College Algebra and Trigonometry

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Volume of An open box:

A open box is made from a 6m × 6m square sheet of wood lamella with squares of length *x* (in meter) removed from each corner. Then the flaps are folded up to form an open box.

a) Write the expression of the volume of the open box.

b) What is the maximum volume of the open box.

Definition of a Polynomial Function:

Let n be a whole number and a_n, a_{n-1}, …… , a_1, a_0 are real numbers ($a_n \neq 0$). Then a function defined by:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is called a polynomial function of degree n.

Polynomial Functions:

$f(x) = 4x^3 - 5x^2 + x - 1$

$f(x) = 5x^4 + 3x^2 - 5$

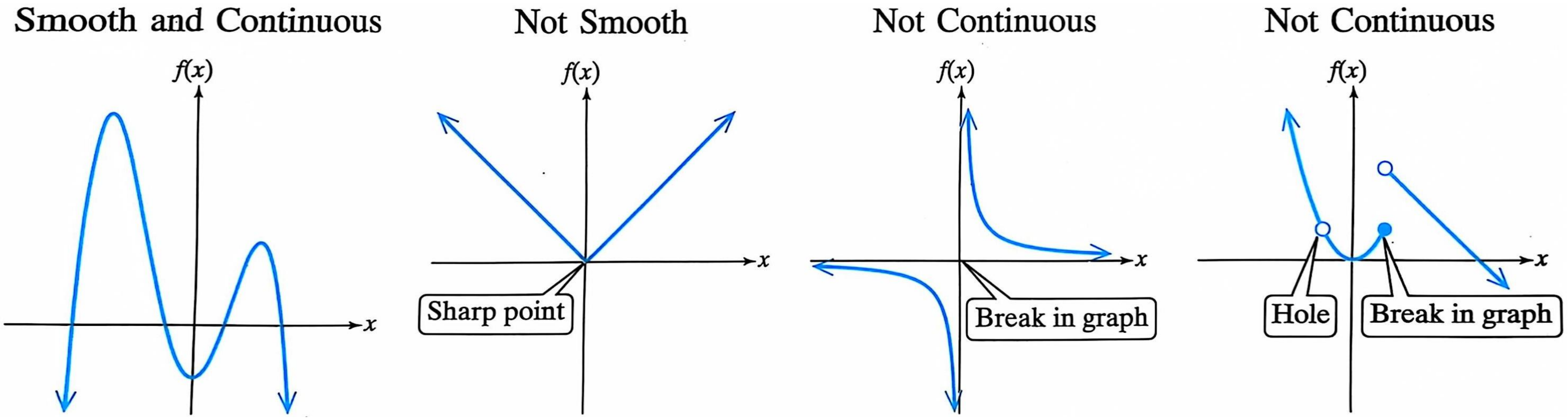
Not Polynomial Functions:

$f(x) = 3\sqrt{x} + 2/x$

$f(x) = 2\sqrt[3]{x} + (3 + i)x$

For Polynomial Functions:

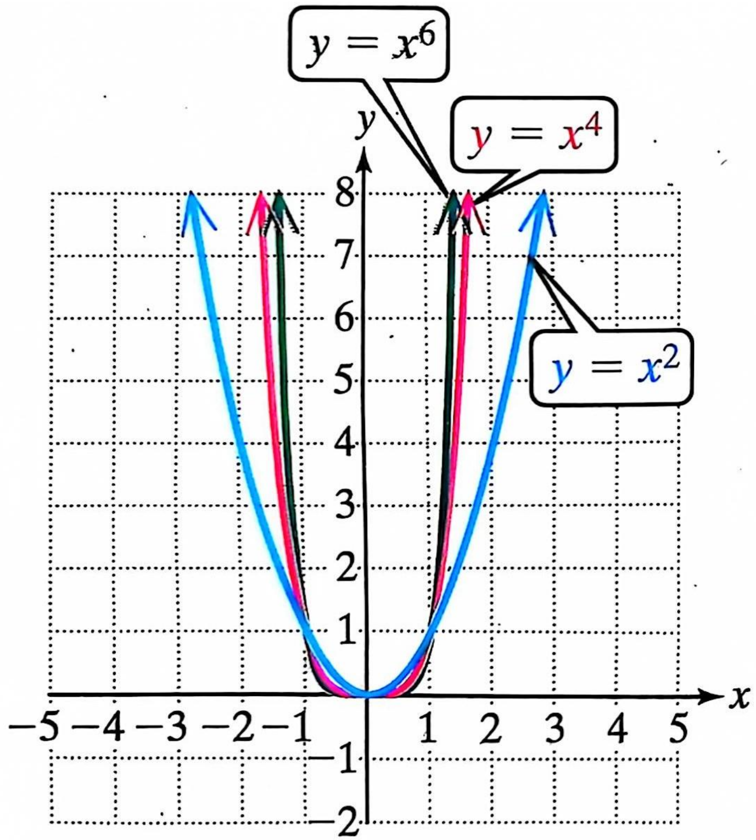
- **The domain of a polynomial function is all real numbers.**
- **The graph of a polynomial function is both continuous and smooth. "Continuous" means no breaks. "Smooth" means no sharp corners or points.**

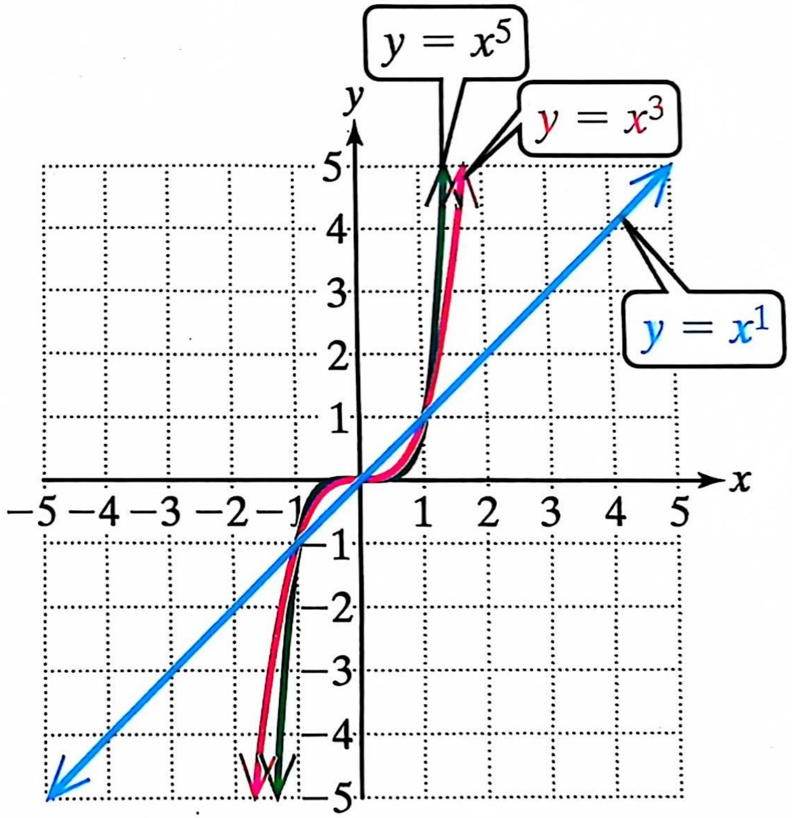


Power Functions:

$$f(x) = ax^n$$

where a is a nonzero real number and n is a positive integer.





① Determine the End Behavior of a Polynomial Function

Notation for Infinite Behavior of $y = f(x)$

$x \to \infty$	is read as "x approaches infinity." This means that x becomes infinitely large in the positive direction.
$x \to -\infty$	is read as "x approaches negative infinity." This means that x becomes infinitely "large" in the negative direction.
$f(x) \to \infty$	is read as "$f(x)$ approaches infinity." This means that the y value becomes infinitely large in the positive direction.
$f(x) \to -\infty$	is read as "$f(x)$ approaches negative infinity." This means that the y value becomes infinitely "large" in the negative direction.

The Leading Term Test

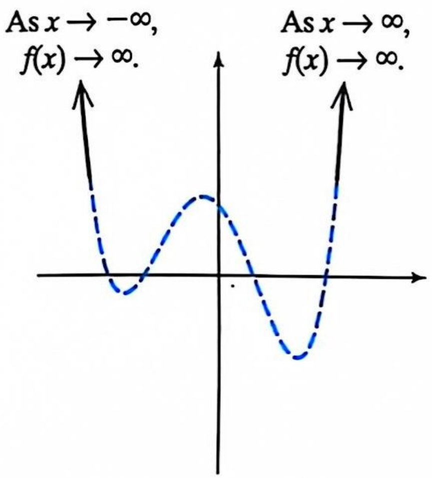
Consider a polynomial function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

As $x \to \infty$ or as $x \to -\infty$, f eventually becomes forever increasing or forever decreasing and will follow the general behavior of $y = a_n x^n$.

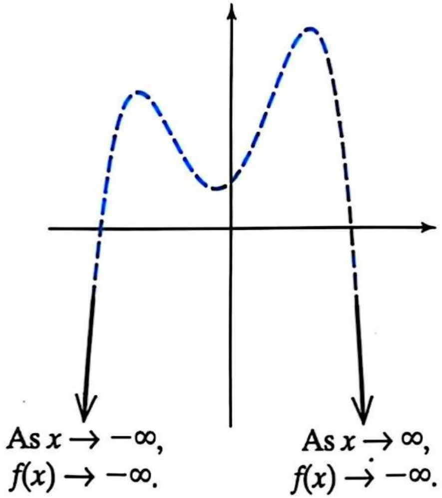
n is even

a_n positive



End behavior:
up left/up right

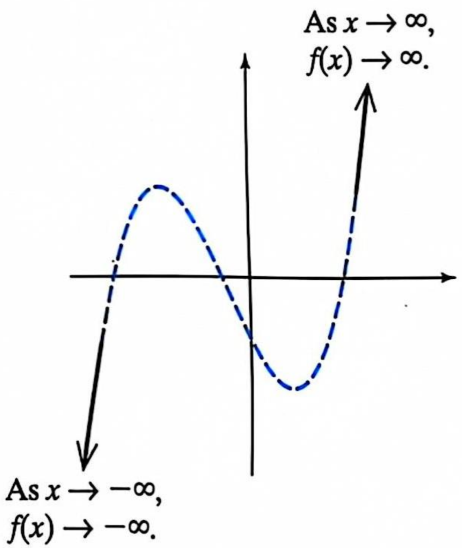
a_n negative



End behavior:
down left/down right

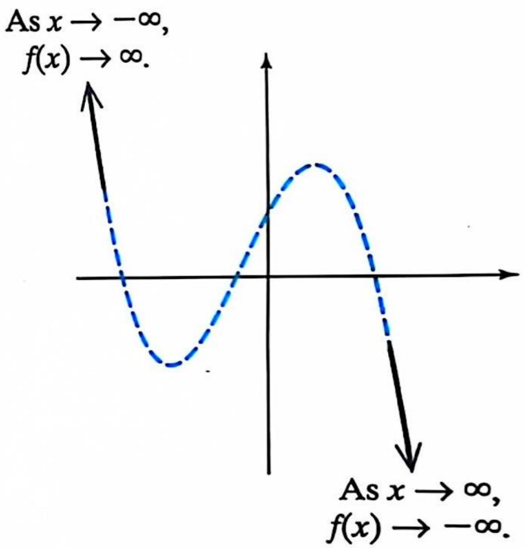
n is odd

a_n positive



End behavior:
down left/up right

a_n negative



End behavior:
up left/down right

Example 1:

Use the leading term test to determine the end behavior of the graph of the following functions.

a) $f(x) = -4x^5 + 6x^4 + 2x$

b) $g(x) = \frac{1}{4}x(2x-3)^3(x+4)^2$

② Identify Zeros and Multiplicities of Zeros

Consider a polynomial function defined by $y = f(x)$.

- **The values of x in the domain of $f(x)$ which make $f(x) = 0$ are called the zeros of the function.**
- **They are the real solutions (roots) of the equation $f(x) = 0$ and correspond to the x-intercepts of the graph of $y = f(x)$.**

Examples 2 & 3:

Find the zeros of the following functions:

a) $f(x) = x^3 + x^2 - 9x - 9$ b) $f(x) = -x^3 + 8x^2 - 16x$

Multiplicities of Zeros

- **If a polynomial function has a factor $(x-c)$ that appears exactly *k* times, then *c* is a zero of multiplicity *k*.**
- **For example, for the function:**

$$f(x) = x^2(x-1)^3(x+2)^6$$

0 is a zero of multiplicity 2.

1 is a zero of multiplicity 3.

-2 is a zero of multiplicity 6.

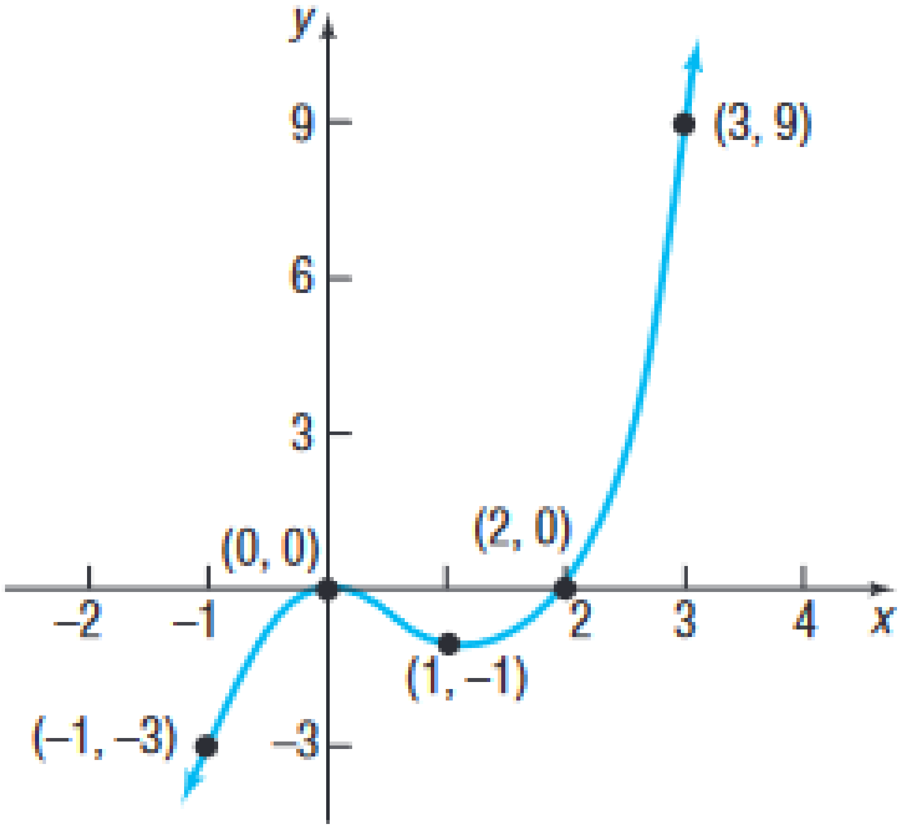
- **The multiplicity of a zero can be used to determine whether the graph of a function touches or crosses the x-axis at the zero.**

Touch Points and Cross Points:

Let f be a polynomial function and let c be a real zero of f. Then the point $(c, 0)$ is an x-intercept of the graph of f. Furthermore,

- **If c is a zero of odd multiplicity, then the graph crosses the x-axis at c. The point $(c, 0)$ is called a cross point.**
- **If c is a zero of even multiplicity, then the graph touches the x-axis at c. The point $(c, 0)$ is called a touch point.**

$$f(x) = x^2(x-2)$$

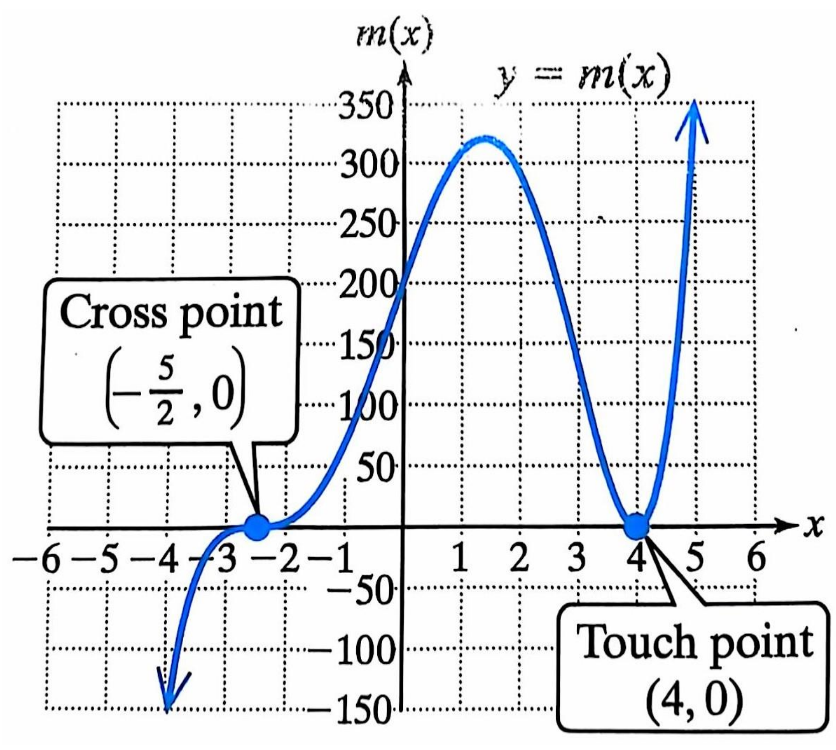


- **The graph of f touches the x-axis at $x = 0$, a zero of multiplicity 2. Hence, $x = 0$ is a touch point.**
- **The graph of f crosses the x-axis at $x = 2$, a zero of multiplicity 1. Hence, $x = 2$ is a cross point.**

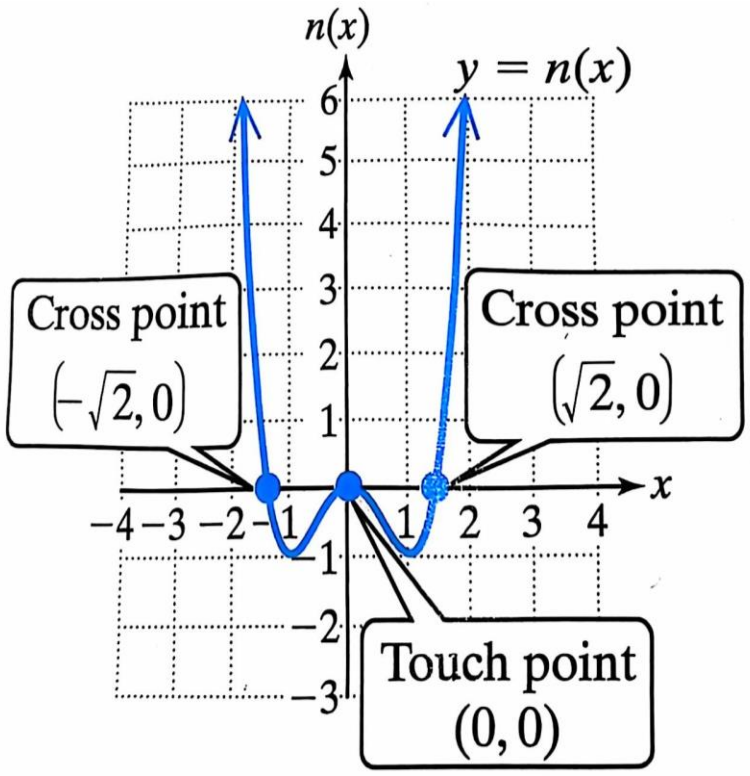
Example 4:

Determine the zeros and their multiplicities for the given functions.

a) $m(x) = \frac{1}{10}(x-4)^2(2x+5)^3$



b) $n(x) = x^4 - 2x^2$



③ Apply the Intermediate Value Theorem

- **In most cases, the real zeros of a polynomial are difficult or impossible determine algebraically.**
- **Difficult :** $f(x) = x^4 + 6x^3 - 26x + 15$
- **Impossible :** $f(x) = 2x^5 - 6x^3 + 11x - 20$

Intermediate Value Theorem:

Let f be a polynomial function. For $a < b$, if $f(a)$ and $f(b)$ have opposite signs, then f has at least one zero on the interval $[a, b]$.

Example 5:

For the given function:

$$f(x) = x^4 + 6x^3 - 26x + 15$$

a) Show that it has one zero on the interval (1, 2).

b) Show all the intervals having zeros.

c) Find all zeros of *f*(*x*).

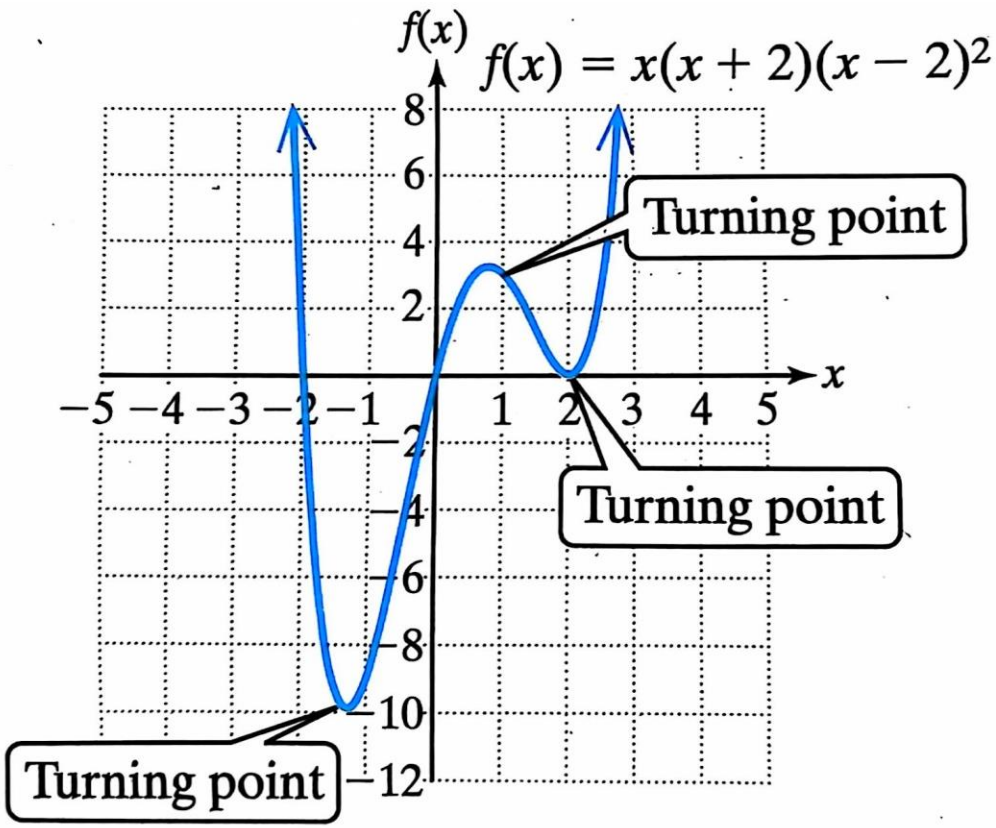
$$f(x) = x^4 + 6x^3 - 26x + 15$$
$$= (x^2 + 4x - 3)(x^2 + 2x - 5)$$
$$= (x + 2 + \sqrt{7})(x + 2 - \sqrt{7})(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$$

$$f(-6) = 171$$
$$f(-5) = 20$$
$$f(-4) = -9$$
$$f(-3) = 12$$
$$f(-2) = 35$$
$$f(-1) = 36$$
$$f(0) = 15$$
$$f(1) = -4$$
$$f(2) = 27$$
$$f(3) = 180$$

④ Sketch a Polynomial Function

Turning Points:

The points correspond to relative maxima and minima.



Number of Turning Points of a Polynomial Function:

Let *f* be a polynomial function of degree *n*. Then the graph of *f* has at most *n*-1 turning points.

Graphing a Polynomial Function

To graph a polynomial function defined by $y = f(x)$,

1. Use the leading term to determine the end behavior of the graph.
2. Determine the y-intercept by evaluating $f(0)$.
3. Determine the real zeros of f and their multiplicities (these are the x-intercepts of the graph of f).
4. Plot the x- and y-intercepts and sketch the end behavior.
5. Draw a sketch starting from the left-end behavior. Connect the x- and y-intercepts in the order that they appear from left to right using these rules:
   - The curve will cross the x-axis at an x-intercept if the corresponding zero has an odd multiplicity.
   - The curve will touch but not cross the x-axis at an x-intercept if the corresponding zero has an even multiplicity.
6. If a test for symmetry is easy to apply, use symmetry to plot additional points. Recall that
   - f is an even function (symmetric to the y-axis) if $f(-x) = f(x)$.
   - f is an odd function (symmetric to the origin) if $f(-x) = -f(x)$.
7. Plot more points if a greater level of accuracy is desired. In particular, to estimate the location of turning points, find several points between two consecutive x-intercepts.

Example 6 & 7:

Graph polynomial functions:

a) $f(x) = x^3 - 9x$

b) $g(x) = -\frac{1}{10}(x-1)(x+2)(x-4)^2$