Lecture 7
Potential Energy & Energy Conservation

Date: 3/27/2025

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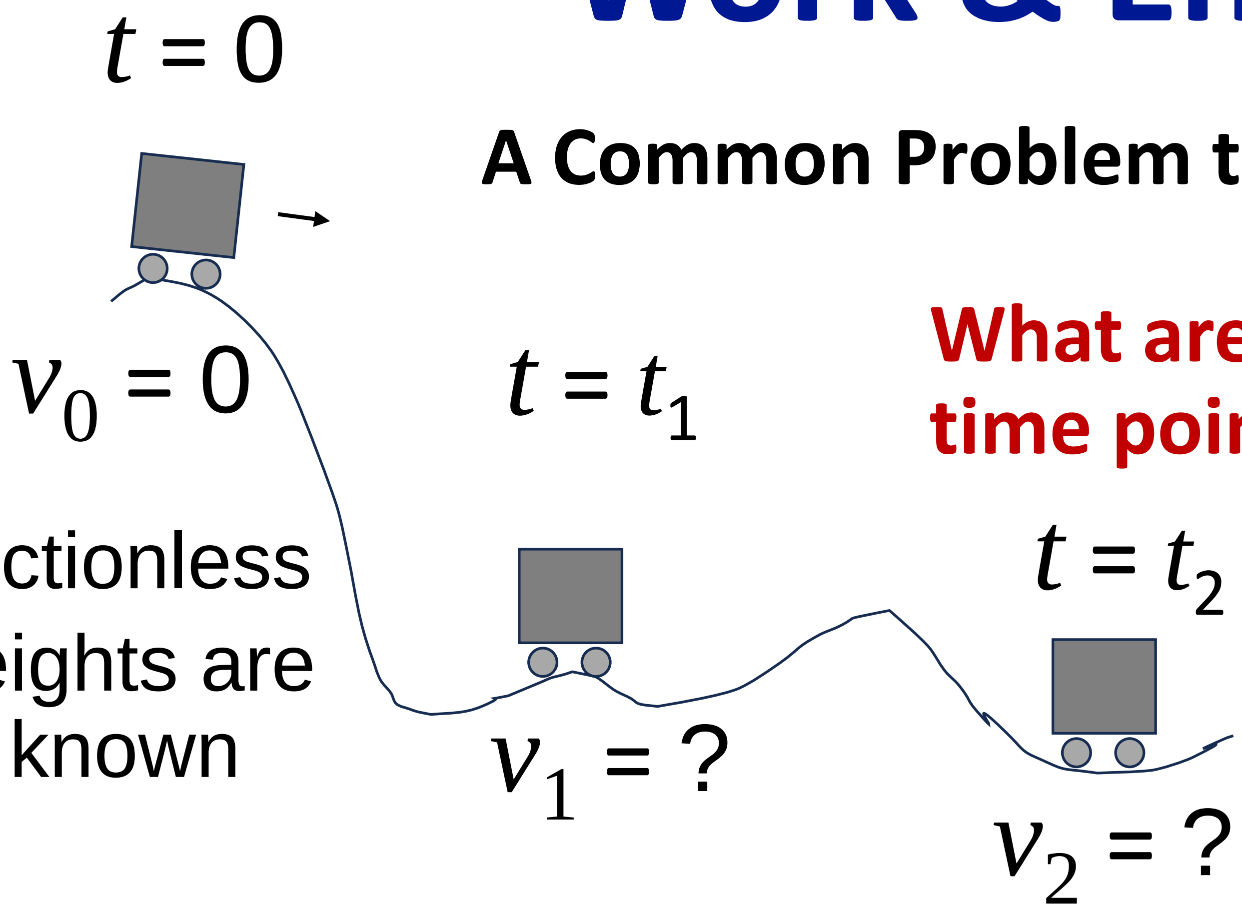
Previously we have learned Work & Energy

A Common Problem to consider:



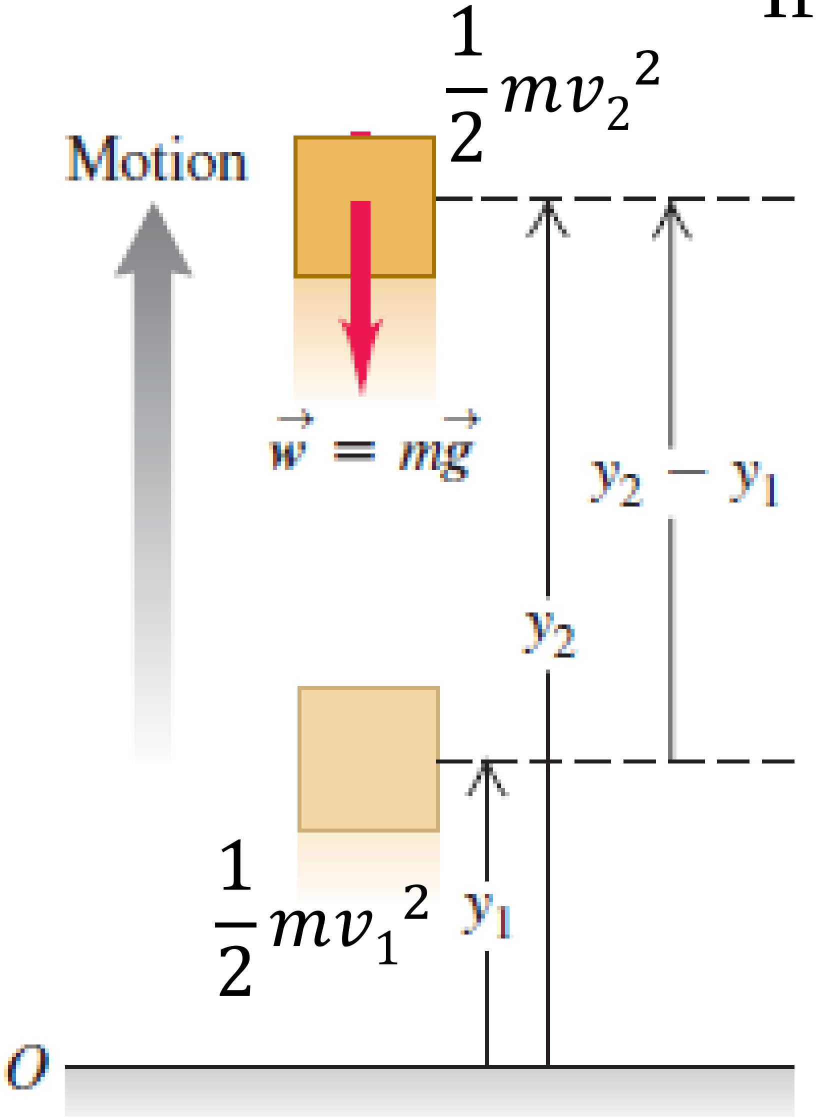
- Frictionless
- Heights are all known

What are the velocity of the cart at time points t_1 and t_2?

Hard to solve with Newton's laws *right*?

This lecture: *potential energy* as a convenient tool

Free Fall by Work-Energy Theorem



If we apply the definition of work here:

$$W_{\text{grav}} = Fs = w(y_1 - y_2) = mgy_1 - mgy_2$$

$$\boxed{U_{\text{grav}} = mgy \qquad \text{(gravitational potential energy)}}$$

$$W_{\text{grav}} = U_{\text{grav},1} - U_{\text{grav},2}$$

$$= -(U_{\text{grav},2} - U_{\text{grav},1}) = -\Delta U_{\text{grav}}$$

When gravity does work, its potential energy decreases – doing work ***consumes*** U_{grav}

Now recall the work-energy theorem

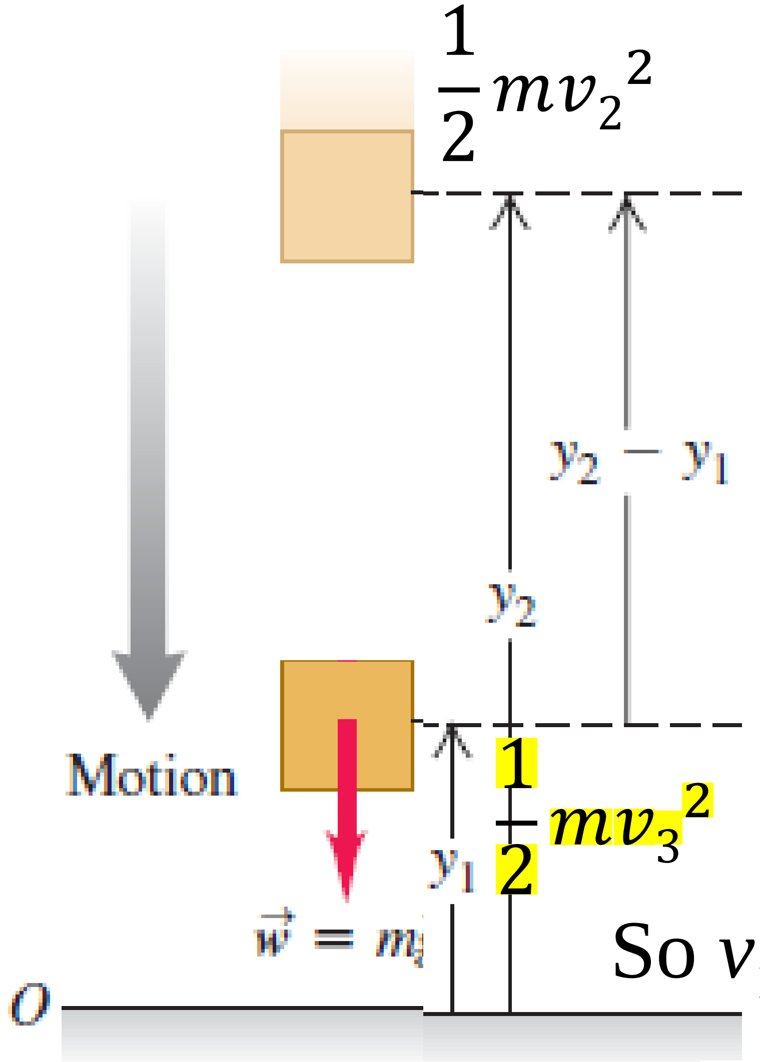
$$W_{grav} = K_2 - K_1 \quad = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2$$

Free Fall by Work-Energy Theorem

When it falls back:

$$W_{grav} = \overbrace{-mg}^{F}\overbrace{(y_1 - y_2)}^{s}$$

$$= -(U_{grav,1} - U_{grav,2})$$



$$W_{grav} = K_3 - K_2 \quad = \frac{1}{2}mv_3{}^2 - \frac{1}{2}mv_2{}^2$$

The final kinetic energy back to y_1:

$$K_3 = \frac{1}{2}mv_3{}^2 = \frac{1}{2}mv_2{}^2 - (U_{grav,1} - U_{grav,2})$$

$$= \frac{1}{2}mv_2{}^2 - mg(y_1 - y_2)$$

So $v_1 = v_3$, $K_3 = K_1$ as the potential energy decreases to U_1

Conservation of Mechanical Energy

To see what gravitational potential energy is good for, suppose the body's weight is the *only* force acting on it, so $\vec{F}_{other} = \mathbf{0}$. The body is then falling freely with no air resistance and can be moving either up or down. Let its speed at point y_1 be v_1 and let its speed at y_2 be v_2. The work–energy theorem, Eq. (6.6), says that the total work done on the body equals the change in the body's kinetic energy: $W_{tot} = \Delta K = K_2 - K_1$. If gravity is the only force that acts, then from Eq. (7.3), $W_{tot} = W_{grav} = -\Delta U_{grav} = U_{grav,1} - U_{grav,2}$. Putting these together, we get

$$\Delta K = -\Delta U_{grav} \quad \text{or} \quad K_2 - K_1 = U_{grav,1} - U_{grav,2}$$

which we can rewrite as

$$K_1 + U_{grav,1} = K_2 + U_{grav,2} \quad \text{(if only gravity does work)} \tag{7.4}$$

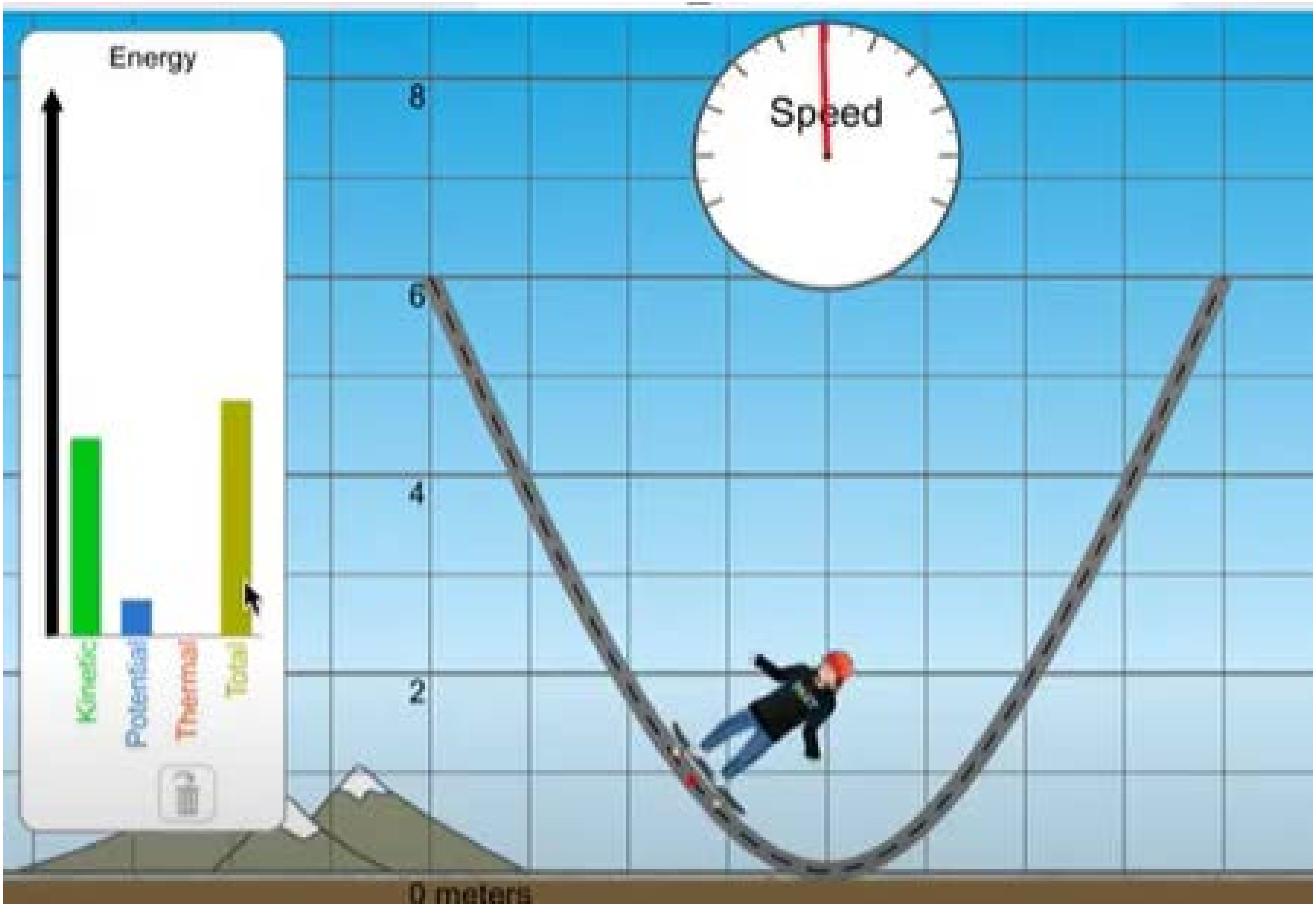
$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2 \quad \text{(if only gravity does work)}$$

Conservation of Mechanical Energy

That is, E is constant; it has the same value at y_1 and y_2. But since the positions y_1 and y_2 are arbitrary points in the motion of the body, the total mechanical energy E has the same value at *all* points during the motion:

$$E = K + U_{\text{grav}} = \text{constant} \qquad \text{(if only gravity does work)}$$

A quantity that always has the same value is called a *conserved* quantity. *When only the force of gravity does work, the total mechanical energy is constant—that is, it is conserved* (Fig. 7.3). This is our first example of the **conservation of mechanical energy.**


Energy
Kinetic
Potential
Thermal
Total
Speed
8
6
4
2
0 meters

When *Forces Other Than Gravity* Do Work

The total work W_{tot} is then the sum of W_{grav} and the work done by $\vec{F}_{other}$. We will call this additional work W_{other}, so the total work done by all forces is $W_{tot} = W_{grav} + W_{other}$. Equating this to the change in kinetic energy, we have

$$W_{other} + W_{grav} = K_2 - K_1$$ but $$W_{grav} = U_{grav,1} - U_{grav,2}$$

We have $$W_{other} + U_{grav,1} - U_{grav,2} = K_2 - K_1$$

which we can rearrange in the form

$$K_1 + U_{grav,1} + W_{other} = K_2 + U_{grav,2}$$ (if forces other than gravity do work)

When *Forces Other Than Gravity* Do Work

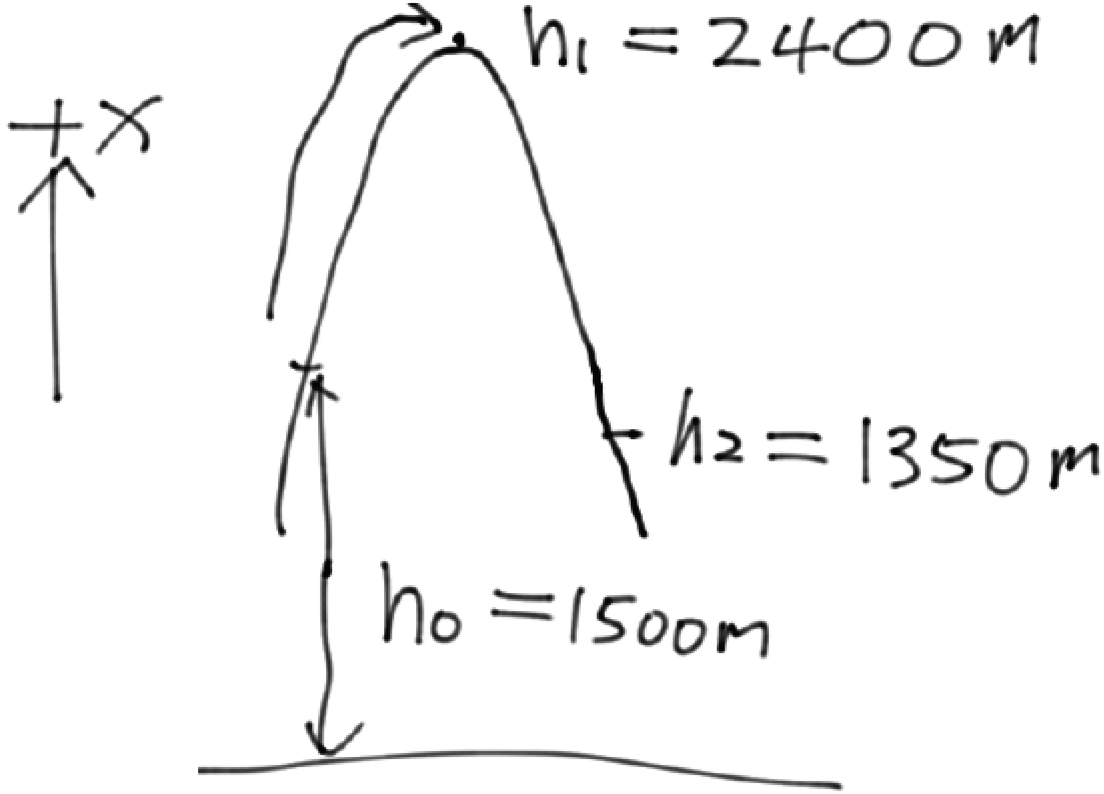
$$K_1 + U_{\text{grav},1} + W_{\text{other}} = K_2 + U_{\text{grav},2}$$ (if forces other than gravity do work)

Finally, using the appropriate expressions for the various energy terms, we obtain

$$\tfrac{1}{2}mv_1^2 + mgy_1 + W_{\text{other}} = \tfrac{1}{2}mv_2^2 + mgy_2 \qquad (7.8)$$ (if forces other than gravity do work)

The meaning of Eqs. (7.7) and (7.8) is this: *The work done by all forces other than the gravitational force equals the change in the total mechanical energy $E = K + U_{\text{grav}}$ of the system, where U_{grav} is the gravitational potential energy.*

7.1 • In one day, a 75-kg mountain climber ascends from the 1500-m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day?

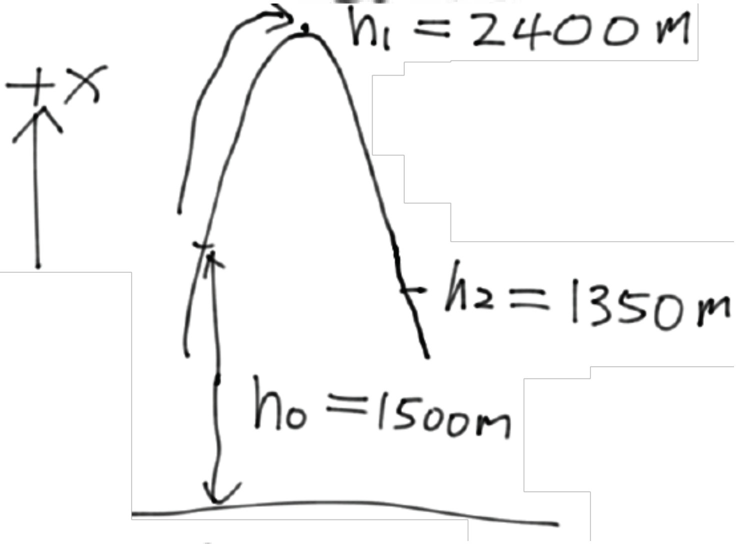


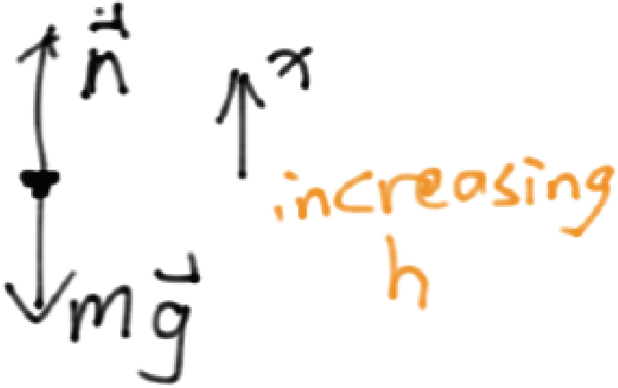
Again, change in gravitational potential

$$\Delta U_{grav} = -W_{grav}$$

so how much work has gravity done on climber

7.1 • In one day, a 75-kg mountain climber ascends from the 1500-m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day?





Go back to free body diagram

$$W_{grav.a} = \vec{F}\cdot\vec{s} = m\vec{g}\cdot\Delta\vec{h}$$

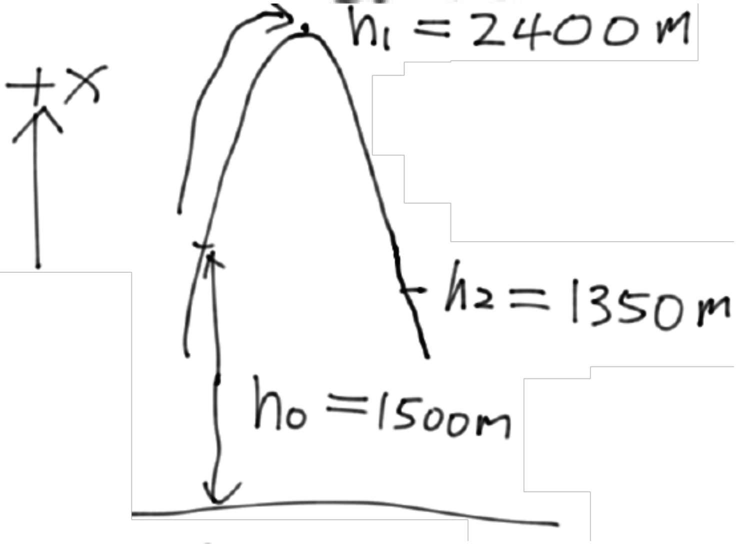
$$= 75\,kg \cdot 9.8\,m/s^2 \cdot (-\hat{i}) \cdot (2400-1500)\,m\,\hat{i}$$

$$\hat{i}\cdot(-\hat{i}) = -1 \qquad = -6.6\times10^{5}\,J$$

so $\Delta U_{grav,a} = -W_{grav.a}$

$$= 6.6\times10^{5}\,J$$

7.1 • In one day, a 75-kg mountain climber ascends from the 1500-m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day?



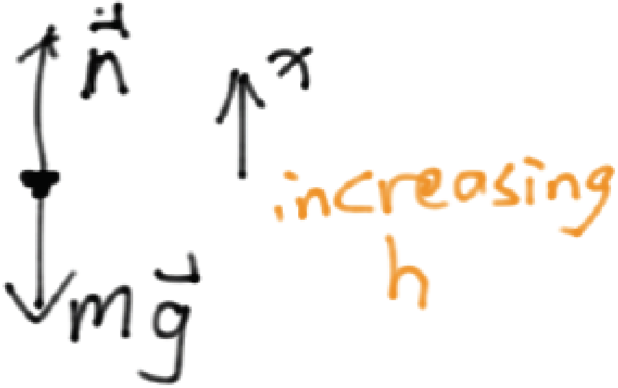
$$\Delta U_{grav.\,b} = -m\vec{g}\cdot\Delta\vec{h}_2$$

$$= -75.0\,kg \cdot 9.8\,m/s^2(-\hat{i})\cdot(1350-2400)m\,\hat{i}$$

$$(-\hat{i}\cdot\hat{i}) = -1$$

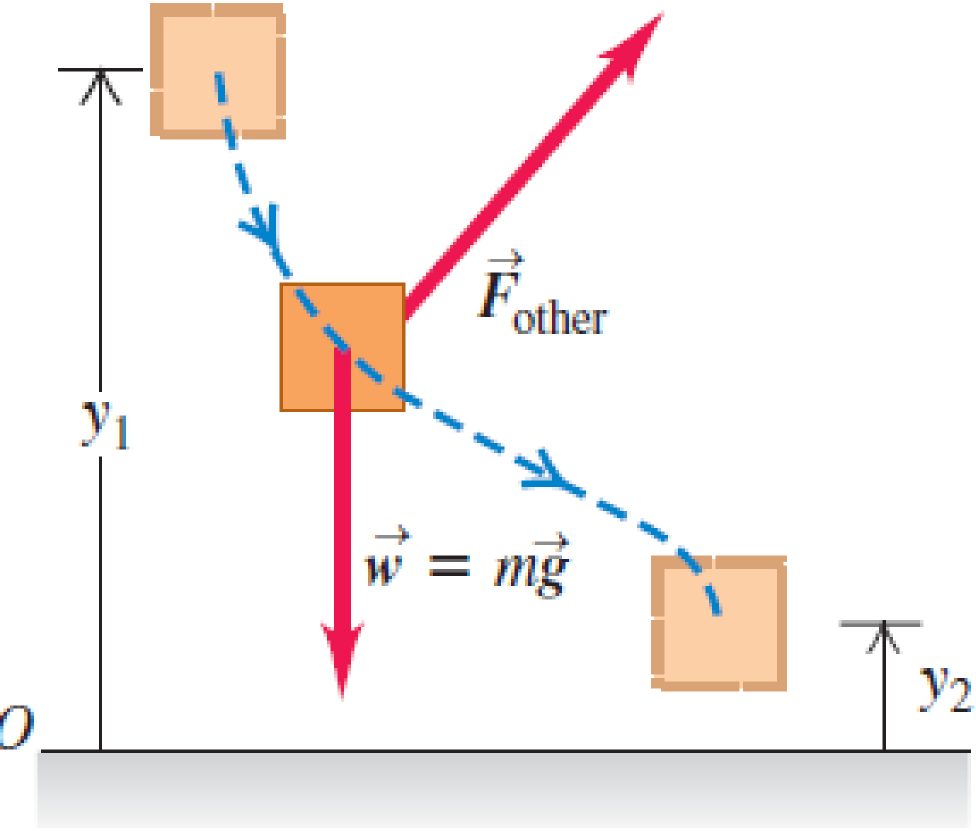
$$= -735\,N\cdot(-1)(-1050)m$$

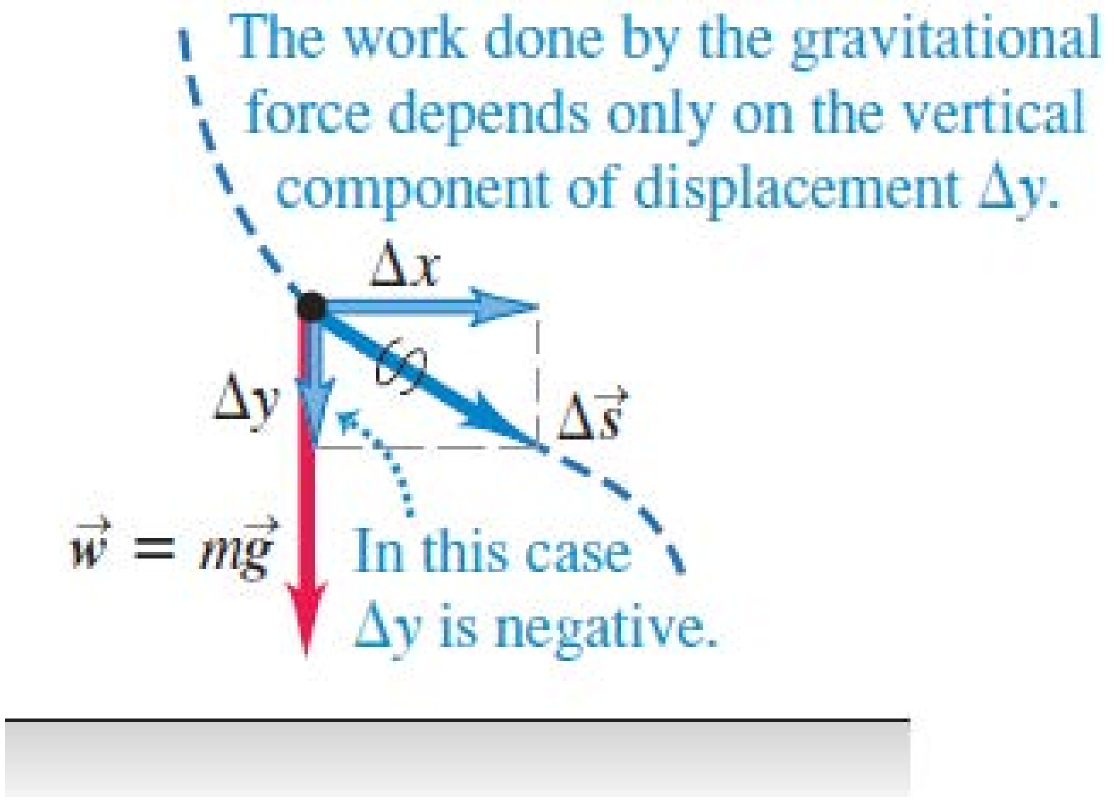
$$= -7.7\times10^5\,J.$$



Check: when h decrease, potential of gravity decreases

Work done by Gravitational Force on Curved Paths





$$\vec{w} = m\vec{g} = -mg\hat{j}$$

$$\Delta\vec{s} = \Delta x\hat{i} + \Delta y\hat{j}$$

Work done by Gravitational Force on Curved Paths

We have $\vec{w} \cdot \Delta\vec{s} = -mg\hat{j} \cdot (\Delta x\hat{i} + \Delta y\hat{j}) = -mg\Delta y$

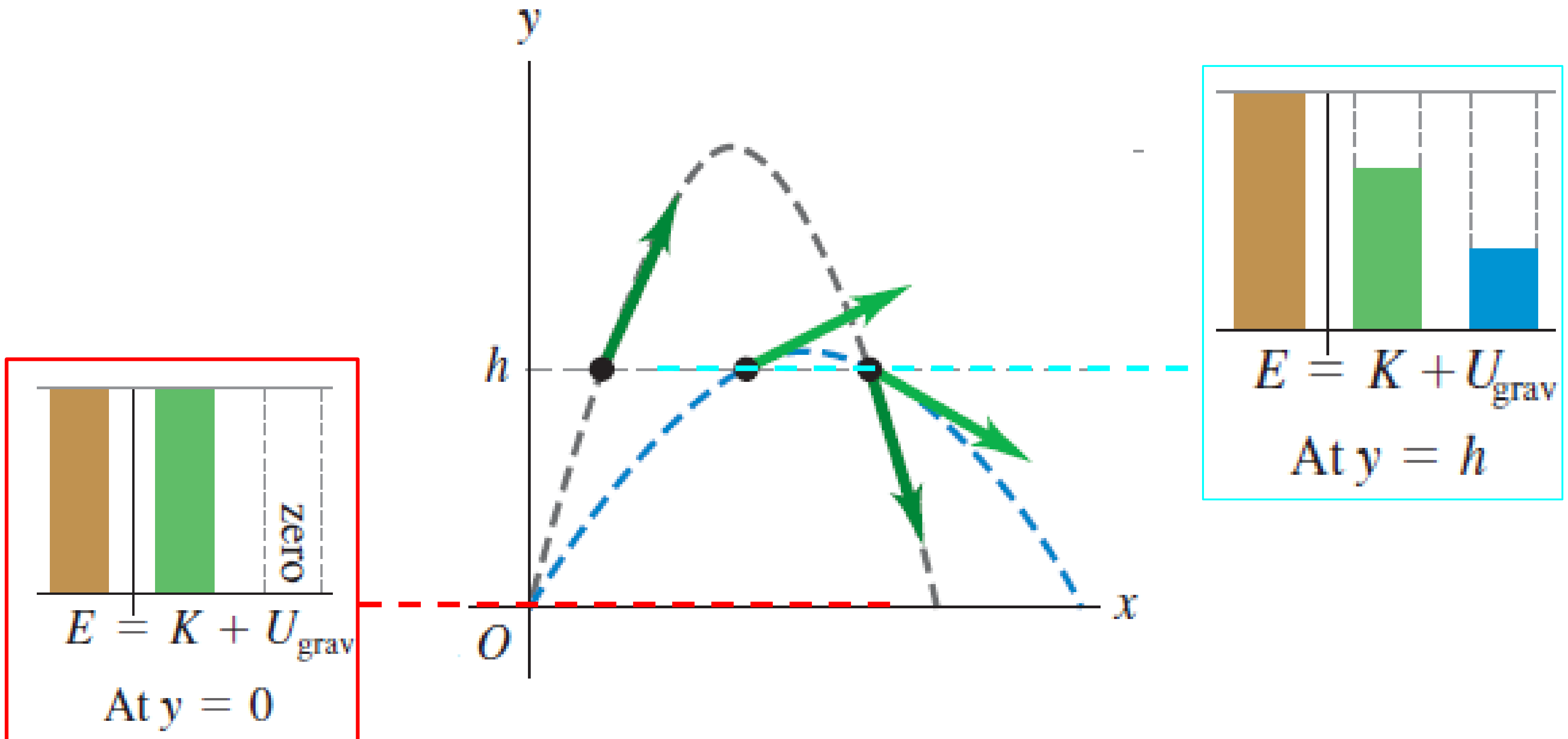
The work done by gravity is the same as though the body had been displaced vertically a distance Δy, with no horizontal displacement. This is true for every segment, so the *total* work done by the gravitational force is $-mg$ multiplied by the *total* vertical displacement $(y_2 - y_1)$:

$$W_{\text{grav}} = -mg(y_2 - y_1) = mgy_1 - mgy_2 = U_{\text{grav},1} - U_{\text{grav},2}$$

So *we can use the same expression for gravitational potential energy whether the body's path is curved or straight.*

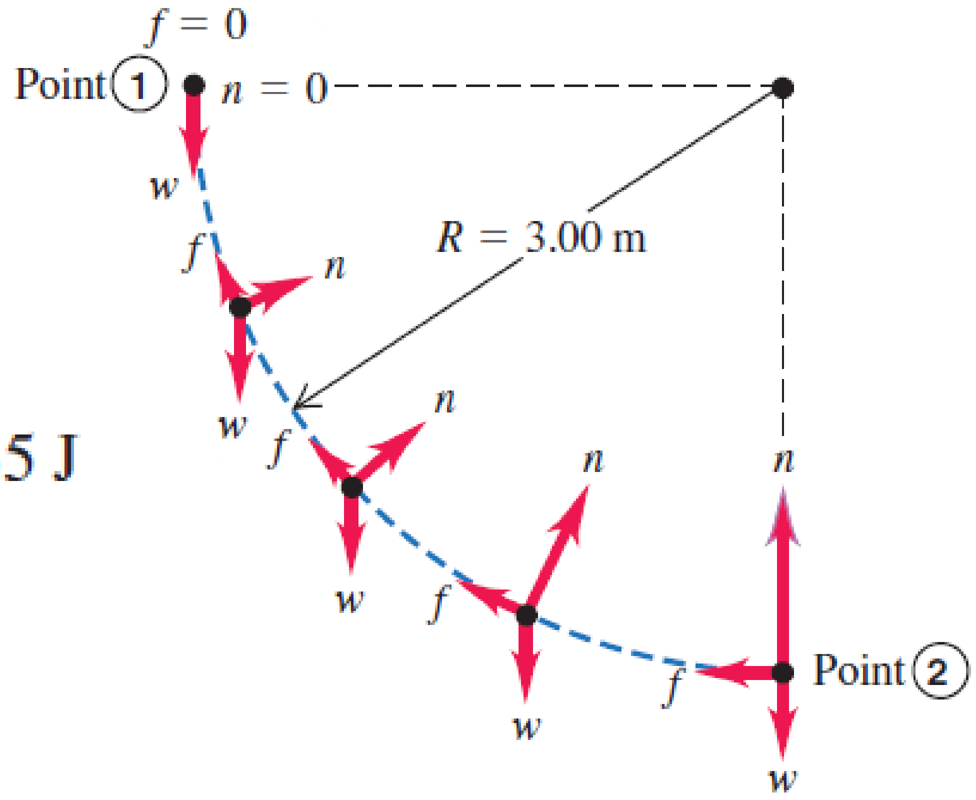
Energy in Projectile Motion

A batter hits two identical baseballs with the same initial speed and from the same initial height but at different initial angles. Prove that both balls have the same speed at any height h if air resistance can be neglected.



A vertical circle with friction

Suppose that the ramp of Example 7.4 is not frictionless, and that Throcky's speed at the bottom is only 6.00 m/s, not the 7.67 m/s we found there. What work was done on him by the friction force?



Mathematically

$$K_1 = 0$$

$$U_{grav,1} = mgR = (25.0\ \text{kg})(9.80\ \text{m/s}^2)(3.00\ \text{m}) = 735\ \text{J}$$

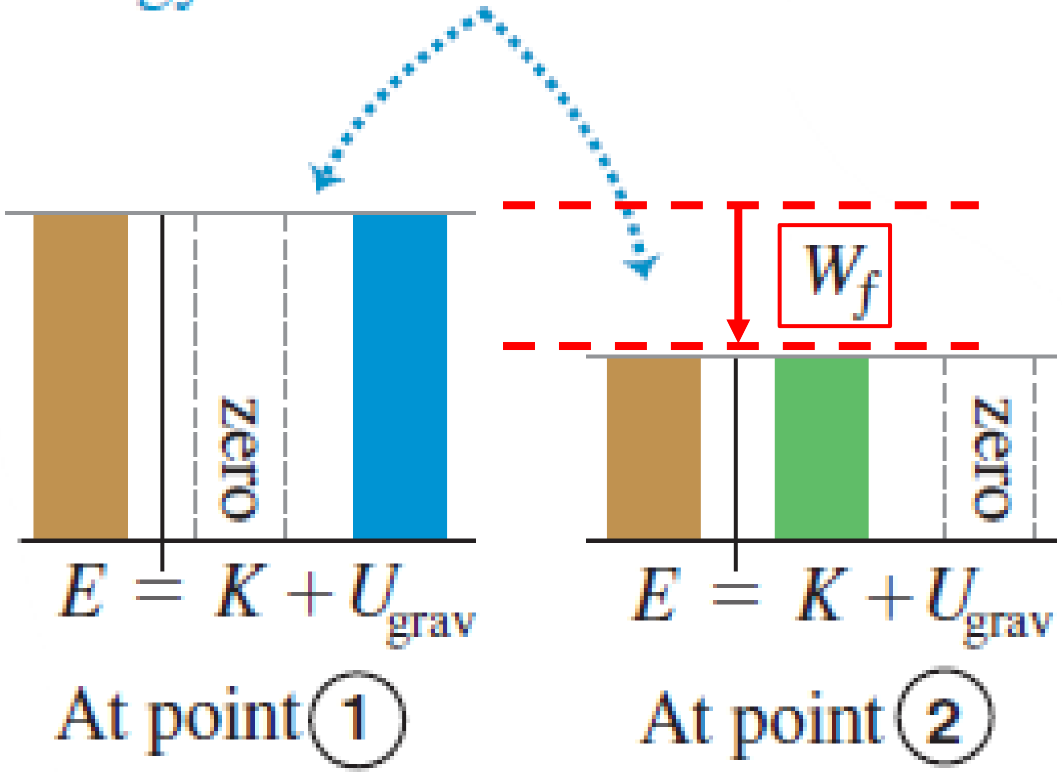
$$K_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}(25.0\ \text{kg})(6.00\ \text{m/s})^2 = 450\ \text{J}$$

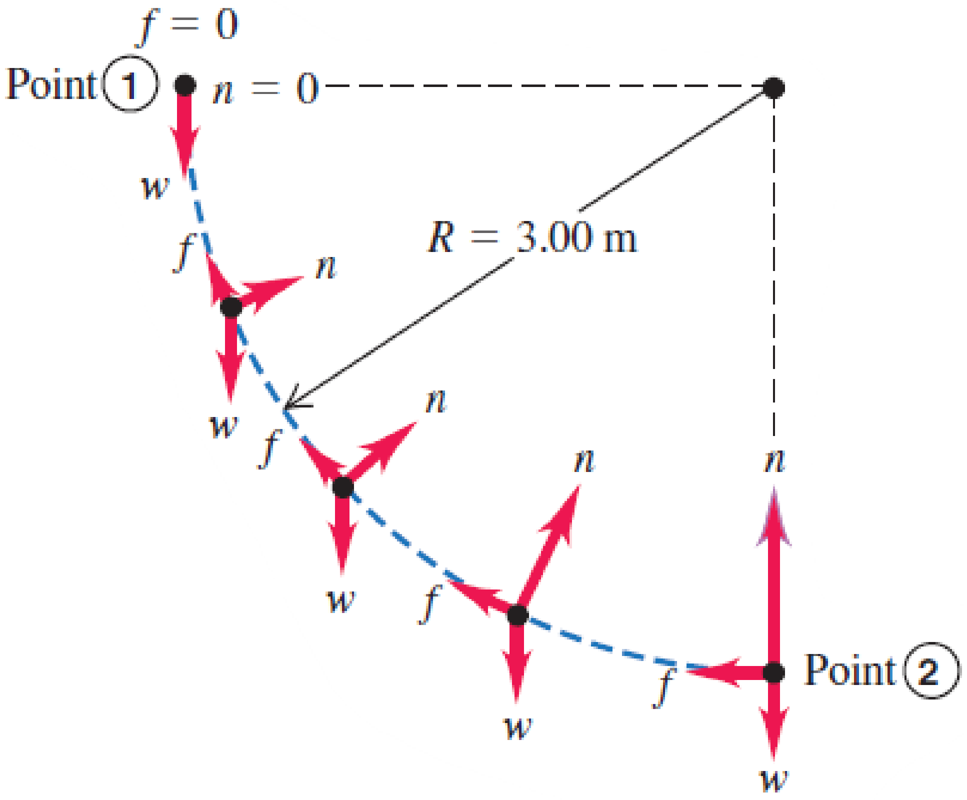
$$U_{grav,2} = 0$$

A vertical circle with friction

The friction force (f) does negative work on Throcky as he descends, so the total mechanical energy decreases.

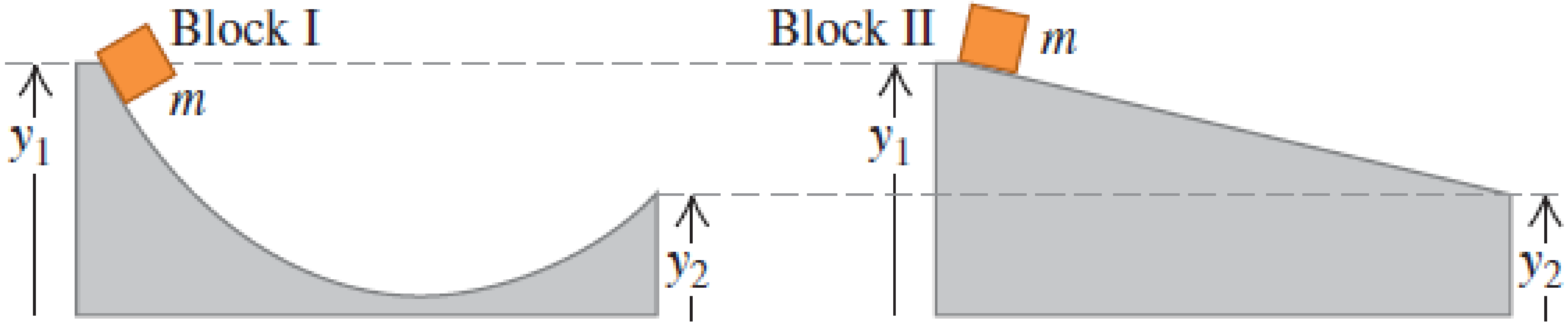
$$W_f = W_{\text{other}} = K_2 + U_{\text{grav},2} - K_1 - U_{\text{grav},1}$$
$$= 450\text{ J} + 0 - 0 - 735\text{ J} = -285\text{ J}$$





An inclined plane without friction

Test Your Understanding of Section 7.1 The figure shows two different frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed? (i) block I; (ii) block II; (iii) the speed is the same for both blocks.

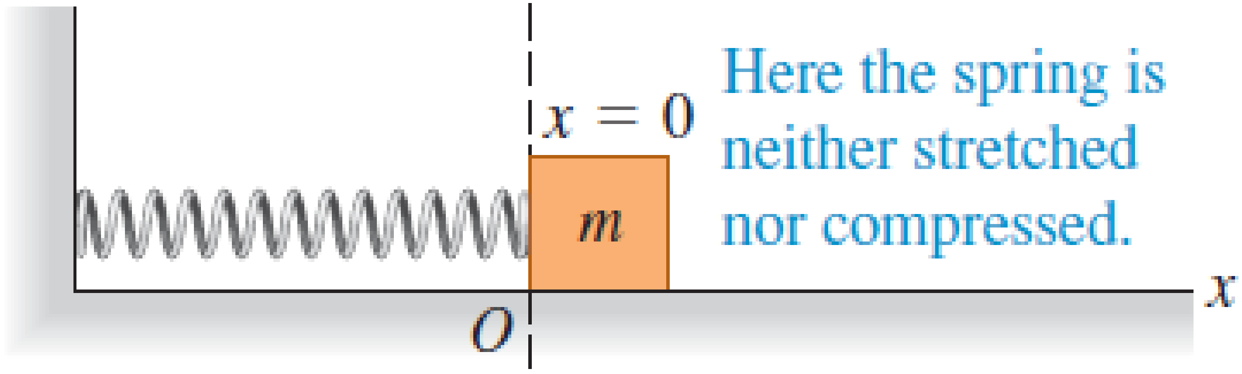


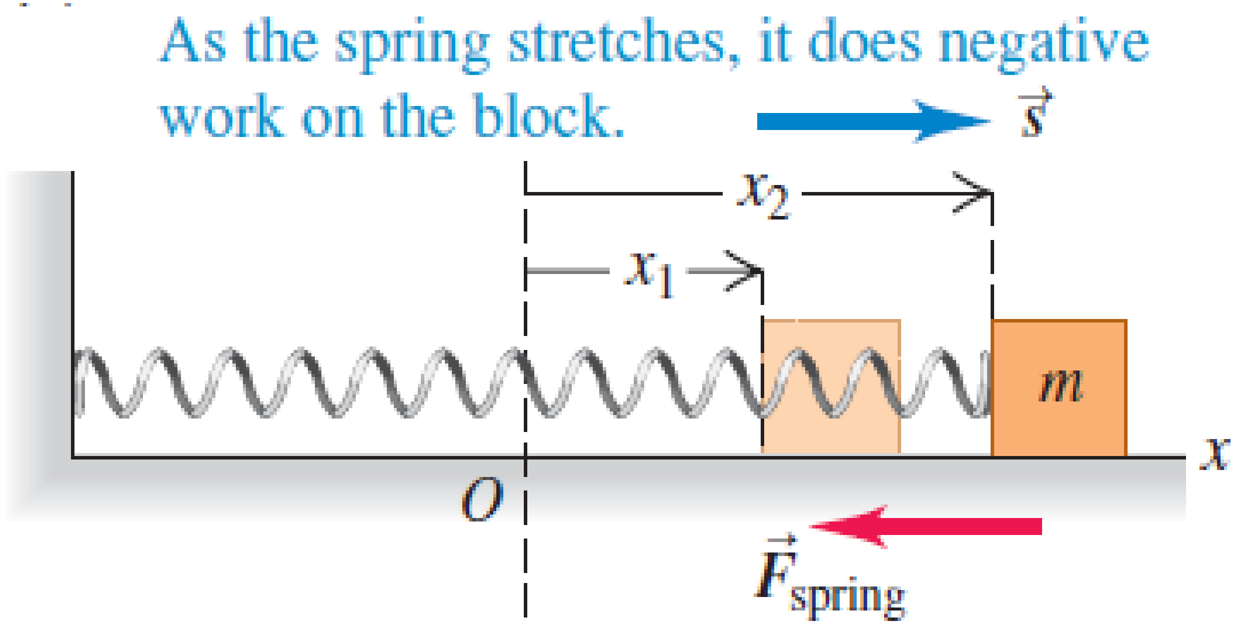
Elastic Potential Energy

We found in Section 6.3 that the work we must do *on* the spring to move one end from an elongation x_1 to a different elongation x_2 is

$$W = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 \qquad \text{(work done } on \text{ a spring)}$$

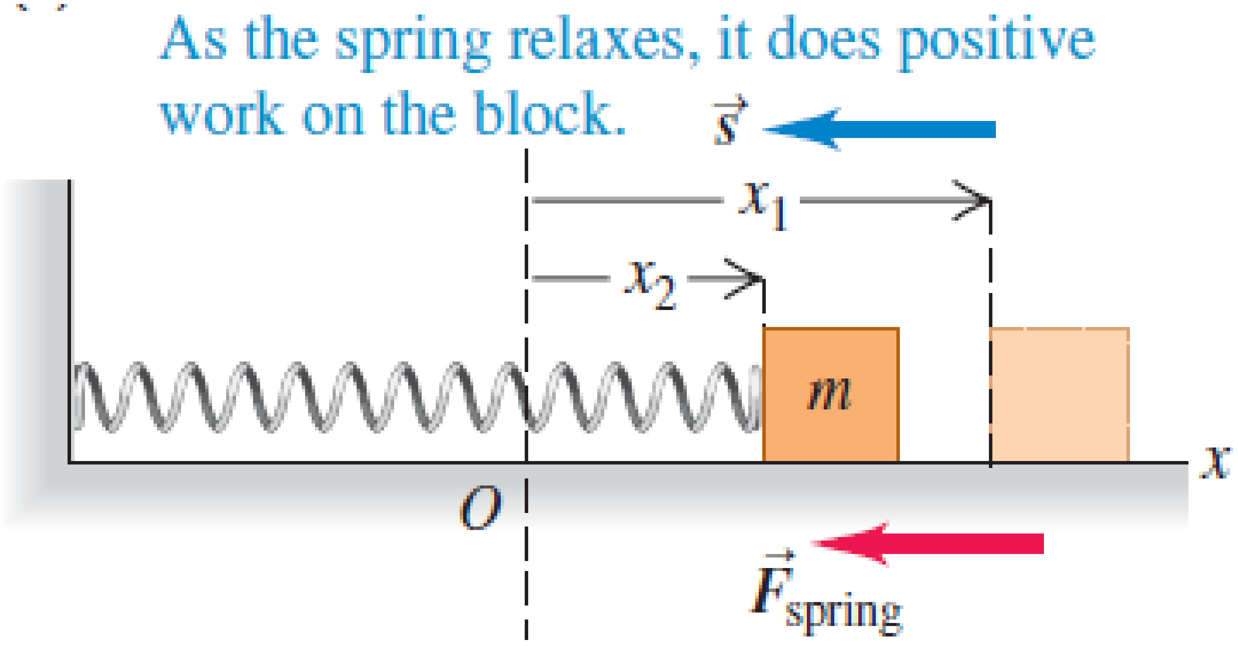
$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \qquad \text{(work done } by \text{ a spring)}$$

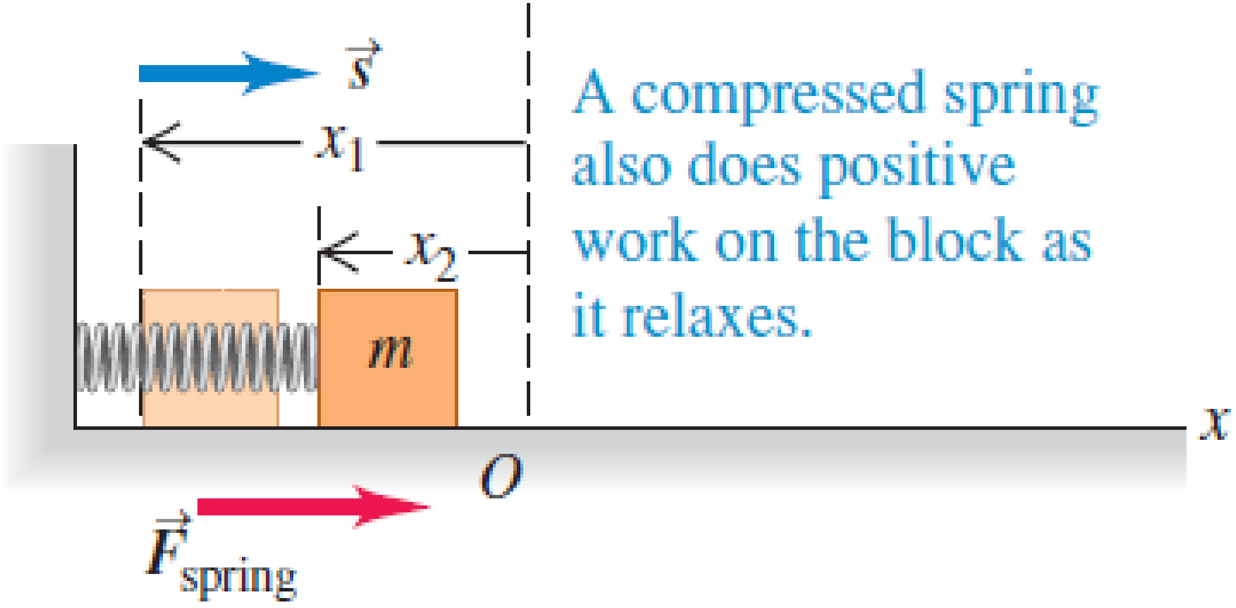




Elastic Potential Energy

$$W_{el} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 = U_{el,1} - U_{el,2} = -\Delta U_{el}$$





Elastic Potential Energy

CAUTION **Gravitational potential energy vs. elastic potential energy** An important difference between gravitational potential energy $U_{grav} = mgy$ and elastic potential energy $U_{el} = \frac{1}{2}kx^2$ is that we do *not* have the freedom to choose $x = 0$ to be wherever we wish. To be consistent with Eq. (7.9), $x = 0$ *must* be the position at which the spring is neither stretched nor compressed. At that position, its elastic potential energy and the force that it exerts are both zero.

The work–energy theorem says that $W_{tot} = K_2 - K_1$, no matter what kind of forces are acting on a body. If the elastic force is the *only* force that does work on the body, then

$$W_{tot} = W_{el} = U_{el,1} - U_{el,2}$$

Elastic Potential Energy

The work–energy theorem, $W_{tot} = K_2 - K_1$, then gives us

$$K_1 + U_{el,1} = K_2 + U_{el,2} \qquad \text{(if only the elastic force does work)} \qquad (7.11)$$

Here U_{el} is given by Eq. (7.9), so

$$\tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kx_2^2 \qquad \text{(if only the elastic force does work)} \qquad (7.12)$$

In this case the total mechanical energy $E = K + U_{el}$—the sum of kinetic and *elastic* potential energy—is *conserved*. An example of this is the motion of the

Situations with Both Gravitational and Elastic Potential Energy

$W_{tot} = W_{grav} + W_{el} + W_{other}$. Then the work–energy theorem gives

$$W_{grav} + W_{el} + W_{other} = K_2 - K_1$$

The work done by the gravitational force is $W_{grav} = U_{grav,1} - U_{grav,2}$ and the work done by the spring is $W_{el} = U_{el,1} - U_{el,2}$. Hence we can rewrite the work–energy theorem for this most general case as

$$K_1 + U_{grav,1} + U_{el,1} + W_{other} = K_2 + U_{grav,2} + U_{el,2} \quad \text{(valid in general)} \quad (7.13)$$

or, equivalently,

$$K_1 + U_1 + W_{other} = K_2 + U_2 \quad \text{(valid in general)} \quad (7.14)$$

Situations with Both Gravitational and Elastic Potential Energy

$$K_1 + U_1 + W_{\text{other}} = K_2 + U_2 \qquad \text{(valid in general)} \qquad (7.14)$$

where $U = U_{\text{grav}} + U_{\text{el}} = mgy + \frac{1}{2}kx^2$ is the *sum* of gravitational potential energy and elastic potential energy. For short, we call U simply "the potential energy."

Equation (7.14) is *the most general statement* of the relationship among kinetic energy, potential energy, and work done by other forces. It says:

The work done by all forces other than the gravitational force or elastic force equals the change in the total mechanical energy $E = K + U$ of the system, where $U = U_{\text{grav}} + U_{\text{el}}$ is the sum of the gravitational potential energy and the elastic potential energy.

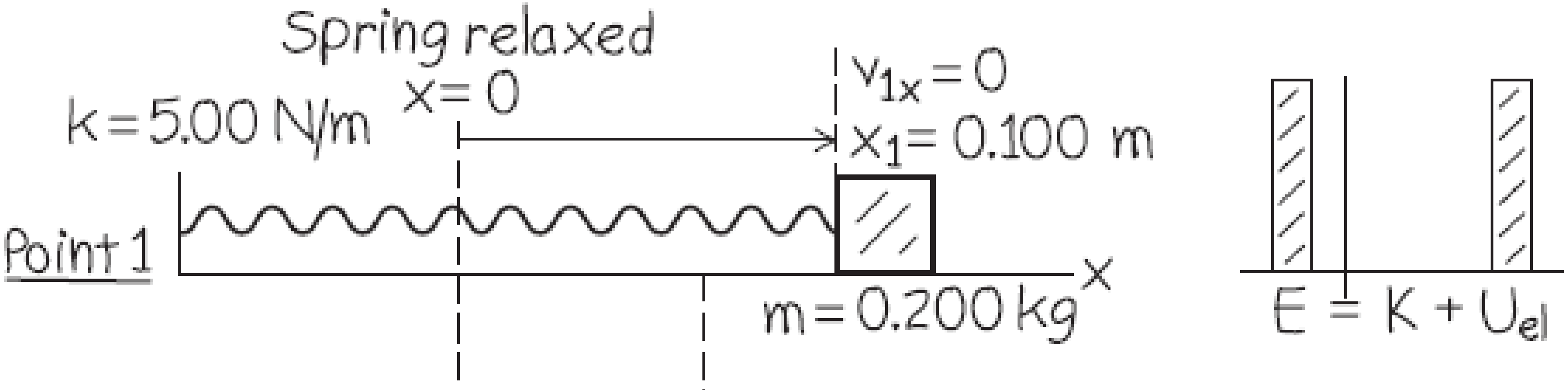
Motion with elastic potential energy

A glider with mass $m = 0.200$ kg sits on a frictionless horizontal air track, connected to a spring with force constant $k = 5.00$ N/m. You pull on the glider, stretching the spring 0.100 m, and release it from rest. The glider moves back toward its equilibrium position ($x = 0$). What is its x-velocity when $x = 0.080$ m?

$$K_1 = \tfrac{1}{2}mv_{1x}^2 = \tfrac{1}{2}(0.200\ \text{kg})(0)^2 = 0$$

$$U_1 = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(5.00\ \text{N/m})(0.100\ \text{m})^2 = 0.0250\ \text{J}$$

Motion with elastic potential energy



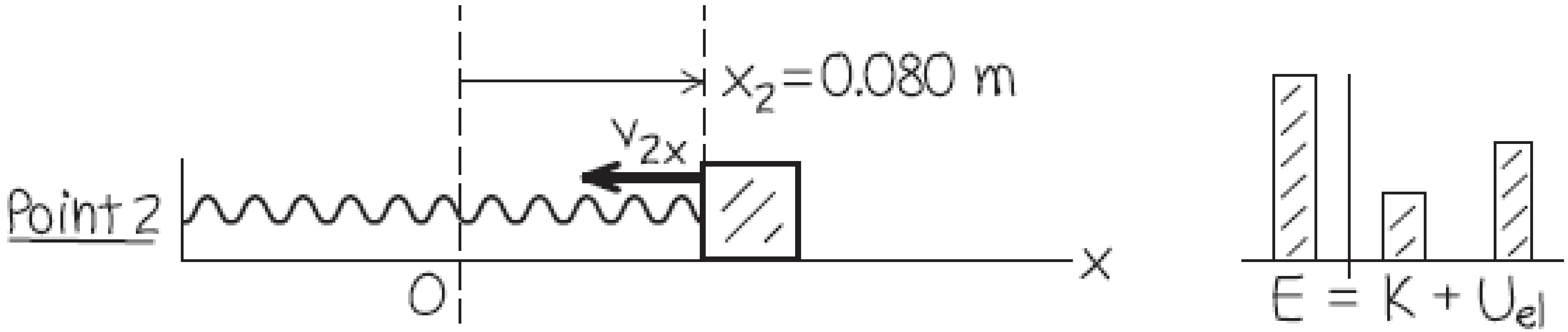
$$K_1 = \tfrac{1}{2}mv_{1x}{}^2 = \tfrac{1}{2}(0.200\ \mathrm{kg})(0)^2 = 0$$

$$U_1 = \tfrac{1}{2}kx_1{}^2 = \tfrac{1}{2}(5.00\ \mathrm{N/m})(0.100\ \mathrm{m})^2 = 0.0250\ \mathrm{J}$$

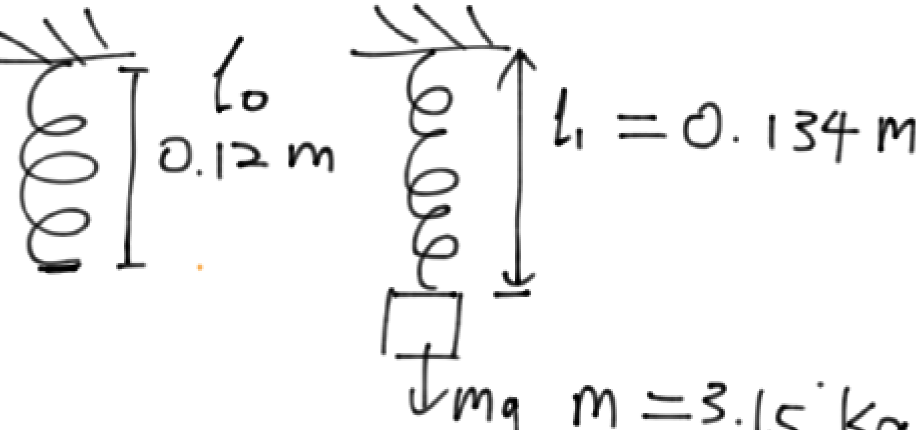
Motion with elastic potential energy

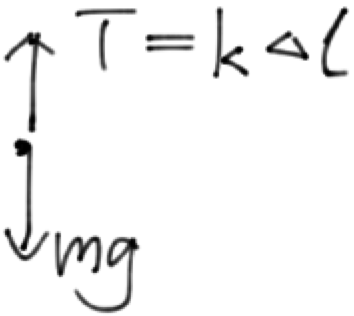
$$K_2 = \tfrac{1}{2}mv_{2x}^{\ 2}$$

$$U_2 = \tfrac{1}{2}kx_2^{\ 2} = \tfrac{1}{2}(5.00\ \mathrm{N/m})(0.080\ \mathrm{m})^2 = 0.0160\ \mathrm{J}$$



7.14 •• An ideal spring of negligible mass is 12.00 cm long when nothing is attached to it. When you hang a 3.15-kg weight from it, you measure its length to be 13.40 cm. If you wanted to store 10.0 J of potential energy in this spring, what would be its *total* length? Assume that it continues to obey Hooke's law.





$m = 3.15$ kg $\quad k = mg/\Delta l$

$$k = \frac{3.15\,kg \cdot 9.8\,m/s^2}{0.134\,m - 0.12\,m} = 2205\ N/m$$

$$U_{el} = \frac{1}{2}kx^2 = 10.0\,J$$

$$x = \sqrt{\frac{2U_{el}}{k}} = \sqrt{\frac{20.0\,J}{2205\,N/m}}$$

$$= 0.095\,m = 9.5\,cm$$

$$l' = l_0 \pm x = 21.5\,cm$$

Motion with elastic potential energy

We use Eq. (7.11) to solve for K_2 and then find v_{2x}:

$$K_2 = K_1 + U_1 - U_2 = 0 + 0.0250\text{ J} - 0.0160\text{ J} = 0.0090\text{ J}$$

$$v_{2x} = \pm\sqrt{\frac{2K_2}{m}} = \pm\sqrt{\frac{2(0.0090\text{ J})}{0.200\text{ kg}}} = \pm 0.30\text{ m/s}$$

We choose the negative root because the glider is moving in the $-x$-direction. Our answer is $v_{2x} = -0.30$ m/s.

Conservative vs Nonconservative Forces

Conservative Forces

A force that offers this opportunity of two-way conversion between kinetic and potential energies is called a **conservative force.**

Nonconservative Forces

Not all forces are conservative. Consider the friction force acting on the crate sliding on a ramp in Example 7.6 (Section 7.1). When the body slides up and then back down to the starting point, the total work done on it by the friction force is *not* zero. When the direction of motion reverses, so does the friction force, and friction does *negative* work in *both* directions.

Conservative vs Nonconservative Forces

The work done by a conservative force *always* has four properties:

1. It can be expressed as the difference between the initial and final values of a *potential-energy* function.
2. It is reversible.
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.

When the *only* forces that do work are conservative forces, the total mechanical energy $E = K + U$ is constant.

Conservative vs Nonconservative Forces

A force that is not conservative is called a **nonconservative force.** The work done by a nonconservative force *cannot* be represented by a potential-energy function. Some nonconservative forces, like kinetic friction or fluid resistance, cause mechanical energy to be lost or dissipated; a force of this kind is called a **dissipative force.** There are also nonconservative forces that *increase* mechanical energy. The fragments of an exploding firecracker fly off with very large kinetic energy, thanks to a chemical reaction of gunpowder with oxygen. The forces unleashed by this reaction are nonconservative because the process is not reversible. (The fragments never spontaneously reassemble themselves into a complete firecracker!)

Example 7.11

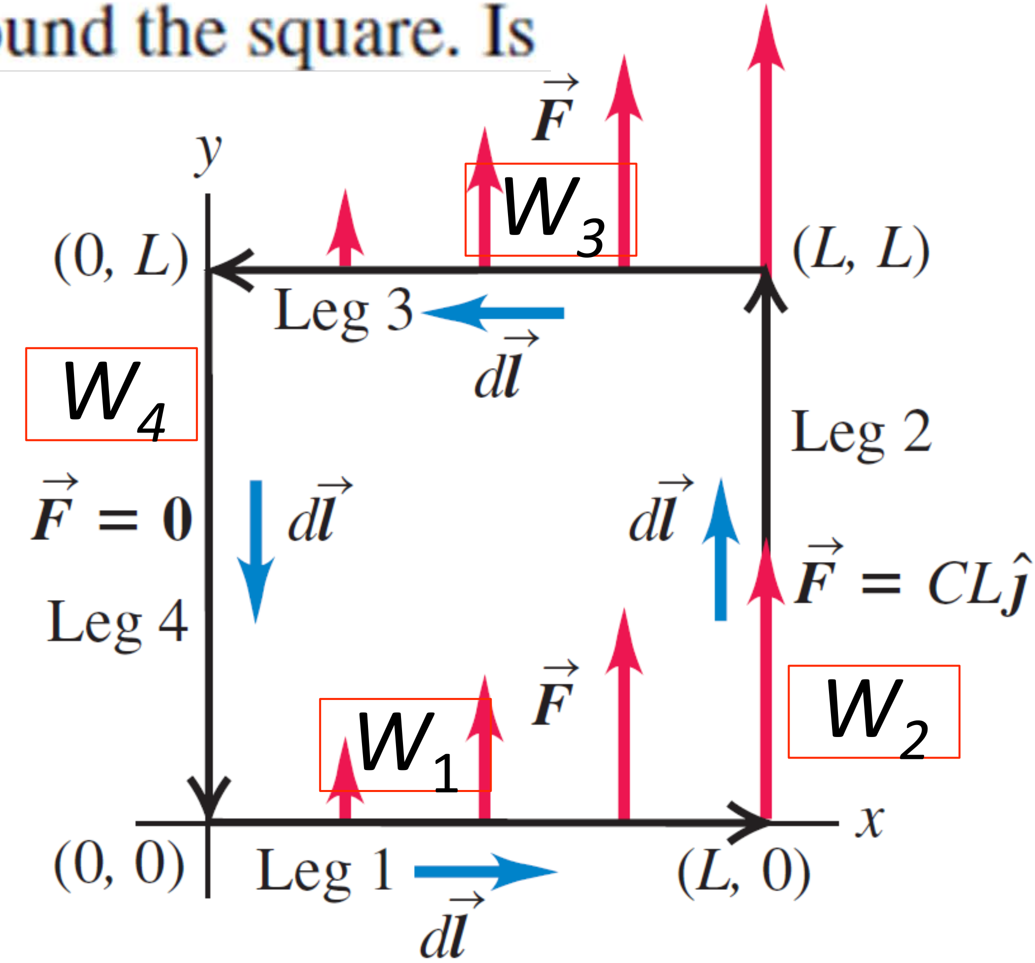
In a region of space the force on an electron is $\vec{F} = Cx\hat{j}$, where C is a positive constant. The electron moves around a square loop in the xy-plane (Fig. 7.20). Calculate the work done on the electron by the force $\vec{F}$ during a counterclockwise trip around the square. Is this force conservative or nonconservative?

The force $\vec{F}$ is not constant

not in the same direction as the displacement.

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l} = W_1 + W_2 + W_3 + W_4$$

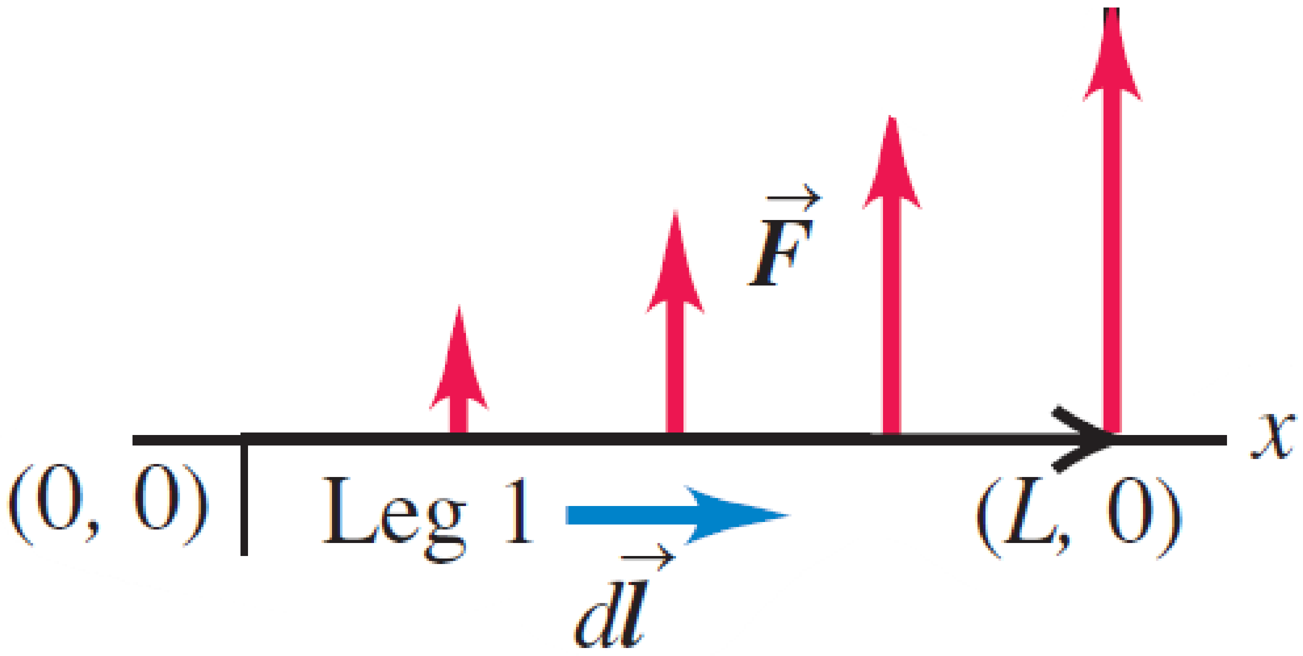
where $d\vec{l}$ is an infinitesimal displacement.



Example 7.11: W_1

EXECUTE: On the first leg, from (0, 0) to (L, 0), the force is everywhere perpendicular to the displacement. So $\vec{F} \cdot d\vec{l} = 0$, and the work done on the first leg is $W_1 = 0$. The force has the same value

$$\boxed{W_1 = 0}$$



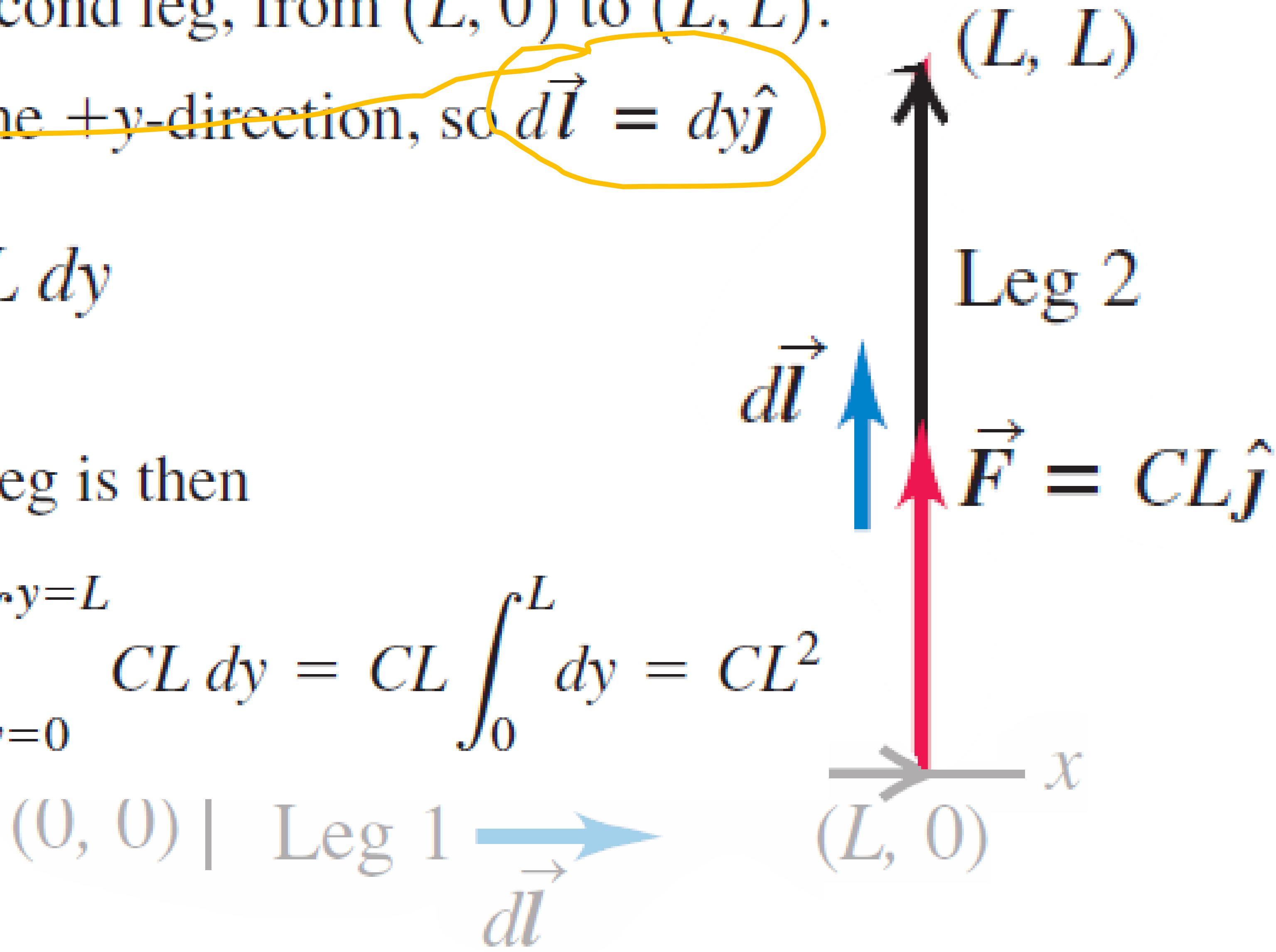
Example 7.11: W_2

$\vec{F} = CL\hat{j}$ everywhere on the second leg, from $(L, 0)$ to (L, L). displacement on this leg is in the $+y$-direction, so $d\vec{l} = dy\hat{j}$

$$\vec{F} \cdot d\vec{l} = CL\hat{j} \cdot dy\hat{j} = CL\,dy$$

The work done on the second leg is then

$$W_2 = \int_{(L,0)}^{(L,L)} \vec{F} \cdot d\vec{l} = \int_{y=0}^{y=L} CL\,dy = CL\int_0^L dy = CL^2$$

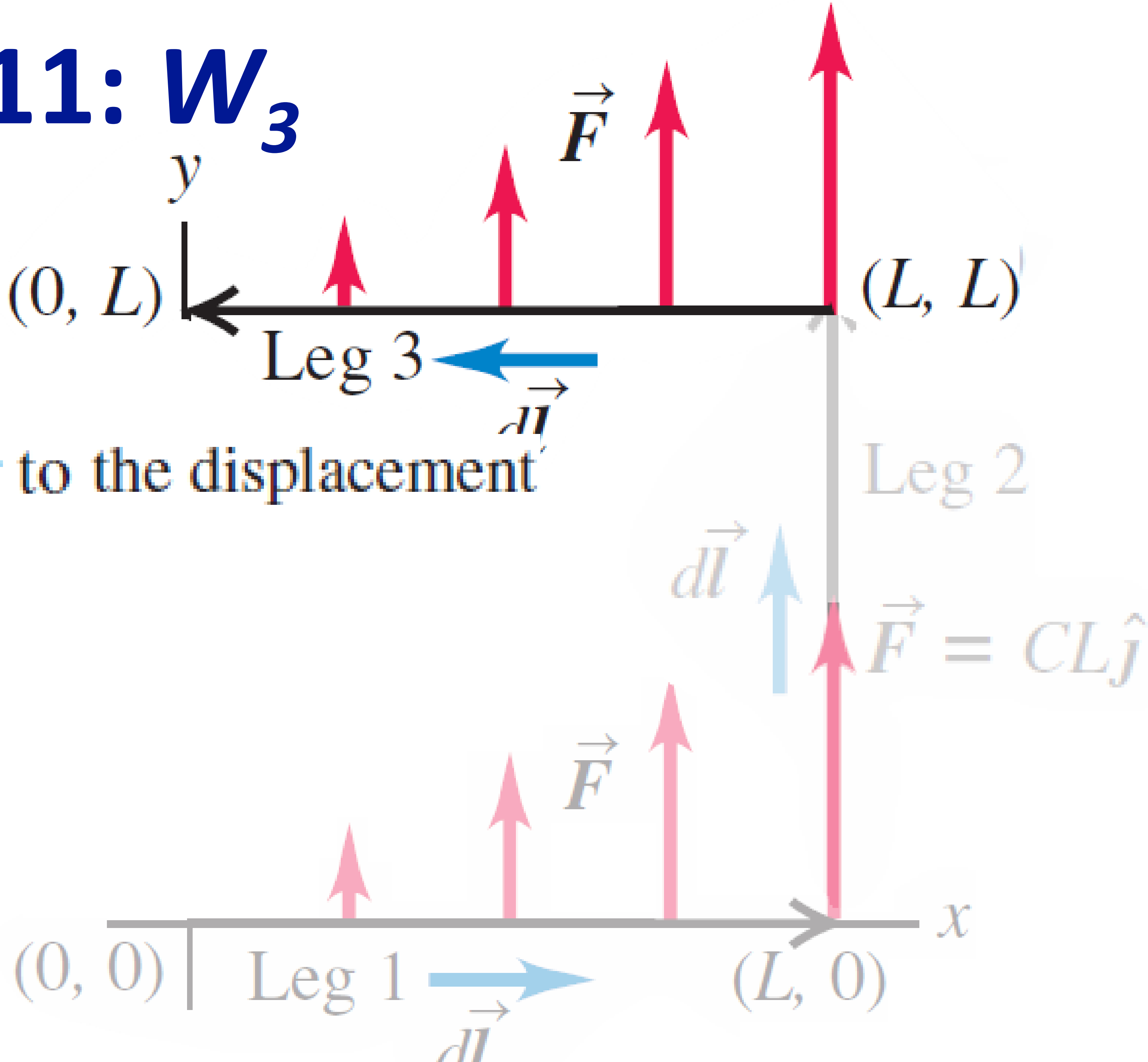


Example 7.11: W_3

from (L, L) to $(0, L)$.

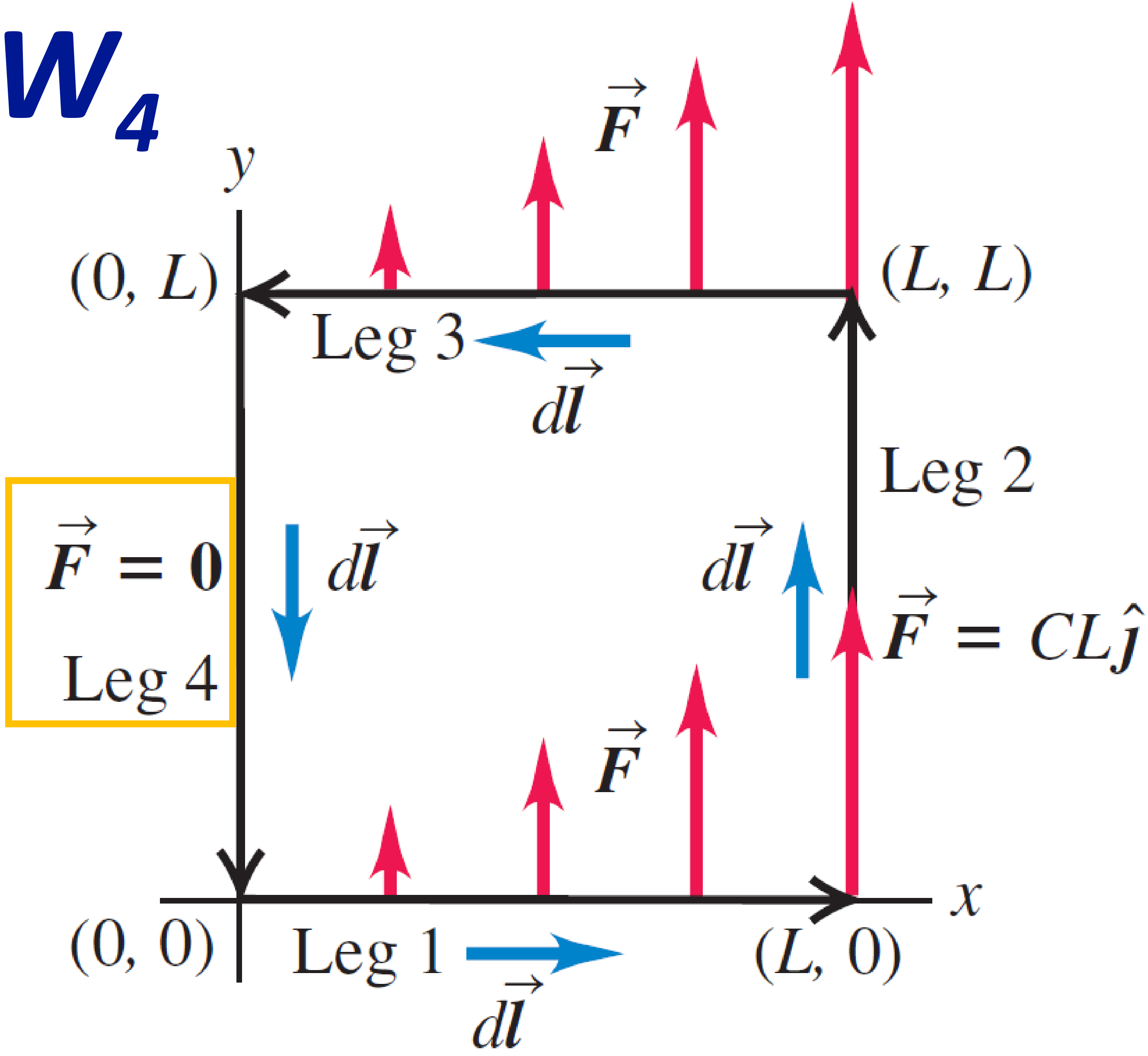
$\vec{F}$ is again perpendicular to the displacement

$$\boxed{\text{so } W_3 = 0}$$



Example 7.11: W_4

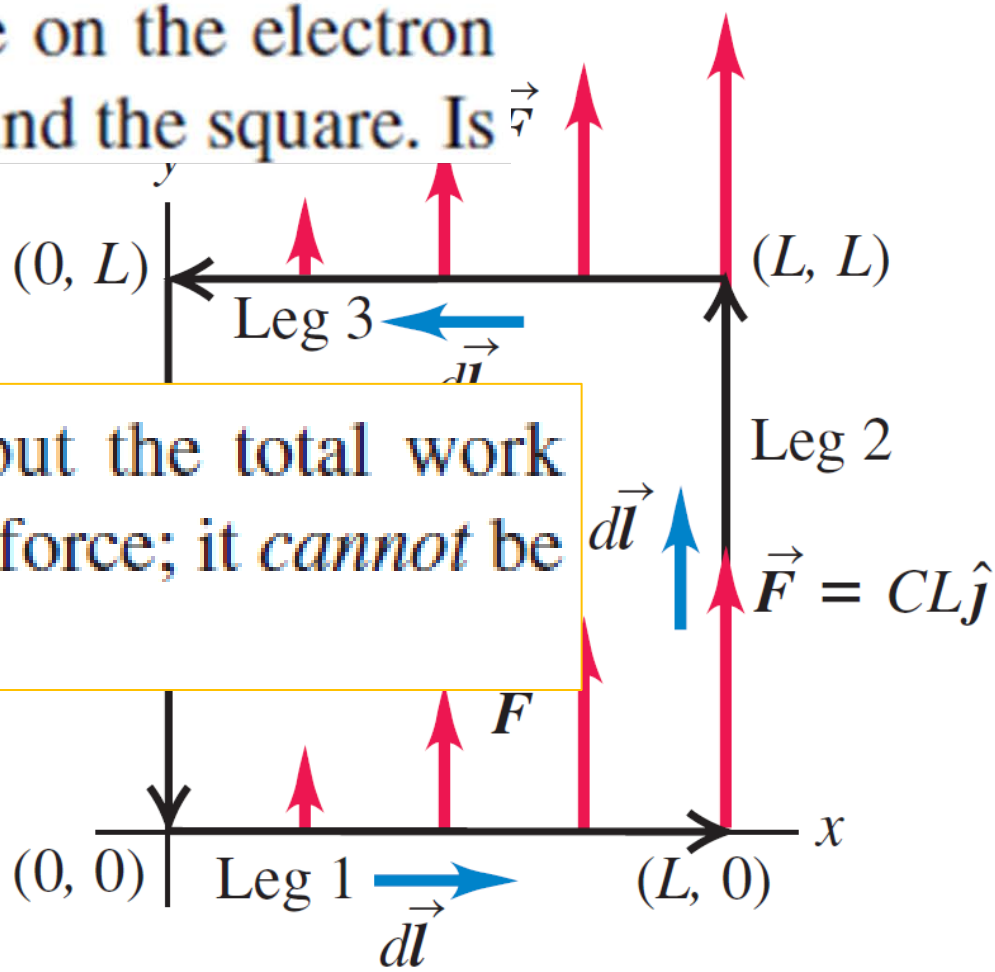
$$\boxed{W_4 = 0}$$



$$W = W_1 + W_2 + W_3 + W_4 = 0 + CL^2 + 0 + 0 = CL^2$$

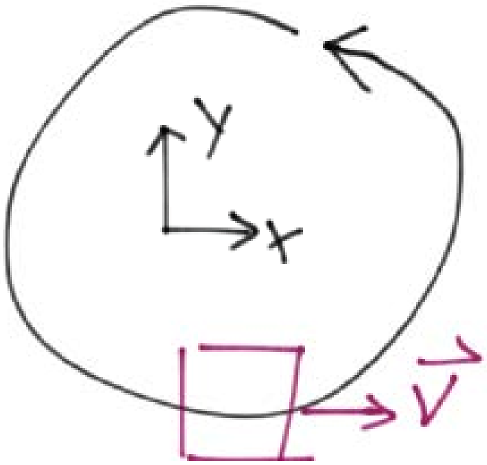
Example 7.11

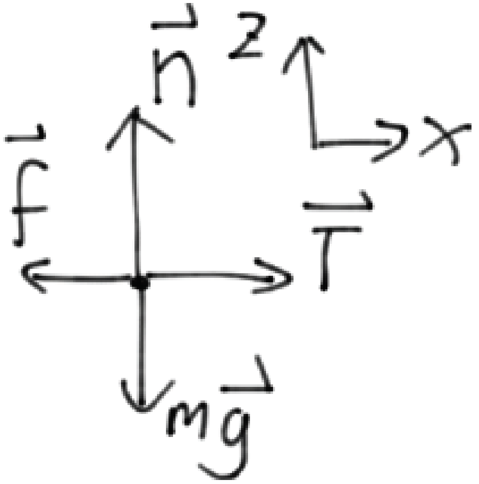
In a region of space the force on an electron is $\vec{F} = Cx\hat{j}$, where C is a positive constant. The electron moves around a square loop in the xy-plane (Fig. 7.20). Calculate the work done on the electron by the force $\vec{F}$ during a counterclockwise trip around the square. Is this force conservative or nonconservative?



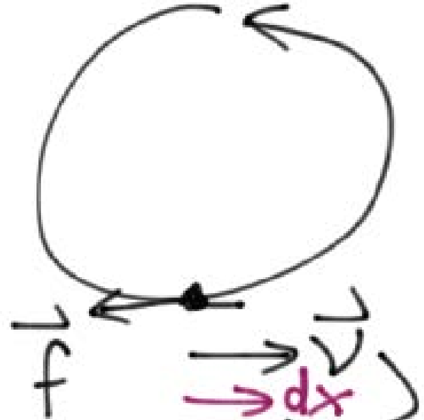
The starting and ending points are the same, but the total work done by $\vec{F}$ is not zero. This is a *nonconservative* force; it *cannot* be represented by a potential-energy function.

7.27 • A 10.0-kg box is pulled by a horizontal wire in a circle on a rough horizontal surface for which the coefficient of kinetic friction is 0.250. Calculate the work done by friction during one complete circular trip if the radius is (a) 2.00 m and (b) 4.00 m. (c) On the basis of the results you just obtained, would you say that friction is a conservative or nonconservative force? Explain.





$$\left.\begin{array}{l} f = \mu_k n \\ n = mg \end{array}\right\} f = \mu_k mg$$



"O" for closed circular motion

and $\vec{v} = \frac{\vec{x}}{dt}$

so $\hat{f}\cdot\hat{x} = -1$

$$W_f = \oint \vec{f}\cdot d\vec{x}$$
$$= -f\cdot 2\pi r$$
$$= -2\pi r\cdot \mu_k mg$$

(a) $W_{f,a} = -2\cdot 3.14\cdot 2.0m\cdot 0.25\cdot 10.0kg\cdot 9.8m/s^2$
$= -307.7J$

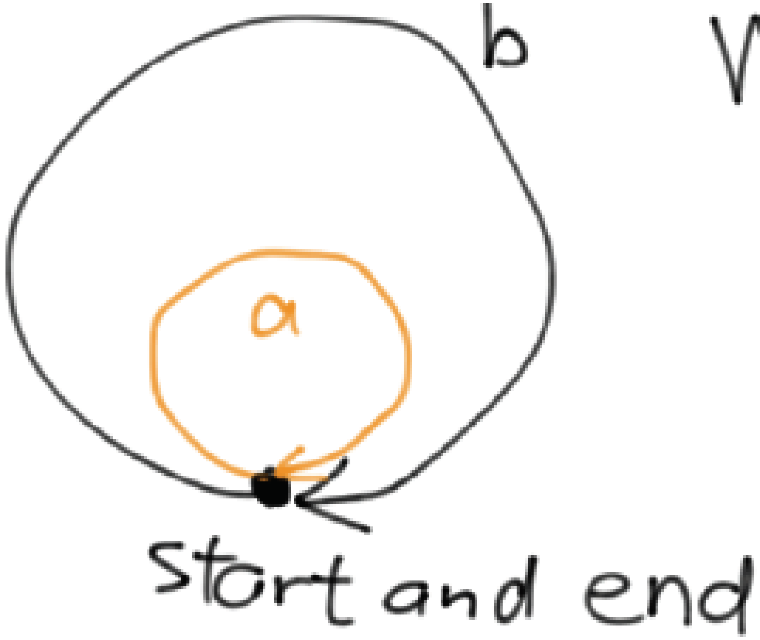
$W_{f,b} = 2W_{fa} = -615.4J$

(c) Non conservative

When answering these types of questions go back to the definition:

The work done by a conservative force *always* has four properties:

1. It can be expressed as the difference between the initial and final values of a *potential-energy* function.
2. It is reversible.
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.



$W_{fa} \neq W_{f_b}$ even when start & end points are the same

Law of Conservation of Energy

Internal Energy

- When a car with locked brakes skids to a stop, the tires and the road surface both become hotter
- Raising the temperature of a body increases its internal energy
- Friction does negative work on the block as it slides; the change in internal energy of the block and surface (both of which get hotter) is positive.

Law of Conservation of Energy

Careful experiments show that the increase in the internal energy is *exactly* equal to the absolute value of the work done by friction.

$$\Delta U_{\text{int}} = -W_{\text{other}}$$

where ΔU_{int} is the change in internal energy.

7.21 When 1 liter of gasoline is burned in an automotive engine, it releases 3.3×10^7 J of internal energy. Hence $\Delta U_{\text{int}} = -3.3 \times 10^7$ J, where the minus sign means that the amount of energy stored in the gasoline has decreased. This energy can be converted to kinetic energy (making the car go faster) or to potential energy (enabling the car to climb uphill).

Law of Conservation of Energy

This remarkable statement is the general form of the **law of conservation of energy.** In a given process, the kinetic energy, potential energy, and internal energy of a system may all change. But the *sum* of those changes is always zero. If there is a decrease in one form of energy, it is made up for by an increase in the other forms (Fig. 7.21). When we expand our definition of energy to include internal energy, Eq. (7.15) says: *Energy is never created or destroyed; it only changes form.* No exception to this rule has ever been found.

Writing $\Delta K = K_2 - K_1$ and $\Delta U = U_2 - U_1$, we can finally express this as

$$\Delta K + \Delta U + \Delta U_{int} = 0 \qquad \text{(law of conservation of energy)} \qquad (7.15)$$

Force and Potential Energy

For the two kinds of conservative forces (gravitational and elastic)

We derived from forces to potential energy

Gravitational:

$$F_y = -mg$$

$$U(y) = mgy$$

Elastic:

$$F_x = -kx$$

$$U(x) = \tfrac{1}{2}kx^2$$

In studying physics, however, you'll encounter situations in which you are ***given*** an expression for the ***potential energy*** as a function of position and have to ***find*** the corresponding ***force***.

Force and Potential Energy

First let's consider motion along a straight line, with coordinate x. We denote the x-component of force, a function of x, by $F_x(x)$, and the potential energy as $U(x)$. This notation reminds us that both F_x and U are *functions* of x. Now we recall that in any displacement, the work W done by a conservative force equals the negative of the change ΔU in potential energy:

$$W = -\Delta U$$

Let's apply this to a small displacement Δx. The work done by the force $F_x(x)$ during this displacement is approximately equal to $F_x(x)\,\Delta x$. We have to say "approximately" because $F_x(x)$ may vary a little over the interval Δx. But it is at least approximately true that

$$F_x(x) = -\frac{dU(x)}{dx} \qquad \text{(force from potential energy, one dimension)}$$

Force and Potential Energy

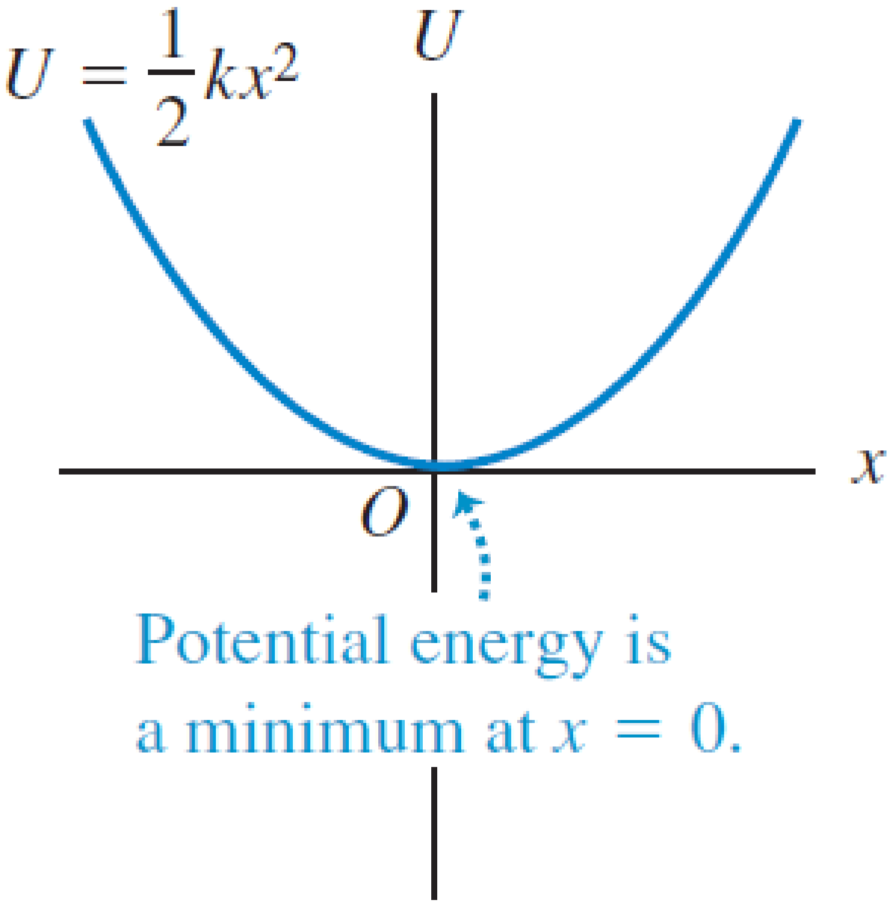
$$F_x(x) = -\frac{dU(x)}{dx} \qquad \text{(force from potential energy, one dimension)} \qquad (7.16)$$

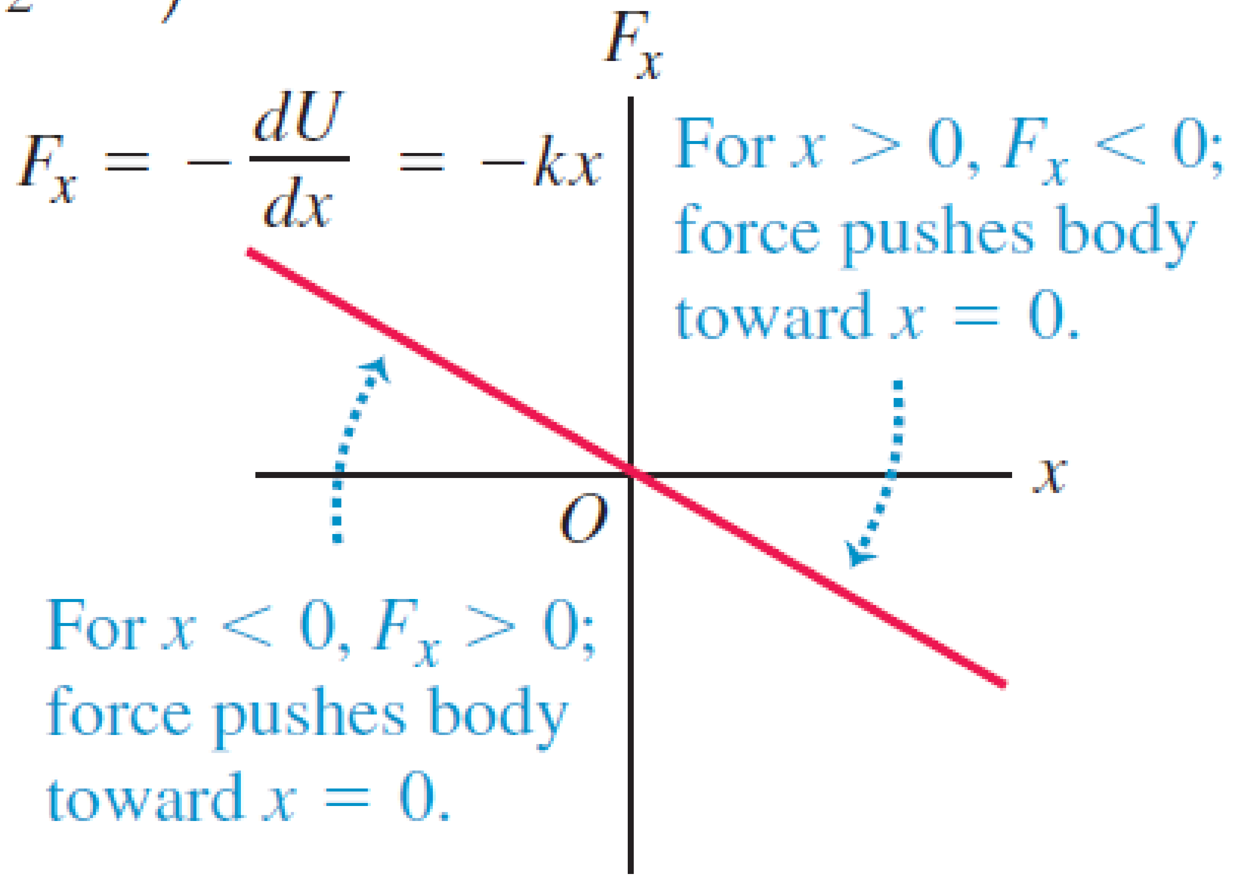
This result makes sense; in regions where $U(x)$ changes most rapidly with x (that is, where $dU(x)/dx$ is large), the greatest amount of work is done during a given displacement, and this corresponds to a large force magnitude. Also, when $F_x(x)$ is in the positive x-direction, $U(x)$ *decreases* with increasing x. So $F_x(x)$ and $dU(x)/dx$ should indeed have opposite signs. The physical meaning of Eq. (7.16) is that *a conservative force always acts to push the system toward lower potential energy.*

Force and Elastic Potential Energy

(a) Spring potential energy and force as functions of x

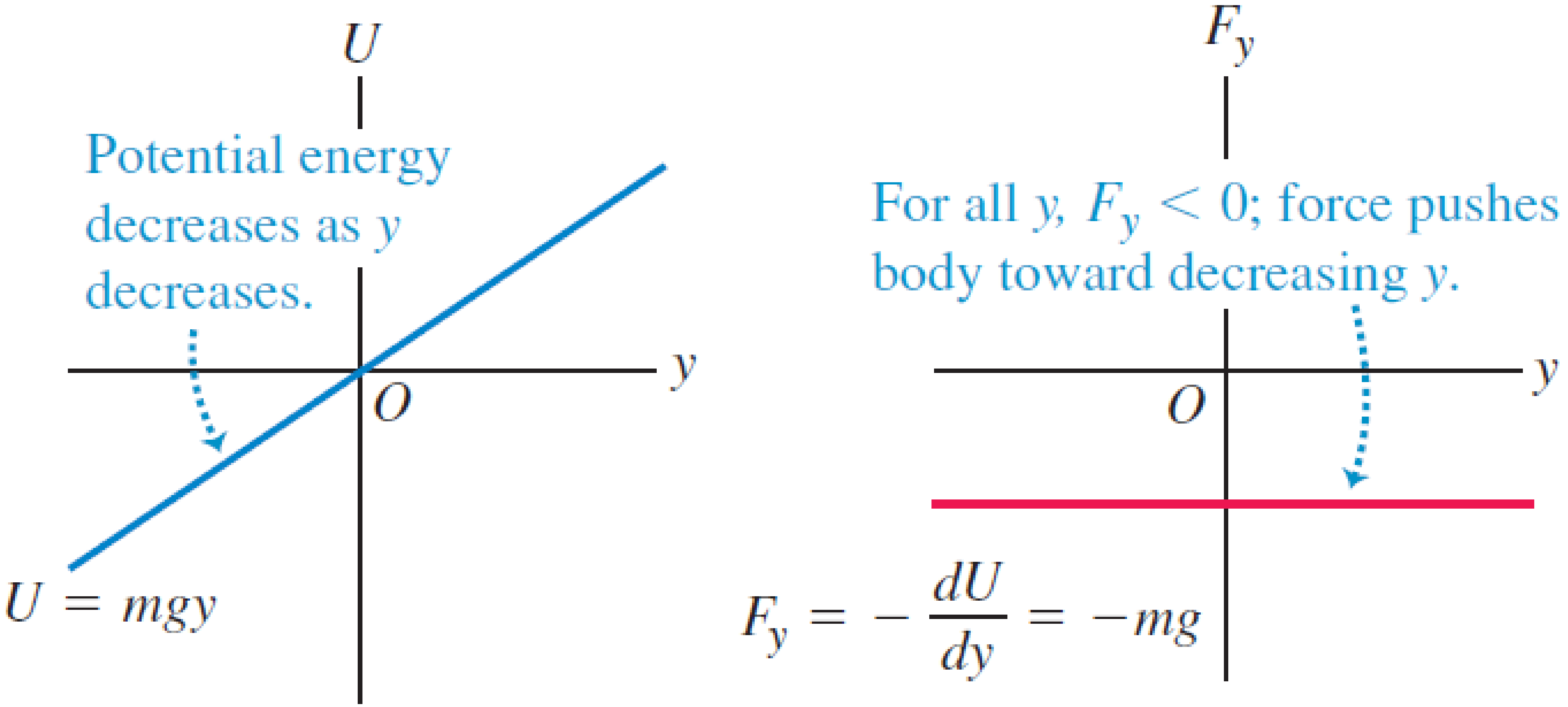
$$F_x(x) = -\frac{d}{dx}\left(\tfrac{1}{2}kx^2\right) = -kx$$





Force and Gravitational Potential Energy

$$F_y = -dU/dy = -d(mgy)/dy = -mg$$



Force and Potential Energy in 3D

The potential-energy change ΔU when the particle moves a small distance Δx in the x-direction is again given by $-F_x \Delta x$; it doesn't depend on F_y and F_z, which represent force components that are perpendicular to the displacement and do no work. So we again have the approximate relationship

$$F_x = -\frac{\Delta U}{\Delta x}$$

The y- and z-components of force are determined in exactly the same way:

$$F_y = -\frac{\Delta U}{\Delta y} \qquad F_z = -\frac{\Delta U}{\Delta z}$$

Approximations

Force and Potential Energy in 3D

$$F_x = -\frac{\Delta U}{\Delta x} \quad F_y = -\frac{\Delta U}{\Delta y} \quad F_z = -\frac{\Delta U}{\Delta z}$$

When we take the limits $\Delta x \to 0,\ \Delta y \to 0\ \ \Delta z \to 0$

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z}$$

This operation is called the **gradient** of U and is often abbreviated as $\vec{\nabla} U$.

$$\vec{\boldsymbol{F}} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) = -\vec{\nabla} U$$

Example 7.14

A puck with coordinates x and y slides on a level, frictionless air-hockey table. It is acted on by a conservative force described by the potential-energy function

$$U(x, y) = \tfrac{1}{2}k(x^2 + y^2)$$

Find a vector expression for the force acting on the puck, and find an expression for the magnitude of the force.

The x- and y-components of $\vec{F}$ are

$$F_x = -\frac{\partial U}{\partial x} = -kx \qquad F_y = -\frac{\partial U}{\partial y} = -ky$$

Example 7.14

The x- and y-components of $\vec{F}$ are

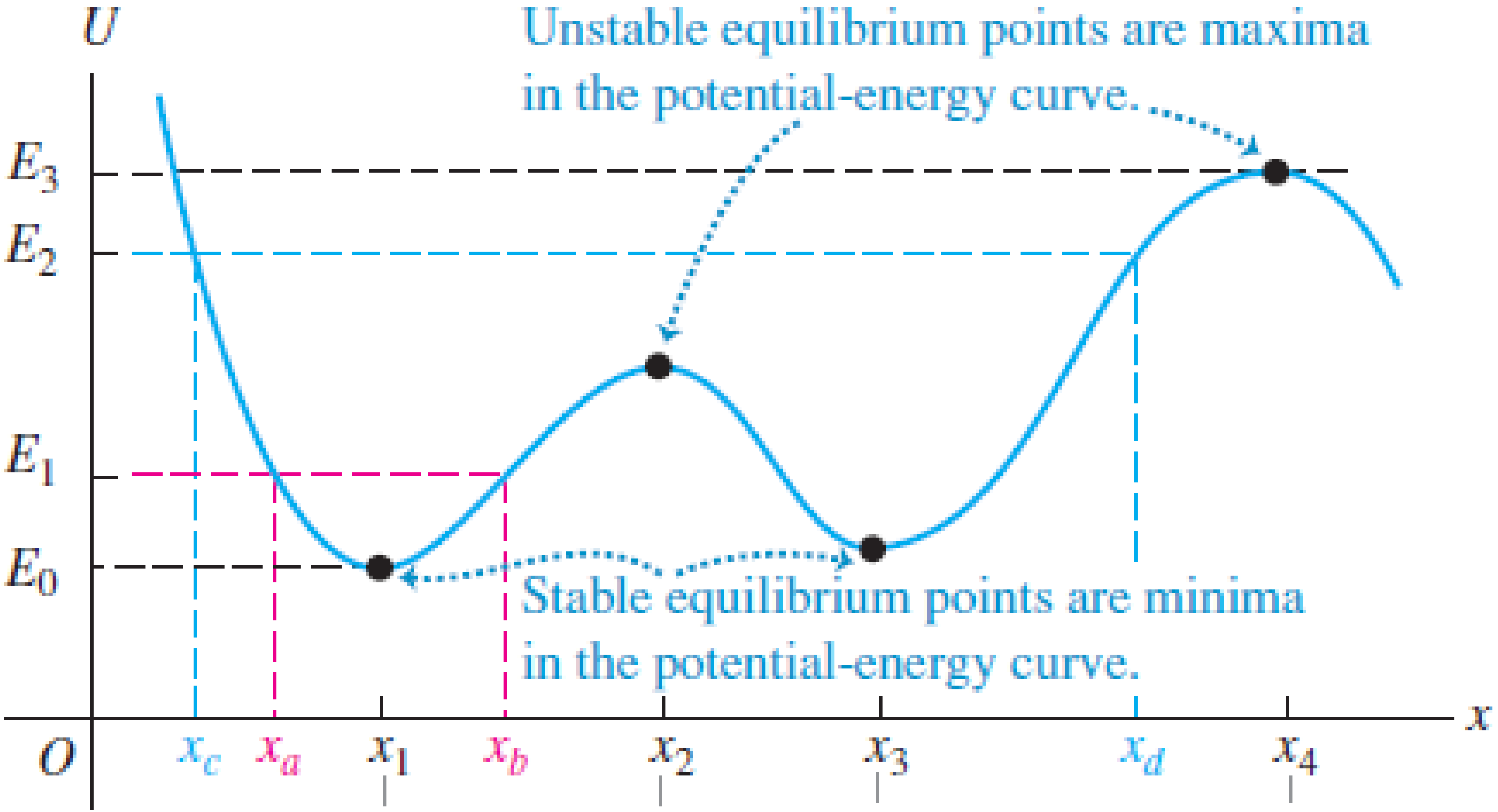
$$F_x = -\frac{\partial U}{\partial x} = -kx \qquad F_y = -\frac{\partial U}{\partial y} = -ky$$

From Eq. (7.18), the vector expression for the force is

$$\vec{F} = (-kx)\hat{\imath} + (-ky)\hat{\jmath} = -k(x\hat{\imath} + y\hat{\jmath})$$

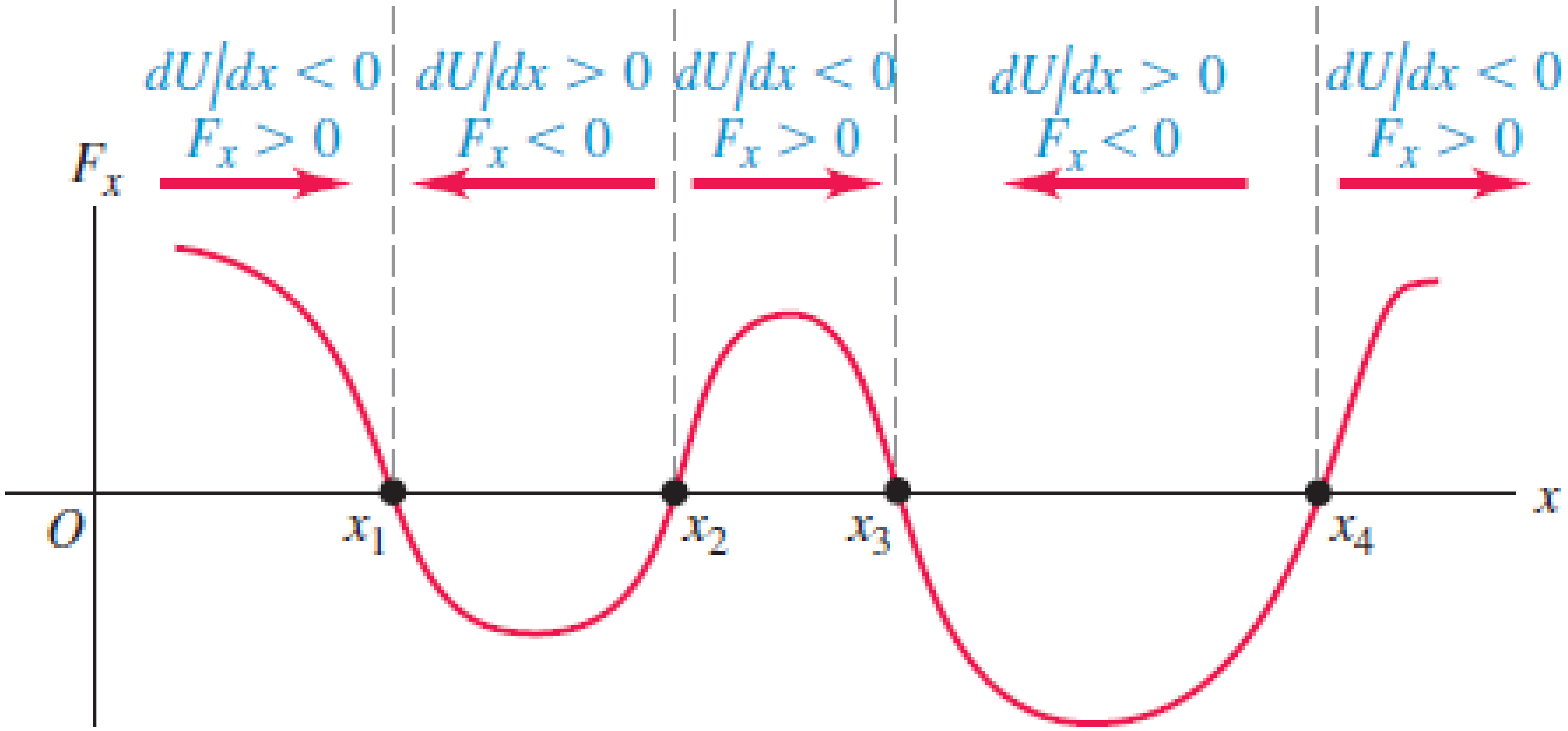
So: $$F = \sqrt{(-kx)^2 + (-ky)^2} = k\sqrt{x^2 + y^2}$$

Energy Diagrams



Energy Diagrams

(b) The corresponding x-component of force $F_x(x) = -dU(x)/dx$



Summary

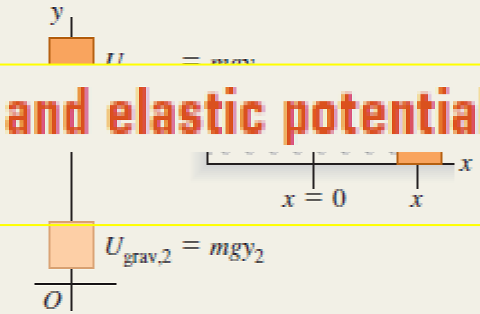
Gravitational potential energy and elastic potential energy:

$W_{grav} = mgy_1 - mgy_2$

$= U_{grav,1} - U_{grav,2}$

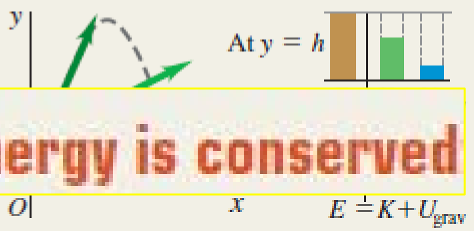
$x_1^2 - \frac{1}{2}kx_2^2$ (7.10)

$= U_{el,1} - U_{el,2} = -\Delta U_{el}$



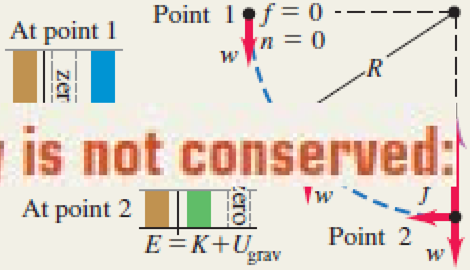
When total mechanical energy is conserved

$K_1 + U_1 = K_2 + U_2$ (7.4), (7.11)



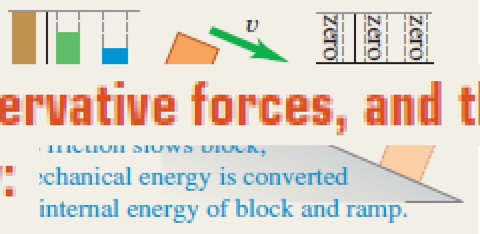
When total mechanical energy is not conserved:

$K_1 + U_1 + W_{other} = K_2 + U_2$ (7.14)



Conservative forces, nonconservative forces, and the law of conservation of energy:

$\Delta K + \Delta U + \Delta U_{int} = 0$ (7.15)



Determining force from potential energy:

$F_x(x)$ dx

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad (7.17)$$

$$F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) \qquad (7.18)$$