CALCULUS

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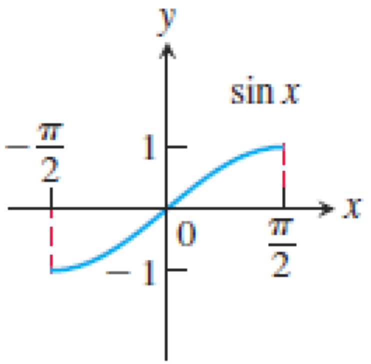
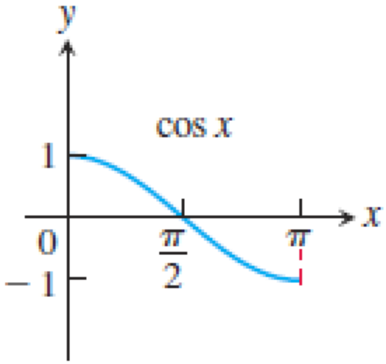
- Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations.

① Defining the Inverse Trigonometric Functions

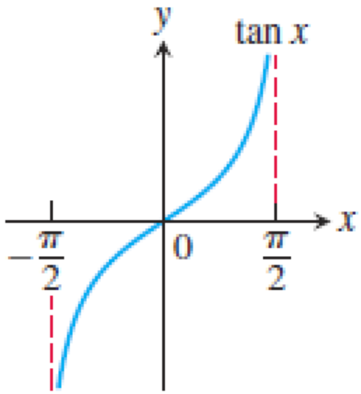
- The six basic trigonometric functions are not one-to-one (since their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse function which is called $\arcsin x$. Similar domain restrictions can be applied to all six trigonometric functions.



$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$

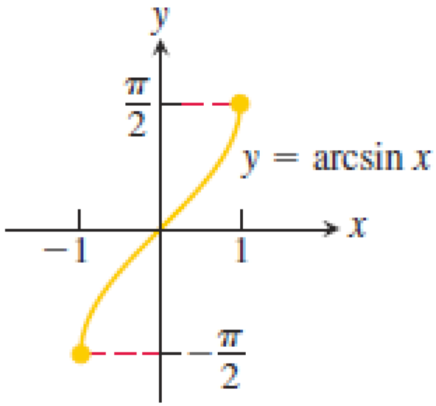


$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$

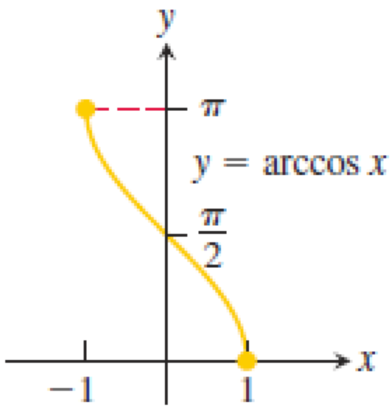
sin$x$ cos$x$ tan$x$

vs

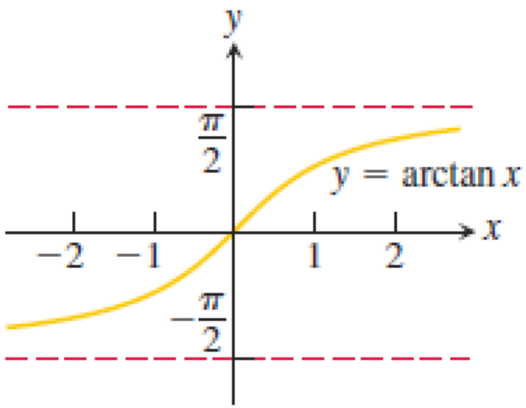
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



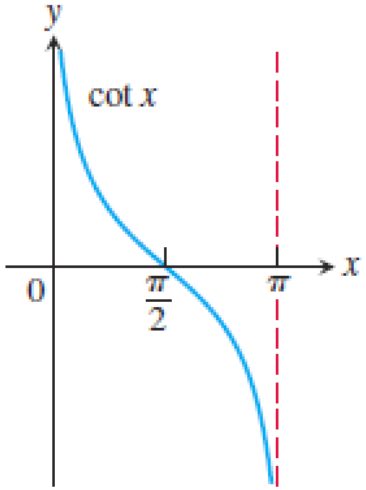
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



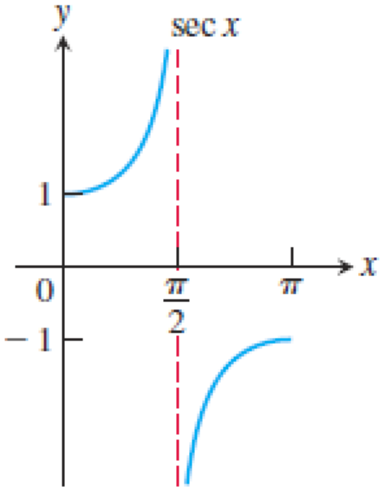
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



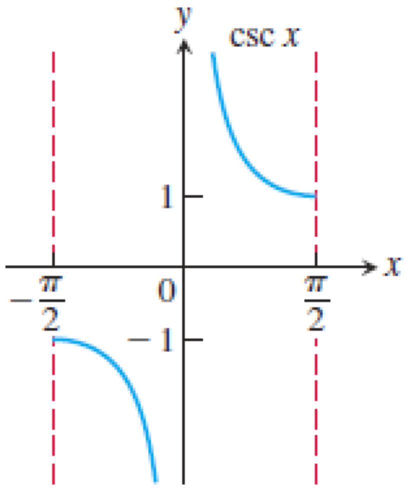
arcsin$x$ arccos$x$ arctan$x$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

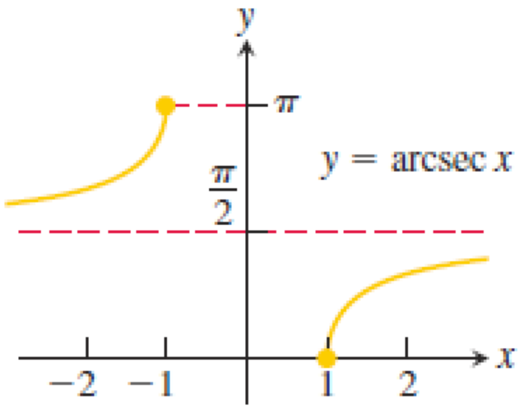


$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

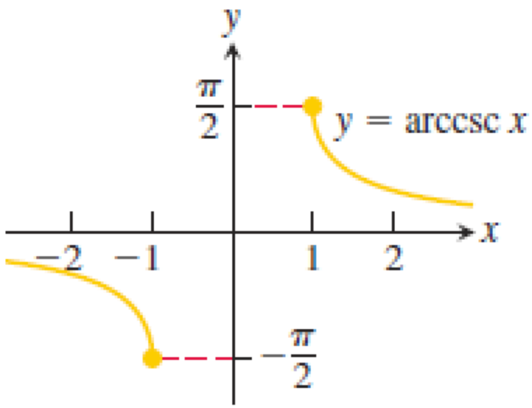
cot$x$ sec$x$ csc$x$

vs

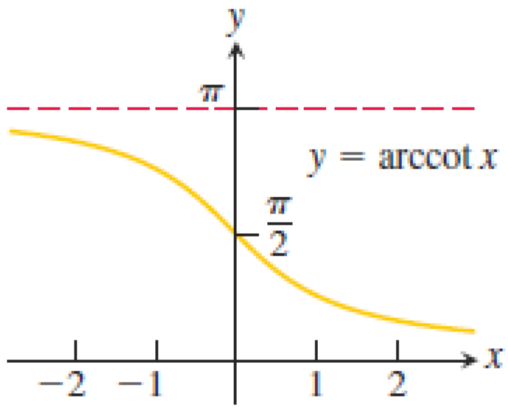
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$



Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$



Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



arccot$x$ arcsec$x$ arccsc$x$

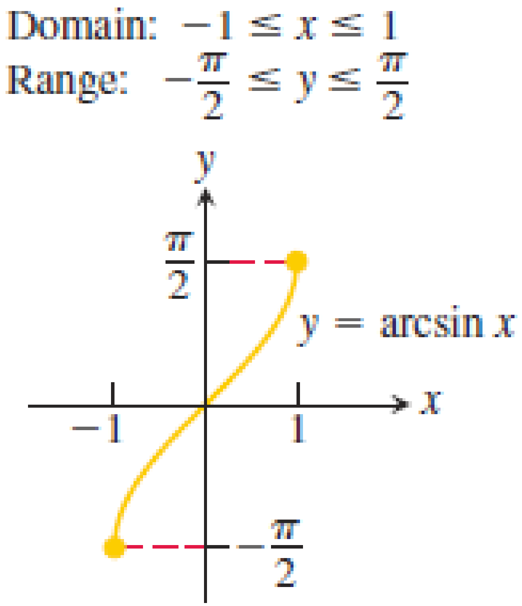
② The inverse functions of sin$x$ and cos$x$

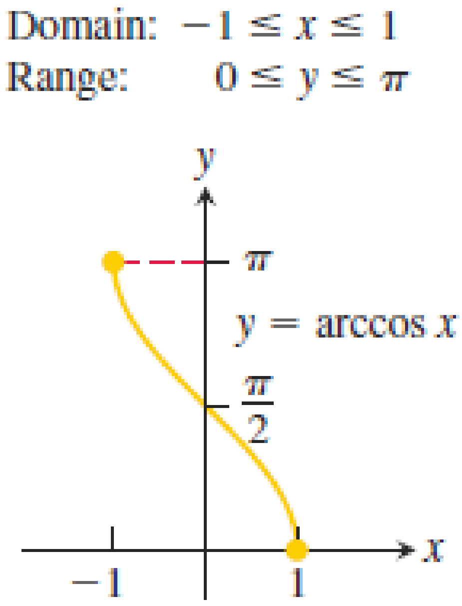
DEFINITION

$y = \sin^{-1}x = \arcsin x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1}x = \arccos x$ is the number in $[0, \pi]$ for which $\cos y = x$.

- $y = \arcsin x$ is an odd function: $\arcsin(-x) = -\arcsin x$.





Example 1 Evaluate

(a) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

(b) $\arccos\left(-\frac{1}{2}\right)$

The common values for the arcsine and arccosine functions.

x	$\arcsin x$	$\arccos x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

$$\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\sin\left(-\frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\sin\left(-\frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\sin\left(-\frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

③ Identities Involving Arcsine and Arccosine

$$\arccos x + \arccos(-x) = \pi$$

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

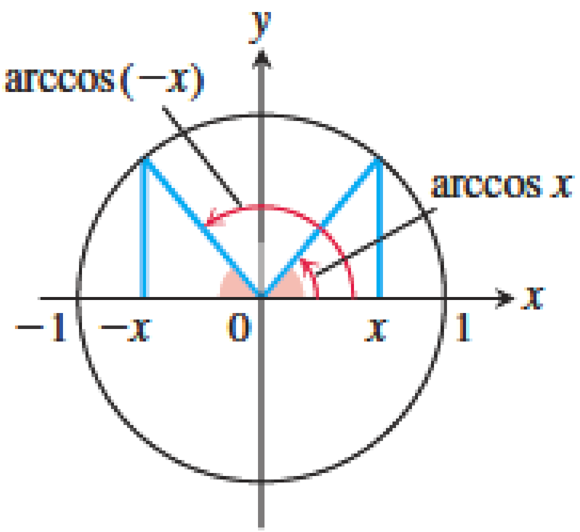


FIGURE 7.27 arccos x and arccos $(-x)$ are supplementary angles (so their sum is π).

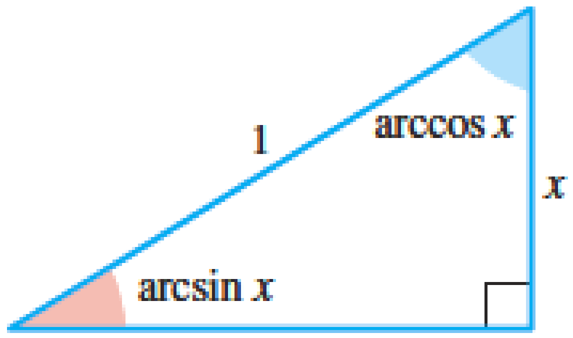


FIGURE 7.28 arcsin x and arccos x are complementary angles (so their sum is $\pi/2$).

④ Inverse functions of tan*x*, cot*x*, sec*x* and csc*x*.

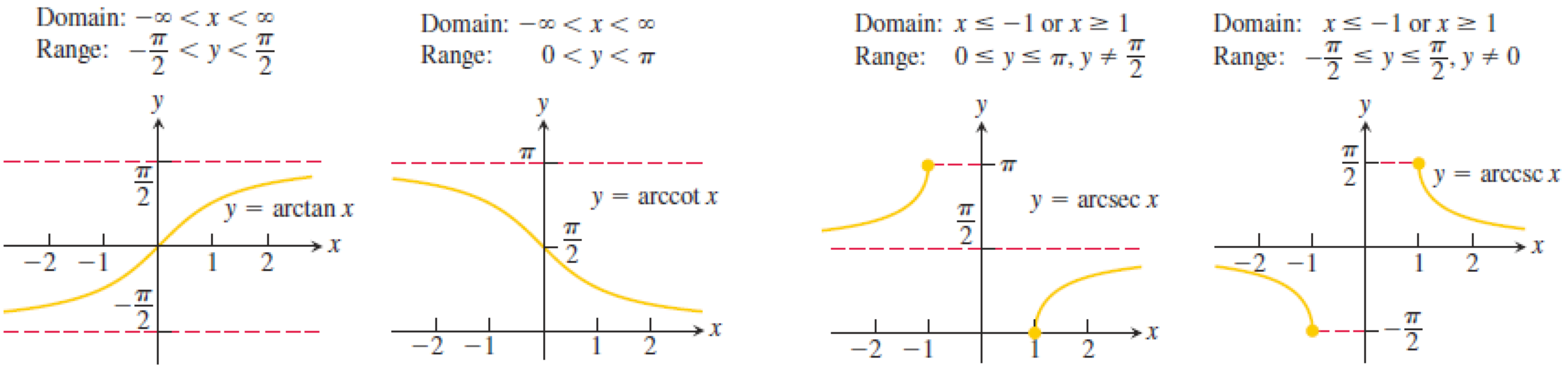
DEFINITION

$y = \tan^{-1}x = \operatorname{arctan}x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \cot^{-1}x = \operatorname{arccot}x$ is the number in $(0, \pi)$ for which $\cot y = x$.

$y = \sec^{-1}x = \operatorname{arcsec}x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

$y = \csc^{-1}x = \operatorname{arccsc}x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.



⑤ The Derivatives of Inverse Trigonometric Functions

- **The Derivative of $\arcsin x = \sin^{-1}x$**

Since the inverse function $y = \sin^{-1}x$ can be expressed as $x = \sin y$, then there is:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

- If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get:

$$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

Example 2 Find

(*a*) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin x^2\right)$ (b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin(\cos x)\right)$

Example 3

If $f(x) = \sin^{-1}(x^2 - 1)$, find:

(a) the domain of $f(x)$,

(b) $f'(x)$,

(c) the domain of $f'(x)$.

- Similarly, we can get $(\arccos x)'$:

  $y=\cos^{-1}x$ can be expressed as $x=\cos y$, then there is:

$$\frac{d}{dx}(x)=\frac{d}{dx}(\cos y) \quad \Rightarrow \quad \frac{dy}{dx}=\frac{-1}{\sin y}=\frac{-1}{\sqrt{1-\cos^2 y}}=\frac{-1}{\sqrt{1-x^2}}$$

- We can get $(\arctan x)'$ and $(\operatorname{arccot} x)'$ in the same way:

  $y=\tan^{-1}x$ and $y=\cot^{-1}x$ can be expressed as $x=\tan y$ *and* $x=\cot y$, then there is:

$$\frac{d}{dx}(x)=\frac{d}{dx}(\tan y) \quad \Rightarrow \quad 1=(\sec^2 y)\frac{dy}{dx}=(1+\tan^2 y)\frac{dy}{dx}=(1+x^2)\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{1}{1+x^2}$$

$$\frac{d}{dx}(x)=\frac{d}{dx}(\cot y) \quad \Rightarrow \quad 1=(-\csc^2 y)\frac{dy}{dx}=-(1+\cot^2 y)\frac{dy}{dx}=-(1+x^2)\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{-1}{1+x^2}$$

- Table for the derivatives of the other inverse trigonometric functions.

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$
2. $\dfrac{d(\arccos u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$
3. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$
4. $\dfrac{d(\operatorname{arccot} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$
5. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$
6. $\dfrac{d(\operatorname{arccsc} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

Example 4

Find the derivatives of the following inverse trigonometric functions.

$$(a)\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin x^3\right)$$

$$(b)\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{1}{x}\right)$$

$$(c)\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan(\ln x)\right)$$

$$(d)\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{arccot}(\tan x)\right)$$

⑥ Integration Formulas

- Based on the derivatives listed in Table 7.3, we can get some useful integrals, as shown in Table 7.4.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

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Example 5 Evaluate the following integrals

$$(a)\ \int \frac{\mathrm{d}x}{\sqrt{9-4x^2}}$$

$$(b)\ \int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$$

$$(c)\ \int \frac{\mathrm{d}x}{\sqrt{4x-x^2}}$$

$$(d)\ \int_{-1}^{0} \frac{\mathrm{d}x}{4x^2+4x+2}$$

$$(e)\ \int \frac{\mathrm{d}x}{\sqrt{e^{2x}-9}}$$

$$(f)\ \int_{\frac{1}{3}}^{\frac{\sqrt{2}}{3}} \frac{dy}{y\sqrt{9y^2-1}}$$