n-k] e^{-j\omega (n-k)} \right) \\
&= G(e^{j\omega}) \cdot H(e^{j\omega})
\end{aligned}
$$

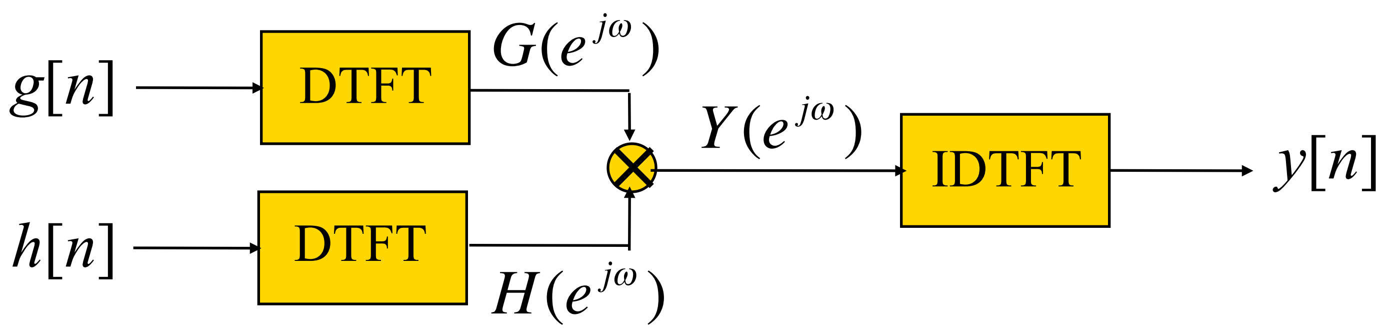
$$g[n] \circledast h[n] \quad \leftrightarrow \quad G(e^{j\omega}) H(e^{j\omega})$$

Convolution *becomes* ***multiplication***

Convolution with DTFT

- Since $g[n] \circledast h[n] \leftrightarrow G(e^{j\omega})H(e^{j\omega})$

  we can calculate a convolution by:
  - finding DTFTs of $g, h \rightarrow G, H$
  - multiply them: $G \cdot H$
  - IDTFT of product is result, $g[n] \circledast h[n]$



DTFT convolution example

- $x[n] = \alpha^n \cdot \mu[n] \Rightarrow X(e^{j\omega}) = \dfrac{1}{1 - \alpha e^{-j\omega}}$
- $h[n] = \delta[n] - \alpha\delta[n-1]$

$$\Rightarrow H(e^{j\omega}) = 1 - \alpha\left(e^{-j\omega\cdot 1}\right)\cdot 1$$

- $y[n] = x[n] \circledast h[n]$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$= \frac{1}{1 - \alpha e^{-j\omega}} \cdot \left(1 - \alpha e^{-j\omega}\right) = 1$$

$\Rightarrow y[n] = \delta[n]$ i.e. ...

DTFT modulation

- Modulation: $x[n] = g[n] \cdot h[n]$
  Could solve if $g[n]$ was just sinusoids...

$$X(e^{j\omega}) = \sum_{\forall n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) e^{j\theta n} d\theta \right) \cdot h[n] e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) \left[\sum_{\forall n} h[n] e^{-j(\omega-\theta)n} \right] d\theta$$

$$\Rightarrow \quad \boxed{g[n] \cdot h[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta}$$

***Dual** of convolution in time*

Parseval's relation

- "Energy" in time and frequency domains are equal:

$$\sum_{\forall n} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})d\omega$$

- If $g = h$, then $g \cdot g^* = |g|^2$ = energy...

Energy density spectrum

- Energy of sequence $\varepsilon_g = \sum_{\forall n} |g[n]|^2$
- By Parseval $\varepsilon_g = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega$
- Define Energy Density Spectrum (EDS)

$$S_{gg}(e^{j\omega}) = |G(e^{j\omega})|^2$$

EDS and autocorrelation

- Autocorrelation of $g[n]$:

$$r_{gg}[\ell] = \sum_{n=-\infty}^{\infty} g[n]g[n-\ell] = g[n] \circledast g[-n]$$

$$\Rightarrow \quad DTFT\{r_{gg}[\ell]\} = G(e^{j\omega})G(e^{-j\omega})$$

- If $g[n]$ is *real*, $G(e^{-j\omega}) = G^*(e^{j\omega})$, so

$$DTFT\{r_{gg}[\ell]\} = \left|G(e^{j\omega})\right|^2 = S_{gg}(e^{j\omega})$$

no phase info.

- Mag-sq of spectrum is DTFT of autoco

3. Discrete FT (DFT)

Discrete FT (DFT)	Discrete finite/pdc $x[n]$	Discrete finite/pdc $X[k]$

- A *finite* or *periodic* sequence has only N unique values, $x[n]$ for $0 \leq n < N$
- Spectrum is completely defined by N distinct frequency samples
- Divide $0..2\pi$ into N equal steps,

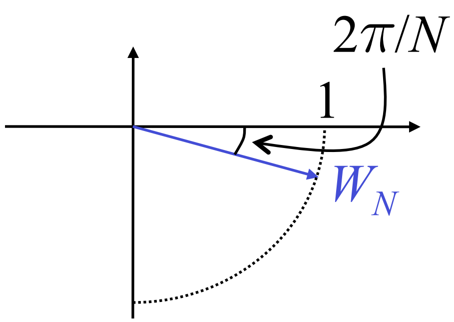
$$\{\omega_k\} = 2\pi k / N$$

DFT and IDFT

- Uniform sampling of DTFT spectrum:

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

- **DFT:** $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$



where $W_N = e^{-j\frac{2\pi}{N}}$ i.e. 1/Nth of a revolution

IDFT

- Inverse DFT **IDFT** $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-nk}$
- Check:

$$x[n] = \frac{1}{N}\sum_k \left(\sum_l x[l] W_N^{kl}\right) W_N^{-nk}$$

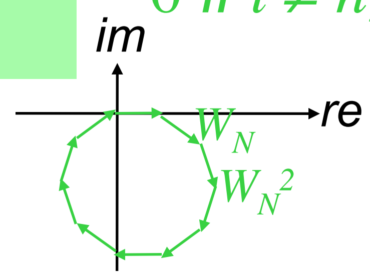
$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l] \sum_{k=0}^{N-1} W_N^{k(l-n)}$$

$$= x[n] \checkmark$$

$0 \le n < N$

Sum of complete set of rotated vectors

$= 0$ *if* $l \neq n$; $= N$ *if* $l = n$



or finite geometric series

$= (1-W_N^{lN})/(1-W_N^{l})$

DFT examples

- *Finite* impulse $x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1..N-1 \end{cases}$

$$\Rightarrow \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = W_N^0 = 1 \quad \forall k$$

- Periodic sinusoid:

$$x[n] = cos\left(\frac{2\pi r n}{N}\right) \qquad (r \in \mathbb{I}) \qquad = \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

$$\Rightarrow X[k] = \tfrac{1}{2}\sum_{n=0}^{N-1}\left(W_N^{-rn} + W_N^{rn}\right)W_N^{kn}$$

$(0 \le k < N)$

$$= \begin{cases} N/2 & k = r, k = N-r \\ 0 & o.w. \end{cases}$$

DFT: Matrix form

- $X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{kn}$ as a matrix multiply:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

- i.e. $\mathbf{X} = \mathbf{D}_N \cdot \mathbf{x}$

Matrix IDFT

- If $\mathbf{X} = \mathbf{D}_N \cdot \mathbf{x}$

  then $\mathbf{x} = \mathbf{D}_N^{-1} \cdot \mathbf{X}$
- i.e. inverse DFT is also just a matrix,

$$\mathbf{D}_N^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

$= {}^1/_N D_N{}^*$

DFT and MATLAB

- MATLAB is concerned with *sequences* not continuous functions like $X(e^{j\omega})$
- Instead, we use the DFT to sample $X(e^{j\omega})$ on an (arbitrarily-fine) grid:
  - `X = freqz(x,1,w);` samples the DTFT of sequence `x` at angular frequencies in `w`
  - `X = fft(x);` calculates the N-point DFT of an N-point sequence `x`

DFT and DTFT

DTFT $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

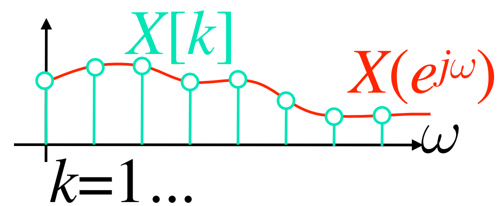
- *continuous freq* ω
- *infinite* $x[n]$, $-\infty < n < \infty$

DFT $$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

- *discrete freq* $k = N\omega/2\pi$
- *finite* $x[n]$, $0 \le n < N$

- DFT 'samples' DTFT at discrete freqs:

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$



DTFT from DFT

- N-point DFT completely specifies the continuous DTFT of the finite sequence

$$X(e^{j\omega}) = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right)e^{-j\omega n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\sum_{n=0}^{N-1} e^{-j\left(\omega-\frac{2\pi k}{N}\right)n}$$

$$\Delta\omega_k = \omega - \frac{2\pi k}{N}$$

"periodic sinc"

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\cdot\frac{\sin N\frac{\Delta\omega_k}{2}}{\sin\frac{\Delta\omega_k}{2}}\cdot e^{-j\frac{(N-1)}{2}\cdot\Delta\omega_k}$$

interpolation

Periodic sinc

$$\sum_{n=0}^{N-1} e^{-j\Delta\omega_k n} = \frac{1-e^{-jN\Delta\omega_k}}{1-e^{-j\Delta\omega_k}}$$

$$= \frac{e^{-jN\Delta\omega_k/2}}{e^{-j\Delta\omega_k/2}} \cdot \frac{e^{jN\Delta\omega_k/2} - e^{-jN\Delta\omega_k/2}}{e^{j\Delta\omega_k/2} - e^{-j\Delta\omega_k/2}}$$

$$= \boxed{e^{-j\frac{(N-1)}{2}\cdot\Delta\omega_k}} \boxed{\frac{\sin N\frac{\Delta\omega_k}{2}}{\sin\frac{\Delta\omega_k}{2}}}$$

pure phase

pure real

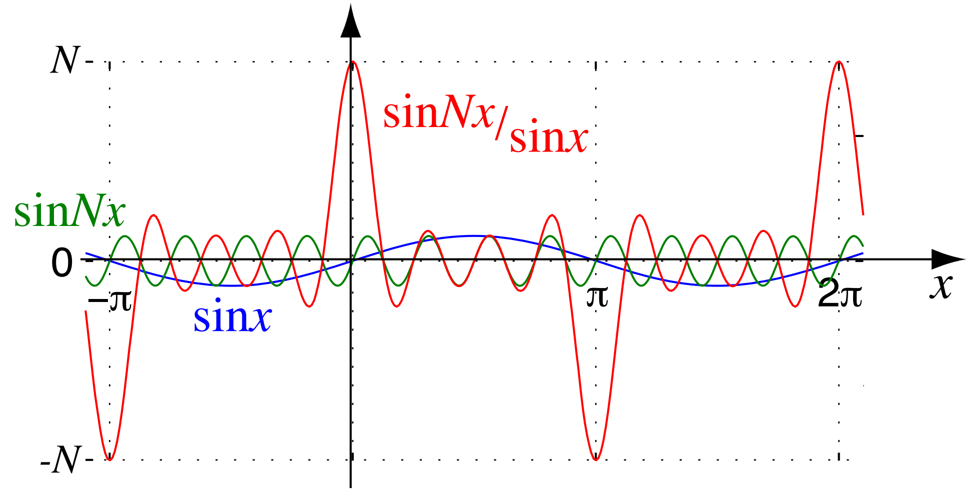
- $= N$ when $\Delta\omega_k = 0$; $= (\text{-})N$ when $\Delta\omega_k/2 = \pi$

  $= 0$ when $\Delta\omega_k/2 = r\cdot\pi/N,\ r = \pm 1, \pm 2, \ldots$

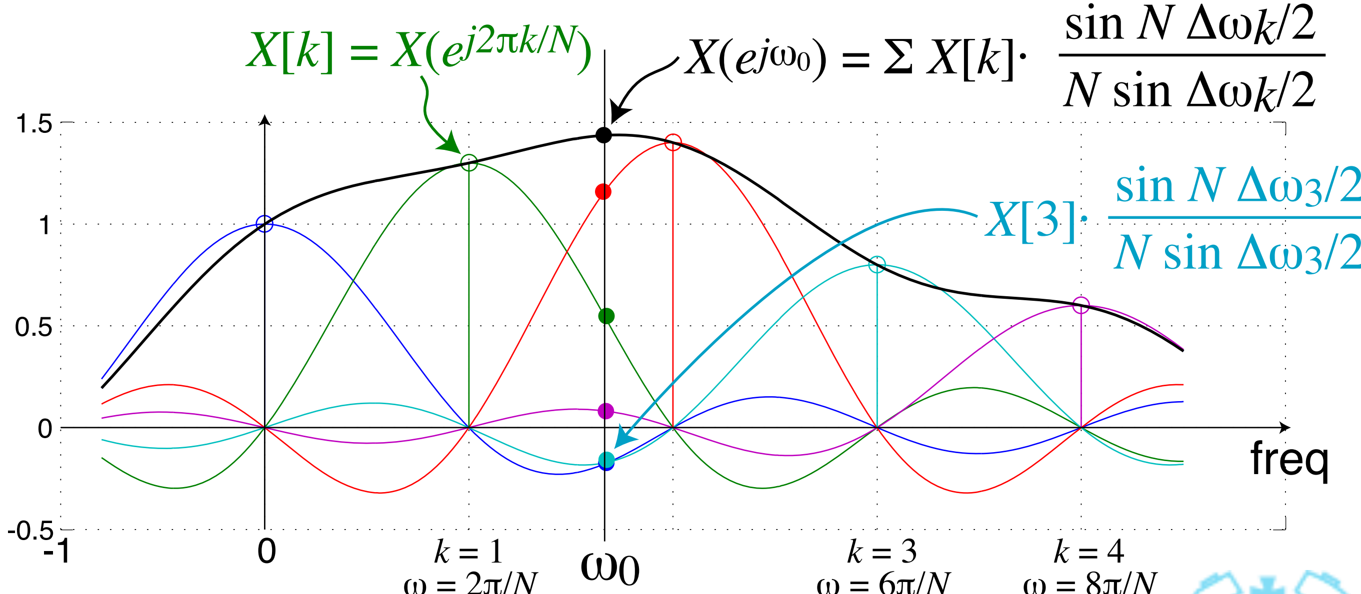
  other values in-between...

Periodic sinc

$$\frac{\sin Nx}{\sin x}$$



DFT→ DTFT
= interpolation
by periodic
sinc



$X[k] \rightarrow X(e^{j\omega})$

DFT from overlength DTFT

- If $x[n]$ has more than N points, can still form $X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$
- IDFT of $X[k]$ will give N point $\tilde{x}[n]$
- How does $\tilde{x}[n]$ relate to $x[n]$?

DFT from overlength DTFT

$$\underset{-A \le n < B}{x[n]} \xrightarrow{\textit{DTFT}} X(e^{j\omega}) \xrightarrow{\textit{sample}} X[k] \xrightarrow{\textit{IDFT}} \underset{0 \le n < N}{\tilde{x}[n]}$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell}\right) W_N^{-nk}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] \left(\boxed{\frac{1}{N}\sum_{k=0}^{N-1} W_N^{k(\ell-n)}}\right)$$

=1 for n-$l = rN$, $r \in$ I

= 0 otherwise

$$\Rightarrow \underset{0 \le n < N}{\tilde{x}[n]} = \sum_{r=-\infty}^{\infty} x[n - rN]$$

all values shifted by exact multiples of N pts to lie in $0 \le n < N$

DFT from DTFT example

- If $x[n] = \{8, 5, 4, 3, 2, 2, 1, 1\}$ (8 point)
- We form $X[k]$ for $k = 0, 1, 2, 3$
  by sampling $X(e^{j\omega})$ at $\omega = 0, \pi/2, \pi, 3\pi/2$
- IDFT of $X[k]$ gives 4 pt $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$ $(N = 4)$
- Overlap only for $r = -1$:

$$\Rightarrow \tilde{x}[n] = \left\{ \begin{matrix} 8 & 5 & 4 & 3 \\ + & + & + & + \\ 2 & 2 & 2 & 1 \end{matrix} \right\} = \{10 \quad 7 \quad 5 \quad 4\}$$

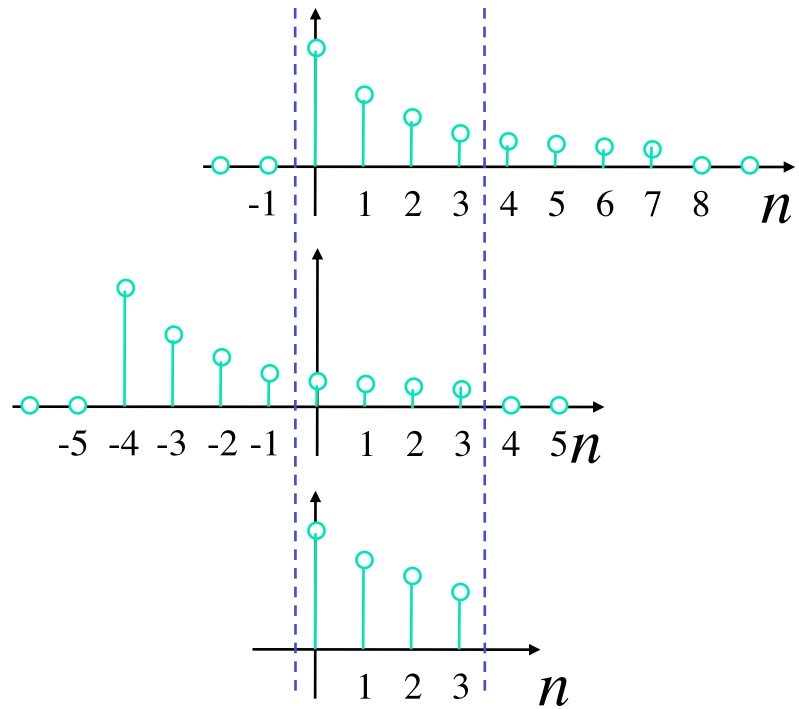
DFT from DTFT example

- $x[n]$

- $x[n+N]$

  $(r = -1)$

- $\tilde{x}[n]$



- $\tilde{x}[n]$ is the time aliased or 'folded down' version of $x[n]$.

Properties: Circular time shift

- DFT properties mirror DTFT, with twists:
- Time shift must stay within N-pt 'window'

$$g[\langle n - n_0 \rangle_N] \quad \leftrightarrow \quad W_N^{kn_0} G[k]$$

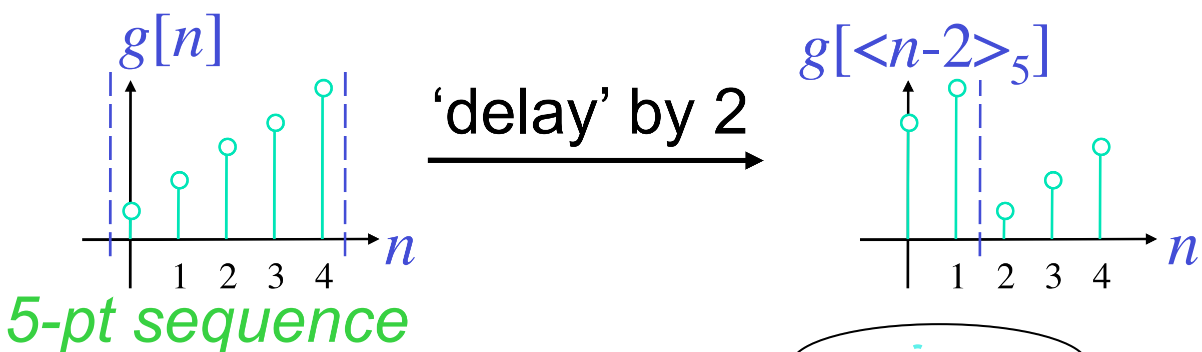
- Modulo-N indexing keeps index between 0 and N-1:

$$g[\langle n - n_0 \rangle_N] = \begin{cases} g[n - n_0] & n \geq n_0 \\ g[N + n - n_0] & n < n_0 \end{cases}$$

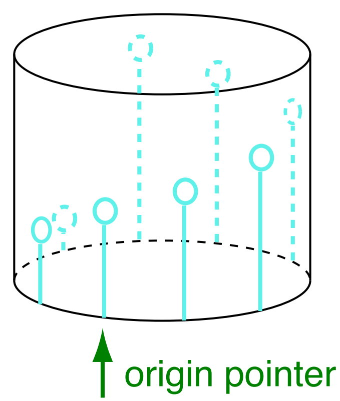
$0 \leq n_0 < N$

Circular time shift

- Points shifted out to the right don’t disappear – they come in from the left

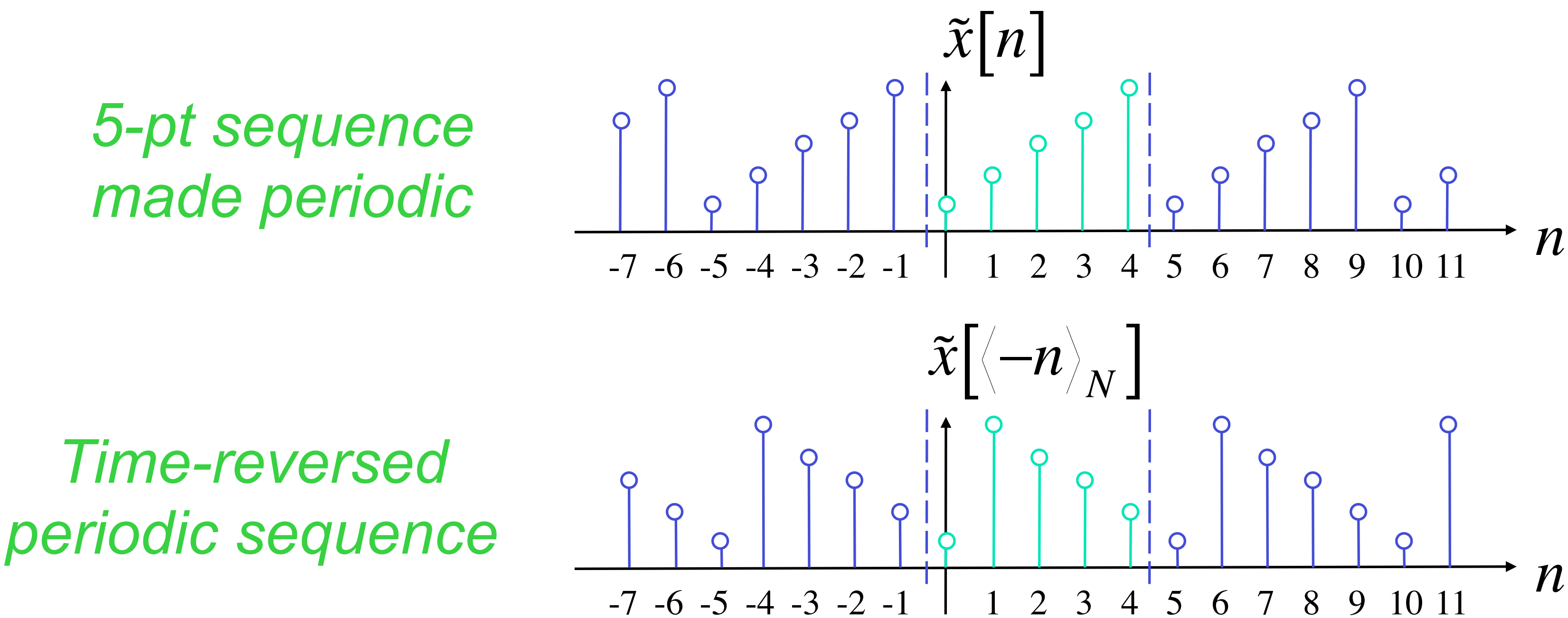


- Like a ‘barrel shifter’:



Circular time reversal

- Time reversal is tricky in 'modulo-N' indexing - **not** reversing the sequence:



- Zero point stays fixed; remainder flips

Duality

- DFT and IDFT are very similar
  - both map an N-pt vector to an N-pt vector
- Duality:

if $\quad g[n] \leftrightarrow G[k]$

then $\quad \boxed{G[n] \leftrightarrow N \cdot g[\langle -k \rangle_N]}$

Circular time reversal

- i.e. if you treat DFT sequence as a time sequence, result is almost symmetric

4. Convolution with the DFT

- IDTFT of product of DTFTs of two N-pt sequences is their $2N$-1 pt convolution
- IDFT of the product of two N-pt DFTs can only give N points!
- Equivalent of $2N$-1 pt result time aliased:
  - i.e. $y_c[n] = \sum_{r=-\infty}^{\infty} y_l[n+rN] \qquad (0 \le n < N)$
  - must be, because $G[k]H[k]$ are exact samples of $G(e^{j\omega})H(e^{j\omega})$
- This is known as **circular convolution**

Circular convolution

- Can also do entire convolution with modulo-N indexing
- Hence, **Circular Convolution**:

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N] \quad \leftrightarrow \quad G[k]H[k]$$

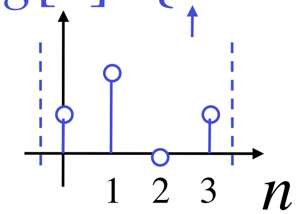
- Written as $g[n] \, Ⓝ \, h[n]$

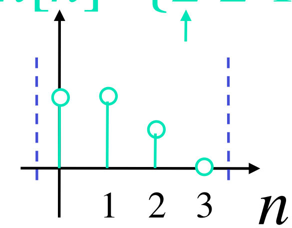
Circular convolution example

- 4 pt sequences:

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N]$$

$g[n]=\{1\ 2\ 0\ 1\}$ $h[n]=\{2\ 2\ 1\ 0\}$



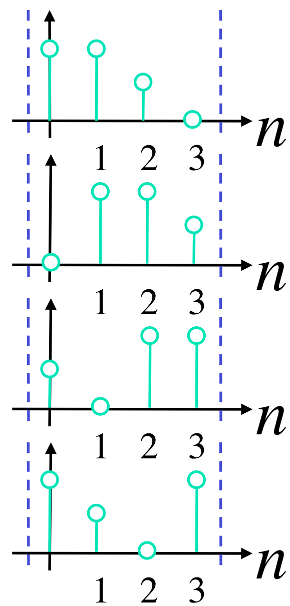


$h[\langle n-0\rangle_4]$ $\cdot\ 1$

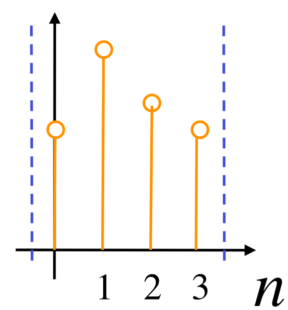
$h[\langle n-1\rangle_4]$ $\cdot\ 2$

$h[\langle n-2\rangle_4]$ $\cdot\ 0$

$h[\langle n-3\rangle_4]$ $\cdot\ 1$



$g[n]$ ④ $h[n]=\{4\ 7\ 5\ 4\}$



check: $g[n] \circledast h[n]$
$=\{2\ 6\ 5\ 4\ 2\ 1\ 0\}$

DFT properties summary

- Circular convolution

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N] \quad \leftrightarrow \quad G[k]H[k]$$

- Modulation

$$g[n]\cdot h[n] \quad \leftrightarrow \quad \tfrac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m \rangle_N]$$

- Duality

$$G[n] \quad \leftrightarrow \quad N\cdot g[\langle -k \rangle_N]$$

- Parseval

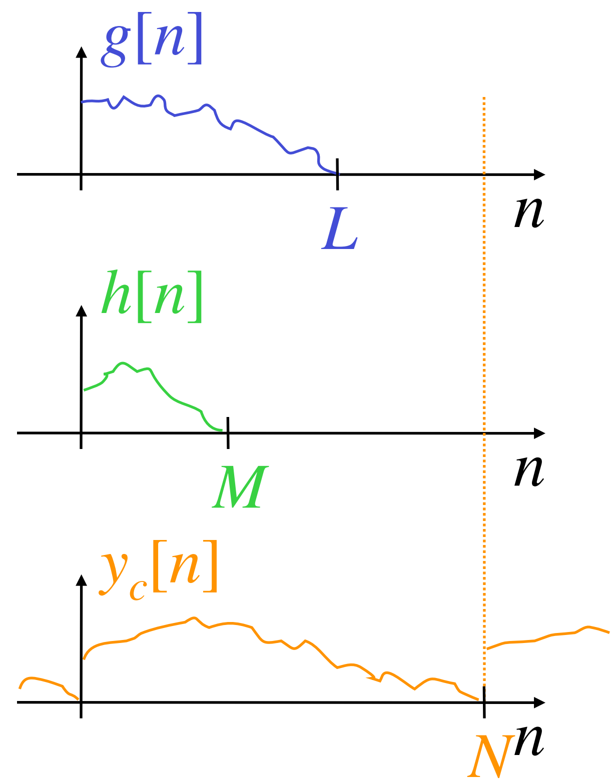
$$\sum_{n=0}^{N-1} \left|x[n]\right|^2 = \tfrac{1}{N}\sum_{k=0}^{N-1} \left|X[k]\right|^2$$

Linear convolution w/ the DFT

- DFT → fast circular convolution
- .. but we need linear convolution
- Circular conv. is time-aliased linear conv.; can aliasing be avoided?
- e.g. convolving L-pt $g[n]$ with M-pt $h[n]$: $y[n] = g[n] \circledast h[n]$ has $L+M$-1 nonzero pts
- Set DFT size $N \geq L+M$-1 → **no aliasing**

Linear convolution w/ the DFT

- Procedure ($N = L + M - 1$):
  - pad L-pt $g[n]$ with (at least) M-1 zeros
    $\rightarrow$ N-pt DFT $G[k], k = 0..N\text{-}1$
  - pad M-pt $h[n]$ with (at least) L-1 zeros
    $\rightarrow$ N-pt DFT $H[k], k = 0..N\text{-}1$
  - $Y[k] = G[k]\cdot H[k],\ k = 0..N\text{-}1$
  - IDFT{$Y[k]$} $= \sum_{r=-\infty}^{\infty} y_L[n+rN] = y_L[n]$ $(0 \le n < N)$



Overlap-Add convolution

- Very long $g[n]$ → break up into segments, convolve piecewise, overlap
  → bound size of DFT, processing delay
- Make $g_i[n] = \begin{cases} g[n] & i \cdot N \leq n < (i+1) \cdot N \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \qquad g[n] = \textstyle\sum_i g_i[n]$$

$$\Rightarrow \quad h[n] \circledast g[n] = \textstyle\sum_i h[n] \circledast g_i[n]$$

- Called Overlap-Add (**OLA**) convolution...

Overlap-Add convolution