CALCULUS

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- In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to infinite limits.
- Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these ideas to analyze the graphs of functions having horizontal or vertical asymptotes.
- The symbol for infinity (∞) does not represent a real number. We use "∞" to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.

① Finite Limits as $x \to \pm\infty$

- $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number ε > 0, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

- $f(x)$ has the **limit L as x approaches negative infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number ε > 0, there exists a corresponding number N such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

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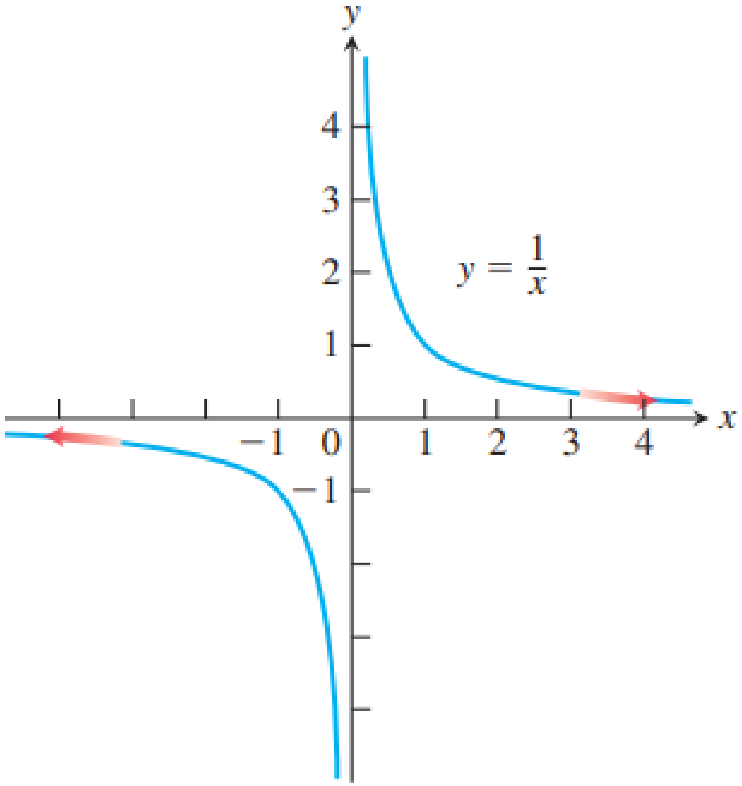
- For example, $f(x)=\frac{1}{x}$ has limit 0 as $x \to \infty$ or $x \to -\infty$, or that 0 is a limit of $f(x)=\frac{1}{x}$ at infinity and negative infinity.



Example 1: Show that:

(a) $\lim\limits_{x\to\infty}\frac{1}{x}=0$ (b) $\lim\limits_{x\to-\infty}\frac{1}{x}=0$

Example 2: Find the limits.

(a) $\lim\limits_{x\to\infty}\left(5+\frac{1}{x}\right)$ (b) $\lim\limits_{x\to\infty}\frac{\pi\sqrt{3}}{x^2}$

② Limits at Infinity of Rational Functions

Recall: For $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, $a_n\neq 0$, n is called the degree of $f(x)$, and a_nx^n is called the leading term.

- To determine the limit of a rational function as $x\to\pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

Example 3 Find the following limits.

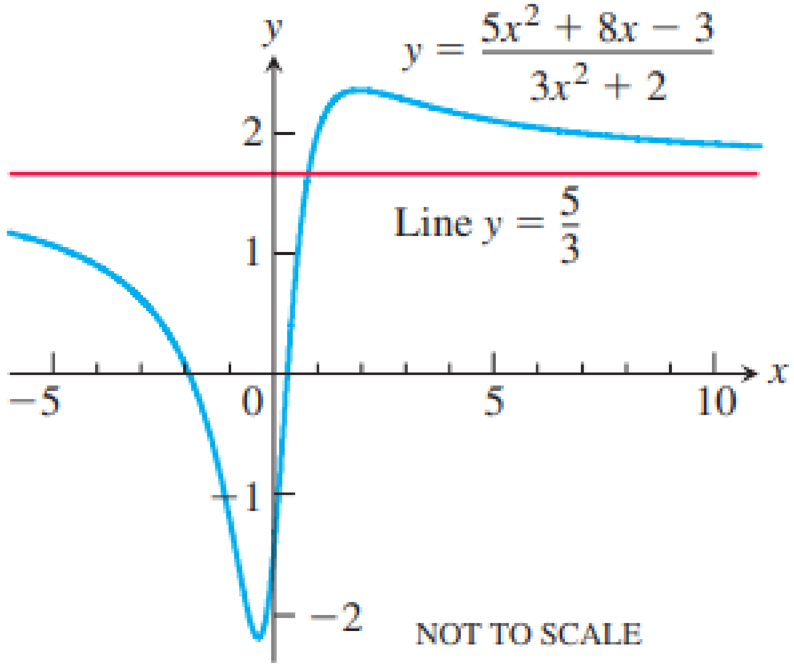
(a) $\lim\limits_{x\to\infty}\dfrac{5x^2+8x-3}{3x^2+2}$ (b) $\lim\limits_{x\to\infty}\dfrac{11x+2}{2x^3-1}$ (c) $\lim\limits_{x\to\infty}(\sqrt{x^2+16}-x)$

③ Horizontal Asymptotes

DEFINITION A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

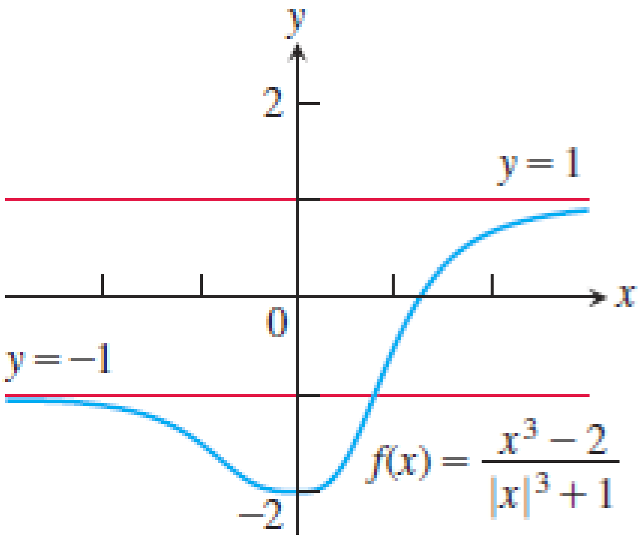
- The line $y = 5/3$ is the horizontal asymptote of the graph of $y = \frac{5x^2+8x-3}{3x^2+2}$ since

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to-\infty}\frac{5x^2+8x-3}{3x^2+2} = \frac{5}{3}$$



Example 4 Find the horizontal asymptotes of the graph of

$$f(x)=\frac{x^3-2}{|x|^3+1}.$$



Example 5 Find the following limits.

(a) $\lim\limits_{x\to\infty} \sin\frac{1}{x}$ (b) $\lim\limits_{x\to\pm\infty} x\sin\frac{1}{x}$

Example 6 Using the Sandwich Theorem, find the horizontal asymptote of the graph

$$y=2+\frac{\sin x}{x}$$

④ Infinite Limits

$$\lim_{x\to 0^+}\frac{1}{x}=\infty \qquad \text{and} \qquad \lim_{x\to 0^-}\frac{1}{x}=-\infty$$

Note: In writing this equation, we are not saying that the limit exists, nor are we saying that there is a real number ∞, for there is no such number. Rather, this expression is just a concise way of saying that **both limits DO NOT EXIST** !!!

Example 6 Find the following limits.

(a) $\lim\limits_{x\to 1^+}\dfrac{1}{x-1}$.   (b) $\lim\limits_{x\to 1^-}\dfrac{1}{x-1}$.   (c) $\lim\limits_{x\to 0}\dfrac{1}{x^2}$.

Example 7 Find the following limits.

(a) $\lim\limits_{x\to 2}\dfrac{(x-2)^2}{x^2-4}$ (b) $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4}$ (c) $\lim\limits_{x\to 2^+}\dfrac{x-3}{x^2-4}$

(d) $\lim\limits_{x\to 2^-}\dfrac{x-3}{x^2-4}$ (e) $\lim\limits_{x\to 2}\dfrac{x-3}{x^2-4}$ (f) $\lim\limits_{x\to 2}\dfrac{2-x}{(x-2)^3}$

Example 8 Find the limit:

$$\lim_{x\to-\infty}\frac{2x^5-6x^3+1}{3x^2+x-8}$$

⑤ Precise Definitions of infinite limits

- $f(x)$ **approaches infinity as x approaches c** and write

$$\lim_{x\to c} f(x) = \infty$$

if, for every $B > 0$, there exists a corresponding number $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta .$$

- $f(x)$ **approaches negative infinity as x approaches c** and write

$$\lim_{x\to c} f(x) = -\infty$$

if, for every $B < 0$, there exists a corresponding number $\delta > 0$ such that

$$f(x) < -B \quad \text{whenever} \quad 0 < |x - c| < \delta .$$

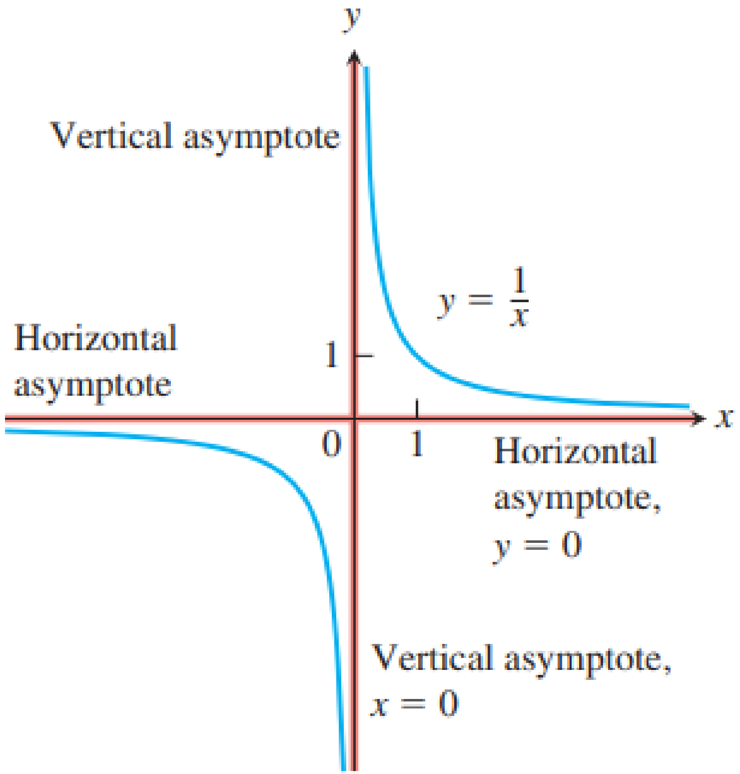
Example 9 Prove that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

⑥ Vertical Asymptotes

DEFINITION A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

For example, the graph of $y = \frac{1}{x}$ has a vertical asymptote $x = 0$.



Example 10 Find the horizontal and vertical asymptotes of the curve

(a) $y = \frac{x+3}{x+2}$ (b) $y = \frac{-8}{x^2-4}$

Example 11 For $y = tanx$ and $y = secx$, both curves have infinitely many vertical asymptotes at $x = \frac{(2k+1)\pi}{2}$, where $cosx = 0$.