CALCULUS

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- Integration by parts is a technique for simplifying integrals of the form $\int u(x)v'(x)\,\mathrm{d}x$. It is useful when u can be differentiated repeatedly and v' can be integrated repeatedly without difficulty.
- The integrals $\int xcosxdx$ and $\int x^2e^xdx$ are such integrals because $u(x)=x$ or $u(x)=x^2$ can be differentiated repeatedly to become zero, and $v'(x) = \cos x$ or $v'(x) = e^x$ can be integrated repeatedly without difficulty.
- Integration by parts also applies to integrals like

$$\int lnxdx \quad \text{and} \quad \int e^x cosxdx\,.$$

In the first case, the integrand $\ln x$ can be rewritten as $(\ln x)(1)$, and $u(x)=\ln x$ is easy to differentiate while $v'(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

① Product Rule in Integral Form

If u and v are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[u(x)v(x)]dx = \int [u'(x)v(x) + u(x)v'(x)]dx$$

$$or \qquad u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

Rearranging the terms of this last equation, we get

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx \quad or \quad \int udv = uv - \int vdu$$

which is called **"Integration by parts" formula**.

② Tip to choose u and dv in $\int f(x)\,\mathrm{d}x$

(a) Try letting u be a factor of $f(x)$ and u' is a function simple than u.

(b) Try letting dv be the most complicated portion of $f(x)\mathrm{d}x$ and $\int \mathrm{d}v$ can be found by a basic integration formula.

Example 1 Evaluate

$$\int x\cos x\,dx$$

Example 2 Evaluate

$$\int \ln x\,dx$$

Example 3 Evaluate

$$\int x^2 e^x dx$$

Example 4 Evaluate

$$\int e^x cosxdx$$

Skill Practice 1 Evaluate

$$\int e^x sinxdx$$

Example 5 Obtain a formula that expresses the integral

$$\int cos^n x dx$$

in terms of an integral of a lower power of $\cos x$ $(n \geq 2)$.

$$\int cos^n x dx = \frac{cos^{n-1} x sin x}{n} + \frac{n-1}{n} \int cos^{n-2} x dx$$

- The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced.
- When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate.
- For example, the result in Example 5 tells us that:

$$\int cos^3xdx = \frac{1}{3}cos^2xsinx + \frac{2}{3}\int cosxdx = \frac{1}{3}cos^2xsinx + \frac{2}{3}sinx + C$$

③ Evaluating Definite Integrals by Parts

Integration by Parts Formula for Definite Integrals

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$

Example 6

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

$$\int_0^4 xe^{-x}dx$$

④ Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

Let $y = f^{-1}(x)$, then $x = f(y)$ and $dx = f'(y)\,dy$. Therefore:

$$\int f^{-1}(x)dx = \int yf'(y)dy = \int yd[f(y)] = yf(y) - \int f(y)dy = xf^{-1}(x) - \int f(y)dy$$

Example 7 **For the integral of lnx :** $y = f^{-1}(x) = \ln x \Rightarrow x = e^y, \quad dx = e^y dy$

$$\int lnxdx = \int ye^y dy = \int yd(e^y) = ye^y - \int e^y dy = ye^y - e^y + C = xlnx - x + C$$

Example 8 **For the integral of $\cos^{-1}x$:** $y = f^{-1}(x) = \cos^{-1}x \Rightarrow x = \cos y, \; dx = -\sin y dy$

$$\int cos^{-1}xdx = xcos^{-1}x - \int cosydy = xcos^{-1}x - \sin(cos^{-1}x) + C$$

Example 9

Use the formula: let $y = f^{-1}(x)$

$$\int f^{-1}(x)dx = \int yd[f(y)] = yf(y) - \int f(y)dy = xf^{-1}(x) - \int f(y)dy$$

to evaluate

$(a)\quad \int sin^{-1}xdx$

$(b)\quad \int tan^{-1}xdx$

$(c)\quad \int log_2 xdx$

Skill Practice 2 Evaluate the following integrals:

$(a)\ \int xe^{2x}dx$ $(b)\ \int_0^{\pi/2} x^3 cosxdx$

Skill Practice 3 **Finding volume**

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$ and the line $x = \ln 2$ about the x-axis.