CALCULUS

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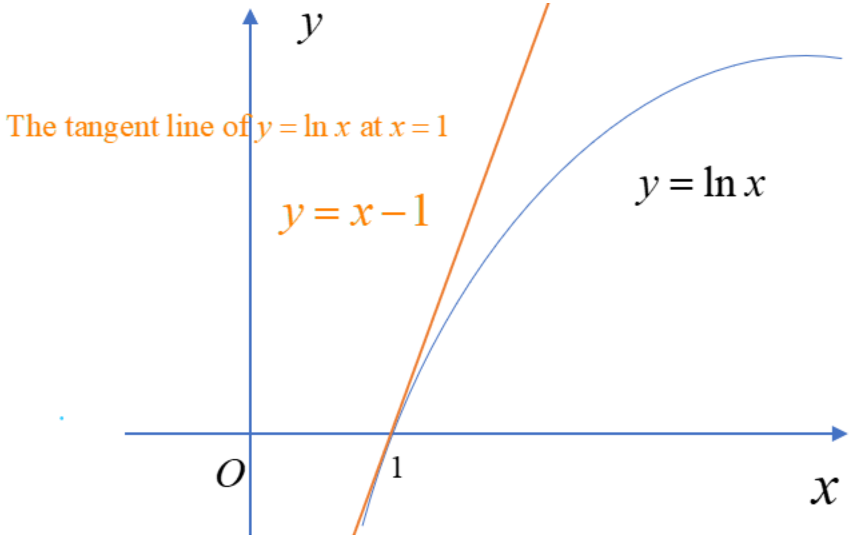
- It is often useful to approximate complicated functions with simpler ones that give the accuracy we want for specific applications and at the same time are easier to work with than the original functions. The approximating functions discussed in this section are called ***linearization***, and they are **based on tangent lines**.
- For example:

  known: $\ln 1 = 0$

  unknown: $\ln 1.001 = ?$

- As we check the graph on the right, we see that when x is close to 1, the value of $y = \ln x$ is very close to the value of $y = x-1$.



- That is $\ln x \approx x-1, \quad x \to 1$

  So, $\ln 1.001 = 1.001-1 = 0.001$

- The approximating function $y = x - 1$ is called the **linearization** of $y = \ln x$ at $x = 1$.

① Linearization

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x-a)$$

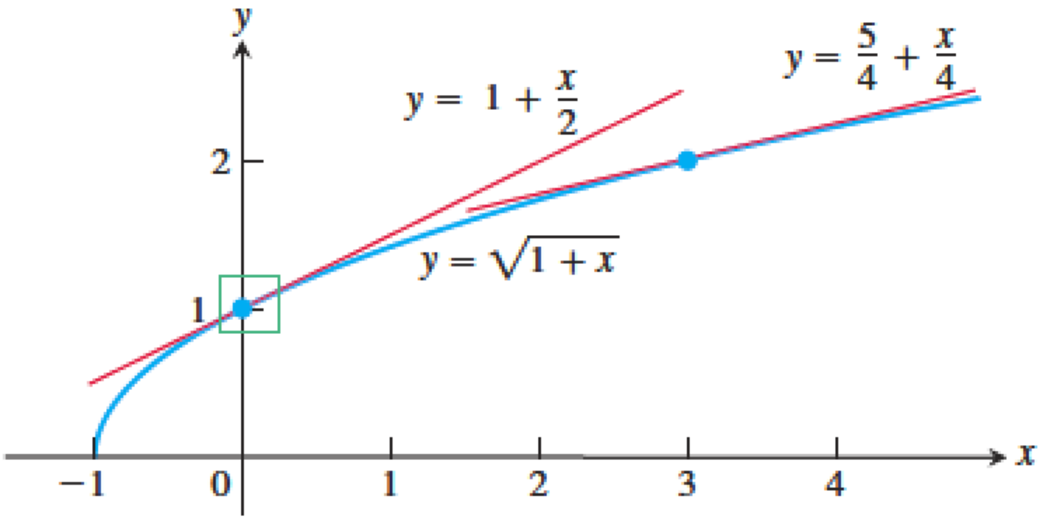
is the **linearization** of f at a.

The approximation $f(x) \approx L(x)$ of f by L is the **standard linear approximation** of f at a.

The point $x = a$ is the **center** of the approximation.

Example 1

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$.

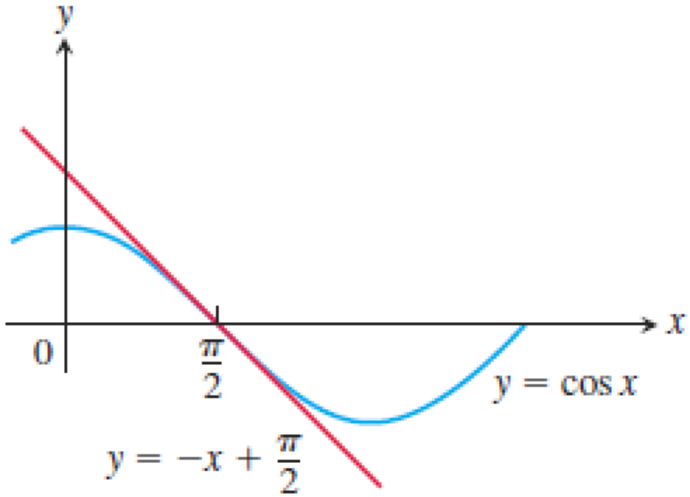


Example 2

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$.

Example 3

Find the linearization of $f(x) = \cos x$ at $x = \frac{\pi}{2}$.



② Differentials

We introduce new variables dx and dy, called ***differentials***, and define them in a way that makes Leibniz's notation for the derivative dy/dx a true ratio.

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Example 4

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and d$x = 0.2$.

- The geometric meaning of differentials is shown in the following Figure.

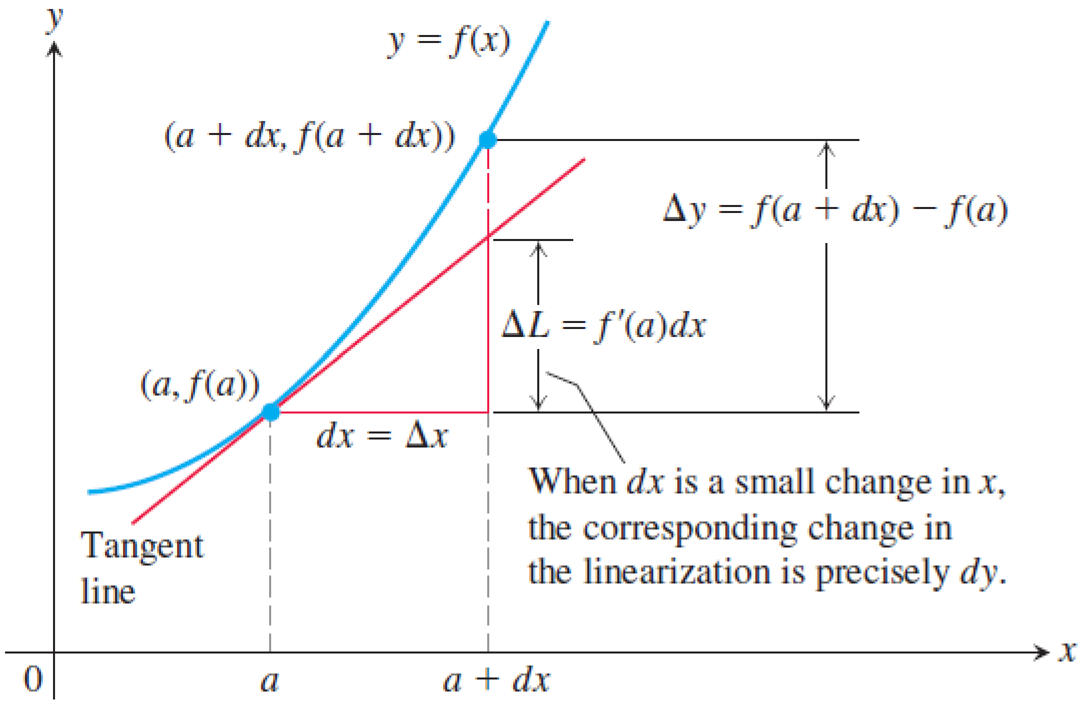
Let $x = a$ and set $\mathrm{d}x = \triangle x$.

The corresponding change in $y = f(x)$ is

$$\triangle y = f(a + \mathrm{d}x) - f(a).$$

The corresponding change in the tangent line L is

$$\begin{aligned}\Delta L &= L(a+\mathrm{d}x) - L(a)\\ &= \underbrace{f(a) + f'(a)\left[(a+\mathrm{d}x) - a\right]}_{L(a+\mathrm{d}x)} - \underbrace{f(a)}_{L(a)}\\ &= f'(a)\,\mathrm{d}x.\end{aligned}$$



- That is, the change in the linearization of f is precisely the value of the differential dy when $x = a$ and dx = $\triangle x$.
- If d$x \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$\mathrm{d}y \div \mathrm{d}x = \frac{f'(x)\mathrm{d}x}{\mathrm{d}x} = f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

- Every differentiation formula like $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ $\quad or \quad$ $\dfrac{d(sinu)}{dx} = cosu\dfrac{du}{dx}$

has a corresponding differential form like $d(u+y) = du + dy$ or $d(\sin u) = \cos u du$.

Example 5 Find differentials of functions.

$$(\mathrm{a})\ \mathrm{d}(\tan 2x); \qquad (\mathrm{b})\ \mathrm{d}\left(\frac{x}{x+1}\right).$$

③ Estimate with Differentials

- Suppose we know the value of a differentiable function $f(x)$ at a point a and want to estimate how much this value will change if it moves to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we know that Δy is approximately equal to the differential dy.

Thus: $$f(a+dx) = f(a) + \Delta y = f(a) + dy = f(a) + f'(a)dx$$

Example 6

The radius r of a circle increases from $a = 10$ m to 10.1 m. Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Example 7 Use differentials to estimate

(a) $7.97^{1/3}$ (b) $\sin(\pi/6 + 0.01)$

3.9 Linearization and Differentials

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④ Error in Differential Approximation

- Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = x$ is an increment of x.

  We have two ways to describe the change in f as x changes from a to $a + x$:

  The true change: $\Delta f = f(a + \Delta x) - f(a)$

  The differential estimate: $df = f'(a)\Delta x$

  How well does df approximate f?

Approximation error = $\boldsymbol{\Delta f - df = f(a+\Delta x) - f(a) - f'(a)\Delta x}$

$$= \left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a) \right) \Delta x = \varepsilon \Delta x$$

$$\Rightarrow \quad \boldsymbol{\Delta f = f'(a)\Delta x + \varepsilon\Delta x}$$

true change = estimate change + error

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + x$, the change Δy in f is given by

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

in which $\varepsilon \to 0$ as $x \to 0$.

Example 8

The radius r of a circle increases from a = 10 m to 10.1 m. Use dA to estimate the increase in the circle's area A. Calculate the approximation error.

⑤ Sensitivity to Change

- The equation $df=f'(x)dx$ tells how *sensitive* the output of f is to a change in input at different values of x. The larger the value of f at x, the greater the effect of a given change dx. As we move from a to a nearby point $a+dx$, we can describe the change in f in three ways: absolute, relative, and percentage.

	True	Estimated
Absolute change	$\Delta f = f(a+dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\dfrac{\Delta f}{f(a)}$	$\dfrac{df}{f(a)}$
Percentage change	$\dfrac{\Delta f}{f(a)} \times 100$	$\dfrac{df}{f(a)} \times 100$

Example 9

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below.

How sensitive will your calculations be to a 0.1-sec error in measuring the time?

Example 10

Newton's second law "$F = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma$" is stated with the assumption that mass is constant, but this is not strictly true because the mass of an object increases with velocity.

In Einstein's corrected formula, mass has the value $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ where the "rest mass" m_0 represents the mass of an object that is not moving and c is the speed of light (300,000 km/sec).

Use the approximation $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2}$ to estimate the increase m in mass resulting from the added velocity y.

⑥ Converting mass to Energy

- In Newtonian physics, $\frac{1}{2}m_0v^2$ is the kinetic energy (KE) of the object. As we get from Example 10.

$$(m - m_0)c^2 \approx \frac{1}{2}m_0v^2$$

$$\Rightarrow \quad (m - m_0)c^2 = (\Delta m)c^2 \approx \frac{1}{2}m_0v^2 - \frac{1}{2}m_0(0)^2 = \Delta(\text{KE})$$

So the change in kinetic energy in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$, the change in mass times the square of the speed of light.

Using $c = 3\times10^8$ m/s , we see that a small change in mass cause a large change in energy.

⑦ Quadratic Approximations

Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties: i) $Q(a) = f(a)$ ii) $Q(a) = f'(a)$ iii) $Q(a) = f''(a)$.

a) Determine the coefficients b_0, b_1, and b_2.

b) Find the quadratic approximation to $f(x) = 1/(1-x)$ at $x = 0$.

Example 11

Find the quadratic approximation

$$g(x) = 1/x$$

at $x = 1$.

Skill Practice 1

Find the linearization of $f(x) = \sqrt[3]{x+3}$ at $x = 5$.

Skill Practice 2

Find the quadratic approximation to

$$h(x) = \sqrt{1+x}$$

at $x = 0$.

Skill Practice 3 Use differentials to estimate

(a) $15.98^{1/2}$ (b) $\cos(\pi/3 - 0.02)$