Lecture 10

Dynamics of Rotational Motion

Date:

Course Instructor:

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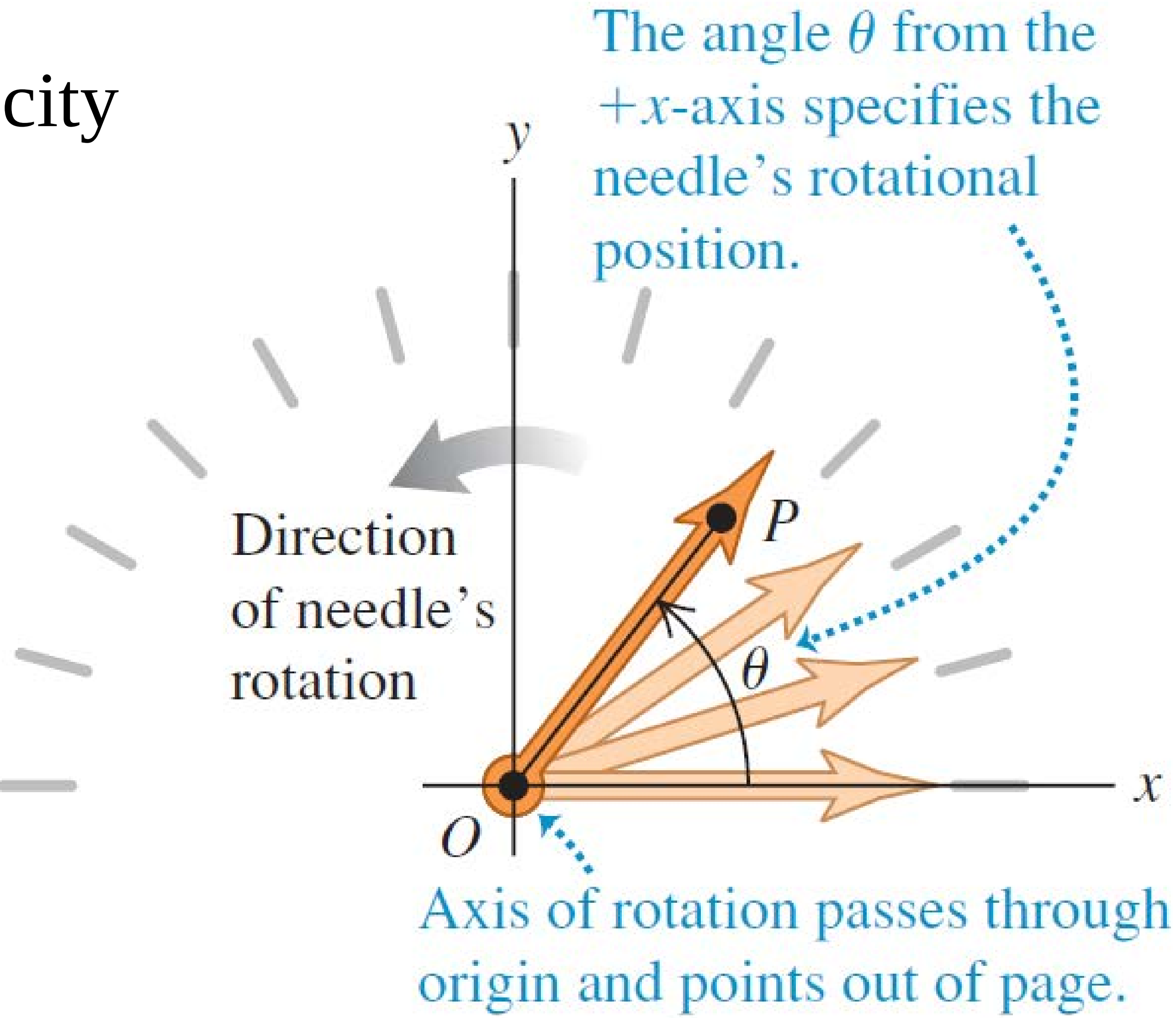
Previous Lectures

Rotation of rigid motion:

- Angular velocity vs linear velocity
- Momentum of Inertia, relate:
  - angular velocity
  - kinetic energy

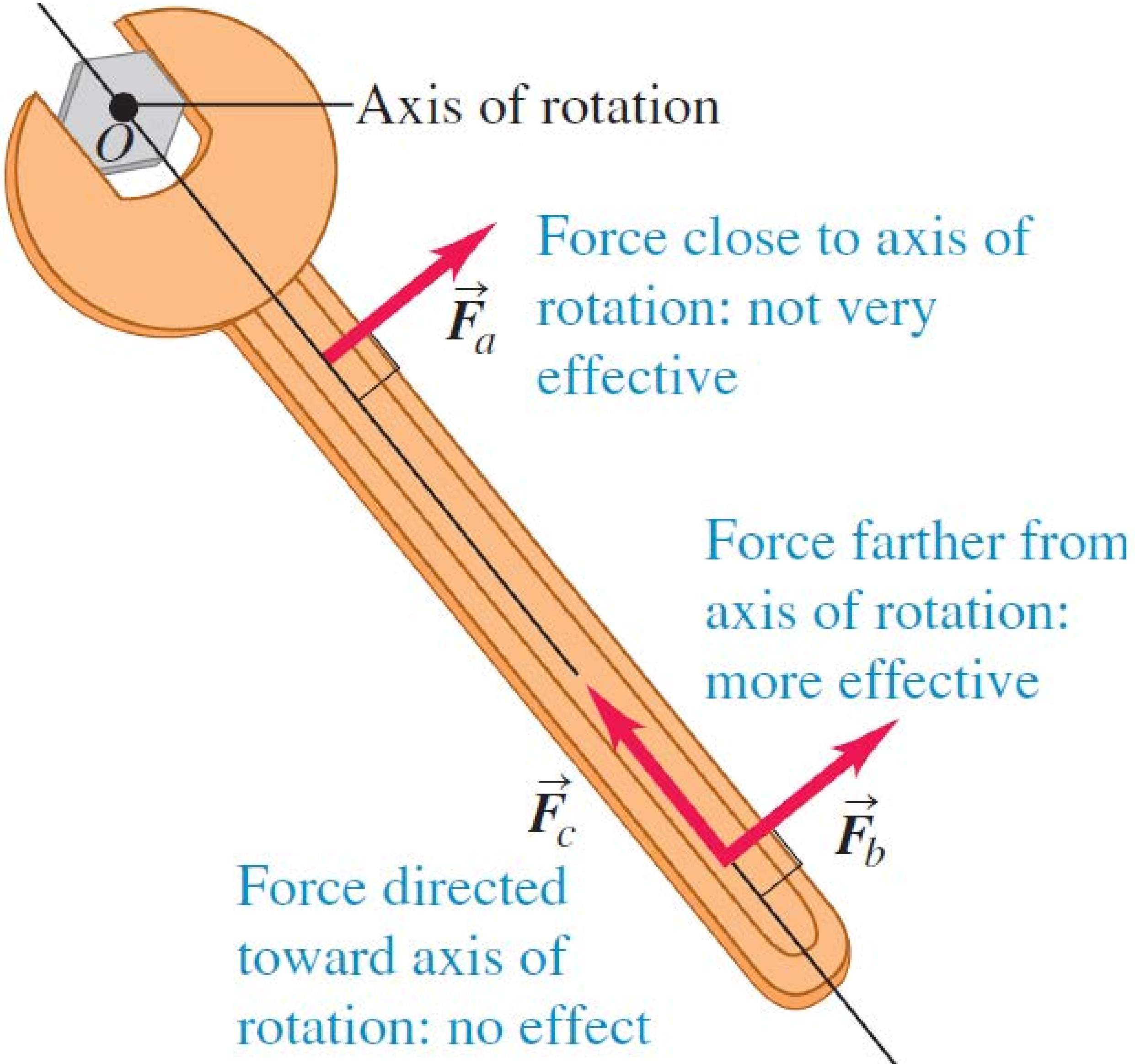
What is missing?

- How does force play a role?
- What if the axis is moving?
  - Think about rolling a wheel



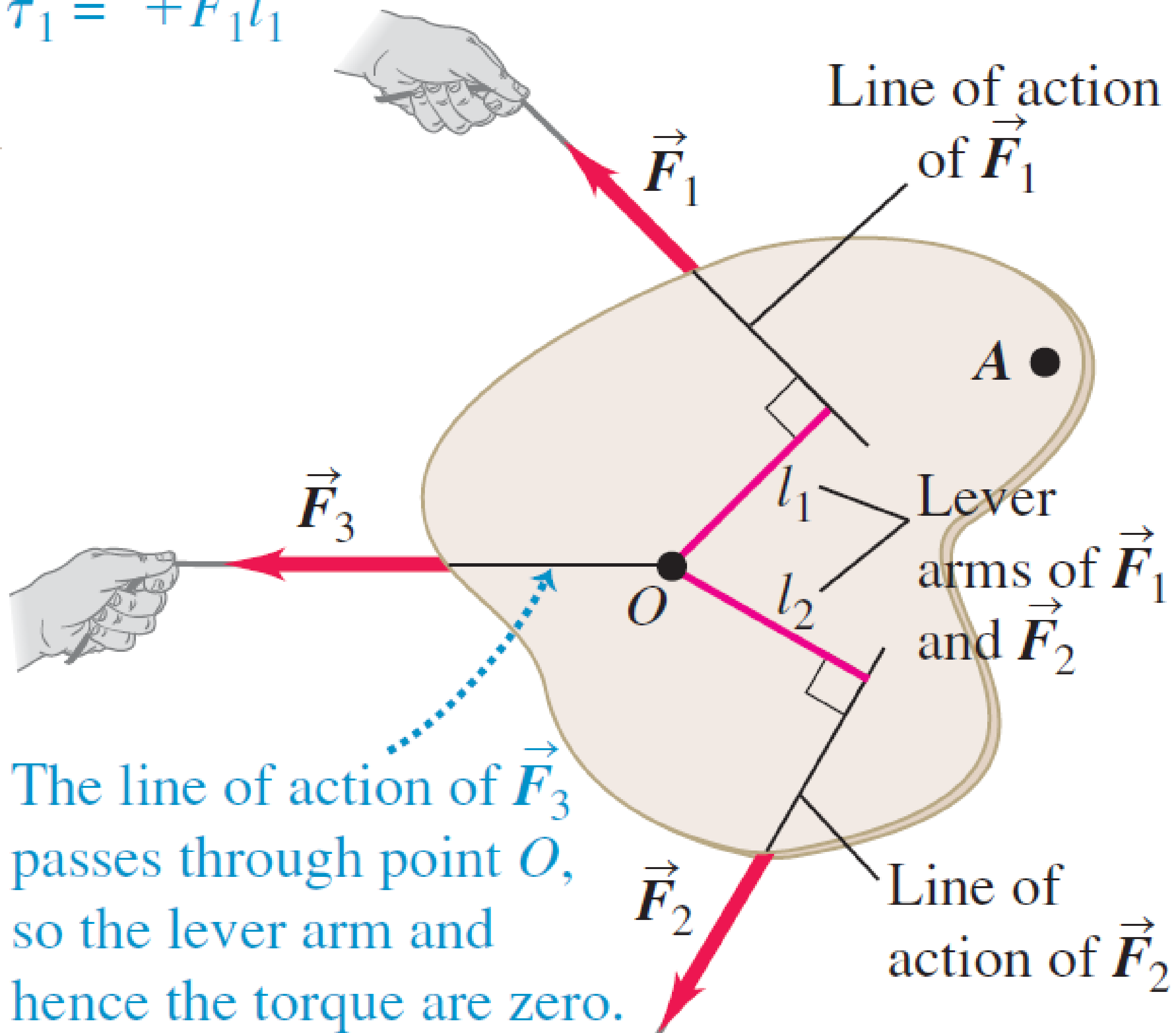
Torque: Relate Force to Rotation

- Forces acting on a body can affect its **translational motion**
- What about rotational motion?



Torque: Relate Force to Rotation

$\vec{F}_1$ tends to cause *counterclockwise* rotation about point O, so its torque is *positive*: $\tau_1 = +F_1 l_1$



Key: *perpendicular* distance l_1 and l_2 between point O and the **line of action** of the force, called ***lever/moment arm***

We define the (magnitude of) **torque τ** as: $\tau = Fl$

F_3 passes through O, so $l = 0$, and $\tau = 0$:
– no tendency for rotation

The line of action of $\vec{F}_3$ passes through point O, so the lever arm and hence the torque are zero.

$\vec{F}_2$ tends to cause *clockwise* rotation about point O, so its torque is *negative*: $\tau_2 = -F_2 l_2$

Torque: Relate Force to Rotation

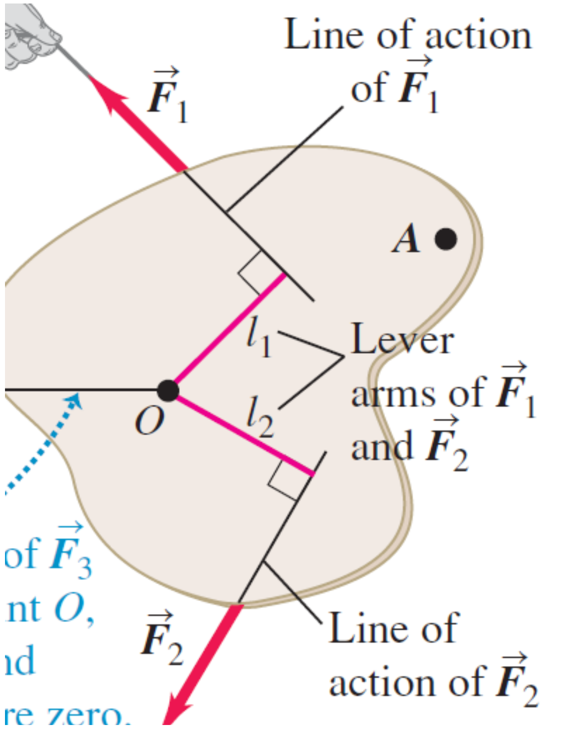
CAUTION **Torque is always measured about a point** Note that torque is *always* defined with reference to a specific point. If we shift the position of this point, the torque of each force may also change. For example, the torque of force $\vec{F}_3$ in Fig. 10.2 is zero with respect to point O, but the torque of $\vec{F}_3$ is *not* zero about point A. It's not enough to refer to "the torque of $\vec{F}$"; you must say "the torque of $\vec{F}$ with respect to point X" or "the torque of $\vec{F}$ about point X."

F_1 tends to cause ***counterclockwise*** rotation about O,
While F_2 tends to cause ***clockwise*** rotation.

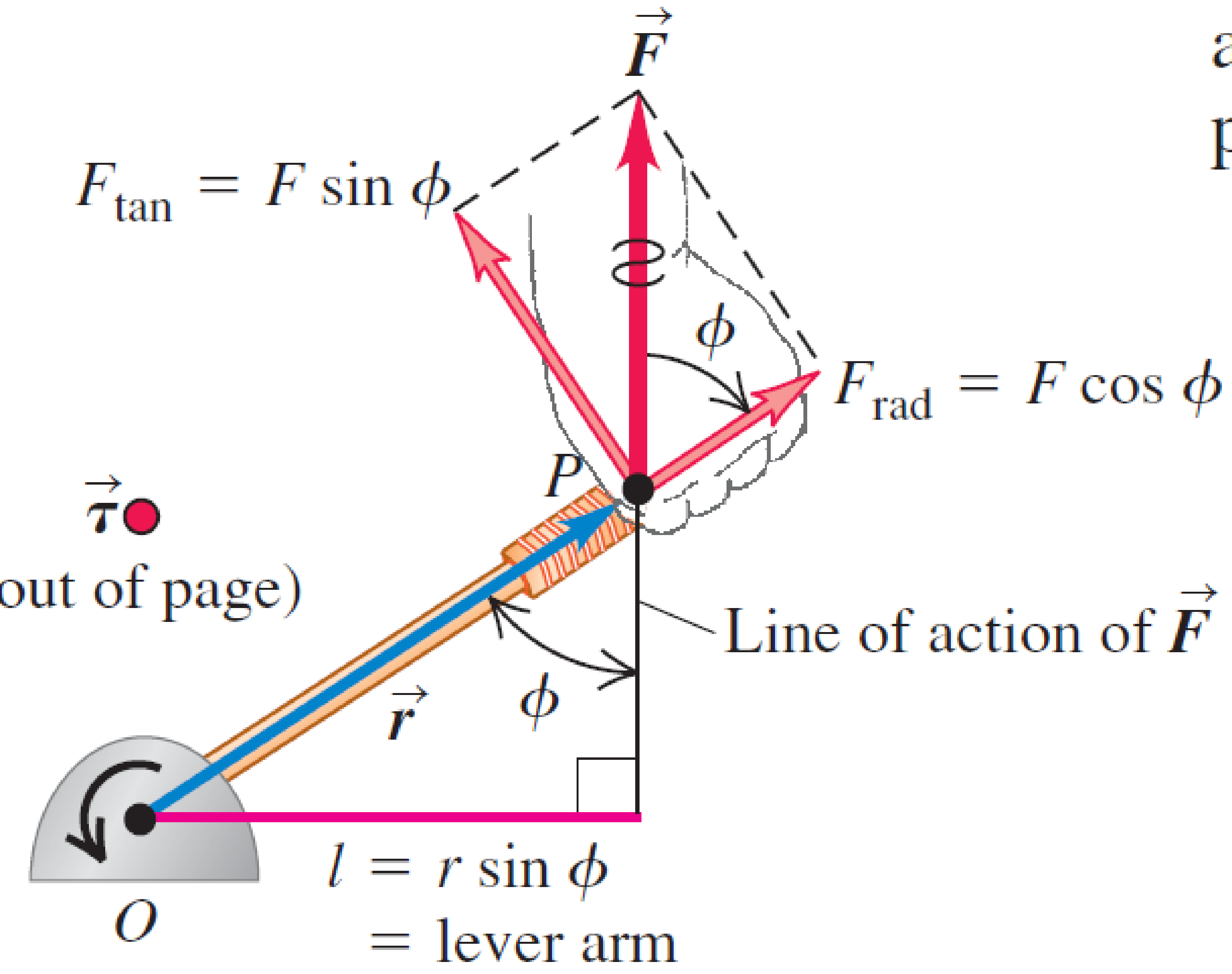
To distinguish between these two possibilities, we need to choose a positive sense of rotation.

Convention: ***counterclockwise torques are positive*** &
clockwise torques are negative:

$$\tau_1 = +F_1 l_1 \qquad \tau_2 = -F_2 l_2$$



Calculate Torque: 3 ways (but same idea)



$\vec{r}$ and $\vec{F}$ are in the plane of the page and the torque vector $\vec{\tau}$ points out of page toward you.

1. Find the lever arm *l* and use $\tau = Fl$
2. Determine the angle between the vectors ***r*** and ***F***; the lever arm is $r \sin\phi$, so $\tau = rF \sin\phi$
3. Decompose ***F*** into tangential and radical terms

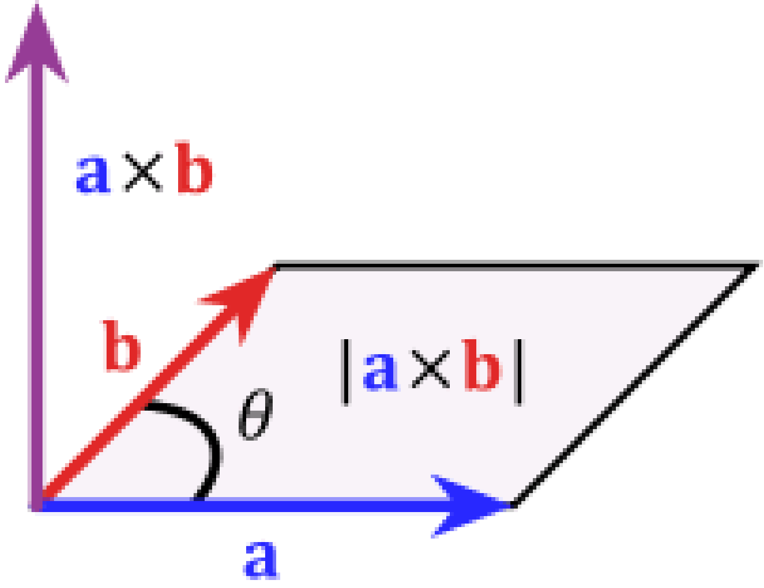
$$\tau = Fl = rF \sin\phi = F_{tan} r$$

Torque as a Vector

When a force $\boldsymbol{F}$ acts at a point with a position vector $\boldsymbol{r}$ with respect to an origin O the torque of the force $\boldsymbol{\tau}$ with respect to O is the *vector* quantity:

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} \qquad \text{(definition of torque vector)}$$

The direction of $\boldsymbol{\tau}$ is perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{F}$



$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{vmatrix}$$

Calcworkshop.com

Torque as a Vector

When a force $\boldsymbol{F}$ acts at a point with a position vector $\boldsymbol{r}$ with respect to an origin O the torque of the force $\boldsymbol{\tau}$ with respect to O is the *vector* quantity:

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{(definition of torque vector)}$$

The direction of $\boldsymbol{\tau}$ is perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{F}$

- In diagrams that involve $\boldsymbol{r}$, $\boldsymbol{F}$ and $\boldsymbol{\tau}$, it's common to have one of the vectors oriented perpendicular to the page.
- We use a dot (·) to represent a vector that points out of the page and a cross (×) to represent a vector that points into the page

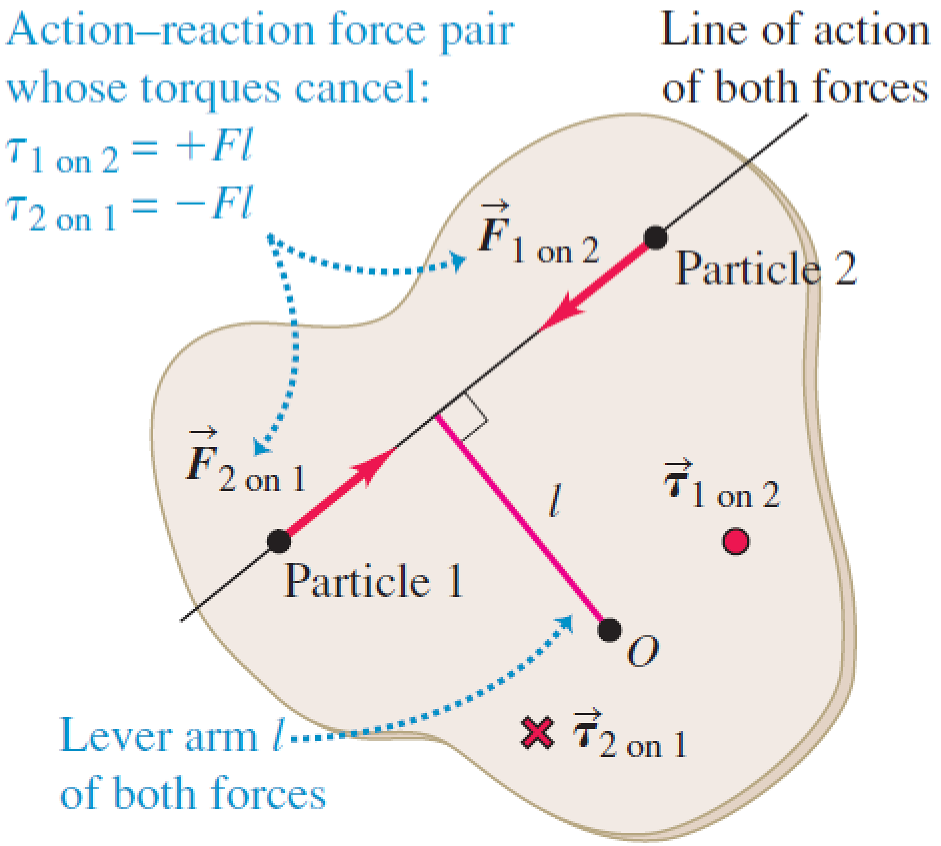
Torque vs Acceleration

$$\sum \tau_z = I\alpha_z$$

(rotational analog of Newton's second law for a rigid body)

The sum of torques here includes only the torques of the *external* forces

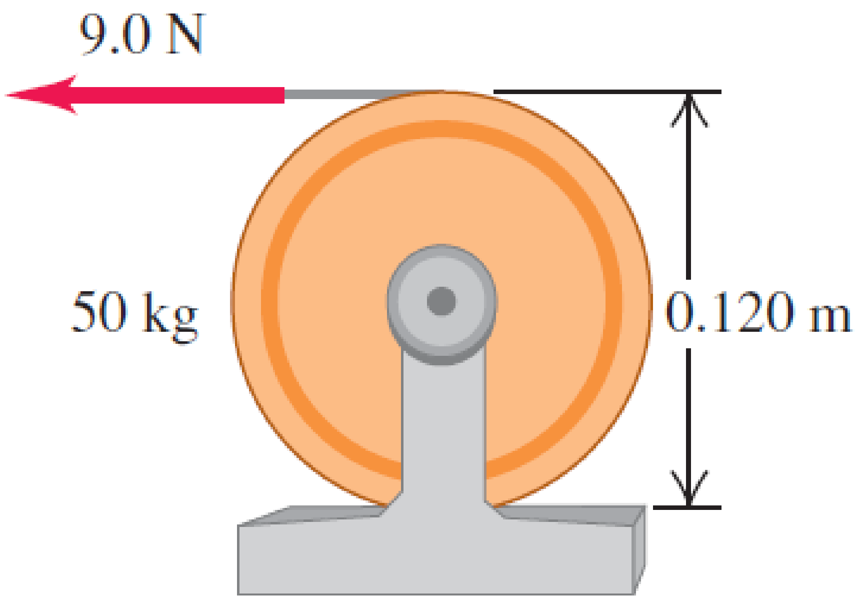
- Newton's third law: the *internal* forces that any pair of particles in the rigid body exert are equal and opposite (F_1 and F_2).
- Their lever arms with respect to any axis are also equal: both are l.
- So the torques for each such pair are equal and opposite, and add to zero.
- Hence *all* the internal torques add to zero



Example 10.2 An unwinding cable

The figure shows the situation analyzed in Example 9.7 using energy methods. What is the cable's acceleration?

lever arm of *F* is equal to the radius R of the cylinder, so the torque is $\tau_z = FR$.



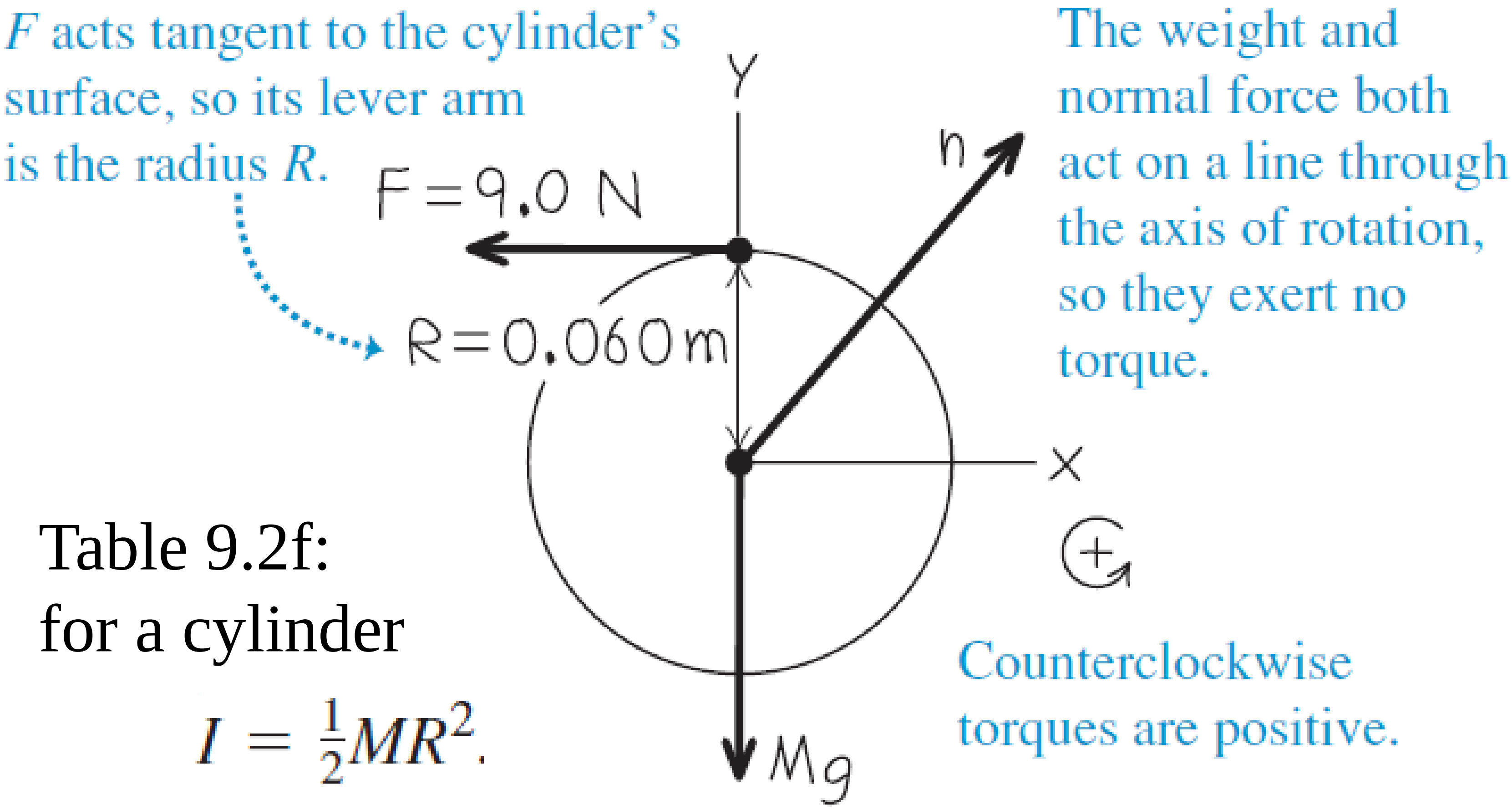


Table 9.2f:
for a cylinder

$$I = \frac{1}{2}MR^2.$$

Example 10.2 An unwinding cable

Now we have

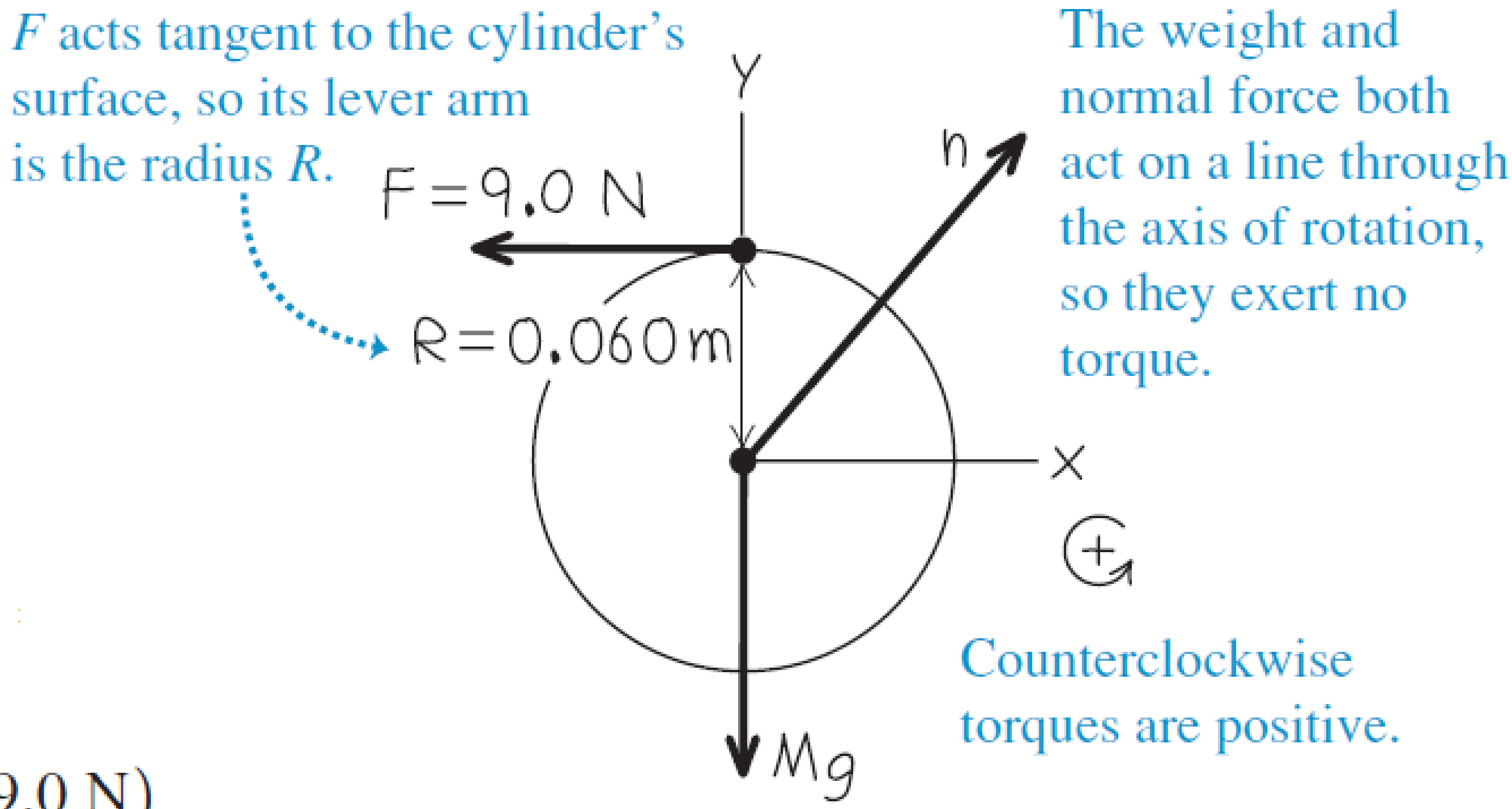
$$I = \tfrac{1}{2}MR^2.$$

$$\tau_z = FR.$$

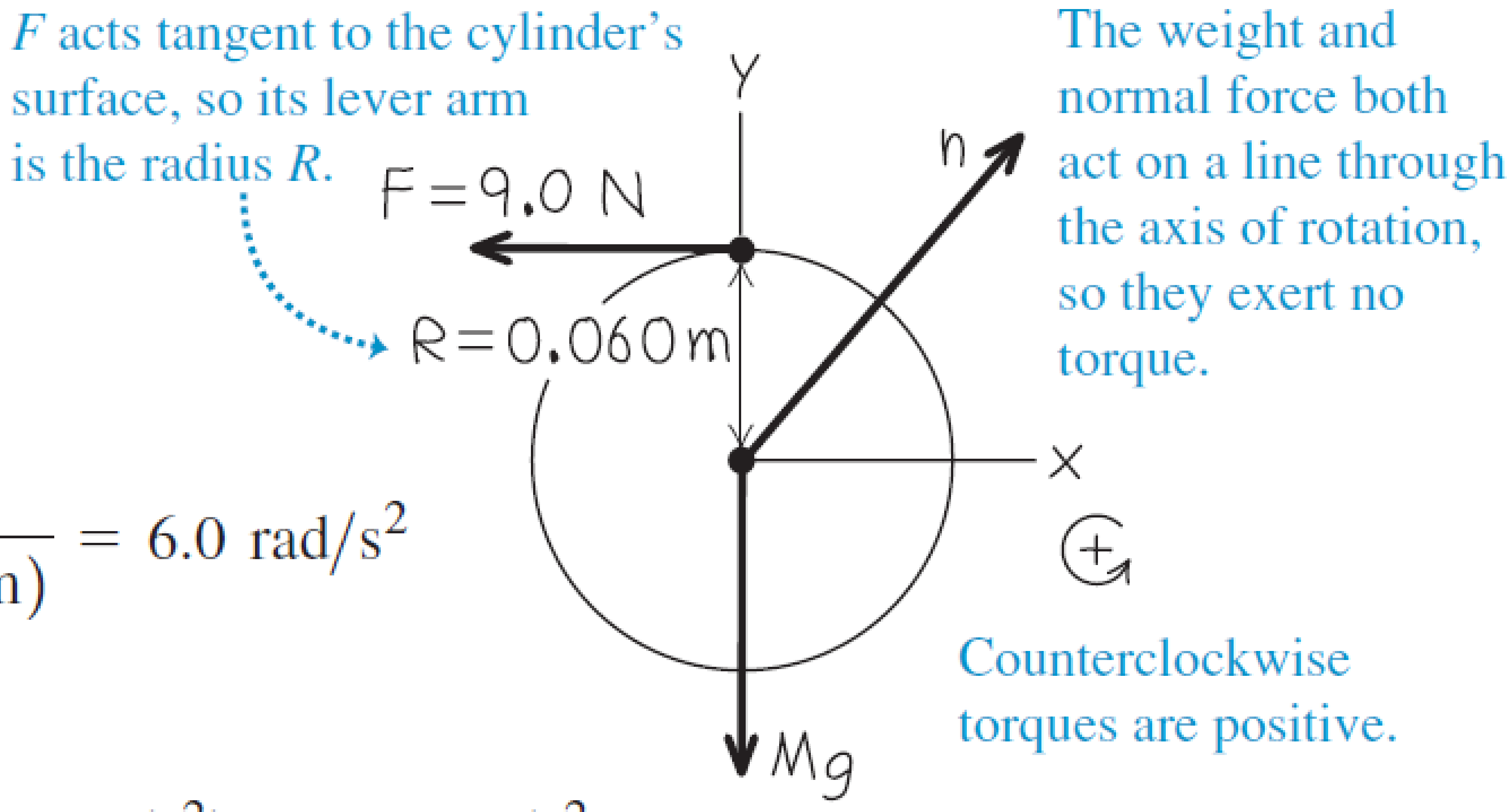
Combine

$$\alpha_z = \frac{\tau_z}{I} = \frac{FR}{MR^2/2}$$

$$= \frac{2F}{MR} = \frac{2(9.0\ \text{N})}{(50\ \text{kg})(0.060\ \text{m})} = 6.0\ \text{rad/s}^2$$



Example 10.2 An unwinding cable



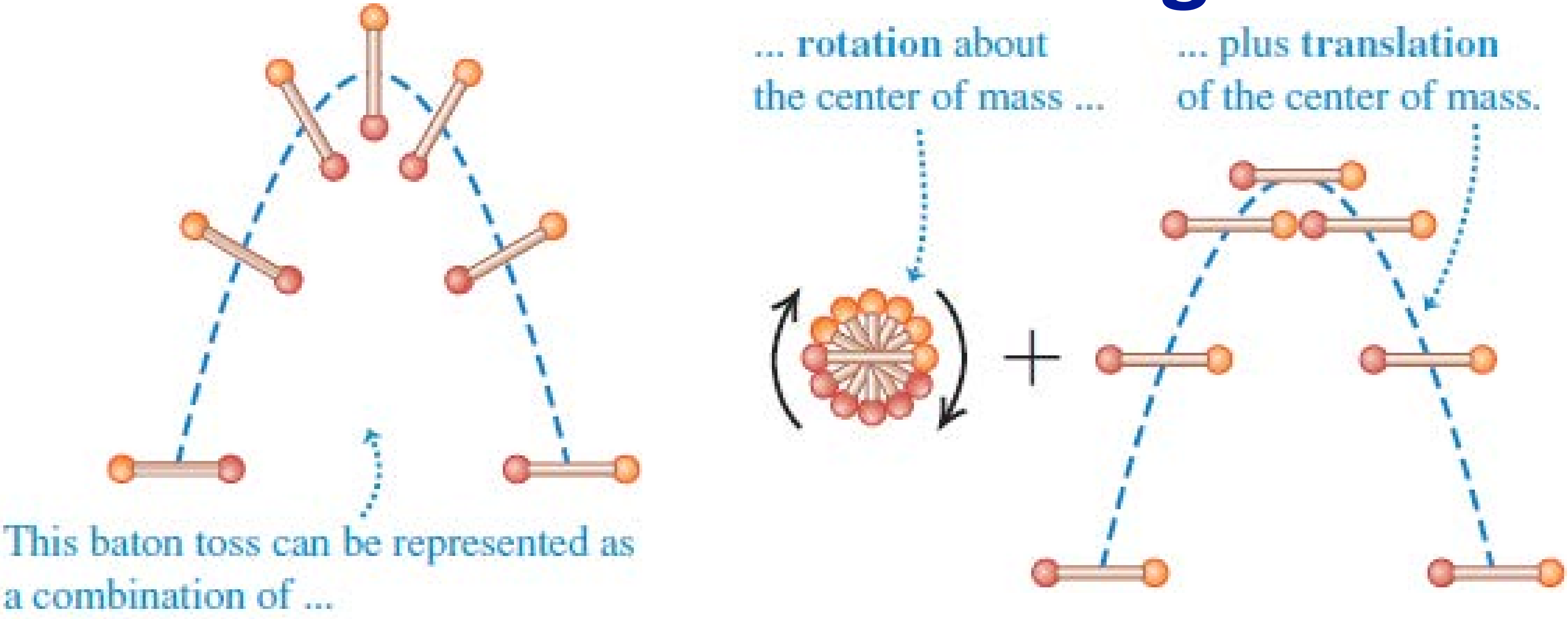
$$\alpha_z = \frac{\tau_z}{I} = \frac{FR}{MR^2/2}$$

$$= \frac{2F}{MR} = \frac{2(9.0\text{ N})}{(50\text{ kg})(0.060\text{ m})} = 6.0\text{ rad/s}^2$$

Use equation 9.14:

$$a_{\tan} = R\alpha_z = (0.060\text{ m})(6.0\text{ rad/s}^2) = 0.36\text{ m/s}^2$$

Rotation About a Moving Axis



$$K = \tfrac{1}{2} M v_{\mathrm{cm}}^2 + \tfrac{1}{2} I_{\mathrm{cm}} \omega^2$$

(rigid body with both translation and rotation)

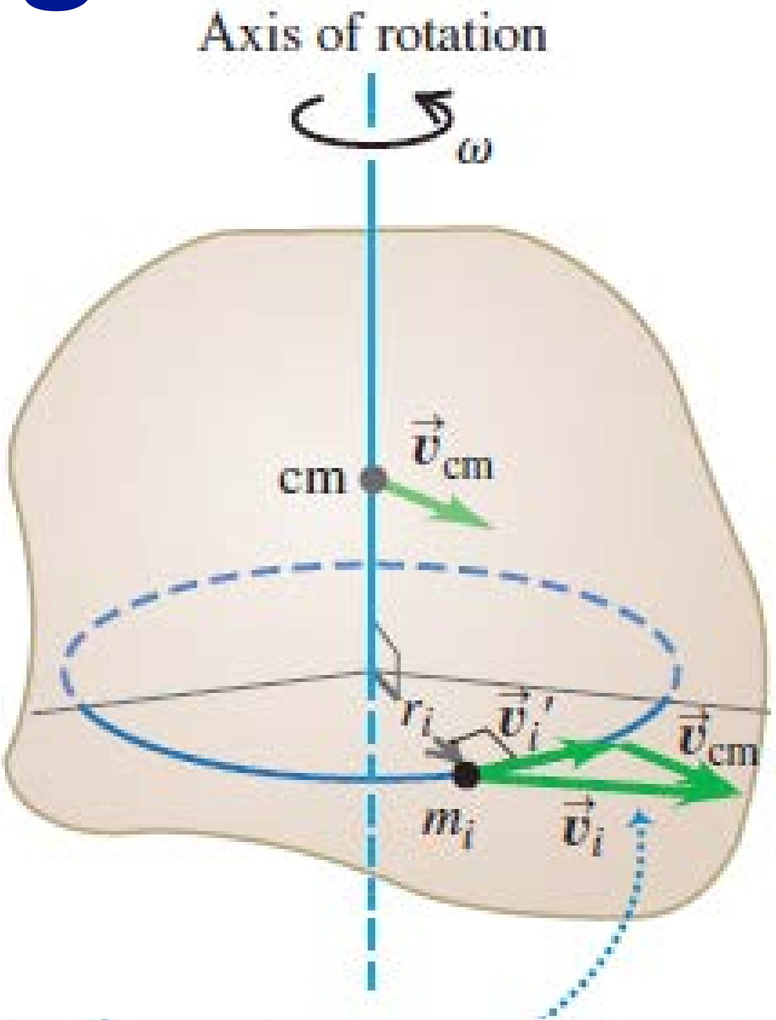
Rotation with Moving Axis: Proof

We again imagine the rigid body to be made up of particles, like the m_i.

The velocity of this particle $\boldsymbol{v}_i$ relative to an inertial frame is the vector sum of the velocity $\boldsymbol{v}_{cm}$ of the center of mass and the velocity $\boldsymbol{v}_i'$ of the particle relative to the center of mass:

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$$

The kinetic energy of this particle in the inertial frame is $\frac{1}{2}mv_i^2$ which we can also express as $\frac{1}{2}m(\vec{v}_i \cdot \vec{v}_i)$.



Velocity $\vec{v}_i$ of particle in rotating, translating rigid body = (velocity $\vec{v}_{cm}$ of center of mass) + (particle's velocity $\vec{v}_i'$ relative to center of mass)

Rotation with Moving Axis: Proof

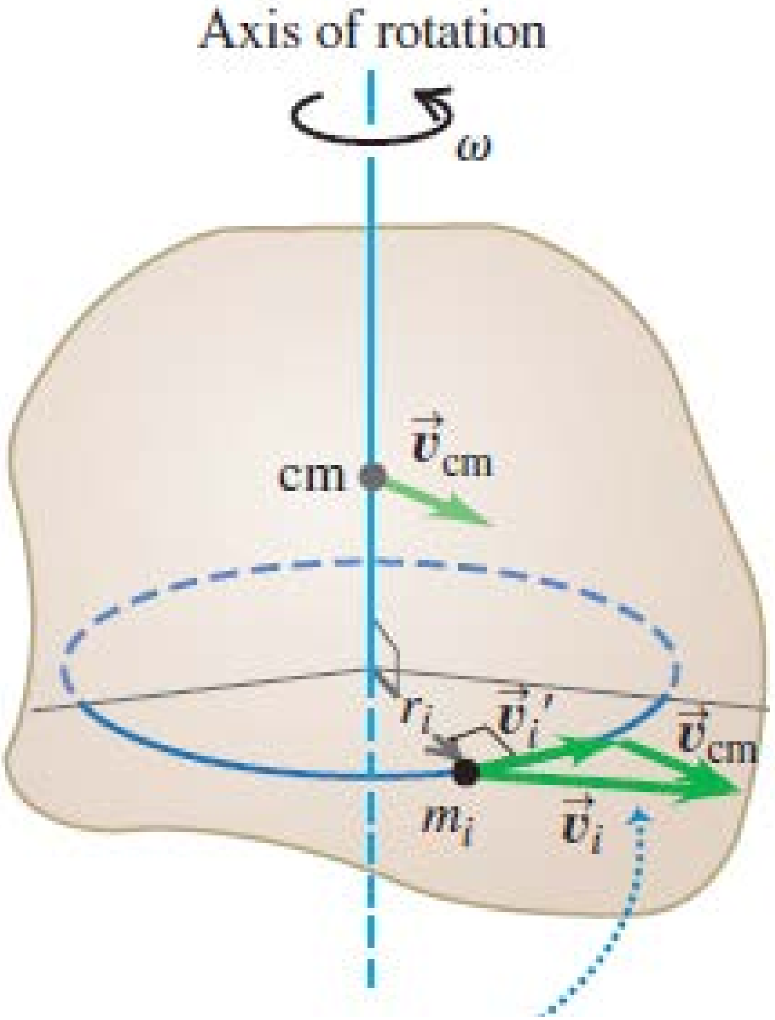
$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$$

The kinetic energy of this particle in the inertial frame is $\frac{1}{2}m{v_i}^2$which we can also express as $\frac{1}{2}m(\vec{\boldsymbol{v}}_i\cdot\vec{\boldsymbol{v}}_i)$.

$$K_i = \tfrac{1}{2}m_i(\vec{v}_{cm} + \vec{v}_i') \cdot (\vec{v}_{cm} + \vec{v}_i')$$

$$= \tfrac{1}{2}m_i(\vec{v}_{cm}\cdot\vec{v}_{cm} + 2\vec{v}_{cm}\cdot\vec{v}_i' + \vec{v}_i'\cdot\vec{v}_i')$$

$$= \tfrac{1}{2}m_i(v_{cm}^2 + 2\vec{v}_{cm}\cdot\vec{v}_i' + v_i'^2)$$



Rotation with Moving Axis: Proof

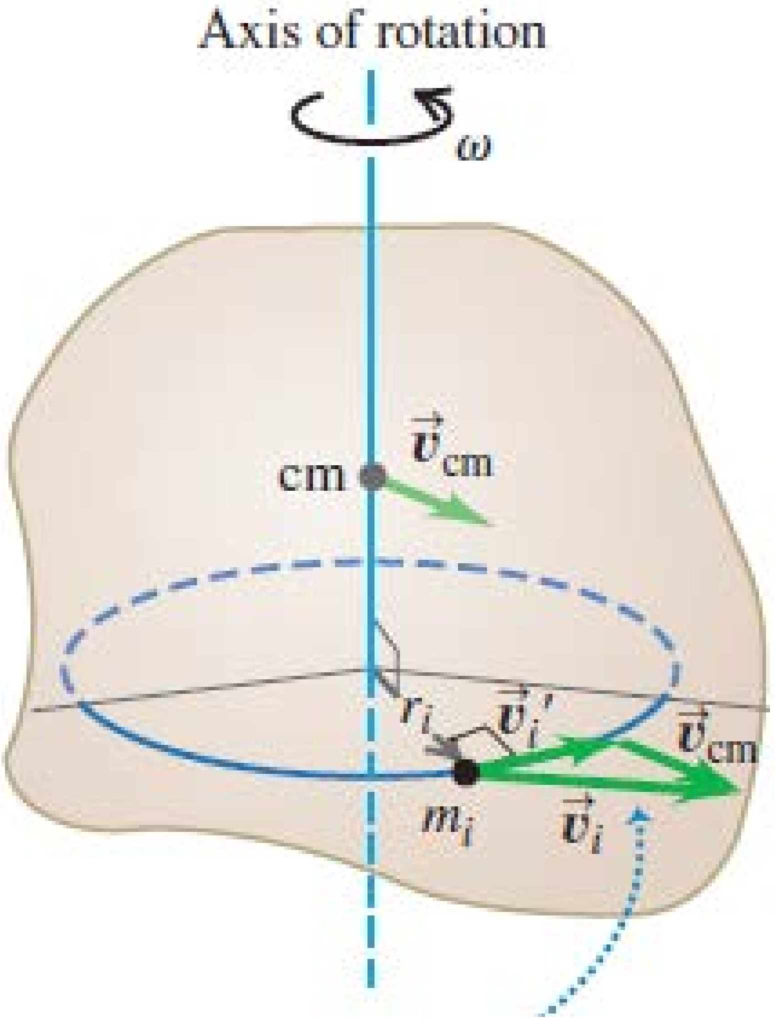
$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$$

The kinetic energy of this particle in the inertial frame is $\frac{1}{2}mv_i^2$which we can also express as $\frac{1}{2}m(\vec{\boldsymbol{v}}_i \cdot \vec{\boldsymbol{v}}_i)$.

$$K_i = \frac{1}{2}m_i(\vec{v}_{cm} + \vec{v}_i') \cdot (\vec{v}_{cm} + \vec{v}_i')$$

$$= \frac{1}{2}m_i(\vec{v}_{cm} \cdot \vec{v}_{cm} + 2\vec{v}_{cm} \cdot \vec{v}_i' + \vec{v}_i' \cdot \vec{v}_i')$$

$$= \frac{1}{2}m_i(v_{cm}^2 + 2\vec{v}_{cm} \cdot \vec{v}_i' + v_i'^2)$$



Rotation with Moving Axis: Proof

The total kinetic energy is the sum $\sum K_i$ for all the particles making up the body. Expressing the three terms in this equation as separate sums, we get

$$K = \sum K_i = \sum\left(\tfrac{1}{2}m_i v_{\mathrm{cm}}^2\right) + \sum\left(m_i \vec{v}_{\mathrm{cm}} \cdot \vec{v}_i'\right) + \sum\left(\tfrac{1}{2}m_i v_i'^2\right)$$

The first and second terms have common factors that can be taken outside the sum:

$$K = \tfrac{1}{2}\left(\sum m_i\right)v_{\mathrm{cm}}^2 + \vec{v}_{\mathrm{cm}} \cdot \left(\sum m_i \vec{v}_i'\right) + \sum\left(\tfrac{1}{2}m_i v_i'^2\right) \qquad (10.10)$$

Now comes the reward for our effort. In the first term, $\sum m_i$ is the total mass M. The second term is zero because $\sum m_i \vec{v}_i'$ is M times the velocity of the center of mass *relative to the center of mass*, and this is zero by definition.

Rotation with Moving Axis: Proof

The second term is zero because $\sum m_i \vec{v}'_i$ is M times the velocity of the center of mass *relative to the center of mass*, and this is zero by definition.

If you don't get it, recall:

These equations are equivalent to the single vector equation obtained by taking the time derivative of Eq. (8.29):

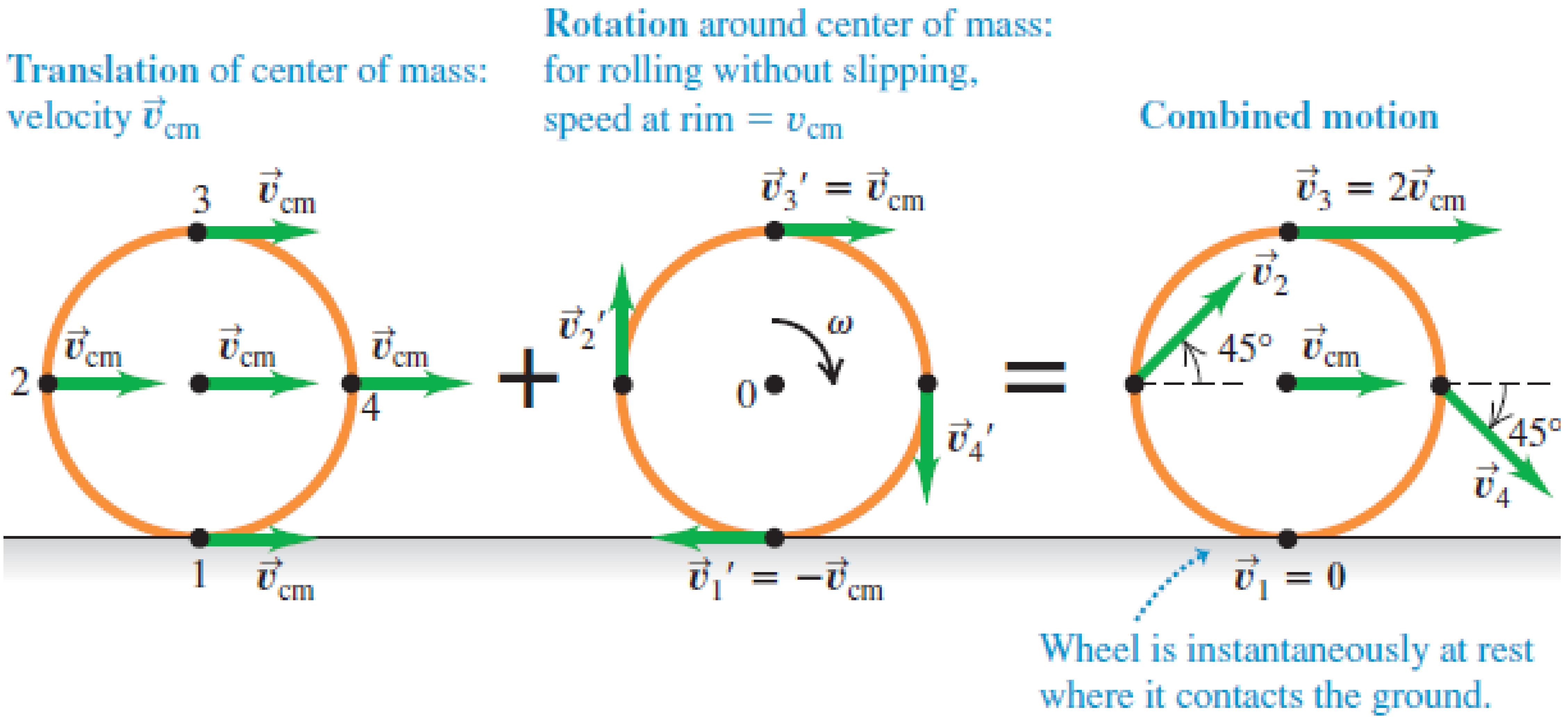
$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \quad (8.31)$$

$$K = \boxed{\tfrac{1}{2}\left(\sum m_i\right)v_{cm}^2} + \vec{v}_{cm}\cdot\left(\sum m_i\vec{v}'_i\right) + \boxed{\sum\left(\tfrac{1}{2}m_i v_i'^2\right)}$$

$$= \boxed{\tfrac{1}{2}Mv_{cm}^2} + \boxed{\tfrac{1}{2}I_{cm}\omega^2}$$

Rolling Without Slipping

$$v_{cm} = R\omega \qquad \text{(condition for rolling without slipping)}$$



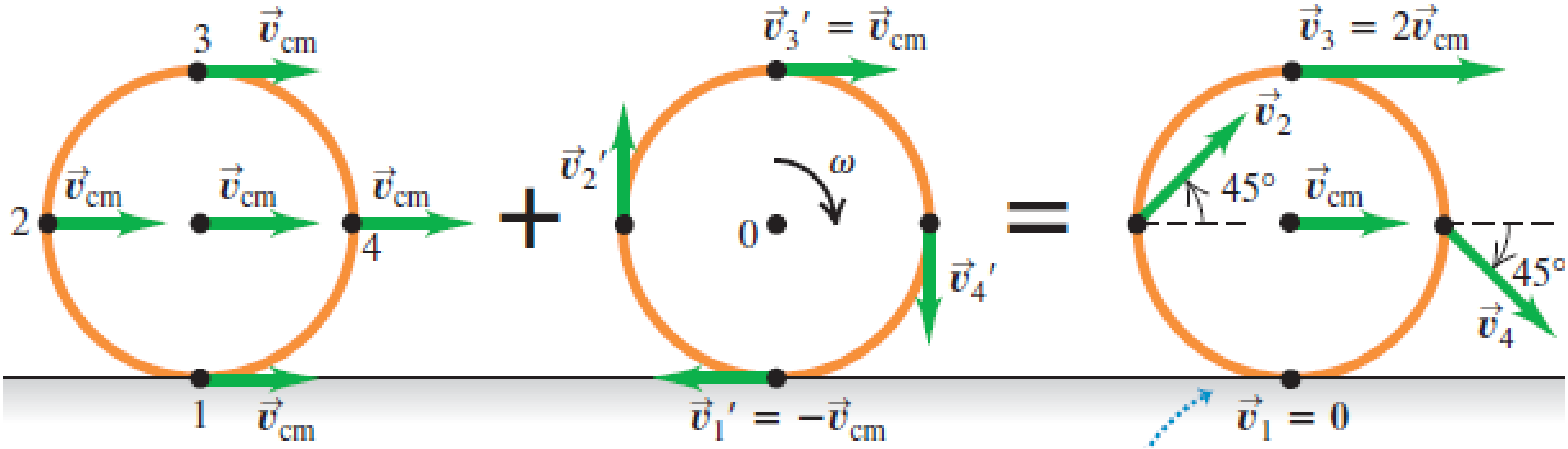
Rolling Without Slipping: Energy

$v_{cm} = R\omega$ (condition for rolling without slipping)

Recall the parallel-axis theorem $I_1 = I_{cm} + MR^2$

I_1 is the moment of inertia of the wheel about an axis through point 1.

$$K = \tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}MR^2\omega^2 = \tfrac{1}{2}I_{cm}\omega^2 + \tfrac{1}{2}Mv_{cm}^2$$



Rolling Without Slipping

CAUTION **Rolling without slipping** Note that the relationship $v_{cm} = R\omega$ holds *only* if there is rolling without slipping. When a drag racer first starts to move, the rear tires are spinning very fast even though the racer is hardly moving, so $R\omega$ is greater than v_{cm} (Fig. 10.14). If a driver applies the brakes too heavily so that the car skids, the tires will spin hardly at all and $R\omega$ is less than v_{cm}.

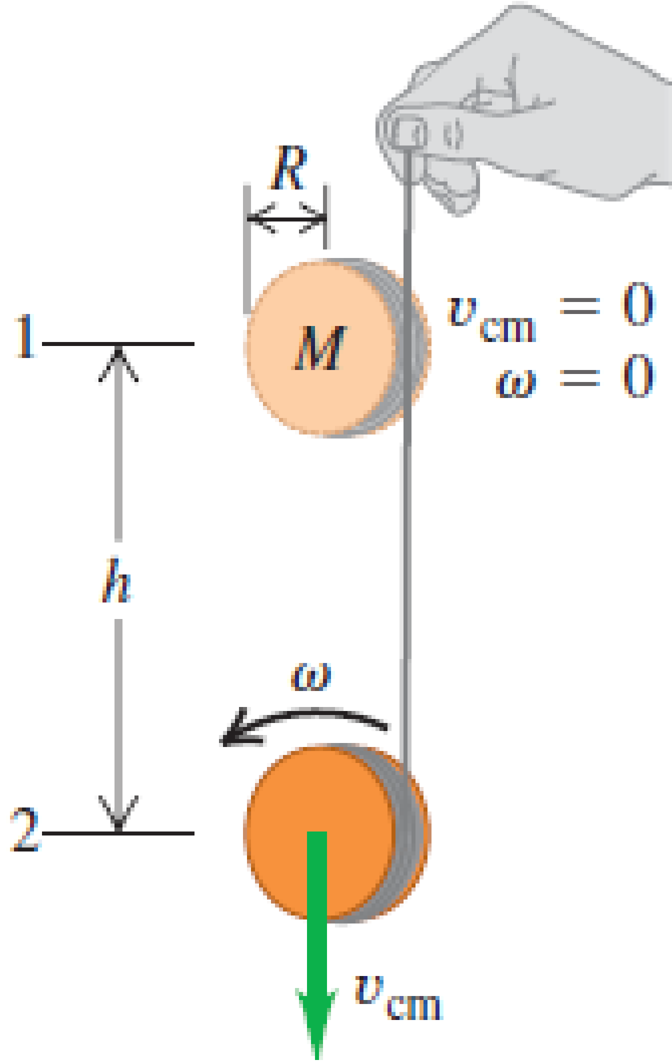
10.14 The smoke rising from this drag racer's rear tires shows that the tires are slipping on the road, so v_{cm} is *not* equal to $R\omega$.

If a rigid body changes height as it moves, we must consider gravitational potential energy

$$U = Mgy_{cm}$$

Example 10.4 Speed of a primitive yo-yo

You make a primitive yo-yo by wrapping a massless string around a solid cylinder with mass M and radius R (Fig. 10.15). You hold the free end of the string stationary and release the cylinder from rest. The string unwinds but does not slip or stretch as the cylinder descends and rotates. Using energy considerations, find the speed v_{cm} of the center of mass of the cylinder after it has descended a distance h.



From Eq. (10.8), the kinetic energy at point 2 is

$$K_2 = \tfrac{1}{2}Mv_{cm}^{\ 2} + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2$$

$$= \tfrac{3}{4}Mv_{cm}^{\ 2}$$

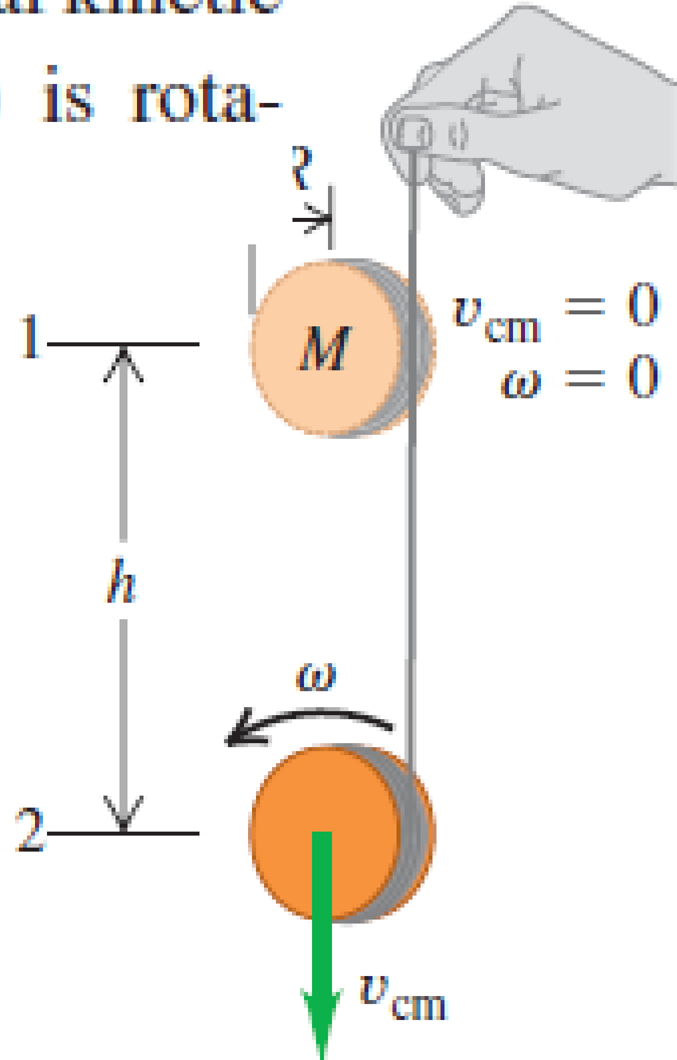
Example 10.4 Speed of a Primitive yo-yo

The kinetic energy is $1\frac{1}{2}$ times what it would be if the yo-yo were falling at speed v_{cm} without rotating. Two-thirds of the total kinetic energy $(\frac{1}{2}Mv_{cm}^2)$ is translational and one-third $(\frac{1}{4}Mv_{cm}^2)$ is rotational. Using conservation of energy,

$$K_1 + U_1 = K_2 + U_2$$

$$0 + Mgh = \frac{3}{4}Mv_{cm}^2 + 0$$

$$v_{cm} = \sqrt{\frac{4}{3}gh}$$



Combined Translation & Rotation: Dynamics

We can also analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. We showed in Section 8.5 that for a body with total mass M, the acceleration $\vec{a}_{cm}$ of the center of mass is the same as that of a point mass M acted on by all the external forces on the actual body:

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \tag{10.12}$$

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

$$\sum \tau_z = I_{cm}\alpha_z \tag{10.13}$$

where I_{cm} is the moment of inertia with respect to an axis through the center of mass and the sum $\sum \tau_z$ includes all external torques with respect to this axis.

Combined Translation & Rotation: Dynamics

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

$$\sum \tau_z = I_{cm}\alpha_z \tag{10.13}$$

where I_{cm} is the moment of inertia with respect to an axis through the center of mass and the sum $\sum \tau_z$ includes all external torques with respect to this axis. It's not immediately obvious that Eq. (10.13) should apply to the motion of a translating rigid body; after all, our derivation of $\sum \tau_z = I\alpha_z$ in Section 10.2 assumed that the axis of rotation was stationary. But in fact, Eq. (10.13) is valid *even when the axis of rotation moves,* provided the following two conditions are met:

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

These conditions are satisfied for many types of rotation (Fig. 10.17). Note that in general this moving axis of rotation is *not* at rest in an inertial frame of reference.

Example 10.6 Acceleration of a yo-yo

For the primitive yo-yo in Example 10.4, find the downward acceleration of the cylinder and the tension in the string.

From Eq. (10.12),

$$\sum F_y = Mg + (-T) = Ma_{\text{cm-}y}$$

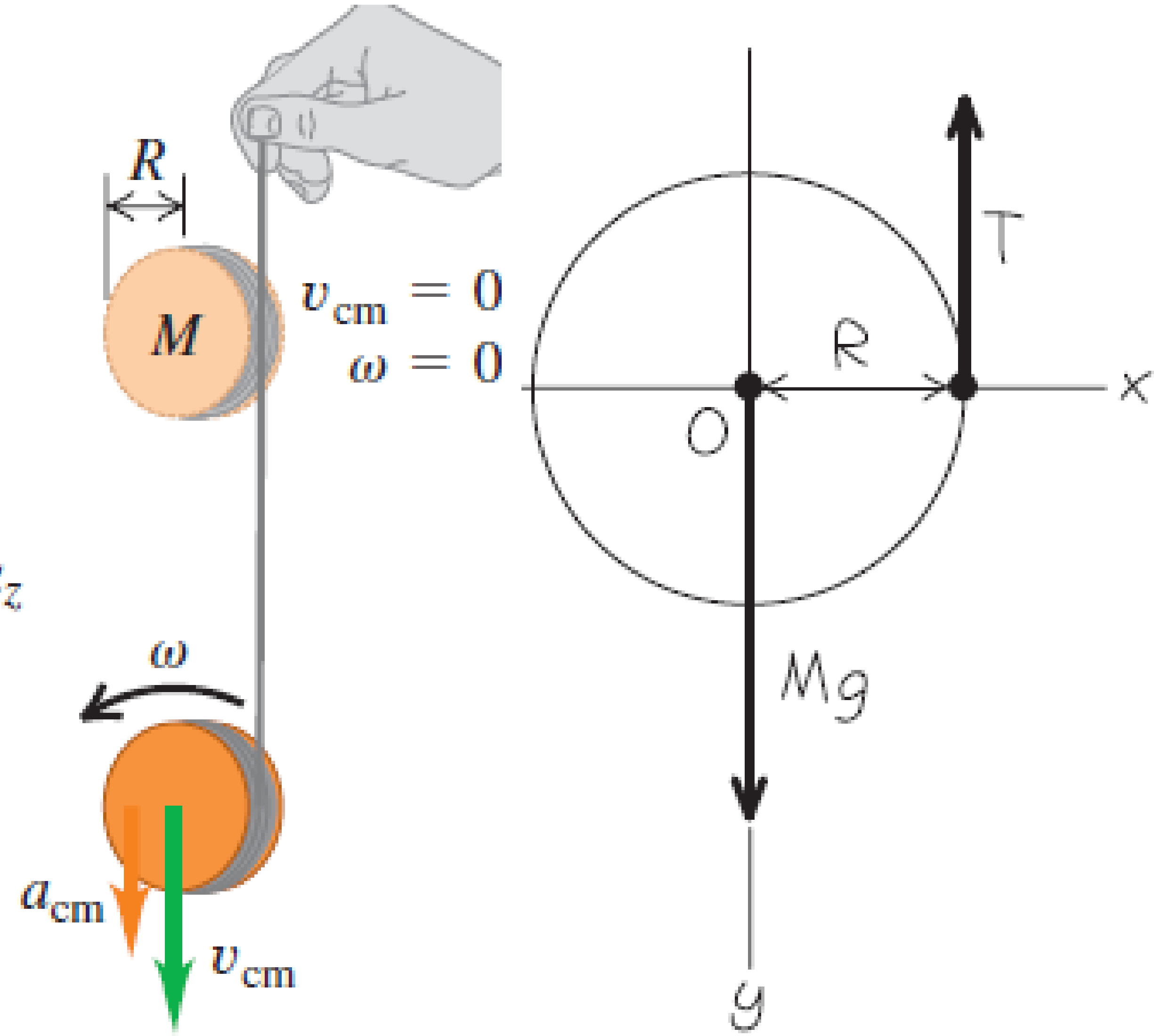
From Eq. (10.13),

$$\sum \tau_z = TR = I_{\text{cm}}\alpha_z = \tfrac{1}{2}MR^2\alpha_z$$

From Eq. (10.11), $v_{\text{cm-}z} = R\omega_z$

Derivative over t: velocity to acceleration

$$a_{\text{cm-}y} = R\alpha_z$$



Example 10.6 Acceleration of a yo-yo

For the primitive yo-yo in Example 10.4, find the downward acceleration of the cylinder and the tension in the string.

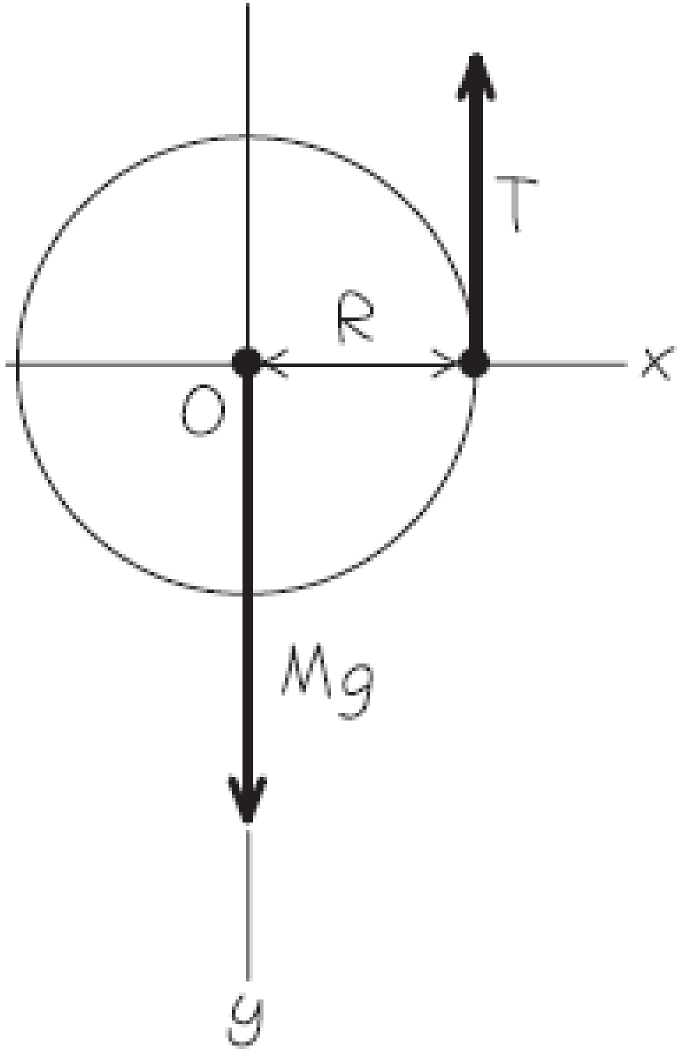
Unknowns

Solve 3 equations, 3 variables

$$\boxed{T}\cancel{R} = \tfrac{1}{2}MR^{\cancel{2}}\boxed{\alpha_z}$$

$$Mg + (-T) = M\boxed{a_{\text{cm-y}}}$$

$$\boxed{a_{\text{cm-y}}} = R\boxed{\alpha_z}$$



Answer:

$$a_{\text{cm-y}} = \tfrac{2}{3}g \qquad T = \tfrac{1}{3}Mg$$

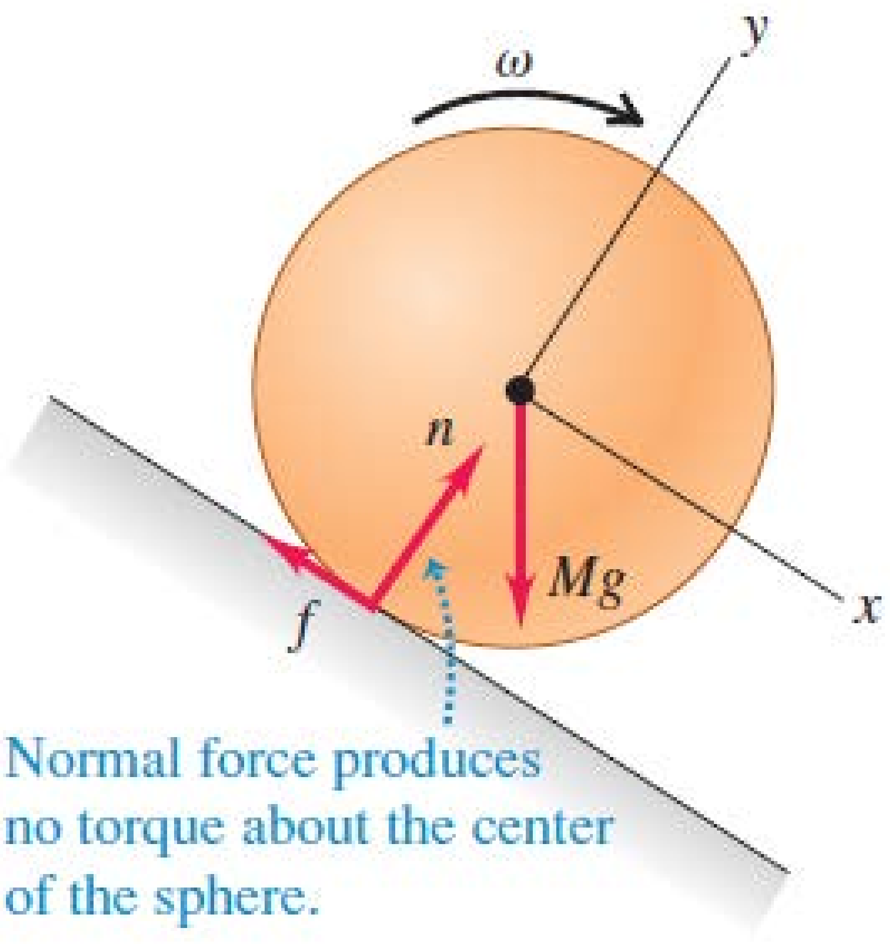
Rolling Friction

Won't be in a test but useful to know

We ignore rolling friction if:

- Both the rolling body and the surface over which it rolls are perfectly rigid
- Torque is zero so there is no sliding at the point of contact, so the friction force does no work

But with *deformation*…

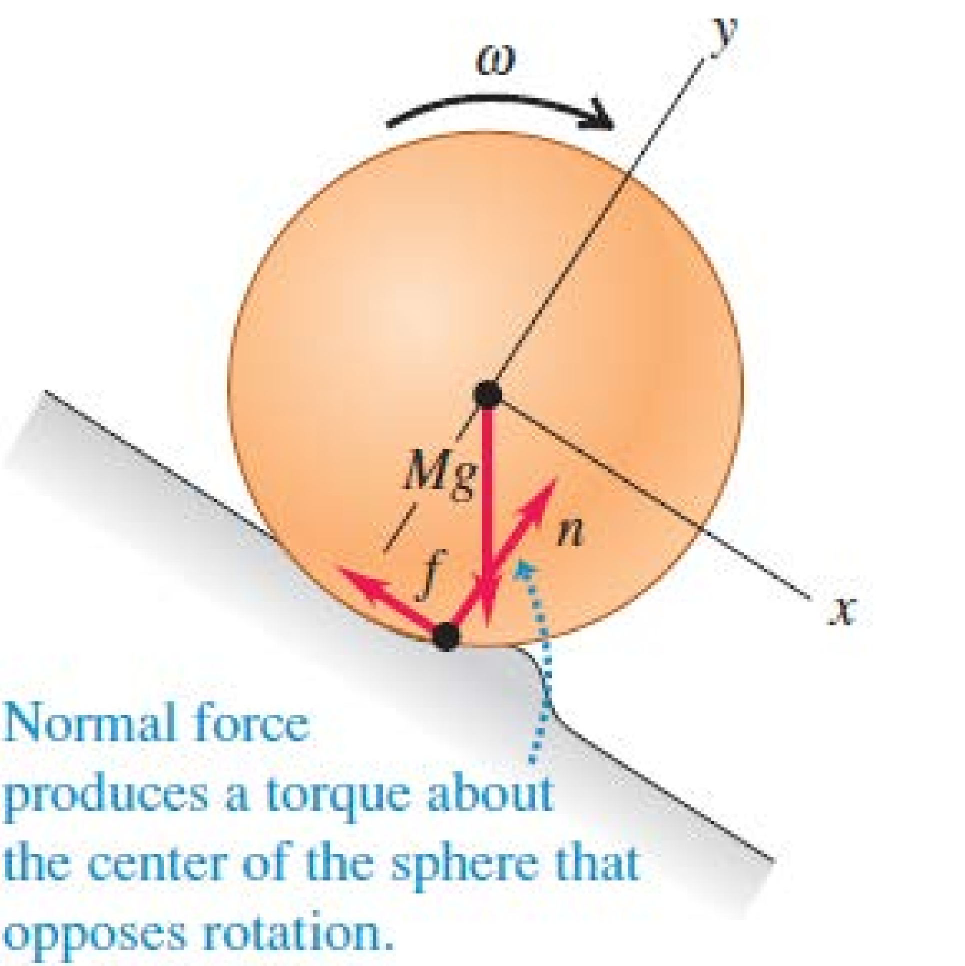


Rolling Friction

Surface "piles up" in front of the sphere and the sphere rides in a shallow trench. Because of these deformations, the **contact forces on the sphere no longer act along a single point, but over an area**; the forces are concentrated on the front of the sphere as shown. As a result, the **normal force now exerts a torque** that **opposes** the rotation. In addition, there is some sliding of the sphere over the surface due to the deformation, causing mechanical energy to be lost. The combination of these two effects is the phenomenon of *rolling friction*.

Example: tires

(b) Rigid sphere rolling on a deformable surface



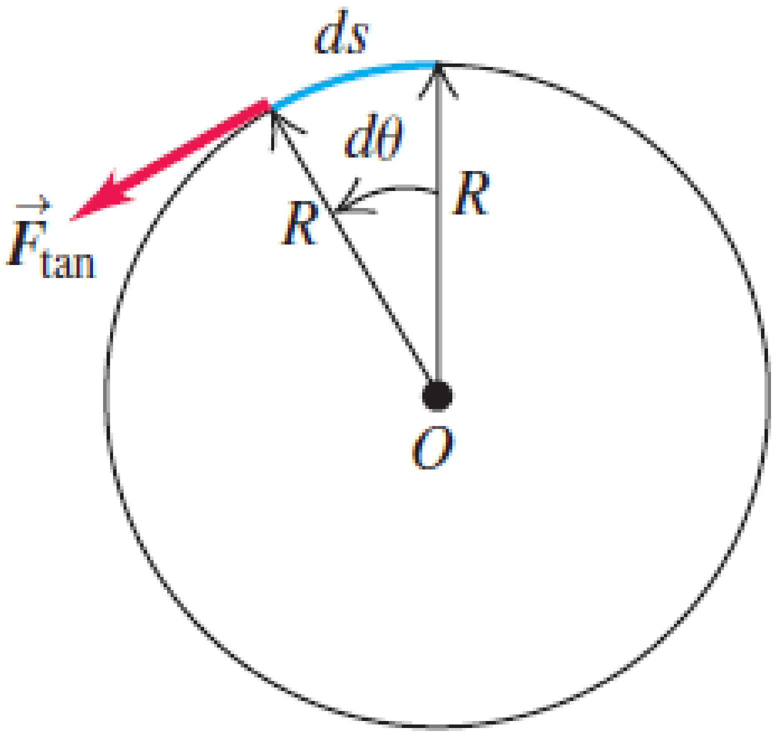
Work in Rotational Motion

Suppose a tangential force $\boldsymbol{F}_{\tan}$ acts on a disk rotates through an infinitesimal angle $d\theta$ during an infinitesimal time interval dt. The work dW done by the force while a point on the rim moves a distance ds is:

$$dW = F_{\tan}\, ds \qquad \text{with} \qquad ds = R\, d\theta \qquad \text{So} \qquad dW = F_{\tan} R\, d\theta$$

Now $F_{\tan}R$ is the *torque* τ_z due to the force $\vec{F}_{\tan}$, s

$$dW = \tau_z\, d\theta$$



The total work W done by the torque during an angu
from θ_1 to θ_2 is:

$$W = \int_{\theta_1}^{\theta_2} \tau_z\, d\theta \qquad \text{(work done by a tor}$$

Work in Rotational Motion

Suppose a tangential force $\boldsymbol{F}_{\tan}$ acts on a disk rotates through an infinitesimal angle $d\theta$ during an infinitesimal time interval dt. The work dW done by the force while a point on the rim moves a distance d*s* is:

$dW = F_{\tan}\, ds$ with $ds = R\, d\theta$ So $dW = F_{\tan} R\, d\theta$

Now $F_{\tan}R$ is the *torque* τ_z due to the force $\vec{F}_{\tan}$, so

$$dW = \tau_z\, d\theta$$

If the torque remains *constant* while the angle changes by a finite amount $\Delta\theta = \theta_2 - \theta_1$, then

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z \Delta\theta \qquad \text{(work done by a constant torque)} \quad (10.21)$$

Work in Rotational Motion

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done.

$$W_{\text{tot}} = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2$$

We expect this from the work-energy theorem, but the derivation is simple.

$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta \qquad \text{(work done by a torque)} \qquad (10.20)$$

As: $$\tau_z \, d\theta = (I\alpha_z)\, d\theta = I\frac{d\omega_z}{dt}\, d\theta = I\frac{d\theta}{dt}\, d\omega_z = I\omega_z \, d\omega_z$$

$$W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega_z \, d\omega_z = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2$$

Power in Rotational Motion

What about the *power* associated with work done by a torque acting on a rotating body? When we divide both sides of the equation by the time interval *dt* during which the angular displacement occurs, we find

$$dW = \tau_z \, d\theta \Rightarrow \frac{dW}{dt} = \tau_z \frac{d\theta}{dt}$$

dW/dt is the rate of doing work, or *power* P,

$d\theta/dt$ is angular velocity ω_z

$$P = \tau_z \omega_z$$

Example 10.8 Calculating power

An electric motor exerts a constant 10-N · m torque on a grindstone, which has a moment of inertia of 2.0 kg · m^2 about its shaft. The system starts from rest. Find the work W done by the motor in 8.0 s and the grindstone kinetic energy K at this time. What average power P_{av} is delivered by the motor?

EXECUTE: We have $\sum \tau_z = 10\ \text{N} \cdot \text{m}$ and $I = 2.0\ \text{kg} \cdot \text{m}^2$, so $\sum \tau_z = I\alpha_z$ yields $\alpha_z = 5.0\ \text{rad/s}^2$. From Eq. (9.11),

$$\Delta\theta = \tfrac{1}{2}\alpha_z t^2 = \tfrac{1}{2}(5.0\ \text{rad/s}^2)(8.0\ \text{s})^2 = 160\ \text{rad}$$

$$W = \tau_z \Delta\theta = (10\ \text{N} \cdot \text{m})(160\ \text{rad}) = 1600\ \text{J}$$

Example 10.8 Calculating power

$$\Delta\theta = \tfrac{1}{2}\alpha_z t^2 = \tfrac{1}{2}(5.0\ \mathrm{rad/s^2})(8.0\ \mathrm{s})^2 = 160\ \mathrm{rad}$$

$$W = \tau_z \Delta\theta = (10\ \mathrm{N \cdot m})(160\ \mathrm{rad}) = 1600\ \mathrm{J}$$

From Eqs. (9.7) and (9.17),

$$\omega_z = \alpha_z t = (5.0\ \mathrm{rad/s^2})(8.0\ \mathrm{s}) = 40\ \mathrm{rad/s}$$

$$K = \tfrac{1}{2} I \omega_z{}^2 = \tfrac{1}{2}(2.0\ \mathrm{kg \cdot m^2})(40\ \mathrm{rad/s})^2 = 1600\ \mathrm{J}$$

The average power is the work done divided by the time interval:

$$P_{\mathrm{av}} = \frac{1600\ \mathrm{J}}{8.0\ \mathrm{s}} = 200\ \mathrm{J/s} = 200\ \mathrm{W}$$

Angular Momentum

The analog of *momentum* of a particle is **angular momentum**, a vector quantity denoted as ***L*** units of angular momentum are $\mathrm{kg \cdot m^2/s}$.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \qquad \text{(angular momentum of a particle)}$$

Conclusions first:

The rate of change of angular momentum of a particle equals the torque of the net force acting on it.

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

Its magnitude is, for a single particle on the right:

$$L = mvr\sin\phi = mvl$$

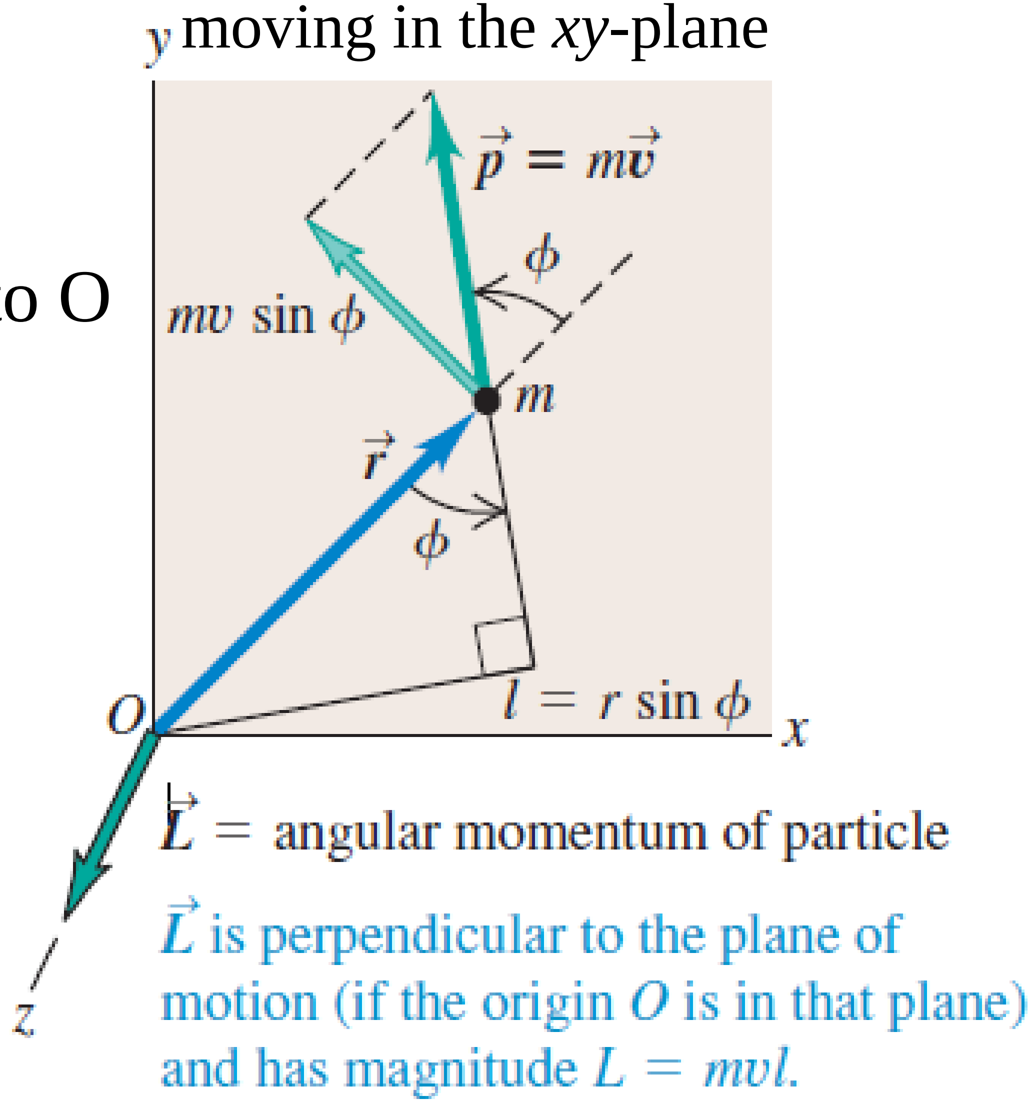
l is the perpendicular distance from the line of $\boldsymbol{v}$ to O

When $\boldsymbol{F}$ acts on a particle, the *rate of change* of $\boldsymbol{L}$ is equal to the torque of the net force. Taking the derivative:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right)$$

$$= (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

A particle with mass m moving in the xy-plane



Angular Momentum

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right)$$

$$= (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

The first term is zero because it contains the vector product of the vector with itself. In the second term we replace with the net force ***F***

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \qquad \text{(for a particle acted on by net force } \vec{F}\text{)} \quad (10.26)$$

Angular Momentum of a Rigid Body

For the i^th particle in a body:

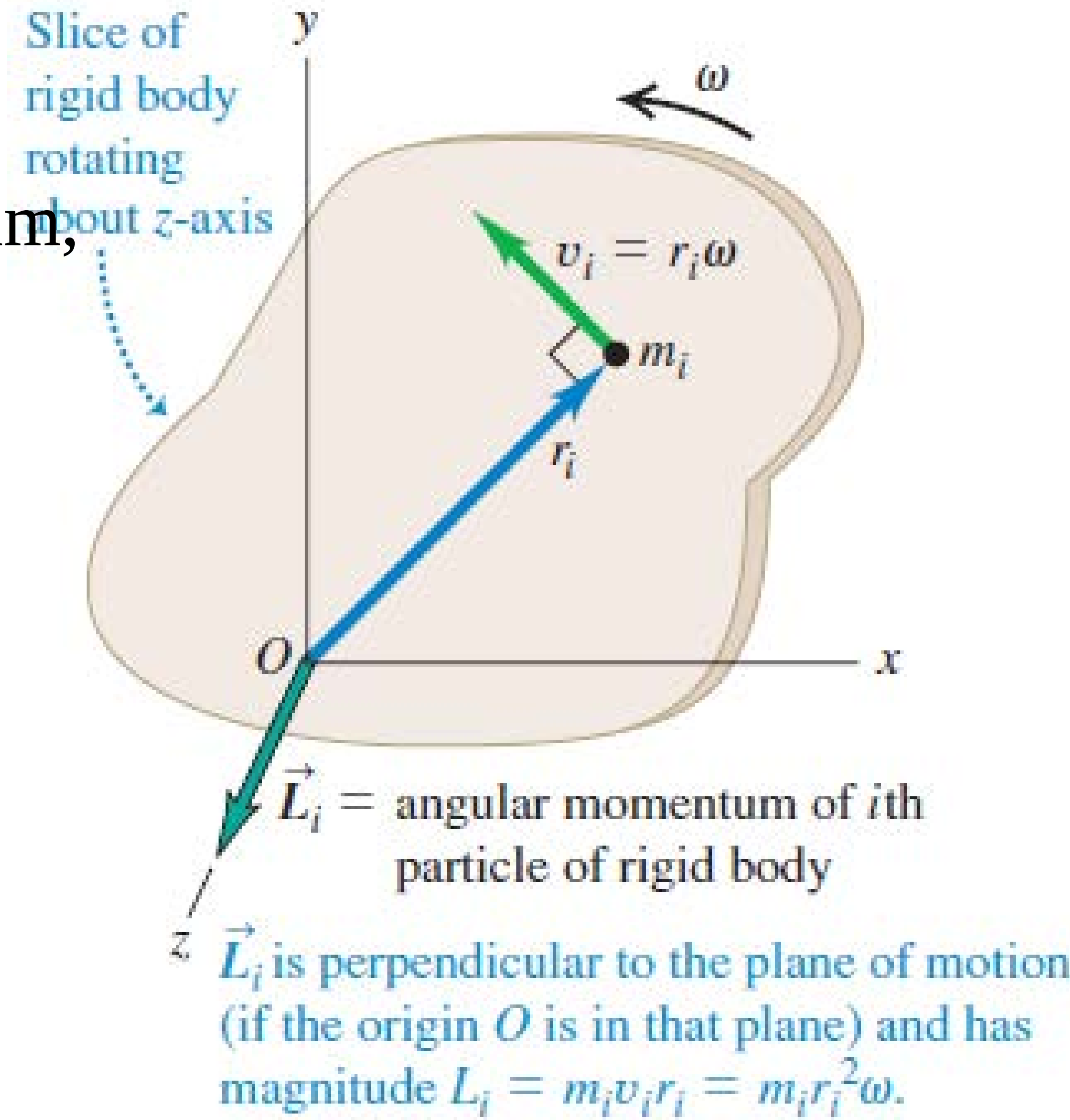
$$L_i = m_i(r_i\omega)r_i = m_i r_i^2\omega$$

The direction of each particle's angular momentum, as given by the right-hand rule, is along $+x$

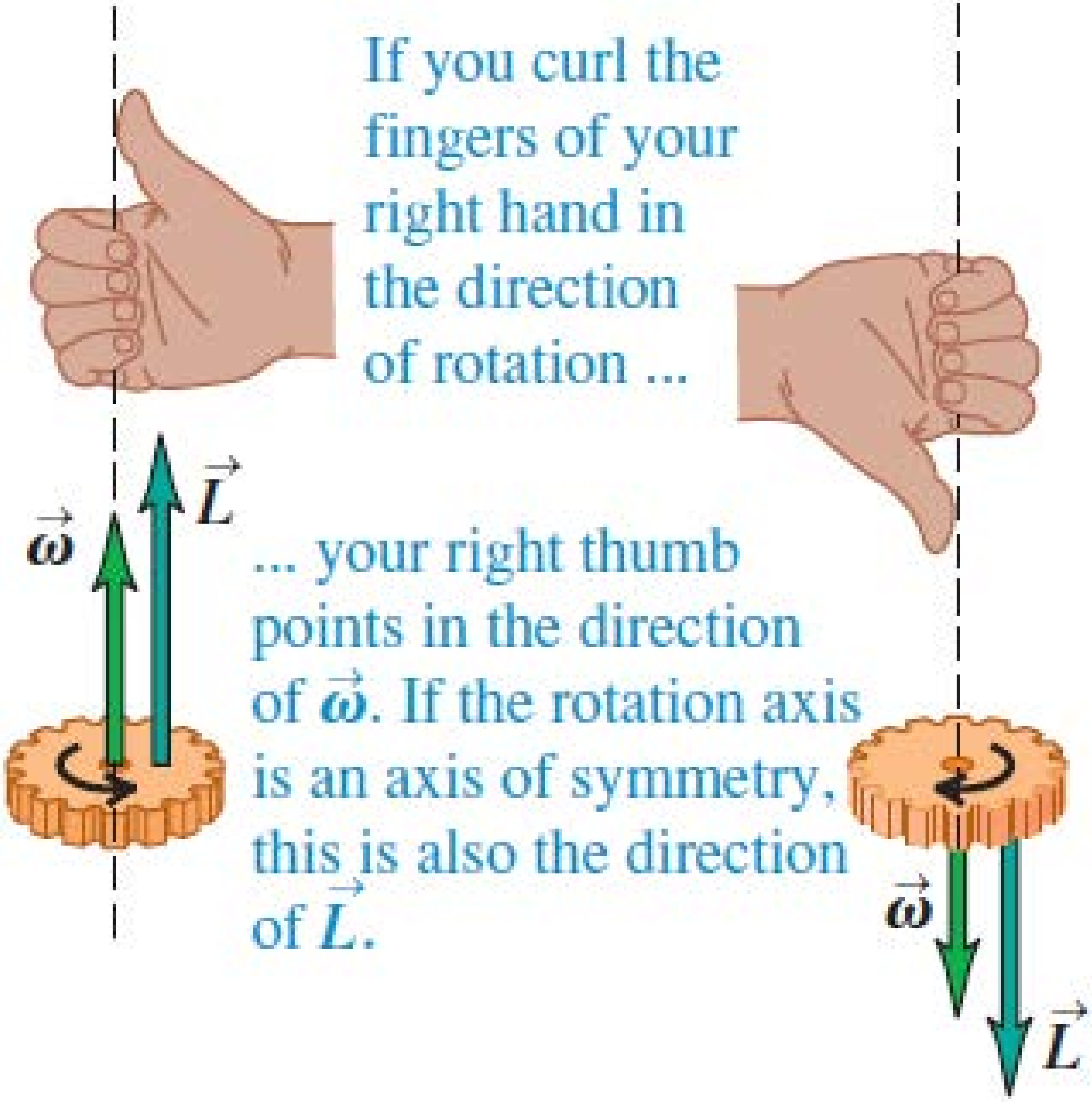
The *total* angular momentum of the slice of the body lying in the *xy*-plane is the sum of the angular momenta L_i of the particles.

$$L = \sum L_i = \left(\sum m_i r_i^2\right)\omega = I\omega$$

We can do this same calculation for the other slices of the body, all parallel to the *xy*-plane.



Angular Momentum of a Rigid Body



The torques of the *internal* forces add to zero if these forces act along the line from one particle to another

$$\vec{L} = I\vec{\omega}$$

If the total angular momentum of the system of particles is L and the sum of the external torques is the sum of torques

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

Example 10.9 Angular momentum & torque

A turbine fan in a jet engine has a moment of inertia of 2.5 kg·m^2 about its axis of rotation. As the turbine starts up, its angular velocity is given by $\omega_z = (40\ \text{rad/s}^3)t^2$

(a) Find the fan's angular momentum as a function of time and its value at t = 3.0 s.

(b) Find the net torque on the fan as a function of time, and find its value at t = 3.0 s。

Solution: The fan rotates about its axis of symmetry (the *z*-axis). Hence the angular momentum vector has only a z- component L_z, which we can determine from the angular velocity ω_z.

EXECUTE: (a) From Eq. (10.28),

$$L_z = I\omega_z = (2.5\ \text{kg}\cdot\text{m}^2)(40\ \text{rad/s}^3)t^2 = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)t^2$$

(We dropped the dimensionless quantity "rad" from the final expression.) At $t = 3.0$ s, $L_z = 900\ \text{kg}\cdot\text{m}^2/\text{s}$.

Example 10.9 Angular momentum & torque

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(a) Find the fan's angular momentum as a function of time and its value at t = 3.0 s.

(b) Find the net torque on the fan as a function of time, and find its value at t = 3.0 s。

(b) From Eq. (10.29),

$$\tau_z = \frac{dL_z}{dt} = (100\ \text{kg}\cdot\text{m}^2/\text{s}^3)(2t) = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)t$$

At $t = 3.0$ s,

$$\tau_z = (200\ \text{kg}\cdot\text{m}^2/\text{s}^3)(3.0\ \text{s}) = 600\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 600\ \text{N}\cdot\text{m}$$

Conservation of Angular Momentum

Principle of conservation of angular momentum

When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved)

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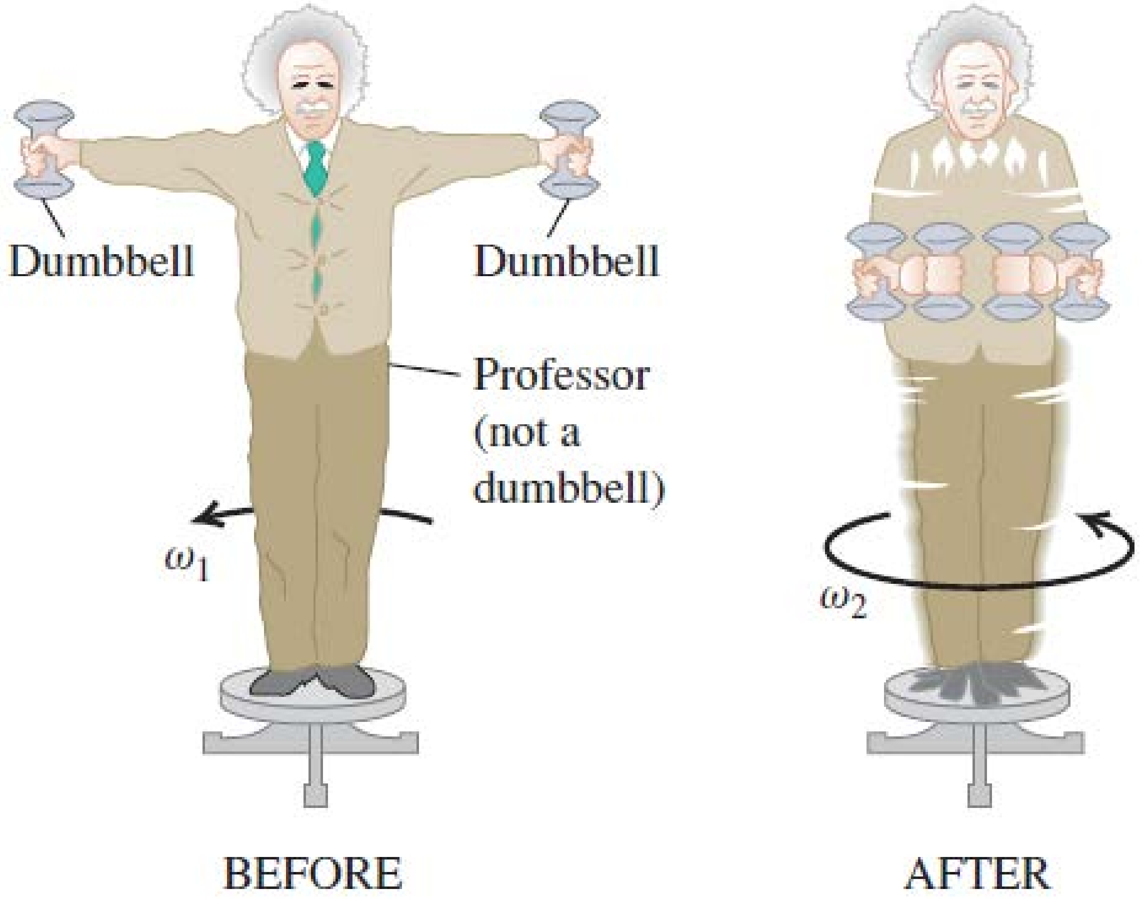
Principle of conservation of angular momentum

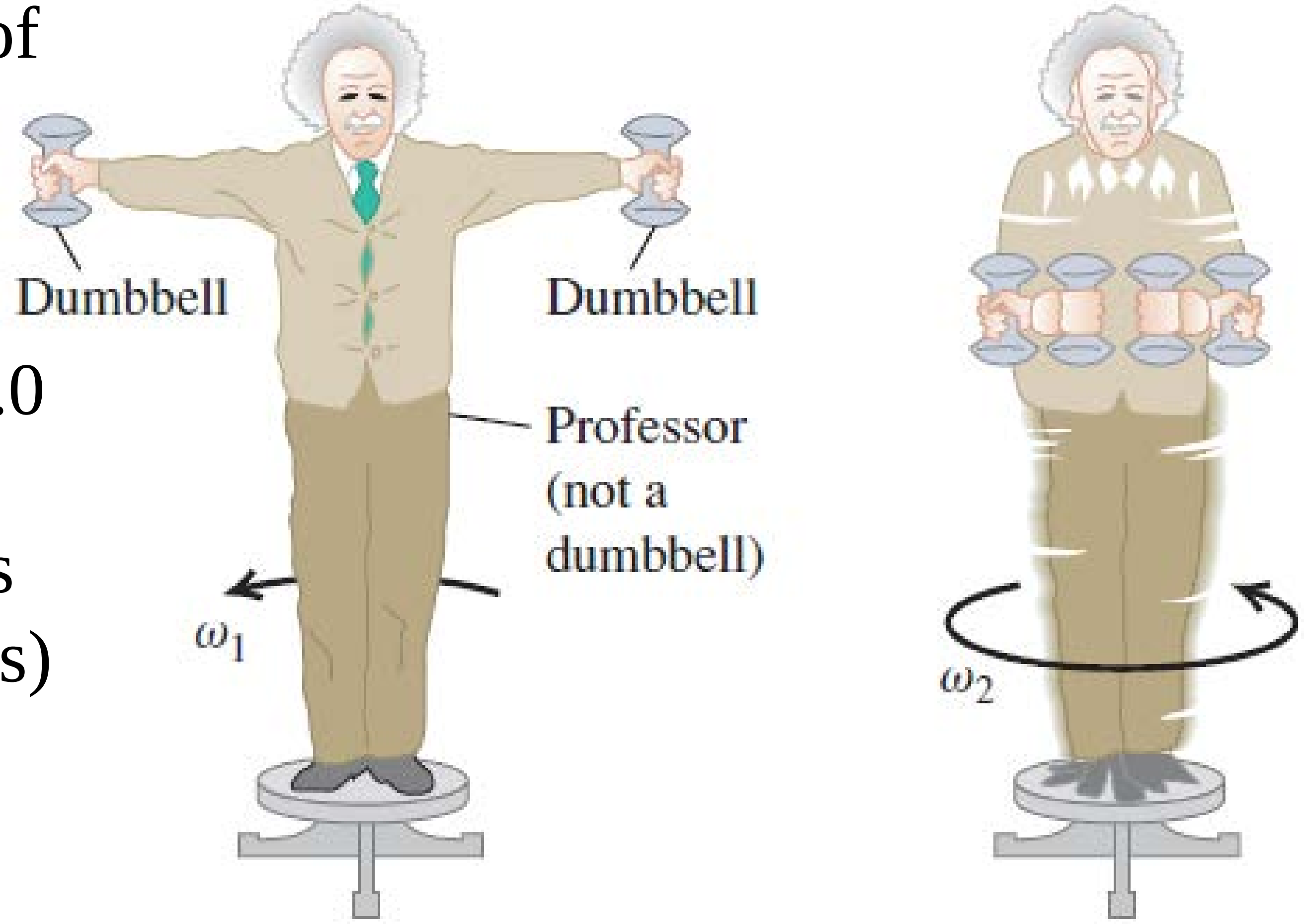
When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved)

$$I_1\omega_{1z} = I_2\omega_{2z}$$



Example 10.10 Anyone can be a ballerina

A physics professor stands at the center of a frictionless turntable with arms outstretched and a 5.0-kg dumbbell in each hand. He is set rotating about the vertical axis, making one revolution in 2.0 s. Find his final angular velocity if he pulls the dumbbells into his stomach. His moment of inertia (without the dumbbells) is $\boxed{3.0\ \text{kg}\cdot\text{m}^2}$ with arms outstretched and $\boxed{2.2\ \text{kg}\cdot\text{m}^2}$ with his hands at his stomach. The dumbbells are $\boxed{1.0\ \text{m}}$ from the axis initially and $\boxed{0.20\ \text{m}}$ at the end.



Example 10.10 Anyone can be a ballerina

The moment of inertia of the system is $I = I_{\text{prof}} + I_{\text{dumbbells}}$

We treat each dumbbell as a particle of mass m that contributes mr^2 to $I_{\text{dumbbells}}$. Initially: $I_1 = \boxed{3.0}\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(\boxed{1.0}\ \text{m})^2 = 13\ \text{kg}\cdot\text{m}^2$

$$\omega_{1z} = \frac{1\ \text{rev}}{2.0\ \text{s}} = 0.50\ \text{rev/s}$$

The final moment of inertia is

$$I_2 = \boxed{2.2}\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(\boxed{0.20}\ \text{m})^2 = 2.6\ \text{kg}\cdot\text{m}^2$$

$$\omega_{2z} = \frac{I_1}{I_2}\omega_{1z} = \frac{13\ \text{kg}\cdot\text{m}^2}{2.6\ \text{kg}\cdot\text{m}^2}(0.50\ \text{rev/s}) = 2.5\ \text{rev/s} = 5\omega_{1z}$$

Example 10.10 Anyone can be a ballerina

EVALUATE: The angular momentum remained constant, but the angular velocity increased by a factor of 5, from $\omega_{1z} = (0.50\ \text{rev/s})(2\pi\ \text{rad/rev}) = 3.14\ \text{rad/s}$ to $\omega_{2z} = (2.5\ \text{rev/s})(2\pi\ \text{rad/rev}) = 15.7\ \text{rad/s}$. The initial and final kinetic energies are then

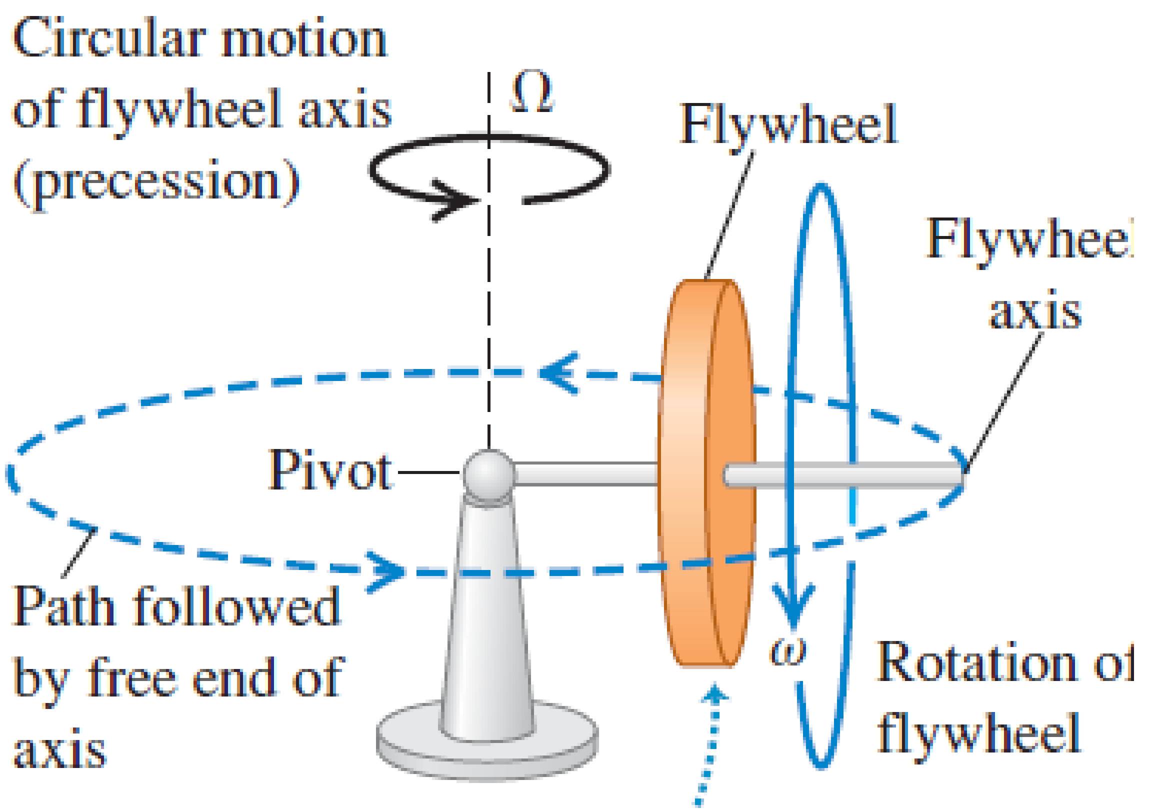
$$K_1 = \tfrac{1}{2}I_1{\omega_{1z}}^2 = \tfrac{1}{2}(13\ \text{kg}\cdot\text{m}^2)(3.14\ \text{rad/s})^2 = 64\ \text{J}$$

$$K_2 = \tfrac{1}{2}I_2{\omega_{2z}}^2 = \tfrac{1}{2}(2.6\ \text{kg}\cdot\text{m}^2)(15.7\ \text{rad/s})^2 = 320\ \text{J}$$

The fivefold increase in kinetic energy came from the work that the professor did in pulling his arms and the dumbbells inward.

Gyroscopes and Precession

For example, consider a toy gyroscope that's supported at one end (Fig. 10.32). If we hold it with the flywheel axis horizontal and let go, the free end of the axis simply drops owing to gravity—*if* the flywheel isn't spinning. But if the flywheel *is* spinning, what happens is quite different. One possible motion is a steady circular motion of the axis in a horizontal plane, combined with the spin motion of the flywheel about the axis. This surprising, nonintuitive motion of the axis is called **precession.**

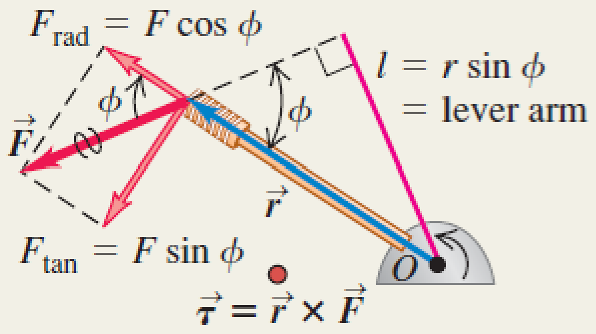


When the flywheel and its axis are stationary, they will fall to the table surface. When the flywheel spins, it and its axis "float" in the air while moving in a circle about the pivot.

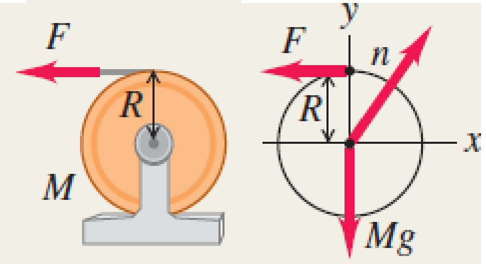
Summary

$$\tau = Fl$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$



$$\sum \tau_z = I\alpha_z$$



$$K = \tfrac{1}{2}Mv_{cm}^2 + \tfrac{1}{2}I_{cm}\omega^2$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm}$$

$$\sum \tau_z = I_{cm}\alpha_z$$

$$v_{cm} = R\omega$$

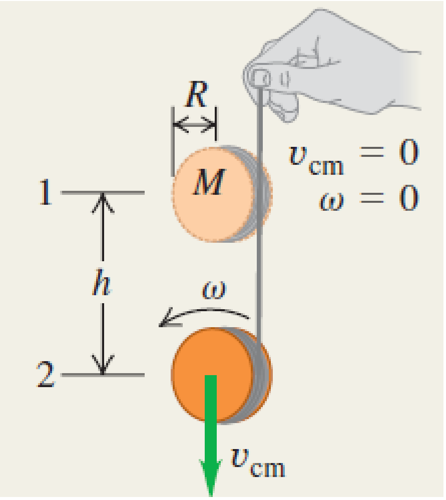
(rolling without slipping)

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

(particle)

$$\vec{L} = I\vec{\omega}$$

(rigid body rotating about axis of symmetry)



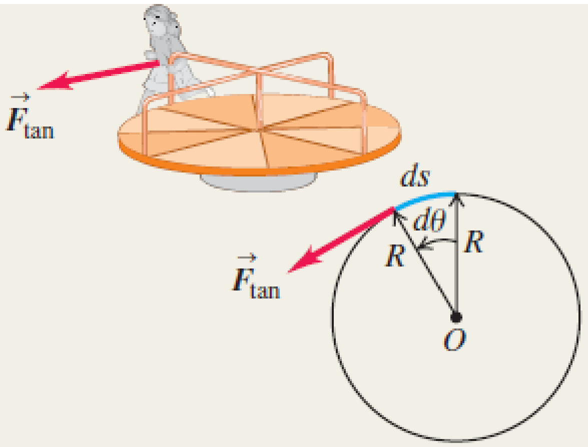
$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta$$

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z \Delta\theta$$

(constant torque only)

$$W_{tot} = \tfrac{1}{2}I\omega_2^2 - \tfrac{1}{2}I\omega_1^2$$

$$P = \tau_z\omega_z$$



$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

$$I_1\omega_{1z} = I_2\omega_{2z} \qquad \text{(zero net external torque)}$$