CALCULUS

Prof. Liang ZHENG

Spring 2025

- Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a\tan u$, $x = a\sin u$, and $x = a\sec u$. These substitutions are effective in transforming integrals involving

$$\sqrt{x^2+a^2}, \qquad \sqrt{a^2-x^2} \quad and \quad \sqrt{x^2-a^2}$$

into integrals we can evaluate directly:

With $x = a\tan\theta \quad \Rightarrow \quad a^2+x^2 = a^2+a^2\tan^2\theta = a^2\sec^2\theta$

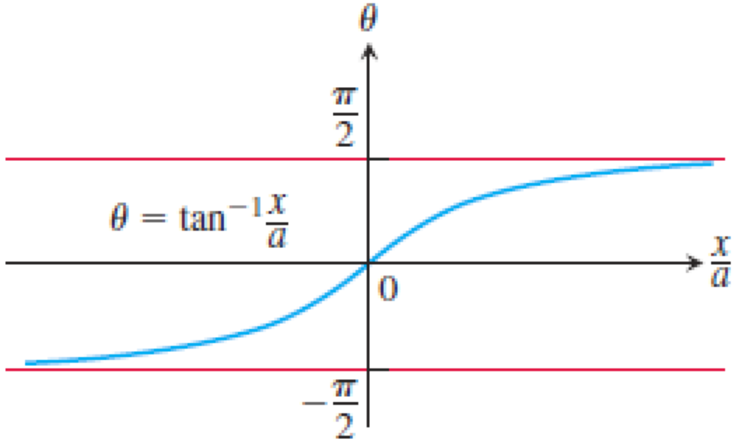
With $x = a\sin\theta \quad \Rightarrow \quad a^2-x^2 = a^2-a^2\sin^2\theta = a^2\cos^2\theta$

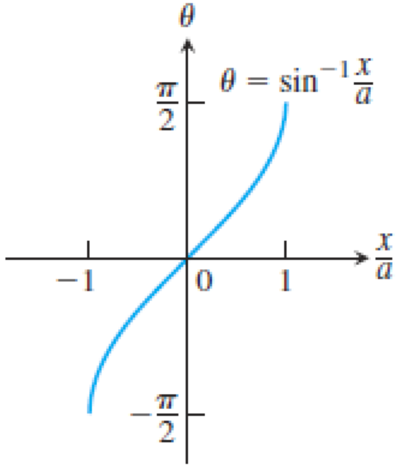
With $x = a\sec\theta \quad \Rightarrow \quad x^2-a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

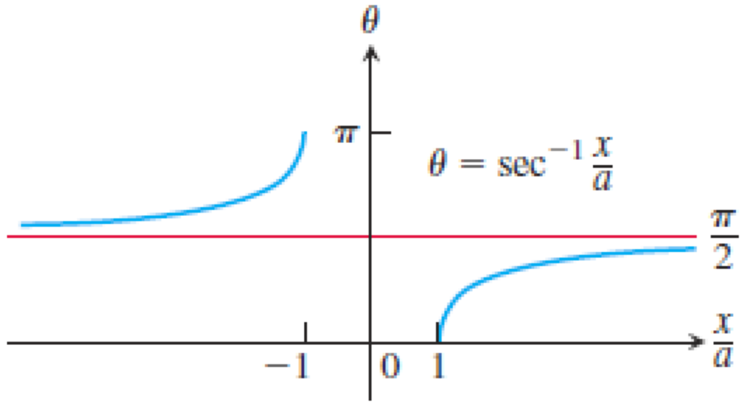
$x = a\tan\theta$ requires $\theta = tan^{-1}\left(\frac{x}{a}\right)$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$x = a\sin\theta$ requires $\theta = sin^{-1}\left(\frac{x}{a}\right)$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$x = a\sec\theta$ requires $\theta = sec^{-1}\left(\frac{x}{a}\right)$ with $\begin{cases} 0 \leq \theta < \frac{\pi}{2} & if \quad \frac{x}{a} \geq 1 \\ \frac{\pi}{2} < \theta \leq \pi & if \quad \frac{x}{a} \leq -1 \end{cases}$







Procedure for a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

① Substitution $x = a\tan\theta$ for integrals involving $\sqrt{x^2 + a^2}$.

Example 1 Evaluate

$$\int \frac{1}{\sqrt{x^2+4}}\,dx$$

Example 2 Evaluate

$$\int_{ln1}^{ln4} \frac{e^t}{\sqrt{e^{2t}+9}}\,dt$$

② Substitution $x = a\sin\theta$ for integrals involving $\sqrt{a^2 - x^2}$.

Example 3 Evaluate

$$\int_{-1/2}^{1/2} \sqrt{1 - x^2}\,dx$$

Example 4 Evaluate

$$\int \frac{x^2}{\sqrt{9 - x^2}}\,dx$$

③ Substitution $x = a\sec\theta$ for integrals involving $\sqrt{x^2 - a^2}$.

Example 5 Evaluate

$$\int \frac{2}{x^3\sqrt{x^2-1}}\,dx, \qquad x > 1$$

Example 6 Evaluate

$$\int \frac{dx}{\sqrt{25x^2-4}}, \qquad x > \frac{2}{5}$$

Skill Practice 1 Evaluate

$$\int \frac{3dx}{\sqrt{1+9x^2}}$$

Skill Practice 2 Evaluate

$$\int_{-1}^{\sqrt{3}} \frac{x+1}{\sqrt{4-x^2}}dx$$

Skill Practice 3 Evaluate

$$\int \frac{x}{\sqrt{4x^2-1}}dx, \qquad x>\frac{1}{2}$$

Skill Practice 4 Evaluate

$$\int \frac{dx}{\sqrt{x^2 - 2x + 5}}$$

Skill Practice 5 Evaluate

$$\int_0^1 \frac{dx}{(4 - x^2)^{3/2}}$$

Skill Practice 6 **Find Area**

Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \frac{1}{3}\sqrt{9 - x^2}$.