CALCULUS

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- In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals by using an antiderivative of the integrand function rather than by taking limits of Riemann sums as we did in Section 5.3.
- Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years. Along the way, we will present an integral version of the Mean Value Theorem, which is another important theorem of integral calculus and is used to prove the Fundamental Theorem.

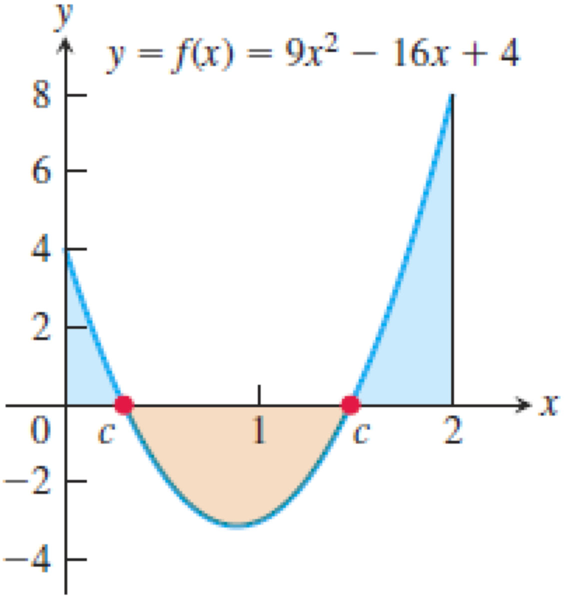
① Mean Value Theorem for Definite integrals

THEOREM 3 — The Mean Value Theorem for Definite integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Proof: **Property 6: $(b-a)\,f_{min} \leq J \leq (b-a)\,f_{max}$**



Example 1 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,\mathrm{d}x = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

② Fundamental Theorem, Part 1

THEOREM 4 —The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x)$$

Proof: Skipped.

Example 2 Use the Fundamental Theorem to find y' if

(a) $y = \int_a^x (t^3 + 1)\,\mathrm{d}t$ (b) $y = \int_x^5 3t\sin t\,\mathrm{d}t$ (c) $y = \int_1^{x^2} \cos t\,\mathrm{d}t$ (d) $y = \int_{1+3x^2}^{6} \frac{1}{2+t}\,\mathrm{d}t$

③ Fundamental Theorem, Part 2 (The Evaluation Theorem)

THEOREM 4 (Continued) — The Fundamental Theorem of Calculus, Part 2

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$$

- To calculate the definite integral of f over an interval $[a, b]$, we need do only two things:

1. Find an antiderivative F of f, and

2. Calculate the number $F(b) - F(a)$, which is equal to $\int_a^b f(x)\,\mathrm{d}x$.

Example 3 Calculate the following integrals:

(a) $\int_0^{\pi} cosxdx$ (b) $\int_0^{\pi/4} secxtanxdx$ (c) $\int_1^4 (\frac{3}{2}\sqrt{x} - \frac{4}{x^2})dx$

④ The Integral of a Rate

- We can interpret Part 2 of the Fundamental Theorem in another way. If F is any antiderivative of f, then $F' = f$. The equation in the theorem can then be rewritten as:

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)$$

- Now $F'(x)$ represents the rate of change of the function $F(x)$ with respect to x, so the last equation asserts that the integral of F' is just the net change in F as x changes from a to b. Thus, we have the following theorem:

THEOREM 5 — The Net Change Theorem

The net change in a differentiable function $F(x)$ over an interval $[a, b]$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,\mathrm{d}x$$

Interpretation of Theorem 5

If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Theorem 5 says that:

$$\int_{t_1}^{t_2} v(t)\,\mathrm{d}t = s(t_2) - s(t_1)$$

So the integral of velocity is the **displacement** over the time interval $[t_1, t_2]$.

Example 4 (Note: what is difference between displacement and distance?)

Revisit Example 1, Section 5.1, a heavy rock blown straight up from the ground by a dynamite blast. The velocity of the rock at any time t during its motion was given as $v(t) = 160 - 10t$ m/s. Find the displacement of the rock during the period $0 \le t \le 24$.

⑤ The Relationship Between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. First,

$$\frac{d}{dx}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x)$$

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise,

$$\int_a^x f(t)\,\mathrm{d}t = F(x) - F(a)$$

so that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant).

In a sense, the processes of integration and differentiation are “inverses” of each other.

⑥ Total Area

- Area is always a nonnegative quantity. The Riemann sum approximations contain terms such as $f(c_k)\,\Delta x_k$ that give the area of a rectangle when $f(c_k)$ is positive. When $f(c_k)$ is negative, then the product $f(c_k)\,\Delta x_k$ is the negative of the rectangle's area.

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.

2. Integrate f over each subinterval.

3. Add the absolute values of the integrals.

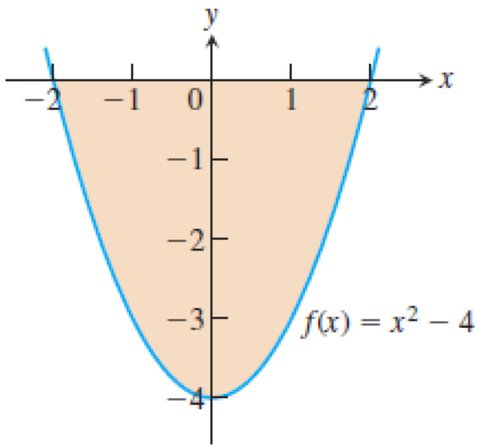
Example 5

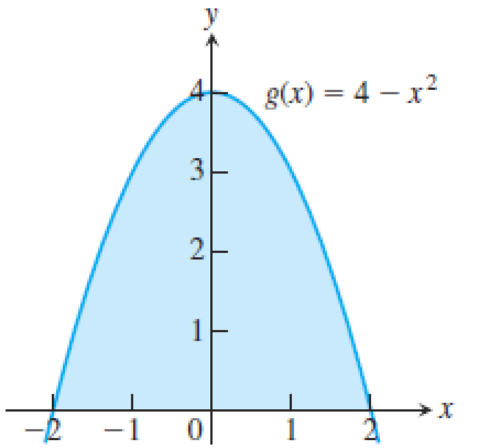
The graphs of $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$ are shown here.

For each function, compute

(a) the definite integral over the interval [–2, 2],

(b) the area between the graph and the x-axis over [–2, 2].





Skill Practice 1

For the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$, compute

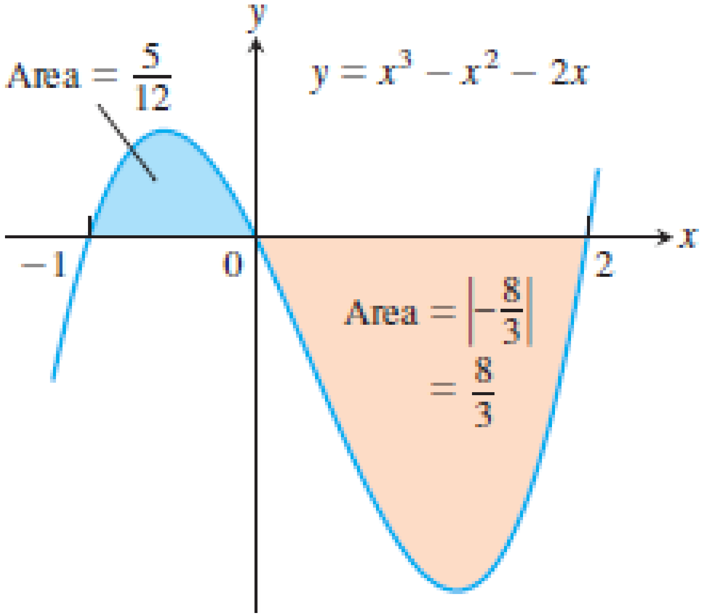
(a) the definite integral over the interval $[0, 2\pi]$,

(b) the area between the graph $f(x)$ and the x-axis over $[0, 2\pi]$.

Skill Practice 2

Find the area of the region between the x-axis and the graph of

$$f(x) = x^3 - x^2 - 2x, \quad -1 \le x \le 2.$$



Skill Practice 3

Evaluate the following definite integrals:

(a) $\int_{-1}^{1}(3x^2 - 2x + 1)dx.$ (b) $\displaystyle\int_{-2}^{2}\frac{2}{(x+1)^3}dx.$ (c) $\int_{-\pi/3}^{\pi/3} 4sin^2\theta d\theta.$