College Algebra and Trigonometry

Prof. Liang ZHENG

Fall 2024

Two different functions can be combined using mathematical operations such as addition, subtraction, multiplication, and division. Also, there is an operation on functions called composition, which can be thought of as a function of a function. When we combine functions, we do so algebraically. Special attention must be paid to the domain and range of the combined functions.

① Adding, Subtracting, Multiplying, and Dividing Functions

Two functions can be added, subtracted, and multiplied. The resulting function domain is therefore the intersection of the domains of the two functions. However, for division, any value of *x* (input) that makes the denominator equal to zero must be eliminated from the domain.

FUNCTION	NOTATION	DOMAIN
Sum	$(f+g)(x) = f(x) + g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Difference	$(f-g)(x) = f(x) - g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$\{\text{domain of } f\} \cap \{\text{domain of } g\} \cap \{g(x) \neq 0\}$

Example 1 Given $f(x)=\sqrt{25-x^2}$ and $g(x)=5$, find $(f+g)(x)$.

Example 2 Given $m(x)=4x,\ n(x)=|x-3|,$ and $p(x)=\dfrac{1}{x+1}$, determine the function values if possible.

(a) $(m-n)(-2)$ (b) $(m\cdot p)(1)$ (c) $\left(\dfrac{p}{n}\right)(3)$

Example 3 Given $f(x) = 2x, g(x) = x^2 - 4x,$ and $h(x) = \sqrt{x-1}$,

a) Find $(f-g)(x)$ and write the domain of $f - g$ in interval notation.

b) Find $(f \cdot h)(x)$ and write the domain of $f \cdot h$ in interval notation.

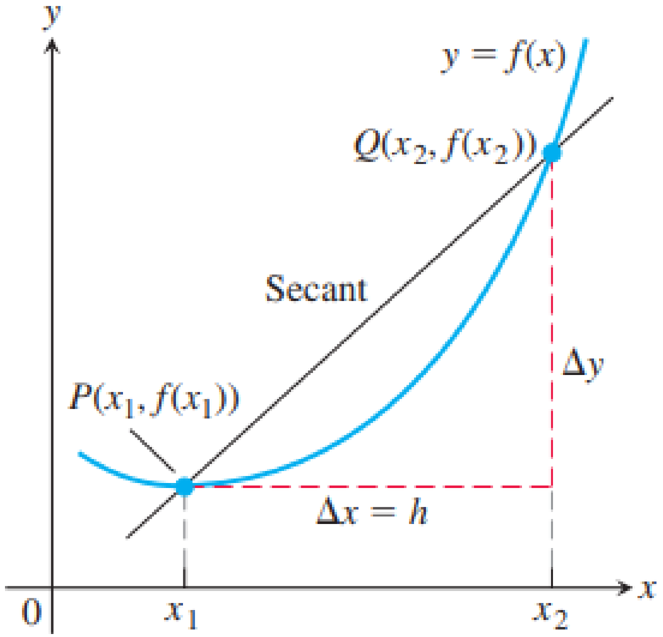
c) Find $\left(\frac{h}{g}\right)(x)$ and write the domain of $\frac{h}{g}$ in interval notation.

② Evaluate a Difference Quotient

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the secant PQ.



Now we look at a related idea. Let P be an arbitrary point $(x, f(x))$ on the function f. Let $h > 0$ and Q be the point $(x + h, f(x + h))$. Then the average rate of change of $y = f(x)$ between P and Q is given

$$\frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{(x+h)-x}=\frac{f(x+h)-f(x)}{h}.$$

The expression

$$\frac{f(x+h)-f(x)}{h} \qquad (h \neq 0)$$

is called the difference quotient of f at x with an increment h and is very important for the foundation of calculus.

Example 4 Given $f(x)=3x-5$,

a) Find $f(x+h)$.

b) Find the difference quotient, $\dfrac{f(x+h)-f(x)}{h}$.

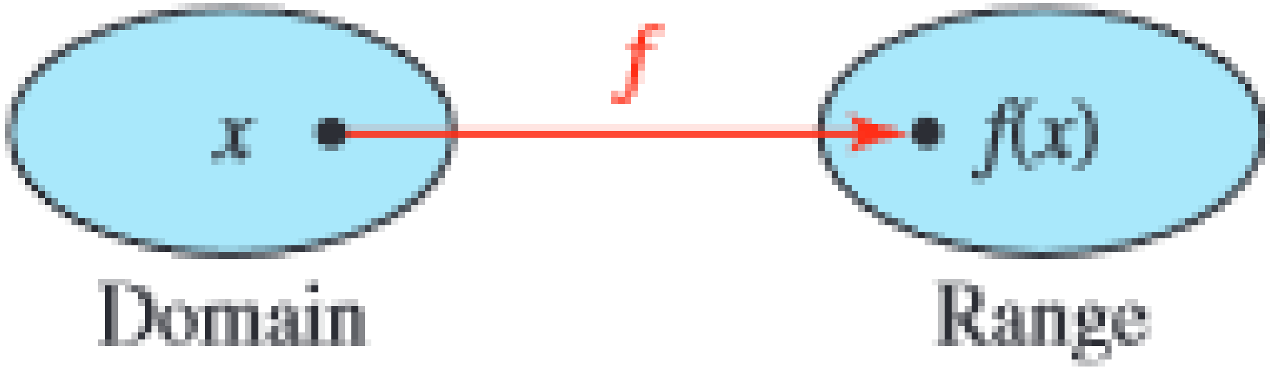
Example 5 Given $f(x)=-2x^2+4x-1$,

a) Find $f(x+h)$.

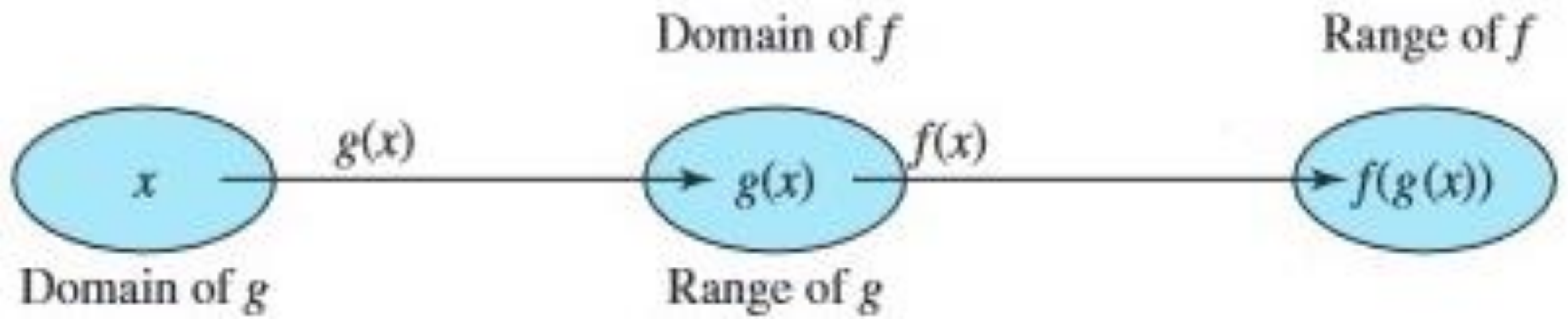
b) Find the difference quotient, $\dfrac{f(x+h)-f(x)}{h}$.

③ Compose and Decompose Functions

Recall that a function maps every element in the domain to exactly one corresponding element in the range as shown in the following figure.



The next operation on functions is called the composition of functions. This involves a substitution process in which the output from one function becomes the input to another function as shown in the following figure.



The symbol that represents composition of functions is a small open circle $\circ$; thus

$$(f\circ g)(x)=f\left(g\left(x\right)\right),$$

which is read aloud as “f of g.”

DEFINITION: Composition of Functions

Given two functions f and g, there are two composite functions that can be formed.

a) The composition of f and g, denoted by $(f \circ g)(x)$, is defined by

$$(f \circ g)(x) = f(g(x)).$$

b) The composition of g and f, denoted by $(g \circ f)(x)$, is defined by

$$(g \circ f)(x) = g(f(x)).$$

NOTATION	WORDS	DEFINITION	DOMAIN
$f \circ g$	f composed with g	$f(g(x))$	The set of all real numbers x in the domain of g such that $g(x)$ is also in the domain of f.
$g \circ f$	g composed with f	$g(f(x))$	The set of all real numbers x in the domain of f such that $f(x)$ is also in the domain of g.

Example 6 Given $f(x)=x^2+2x$ and $g(x)=x-4$, find

(a) $f(g(6))$ (b) $g(f(-3))$ (c) $(f\circ g)(0)$ (d) $(g\circ f)(5)$

Example 7 Given $f(x)=2x-6$ and $g(x)=\dfrac{1}{x+4}$, write a rule for each function and write the domain in interval notation.

(a) $(f\circ g)(x)$ (b) $(g\circ f)(x)$

Example 8 Given $f(x)=\dfrac{1}{x-5}$ and $g(x)=\sqrt{x-2}$, find $(f\circ g)(x)$ and write the domain in interval notation.

Example 9 Given $f(x)=\dfrac{x}{x-2}$ and $g(x)=\dfrac{6}{x^2-1}$, find $(f\circ g)(x)$ and write the domain in interval notation.

The composition of two functions creates a new function in which the output from one function becomes the input to the other. We can also reverse this process. That is, we can decompose a composite function into two or more simple functions.

Example 10 Given $h(x)=\left|2x^2-5\right|$, find two functions f and g such that $h=(f\circ g)(x)$.

Example 11 Given $h(x)=\sqrt{5x+1}$, find two functions f and g such that $h=(f\circ g)(x)$.

Example 12:

The graphs of f and g are shown.

Evaluate the following functions:

a) $(f + g)(1)$

b) $(fg)(0)$

c) $(g - f)(-3)$

d) $(f \circ g)(3)$

e) $(g \circ f)(3)$

f) $f(g(1))$