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CALCULUS

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3.6 The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$?

- F is a composite function. In fact, if we let $y = f(u) = \sin u$ and let $u = g(x) = x^2 - 4$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$. We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

- It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This is one of the most important differentiation rules and is called the ***Chain Rule***.

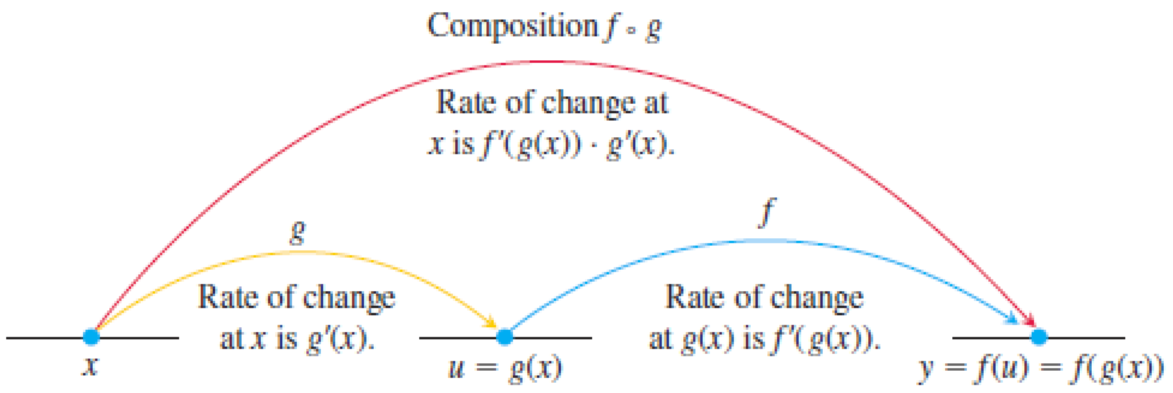

FIGURE 3.25 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

① Derivative of a Composite Function

Example 1

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composition of the functions $y = \frac{1}{2}u$ and $u = 3x$.

We can easily find out that:

$$\frac{dy}{dx} = \frac{3}{2} = \frac{1}{2} \cdot 3 = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 2

The function $y = (3x^2 + 1)^2$ is the composition of the functions $y = u^2$ and $u = 3x^2+1$.

We see that:

$$\frac{dy}{dx} = \frac{d}{dx}[(3x^2 + 1)^2] = \frac{d}{dx}\left(9x^4 + 6x^2 + 1\right) = 36x^3 + 12x$$

$$= 12(3x^2 + 1)x = 12ux = (2u)(6x) = \frac{dy}{du} \cdot \frac{du}{dx}$$

THEOREM — The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

where dy/du is evaluated at $u = g(x)$.

- **A complete Proof of the Chain Rule is provided in Section 3.9 on Page 169.**

Example 3 An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

② "Outside-Inside" Rule

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to write the Chain Rule using functional notation. If $y = f(g(x))$, then

$$y' = f'(g(x)) \cdot g'(x)$$

Example 4

Differentiate $f(x) = \sin(x^2 + x)$ with respect to x.

Skill Practice 1

Differentiate $f(x) = \cos(x^3 - 3x)$ with respect to x.

③ Repeated use of the Chain Rule

Sometimes we have to use the Chain Rule two or more times to find a derivative.

Example 5

Find the derivative of $g(t) = \tan(5 - sin2t)$.

Skill Practice 2

Find the derivative of

$$f(x) = \sqrt{3 - 2sin^2x}$$

④ The Chain Rule with Powers of a Function

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{\mathrm{d}(u^n)}{\mathrm{d}x} = nu^{n-1} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

Example 6

Use the Power Chain Rule to determine the derivative of the following power functions.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3 - x^4)^7$ (b) $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{3x-2})$ (c) $\frac{\mathrm{d}}{\mathrm{d}x}(sin^5 x)$

Example 7

Show that the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

Example 8

The formulas for the derivatives of both sin *x* and cos *x* were obtained under the assumption that *x* is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since 180° = π radians, $x° = \pi x/180$ radians where $x°$ is the size of the angle measured in degrees.

By the Chain Rule:

$$\frac{d}{dx}(sinx°) = \frac{d}{dx}\left(sin\frac{\pi x}{180}\right) = \frac{\pi}{180}cos\frac{\pi x}{180} = \frac{\pi}{180}cosx°$$

Skill Practice 3

Find the derivatives of (a) $f(x) = \sqrt{3cosx - 2sinx}$ (b) $f(x) = \dfrac{tan6x}{(x+1)^2}$

Skill Practice 4

Find the second derivatives of (a) $f(x) = 2tan\dfrac{x}{2}$ (b) $f(x) = \dfrac{3x}{1-\sqrt{x}}$

Skill Practice 5

Consider the function:

$$f(x) = \begin{cases} x^2\cos\left(\dfrac{2}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

a. Show that f is continuous at $x = 0$.

b. Determine f' for $x \neq 0$.

c. Show that f is differentiable at $x = 0$.

d. Show that f' is not continuous at $x = 0$.