3. Kinematics in 3D

Treat the components individually then the resultant: $\sqrt{x^2 + y^2 + z^2}$.

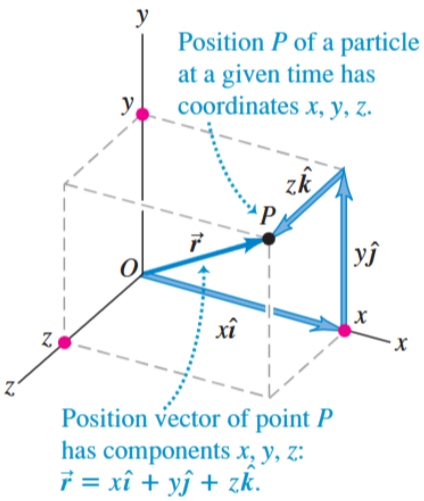
The direction of the resultant is
$\tan\theta = \frac{y}{x}$

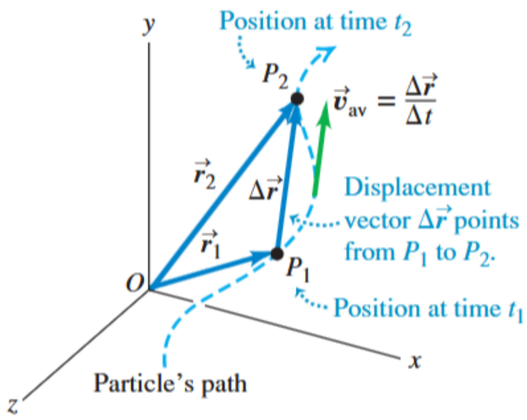
Position and Velocity Vectors

Position

The position vector of the particle at this instant is a vector that goes from the origin.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$





During a time interval Δt the particle moves from P_1 to P_2 ($\vec{r}_1$ to $\vec{r}_2$).

The change in position (the displacement) is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$

$$\vec{v}_{av-x} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t}$$

The **magnitude** of the displacement or the **distance**:

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Velocity Vectors and Instantaneous Vectors

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

It follows that the components v_x, v_y and v_z of the instantaneous velocity v_x are simply the time derivates of the coordinates.

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt}$$

The **magnitude** of the velocity or **the speed:**

$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The instantaneous velocity $\vec{v}$ is tangent to the path at each point. Often "direction" of a velocity refers to the angle α.

$$\tan \alpha = \frac{v_y}{v_x}$$

Acceleration Vector

$$a_{av-x} = \frac{v_{-}v_{1x}}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t}$$

Projectile Motion

Assuming there is no air resistance, the gravitational acceleration will affect the vertical y component, and since the horizontal x component has no acceleration, the final and the initial velocities are the same, the equations of motions in the $x - y$ components become:

$$x = x_0 + v_{ox} t$$
$$v_x = v_{ox}$$

$$y = y_0 + v_{0y}\, t - \frac{1}{2} g t^2$$
$$v_y = v_{0y} - gt$$

where

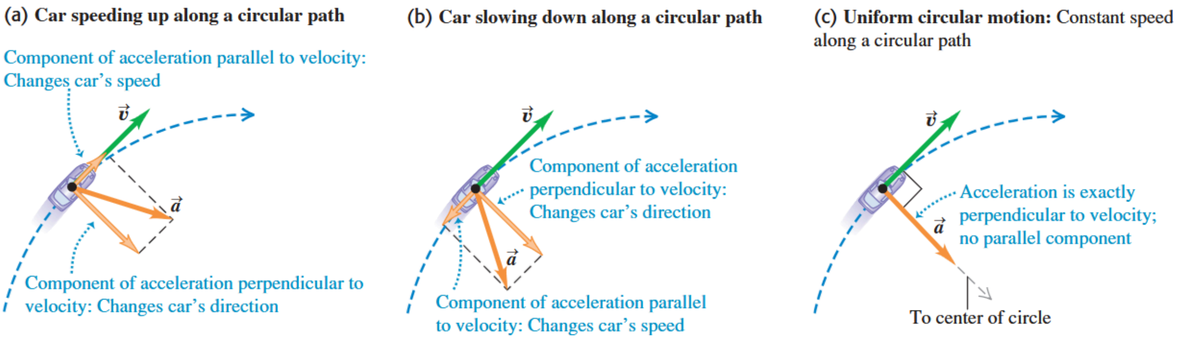
$$v_{0x} = v_0 \cos \theta \qquad v_{0y} = v_0 \sin \theta$$

Height and Range derivations:

Motion in a Circle

Uniform Circular Motion

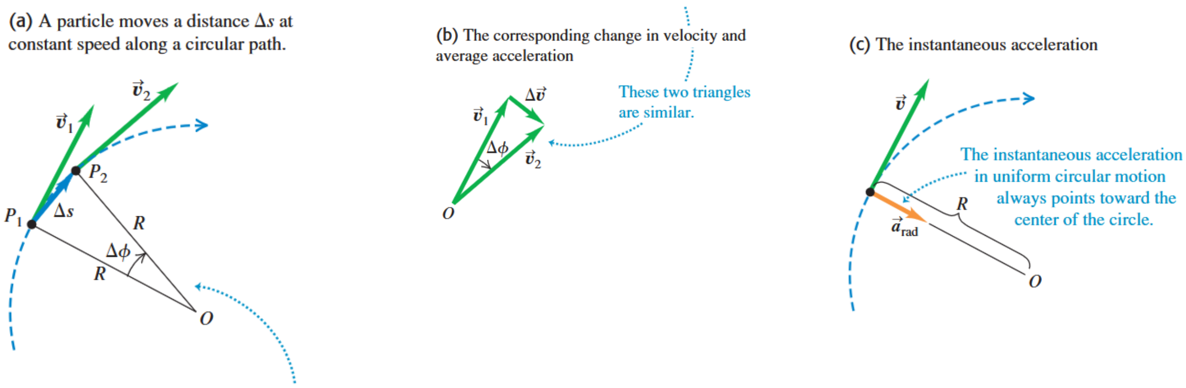
When a particle moves in a circle with *constant speed*, the motion is called uniform circular motion



3.27 A car moving along a circular path. If the car is in uniform circular motion as in (c), the speed is constant and the acceleration is directed toward the center of the circular path (compare Fig. 3.12).

The centripetal acceleration of an object in an uniform circular motion (constant velocity), which is also equal to the normal (or resultant) acceleration of the body:

$$a_{\text{rad}} = \frac{v^2}{R} = a$$



The angles labeled $\Delta\phi$ in Figs. 3.28a and 3.28b are the same because $\vec{v}_1$ is perpendicular to the line OP_1 and $\vec{v}_2$ is perpendicular to the line OP_2. Hence the triangles in Figs. 3.28a and 3.28b are *similar.* The ratios of corresponding sides of similar triangles are equal, so

$$\frac{|\Delta\vec{v}|}{v_1} = \frac{\Delta s}{R} \qquad \text{or} \qquad |\Delta\vec{v}| = \frac{v_1}{R}\Delta s$$

The magnitude a_{av} of the average acceleration during Δt is therefore

$$a_{av} = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

The magnitude a of the *instantaneous* acceleration $\vec{a}$ at point P_1 is the limit of this expression as we take point P_2 closer and closer to point P_1:

$$a = \lim_{\Delta t \to 0} \frac{v_1}{R}\frac{\Delta s}{\Delta t} = \frac{v_1}{R}\lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t}$$

If the time interval Δt is short, Δs is the distance the particle moves along its curved path. So the limit of $\Delta s/\Delta t$ is the speed v_1 at point P_1. Also, P_1 can be any point on the path, so we can drop the subscript and let v represent the speed at any point. Then

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In a time T of the motion, the time for one revolution (one complete trip around the circle). In a time T the particle travels a distance equal to the circumference $2\pi R$ of the circle, so:

$$v = \frac{2\pi R}{T} \qquad \text{and} \qquad a_{\text{rad}} = \frac{4\pi^2 R}{T^2}$$

Non-uniform Circular Motion

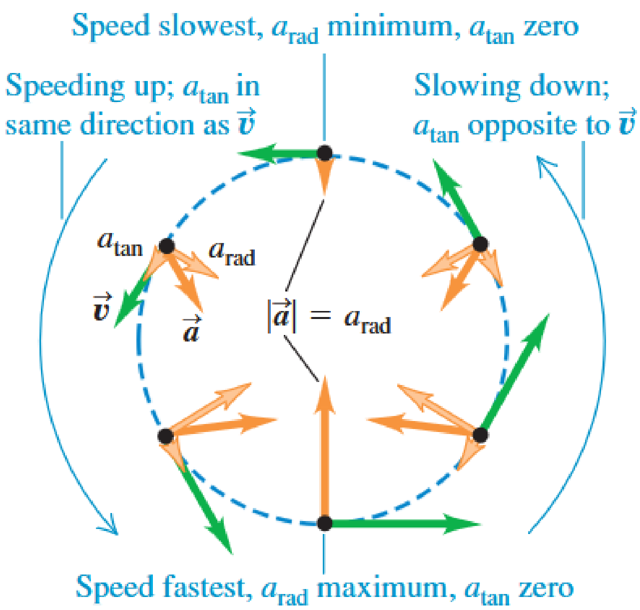
If the speed varies, the motion becomes a non-uniform circular motion.

In it, there is still the *radial* component of acceleration $a_{\text{rad}} = \frac{v^2}{R}$, which is always perpendicular to the instantaneous velocity and directed toward the centre of the circle.

But since the speed v has different values at different points, the value of a_{rad} is **not constant**. The radial (centripetal) acceleration is the greatest at the point in the circle where the speed is the greatest.

$$a_{\text{rad}} = \frac{v^2}{R} \qquad a_{\tan} = \frac{d|\vec{v}|}{dt}$$

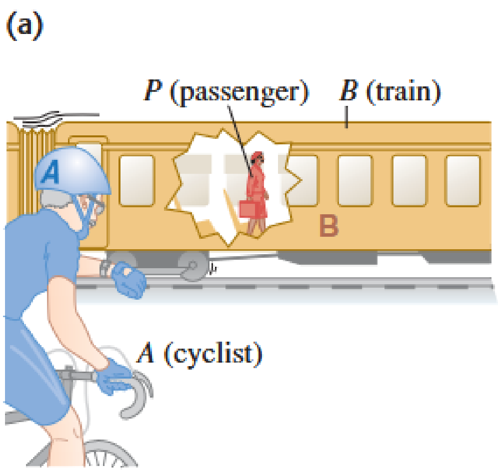
The tangential component is in the same direction as the velocity if the particle is speeding up, and in the opposite direction if the particle is slow down.



Relative Velocity

The velocity seen by a particular observer is called the velocity *relative* to the observer, or simply **relative velocity**.

(vector addition)



$$x_{P/A} = x_{P/B} + x_{B/A}$$

$$v_{P/A-x} = v_{P/B-x} + v_{B/A-x}$$