CALCULUS

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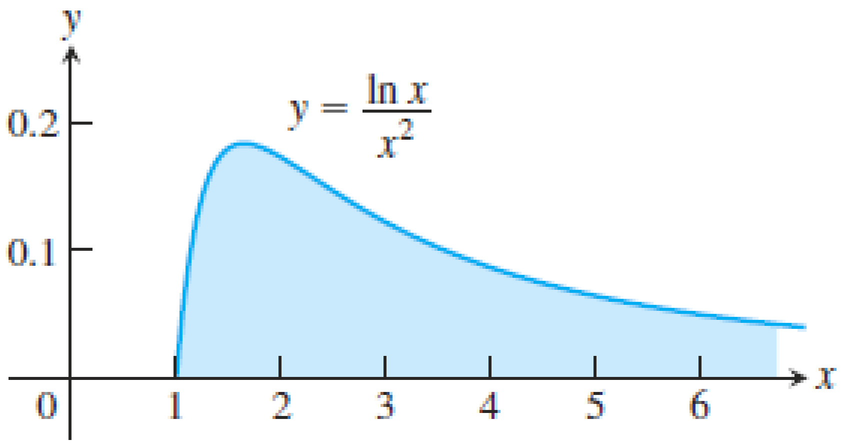
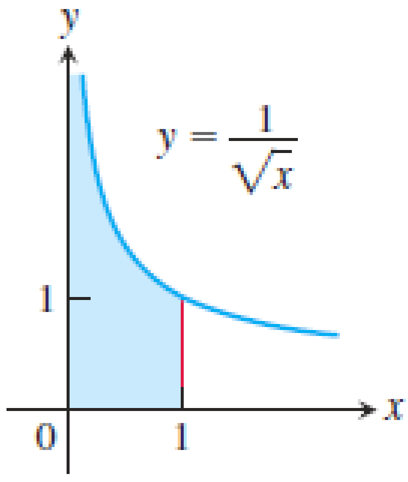
Spring 2025

- In practice, we may encounter problems that either the domain of integration is finite or the range of the integrand is infinite.
- The integral for the area under the curve $y=(\ln x)/x^2$ from $x=1$ to $x=\infty$ is an example for which the domain is infinite (Fig. a). The integral for the area under the curve of $y=1/\sqrt{x}$ between $x=0$ and $x=1$ is an example that the range of the integrand is infinite (Fig. b).
- In either case, the integrals are said to be improper and are calculated as limits.



(a)



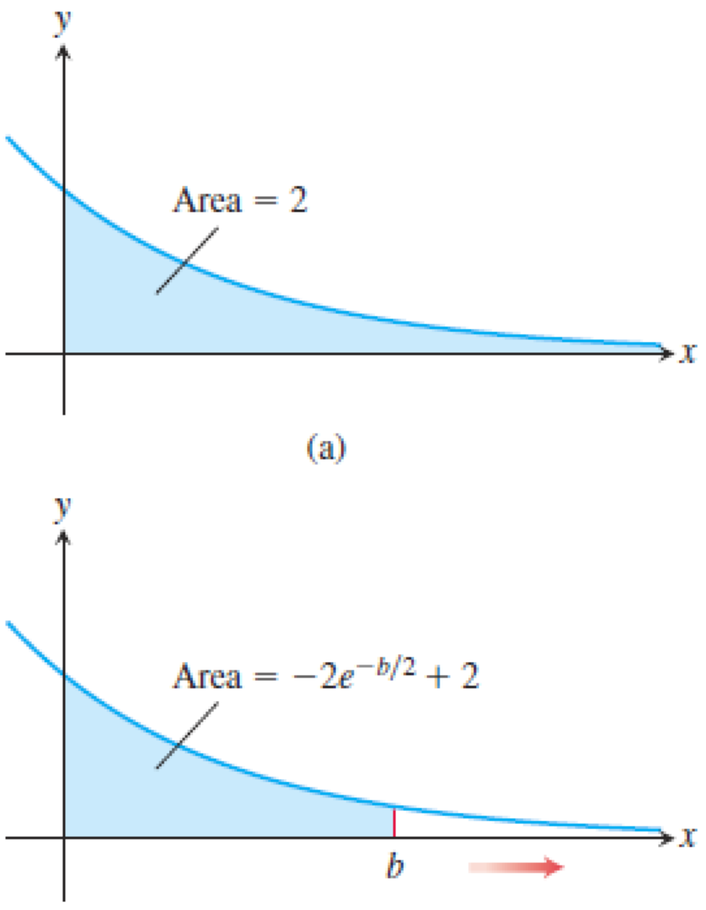
(b)

① Infinite Limits of Integration

- Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant (Fig. a). We want to find the area of this infinite region.
- First, we will find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Fig. b), which can be expressed as:

$$A(b) = \int_0^b e^{-x/2}dx = -2e^{-x/2}\Big|_0^b = 2 - 2e^{-b/2}$$

$$\xRightarrow{take\ the\ limit} \quad \lim_{b\to\infty} A(b) = \lim_{b\to\infty} (2 - 2e^{-b/2}) = 2$$



(a) (b)

Improper Integrals of Type I.

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

- The improper integral is **converged** if the limit exists and is finite. The improper integral is **diverged** if the limit doesn't exist.
- The choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^{\infty} f(x)dx$ with any convenient choice.
- Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration.

Example 1

Is the area under the curve $y = \frac{lnx}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Example 2 Evaluate $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

② The Integral $\int_1^\infty x^{-p}dx$

- The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$.

Example 3

For what values of p does the following integral converge? When the integral does converge, what is its value?

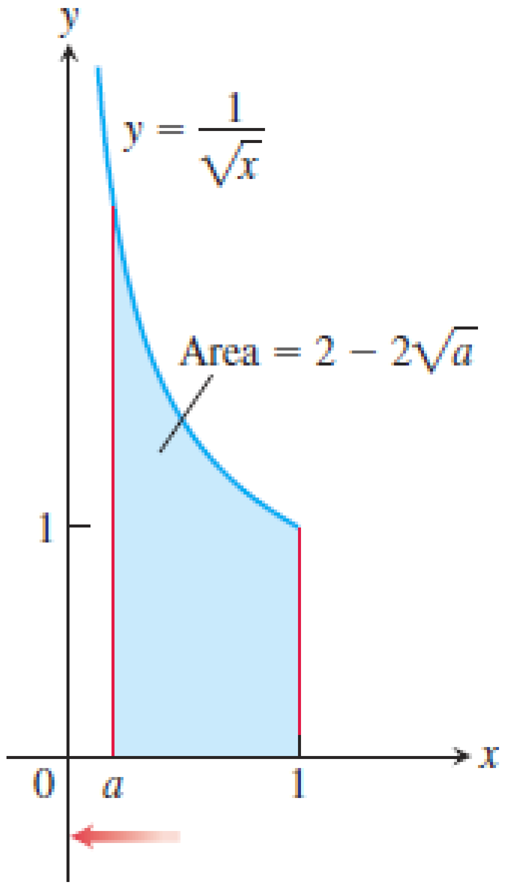
$$\int_1^\infty \frac{1}{x^p}\,dx$$

③ Integrands with Vertical Asymptotes

- Another type of improper integral arises when the integrand has a vertical asymptote — an infinite discontinuity — at a limit of integration or at some point between the limits of integration.
- Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure on the right). First, we find the area of the portion from a to 1:

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\Big|_a^1 = 2 - 2\sqrt{a}$$

$$\xrightarrow{a\to0^+} \quad \lim_{a\to0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to0} \left(2 - 2\sqrt{a}\right) = 2$$



Improper Integrals of Type II.

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example 4

Investigate the convergence of:

$$\int_0^1 \frac{dx}{1-x}$$

Example 5

Evaluate:

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}$$

④ Tests for Convergence and Divergence

- When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value.
- The principal tests for convergence or divergence are the **Direct Comparison Test** and the **Limit Comparison Test**.

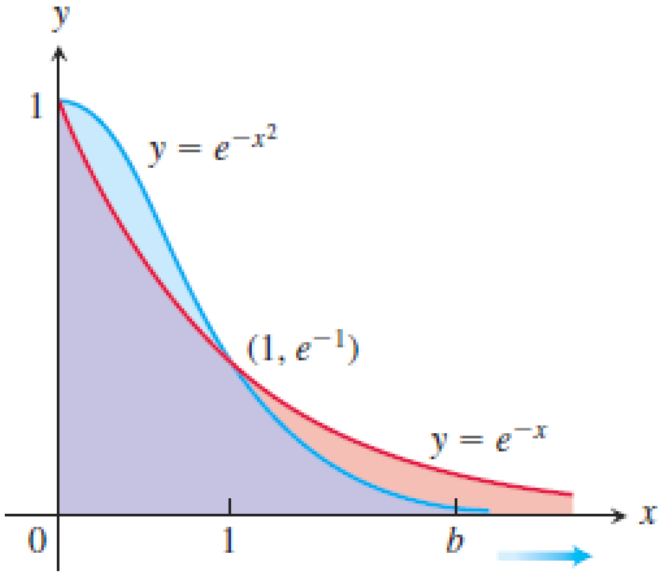
THEOREM 2—Direct Comparison Test
Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. If $\int_a^{\infty} g(x)\,dx$ converges, then $\int_a^{\infty} f(x)\,dx$ also converges.

2. If $\int_a^{\infty} f(x)\,dx$ diverges, then $\int_a^{\infty} g(x)\,dx$ also diverges.

Example 6

Does $\int_1^{\infty} e^{-x^2} dx$ converge?



Example 7

Do the following integrals converge or diverge?

$$(a)\quad \int_1^{\infty} \frac{sin^2 x}{x^2} dx$$

$$(b)\quad \int_1^{\infty} \frac{1}{\sqrt{x^2 - 1/9}} dx$$

$$(c)\quad \int_0^{\pi/2} \frac{cosx}{\sqrt{x}} dx$$

THEOREM 3—Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^{\infty} f(x)\,dx \qquad \text{and} \qquad \int_a^{\infty} g(x)\,dx$$

either *both converge* or *both diverge*.

Example 8 Show that $\int_1^{\infty} \frac{1}{1+x^2}\,dx$ converges by comparison with $\int_1^{\infty} \frac{1}{x^2}\,dx$. Find and compare these two values.

Example 9 Investigate the convergence of $\displaystyle\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$

Skill Practice 1 Evaluate:

$$(a)\ \int_{-\infty}^{0} \frac{x}{(x^2+1)^{3/2}}\,dx \qquad (b)\ \int_{0}^{\infty} \frac{1}{(x+1)\sqrt{x}}\,dx$$

Skill Practice 2

Investigate the convergence of:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$