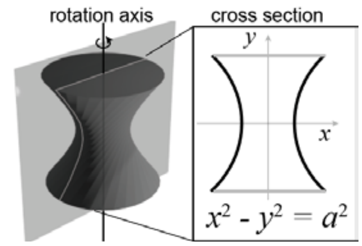
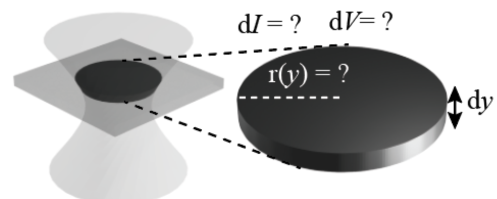
Name: ______________________ Student ID: ________________________

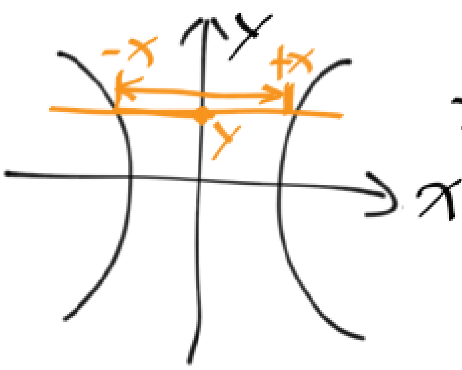
Quiz 8 Rotations



A hyperboloid with mass M (and uniform density) has a cross-section defined as $x^2 - y^2 = a^2$ with height range $-a < y < a$, where a is a known constant as shown in the figure. Use calculus, derive the moment of inertial I of the hyperboloid when rotating around its central axis. Express your answer with M and a.

1. As usual, the total moment of inert I is the integral ("sum") of the slices across y, each with a thickness dy, from $-a$ to a. Assume we know the density of the material ρ, derive the momentum of inertia dI of such a slice at an arbitrary height y in terms of y, a, and ρ. Hint: to calculate dI, you need to derive the mass dm of the slide, which needs to derive dV of the slice. To get dV, you need to calculate the radius of the disk-shape slice as a function of r(y). So the order of calculations goes r(y) -> dV -> dm -> dI.



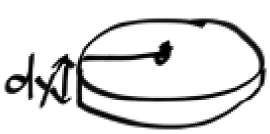


$x^2 - y^2 = a^2$

given y, $x^2 = y^2 + a^2$

$x = \pm\sqrt{y^2+a^2}$

for the slice dy at y



$r = \sqrt{y^2+a^2}$ so volume of the slice

$dV = \pi r^2 dy = \pi(y^2+a^2)dy$

mass $dm = \rho dV = \pi\rho(y^2+a^2)dy$

$dI = \frac{1}{2}(dm)r^2$ for a disk (remember?)

$= \frac{1}{2}\pi\rho(y^2+a^2)dy \cdot (y^2+a^2)$

$= \frac{\pi\rho}{2}(y^2+a^2)^2 dy = \frac{\pi\rho}{2}(y^4+2a^2y^2+a^4)dy$

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2. Now “add” up the contributions from the slices from y = -a to y = a. Write out the integral

Now: "add" all slices from $y=-a$ to $y=+a$ by integration

$$I = \int dI = \int_0^a \pi \rho (y^4 + 2a^2y^2 + a^4)\,dy$$

$$= \pi \rho \left(\frac{y^5}{5} + \frac{2}{3}a^2y^3 + a^4 y \right)\Big|_0^a = \frac{28}{15}\pi \rho a^5$$

3. Finally, provided that the volume of the hyperboloid is (8/3) πa^3, calculate ρ of the shape and express the moment of inertia in terms of M and a.

$$2\pi a \left(\frac{4}{3}a^2\right) = \frac{8}{3}\pi a^3$$

$$\text{so } \rho = \frac{M}{V} = \frac{M}{\frac{8}{3}\pi a^3}$$

$$I = \frac{28}{15}\cancel{\pi}\overset{a^2}{\cancel{a^5}} \cdot \frac{3M}{8\cancel{\pi}\cancel{a^3}} = 0.7\,Ma^2$$

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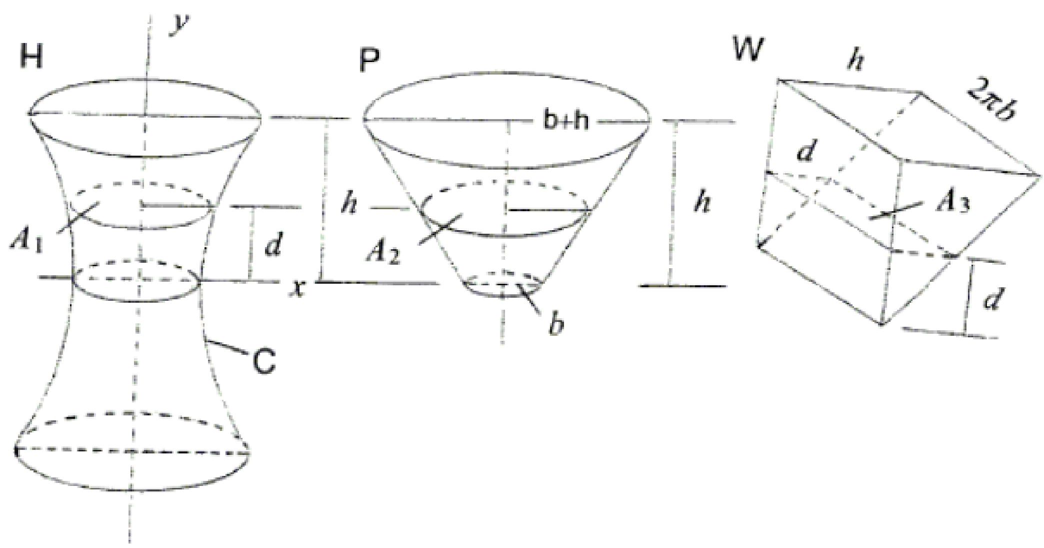
C: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, -h < y < h,\ x^2 = \frac{a^2}{b^2}(b^2 + y^2)$



Denote the solid bounded by the surface and two planes $y = \pm h$ by H. At the level d above the x-axis, the cross-section of H is a circle of radius $\frac{a}{b}\sqrt{b^2 + d^2}$. So, $A_1 = \frac{\pi a^2}{b^2}(b^2 + d^2)$.

We place two solids P and W parallel to H on the x-axis. Their dimensions are chosen such that $A_2 = \pi(b+d)^2$ and $A_3 = 2\pi bd$. Thus, $A_1 = \frac{\pi a^2}{b^2}[(b+d)^2 - 2bd] = \frac{a^2}{b^2}(A_2 - A_3)$. Hence, by (**), the volume of the hyperboloid H is $2 \times \frac{a^2}{b^2}[\text{volume of } P - \text{volume of } W\ ]$, which is

$$V_H = \frac{2a^2}{b^2}\{\frac{\pi h}{3}[(b+h)^2 + b(b+h) + b^2] - \pi h^2 b\} = \frac{2\pi h a^2}{b^2}\left(b^2 + \frac{h^2}{3}\right).$$

a = b = h, so $V = (8/3)\,\pi a^3$