CALCULUS

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Spring 2025

- We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns.

① Concavity

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

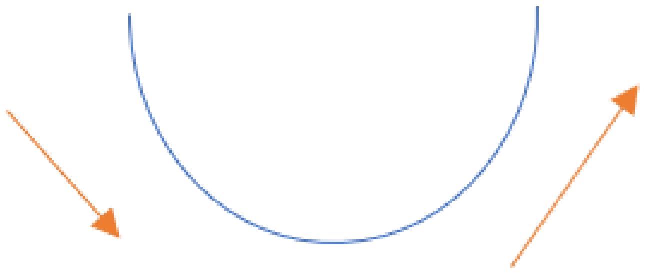
(b) **concave down** on an open interval I if f' is decreasing on I.

A function whose graph is concave up is also often called convex.

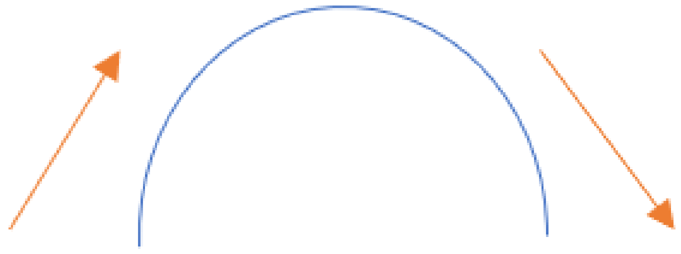
② The Second Derivative Test for Concavity

- Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up (convex).
2. If $f'' < 0$ on I, the graph of f over I is concave down (concave).



Concave up

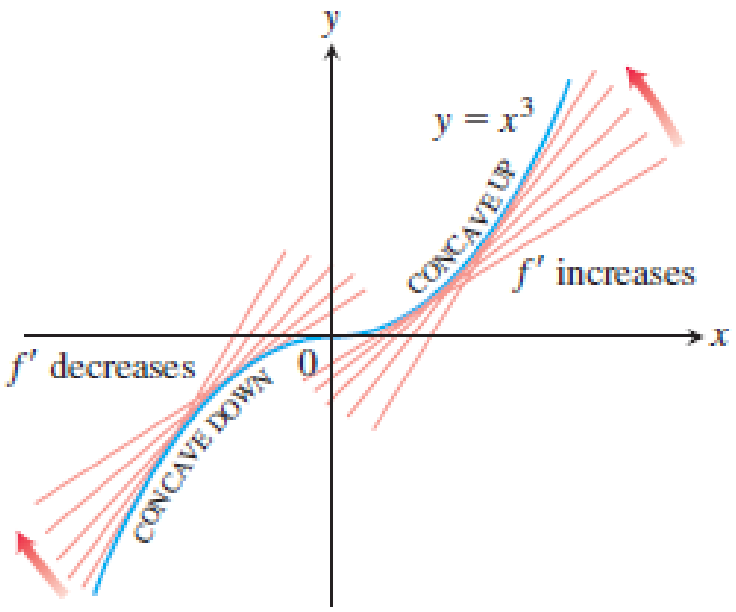


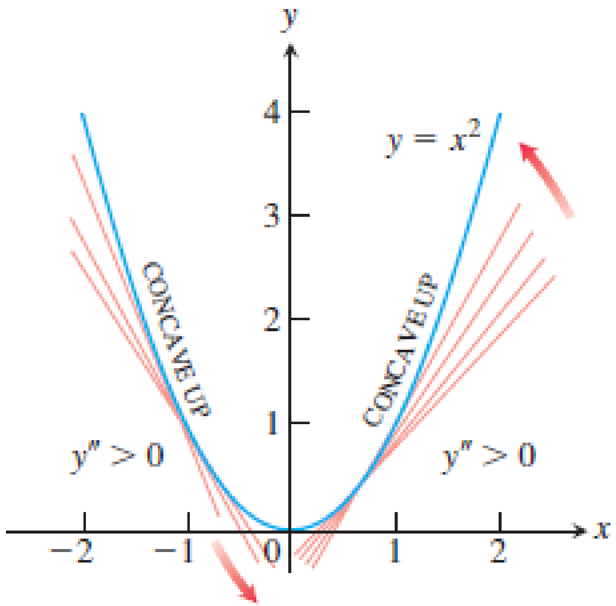
Concave down

Example 1

(a) The curve $y = x^3$ is concave down on $(-\infty, 0)$, where $y'' = 6x < 0$, and concave up on $(0, \infty)$, where $y'' = 6x > 0$.

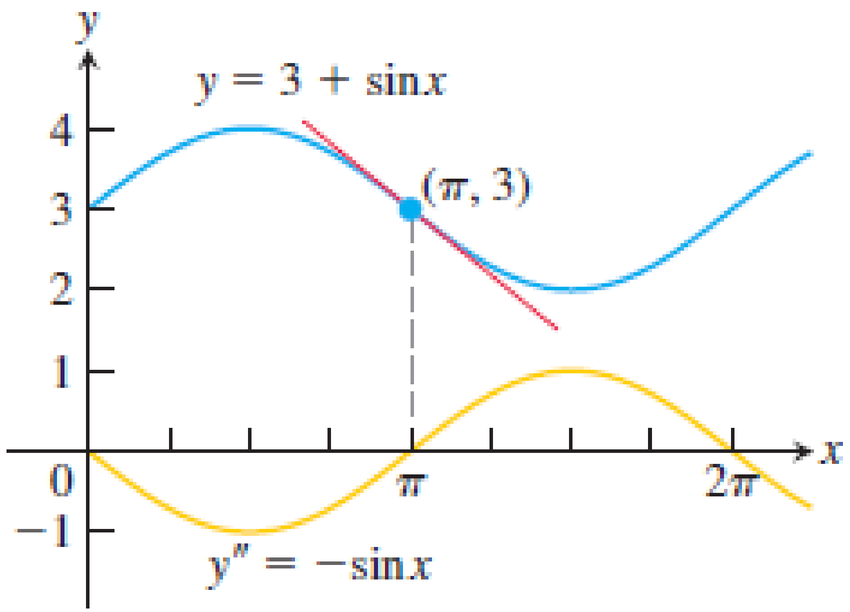
(b) The curve $y = x^2$ is concave up on $(-\infty, \infty)$ because its second derivative $y''= 2$ is always positive.





Example 2

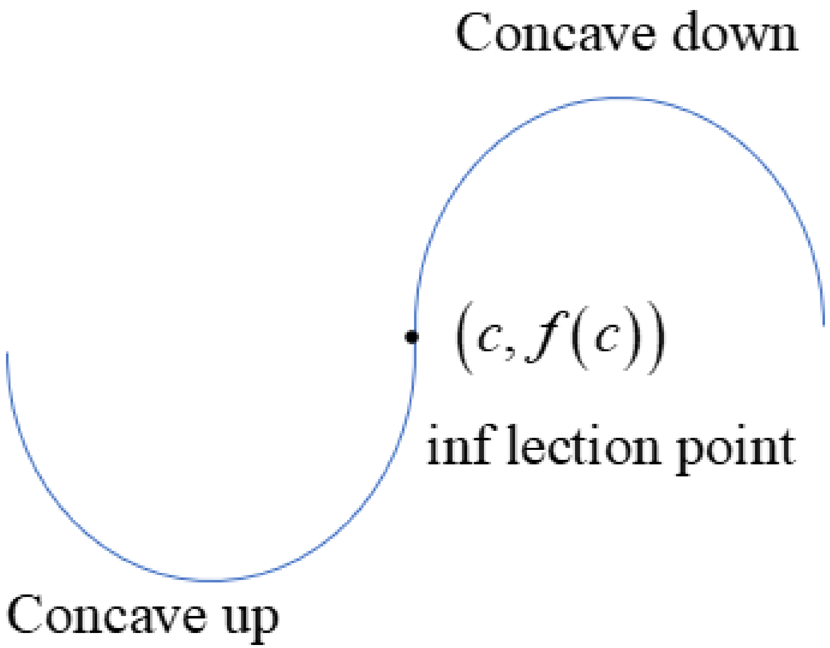
Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

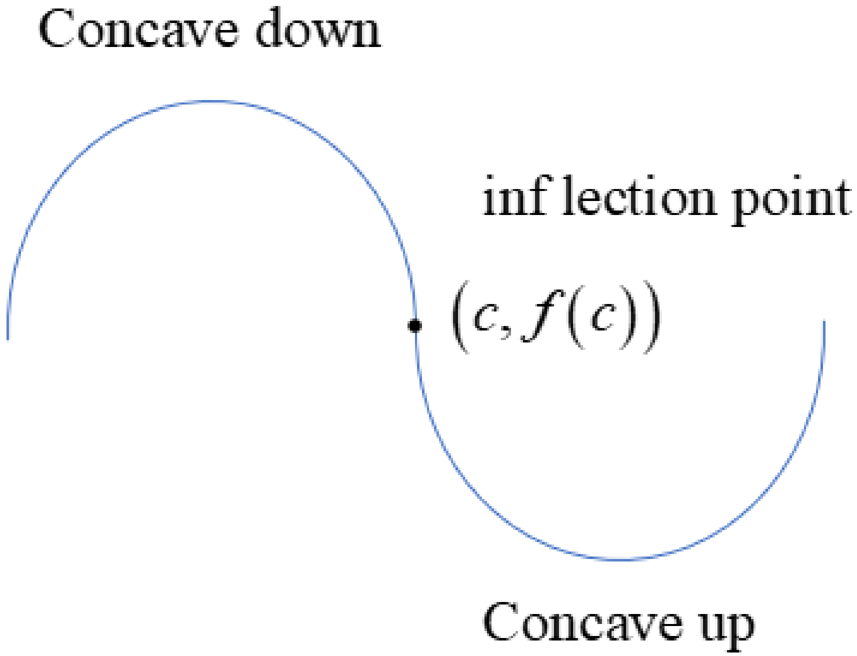


- The curve $y = 3 + \sin x$ changes concavity at the point $(\pi, 3)$. Since the first derivative $y' = \cos x$ exists for all x, we see that the curve has a tangent line of slope -1 at the point $(\pi, 3)$. This point is called a **point of inflection** of the curve.

③ Points of Inflection

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.





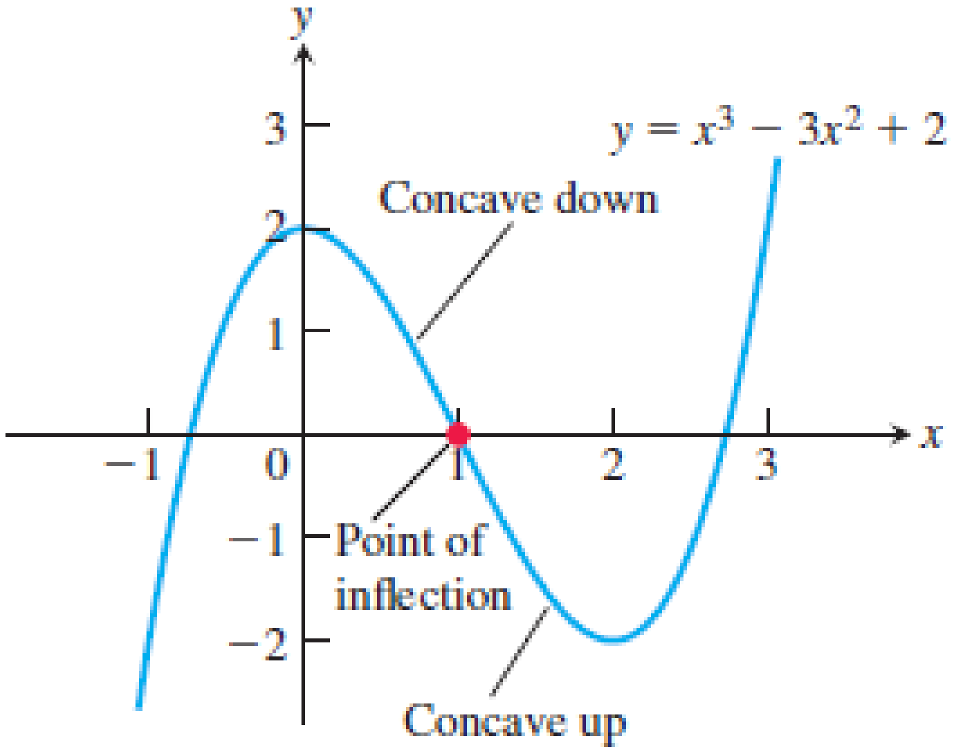
- **At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.**

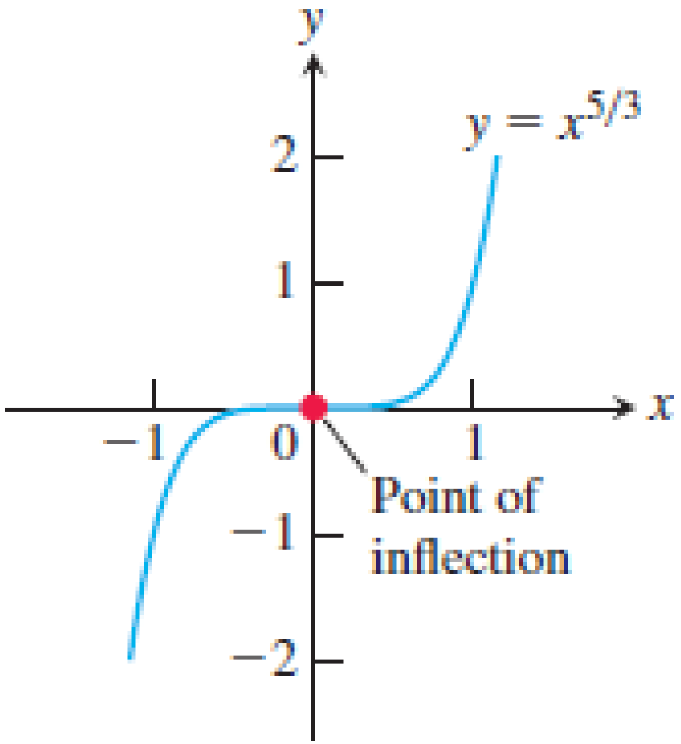
Example 3

Determine the concavity and find the inflection points of the function

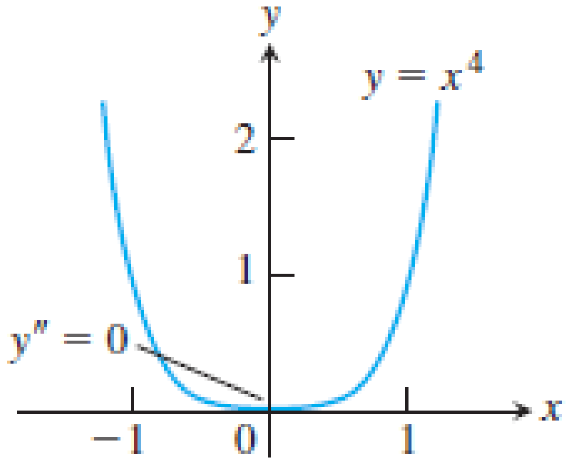
$$f(x) = x^3 - 3x^2 + 2.$$

Example 4 Find the inflection point of $f(x) = x^{5/3}$.





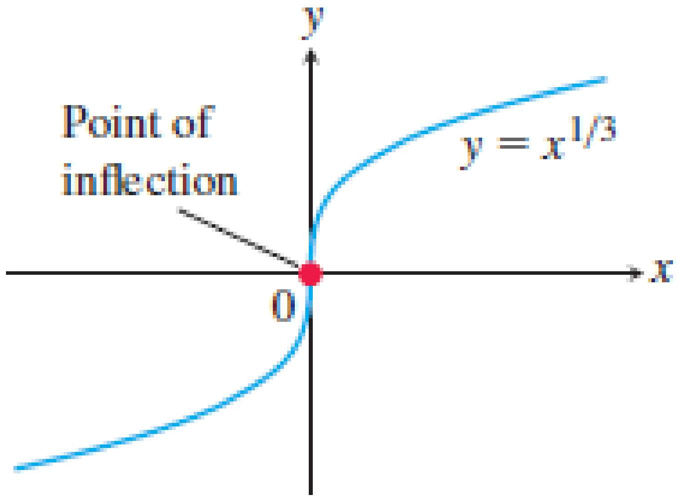
Example 5 The curve $y = x^4$ has no inflection point at $x = 0$.



Example 6 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}$$

However, both y' and y'' fail to exist at $x = 0$, and there is a vertical tangent there.



④ Second Derivative Test for Local Extrema

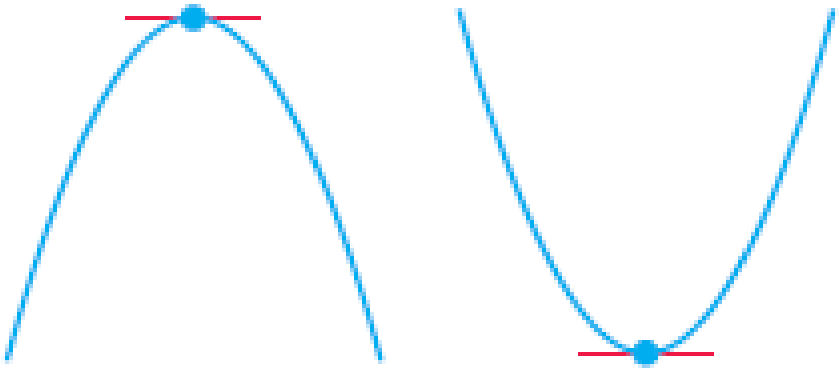
THEOREM 5 – Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

⑤ Sketch a graph of the function that captures its key features:

Procedures to sketch the graph of $f(x)$:

1. Identify the domain of f and any symmetries the curve may have.
2. Find f' and f''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

4.4 Concavity and Curve Sketching

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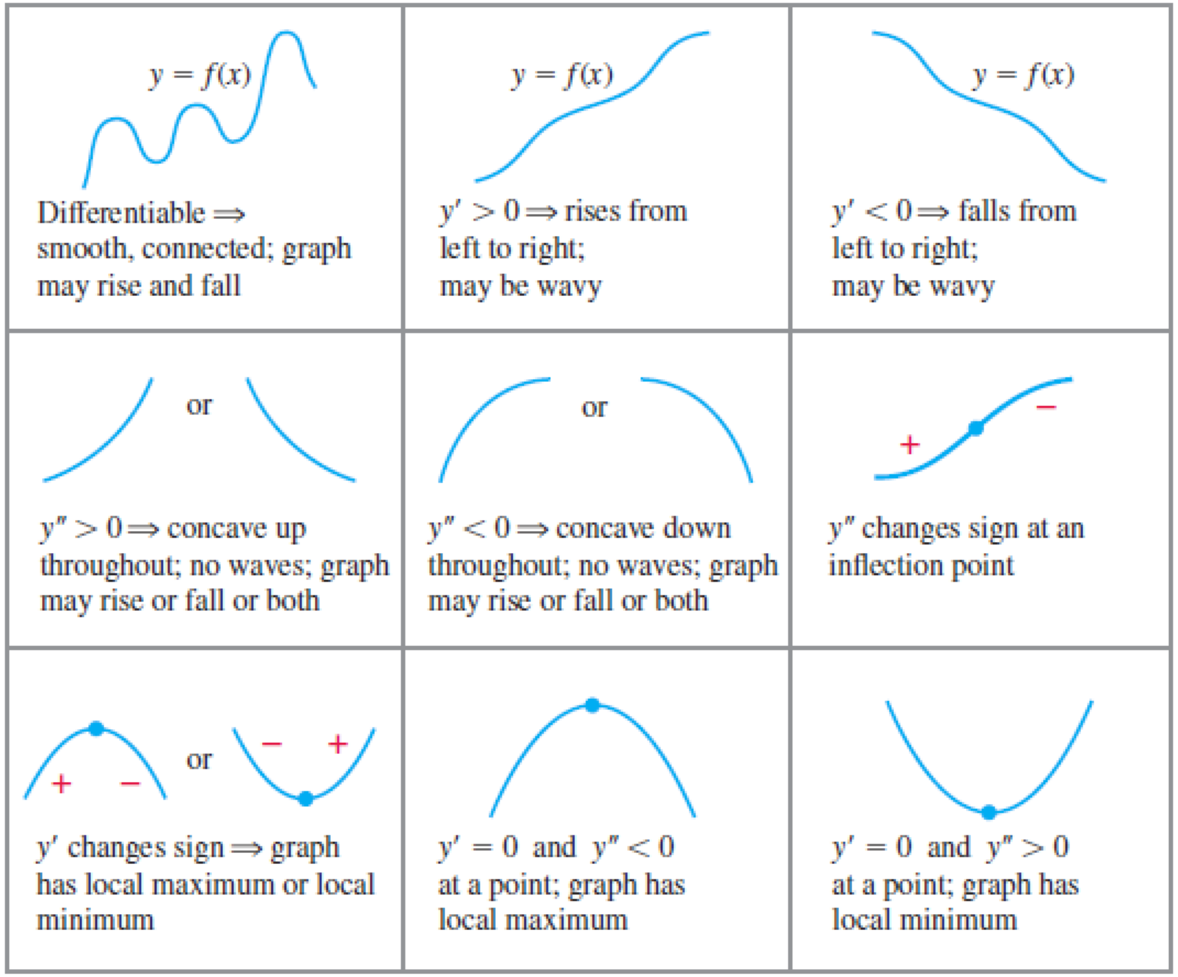
⑤ Graphical Behavior of Functions from Derivatives

- This figure indicates how the first two derivatives of a function affect the shape of its graph.



Example 7 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

Example 8 Sketch the graph of

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

	$x < -\sqrt{3}$	$-\sqrt{3} < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \sqrt{3}$	$x > \sqrt{3}$
$f'(x)$	< 0	< 0	> 0	> 0	< 0	< 0
$f''(x)$	< 0	> 0	> 0	< 0	< 0	> 0

Example 9 Sketch the graph of

$$f(x) = \frac{x^2+4}{2x}$$

	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$x > 2$
$f'(x)$	> 0	< 0	< 0	< 0
$f''(x)$	< 0	< 0	> 0	> 0