Chapter 4 Oversimplified

💡 Please use this only as a reference and as up to your own interpretation.

Space

💡 The **space** R^n consists of all column vectors v with n components.

Just imagine a box (three components) any point inside that box is some kind of vector.

Vector space is a set of vectors together with rules for vector addition and scalar multiplication.

e.g.

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in R^3 as it has 3 components

---

Subspace

💡 A **subspace** is a set of vectors that (including the zero vector) that satisfies:

- $v + w$ is in the subspace (given that v and w are in the subspace)
- cv is in the subspace (any scalar multiplication of v)
- which also means $cv + dw$ is in the subspace (all linear combinations of v and w)

It is important to remember that a subspace **must contain the zero vector (the origin),** otherwise it is not a subspace.

Subspace of V containing vectors $\vec{v}$ and $\vec{w}$ must contain all their linear combinations.

- $C(A)$ is a subspace of R^n and null space $N(A)$ is subspace of R^n.

- Subspace of V spanned by a set of vectors:
  - let SS be the set containing all combinations of vectors in S, then SS is a subspace of V spanned by S, which is the smallest subspace of Vcontaining S.
  - The smallest subspace is the all combinations of vectors in a set

---

Column Space

> 💡 Column space (of a matrix) is all the possible linear combinations of the columns (of the matrix).
>
> The system
> $Ax = b$ is solvable only if b is in the column space of A

How to find the Column Space

- To find the column space, you must row reduce (elimination) the matrix. (In order to acquire the pivots, all components under the pivots must be zero).

$$e.g.\ A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}$$

$$\text{(row 2 - row 1) \& (row 3 - row 1)} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix}$$

$$(\text{row } 3 + \text{row } 2) \rightarrow U = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- From here you can see that there is only **two pivots (col 1 and col 2)**, meaning that out of the 4 columns, only two of them make up a space.
- The rest can be expressed in terms of the first two (meaning that they are dependent and free columns).
- Hence, the column space of the matrix is the span (all scalar multiple) of the first two columns:

$$C(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}$$

- Check:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix} \text{(col 3)}$$

$$(-2) \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix} \text{(col 4)}$$

Null space

> 💡 A null space is a space that contains all solutions x that produces $Ax = 0$.
>
> Basically, space of vectors (in
> R^n) that multiplies the matrix and produces 0.

How to find the Null space

$$Ax = 0$$

- The null space is simply the (all the possible) solutions to the x's that produce the zero vector.

$$e.g. \quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

- Similar to finding the column space, we must first turn the matrix into an echelon form.

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$(\text{row 2 - row 1}) \to \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$(\text{row 3 - row 2}) \to U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Here, you can see that there is only two pivots (col 1 and col 3, or the variables x_1 and x_3). The rest of the variables (x_2, x_4, x_5) are **free variables.**
- With $r = 2$ (number of pivots) there are $n - r = 3$ free variables (n: number of columns), or in other words, the basis for the null space is 3.
- Each free variable leads to a special solution to $A\overrightarrow{x} = \overrightarrow{0}$, $R\overrightarrow{x} = \overrightarrow{0}$.
- When finding the null space, you must find the **Row Reduced Echelon Form** (RREF) of the matrix (condition that all pivots are 1's with zeros above and below).

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(\text{row 1 - (2) row 2}) \ \to R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Now, you need to express the pivot variables in terms of the free variables, similar to $Ax = b \to x = A^{-1}b$

$$
\begin{aligned}
x_1 + 2x_2 &= 0 \\
x_3 + 2x_4 + 3x_5 &= 0
\end{aligned}
$$

$$
\rightarrow \begin{aligned}
x_1 &= -2x_2 \\
x_3 &= -2x_4 - 3x_5
\end{aligned}
$$

- The free variables can be expressed in terms of themselves since they are dependent on the pivot variables, and do not have their own solutions.

$$
\begin{aligned}
x_1 &= -2x_2 \\
x_2 &= x_2 \\
x_3 &= -2x_4 - 3x_5 \\
x_4 &= x_4 \\
x_5 &= x_5
\end{aligned}
\quad \rightarrow \quad
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}
$$

- The null space is the columns made up by the free variables, or in other words, the special solutions:

$$
N(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}
$$

- Any one of these columns (or vectors) multiply matrix A to produce the zero vector. (You can check).

Complete Solution

$$
x = x_p + x_n
$$

- where x_p is particular solution for $Ax_p = b$ and x_n is special solution (null space) for $Ax_n = 0$.
- You can reason that when a matrix does not have a full set of pivots, in other words, there are free variables **dependent** on the pivot variables, the free variables do not contribute to the final answer (b). So they are assumed to produce 0.
- Finding the Complete Solution is similar to finding the null space. Only you need to substitute the zero vector with the final answer.
- In fact, the complete solution to $A\overrightarrow{x} = \overrightarrow{b}$ is $\overrightarrow{x}$ = particular solution + all null space solutions.

$$
Ax = b
$$

$$
\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}
$$

- You should augment A and b for easier elimination to the Row Reduced Echelon Form.

$$\begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 3 \\ 2 & 3 & 5 & 2 & 5 \end{bmatrix}$$

$$\text{(row 2 - row 1) \& (row 3 - row 1)} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & -2 & 1 \end{bmatrix}$$

$$(\text{row } 3 + \text{row } 2) \rightarrow U = \begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(\frac{1}{2} \text{ row } 1\right) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(row 1 - (2) row 2)} \rightarrow R = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Here x_1 and x_2 are pivot variables, while the rest are free variables.
- Now similar to finding the null space, express the pivot variables in terms of the free variables and the real numbers (components of b).

$$\begin{aligned} x_1 + x_3 - 2x_4 &= 4 \\ x_2 + x_3 + 2x_4 &= -1 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 &= 4 - x_3 + 2x_4 \\ x_2 &= -1 - x_3 - 2x_4 \end{aligned}$$

$$\begin{aligned} x_1 &= 4 - x_3 + 2x_4 \\ x_2 &= -1 - x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \quad \rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

---

and this is the complete solution for x where $\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ is the x_p, and $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ is the x_n

---

Independence

💡 The columns of a matrix are **linearly independent** when only $x = 0$ produces $Ax = 0$.

Rank of A = number of pivots = number of nonzero rows in $\mathbb{R}$.

Or the matrix has a Full Rank, or in other words, a full set of pivots and **no free variables**.

That is, simply, when all the columns of a matrix are independent of one another. Every column is included in the Column Space.

Basis

A basis for V is sequence of vectors which are independent and span V.

$$e.g.\ N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{basis} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

- this null space has the following basis (the fundamental or the rudimentary vectors of a space, or in other words, just the vectors in a given space).
- the word "span" just refers to the all possible scalar multiples of the basis vectors.

Dimension

The dimension of a space is the number of vectors in every basis.

$$e.g.\ C(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}$$

- this column space has a dimension of 2 (number of vectors) in a $\mathbb{R}^3$ (number of components) space.

- **The dimension of $C(A)$ equals the rank of A, with the r pivot columns being a basis.**
- **The dimension of $N(A)$ is $n - r$ with the $n - r$ special solutions to $A\vec{x} = \vec{0}$ for a basis.**