# CALCULUS

Prof. Liang ZHENG

Spring 2025

- We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? These and other questions can be answered by applying the Mean Value Theorem. First, we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

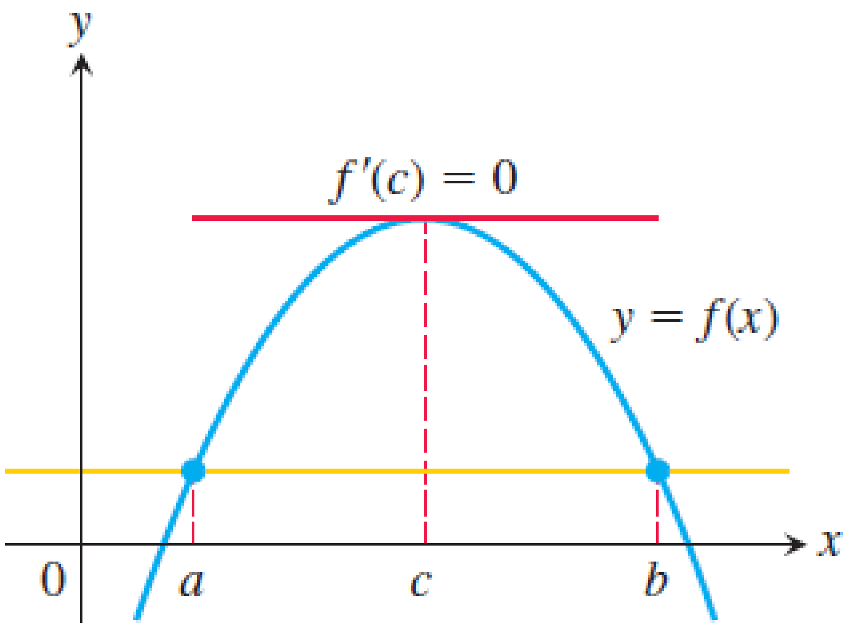
## ① Rolle's Theorem

**THEOREM 3 − Rolle's Theorem**

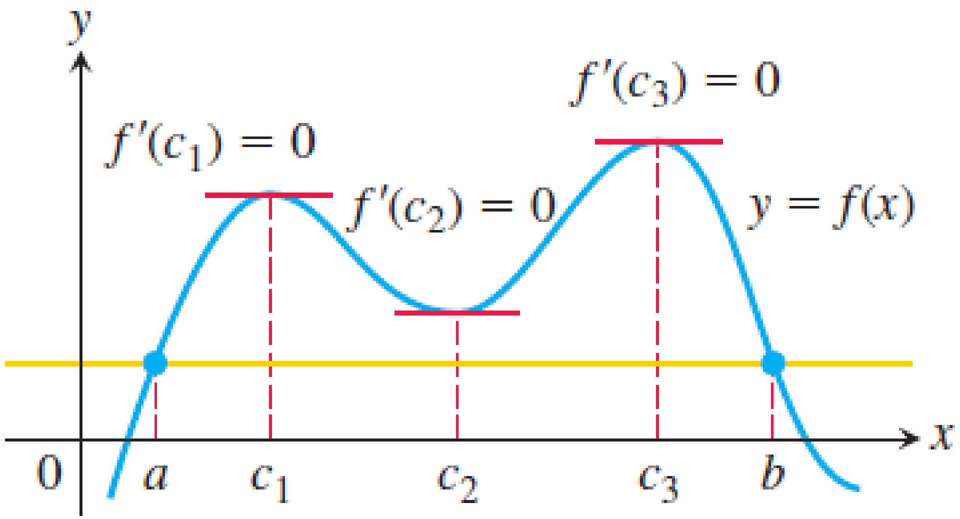
Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If $f(a) = f(b)$, then there is at least one number $c$ in $(a, b)$ at which $f'(c) = 0$.

**Proof:**

- Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (Fig. a), or it may have more (Fig. b).

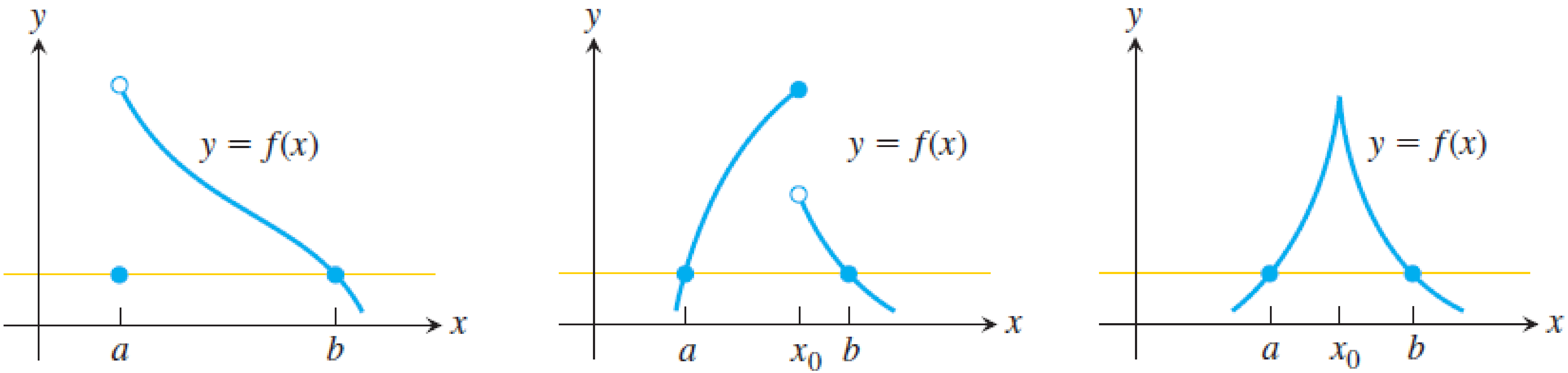



(a)




(b)

**Note 1:** The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent.




(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

**Example 1** Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

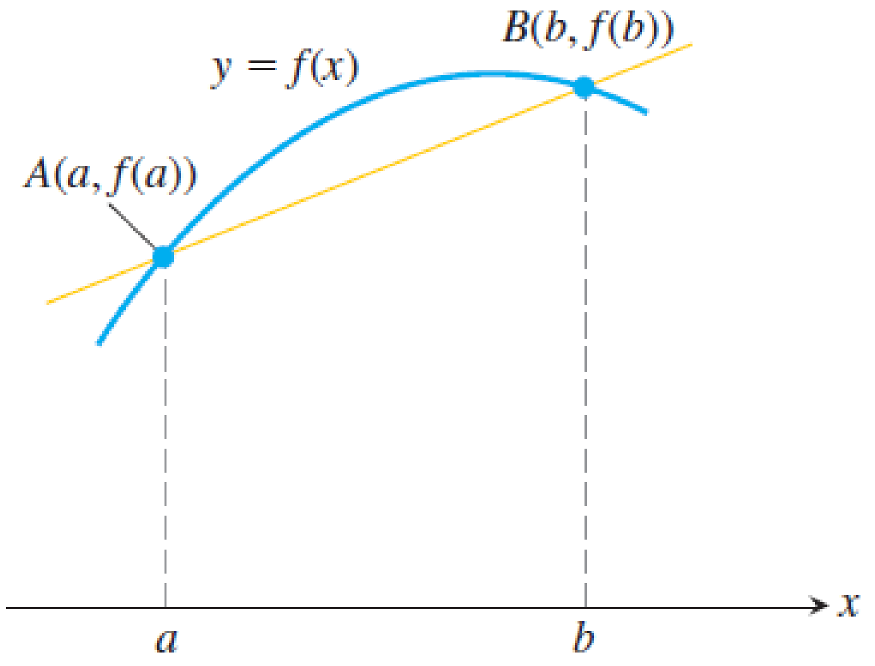
## ② The Mean Value Theorem (first stated by Joseph-Louis Lagrange)

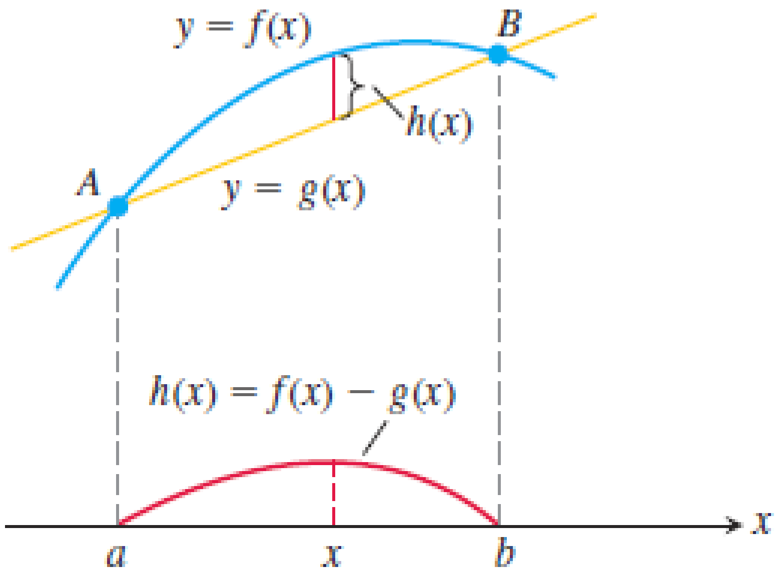
- The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange (1736-1813), is a slanted version of Rolle's Theorem. The Mean Value Theorem guarantees that there is a point where the tangent line is parallel to the secant line that joins *A* and *B*.

### THEOREM 4 – The Mean Value Theorem

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$






**Proof:**

The secant line through the points $A(a, f(a))$ and $B(b, f(b))$ can be expressed by the function $g(x)$:

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

The vertical difference between the graphs of $f$ and $g$ at $x$ is:

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \end{aligned}$$

The function $h(x)$ satisfies all the hypotheses of Rolle's Theorem on $[a, b]$.

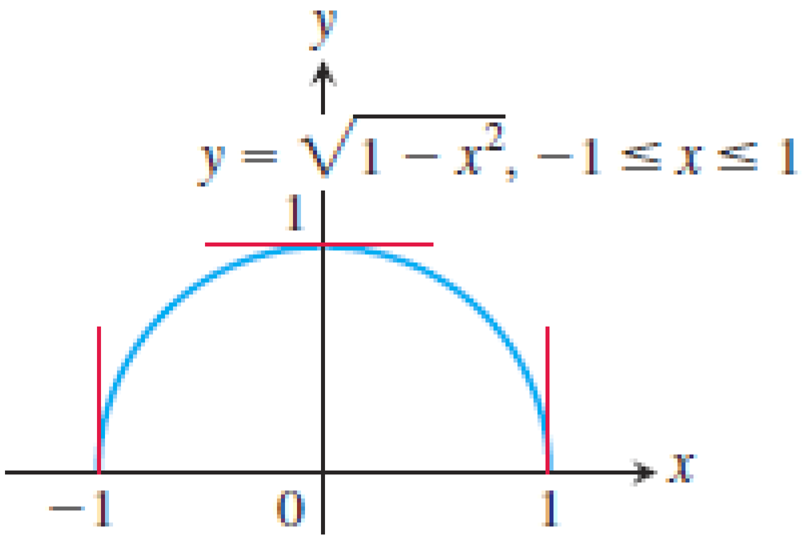
It is continuous on $[a, b]$ and differentiable on $(a, b)$, and $h(a) = h(b) = 0$.

Therefore, $h'(c) = 0$ at some point $c \in (a, b)$.

**Note:**

The hypotheses of the Mean Value Theorem do not require $f$ to be differentiable at either $a$ or $b$. One-sided continuity at $a$ and $b$ is enough.

For example, the function $f(x) = (1 - x^2)^{1/2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though $f$ is not differentiable at $-1$ and $1$.




## A Physical Interpretation

We can think of the number $(f(b) - f(a))/(b - a)$ as the average change in $f$ over $[a, b]$ and $f(c)$ as an instantaneous change. Then the Mean Value Theorem says that the instantaneous change at some interior point is equal to the average change over the entire interval.

## ③ Mathematical Consequences

### COROLLARY 1

If $f'(x) = 0$ at each point $x$ of an open interval $(a, b)$, then $f(x) = C$ for all $x \in (a, b)$, where $C$ is a constant.

### COROLLARY 2

If $f'(x) = g'(x)$ at each point $x$ in an open interval $(a, b)$, and then there exists a constant $C$ such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

### EXAMPLE 2

Find function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0, 2).

## Skill Practice 1

Find all possible functions with the given derivative.

(a) $y' = 2x$ (b) $y' = 3x^2 - 1/(x^{1/2})$ (c) $y' = \cos(t/2)$

## Skill Practice 2

Show that the following functions have exactly one zero in the given interval.

(a) $f(x) = x^3 + 4x^{-2} + 7$, $(-\infty, 0)$. (b) $r(\theta) = \tan\theta - \cot\theta - \theta$, $(0, \pi/2)$.

## Skill Practice 3

A male marathoner ran the 42.2-km Shenzhen Marathon in 2.2 hours. Show that at least twice he was running at exactly 18 kmh, assuming his initial and final speeds are zero.