CALCULUS

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- In Chapter 2, we discussed how to determine the slope of a curve at a point and how to measure the rate at which a function changes. Now that we have studied limits, we can define these ideas precisely and see that both are interpretations of the ***derivative*** of a function at a point.
- Then we extend this concept from a single point to the **derivative function**, and we develop rules for finding this derivative function easily, without having to calculate any limits directly.

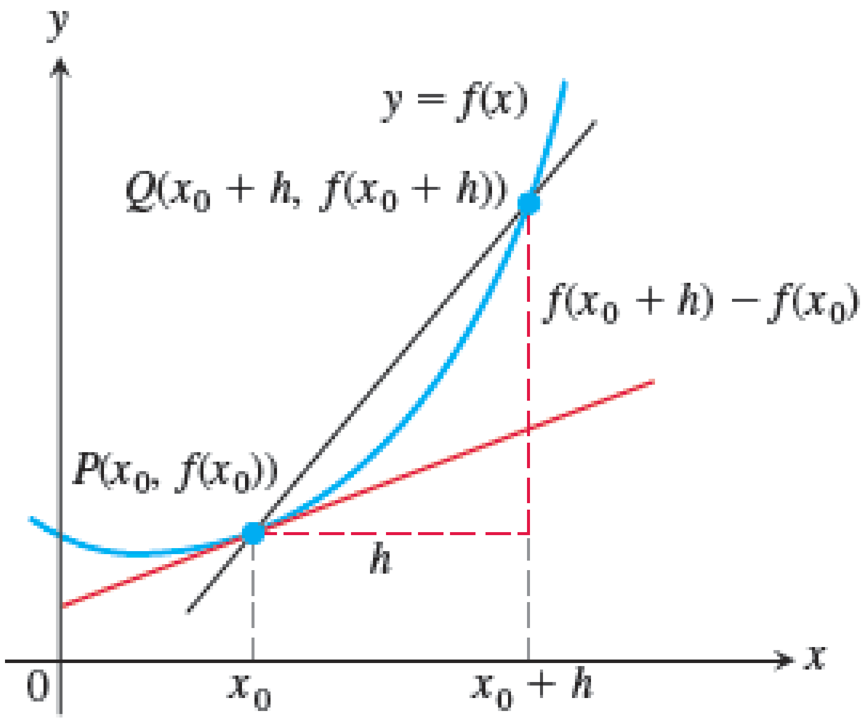
① Finding a Tangent to the Graph of a Function

- To find a tangent line to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant line through P and a nearby point $Q(x_0 + h, f(x_0 + h))$.

- Then we investigate the limit of the slope as $h \to 0$.

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

- If the limit exists, we call it the slope of the curve at $P\ (x_0, f(x_0))$ and define the tangent line at P to be the line through P having this slope.



DEFINITION 1 The slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists. The tangent line to the curve at P is the line through P with this slope. That is, the tangent line to the curve at P is

$$y - f(x_0) = m(x - x_0).$$

Example 1

(a) Find the slope of $y = \frac{1}{x}$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-\frac{1}{4}$?

(c) What happens to the tangent to the curve at the point $\left(a, \frac{1}{a}\right)$ as a changes?

Example 2

Let $f(x) = x^2$. Find the tangent line of $f(x)$ at $x = 1$.

② Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0+h)-f(x_0)}{h}, \qquad h \neq 0$$

is called **the difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION: **The derivative of a function f at a point x_0**, denoted as $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

provided this limit exists.

- If we interpret the difference quotient as the slope of a secant line, then the derivative gives **the slope of the curve** $y = f(x)$ at the point $P\left(x_0, f(x_0)\right)$.
- If we interpret the difference quotient as an average rate of change, the derivative gives the function's **instantaneous rate of change** with respect to x at the point $x = x_0$.

Example 3

Using the definition to find the derivative of $y = -2x^2$ at $x = 1$ and find the equation of the tangent line at the point $(1, -2)$.

Summary

- We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas refer to the same limit.
- The following are all interpretations for the limit of the difference quotient

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$

2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$

3. The rate of change of $f(x)$ with respect to x at the $x = x_0$

4. The derivative $f'(x_0)$ at $x = x_0$

Skill Practice 1: Object dropped from a tower

An object is dropped from the top of a 490-m-high tower. Its height above ground after t sec is $h(t) = 490 - 4.9t^2$ m. How fast is it when it hits the ground?

Skill Practice 2:

Find an equation of the straight line having slope 1/4 that is tangent to the curve $y = \sqrt{x}$?