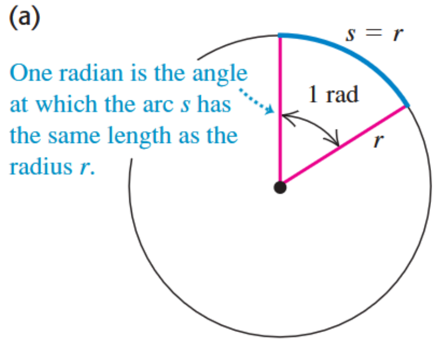
# 9 Rotation of Rigid Bodies

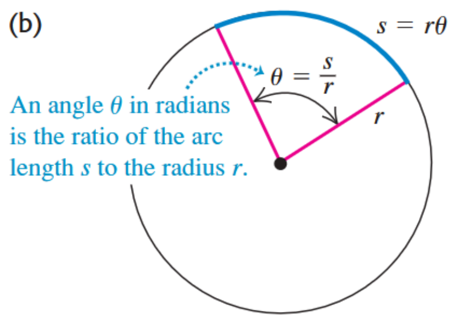
## Angular Velocity and Acceleration

One radian (1 rad) is the angle subtended at the centre of a circle by an arc length of the radius of the circle:

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta$$

The angle $\theta$ is subtended by an arc of length $s$ on a circle of radius $r$.




Angle in radians is the ratio of two length, so it is a **pure number**:

$$1\ \text{rad} = \frac{360^\circ}{\pi}$$

### Angular Velocity

> The rotational motion of rigid bodies are measured in terms of the rate of change of $\theta$

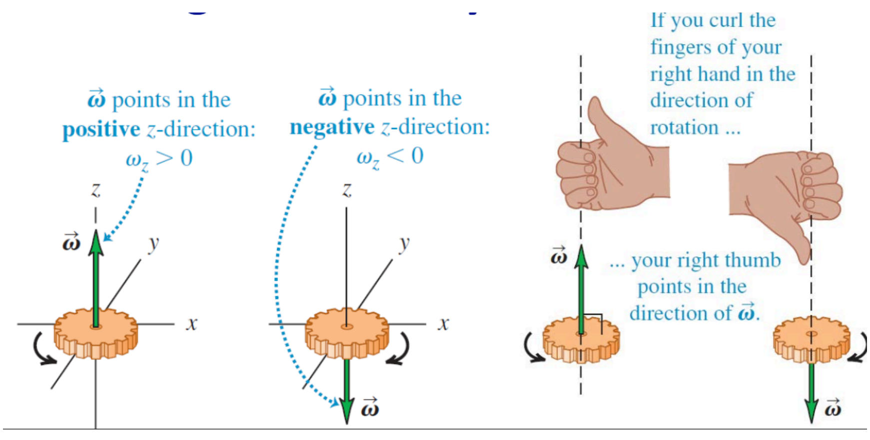
the ratio of the angular displacement $\Delta\theta = \theta_2 - \theta_1$ to $\Delta t$:

$$\omega_{av-z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

instantaneous angular velocity:

$$\omega_z = \lim_{t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular velocity $w_z$ can be positive (anti-clockwise) or negative (clockwise). The angular speed $\omega$ is the magnitude of the angular velocity.



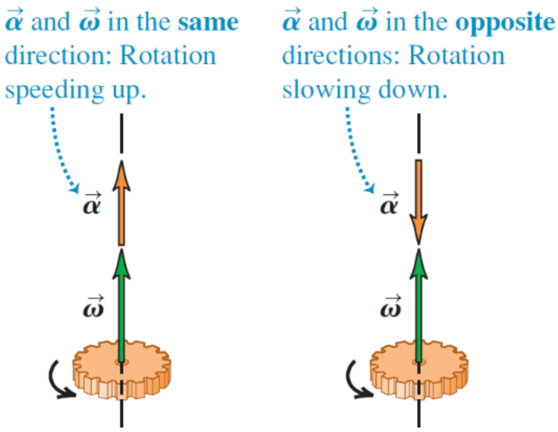

## Converting $T$ time period to Angular Velocity

$$\omega = \frac{2\pi}{T}$$

## Angular Acceleration

$$a_{av-z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}$$

$$a_z = \lim_{t \to 0} \frac{\Delta w_z}{\Delta t} = \frac{dw_z}{dt} = \frac{d^2\theta}{dt^2}$$




## Rotation with Constant Angular Acceleration

$$w_z = w_{0z} + a_z t$$

$$\theta = \theta_0 + w_{0z} t + \frac{1}{2} a_z t^2$$

$$w_z^2 = w_{0z}^2 + 2a_z(\theta - \theta_0)$$

$$\theta - \theta_0 = \frac{1}{2}(w_{0z} + w_z)t$$

## Relating Linear and Angular Kinematics

linear tangential velocity:

$$v_{tan} = r\omega_z$$

linear tangential acceleration:

$$a_{tan} = r\alpha_z$$

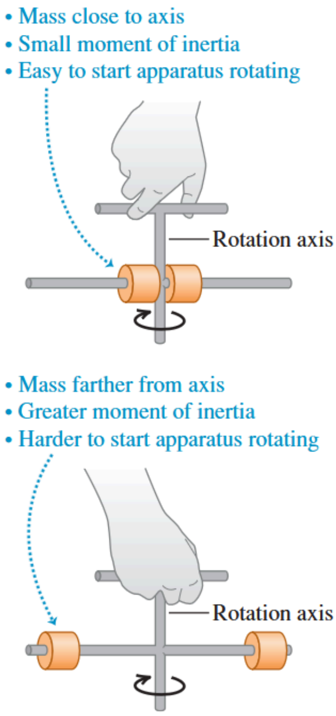
linear radial acceleration:

$$a_{rad} = \omega r^2$$

## Energy in Rotational Motion

in translational sense, the property of mass to resist motion called its *inertia*

in rotation, the same/similar is true: the *moment of inertia* is the property of mass to resist rotational motion:




$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots = \sum_i m_i r_i^2$$

closer the (distribution) of mass is to the rotational axis, easier it is to rotate the system, and further the distance to the rotational axis, harder it is rotate the system.

akin to kinetic energy in translational motion, kinetic energy in rotational motion is:

$$K = \frac{1}{2} I \omega^2$$

the total kinetic energy therefore:

$$K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

**CAUTION** **Moment of inertia depends on the choice of axis** The results of parts (a) and (b) of Example 9.6 show that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to just say, "The moment of inertia of this body is 0.048 kg · m$^2$." We have to be specific and say, "The moment of inertia of this body *about the axis through B and C* is 0.048 kg · m$^2$."

## Parallel-Axis Theorem

$$I_P = I_{cm} + Md^2$$

the new moment of inertia is the center of mass inertia $I_{cm}$ plus the total mass times the square of the distance from the center position

## Moment of Inertia Calculations

Using the definition of moment of inertia:

$$I = m_1 r_1^2 + m_2 r_2^2 + ... = \sum_i m_i r_i^2$$

The moment of inertia of an object:

$$I = \int r^2 \, dm$$

e.g. finding the moment of inertia of a disk https://www.youtube.com/watch?v=0aLGCbL3htQ

by ratio, the infinitesimal mass over its area should be of the same proportion to the total mass by the total area:

$$\frac{dm}{2\pi r\ dr} = \frac{M}{\pi R^2}$$

we get:

$$dm = \frac{2Mr\ dr}{R^2}$$

$$I = \frac{2M}{R^2}\int_0^R r^3\ dr = \frac{2M}{R^2}\left(\frac{1}{4}R^4\right) = \frac{1}{2}MR^2$$