Fundamentals of Electric Circuits

CHAPTER 3 Methods of Analysis

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Revisiting
2.3 Nodes, Branches, and Loops

Some basic network topology concepts:

- A **branch** represents a single element such as a voltage source or a resistor.
- A **node** is the point of connection between two or more branches.

  If a short circuit (connecting wire) connects two nodes, the two nodes constitute a single node.

- A **loop** is any closed path in a circuit.
- A loop is said to be ***independent (mesh)*** if it does not contain any other loop.

  Independent loops (meshes) result in independent sets of equations.

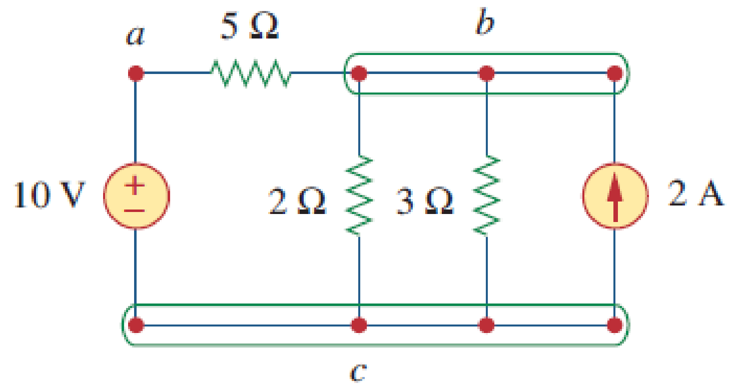


Figure 2.10
Nodes, branches, and loops.

5 branches, 3 nodes and 3 independent loops

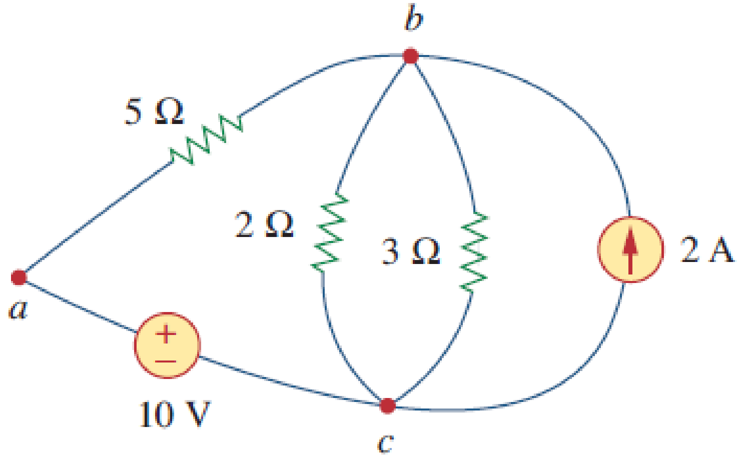


Figure 2.11
The three-node circuit of Fig. 2.10 is redrawn.

Revisiting

- A network with ***b* branches, *n* nodes, and *l* independent loops**

$$\boxed{b = l + n - 1}$$

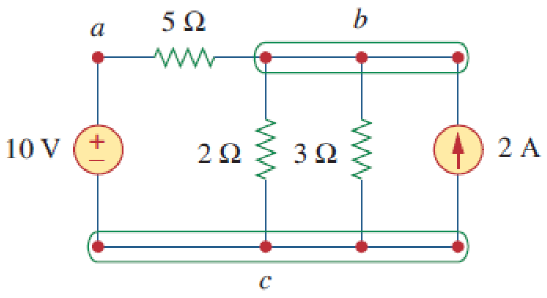


Figure 2.10
Nodes, branches, and loops.

How many basic equations???

- With Ohm's Law, *b* equations
- With KCL, *n*-1 equations
- With KVL, *l* equations

Do we need all of them?!

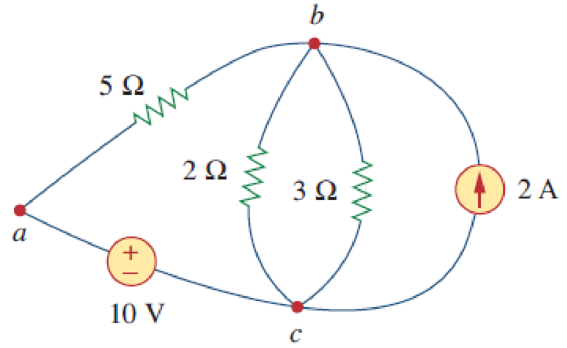


Figure 2.11
The three-node circuit of Fig. 2.10 is redrawn.

Objective

- With Ohm’s and Kirchhoff’s law established, they may now be applied to circuit analysis.
- Two techniques will be presented in this chapter:
  - Nodal analysis, which is based on Kirchhoff current law (KCL)
  - Mesh analysis, which is based on Kirchhoff voltage law (KVL)
- Any linear circuit can be analyzed using these two techniques.
- The analysis will result in a set of simultaneous equations

CHAPTER 3 Method of Analysis

3.2 Nodal Analysis

3.3 Nodal Analysis with Voltage Sources

3.4 Mesh Analysis

3.5 Mesh Analysis with Current Sources

3.6 Nodal and Mesh Analysis by Inspection

3.7 Nodal Versus Mesh Analysis

3.2 Nodal Analysis without Voltage Sources

Purpose:

Nodal analysis provides a general procedure for analyzing circuits using ***node voltages*** as the circuit variables.

Given a circuit with *n* nodes without voltage sources, the nodal analysis is accomplished via three steps:

1. Select a node as the reference node. Assign voltages $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots \boldsymbol{v_{n-1}}$ to the remaining *n*-1 nodes.

   The voltages are refenced with respect to the reference node.

2. Apply KCL to each of the *n*-1 non-reference nodes.

   Use Ohm's law to express the branch currents in terms of node voltages.

3. Solve the resulting n-1 simultaneous equations to obtain the unknown node voltages.

Reference node

The reference node is commonly called the ground since it is assumed to have zero potential.

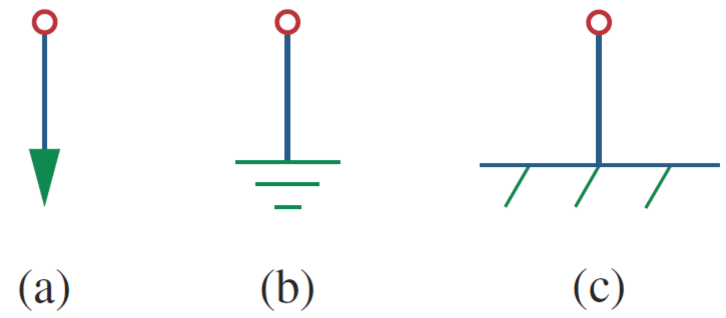
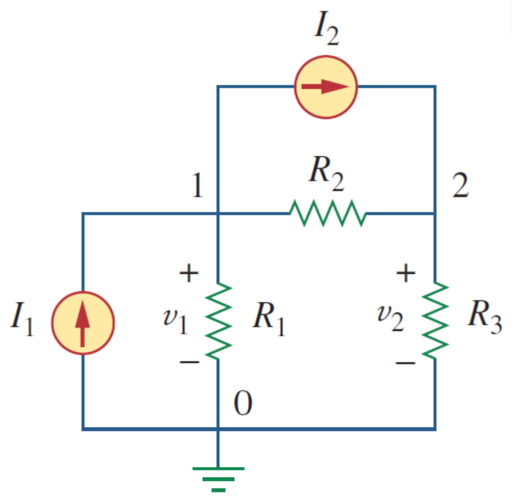


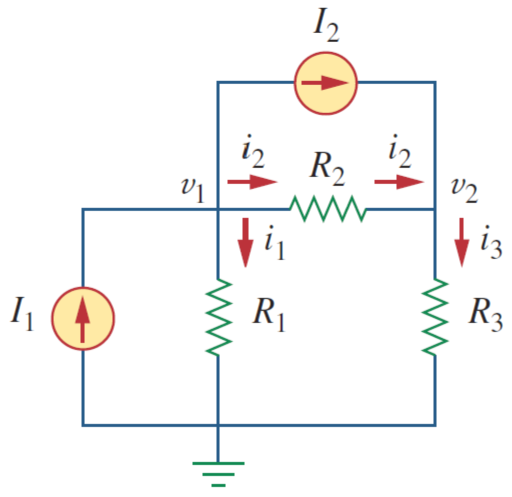
Figure 3.1
Common symbols for indicating a reference node, (a) common ground, (b) ground, (c) chassis ground.

Applying Nodal Analysis

- Let's apply nodal analysis to this circuit to see how it works.
- **Step 1**:
- This circuit has a node that is designed as ground. We will use that as the reference node (Node 0)
- The remaining two nodes are assigned voltages v_1 and v_2.

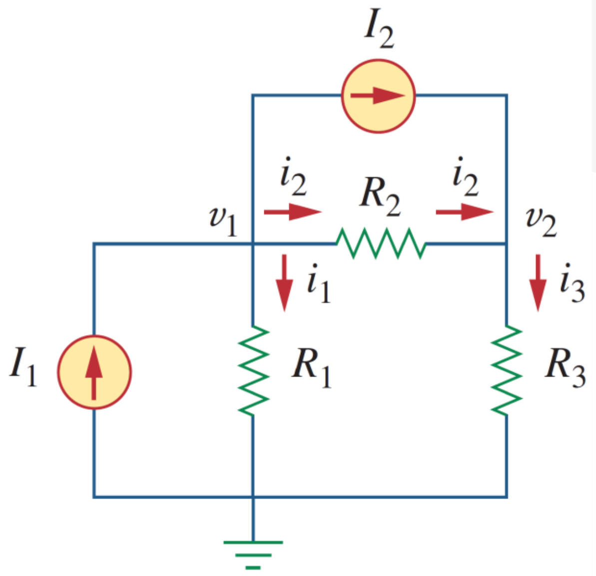


(a)



(b)

Applying Nodal Analysis



Step 2:

Now apply KCL to each nonreference node:

At **node 1**

$$I_1 = I_2 + i_1 + i_2$$

At **node 2**

$$I_2 + i_2 = i_3$$

- Use Ohm's law to express the unknown currents $\boldsymbol{i_1}$, $\boldsymbol{i_2}$, *and* $\boldsymbol{i_3}$ in terms of node voltages.
- In doing so, keep in mind that current flows from **high** potential to **low** potential
- From this we get:

$$i_1 = \frac{v_1 - 0}{R_1} \quad \text{or} \quad i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2} \quad \text{or} \quad i_2 = G_2(v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3} \quad \text{or} \quad i_3 = G_3 v_2$$

Substitute into the node equations →

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

or

$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2)$$

$$I_2 + G_2(v_1 - v_2) = G_3 v_2$$

Applying Nodal Analysis

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

or

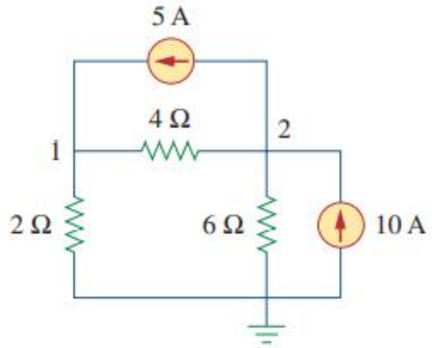
$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2)$$
$$I_2 + G_2(v_1 - v_2) = G_3 v_2$$

Step 3: Solve the resulting n-1 simultaneous equations to obtain the unknown node voltages.
write in matrix form, solve n-1 (n=3) equations by using substitution method, the elimination method, cramer's rule, matrix inversion, or software

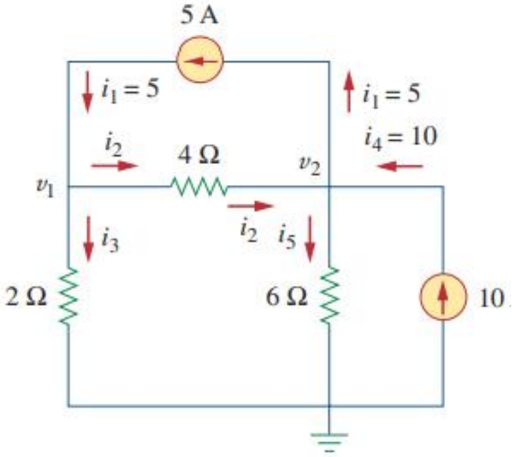
$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

Example

Example 3.1 Calculate the node voltages in the circuit shown in Fig. 3.3(a).



(a)



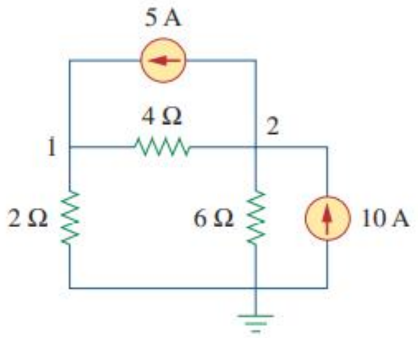
(b)

Figure 3.3
For Example 3.1: (a) original circuit, (b) circuit for analysis.

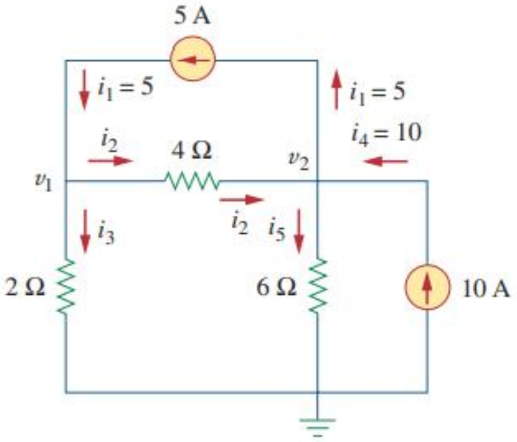
Example

Example 3.1

Calculate the node voltages in the circuit shown in Fig. 3.3(a).



(a)



(b)

Figure 3.3
For Example 3.1: (a) original circuit, (b) circuit for analysis.

Solution:

Consider Fig. 3.3(b), where the circuit in Fig. 3.3(a) has been prepared for nodal analysis. Notice how the currents are selected for the application of KCL. Except for the branches with current sources, the labeling of the currents is arbitrary but consistent. (By consistent, we mean that if, for example, we assume that i_2 enters the 4-Ω resistor from the left-hand side, i_2 must leave the resistor from the right-hand side.) The reference node is selected, and the node voltages v_1 and v_2 are now to be determined.

At node 1, applying KCL and Ohm's law gives

$$i_1 = i_2 + i_3 \quad \Rightarrow \quad 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

Multiplying each term in the last equation by 4, we obtain

$$20 = v_1 - v_2 + 2v_1$$

or

$$3v_1 - v_2 = 20 \tag{3.1.1}$$

At node 2, we do the same thing and get

$$i_2 + i_4 = i_1 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

Multiplying each term by 12 results in

$$3v_1 - 3v_2 + 120 = 60 + 2v_2$$

or

$$-3v_1 + 5v_2 = 60 \tag{3.1.2}$$

Now we have two simultaneous Eqs. (3.1.1) and (3.1.2). We can solve the equations using any method and obtain the values of v_1 and v_2.

METHOD 1 Using the elimination technique, we add Eqs. (3.1.1) and (3.1.2).

$$4v_2 = 80 \quad \Rightarrow \quad v_2 = 20 \text{ V}$$

Substituting $v_2 = 20$ in Eq. (3.1.1) gives

$$3v_1 - 20 = 20 \quad \Rightarrow \quad v_1 = \frac{40}{3} = 13.333 \text{ V}$$

METHOD 2 To use Cramer's rule, we need to put Eqs. (3.1.1) and (3.1.2) in matrix form as

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix} \tag{3.1.3}$$

The determinant of the matrix is

$$\Delta = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 \quad 3 = 12$$

We now obtain v_1 and v_2 as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.333 \text{ V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\Delta} = \frac{180 + 60}{12} = 20 \text{ V}$$

giving us the same result as did the elimination method.

If we need the currents, we can easily calculate them from the values of the nodal voltages.

$$i_1 = 5 \text{ A}, \qquad i_2 = \frac{v_1 - v_2}{4} = -1.6668 \text{ A}, \qquad i_3 = \frac{v_1}{2} = 6.666 \text{ A}$$

$$i_4 = 10 \text{ A}, \qquad i_5 = \frac{v_2}{6} = 3.333 \text{ A}$$

Example

Practice Problem 3.1

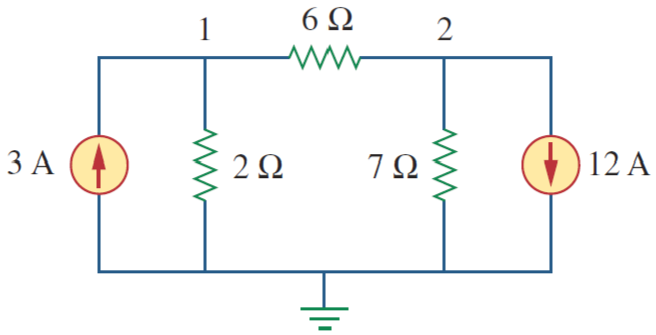


Figure 3.4
For Practice Prob. 3.1.

Obtain the node voltages in the circuit of Fig. 3.4.

Answer: $v_1 = -6$ V, $v_2 = -42$ V.

Example

Example 3.2 Determine the voltages at the nodes in Fig. 3.5(a).

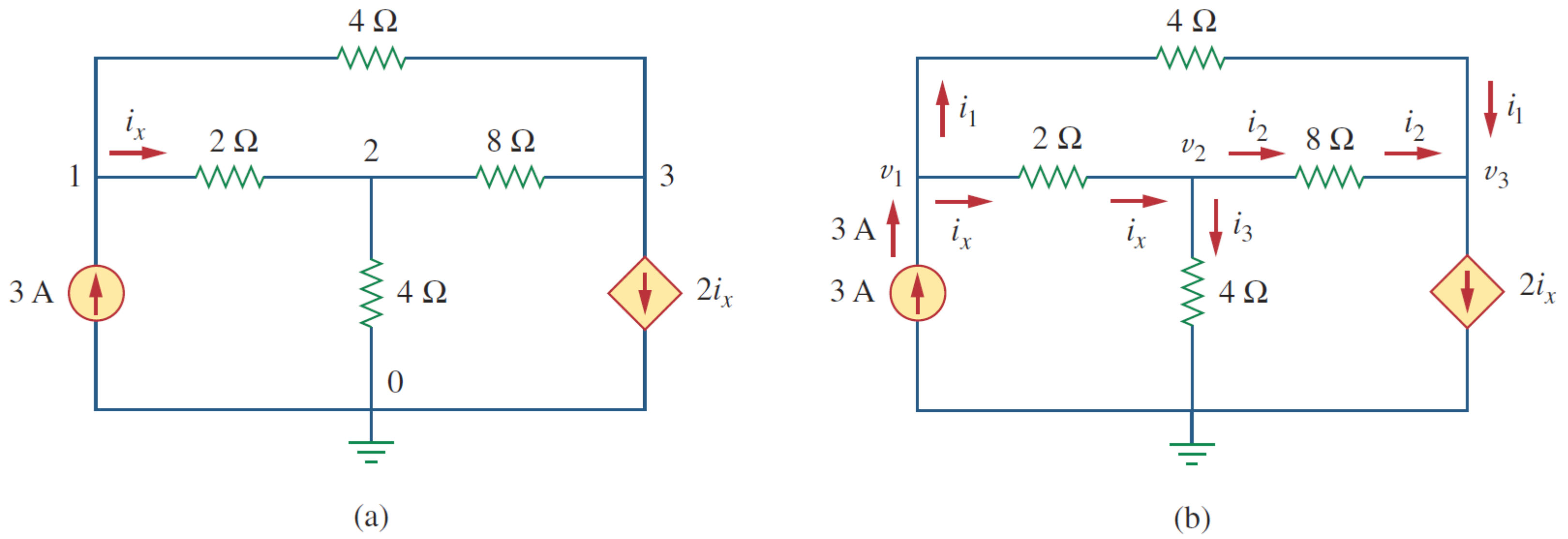


Figure 3.5
For Example 3.2: (a) original circuit, (b) circuit for analysis.

Example

Practice Problem 3.2

Find the voltages at the three nonreference nodes in the circuit of Fig. 3.6.

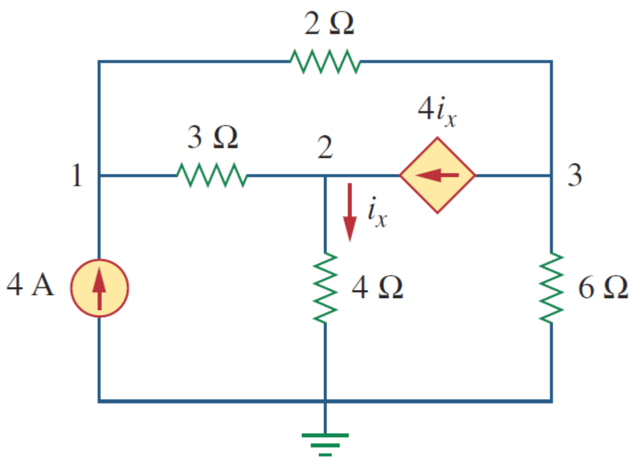


Figure 3.6
For Practice Prob. 3.2.

Answer: $v_1 = 32$ V, $v_2 = -25.6$ V, $v_3 = 62.4$ V.

3.2 Nodal Analysis with Voltage Sources

Depending on what nodes the voltage source is connected to, there are two possibilities:

- **Case 1: Between the reference node and a non-reference node**
  - Set the voltage at the non-reference node to the voltage of the voltage source
  - In Figure 3.7, $\boldsymbol{v_1 = 10V}$
- **Case 2: Between two nonreference nodes**
  - The two nodes form *a generalized node* or *supernode*.
  - Apply both **KCL** and **KVL** to determine the node voltages
  - In Figure 3.7, node 2 and node 3 form a supernode.

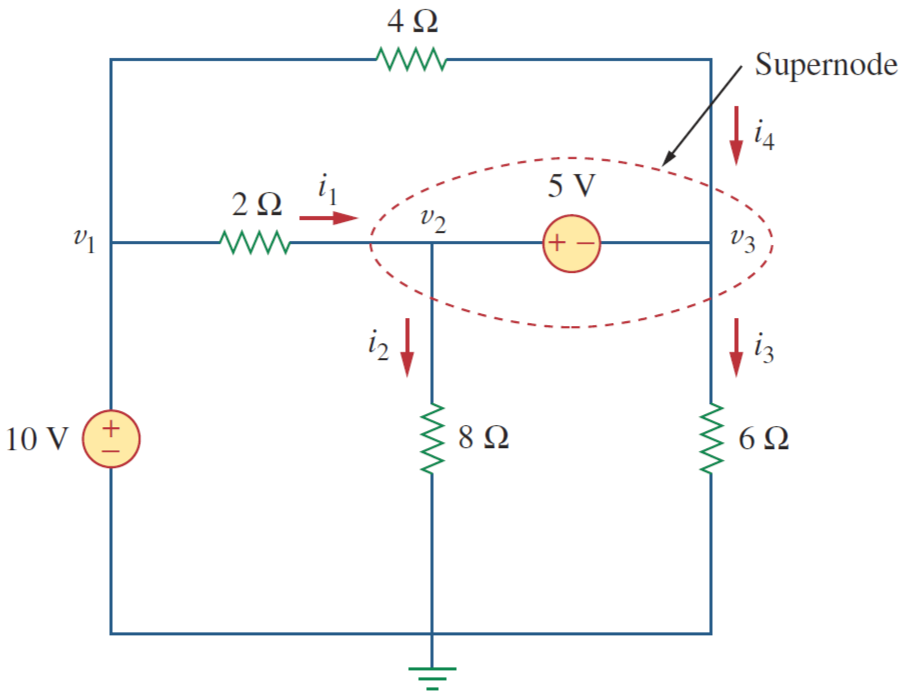


Figure 3.7
A circuit with a supernode.

Supernode

- A supernode is formed by enclosing a voltage source (dependent or independent) connected between two non-reference nodes and any elements connected in parallel with it.

- How to deal with supernode?
  - Nodal analysis requires applying KCL
  - The current through a voltage source cannot be known in advance (Ohm's law does not apply)
  - However, KCL must be satisfied at a supernode like any other node

- In Figure 3.7, node 2 and node 3 form a supernode
- The current balance KCL would be: $i_1 + i_4 = i_2 + i_3$
- Or this can be expressed as: $\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$

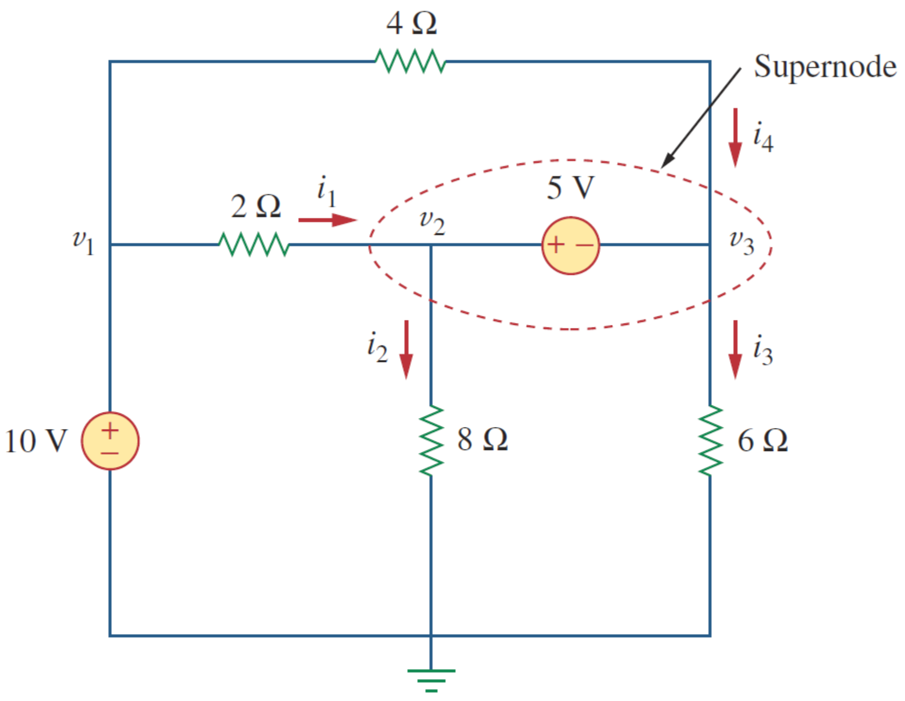


Figure 3.7
A circuit with a supernode.

Analysis with a supernode

In order to apply KVL to the supernode, the circuit is redrawn.

Going around the loop in the clockwise direction gives:

$$-v_2 + 5 + v_3 = 0 \ \Rightarrow v_2 - v_3 = 5$$

Note the following properties of a supernode:

1. The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages
2. A supernode has no voltage of its own
3. A supernode requires the application of both KCL and KVL

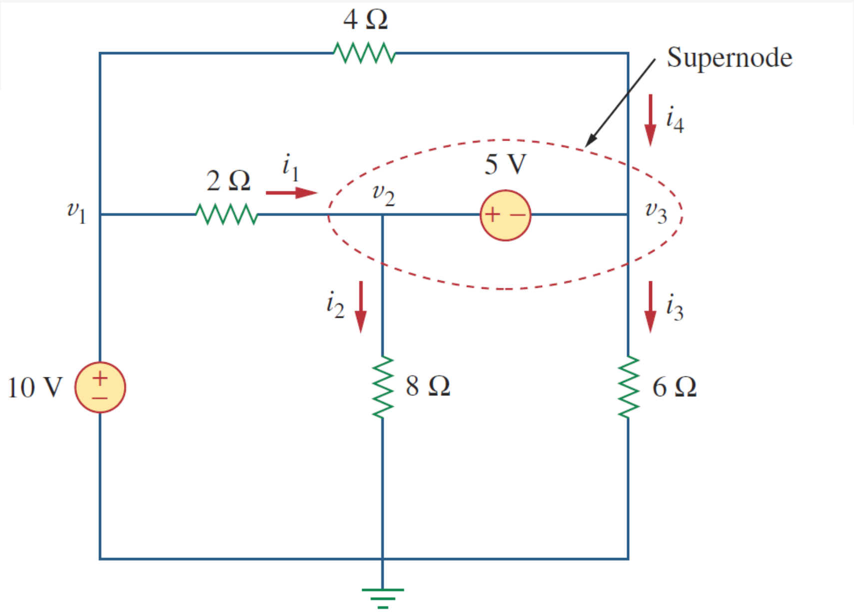


Figure 3.7
A circuit with a supernode.

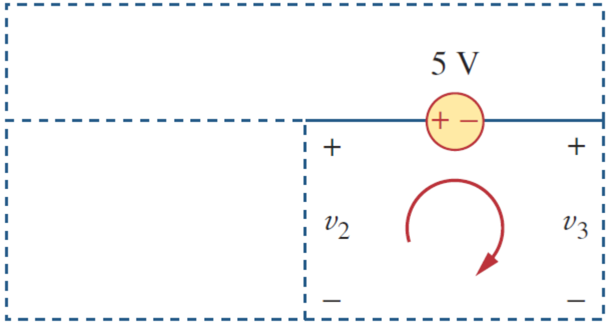


Figure 3.8
Applying KVL to a supernode.

Analysis with a supernode

Example 3.3 For the circuit shown in Fig. 3.9, find the node voltages.

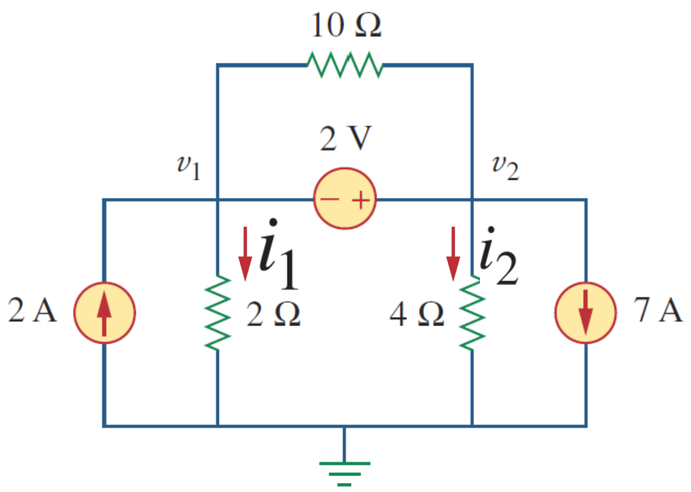


Figure 3.9
For Example 3.3.

Analysis with a supernode

Practice Problem 3.3 Find v and i in the circuit of Fig. 3.11.

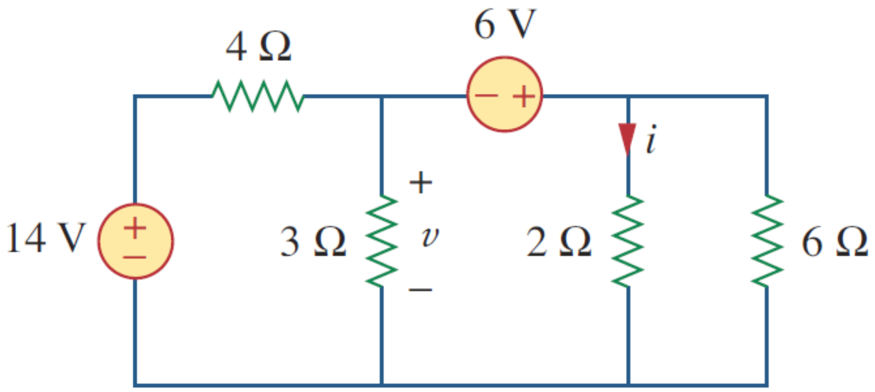


Figure 3.11
For Practice Prob. 3.3.

Answer: -400 mV, 2.8 A.

Analysis with a supernode

Find the node voltages in the circuit of Fig. 3.12.

Example 3.4

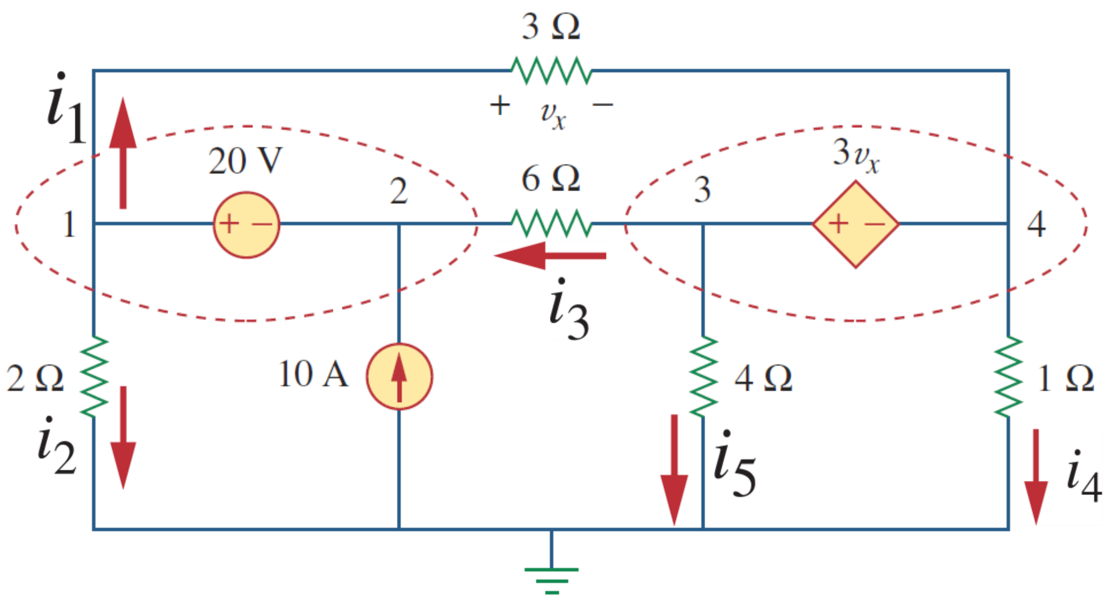


Figure 3.12
For Example 3.4.

Solution:
Nodes 1 and 2 form a supernode; so do nodes 3 and 4. We apply KCL to the two supernodes as in Fig. 3.13(a). At supernode 1-2,

$$i_3 + 10 = i_1 + i_2$$

Expressing this in terms of the node voltages,

$$\frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_1}{2}$$

or

$$5v_1 + v_2 - v_3 - 2v_4 = 60 \tag{3.4.1}$$

At supernode 3-4,

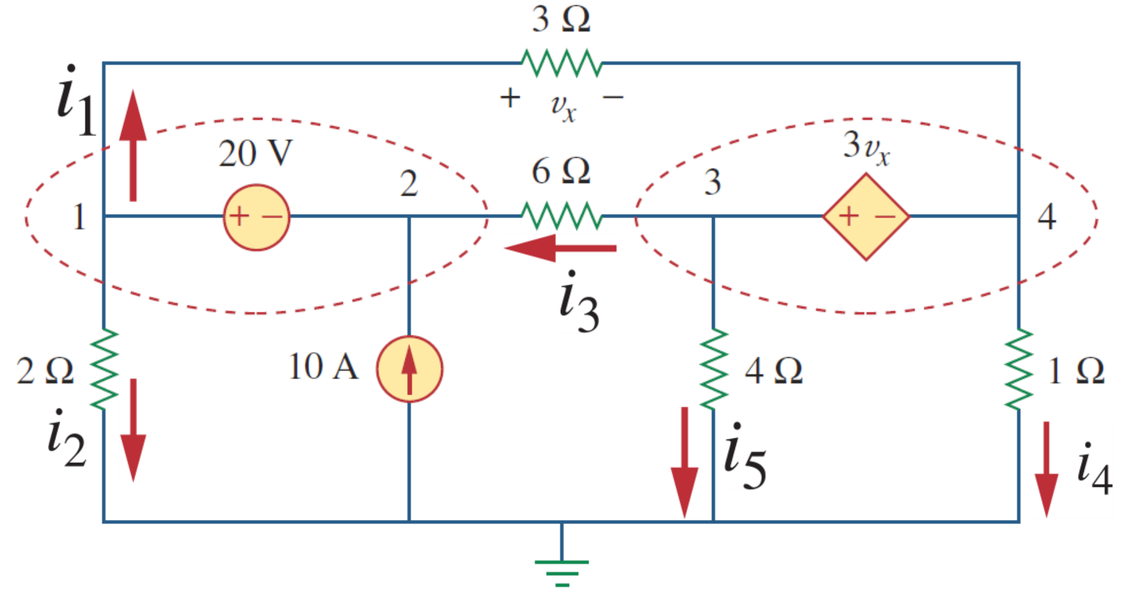
$$i_1 = i_3 + i_4 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$$

or

$$4v_1 + 2v_2 - 5v_3 - 16v_4 = 0 \tag{3.4.2}$$

Analysis with a supernode

Find the node voltages in the circuit of Fig. 3.12. **Example 3.4**



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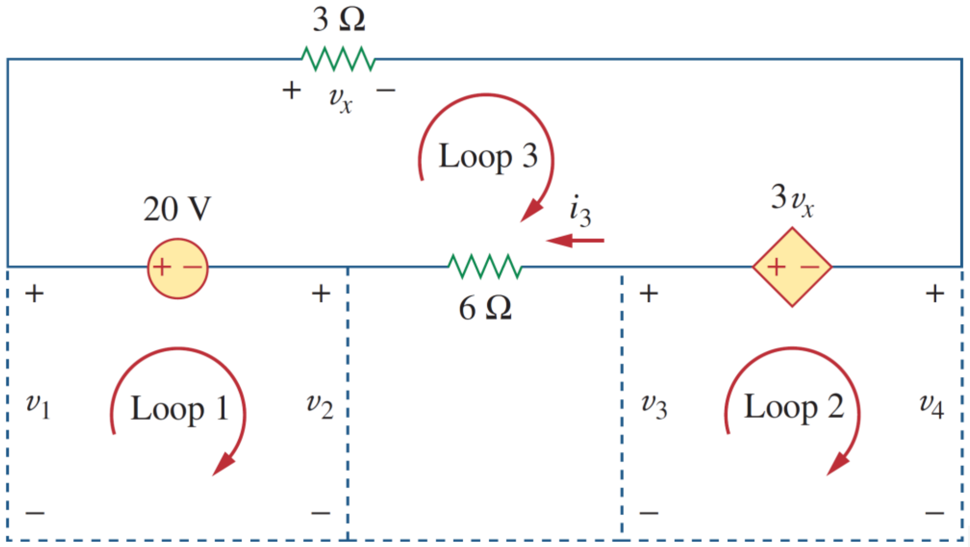


$$5v_1 + v_2 - v_3 - 2v_4 = 60 \quad \textbf{(3.4.1)}$$

$$4v_1 + 2v_2 - 5v_3 - 16v_4 = 0 \quad \textbf{(3.4.2)}$$

We now apply KVL to the branches involving the voltage sources as shown in Fig. 3.13(b). For loop 1,

$$-v_1 + 20 + v_2 = 0 \quad \Rightarrow \quad v_1 - v_2 = 20 \quad \textbf{(3.4.3)}$$

For loop 2,

$$-v_3 + 3v_x + v_4 = 0$$

But $v_x = v_1 - v_4$ so that

$$3v_1 - v_3 - 2v_4 = 0 \quad \textbf{(3.4.4)}$$

For loop 3,

$$v_x - 3v_x + 6i_3 - 20 = 0$$

But $6i_3 = v_3 - v_2$ and $v_x = v_1 - v_4$. Hence,

$$-2v_1 - v_2 + v_3 + 2v_4 = 20 \quad \textbf{(3.4.5)}$$

Analysis with a supernode

Find v_1, v_2, and v_3 in the circuit of Fig. 3.14 using nodal analysis. **Practice Problem 3.4**

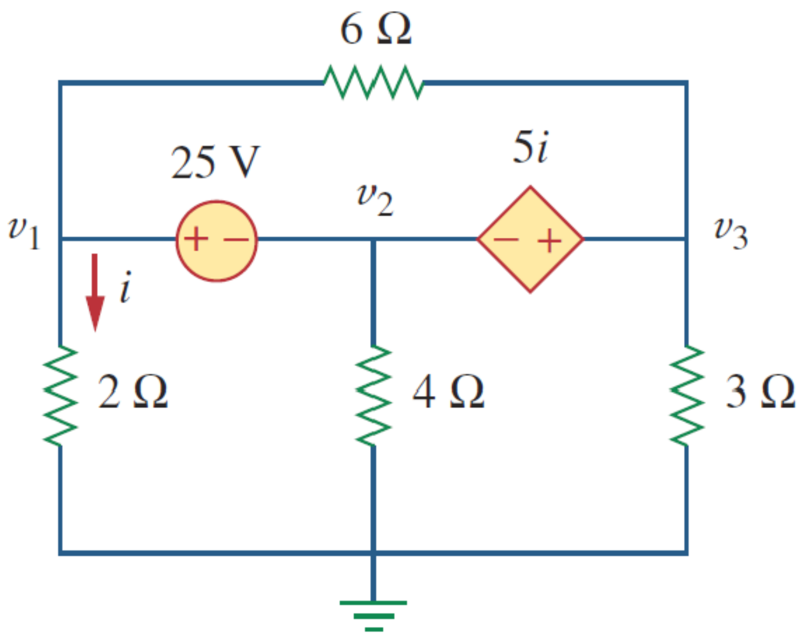
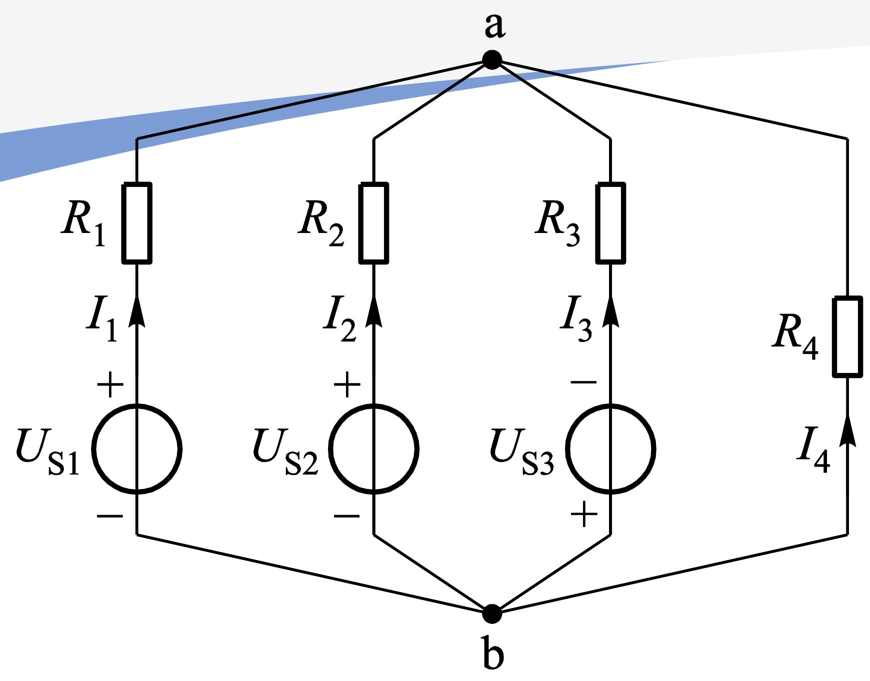


Figure 3.14
For Practice Prob. 3.4.

Answer: $v_1 = 7.608$ V, $v_2 = -17.39$ V, $v_3 = 1.6305$ V.

If the voltage source is in series with one resistor



（1）Set node b as the reference node

$$V_b = 0\text{V}$$

（2）Apply KCL to node a

$$I_1 + I_2 + I_3 + I_4 = 0$$

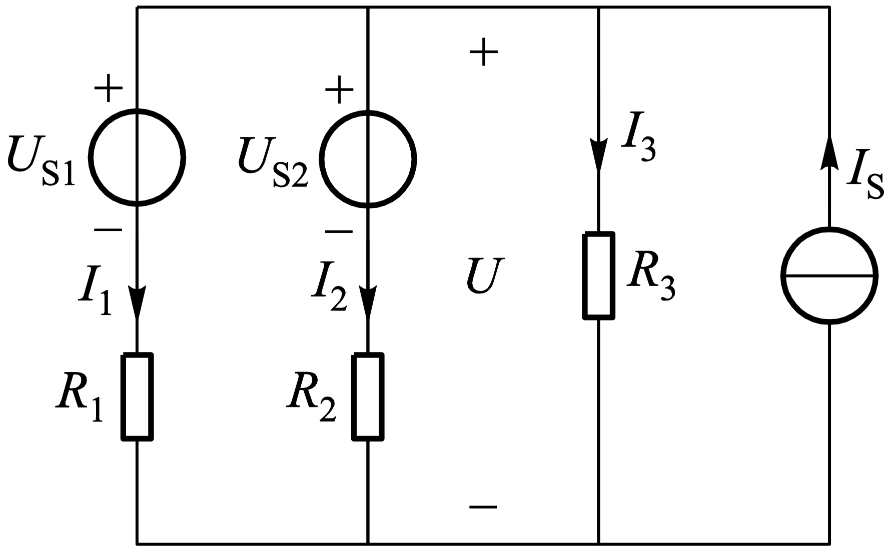
（3）Represent the currents in terms of node voltages

$$I_1 = \frac{U_{S1} - V_a}{R_1} \qquad I_2 = \frac{U_{S2} - V_a}{R_2} \qquad I_3 = \frac{-U_{S3} - V_a}{R_3} \qquad I_4 = \frac{-V_a}{R_4}$$

Substitute

$$\frac{U_{S1} - V_a}{R_1} + \frac{U_{S2} - V_a}{R_2} + \frac{-U_{S3} - V_a}{R_3} + \frac{-V_a}{R_4} = 0$$

Example U_{S1}=40V， U_{S2}=15V， I_S=5A， R_1=2Ω， R_2=3Ω， R_3=1.2Ω。Find I_1、I_2、I_3。



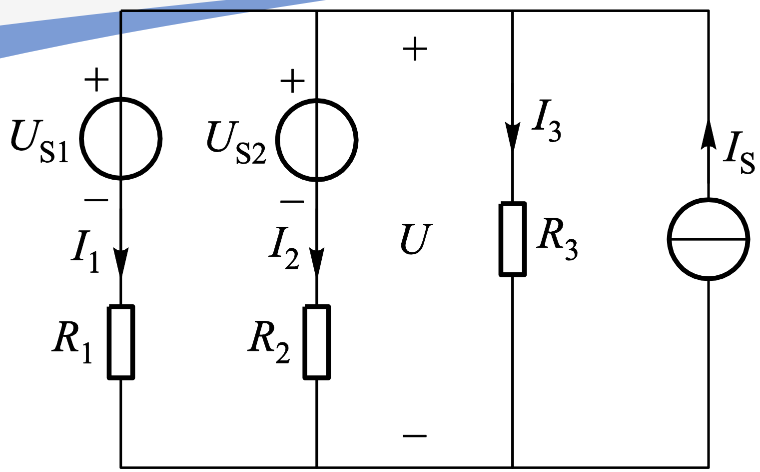
$$U=\frac{\frac{U_{S1}}{R_1}+\frac{U_{S2}}{R_2}+I_S}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}=\frac{\frac{40}{2}+\frac{15}{3}+5}{\frac{1}{2}+\frac{1}{3}+\frac{1}{1.2}}=18\text{V}$$

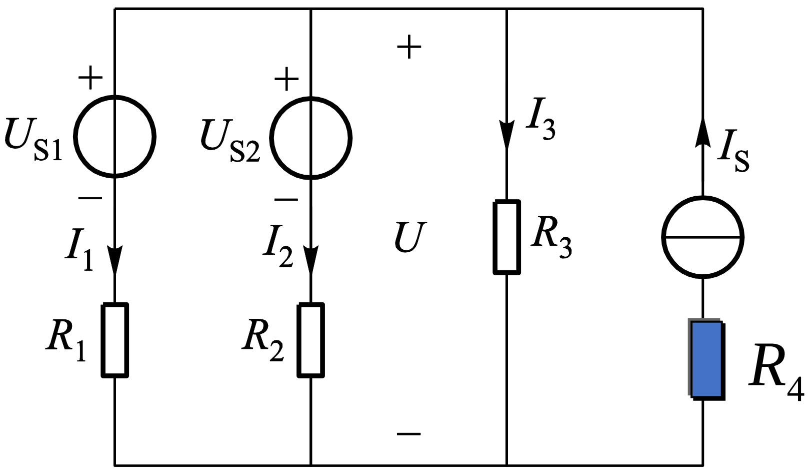
$$I_1=\frac{U-U_{S1}}{R_1}=\frac{18-40}{2}=-11\text{A}$$

$$I_2=\frac{U-U_{S2}}{R_2}=\frac{18-15}{3}=1\text{A}$$

$$I_3=\frac{U}{R_3}=\frac{18}{1.2}=15\text{A}$$

Question：





$$U = \frac{\dfrac{U_{S1}}{R_1} + \dfrac{U_{S2}}{R_2} + I_S}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$$

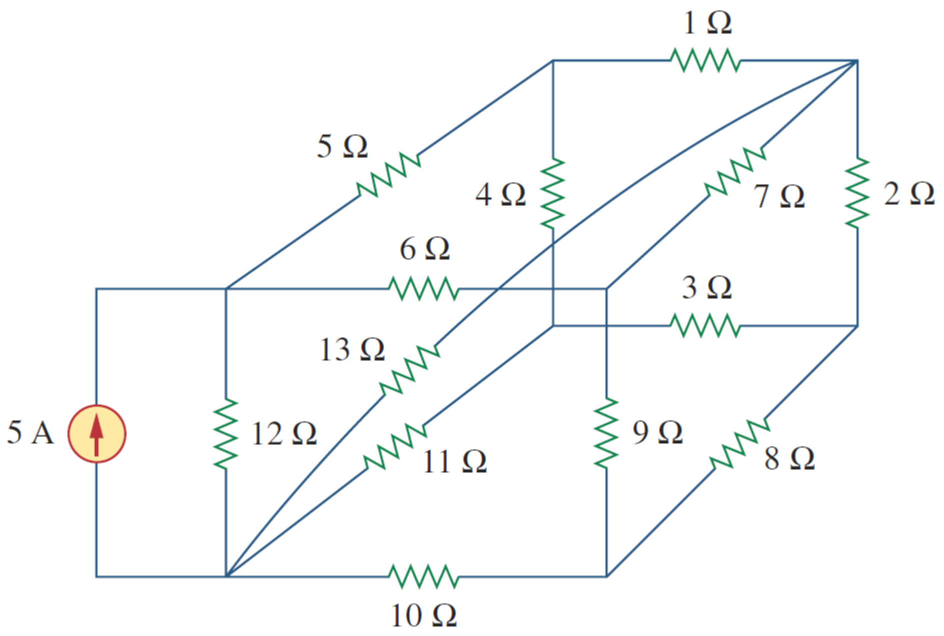
$$U = \frac{\dfrac{U_{S1}}{R_1} + \dfrac{U_{S2}}{R_2} + I_S}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$$

3.4 Mesh Analysis

- Another general procedure for analyzing circuits is to use the ***mesh currents*** as the circuit variables.
- Use mesh currents instead of element currents
- Recall:
  - A loop is a closed path with no node passed more than once
  - A mesh is a loop that does not contain any other loop within it
- Mesh analysis uses KVL to find unknown currents.
- Mesh analysis can only be applied to a circuit that is planar.

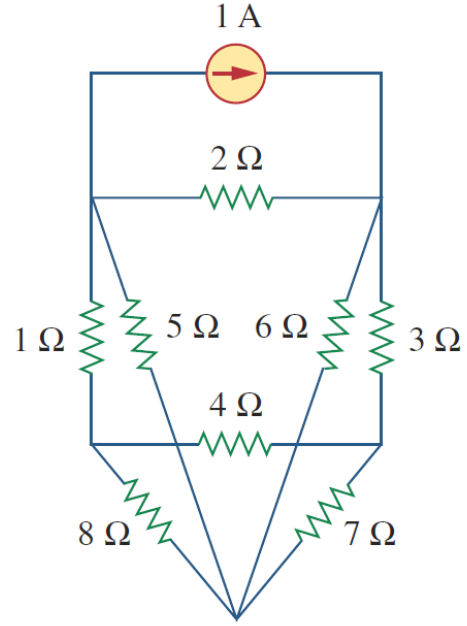
Planar vs Nonpalanar

A **planar** circuit can be drawn in a plane with no branches crossing one another; otherwise it is **nonplanar**.

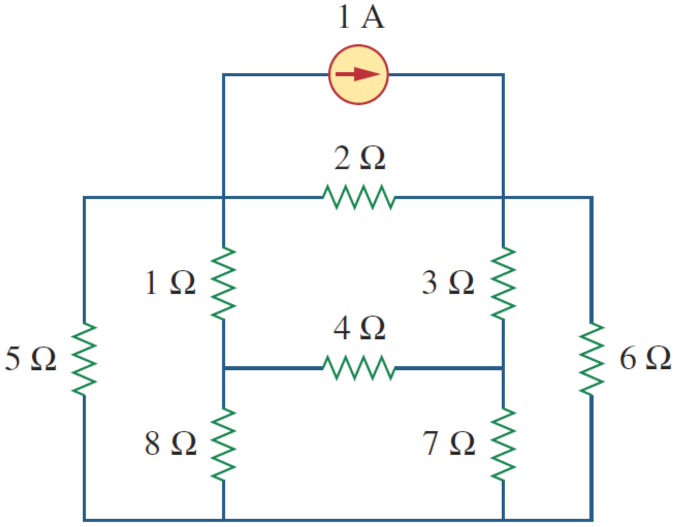


The figure on the left is a **nonplanar circuit**:

The branch with the 13Ω resistor prevents the circuit from being drawn without crossing branches



(a)



(b)

The figure on the right is **a planar circuit**: It can be redrawn to avoid crossing branches

Mesh Analysis

- Recall:
  - A loop is a closed path with no node passed more than once
  - A mesh is a loop that does not contain any other loop within it

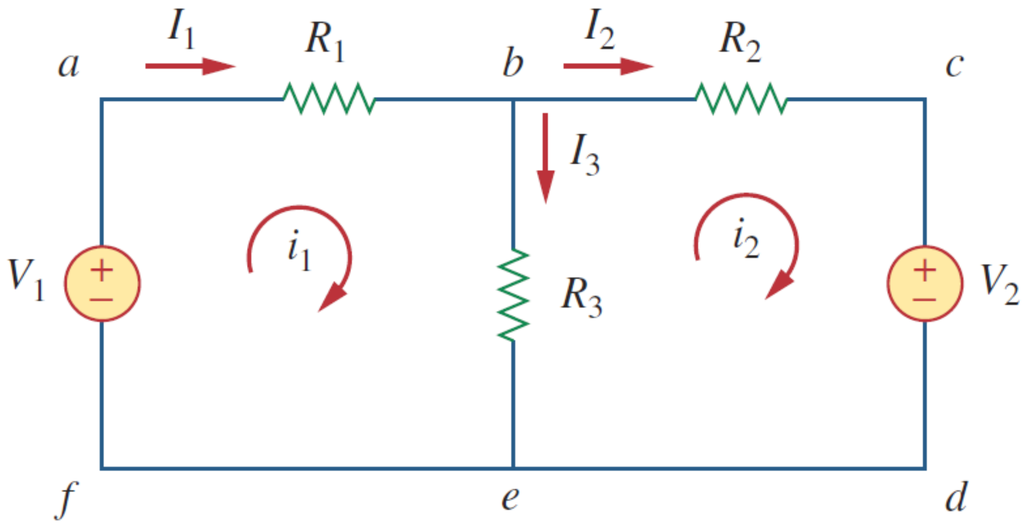


Figure 3.17
A circuit with two meshes.

- Path *abefa* and *bcdeb* are meshes, path *abcdefa* is a loop, but not a mesh.
- The current through a mesh is known as mesh current

Mesh Analysis

Mesh currents are envisioning circulating currents in each of the meshes.

Mesh currents are different from branch currents!

(a) If an element is located on a single mesh (such as $R1$, $R2$, $V1$, $V2$), it carries the same current as the mesh current

(b) If an element is located on the boundary of two meshes (such as R3), it will carry a current that is the algebraic sum of the two mesh currents

All branch currents in a circuit can be represented in terms of mesh currents!

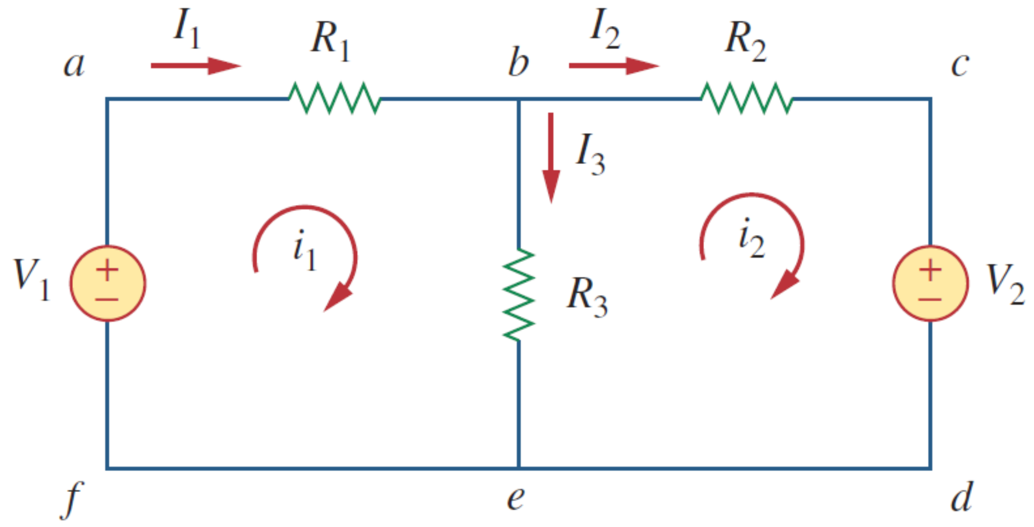


Figure 3.17
A circuit with two meshes.

$$I_1 = i_1, \qquad I_2 = i_2, \qquad I_3 = i_1 - i_2$$

Mesh Analysis Steps

Apply mesh analysis to planar circuits that **do not contain current sources**. In a circuit contains *n* meshes, follow these steps:

Step 1: Assign mesh currents $\boldsymbol{i_1}$, $\boldsymbol{i_2}$,…$\boldsymbol{i_n}$ to the *n* meshes

Step 2: Apply KVL to each of the *n* mesh currents, use Ohm’s law to express the voltages in terms of mesh currents

Step 3: Solve the resulting ***n* simultaneous equations** to get the mesh currents

Mesh Analysis Example

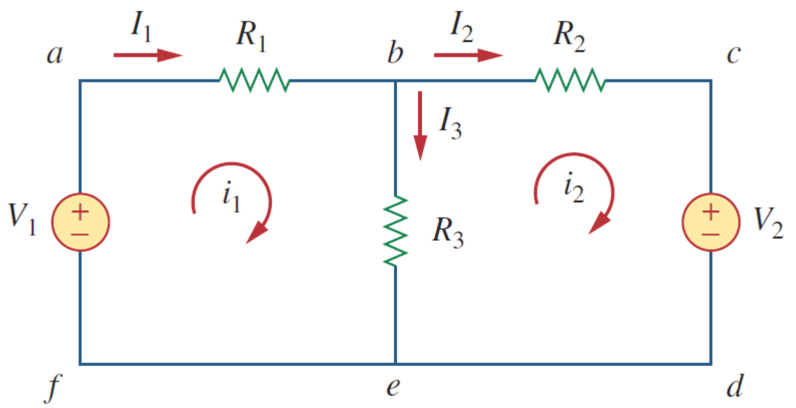


Figure 3.17
A circuit with two meshes.

Step 1: Assign mesh current

- The above circuit has two meshes (*abefa* and *bcdeb*)
- Mesh currents $\boldsymbol{i_1}$ and $\boldsymbol{i_2}$ are assigned to the two meshes.
- The direction of the mesh current is aribitrary (clockwise or counterclockwise), which does not affect the solution.
- Conventially, we use clockwise.

Step 2: Applying KVL to the meshes

Apply KVL to mesh1

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0$$
$$\Downarrow$$
$$\boldsymbol{(R_1 + R_3)i_1 - R_3 i_2 = V_1}$$

Apply KVL to mesh2

$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0$$
$$\Downarrow$$
$$\boldsymbol{-R_3 i_1 + (R_2 + R_3)i_2 = -V_2}$$

Step 3: Solve for the mesh currents

Note that the coefficient of i_1 is the sum of the resistances in the first mesh, while the coefficient of i_2 is the negative of the resistance common to meshes 1 and 2.

Example

For the circuit in Fig. 3.18, find the branch currents I_1, I_2, and I_3 using mesh analysis.

Example 3.5

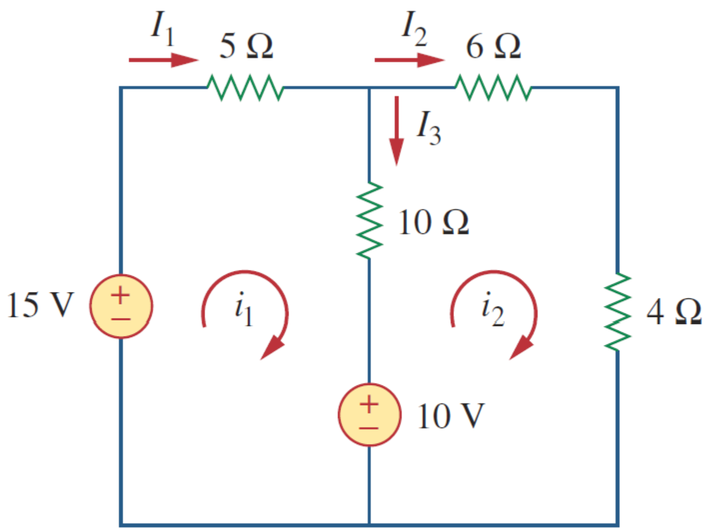


Figure 3.18
For Example 3.5.

Solution:

We first obtain the mesh currents using KVL. For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

or

$$3i_1 - 2i_2 = 1 \tag{3.5.1}$$

For mesh 2,

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

or

$$i_1 = 2i_2 - 1 \tag{3.5.2}$$

$$I_1 = i_1 = 1 \text{ A}, \qquad I_2 = i_2 = 1 \text{ A}, \qquad I_3 = i_1 - i_2 = 0$$

Example

Practice Problem 3.5 Calculate the mesh currents i_1 and i_2 of the circuit of Fig. 3.19.

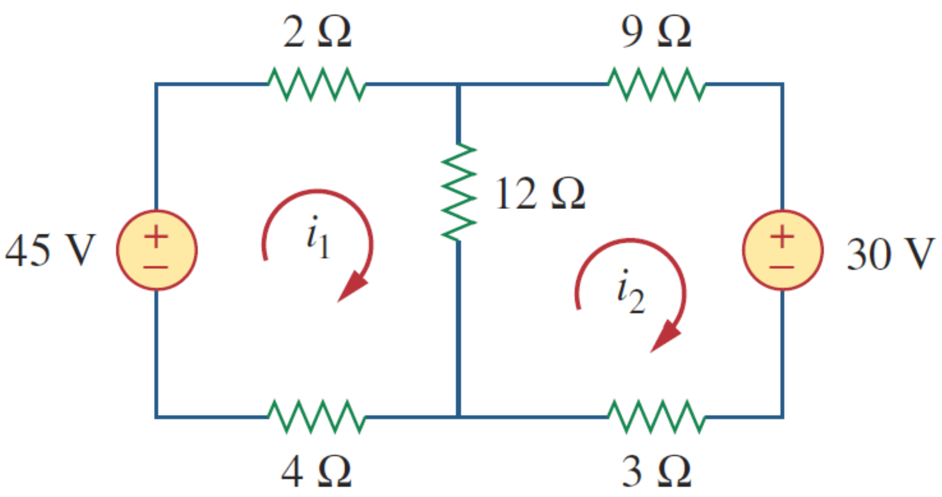


Figure 3.19
For Practice Prob. 3.5.

Answer: $i_1 = 2.5$ A, $i_2 = 0$ A.

Example 3.6

Use mesh analysis to find the current I_o in the circuit of Fig. 3.20.

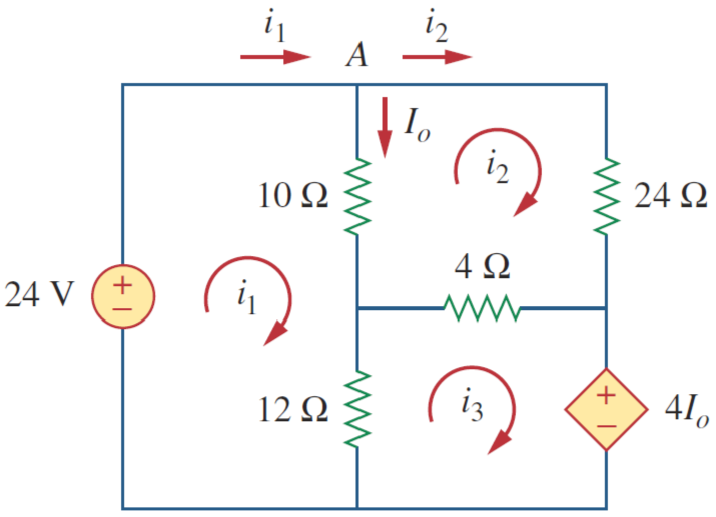


Figure 3.20
For Example 3.6.

Solution:

We apply KVL to the three meshes in turn. For mesh 1,

$$-24 + 10(i_1 - i_2) + 12(i_1 - i_3) = 0$$

or

$$11i_1 - 5i_2 - 6i_3 = 12 \tag{3.6.1}$$

For mesh 2,

$$24i_2 + 4(i_2 - i_3) + 10(i_2 - i_1) = 0$$

or

$$-5i_1 + 19i_2 - 2i_3 = 0 \tag{3.6.2}$$

For mesh 3,

$$4I_o + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$

But at node A, $I_o = i_1 - i_2$, so that

$$4(i_1 - i_2) + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$

or

$$-i_1 - i_2 + 2i_3 = 0 \tag{3.6.3}$$

Example

Practice Problem 3.6 Using mesh analysis, find I_o in the circuit of Fig. 3.21.

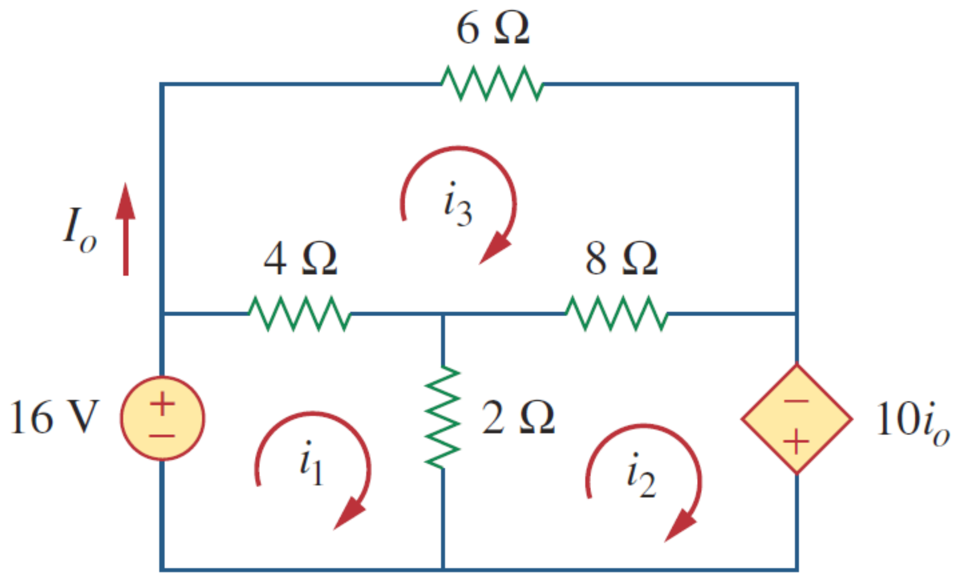


Figure 3.21
For Practice Prob. 3.6.

Answer: −4 A.

3.5 Mesh Analysis with Current Sources

The presence of a current sources (dependent or independent) makes the mesh analysis simpler because it reduces the number of equations.

There are two possible cases:

Case1: If the current source is located on only one mesh, the current for that mesh is defined by the source.

For example in Figure 3.22, the current $\boldsymbol{i_2}$ is equal to -5A, i_2=-5A,

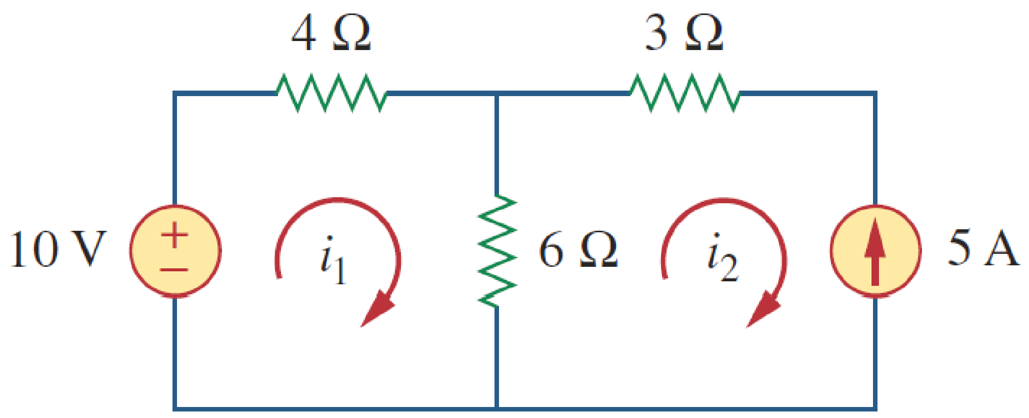


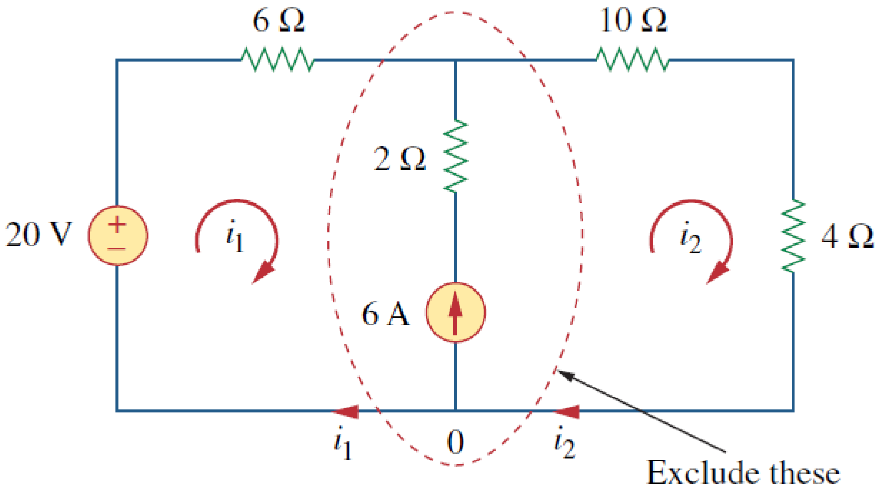
Figure 3.22
A circuit with a current source.

$$-10 + 4i_1 + 6(i_1 - i_2) = 0 \quad \Rightarrow \quad i_1 = -2\text{ A}$$

Supermesh

Case2: Similar to the case of nodal analysis where a voltage source shared two non-reference nodes, current sources (dependent or independent) that are shared by more than one mesh need special treatment.

- A supermesh results when two meshes have a (dependent or independent) current source in common.
- The supermesh is constructed by merging the two meshes and excluding the shared source and any elements connected in series with it.
- A supermesh is required because mesh analysis uses KVL, but the voltage across a current source cannot be known in advance.



(a)

Creating a Supermesh

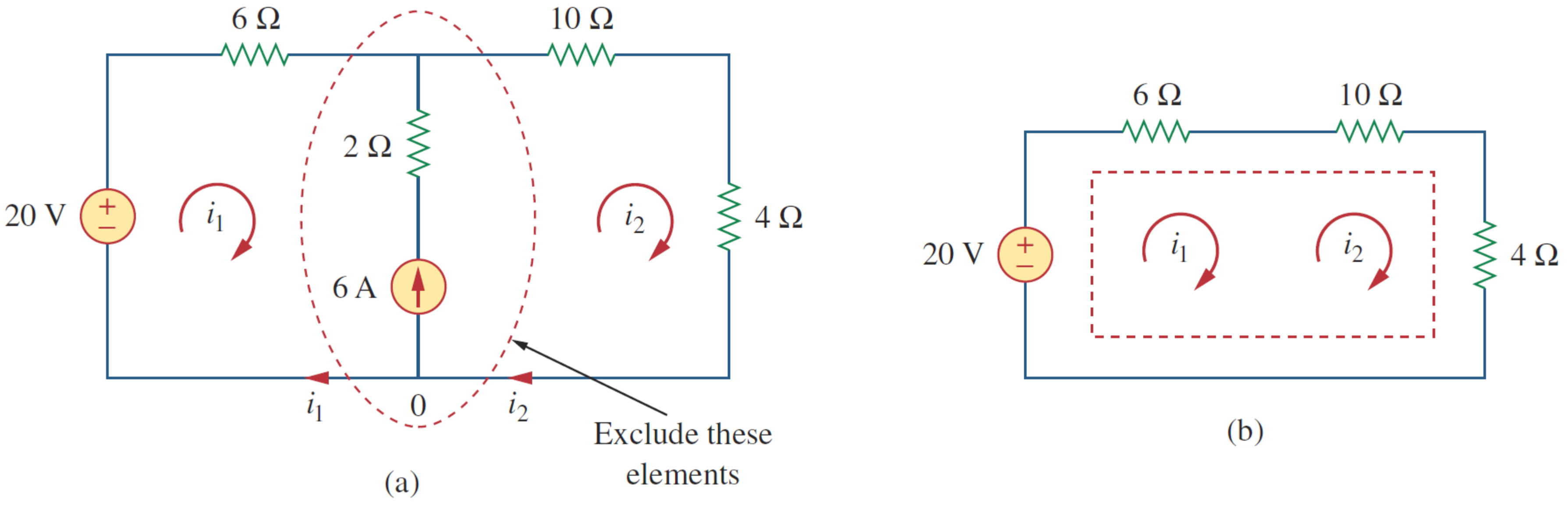


Figure 3.23
(a) Two meshes having a current source in common, (b) a supermesh, created by excluding the current source.

In this example, a 6A current source is shared between mesh 1 and mesh 2.

- The supermesh is formed by merging the two meshes.
- The current source and the 2Ω resistor in series with it are removed.

Creating a Supermesh

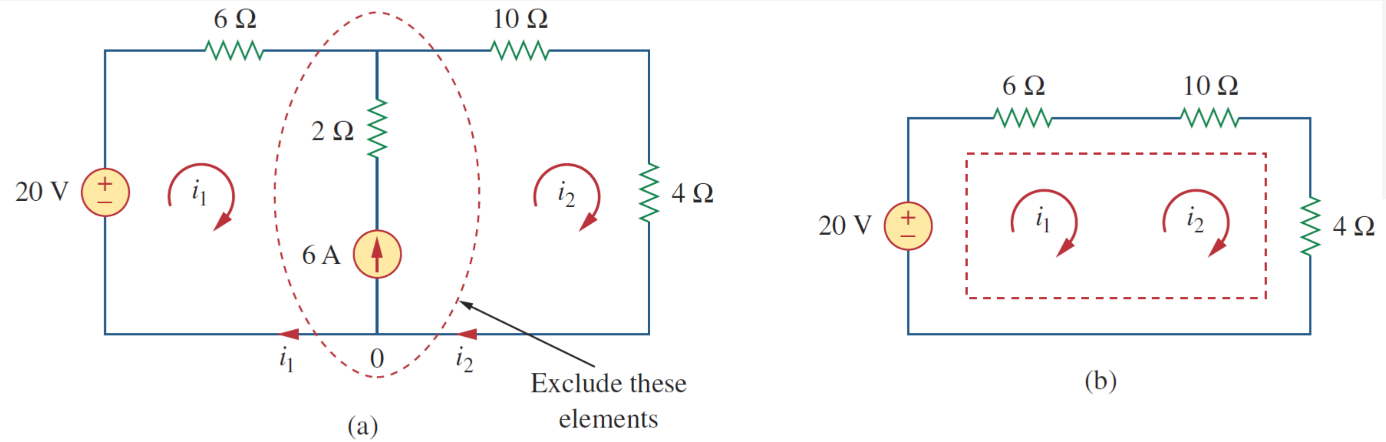


Figure 3.23
(a) Two meshes having a current source in common, (b) a supermesh, created by excluding the current source.

- Apply KVL to the supermesh

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0 \qquad \text{or} \qquad 6i_1 + 14i_2 = 20$$

- We next apply KCL to the node in the branch where the two meshes intersect.

$$i_2 = i_1 + 6$$

- Solving these two equations we get:

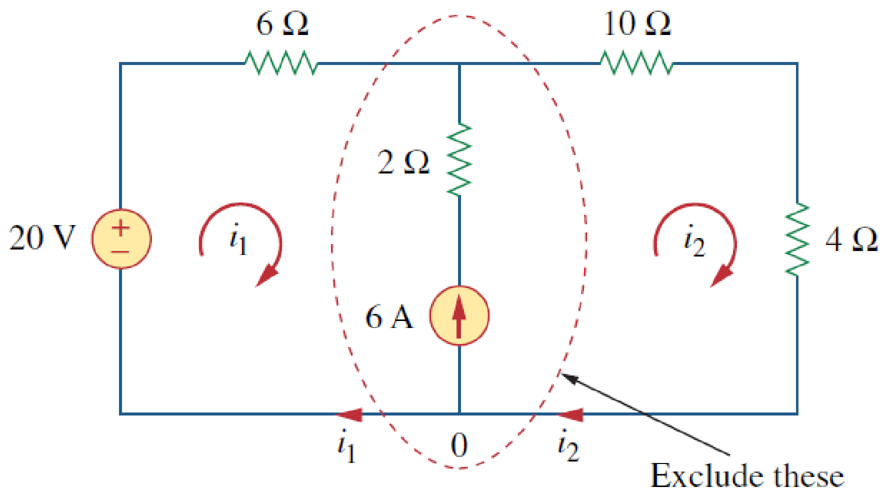
$$i_1 = -3.2\text{A} \qquad i_2 = 2.8\text{A}$$

- Note that the supermesh required using both KVL and KCL

Note the following properties of a supermesh:

1. The current source in the supermesh provides the constraint equation necessary to solve for the mesh currents.

2. A supermesh requires the application of both KVL and KCL.



(a)

Example

For the circuit in Fig. 3.24, find i_1 to i_4 using mesh analysis.

Example 3.7

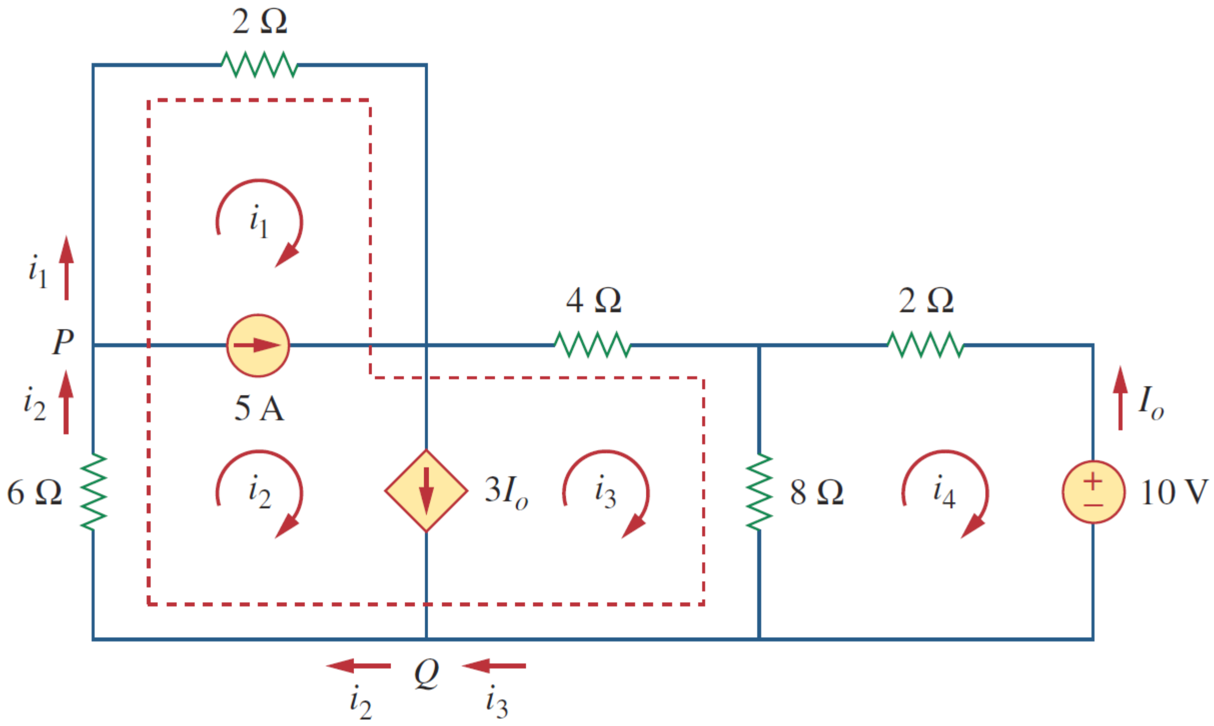


Figure 3.24
For Example 3.7.

Solution:

Note that meshes 1 and 2 form a supermesh since they have an independent current source in common. Also, meshes 2 and 3 form another supermesh because they have a dependent current source in common. The two supermeshes intersect and form a larger supermesh as shown. Applying KVL to the larger supermesh,

$$2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$$

or

$$i_1 + 3i_2 + 6i_3 - 4i_4 = 0 \qquad \textbf{(3.7.1)}$$

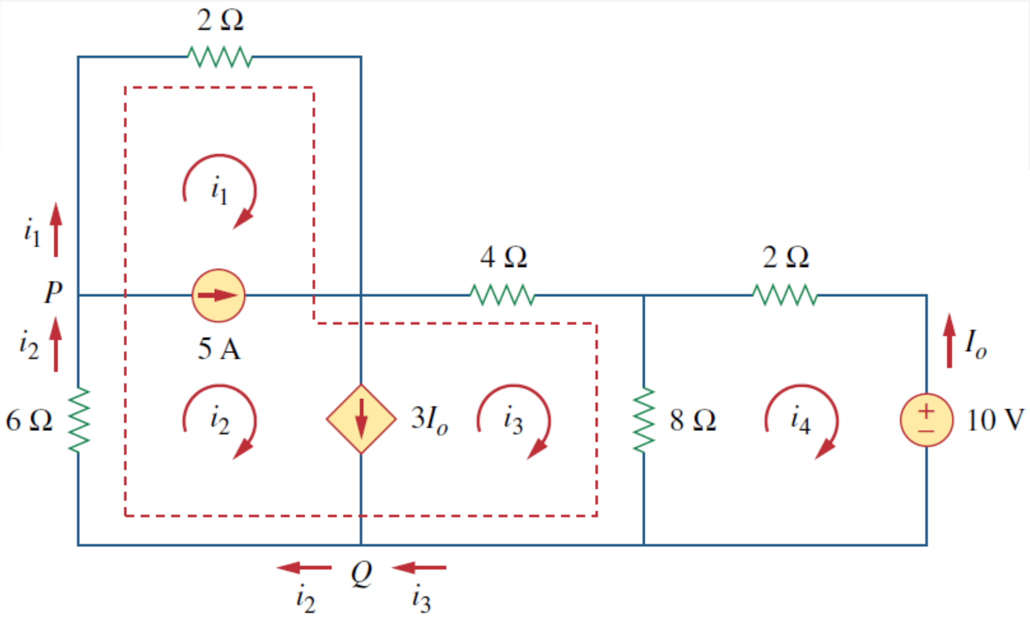
For the independent current source, we apply KCL to node P:

$$i_2 = i_1 + 5 \qquad \textbf{(3.7.2)}$$

For the dependent current source, we apply KCL to node Q:

$$i_2 = i_3 + 3I_o$$

If a circuit has two or more supermeshes that intersect, they should be combined to form a larger supermesh.



But $I_o = -i_4$, hence,

$$i_2 = i_3 - 3i_4 \qquad \textbf{(3.7.3)}$$

Applying KVL in mesh 4,

$$2i_4 + 8(i_4 - i_3) + 10 = 0$$

or

$$5i_4 - 4i_3 = -5 \qquad \textbf{(3.7.4)}$$

From Eqs. (3.7.1) to (3.7.4),

$$i_1 = -7.5 \text{ A}, \qquad i_2 = -2.5 \text{ A}, \qquad i_3 = 3.93 \text{ A}, \qquad i_4 = 2.143 \text{ A}$$

Example

Practice Problem 3.7 Use mesh analysis to determine i_1, i_2, and i_3 in Fig. 3.25.

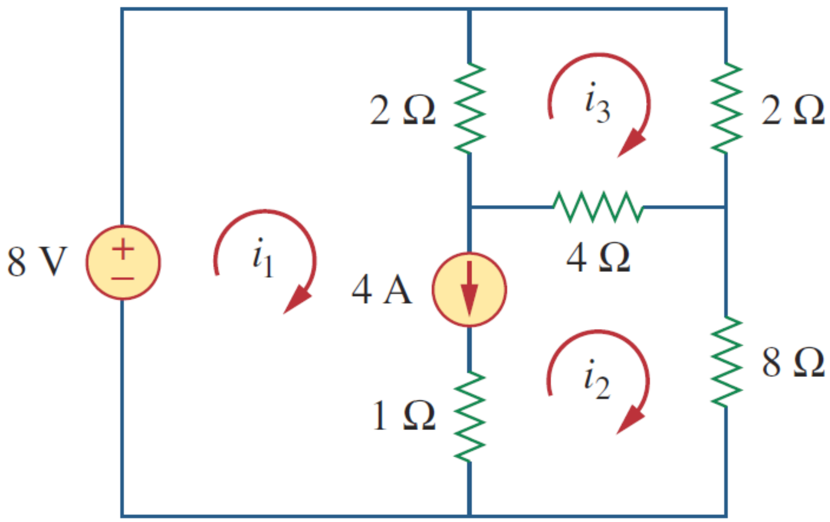


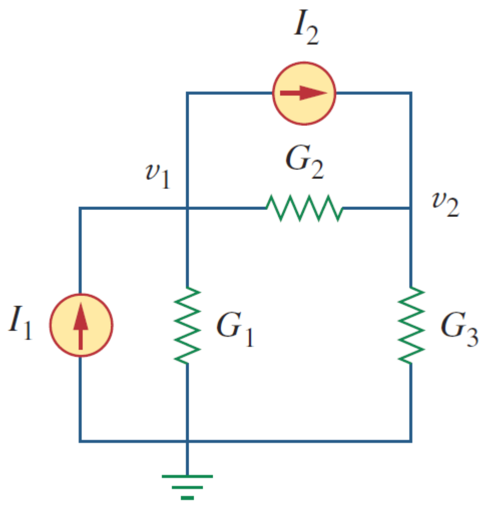
Figure 3.25
For Practice Prob. 3.7.

Answer: $i_1 = 4.632$ A, $i_2 = 631.6$ mA, $i_3 = 1.4736$ A.

3.6 Nodal and Mesh Analysis by Inspection

Nodal Analysis by Inspection

- There is a faster way to construct a matrix for solving a circuit by nodal analysis
- It requires that **all current sources in the circuit are independent**



$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

Each of the ***diagonal terms*** is the sum of the conductances directly connected to node 1 and node 2

The ***off-diagonal terms*** are the negatives of the conductances connected between the nodes

Each term on the ***right-hand side*** is the algebraic sum of the currents entering the node.

Nodal Analysis by Inspection

In general, **if a circuit has only independent current sources and linear resistors**, the node-voltage equations with N nonreference nodes can be written as:

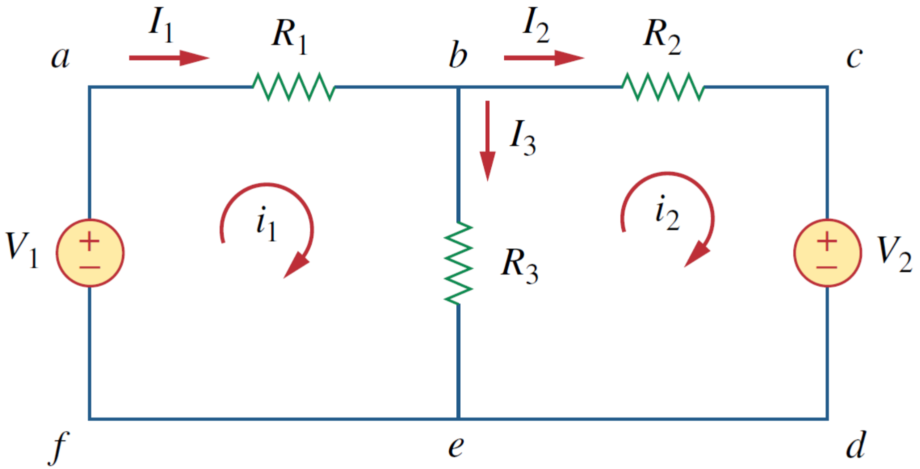
Conductance matrix

$$\begin{bmatrix} G_{11} & G_{12} & \dots & G_{1N} \\ G_{21} & G_{22} & \dots & G_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ G_{N1} & G_{N2} & \dots & G_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} \qquad \mathbf{Gv} = \mathbf{i}$$

- G_{kk} Each diagonal term is the sum of conductances connected to the node k (self conductance)
- $G_{jk} = G_{kj}$ The off-diagonal terms are negative of the sum of all conductances directly connecting nodes j and k, with $j \neq k$. (mutual conductance)
- V_k Unknown voltage at node k
- i_k Sum of all independent current sources directly connected to node k , with current entering the node treated as positive

Mesh Analysis by Inspection

- There is a similarly fast way to construct a matrix for solving a circuit by mesh analysis
- It requires that all voltage sources in the circuit are independent



$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

Each of the ***diagonal terms*** is the sum of the resistances in the related mesh

The ***off-diagonal*** terms are the negatives of the resistances common to meshes 1 and 2

Each term on the ***right-hand side*** is the algebraic sum taken clockwise of all independent voltage sources in the related mesh, **with voltage rise treated as positive**

Mesh Analysis by Inspection

In general, for a circuit with N meshes, the mesh-current equations may be written as:

Resistance matrix

$$\begin{bmatrix} R_{11} & R_{12} & \dots & R_{1N} \\ R_{21} & R_{22} & \dots & R_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ R_{N1} & R_{N2} & \dots & R_{NN} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \qquad \mathbf{Ri} = \mathbf{v}$$

- $\boldsymbol{R_{kk}}$ Each diagonal term is the sum of resistances in mesh k.
- $\boldsymbol{R_{jk}} = \boldsymbol{R_{kj}}$ The off-diagonal terms are negative of the sum of all resistances in common with meshes j and k, $\boldsymbol{j \neq k}$.
- $\boldsymbol{i_k}$ Unknown mesh currents in the clockwise direction.
- $\boldsymbol{V_k}$ Sum taken clockwise of all independent voltage sources in mesh k, with voltage rise treated as positive.

Example

Example 3.8

Write the node-voltage matrix equations for the circuit in Fig. 3.27 by inspection.

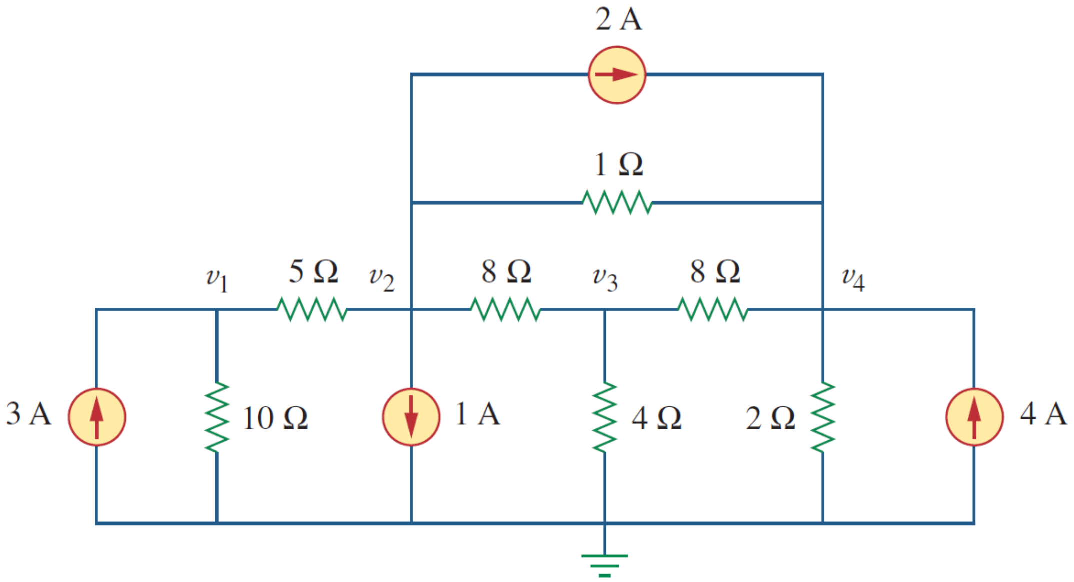


Figure 3.27
For Example 3.8.

Solution:

The circuit in Fig. 3.27 has four nonreference nodes, so we need four node equations. This implies that the size of the conductance matrix **G**, is 4 by 4. The diagonal terms of **G**, in siemens, are

$$G_{11} = \frac{1}{5} + \frac{1}{10} = 0.3, \qquad G_{22} = \frac{1}{5} + \frac{1}{8} + \frac{1}{1} = 1.325$$

$$G_{33} = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = 0.5, \qquad G_{44} = \frac{1}{8} + \frac{1}{2} + \frac{1}{1} = 1.625$$

The off-diagonal terms are

$$G_{12} = -\frac{1}{5} = -0.2, \qquad G_{13} = G_{14} = 0$$

$$G_{21} = -0.2, \qquad G_{23} = -\frac{1}{8} = -0.125, \qquad G_{24} = -\frac{1}{1} = -1$$

$$G_{31} = 0, \qquad G_{32} = -0.125, \qquad G_{34} = -\frac{1}{8} = -0.125$$

$$G_{41} = 0, \qquad G_{42} = -1, \qquad G_{43} = -0.125$$

The input current vector **i** has the following terms, in amperes:

$$i_1 = 3, \qquad i_2 = -1 - 2 = -3, \qquad i_3 = 0, \qquad i_4 = 2 + 4 = 6$$

Thus the node-voltage equations are

$$\begin{bmatrix} 0.3 & -0.2 & 0 & 0 \\ -0.2 & 1.325 & -0.125 & -1 \\ 0 & -0.125 & 0.5 & -0.125 \\ 0 & -1 & -0.125 & 1.625 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 6 \end{bmatrix}$$

Example

Practice Problem 3.8

By inspection, obtain the node-voltage equations for the circuit in Fig. 3.28.

Answer:

$$\begin{bmatrix} 1.25 & -0.2 & -1 & 0 \\ -0.2 & 0.2 & 0 & 0 \\ -1 & 0 & 1.25 & -0.25 \\ 0 & 0 & -0.25 & 1.25 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -3 \\ 2 \end{bmatrix}$$

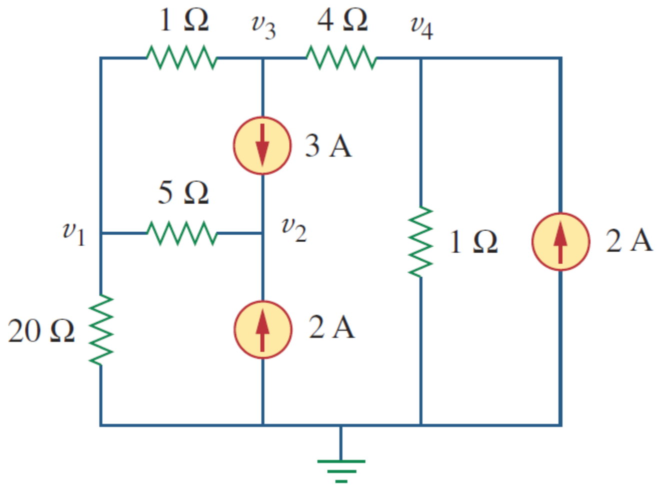


Figure 3.28
For Practice Prob. 3.8.

Example

By inspection, write the mesh-current equations for the circuit in Fig. 3.29.

Example 3.9

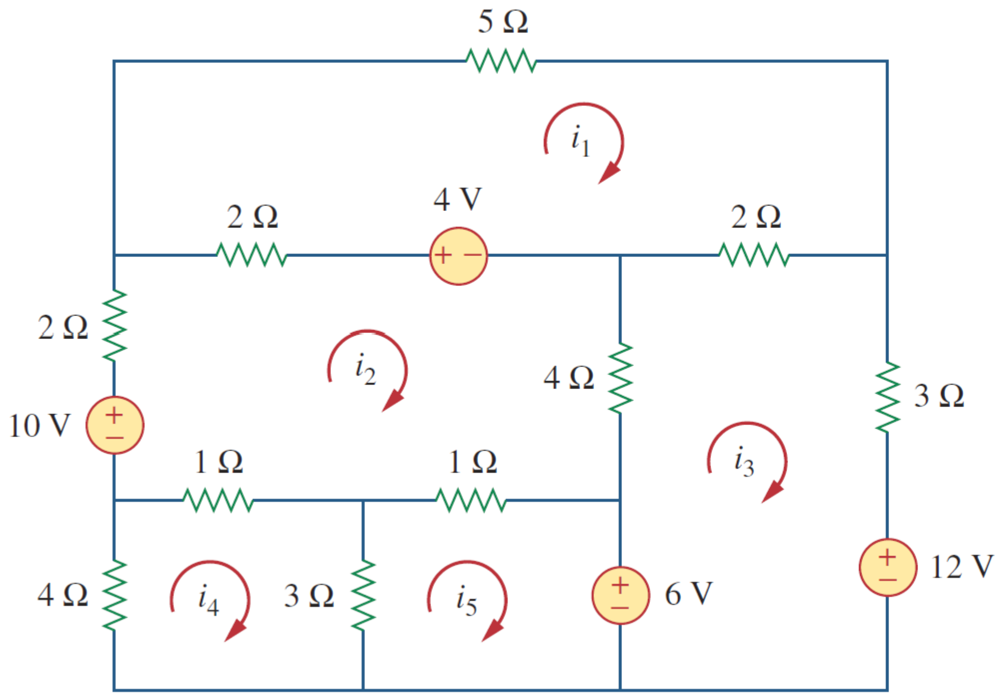


Figure 3.29

Solution:

We have five meshes, so the resistance matrix is 5 by 5. The diagonal terms, in ohms, are:

$$R_{11} = 5 + 2 + 2 = 9, \qquad R_{22} = 2 + 4 + 1 + 1 + 2 = 10,$$
$$R_{33} = 2 + 3 + 4 = 9, \qquad R_{44} = 1 + 3 + 4 = 8, \qquad R_{55} = 1 + 3 = 4$$

The off-diagonal terms are:

$$R_{12} = -2, \qquad R_{13} = -2, \qquad R_{14} = 0 = R_{15},$$
$$R_{21} = -2, \qquad R_{23} = -4, \qquad R_{24} = -1, \qquad R_{25} = -1,$$
$$R_{31} = -2, \qquad R_{32} = -4, \qquad R_{34} = 0 = R_{35},$$
$$R_{41} = 0, \qquad R_{42} = -1, \qquad R_{43} = 0, \qquad R_{45} = -3,$$
$$R_{51} = 0, \qquad R_{52} = -1, \qquad R_{53} = 0, \qquad R_{54} = -3$$

The input voltage vector **v** has the following terms in volts:

$$v_1 = 4, \qquad v_2 = 10 - 4 = 6,$$
$$v_3 = -12 + 6 = -6, \qquad v_4 = 0, \qquad v_5 = -6$$

Thus, the mesh-current equations are:

$$\begin{bmatrix} 9 & -2 & -2 & 0 & 0 \\ -2 & 10 & -4 & -1 & -1 \\ -2 & -4 & 9 & 0 & 0 \\ 0 & -1 & 0 & 8 & -3 \\ 0 & -1 & 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -6 \\ 0 \\ -6 \end{bmatrix}$$

Example

Practice Problem 3.9

By inspection, obtain the mesh-current equations for the circuit in Fig. 3.30.

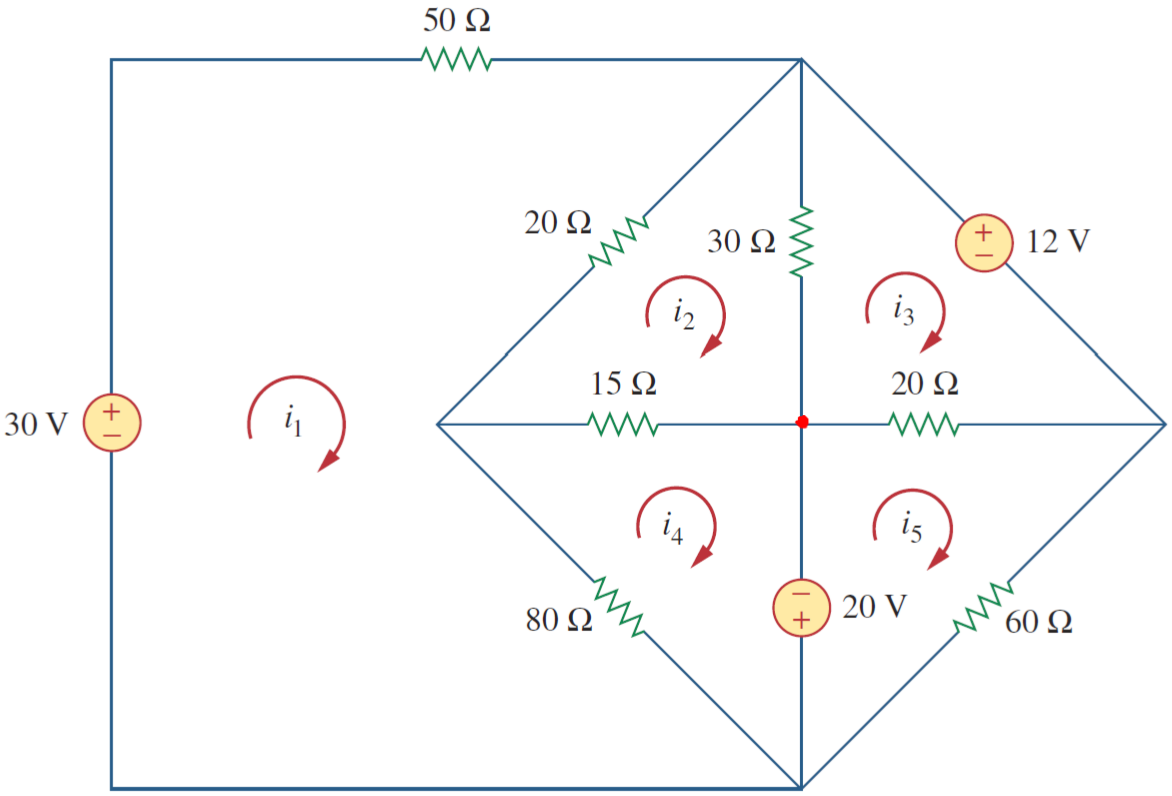


Figure 3.30
For Practice Prob. 3.9.

Answer:

$$\begin{bmatrix} 150 & -\overset{2}{\cancel{4}}0 & 0 & -80 & 0 \\ -\overset{2}{\cancel{4}}0 & 65 & -30 & -15 & 0 \\ 0 & -30 & 50 & 0 & -20 \\ -80 & -15 & 0 & 95 & 0 \\ 0 & 0 & -20 & 0 & 80 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ -12 \\ 20 \\ -20 \end{bmatrix}$$

3.7 Nodal Versus Mesh Analysis

- In principle both the nodal and mesh analysis are useful for any given circuit.
- Which one is better when solving a circuit problem?
- There are two factors that dictate the best choice:
  - The first factor is the nature of the particular network
  - The second factor is the information required
- The key is to select the method that results in the smaller number of equations.

Mesh analysis when…

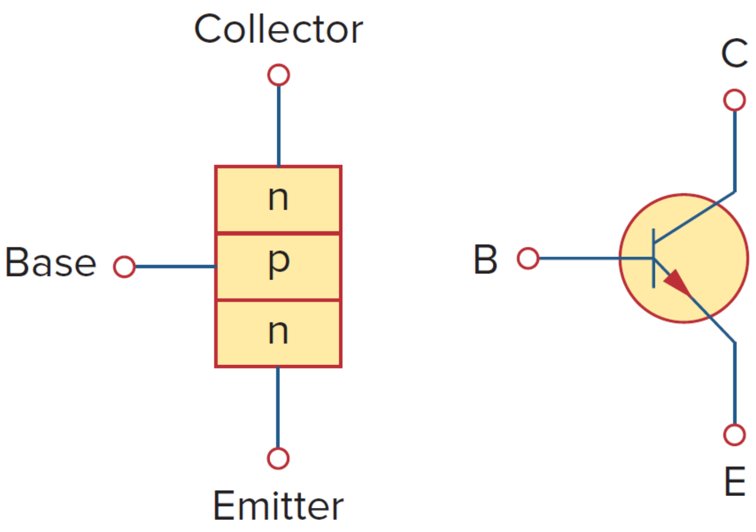
- If the network contains:
  - Many series connected elements
  - Voltage sources
  - Supermeshes
  - A circuit with fewer meshes than nodes
- If branch or mesh currents are required.
- Mesh analysis is the only suitable method for transistor circuits
- It is not appropriate for operational amplifiers because there is no direct way to obtain the voltage across an op-amp.

Nodal analysis if…

- If the network contains:
  - Many parallel connected elements
  - Current sources
  - Supernodes
  - A circuit with fewer nodes than meshes
- If node voltages are required.
- Non-planar circuits can only be solved using nodal analysis

Application: DC transistor circuit

- Here we will use the approaches learned in this chapter to analyze a transistor circuit.
- Consider one type of transistors commonly used: Bipolar Junction Transistor (BJT).
- A BJT is a three terminal device, where the input current into one terminal **(the base)** affects the current flowing out of a second terminal (**the collector**).
- The third terminal (the emitter) is the common terminal for both currents



NPN BJT

KCL and KVL for a BJT

- The currents from each terminal can be related to each other as follows:

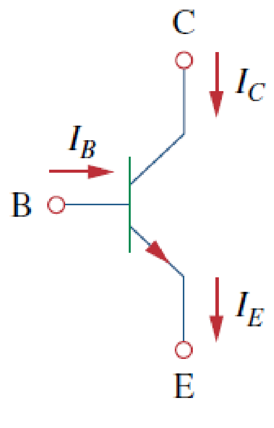
$$I_E = I_B + I_C$$

- The BJT can operate in one of three modes depending on the applied voltages/currents: active, cutoff, and saturation.

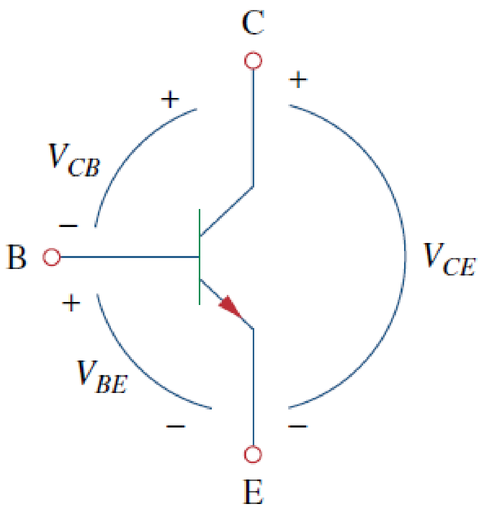
- "active mode" is used for amplifying signals.

- When transistors operate in the active mode, typically $V_{BE} \simeq 0.7$ V, the base and collector current can be related to each other by the parameter β, which can range from 50-1000

$$I_c = \beta I_B$$

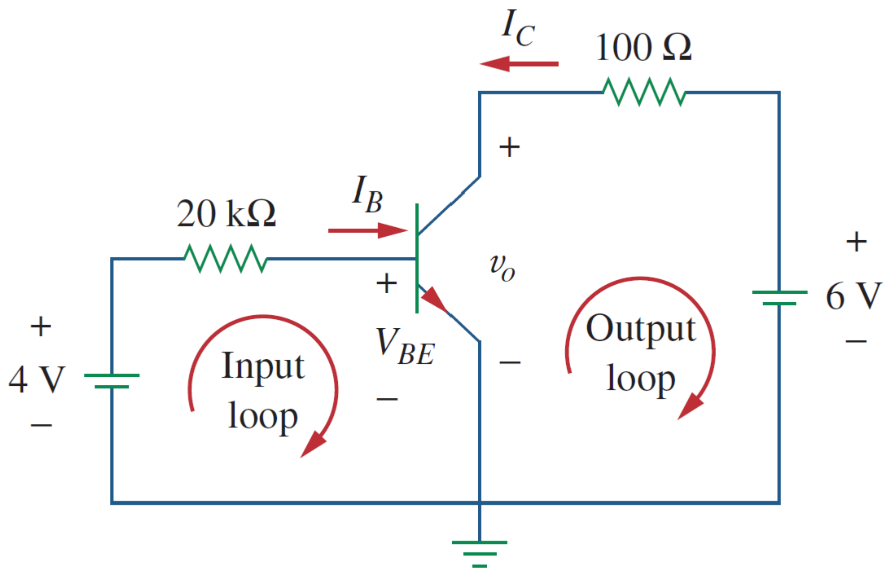


(a)



(b)

Find I_B, I_C, and v_o in the transistor circuit of Fig. 3.41. Assume that the transistor operates in the active mode and that $\beta = 50$.



Method for transistor circuits: Mesh analysis

Solution:

For the input loop, KVL gives

$$-4 + I_B(20 \times 10^3) + V_{BE} = 0$$

Since $V_{BE} = 0.7$ V in the active mode,

$$I_B = \frac{4 - 0.7}{20 \times 10^3} = 165\ \mu\text{A}$$

But

$$I_C = \beta I_B = 50 \times 165\ \mu\text{A} = 8.25\ \text{mA}$$

For the output loop, KVL gives

$$-v_o - 100 I_C + 6 = 0$$

or

$$v_o = 6 - 100 I_C = 6 - 0.825 = 5.175\ \text{V}$$

Note that $v_o = V_{CE}$ in this case.

DC model of a BJT

- In the active mode, the BJT can be modelled as a dependent current-controlled current source.

$V_{BE} \simeq 0.7 \text{ V}$

$I_c = \beta I_B$

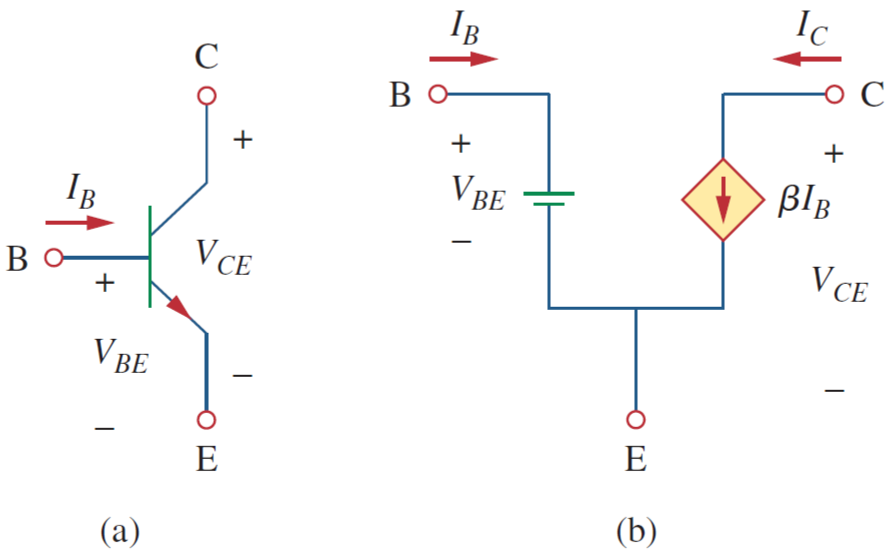


Figure 3.40
(a) An *npn* transistor, (b) its dc equivalent model.