CALCULUS

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- There are two methods for evaluating a definite integral by substitution. One method is to find an antiderivative using substitution and then to evaluate the definite integral by applying the Evaluation Theorem. The other method extends the process of substitution directly to *definite* integrals by changing the limits of integration. We will use the new formula that we introduce here to compute the area between two curves.

① The Substitution Formula

THEOREM 7 – Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Proof:

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_a^b f\big(g(x)\big) \cdot d[g(x)] = \int_{g(a)}^{g(b)} f(u)du$$

Example 1 Evaluate

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$$

Example 2 Evaluate

(a) $\int_{\pi/4}^{\pi/2} cot\theta csc^2\theta d\theta$ (b) $\int_0^{\frac{\pi}{2}} \frac{2sin\theta cos\theta}{(1+sin^2\theta)^3}d\theta$

② Definite Integrals of Symmetric Functions

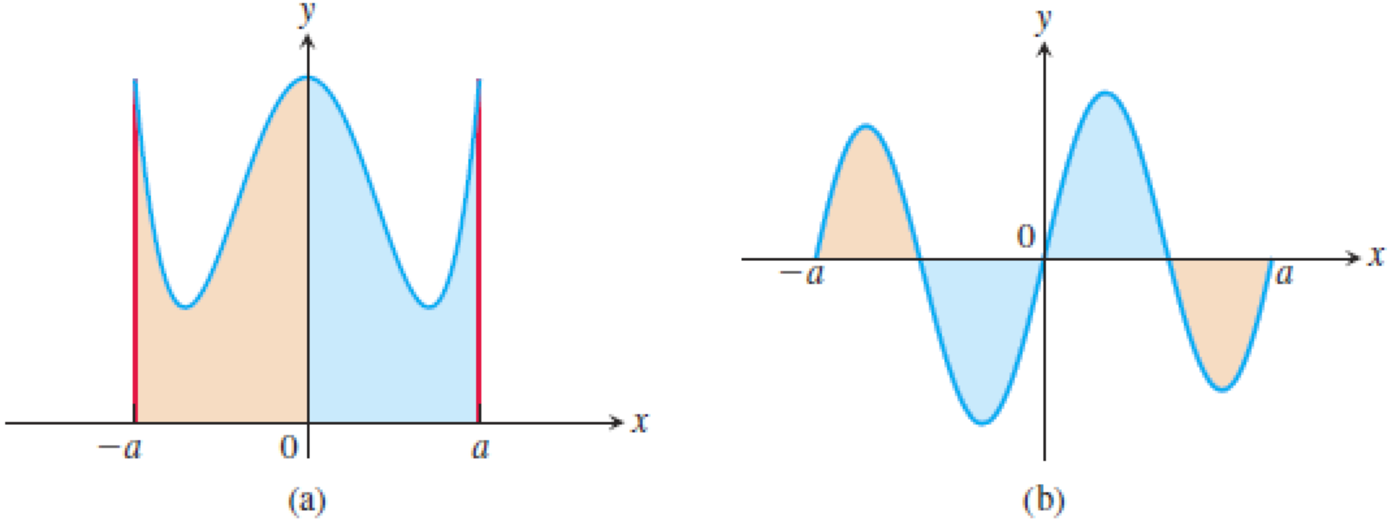


FIGURE 5.24 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

Recall:

$f(x)$ is even $\Leftrightarrow$ $f(-x) = f(x)$.

$f(x)$ is odd $\Leftrightarrow$ $f(-x) = -f(x)$.

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Proof:

Example 3 Evaluate

$$\int_{-2}^{2}\left(5x^4-3x^2+6\right)dx$$

Example 4 Evaluate

$$\int_{-\pi/2}^{\pi/2}\left(sin^3xcos2x-2x^3\right)dx$$

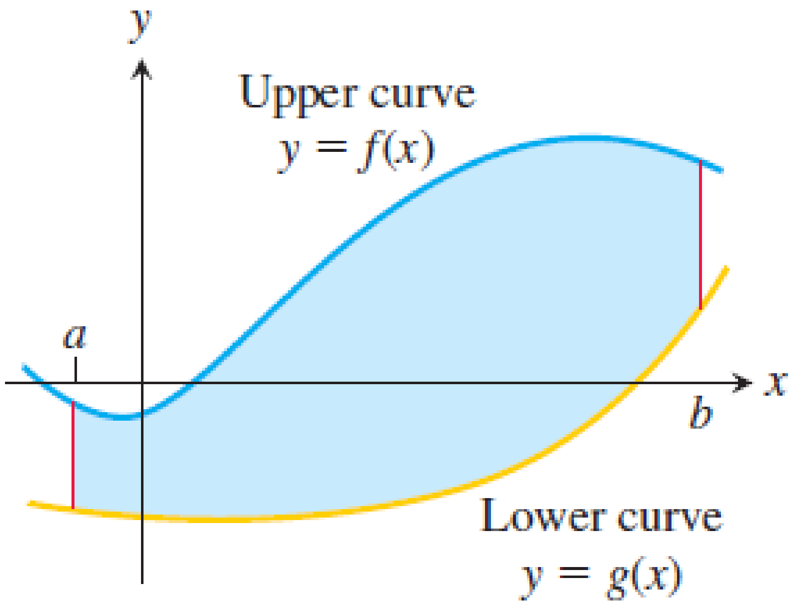
Example 5 Evaluate

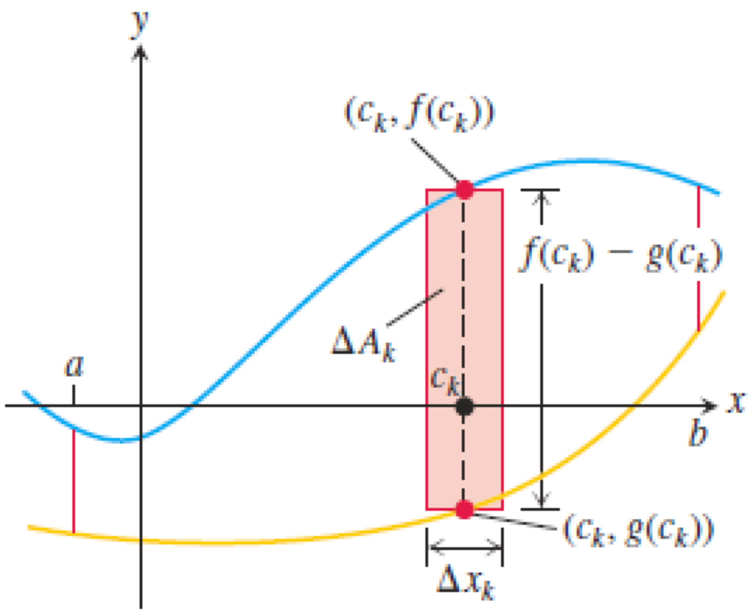
$$\int_{-1}^{1}\left(sinx^5+cos2x-6x^3cosx^3+3x^2\right)dx$$

③ Areas Between Curves

DEFINITION If f and g are continuous with $f(x) \ge g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

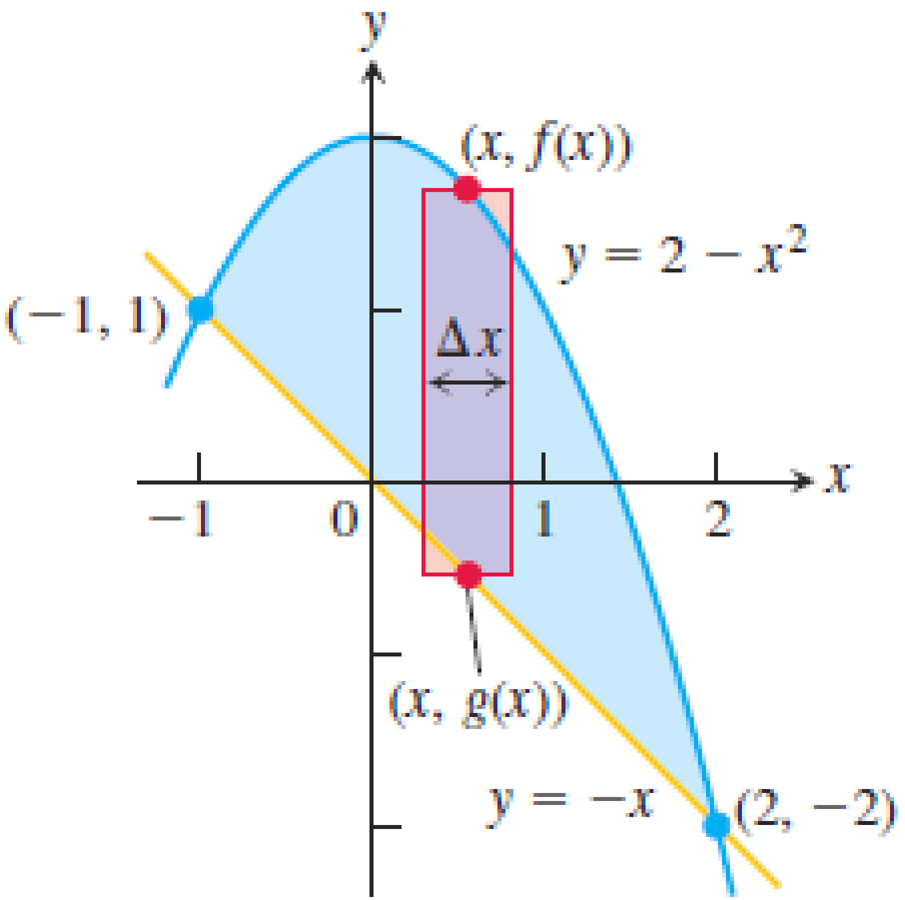
$$A = \int_a^b [f(x) - g(x)]\,dx$$





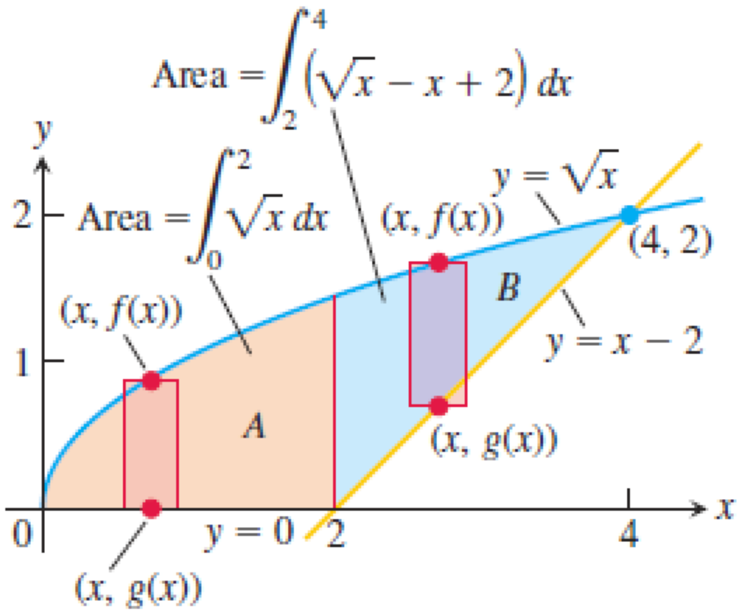
Example 6

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



Example 7

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



Example 8

Find the area of the region bounded below by the line $y = 2 - x$ and above by the curve $y = \sqrt{2x - x^2}$.