Master ENMEO - fundamentals of coherent optics

Determination of the dispersion relation of a slab waveguide

In a waveguide, the dispersion relation expresses the relation between the propagation constant β and the pulsation ω (or a related quantity like the frequency ν, the wavelength λ, the normalized spatial frequency V...).

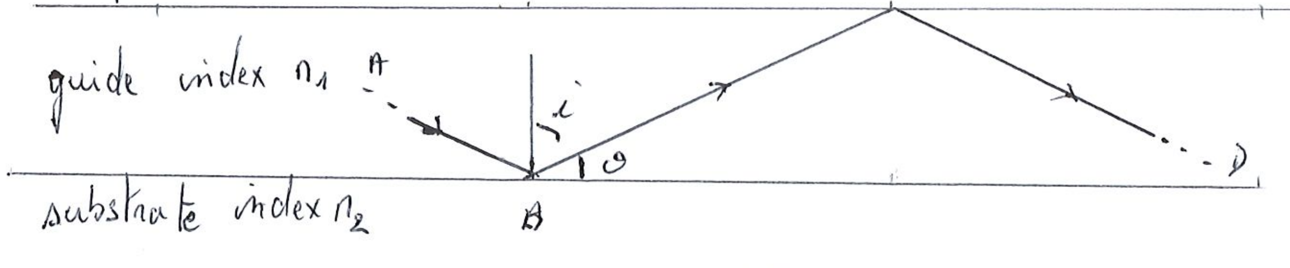
It takes the form of an equation including an integer (m or ν).

This integer being fixed, this equation can have zero, one or several solutions.

Each solution is associated to one particular transverse electromagnetic mode of the waveguide.

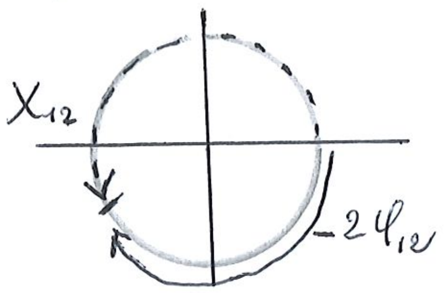
Note: a transverse mode is a transverse distribution of the electromagnetic field in the guide, INVARIANT along the propagation axis.

I Determination of the dispersion equation of a slab waveguide, considering the propagation of a ray



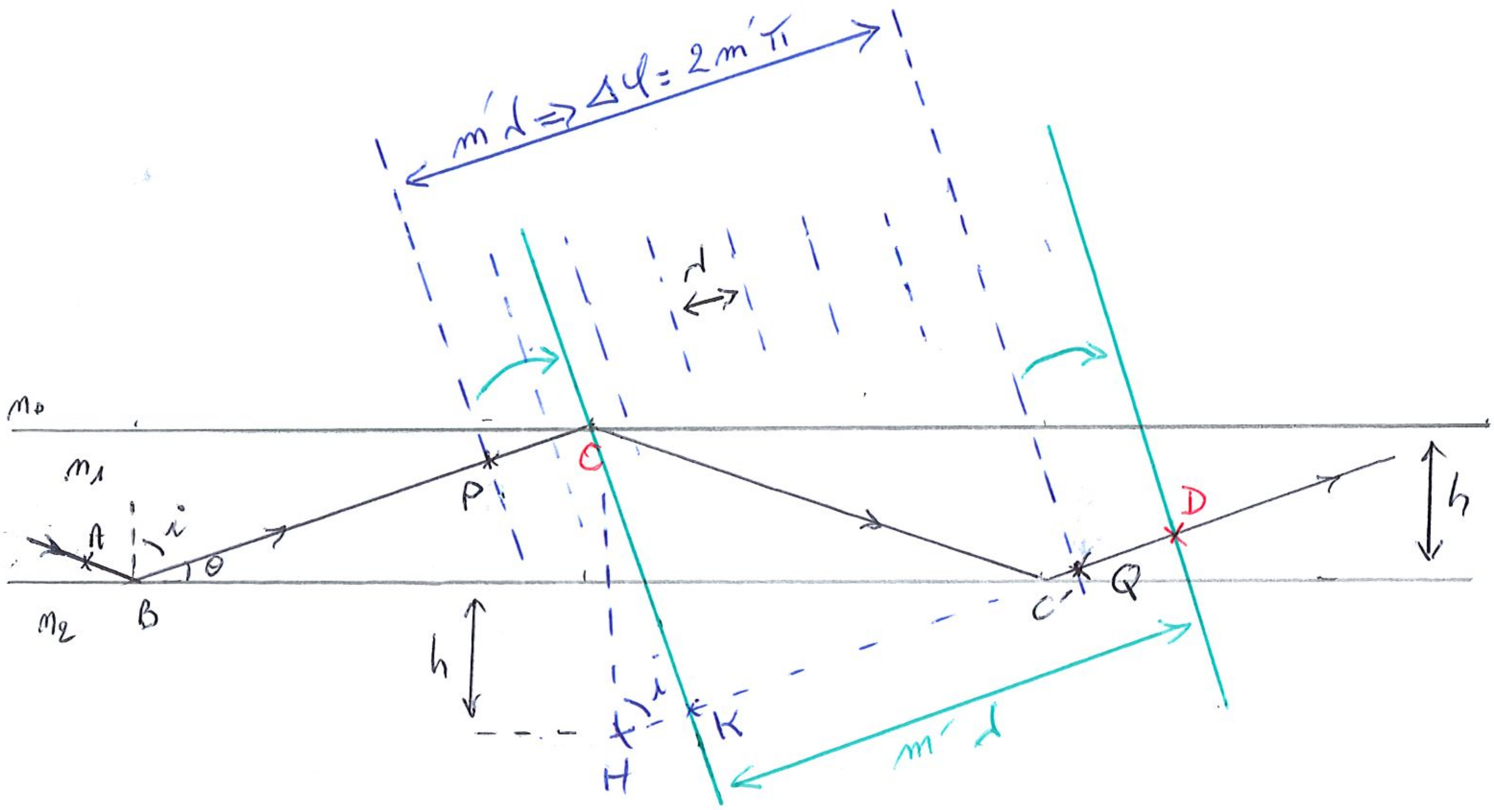
Let us consider the ray ABCD. Its reflexion angle is i. It experiences a phase shift due to the propagation and two additional phase delays: the Goos Hanchen phase delay in B and the Goos Hanchen phase delay in C. These two phase delays are larger than π

Let us note $X_{12} > \pi$, the Goos Hanchen phase delay in B, at the interface between medium "1" and medium "2".



Because it will simplify the relations in the following, we write that this delay is equivalent to a negative phase advance arbitiary called $-2\varphi_{12}$.

Similarly, the Goos Hanchen phase delay in C will be noted $-2\varphi_{10}$



Let us consider that the ray ABCC'D.. is associated to a given mode.
On this ray, the two points P and Q are in phase $\Rightarrow \Delta\varphi_{PQ} = 2m''\pi$
$m'' \in \mathbb{N}$

the path $PQ = \Delta l = PCC'Q = CC'D = HC'D = HD = HK + KD$

Let us first assume that there are no Goos Hanchen delays in this case,
$KD = m'd \; (m' \in \mathbb{N})$ and $\Delta l = HK + m'd = HC \cos i + m'd$
$= 2h \cos i + m'd$

The phase shift due to the path is $k\,\Delta l$ (k = modulus of the wave vector in the guide ($k = k_0 n_1 = \frac{2\pi}{d} n_1$)

Let us now take the Goos Hanchen delays into account.
The phase shift between P and Q is

$$\Delta\varphi_{PQ} = k\Delta l - 2\varphi_{10} - 2\varphi_{12} = 2m''\pi \quad \text{(see above)}$$

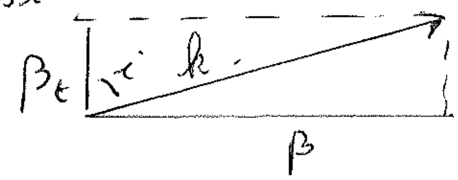
$$\Leftrightarrow \quad k(2h\cos i + m'd) - 2\varphi_{10} - 2\varphi_{12} = 2m''\pi$$

$$\Leftrightarrow \quad 2kh\cos i + 2m'\pi - 2\varphi_{10} - 2\varphi_{12} = 2m''\pi$$

$$\Leftrightarrow \quad kh\cos i = \varphi_{10} + \varphi_{12} + (\underbrace{m'' - m'}_{m})\pi \qquad m \in \mathbb{N}$$

$$\Leftrightarrow \quad kh\cos i = \varphi_{10} + \varphi_{12} + m\pi$$

Much more easily: if we consider that a mode is a stable <u>transverse</u> interference pattern, we can consider the transverse propagation constant $\beta_t = k\cos i$



$k = k_0 n_1$

Over a "transverse round trip" of $2h$, the phase shift is

$$\Delta\varphi = 2h\beta_t - 2\varphi_{10} - 2\varphi_{12}$$

The condition for having an interference pattern is $\Delta\varphi = 2m\pi \quad m \in \mathbb{N}$

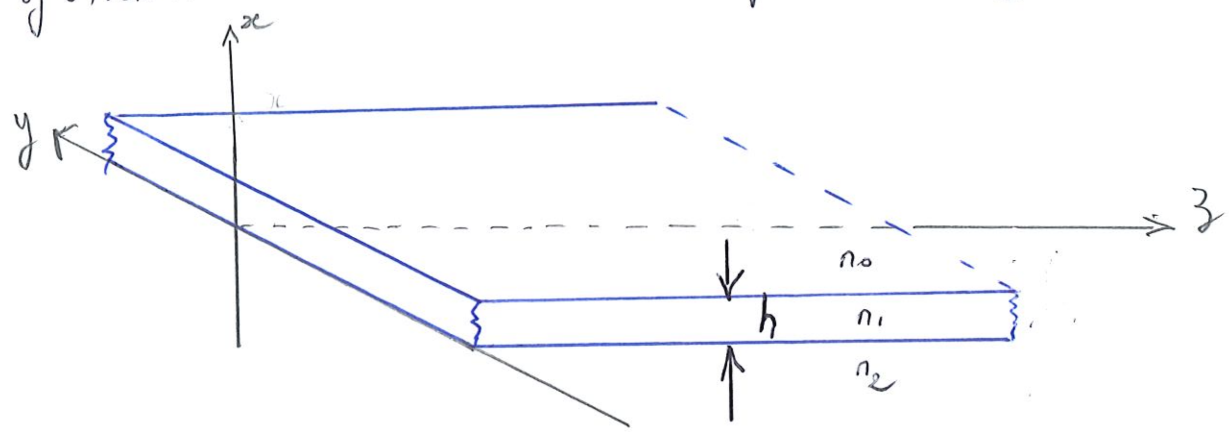
$$\Rightarrow \quad 2h\beta_t - 2\varphi_{10} - 2\varphi_{12} = 2m\pi$$

$$\Leftrightarrow \quad \beta_t h = kh\cos i = \varphi_{10} + \varphi_{12} + m\pi$$

This is a dispersion relation: if we find i, we can deduce β versus the wavelength (in k). However, φ_{10} and φ_{12} remain unknown. In order to solve this problem, we must write Maxwell equations in the waveguide.

II Determination of the dispersion equation of a slab waveguide considering the solutions of Maxwell equations

Let us consider transverse modes of a slab waveguide of thickness h in the x direction and infinite in the y direction.



In harmonic regime, the electric field is $\vec{\mathcal{E}}(x\,y\,z) = \mathrm{Re}\left[\vec{E}(x,y)\, e^{j(\omega t - \beta z)}\right]$

$$\vec{E}(x\,y) = \begin{cases} E_x(x,y)\ \vec{e_x} \\ E_y(x,y)\ \vec{e_y} \\ E_z(x\,y)\ \vec{e_z} \end{cases}$$

Associated magnetic field $\vec{\mathcal{H}}(x,y,z) = \mathrm{Re}\left[\vec{H}(x,y)\, e^{j(\omega t - \beta z)}\right]$

In the dielectric media of indices n_0, n_1 and n_2 which are linear isotropic homogeneous media with no electric free charges and no current densities, the Maxwell equations are:

$$\mathrm{curl}\,\vec{\mathcal{E}} = -\frac{\partial \vec{B}}{\partial t} \quad \text{with } \vec{B} = \mu\vec{\mathcal{H}} \qquad (\mu = \mu_0 = 4\pi\, 10^{-7}\ \mathrm{H/m}) \quad (1)$$

$$\mathrm{curl}\,\vec{\mathcal{H}} = \varepsilon\frac{\partial \vec{\mathcal{E}}}{\partial t} \quad \text{with } \varepsilon = \varepsilon_0\varepsilon_r \qquad \varepsilon_0 = \frac{1}{36\pi}10^{-9}\ \mathrm{F/m}) \quad (2)$$

$\varepsilon_r = n_0^2$, n_1^2 or n_2^2, depending on the medium

$(\mathrm{curl} = \nabla \wedge\ )$

* If X is any component of any field, the harmonic form of the fields leads to $\frac{\partial X}{\partial t} = j\omega X$ and $\frac{\partial X}{\partial z} = -j\beta z$

* the infinite extension of the slab in the y direction results in the fact that the components of the fields do not depend on $y \Rightarrow \frac{\partial X}{\partial y} = 0$

* Let us consider TE modes $\Rightarrow E_z = 0$

In the above conditions, relations (1) and (2) lead to: see the demonstration in Annexe 1 pages 11 → 13

$$\vec{E}(x,y) = \vec{E}(x) = \begin{cases} E_x = 0 \\ E_y \neq 0 \\ E_z = 0 \end{cases} \quad \text{and} \quad \vec{H}(x,y) = \vec{H}(x) = \begin{cases} H_x = \frac{-\beta}{\omega\mu_0} E_y \\ H_y = 0 \\ H_z = \frac{j}{\omega\mu_0} \frac{\partial E_y}{\partial x} \end{cases}$$

$$\Downarrow$$

$$\vec{\mathcal{E}}(x,y,z) = E_y(x)\, e^{j(\omega t - \beta z)}\, \vec{e_y}$$

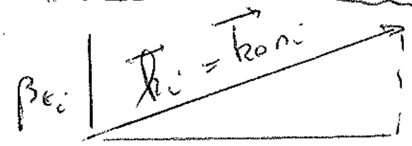
our goals are to determine E_y (distribution of the field amplitude along x) and the associated propagation constant β, for a given transverse mode.

From (1) and (2), we find the propagation equation (or Helmoltz equation),

$$\Delta \vec{\mathcal{E}} + k_0^2 n_i^2 \vec{\mathcal{E}} = \vec{0} \quad (3) \quad \text{with } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ vectorial laplacian}$$

using (3) and the expression of $\vec{\mathcal{E}}$ we obtain: see the demonstration of (3) and (4) in Annexe 2 pages 14 - 15

$$(4) \qquad \frac{\partial^2 E_y}{\partial x^2} + \underbrace{(k_0^2 n_i^2 - \beta^2)}_{\beta_{t_i}^2} E_y = 0$$



$\beta = k_0 n_e$

n_e = effective index of the mode

β_{t_i} = transverse propagation constante

$$\beta_{t_i}^2 = k_0^2 (n_i^2 - n_e^2)$$

The solution of the Helmoltz equation is:

$E_y = A_i e^{-\gamma_i x} + B_i e^{\gamma_i x}$ where $\gamma_i = j \beta_{t_i}$

$\Rightarrow \beta_{t_i}^2 = -\gamma_i^2$

* if γ_i is real $\rightarrow E_y = A_i e^{-\gamma_i x} + B_i e^{\gamma_i x}$ sol1
* if γ_i is pure imaginary $\rightarrow E_y = C_i \cos(\beta_{t_i} x + \phi)$ (β_{t_i} real) sol2

To be guided, the field must remain confined in the guide $(0 \leq x \leq h)$

- in the superstrate $h < x < \infty$, $n = n_0$: $\gamma_i = \gamma_0$, sol2 is not possible because the field could extend sinusoidally towards the infinite

$\Rightarrow$ the solution must be sol1, with $B_i = B_0 = 0$
(if $B_0 \neq 0 \rightarrow$ the field should be infinite for x infinite)

$\Rightarrow \underline{E_y = A_0 e^{-\gamma_0 x}}$ with γ_0 real $\overset{(*)}{\Rightarrow} \beta_{t_0} = \sqrt{k_0^2 n_0^2 - \beta^2}$ imaginary

$\Rightarrow k_0^2 n_0^2 - \beta^2 < 0 \Rightarrow k_0 n_0 < \beta = k_0 n_e$

$\Rightarrow n_0 < n_e$

(*) $\gamma_0 = \sqrt{\beta^2 - k_0^2 n_0^2} = k_0 \sqrt{n_e^2 - n_0^2}$

- in the substrate $x < 0$, $n = n_2$: $\gamma_i = \gamma_2$, sol2 is not possible because the field would extend sinusoidally towards the infinite (for $x \rightarrow -\infty$)

$\Rightarrow$ the solution must be sol 1 with $A_i = A_2 = 0$
(if $A_2 \neq 0$ the field should be infinite of $x \rightarrow -\infty$)

$\Rightarrow \underline{E_y = B_2 e^{+\gamma_2 x}}$ with γ_2 real $\overset{(*)}{\Rightarrow} \beta_{t_2} = \sqrt{k_0^2 n_2^2 - \beta^2}$ imaginary

$\Rightarrow k_0^2 n_2^2 - \beta^2 < 0 \Rightarrow k_0 n_2 < \beta = k_0 n_e$

$\Rightarrow n_2 < n_e$

(*) $\gamma_2 = \sqrt{\beta^2 - k_0^2 n_2^2} = k_0 \sqrt{n_e^2 - n_2^2}$

- in the guide $\quad \gamma_i = \gamma_1$. Sol 1 is not possible because
$0 < x < h$ it should allow an exponential increase of the
$n = n_1$ field when x increased, which is not realistic

$\Rightarrow$ the solution must be Sol 2 $\Rightarrow \gamma_1$ is imaginary

$\underline{E_y = C \cos(\beta_{t_1} x + \phi)}$ with $\beta_{t_1} = -j\gamma_1$ real

$\underline{\beta_{t_1} \text{ is noted } p}$. $\quad p = \sqrt{k_0^2 n_1^2 - \beta^2} = k_0 \sqrt{n_1^2 - n_e^2}$ real

$$\Rightarrow \quad k_0^2 n_1^2 - \beta^2 > 0 \Rightarrow \beta = k_0 n_e < k_0 n_1$$
$$\Rightarrow \quad n_e < n_1$$

We refind the guiding condition $\quad \max(k_0 n_2, k_0 n_0) < \beta < k_0 n_1$

or $\quad \max(n_2, n_0) < n_e < n_1$

Now we know the functions governing E_y but we have to determine γ_0, γ_2 and p , i.e. we must determine β.

We are now going to write the continuity conditions of E_y and of its derivative with respect to x at the boundaries ($x = 0$ and $x = h$)

<u>Note</u>: we arbitrary decide to have $E_y(x=0) = B_2$ and $E_y(x=h) = A_0$

For this we write $E_y = B e^{\gamma_2 x}$ in the substrate (= previous expression)

and we write $E_y = A_0 \, e^{-\gamma_0 (x-h)}$ in the superstrate.

"new A_0"

The previous expression was $E_y = A_0 e^{-\gamma x}$

"old A_0"

$\Rightarrow A_{0\,new} \, e^{\gamma_0 h} = A_{0\,old}$: the two are constants giving the amplitude

This simplifies the calculations but it does not change the final results.

summary

in the superstrate : $E_y = A_0 e^{-\gamma_0 (x-h)}$; $\frac{dE_y}{dx} = -\gamma_0 A_0 e^{-\gamma_0 (x-h)}$

interface superstrate / film : $x = h$

in the film (guide) : $E_y = C \cos(px + \phi)$; $\frac{dE_y}{dx} = -pC \sin(px + \phi)$

interface film / substrate : $x = 0$

in the substrate : $E_y = B_2 e^{+\gamma_2 x}$; $\frac{dE_y}{dx} = \gamma_2 B_2 e^{\gamma_2 x}$

continuity → ↓	for E_y	for $\frac{dE_y}{dx}$
in $x = h$:	$A_0 = C \cos(ph + \phi)$ (5)	$-\gamma_0 A_0 = -pC \sin(ph + \phi)$ (7)
in $x = 0$:	$C \cos\phi = B_2$ (6)	$-pC \sin\phi = \gamma_2 B_2$ (8)

Now let us calculate $\frac{(8)}{(6)} \Rightarrow \frac{\sin\phi}{\cos\phi} = \tan\phi = -\frac{\gamma_2}{p}$ (9)

and $\frac{(7)}{(5)} \Rightarrow \frac{\sin(ph+\phi)}{\cos(ph+\phi)} = \tan(ph + \phi) = +\frac{\gamma_0}{p}$ (10)

Let us remind that $\gamma_0 = k_0 \sqrt{n_e^2 - n_0^2}$; $\gamma_2 = k_0 \sqrt{n_e^2 - n_2^2}$; $p = k \sqrt{n_1^2 - n_e^2}$

We can rely (9) and (10) by the following technique:

$$ph = \underbrace{(ph + \phi)}_{\text{found with (10)}} - \underbrace{\phi}_{\text{found with (9)}}$$

Let us introduce two angles (= phase delays) φ_{10} and φ_{12}

such that $\tan(ph+\phi) = \tan\varphi_{10} = \frac{\gamma_0}{p} \geq 0$ $\quad 0 \leq \varphi_{10} \leq \pi/2$

and $-\tan\phi = \tan\varphi_{12} = \frac{\gamma_2}{p} \geq 0$ $\quad 0 \leq \varphi_{12} \leq \frac{\pi}{2}$

An angle is defined by its tangent modulo π so

$$ph + \phi = \varphi_{10} + n\pi \quad (n \in \mathbb{N})$$
$$-\phi = \varphi_{12} + n'\pi \quad (n' \in \mathbb{N})$$

$$\beta t \rightarrow ph = (ph+\phi) - \phi = \varphi_{10} + \varphi_{12} + \underbrace{(n+n')}_{m}\pi = \varphi_{10} + \varphi_{12} + m\pi \quad (m \in \mathbb{N}) \qquad (9)$$

we recover the dispersion equation found with rays considerations (p3) !!!

φ_{10} and φ_{12} correspond to the Goos Hanchen phase delays. But now we are able to get them:

Let us note that the quantities ph, φ_{10} and φ_{12} being positive and with $0 \leq \varphi_{10} + \varphi_{12} \leq \pi$, the integer m can only take integer values positive or null.

Note: To get $m = -1$ we must have in (9): $ph = 0$; $\varphi_{10} = \varphi_{12} = \frac{\pi}{2}$

$\Rightarrow ph = 0 \Rightarrow p = 0 \Rightarrow n_e = n_1$

and with $-\tan\phi = \tan\varphi_{12}$ we should have $\phi = -\frac{\pi}{2}$

Then, with $p=0$ and $\phi = -\frac{\pi}{2}$ (5) becomes $A_0 = C\cos(0\times h - \frac{\pi}{2}) = 0$

and (6) becomes $C\cos(-\frac{\pi}{2}) = B_2 = 0$

and everywhere in the guide $E_y = C\cos(0 \times x - \frac{\pi}{2}) = 0$

the field is null everywhere

Now, let us replace p by its value as a function of the indices:

$$k_0 \sqrt{n_1^2 - n_e^2} \times h = \varphi_{10} + \varphi_{12} + m\pi \quad \text{with } m = 0, 1, 2 \ldots \quad (10)$$

and

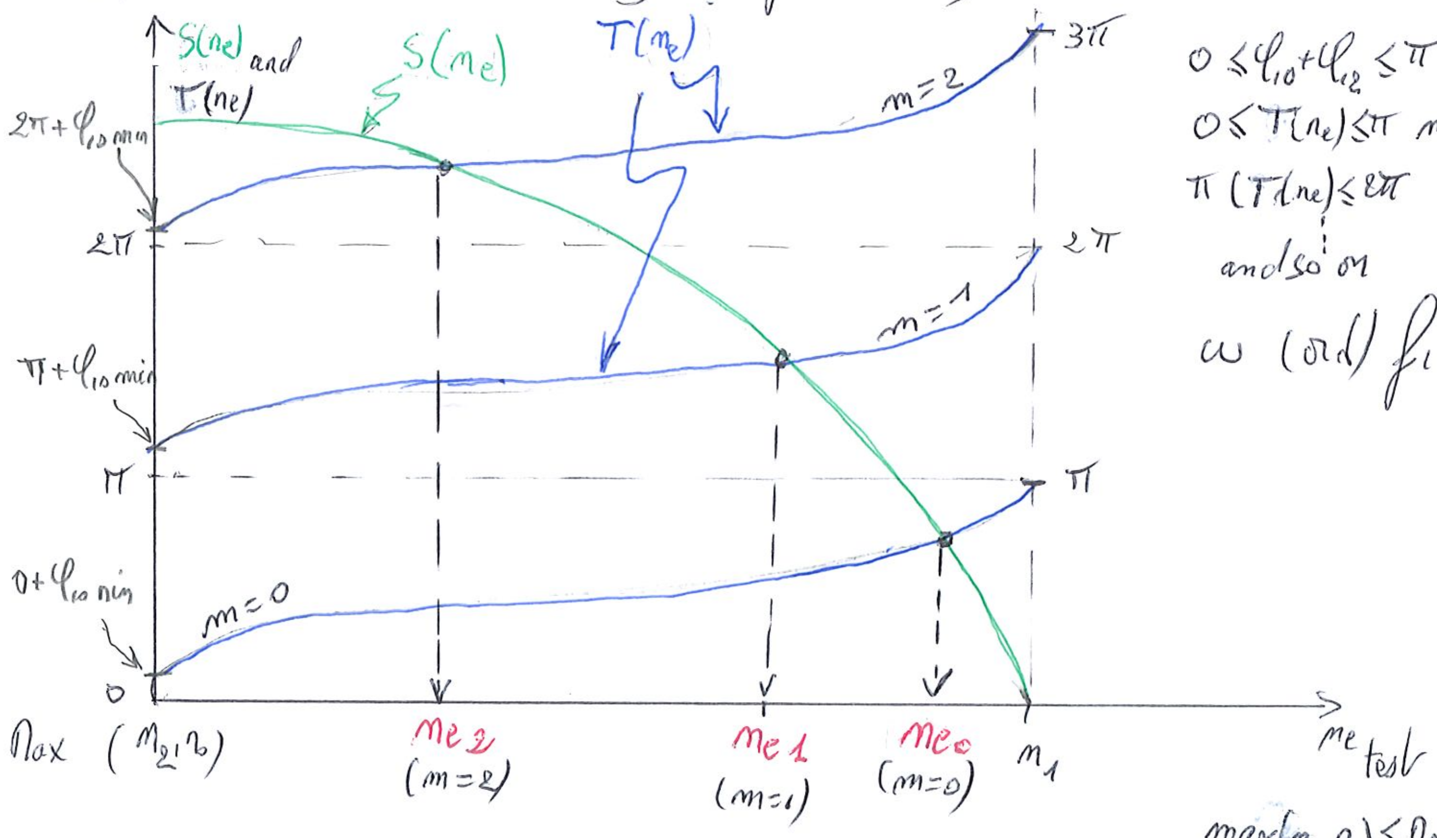
$$\varphi_{10} = ph + \phi = \mathrm{Atan}\,\frac{\gamma_0}{p} = \mathrm{Atan}\sqrt{\frac{n_e^2 - n_0^2}{n_1^2 - n_e^2}}$$

$$\varphi_{12} = -\phi = \mathrm{Atan}\,\frac{\gamma_2}{p} = \mathrm{Atan}\sqrt{\frac{n_e^2 - n_2^2}{n_1^2 - n_e^2}}$$

The relation (10) is the dispersion equation of TE modes

n_e (which allows to recover β with $\beta = k_0 n_e$) cannot be directly calculated because the equation is a transcendent equation.

It can be solved numerically or graphically : $\underset{\uparrow ph}{S(n_e)} = \underset{\uparrow \varphi_{10} + \varphi_{12} + m\pi}{T(n_e)}$



$0 \leq \varphi_{10} + \varphi_{12} \leq \pi$

$0 \leq T(n_e) \leq \pi \quad m = 0$

$\pi \leq T(n_e) \leq 2\pi \quad m = 1$

and so on

ω (or λ) fixed

$\max(n_2, n_0) \leq n_e \leq n_1$

end

Annexe 1

Calculation of the relationships between the components of the electric field (E_x, E_y and E_z) and the magnetic field (H_x, H_y and H_z), for a TE mode in a slab waveguide with an infinite extend in the y direction.

As stated pages 4 and 5, in an harmonic regime, the electric field of a given transverse mode is $\vec{\mathcal{E}}(x,y,z) = \mathrm{Re}\left[\underbrace{\vec{E}(x,y)}_{\text{transverse mode}} e^{j(\omega t - \underbrace{\beta}_{\text{propagation constant}} z)}\right]$

The associated magnetic field is $\vec{\mathcal{H}}(x,y,z) = \mathrm{Re}\left[\vec{H}(x,y)\, e^{j(\omega t - \beta z)}\right]$

In this dielectric waveguide, the Maxwell equations relating $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ are

$$\mathrm{curl}\,\vec{\mathcal{E}} = -\frac{\partial \vec{B}}{\partial t} \quad \text{with } \vec{B} = \mu \vec{\mathcal{H}} \text{ and } \mu = \mu_0 \qquad (1)$$

$$\mathrm{curl}\,\vec{\mathcal{H}} = \varepsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \quad \text{with } \varepsilon = \varepsilon_0 \varepsilon_r \qquad (2)$$

* the harmonic form of the fields allows to write, for any component X:
  $\frac{\partial X}{\partial t} = j\omega X$ and $\frac{\partial X}{\partial z} = -j\beta z X$
* As the extension of the slab is assumed to be infinite in the y direction, the components do no depend on $y \Rightarrow \frac{\partial X}{\partial y} = 0$
* we consider TE modes $\Rightarrow E_z = 0$

Let us now develop relations (1) and (2) (with the operator curl $= \nabla \wedge$.)

$$\operatorname{curl}\vec{\mathcal{E}} = \begin{vmatrix} \frac{\partial}{\partial x} & \mathcal{E}_x & \vec{e_x} \\ \frac{\partial}{\partial y} & \mathcal{E}_y & \vec{e_y} \\ \frac{\partial}{\partial z} & \mathcal{E}_z & \vec{e_z} \end{vmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}\mathcal{E}_z - \frac{\partial}{\partial z}\mathcal{E}_y \\ -\frac{\partial}{\partial x}\mathcal{E}_z + \frac{\partial}{\partial z}\mathcal{E}_x \\ \frac{\partial}{\partial x}\mathcal{E}_y - \frac{\partial}{\partial y}\mathcal{E}_x \end{pmatrix}$$

$$= \begin{pmatrix} j\beta\mathcal{E}_y \\ -j\beta\mathcal{E}_x \\ \frac{\partial}{\partial x}\mathcal{E}_y \end{pmatrix} = \begin{pmatrix} j\beta E_y \\ -j\beta E_x \\ \frac{\partial}{\partial x}E_y \end{pmatrix} e^{j(\omega t-\beta z)} \qquad (3)$$

$$-\frac{\partial\vec{\mathcal{B}}}{\partial t} = \begin{pmatrix} -j\omega\mu\mathcal{H}_x \\ -j\omega\mu\mathcal{H}_y \\ -j\omega\mu\mathcal{H}_z \end{pmatrix} = \begin{pmatrix} -j\omega\mu H_x \\ -j\omega\mu H_y \\ -j\omega\mu H_z \end{pmatrix} e^{j(\omega t-\beta z)} \qquad (4)$$

From (1) : $(3)=(4) \Rightarrow$

$$\begin{cases} j\beta E_y = -j\omega\mu H_x & (5) \\ -j\beta E_x = -j\omega\mu H_y & (6) \\ \frac{\partial}{\partial x}E_y = -j\omega\mu H_z & (7) \end{cases}$$

similarly

$$\operatorname{curl}\vec{\mathcal{H}} = \begin{vmatrix} \frac{\partial}{\partial x} & \mathcal{H}_x & \vec{e_x} \\ \frac{\partial}{\partial y} & \mathcal{H}_y & \vec{e_y} \\ \frac{\partial}{\partial z} & \mathcal{H}_z & \vec{e_z} \end{vmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}\mathcal{H}_z - \frac{\partial}{\partial z}\mathcal{H}_y \\ -\frac{\partial}{\partial x}\mathcal{H}_z + \frac{\partial}{\partial z}\mathcal{H}_x \\ \frac{\partial}{\partial x}\mathcal{H}_y - \frac{\partial}{\partial y}\mathcal{H}_x \end{pmatrix} = \begin{pmatrix} j\beta H_y \\ -j\beta H_x \\ \frac{\partial}{\partial x}H_y \end{pmatrix} e^{j(\omega t-\beta z)} \qquad (8)$$

$$\varepsilon\frac{\partial}{\partial t}\vec{\mathcal{E}} = \begin{pmatrix} j\omega\varepsilon\mathcal{E}_x \\ j\omega\varepsilon\mathcal{E}_y \\ j\omega\varepsilon\mathcal{E}_z = 0 \end{pmatrix} = \begin{pmatrix} j\omega\varepsilon E_x \\ j\omega\varepsilon E_y \\ 0 \end{pmatrix} e^{j(\omega t-\beta z)} \qquad (9)$$

From (2) : $(8)=(9) \Rightarrow$

$$\begin{cases} j\beta H_y = j\omega\varepsilon E_x & (10) \\ -j\beta H_x = j\omega\varepsilon E_y & (11) \\ \frac{\partial H_y}{\partial x} = 0 & (12) \end{cases}$$

From (12) : $\frac{\partial H_y}{\partial x} = 0 \Rightarrow H_y = \text{cte}$ versus x

$\Rightarrow$ Far from the guide (x very large), thus at the ∞, $H_y = \text{cte}$

$\Rightarrow$ the only physical solution is $H_y = 0$ (13)

Thus from (6) $\Rightarrow E_x = 0$ (14)

from (5) $\Rightarrow H_x = \frac{-\beta}{\omega \mu_0} E_y$ (15)

from (7) $\Rightarrow H_z = \frac{j}{\omega \mu_0} \frac{\partial E_y}{\partial x}$ (16)

Finally, with (13), (14), (15), (16) we can write:

$$\vec{E}(x, y) = \vec{E}(x) = \begin{cases} E_x = 0 \\ E_y \neq 0 \\ E_z = 0 \end{cases}$$

and
$$\vec{H}(x, y) = \vec{H}(x) = \begin{cases} H_x = \frac{-\beta}{\omega \mu_0} E_y \\ H_y = 0 \\ H_z = \frac{j}{\omega \mu_0} \frac{\partial E_y}{\partial x} \end{cases}$$

?

Annexe 2

Demonstration of the Helmoltz equation (3) of pages 5

We start from the 2 Maxwell equations

$$\text{curl } \vec{\mathcal{E}} = -\frac{\partial \vec{\mathcal{B}}}{\partial t} \quad (1) \qquad \text{and} \quad \text{curl } \vec{\mathcal{H}} = \varepsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \quad (2)$$

For a vector $\vec{X}$ we know that $\text{curl}(\text{curl}(\vec{X})) = \text{grad}(\text{div}\vec{X}) - \Delta\vec{X}$

thus $\text{curl}(\text{curl}(\vec{\mathcal{E}})) = \text{grad}(\text{div}\vec{\mathcal{E}}) - \Delta\vec{\mathcal{E}}$

with $\text{div}\vec{\mathcal{E}} = \frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} + \frac{\partial \mathcal{E}_z}{\partial z} = 0$

since $E_x = 0$, $\mathcal{E}_z = 0$ and $\frac{\partial(\text{component})}{\partial y} = 0$

$$\Rightarrow \text{curl}(\text{curl}(\vec{\mathcal{E}})) = -\Delta\vec{\mathcal{E}} \quad (3)$$

With (1) and (2)

$$\begin{aligned}\text{curl}(\text{curl}(\vec{\mathcal{E}})) &= \text{curl}\left(-\frac{\partial \vec{\mathcal{B}}}{\partial t}\right) \\ &= \text{curl}(-j\omega\vec{\mathcal{B}}) = \text{curl}(-j\omega\mu_0\vec{\mathcal{H}}) \\ &= -j\omega\mu_0 \,\text{curl}(\vec{\mathcal{H}}) \\ &= -j\omega\mu_0 \cdot \left(\varepsilon\frac{\partial\vec{\mathcal{E}}}{\partial t}\right) \\ &= -j\omega\mu_0\,\varepsilon\, j\omega\,\vec{\mathcal{E}} \\ &= \omega^2\mu_0(\varepsilon_0 n_i^2)\vec{\mathcal{E}}\end{aligned}$$

knowing that $\omega = k_0 c$ and $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \Rightarrow \omega^2\mu_0\varepsilon_0 = k_0^2$

Finally $\text{curl}(\text{curl}(\vec{\mathcal{E}})) = k_0^2 n_i^2 \vec{\mathcal{E}}$ (4)

With (3) and (4): $-\Delta\vec{\mathcal{E}} = k_0^2 n_i^2 \vec{\mathcal{E}}$

The Helmoltz equation is then $\Delta\vec{\mathcal{E}} + k_0^2 n_i^2 \vec{\mathcal{E}} = \vec{0}$ (5)

Let us calculate $\Delta \vec{\mathcal{E}}$

$$\Delta \vec{\mathcal{E}} = \begin{pmatrix} \frac{\partial^2 \mathcal{E}_x}{\partial x^2} + \frac{\partial^2 \mathcal{E}_x}{\partial y^2} + \frac{\partial^2 \mathcal{E}_x}{\partial z^2} \\ \frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial y^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2} \\ \frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 0 \quad \text{since } \mathcal{E}_x = 0 \\ \frac{\partial^2 \mathcal{E}_y}{\partial x^2} + 0 + (-j\beta)^2 \mathcal{E}_y \\ 0 \quad \text{since } \mathcal{E}_z = 0 \end{pmatrix}$$

Thus $\Delta \vec{\mathcal{E}} = \left(\frac{\partial^2 \mathcal{E}_y}{\partial x^2} - \beta^2 \mathcal{E}_y \right) \vec{e_y}$ (6)

With (6), (5) becomes $\frac{\partial^2 \mathcal{E}_y}{\partial x^2} - \beta^2 \mathcal{E}_y + k_0^2 n_i^2 \mathcal{E}_y = 0$ (7)

knowing that $\mathcal{E}_y = E_y e^{j(\omega t - \beta z)}$, we divide $\mathcal{E}_y$ by $e^{+j(\omega t - \beta z)}$

and (7) becomes $\frac{\partial^2 E_y}{\partial x^2} - \beta^2 E_y + k_0^2 n_i^2 E_y = 0$

$$\Leftrightarrow \quad \frac{\partial^2 E_y}{\partial x^2} + \underbrace{\left(k_0^2 n_i^2 - \beta^2 \right)}_{\beta_{e_i}^2} E_y = 0$$