CALCULUS

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- Exponential functions model a wide variety of phenomena of interest in science, engineering, mathematics, and economics. In this chapter, we use the methods of calculus to obtain rigorous and precise definitions and properties of the exponential functions.
- We first define the natural logarithm function $y = \ln x$ as a certain integral, and then the natural exponential function $y = e^x$ as its inverse function. We also introduce inverse trigonometric functions, as well as hyperbolic functions and their inverses, and investigate their applications.
- Along with the trigonometric functions, all of these functions belong to the category of ***transcendental functions***.

① One-to-One Functions

DEFINITION

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

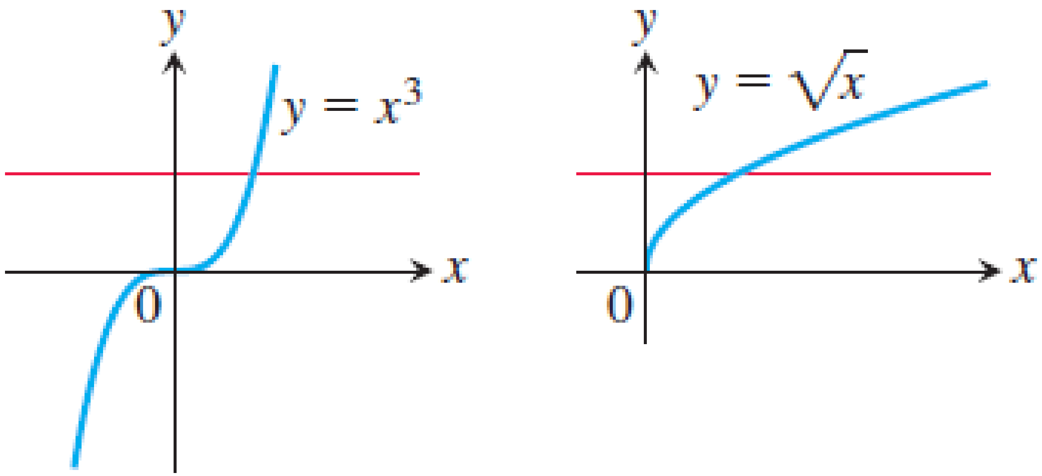
Example 1

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

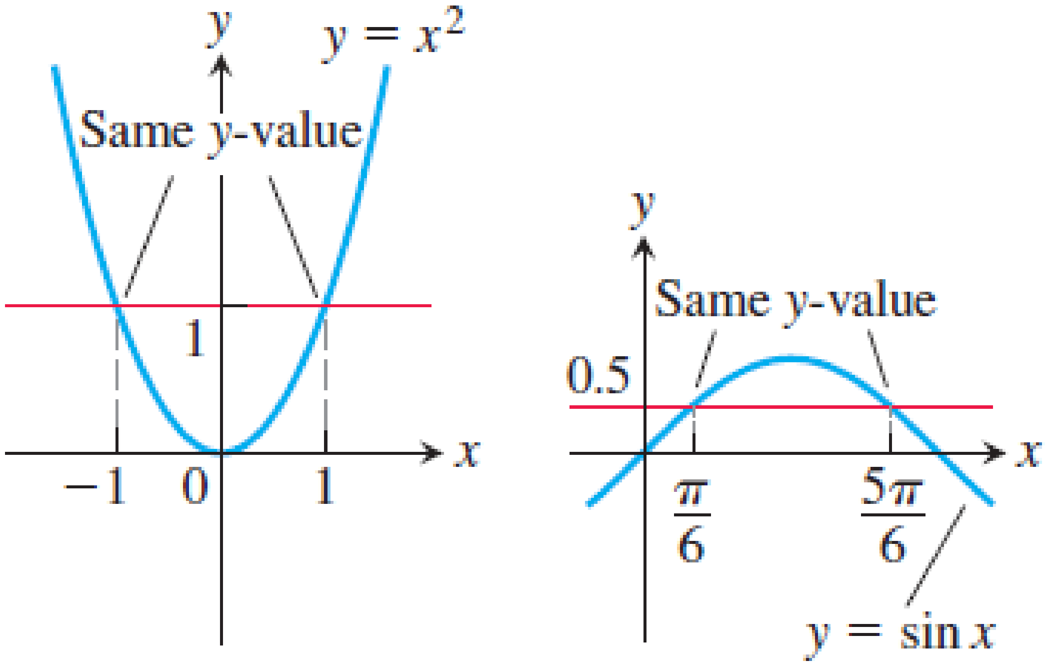
(b) $f(x) = x^2$ is NOT one-to-one on its domain because $(a)^2 = (-a)^2$ as long as a is a real number.

● The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is **one-to-one** if and only if its graph intersects each horizontal line at most once.



(a) One-to-one: Graph meets each horizontal line at most once.



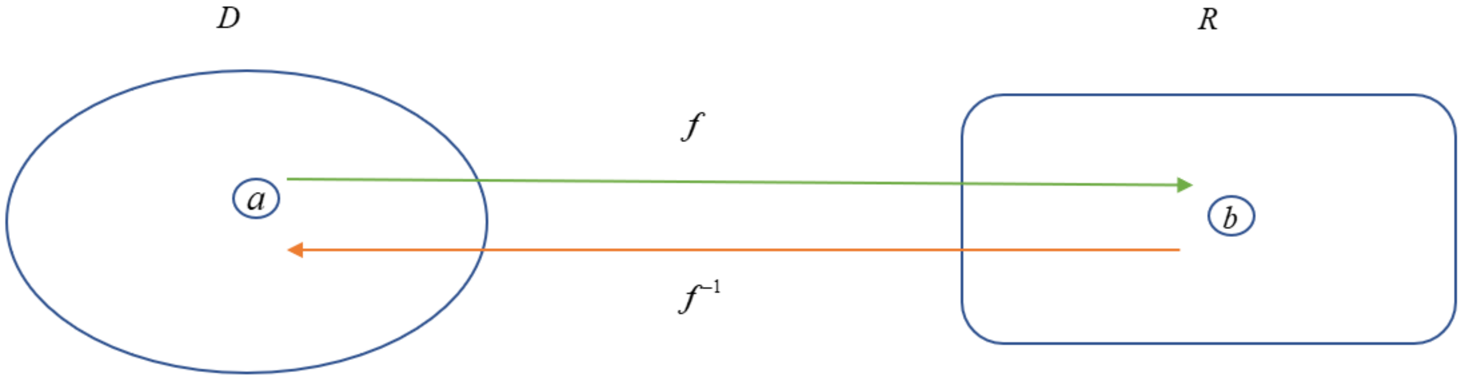
(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

② Inverse Functions

DEFINITION

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by $f^{-1}(b) = a$ if $f(a) = b$.

The domain of f^{-1} is R and the range of f^{-1} is D.



③ Finding Inverses

- The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x.

2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$.

Example 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Example 4 Find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of x.

④ Derivatives of Inverses of Differentiable Functions

THEOREM 1 – The Derivative Rule for Inverses

If f has an interval I as domain and $f^{-1}(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)} = \frac{1}{f'(a)}$$

or

$$\left.\frac{\mathrm{d}f^{-1}}{\mathrm{d}x}\right|_{x=b} = \frac{1}{\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=f^{-1}(b)}}.$$

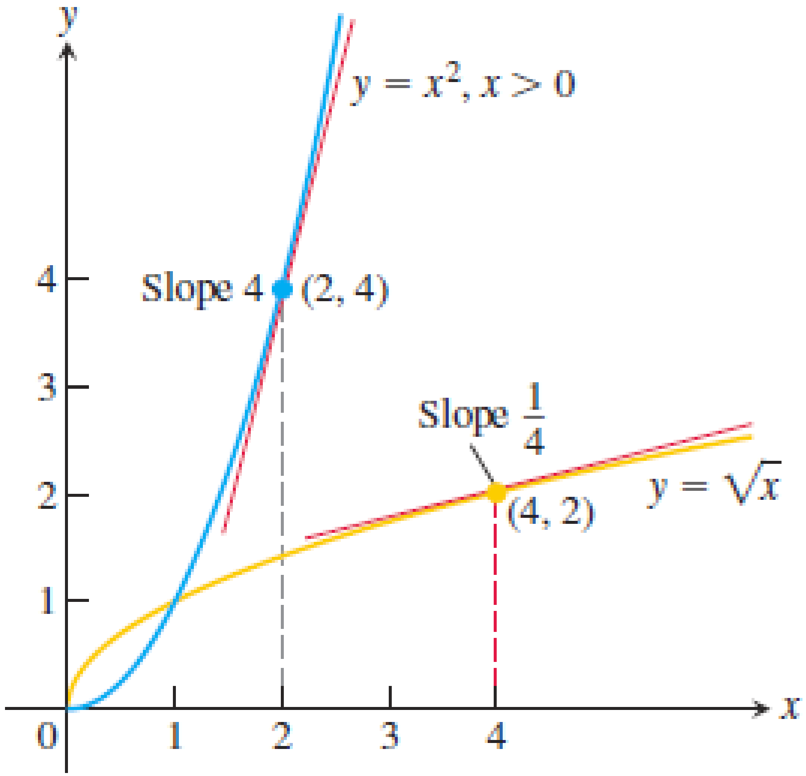
Example 5 Verification of Theorem 1

The function $y = x^2$ $(x \geq 0)$ and its inverse function $f^{-1}(x) = \sqrt{x}$ have derivatives

$f'(x) = 2x$ and $(f^{-1})'(x) = \dfrac{1}{2\sqrt{x}}$.

$$f'\left(f^{-1}(x)\right) = f'(\sqrt{x}) = 2\sqrt{x}$$

$$\Rightarrow \quad (f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

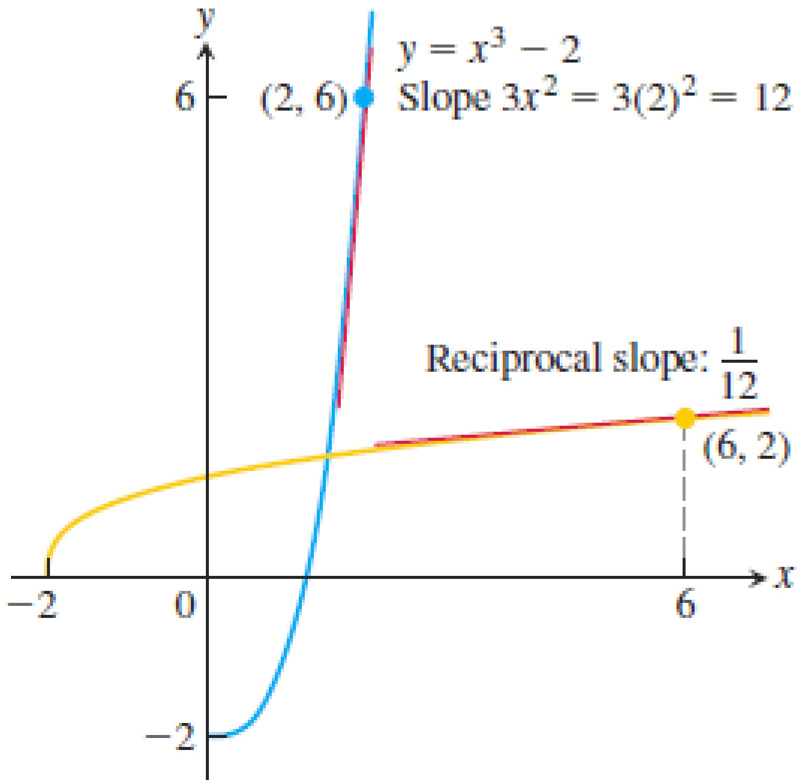


Look at a specific point (2, 4).

Theorem 1 is verified.

Example 6

Let $f(x) = x^3 - 2$ $(x > 0)$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.



Example 7

Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find $(f^{-1})'(0)$.

Skill Practice

$$f(x) = \frac{x+2}{1-x}$$

(a) Find $f^{-1}(x)$.

(b) Evaluate $f'(x)$ at $x = 1/2$ and $(f^{-1})'(x)$ at $x = f(1/2) = 5$ to show that at these points $(f^{-1})'(5) = 1/f'(1/2)$.