CALCULUS

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Spring 2025

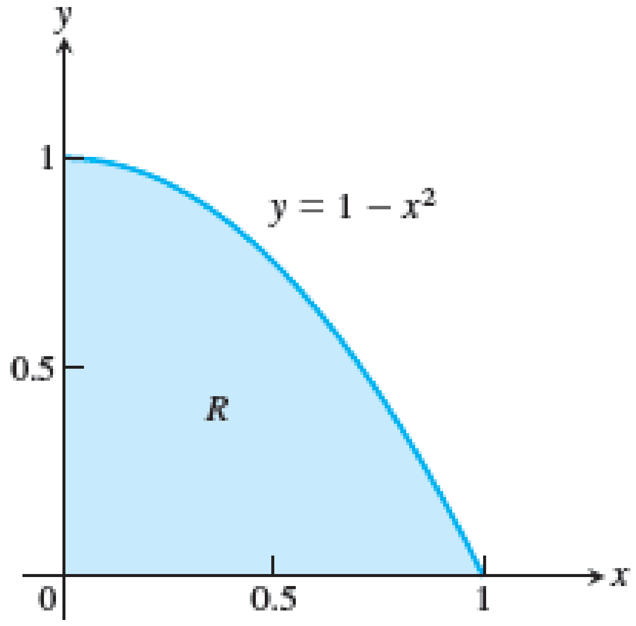
- In this chapter we develop a method, called ***integration***, to calculate the areas and volumes of more general shapes. The ***definite integral*** is the key tool in calculus for defining and calculating areas and volumes, as well as other physical quantities.
- Like the derivative, the definite integral is defined as a limit. The definite integral is a limit of increasingly fine approximations. The idea is to approximate a quantity (such as the area of a curvy region) by dividing it into many small pieces, each of which we can approximate by something simple (such as a rectangle). We take a limit as the number of terms increases to infinity, and when the limit exists, the result is a definite integral.
- The process of computing the definite integrals is closely connected to finding the antiderivatives.

- The basis for formulating definite integrals is the construction of approximations by finite sums. In this section we consider three examples of this process: finding the area under a graph, the distance traveled by a moving object, and the average value of a function.
- Although we have yet to define precisely what we mean by the area of a general region in the plane, or the average value of a function over a closed interval, we do have intuitive ideas of what these notions mean.
- We begin our approach to integration by *approximating* these quantities with simpler finite sums related to these intuitive ideas, then consider what happens when we take more and more terms in the summation process. In subsequent sections we look at taking the limit of these sums as the number of terms goes to infinity, which leads to a precise definition of the definite integral.

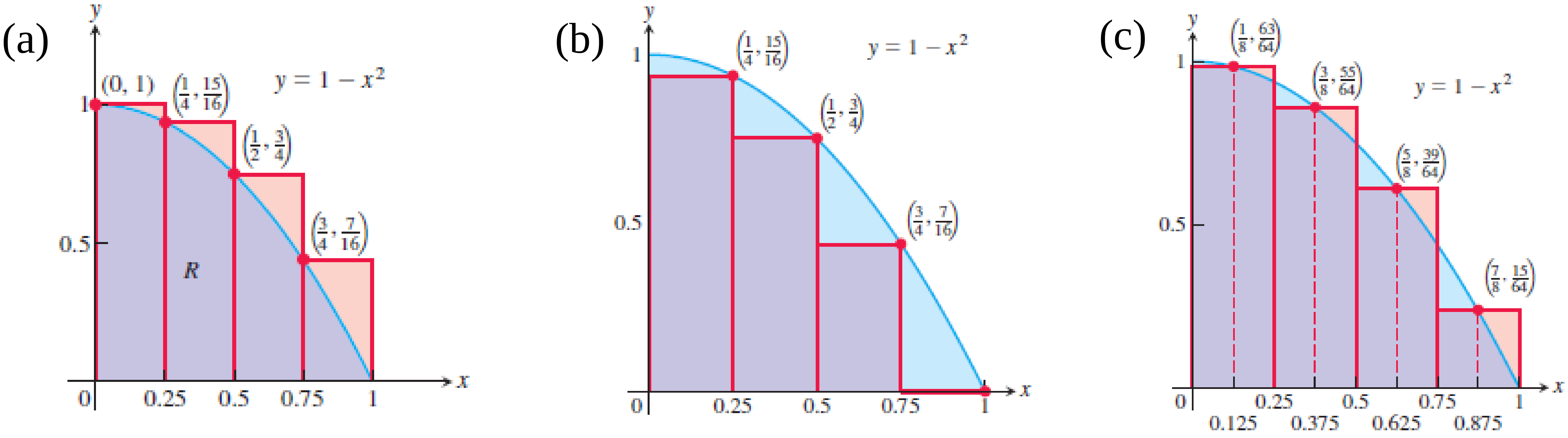
① Area

- Suppose we want to find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$.

- Unfortunately, there is no simple geometric formula for calculating the area of this complex region R. How can we find the area of R?
- While we do not yet have a method for determining the exact area of R, we can approximate it in a simple way.



$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \ldots + f(c_n)\,\Delta x$$



(a) Upper sum $\quad \text{Area} \approx \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{15}{16} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{7}{16} = \frac{25}{32} = 0.78125$

(b) Lower sum $\quad \text{Area} \approx \frac{1}{4} \cdot \frac{15}{16} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{7}{16} + \frac{1}{4} \cdot 0 = \frac{17}{32} = 0.53125$

(c) Midpoint Sum $\quad \text{Area} \approx \frac{1}{4} \cdot \frac{63}{64} + \frac{1}{4} \cdot \frac{55}{64} + \frac{1}{4} \cdot \frac{36}{64} + \frac{1}{4} \cdot \frac{15}{64} = \frac{43}{64} = 0.671875$

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

Exact value of the area: $\boldsymbol{Area = \frac{2}{3}}$

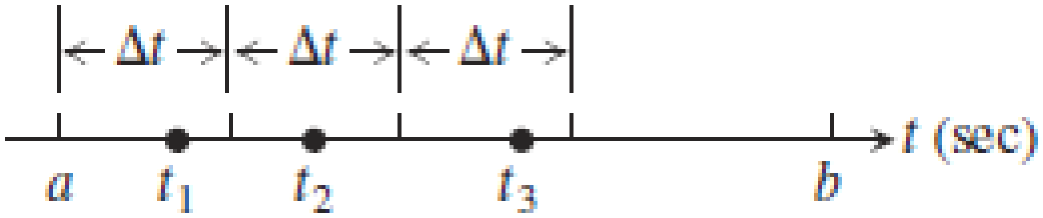
② Distance Traveled

- Suppose a car having the velocity function $\upsilon(t)$ moves straight down a highway without changing direction. how far it traveled between times $t = a$ and $t = b$?
- When an antiderivative for the velocity $\upsilon(t)$ is unknown, we can approximate the distance traveled by using finite sums in a way similar to the area estimates as before.
- We subdivide the interval $[a, b]$ into short time intervals and assume that the velocity on each subinterval is fairly constant. Then we approximate the distance traveled on each time subinterval with the usual distance formula:

$$\text{distance} = \text{velocity} \times \text{time}$$

and add the results across $[a, b]$.

- Suppose the subdivided interval looks like



with the subintervals all of equal length Δt.

- If Δt is so small that the velocity barely changes over a short time interval of duration Δt, then the sum of the distances traveled over all the time intervals is

$$D \approx \upsilon(t_1)\,\Delta t + \upsilon(t_2)\,\Delta t + \ldots + \upsilon(t_n)\,\Delta t$$

where n is the total number of subintervals.

- This sum is only an approximation to the true distance D, but the approximation increases in accuracy as we take more and more subintervals.

Example 1

The velocity function of a projectile fired straight into the air is

$$\upsilon(t) = 160 - 9.8t \text{ m/sec.}$$

Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m?

Number of subintervals	Length of each subinterval	Upper sum	Lower sum
3	1	450.6	421.2
6	1/2	443.25	428.55
12	1/4	439.58	432.23
24	1/8	437.74	434.06
48	1/16	436.82	434.98
96	1/32	436.36	435.44
192	1/64	436.13	435.67

- Midpoint Sum reaches at the exact value at the first step (n=3).

③ Average Value of a Nonnegative Continuous Function

- The average value of a collection of n numbers $x_1, x_2, \ldots, x_n$ is obtained by adding them together and dividing by n. But what is the average value of a continuous function f on an interval $[a, b]$?

- Fig. 5.6 illustrates the way to calculate the average value of a nonnegative continuous function when it is (a) constant and (b) not constant.

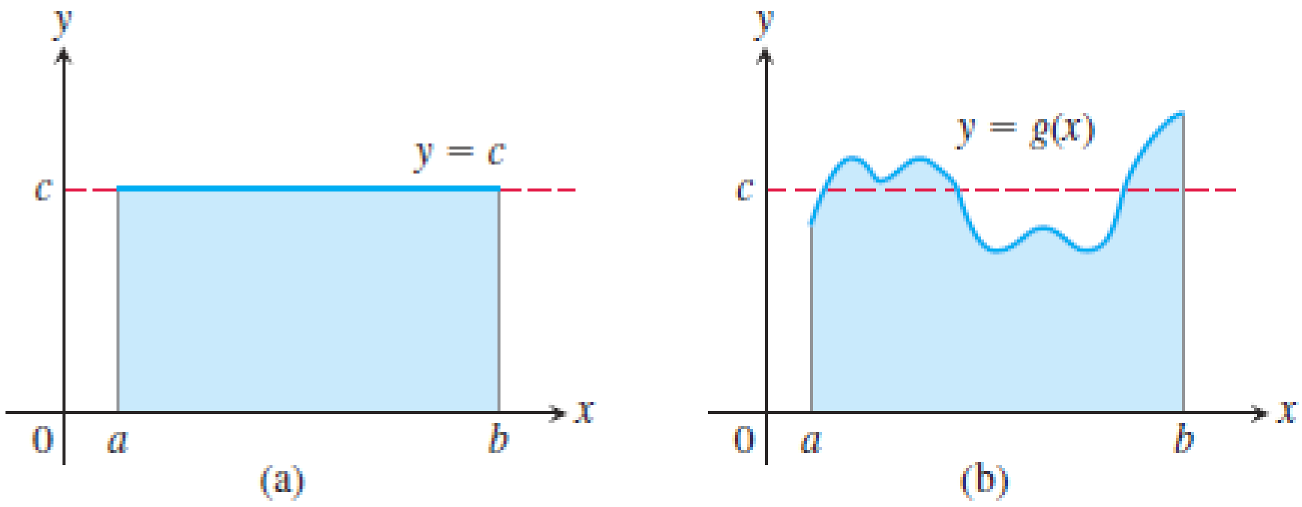


FIGURE 5.6 (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$. (b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b - a$.

Example 2 Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

The average function value on this interval is:

$$\frac{A}{\pi} \approx \frac{1}{8}\left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right) = 0.75342$$

TABLE 5.5 Average value of $\sin x$ on $0 \le x \le \pi$

Number of subintervals	Upper sum estimate	Midpoint Sum Estimate
8	0.75342	0.64073
16	0.69707	0.63764
32	0.65212	0.63687
50	0.64657	
100	0.64161	
1000	0.63712	

Exact Value:

$$\overline{f(x)} = \frac{2}{\pi} = 0.63662$$

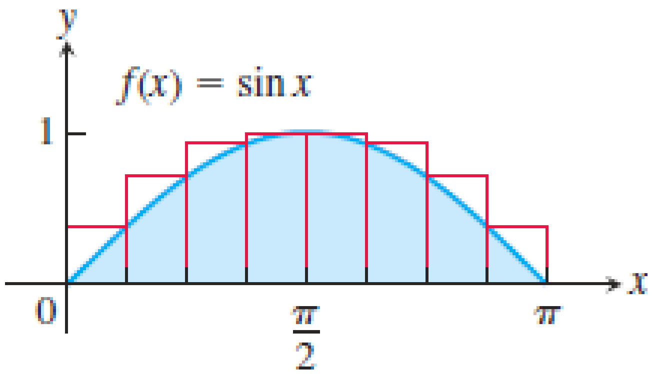


FIGURE 5.7 Approximating the area under $f(x) = \sin x$ between 0 and π to compute the average value of $\sin x$ over $[0, \pi]$, using eight rectangles (Example 3).

Summary

- The area under the graph of a positive function, the distance traveled by a moving object that doesn't change direction, and the average value of a nonnegative function f over an interval can all be approximated by finite sums constructed in a certain way.
- First, we subdivide the interval into subintervals, treating f as if it were constant over each subinterval. Then, we multiply the width of each subinterval by the value of f at some point within it, and add these products together.
- If the interval $[a, b]$ is subdivided into n subintervals of equal widths $\Delta x = (b-a)/n$ and if $f(c_k)$ is the value of f at the point c_k in the kth subinterval, this process gives a finite sum of the form

$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x.$$

- The choices for the c_k could maximize or minimize the value of f in the kth subinterval, or give some value in between. The true value lies somewhere between the approximations given by upper sums and lower sums. The finite sum approximations improved as we took more subintervals of thinner width.

Skill Practice 1

Use finite approximations to estimate the area under the graph of the function $f(x) = x^2$ on the interval [0, 1].

a) a lower sum with four/eight rectangles of equal width. (7/32 and 35/128)

b) an upper sum with four/eight rectangles of equal width. (15/32 and 51/128)

c) a midpoint sum with four/eight rectangles of equal width. (21/64 and 85/256)

Question: What is the average value of this function on this interval?