CALCULUS

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DEFINITION: **The derivative of the function *f*(*x*) with resect to the variable *x*** is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

- Let $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, **an alternative definition of the derivative** is as follows:

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}$$

① Calculating Derivatives from the definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation $\frac{d}{dx}f(x)$ as another way to denote the derivative $f'(x)$.

Example 1 Differentiate

$$f(x) = \frac{x}{x-1}$$

Example 2 **(a)** Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

③ Notation

- There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative include

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D_x f(x)$$

- To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}$$

③ Differentiable on an Interval; One-Sided Derivatives

- A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h) - f(x)}{h} \qquad \text{Right-hand derivative at } a$$

$$f'(b^-) = \lim_{h\to 0^-} \frac{f(b+h) - f(x)}{h} \qquad \text{Left-hand derivative at } b$$

exist at the endpoints.

Example 3

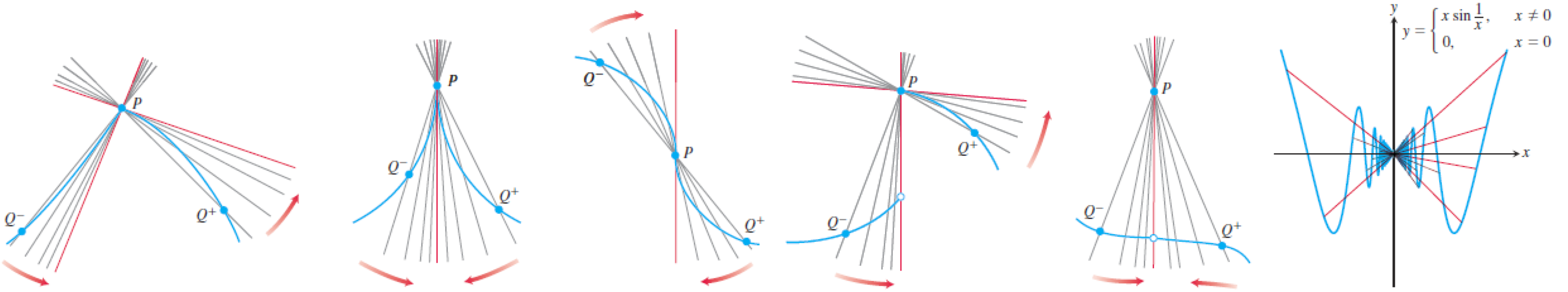
Show that the function $f(x) = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Example 4

Let $f(x) = \sqrt{x}$. Show that the function $f(x)$ has no derivative at $x = 0$.

④ When Does a Function Not Have a Derivative at a Point?

- Differentiability is a "smoothness" condition on the graph of f. A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has



1. a *corner*, where the one-sided derivatives differ
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other
3. a *vertical tangent line*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)
4. a *discontinuity* (two examples shown)
5. wild oscillation

Example 5

Show that the following function has no derivative at $x = 0$.

$$f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

⑤ Differentiable Functions are Continuous

THEOREM 1 If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof is skipped here. It can be found in the textbook.

Note: The converse of Theorem 1 is false. See Examples 3 and 4.

Skill Practice 1

Using the definition, calculate the derivatives of the function. Then find the values of the derivatives as specified

$$f(x) = \frac{x}{1+x}\ , \qquad f'(0) \text{ and } f'(-1/2)\ .$$

Skill Practice 2

Differentiate the function. Then find an equation of the tangent line at the indicated point on the graph of the function.

$$y = f(x) = 1 + \sqrt{x-2}, \qquad (x, y) = (6, 3)$$

Skill Practice 3

Determine if the piecewise-defined function is differentiable at $x = 0$.

$$f(x) = \begin{cases} -x^3 + 2x - 1 & x \geq 0 \\ \dfrac{x^2 + x - 1}{x + 1} & x < 0 \end{cases}$$

Skill Practice 4

Show that the following function is differentiable at $x = 0$. Then find $f'(0)$.

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$