Lecture 6

Work and Energy

Date: 3/20/2025

Course Instructor:
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Previously we have learned Newton's Laws

For problem-solving, are Newton's laws versatile?

Example of a major stumbling block: Varying forces

More effective to analyze with Work and Energy

What is "Work"?

6.1 These people are doing work as they push on the stalled car because they exert a force on the car as it moves.

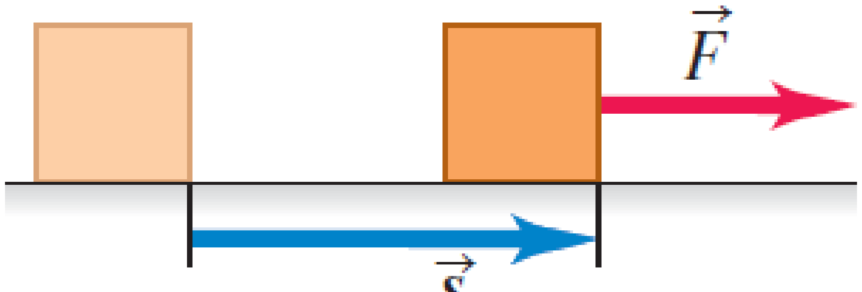
Intuition

Definition of Work

The three examples of work described above—pulling a sofa, lifting encyclopedias, and pushing a car—have something in common. In each case you do work by exerting a *force* on a body while that body *moves* from one place to another—that is, undergoes a *displacement* (Fig. 6.1). You do more work if the force is greater (you push harder on the car) or if the displacement is greater (you push the car farther down the road).

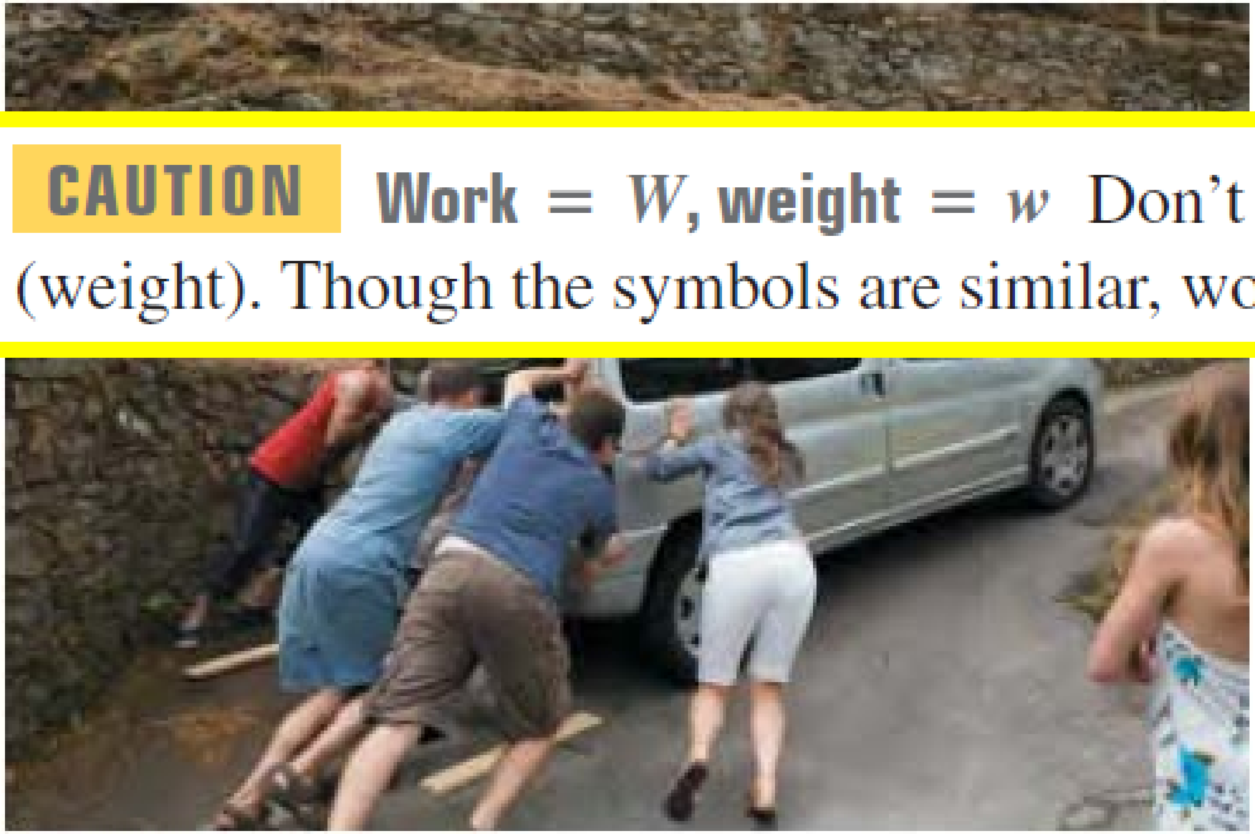
constant force $\vec{F}$ acts on it in the same direction as the displacement $\vec{s}$ (Fig. 6.2). We define the **work** W done by this constant force under these circumstances as the product of the force magnitude F and the displacement magnitude s:

$$W = Fs \qquad \text{(constant force in direction of straight-line displacement)} \quad (6.1)$$



What is Work?

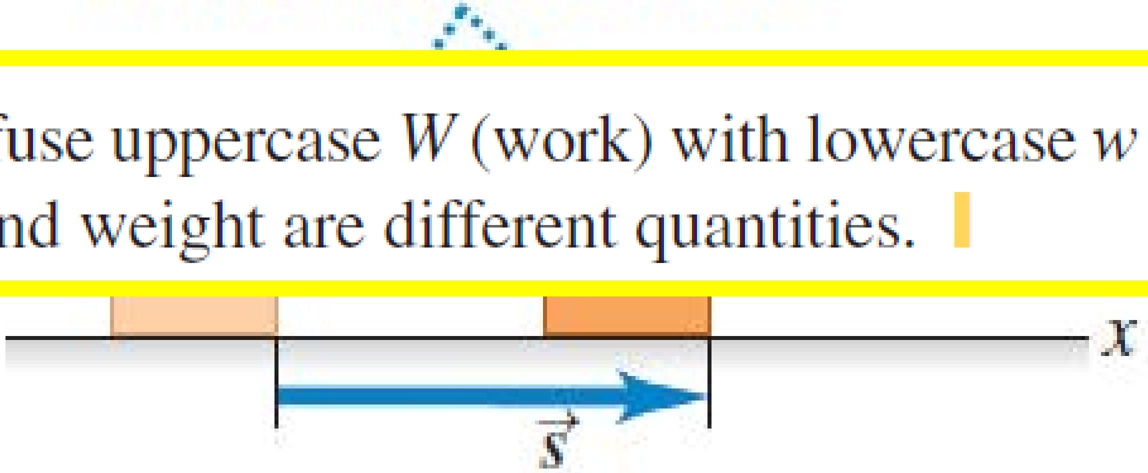
6.1 These people are doing work as they push on the stalled car because they exert a force on the car as it moves.



CAUTION **Work** $= W$, **weight** $= w$ Don't confuse uppercase W (work) with lowercase w (weight). Though the symbols are similar, work and weight are different quantities.

Intuition

If a body moves through a displacement $\vec{s}$ while a constant force $\vec{F}$ acts on it in the same direction ...



... the work done by the force on the body is $W = Fs$.

Definition

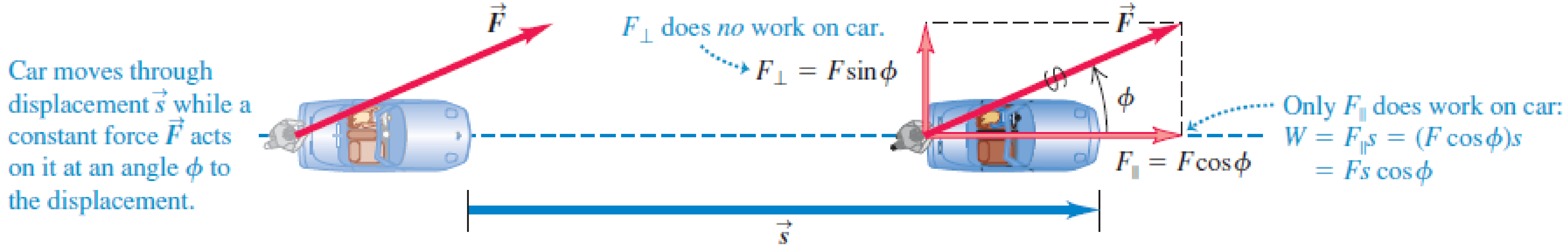
Notation and Unit for Work

CAUTION **Work $= W$, weight $= w$** Don't confuse uppercase W (work) with lowercase w (weight). Though the symbols are similar, work and weight are different quantities.

The SI unit of work is the **joule** (abbreviated J, pronounced "jool," and named in honor of the 19th-century English physicist James Prescott Joule). From Eq. (6.1) we see that in any system of units, the unit of work is the unit of force multiplied by the unit of distance. In SI units the unit of force is the newton and the unit of distance is the meter, so 1 joule is equivalent to 1 *newton-meter* ($N \cdot m$):

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}) \quad \text{or} \quad 1 \text{ J} = 1 \text{ N} \cdot \text{m}$$

Quantification of Work W



$$W = Fs\cos\phi \qquad \text{(constant force, straight-line displacement)}$$

$F_\perp$ perpendicular to displacement $\vec{s}$ does **NO WORK**!

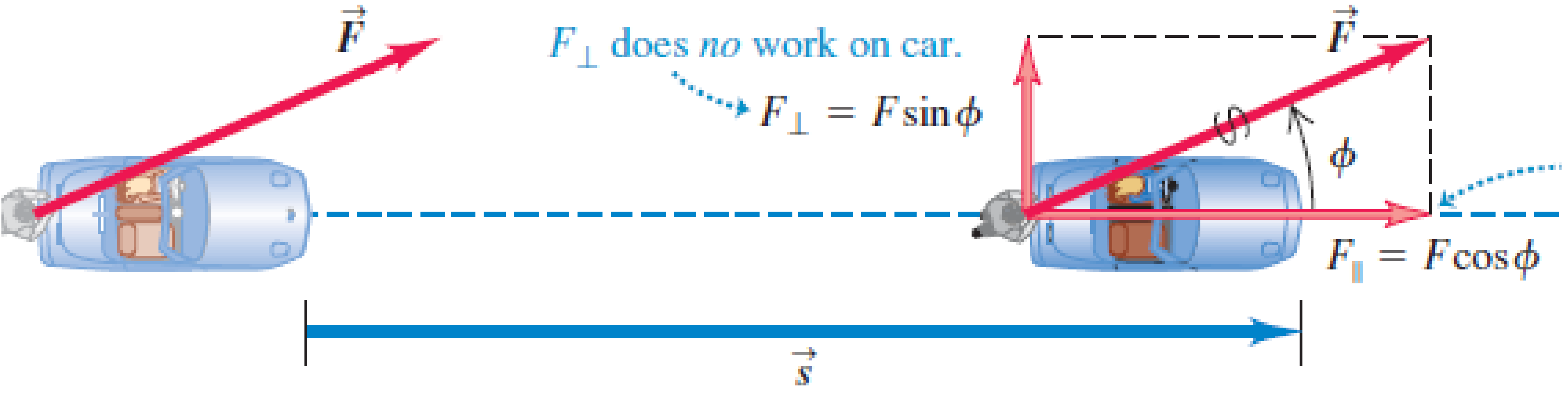
$$W = \vec{F}\cdot\vec{s} \qquad \text{(constant force, straight-line displacement)} \tag{6.3}$$

For *varying forces* or *non-straight-line motion*:

$$\mathrm{d}W = \vec{F}(t)\cdot d\vec{s} = F(t)\cdot ds\cdot\cos\phi$$

Example 6.1

(a) Steve exerts a steady force of magnitude 210 N (about 47 lb) on the stalled car in Fig. 6.3 as he pushes it a distance of 18 m. The car also has a flat tire, so to make the car track straight Steve must push at an angle of 30° to the direction of motion. How much work does Steve do?



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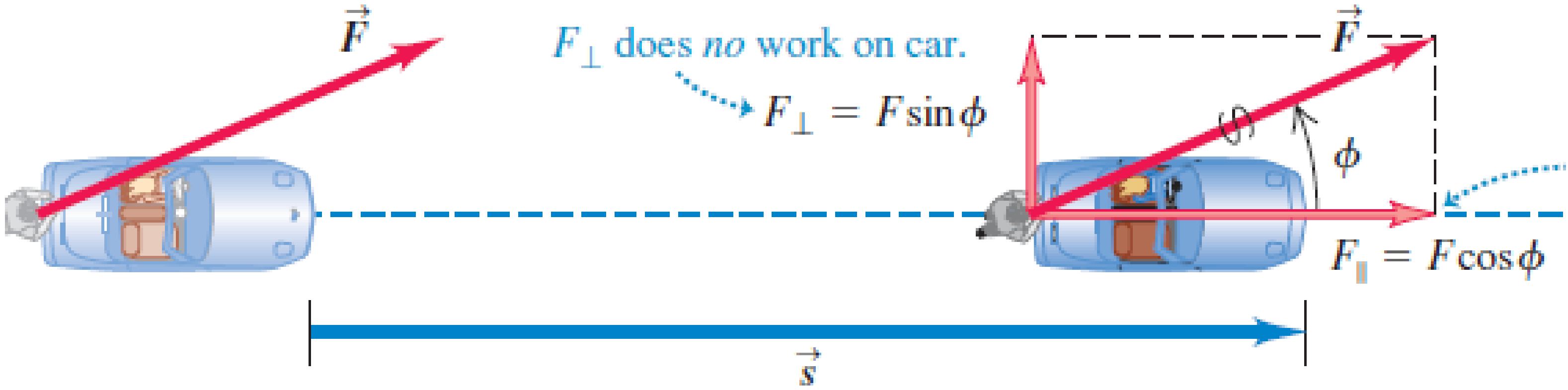

EXECUTE: (a) From Eq. (6.2),

$$W = Fs\cos\phi = (210\ \text{N})(18\ \text{m})\cos 30° = 3.3 \times 10^3\ \text{J}$$

Example 6.1

(a) Steve exerts a steady force of magnitude 210 N (about 47 lb) on the stalled car in Fig. 6.3 as he pushes it a distance of 18 m. The car also has a flat tire, so to make the car track straight Steve must push at an angle of 30° to the direction of motion. How much work does Steve do? (b) In a helpful mood, Steve pushes a second stalled car with a steady force $\vec{F} = (160\ \text{N})\hat{\imath} - (40\ \text{N})\hat{\jmath}$. The displacement of the car is $\vec{s} = (14\ \text{m})\hat{\imath} + (11\ \text{m})\hat{\jmath}$. How much work does Steve do in this case?

In part (b) both $\vec{F}$ and $\vec{s}$ are given in terms of components, so it's best to calculate the scalar product using Eq. (1.21): $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$.

Example 6.1

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$$
\begin{aligned}
W &= \vec{F} \cdot \vec{s} = F_x x + F_y y \\
&= (160\ \text{N})(14\ \text{m}) + (-40\ \text{N})(11\ \text{m}) \\
&= 1.8 \times 10^3\ \text{J}
\end{aligned}
$$

6.8 •• A loaded grocery cart is rolling across a parking lot in a strong wind. You apply a constant force $\vec{F} = (30\text{ N})\hat{\imath} - (40\text{ N})\hat{\jmath}$ to the cart as it undergoes a displacement $\vec{s} = (-9.0\text{ m})\hat{\imath} - (3.0\text{ m})\hat{\jmath}$. How much work does the force you apply do on the grocery cart?

$$W = \vec{F}\cdot\vec{s} = 30\text{N}\cdot(-9.0\text{m}) + (-40\text{N})\cdot(-3.0\text{m})$$

$$= -270.0\text{J} + 120.0\text{J}$$

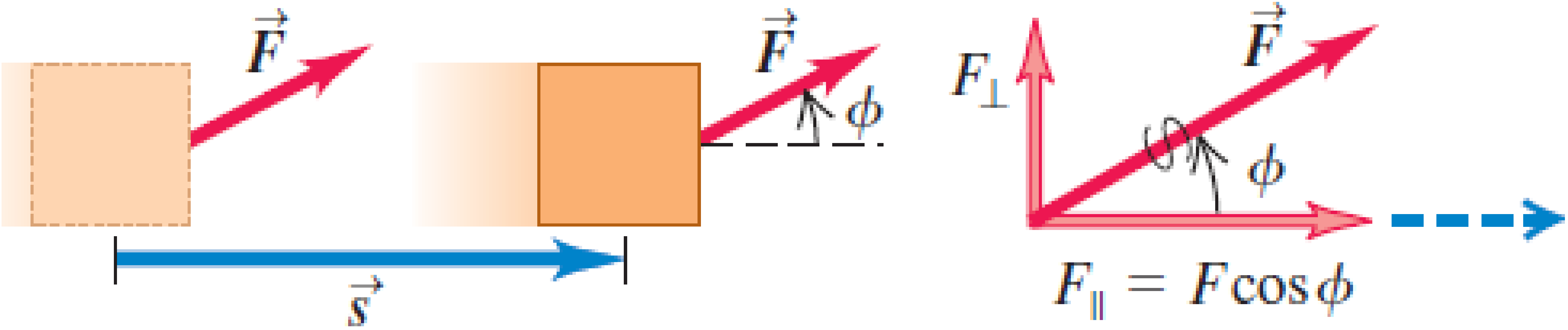
$$= -150.0\text{ J}$$

Work Can be Zero or Negative!

Force $\vec{F}$ has a component in direction of displacement:

$W = F_{\parallel}s = (F\cos\phi)s$

Work is *positive*.

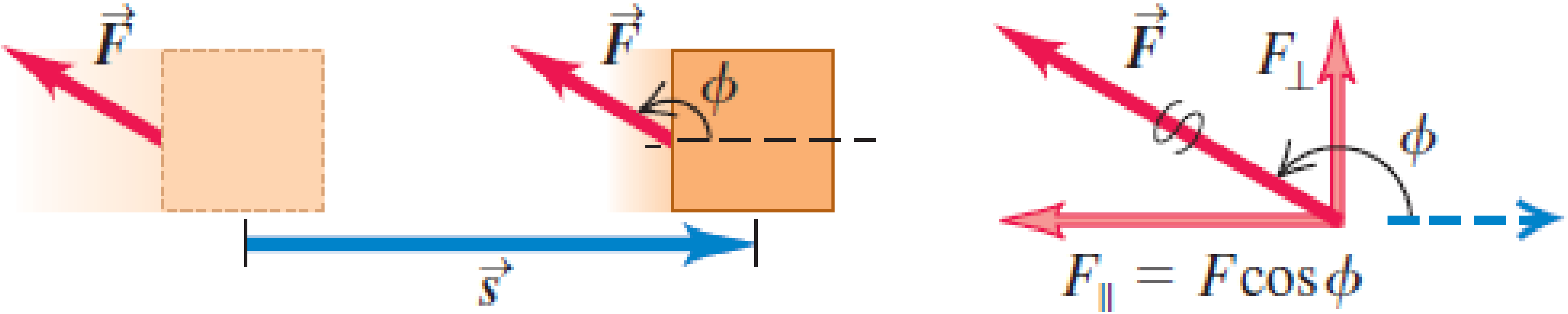


Work Can be Zero or Negative!

Force $\vec{F}$ has a component opposite to direction of displacement:

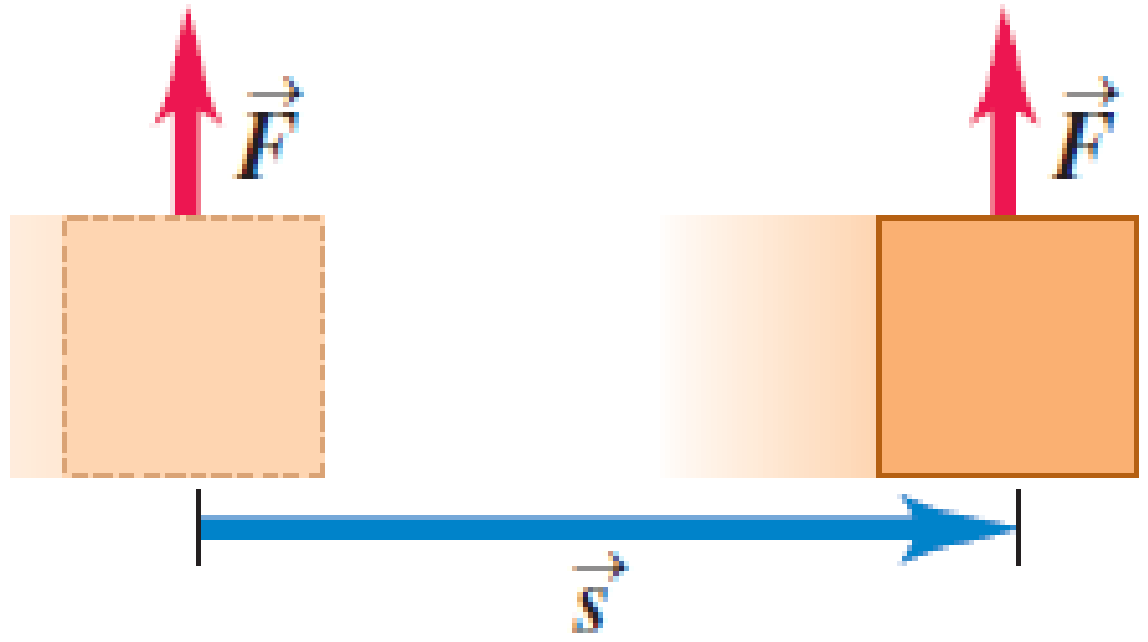
$W = F_{\parallel}s = (F\cos\phi)s$

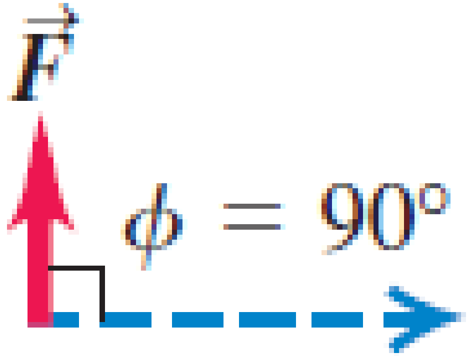
Work is *negative* (because $F\cos\phi$ is negative for $90° < \phi < 180°$).



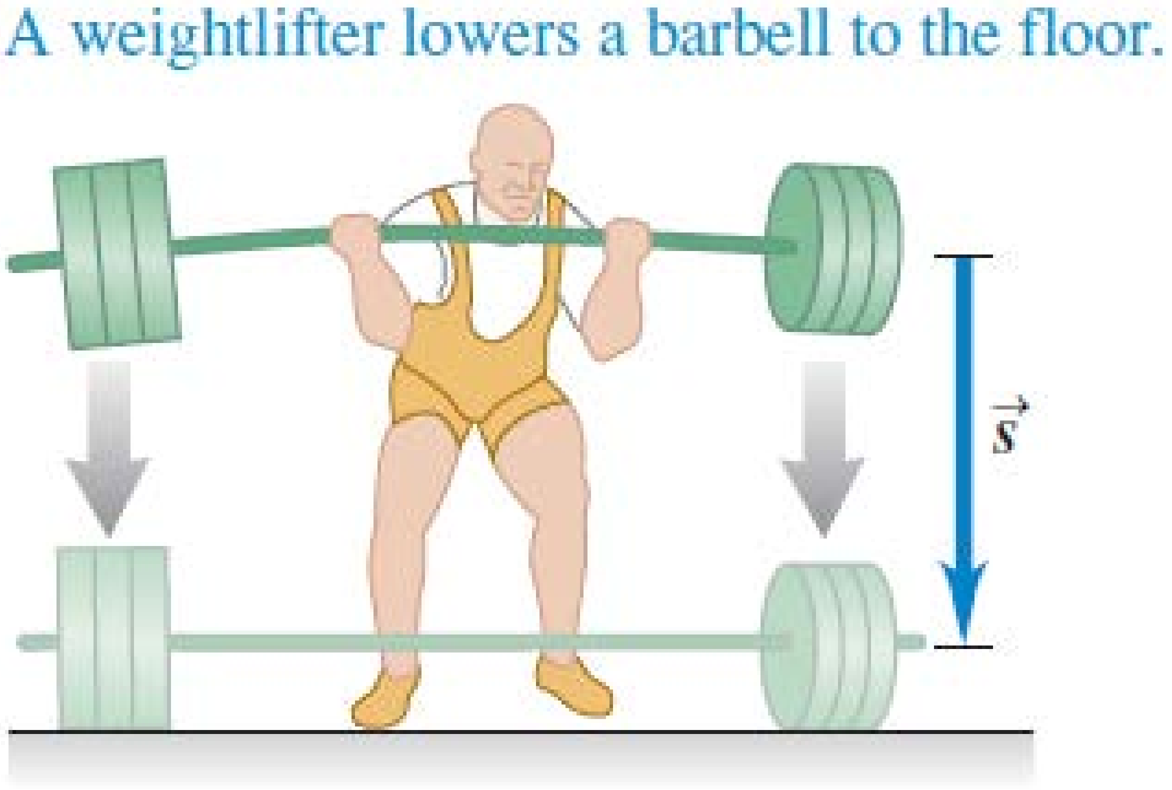
Work Can be Zero or Negative!

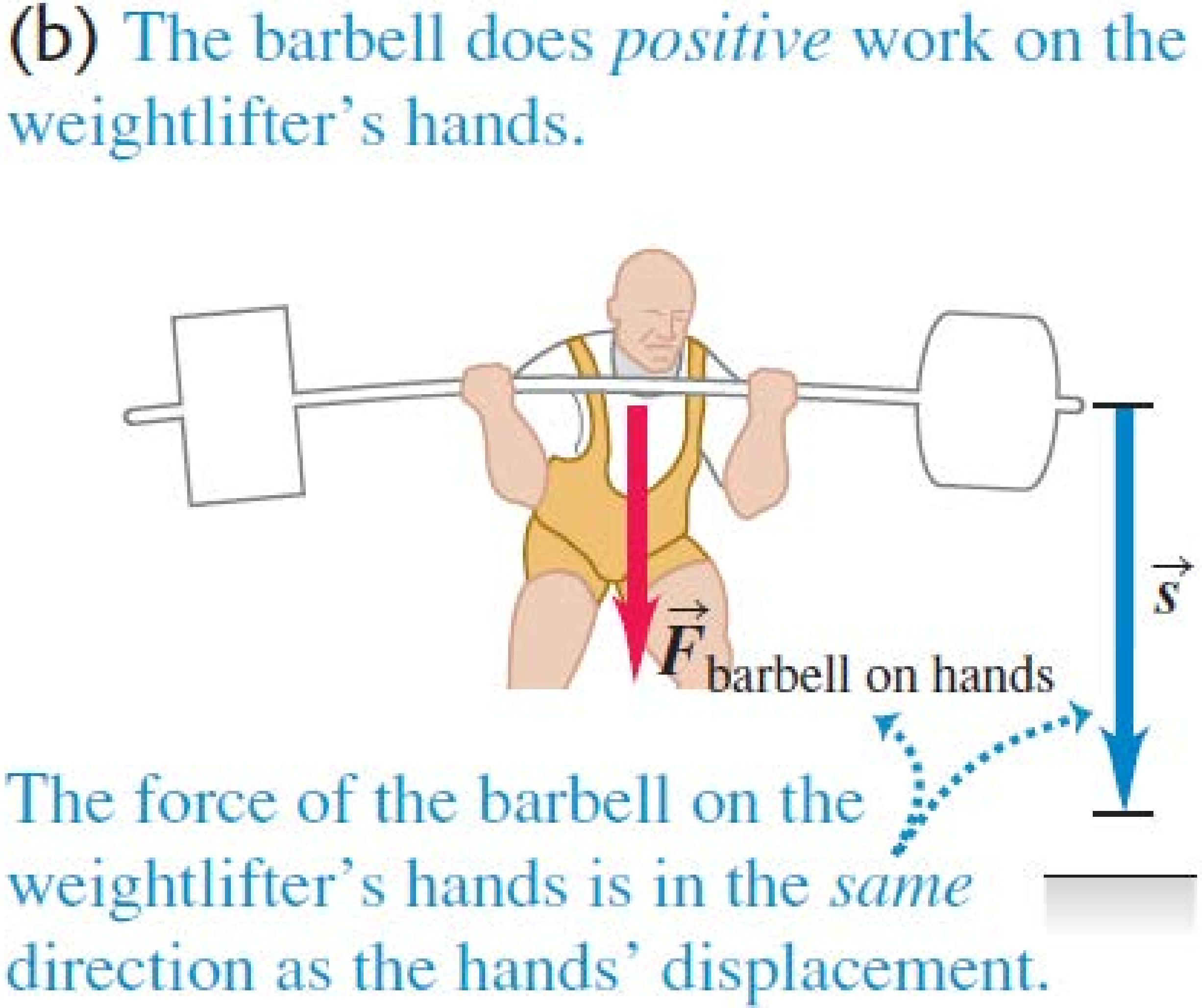
Force $\vec{F}$ (or force component $F_\perp$) is perpendicular to direction of displacement: The force (or force component) does *no* work on the object.





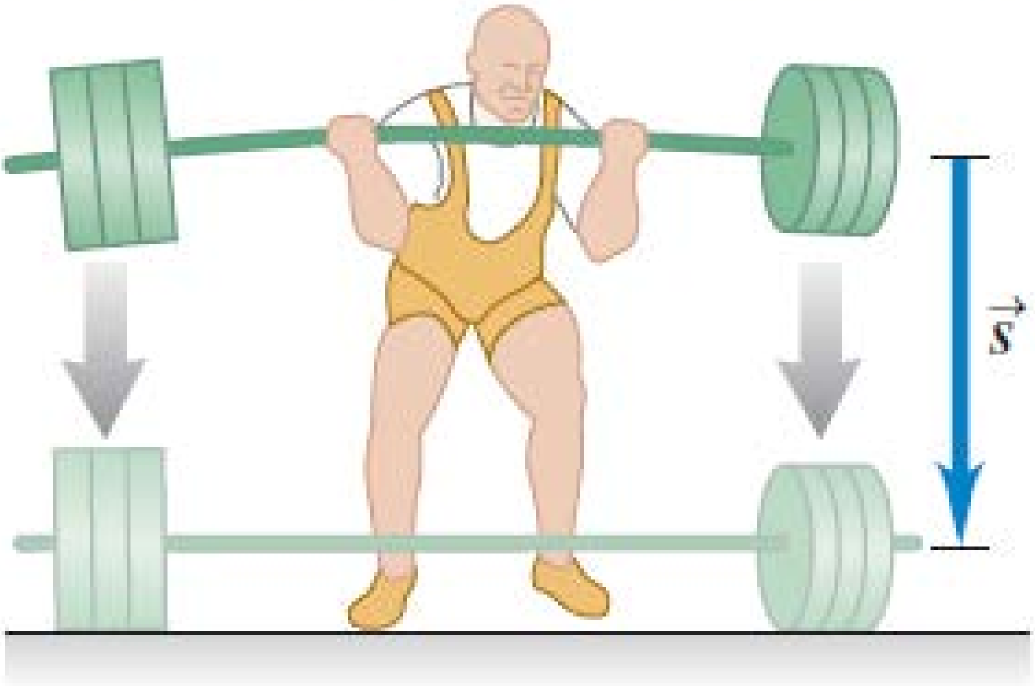
Example for Negative Work



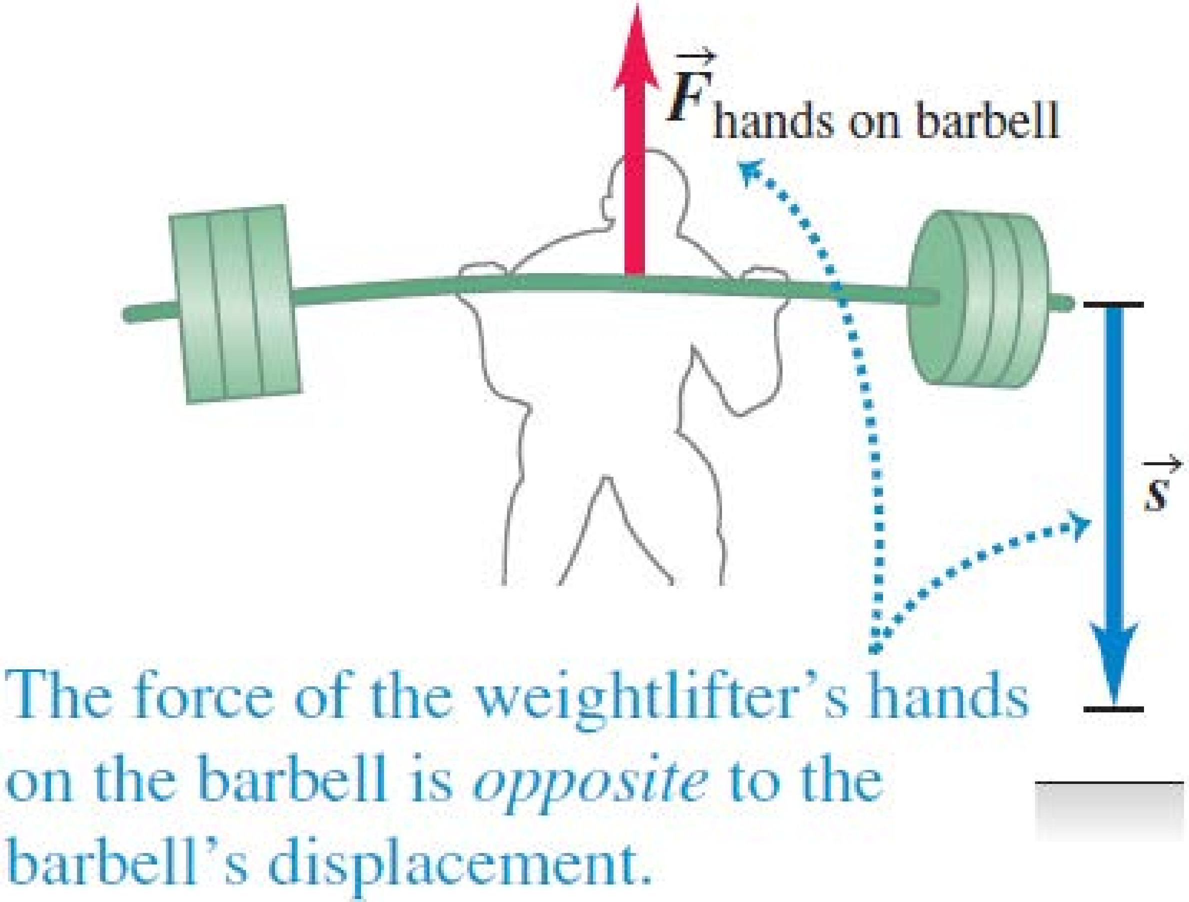


Example for Positive Work

A weightlifter lowers a barbell to the floor.



(c) The weightlifter's hands do *negative* work on the barbell.



Example for Positive Work

CAUTION **Keep track of who's doing the work** We always speak of work done *on* a particular body *by* a specific force. Always be sure to specify exactly what force is doing the work you are talking about. When you lift a book, you exert an upward force on the book and the book's displacement is upward, so the work done by the lifting force on the book is positive. But the work done by the *gravitational* force (weight) on a book being lifted is *negative* because the downward gravitational force is opposite to the upward displacement.

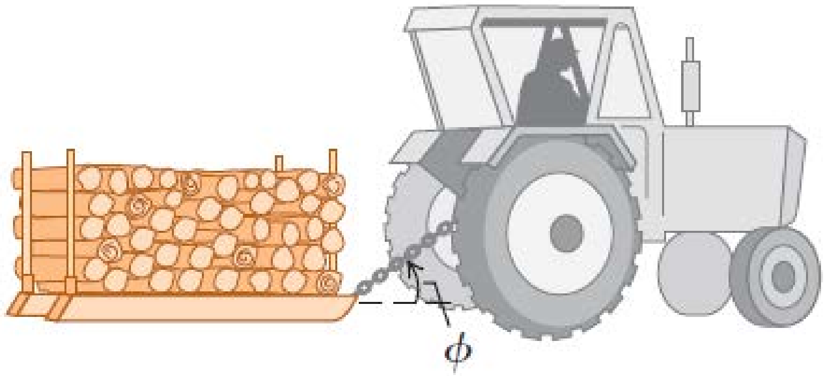
Total Work Done by Multiple Forces

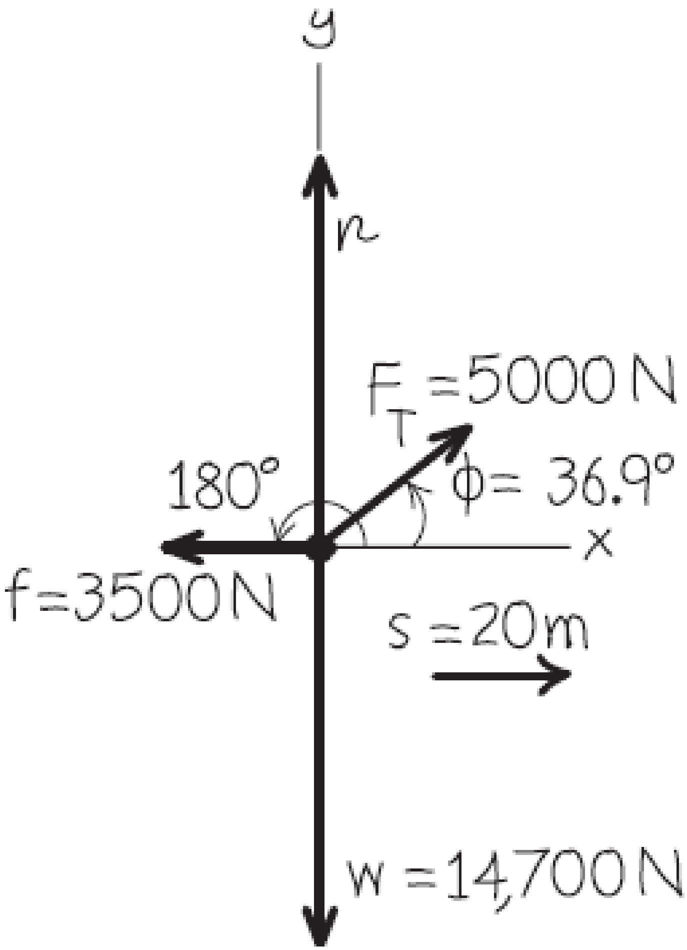
Two approaches:

1. Compute work done by **individual forces** first and **algebraic sum** of them is the total work
2. Compute the **net force** first, and then the total work

Example 6.2: Work by Multiple Forces

A farmer hitches her tractor to a sled loaded with firewood and pulls it a distance of 20 m along level ground (Fig. 6.7a). The total weight of sled and load is 14,700 N. The tractor exerts a constant 5000-N force at an angle of 36.9° above the horizontal. A 3500-N friction force opposes the sled's motion. Find the work done by each force acting on the sled and the total work done by all the forces.





Free-body diagram for sled

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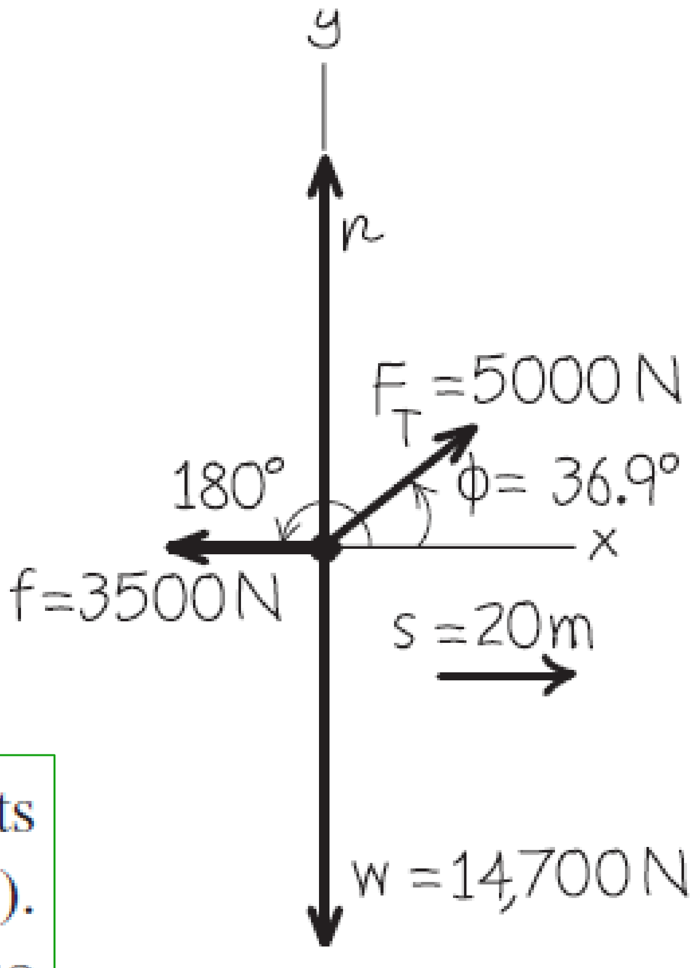

Free-body diagram for sled

Solution by approach 1:

- Compute work done by individual forces first and algebraic sum of them is the total work

EXECUTE: (1) The work W_w done by the weight is zero because its direction is perpendicular to the displacement (compare Fig. 6.4c). For the same reason, the work W_n done by the normal force is also zero. (Note that we don't need to calculate the magnitude n to conclude this.) So $W_w = W_n = 0$.

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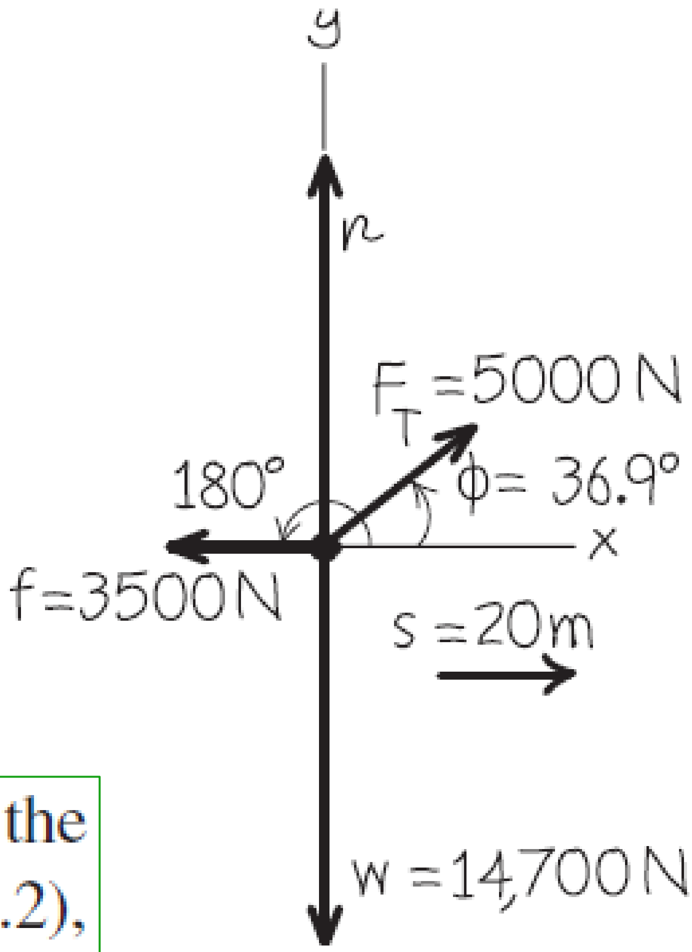
Solution by approach 1:

- Compute work done by individual forces first and algebraic sum of them is the total work

That leaves the work W_T done by the force F_T exerted by the tractor and the work W_f done by the friction force f. From Eq. (6.2),

$$W_T = F_T s \cos\phi = (5000 \text{ N})(20 \text{ m})(0.800) = 80{,}000 \text{ N} \cdot \text{m}$$
$$= 80 \text{ kJ}$$



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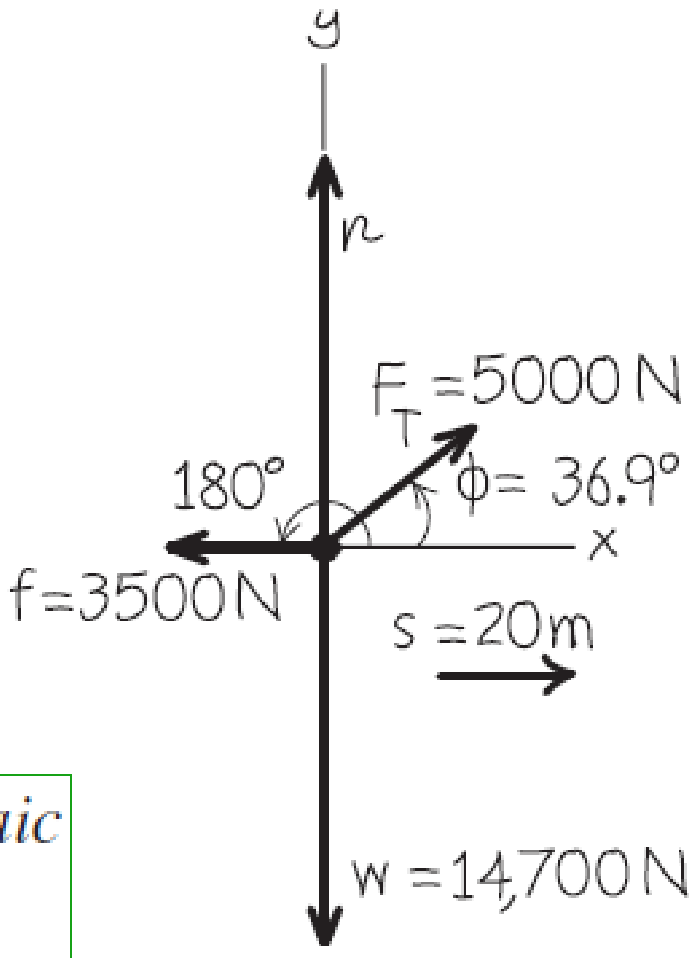

Free-body diagram for sled

The total work W_{tot} done on the sled by all forces is the *algebraic* sum of the work done by the individual forces:

$$W_{tot} = W_w + W_n + W_T + W_f = 0 + 0 + 80\text{ kJ} + (-70\text{ kJ}) = 10\text{ kJ}$$

Example 6.2: Work by Multiple Forces

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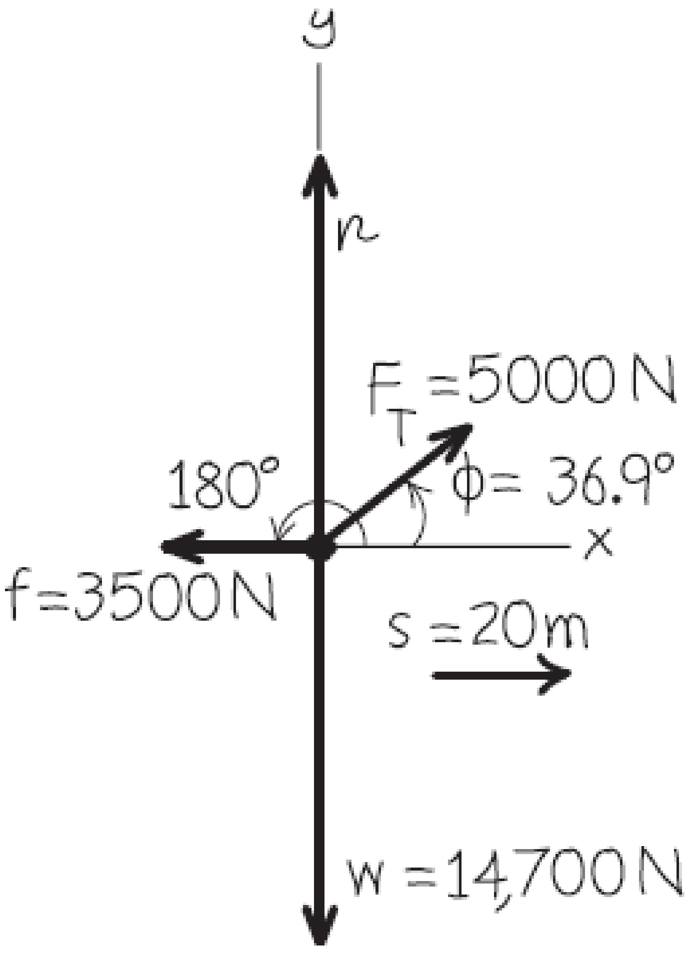
Solution by approach 2:

- Compute the net force first, and then the total work

(2) In the second approach, we first find the *vector* sum of all the forces (the net force) and then use it to compute the total work. The vector sum is best found by using components. From Fig. 6.7b,

$$\sum F_x = F_T \cos\phi + (-f) = (5000\ \text{N}) \cos 36.9° - 3500\ \text{N}$$
$$= 500\ \text{N}$$



Free-body diagram for sled

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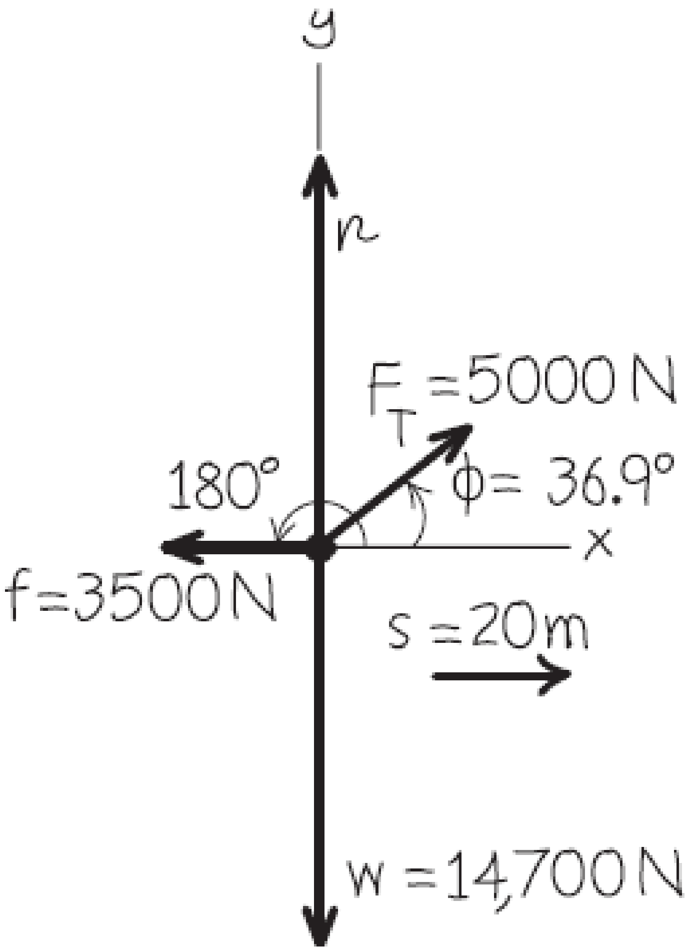
Solution by approach 2:

- Compute the net force first, and then the total work

$$\sum F_y = F_{\mathrm{T}} \sin\phi + n + (-w)$$
$$= (5000\ \mathrm{N}) \sin 36.9° + n - 14{,}700\ \mathrm{N}$$



Free-body diagram for sled

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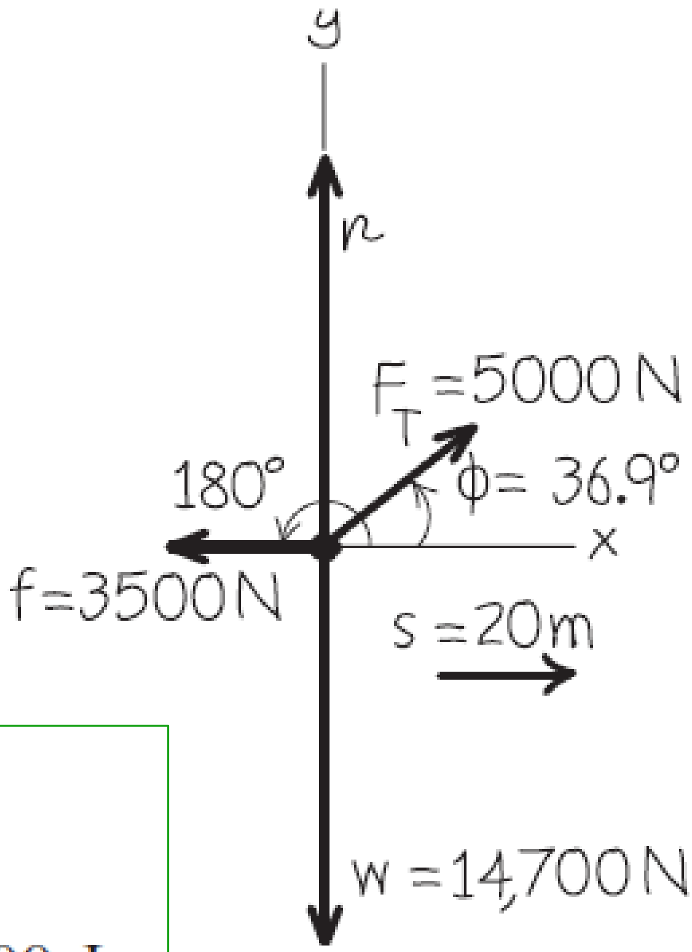
Solution by approach 2:

- Compute the net force first, and then the total work

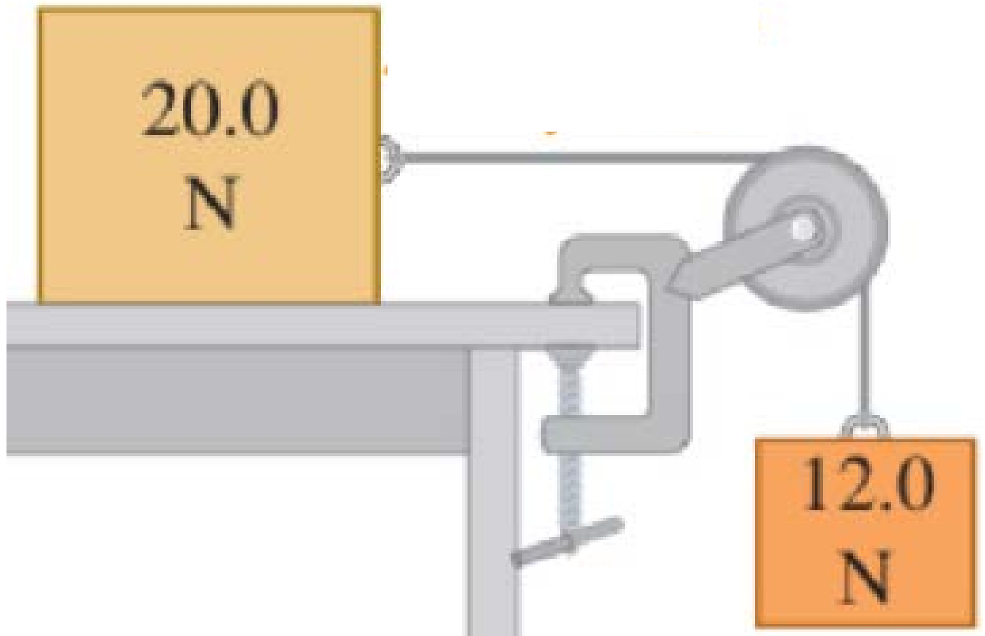
The total work is therefore the work done by the total x-component:

$$W_{\text{tot}} = \left(\sum \vec{F}\right) \cdot \vec{s} = \left(\sum F_x\right)s = (500\ \text{N})(20\ \text{m}) = 10{,}000\ \text{J}$$
$$= 10\ \text{kJ}$$

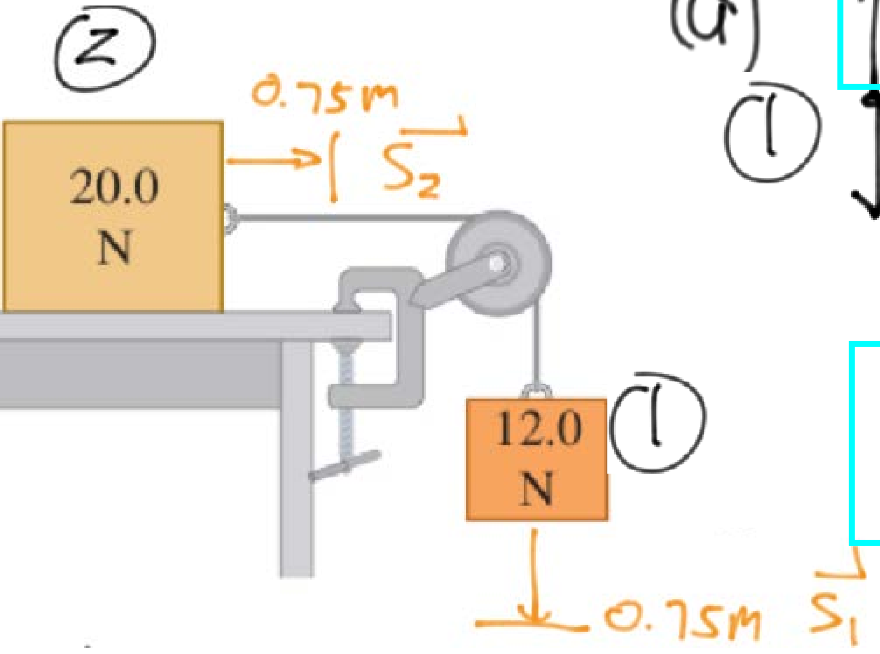


Free-body diagram for sled

6.7 • Two blocks are connected by a very light string passing over a massless and frictionless pulley (Fig. E6.7). Traveling at constant speed, the 20.0-N block moves 75.0 cm to the right and the 12.0-N block moves 75.0 cm downward. During this process, how much work is done (a) on the 12.0-N block by (i) gravity and (ii) the tension in the string? (b) On the 20.0-N block by (i) gravity, (ii) the tension in the string, (iii) friction, and (iv) the normal force? (c) Find the total work done on each block.



6.7 • Two blocks are connected by a very light string passing over a massless and frictionless pulley (Fig. E6.7). Traveling at constant speed, the 20.0-N block moves 75.0 cm to the right and the 12.0-N block moves 75.0 cm downward. During this process, how much work is done (a) on the 12.0-N block by (i) gravity and (ii) the tension in the string



(a) 1 $\boxed{\uparrow \vec{T}_1}$ $\downarrow \vec{w}_1$

$\boxed{\downarrow \vec{s}_1}$

$|\vec{w}_1| = |\vec{T}_1|$

$W_{1,g} = \vec{F}_1 \cdot \vec{s}_1$

$= \vec{w}_1 \cdot \vec{s}_1$

$= w_1 \cdot s_1$

$= 12.0\,N \cdot 0.75\,m$

$= 9.0J$ (i)

$W_{1,T} = \vec{T}_1 \cdot \vec{s}_1$

$\boxed{\hat{T} \cdot \hat{s}_1 = -1}$

$= -T_1 s_1$

$= -12.0N \cdot 0.75m$

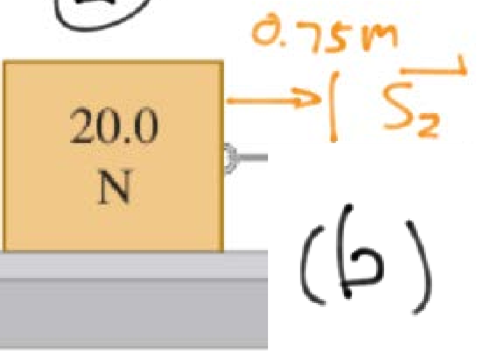
$= -9.0\,J$

(ii)

speed, the 20.0-N block moves 75.0 cm to the right and the 12.0-N block moves 75.0 cm downward. During this process, how much work is done (a) on the 12.0-N block by (i) gravity and (ii) the tension in the string? (b) On the 20.0-N block by (i) gravity, (ii) the tension in the string, (iii) friction, and (iv) the normal force? (c) Find the total work done on each block.

(2)

Again, along a string: $|T_2| = |T_1|$

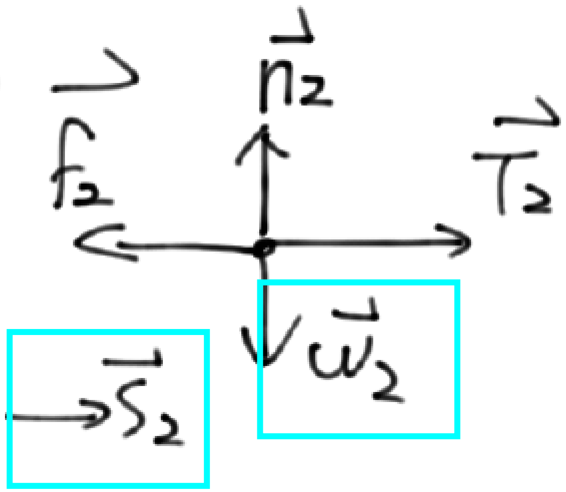


90° degrees

$$\boxed{(i)\quad \vec{w}_2 \cdot \vec{S}_2 = 0}$$

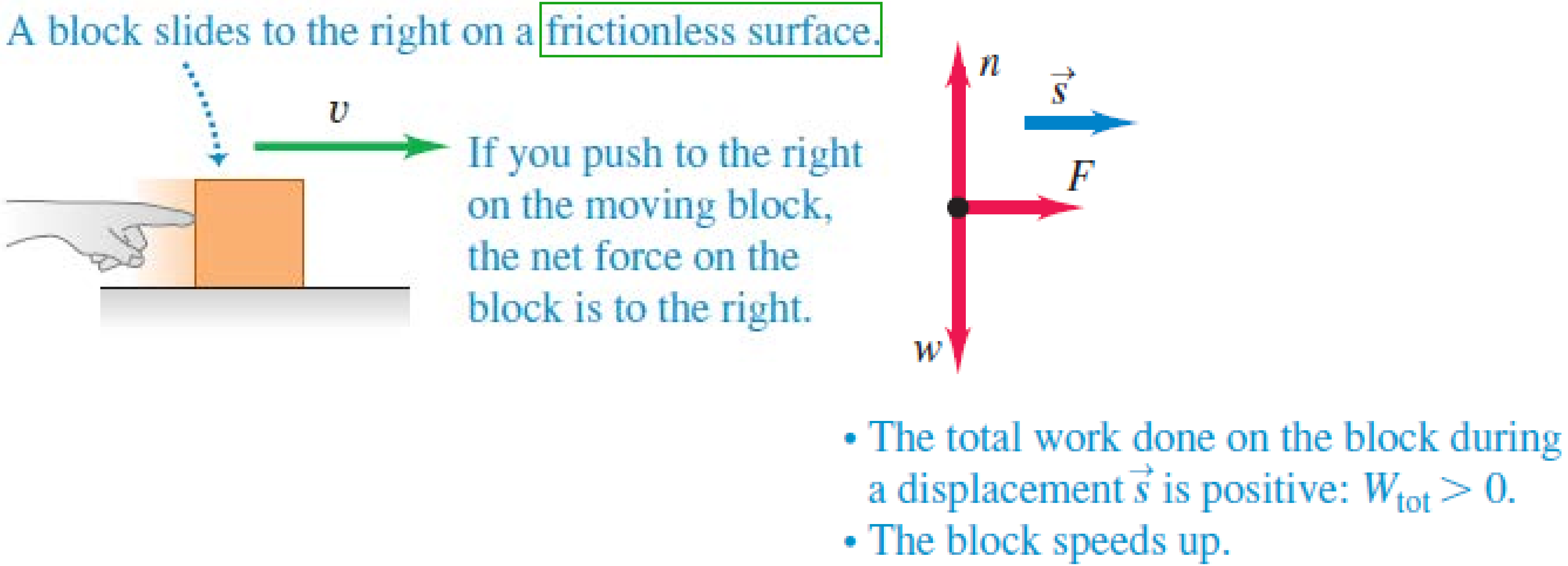
$$\vec{W}_{2.T} = \vec{T}_2 \cdot \vec{S}_2$$
$$= T_2 \cdot S_2 = 12.0\,N \cdot 0.75\,m$$
$$= 9.0\ J = -W_{2\,f}$$

(b)



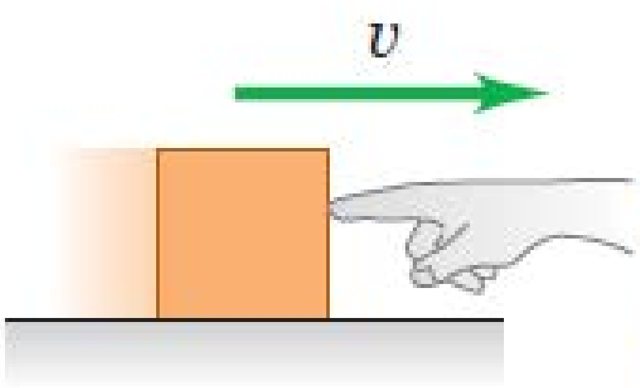
(c) Total work for both: 9.0 J – 9.0 J = 0

Work vs Speed

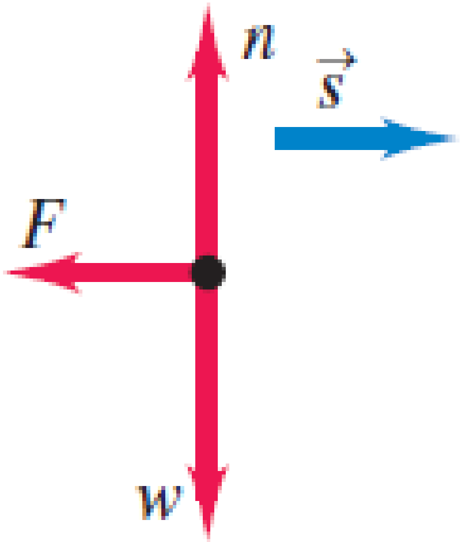


The relationship between the total work done on a body and how the body's speed changes.

Work vs Speed



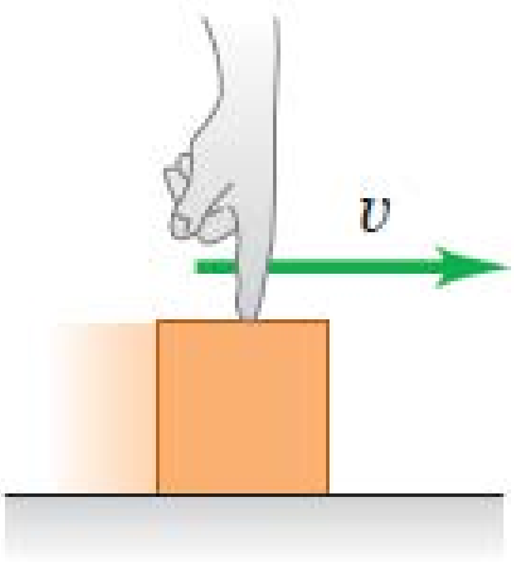
If you push to the left on the moving block, the net force on the block is to the left.



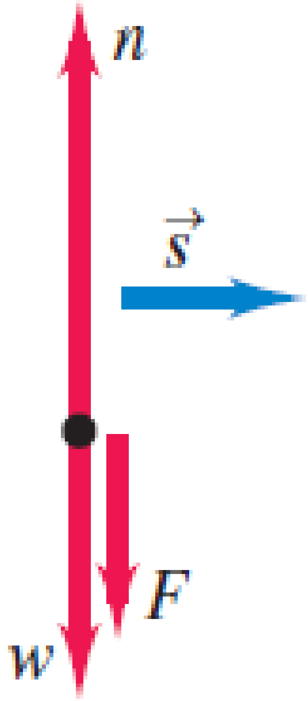
- The total work done on the block during a displacement $\vec{s}$ is negative: $W_{tot} < 0$.
- The block slows down.

The relationship between the total work done on a body and how the body's speed changes.

Work vs Speed



If you push straight down on the moving block, the net force on the block is zero.



- The total work done on the block during a displacement $\vec{s}$ is zero: $W_{tot} = 0$.
- The block's speed stays the same.

The relationship between the total work done on a body and how the body's speed changes.

Work vs Speed: Quantitative

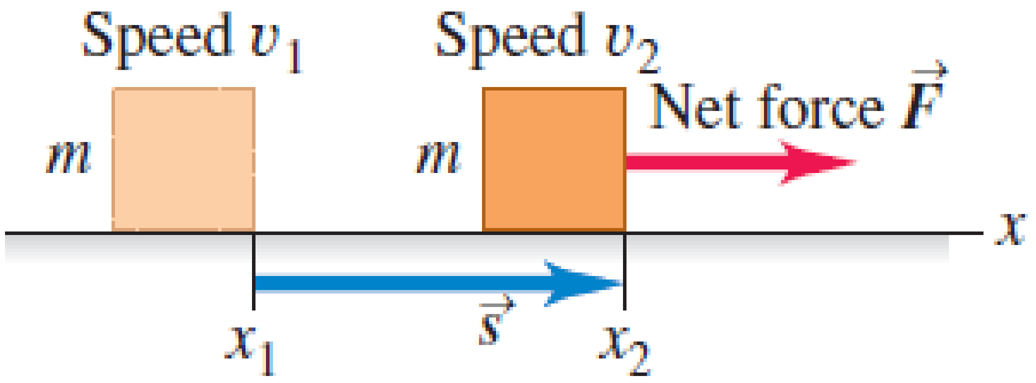
Can we relate work *W* to change of velocity?

Using a constant-acceleration equation,

$$v_2^2 = v_1^2 + 2a_x s$$

$$a_x = \frac{v_2^2 - v_1^2}{2s}$$

6.9 A constant net force $\vec{F}$ does work on a moving body.



When we multiply this equation by m and equate ma_x to the net force F, we find

$$F = ma_x = m\frac{v_2^2 - v_1^2}{2s} \qquad \text{and}$$

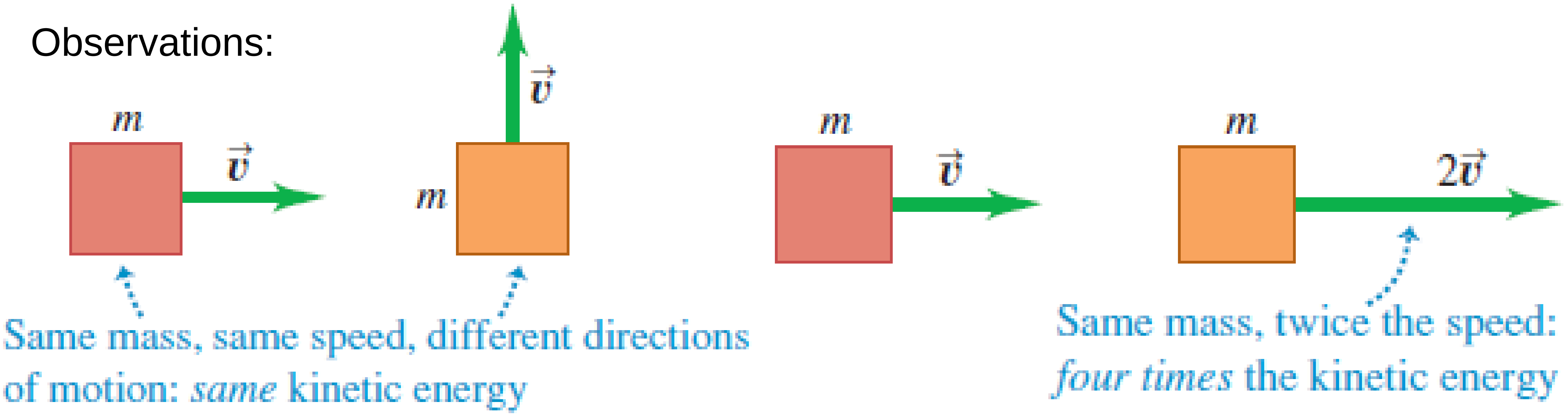
$$Fs = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \qquad (6.4)$$

Definition of Kinetic Energy

The product Fs is the work done by the net force F and thus is equal to the total work W_{tot} done by all the forces acting on the particle. The quantity $\frac{1}{2}mv^2$ is called the **kinetic energy** K of the particle:

$$K = \frac{1}{2}mv^2 \qquad \text{(definition of kinetic energy)} \qquad (6.5)$$

Observations:



Kinetic Energy & Work–Energy Theorem

The work done by the net force on a particle equals the change in the particle's kinetic energy:

$$W_{tot} = K_2 - K_1 = \Delta K \qquad \text{(work–energy theorem)} \qquad (6.6)$$

This result is the **work–energy theorem.**

From Eq. (6.4) or Eq. (6.6), kinetic energy and work must have the same units. Hence the joule is the SI unit of both work and kinetic energy (and, as we will see later, of all kinds of energy). To verify this, note that in SI units the quantity $K = \frac{1}{2}mv^2$ has units $\text{kg} \cdot (\text{m/s})^2$ or $\text{kg} \cdot \text{m}^2/\text{s}^2$; we recall that $1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$, so

$$1\ \text{J} = 1\ \text{N} \cdot \text{m} = 1\ (\text{kg} \cdot \text{m/s}^2) \cdot \text{m} = 1\ \text{kg} \cdot \text{m}^2/\text{s}^2$$

So far we've derived the work–energy theorem for the special case of straight-line motion with constant forces

Problem Solving with Work & Energy

IDENTIFY *the relevant concepts:* The work–energy theorem, $W_{tot} = K_2 - K_1$, is extremely useful when you want to relate a body's speed v_1 at one point in its motion to its speed v_2 at a different point. (It's less useful for problems that involve the *time* it takes a body to go from point 1 to point 2 because the work–energy theorem doesn't involve time at all. For such problems it's usually best to use the relationships among time, position, velocity, and acceleration described in Chapters 2 and 3.)

For exams, first identify WHEN to use work-energy theorem instead of simply Newton's Laws.

For real problems you encounter in your future career, you wouldn't know ahead of time what theories to use!

Problem Solving with Work & Energy

SET UP *the problem* using the following steps:

1. Identify the initial and final positions of the body, and draw a free-body diagram showing all the forces that act on the body.
2. Choose a coordinate system. (If the motion is along a straight line, it's usually easiest to have both the initial and final positions lie along one of the axes.)
3. List the unknown and known quantities, and decide which unknowns are your target variables. The target variable may be the body's initial or final speed, the magnitude of one of the forces acting on the body, or the body's displacement.

Problem Solving with Work & Energy

EXECUTE *the solution:* Calculate the work W done by each force. If the force is constant and the displacement is a straight line, you can use Eq. (6.2) or Eq. (6.3). (Later in this chapter we'll see how to handle varying forces and curved trajectories.) Be sure to check signs; W must be positive if the force has a component in the direction of the displacement, negative if the force has a component opposite to the displacement, and zero if the force and displacement are perpendicular.

Add the amounts of work done by each force to find the total work W_{tot}. Sometimes it's easier to calculate the vector sum of the forces (the net force) and then find the work done by the net force; this value is also equal to W_{tot}.

Problem Solving with Work & Energy

Write expressions for the initial and final kinetic energies, K_1 and K_2. Note that kinetic energy involves *mass*, not *weight*; if you are given the body's weight, use $w = mg$ to find the mass.

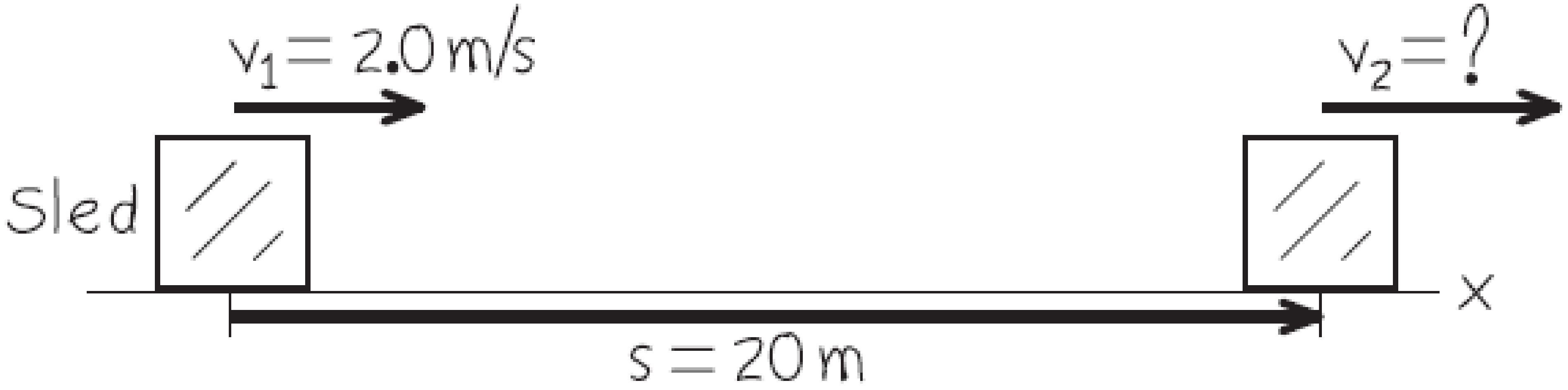
Finally, use Eq. (6.6), $W_{tot} = K_2 - K_1$, and Eq. (6.5), $K = \frac{1}{2}mv^2$, to solve for the target variable. Remember that the right-hand side of Eq. (6.6) represents the change of the body's kinetic energy between points 1 and 2; that is, it is the *final* kinetic energy minus the *initial* kinetic energy, never the other way around. (If you can predict the sign of W_{tot}, you can predict whether the body speeds up or slows down.)

Problem Solving with Work & Energy

EVALUATE *your answer:* Check whether your answer makes sense. Remember that kinetic energy $K = \frac{1}{2}mv^2$ can never be negative. If you come up with a negative value of K, perhaps you interchanged the initial and final kinetic energies in $W_{tot} = K_2 - K_1$ or made a sign error in one of the work calculations.

Example 6.3: Simplest Example

Let's look again at the sled in Fig. 6.7 and our results from Example 6.2. Suppose the sled's initial speed v_1 is 2.0 m/s. What is the speed of the sled after it moves 20 m?



<u>*From 6.2:*</u>

$$W_{tot} = \left(\sum \vec{F}\right) \cdot \vec{s}$$
$$= 10\ \text{kJ}$$

Example 6.3: Simplest Example

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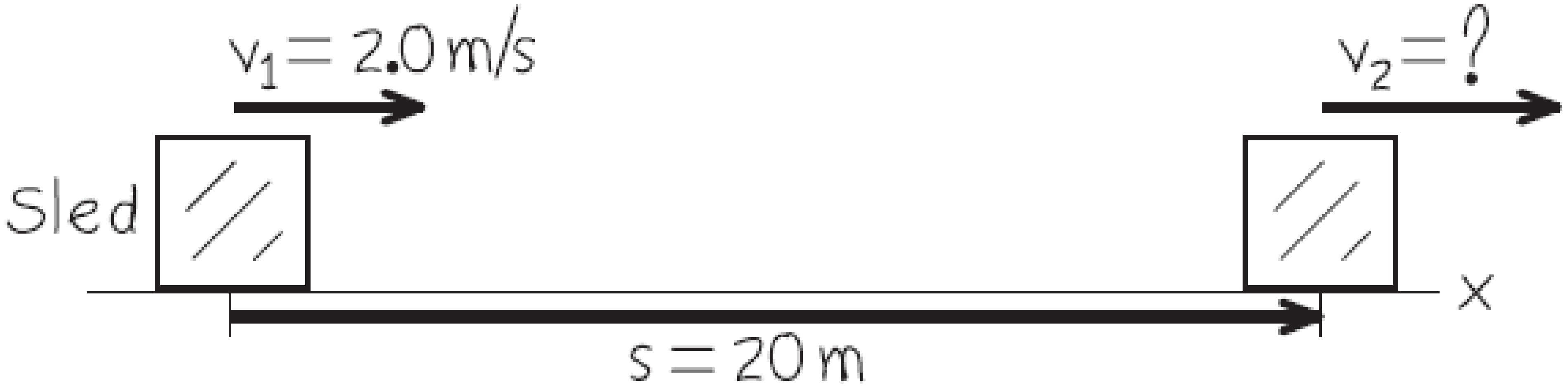


IDENTIFY and SET UP: We'll use the work–energy theorem, Eq. (6.6), $W_{tot} = K_2 - K_1$, since we are given the initial speed $v_1 = 2.0\ \text{m/s}$ and want to find the final speed v_2.

Example 6.3: Simplest Example

Let's look again at the sled in Fig. 6.7 and our results from Example 6.2. Suppose the sled's initial speed v_1 is 2.0 m/s. What is the speed of the sled after it moves 20 m?

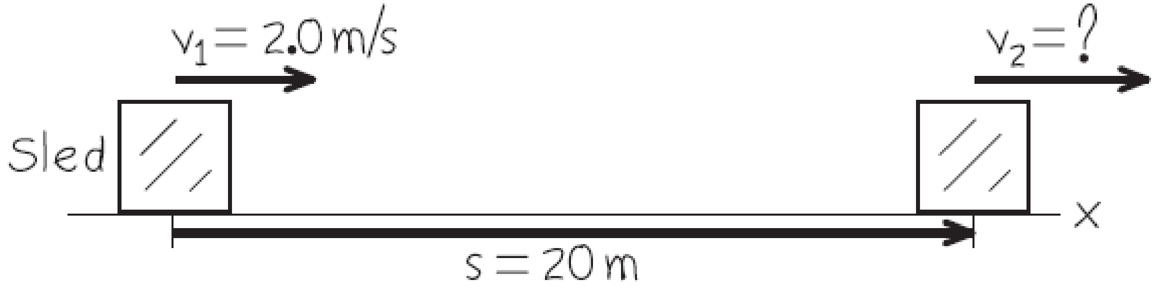
$$W_{tot} = K_2 - K_1 = \Delta K \quad = \tfrac{1}{2} m v_2^2 - \tfrac{1}{2} m v_1^2$$

$$m = \frac{w}{g} = \frac{14{,}700 \text{ N}}{9.8 \text{ m/s}^2} = 1500 \text{ kg}$$

Then the initial kinetic energy K_1 is

$$K_1 = \tfrac{1}{2} m v_1^2 = \tfrac{1}{2}(1500 \text{ kg})(2.0 \text{ m/s})^2$$

$$= 3000 \text{ J}$$



Example 6.3: Simplest Example

Let's look again at the sled in Fig. 6.7 and our results from Example 6.2. Suppose the sled's initial speed v_1 is 2.0 m/s. What is the speed of the sled after it moves 20 m?

Then the initial kinetic energy $K_1 = \frac{1}{2} m v_1^2$

$$= 3000 \text{ J}$$

The final kinetic energy K_2 is

$$\boxed{m = \frac{w}{g} = \frac{14{,}700 \text{ N}}{9.8 \text{ m/s}^2} = 1500 \text{ kg}} \quad K_2 = \frac{1}{2} m v_2^2 = \frac{1}{2}(1500 \text{ kg}) v_2^2$$

Answer:

$$v_2 = 4.2 \text{ m/s}$$

The work–energy theorem, Eq. (6.6), gives

$$K_2 = K_1 + W_{\text{tot}} = 3000 \text{ J} + 10{,}000 \text{ J} = 13{,}000 \text{ J}$$

6.13 •• Animal Energy. BIO Adult cheetahs, the fastest of the great cats, have a mass of about 70 kg and have been clocked running at up to 72 mph (32 m/s). (a) How many joules of kinetic energy does such a swift cheetah have? (b) By what factor would its kinetic energy change if its speed were doubled?

Kinetic energy $K = \frac{1}{2}mv^2$

(a) $m = 70\ \text{kg}$, $v = 32\ \text{m/s}$

$$K = 0.5 \times 70\ \text{kg} \times (32\ \text{m/s})^2$$
$$= 35840\ \text{J}$$

(b) $\text{factor} = \dfrac{K_2}{K_1}$

$$= \frac{\cancel{\frac{1}{2}m}v_2^2}{\cancel{\frac{1}{2}m}v_1^2} = \left(\frac{v_2}{v_1}\right)^2$$

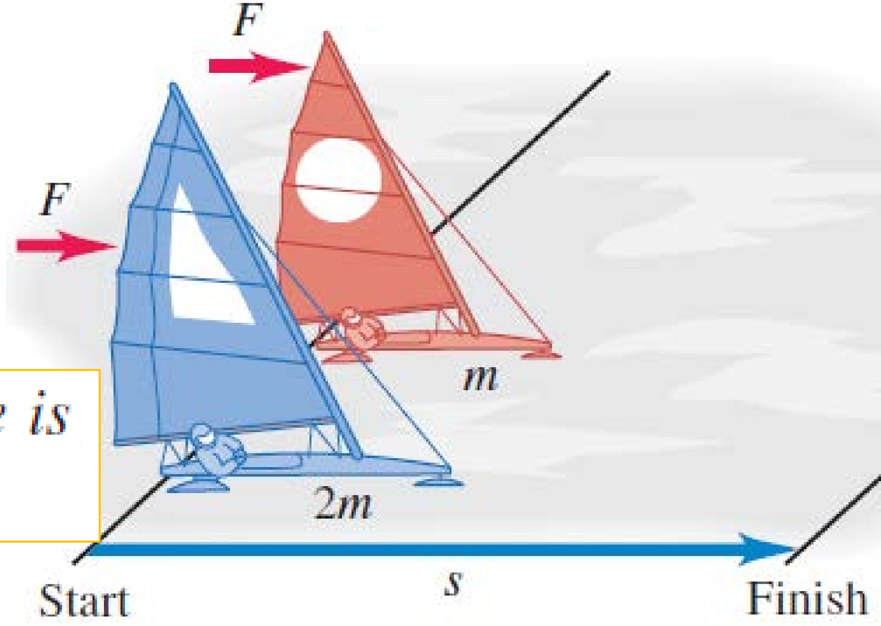
so when v doubles, K increases by 4 times

Example 6.5

Two iceboats like the one in Example 5.6 (Section 5.2) hold a race on a frictionless horizontal lake (Fig. 6.14). The two iceboats have masses m and $2m$. The iceboats have identical sails, so the wind exerts the same constant force $\vec{F}$ on each iceboat. They start from rest and cross the finish line a distance s away. Which iceboat crosses the finish line with greater kinetic energy?

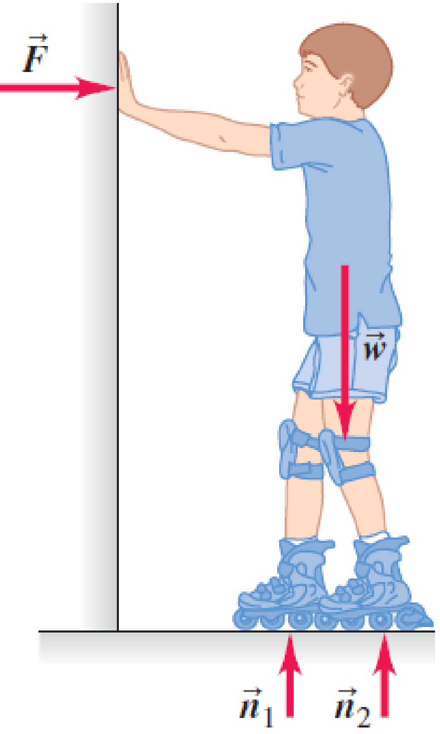
If you use the definition of kinetic energy, $K = \frac{1}{2}mv^2$ the answer to this problem isn't obvious.

The key is to remember that *the kinetic energy of a particle is equal to the total work done to accelerate it from rest.*



Hence the total work done between the starting line and the finish line is the *same* for each iceboat, $W_{tot} = Fs$. *same* kinetic energy at the finish line!

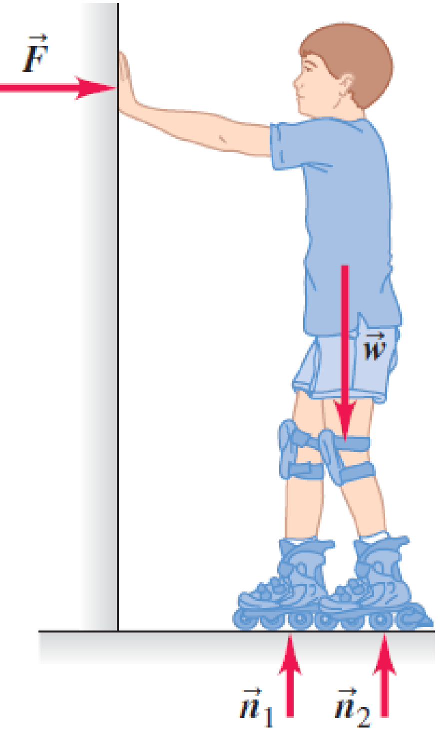
Work & Kinetic Energy in Composite Systems



Suppose a boy stands on frictionless roller skates on a level surface, facing a rigid wall (Fig. 6.15). He pushes against the wall, which makes him move to the right. The forces acting on him are his weight $\vec{w}$, the upward normal forces $\vec{n}_1$ and $\vec{n}_2$ exerted by the ground on his skates, and the horizontal force $\vec{F}$ exerted on him by the wall. There is no vertical displacement, so $\vec{w}$, $\vec{n}_1$, and $\vec{n}_2$ do no work. Force $\vec{F}$ accelerates him to the right, but the parts of his body where that force is applied (the boy's hands) do not move while the force acts. Thus the force $\vec{F}$ also does no work. Where, then, does the boy's kinetic energy come from?

6.15 The external forces acting on a skater pushing off a wall. The work done by these forces is zero, but the skater's kinetic energy changes nonetheless.

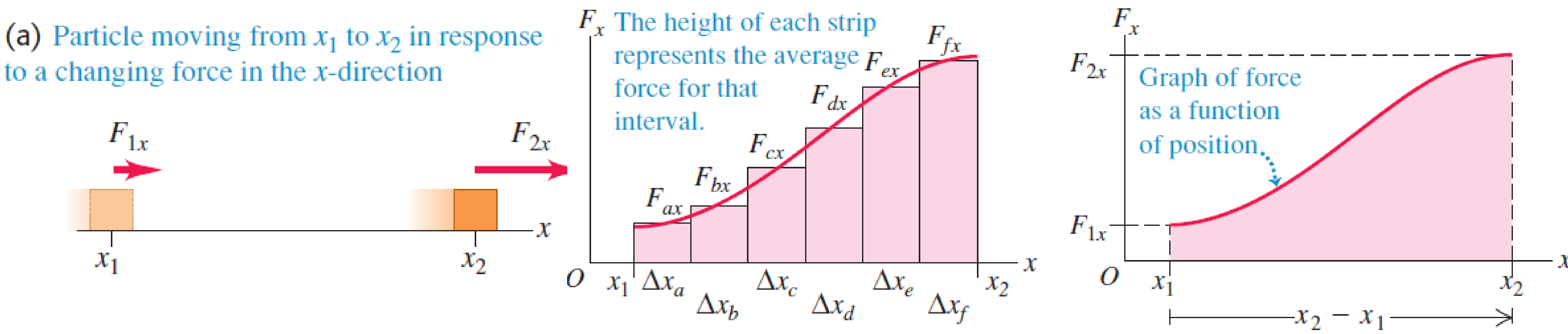
Work & Kinetic Energy in Composite Systems



The explanation is that it's not adequate to represent the boy as a single point mass. Different parts of the boy's body have different motions; his hands remain stationary against the wall while his torso is moving away from the wall. The various parts of his body interact with each other, and one part can exert forces and do work on another part. Therefore the *total* kinetic energy of this *composite* system of body parts can change, even though no work is done by forces applied by bodies (such as the wall) that are outside the system. In Chapter 8 we'll consider further the motion of a collection of particles that interact with each other. We'll discover that just as for the boy in this example, the total kinetic energy of such a system can change even when no work is done on any part of the system by anything outside it.

6.15 The external forces acting on a skater pushing off a wall. The work done by these forces is zero, but the skater's kinetic energy changes nonetheless.

Work and Energy with Varying Forces

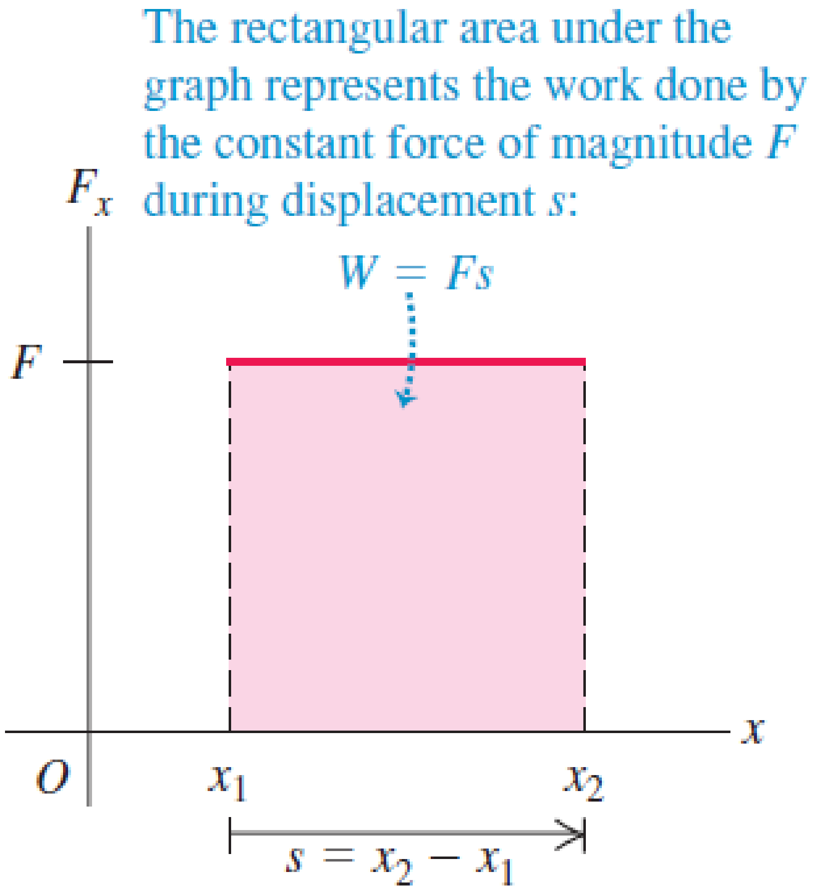


$$\boxed{W = F_{ax}\Delta x_a + F_{bx}\Delta x_b + \cdots}$$

In the limit that the number of segments becomes very large and the width of each becomes very small, this sum becomes the *integral* of F_x from x_1 to x_2:

$$W = \int_{x_1}^{x_2} F_x\, dx \qquad \text{(varying } x\text{-component of force, straight-line displacement)} \qquad (6.7)$$

Special Case: Constant F along Straight Line



$$W = \int_{x_1}^{x_2} F_x \, dx$$

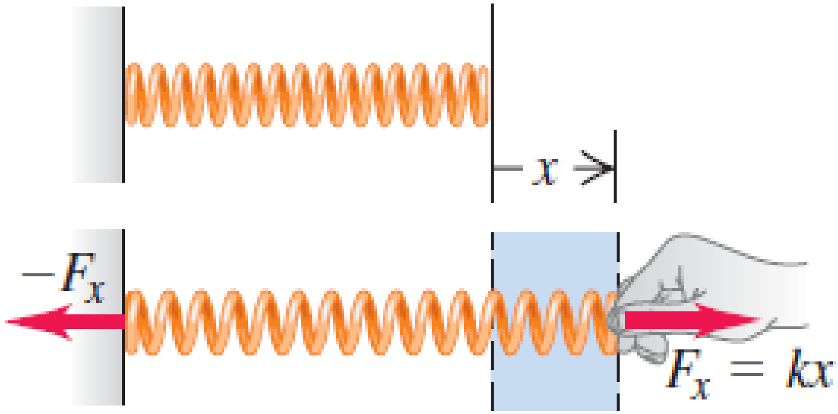
F_x *is a constant with respect to* x

$$= F_x \int_{x_1}^{x_2} dx = F_x(x_2 - x_1)$$

But $x_2 - x_1 = s$, the total displacement of the particle. So in the case of a constant force F, Eq. (6.7) says that $W = Fs$, in agreement with Eq. (6.1).

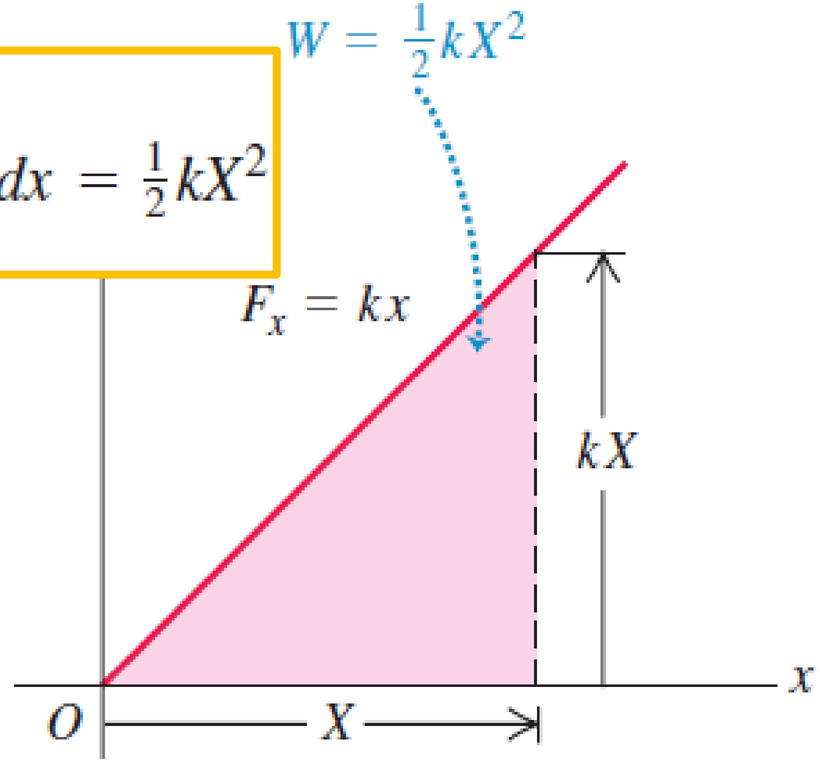
Special Case: Spring

6.18 The force needed to stretch an ideal spring is proportional to the spring's elongation: $F_x = kx$.



$$\boxed{W = \int_0^X F_x\, dx = \int_0^X kx\, dx = \tfrac{1}{2}kX^2}$$

The area under the graph represents the work done on the spring as the spring is stretched from $x = 0$ to a maximum value X:



$$F_x = kx \qquad \text{(force required to stretch a spring)}$$

where k is a constant called the **force constant** (or spring constant) of the spring. The units of k are force divided by distance: N/m in SI units

Special Case: Spring

Equation (6.9) assumes that the spring was originally unstretched. If initially the spring is already stretched a distance x_1, the work we must do to stretch it to a greater elongation x_2 (Fig. 6.20a) is

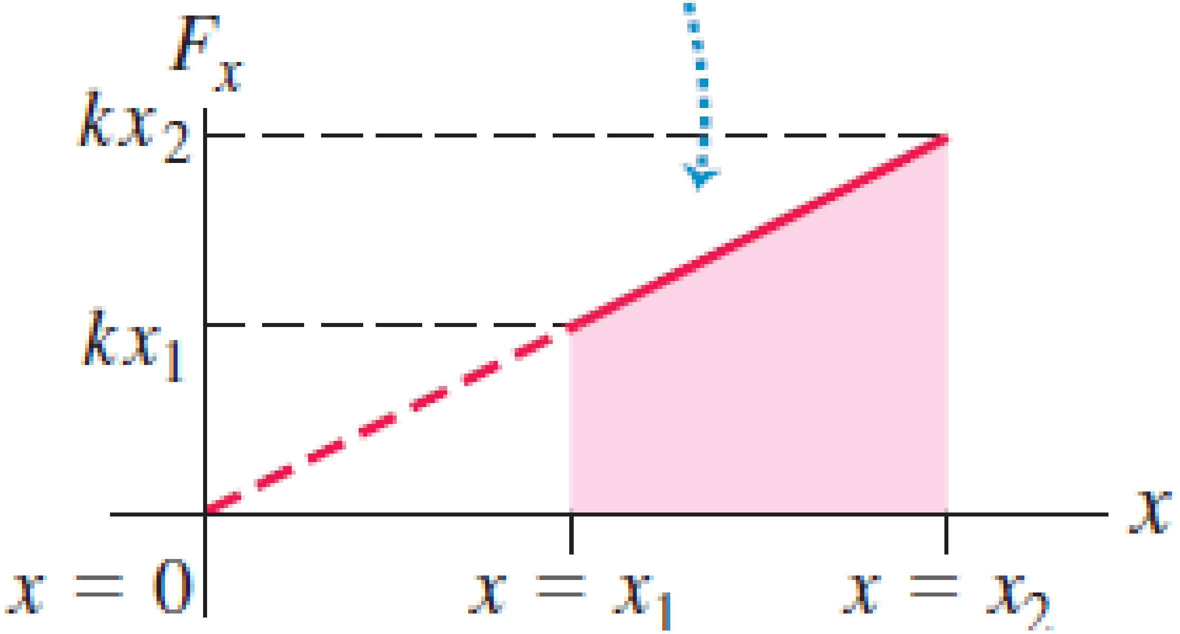
$$W = \int_{x_1}^{x_2} F_x \, dx = \int_{x_1}^{x_2} kx \, dx = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 \qquad (6.10)$$

CAUTION **Work done *on* a spring vs. work done *by* a spring** Note that Eq. (6.10) gives the work that *you* must do *on* a spring to change its length. For example, if you stretch a spring that's originally relaxed, then $x_1 = 0$, $x_2 > 0$, and $W > 0$: The force you apply to one end of the spring is in the same direction as the displacement, and the work you do is positive. By contrast, the work that the *spring* does on whatever it's attached to is given by the *negative* of Eq. (6.10). Thus, as you pull on the spring, the spring does negative work on you. Paying careful attention to the sign of work will eliminate confusion later on!

Special Case: Spring

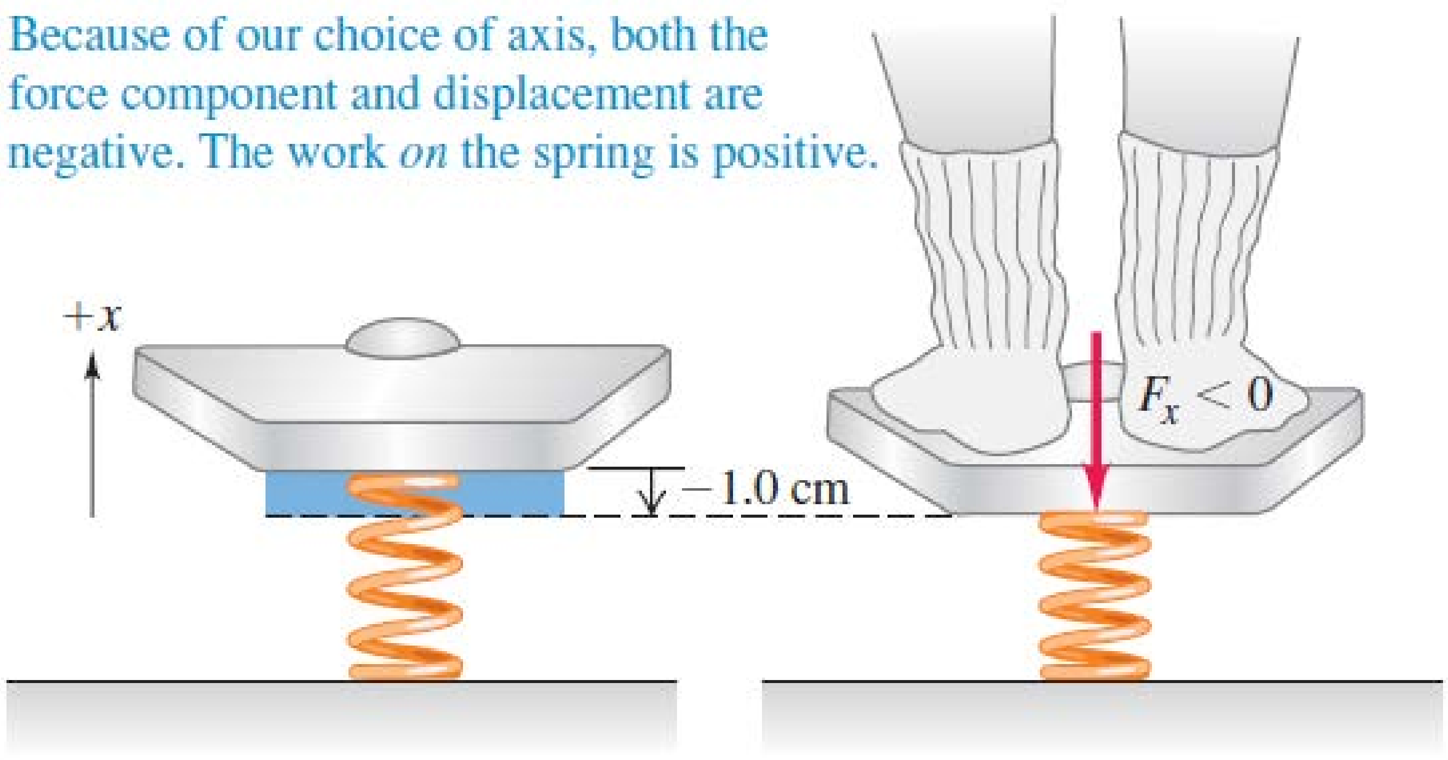
$$W = \int_{x_1}^{x_2} F_x \, dx = \int_{x_1}^{x_2} kx \, dx = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2$$

The trapezoidal area under the graph represents the work done on the spring to stretch it from $x = x_1$ to $x = x_2$: $W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$



Example 6.6: Work done on a spring scale

A woman weighing 600 N steps on a bathroom scale that contains a stiff spring (Fig. 6.21). In equilibrium, the spring is compressed 1.0 cm under her weight. Find the force constant of the spring and the total work done on it during the compression.



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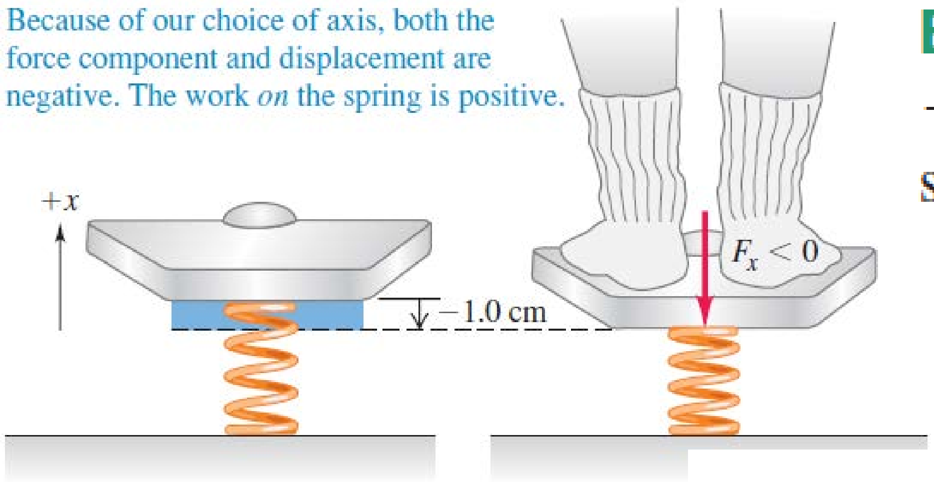


EXECUTE: The top of the spring is displaced by $x = -1.0\text{ cm} = -0.010\text{ m}$, and the woman exerts a force $F_x = -600\text{ N}$ on the spring. From Eq. (6.8) the force constant is then

$$k = \frac{F_x}{x} = \frac{-600\text{ N}}{-0.010\text{ m}} = 6.0 \times 10^4\text{ N/m}$$

Then, using $x_1 = 0$ and $x_2 = -0.010\text{ m}$ in Eq. (6.10), we have

$$W = \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(6.0 \times 10^4\text{ N/m})(-0.010\text{ m})^2 - 0 = 3.0\text{ J}$$

Work–Energy Theorem for Straight-Line Motion, Varying Forces

Derive W_{total} vs *velocity* change in a general setup :

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x \frac{dv_x}{dx}$$

First go back to definition of acceleration since it is related to force

$$W_{\text{tot}} = \int_{x_1}^{x_2} F_x \, dx = \int_{x_1}^{x_2} m a_x \, dx = \int_{x_1}^{x_2} m v_x \frac{dv_x}{dx} \, dx$$

Now $(dv_x/dx)dx$ is the change in velocity dv_x during the displacement dx, so in Eq. (6.12) we can substitute dv_x for $(dv_x/dx)\,dx$.

Work–Energy Theorem for Straight-Line Motion, Varying Forces

Derive W_{total} vs *velocity* change in a general setup :

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x\frac{dv_x}{dx}$$

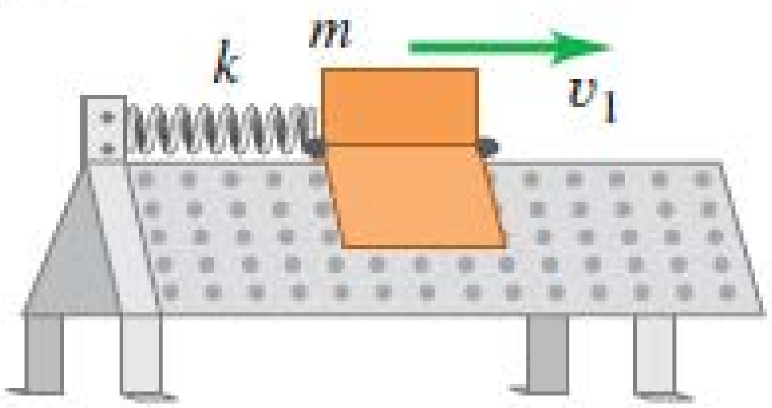
First go back to definition of acceleration since it is related to force

$$W_{\text{tot}} = \int_{x_1}^{x_2} F_x\,dx = \int_{x_1}^{x_2} ma_x\,dx = \int_{x_1}^{x_2} mv_x\frac{dv_x}{dx}\,dx$$

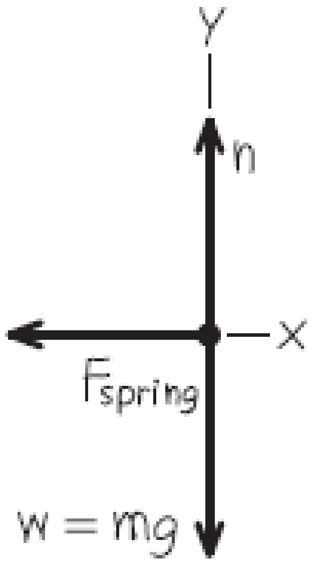
so $W_{\text{tot}} = \displaystyle\int_{v_1}^{v_2} mv_x\,dv_x$ ➡ $W_{\text{tot}} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

Example 6.7: Spring and *Air-Track*

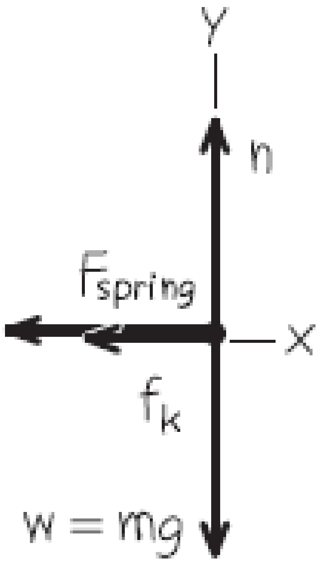
An air-track glider of mass 0.100 kg is attached to the end of a horizontal air track by a spring with force constant 20.0 N/m (Fig. 6.22a). Initially the spring is unstretched and the glider is moving at 1.50 m/s to the right. Find the maximum distance d that the glider moves to the right (a) if the air track is turned on, so that there is no friction, and (b) if the air is turned off, so that there is kinetic friction with coefficient $\mu_k = 0.47$.



(b) No friction



(c) With friction



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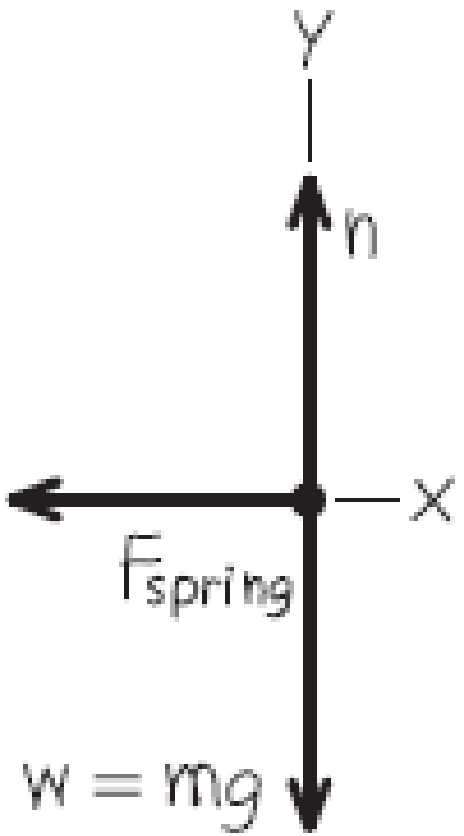
No friction



EXECUTE: (a) Equation (6.10) says that as the glider moves from $x_1 = 0$ to $x_2 = d$, it does an amount of work $W = \frac{1}{2}kd^2 - \frac{1}{2}k(0)^2 = \frac{1}{2}kd^2$ on the spring.

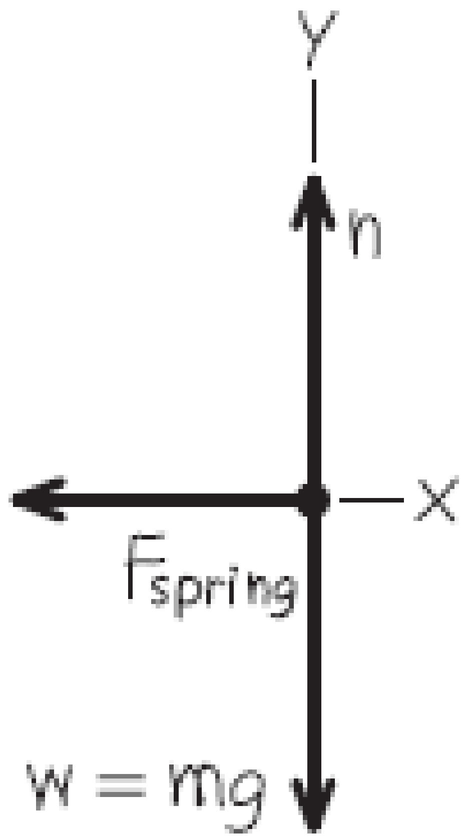
d is the distance when v = 0 or K = 0

The work done __by__ the spring equals all kinetic energy $\frac{1}{2}mv^2$

Example 6.7: Spring and *Air-Track*

An air-track glider of mass 0.100 kg is attached to the end of a horizontal air track by a spring with force constant 20.0 N/m (Fig. 6.22a). Initially the spring is unstretched and the glider is moving at 1.50 m/s to the right. Find the maximum distance d that the glider moves to the right (a) if the air track is turned on, so that there is no friction, and (b) if the air is turned off, so that there is kinetic friction with coefficient $\mu_k = 0.47$.

No friction



EXECUTE: (a) Equation (6.10) says that as the glider moves from $x_1 = 0$ to $x_2 = d$, it does an amount of work $W = \frac{1}{2}kd^2 - \frac{1}{2}k(0)^2 = \frac{1}{2}kd^2$ on the spring.

$$\boxed{-\frac{1}{2}kd^2 = 0 - \frac{1}{2}mv_1{}^2}$$

Example 6.7: Spring and *Air-Track*

$$\boxed{-\tfrac{1}{2}kd^2 = 0 - \tfrac{1}{2}m{v_1}^2}$$

We solve for the distance d the glider moves:

$$d = v_1\sqrt{\frac{m}{k}} = (1.50\ \text{m/s})\sqrt{\frac{0.100\ \text{kg}}{20.0\ \text{N/m}}}$$

$$= 0.106\ \text{m} = 10.6\ \text{cm}$$

Example 6.7: Spring and *Air-Track*

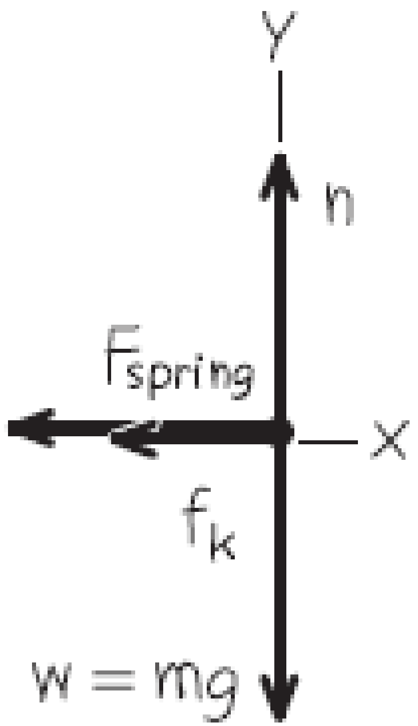
(b) If the air is turned off, we must include the work done by the kinetic friction force. The normal force n is equal in magnitude to the weight of the glider, since the track is horizontal and there are no other vertical forces. Hence the kinetic friction force has constant magnitude $f_k = \mu_k n = \mu_k mg$. The friction force is directed opposite to the displacement, so the work done by friction is

$$W_{\text{fric}} = f_k d \cos 180° = -f_k d = -\mu_k mgd$$

The total work is the sum of W_{fric} and the work done by the spring, $-\frac{1}{2}kd^2$. The work–energy theorem then says that

$$-\mu_k mgd - \frac{1}{2}kd^2 = 0 - \frac{1}{2}mv_1{}^2$$

(c) With friction



Example 6.7: Spring and *Air-Track*

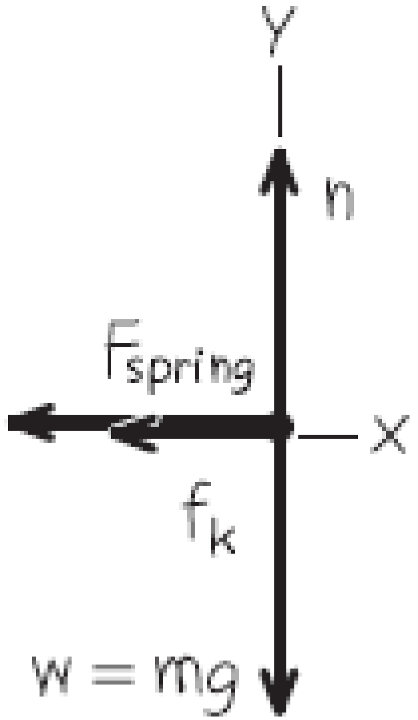
$$-\mu_k mgd - \tfrac{1}{2}kd^2 = 0 - \tfrac{1}{2}mv_1{}^2$$

$$\tfrac{1}{2}kd^2 + \mu_k mgd - \tfrac{1}{2}mv_1{}^2 = 0$$

This is a quadratic equation for d. The solutions are

$$d = -\frac{\mu_k mg}{k} \pm \sqrt{\left(\frac{\mu_k mg}{k}\right)^2 + \frac{mv_1{}^2}{k}}$$

(c) With friction



Complicated, but all variables are known!

Example 6.7: Spring and *Air-Track*

This is a quadratic equation for d. The solutions are

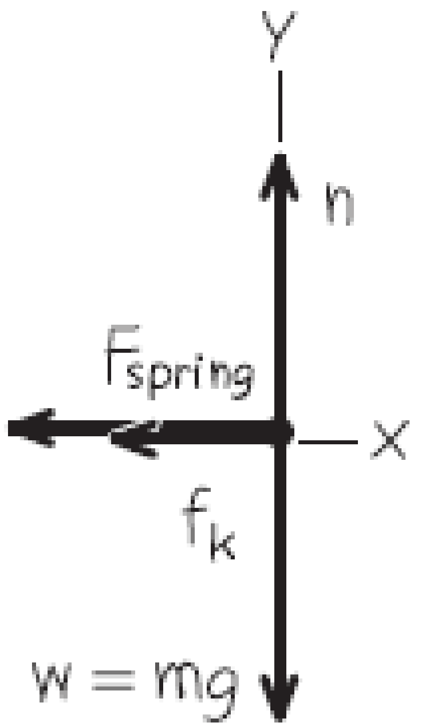
$$d = -\frac{\mu_K mg}{k} \pm \sqrt{\left(\frac{\mu_K mg}{k}\right)^2 + \frac{mv_1^2}{k}}$$

We have

$$\frac{\mu_K mg}{k} = \frac{(0.47)(0.100\ \text{kg})(9.80\ \text{m/s}^2)}{20.0\ \text{N/m}} = 0.02303\ \text{m}$$

$$\frac{mv_1^2}{k} = \frac{(0.100\ \text{kg})(1.50\ \text{m/s})^2}{20.0\ \text{N/m}} = 0.01125\ \text{m}^2$$

(c) With friction



Example 6.7: Spring and *Air-Track*

This is a quadratic equation for d. The solutions are

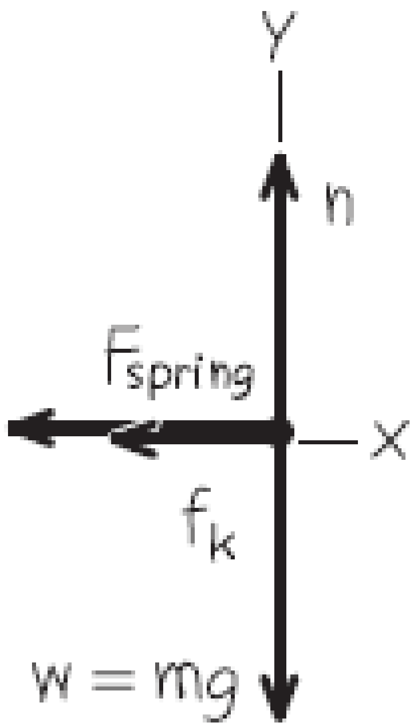
$$d = -\frac{\mu_K mg}{k} \pm \sqrt{\left(\frac{\mu_K mg}{k}\right)^2 + \frac{m{v_1}^2}{k}}$$

so

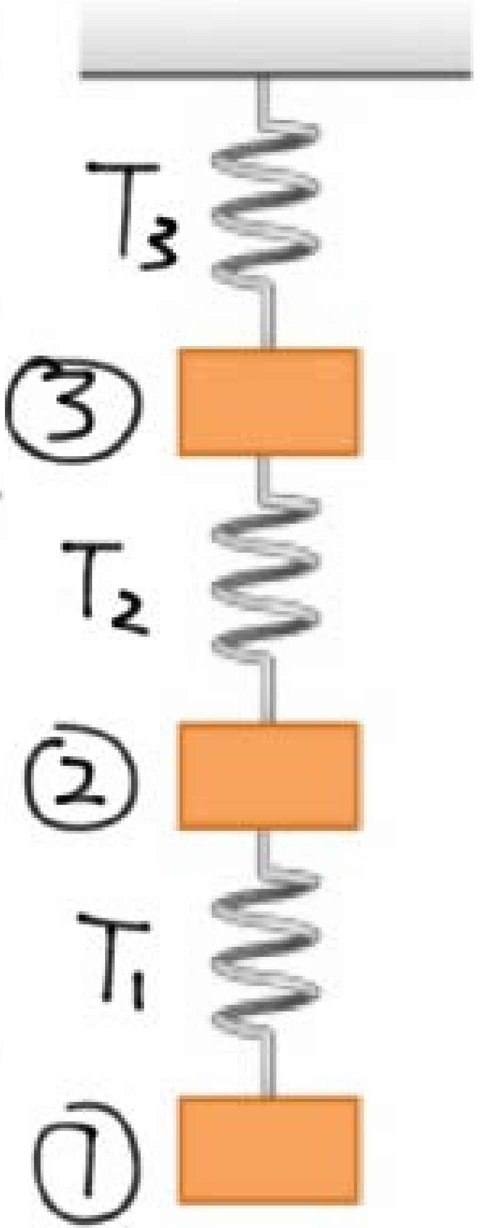
$$d = -(0.02303 \text{ m}) \pm \sqrt{(0.02303 \text{ m})^2 + 0.01125 \text{ m}^2}$$
$$= 0.086 \text{ m} \quad \text{or} \quad -0.132 \text{ m}$$

The quantity d is a positive displacement, so only the positive value of d makes sense. Thus with friction the glider moves a distance $d = 0.086 \text{ m} = 8.6 \text{ cm}$.

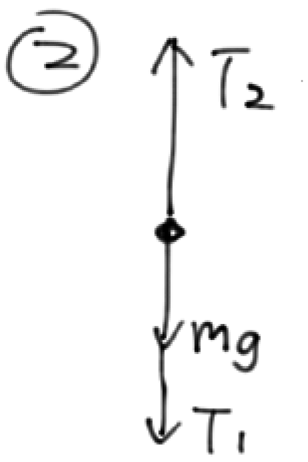
(c) With friction

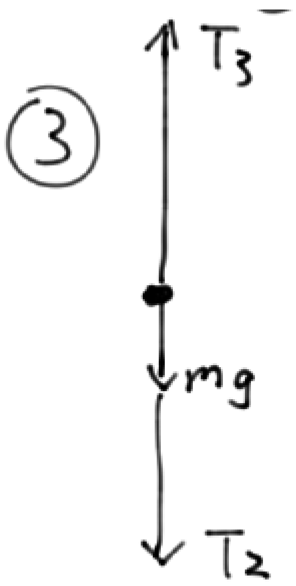


6.33 • Three identical 6.40-kg masses are hung by three identical springs, as shown in Fig. E6.33. Each spring has a force constant of 7.80 kN/m and was 12.0 cm long before any masses were attached to it. (a) Draw a free-body diagram of each mass. (b) How long is each spring when hanging as shown? (*Hint:* First isolate only the bottom mass. Then treat the bottom two masses as a system. Finally, treat all three masses as a system.)



6.33 • Three identical 6.40-kg masses are hung by three identical springs, as shown in Fig. E6.33. Each spring has a force constant of 7.80 kN/m and was 12.0 cm long before any masses were attached to it. (a) Draw a free-body diagram of each mass. (b) How long is each spring when





$T_1 = mg$
$= k \cdot \Delta l_1$

$T_2 = mg + T_1$
$= k \Delta l_2$

$T_3 = mg + T_2$
$= k \cdot \Delta l_3$

(b)

$$\Delta l_1 = \frac{mg}{k} = \frac{6.4\,kg \times 9.8\,m/s^2}{7.8 \times 10^3\,N/m} = 8.04 \times 10^{-3}\,m$$

$$l_1 = l_{1,0} + \Delta l_1 = 0.128\,m$$

$$\Delta l_2 = \frac{T_2}{k} = \frac{mg + T_1}{k} = \frac{2mg}{k}$$

$$l_2 = 0.136\,m$$

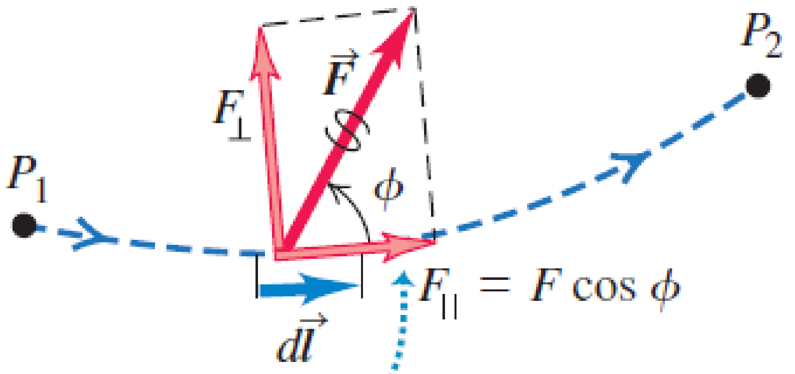
$$\Delta l_3 = \frac{T_3}{k} = \frac{3mg}{k}$$

$$l_3 = 0.144\,m$$

Motion Along a Curve

$$dW = F\cos\phi \, dl = F_{\parallel} \, dl = \vec{F} \cdot d\vec{l}$$

where $F_{\parallel} = F\cos\phi$ is the component of $\vec{F}$ in the direction parallel to $d\vec{l}$ (Fig. 6.23b). The total work done by $\vec{F}$ on the particle as it moves from P_1 to P_2 is



Only the component of $\vec{F}$ parallel to the displacement, $F_{\parallel} = F\cos\phi$, contributes to the work done by $\vec{F}$.

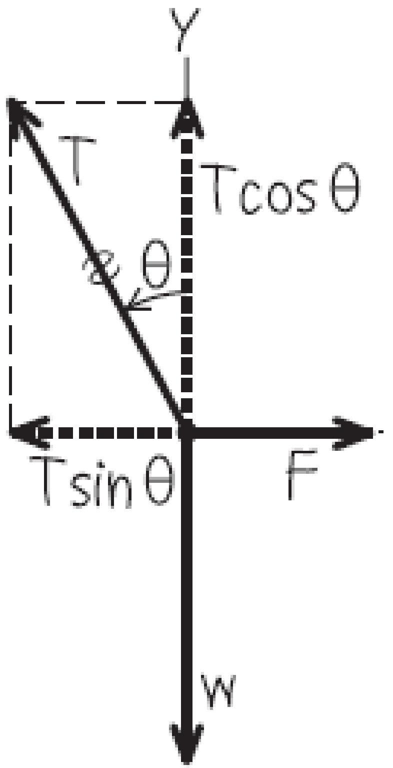
$$W = \int_{P_1}^{P_2} F\cos\phi \, dl = \int_{P_1}^{P_2} F_{\parallel} \, dl = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$$

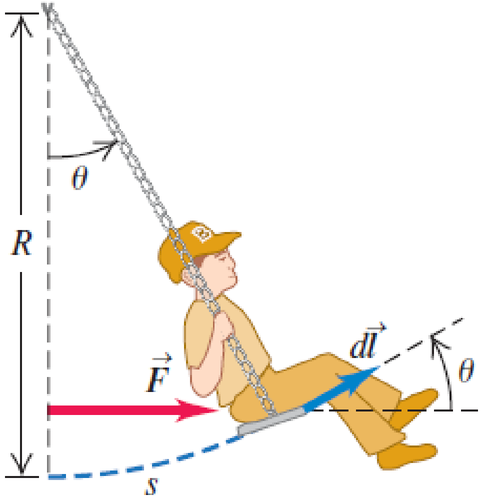
Example 6.8: Motion on a curved path

At a family picnic you are appointed to push your obnoxious cousin Throckmorton in a swing (Fig. 6.24a). His weight is w, the length of the chains is R, and you push Throcky until the chains make an angle θ_0 with the vertical. To do this, you exert a varying horizontal force $\vec{F}$ that starts at zero and gradually increases just enough that Throcky and the swing move very slowly and remain very nearly in equilibrium throughout the process. What is the total work done on Throcky by all forces? What is the work done by the tension T in the chains? What is the work you do by exerting the force $\vec{F}$? (Neglect the weight of the chains and seat.)

Example 6.8: Motion on a curved path

To compute the work done by $\vec{F}$, we need to know how this force varies with the angle θ. The net force on Throcky is zero, so $\sum F_x = 0$ and $\sum F_y = 0$. From Fig. 6.24b,

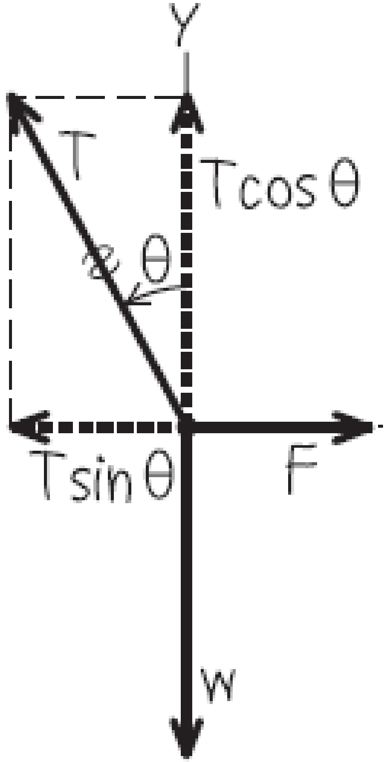


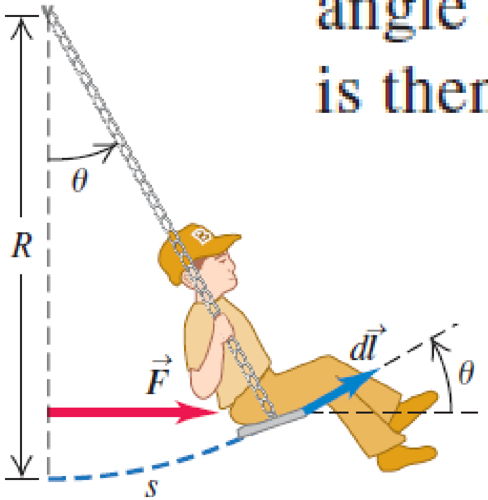


$$\sum F_x = F + (-T\sin\theta) = 0$$
$$\sum F_y = T\cos\theta + (-w) = 0$$

Example 6.8: Motion on a curved path

To compute the work done by $\vec{F}$, we need to know how this force varies with the angle θ. The net force on Throcky is zero, so $\sum F_x = 0$ and $\sum F_y = 0$. From Fig. 6.24b,





The point where $\vec{F}$ is applied moves through the arc s (Fig. 6.24a). The arc length s equals the radius R of the circular path multiplied by the length θ (in radians), so $s = R\theta$. Therefore the displacement $d\vec{l}$ corresponding to a small change of angle $d\theta$ has a magnitude $dl = ds = R\,d\theta$. The work done by $\vec{F}$ is then

$$W = \int \vec{F} \cdot d\vec{l} = \int F\cos\theta\ ds$$

Example 6.8: Motion on a curved path

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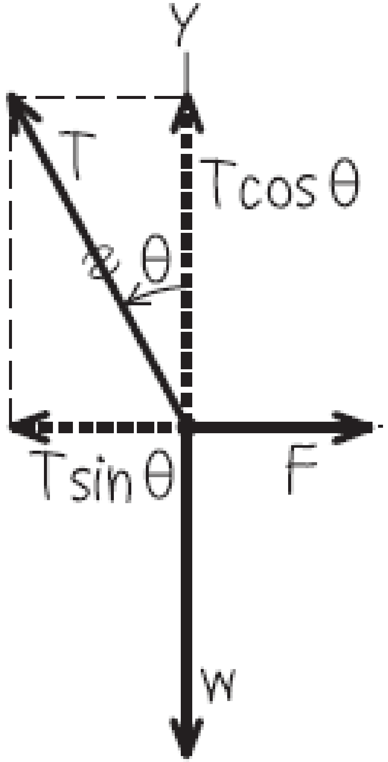
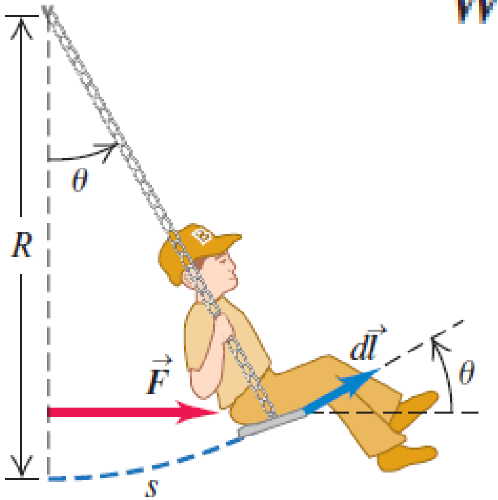




Now we express F and ds in terms of the angle θ, whose value increases from 0 to θ_0:

$$W = \int_0^{\theta_0} (w \tan\theta)\cos\theta \; (R\; d\theta) = wR\int_0^{\theta_0} \sin\theta\, d\theta$$

$$= wR(1 - \cos\theta_0)$$

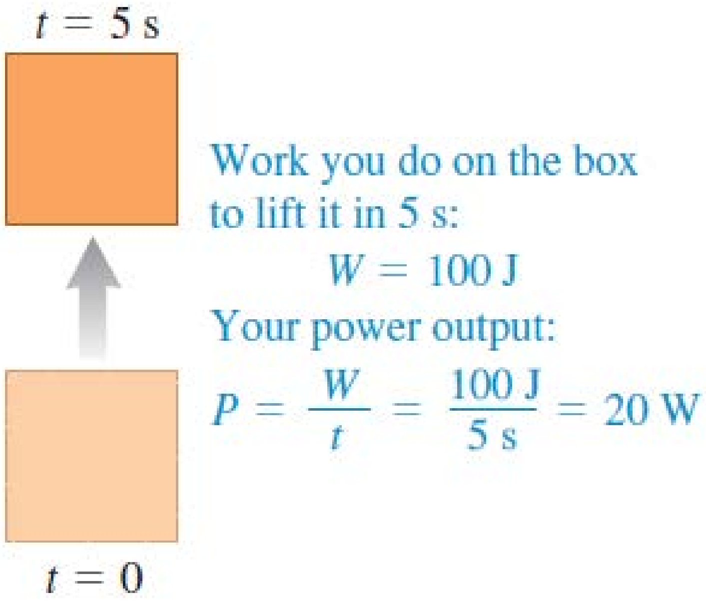
Power: Work Done per Unit Time

$$P_{av} = \frac{\Delta W}{\Delta t} \qquad \text{(average power)} \qquad (6.15)$$

The rate at which work is done might not be constant. We can define **instantaneous power** P as the quotient in Eq. (6.15) as Δt approaches zero:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \qquad \text{(instantaneous power)}$$

$$P = \vec{F} \cdot \vec{v} \qquad \text{(instantaneous rate at which force } \vec{F} \text{ does work on a particle)}$$



The SI unit of power is the **watt** (W), named for the English inventor James Watt. One watt equals 1 joule per second: 1 W = 1 J/s (Fig. 6.25).

Power: Work Done per Unit Time

In mechanics we can also express power in terms of force and velocity. Suppose that a force $\vec{F}$ acts on a body while it undergoes a vector displacement $\Delta\vec{s}$. If $F_\parallel$ is the component of $\vec{F}$ tangent to the path (parallel to $\Delta\vec{s}$), then the work done by the force is $\Delta W = F_\parallel \Delta s$. The average power is

$$P_{\text{av}} = \frac{F_\parallel \Delta s}{\Delta t} = F_\parallel \frac{\Delta s}{\Delta t} = F_\parallel v_{\text{av}} \tag{6.17}$$

Instantaneous power P is the limit of this expression as $\Delta t \to 0$:

$$P = F_\parallel v \tag{6.18}$$

where v is the magnitude of the instantaneous velocity. We can also express Eq. (6.18) in terms of the scalar product:

$$P = \vec{F} \cdot \vec{v} \quad \text{(instantaneous rate at which force } \vec{F} \text{ does work on a particle)} \tag{6.19}$$

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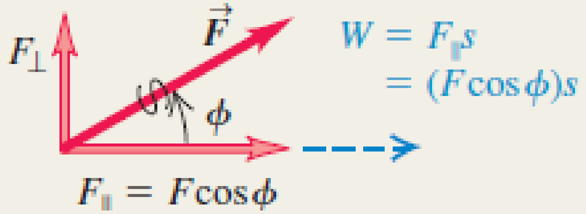
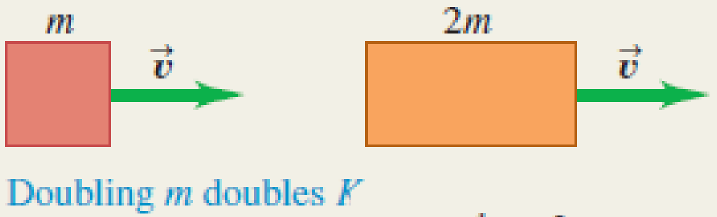
$$P = \vec{F}\cdot\vec{v} \quad \text{(instantaneous rate at which force } \vec{F} \text{ does work on a particle)} \tag{6.19}$$

Summary

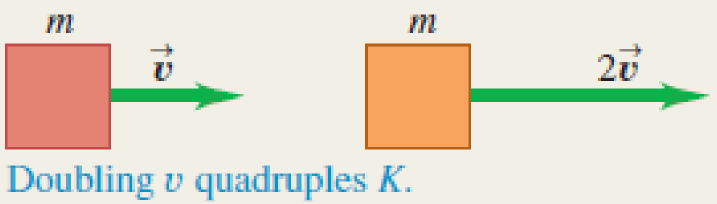
$$W = \vec{F} \cdot \vec{s} = Fs\cos\phi \qquad (6.2), (6.3)$$

ϕ = angle between $\vec{F}$ and $\vec{s}$



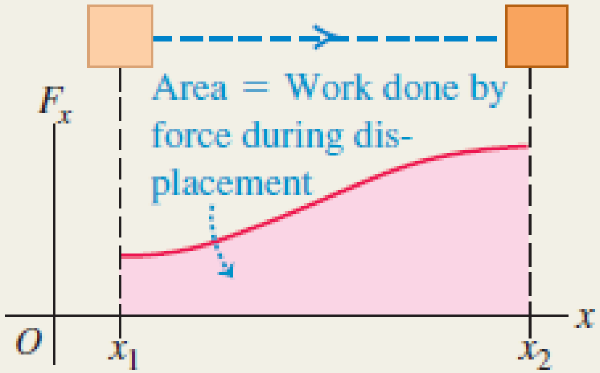


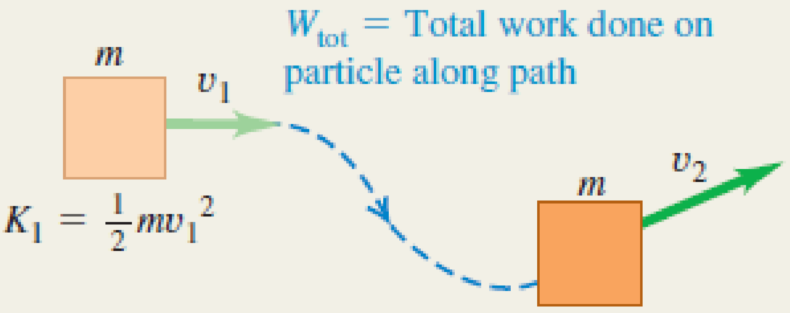
$$K = \tfrac{1}{2}mv^2$$



$$W = \int_{x_1}^{x_2} F_x\,dx \qquad (6.7)$$

$$W = \int_{P_1}^{P_2} F\cos\phi\,dl = \int_{P_1}^{P_2} F_\parallel\,dl = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l} \qquad (6.14)$$





$$W_{tot} = K_2 - K_1 = \Delta K \qquad K_2 = \tfrac{1}{2}mv_2^2 = K_1 + W_{tot}$$

$$P_{av} = \frac{\Delta W}{\Delta t} \qquad (6.15)$$

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \qquad (6.16)$$

$$P = \vec{F} \cdot \vec{v} \qquad (6.19)$$