CALCULUS

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- The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed directly if we can find an antiderivative of the function.
- Since any two antiderivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any antiderivative F of f,

$$\int f(x)dx = F(x) + C$$

where C is any arbitrary constant.

- The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation:

$$\int_a^b f(x)dx = F(b) - F(a) = [F(b)+C] - [F(a)+C] = \Big[F(x)+C\Big]_a^b = \Big[\int f(x)\mathrm{d}x\Big]_a^b$$

- We must distinguish carefully between definite and indefinite integrals.

A definite integral $\int_a^b f(x)dx$ is a number.

An indefinite integral $\int f(x)dx$ is a function plus an arbitrary constant C.

- So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives of functions we can't easily recognize as derivatives, such as:

$$\int \frac{x}{\sqrt{2x+1}}dx \ , \qquad \int sin^2 x cos x dx \ , \qquad \int \theta\sqrt{1-\theta^2}d\theta$$

① Substitution: Running the Chain Rule Backwards

- Substitution applies to integrals of form $\int f[g(x)]g'(x)dx$

If we let $u = g(x)$, then $du = g'(x)dx$. Therefore, we get

$$\int f[g(x)]g'(x)dx = \int f(u)du$$

Then, we integrate with respect to u and replace u by $g(x)$.

THEOREM 6 — The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f[g(x)]g'(x)dx = \int f(u)du$$

The Substitution method to evaluate $\int f(g(x))g'(x)dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\, dx = g(x)\, dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

Example 1 Find $\int \sin(x^2)\, 2x\, dx$.

Example 2 Find $\int (x^3 + x)^5 (3x^2+1)\, dx$.

Example 3 Find $\int \sqrt{2x+1}\, dx$.

Example 4 Find $\int x^3 \cos(x^4+1)\,\mathrm{d}x$.

Example 5 Find $\int \cos(7\theta+3)\,\mathrm{d}\theta$.

Example 6 Find $\int x\sqrt{2x+1}\,\mathrm{d}x$.

Example 7 Find

(a) $\int sin^2 x dx$ (b) $\int cos^2 x dx$ (c) $\int (1-2sin^2 x) sin2x dx$

② Try Different Substitutions

- The success of the substitution method depends on finding a substitution that changes an integral we cannot evaluate directly into one that we can. Finding the right substitution gets easier with practice and experience.
- If your first substitution fails, try another substitution, possibly coupled with other algebraic or trigonometric simplifications to the integrand.

Example 8 Evaluate

$$\int \frac{2x}{\sqrt[3]{x^2+1}}\,dx$$

Skill Practice 1

Evaluate

$$(a)\ \int \frac{2x}{\sqrt{x^2+1}}dx \qquad (b)\ \int_0^{\frac{\pi}{2}} \sqrt{sin\theta}cos^3\theta d\theta$$

Skill Practice 2

Solve the initial value problem:

$$\frac{dy}{dx} = \frac{4x}{\sqrt[3]{x^2+8}}, \qquad y(0) = 0$$