Fundamentals of Electric Circuits

CHAPTER 7 Sinusoids and Phasors

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CHAPTER 7 Sinusoids and Phasors

7.2 Alternating Current

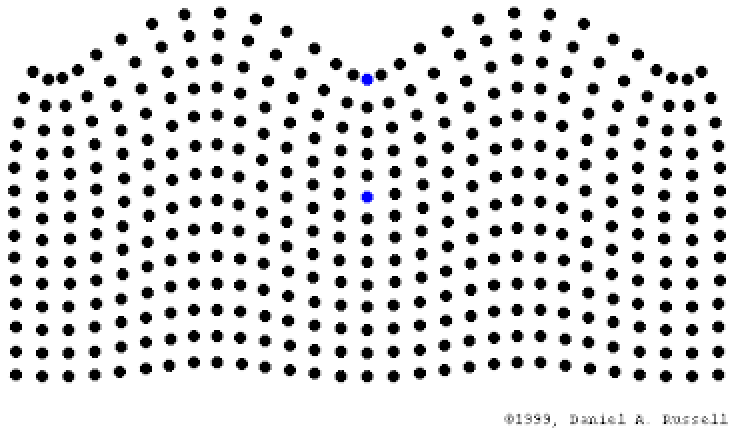
- Alternating Current, or AC, is the dominant form of electrical power that is delivered to homes and industry.
- In the late 1800’s, there was a battle between proponents of DC and AC.
- AC won out due to its efficiency for long distance transmission.
- AC is a sinusoidal current, meaning the current reverses at regular time intervals and has alternating positive and negative values.

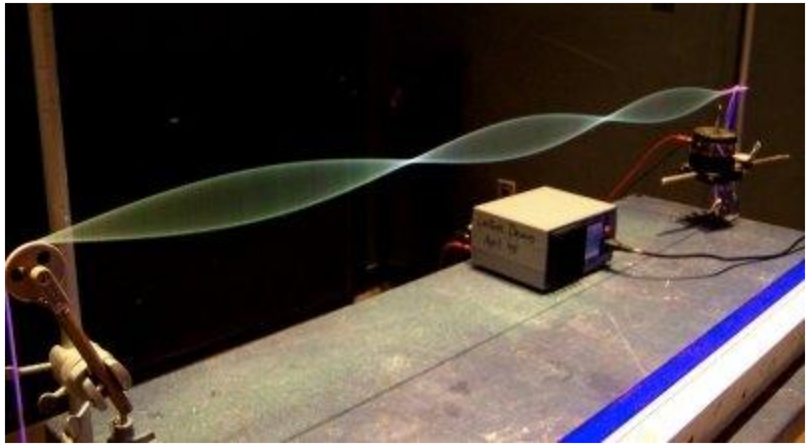
The dramatic story of the cutthroat race between electricity titans Thomas A. Edison and George Westinghouse to determine whose electrical system would power the modern world.

Sinusoids

- Sinusoids are interesting to us because there are a number of natural phenomenon that are sinusoidal in nature, such as the vibration of a string, the ripples on the ocean surface.

- It is also a very easy signal to generate and transmit.

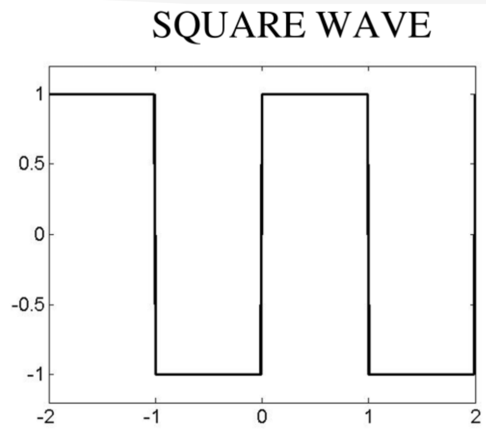


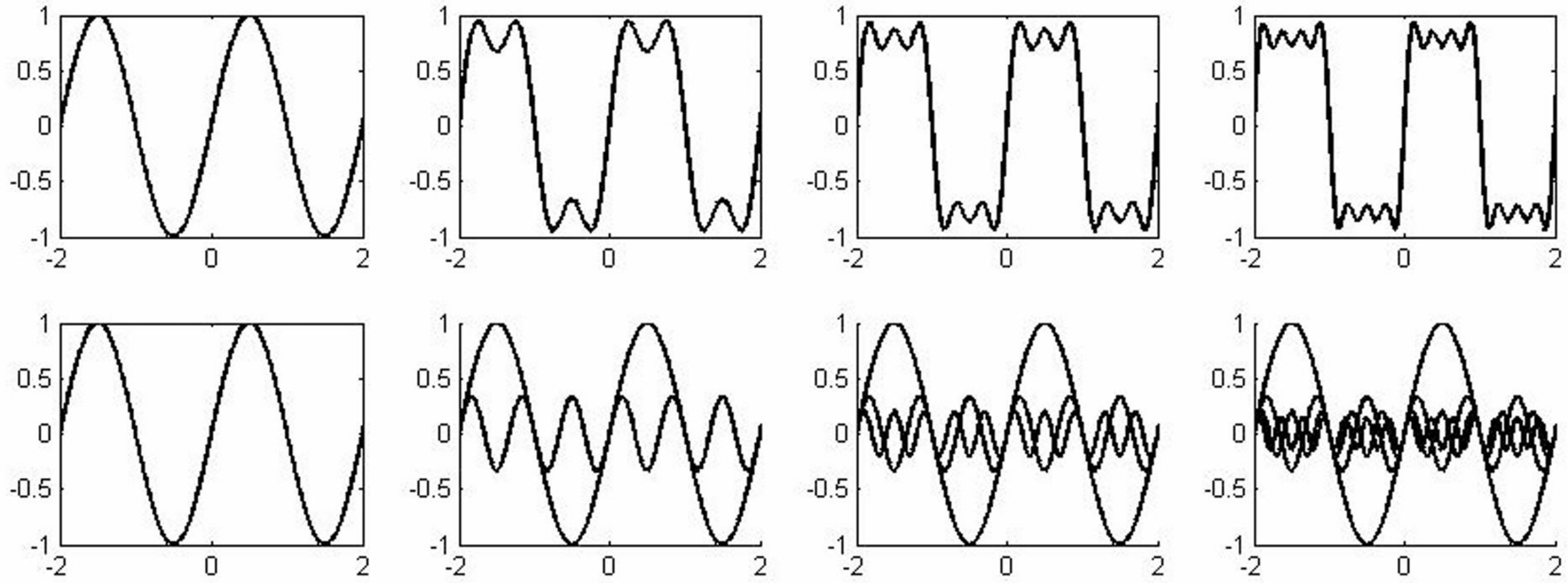


https://sciencedemonstrations.fas.harvard.edu/presentations/vibrating-string

Sinusoids

- Also, through **Fourier's theorem**, any function can be represented by a sum of sinusoids of various amplitudes and frequencies. $square(x) = \sin \pi x + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \frac{\sin 7\pi x}{7} + \ldots$





summed waveform

component sine waves

- Lastly, they are very easy to handle mathematically.

Sinusoids

- A sinusoidal voltage may be represented as:

$$v(t) = V_m \sin \omega t$$

V_m = the *amplitude* of the sinusoid
ω = the *angular frequency* in radians/s
ωt = the *argument* of the sinusoid

- Sinusoids may be expressed as sine or cosine
- From the waveform shown below, one characteristic is clear: The function repeats itself every T seconds.
- This is called the period T.

$$T = \frac{2\pi}{\omega}$$

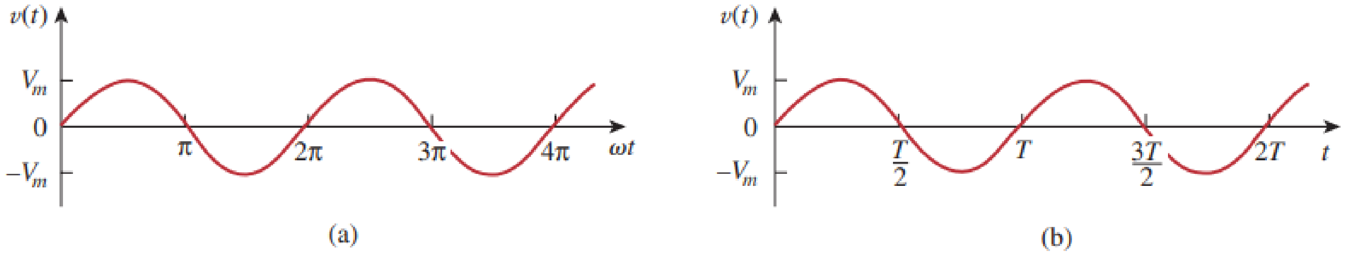


Figure 9.1
A sketch of $V_m \sin \omega t$: (a) as a function of ωt, (b) as a function of t.

Sinusoids

- The reciprocal of the period is the number of cycles per second, known as the frequency.

$$f = \frac{1}{T}$$

- The unit is Hertz (Hz).
- Angular frequency is often used.

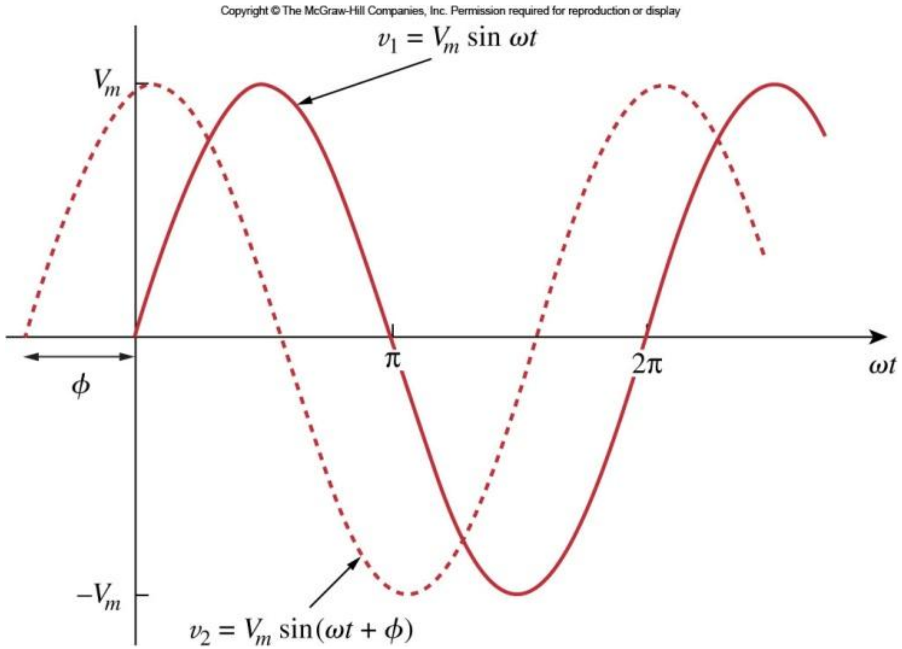
$$\omega = 2\pi f$$

- The unit is radians per second (rad/s).

Phase

- ϕ is the phase;
- Consider the two sinusoids with different phase:

$$v_1(t) = V_m \sin \omega t \quad \text{and} \quad v_2(t) = V_m \sin(\omega t + \phi)$$



If $\phi > 0$, $V2$ *leads* $V1$ *by* ϕ

or $V1$ *lags* $V2$ *by* ϕ

If $\phi = 0$, the $V1$ and $V2$ are said to be *in phase,* they reach their maximum and minimum at the same time

Example

Find the amplitude, phase, period, and frequency of the sinusoid

Example 9.1

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.
The phase is $\phi = 10^\circ$.
The angular frequency is $\omega = 50$ rad/s.
The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.
The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

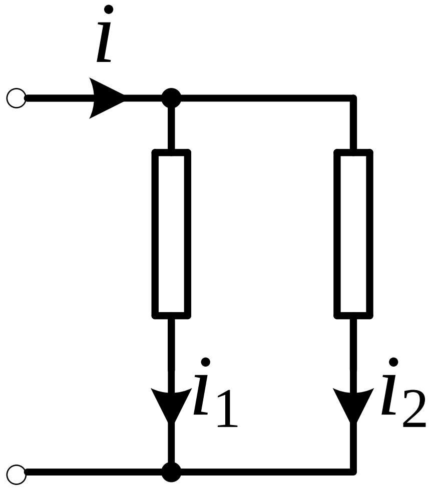
7.3 Phasors

Question:

$$i_1 = 6\cos(314t + 30^\circ)$$

$$i_2 = 8\cos(314t - 60^\circ)$$

$$i = i_1 + i_2 = ?$$



$$\begin{aligned} i &= i_1 + i_2 \\ &= 6\cos(314t + 30^\circ) + 8\cos(314t - 60^\circ) \\ &= \left(3\sqrt{3} + 4\right)\cos(314t) + \left(4\sqrt{3} - 3\right)\sin(314t) \\ &= 10\cos(314t - 23.1^\circ) \end{aligned}$$

Complicated!

7.3 Phasors

Especially, if inductors or capacitors are contained in ac circuits:

$$i = C\frac{du}{dt} \qquad u = L\frac{di}{dt}$$

Therefore, when analyzing the circuits in the time domain, it is necessary to establish differential and integral equations.

Ac circuit $\xrightarrow{\textbf{analyze}}$ **circuit equations (differential and integral)** $\xrightarrow{\textbf{solve}}$ **time domain response**

7.3 Phasors

Consider:

1. Can a simple method be used to avoid tedious trigonometric function operations?

2. The integral and derivative of a sinusoidal function, and the signed summation of sinusoids of the same frequency is a sinusoid of the same frequency. So the key is to determine the amplitude and phase.

$$i_1 = 6\cos(314t + 30^\circ)$$

$$i_2 = 8\cos(314t - 60^\circ)$$

$$i = i_1 + i_2 = 10\cos(314t - 23.1^\circ)$$

7.3 Phasors

- A powerful method for representing sinusoids is the phasor.
- A phasor is a complex number that represents the amplitude and phase of a sinusoid.
- But in order to understand how they work, we need to cover some complex numbers first.
- A complex number *z* can be represented in rectangular form as:

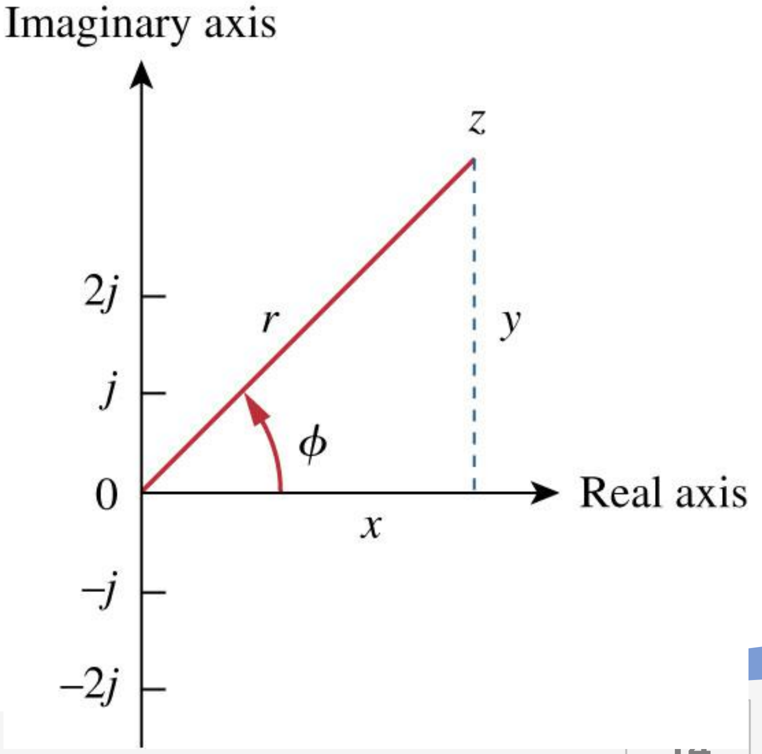
$$z = x + jy$$

where $j = \sqrt{-1}$, x is the real part of z, y is the imaginary part of z.

- The complex number z can also be written in polar form or exponential form as:

$$z = r\angle\emptyset = re^{j\phi}$$

where r is the magnitude of z, and ∅ is the phase of z.



Complex Numbers

$z = x + jy$ rectangular form

$z = r\angle\phi$ polar form

$z = re^{j\phi}$ exponential form

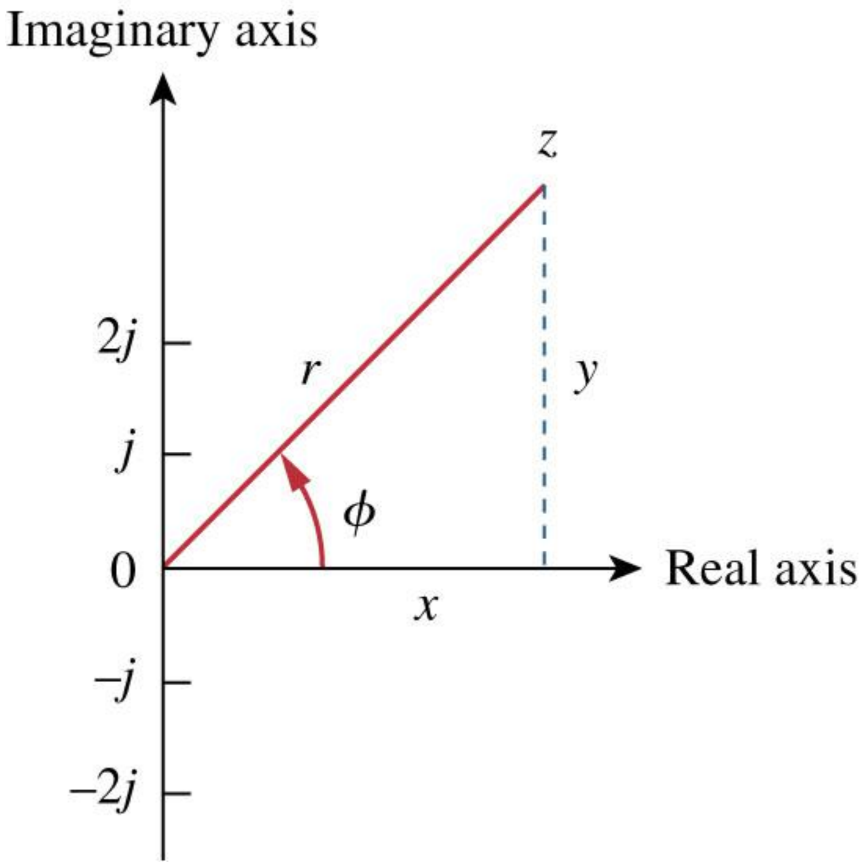
The different forms can be interconverted.

- Starting with rectangular form, one can go to polar:

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x}$$

- Likewise, from polar to rectangular form:

$$x = r\cos\phi \qquad y = r\sin\phi$$



Complex Numbers

- The following mathematical operations are important.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

- Addition and subtraction of complex numbers are better performed in **rectangular form**!

Multiplication:

$$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle(\phi_1 - \phi_2)$$

- Multiplication and division of complex numbers are better performed in **polar form**!

Phasors

The idea of a phasor representation is based on **Euler's identity**:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

Given a sinusoid: $v(t) = V_m \cos(\omega t + \varphi)$

Complex exponential function $V_m e^{j(\omega t+\varphi)}$

$$V_m e^{j(\omega t+\varphi)} = V_m \cos(\omega t + \varphi) + jV_m \sin(\omega t + \varphi)$$

so $v(t) = V_m \cos(\omega t + \varphi) = \mathrm{Re}[V_m e^{j(\omega t+\varphi)}]$

$$= \mathrm{Re}[V_m e^{j\varphi} e^{j\omega t}] = \mathrm{Re}[V e^{j\omega t}]$$

$V = V_m e^{j\varphi} = V_m \angle \varphi$ ——**phasor**

- We can represent a sinusoid as the real component of a complex exponential function in the complex plane.
- A phasor ***V*** is complex number.
- The magnitude of the phasor ***V*** is the amplitude of the sinusoid.
- The phasor ***V***, is at an angle φ with respect to the positive real axis.

Phasors

The transformation between time domain to phasor (frequency) domain is:

$$v(t) = \begin{array}{c} V_m \cos(\omega t + \varphi) \\ \text{(Time − domain} \\ \text{representation)} \end{array} \Leftrightarrow \begin{array}{c} \boldsymbol{V} = V_m \angle \varphi \\ \text{(Phasor − domain} \\ \text{representation)} \end{array}$$

$$v(t) = V_m \cos(\omega t + \varphi)$$

$$= \mathrm{Re}[V_m e^{j(\omega t + \varphi)}] = \mathrm{Re}[\mathbf{V} e^{j\omega t}]$$

$$\mathbf{V} = V_m e^{j\varphi} = V_m \angle \varphi$$

$e^{j\omega t}$

Phasors

$$v(t) = V_m \cos(\omega t + \varphi)$$

$$= \mathrm{Re}[V_m e^{j(\omega t+\varphi)}] = \mathrm{Re}[V e^{j\omega t}] \qquad V = V_m e^{j\varphi} = V_m \angle \varphi$$

The differences between *v*(*t*) and *V* :

(1) $v(t)$ is the instantaneous or time domain representation, while $\boldsymbol{V}$ is the frequency or phasor domain representation.

(2) $v(t)$ is the time dependent, while $\boldsymbol{V}$ is not.

(3) $v(t)$ is always real with no complex term, while $\boldsymbol{V}$ is complex.

Phasors

The meaning of the complex exponential function

$$\boxed{V_m e^{j(\omega t+\varphi)}}$$

$$v(t) = V_m \cos(\omega t + \varphi)$$
$$= \mathrm{Re}[V_m e^{j(\omega t+\varphi)}] = \mathrm{Re}[V e^{j\omega t}]$$

$$V = V_m e^{j\varphi} = V_m \angle \varphi$$

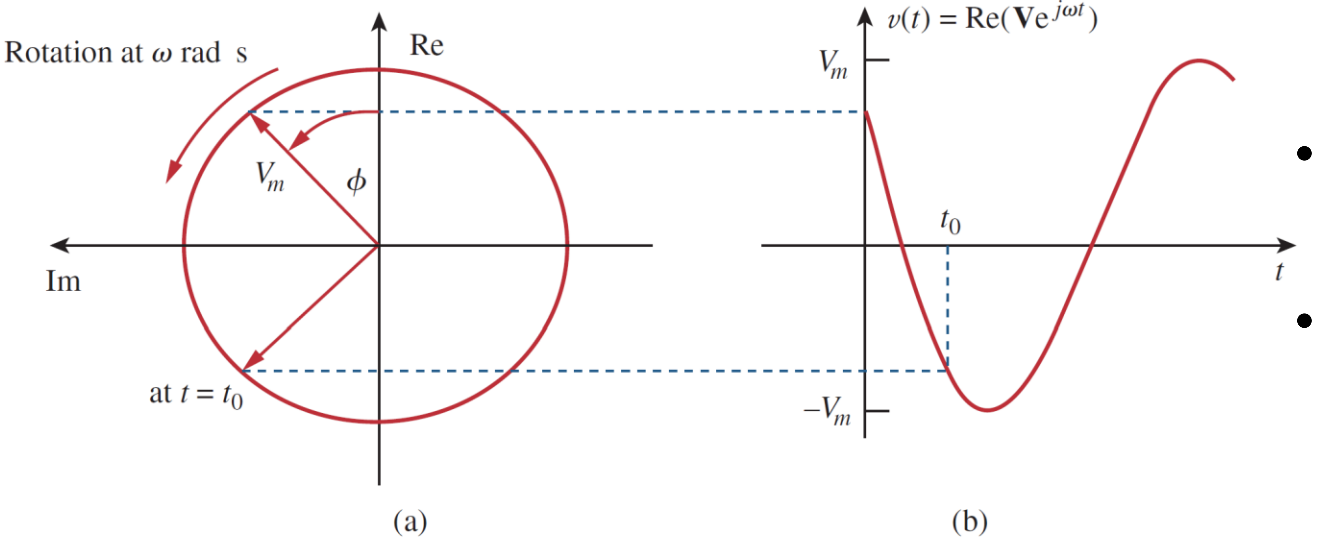


Figure 9.7
Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

- The value of the function at time t=0 is the phasor ***V*** of the sinusoid $v(t)$.
- $v(t)$ is the projection of the vector on the real axis.
- It is a rotating vector. As time increases, the vector rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction.
- When dealing with phasor, the term $e^{j\omega t}$ is not included. But keep in mind the frequency of circuit response is ω.

Phasors

- A phasor may be expressed in rectangular form, polar form, or exponential form. A phasor has magnitude and phase ("direction"), they can also be graphically represented (phasor diagram).

- The frequency is not shown in the phasor diagram because ω is constant.

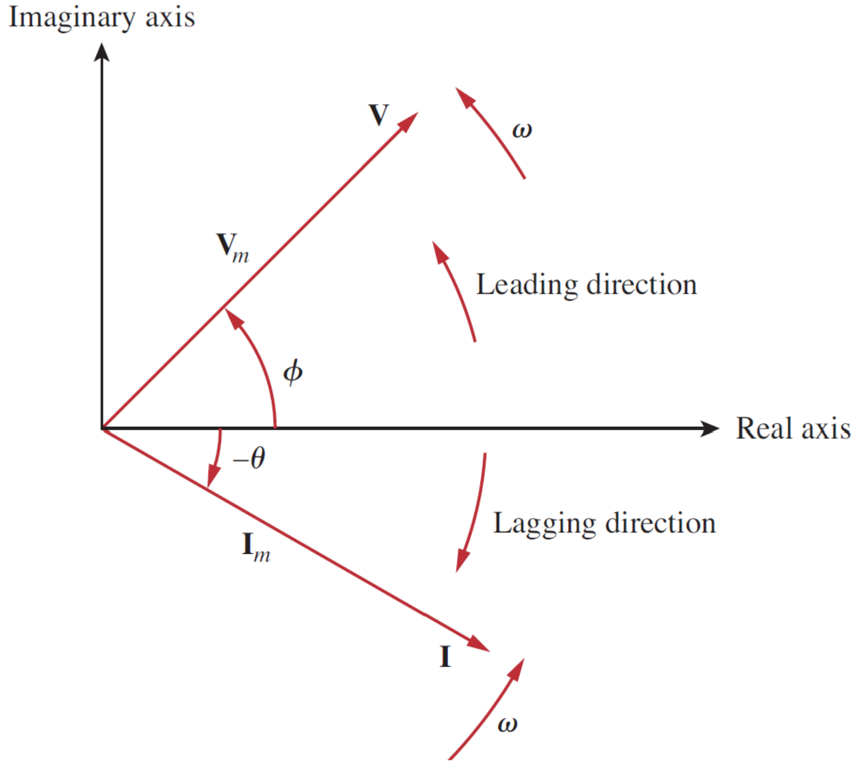


Figure 9.8
A phasor diagram showing $\mathbf{V} = V_m \underline{/\phi}$ and $\mathbf{I} = I_m \underline{/-\theta}$.

Sinusoid-Phasor Transformation

- Here is a handy table for transforming various time domain sinusoids into phasor domain:

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle (\phi - 90°)$
$I_m \cos(\omega t + \phi)$	$I_m \angle \phi$
$I_m \sin(\omega t + \phi)$	$I_m \angle (\phi - 90°)$

From *time-domain* to *phasor-domain*, express it in cosine form and take the magnitude and the phase;

From *phasor-domain* to *time-domain*, with the magnitude and phase of the phasor, with the frequency ωt

The symbols

Instantaneous value → u、i

Maximum value → U_{m}

Phasor (complex number) → $\boldsymbol{U}$ $\dot{U}$

Sinusoid-Phasor Transformation

- **The addition and subtraction of sinusoids, as well as calculus operations, can be performed using phasors.**

1. The addition of two sinusoids

$$i_1 = I_1\cos(\omega t+\varphi_1) \qquad i_1 = \mathrm{Re}[\dot{I}_1 e^{j\omega t}]$$

$$i_2 = I_2\cos(\omega t+\varphi_2) \qquad i_2 = \mathrm{Re}[\dot{I}_2 e^{j\omega t}]$$

$$i = i_1 + i_2$$

so

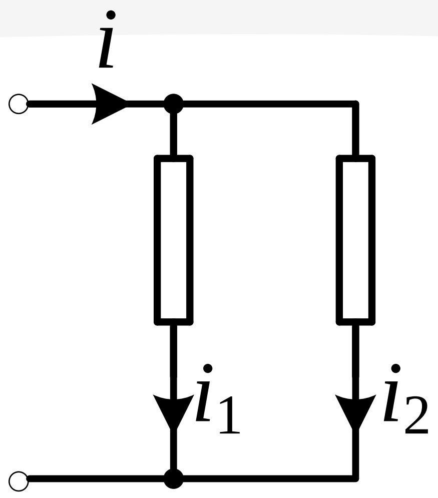
$$i = i_1 + i_2 = \mathrm{Re}[\dot{I}_1 e^{j\omega t}] + \mathrm{Re}[\dot{I}_2 e^{j\omega t}]$$

$$= \mathrm{Re}[(\dot{I}_1 + \dot{I}_2) e^{j\omega t}] = \mathrm{Re}[\dot{I} e^{j\omega t}]$$

$$\boxed{\dot{I} = (\dot{I}_1 + \dot{I}_2)}$$

The addition and subtraction operation of sinusoids at the same frequency becomes the addition and subtraction operation of corresponding phasors.

Question：



$$i_1 = 6\cos(314t + 30^\circ)$$

$$i_2 = 8\cos(314t - 60^\circ)$$

$$i = i_1 + i_2 = ?$$

$$\dot{I}_1 = 6\angle 30^\circ$$

$$\dot{I}_2 = 8\angle -60^\circ$$

$$\dot{I} = \dot{I}_1 + \dot{I}_2 = 6\angle 30^\circ + 8\angle -60^\circ$$

$$= \left(5.2 + j3\right) + (4 - j6.9)$$

$$= 10\angle -23.1^\circ$$

$$i = 10\cos(314t - 23.1^\circ)$$

Sinusoid-Phasor Transformation

Applying a derivative to a phasor yields: $v(t) = V_m \cos(\omega t + \varphi)$

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \varphi) = \omega V_m \cos(\omega t + \varphi + 90^0) = \mathrm{Re}[\omega V_m e^{j(\omega t + \varphi + 90^0)}]$$

$$= \mathrm{Re}[\omega V_m e^{j\varphi} e^{j90^0} e^{j\omega t}] = \mathrm{Re}[j\omega V_m e^{j\varphi} e^{j\omega t}] = \mathrm{Re}[j\omega \mathbf{V} e^{j\omega t}]$$

$\frac{dv}{dt}$	$\Leftrightarrow$	$j\omega V$
(Time domain)		**(Phasor domain)**

- The derivative of $v(t)$ is transformed to the phasor domain as $j\omega \mathbf{V}$

Sinusoid-Phasor Transformation

Applying an integral to a phasor yields: $v(t) = V_m \cos(\omega t + \varphi)$

$$\int v dt = \frac{V_m}{\omega}\sin(\omega t + \varphi) = \frac{V_m}{\omega}\cos(\omega t + \varphi - 90^0) = \mathrm{Re}[\frac{V_m}{\omega} e^{j(\omega t + \varphi - 90^0)}]$$

$$= \mathrm{Re}[\frac{V_m}{\omega} e^{j\varphi} e^{-j90^0} e^{j\omega t}] = \mathrm{Re}[-j\frac{V_m}{\omega} e^{j\varphi} e^{j\omega t}] = \mathrm{Re}[\frac{\mathbf{V}}{j\omega} e^{j\omega t}]$$

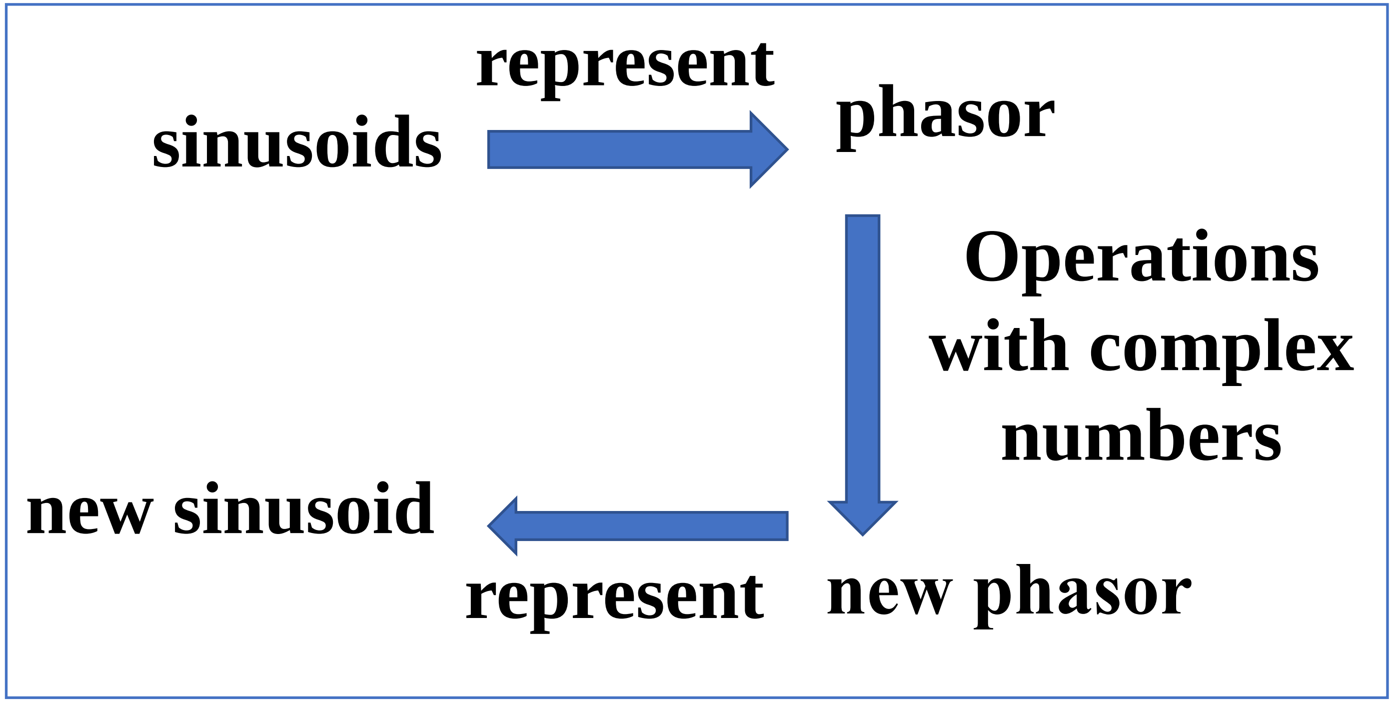
$\int v dt$	$\Leftrightarrow$	$\frac{\mathbf{V}}{j\omega}$
(Time domain)		**(Phasor domain)**

- Similarly, the integral of $v(t)$ is transformed to the phasor domain as $\mathbf{V} / j\omega$
- Phasor analysis applies only when the frequency is constant.

Phasor method

Represent sinusoidal voltages and currents as phasors, and then computes voltages and currents of AC circuit in phasor form—phasor method

The process



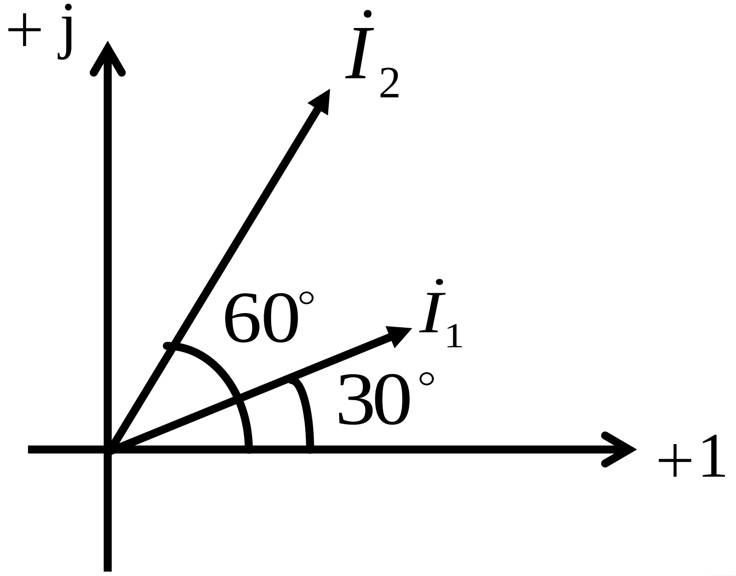
Note: The phasor method is only applicable to linear circuits excited by sinusoids of the same frequency.

Example

Given $i_1 = 2\cos(\omega t + 30^\circ)\boldsymbol{A}$, $i_2 = 5\cos(\omega t + 60^\circ)\boldsymbol{A}$, draw the two sinusoids in the phasor diagram.

【Solution】 The amplitude is $I_1 = 2\text{A}$, $I_2 = 5\text{A}$。

The phase is $+30^\circ$、$+60^\circ$,



Note: Only the sinusoids of the same frequency can be drawn on the same phasor diagram.

Example Given $i_1 = 2\cos(\omega t + 30^\circ)\boldsymbol{A}$, $i_2 = 5\cos(\omega t - 60^\circ)\boldsymbol{A}$, write the two sinusoids in rectangular form, polar form and exponential form.

【solution】First, write the polar form and exponential form.

$$\dot{I}_1 = 2e^{j30^\circ} = 2\angle 30^\circ \text{ A} \qquad \dot{I}_2 = 5e^{-j60^\circ} = 5\angle -60^\circ \text{ A}$$

$$\dot{I}_1 = 2e^{j30^\circ} = 2(\cos 30^\circ + j\sin 30^\circ) = (1.731 + j1)A$$

$$\dot{I}_2 = 5e^{-j60^\circ} = 5\left(\cos 60^\circ - j\sin 60^\circ\right) = \left(2.5 - j4.33\right)A$$

Example $\dot{I}_1 = (1.73 + \mathrm{j}1)\mathrm{A}$, $\dot{I}_2 = (2.5 - \mathrm{j}4.33)\mathrm{A}$ 。

write the two sinusoids in polar form and exponential form, and draw the two sinusoids in the phasor diagram.

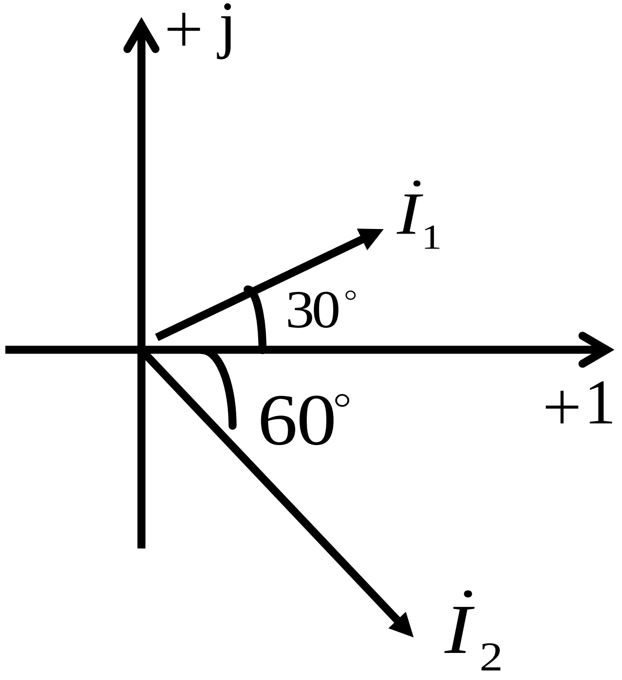
【solution】

$$\dot{I}_1 = 1.73 + \mathrm{j}1 = \sqrt{1.73^2 + 1^2}\angle\arctan\frac{1}{1.73}$$

$$= 2\angle 30^\circ\,\mathrm{A}$$

$$\dot{I}_2 = 2.5 - \mathrm{j}4.33 = \sqrt{2.5^2 + 4.33^2}\angle\arctan\frac{-4.33}{2.5}$$

$$= 5\angle -60^\circ\,\mathrm{A}$$



Example

Given $i_1 = 3\cos(\omega t)\boldsymbol{A}$, $i_2 = 4\cos(\omega t + 90^\circ)\boldsymbol{A}$,

Find $i = i_1 + i_2$

【Solution 1】 **Phasor formula**

$$\dot{I}_1 = 3\angle 0^\circ = 3\text{A}$$

,

$$\dot{I}_2 = 4\angle 90^\circ = 4(\cos 90^\circ + \text{j}\sin 90^\circ) = \text{j}4\text{A}$$

$$\dot{I} = \dot{I}_1 + \dot{I}_2 = 3 + \text{j}4 = \sqrt{3^2 + 4^2}\angle\arctan\frac{4}{3} = 5\angle 53.1^\circ\,\text{A}$$

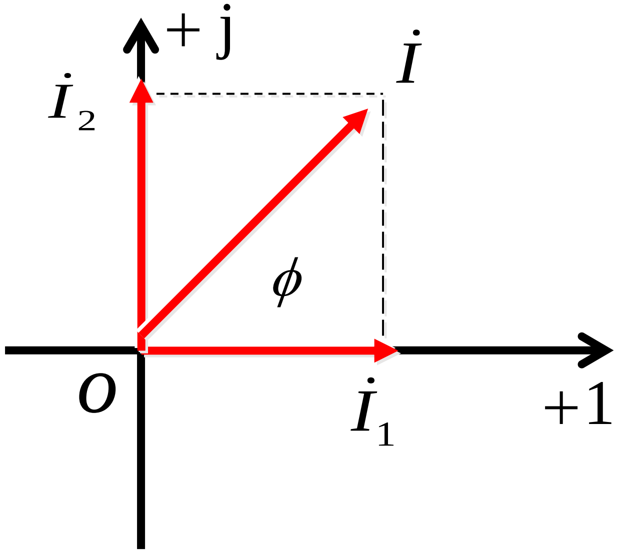
so $i = i_1 + i_2 = 5\cos(\omega t + 53.1^\circ)\boldsymbol{A}$

Example

Given $i_1 = 3\cos(\omega t)\boldsymbol{A}$, $i_2 = 4\cos(\omega t + 90^\circ)\boldsymbol{A}$,

Find $i = i_1 + i_2$

【Solution 2】 Phasor diagram



$$I = \sqrt{I_1^{\ 2} + I_2^{\ 2}} = \sqrt{3^2 + 4^2} = 5\text{A}$$

$$\varphi = \arctan\frac{I_2}{I_1} = \arctan\frac{4}{3} = 53.1^\circ$$

$$\dot{I} = 5\angle 53.1^\circ\ \text{A}$$

$$i = i_1 + i_2 = 5\cos(\omega t + 53.1^\circ)\boldsymbol{A}$$

Example 9.6

Example

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ A and $i_2(t) = 5\sin(\omega t - 20^\circ)$ A, find their sum.

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\underline{/30^\circ}$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is

$$\mathbf{I}_2 = 5\underline{/-110^\circ}$$

If we let $i = i_1 + i_2$, then

$$\begin{aligned}\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 &= 4\underline{/30^\circ} + 5\underline{/-110^\circ} \\ &= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698 \\ &= 3.218\underline{/-56.97^\circ}\text{ A}\end{aligned}$$

7.4 Phasor Relationships for Circuit Elements

- Each circuit element has a relationship between its current and voltage.
- The voltage-current relationship can be transformed from the time-domain to the frequency-domain for each element.

Phasor Relationships for Resistors

- For the resistor, the voltage and current are related via Ohm's law.

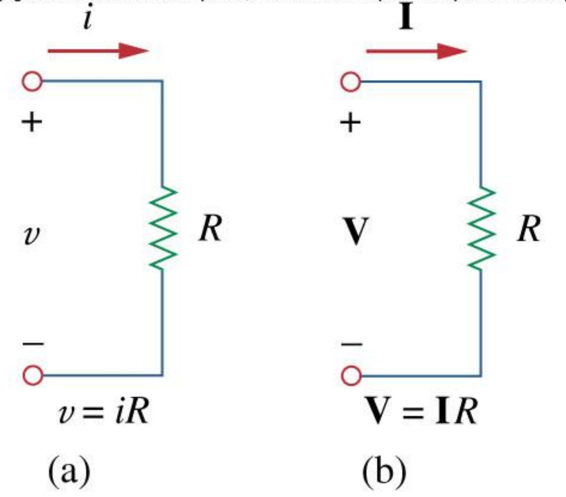
$$i = \mathrm{I_m}\cos(\omega t + \varphi)$$

$$u = Ri = R\,\mathrm{I_m}\cos(\omega t + \varphi)$$

$$\boldsymbol{I} = I_m \angle \varphi$$

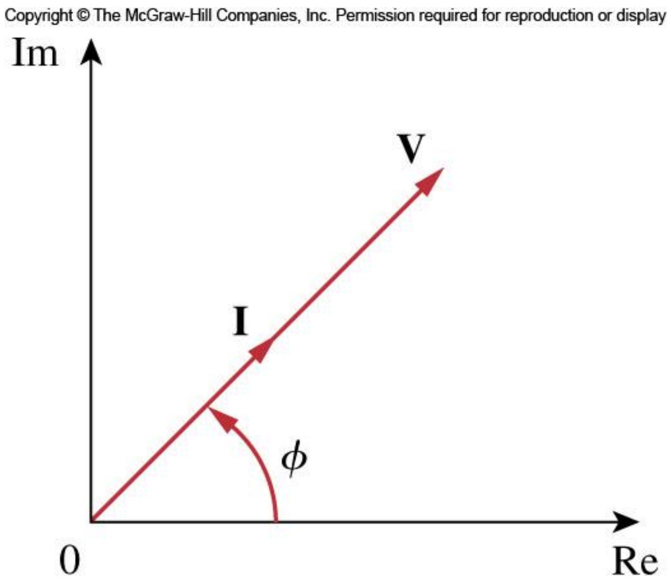
$$\boldsymbol{V} = RI_m \angle \varphi$$

$$\boldsymbol{V} = R\boldsymbol{I}$$



Phasor Relationships for Resistors

$$\boxed{\mathbf{V} = R\mathbf{I}}$$



- The voltage-current relation for the resistor in the phasor domain continues to be Ohm's law.
- The voltage and current are in phase.

Phasor Relationships for Inductors

- Assume the current through the inductor is:

$$i = \mathrm{I_m}\cos(\omega t + \varphi)$$

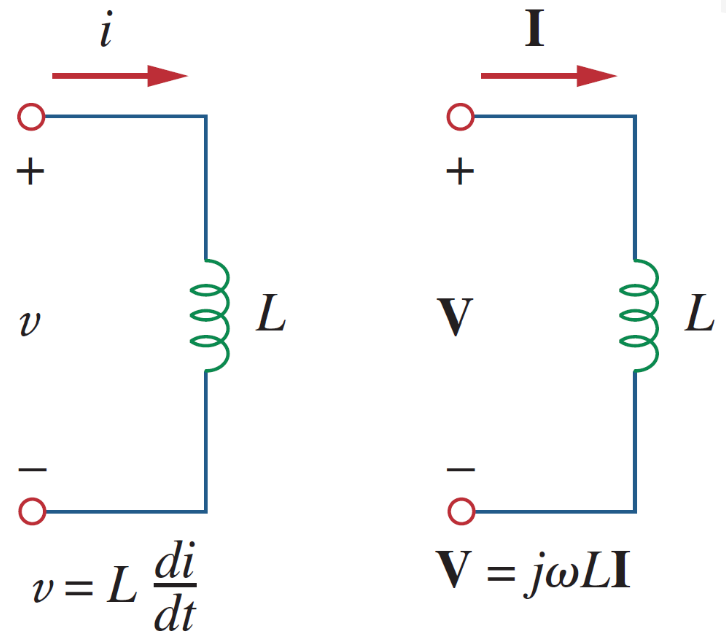
The voltage across the inductor is

$$v = L\frac{di}{dt} = \omega L\,\mathrm{I_m}\cos(\omega t + \varphi + \mathbf{90}^{o})$$

$$\boldsymbol{I} = I_m \angle \varphi$$

$$\boldsymbol{V} = \omega L I_m \angle(\varphi + 90^{o})$$

$$\boxed{\boldsymbol{V} = j\omega L \boldsymbol{I}}$$

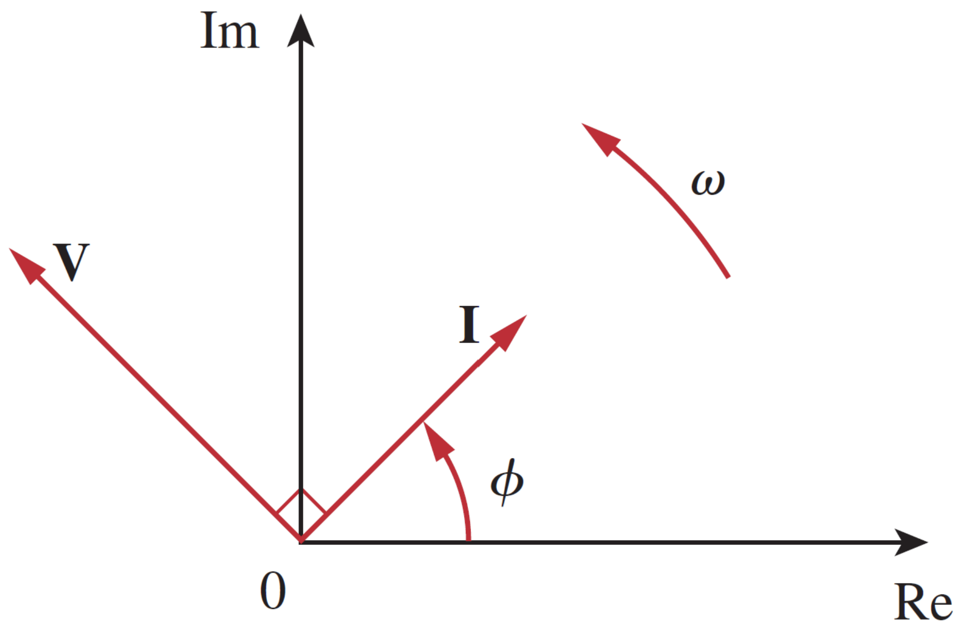


The voltage has the magnitude of $\omega L I_m$ and a phase of $\varphi + 90^{\circ}$.

Phasor Relationships for Inductors

$$\boxed{\mathbf{V} = j\omega L\mathbf{I} = jX_L\mathbf{I}}$$

Inductive reactance



- The voltage and current are **90°** out of phase.
- The voltage leads the current by **90°** or the current lags the voltage by **90°**.

Phasor Relationships for Capacitors

- Assume the voltage across the capacitor is:

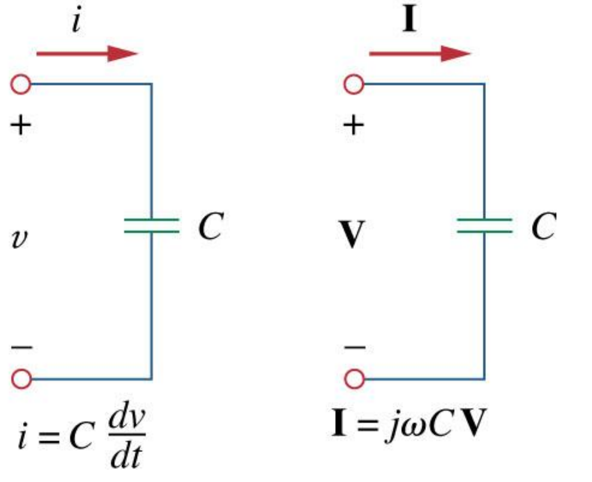
$$v = V_{\mathrm{m}} \cos(\omega t + \varphi)$$

$$i = C\frac{du}{dt} = \omega C V_{\mathrm{m}} \cos(\omega t + \varphi + \mathbf{90}^{o})$$

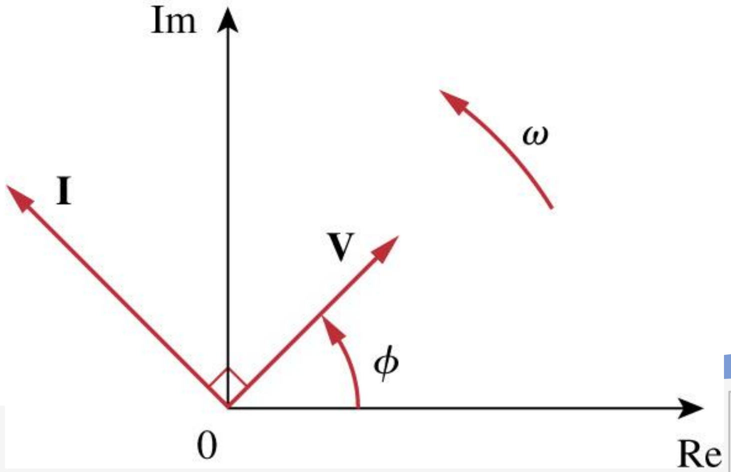
Capacitive reactance

$$\boxed{\boldsymbol{I} = j\omega C\boldsymbol{V}}$$

$$\boxed{\boldsymbol{V} = \frac{\boldsymbol{I}}{j\omega C} = -j\frac{1}{\omega C}\boldsymbol{I} = -jX_c\boldsymbol{I}}$$



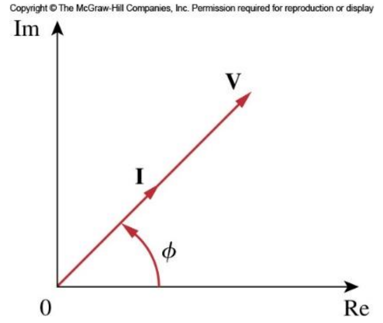
(a) (b)

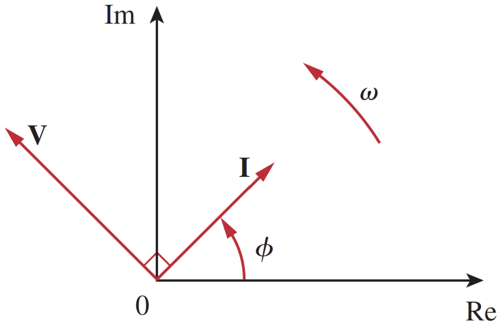


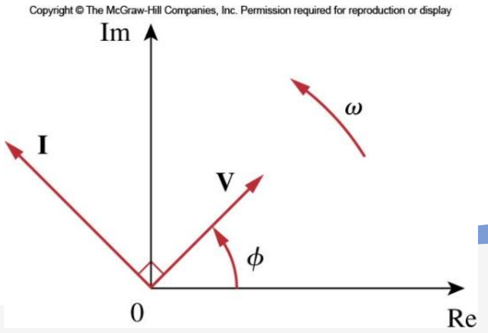
- The voltage and current are **90°** out of phase.
- The current leads the voltage by **90°**

Summary of Voltage-current relationships

Element	Time domain	Frequency domain
R	$v = Ri$	$\boldsymbol{V} = R\boldsymbol{I}$
L	$v = L\dfrac{di}{dt}$	$\boldsymbol{V} = j\omega L\boldsymbol{I}$
C	$i = C\dfrac{dv}{dt}$	$\boldsymbol{V} = \dfrac{\boldsymbol{I}}{j\omega C}$







Example

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\angle 45°$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\angle 45°}{j60 \times 0.1} = \frac{12\angle 45°}{6\angle 90°} = 2\angle -45° \text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°) \text{ A}$$

Example

If voltage $v = 10 \cos(100t + 30^\circ)$ is applied to a 50 μF capacitor, calculate the current through the capacitor.

Answer: $50 \cos(100t + 120^\circ)$ mA.

7.5 Impedance and Admittance

- It is possible to expand Ohm's law to capacitors and inductors.
- In time domain, this would be tricky as the ratio of voltage to current are always changing.
- But in frequency domain it is straightforward.
- The impedance $\boldsymbol{Z}$ of a circuit is the ratio of the phasor voltage $\boldsymbol{V}$ to the phasor current $\boldsymbol{I}$, measured in **ohms**.

$$Z = \frac{\boldsymbol{V}}{\boldsymbol{I}} \qquad \text{or} \qquad \boldsymbol{V} = Z\boldsymbol{I}$$

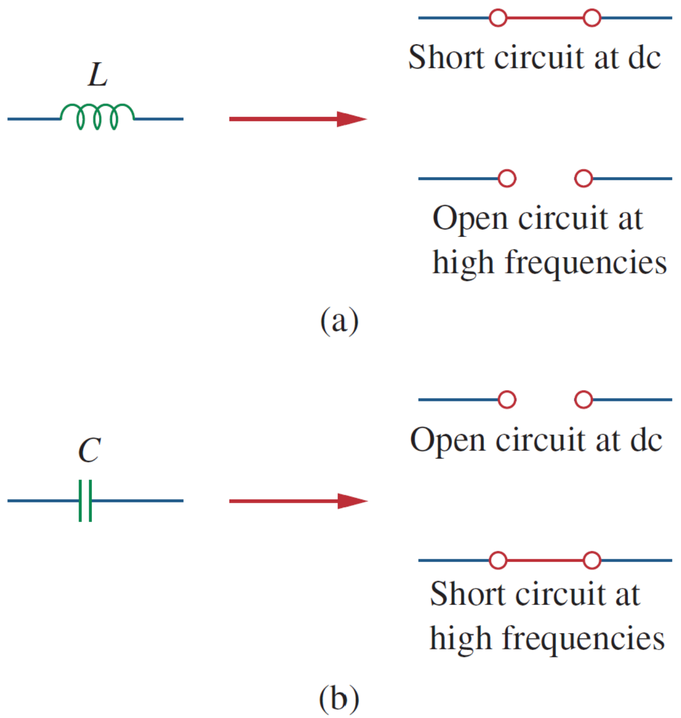
- Although the impedance is the ratio of two phasors, it is not a phasor, because it is not a sinusoidally varying quantity!!!

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Impedance

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$



- It is important to realize that in frequency domain, the values obtained for impedance are only valid at that frequency.
- Changing to a new frequency will require recalculating the values.

Two extreme cases:

- When $\omega=0$

$Z_L \to 0$ and $Z_C \to \infty$

The inductor acts like a short circuit, while the capacitor acts like an open circuit.

- When $\omega=\infty$

$Z_L \to \infty$ and $Z_C \to 0$

The inductor is an open circuit to high frequencies, while the capacitor is a short circuit.

Impedance

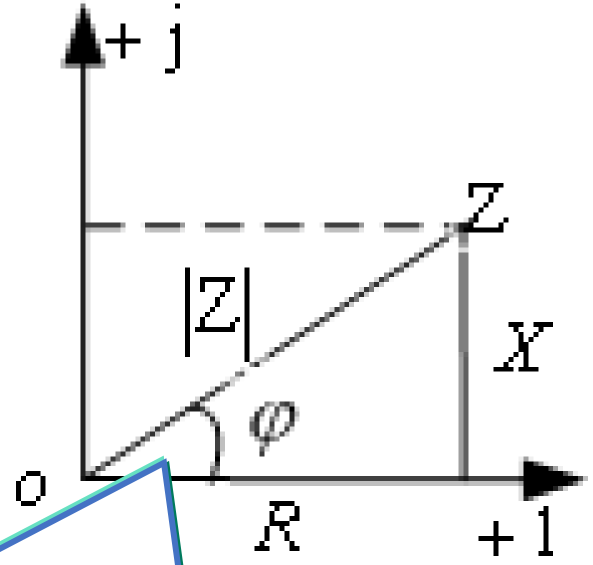
- As a complex quantity, the impedance may be expressed in rectangular form .

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = R + jX$$

- The real part R is the resistance.
- The imaginary part X is called the reactance.
- May also be expressed in polar from:

$$Z = R + \mathrm{j}X = |Z|e^{\mathrm{j}\varphi} = |Z|\angle\varphi$$

$$|Z| = \sqrt{R^2 + X^2} \quad , \varphi = \arctan\frac{X}{R}$$



φ is the phase difference between voltage and current

Admittance

- Admittance, being the reciprocal of the impedance, is also a complex number.
- The real part G is called the conductance.
- The imaginary part B is called the susceptance.
- These are all expressed in Siemens (S).

$$Y = \frac{\mathbf{1}}{Z} = G + jB$$

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

7.6 Kirchoff's Laws in the Frequency Domain

- Kirchoff's laws can be applied to phasors as well.
- For KVL, let v_1, v_2, …, v_n be voltages around a closed loop, then

$$v_1 + v_2 + \ldots + v_n = 0$$

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \ldots + V_{mn}\cos(\omega t + \theta_n) = 0$$

$$\mathrm{Re}[V_{m1}e^{j\theta_1}e^{j\omega t}] + \mathrm{Re}[V_{m2}e^{j\theta_2}e^{j\omega t}] + \ldots + \mathrm{Re}[V_{mn}e^{j\theta_n}e^{j\omega t}] = 0$$

$$\mathrm{Re}[\left(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \ldots + V_{mn}e^{j\theta_n}\right)e^{j\omega t}] = 0$$

$$\mathrm{Re}[\left(\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_n\right)e^{j\omega t}] = 0$$

$$\boxed{\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_n = 0}$$

Kirchhoff's voltage law holds for phasors.

$$\boxed{\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_n = \mathbf{0}}$$

Kirchhoff's voltage law holds for phasors.

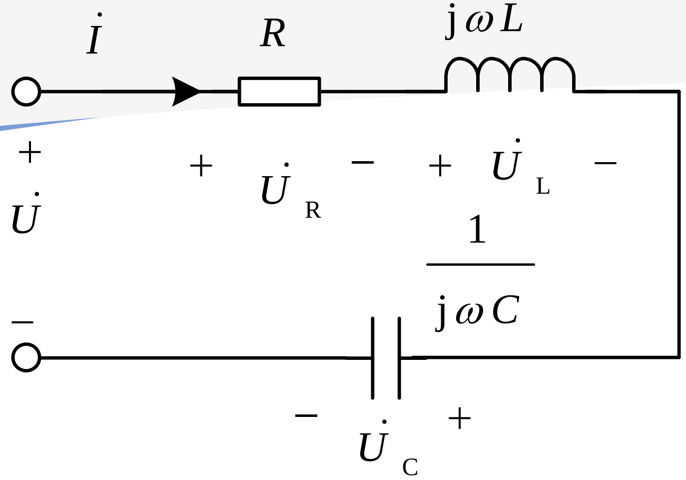
Similarly,

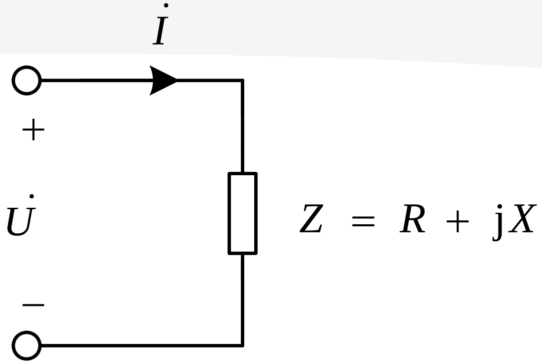
$$\boxed{\mathbf{I}_1 + \mathbf{I}_2 + \ldots + \mathbf{I}_n = \mathbf{0}}$$

Kirchhoff's current law holds for phasors.

- A circuit transformed to frequency domain can be evaluated using the methodology developed from KVL and KCL, such as impedance combination, nodal and mesh analysis, superposition, and source transformation.

Example

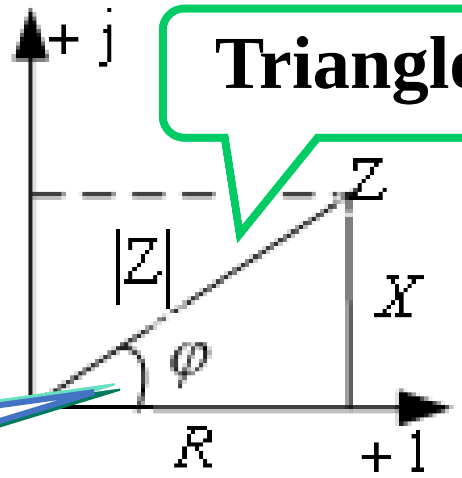




$$\dot{U} = \dot{U}_{\mathrm{R}} + \dot{U}_{\mathrm{L}} + \dot{U}_{C} = R\dot{I} + \mathrm{j}X_{\mathrm{L}}\dot{I} - \mathrm{j}X_{\mathrm{C}}\dot{I}$$

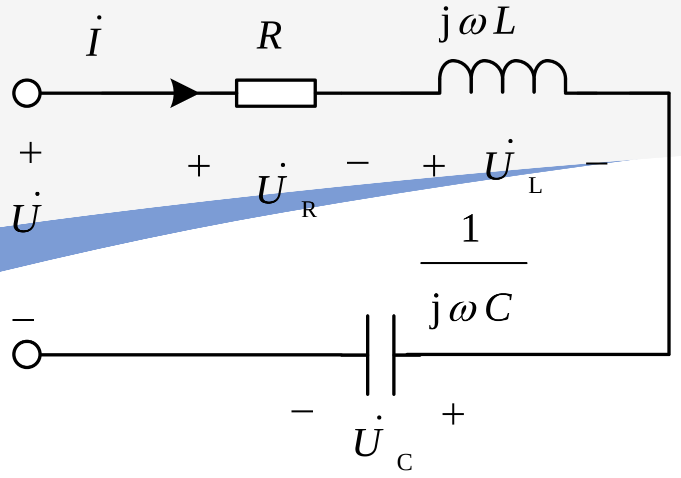
$$= \dot{I}[R + \mathrm{j}(X_{\mathrm{L}} - X_{\mathrm{C}})] = \dot{I}Z$$

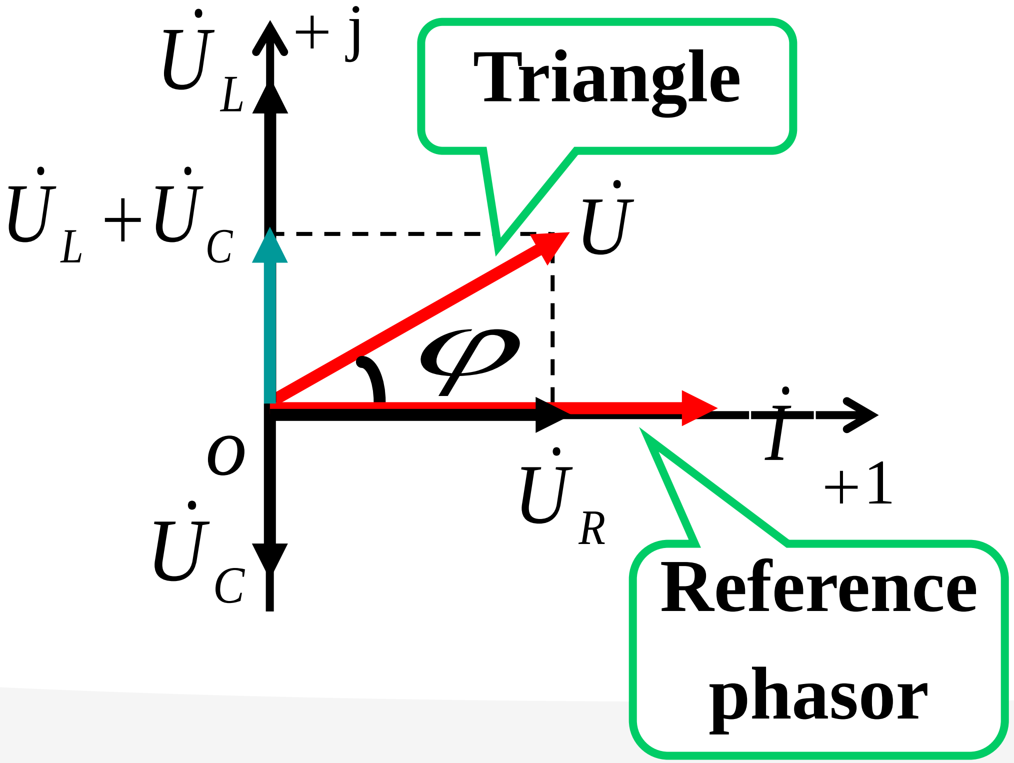
where $\boxed{Z = R + \mathrm{j}(X_{L} - X_{C}) = R + \mathrm{j}X}$



Triangle

φ is the phase difference between voltage and current



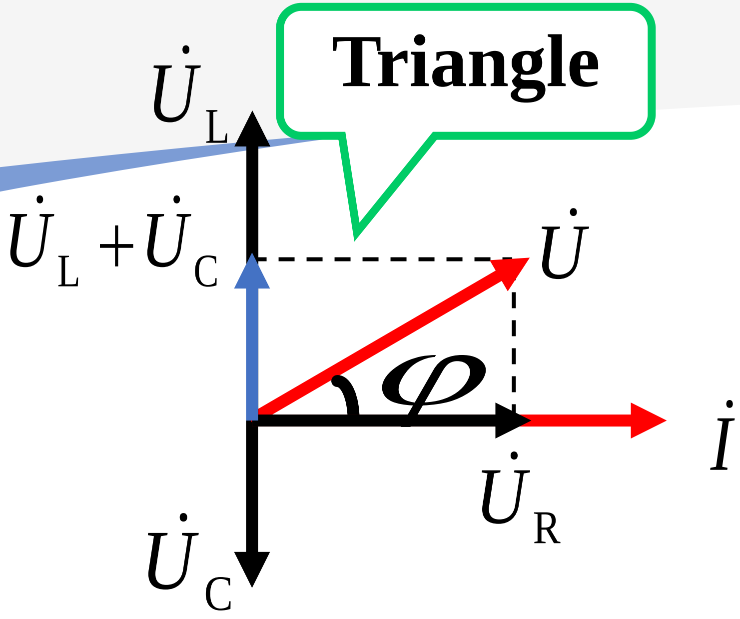


The phasor relationship can of the circuit can also be determined by ***the phasor diagram***

Provided $X_{\mathrm{L}} > X_{\mathrm{C}}$

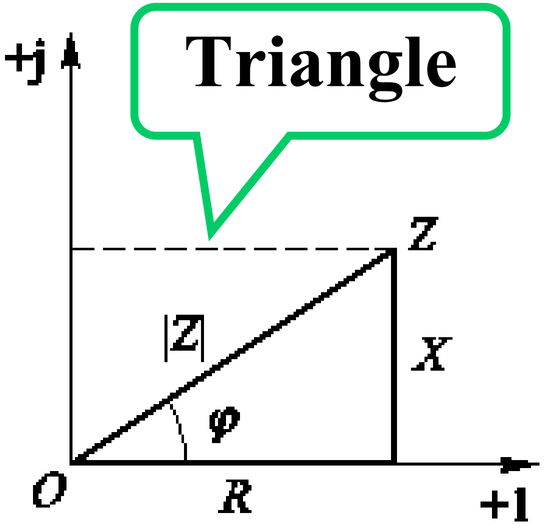
Use current as the reference phasor

$$\dot{U} = \dot{U}_{\mathrm{R}} + \dot{U}_{\mathrm{L}} + \dot{U}_{C}$$

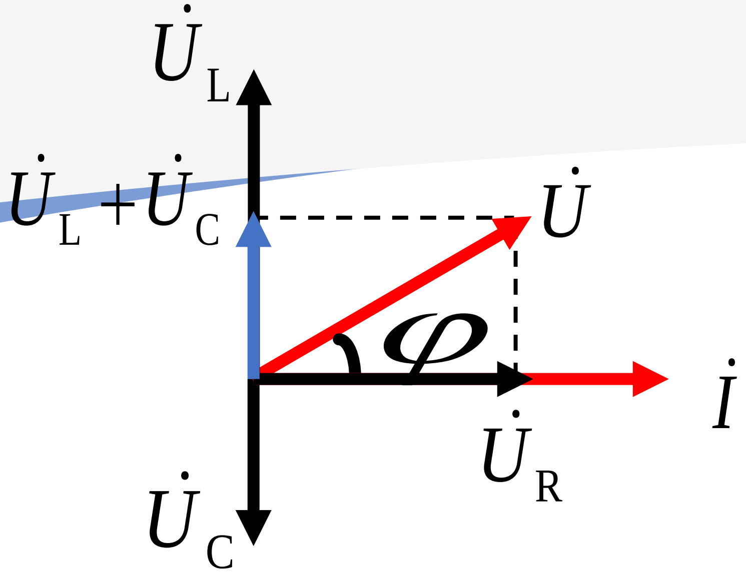


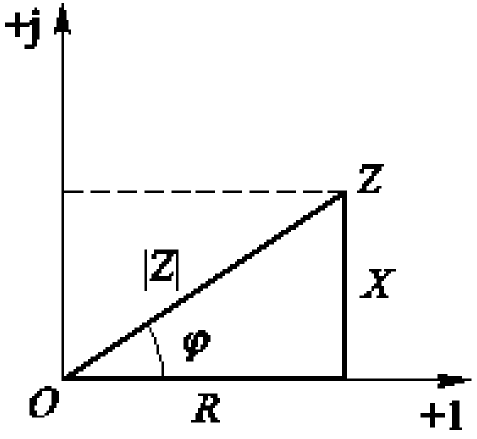
Find the phase of the impedance:

$$U=\sqrt{U_R^2+(U_L-U_C)^2}$$
$$=\sqrt{(RI)^2+(X_LI-X_CI)^2}$$
$$=I\sqrt{R^2+(X_L-X_C)^2}$$
$$=I|Z|$$



$$\varphi=\arctan\frac{U_L-U_C}{U_R}=\arctan\frac{X_L-X_C}{R}$$





When the frequency of the power supply is constant, the impedance angle determines the property of the circuit.

$$\boxed{\varphi = \arctan \frac{X_L - X_C}{R}}$$

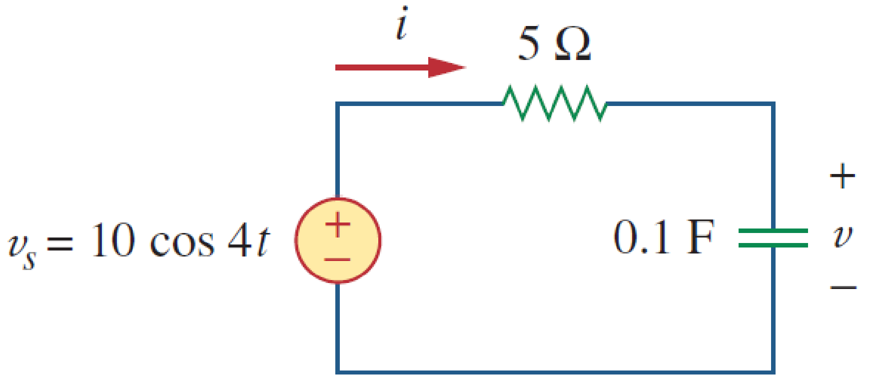
If $X_L > X_C$, $\varphi > 0$, the current lags voltage, the circuit is inductive

If $X_L < X_C$, $\varphi < 0$, the current leads voltage, the circuit is capacitive

If $X_L = X_C$, $\varphi = 0$, the current and the voltage are in phase, the circuit is resistive

Example

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.



Solution:

From the voltage source $10 \cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \tag{9.9.1}$$

$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1}$$

$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V} \tag{9.9.2}$$

Converting **I** and **V** in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$

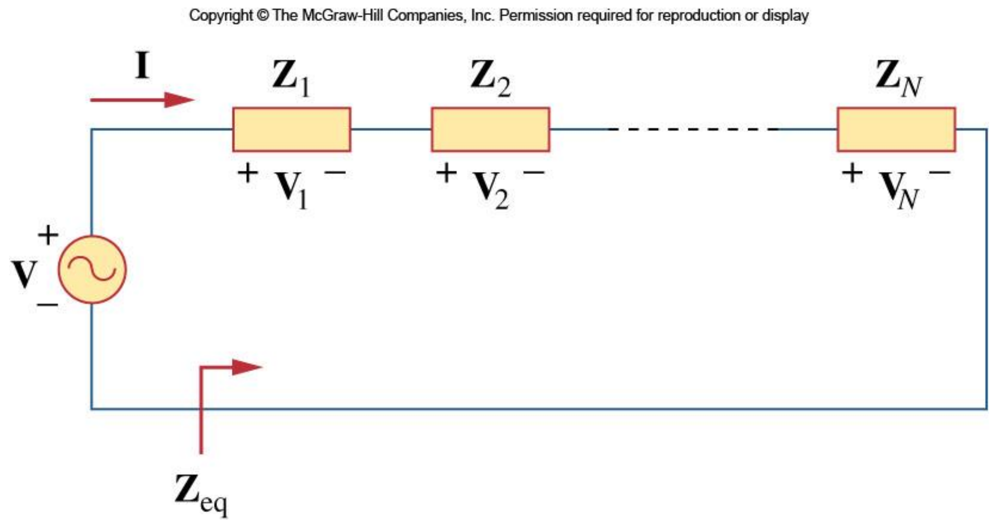
$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

7.7 Impedance Combinations

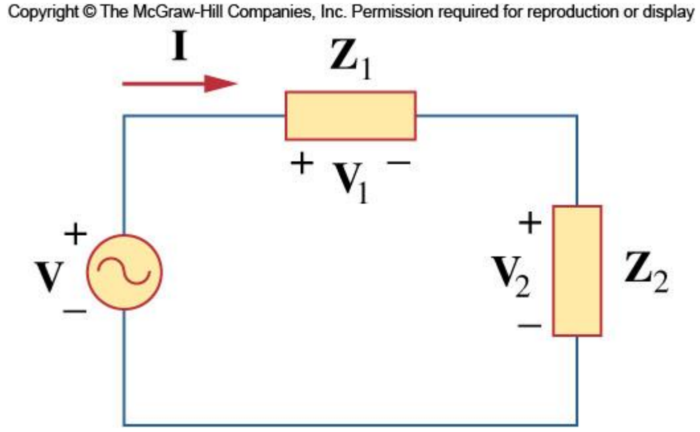
- The equivalent impedance of series-connected impedances is the sum of the individual impedances.

$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots + Z_N$$



- If N=2, it acts like a voltage divider.

$$V_1 = \frac{Z_1}{Z_1+Z_2}V \qquad V_2 = \frac{Z_2}{Z_1+Z_2}V$$



Impedance Combinations

- The equivalent impedance of N parallel-connected impedances:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots + \frac{1}{Z_N}$$

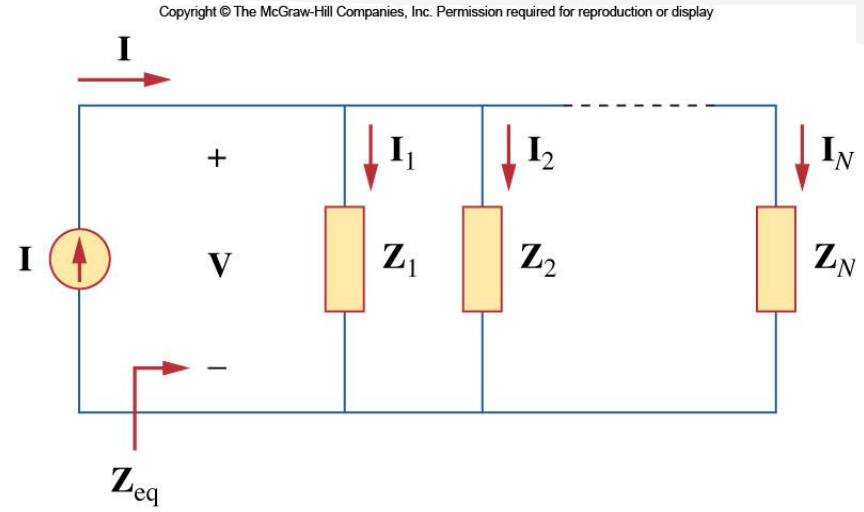
- Expressed by admittance:

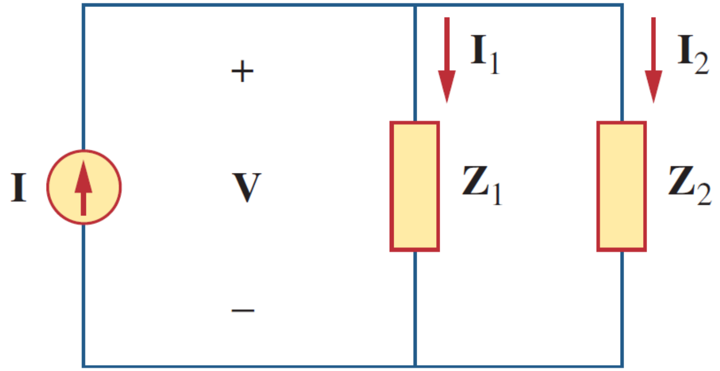
$$Y_{eq} = Y_1 + Y_2 + Y_3 + \cdots + Y_N$$

- The equivalent admittance of parallel-connected admittances is the sum of the individual admittances.

- If N=2, it acts like a current divider:

$$I_1 = \frac{Z_2}{Z_1+Z_2} I \qquad I_2 = \frac{Z_1}{Z_1+Z_2} I$$





Impedance Combinations

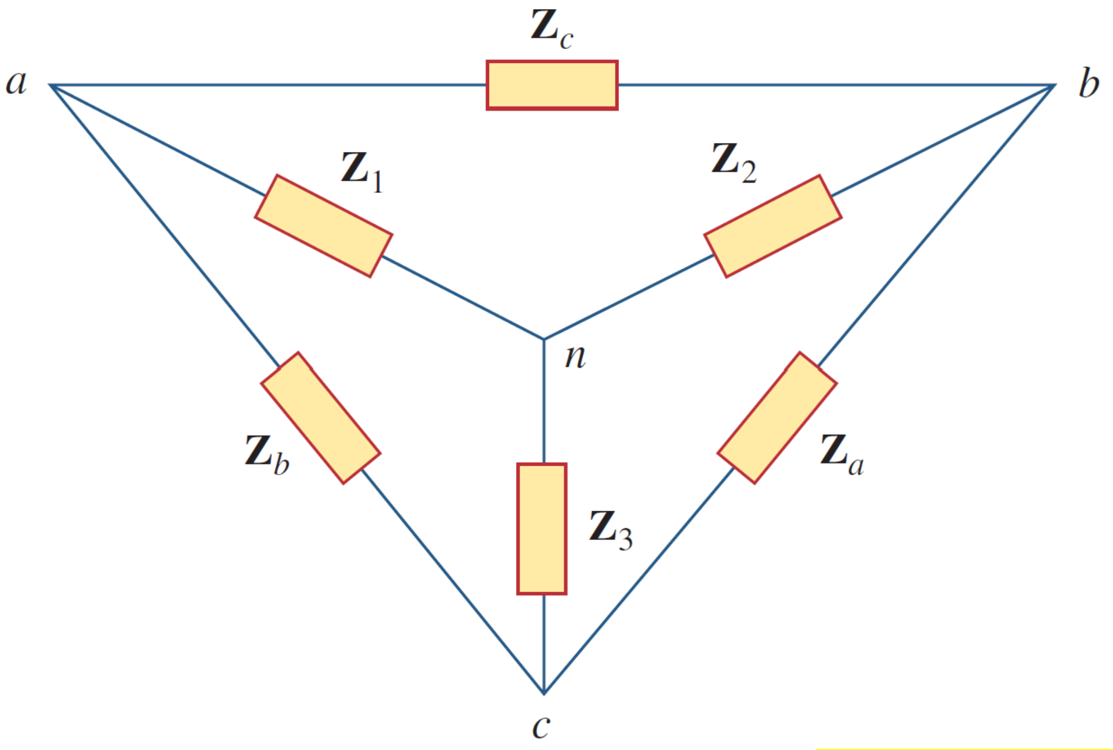
- The Delta-Wye and Wye-Delta transformations are also valid for impedances.

Delta-Wye

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$



Wye-Delta

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

Conversion rule (from delta to wye): Each impedance in the Y network is the product of the impedances in the two adjacent Δ branches, divided by the sum of the three Δ impedances.

Conversion rule (from wye to delta): Each impedance in the Δ network is the sum of all possible products of Y impedances taken two at a time, divided by the opposite Y impedance.

Example 9.10

Example

Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at $\omega = 50$ rad/s.

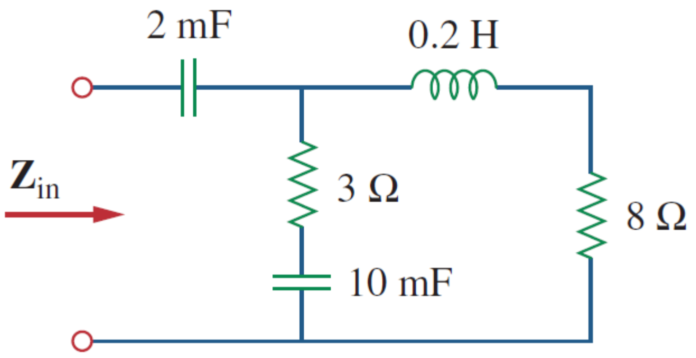


Figure 9.23
For Example 9.10.

Solution:

Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{\text{in}} = \mathbf{Z}_1 + \mathbf{Z}_2 \| \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{\text{in}} = 3.22 - j11.07\ \Omega$$

Example

Practice Problem 9.10

Determine the input impedance of the circuit in Fig. 9.24 at $\omega = 10$ rad/s.

Answer: $(149.52 - j195)$

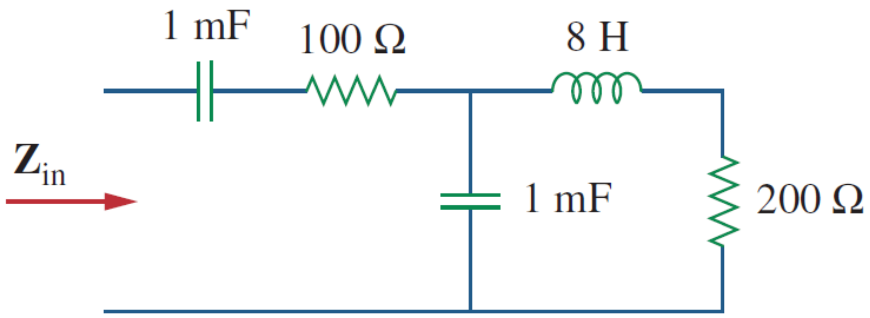


Figure 9.24
For Practice Prob. 9.10.

The steps to analyze a sinusoidal AC circuit using the phasor method:

（1）Draw the phasor model of the circuit；

（2）Write circuit equations in phasor form using appropriate methods；

（3）Calculate the unknown phasor；

（4）Write the expressions for instantaneous voltages or currents

Example

Determine $v_o(t)$ in the circuit

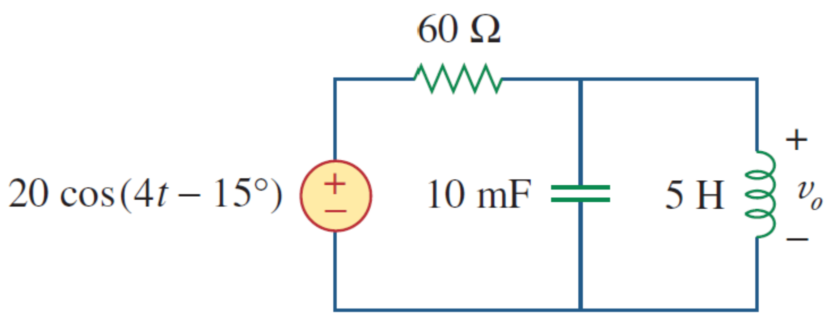


Figure 9.25
For Example 9.11.

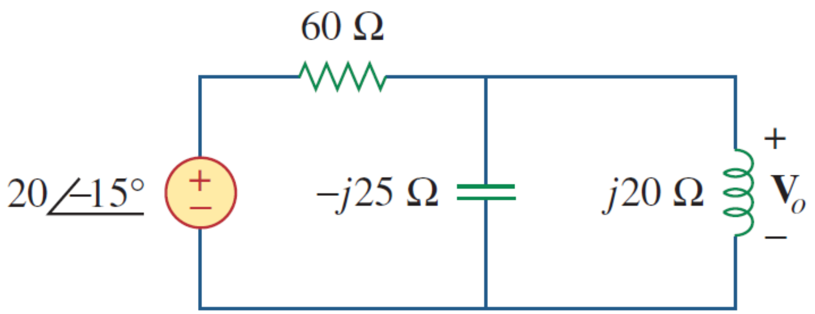


Figure 9.26
The frequency domain equivalent of the circuit in Fig. 9.25.

Solution:

To do the analysis in the frequency domain, we must first transform the time domain circuit in Fig. 9.25 to the phasor domain equivalent in Fig. 9.26. The transformation produces

$$v_s = 20\cos(4t - 15°) \quad \Rightarrow \quad \mathbf{V}_s = 20\underline{/-15°}\text{ V}, \qquad \omega = 4$$

$$10\text{ mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5\text{ H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

Let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\underline{/-15°})$$

$$= (0.8575\underline{/30.96°})(20\underline{/-15°}) = 17.15\underline{/15.96°}\text{ V}$$

We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96°)\text{ V}$$

Example

Practice Problem 9.11 Calculate v_o in the circuit of Fig. 9.27.

Answer: $v_o(t) = 35.36\cos(10t - 105°)$ V.

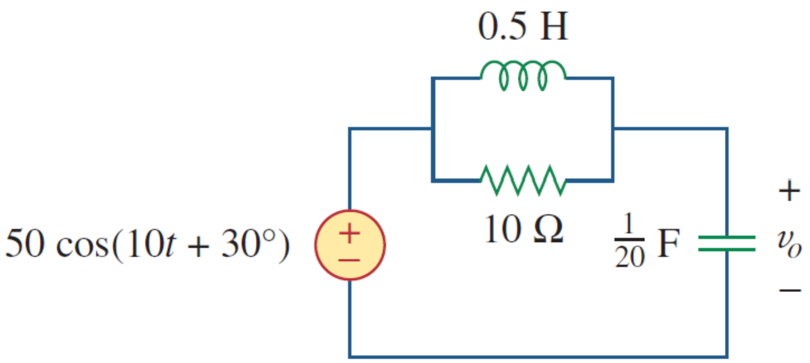


Figure 9.27
For Practice Prob. 9.11.

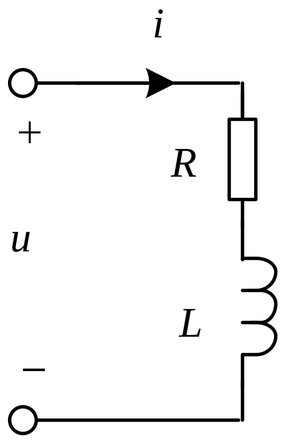
Example

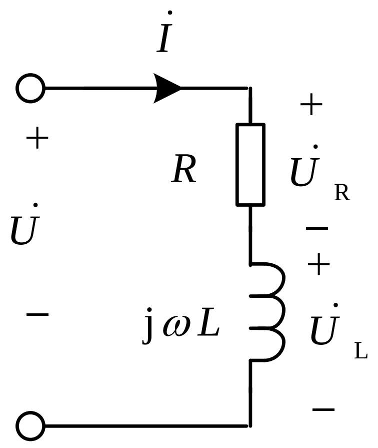
For the inductive load, $R = 1333.3\Omega$, $L = 1.95\mathrm{H}$ 。

The applied source voltage is $u = 220\cos\omega t$, $f = 50\mathrm{Hz}$

(1) Find the inductive reactance X_L and input impedance Z_L; (2) Find the current $\dot{I}$、I、i ;

(3) Find the voltage $\dot{U}_R$ U_R $\dot{U}_L$ U_L (4) Draw the phasor diagram.





【Solution】

（1） $X_L = \omega L = 2\pi f L$

$$= 2 \times 3.14 \times 50 \times 1.95$$

$$= 314 \times 1.95 = 612.3\Omega$$

For the inductive load, $R = 1333.3\Omega$, $L = 1.95\text{H}$。

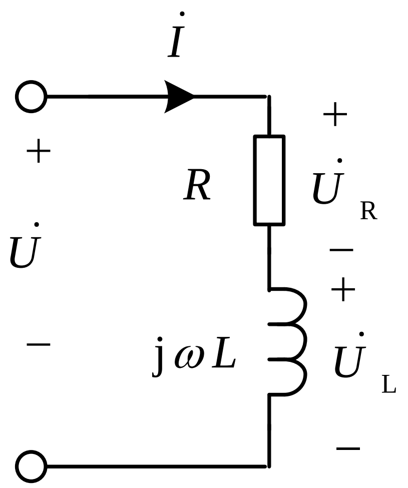
The applied source voltage is $u = 220\cos\omega t$ $\quad f = 50\text{Hz}$

(1) Find the inductive reactance X_L and impedance Z_L; (2) Find the current $\dot{I}$、I、i;

(3) Find the voltage $\dot{U}_R$ U_R $\dot{U}_L$ U_L (4) Draw the phasor diagram.

$$\boxed{X_{\text{L}} = 612.3\Omega}$$

$$Z_L = R + jX_L = 1333.3 + j612.3$$
$$= 1467.2\angle 24.7°\Omega$$

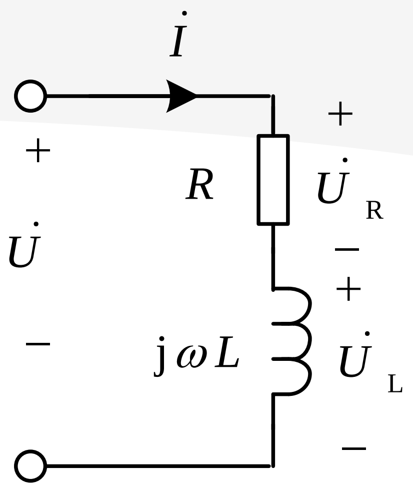


（2） $\dot{I} = \frac{\dot{U}}{Z} = \frac{220\angle 0°}{1467.2\angle 24.7°} = 0.15\angle -24.7° A$

$$I = 0.15\text{A}$$

$$i = 0.15\cos\left(\omega t - 24.7°\right) A$$

$$\boxed{\dot{I} = 0.15\angle -24.7^\circ \text{ A}}$$



(3) Find the voltage $\dot{U}_R$ U_R $\dot{U}_L$ U_L

$$\dot{U}_R = R\dot{I} = 1333.3 \times 0.15\angle -24.7^\circ = 200\angle -24.7^\circ \text{ V}$$

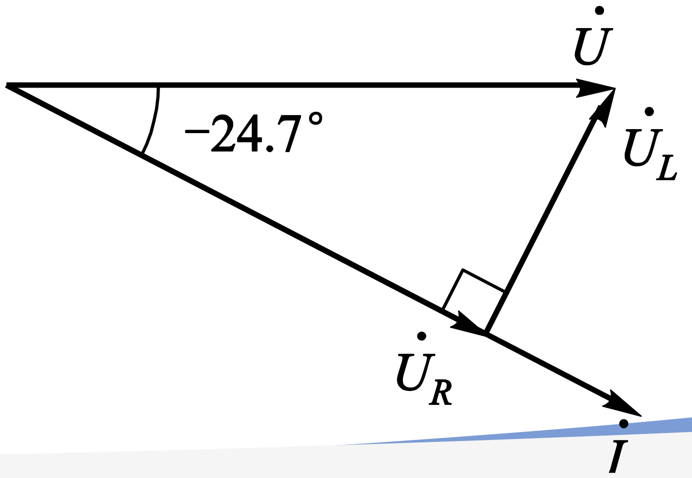
$$U_R = 200\text{V}$$

$$\dot{U}_L = jX_L\dot{I} = j \times 612.3 \times 0.15\angle -24.7^\circ$$

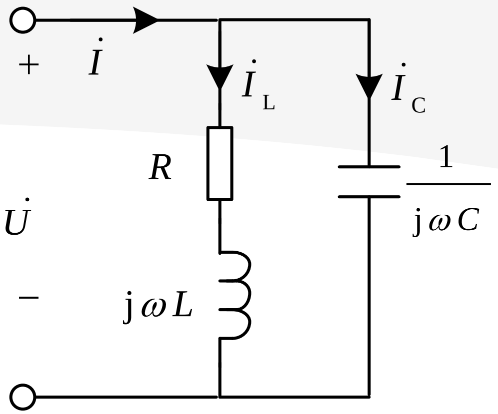
$$= 91.8\angle 65.3^\circ \text{ V}$$

$$U_L = 91.8\text{V}$$

(4) Draw the phasor diagram



(5) An additional capacitor is paralleled $C = 0.26\mu\text{F}$。



Find the capacitive reactance X_C, and the new

$\dot{I}$、$\dot{I}_L$; and draw the phasor diagram. $Z_L = R + jX_L = 1333.3 + j612.3$
$= 1467.2\angle 24.7°\Omega$

$$X_C = \frac{1}{\omega C}$$

$$= \frac{1}{314 \times 0.26 \times 10^{-6}}$$

$$= 12248.9\Omega$$

$$\approx 12.3\text{k}\Omega$$

$$Z = Z_L // \frac{1}{j\omega C} = \frac{1.47\angle 24.7^\circ \times (-j12.3)}{1.33 + j0.612 - j12.3} = \frac{18.1\angle -65.3^\circ}{1.33 - j11.7}$$

$$= \frac{18.1\angle -65.3^\circ}{11.8\angle -83.6} = 1.53\angle 18.3^\circ \text{k}\Omega$$

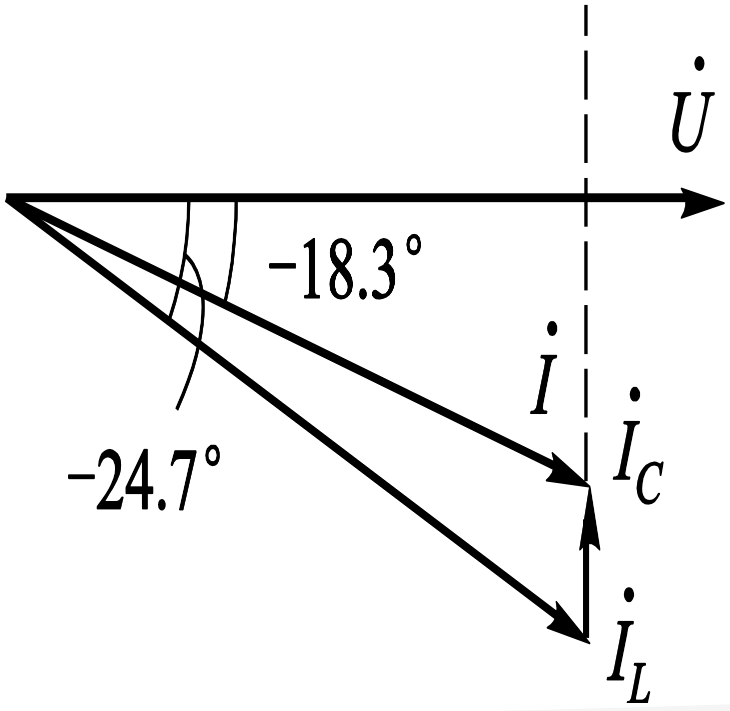
$$\boxed{Z = 1.53\angle 18.3^\circ\ \text{k}\Omega}\qquad \boxed{Z_\text{L} \approx 1.47\angle 24.7^\circ\ \text{k}\Omega}$$

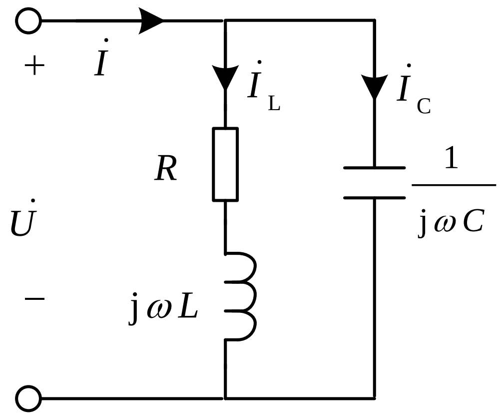
$$\dot{I} = \frac{\dot{U}}{Z} = \frac{220\angle 0^\circ}{1.53\times 10^3\angle 18.3^\circ} = 0.144\angle -18.3^\circ\ \text{A} = 144\angle -18.3^\circ\ \text{mA}$$

$$\dot{I}_\text{L} = \frac{\dot{U}}{Z_\text{L}} = \frac{220\angle 0^\circ}{1.47\times 10^3\angle 24.7^\circ}$$

$$= 0.15\angle -24.7^\circ\ \text{A}$$

$$= 150\angle -24.7^\circ\ \text{mA}$$





Example

Find current **I** in the circuit of Fig. 9.28.

Example 9.12

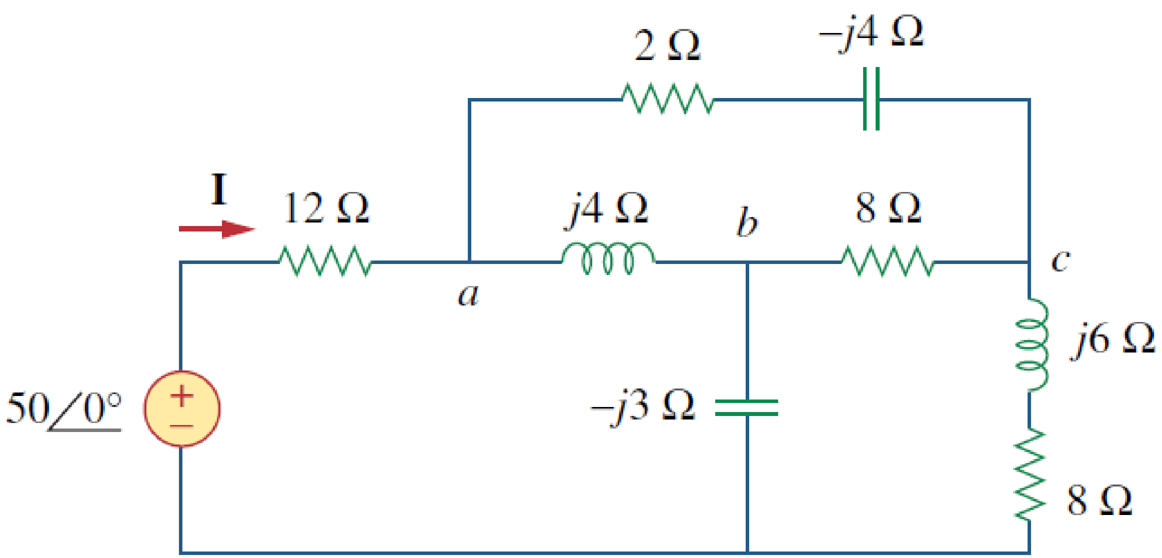
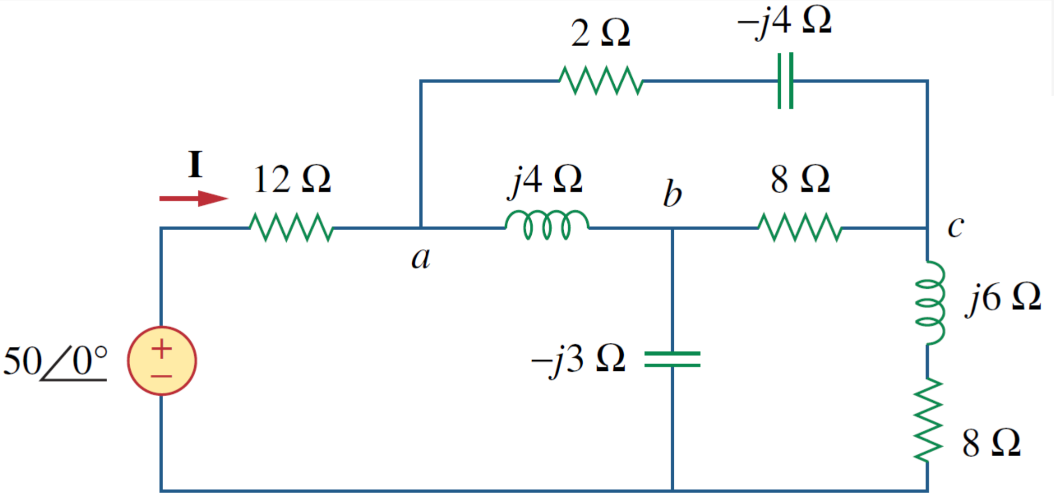
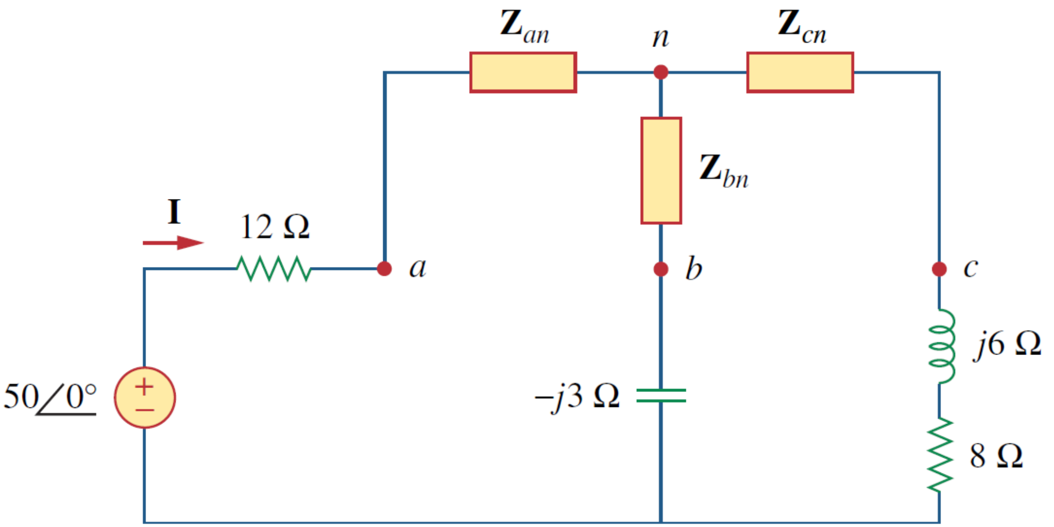


Figure 9.28
For Example 9.12.

Example





Solution:

The delta network connected to nodes a, b, and c can be converted to the Y network of Fig. 9.29. We obtain the Y impedances as follows using Eq. (9.68):

$$\mathbf{Z}_{an} = \frac{j4(2 - j4)}{j4 + 2 - j4 + 8} = \frac{4(4 + j2)}{10} = (1.6 + j0.8)\ \Omega$$

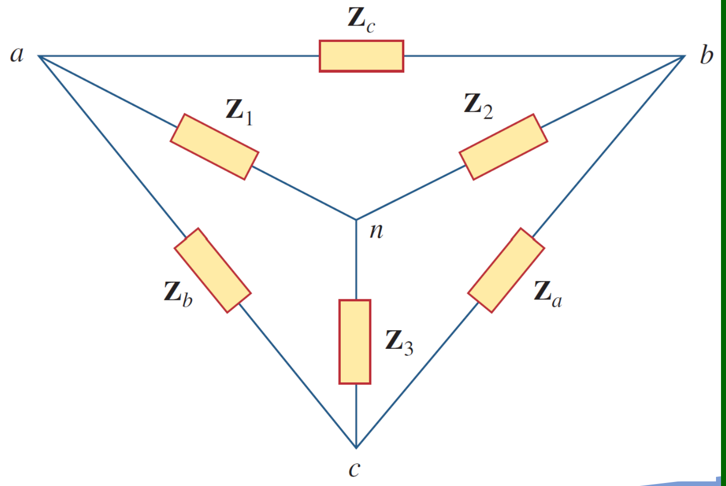
$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2 - j4)}{10} = (1.6 - j3.2)\ \Omega$$

Delta-Wye

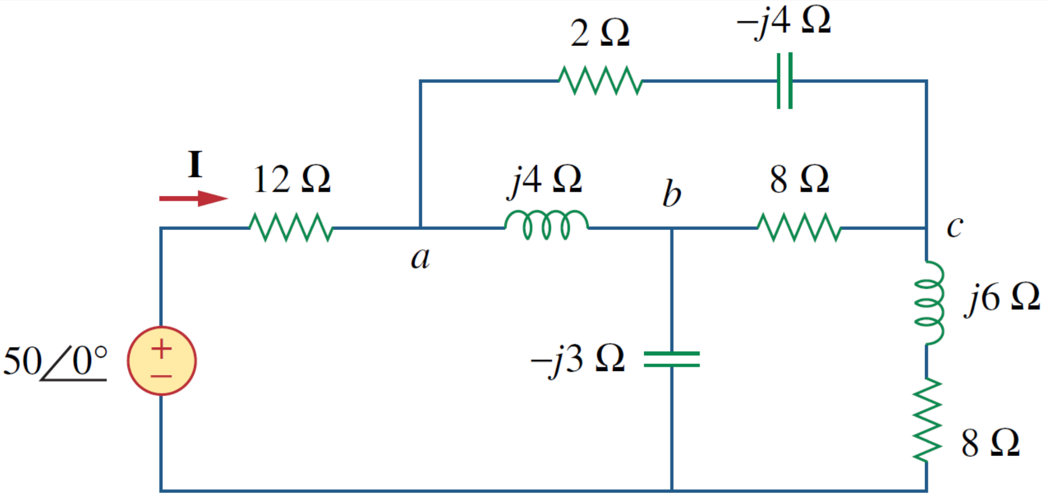
$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

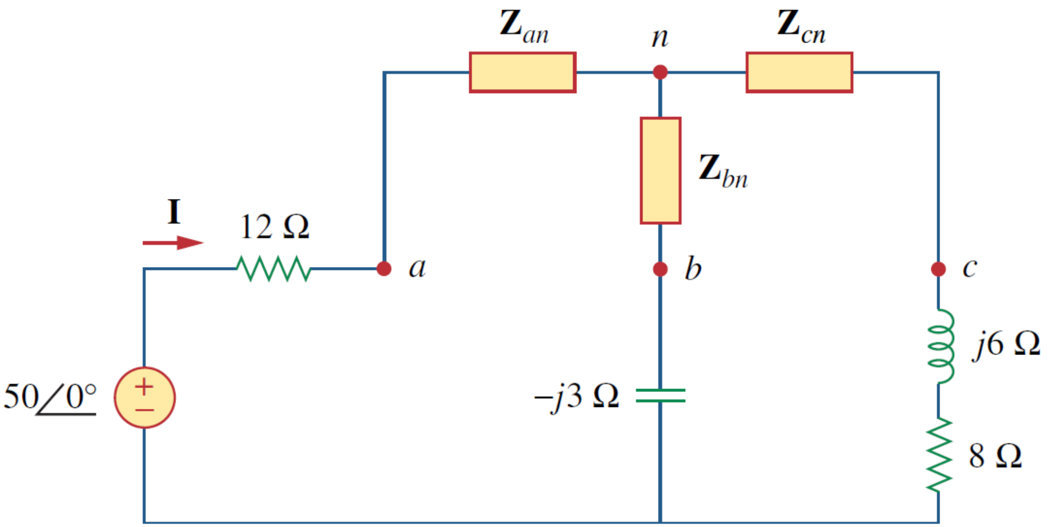
$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$



Example





The total impedance at the source terminals is

$$\begin{aligned}\mathbf{Z} &= 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \parallel (\mathbf{Z}_{cn} + j6 + 8) \\ &= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8) \\ &= 13.6 + j0.8 + \frac{j0.2(9.6 + j2.8)}{9.6 + j3} \\ &= 13.6 + j1 = 13.64\underline{/4.204^\circ}\ \Omega\end{aligned}$$

The desired current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{50\underline{/0^\circ}}{13.64\underline{/4.204^\circ}} = 3.666\underline{/-4.204^\circ}\ \text{A}$$

Example

Find current **I** in the circuit of Fig. 9.28.

Example 9.12

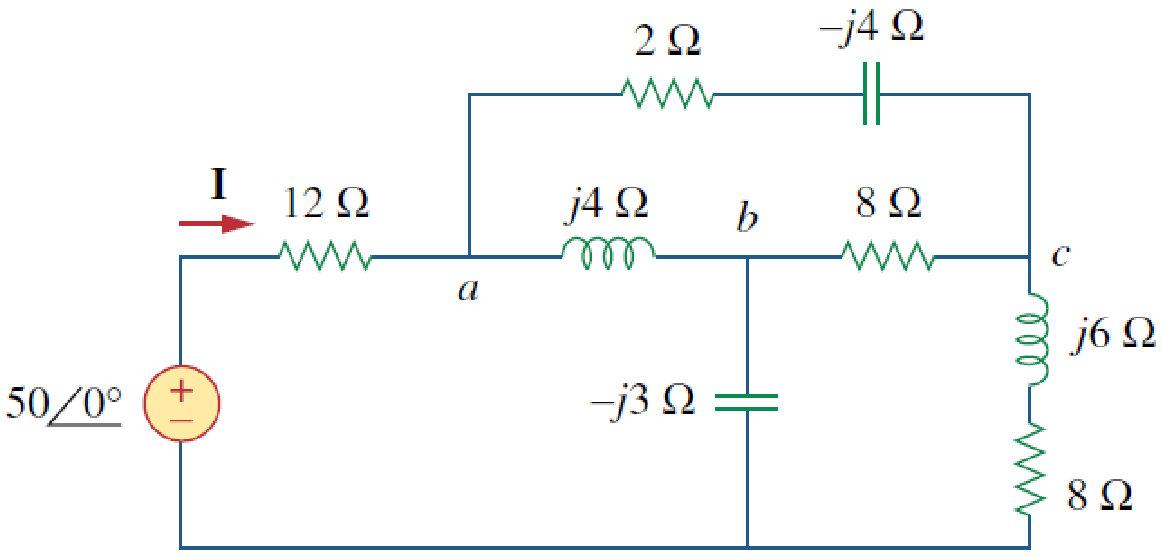


Figure 9.28
For Example 9.12.

Other solutions?

Example

Find current **I** in the circuit of Fig. 9.28.

Example 9.12

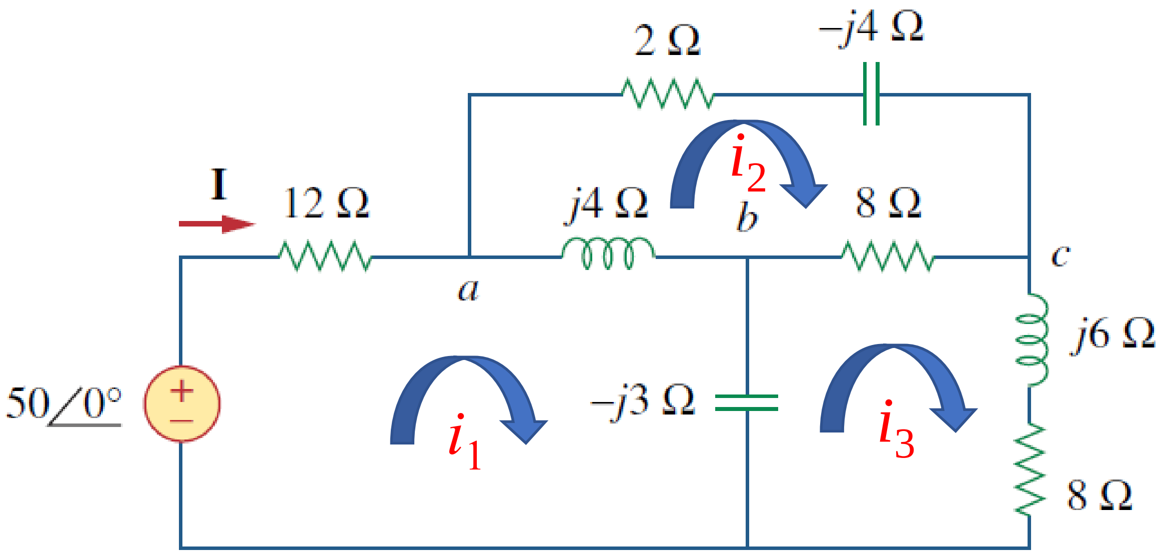


Figure 9.28
For Example 9.12.

Mesh analysis