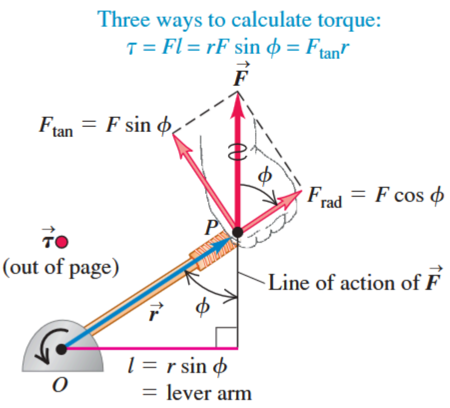
10. Dynamics of Rotational Motion

Torque

$$\tau = F\,l$$

l is the perpendicular distance from the point where the torque is, so if force where to be applied not strictly from perpendicular direction:

$$\tau = rF\sin\phi = F_{\tan}\,r$$



10.3 Three ways to calculate the torque of the force $\vec{F}$ about the point O. In this figure, $\vec{r}$ and $\vec{F}$ are in the plane of the page and the torque vector $\vec{\tau}$ points out of the page toward you.

torque is always measured about a point, and torque as a vector:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Torque and Acceleration for a Rigid Body

similar to Newton's second law, torque could be thought of as the equivalent of a force in rotational systems:

$$F_{1,\tan} = m_1 a_{1,\tan} \tag{10.4}$$

We can express the tangential acceleration of the first particle in terms of the angular acceleration α_z of the body using Eq. (9.14): $a_{1,\tan} = r_1\alpha_z$. Using this relationship and multiplying both sides of Eq. (10.4) by r_1, we obtain

$$F_{1,\tan} r_1 = m_1 r_1^2 \alpha_z \tag{10.5}$$

$$\tau_{1z} = I_1 \alpha_z = m_1 r_1^2 \alpha_z$$

We write an equation like this for every particle in the body and then add all these equations:

$$\tau_{1z} + \tau_{2z} + \cdots = I_1\alpha_z + I_2\alpha_z + \cdots = m_1 r_1^2 \alpha_z + m_2 r_2^2 \alpha_z + \cdots$$

so for a system in rotational motion, the sum of all the torque is equal to the angular acceleration of the moment of inertia of the system:

$$\sum \tau_z = I\alpha_z$$

usually in questions, you equal the sum of the torque to the forces in system:

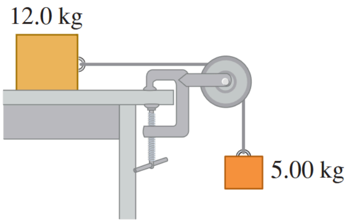
e.g. here the main derivation is:

$$\sum \tau_z = T_1 r + T_2 r$$

$$\Rightarrow T_1 r + T_2 r = I\alpha_z$$

$$\Rightarrow T_1 r + T_2 r = I\frac{a}{r}$$

Figure **E10.17**



and finding the tensions:

$$T_1 = m_1 a \qquad T_2 = m_2 g - m_2 a$$

so the acceleration of the system is:

$$a = \frac{m_2 g}{m_1 + m_2 + \frac{1}{2}M}$$

Rigid-Body Rotation About a Moving Axis

the total kinetic energy in a system where both linear and rotational motions are present:

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

usually accompanied by $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$ in most questions

Rolling without slipping

$$v_{cm} = R\omega$$

Work and Power in Rotational Motion

the total work done by the torque for an angular displacement from θ_1 to θ_2:

$$W = \int_{\theta_1}^{\theta_2} \tau \, d\theta$$

and similar to work done in a linear, or transitional, motion:

$$W_{tot} = \frac{1}{2}I\omega_2^2 - \frac{1}{2}Iw_1^2$$

the total work done in an rotational motion is equal to the difference in the kinetic energy between the final and initial energies.

and power:

$$P = \tau_z \omega_z$$

Angular Momentum

$$\overrightarrow{L} = \overrightarrow{r} \times m\overrightarrow{v}$$

the angular momentum of a rotational motion is the **cross product** of the position vector and the linear momentum, which is also the product of the moment of inertia of the system and the its angular velocity (like linear momentum $p = mv$) :

$$\overrightarrow{L} = I\,\overrightarrow{\omega}$$

$$\sum \tau = \frac{d(\overrightarrow{L})}{dt} = \frac{d(Iw)}{dt} = I\alpha_z$$

Conservation of Angular Momentum

again, much similar to linear momentum, the angular momentum is always conserved.

$$\overrightarrow{L}_1 = \overrightarrow{L}_2$$

$$I\omega_{z,1} = I\omega_{z,2}$$

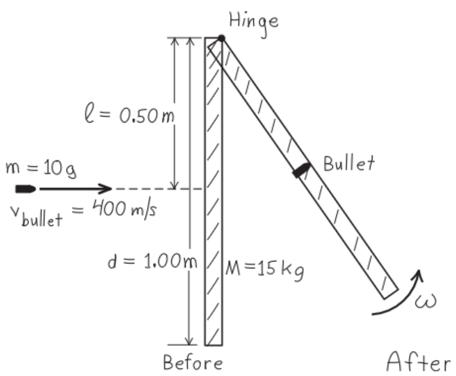
commonly, questions include both the linear and angular momentum, so:

$$\vec{r} \times m\vec{v}_1 + I\omega_{z,1} = \vec{r} \times m\vec{v}_2 + I\omega_{z,2}$$

and for rotational collision, similar to elastic and inelastic collisions in linear systems, if the bodies collide and they are stuck together, they would have the same angular velocity:

$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega$$

e.g.



$$L = mvl = (0.010\ \text{kg})(400\ \text{m/s})(0.50\ \text{m}) = 2.0\ \text{kg}\cdot\text{m}^2/\text{s}$$

The final angular momentum is $I\omega$, where $I = I_{\text{door}} + I_{\text{bullet}}$. From Table 9.2, case (d), for a door of width $d = 1.00$ m,

$$I_{\text{door}} = \frac{Md^2}{3} = \frac{(15\ \text{kg})(1.00\ \text{m})^2}{3} = 5.0\ \text{kg}\cdot\text{m}^2$$

The moment of inertia of the bullet (with respect to the axis along the hinges) is

$$I_{\text{bullet}} = ml^2 = (0.010\ \text{kg})(0.50\ \text{m})^2 = 0.0025\ \text{kg}\cdot\text{m}^2$$

Conservation of angular momentum requires that $mvl = I\omega$, or

$$\omega = \frac{mvl}{I} = \frac{2.0\ \text{kg}\cdot\text{m}^2/\text{s}}{5.0\ \text{kg}\cdot\text{m}^2 + 0.0025\ \text{kg}\cdot\text{m}^2} = 0.40\ \text{rad/s}$$

here, the collision is "inelastic", so the bullet sticks to the door.

the moment of inertia of the bullet and the door gets combined and they have the same angular velocity.