CALCULUS

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- This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called ***partial fractions***, which are easily integrated. For instance, the rational function $\frac{5x-3}{x^2-2x-3}$ can be rewritten as

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}.$$

To integrate the rational function $\frac{5x-3}{x^2-2x-3}$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx = 2\ln|x+1| + 3\ln|x-3| + C$$

① General Description of the Method

- Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:
  - The degree of $f(x)$ must be less than the degree of $g(x)$, i.e., the function must be proper. If it isn't, divide $f(x)$ by g(x) and work with the remainder term.
  - We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.
- Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of $g(x)$ are known.

(a) Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m} + \cdots.$$

Example 1 Use partial fractions to evaluate

$$\int \frac{6x+9}{(x+2)^2}\,dx$$

Skill Practice 1 Evaluate

$$\int \frac{3x}{(x-1)^2}\,dx$$

(b) If $g(x)$ has a factor $(x^2+px+q)^n$, where x^2+px+q is an irreducible quadratic factor of $g(x)$ so that it has no real roots, then

$$\frac{f(x)}{g(x)} = \frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n} + \cdots.$$

Example 2 Use partial fractions to evaluate

$$\int \frac{5x\,dx}{(x-1)(x^2+2x+2)}$$

Skill Practice 2 Evaluate

$$\int \frac{x+1}{x(x^2+1)}\,dx$$

(c) When $f(x)/g(x)$ is an improper fraction, meaning that the degree of $f(x)$ is larger than the degree of $g(x)$, we shall decompose it to be a polynomial plus a proper fraction.

Example 3 Evaluate the improper fraction

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$$

Skill Practice 3 Evaluate the improper fraction

$$\int \frac{x^3 + 3x}{x^2 + 1} dx$$

(d) When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

Example 4 Find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Skill Practice 4 Evaluate

$$\int \frac{x+3}{x^3-2x^2-3x}\,dx$$

② Other Ways to Determine the Coefficients

- Another way to determine the constants that appear in partial fractions is to differentiate, and assign selected numerical values to x.

Example 5 Find A, B, and C in the expansion

$$\frac{x-1}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

by clearing fractions, differentiating the result, and substituting $x=-1$.

Skill Practice 5 Evaluate

$$\int \frac{x^2+x}{(x-1)^3}\,dx$$

- In some problems, assigning small values to x, such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

Example 6 Evaluate

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx$$

by assigning numerical values to x.

Skill Practice 6 Evaluate

$$\int \frac{dx}{x(x^2+1)^2}$$

Skill Practice 7 Use partial fractions to evaluate

$$\int_2^5 \frac{4}{x^2+2x-3}\,dx$$

Skill Practice 8 Use partial fractions to evaluate

$$\int_2^3 \frac{x^3}{x^2-2x+1}\,dx$$

Skill Practice 9 Evaluate

$$\int_3^8 \frac{1}{x\sqrt{x+1}}\,dx$$