CALCULUS

Prof. Liang ZHENG

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① Instantaneous Rates of Change

DEFINITION:

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists.

Example 1 The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

② Motion Along a Line: Displacement, Velocity, Speed, Acceleration

- Suppose that an object (or body, considered as a whole mass) is moving along a coordinate line (an s-axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t: $s = f(t)$.
- Then the velocity (**instantaneous velocity**) is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

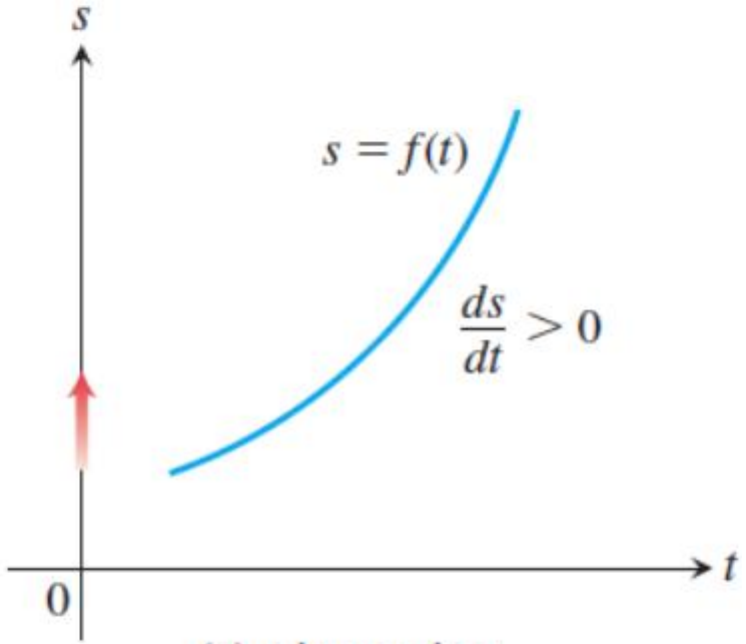
and the **speed** is the absolute value of velocity

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

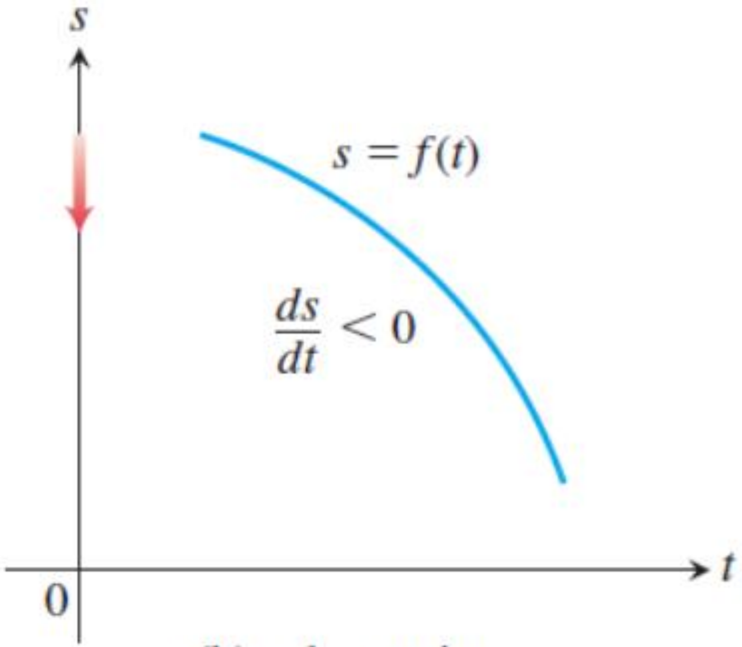
- The **acceleration** is the derivative of velocity with respect to time:

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

- Besides telling how fast an object is moving along the given axis, **its velocity tells the direction of motion**. When the object is moving upward for positive velocity and downward for negative velocity.



(a) s increasing: positive slope so moving upward

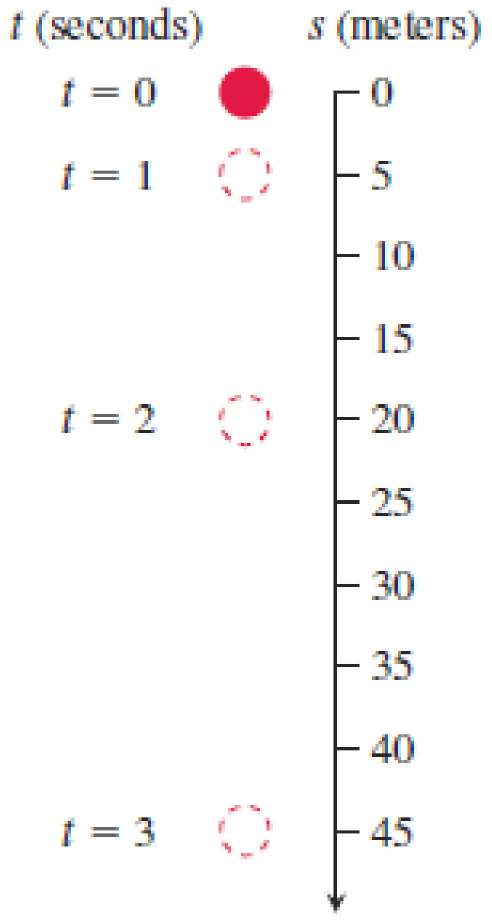


(b) s decreasing: negative slope so moving downward

Example 2 The metric free-fall equation is $s = \frac{1}{2}gt^2$. If the free fall of a heavy ball bearing released from rest at time $t = 0$ sec, then

(a) How many meters does the ball fall in the first 3 sec?

(b) What is its velocity, speed, and acceleration when $t = 3$?



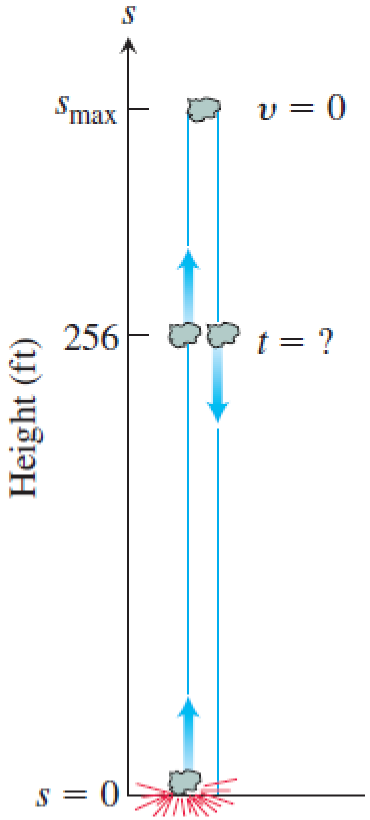
Example 3 A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

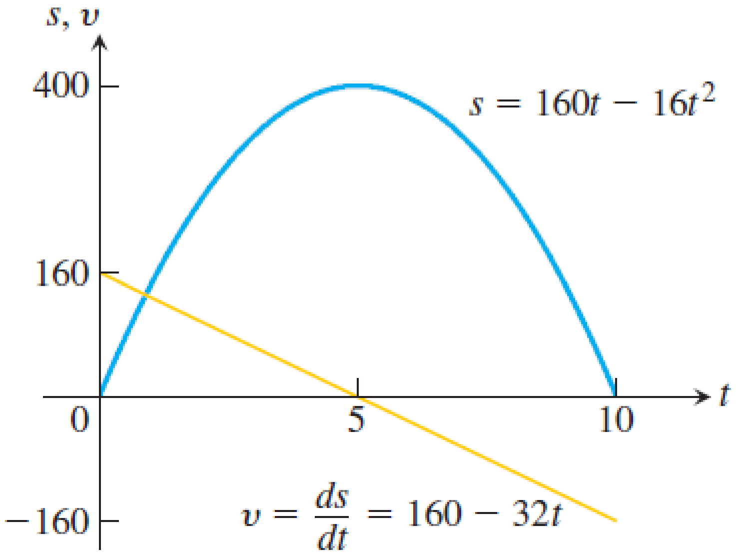
(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?





Skill Practice 1: Draining a tank

It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m}$$

(a) Find the rate dy/dt (m/h) at which the tank is draining at time t.

(b) When is the fluid level in the tank falling fastest? Slowest? What are the values of dy/dt at these times?

(c) Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.