CALCULUS

Prof. Liang ZHENG

Spring 2025

- Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

① Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int sin^m x cos^n x dx$$

where *m* and *n* are nonnegative integers (positive or zero). We can divide the appropriate substitution into four cases according to *m* and *n* being odd or even.

Case 1 If ***m* is odd and *n* is even**, we write m as $2k + 1$ and use the identity

$$\sin^2 x = 1 - \cos^2 x$$

to obtain

$$\int \sin^m x \cos^n x \, dx = \int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x (\sin x) dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x (-d \cos x)$$

$$\overset{u=\cos x}{=} -\int (1-u^2)^k u^n du.$$

Example 1 Evaluate

$$\int \sin^3 x \cos^2 x dx$$

Case 2 If ***n* is odd and *m* is even**, we write n as $2k+1$ and use the identity

$$\cos^2 x = 1 - \sin^2 x$$

to obtain

$$\begin{aligned}\int \sin^m x \cos^n x \, dx &= \int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \cos^{2k} x \cos x dx \\ &= \int \sin^m x (\cos^2 x)^k d(\sin x) = \int (\sin x)^m (1-\sin^2 x)^k d(\sin x) \\ &= \int u^m (1-u^2)^k du\end{aligned}$$

Example 2 Evaluate

$$\int sin^2 x cos^5 x dx$$

Case 3 If **both *m* and *n* are even**, we substitute

$$sin^2x = \frac{1 - cos2x}{2} \quad and \quad cos^2x = \frac{1 + cos2x}{2}$$

to reduce the integrand to one in lower powers of cos $2x$.

Case 4 If **both *m* and *n* are odd**,

we can choose either sin$x$ or cos$x$ as the substitution function.

Example 3 Evaluate

$$(a) \int sin^2xcos^2xdx \qquad (b) \int sin^5xcos^3xdx$$

② Eliminating Square Roots

Use the identities:

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad and \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

to eliminate a square root.

Example 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx$$

Example 5 Evaluate

$$\int_{-\pi/2}^{\pi/2} \sqrt{1-\cos 2x}\,dx$$

③ Integrals of Powers of $\tan x$ and $\sec x$

- We know how to integrate the tangent and secant functions and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.
- Some derivatives to recall:

$$(tanx)' = sec^2x \quad and \quad (secx)' = secxtanx$$

Example 6 Evaluate

$(a)\ \int tan^4xsec^4xdx \qquad (b)\ \int tan^3xsec^5xdx$

Example 7 Evaluate

$$(a)\ \int tan^3xdx$$

$$(b)\ \int tan^4xdx$$

Example 8 Evaluate

$$(a)\ \int sec^3xdx$$

$$(b)\ \int sec^4xdx$$

④ Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\,dx, \qquad \int \sin mx \cos nx\,dx, \qquad \int \cos mx \cos nx\,dx$$

arise in many applications involving periodic functions. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

Example 9 Evaluate

$$\int sin3xcos5xdx$$

Example 10 Evaluate

$$\int cosxcos3xdx$$

Skill Practice 1 Evaluate

$$\int cos^3 4x dx$$

Skill Practice 2 Evaluate

$$\int \frac{sec^3 x}{tanx} dx$$

Skill Practice 3 **Average Function Value**

Find the average value of the function on [0, π/3]:

$$f(\theta) = \frac{1}{1 - sin\theta}$$