CALCULUS

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- This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

① Powers, Multiples, Sums, and Differences

(a) Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

(b) Derivative Constant multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

(c) Derivative Sum and Difference Rules

If u and v are differentiable functions of x, then their sum $u + v$ and difference $u - v$ are differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

(d) Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Note: $z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$

(e) Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

For all x where the powers x^n and x^{n-1} are defined.

Example 1 Differentiate the following powers of x.

(a) x^3 (b) $x^{\frac{2}{3}}$ (c) $x^{\sqrt{2}}$ (d) $\dfrac{1}{x^4}$ (e) $x^{-\frac{4}{3}}$ (f) $\sqrt{x^{2+\pi}}$

Example 2 Differentiate

(a) $3x^2$ (b) $-f(x)$

Example 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Example 4 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

② Products and Quotients

If u and v are differentiable functions of x, then $\dfrac{\mathrm{d}}{\mathrm{d}x}(uv) = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + \dfrac{\mathrm{d}u}{\mathrm{d}x}v.$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{\dfrac{\mathrm{d}u}{\mathrm{d}x}v - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

or in function notation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Example 5 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Example 6 Find the derivative of $y = \dfrac{t^2 - 1}{t^3 + 1}$

Example 7 Find the derivative of $y = \dfrac{(x-1)(x^2 - 2x)}{x^4}$

③ Second- and Higher-Order Derivatives

- If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$.
- The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = y'' = D^2y$$

- Similarly, there is:

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

Example 8 Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

- All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero.

Skill Practice 1

Find the derivatives of all orders of $y = x^6/120$.

Skill Practice 2

Find all points (x, y) on the graph of $f(x) = (x-2)/x$ with tangent lines parallel to the line $y = 2x + 5$.