Spatial Optics

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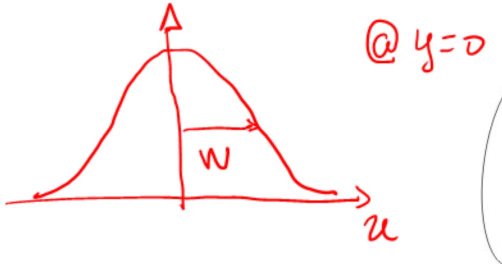
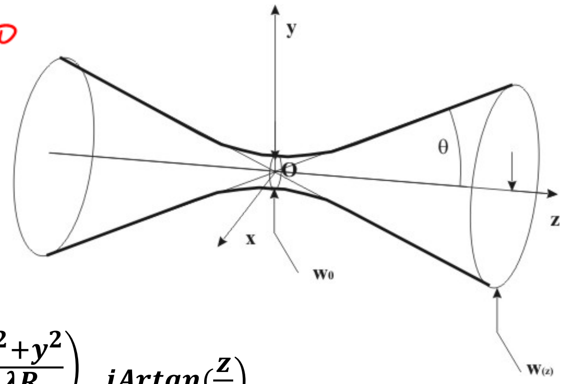
Gaussian beams





Why?

The only realistic solution for a free beam propagation

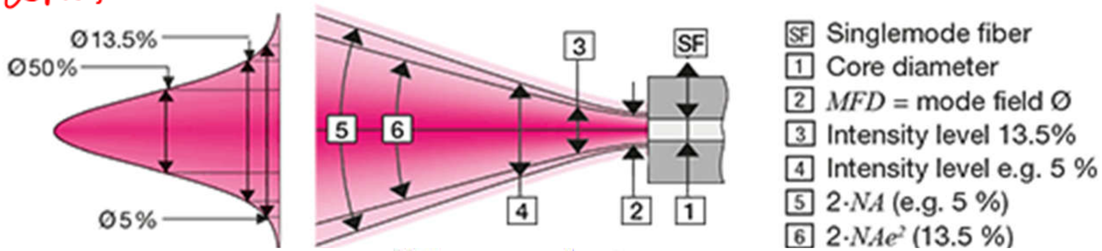
$$E_z(x,y) = E_0 \frac{e^{-(\frac{x^2+y^2}{w^2})} e^{-jkz} e^{-j\pi(\frac{x^2+y^2}{\lambda R})} e^{jArtan(\frac{z}{\alpha})}}{\sqrt{1+\frac{z^2}{\alpha^2}}}$$

Optical field

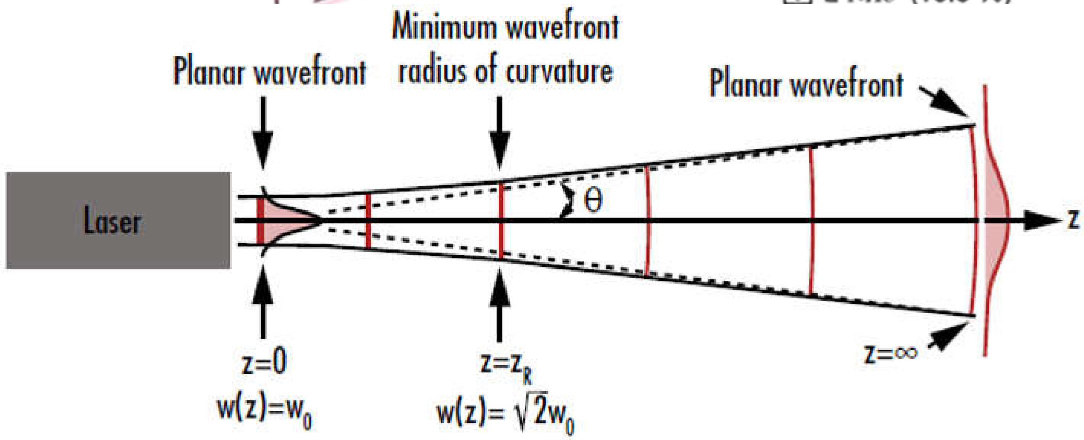
propag axis

position

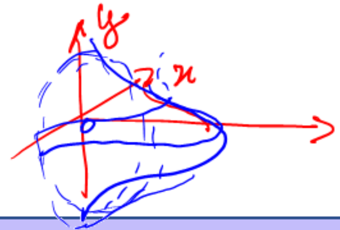
Modélisation of the single mode Optical fibre beams



Modélisation of the single mode Laser beams



Propagation of a gaussian beam



z propag

Gaussian beam

transverse coordinates

$$E_0(x,y) = \boldsymbol{E}_0\, e^{-(\frac{x^2+y^2}{w_0^2})}$$

z=0

propagation axis z

propagation

$$E_z(x,y) = \boldsymbol{E}_0 \frac{e^{-(\frac{x^2+y^2}{w^2})}\, e^{-jkz}\, e^{-j\pi(\frac{x^2+y^2}{\lambda R})}\, e^{jArtan(\frac{z}{\alpha})}}{\sqrt{1+\frac{z^2}{\alpha^2}}}$$

z≠0

with

W0 radius of the input beam

$$\alpha = \frac{\pi . w_0^2}{\lambda}$$

$$\boldsymbol{R} = z + \frac{\alpha^2}{z}$$

$$\boldsymbol{w}^2 = \boldsymbol{w}_0^2\left(1 + \frac{z^2}{\alpha^2}\right)$$

Two aspects :

*** Demonstration of the analytic solution**) Data processing

*** Analysis of the formula**) Applications

Demonstration of the gaussian beam formula

general data processing scheme.

General principle

$$in(x,y) \xrightarrow{*\,PSF} out(x,y) = in * PSF = FT^{-1}(\widetilde{in}.TF)$$

FT ❶ LTI system Σ (Free space) ❸ FT^{-1}

$$\widetilde{in}(N_x, N_y) \xrightarrow{.TF\ ❷} \widetilde{out}(N_x, N_y) = \widetilde{in}(N_x, N_y).\mathrm{TF}$$

❶ **Input field spectrum**

❷ **Transfert function and output spectrum determination**

❸ **Output field derivation**

❶ Input field spectrum calculation

$$E_0(t) = E_0\, e^{-\pi t^2} \quad \xrightarrow{\text{FT}} \quad \widetilde{E_0}(v) = E_0\, e^{-\pi v^2}$$

$$f(t) \quad \xrightarrow{\text{FT}} \quad \tilde{f}(v)$$

Scaling the Function

cte.

$$f(kt) \quad \xrightarrow{\text{FT}} \quad \frac{1}{k}\tilde{f}\left(\frac{v}{k}\right)$$

$$f(kx) \quad \xrightarrow{\text{FT}} \quad \frac{1}{k}\tilde{f}\left(\frac{N_x}{k}\right)$$

$$f(kx, ky) \quad \xrightarrow{\text{FT}} \quad \frac{1}{k^2}\tilde{f}\left(\frac{N_x}{k}, \frac{N_y}{k}\right)$$

$$E_0(x, y) = E_0\, e^{-\left(\frac{x^2+y^2}{w_0^2}\right)} \quad \xrightarrow{\text{FT}} \quad \widetilde{E}_0\left(N_x, N_y\right) = E_0\, \pi w_0^2 e^{-\pi^2 w_0^2 \left({N_x}^2 + {N_y}^2\right)}$$

$$f(t) \xrightarrow{FT} \tilde{F}(\nu) = \int_{-\infty}^{+\infty} f(t)\, e^{-j2\pi\nu t}\, dt$$

$$t \longrightarrow x, y$$

$$\nu \longrightarrow N_x \quad N_y \qquad N_x = \frac{k_x}{2\pi}$$

$$f(x,y) \longrightarrow \tilde{F}(N_x N_y) = \iint f(x,y)\, e^{j2\pi(xN_x + yN_y)}\, dx\, dy$$

$$\underset{z=0}{E_0}(x,y) = e^{-\frac{x^2+y^2}{w_0^2}} \longrightarrow$$

$$f(t) \longrightarrow \tilde{F}(\nu) = \int f(t)\, e^{-j2\pi\nu t}\, dt$$

$$f(k.t) \longrightarrow \tilde{F}_1(\nu) = \int f(\overbrace{kt}^{u})\, e^{-j2\pi\frac{\nu}{k}\overbrace{tk}^{u}}\, \frac{d\overbrace{kt}^{u}}{k}$$

$$f(kt) \longrightarrow \tilde{F}_1(\nu) = \int f(u)\, e^{-j2\pi\frac{\nu}{k}u}\, \frac{du}{k} = \frac{1}{k}\tilde{F}\left(\frac{\nu}{k}\right)$$

$$e^{-\frac{x^2+y^2}{w_0^2}}$$

$$e^{-\pi t^2} \xrightarrow{FT} e^{-\pi \nu^2}$$

$$e^{-\pi x^2} \longrightarrow e^{-\pi N_x^2}$$

$$e^{-\pi y^2} \longrightarrow e^{-\pi N_y^2}$$

$$e^{-\frac{x^2}{w_0^2}} = e^{-\pi (h x)^2} \longrightarrow \frac{1}{h} e^{-\pi \left(\frac{N_x}{h}\right)^2} = \sqrt{\pi}\, w_0\, e^{-\pi^2 w_0^2 N_x^2}$$

$$h^2 \pi = \frac{1}{w_0^2}$$

$$h = \frac{1}{\sqrt{\pi}\, w_0}$$

$$e^{-\frac{y^2}{w_0^2}} \longrightarrow \sqrt{\pi}\, w_0\, e^{-\pi^2 w_0^2 N_y^2}$$

$$E = e^{-\frac{x^2}{w_0^2}} \cdot e^{-\frac{y^2}{w_0^2}} \xrightarrow{FT} \pi w_0^2\, e^{-\pi^2 w_0^2 (N_x^2 + N_y^2)}$$

❷ Transfert function an output spectrum determination

$$PW_{z=0}(x,y) = e^{-j(k_x.x+k_y.y)} \qquad PW_{z\neq 0}(x,y) = e^{-j(k_x.x+k_y.y)}\boxed{e^{-j(k_z.z)}}$$

Transfert function

$$e^{-j\vec{k}.\overrightarrow{\mathbf{OM}}} \qquad \vec{k} = \frac{2\pi}{\lambda}\vec{n} \qquad \vec{n} = \begin{matrix}\sin(\alpha)\\ \sin(\beta)\\ \cos(\gamma)\end{matrix} \qquad N_x = \frac{k_x}{2\pi} = \frac{\sin(\alpha)}{\lambda}$$

$$N_y = \frac{\sin(\beta)}{\lambda}$$

$$\cos(\gamma) = \sqrt{1-(\sin^2(\alpha)+\sin^2(\beta))} = 1 - 1/2(\sin^2(\alpha)+\sin^2(\beta))$$

$$k_z = \frac{2\pi}{\lambda}\left(1-\frac{1}{2}\lambda^2({N_x}^2+{N_y}^2)\right) \qquad TF(N_x,N_y) =.e^{-j(k_z.z)} = e^{-j\frac{2\pi}{\lambda}z}.e^{+j\frac{\pi}{\lambda}\lambda^2({N_x}^2+{N_y}^2)z}$$

Transfert function

$$\tilde{E}_0(N_x,N_y) = E_0\,\pi w_0^2 e^{-\pi^2 w_0^2({N_x}^2+{N_y}^2)}$$

Propagation = x Transfert function

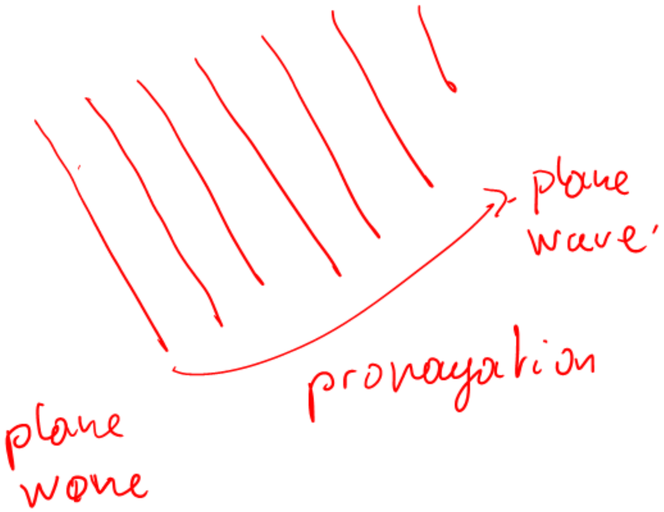
$$\tilde{E}_z(N_x,N_y) = E_0\,\pi w_0^2 e^{-\pi^2 w_0^2({N_x}^2+{N_y}^2)}.e^{-j\frac{2\pi}{\lambda}z}.e^{+j\pi\lambda z({N_x}^2+{N_y}^2)}$$

eigen function for free space propagation

plane waves " $e^{-j\vec{k}\cdot\vec{OM}}$

$\vec{OM}$ → position vector

$\vec{k}$: wave vector



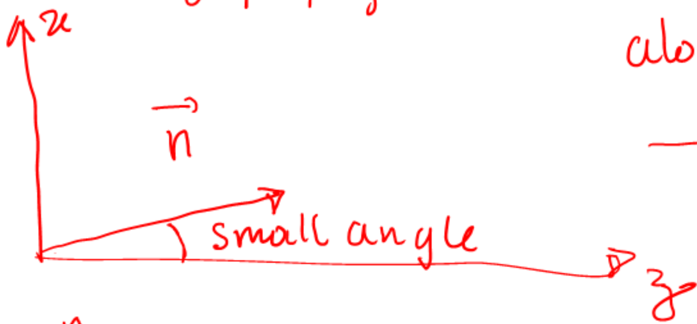
$$\vec{k} = \frac{2\pi}{\lambda}\vec{n} \qquad \vec{OM} = \left|\begin{matrix} x \\ y \\ z \end{matrix}\right.$$

x, y) tranverse coordinates

z) main propagation axis

$\vec{n}$ denotes unit vector that give the direction of the plane wave of propag.

Function is significant for $x, y \ll z$ paraxial

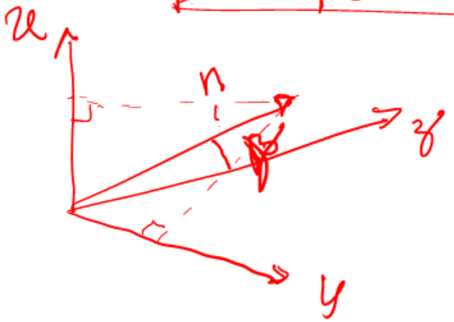


along z axis $\vec{n}$ # 1

x and y : $\vec{n}$ ~# 0 small

$$\vec{n} = \left|\begin{matrix} \sin\alpha \\ \sin\beta \\ \cos\gamma \end{matrix}\right.$$

$\alpha\ \beta\ \gamma$ are small.



$$\vec{k} = \frac{2\pi}{\lambda}\left|\begin{matrix} \sin\alpha \\ \sin\beta \\ \cos\gamma \end{matrix}\right.$$

Plane wave $= PW_z(x;y) = e^{-j\frac{2\pi}{\lambda}\left(x\sin\alpha + y\sin\beta + z\cos\gamma\right)}$

$$= e^{-j2\pi\left(x\left(\frac{\sin\alpha}{\lambda}\right) + y\left(\frac{\sin\beta}{\lambda}\right)\right)} \cdot e^{-j2\pi\frac{z\cos\gamma}{\lambda}}$$

eigein functions + Fourier T. $\quad e^{-j2\pi(x\underbrace{N_x + y N_y}_{\text{Spatial frequency}})}$

$PW_0(x;y) = e^{-j2\pi(xN_x + yN_y)}$ with $N_x = \frac{\sin\alpha}{\lambda}$

$$PW_z(x;y) = PW_0(x;y) \cdot \underbrace{e^{-j2\pi\frac{z\cos\gamma}{\lambda}}}_{\text{Transfert Function}(N_x,N_y)} \qquad N_y = \frac{\sin\beta}{\lambda}$$

As $\vec{n}$ is a unit vector

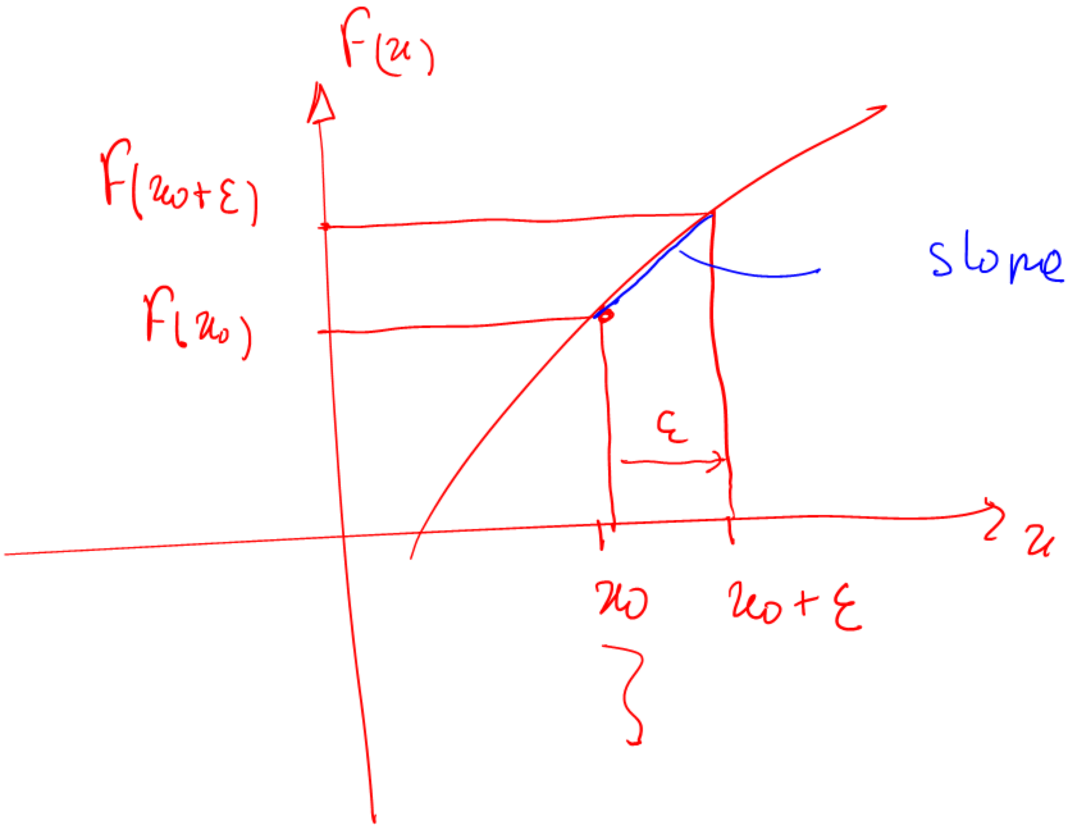
$$\sin^2\alpha + \sin^2\beta + \cos^2\gamma = 1 \qquad \sqrt{1+\varepsilon} \# 1 + \frac{\varepsilon}{2}$$

$$\cos\gamma = \sqrt{1-(\sin^2\alpha + \sin^2\beta)} \# 1 - \frac{1}{2}(\sin^2\alpha + \sin^2\beta)$$

$$\# 1 - \frac{1}{2}\lambda^2(N_x^2 + N_y^2)$$

Transfert Function:

$$\boxed{TF_z(N_x;N_y) = e^{-j\frac{2\pi z}{\lambda}} \cdot e^{+j\pi\lambda z(N_x^2 + N_y^2)}}$$



$$\text{slope} = \frac{\partial F}{\partial x}(x_0) = \frac{F(x_0+\varepsilon) - F(x_0)}{\varepsilon}$$

$$F(x_0+\varepsilon) = F(x_0) + \varepsilon \cdot \frac{\partial F}{\partial x}(x_0) + \frac{1}{2}\varepsilon^2 \frac{\partial^2 F}{\partial x^2}$$

$\vdots$

$$F = \sqrt{x} \qquad F(1) = 1$$

$$\frac{\partial F}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x}} \qquad \left(\frac{\partial F}{\partial x}\right)(1) = \frac{1}{2}$$

$$F(x_0+\varepsilon) = F(x_0) + \left(\frac{\partial F}{\partial x}\right)(x_0)$$

$$\sqrt{1+\varepsilon} = 1 + \frac{1}{2}\cdot\varepsilon$$

❸ Output field derivation

$$\tilde{E}_z(N_x, N_y) = E_0\, \pi w_0^2 e^{-\pi^2 w_0^2 (N_x{}^2 + N_y{}^2)} . e^{-j\frac{2\pi}{\lambda}z} . e^{+j\pi\lambda z (N_x{}^2 + N_y{}^2)}$$

$$= E_0\, \pi w_0^2 e^{-\pi((\pi w_0^2 - j\lambda z))(N_x{}^2 + N_y{}^2)} . e^{-j\frac{2\pi}{\lambda}z}$$

k'2

FT-1

$$E_z(x,y) = E_0 \frac{\pi w_0^2}{\pi w_0^2 - j\lambda z} e^{-\pi \frac{(x^2+y^2)}{\pi w_0^2 - j\lambda z}} . e^{-j\frac{2\pi}{\lambda}z}$$

$$\alpha = \frac{\pi . w_0^2}{\lambda}$$

$$E_z(x,y) = E_0 \frac{e^{-\left(\frac{x^2+y^2}{w^2}\right)} e^{-jkz} e^{-j\pi\left(\frac{x^2+y^2}{\lambda R}\right)} e^{jArtan\left(\frac{z}{\alpha}\right)}}{\sqrt{1+\frac{z^2}{\alpha^2}}}$$

$$R = z + \frac{\alpha^2}{z}$$

$$w^2 = w_0^2\left(1+\frac{z^2}{\alpha^2}\right)$$

$$\tilde{E}_z(N_x, N_y) = \underbrace{E_0\, \pi w_0^2\, e^{-\pi^2 w_0^2 (N_x^2+N_y^2)}}_{\text{input spectrum}} \; \underbrace{e^{-j\frac{2\pi z}{\lambda}}\, e^{+j\pi\lambda z (N_x^2+N_y^2)}}_{TF_z(N_x, N_y)}$$

$$= \underbrace{E_0\, \pi w_0^2\, e^{-j\frac{2\pi z}{\lambda}}}_{c^{te} \text{ as } f^t \text{ of } N_x, N_y} \; e^{-\pi (N_x^2+N_y^2) \overbrace{(\pi w_0^2 - j\lambda z)}^{}}_{}$$

$$\underbrace{(\pi w_0^2 - j\lambda z)}_{h'^2}$$

$$\Bigg\Downarrow FT^{-1} \qquad\qquad e^{-\pi\left((h' N_x)^2 + (h' N_y)^2\right)}$$

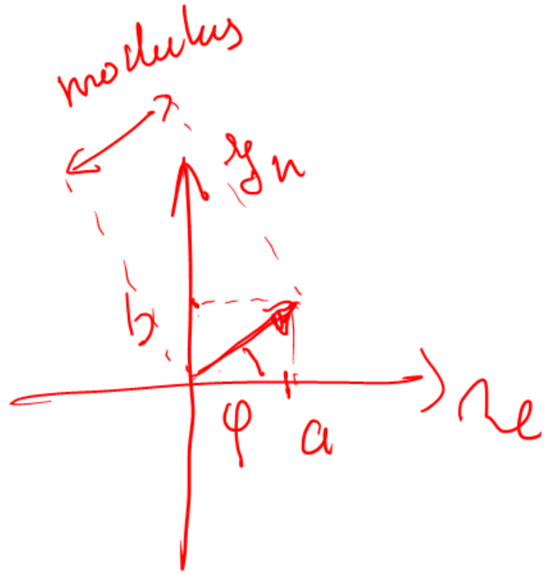
$$\|$$

$$E_z(x, y) = E_0\, \pi w_0^2\, e^{-j\frac{2\pi z}{\lambda}} \cdot \frac{1}{h'^2}\, e^{-\pi\left(\left(\frac{x}{h'}\right)^2 + \left(\frac{y}{h'}\right)^2\right)}$$

$$E_z(x, y) = \frac{E_0\, \pi w_0^2\, e^{-j\frac{2\pi z}{\lambda}}\, e^{-\pi \frac{x^2+y^2}{(\pi w_0^2 - j\lambda z)}}}{\pi w_0^2 - j\lambda z}$$

$$CN = a + jb = \text{modulus} \cdot e^{j\varphi}$$

$$= \sqrt{a^2+b^2} \cdot e^{j\,\text{Arctan}(b/a)}$$



$$* \quad \frac{\pi W_0^2}{\pi W_0^2 - j\lambda z} = \frac{1}{1 - j\, z/\alpha}$$

with $\alpha = \dfrac{\pi W_0^2}{\lambda}$ it's a length = the Fresnel length

$$= \frac{1}{1 - j\,\dfrac{z\lambda}{\pi W_0^2}}$$

$$= \frac{1}{\sqrt{1 + z^2/\alpha^2} \cdot e^{-j\arctan(z/\alpha)}} = \frac{e^{+j\arctan z/\alpha}}{\sqrt{1 + z^2/\alpha^2}}$$

$$* \quad e^{-\pi \frac{x^2+y^2}{(\pi w_0^2 - j\lambda z)}} = e^{-\frac{\cancel{\pi}(x^2+y^2)}{\cancel{\pi} w_0^2 \left(1 - j\frac{z}{\alpha}\right)}} = e^{-\frac{(x^2+y^2)\left(1+j\frac{z}{\alpha}\right)}{w_0^2\left(1+z^2/\alpha^2\right)}}$$

$$= e^{-\frac{(x^2+y^2)}{w_0^2(1+z^2/\alpha^2)}} \cdot e^{-\frac{j\pi\, z/\alpha\, (x^2+y^2)}{\underbrace{\frac{\pi w_0^2}{\lambda}}\left(1+z^2/\alpha^2\right)\lambda}}$$

$$= e^{-\frac{x^2+y^2}{w^2}} \cdot e^{-j\frac{\pi(x^2+y^2)}{\alpha\left(\frac{\alpha}{z}+z/\alpha\right)\lambda}}$$

$$= e^{-\frac{x^2+y^2}{w^2}} \underbrace{e^{-j\frac{\pi(x^2+y^2)}{\lambda R}}}_{\text{spheric wave}}$$

with $\quad w^2 = w_0^2\left(1 + z^2/\alpha^2\right)$

and $\quad R = z + \frac{\alpha^2}{z}$

$$\left\{ E_z(x,y) = \frac{E_0 \, \pi w_0^2 \, e^{-j\frac{2\pi z}{\lambda}} \, e^{-\pi \frac{x^2+y^2}{(\pi w_0^2 - j\lambda z)}}}{\pi w_0^2 - j\lambda z} \right.$$

$$\boxed{E_z(x,y) = \frac{E_0 \, \overbrace{e^{j\arctan(z/\alpha)}}^{\text{Gouy phase shift}} \, \overbrace{e^{-j\frac{2\pi z}{\lambda}}}^{\text{plane wave phase}} \, \overbrace{e^{-\frac{x^2+y^2}{w^2}}}^{\text{gaussian f t}} \, \overbrace{e^{-j\pi\frac{x^2+y^2}{\lambda R}}}^{\text{spherical wave phase}}}{\underbrace{\left(\sqrt{1+z^2/\alpha^2}\right)}_{\text{attenuation}}}}$$

phase term

modulus term

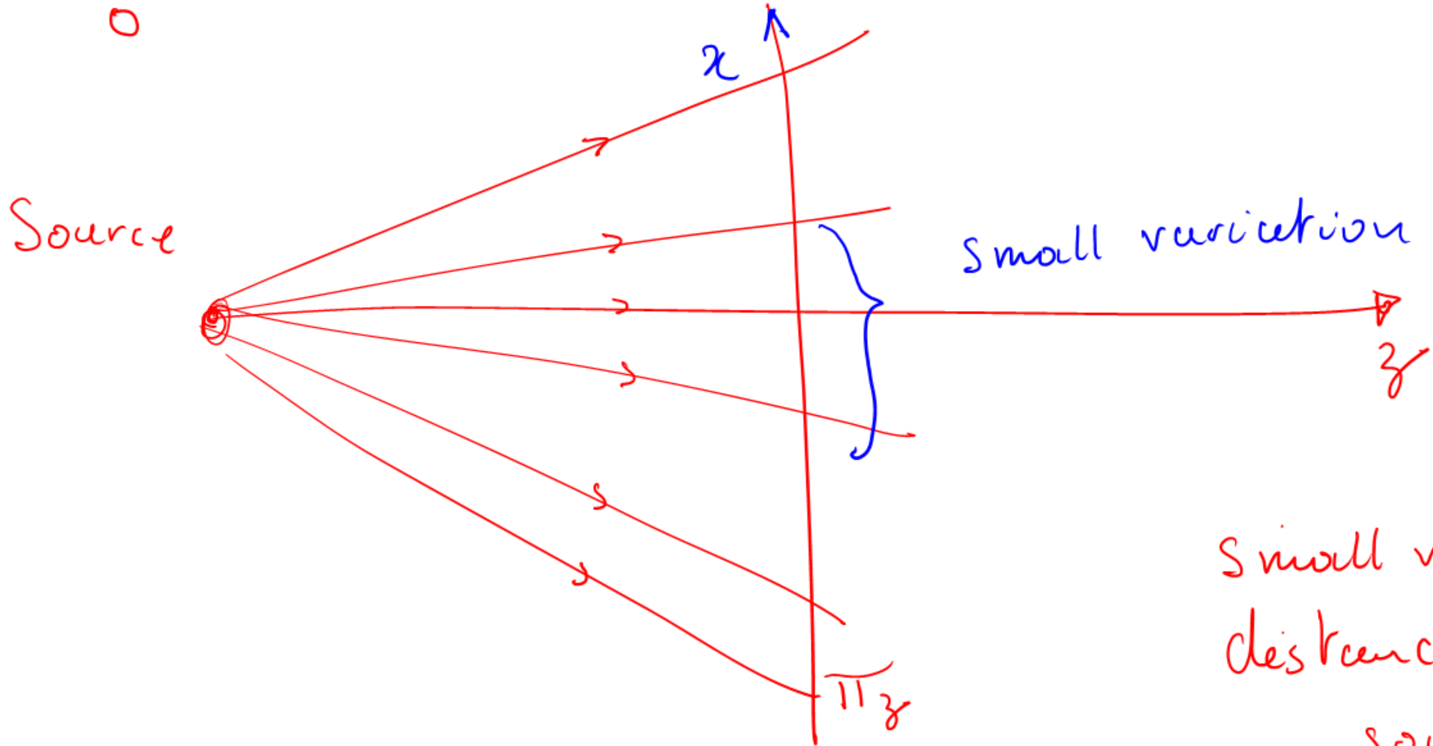
with $\alpha = \frac{\pi w_0^2}{\lambda}$

$w^2 = w_0^2 \left(1 + z^2/\alpha^2\right)$

$R = z + \frac{\alpha^2}{z}$

Demonstration of $e^{-j\pi\frac{x^2+y^2}{\lambda R}}$

paraxial approximation



small variation of x

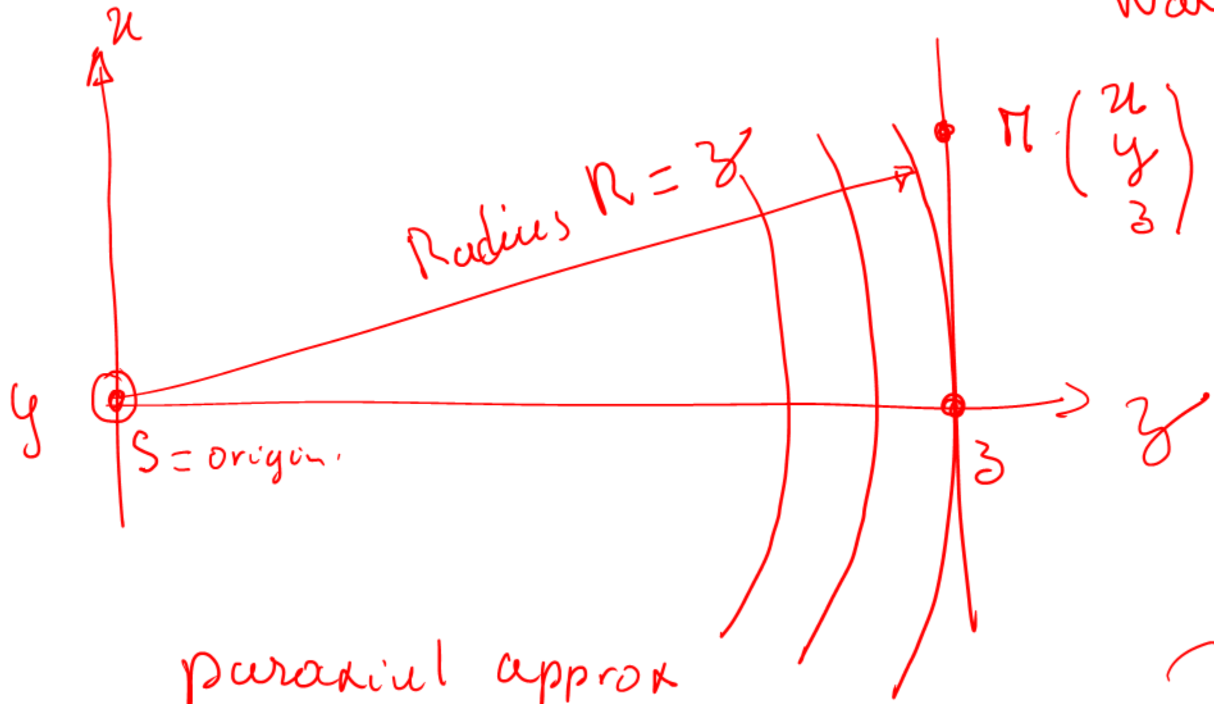
Small variation of the distance

source $\longrightarrow$ observation

modulus of the wave is $\simeq c^{te}$

$$SW_{\Pi_z} = e^{-j\varphi}$$

to derive the phase $\longrightarrow 2\pi \dfrac{\text{optical path}}{\text{wavelength}} = 2\pi \dfrac{\delta}{\lambda}$



$$\delta = \sqrt{x^2+y^2+z^2}$$

$$\varepsilon << 1$$

paraxial approx

$z >> x$ and y

$$\sqrt{1+\varepsilon} \approx 1 + \frac{\varepsilon}{2}$$

$$\delta = z \underbrace{\sqrt{1+\frac{x^2+y^2}{z^2}}}$$

$$\approx z\left(1+\frac{x^2+y^2}{2z^2}\right) = \underline{z} + \frac{x^2+y^2}{2z}$$

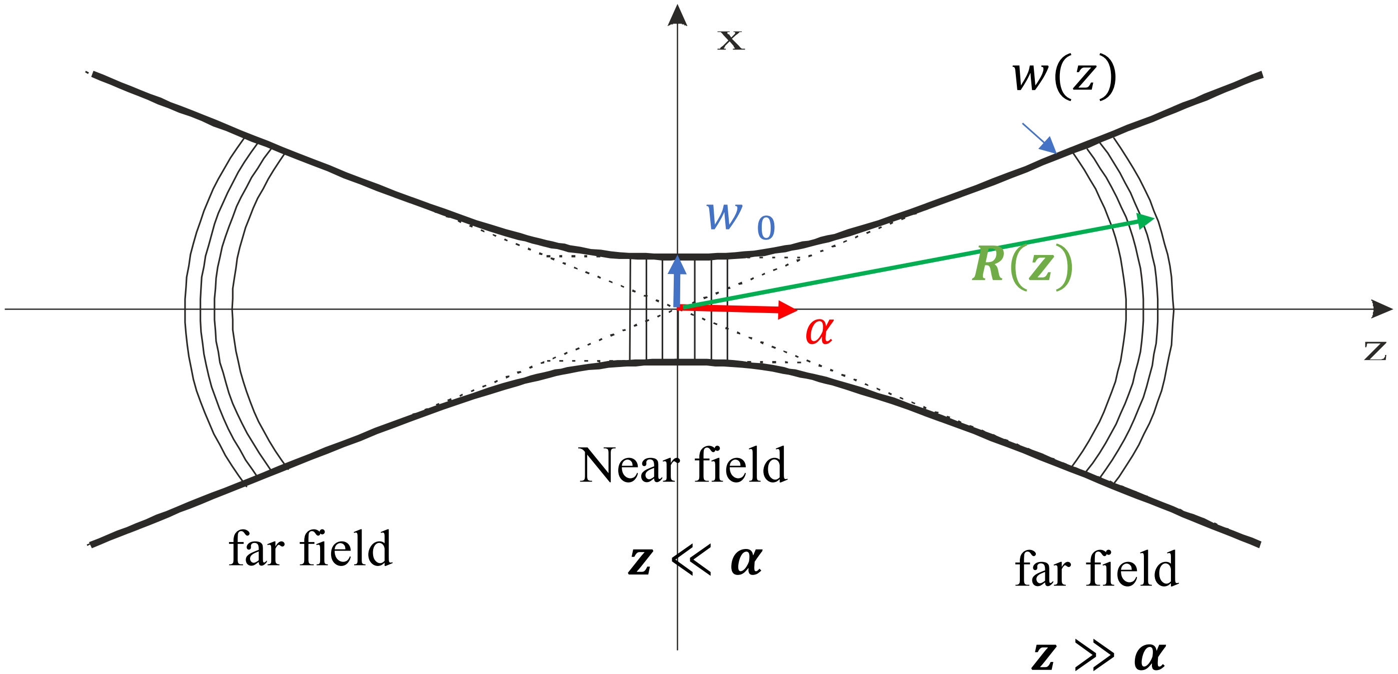
$$SW_z(x,y) = e^{-j\frac{2\pi z}{\lambda}} e^{-j\pi\frac{x^2+y^2}{\lambda z}} = e^{-j\frac{2\pi z}{\lambda}} e^{-j\pi\frac{x^2+y^2}{\lambda R}}$$

Analysis of the gaussian beam formula

Quite in all part of the formula comparison between z and $\boldsymbol{\alpha = \frac{\pi w_0^2}{\lambda}}$

$$\boldsymbol{w^2(z) = w_0^2\sqrt{1+\frac{z^2}{\alpha^2}}}$$

$$\boldsymbol{R} = z + \frac{\alpha^2}{z}$$



Eq (1)

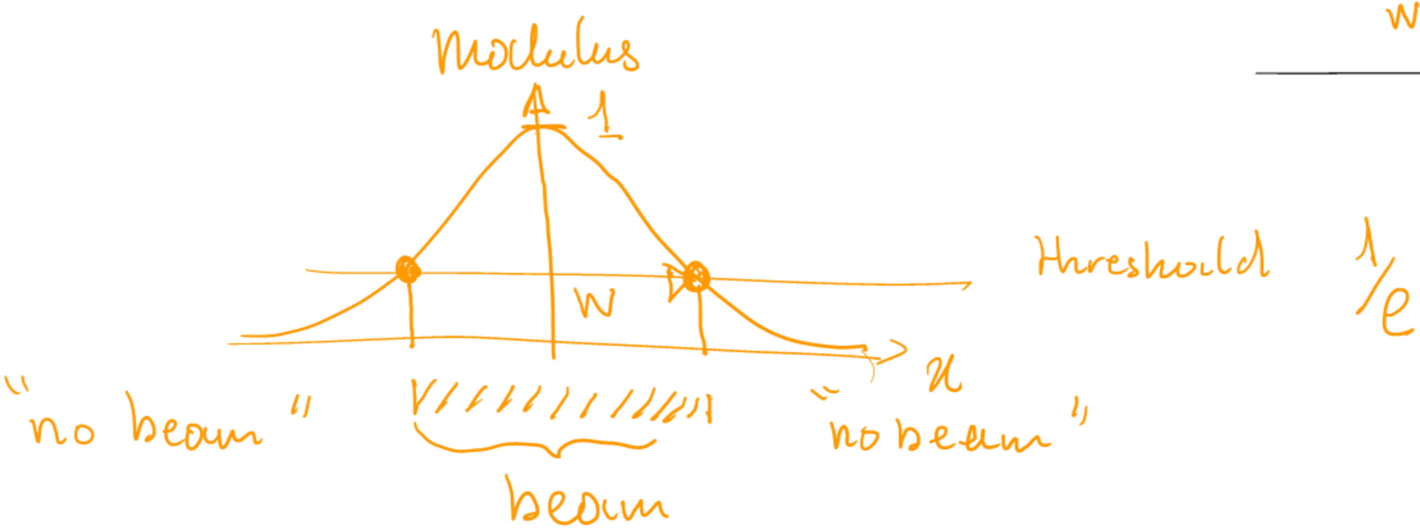
$$E_z(x,y) = \frac{E_0 \, e^{-j\frac{2\pi z}{\lambda}} \, e^{+j\arctan z/\alpha} \, e^{+j\pi\frac{x^2+y^2}{\lambda R}} \, e^{-\frac{x^2+y^2}{w^2}}}{\sqrt{1+z^2/\alpha^2}}$$

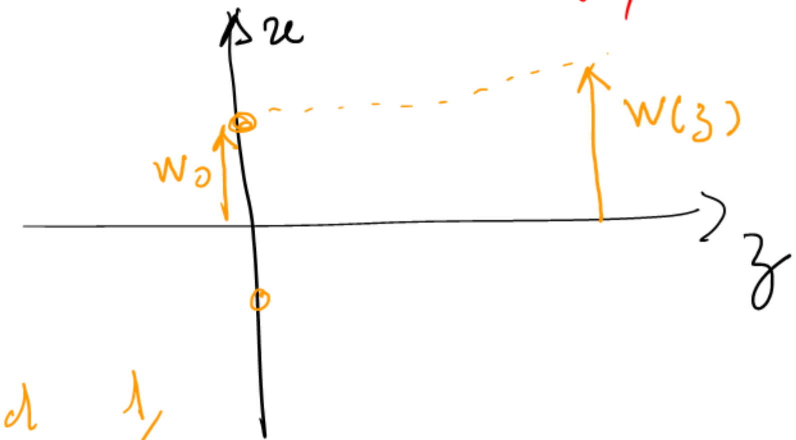
with $\alpha = \dfrac{\pi w_0^2}{\lambda}$

$R = z + \dfrac{\alpha^2}{z}$

$w^2 = w_0^2\left(1 + z^2/\alpha^2\right)$

$E_z(x,y) = \text{modulus} \; e^{+j\,\text{phase}}$



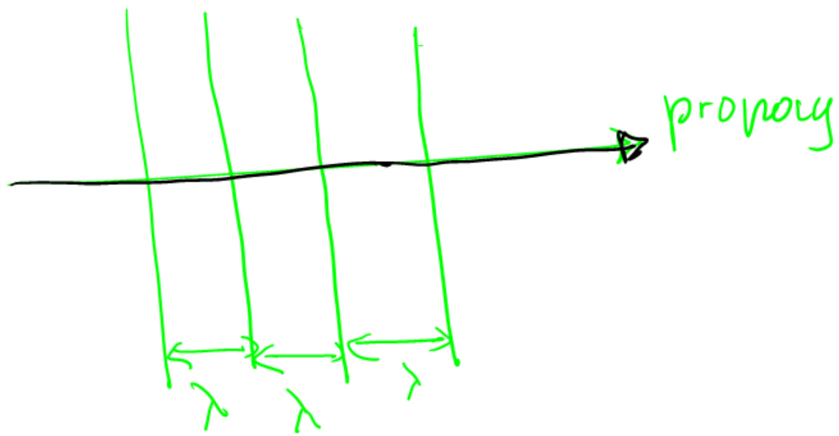


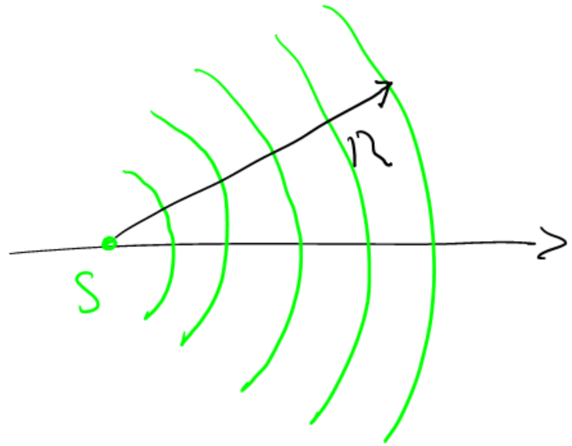
Phase term.

Representation of the wave fronts ($\varphi = c^{te}$)

plane wave

spherical wave.





For a plane wave

$R \to \infty$

1. $$E_z(x,y) = \frac{E_0 \, e^{-j\frac{2\pi z}{\lambda}} \, e^{+j\arctan z/\alpha} \, e^{+j\pi\frac{x^2+y^2}{\lambda R}} \, e^{-\frac{x^2+y^2}{w^2}}}{\sqrt{1+z^2/\alpha^2}}$$

$$\alpha = \frac{\pi w_0^2}{\lambda}$$

$$w^2 = w_0^2\left(1+z^2/\alpha^2\right)$$

$$R = z + \frac{\alpha^2}{z}$$

α

0 — z

$z \ll \alpha$	$z \gg \alpha$
$w \# w_0$	$w \# w_0 \, z/\alpha$
$R \# \infty$	$R \# z$
plane wave	spherical wave
$\boxed{\text{Near Field}}$	Far Field

Near Field

$W \# W_0$

$R = \infty$

① becomes ?

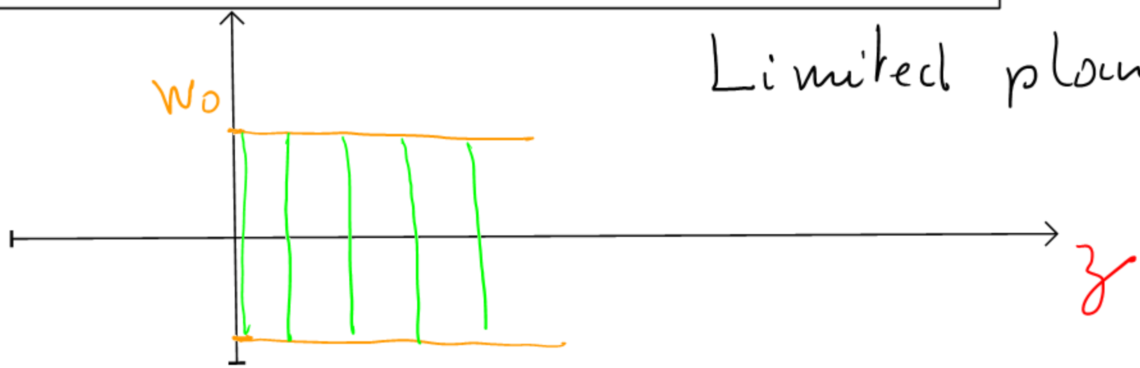
$e^{j \operatorname{Atan} z/\alpha} \# e^{j0} \# 1$

$\sqrt{1+z^2/\alpha^2} \# 1$

$e^{-j\pi \frac{x^2+y^2}{\lambda R}} \# e^{-\dots \frac{1}{\infty}} = e^{0} = 1$

$e^{-\frac{x^2+y^2}{W^2}} \# e^{-\frac{x^2+y^2}{W_0^2}}$

$$\boxed{E_z(x,y) = E_0\, e^{-j 2\pi \frac{z}{\lambda}}\, e^{-\frac{x^2+y^2}{W_0^2}}}$$



Limited plane wave

Far Field

$z \gg \alpha$

$$w^2 = w_0^2\left(\underbrace{1}_{\simeq 0} + z^2/\alpha^2\right)$$

$$\sqrt{1 + z^2/\alpha^2} \simeq \frac{z}{\alpha}$$

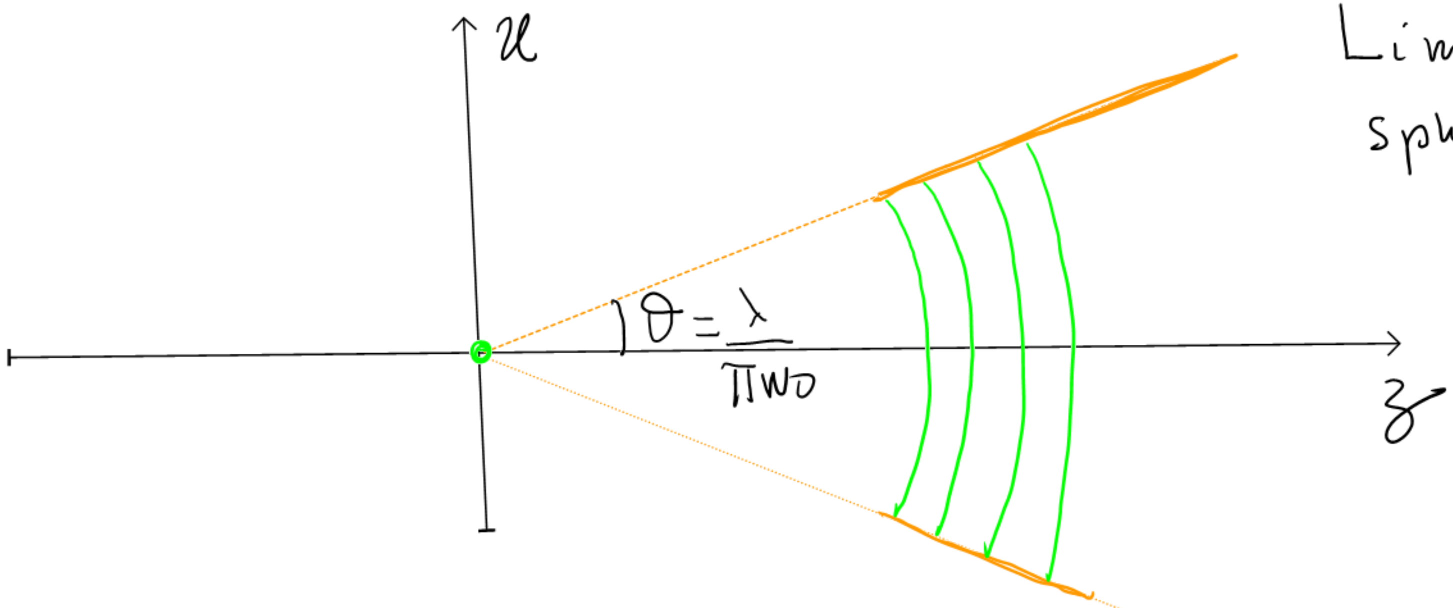
$$e^{j\arctan z/\alpha} \simeq e^{j\frac{\pi}{2}} = j$$

$$w = w_0\, z/\alpha = \frac{w_0\, z}{\pi w_0^2/\lambda} = \frac{\lambda}{\pi w_0}\cdot z$$

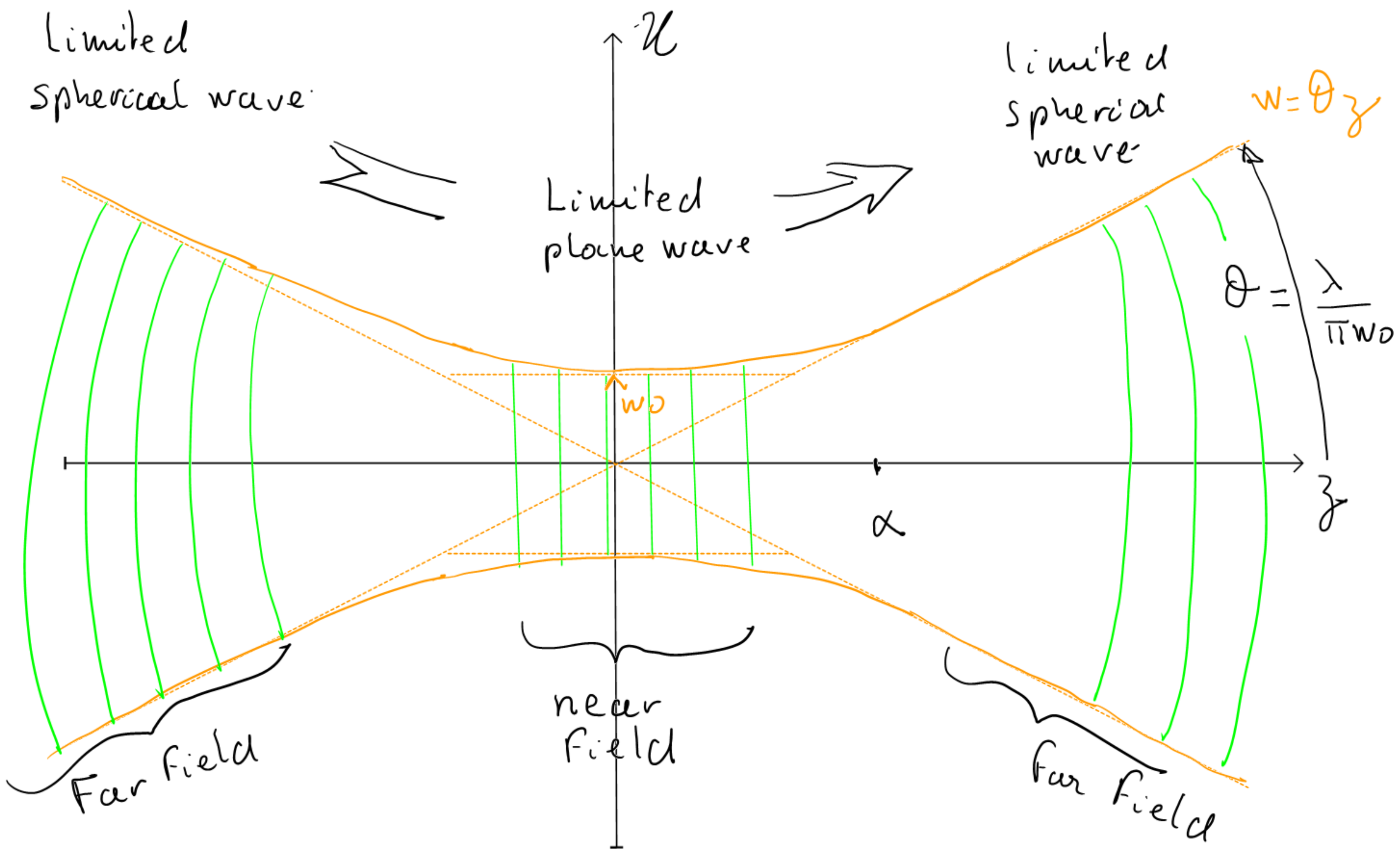
$$R = z + \underbrace{\frac{\alpha^2}{z}}_{\simeq 0} \simeq z$$

$$\boxed{E_z(x,y) = j\,\frac{E_0\,\alpha}{z}\, e^{-j\frac{2\pi z}{\lambda}} \cdot e^{-\frac{x^2+y^2}{(\theta z)^2}} \cdot e^{-j\pi\frac{x^2+y^2}{\lambda z}}}$$

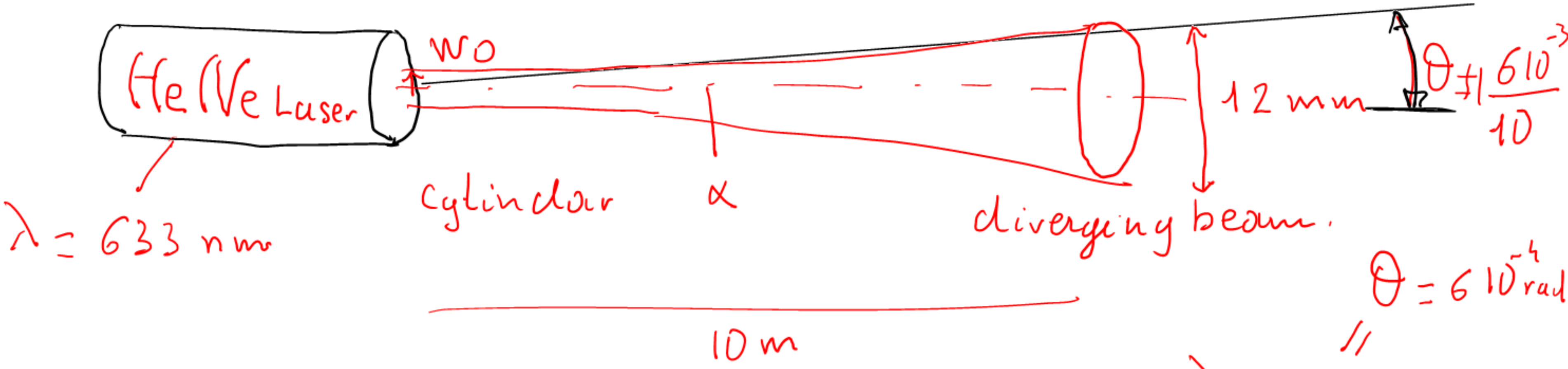
with $\theta = \frac{\lambda}{\pi w_0}$



Limited spherical wave



Limited spherical wave
limited spherical wave
Limited plane wave
x
z
α
w_0
$w = \theta z$
$\theta = \frac{\lambda}{\pi w_0}$
Far field
near field
Far field



$$\theta \approx 1\,\frac{6\,10^{-3}}{10}$$

$$\theta = 6\,10^{-4}\,\text{rad}$$

$$\frac{\lambda}{\pi w_0} = \theta$$

$$w_0 = \frac{\lambda}{\pi\theta} \approx \frac{6\,10^{-7}}{3 . 6\,10^{-4}} = \frac{1}{3}\,10^{-3}\,\text{m} \simeq 300\,\mu\text{m}$$

Intensity

Near field

$z \ll \alpha$

$$E_z(x,y) = E_0 \frac{e^{-\left(\frac{x^2+y^2}{w_0^2}\right)} e^{-jkz}}{1}$$

far field

$z \gg \alpha$

$$E_z(x,y) = E_0 \frac{e^{-\left(\frac{x^2+y^2}{(\theta z)^2}\right)} e^{-jkz} e^{-j\pi\left(\frac{x^2+y^2}{\lambda z}\right)} j}{\frac{z}{\alpha}}$$

$$\theta = \frac{\lambda}{\pi w_0}$$