CALCULUS

Prof. Liang ZHENG

Spring 2025

- Exponential functions increase or decrease very rapidly with changes in the independent variable. They describe growth or decay in many natural and human-made situations. The variety of models based on these functions partly accounts for their importance.

① Exponential Change (Growth or Decay)

- In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. Examples of such quantities include the size of a population, the amount of a decaying radioactive material, and the temperature difference between a hot object and its surrounding medium. Such quantities are said to undergo **exponential change**.

- If the amount present at time $t = 0$ is called y_0 , then we can find y as a function of t by solving the following initial value problem:

Differential equation: $\dfrac{dy}{dt} = ky$

Initial condition: $y = y_0$ when $t = 0$.

$$\frac{dy}{dt} = ky \quad \Rightarrow \quad \frac{dy}{y} = k\,dt \quad \Rightarrow \quad y = Ae^{kt} \quad \overset{\text{I.C.}}{\Rightarrow} \quad y = y_0 e^{kt}$$

- The quantity y is said to undergo exponential growth if $k > 0$ and exponential decay if $k < 0$.

② Separable Differential Equations

A first-order differential equation is an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

where f is a function of *both* the independent and dependent variables.

The **general solution** is a solution $y(x)$ that contains all possible solutions and it always contains an arbitrary constant.

The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ is **separable** if f can be expressed as a product of a function of x and a function of y. That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y) \quad \Rightarrow \quad \frac{\mathrm{d}y}{h(y)} = g(x)dx \quad \Rightarrow \quad \int \frac{1}{h(y)}\mathrm{d}y = \int g(x)dx$$

Example 1 Solve the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1+y)\mathrm{e}^{x}, \qquad y > -1$$

Example 2 Solve the equation:

$$y(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} = x(1+y^2)$$

Example 3 Solve the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2\mathrm{e}^{-y} \qquad y(0) = 0$$

Unlimited Population Growth

- When the number of individuals becomes large enough, the population can be approximated by a continuous function. Assuming that both the birth rate and the death rate of the population are stable and proportional to the number $y(t)$ of individuals present at any instant t, then the growth rate dy/dt is the birth rate minus the death rate.

$$\frac{dy}{dt} = k_1 y - k_2 y = (k_1 - k_2)y = ky$$

$$\Rightarrow \quad y = y_0 e^{kt}$$

- When k is positive, the equation $dy/dt = ky$ models ***unlimited population growth***.

Example 4 **Unlimited Population Growth**

The biomass of a yeast culture in an experiment is initially 27 grams. After 30 minutes the mass is 36 grams. Assuming that the equation for unlimited population growth gives a good model for the growth of the yeast when the mass is below 100 grams, how long will its take for the mass to double from its initial value? ($\approx$ 72.3 mins)

Solution:

The exponential growth model can be used for unlimited population growth:

$$\frac{dy}{dt} = ky \quad \Rightarrow \quad y = y_0 e^{kt}$$

Radioactivity

- Some atoms are unstable and can spontaneously emit mass or radiation. This process is called **radioactive decay**, and an element whose atoms go spontaneously through this process is called **radioactive**.
- If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-kt}, \qquad k > 0$$

- The **half-life** of a radioactive element is the time expected to pass until half of the radioactive nuclei present in a sample decay.

$$\text{Half} - \text{life} \;\; t = \frac{ln2}{k}$$

Example 5 Radioactive Decay

The decay of radioactive elements can sometimes be used to date events from Earth's past. After the death of an organism, since no new carbon is ingested, the proportion of carbon-14 in the organism remains decreases as the carbon-14 decays. Scientists who do carbon-14 dating often use a figure of 5730 years for its half-life.

Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed. ($\approx$ 871 years)

Heat Transfer: Newton's Law of Cooling

- Hot soup left in a tin cup cools to the temperature of the surrounding air. A hot silver spoon immersed in a large tub of water cools to the temperature of the surrounding water…...
- In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium.
- This observation is called Newton's Law of Cooling, although it applies to warming as well.

- If H is the temperature of the object at time t and H_S is the constant surrounding temperature, then the differential equation is:

$$\frac{dH}{dt} = -k(H - H_S) \quad \Rightarrow \quad H = H_S + (H_0 - H_S)e^{-kt}$$

Example 6

A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C? (≈ 13.3 mins)

Skill Practice 1

Solve the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\frac{e^{2x-y}}{e^{x+y}} \qquad y(0) = \frac{1}{2}$$

Skill Practice 2 The age of Crater Lake

The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

($\approx$ 6693 years)