Linear Algebra Mid Term Exam

🤓 Disclaimer:

I only made this as a reference, may be for a cross-check or for the general idea, from what
I remember from the exam. Although I checked multiple times, there could still be errors and mistakes. **The questions could have also been different.**

Also, any of the method used is only from my own preference and there are (probably) different methods to solve the questions.

Question 2

v and w fill a plane in xyz space

$$v = (1, 2, 3) \text{ and } w = (3, -2, 1)$$

1. Find the linear combination equation of v and w with components c and d

$$c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{matrix} c + 3d \\ 2c - 2d \\ 3c + d \end{matrix}$$

2. Verify that the zero vector $(0, 0, 0)$ lie on the plane of $cv + dw$

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- You can turn the matrix system into an upper triangular system:

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 0 & -8 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c + 3d = 0 \\ 2c - 2d = 0 \\ -8d = 0 \end{matrix}$$

- Here we can see that $d = 0$ and by back-substitution also $c = 0$
- So the zero vector $(0, 0, 0)$ exists on the plane of $cv + dw$ when $c = 0$ and $d = 0$

3. Verify that the vector $n = (1, 1, -1)$ is perpendicular to the plane

- If the dot product of two vectors is equal to zero, they are perpendicular.

$$v + w = (4, 0, 4)$$

$$(v + w) \cdot n = 4 + 0 + (-4) = 0$$

- So n is perpendicular to the plane.

4. Describe the plane by using three unknowns (very unsure to be correct)

- It is not possible to find three unknowns using the plane $v + w$ since there will be only **two pivots.**

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & -8 \\ 0 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & -8 \\ 0 & 0 \end{bmatrix}$$

- Also by matrix operation rule $(m \times n)(n \times p) = m \times p$, it is not possible to multiply the 3×2 system (plane of v and w) by 3×1 system (three unknowns)

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} \text{cannot be multipled by e.g. } \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Question 3

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & -2 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

1. Find the Upper Triangular System $Ux = c$ and Compute the solution for x

- We have to combine A and b into an augmented matrix and apply elimination to find the upper triangular system $Ux = c$

$$Ab = \begin{bmatrix} 1 & 3 & 1 & 1 \\ -1 & -2 & 1 & 4 \\ 3 & 7 & 5 & -1 \end{bmatrix}$$

- $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to eliminate the -1 (in in row#2) under the first pivot 1 (in row #1) by adding row#1 unto row#2:

$$E_{21}Ab = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 3 & 7 & 5 & -1 \end{bmatrix}$$

- $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ to eliminate the 3 (in in row#3) under the first pivot 1 (in row #1) by adding -3 of row#1 to row#3:

$$E_{31}Ab = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 2 & -4 \end{bmatrix}$$

- $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ to eliminate the -2 (in in row#3) under the second pivot 1 (in row #2) by adding 2 of row#2 to row#3:

$$E_{32}Ab = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 6 & 6 \end{bmatrix} = Uc$$

- Since we now have the Upper Triangular system Uc we can deduce $Ux = c$:

$$Ux = c <=> \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$$

- Which, after matrix-vector multiplication, gives:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= 5 \\ 6x_3 &= 6 \end{aligned}$$

- We can use back-substitution to easily find $x = (x_1, x_2, x_3)$:

$$\begin{aligned} x_1 &= -9 \\ x_2 &= 2 \\ x_3 &= 1 \end{aligned}$$

2. Find the inverse of matrix A (find A^{-1})

- Use Gauss-Jordan method to find the inverse (turn the left side into an identity matrix):

$$\begin{bmatrix} 1 & 3 & 1 & & 1 & 0 & 0 \\ -1 & -2 & 1 & & 0 & 1 & 0 \\ 3 & 7 & 5 & & 0 & 0 & 1 \end{bmatrix}$$

- We can use the Upper Triangular system U from part 1. If we turn the three elimination steps from part 1 into one elimination matrix, we have $E_{32}E_{31}E_{21} = E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$ and we can write this on the right side with U on the left side.

$$\begin{bmatrix} 1 & 3 & 1 & & 1 & 0 & 0 \\ 0 & 1 & 2 & & 1 & 1 & 0 \\ 0 & 0 & 6 & & -1 & 2 & 1 \end{bmatrix}$$

- From here we can divide row#3 and eliminate the components above it (subtract 2 of new row#3 from row#2 and 1 of new row#3 from row#1):

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{6} & \frac{2}{6} & \frac{1}{6} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{7}{8} & -\frac{2}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{8}{6} & \frac{2}{6} & -\frac{1}{6} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{2}{6} & \frac{1}{6} \end{array}\right]$$

- Finally we have to subtract 3 of new row#2 from row#1 to eliminate the 3:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{17}{6} & -\frac{8}{6} & \frac{2}{6} \\ 0 & 1 & 0 & \frac{8}{6} & \frac{2}{6} & -\frac{1}{6} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{2}{6} & \frac{1}{6} \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} -\frac{17}{6} & -\frac{8}{6} & \frac{2}{6} \\ \frac{8}{6} & \frac{2}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{6} & \frac{1}{6} \end{bmatrix}$$

- You can multiply A and A^{-1} to check if it's correct since $AA^{-1} = I$

3. Find the Factorization $A = LU$

- From part 1 we know that the elimination matrix to produce U from A is:

$$E_{32}E_{31}E_{21} = E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

- L is just the inverse of the elimination done on A or simply the inverse of E:

$$L = E^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$

- I simply use the Gauss-Jordan method in part 2 to find the inverse of E which is L:

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ -1 & 2 & 1 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 1 & 0\\ 0 & 2 & 1 & 1 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 1 & 0\\ 0 & 0 & 1 & 3 & -2 & 1\end{array}\right] \longrightarrow L = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 3 & -2 & 1\end{bmatrix}$$

- You can also multiply $EL = I$ or $LU = A$ to check if it's correct.
  Now we just have to write $A = LU$ which is:

$$\begin{bmatrix}1 & 3 & 1\\ -1 & -2 & 1\\ 3 & 7 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 3 & -2 & 1\end{bmatrix} \begin{bmatrix}1 & 3 & 1\\ 0 & 1 & 2\\ 0 & 0 & 6\end{bmatrix}$$