CALCULUS

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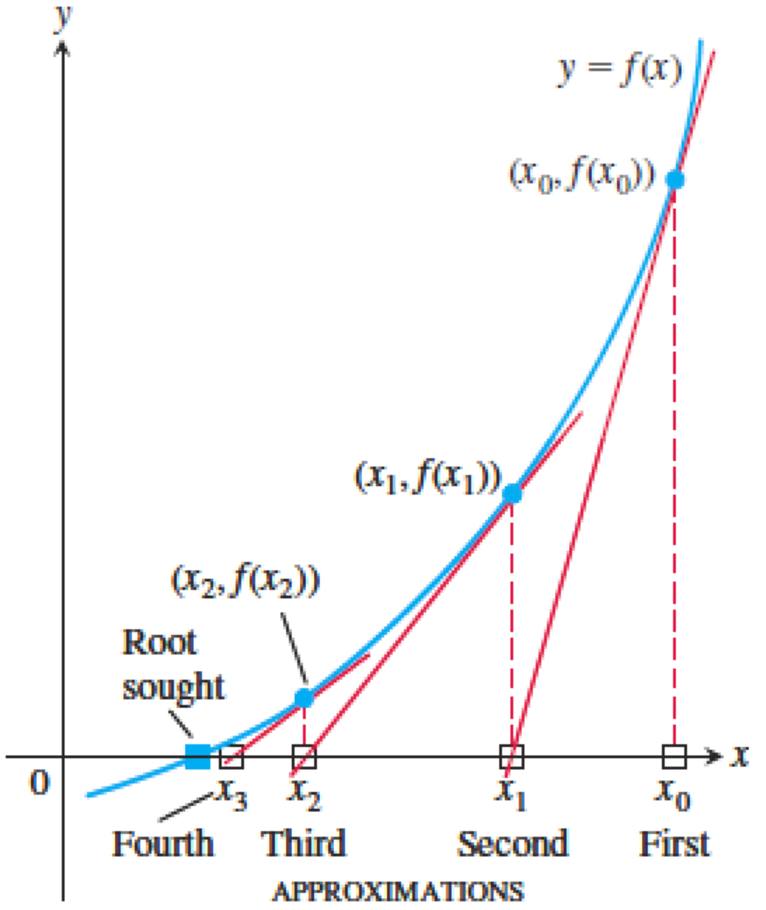
Newton's Method

- For thousands of years, one of the main goals of mathematics has been to find solutions to equations. For linear equations ($ax + b = 0$), and for quadratic equations ($ax^2 + bx + c = 0$), we can explicitly solve for a solution. However, for most equations there is no simple formula that gives the solutions.
- In this section we study a numerical method called **Newton's method** or the **Newton–Raphson method**, which is a technique to approximate the solutions to an equation $f(x) = 0$. Newton's method estimates the solutions using tangent lines of the graph of $y = f(x)$ near the points where f is zero. Newton's method is both powerful and efficient, and it has numerous applications in engineering and other fields where solutions to complicated equations are needed.

① Procedure for Newton's Method

- The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution.
- The initial estimate, x_0, may be found by just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling the point x_1 where the tangent meets the x-axis.
- The number x_1 is usually a better approximation to the solution than is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x-axis is the next approximation in the sequence.
- It continues, using each approximation to generate the next, until it is close enough to the root to stop.



- A formula can be derived for generating the successive approximations in the following way. Given the approximation x_n, the point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)(x - x_n)$$

- We can find where it crosses the x-axis by setting y = 0.

$$0 = f(x_n) + f'(x_n)(x - x_n) \quad \Rightarrow \quad x = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{if} \quad f'(x_n) \neq 0$$

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.

2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{if} \quad f'(x_n) \neq 0.$$

② Applying Newton's Method

EXAMPLE 1

Find the positive root of the equation

$$f(x) = x^2 - 2 = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{x_n}{2} + \frac{1}{x_n}$$

$\sqrt{2} = 1.4142136$	Error
$x_0 = 1$	–0.4142136
$x_1 = 3/2 = 1.5$	0.0857864
$x_2 = 17/12 = 1.4166667$	0.0024541
$x_3 = 577/408 = 1.4142157$	0.0000021

EXAMPLE 2

Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{2x_n^3 + 1}{3x_n^2 - 1}$$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	−1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	−1.8672E-13	4.2646 32999	1.3247 17957

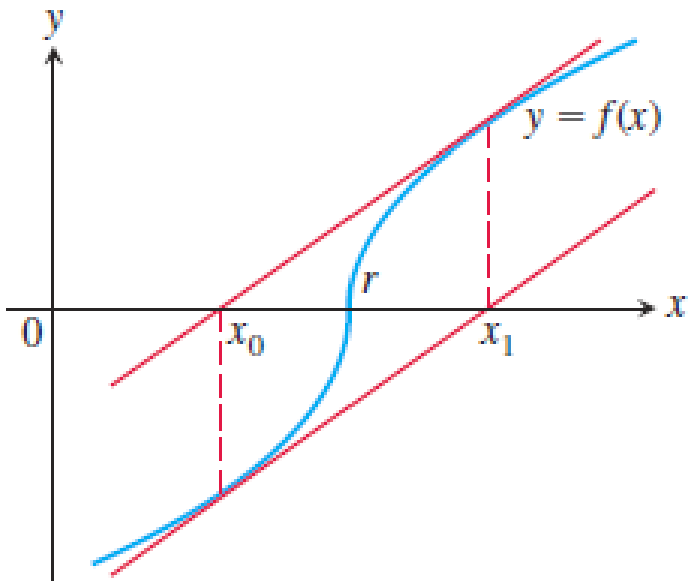
③ Convergence of the Approximations

- In practice, Newton's method usually gives convergence with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for x_0. You can test that you are getting closer to a zero of the function by checking that $|f(x_n)|$ is approaching zero, and you can check that the approximations are converging by evaluating $|x_n - x_{n+1}|$.

- *Newton's method does not always converge*. For instance,

$$f(x) = \begin{cases} -\sqrt{r-x}\,, & x < r \\ \sqrt{x-r}\,, & x \geq r \end{cases}$$

It begins with $x_0 = r - h$, then $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.



- *If Newton's method does converge, it converges to a root*. However, it may not be the root you have in mind. The Figure below shows two ways this can happen.

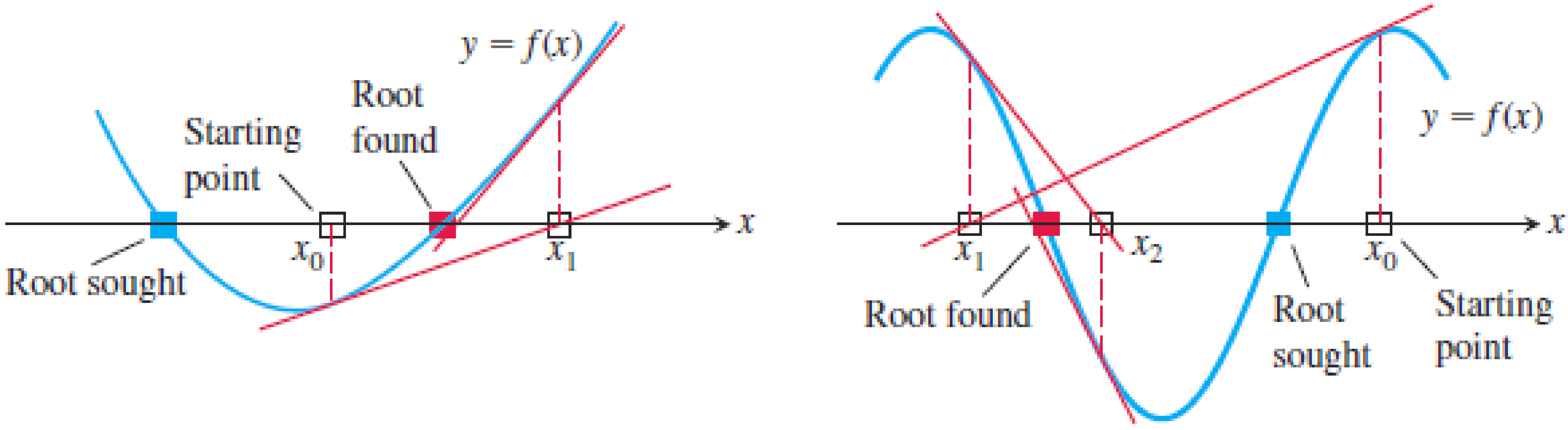


FIGURE 4.52 If you start too far away, Newton's method may miss the root you want.

Skill Practice 1

Use the Intermediate Value Theorem to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then use Newton's method to find the root to five decimal places. (1.17951)

$$x = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{2x_n^3 + 4}{3x_n^2 + 2}$$

x	$f(x)$
$x_0 = 1$	–1
$x_1 = 6/5 = 1.2$	0.128
$x_2 = 932/790 = 1.17975$	0.001488
$x_3 = 577/408 = 1.17951$	0.000006

Skill Practice 2

The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find the x-coordinate of the point to five decimal places. (1.16556)

$$f(x) = tanx - 2x \qquad x = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{x_n tan^2 x_n + x_n - tanx_n}{tan^2 x_n - 1}$$

x	$f(x)$
$x_0 = \pi/3 = 1.04720$	–0.36234
$x_1 = 1.22837$	0.34855
$x_2 = 1.17763$	0.05577
$x_3 = 1.16604$	0.00213
$x_4 = 1.16556$	0.00001