CALCULUS

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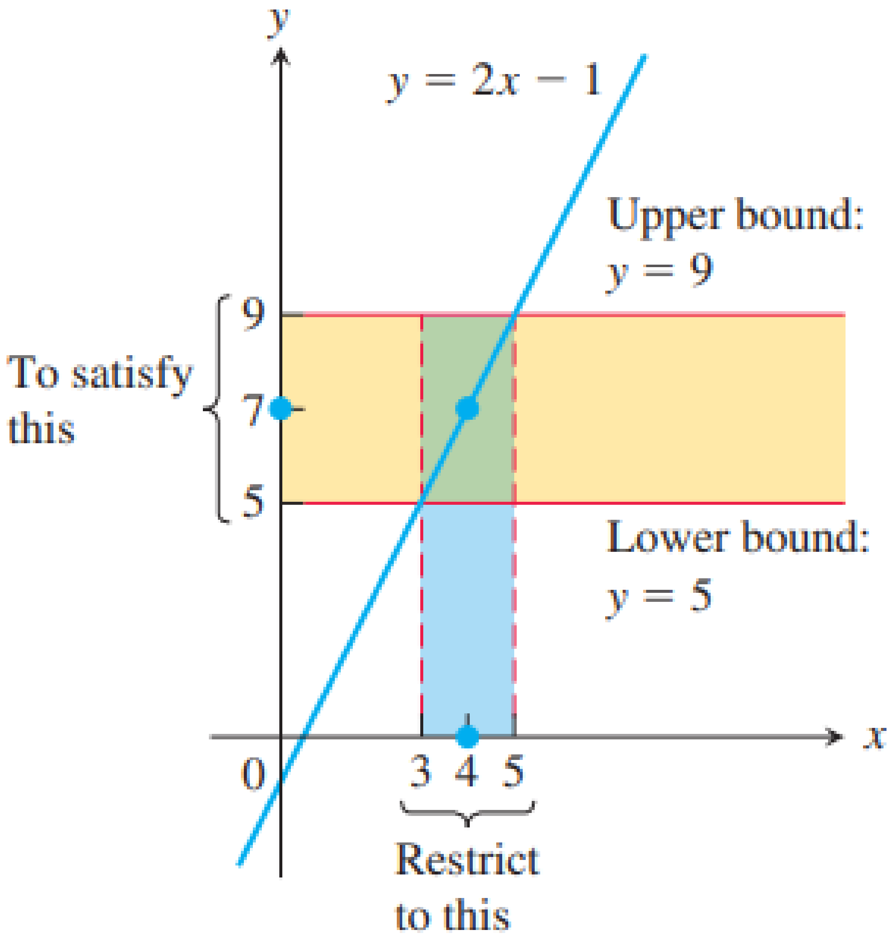
Example 1

Consider the function $y = 2x–1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so that $\lim_{x\to 4}(2x-1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x–1$ differs from 7 by, say, less than 2 units?



- In the previous example we determined how close x must be to a particular value c to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L.
- To show that the limit of $f(x)$ as $x \to c$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than any prescribed error, no matter how small, by holding x close enough to c.
- To describe arbitrary prescribed errors, we introduce two constants, δ (delta) and ε (epsilon). These Greek letters are traditionally used to represent small changes in a variable or a function.

① Precise Definition of Limit

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x\to c} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

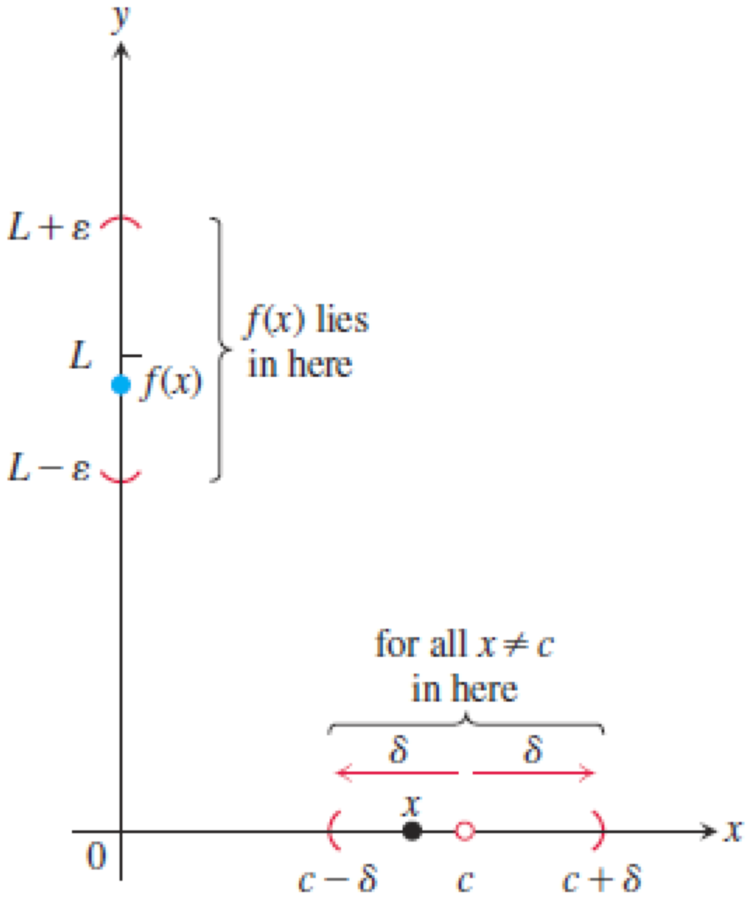


FIGURE 2.17 The relation of δ and ε in the definition of limit.

Example 2 Show that $\lim\limits_{x \to 1}(5x - 3) = 2$.

Example 3 Prove the following results in Section 2.2.

(a) $\lim\limits_{x \to c} x = c$.

(b) $\lim\limits_{x \to c} k = k$ (k constant).

② Finding Deltas Algebraically for Given Epsilons

How to Find Algebraically a δ for a Given f, L, c, and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$. Note that we do not require the inequality to hold at $x = c$. It may hold there or it may not, but the value of f at $x = c$ does not influence the existence of a limit.
2. *Find a value of* $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq c$ in this δ-interval.

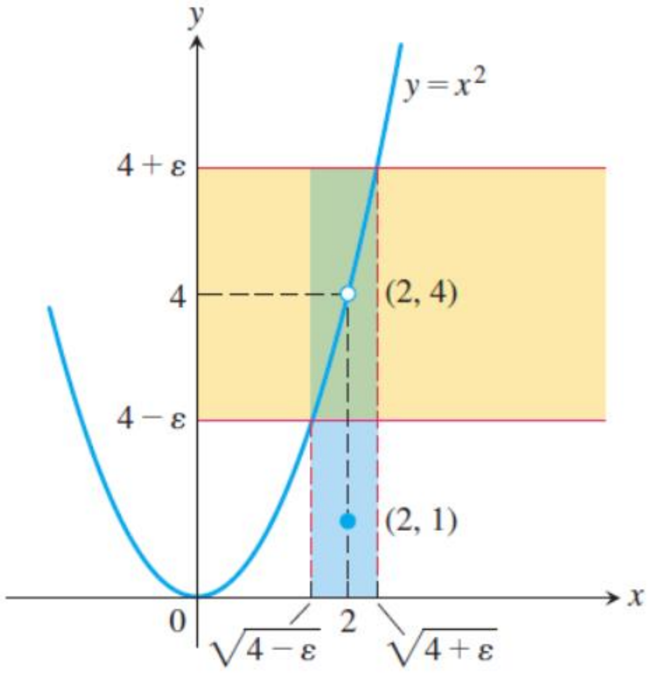
Example 4

For $\lim_{x\to 5}\sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that

$$\left|\sqrt{x-1} - 2\right| < 1 \qquad \text{whenever} \qquad 0 < |x-5| < \delta$$

Example 5 Prove that $\lim_{x\to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$



Solution:

Need to show that given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - 4| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-2| < \delta$$

Skill Practice 1

Prove:

$$\lim_{x\to 9}\sqrt{x-5}=2$$

Skill Practice 2

(1) Find $L=\lim_{x\to -2}\sqrt{1-4x}$

(1) Given ε = 1, find a δ > 0 such that

$$\left|\sqrt{1-4x}-L\right|<1 \quad \text{whenever} \quad 0<|x-c|<\delta$$