Chapter 5.1 Oversimplified

> 💡 Please use this only as a reference and up to your own interpretation.

Eigenvectors

> 💡 Eigenvectors are vectors that do not change direction (same direction means a multiple of a scalar) when multiplied by a matrix, say A.
>
> $$e.g.\ A = \begin{bmatrix} -2 & 2 \\ 0 & 2 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
>
> $$Av = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad Aw = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
>
> - Here Aw is a scalar multiple of w (a multiple of 2 to be exact), hence the vector w is an **eigenvector** of matrix A.

Eigenvalues

> 💡 From the equation $Ax = \lambda x$ the eigenvalues are the possible values of the λ.
>
> In the example above, one of the λ is equal to 2.
>
> Basically, eigenvalues are the magnitude for scalar multiples of the eigenvectors.

How to Find the Eigenvalues

- There is a condition that the number λ is an eigenvalue if and only if $A - \lambda I$ is singular.
- In other words, $\det(A - \lambda I) = 0$ (must be).
- So use the equation:

$$\det(A - \lambda I) = 0$$

$$e.g.\ \ A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

- First find the matrix $A - \lambda I$:

$$A - \lambda I = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{bmatrix}$$

- Next find the determinant of $A - \lambda I$:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \det = ad - bc$$

$$\det\begin{pmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{pmatrix} \to (-1-\lambda)(-\lambda) - (3)(2)$$

$$(\lambda + \lambda^2) - (6) = \lambda^2 + \lambda - 6$$

- Now there is a quadratic equation to be solved:

$$\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$$

$$\lambda = -3, \quad \lambda = 2$$

- The solutions to the λ are the **eigenvalues**.

How to find the Eigenvectors

- To find the Eigenvectors using the Eigenvalues we use the formula $(A - \lambda I)x = 0$.
- Using the previous example Eigenvalues:

$$\lambda_1 = -3 \text{ and } \lambda_2 = 2$$

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}, \quad \lambda_1 I = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$A - \lambda_1 I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

- We equal the matrix to 0 and solve, just like finding the null space. In fact, the eigenvectors are in null space:

$$A - \lambda_1 I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- From here you can obviously just eyeball and say that $x_1 = 3$ and $x_2 = -2$, which is true and correct.
- However, if you want a more robust approach then finding the Null space is better:

$$(\text{row 2 - row 1}) \to \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\to \begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_2 &= x_2 \end{aligned} \quad \to \quad \begin{aligned} x_1 &= -\frac{3}{2} \\ x_2 &= x_2 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} \right\}$$

$$\text{eigenvector} = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} \text{ when } \lambda = 2$$

- Which means any **scalar multiple** of the null space basis $\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$ is an eigenvector when $\lambda = -3$.
- For $\lambda = 2$:

$$A - \lambda_2 I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{aligned} -3x_1 + 3x_2 &= 0 \\ x_2 &= x_2 \end{aligned}$$

$$\rightarrow \begin{aligned} x_1 &= x_2 \\ x_2 &= x_2 \end{aligned} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\text{eigenvector} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ when } \lambda = 2$$

Determinant and Trace

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- The product of the eigenvalues is equals the determinant.

$$\lambda_1 \times \lambda_2 \times \ldots \times \lambda_n = \text{determinant}$$

- The sum of the n eigenvalues equals the sum of the n diagonal entries (which is also called the **trace**)

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = a_{11} + a_{22} + \ldots a_{nn}$$