CALCULUS

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- Historically, logarithms played important roles in arithmetic computations, making possible the great seventeenth-century advances in offshore navigation and celestial mechanics. In this section we define the natural logarithm as an integral using the Fundamental Theorem of Calculus. While this indirect approach may at first seem strange, it provides an elegant and rigorous way to obtain the key characteristics of logarithmic and exponential functions.

① Definition of the Natural Logarithm Function

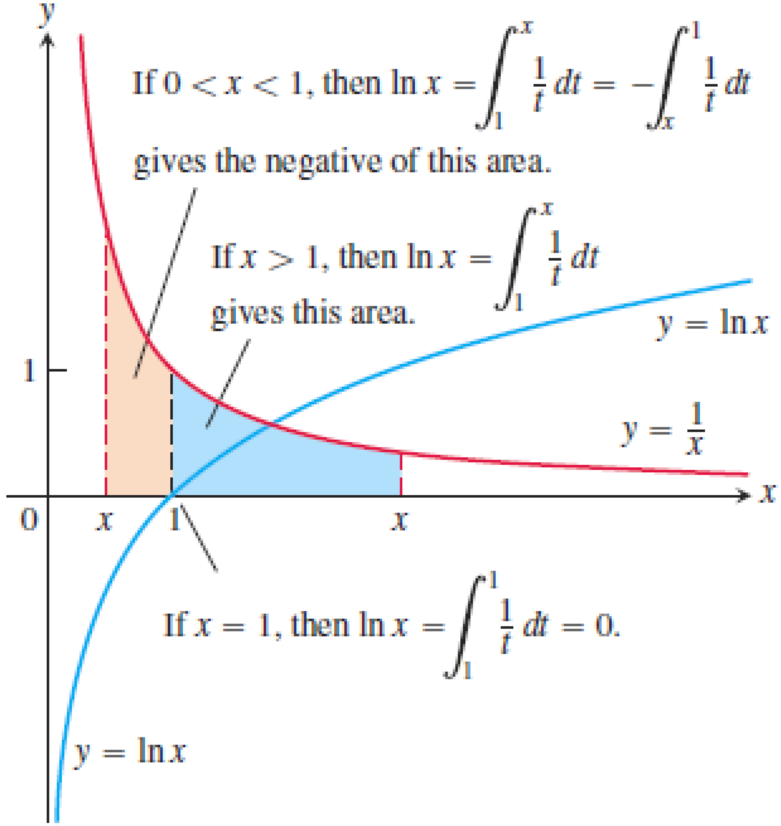
DEFINITION The **natural logarithm** is the function given by

$$lnx = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

- From the Fundamental Theorem of Calculus, $\ln x$ is a continuous function.
- Geometrically, for $x > 1$, then $\ln x$ is the area under the curve $y = \dfrac{1}{t}$ from $t = 1$ to $t = x$.

  For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from $t = x$ to $t = 1$.

  For $x \le 0$, the function is not defined.



- $\ln 1 = \int_1^1 \frac{1}{t}\mathrm{d}t = 0$.
- By using rectangles to obtain finite approximations of the area under the graph of $y = \frac{1}{t}$ and over the interval between $t = 1$ and $t = x$, as in Section 5.1, we can approximate the values of the function ln x. Several values are given in the following Table.

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

- There is an important number between $x = 2$ and $x = 3$ whose natural logarithm equals 1. This irrational number, called ***e***, with the value of 2.718281828459… , is defined as:

$$\ln e = \int_1^e \frac{1}{t}\mathrm{d}t = 1$$

② The Derivative of $y = \ln x$

- By the first part of the Fundamental Theorem of Calculus, we get

$$(\ln x)' = \left(\int_1^x \frac{1}{t}\,\mathrm{d}t\right)' = \frac{1}{x}, \quad x > 0.$$

- The Chain Rule about natural logarithm is

$$(\ln u)' = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x},$$

where $u = u(x)$.

Example 1 Find the derivatives.

(a) $\ln 2x$; (b) $\ln(x^2+3)$; (c) $\ln|x|$.

③ Properties of Logarithms

THEOREM 2 — Algebraic Properties of the Natural Logarithm

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:* $\ln(bx) = \ln b + \ln x$

2. *Quotient Rule:* $\ln\frac{b}{x} = \ln b - \ln x$

3. *Reciprocal Rule:* $\ln\frac{1}{x} = -\ln x$ Rule 2 with $b = 1$

4. *Power Rule:* $\ln x^r = r\ln x$ (r is rational)

Example 2

$$\text{(a)}\quad \ln 4+\ln\sin x=\ln(4\sin x)$$

$$\text{(b)}\quad \ln\frac{x+1}{2x-3}=\ln(x+1)-\ln(2x-3)$$

$$\text{(c)}\quad \ln\sqrt[3]{x+1}=\ln(x+1)^{\frac{1}{3}}=\frac{1}{3}\ln(x+1)$$

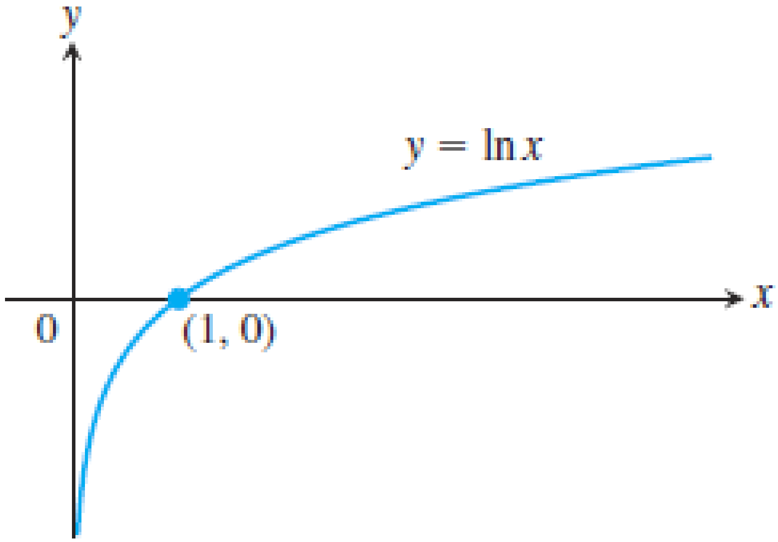
Example 3 Find y' if

$$\text{(a)}\quad y=\frac{1}{2}\ln\frac{1+x}{1-x};$$

$$\text{(b)}\quad y=\ln\sqrt{\frac{(1+x)^5}{(x+2)^7}}.$$

④ The Graph and Range of ln*x*

- The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so ln*x* is an increasing function of *x*. The second derivative, $-1/x^2$, is negative, so the graph of ln*x* is concave down. Domain: $(0, \infty)$, Range: $(-\infty, \infty)$.



Estimate of ln2

(1) $2 < e = 2.718 < 2\sqrt{2} = 2.828 \quad \Rightarrow \quad \ln 2 < 1 = \ln e < \ln 2\sqrt{2} = \frac{3}{2}\ln 2$

$\Rightarrow \quad \frac{2}{3} < \ln 2 < 1$ AND ln2 is slightly greater than $2/3 \approx 0.667$

True Value: ln2 = 0.69314718…

⑤ The integral $\int \frac{1}{u} du$

If u is a differentiable function that is never zero, then

$$\int \frac{1}{u} du = ln|u| + C \quad or \quad \int \frac{f'(x)}{f(x)} dx = ln|f(x)| + C$$

Example 5 Evaluate

(a) $\int \frac{2x}{x^2 - 5} \mathrm{d}x$

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3 + 2\sin\theta} \mathrm{d}\theta$

⑥ The Integrals of tan*x*, cot*x*, sec*x*, and csc*x*.

$$\int tanxdx = \int \frac{sinx}{cosx}dx = \cdots$$

$$\int cotxdx = \int \frac{cosx}{sinx}dx = \cdots$$

$$\int secxdx = \int secx\frac{secx + tanx}{secx + tanx}dx = \cdots$$

$$\int cscxdx = \int cscx\frac{cscx + cotx}{cscx + cotx}dx = \cdots$$

Example 6 Evaluate

$$\int_0^{\frac{\pi}{6}} tan2xdx$$

⑦ Logarithmic Differentiation

- The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

Example 7 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, if

$$y=\frac{\left(x^2+1\right)\left(x+3\right)^{\frac{1}{2}}}{x-1}, \qquad x>1.$$

Skill Practice 1 Solve for t

$$ln(t+1) = ln12 - lnt$$

Skill Practice 2 Find y′ if

$$y = \int_{x^2}^{x^3} lntdt \qquad (x > 0)$$

Skill Practice 3 Evaluate

$$y = \int_{e}^{e^2} \frac{dx}{2x\sqrt{lnx}}$$