CALCULUS

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- Many problems require that we recover a function from its derivative, or from its rate of change. For instance, the laws of physics tell us the acceleration of an object falling from an initial height, and we can use this to compute its velocity and its height at any time.
- More generally, starting with a function f, we want to find a function F whose derivative is f. If such a function F exists, it is called an ***antiderivative*** of f. Antiderivatives are the link connecting the two major elements of calculus: derivatives and definite integrals.

① Finding Antiderivatives

DEFINITION

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

- The process of recovering a function $F(x)$ from its derivative $f(x)$ is called ***antidifferentiation***. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

Example 1 Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ (b) $g(x) = \cos x$ (c) $h(x) = sec^2 x + \frac{1}{2\sqrt{x}}$

Example 2 Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

② General Antiderivative

THEOREM 8

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 3 Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$ (b) $g(x) = \frac{1}{\sqrt{x}}$ (c) $h(x) = \sin 2x$ (d) $i(x) = \cos \frac{x}{2}$

◆ Table of Antiderivative formulas, k a nonzero constant

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

③ Antiderivative linearity rules

	Function	General antiderivative
1. *Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, $\quad k$ a constant
2. *Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Example 4 Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin x$$

Example 5 Find the general antiderivative of

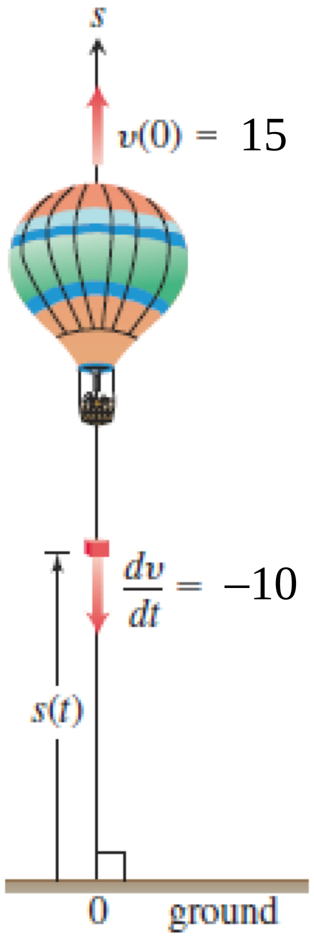
$$f(x) = \frac{4}{3}\sqrt[3]{x} + 2cos2x$$

④ Antiderivatives and Motion

Example 6

A hot-air balloon ascending at the rate of 15 m/s is at a height 50 m above the ground when a package is dropped. How long does it take the package to reach the ground?

Note: The acceleration of gravity near the surface of the earth is taken as 10 m/s^2.



④ Indefinite Integrals

DEFINITION

The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2xdx = x^2 + c \qquad \int cosxdx = sinx + c \qquad \int \left(sec^2x + \frac{1}{2\sqrt{x}}\right)dx = tanx + \sqrt{x} + c$$

Example 7 Evaluate

$$\int (3x^2 - 2x + 5)dx$$

Example 8 Evaluate

$$\int (\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}})dx$$

Example 9 Evaluate

$$\int \frac{4+\sqrt{t}}{t^3}dt$$

Skill Practice 1 Evaluate

$(a)\ \int 15cos5\theta d\theta$ $(b)\ \int x^{-3}(2-x)dx$

Skill Practice 2

Find a curve $y = f(x)$ with the following properties:

(1) $\frac{d^2y}{dx^2} = 6x$

(2) Its graph passes through the point (0, 1) and has a horizontal tangent there.