# CALCULUS

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- In sketching the graph of a differentiable function, it is useful to know where it increases and where it decreases over an interval. This section gives a test to determine where it increases and where it decreases.

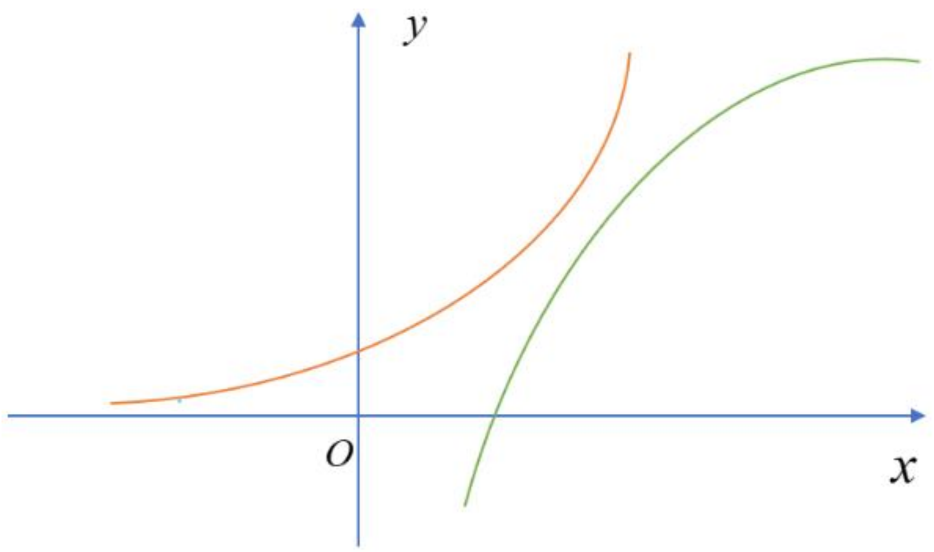
## ① Increasing Functions and Decreasing Functions

**Definition:**

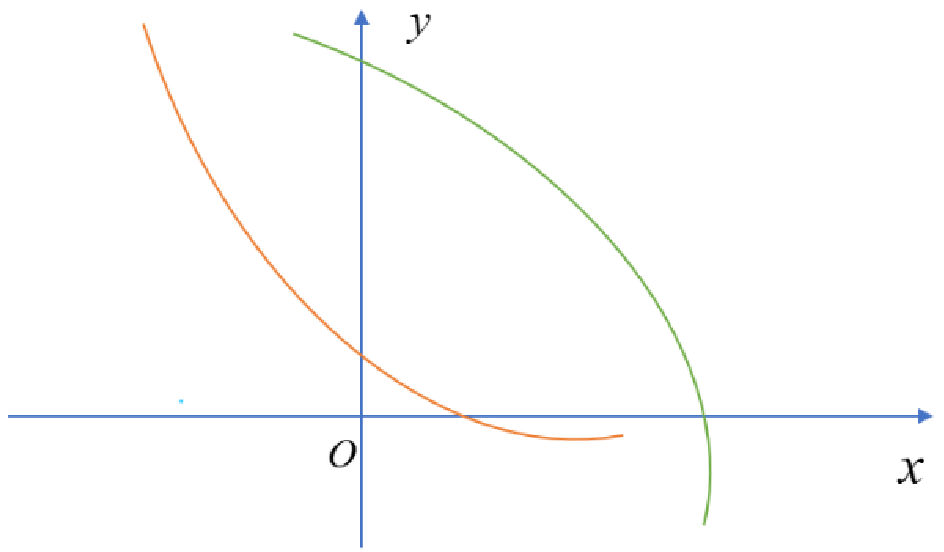
(a) $f(x)$ is increasing on $I$ $\Leftrightarrow$ $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ .

(b) $f(x)$ is decreasing on $I$ $\Leftrightarrow$ $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ .

- A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.




increasing function




decreasing function

**Corollary 3:**

Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

- If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ is increasing on $[a, b]$.
- If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.

## Example 1

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which $f$ is increasing and on which $f$ is decreasing.

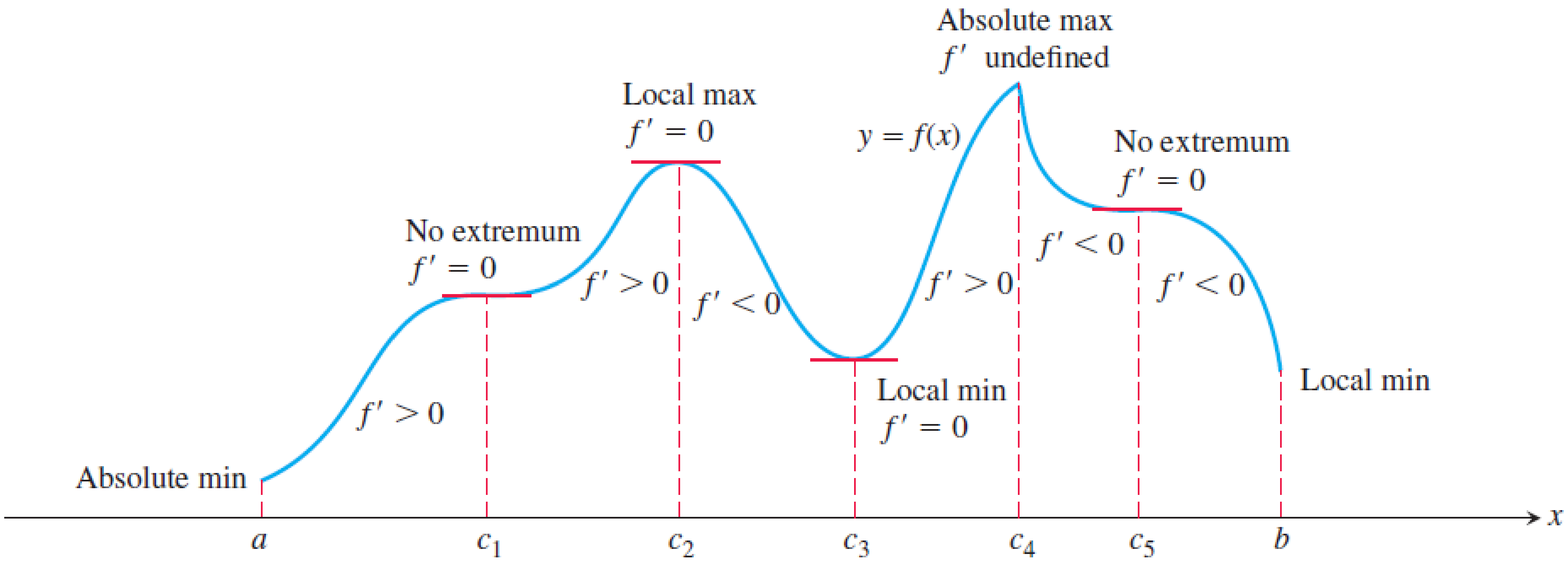
## Example 2

Determine where the following function is increasing and where it is decreasing.

$$f(x) = x^3 - \frac{3}{2}x^2 - 6x + 3$$

## ③ First Derivative Test for Local Extrema

- At the points where $f$ has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.). Similarly, at points where $f$ has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. In summary, at a local extreme point, the sign of $f'(x)$ changes.



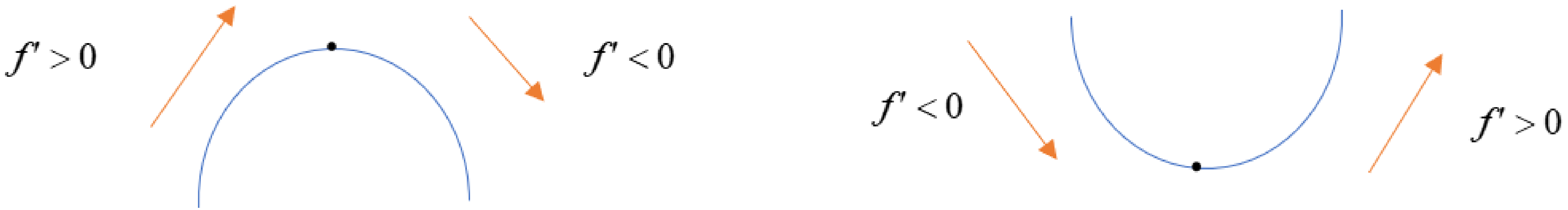
## ● First Derivative Test for Local Extrema

Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself. Moving across this interval from left to right,

**1.** if $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$;

**2.** if $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$;

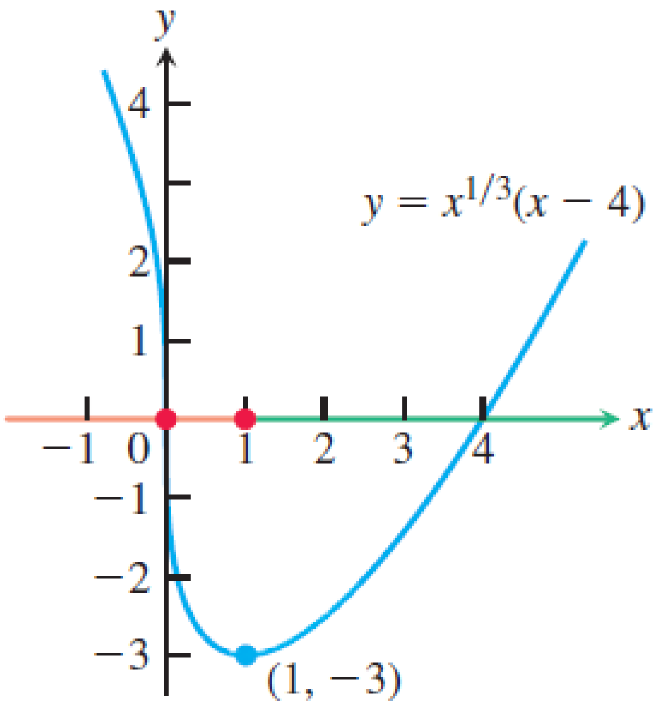
**3.** if $f'$ does not change sign at $c$, then $f$ has no local extremum at $c$.



**Example 3** Find the critical points of

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$$

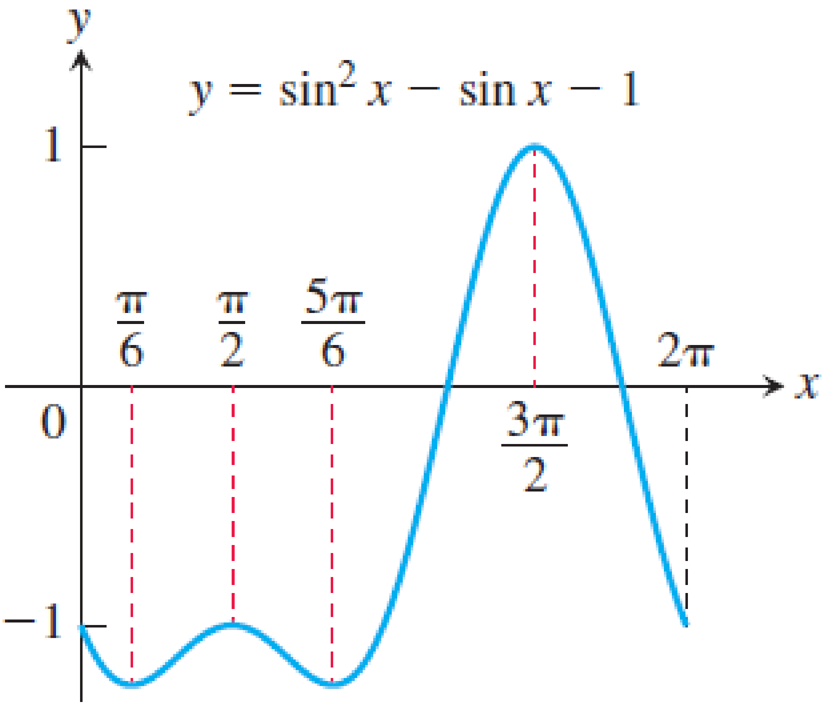
Identify the open intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.



**Example 4** Within the interval $[0, 2\pi]$ find the critical points of

$$f(x) = \sin^2 x - \sin x - 1.$$

Identify the open intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.



**Skill Practice 1** For the function

$$f(x) = x\sqrt{1 - x^2}$$

**a.** What are the critical points of $f$?

**b.** On what open intervals is $f$ increasing or decreasing?

**c.** At what points, if any, does $f$ assume local maximum and minimum values?

**Skill Practice 2**

Determine the values of constants $a$, $b$, $c$, and $d$ so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.