CALCULUS

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3.7 Implicit Differentiation

- Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. For example,

$$y = x^2 + x + 1 \quad \Rightarrow \quad y' = 2x + 1$$
$$y = \sin x + x^3 \quad \Rightarrow \quad y' = \cos x + 3x^2$$

- However, for the equations like $x^2 + y^2 - 25 = 0$ or $x^3 + y^3 - 9xy = 0$, they define an implicit relation between the variables x and y, meaning that a value of x determines one or more values of y.
- When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by **implicit differentiation**.

3.7 Implicit Differentiation

① Implicit Differentiation

1) Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2) Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Example 1 Find dy/dx if $y^2 = x$.

Example 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Example 3 Find dy/dx if $y^2 = x^2 + \sin xy$.

② Derivatives of Higher Order

- Implicit differentiation can also be used to find higher derivatives.

Example 4 Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Example 5 Find d^2y/dx^2 if $x^{2/3} + y^{2/3} = 1$.

Example 6 Find d^2y/dx^2 if $x^2 - y^2 + \sin x - y = 0$.

③ Tangent and Normal lines

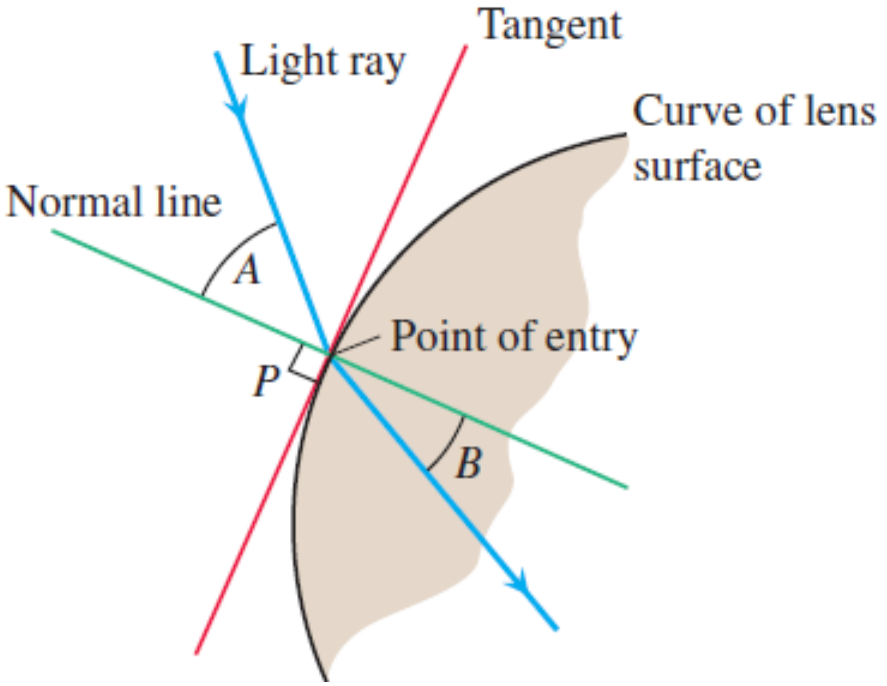
Recall:

- Two lines are orthogonal $\Rightarrow$ the product of their slopes is -1.
- The **normal** is the line perpendicular to the tangent of the curve at the point of entry.

- For a given curve $y = f(x)$, the tangent line and normal line to the curve at point (x_0, y_0) are

Tangent line: $y - y_0 = f'(x_0)(x - x_0)$.

Normal line: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$.



Example 7

The folium of Descartes is expressed as:

$$x^3 + y^3 - 9xy = 0$$

a) Show that the point (2, 4) lies on the curve of the folium of Descartes.

b) Find the tangent and normal to the curve there.

c) At what point other than the origin does the folium have a horizontal tangent?

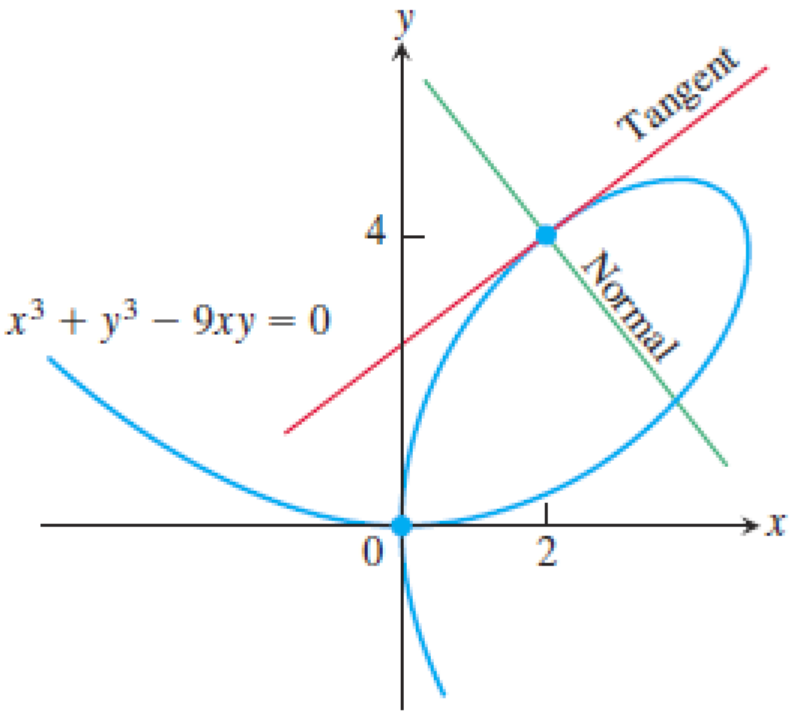


FIGURE 3.32 Example 5 shows how to find equations for the tangent and normal to the folium of Descartes at (2,4).

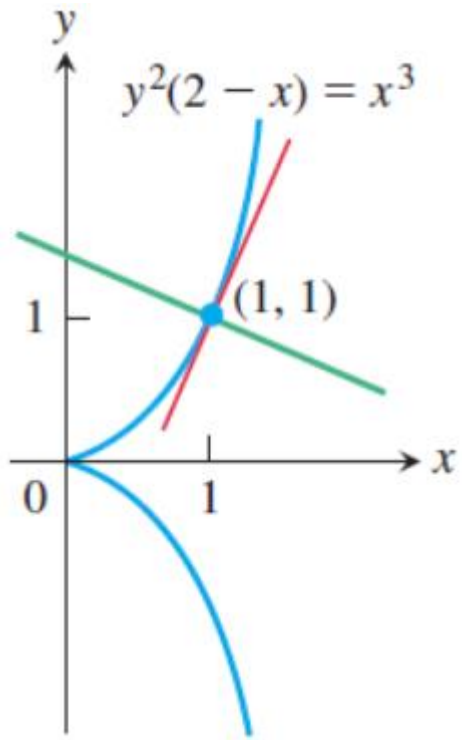
Skill Practice 1

The cissoid of Diocles (from about 200 B.C.) is expressed as:

$$y^2(2-x) = x^3$$

a) Find equations for the tangent and normal to the cissoid of Diocles at (1, 1) and (1, –1).

b) Find y''.



Skill Practice 2 Find dy/dx for

(a) $y^3 = 3xy + x^3$ (b) $y^2 = x\sin x$

Skill Practice 3 Find d^2y/dx^2 for

(a) $xy + y^2 = 1$ (b) $\sin y + 2\cos y = x$