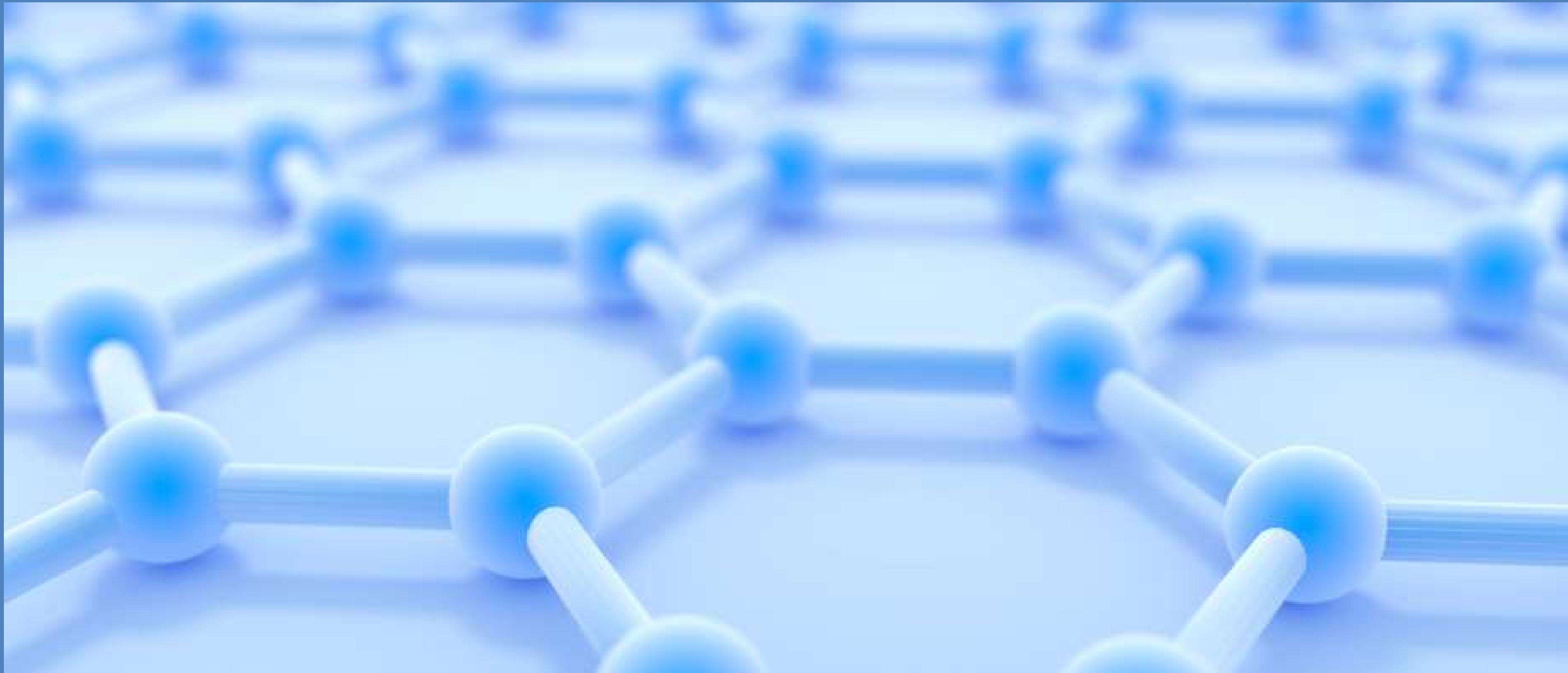


Photonics
aa 2021/2022

Prof. Maria Antonietta Vincenti
Università degli Studi di Brescia

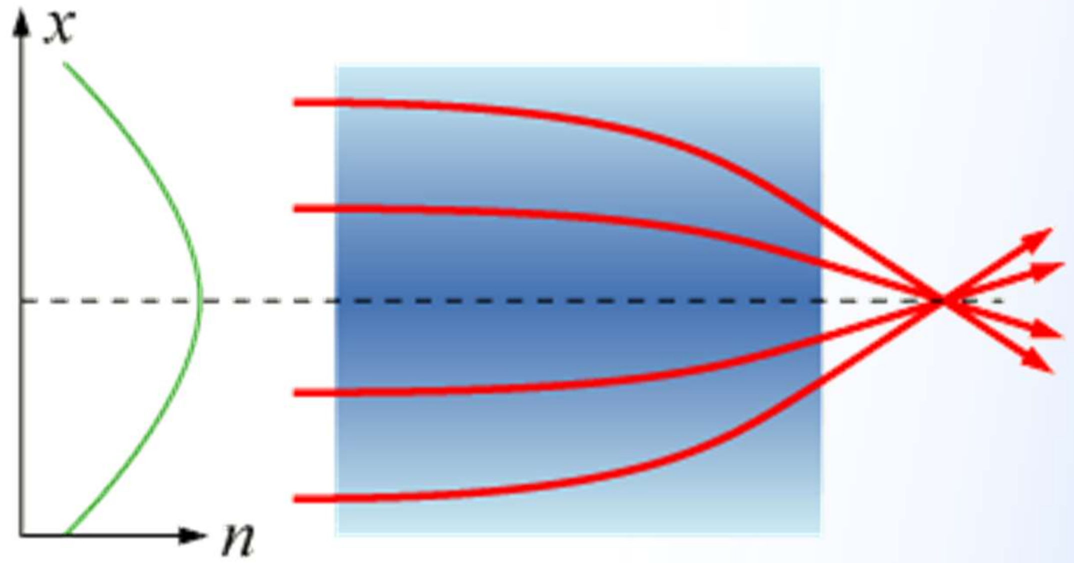
Ray Optics:
Graded index optics
and
Matrix optics

Graded index Optics

Graded-index material (**GRIN**) → refractive index changes with position with a function n(**r**);

In GRIN light follows curved trajectories;

GRIN are usually fabricated adding dopants in controlled concentrations;

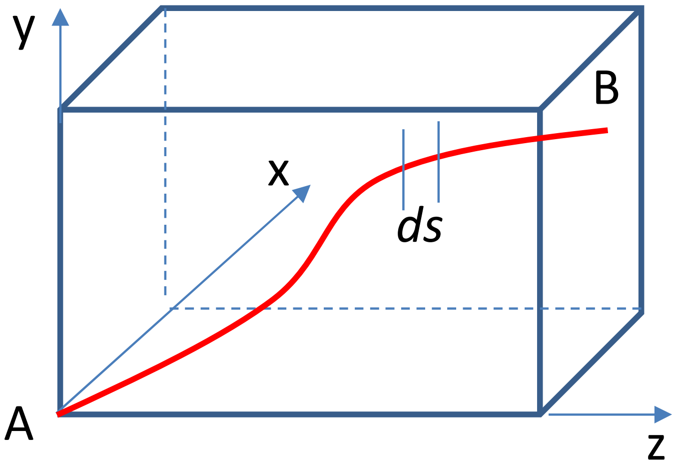


Graded index Optics

To determine the trajectory of rays we can use the Fermat's principle

$$\delta \int_A^B n(\boldsymbol{r})dS = 0$$

If we assume the trajectory is described by the function x(s), y(s) and z(s) where s is the length of the trajectory then we can show that these three functions should satisfy a set of partial differential equations that can be written in the following compact form:



$$\frac{d}{ds}\left(n\frac{dr}{ds}\right) = \nabla n$$

THE RAY EQUATION

Graded index Optics

In the paraxial approximation the trajectory can be considered almost parallel to the z axis, therefore the ray equation can be simplified in the following form:

$$\frac{d}{dz}\left(n\frac{dx}{dz}\right) \approx \frac{dn}{dx} \qquad \frac{d}{dz}\left(n\frac{dy}{dz}\right) \approx \frac{dn}{dy}$$

PARAXIAL RAY EQUATIONS

If n= n(x,y,z), the two equations can be solved for the trajectories x(z) and y(z).

NOTE: when the medium is homogeneous n is independent on the cartesian coordinates and it follows that $\frac{d^2x}{dz^2} = 0$ and $\frac{d^2y}{dz^2} = 0$. In this case x and y are linear functions of z and so trajectories are straight lines.

Graded index components

UNIVERSITY OF BRESCIA

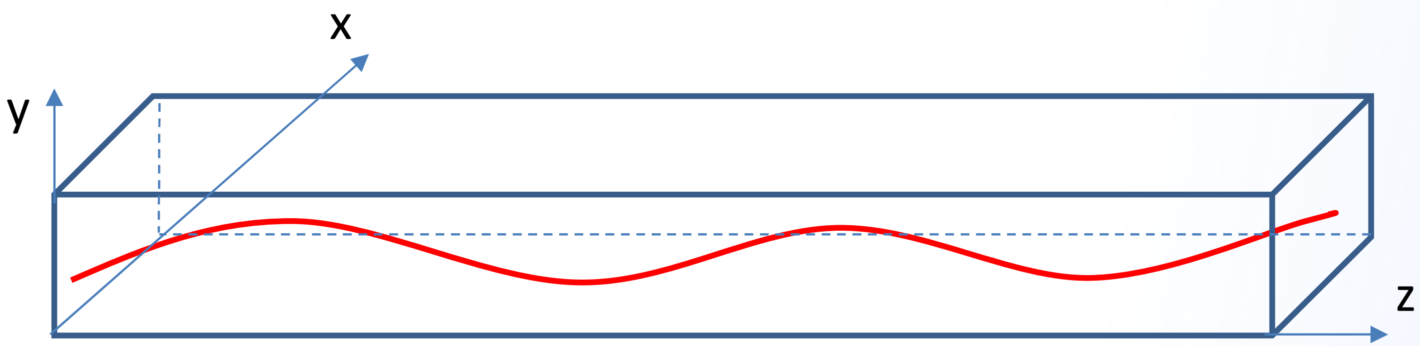
Graded-index slab

Let's consider a slab of material with refractive index n=n(y) and constant in the x and z directions. The trajectories of the rays can be described with a further simplified paraxial ray equation:

$$\frac{d}{dz}\left(n\frac{dy}{dz}\right) \approx \frac{dn}{dy}$$

$$\Downarrow$$

$$\frac{d^2y}{dz^2} = \frac{1}{n(y)}\frac{dn(y)}{dy}$$



Graded index components

Graded-index fiber

A graded index fiber is a glass cylinder with refractive index that varies along its radius. If we assume the refractive index having a distribution:

$$n^2 = n_0^2[1 - \alpha^2(x^2 + y^2)]$$

And that $\alpha^2(x^2 + y^2) \ll 1$, then the paraxial ray equations assume the following form:

$$\frac{d^2x}{dz^2} \approx -\alpha^2 x, \ \frac{d^2y}{dz^2} \approx -\alpha^2 y$$

With x and y being harmonic functions of z with period $2\pi/\alpha$. The initial position on the x and y axis determine the amplitudes and phases of the harmonic functions.

Graded index components

For example:

If $x_0 = 0$ the solution of the paraxial ray equations is:

$$x(z) = \frac{\theta_{x0}}{\alpha}\sin(\alpha z),\ y(z) = \frac{\theta_{y0}}{\alpha}\sin(\alpha z) + y_0\cos(\alpha z)$$

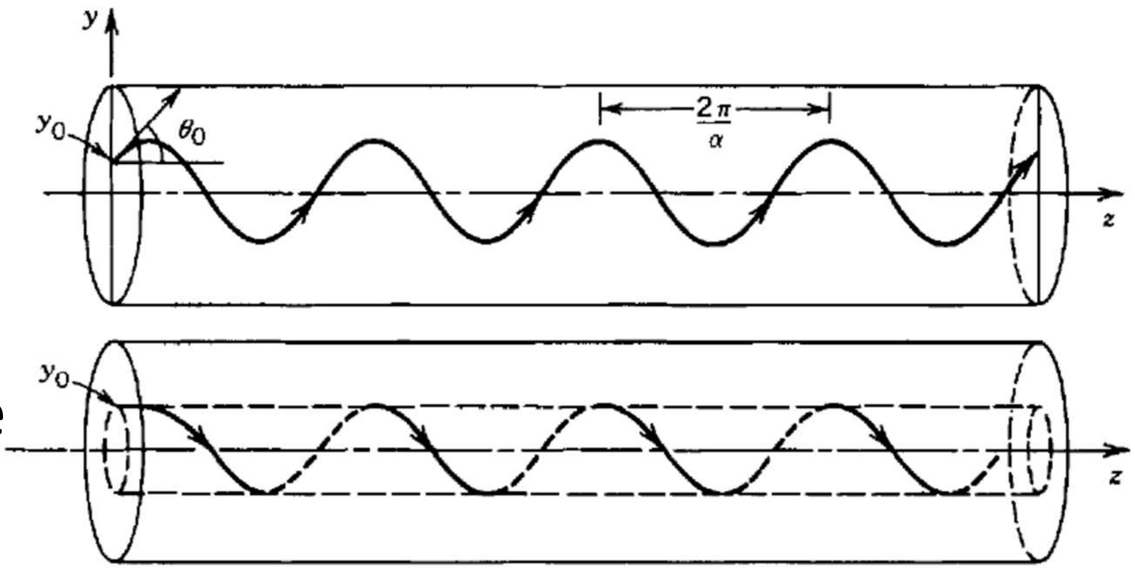
If $\theta x_0 = 0$ the ray lies on a plane that passes through the axis of the cylinder (y-z plane) and continues to propagate on the same plane following a sinusoidal trajectory.

If $\theta y_0 = 0$ and $\theta x_0 = \alpha y_0$ then

$x(z) = y_0 \sin(\alpha z)$,

$y(z) = y_0 \cos(\alpha z)$,

so the ray follows a helical trajectory that lies on the surface of a cylinder of radius y_0



Matrix Optics

Is a **technique for tracing paraxial rays**.

Rays are assumed to travel **within a single plane.** This technique is therefore appropriate to describe planar geometries or circular geometries where the rays propagate in a plane (see optical fiber ex).

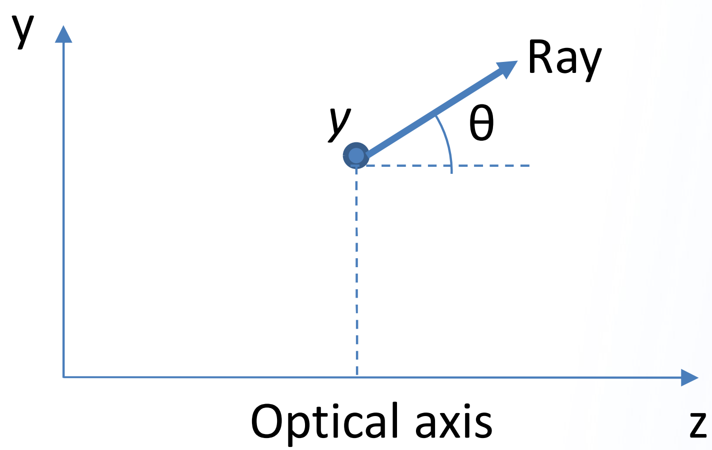
The ray is described by its position and its angle with respect to the optical axis. Positions at input and output planes are related by two algebraic equations.

The optical system is described by a 2x2 matrix called the **ray transfer matrix.**

The convenience of the matrix description is in the ability to describe complex optical system with a product of several matrices.

The ray transfer matrix

Let’s consider a circularly symmetric optical system formed by a succession of reflecting and refracting surfaces centered on the same optical axis. Let’s assume that the optical axis coincides with the z axis and all rays travel on a plane that includes the z axis, i.e. the y-z plane.
A generic ray traveling on this plane is perfectly described by its coordinate *y* and the angle θ.



The ray transfer matrix

An optical system is a set of optical components placed between the planes z_1 and z_2 (input and output planes). The systems steers the ray from the position (y_1, θ_1) to the position (y_2, θ_2). In the paraxial approximation the relation between these two positions is linear and can be written as:

$$\boxed{\begin{aligned} y_2 &= Ay_1 + B\theta_1 \\ \theta_2 &= Cy_1 + D\theta_1 \end{aligned}}$$

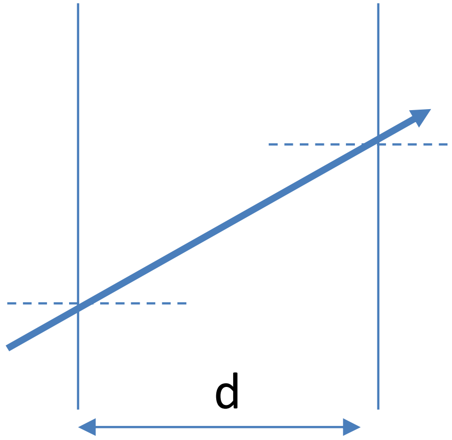
Where A, B, C, D are the element of the **ray transfer matrix**

$$\mathrm{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

So that we can write $\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \mathrm{M} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$.

Matrices of simple optical components

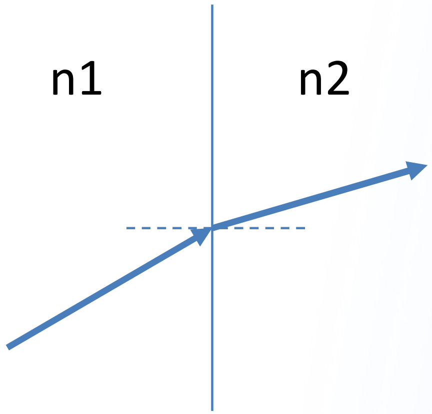
Free space propagation:



$$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

Refraction at

a planar boundary:

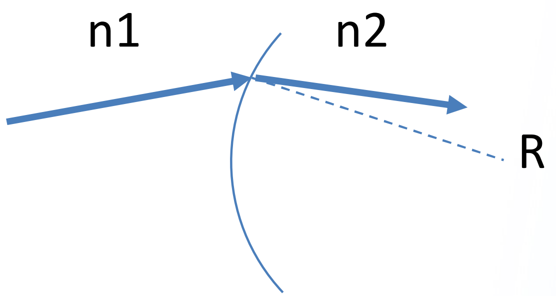


$$M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n1}{n2} \end{bmatrix}$$

Matrices of simple optical components

Refraction at

a spherical boundary:



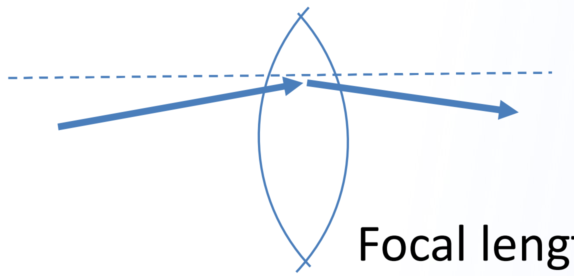
Convex R>0, Concave R<0

$y_2 \sim y_1$

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$$

Refraction through

a thin lens:



Focal length f

Convex f>0, Concave f<0

$y_2 = y_1$

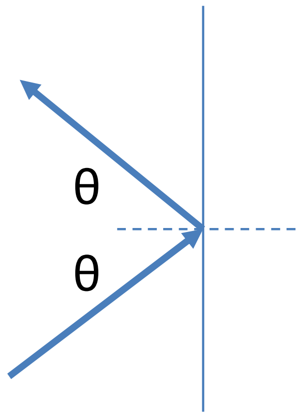
$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

UNIVERSITY OF BRESCIA

Matrices of simple optical components

Reflection at

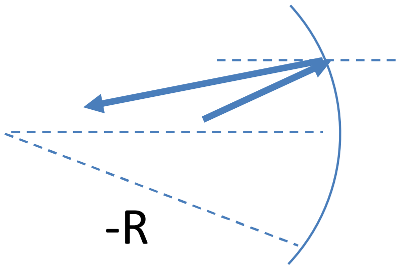
a planar mirror:



$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection at

a spherical mirror:



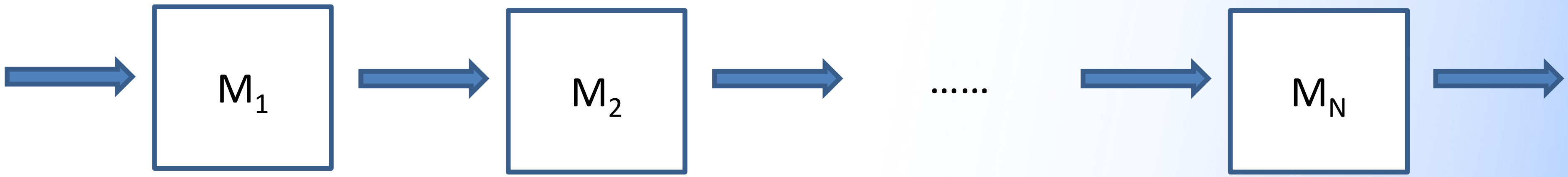
$$M = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$

Convex R>0, Concave R<0

Matrices of Cascaded optical components

A cascade of N optical components with ray matrices M_1, M_2, M_3, …M_N is equivalent to a single optical system with ray transfer matrix

$$\boxed{M = M_N \ldots M_2 \, M_1}$$

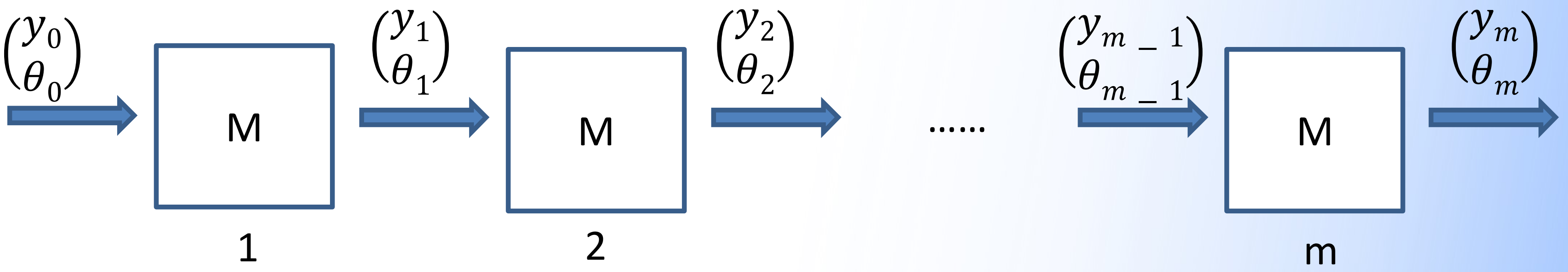


Periodic Optical Systems

A periodic optical system is composed of a cascade of identical unit systems. Even an optical fiber can be considered a periodic system if the length is divided into contiguous identical elements.

If each element of the system is characterized by the same ray transfer matrix $\mathrm{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ we can simply apply the same matrix *m* times, where *m* is the number of units in the system. In other words we can write:

$$\boxed{\begin{pmatrix} y_m \\ \theta_m \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^m \begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix}}$$



Periodic Optical Systems

We can also derive the equation that govern the dynamic of the position y_m without knowing anything about the angle θ_m.
If we write:

$$\theta_m = \frac{y_{m+1} - Ay_m}{B}$$

And then replace m with m+1 we get:

$$\theta_{m+1} = \frac{y_{m+2} - Ay_{m+1}}{B}$$

Replacing those two equation in $\theta_{m+1} = Cy_m + D\theta_m$ we get:

$$\boxed{y_{m+2} = 2by_{m+1} - F^2\, y_m}$$

$$\boxed{b = \frac{A+D}{2}, F^2 = AD - BC = \det[\mathrm{M}]}$$

Recurrence relation for ray position

Periodic Optical Systems

$$\boxed{y_{m+2} = 2by_{m+1} - F^2\, y_m}$$

Recurrence relation for ray position

A possible solution for this equation could be of the following form:

$$y_m = y_0 h^m$$

Where *h* is a constant that satisfies the following relation:

$$h^2 - 2bh + F^2 = 0$$

So that $h = b \mp j\sqrt{F^2 - b^2} = 0$. If we then define $\varphi = \cos^{-1}(b/F)$ we can re-write $b = Fcos\varphi, \sqrt{F^2 - b^2} = Fsin\varphi$ and therefore

$$h = F(cos\varphi \mp jsin\varphi) = Fexp(\mp j\varphi)$$

UNIVERSITY OF BRESCIA

Periodic Optical Systems

Replacing we get

$$y_m = y_0 F^m exp(\mp jm\varphi)$$

A general solution can be a combination of positive and negative exponentials. We also recall that the sum of two exponential can also be written as a harmonic circular function so that:

$$y_m = y_{max} F^m sin(m\varphi + \varphi_0)$$

With y_{max} and φ_0 constants that can be determined from the initial conditions.

Regardless of the system it can be shown that det(M)=n_1/n_2, where n_1 and n_2 are refractive indices of the input and output medium of the system. In the particular case where $n_1=n_2$ then

$$y_m = y_{max} sin(m\varphi + \varphi_0)$$

ray position in a periodic system

Periodic Optical Systems

y_m is **harmonic** if $\varphi = \cos^{-1} b$ is real. This requires that:

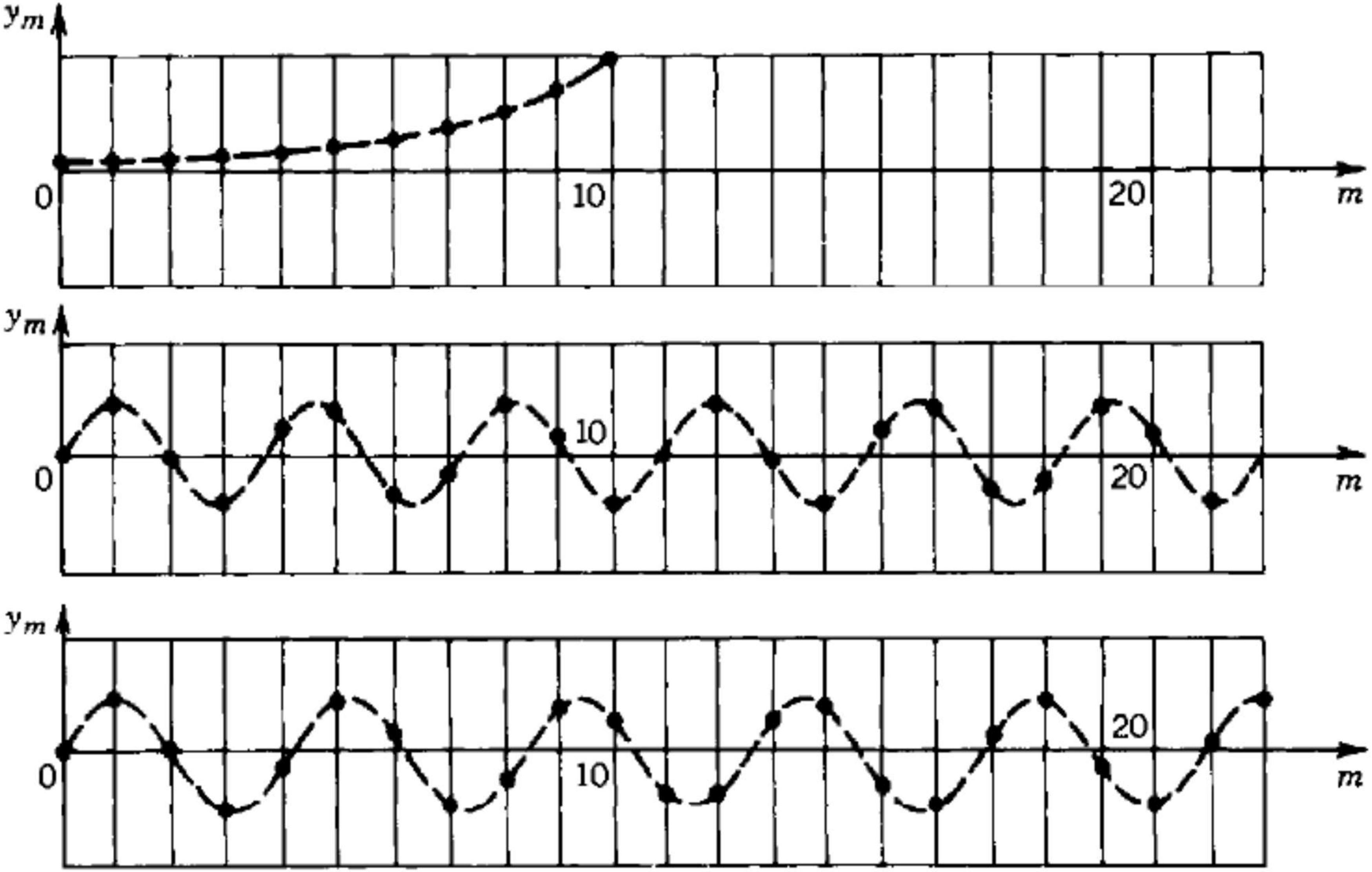
$$\boxed{|b| \leq 1 \quad \text{or} \quad \frac{1}{2}|A + D| \leq 1}$$

Stability condition

If $|b| > 1$ then the solution is a hyperbolic function, which increases without bound.

The harmonic function $y_m = y_{max} sin(m\varphi + \varphi_0)$ is **periodic** in m if we can find an integer s such that $y_{m+s} = y_m$. The smallest integer is the **period**. This also means that **the ray retraces its path** after s stages. This condition is satisfied if $s\varphi = 2\pi q$, where q is an integer.

Periodic Optical Systems



UNSTABLE

STABLE PERIODIC

STABLE NON PERIODIC