CALCULUS

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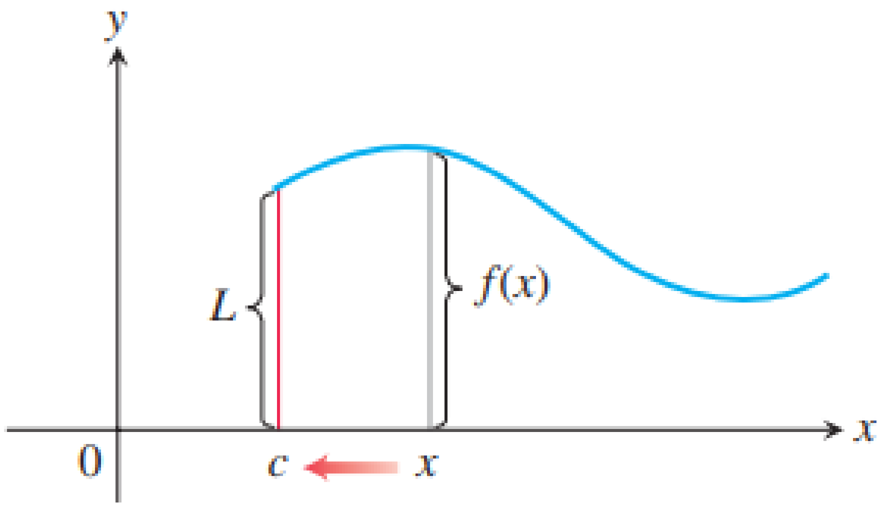
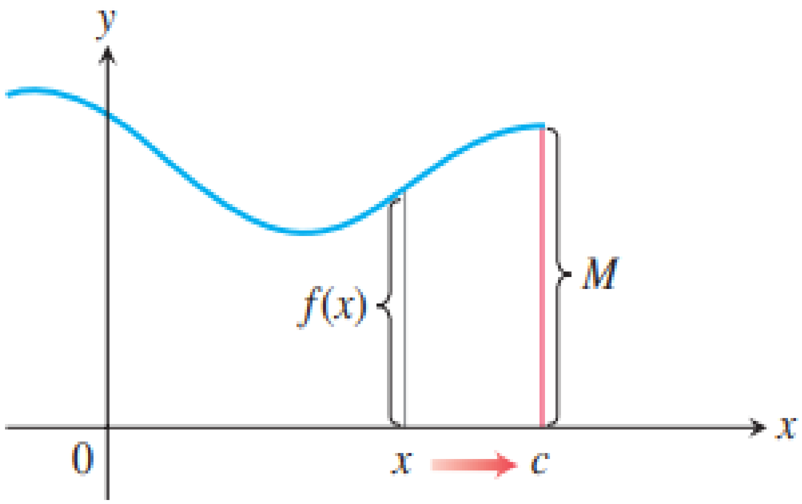
- In this section, we give the definition of the limit of a function at a boundary point of its domain. This definition is consistent with limits at boundary points of regions in the plane and in space, as we will see in Chapter 14.
- When the domain of f is an interval lying to the left of c, such as $(a, c]$ or (a, c), then we say that f has a limit at c if it has a left-hand limit at c.
- Similarly, if the domain of f is an interval lying to the right of c, such as $[c, b)$ or (c, b), then we say that f has a limit at c if it has a right-hand limit at c.

- $f(x)$ is only defined on an interval (c, b), where $c < b$, and the values of $f(x)$ become arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. The **left-hand limit** can be defined in a similar way.



(a) $\lim_{x \to c^+} f(x) = L$

Right-hand limit



(b) $\lim_{x \to c^-} f(x) = M$

Left-hand limit

① Approaching a Limit from One Side

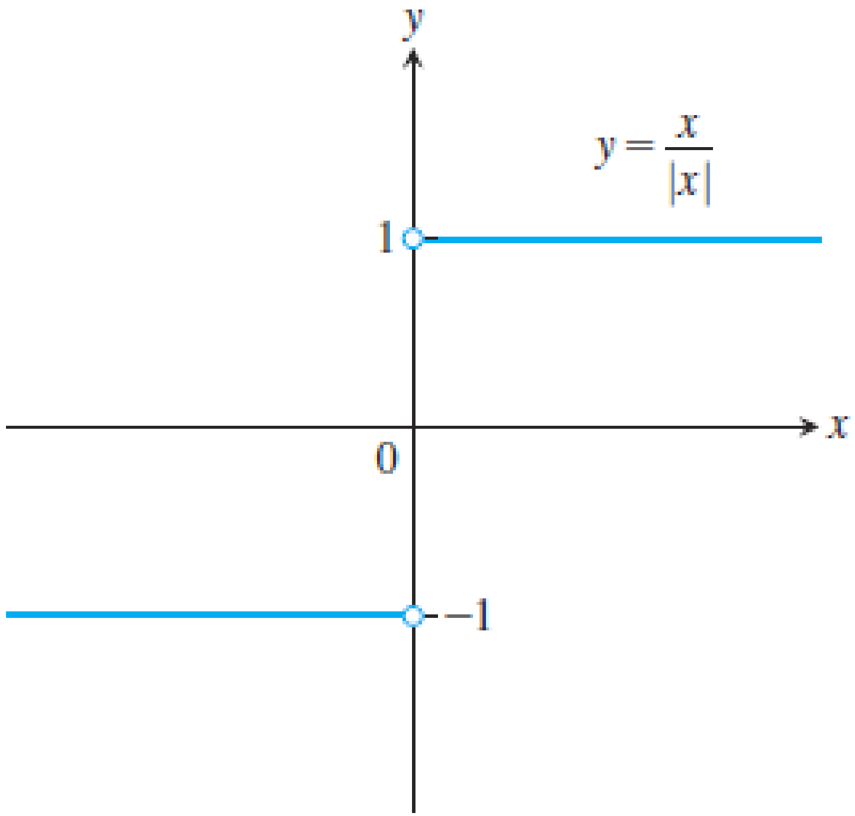
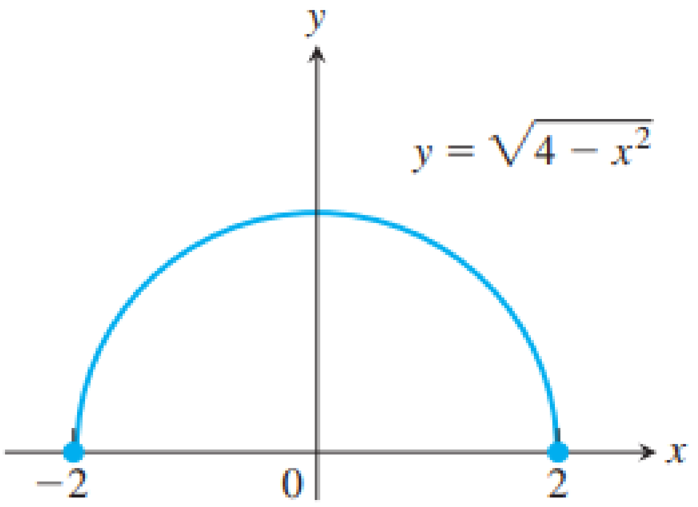


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

The function $f(x) = x/|x|$ has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Example 1 The domain of $f(x) = \sqrt{4 - x^2}$ is [–2, 2]; its graph is the semicircle shown here. We have

$$\lim_{x \to -2^+} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} f(x) = 0$$



This function has a two-sided limit at each point in (–2, 2). It has a left-hand limit at x = 2 and a right-hand limit at x = –2. The function does not have a left-hand limit at x = –2 or a right-hand limit at x = 2. It does not have a two-sided limit at either –2 or 2 because f is not defined on both sides of these points.

THEOREM 6 Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

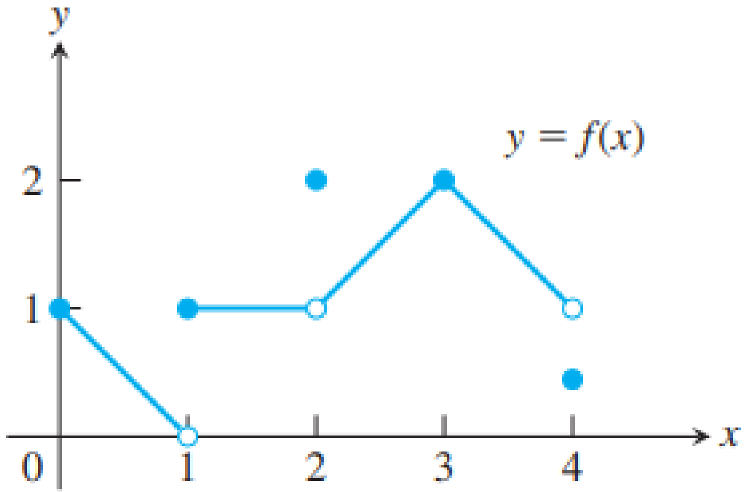
$$\lim_{x\to c} f(x) = L \quad\Leftrightarrow\quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L$$

- Theorem 6 applies at interior points of a function's domain. At a boundary point of its domain, a function has a limit when it has an appropriate one-sided limit.

Example 2

Find the limits for the following function at points

$x = 0$, $x = 1$, $x = 2$, $x = 3$ and $x = 4$.



② Precise Definitions of One-Sided Limits

DEFINITIONS (a) Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c. We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

Example 3 Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Example 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side.

Example 5 Find $\lim_{x \to -2^-} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$.

Example 6 Find $\lim_{x\to 1^+}\frac{|x-1|}{x-1}$.

Example 7 Find $\lim_{x\to 1^-}\frac{|x-1|}{x-1}$.

Example 8 Find $\lim_{x\to 0^+}\frac{\sqrt{x+4}-2}{x}$.

③ Limits Involving $\frac{\sin\theta}{\theta}$

- A central fact about $\frac{\sin\theta}{\theta}$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.32 and confirm it algebraically using the Sandwich Theorem.
- You will see the importance of this limit in Section 3.5, where instantaneous rates of change of the trigonometric functions are studied.

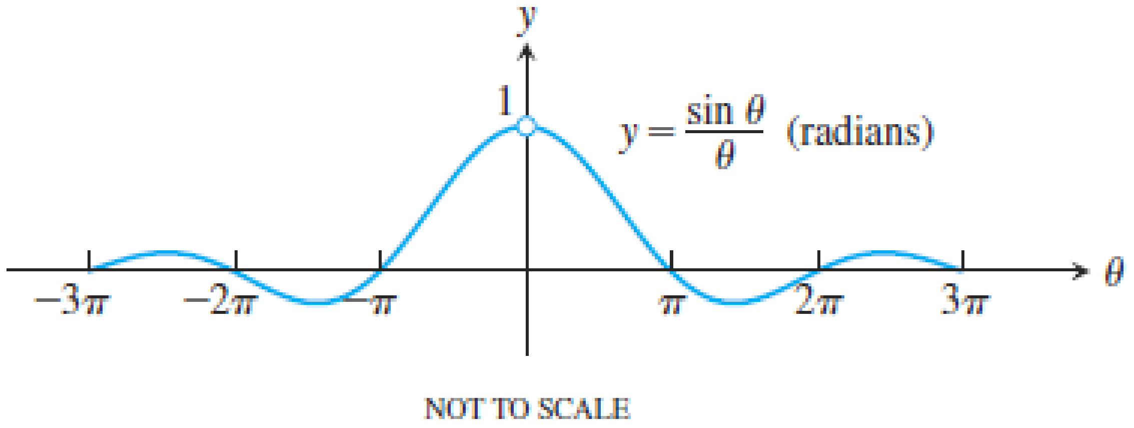


FIGURE 2.32 The graph of $f(\theta) = (\sin\theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

Theorem 7: $\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$ (θ in radian)

Proof: Let $0 < \theta < \frac{\pi}{2}$. Note that

area ΔOAP < area sector OAP < area ΔOAT.

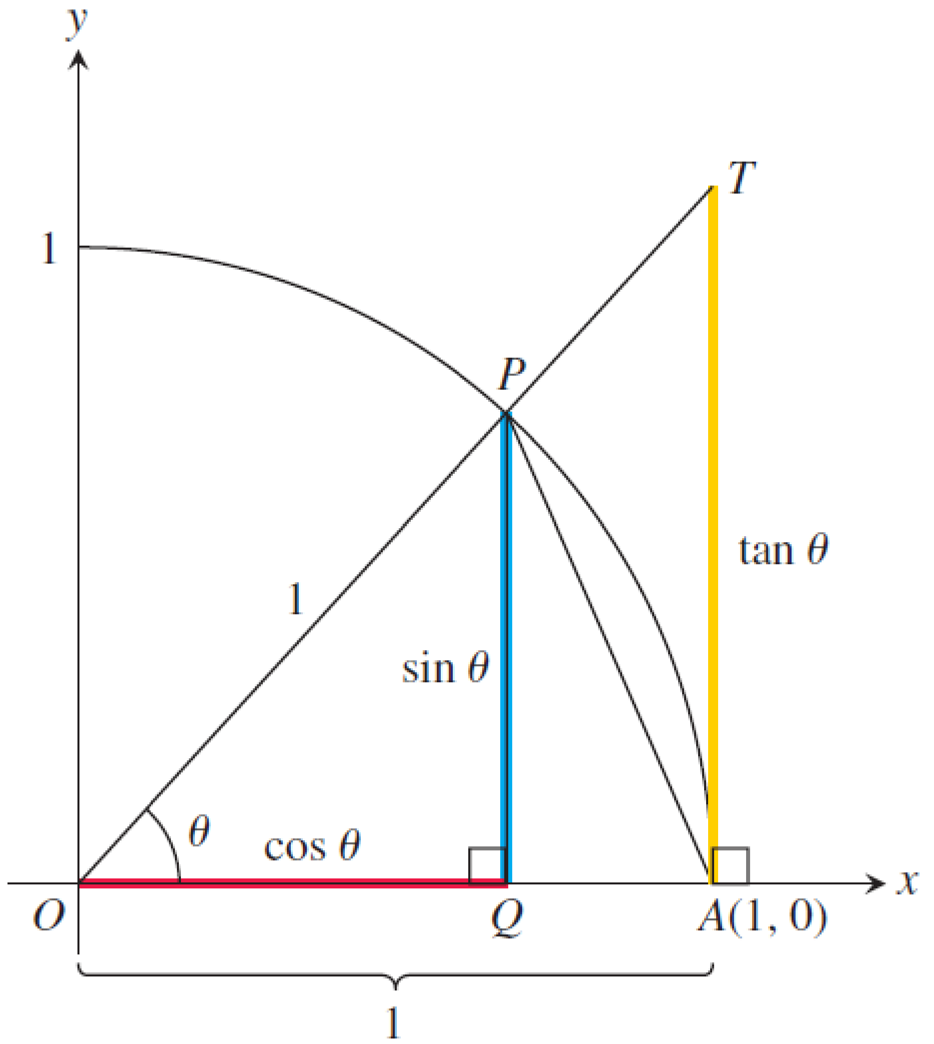
Since: area $\Delta OAP = \frac{1}{2} sin\theta$

area sector $OAP = \frac{1}{2} r^2\theta = \frac{1}{2}\theta$

area $\Delta OAT = \frac{1}{2} tan\theta$

Thus: $\frac{1}{2} sin\theta < \frac{1}{2}\theta < \frac{1}{2} tan\theta \Rightarrow 1 < \frac{\theta}{sin\theta} < \frac{1}{cos\theta}$

$\Rightarrow 1 > \frac{sin\theta}{\theta} > cos\theta$



Example 9 Show that (a) $\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{\theta}=0$ and (b) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{5x}=\dfrac{2}{5}$.

Example 10 Find $\lim\limits_{t\to 0}\dfrac{\tan t\sec 2t}{3t}$.

Example 11 Show that for nonzero constants A and B.

$$\lim_{\theta\to 0}\frac{sinA\theta}{sinB\theta}=\frac{A}{B}$$

Skill Practice 1 Determine:

$$\lim_{h\to 0^+}\frac{\sqrt{h^2+6h+25}-5}{h}$$

Skill Practice 2 Find:

$$\lim_{x\to 0}\frac{cos^2x-cosx}{x^2}$$