① Identify Key Elements of a Polynomial

DEFINITION Polynomial

A polynomial in the variable x is an expression of the form:

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

The coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are real numbers, where $a_n \neq 0$, and the exponents n, $n-1$, $n-2$, … , 1 are natural numbers.

- The term $a_n x^n$ is called the leading term.
- The coefficient a_n is the leading coefficient.
- The exponent n is the degree of the polynomial.

- A polynomial with ONE term is called monomial.

  TWO terms is called binomial.

  THREE terms is called trinomial.

For example, the polynomial

$$7x^5 + 4x^3 + 7x + 1,$$

has the leading term $7x^5$, the leading coefficient 7, and its degree is 5.

② Addition, Subtraction and Multiplication of Polynomials

Example 1 Add or subtract as indicated, and simplify.

a) $\left(-4w^{3}-5w^{2}+6w+3\right)+\left(8w^{2}-4w+2\right)$

b) $\left(6.1x^{3}+2.9x^{2}-4.5x+2.1\right)-\left(2.6x^{3}-4.1x^{2}+2.1x+1.1\right)$

Example 2 Multiply and simplify.

a) $(4x+2)\left(x^{2}-6x+\dfrac{1}{2}\right)$

b) $(3y-6)\left(y^{2}+4y+\dfrac{2}{3}\right)$

③ Identify and Simplify Special Case Products

- The expressions of the form $a - b$ and $a + b$ are called conjugates.

Example 3 Multiply and simplify.

a) $(a+b)(a-b)$

b) $(a+b)^2$

c) $(a-b)^2$

Special Case Products

Product of Conjugates : $(a+b)(a-b)=a^2-b^2$

Square of a Binomial: $(a+b)^2=a^2+2ab+b^2$

$$(a-b)^2=a^2-2ab+b^2$$

Example 4 Multiply and simplify.

a) $(2x+5)(2x-5)$

b) $\left(\frac{1}{3}c^2-\frac{1}{2}d\right)\left(\frac{1}{3}c^2+\frac{1}{2}d\right)$

c) $(3x-7)^2$

d) $(5t^2+2v^2)^2$

④ Multiply Radical Expressions Involving Multiple Terms

- The process used to multiply polynomials can be extended to algebraic expressions that are not polynomials.

Example 5 Multiply and simplify.

a) $3\sqrt{5}\left(2\sqrt{5}+4\sqrt{2}+1\right)$

b) $\left(3\sqrt{x}+5\right)\left(2\sqrt{x}-7\right)$

c) $\left(3x+\sqrt{2}\right)^2$

d) $\left(4\sqrt{5}+\sqrt{6}\right)\left(4\sqrt{5}-\sqrt{6}\right)$

In last section we learned how to multiply polynomials. In this section, we reverse this process. The goal is to decompose a polynomial into a product of factors. This process is called factoring. Factoring is important in a variety of applications. In particular, factoring is often used to solve equations.

① Factor Out the Greatest Common Factor

The first step to factor a polynomial is always to factor out the greatest common factor.

- The Greatest Common Factor (GCF) of a polynomial is the expression of highest degree that divides evenly into each term of the polynomial.

For example, the GCF of

$$12x^5 + 18x^4 - 24x^3,$$

is $6x^3$.

To factor out the GCF, we use the distributive property.

Example 1 Factor out the greatest common factor.

a) $12x^5+18x^4-24x^3$

b) $3y(2y-5)+(2y-5)$

c) $-4x^2-8x+12$

② Factor by Group

To factor a polynomial containing four terms, we often try factoring by grouping.

Example 2 Factoring by grouping.

a) $2ax-6ay+5x-15y$

b) $m^2-3n+3m-mn$

③ Factor Quadratic Trinomials

Now we want to factor quadratic trinomials. These are trinomials of the form

$$ax^2 + bx + c$$

where the coefficients a, b, and c are integers and $a \neq 0$. To understand the basis to factor a trinomial, suppose that

$$ax^2 + bx + c = (a_1x + d_1)(a_2x + d_2)$$

$$= a_1a_2x^2 + (a_1d_2 + a_2d_1)x + d_1d_2.$$

So, we have

$$\begin{cases} a_1 a_2 = a \\ d_1 d_2 = c \\ a_1 d_2 + a_2 d_1 = b \end{cases}.$$

In particular, if $a = 1$, we get

$$\begin{cases} a_1 = a_2 = 1 \\ d_1 d_2 = c \\ d_1 + d_2 = b \end{cases}.$$

Example 3 Factor.

a) $x^2 - 8x + 12$

b) $27y + 10y^2 + 5$

c) $10x^3 + 105x^2y - 55xy^2$

Recall

Square of a Binomial: $(a+b)^2 = a^2 + 2ab + b^2$

$$(a-b)^2 = a^2 - 2ab + b^2$$

If we can recognize a perfect square trinomial, then we factor it as the square of a binomial.

Example 4 Factor.

a) $4x^2 - 28x + 49$

b) $81c^4 + 90c^2d + 25d^2$

④ Factor Binomials

- Recall $a^2 - b^2 = (a+b)(a-b)$

Example 5 Factor completely.

a) $x^2 - 100$

b) $32y^4 - 162$

c) $x^2 - 6x - 16$

⑤ Apply a General Strategy to Factor Polynomials

Summary of Factoring a Polynomial

1) The first step to factor a polynomial is always to factor out the GCF.

2) If a polynomial contains four terms, we often try factoring by grouping.

3) If a polynomial is the form of $ax^2 + bx + c$, all possible factors of a must be test with all factors of c until the correct middle term is found.

4) If we can recognize a perfect square trinomial, then we factor it as the square of a binomial.

5) If the binomial is a difference of square, factor as $a^2 - b^2 = (a + b)(a - b)$.

Example 6:

Factor Polynomials by applying general strategies.

a) $x^2 - 8x - 48$

b) $9x^2 + 6x - 80$

c) $x^2 + 2xy + y^2 - 25$

① Determine Restricted Values for a Rational Expression

Recall that a rational number is the ratio of two integers with the denominator not equal to zero. Similarly, the ratio, or quotient, of two polynomials is a rational expression.

Rational Numbers: $\frac{3}{7}$ $\frac{5}{9}$ $\frac{9}{11}$

Rational Expressions: $\frac{3x+2}{x-5}$ $\frac{5x^2}{x^2+1}$ $\frac{9}{3x-2}$

Since a rational expression may have a variable in the denominator, we must be careful to exclude values of variable that make the denominator zero. In the first and third rational expressions, there are values of the variable that the denominator zero; these values are not permitted:

$$\frac{3x+2}{x-5} \qquad x \neq 5 \qquad\qquad \frac{9}{3x-2} \qquad x \neq \frac{2}{3}$$

In the second rational expression, there are no real numbers that will correspond to a zero denominator.

Example 1 Determine the restrictions on the variable for each rational expression.

a) $\dfrac{x-3}{x+2}$

b) $\dfrac{x}{x^2-49}$

c) $\dfrac{4}{5x^2y}$

② Simplify Rational Expression

A rational expression is reduced to lowest terms, or simplified, if the numerator and denominator have no common factors other than ± 1.

Simplifying a rational expression

1) Factor the numerator and denominator completely.

2) State any restrictions.

3) Cancel the common factors in the numerator and denominator.

Example 2 Simplify.

a) $\dfrac{x^2-16}{x^2-x-12}$

b) $\dfrac{8+2\sqrt{7}}{4}$

Example 3 Simplify $\dfrac{14-2x}{x^2-7x}$.

③ Multiplying and Dividing Rational Expressions

The same rules that apply to multiplying and dividing rational numbers also apply to rational expressions.

PROPERTY	RESTRICTION	DESCRIPTION
$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$	$b\neq 0, d\neq 0$	Multiply numerators and denominators, respectively.
$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$	$b\neq 0, d\neq 0, c\neq 0$	Dividing is equivalent to multiplying by a reciprocal.

Example 4 Multiply and simplify

$$\frac{2xy}{x^2y+3xy}\cdot\frac{x^2+6x+9}{4x+12}.$$

Example 5 Divide and simplify

$$\frac{x^2-4}{x}\div\frac{3x^3-12x}{5x^3}.$$

④ Adding and Subtracting Rational Expressions

The same rules that apply to adding and subtracting rational numbers also apply to rational expressions.

PROPERTY	RESTRICTION	DESCRIPTION
$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$	$b \neq 0$	Adding or subtracting rational expressions when the denominators are the same
$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$	$b \neq 0$ and $d \neq 0$	Adding or subtracting rational expressions when the denominators are different

Example 6. Perform the indicated operation and simplify.

a) $\dfrac{x+7}{(x+2)^2}+\dfrac{3x+1}{(x+2)^2}$ b) $\dfrac{6x+7}{2x-1}-\dfrac{2x+9}{2x-1}$

Example 7. Perform the indicated operation and simplify.

a) $\dfrac{3-x}{2x+1}+\dfrac{x}{x-1}$ b) $\dfrac{1}{x^2}-\dfrac{2}{x+1}$

⑤ Rationalize the denominator of a Radical Expression

Example 8:

Rationalizing the denominator.

a) $\dfrac{5}{\sqrt{x}}$ Soln: $\dfrac{5}{\sqrt{x}} = \dfrac{5\cdot\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}} = \dfrac{5\sqrt{x}}{x}$

b) $\dfrac{2}{\sqrt{7}-\sqrt{5}}$ Soln: $\dfrac{2}{\sqrt{7}-\sqrt{5}} = \dfrac{2\cdot(\sqrt{7}+\sqrt{5})}{(\sqrt{7}-\sqrt{5})\cdot(\sqrt{7}+\sqrt{5})}$

$$= \frac{2(\sqrt{7}+\sqrt{5})}{2} = \sqrt{7}+\sqrt{5}$$