CALCULUS

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- In this chapter, we develop the concepts of limit and continuity to describe the way a function varies. The following sections will be covered.

① Average and Instantaneous Speed

Average Speed

When $f(t)$ measures the distance traveled at time t, The average speed over an interval of time $[t_1, t_2]$ can be expressed as:

$$\bar{v} = \frac{\text{traveled distance}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Example 1:

A rock breaks loose from the top of a tall cliff. The distance fallen in meters is denoted by y after t seconds, which can be expressed by $y = gt^2/2$. where $g = 9.8\ m/s^2$ is the gravitational acceleration. What is its average speed

(a) during the first 2 seconds of fall?

(b) during the 1-s interval between sec 1 and sec 2?

Instantaneous Speed

What if we want to determine the speed of a falling object at a single instant t_0, instead of the average speed over an interval of time?

Example 2:

Find the speed of the falling rock in Example 1 at $t = 1\ s$ and $t = 2\ s$.

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}$$

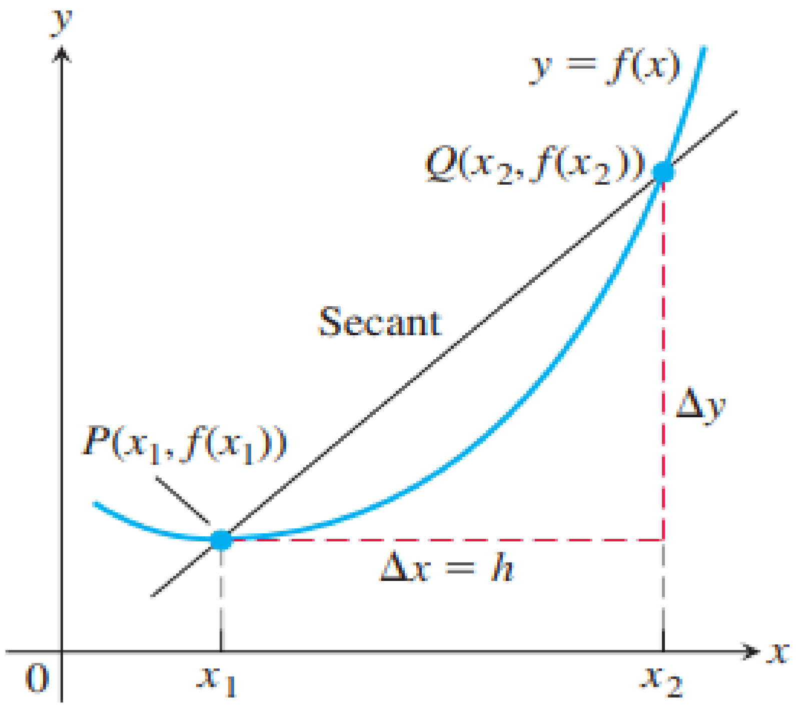
Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049	19.60049

② Average Rates of Change and Secant Lines

Definition: Average Rate of Change

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$



The average rate of change of f from x_1 to x_2 is identical with the slope of secant line PQ.

Consider what happens as point Q approaches point P along the curve, so the length h of the interval over which the change occurs approaches zero. It leads to the definition the slope of a curve at a point.

③ Defining the Slope of a Curve

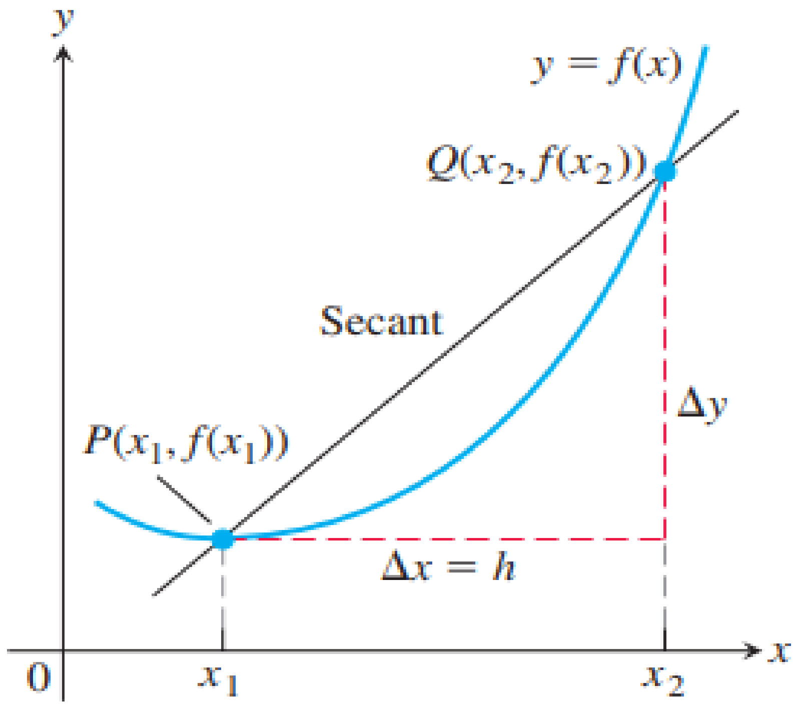
- We know what is meant by the slope of a straight line. But what is meant by the slope of a curve at a point P on the curve?
- If there is a tangent line to the curve at P, it would be reasonable to identify the slope of the tangent as the slope of the curve at P.
- So we need a precise meaning for the tangent at a point on a curve.
- For circles, tangency is straightforward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the radius at P. Such a line just touches the circle.
- But what does it mean to say that a line L is tangent to some other curve C at a point P?

- To define tangency for general curves, we need an approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve.
- Here are the procedures of this approach.

(1) Start with the slope of the secant PQ.

(2) Investigate the limiting value of the secant slope as Q approaches P along the curve.

(3) If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.



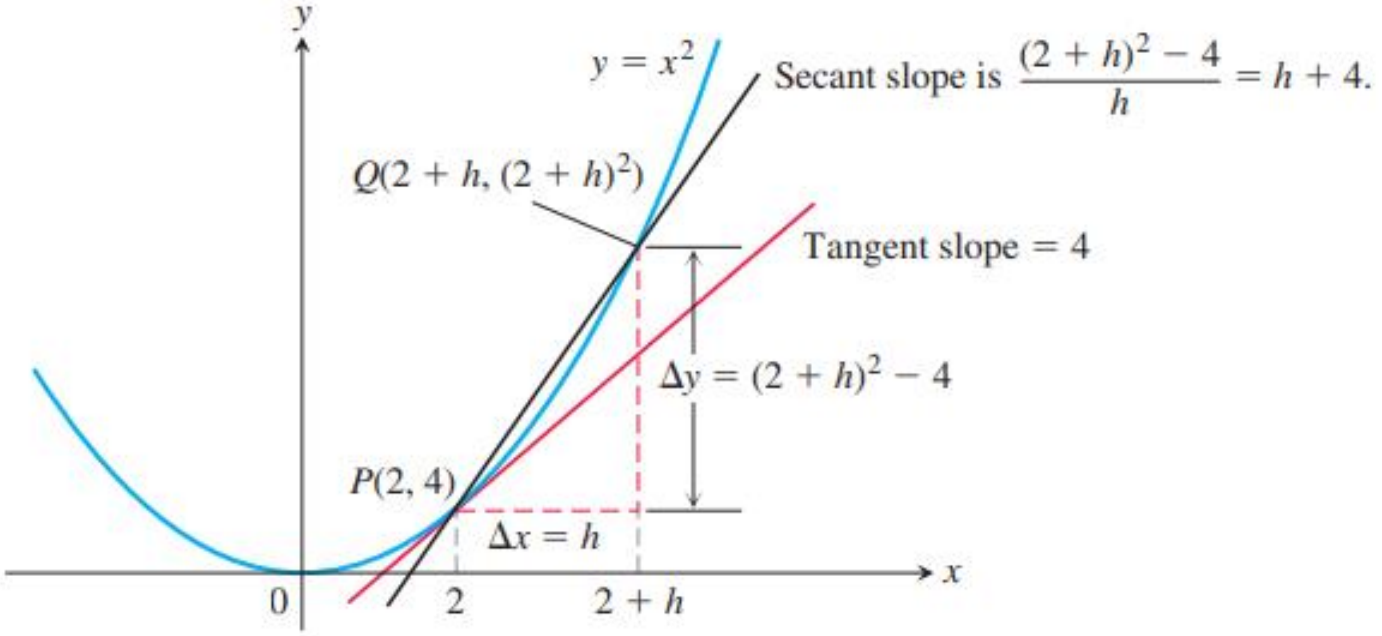
Example 3 Find the slope of the tangent line to the parabola $y = x^2$ at the point $P(2, 4)$.

Solution: The secant line is through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$.

$$\text{Slope of } PQ = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h}$$

$$= \frac{h^2 + 4h}{h} = h + 4$$

As Q approaches P along the curve, h approaches zero and the slope $h + 4$ approaches 4. Then the parabola's slope at P is taken as 4.



① Limits of Function Values

Example 1 How does the function

$$f(x)=\frac{x^2-1}{x-1}$$

behave near $x=1$?

So, when x goes to 1 from both sides of 1, $f(x)$ gets closer and closer to 2.

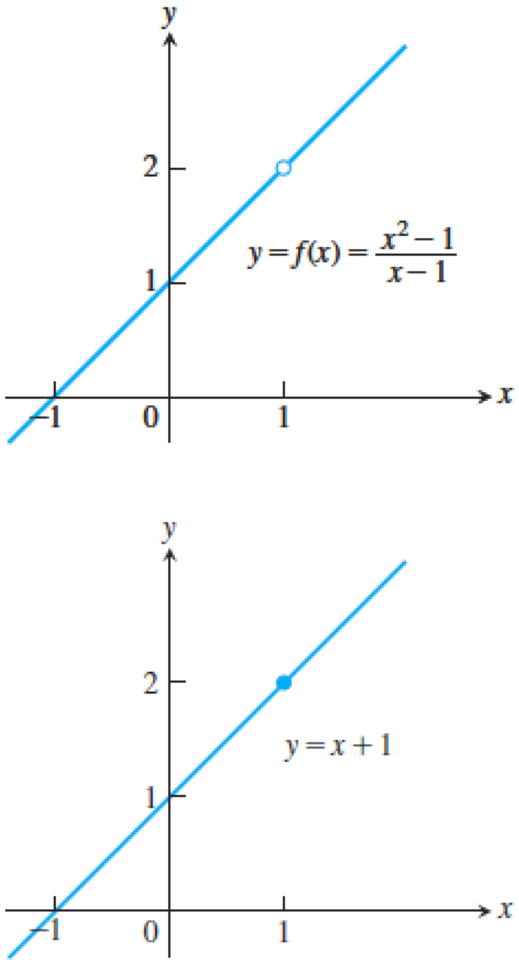


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

② An Informal Description of the Limit of a Function

Suppose that $f(x)$ is defined on an open interval about c, except possibly at c itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, other than c itself, then we say that f approaches the limit L as x approaches c, and write as:

$$\lim_{x \to c} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches c is L."

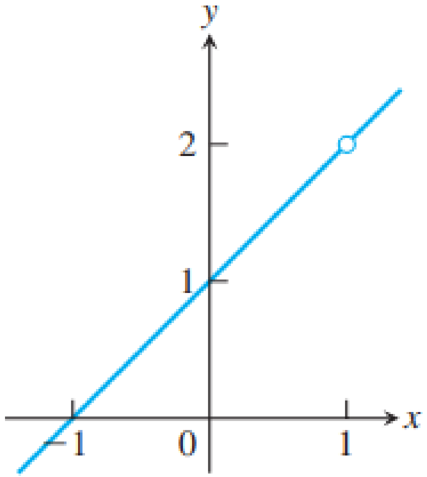
Thus, the limit in Example 1 is: $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

Our definition here is informal, because phrases like arbitrarily close and sufficiently close are imprecise; their meaning depends on the context.

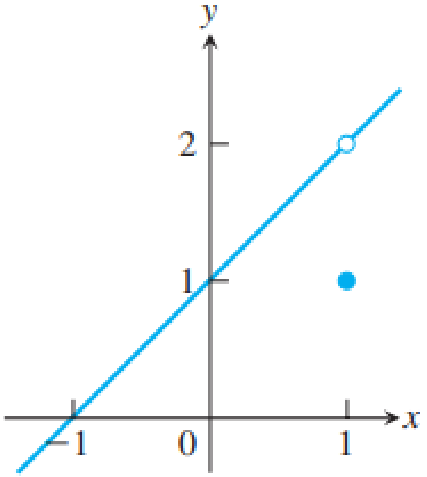
Note 1 The limit value of a function does not depend on how the function is defined at the point being approached.

For example, $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} g(x) = \lim\limits_{x\to 1} h(x) = 2$, where

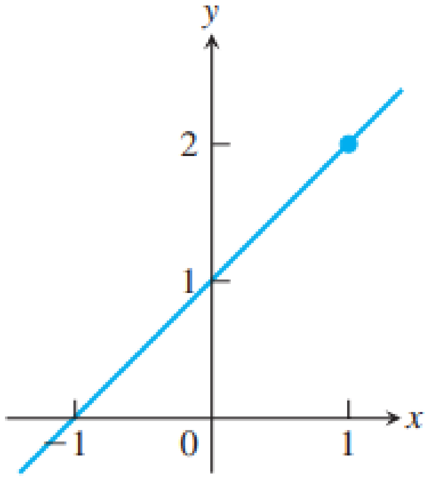
$$\text{(a) } f(x) = \frac{x^2-1}{x-1}, \quad \text{(b) } g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1; \\ 1, & x = 1. \end{cases}, \quad \text{(c) } h(x) = x+1.$$



(a) $f(x) = \dfrac{x^2-1}{x-1}$



(b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$



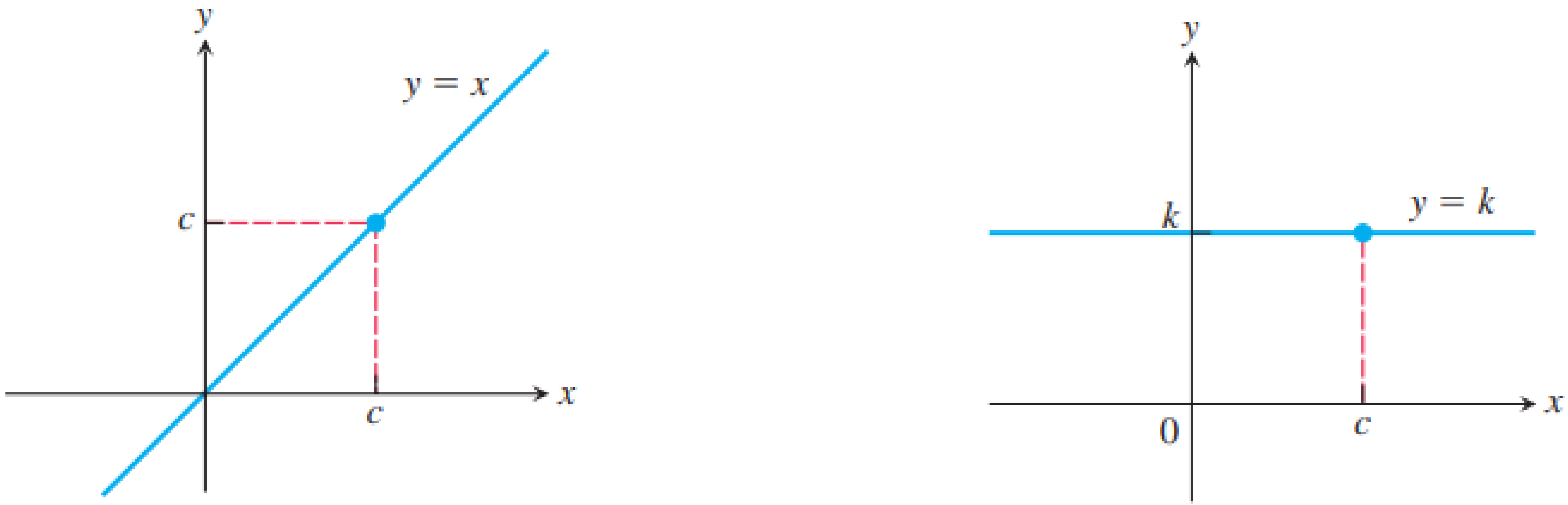
(c) $h(x) = x + 1$

Note 2 If we know the graph of a function, we can easy to determine the limit of the function.

Example 2

(a) If $f(x)=x$, then $\lim_{x\to c} f(x)=c$.

(b) If $f(x)=k$, and k is a constant, then $\lim_{x\to c} f(x)=k$.

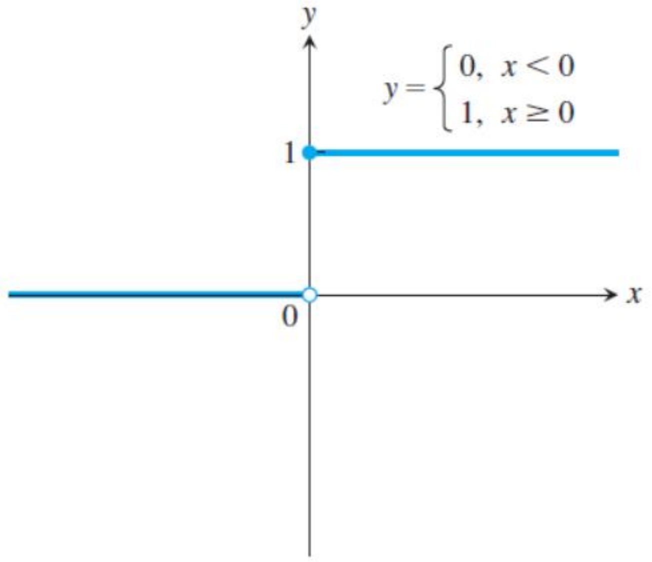


(a) Identity function

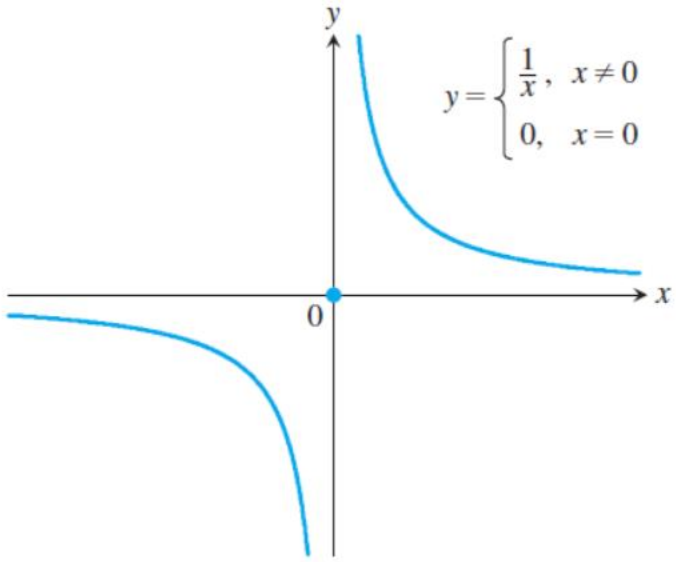
(b) Constant function

Example 3 Discuss the behavior of the following functions as $x \to 0$, explaining why they have no limit as $x \to 0$.

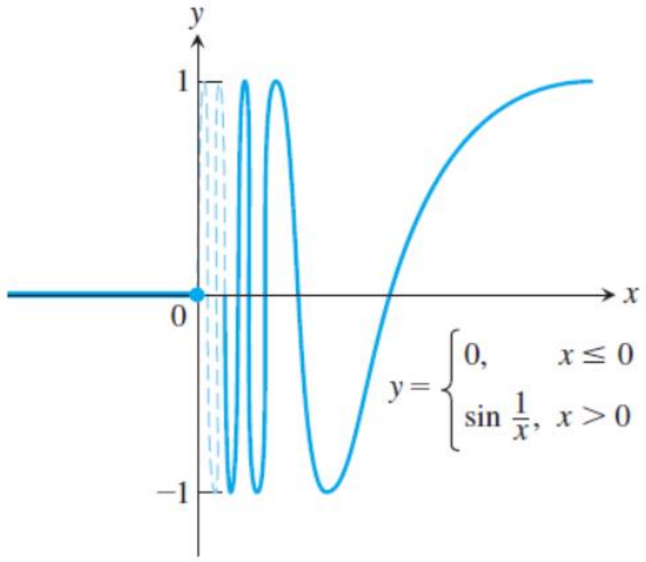
a) $u(x)=\begin{cases}0, & x<0;\\ 1, & x\geq 0.\end{cases}$ b) $g(x)=\begin{cases}\dfrac{1}{x}, & x\neq 0;\\ 0, & x=0.\end{cases}$ c) $f(x)=\begin{cases}0, & x\leq 0;\\ \sin\dfrac{1}{x}, & x>0.\end{cases}$



(a) Unit step function $U(x)$



(b) $g(x)$



(c) $f(x)$

③ The Limit Laws

THEOREM 1—Limit Laws

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

Example 4 Use the observations $\lim_{x \to c} x = c$ and $\lim_{x \to c} k = k$ (Example 2) and the fundamental rules of limits to find the following limits.

(a) $\lim_{x \to c} \left(x^3 + 4x^2 - 3 \right)$

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim_{x \to c} \sqrt{4x^2 - 3}$

④ Evaluate Limits of Polynomials and Rational Functions

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 5 Find the following limits.

(a) $\lim\limits_{x \to c}\left(x^3 + 4x^2 - 3\right)$ (b) $\lim\limits_{x \to -1} \dfrac{x^3 + 4x^2 - 1}{x^2 + 5}$ (c) $\lim\limits_{x \to -1} \dfrac{2x^2 - x + 5}{x^2 + 3x + 1}$

⑤ Eliminating Common Factors from Zero Denominators

Example 6 Evaluate

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}.$$

Example 7 Evaluate

$$\lim_{x\to 4}\frac{x^2-16}{x-4}.$$

Example 8 Evaluate

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}.$$

⑥ The Sandwich Theorem

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

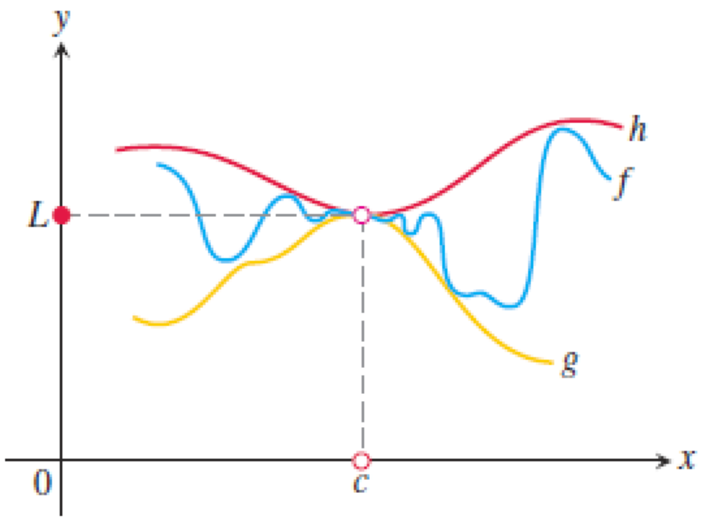


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

Example 9 Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$$

for all $x \ne 0$, find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Example 10 Show that

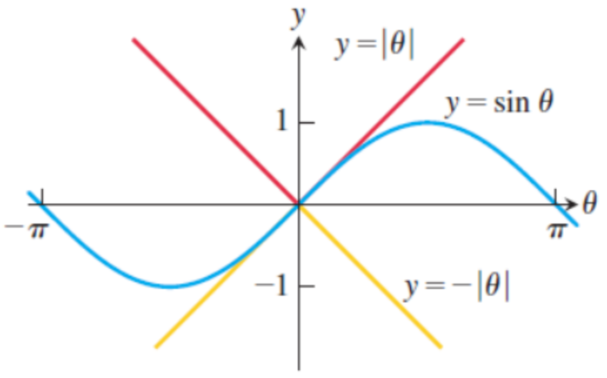
$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

EXAMPLE 11 The Sandwich Theorem helps us establish rules:

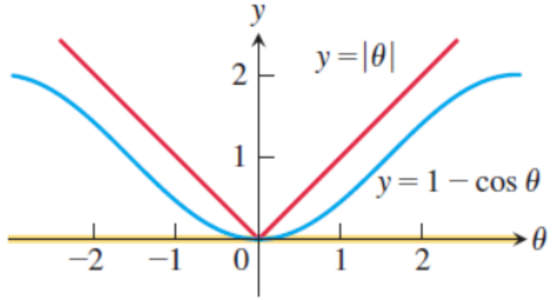
(a) $\lim_{\theta\to 0} \sin\theta = 0$

(b) $\lim_{\theta\to 0} \cos\theta = 1$

(c) For any function f, $\lim_{x\to c} |f(x)| = 0$ implies $\lim_{x\to c} f(x) = 0$.



(a)



(b)