College Algebra and Trigonometry

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① Definition of a Rational Function

Let $p(x)$ and $q(x)$ be polynomials where $q(x) \neq 0$. A function f defined by

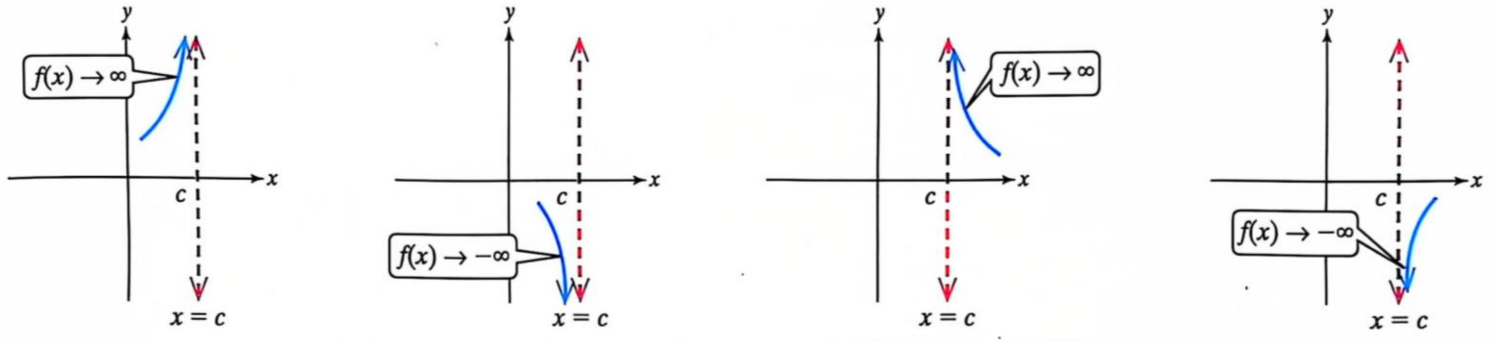
$$f(x) = \frac{p(x)}{q(x)}$$ is called a **rational function.**

Note: The domain of a rational function is all real numbers excluding the real zeros of $q(x)$.

② Identify the Vertical Asymptote of a Rational Function

Definition of a Vertical Asymptote

The line $x = c$ is a **vertical asymptote** of the graph of a function f if $f(x)$ approaches infinity or negative infinity as x approaches c from either side.



Notation
$x \to c^+$
$x \to c^-$
$y \to \infty$
$y \to -\infty$

Identify Vertical Asymptotes of a Rational Function

Consider a rational function defined by $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ have no common factors other than 1. If c is a real zero of $q(x)$, then $x = c$ is a vertical asymptote of the graph of $f(x)$.

Example 2:

Identify the vertical asymptotes.

a) $f(x) = \frac{2}{x-3}$

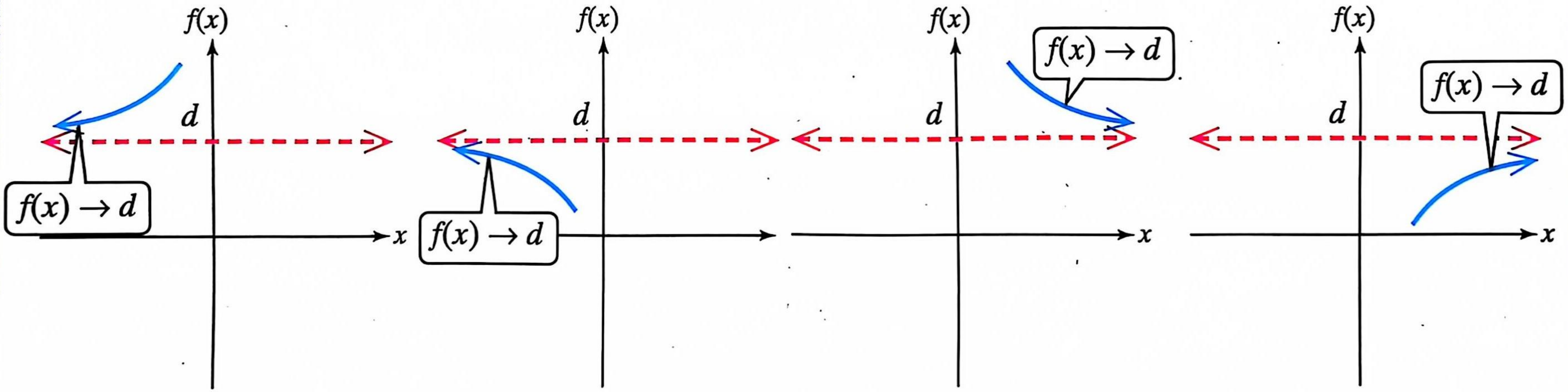
b) $g(x) = \frac{x-4}{3x^2+5x-2}$

c) $h(x) = \frac{4x^2}{x^2+4}$

③ Identify Horizontal Asymptotes

Definition of a Horizontal Asymptote

The line $y = d$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches d as x approaches infinity or negative infinity.



Identifying Horizontal Asymptotes of a Rational Function

Let f be a rational function defined by

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_1 x + b_0}$$

The definition of $f(x)$ indicates that n is the degree of the numerator and m is the degree of the denominator.

1. If $n > m$, then f has no horizontal asymptote.
2. If $n < m$, then the line $y = 0$ (the x-axis) is the horizontal asymptote of f.
3. If $n = m$, then the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote of f.

Example 3:

Find the horizontal asymptotes.

a) $f(x)=\frac{8x^2+1}{x^4+1}$ b) $g(x)=\frac{2x^3-6x}{x^2+4}$ c) $h(x)=\frac{8x^2+9x-5}{2x^2+1}$

Example 4:

Given $h(x)=\frac{8x^2+9x-5}{2x^2+1}$, **Determine the point where the graph of** $h(x)$ **crosses its horizontal asymptote.**

④ Identify Slant Asymptotes

Identify Slant Asymptotes of a Rational Function

- A rational function will have a slant asymptote if the degree of the numerator is exactly one greater than the degree of the denominator.
- To find the equation of a slant asymptote, divide the numerator by the denominator. The quotient will be linear and the slant asymptote will be of the form y = quotient.

Example 5:

Determine the asymptotes of

$$f(x) = \frac{2x^2-5x-3}{x-2}$$

⑤ Graph Rational Functions

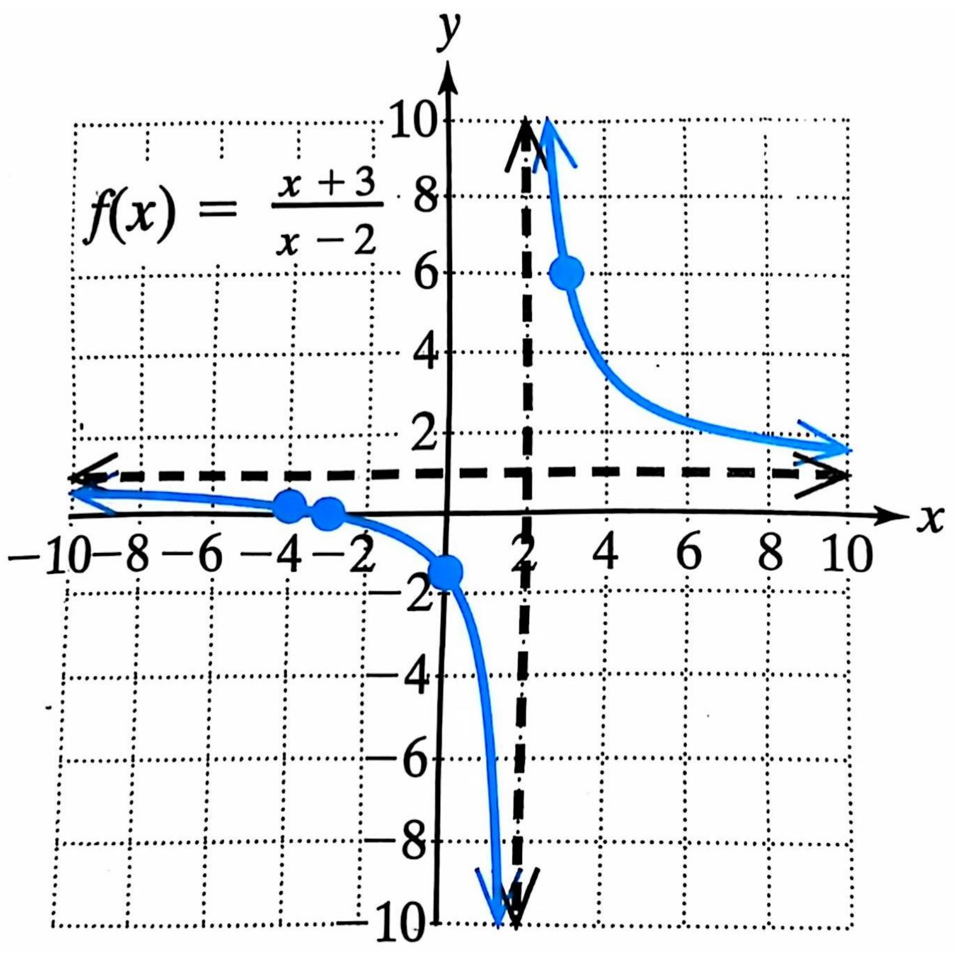
Graphing a Rational Function

Consider a rational function f defined by $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with no common factors.

1. Determine the y-intercept by evaluating $f(0)$.
2. Determine the x-intercept(s) by finding the real solutions of $f(x) = 0$. The value $f(x)$ equals zero when the numerator $p(x) = 0$.
3. Identify any vertical asymptotes and graph them as dashed lines.
4. Determine whether the function has a horizontal asymptote or a slant asymptote (or neither), and graph the asymptote as a dashed line.
5. Determine where the function crosses the horizontal or slant asymptote (if applicable).
6. If a test for symmetry is easy to apply, use symmetry to plot additional points. Recall:
   - f is an even function (symmetric to the y-axis) if $f(-x) = f(x)$.
   - f is an odd function (symmetric to the origin) if $f(-x) = -f(x)$.
7. Plot at least one point on the intervals defined by the x-intercepts, vertical asymptotes, and points where the function crosses a horizontal or slant asymptote.
8. Sketch the function based on the information found in steps 1–7.

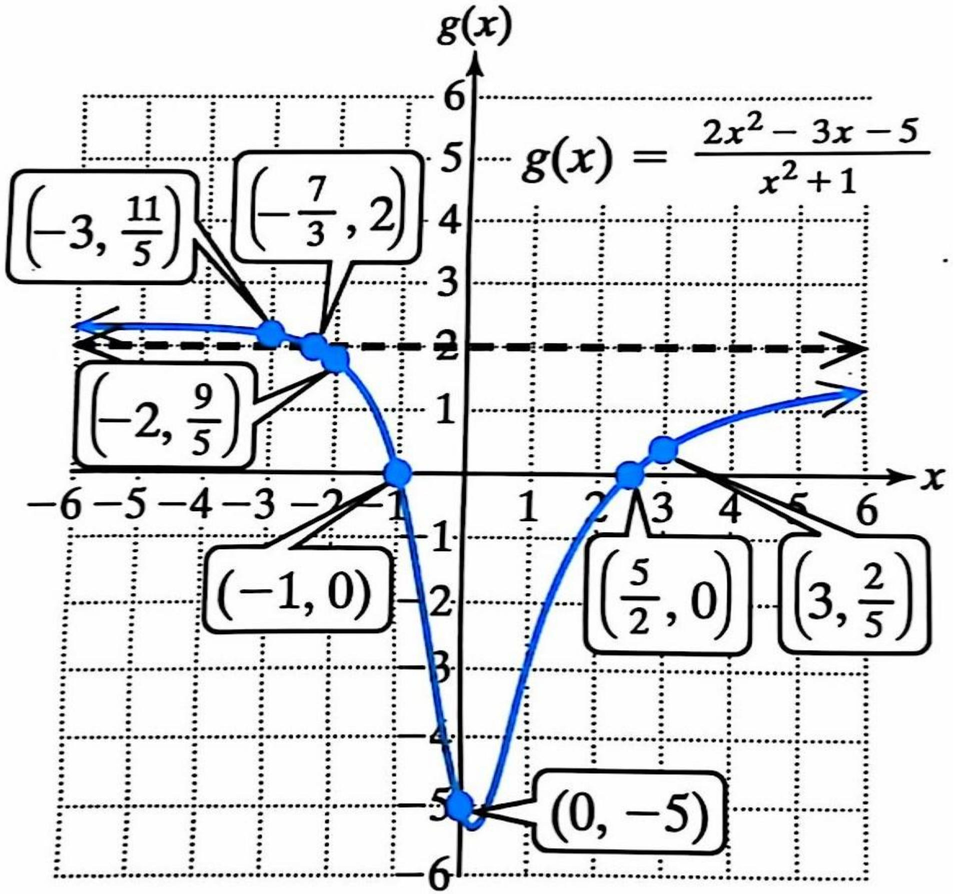
Example 7:

Graph $f(x) = \frac{x+3}{x-2}$



Example 9:

Graph $f(x) = \frac{2x^2-3x-5}{x^2+1}$



① Solve Polynomial Inequalities

Definition of a Polynomial Inequality

Let $f(x)$ be a polynomial. Then an inequality of the form

$f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$ is called a **polynomial inequality.**

Note: A polynomial inequality is nonlinear if $f(x)$ is a polynomial of degree greater than 1.

Procedure to Solve a Nonlinear Inequality

1. Express the inequality as $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$. That is, rearrange the terms of the inequality so that one side is set to zero.
2. Find the real solutions of the related equation $f(x) = 0$ and any values of x that make $f(x)$ undefined. These are the "boundary" points for the solution set to the inequality.
3. Determine the sign of $f(x)$ on the intervals defined by the boundary points.
   - If $f(x)$ is positive, then the values of x on the interval are solutions to $f(x) > 0$.
   - If $f(x)$ is negative, then the values of x on the interval are solutions to $f(x) < 0$.
4. Determine whether the boundary points are included in the solution set.
5. Write the solution set in interval notation or set-builder notation.

Example 1:

Solve the quadratic inequality

$$3x(x-1) > 10 - 2x$$

Example 2:

Solve the cubic inequality

$$x^3 - 3x - 2 \leq 0$$

Example 3:

Solve the quartic inequality

$$x^4 - 12x \geq 8x^2 - x^3$$

② Solve Rational Inequalities

Definition of a Rational Inequality

Let $f(x)$ be a rational expression, then an inequality of the form $f(x) < 0$, $f(x) > 0$, $f(x) \le 0$, $f(x) \ge 0$ is called a rational inequality.

Example 4:

Solve the rational inequalities:

a) $\frac{4x-5}{x-2} \le 3$

b) $\frac{1}{x-2} \ge \frac{1}{x+3}$