Lecture 9

Rotations of Rigid Bodies

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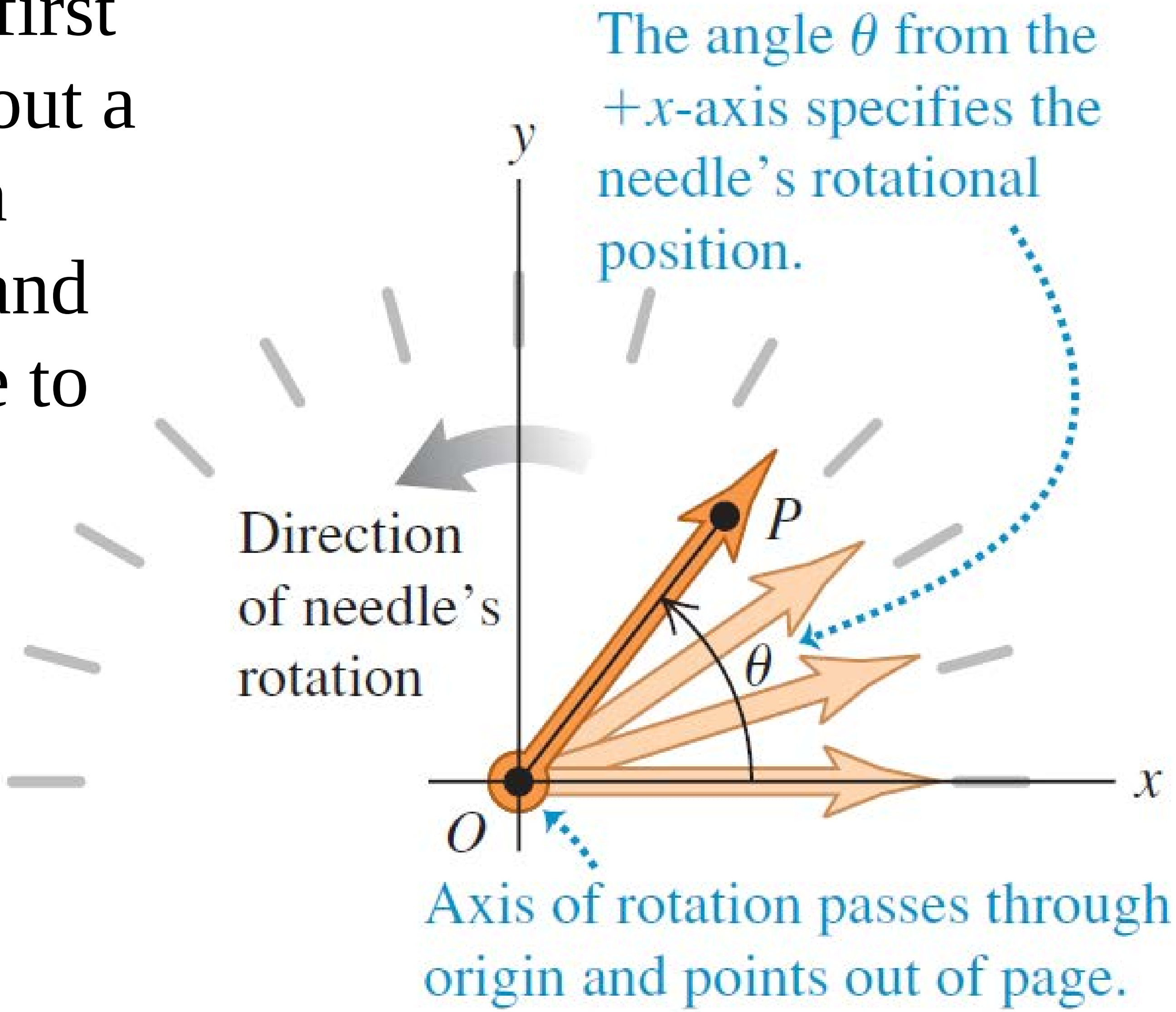
Course Instructor:
Jingtian Hu (胡竞天)

Angular Velocity and Acceleration

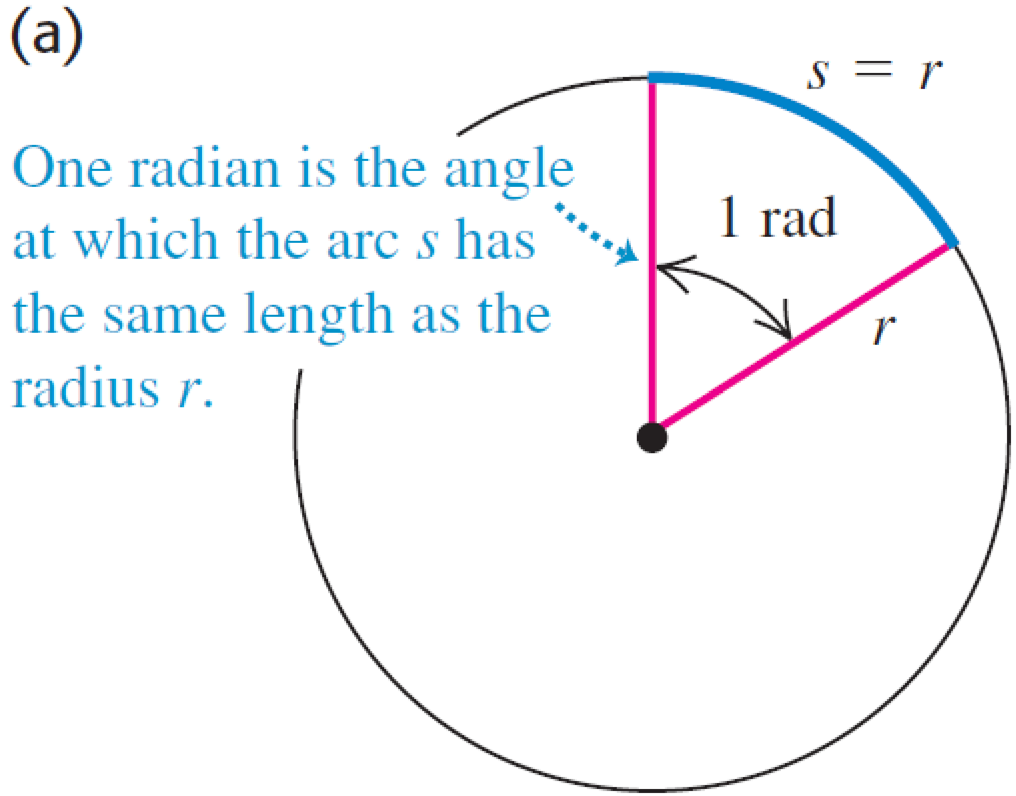
For rotational motion, let's think first about a rigid body that rotates about a *fixed axis*, an axis that is at rest in some inertial frame of reference and does not change direction relative to that frame

The angle that this line makes with the $+x$ describes the rotational position of the body; we will use this single quantity as a *coordinate* for rotation.



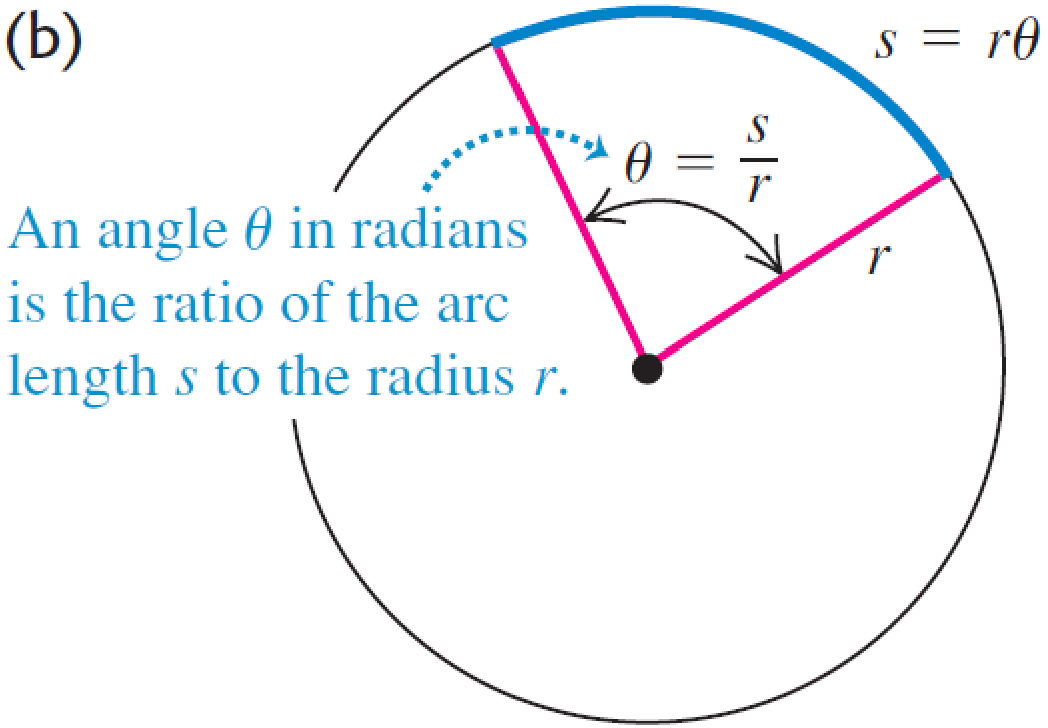
Angular Velocity and Acceleration

To describe rotational motion, the most natural way to measure the angle is not in degrees, but in **radians.**



Angular Velocity and Acceleration

One radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle. In Fig. 9.2b an angle is subtended by an arc of length *s* on a circle of radius *r*. The **value of the angle** (in radians) is equal to *s* divided by *r*:



$$\theta = \frac{s}{r} \qquad \text{or} \qquad s = r\theta$$

The coordinate θ specifies the rotational position of a rigid body at a given instant

Angular Velocity and Acceleration

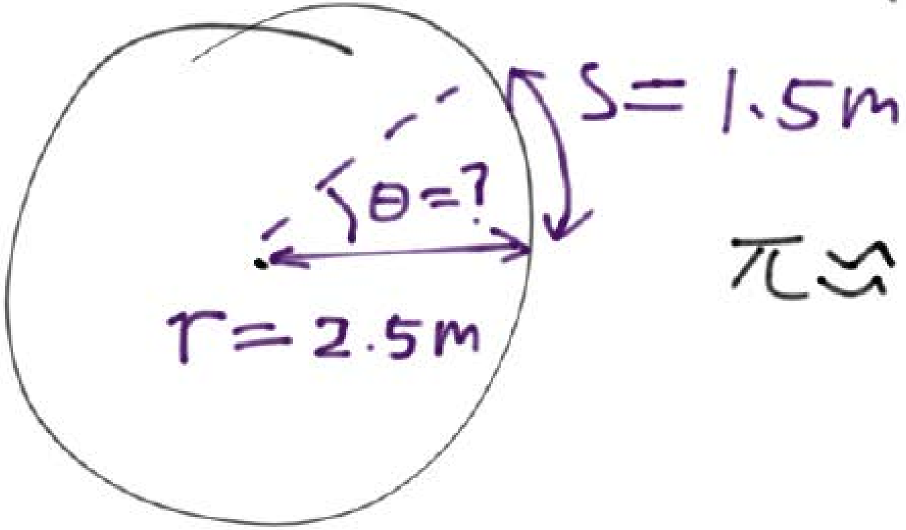
An angle in radians is the ratio of two lengths, so it is a pure number, without dimensions. If $s = 3.0\text{ m}$ and $r = 2.0\text{ m}$, then $\theta = 1.5$, but we will often write this as 1.5 rad to distinguish it from an angle measured in degrees or revolutions.

The circumference of a circle (that is, the arc length all the way around the circle) is 2π times the radius, so there are 2π (about 6.283) radians in one complete revolution (360°). Therefore

$$1\text{ rad} = \frac{360°}{2\pi} = 57.3°$$

Similarly, $180° = \pi$ rad, $90° = \pi/2$ rad, and so on. If we had insisted on measuring the angle θ in degrees, we would have needed to include an extra factor of $(2\pi/360)$ on the right-hand side of $s = r\theta$ in Eq. (9.1).

9.1 • (a) What angle in radians is subtended by an arc 1.50 m long on the circumference of a circle of radius 2.50 m? What is this angle in degrees? (b) An arc 14.0 cm long on the circumference of a circle subtends an angle of 128°. What is the radius of the circle? (c) The angle between two radii of a circle with radius 1.50 m is 0.700 rad. What length of arc is intercepted on the circumference of the circle by the two radii?



(a) $\theta = \frac{s}{r} = 0.6 \text{ rad}$

$\pi \approx 3.14159\cdots$

$= 0.6 \text{ rad} \cdot \frac{180°}{\pi \cdot \text{rad}}$

$\approx 34.4°$

angle in degrees? (b) An arc 14.0 cm long on the circumference of a circle subtends an angle of 128°. What is the radius of the circle?

(b)

$S = 0.14\ m$

$\theta = 128°$

$r = ?$

$\boxed{\theta \text{ must be in rad. not degrees}}$

From $\theta = \frac{s}{r}$

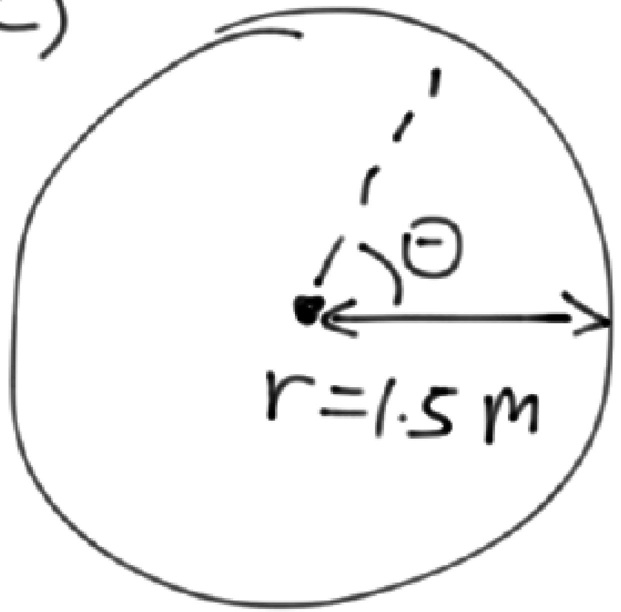
we know $r = \frac{s}{\theta}$

$$= \frac{0.14\ m}{128° \cdot 3.14\ rad/180°}$$

$$\cong 0.0627\ m = 6.27\ cm$$

(c) The angle between two radii of a circle with radius 1.50 m is 0.700 rad. What length of arc is intercepted on the circumference of the circle by the two radii?

(c)



$\theta = 0.7 \text{ rad}$

$s = \theta \cdot r = 1.05 \text{ m}$

Angular Velocity: Definition

We can describe the rotational *motion* of such a rigid body in terms of the rate of change of θ.

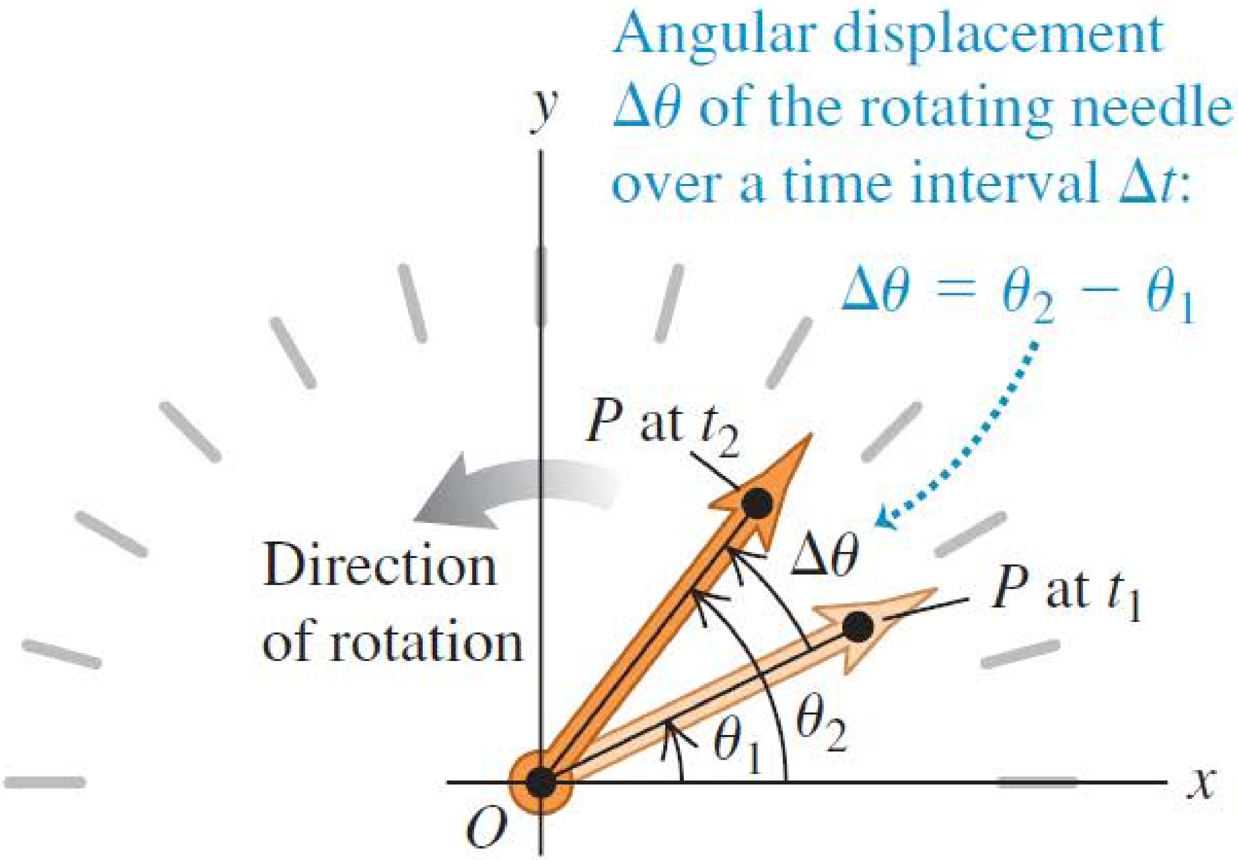
average angular velocity $\omega_{\text{av-}z} = \dfrac{\theta_2 - \theta_1}{t_2 - t_1} = \dfrac{\Delta\theta}{\Delta t}$

angular displacement $\Delta\theta = \theta_2 - \theta_1$

The **instantaneous angular velocity** $\boldsymbol{\omega_z}$ is the limit of $\boldsymbol{\omega_{\text{av-z}}}$ as Δt approaches zero:

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(definition of angular velocity)}$$

Angular Velocity: Example



Angular Velocity: Properties

- Angular velocity $\boldsymbol{\omega}_{\mathbf{z}}$ can be positive or negative.
- The angular speed $\boldsymbol{\omega}$ is the magnitude of angular velocity.

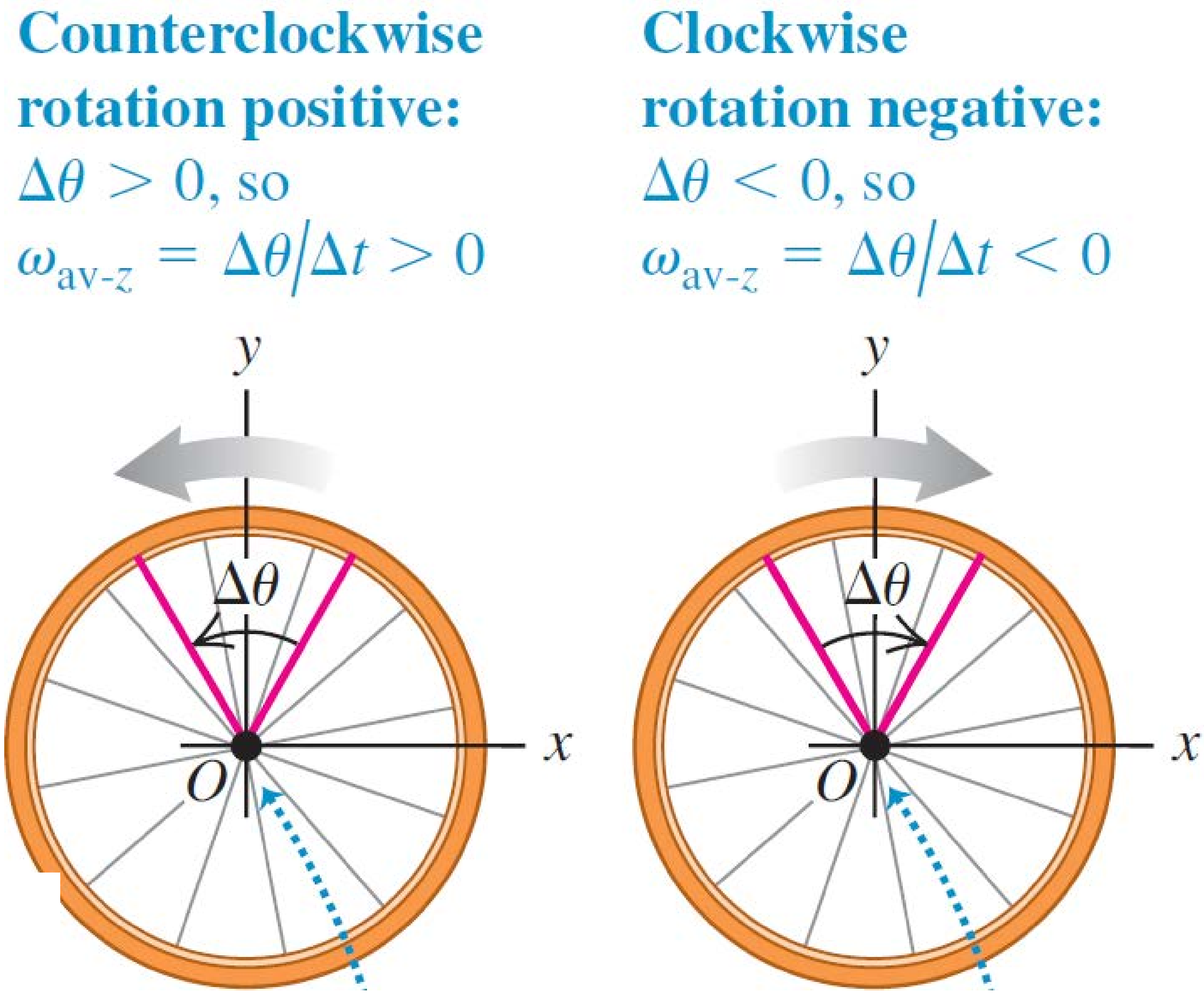
CAUTION **Angular velocity vs. linear velocity** Keep in mind the distinction between angular velocity ω_z and ordinary velocity, or *linear velocity*, v_x (see Section 2.2). If an object has a velocity v_x, the object as a whole is *moving* along the x-axis. By contrast, if an object has an angular velocity ω_z, then it is *rotating* around the z-axis. We do *not* mean that the object is moving along the z-axis.

- *At any instant, every part of a rotating rigid body has the same angular velocity.*

$$1 \text{ rev/s} = 2\pi \text{ rad/s} \qquad \text{and} \qquad 1 \text{ rev/min} = 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

Angular Velocity: Properties

- Angular velocity $\boldsymbol{\omega}_z$ can be positive or negative.
- The angular speed $\boldsymbol{\omega}$ is the magnitude of angular velocity.



Axis of rotation (z-axis) passes through origin and points out of page.

Example 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

(a) Find θ, in radians and in degrees, at $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the distance that a particle on the flywheel rim moves over the time interval from $t_1 = 2.0$ s to $t_2 = 5.0$ s. (c) Find the average angular velocity, in rad/s and in rev/min, over that interval. (d) Find the instantaneous angular velocities at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

Example 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

(a) Find θ, in radians and in degrees, at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

We substitute the values of t into the equation for θ:

$$\theta_2 = (2.0\ \text{rad/s}^3)(5.0\ \text{s})^3 = 250\ \text{rad}$$

$$= (250\ \text{rad})\frac{360^\circ}{2\pi\ \text{rad}} = 14{,}000^\circ$$

Example 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0 \text{ rad/s}^3)t^3$$

(b) Find the distance that a particle on the flywheel rim moves over the time interval from $t_1 = 2.0$ s to $t_2 = 5.0$ s.

(b) During the interval from t_1 to t_2 the flywheel's angular displacement is $\Delta\theta = \theta_2 - \theta_1 = 250 \text{ rad} - 16 \text{ rad} = 234 \text{ rad}$. The radius r is half the diameter, or 0.18 m. To use Eq. (9.1), the angles *must* be expressed in radians:

$$s = r\theta_2 - r\theta_1 = r\Delta\theta = (0.18 \text{ m})(234 \text{ rad}) = 42 \text{ m}$$

Example 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

(c) Find the average angular velocity, in rad/s and in rev/min, over that interval.

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250\ \text{rad} - 16\ \text{rad}}{5.0\ \text{s} - 2.0\ \text{s}} = 78\ \text{rad/s}$$

Example 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

(d) Find the instantaneous angular velocities at $t_1 = 2.0\ \text{s}$ and $t_2 = 5.0\ \text{s}$.

$$\omega_z = \frac{d\theta}{dt} = \frac{d}{dt}[(2.0\ \text{rad/s}^3)t^3] = (2.0\ \text{rad/s}^3)(3t^2)$$

$$= (6.0\ \text{rad/s}^3)t^2$$

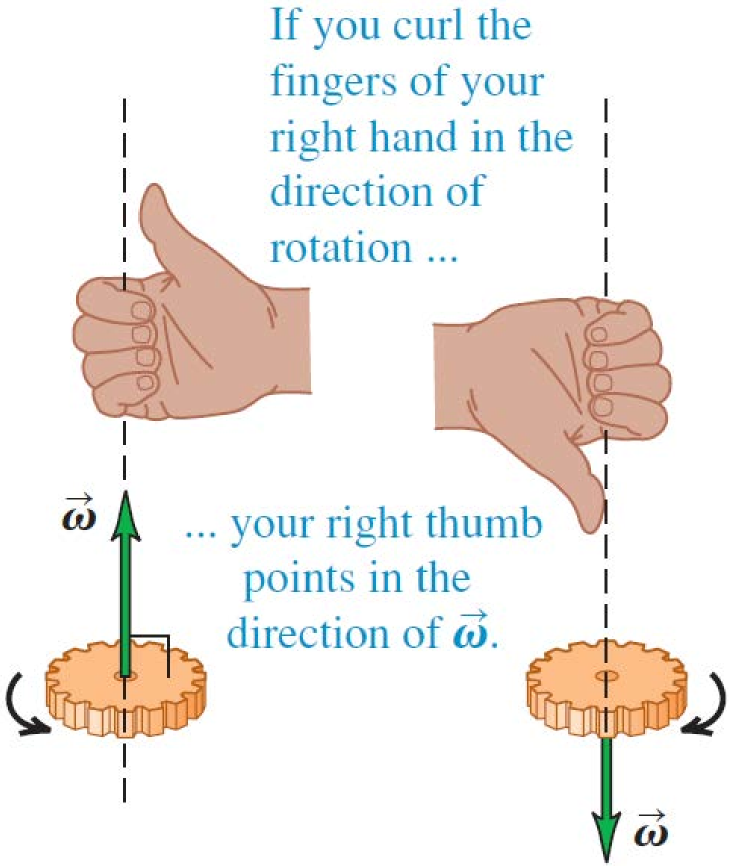
At times $t_1 = 2.0\ \text{s}$ and $t_2 = 5.0\ \text{s}$ we have

$$\omega_{1z} = (6.0\ \text{rad/s}^3)(2.0\ \text{s})^2 = 24\ \text{rad/s}$$

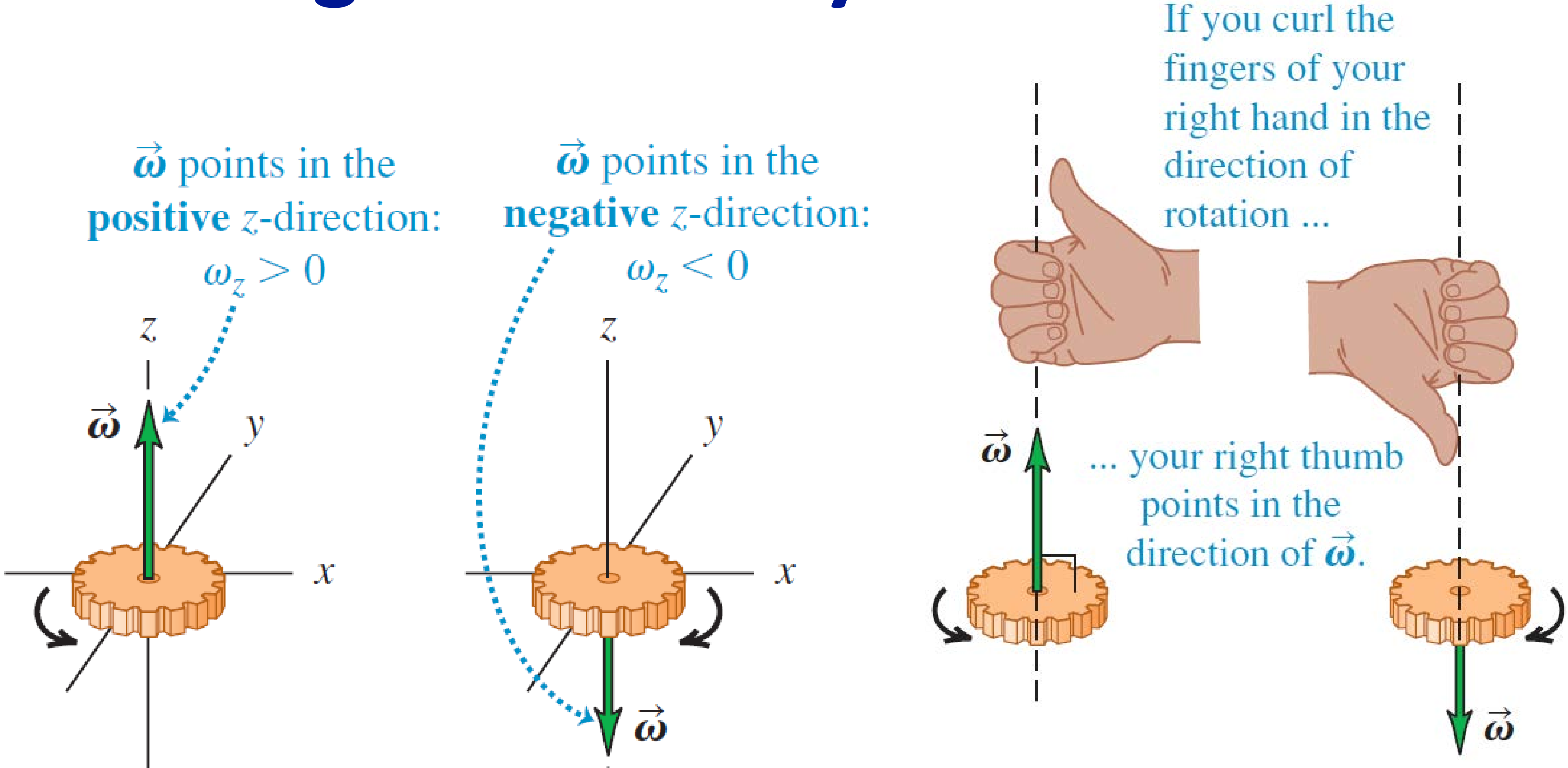
$$\omega_{2z} = (6.0\ \text{rad/s}^3)(5.0\ \text{s})^2 = 150\ \text{rad/s}$$

Angular Velocity As a Vector

Just as v_x is the x-component of the velocity vector $\vec{v}$, ω_z is the z-component of an angular velocity *vector* $\vec{\omega}$ directed along the axis of rotation. As Fig. 9.5a shows, the direction of $\vec{\omega}$ is given by the right-hand rule that we used to define the vector



Angular Velocity As a Vector



Angular Acceleration

If ω_{1z} and ω_{2z} are the instantaneous angular velocities at times t_1 and t_2, we define the **average angular acceleration** $\alpha_{\text{av-}z}$ over the interval $\Delta t = t_2 - t_1$ as the change in angular velocity divided by Δt (Fig. 9.6):

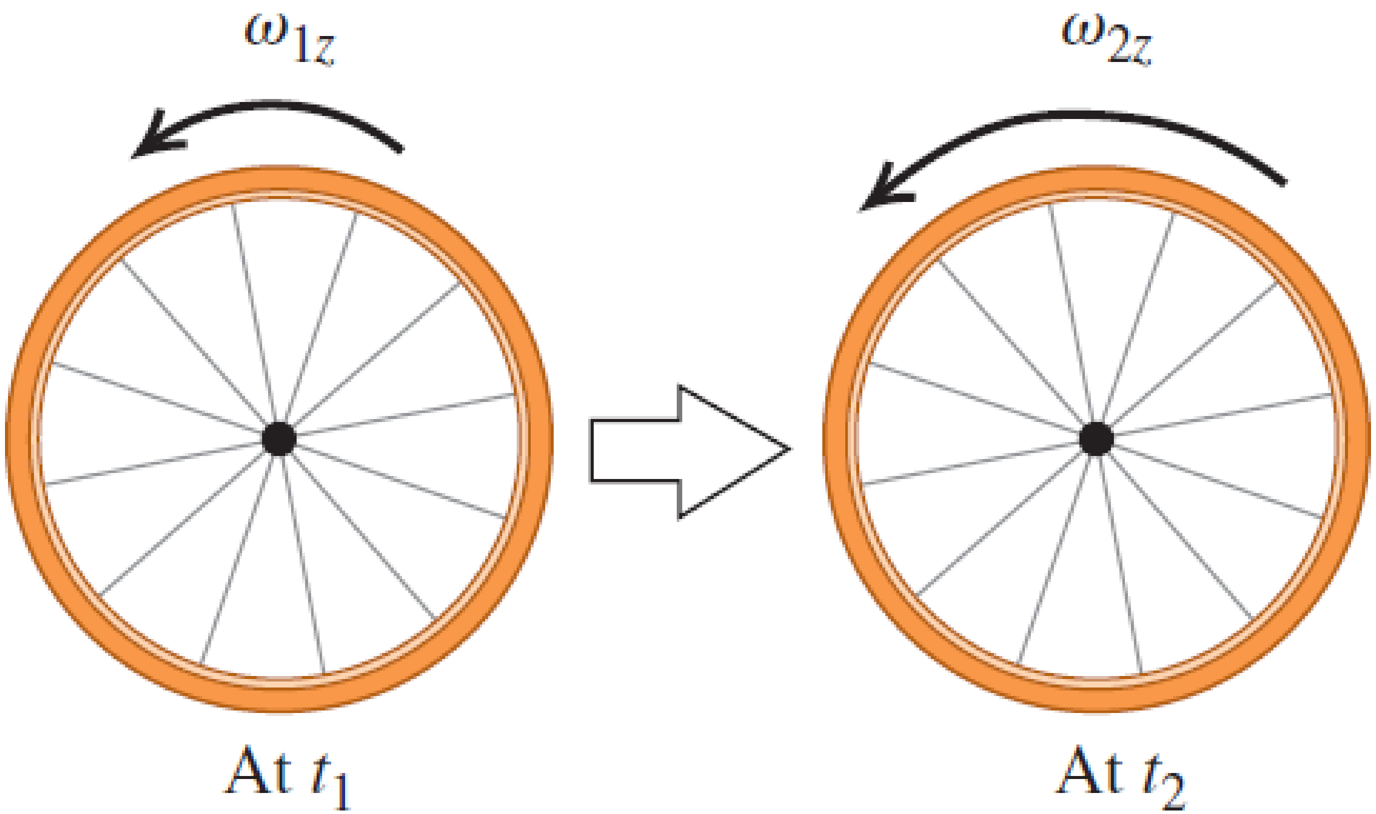
$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t} \tag{9.4}$$

Angular Acceleration

The average angular acceleration is the change in angular velocity divided by the time interval:

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}$$

$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}$$



Example 9.2 Calculating angular acceleration

For the flywheel of Example 9.1, (a) find the average angular acceleration between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the instantaneous angular accelerations at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0 \text{ rad/s}^3)t^3$$

EXECUTE: (a) From Example 9.1, the values of ω_z at the two times are

$$\omega_{1z} = 24 \text{ rad/s} \qquad \omega_{2z} = 150 \text{ rad/s}$$

$$\alpha_{\text{av-}z} = \frac{150 \text{ rad/s} - 24 \text{ rad/s}}{5.0 \text{ s} - 2.0 \text{ s}} = 42 \text{ rad/s}^2$$

Example 9.2 Calculating angular acceleration

For the flywheel of Example 9.1, (a) find the average angular acceleration between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the instantaneous angular accelerations at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0\ \text{rad/s}^3)t^3$$

(b) From Eq. (9.5), the value of α_z at any time t is

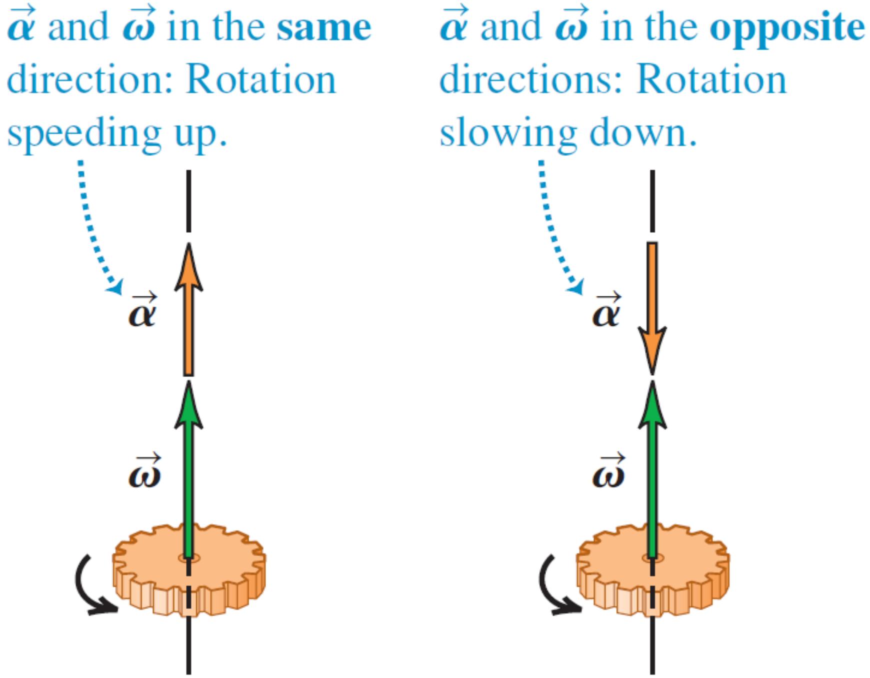
$$\alpha_z = \frac{d\omega_z}{dt} = \frac{d}{dt}[(6.0\ \text{rad/s}^3)(t^2)] = (6.0\ \text{rad/s}^3)(2t) = (12\ \text{rad/s}^3)t$$

$$\alpha_{1z} = (12\ \text{rad/s}^3)(2.0\ \text{s}) = 24\ \text{rad/s}^2$$

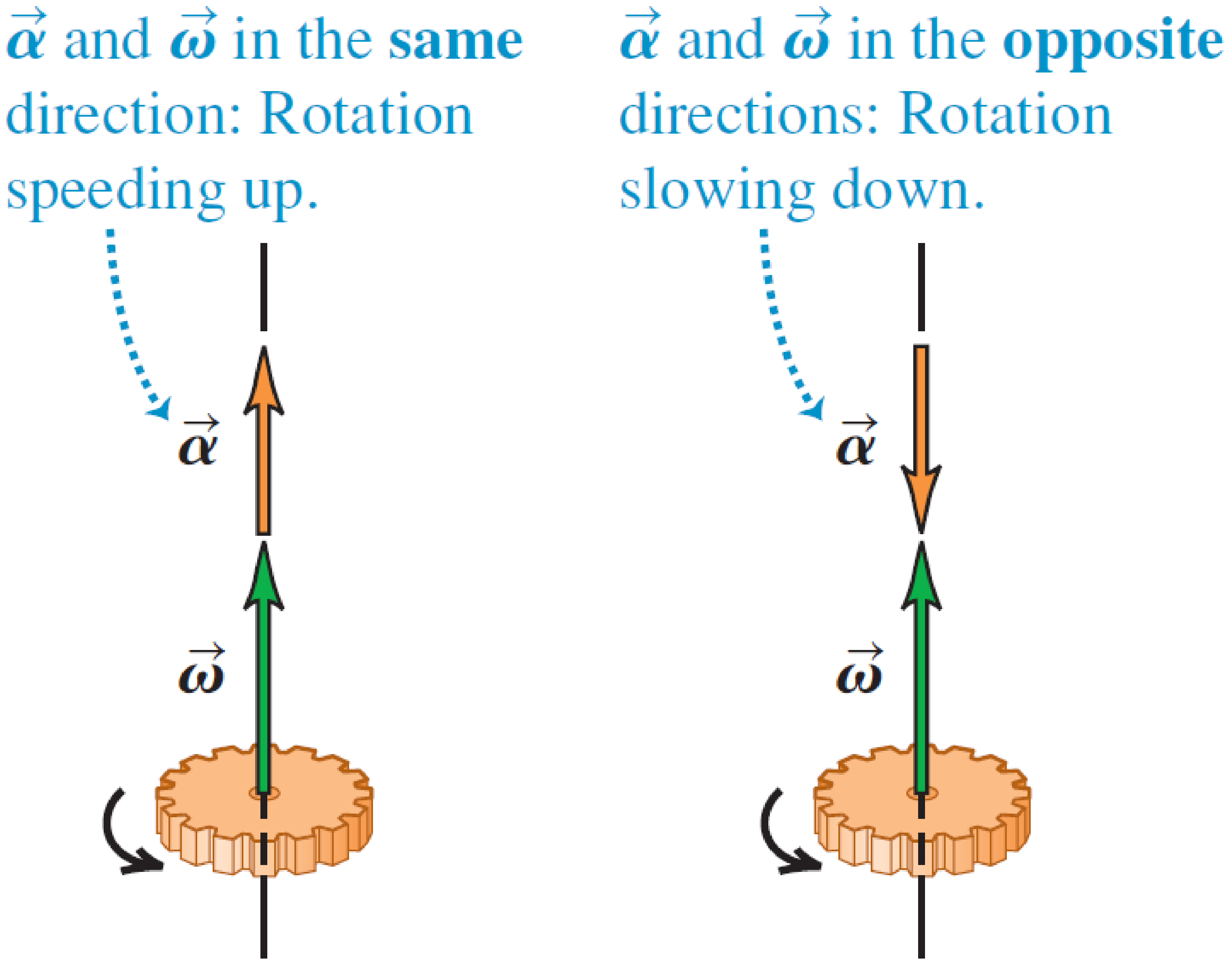
$$\alpha_{2z} = (12\ \text{rad/s}^3)(5.0\ \text{s}) = 60\ \text{rad/s}^2$$

Angular Acceleration As a Vector

Just as we did for angular velocity, it's useful to define an angular acceleration *vector* $\vec{\boldsymbol{\alpha}}$. Mathematically, $\vec{\boldsymbol{\alpha}}$ is the time derivative of the angular velocity vector $\vec{\boldsymbol{\omega}}$. If the object rotates around the fixed z-axis, then $\vec{\boldsymbol{\alpha}}$ has only a z-component; the quantity α_z is just that component. In this case, $\vec{\boldsymbol{\alpha}}$ is in the same direction as $\vec{\boldsymbol{\omega}}$ if the rotation is speeding up and opposite to $\vec{\boldsymbol{\omega}}$ if the rotation is slowing down



Angular Acceleration As a Vector



Rotation with Constant Angular Acceleration

- By definition $\alpha_z = \dfrac{\omega_z - \omega_{0z}}{t - 0}$ is a constant

$$\omega_z = \omega_{0z} + \alpha_z t \qquad \text{(constant angular acceleration only)} \qquad (9.7)$$

With constant angular acceleration, the angular velocity changes at a uniform rate, so its average value between 0 and t is the average of the initial and final values

$$\omega_{\text{av-}z} = \frac{\omega_{0z} + \omega_z}{2}$$

But by definition again $\omega_{\text{av-}z} = \dfrac{\theta - \theta_0}{t - 0}$

Combine the two equations and we can get (recall constant acceleration):

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \qquad \text{(constant angular acceleration only)}$$

Rotation with Constant Angular Acceleration

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \qquad \text{(constant angular acceleration only)} \qquad (9.10)$$

To obtain a relationship between θ and t that doesn't contain ω_z, we substitute Eq. (9.7) into Eq. (9.10):

$$\theta - \theta_0 = \tfrac{1}{2}[\omega_{0z} + (\omega_{0z} + \alpha_z t)]t \qquad \text{or}$$

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \qquad \text{(constant angular acceleration only)} \qquad (9.11)$$

Rotation with Constant Angular Acceleration

Following the same procedure as for straight-line motion

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad \text{(constant angular acceleration only)} \qquad (9.12)$$

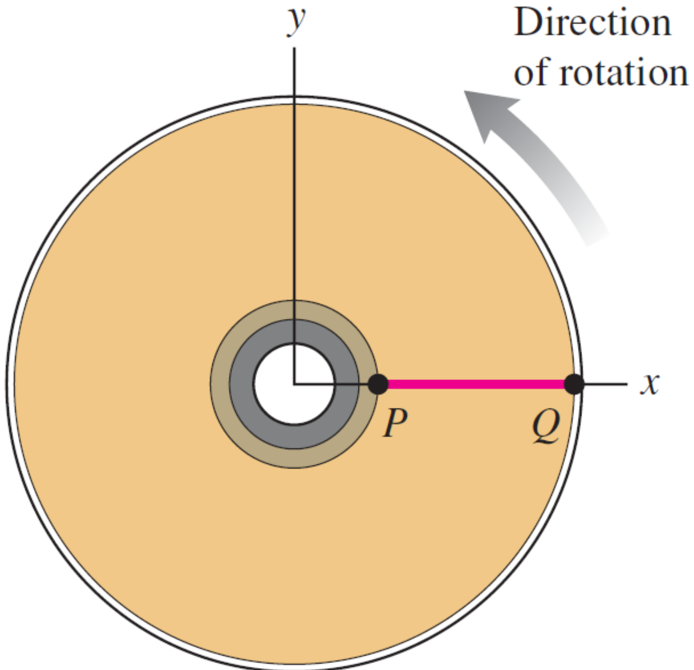
CAUTION **Constant angular acceleration** Keep in mind that all of these results are valid *only* when the angular acceleration α_z is *constant;* be careful not to try to apply them to problems in which α_z is *not* constant. Table 9.1 shows the analogy between Eqs. (9.7), (9.10), (9.11), and (9.12) for fixed-axis rotation with constant angular acceleration and the corresponding equations for straight-line motion with constant linear acceleration.

Rotation with Constant Angular Acceleration

Straight-Line Motion with Constant Linear Acceleration		Fixed-Axis Rotation with Constant Angular Acceleration	
$a_x = \text{constant}$		$\alpha_z = \text{constant}$	
$v_x = v_{0x} + a_x t$	(2.8)	$\omega_z = \omega_{0z} + \alpha_z t$	(9.7)
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	(2.12)	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$	(9.11)
$v_x^2 = v_{0x}^2 + 2a_x (x - x_0)$	(2.13)	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z (\theta - \theta_0)$	(9.12)
$x - x_0 = \frac{1}{2}(v_{0x} + v_x) t$	(2.14)	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z) t$	(9.10)

Example 9.3 Constant angular acceleration

You have finished watching a movie on Blu-ray and the disc is slowing to a stop. The disc's angular velocity at $t = 0$ is 27.5 rad/s, and its angular acceleration is a constant $-10.0\ \text{rad/s}^2$. A line PQ on the disc's surface lies along the $+x$-axis at $t = 0$ (Fig. 9.8). (a) What is the disc's angular velocity at $t = 0.300$ s? (b) What angle does the line PQ make with the $+x$-axis at this time?



(a) at $t = 0.300$ s we have

$$\omega_z = \omega_{0z} + \alpha_z t = 27.5\ \text{rad/s} + (-10.0\ \text{rad/s}^2)(0.300\ \text{s}$$

$$= 24.5\ \text{rad/s}$$

Example 9.3 Constant angular acceleration

You have finished watching a movie on Blu-ray and the disc is slowing to a stop. The disc's angular velocity at $t = 0$ is 27.5 rad/s, and its angular acceleration is a constant $-10.0\ \text{rad/s}^2$. A line PQ on the disc's surface lies along the $+x$-axis at $t = 0$ (Fig. 9.8). (a) What is the disc's angular velocity at $t = 0.300$ s? (b) What angle does the line PQ make with the $+x$-axis at this time?

(b)

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2$$

$$= 0 + (27.5\ \text{rad/s})(0.300\ \text{s}) + \tfrac{1}{2}(-10.0\ \text{rad/s}^2)(0.300\ \text{s})^2$$

$$= 7.80\ \text{rad} = 7.80\ \text{rad}\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = 1.24\ \text{rev}$$

9.11 •• The rotating blade of a blender turns with constant angular acceleration 1.50 rad/s^2. (a) How much time does it take to reach an angular velocity of 36.0 rad/s, starting from rest? (b) Through how many revolutions does the blade turn in this time interval?

(a) $\omega_0 = 0$, $\omega_1 = 36$ rad/s $\alpha = 1.5$ rad/s^2

so $\omega_1 - \omega_0 = \alpha \Delta t \Rightarrow \Delta t = \dfrac{36 \text{ rad/s}}{1.5 \text{ rad/s}^2}$

$= 24$ s

an angular velocity of 36.0 rad/s, starting from rest? (b) Through how many revolutions does the blade turn in this time interval?

(b) $x = \frac{1}{2}at^2$

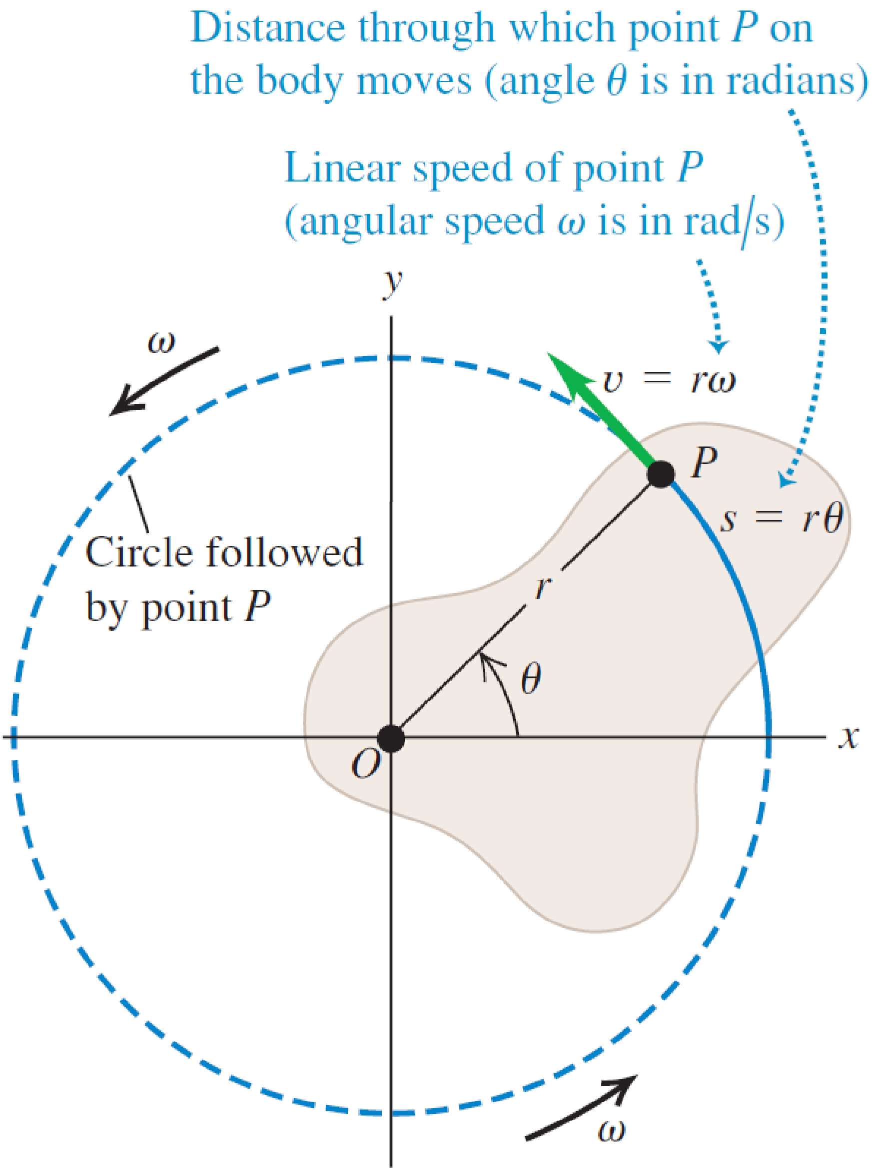
$\theta = \frac{1}{2}\alpha t^2$ starting from rest

$= 0.5 \cdot 1.5\ rad/s^2 \cdot 24 s^2 = 432\ rad$

number of revs:

$\frac{432\ rad}{2\pi\ rad/rev} = 68.75\ rev$

Relating Linear and Angular Kinematics



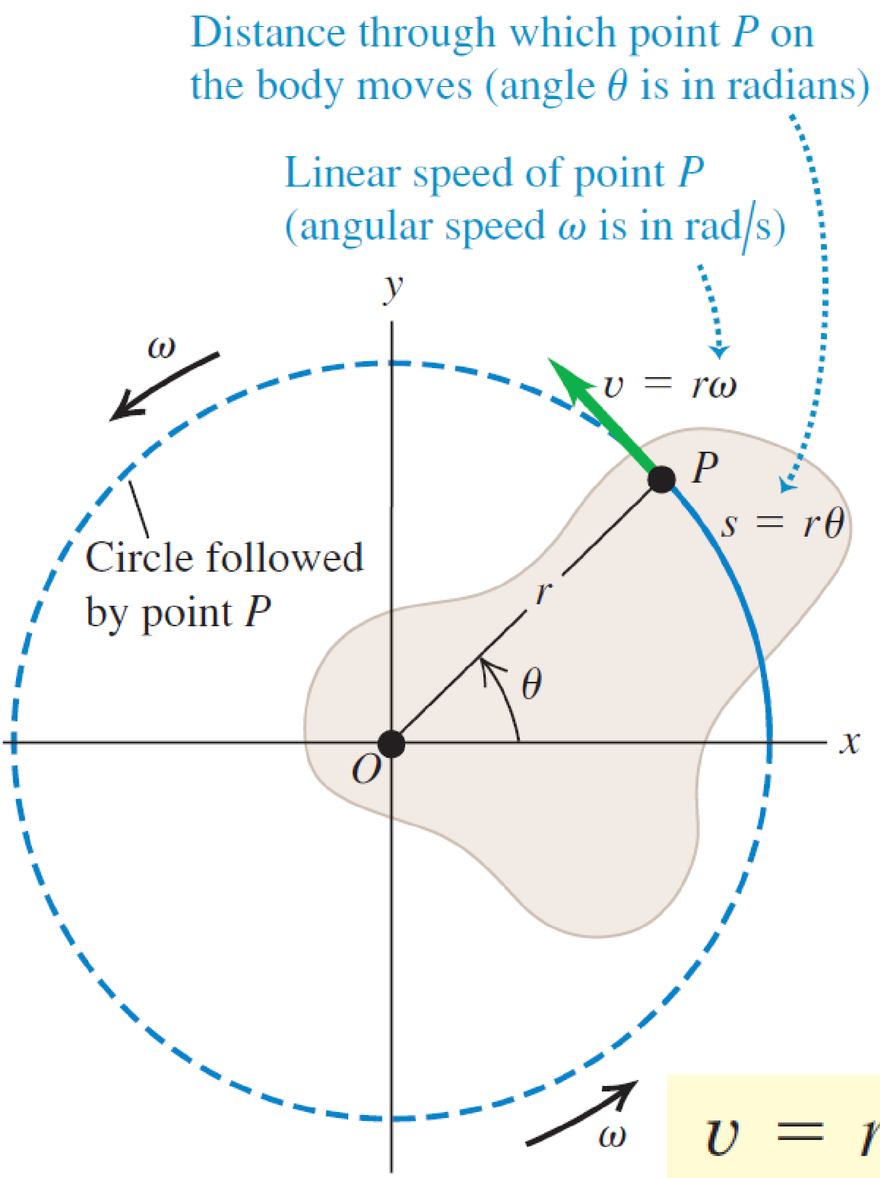
How do we find the linear speed & acceleration of a particular point in a rigid body? We need to answer this question to proceed with our study of rotation.

For example, to find the kinetic energy of a rotating body, we have to start from $K = \frac{1}{2}mv^2$ for a particle

Point P is a constant distance r from the axis of rotation, so it moves in a circle of radius r. At any time, the angle θ (in radians) and the arc length s are related by

$$s = r\theta$$

Relating Linear and Angular Kinematics



Point P is a constant distance r from the axis of rotation, so it moves in a circle of radius r. At any time, the angle θ (in radians) and the arc length s are related by

$$s = r\theta \quad \Rightarrow \quad \left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

ds/dt:

- Instantaneous linear speed of the particle
- Absolute value of the rate of change of arc length

dθ/dt:

- Instantaneous angular speed

$v = r\omega$ (relationship between linear and angular speeds)

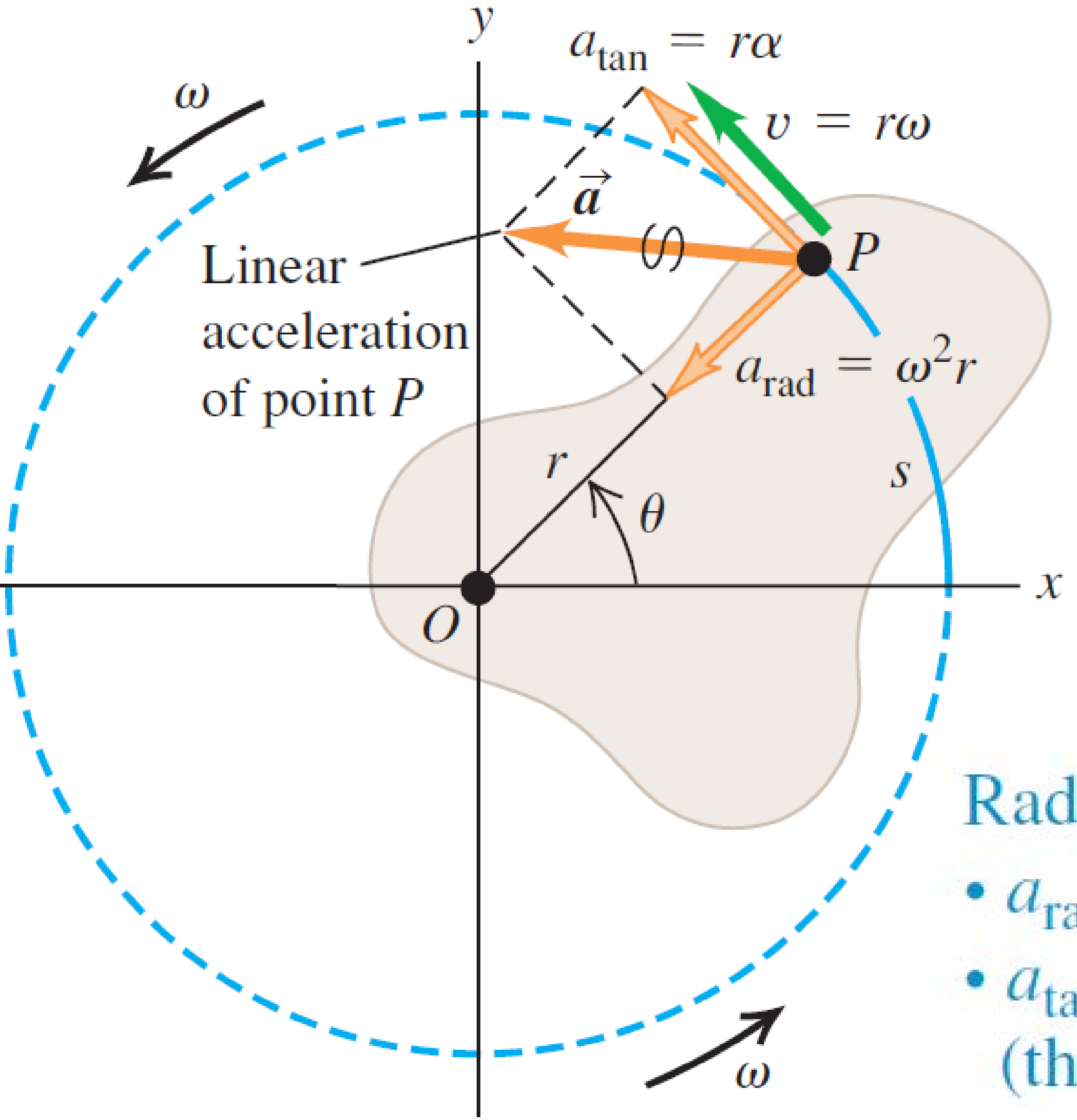
Relating Linear and Angular Kinematics

$$v = r\omega \qquad \text{(relationship between linear and angular speeds)} \qquad (9.13)$$

The farther a point is from the axis, the greater its linear speed. The *direction* of the linear velocity *vector* is tangent to its circular path at each point (Fig. 9.9).

CAUTION **Speed vs. velocity** Keep in mind the distinction between the linear and angular *speeds* v and ω, which appear in Eq. (9.13), and the linear and angular *velocities* v_x and ω_z. The quantities without subscripts, v and ω, are never negative; they are the magnitudes of the vectors $\vec{v}$ and $\vec{\omega}$, respectively, and their values tell you only how fast a particle is moving (v) or how fast a body is rotating (ω). The corresponding quantities with subscripts, v_x and ω_z, can be either positive or negative; their signs tell you the direction of the motion.

Linear Acceleration in Rigid-Body Rotation



Recall $v = r\omega$

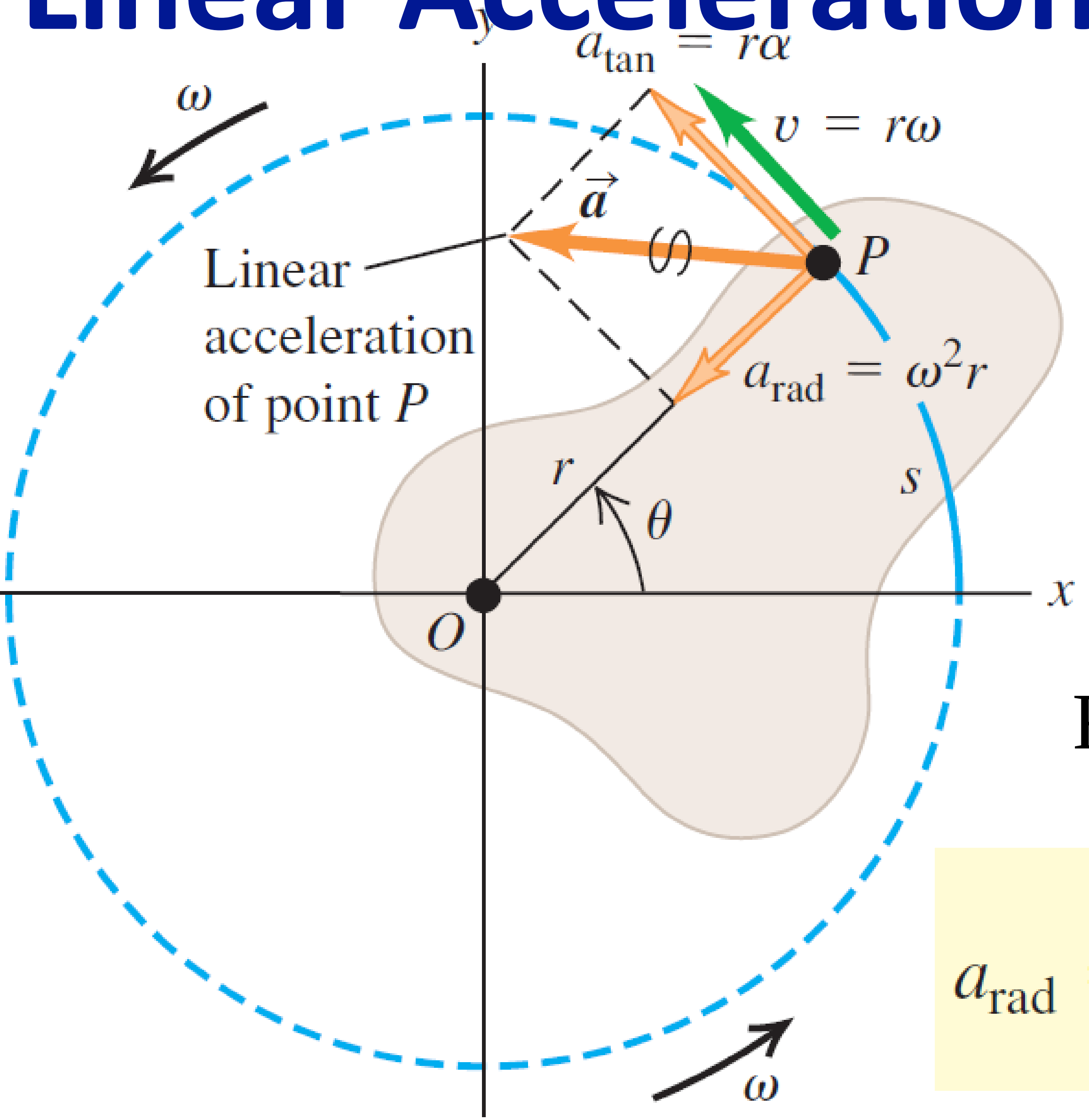
$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$ by definition

(tangential acceleration of a point on a rotating body)

Radial and tangential acceleration components:

- $a_{rad} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).

Linear Acceleration in Rigid-Body Rotation



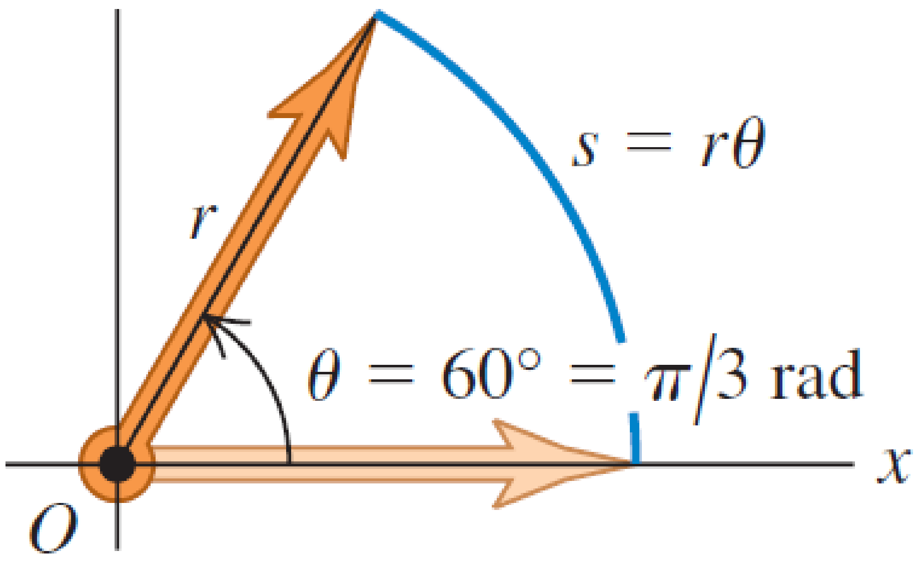
Recall Section 3.4

$$a_{rad} = \frac{v^2}{r} = \omega^2 r$$ (centripetal acceleration of a point on a rotating body)

This is true at each instant, *even when ω and v are not constant.*

Linear Acceleration in Rigid-Body Rotation

CAUTION **Use angles in radians in all equations** It's important to remember that Eq. (9.1), $s = r\theta$, is valid *only* when θ is measured in radians. The same is true of any equation derived from this, including Eqs. (9.13), (9.14), and (9.15). When you use these equations, you *must* express the angular quantities in radians, not revolutions or degrees (Fig. 9.11).



In any equation that relates linear quantities to angular quantities, the angles MUST be expressed in radians ...

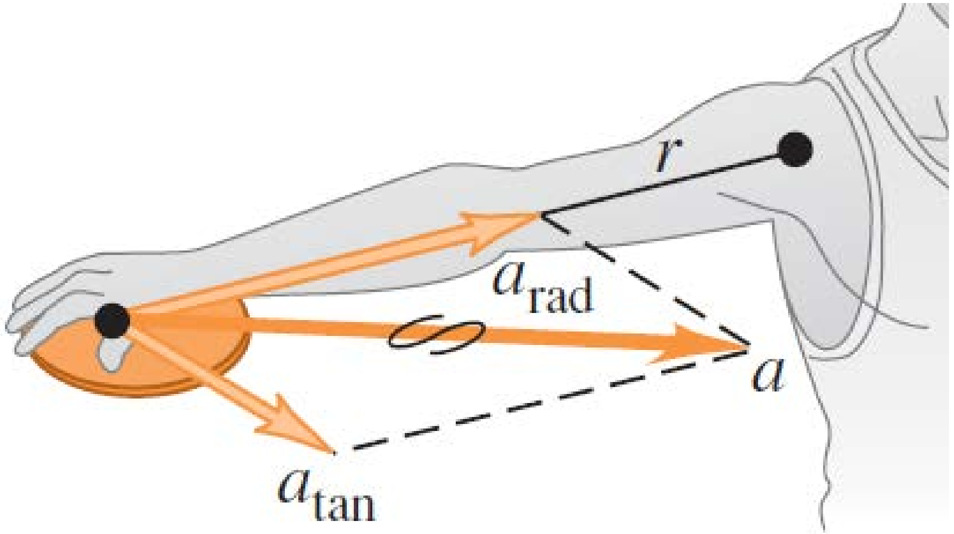
RIGHT! ▶ $s = (\pi/3)r$

... never in degrees or revolutions.

WRONG ▶ $s = \cancel{60r}$

Example 9.4 Throwing a Disk

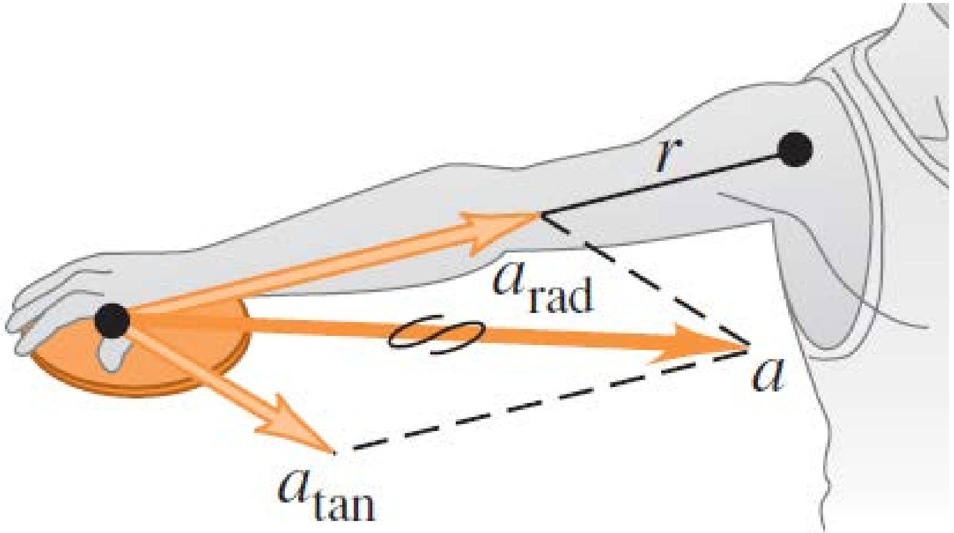
An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. At this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

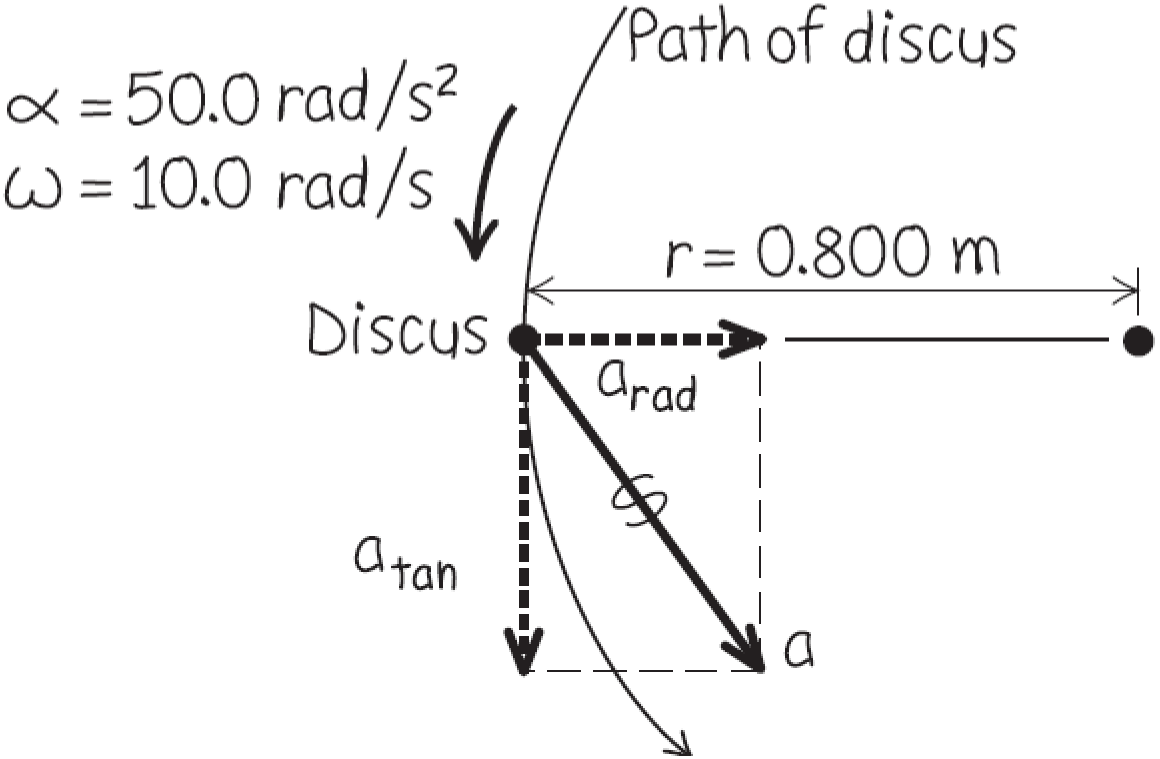


We treat the discus as a **particle traveling in a circular path** Given r = 0.800 m, v = 10.0 rad/s, and α = 50.0 rad/s^2. We will calculate a_{tan} and a_{rad} and then find the magnitude a using the Pythagorean theorem.

Example 9.4 Throwing a Disk

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. At this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

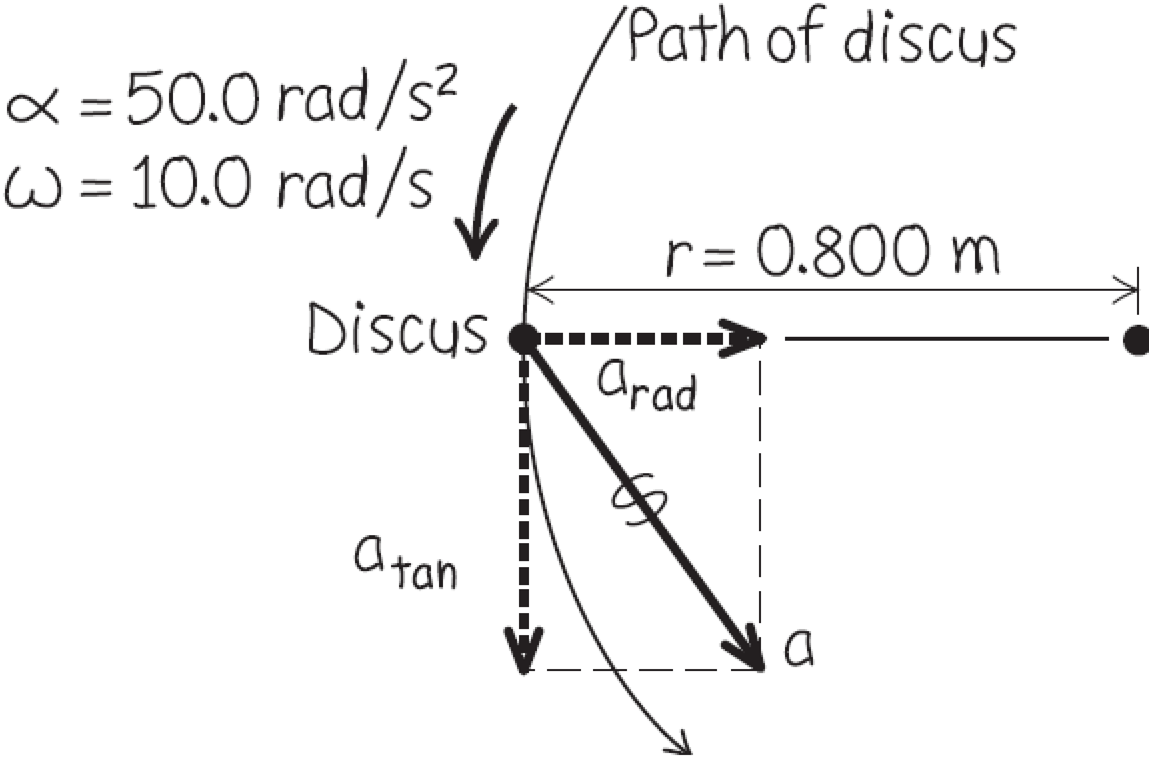




Example 9.4 Throwing a Disk

The calculation is simple, but let's go slowly in case you have questions:

$$a_{\tan} = r\alpha$$



Energy in Rotational Motion

Kinetic energy is defined in terms of the body's **angular speed** and a new quantity, called ***moment of inertia* I.**

Use this quantity like mass for angular motions (but unit is different!)

Now we apply what we know about kinetic energy: K = ½ mv^2

Think of a body as being made up of a large number of particles, with masses m_1, m_2, ….. at distances r_1, r_2, …

Suppose the body is rotating at ω, for i^{th} particle: $v_i = \omega_i r$

The kinetic energy of the *i*th particle can be expressed as

$$\frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

Energy in Rotational Motion

The *total* kinetic energy of the body is the sum of the kinetic energies of all its particles:

$$K = \tfrac{1}{2}m_1 r_1^2\omega^2 + \tfrac{1}{2}m_2 r_2^2\omega^2 + \cdots = \sum_i \tfrac{1}{2}m_i r_i^2\omega^2$$

Taking the common factor $\omega^2/2$ out of this expression, we get

$$K = \tfrac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \cdots)\omega^2 = \tfrac{1}{2}\Big(\sum_i m_i r_i^2\Big)\omega^2$$

moment of inertia

$$\boxed{I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2}$$

In analogy to K = ½ mv^2

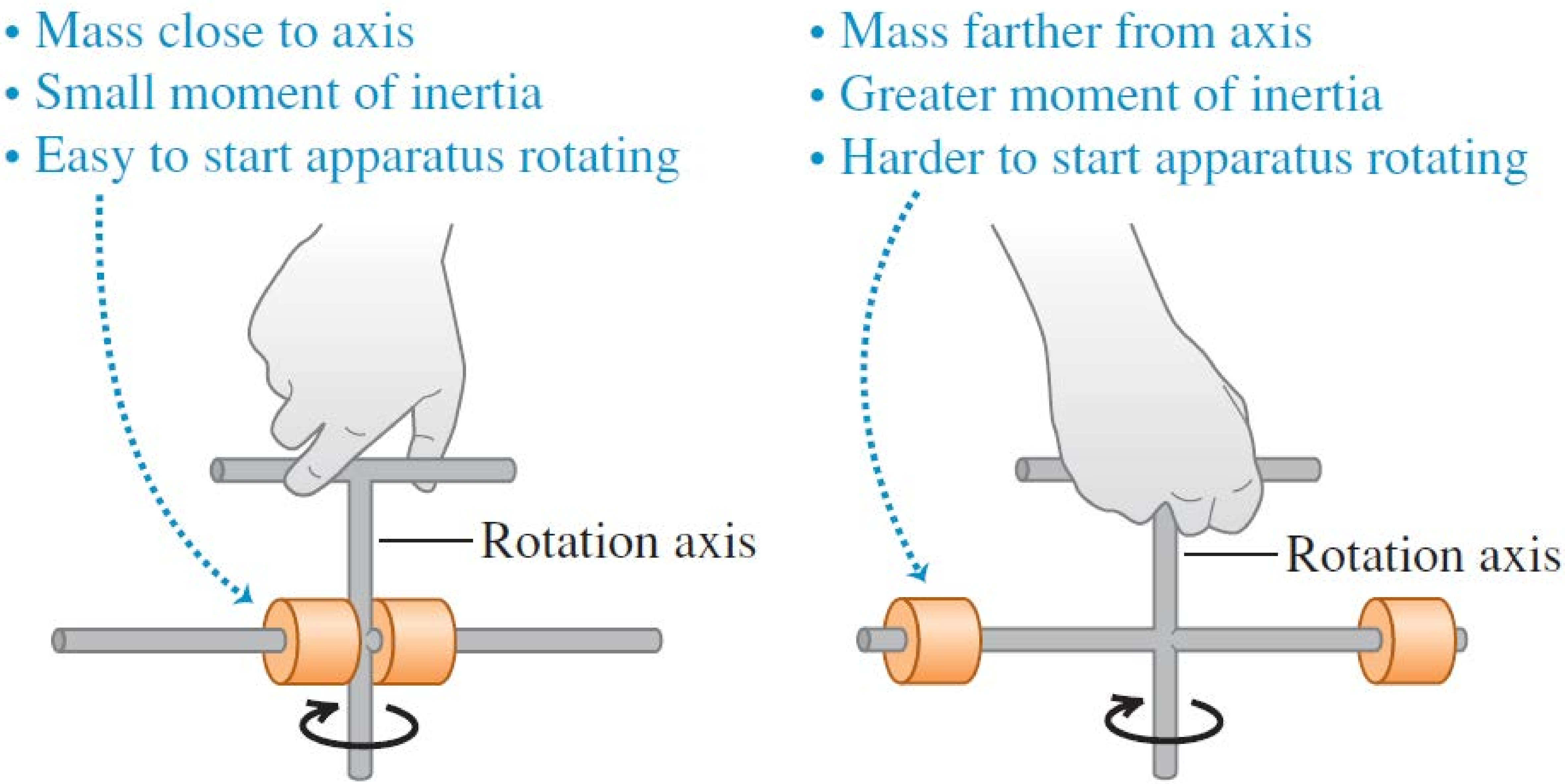
Kinetic Energy vs Moment of Inertia

In terms of moment of inertia I, the **rotational kinetic energy** K of a rigid body is

$$K = \frac{1}{2}I\omega^2 \qquad \text{(rotational kinetic energy of a rigid body)} \qquad (9.17)$$

Equation (9.17) gives a simple interpretation of moment of inertia: *The greater the moment of inertia, the greater the kinetic energy of a rigid body rotating with a given angular speed* ω.

Energy in Rotational Motion

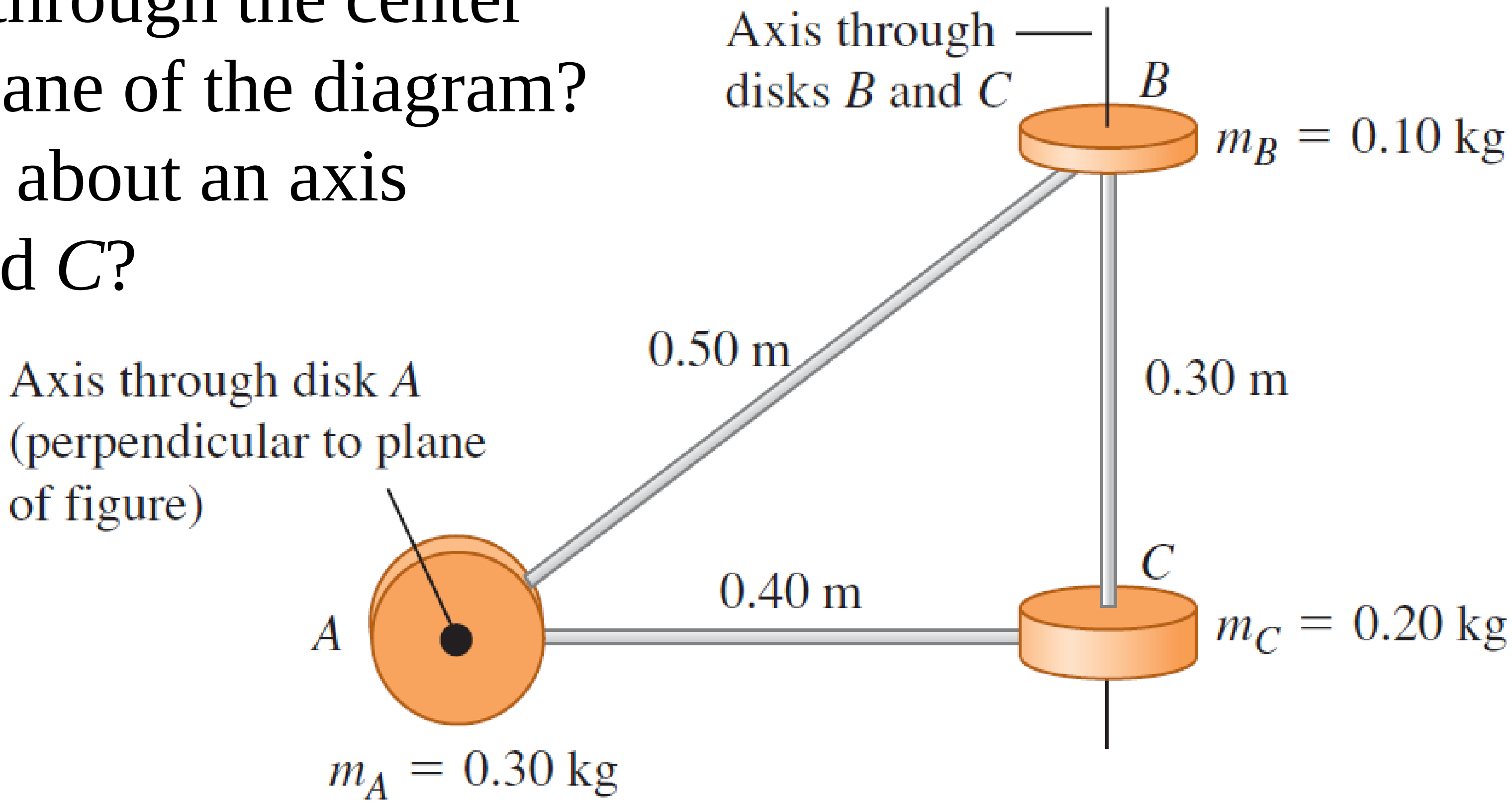


Example 9.6 Moments of inertia for different rotation axes

A machine part (Fig. 9.15) consists of three disks linked by lightweight struts. (a) What is this body's moment of inertia about an axis through the center of disk *A*, perpendicular to the plane of the diagram? (b) What is its moment of inertia about an axis through the centers of disks *B* and *C*?

(c) What is the body's kinetic energy if it rotates about the axis through *A* with angular speed ω = 4.0 rad/s

Assume we can ignore the size of the disks



Example 9.6 Moments of inertia for different rotation axes

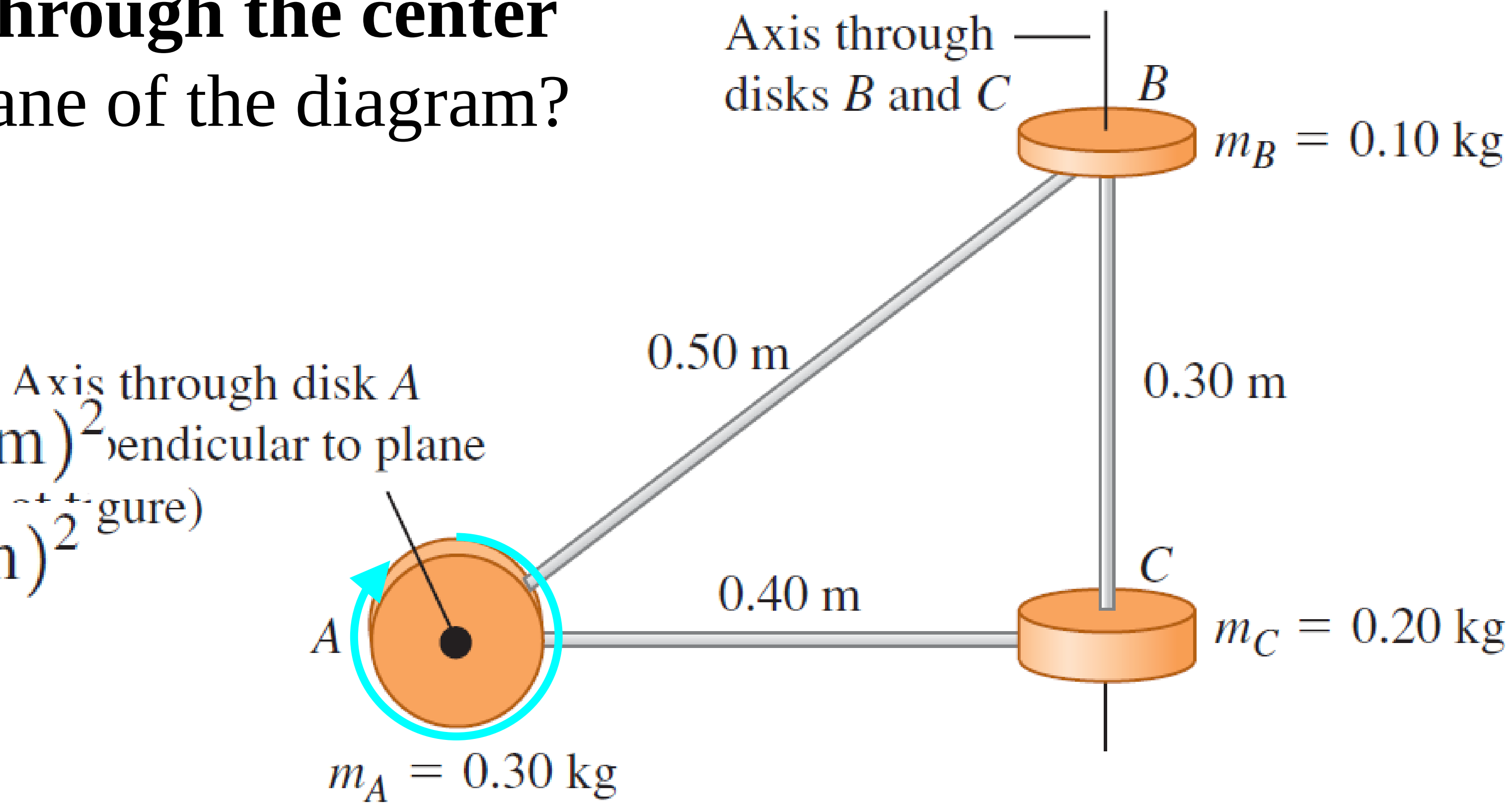
A machine part (Fig. 9.15) consists of three disks linked by lightweight struts. (a) What is this body's moment of inertia about **an axis through the center of disk *A*,** perpendicular to the plane of the diagram?

Assume we can ignore the size of the disks

For A: r = 0, so I = 0

Only *B* and *C* contribute to I:

$$
\begin{aligned}
I_A = \sum m_i r_i^2 &= (0.10\ \text{kg})(0.50\ \text{m})^2 \\
&\quad + (0.20\ \text{kg})(0.40\ \text{m})^2 \\
&= 0.057\ \text{kg}\cdot\text{m}^2
\end{aligned}
$$



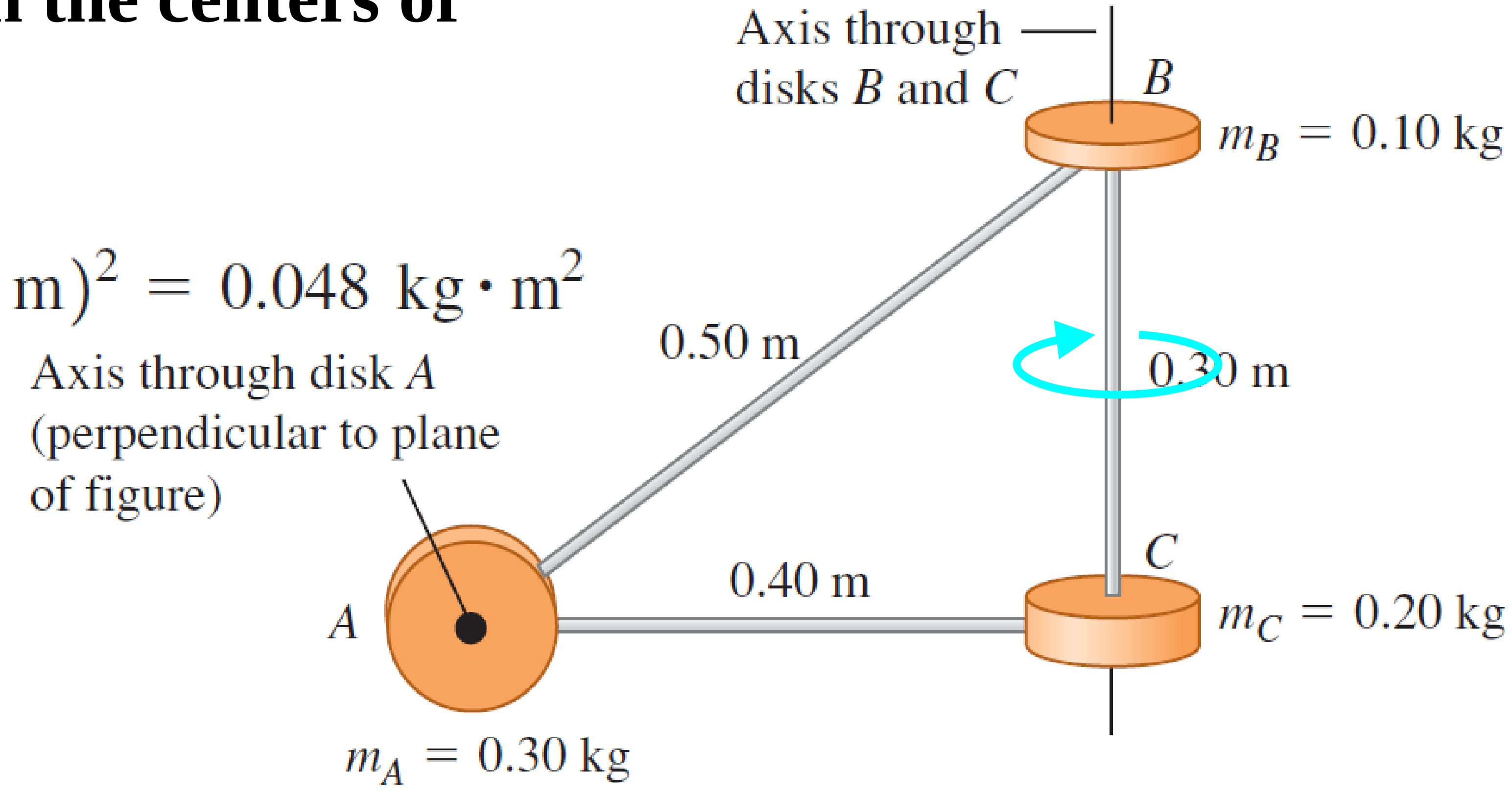
Example 9.6 Moments of inertia for different rotation axes

A machine part (Fig. 9.15) consists of three disks linked by lightweight struts. **(b)What is its moment of inertia about an axis through the centers of disks B and C?**

Assume we can ignore the size of the disks

For B and C: $r = 0$, so $I = 0$

$$I_{BC} = \sum m_i r_i^2 = (0.30 \text{ kg})(0.40 \text{ m})^2 = 0.048 \text{ kg} \cdot \text{m}^2$$

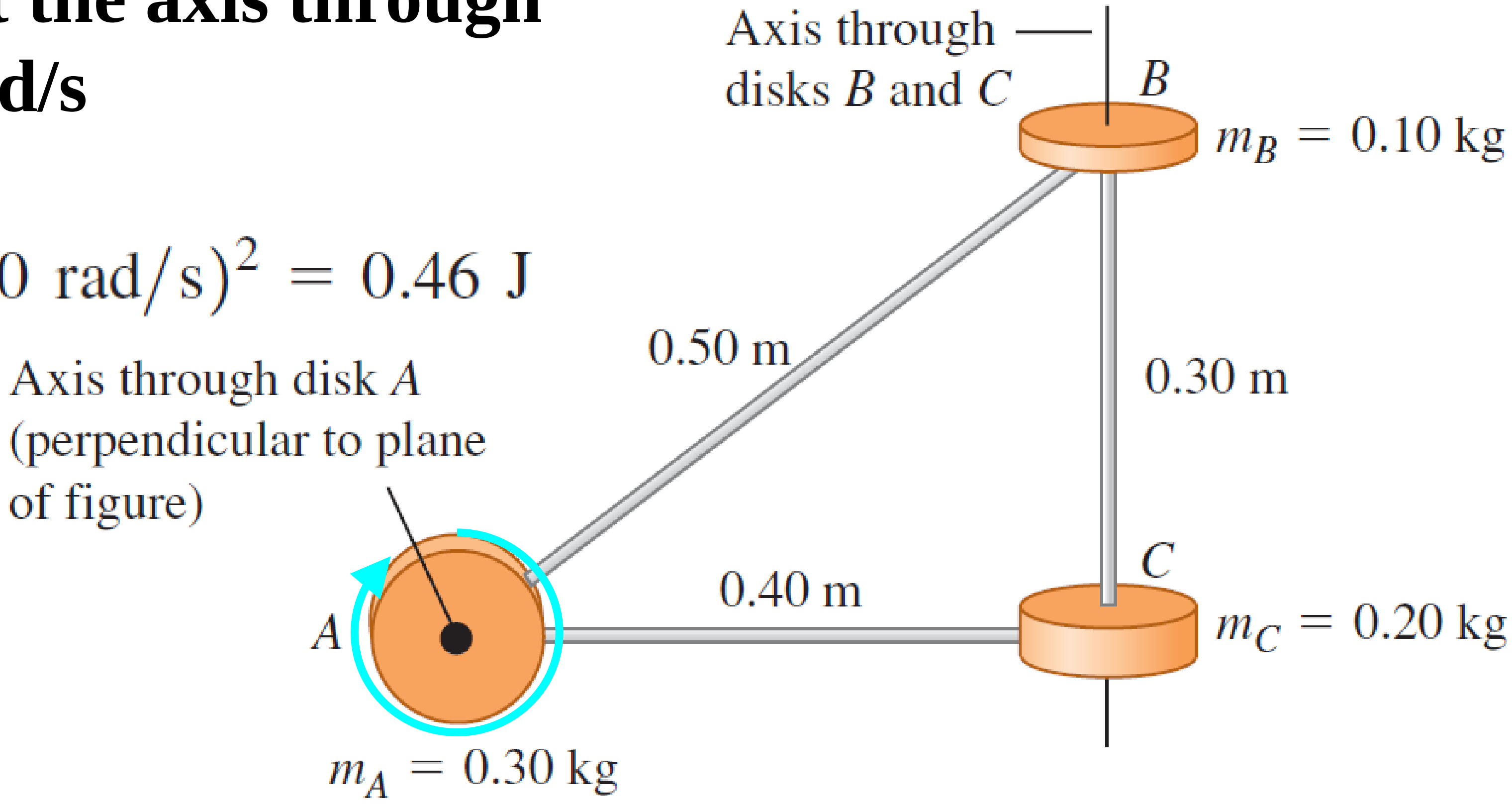


Example 9.6 Moments of inertia for different rotation axes

A machine part (Fig. 9.15) consists of three disks linked by lightweight struts. (c) **What is the body's kinetic energy if it rotates about the axis through *A* with angular speed ω = 4.0 rad/s**

Assume we can ignore the size of the disks

$$K_A = \tfrac{1}{2} I_A \omega^2 = \tfrac{1}{2}(0.057\ \text{kg} \cdot \text{m}^2)(4.0\ \text{rad/s})^2 = 0.46\ \text{J}$$



Example 9.6 Moments of inertia for different rotation axes

CAUTION **Moment of inertia depends on the choice of axis** The results of parts (a) and (b) of Example 9.6 show that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to just say, "The moment of inertia of this body is $0.048 \text{ kg} \cdot \text{m}^2$." We have to be specific and say, "The moment of inertia of this body *about the axis through B and C* is $0.048 \text{ kg} \cdot \text{m}^2$."

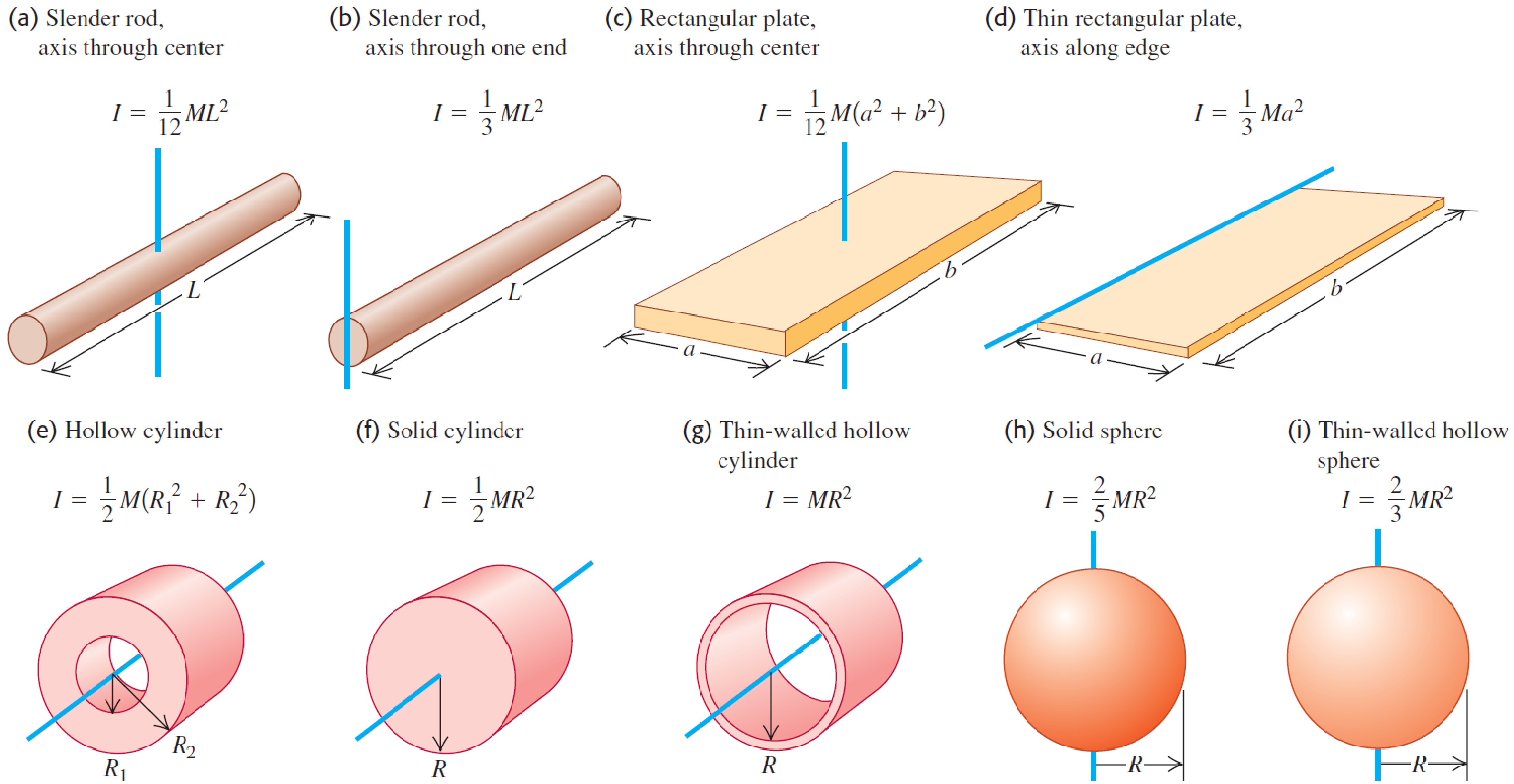
Example 9.6 Moments of inertia for different rotation axes

CAUTION **Moment of inertia depends on the choice of axis** The results of parts (a) and (b) of Example 9.6 show that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to just say, "The moment of inertia of this body is $0.048\ \mathrm{kg \cdot m^2}$." We have to be specific and say, "The moment of inertia of this body *about*

CAUTION **Computing the moment of inertia** You may be tempted to try to compute the moment of inertia of a body by assuming that all the mass is concentrated at the center of mass and multiplying the total mass by the square of the distance from the center of mass to the axis. Resist that temptation; it doesn't work! For example, when a uniform thin rod of length L and mass M is pivoted about an axis through one end, perpendicular to the rod, the moment of inertia is $I = ML^2/3$ [case (b) in Table 9.2]. If we took the mass as concentrated at the center, a distance $L/2$ from the axis, we would obtain the *incorrect* result $I = M(L/2)^2 = ML^2/4$.

Perhaps your Favorite Table

Table 9.2 Moments of Inertia of Various Bodies



Example 9.8 An unwinding cable II

We wrap a light, nonstretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Find expressions for the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.

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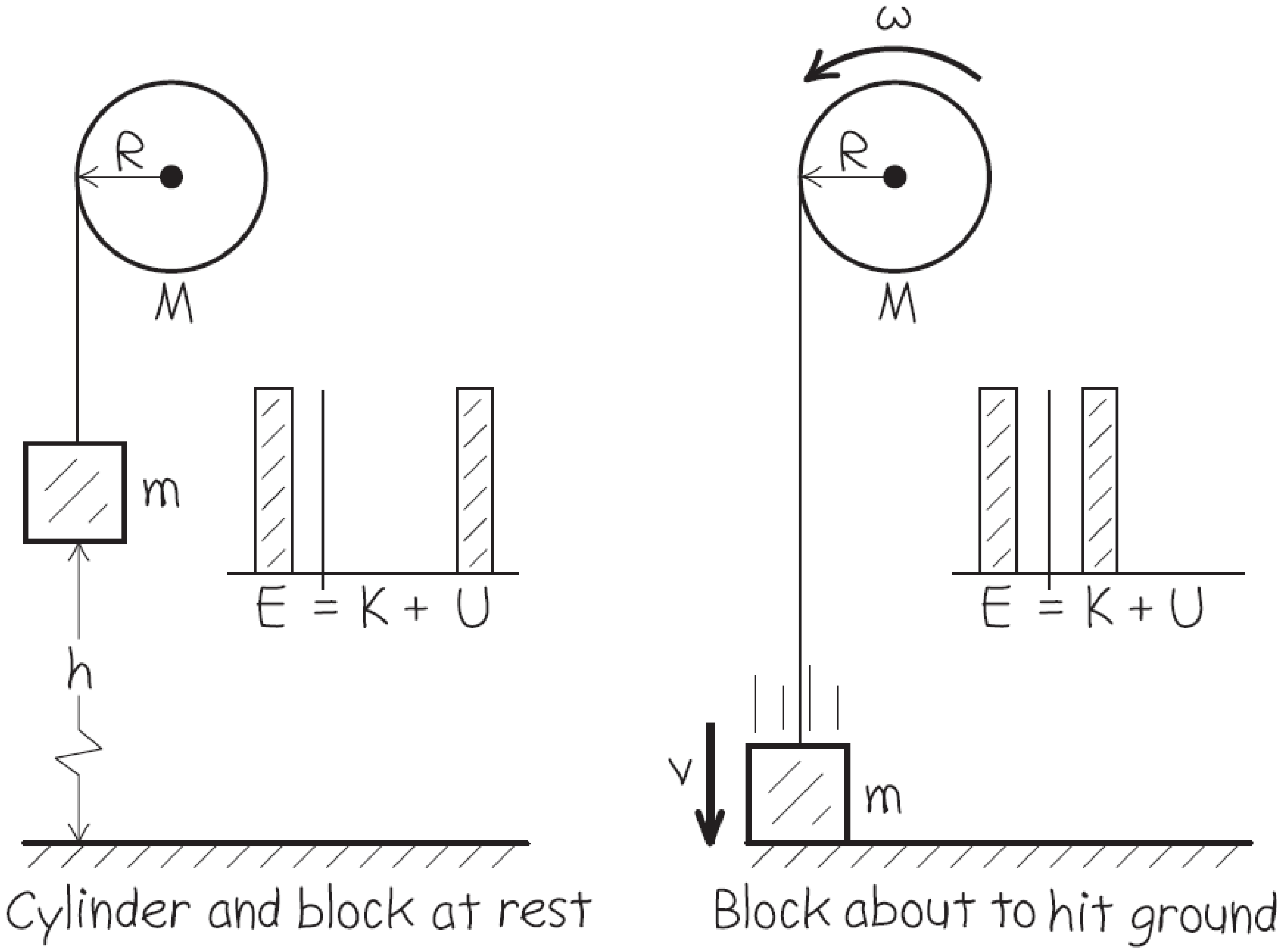


Example 9.8 An unwinding cable II

initial kinetic energy is $K_1 = 0.$

$$U_1 = mgh$$

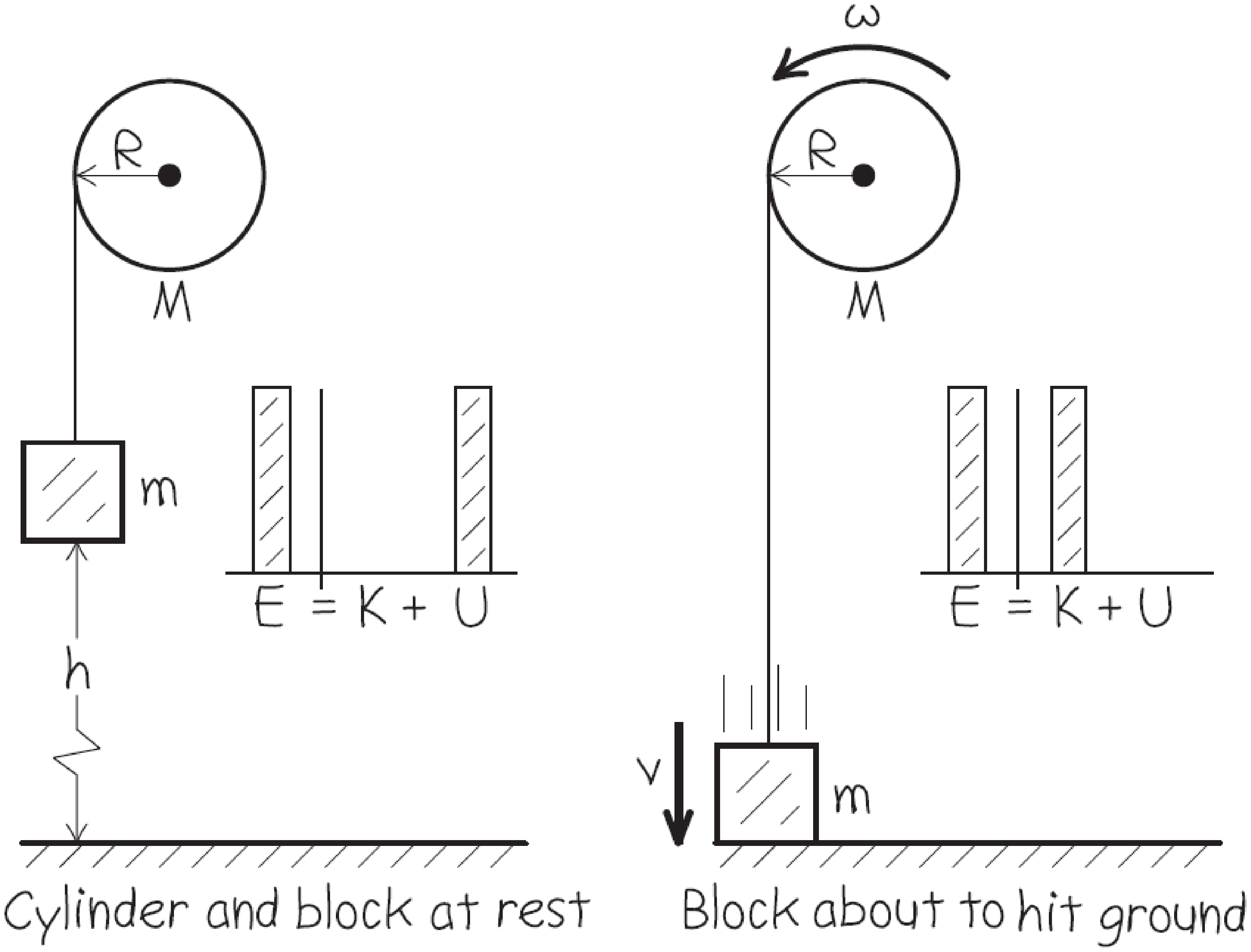
Just before the block hits the floor

$$U_2 = 0.$$

$$K_2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Energy Conservation:

$$K_1 + U_1 = K_2 + U_2$$



Example 9.8 An unwinding cable II

initial kinetic energy is $K_1 = 0.$

$$U_1 = mgh$$

$$U_2 = 0.$$

$$K_2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

Energy Conservation: $K_1 + U_1 = K_2 + U_2$

$$0 + mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + 0 = \tfrac{1}{2}\left(m + \tfrac{1}{2}M\right)v^2$$

$$v = \sqrt{\frac{2gh}{1 + M/2m}}$$

The final angular speed of the cylinder is $\omega = v/R$.

Gravitational Potential Energy for an Extended Body

In Example 9.8 the cable was of negligible mass, so we could ignore its kinetic energy as well as the gravitational potential energy associated with it. If the mass is *not* negligible, we need to know how to calculate the *gravitational potential energy* associated with such an extended body. If the acceleration of gravity g is the same at all points on the body, the gravitational potential energy is the same as though all the mass were concentrated at the center of mass of the body. Suppose we take the y-axis vertically upward. Then for a body with total mass M, the gravitational potential energy U is simply:

$$U = Mgy_{\mathrm{cm}} \qquad (9.18)$$

where y_{cm} is the y-coordinate of the center of mass. This expression applies to any extended body, whether it is rigid or not.

Gravitational Potential Energy for an Extended Body

To prove Eq. (9.18), we again represent the body as a collection of mass elements m_i. The potential energy for element m_i is $m_i g y_i$, so the total potential energy is

$$U = m_1 g y_1 + m_2 g y_2 + \cdots = (m_1 y_1 + m_2 y_2 + \cdots) g$$

But from Eq. (8.28), which defines the coordinates of the center of mass,

$$m_1 y_1 + m_2 y_2 + \cdots = (m_1 + m_2 + \cdots) y_{\mathrm{cm}} = M y_{\mathrm{cm}}$$

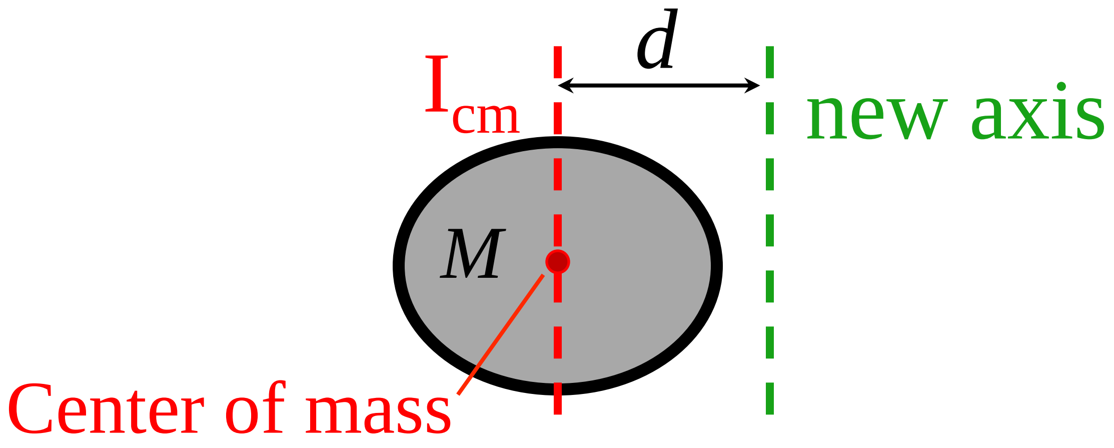
where $M = m_1 + m_2 + \cdots$ is the total mass. Combining this with the above expression for U, we find $U = M g y_{\mathrm{cm}}$ in agreement with Eq. (9.18).

Parallel-Axis Theorem

A body doesn't have just one moment of inertia.

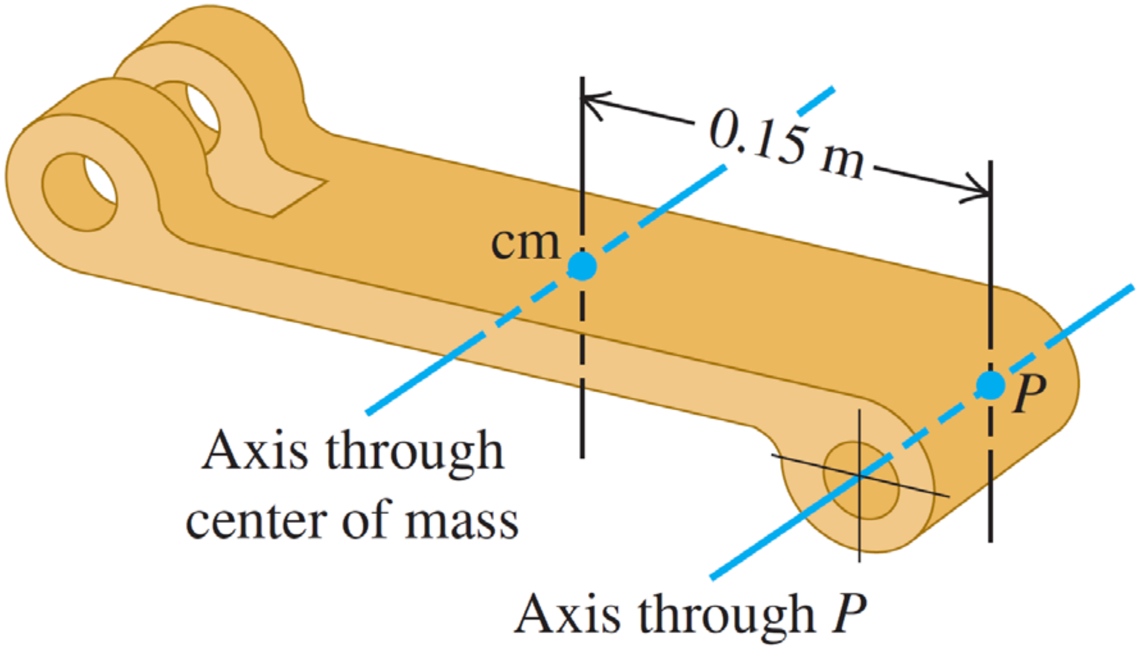
But there is a simple relationship between the moment of inertia of a body of mass M about an axis through its center of mass and the moment of inertia about any other axis parallel to the original one but displaced from it by a distance d.

$$I_P = I_{cm} + Md^2 \qquad \text{(parallel-axis theorem)}$$



Example 9.9 Using the parallel-axis theorem

A part of a mechanical linkage (Fig. 9.20) has a mass of 3.6 kg. Its moment of inertia I_P about an axis 0.15 m from its center of mass is $I_P = 0.132\ \text{kg} \cdot \text{m}^2$. What is the moment of inertia I_{cm} about a parallel axis through the center of mass?



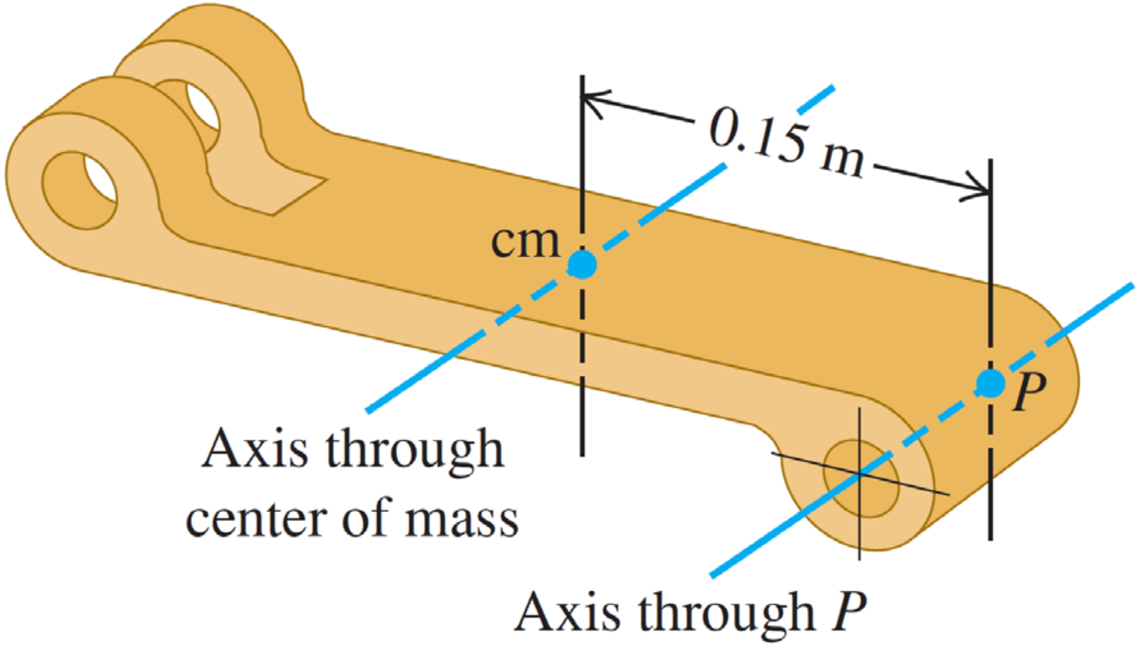
$$I_P = I_{\text{cm}} + Md^2$$

(parallel-axis theorem)

Example 9.9 Using the parallel-axis theorem

IDENTIFY, SET UP, and EXECUTE: We'll determine the target variable I_{cm} using the parallel-axis theorem, Eq. (9.19). Rearranging the equation, we obtain

$$I_{cm} = I_P - Md^2 = 0.132\ \text{kg}\cdot\text{m}^2 - (3.6\ \text{kg})(0.15\ \text{m})^2$$
$$= 0.051\ \text{kg}\cdot\text{m}^2$$



Moment-of-Inertia Calculations

If a rigid body is a **continuous distribution of mass**—like a solid cylinder or a solid sphere—it **cannot** be represented by a few point masses. Instead….

Imagine dividing the body into elements of mass *dm* at a distance *r*, as before. Then the moment of inertia is:

$$I = \int r^2 \, dm$$

If body is uniform in density ρ, $dm = \rho dV$

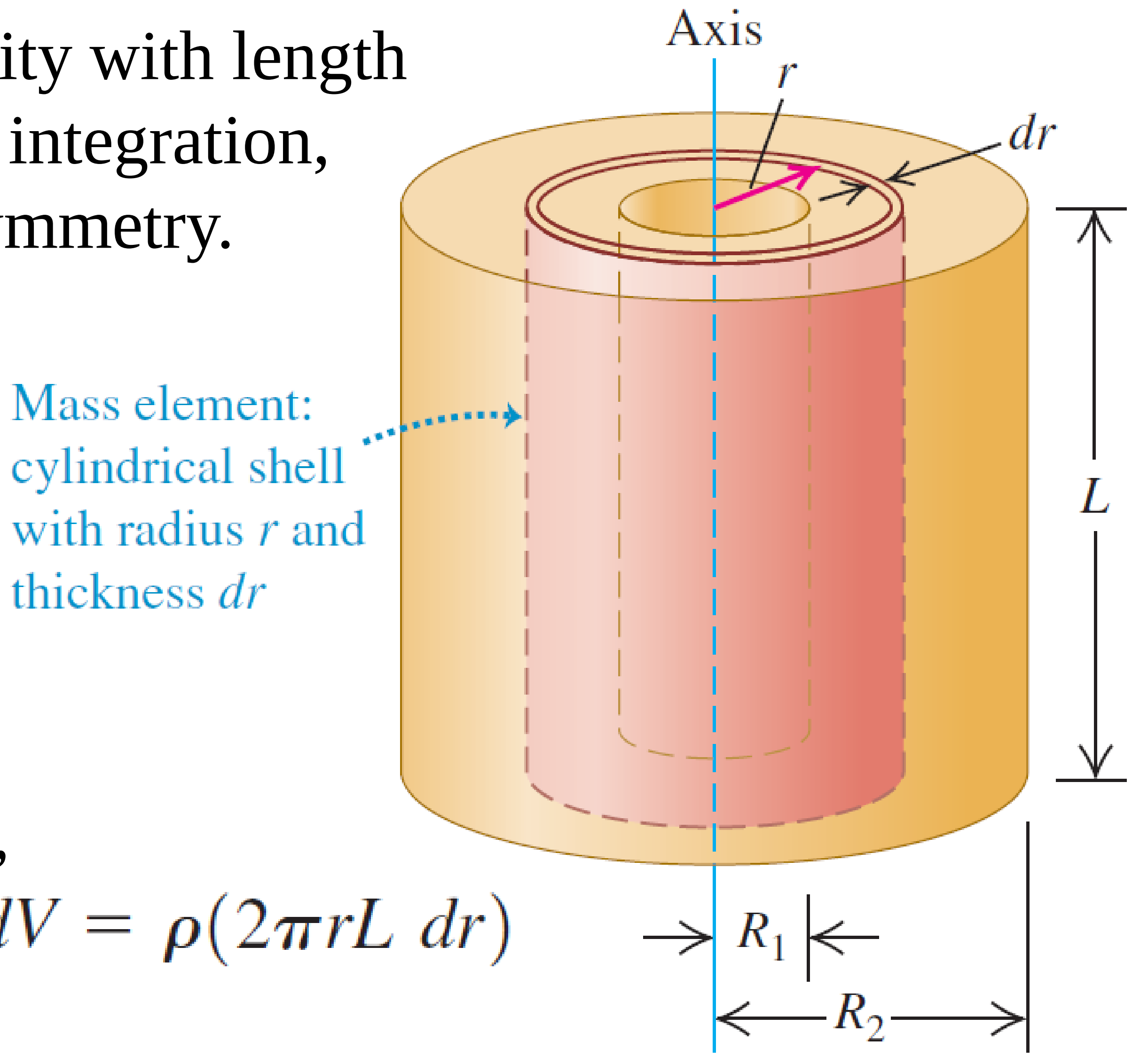
$$I = \rho \int r^2 \, dV$$

Example 9.10 Hollow or solid cylinder, rotating about axis of symmetry

Given hollow cylinder of uniform mass density with length L, inner radius R_1 and outer radius R_2. Using integration, find its moment of inertia about its axis of symmetry.

Tips for taking the integral.

1. Choose a dV or dM, for this question:
   - a thin cylindrical shell of radius *r*, thickness *dr*, and length *L*.
   - All parts of this shell are at **the same distance *r*** from the axis (d*r* ~= 0)
   - A flat sheet with thickness d*r*, length *L*, and width 2π*r*, with a mass $dm = \rho\ dV = \rho(2\pi rL\ dr)$



Example 9.10 Hollow or solid cylinder, rotating about axis of symmetry

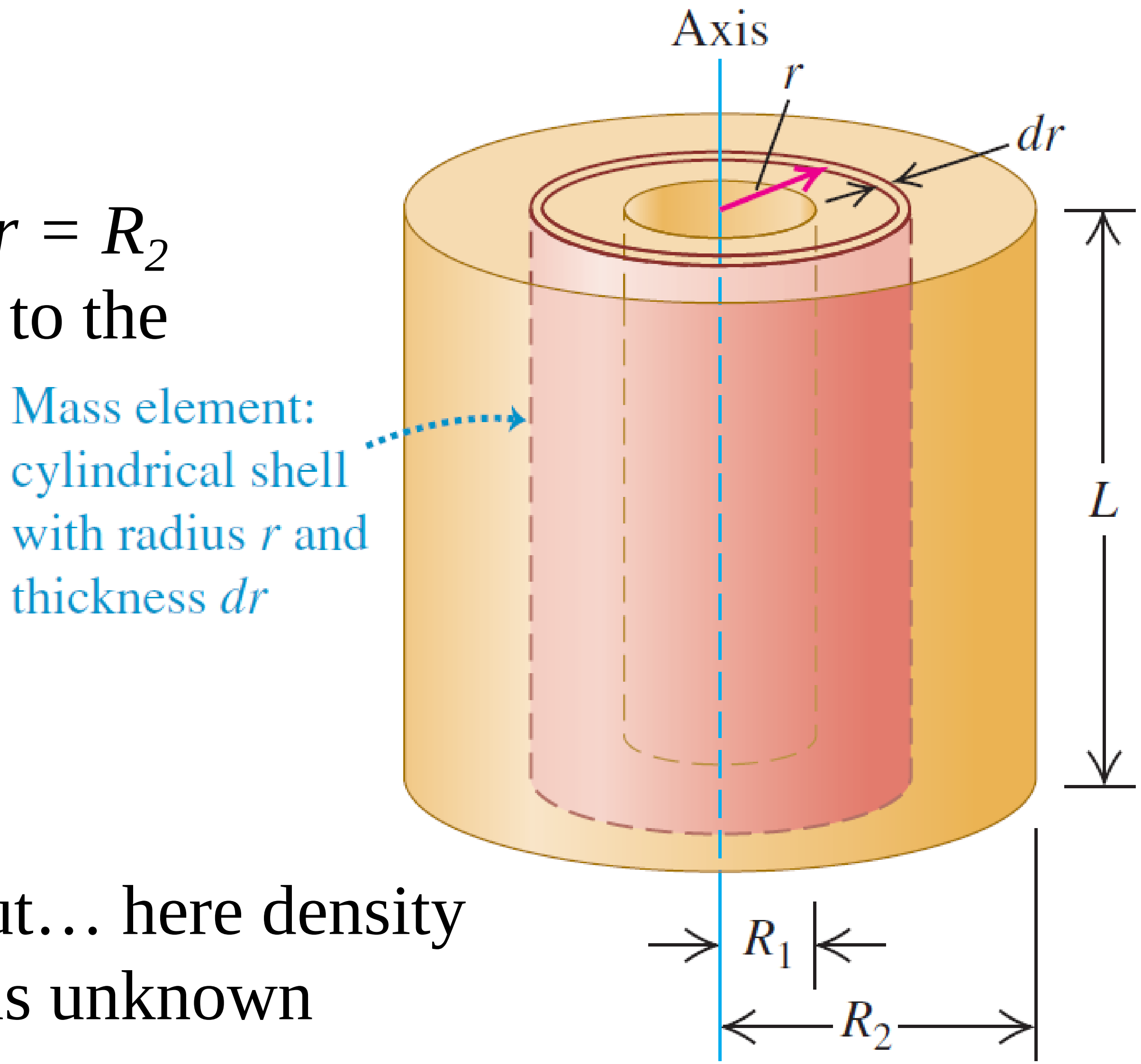
$$dm = \rho\, dV = \rho(2\pi r L\, dr)$$

Now we need to integrate from $r = R_1$ to $r = R_2$ (add up the contribution of the thin shells to the momentum of inertia, slice by slice)

For a single slice: $dI = r^2\, dm$

$$I = \int r^2\, dm = \int_{R_1}^{R_2} r^2 \rho(2\pi r L\, dr)$$

$$= 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr = \frac{2\pi\rho L}{4}(R_2^4 - R_1^4)$$



But… here density ρ is unknown

Example 9.10 Hollow or solid cylinder, rotating about axis of symmetry

Let's express this result in terms of the total mass M of the body, which is its density ρ multiplied by the total volume V. The cylinder's volume and mass are, by definition

$$V = \pi L(R_2^2 - R_1^2)$$

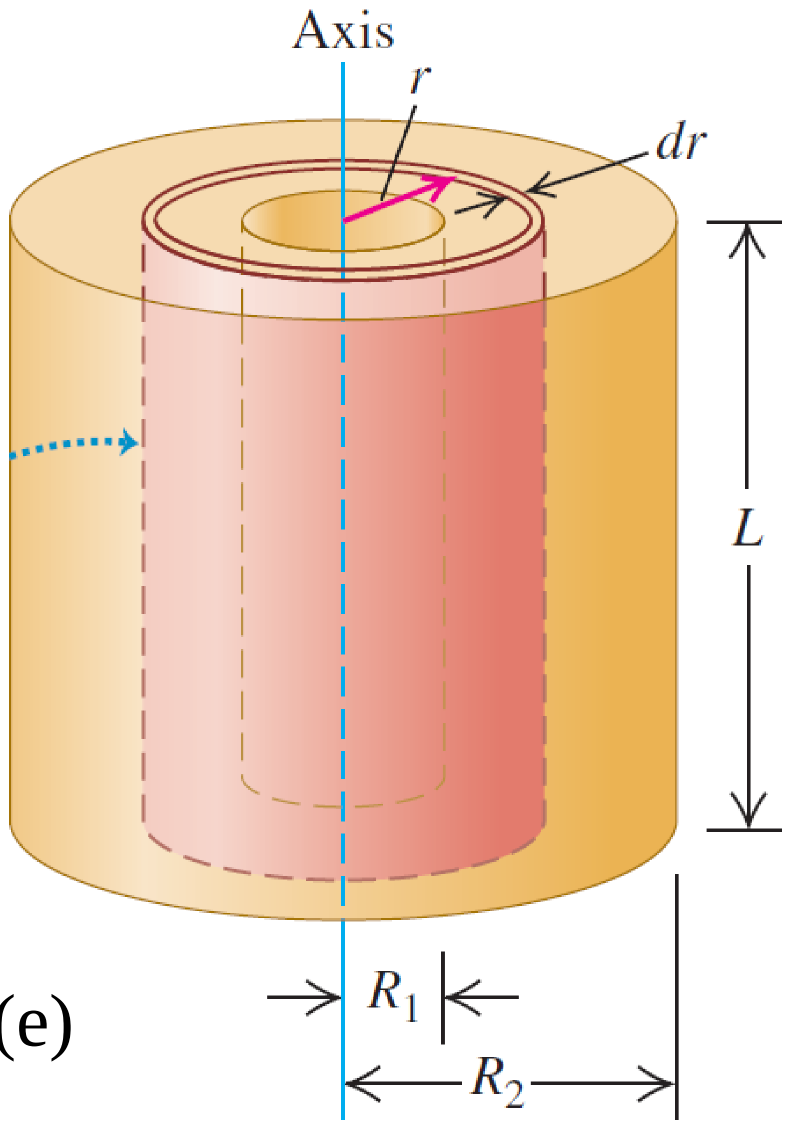
$$M = \rho V = \pi L\rho(R_2^2 - R_1^2)$$

Here we got density ρ

$$I = \frac{2\pi\rho L}{4}(R_2^4 - R_1^4) = \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

last step we used the identity $a^2 - b^2 = (a - b)(a + b)$.

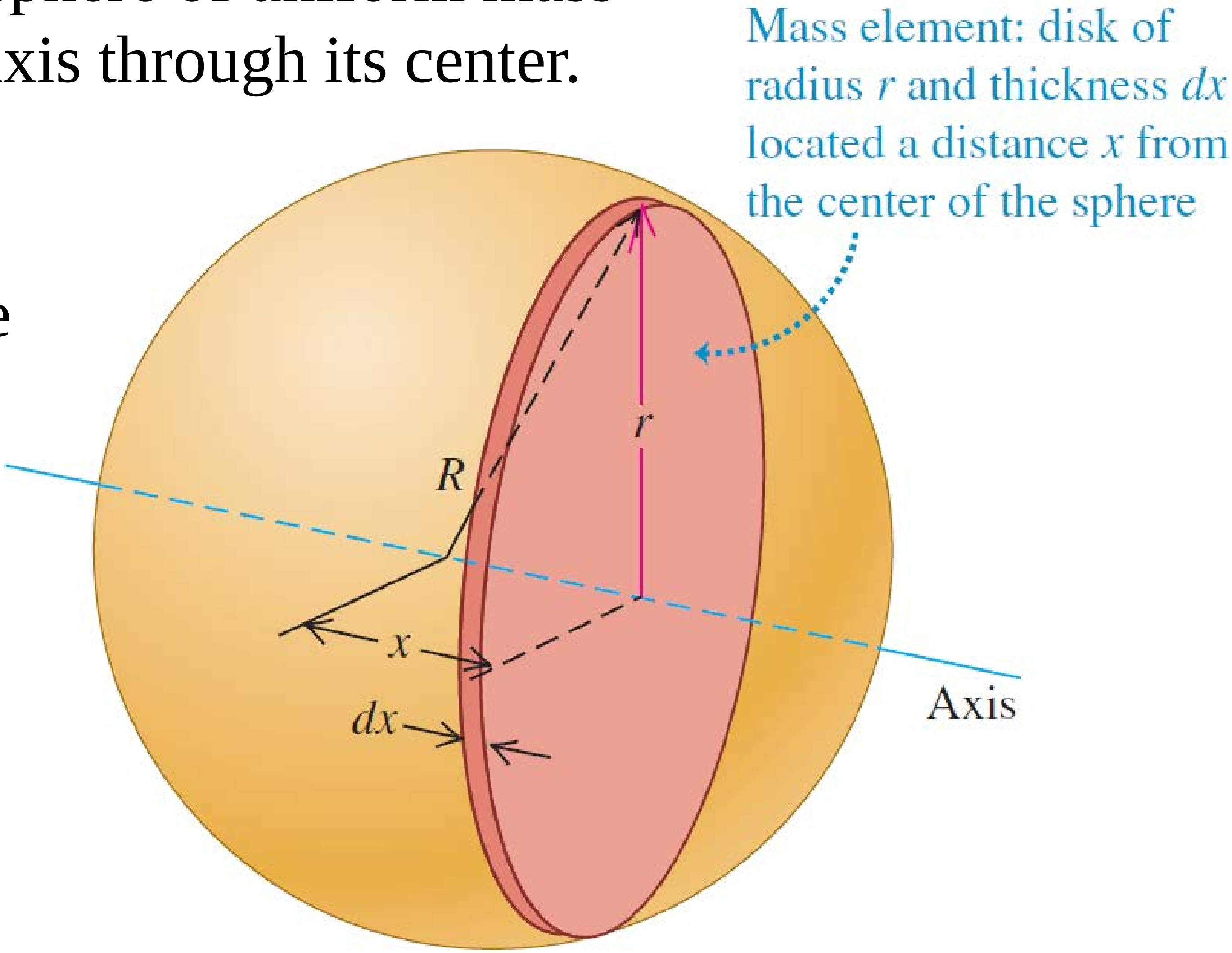
We finally get: $I = \frac{1}{2}M(R_1^2 + R_2^2)$ Table 9.2(e)



Example 9.11 Uniform sphere with radius R, axis through center

Find the moment of inertia of a solid sphere of uniform mass density (like a billiard ball) about an axis through its center.

To start: we divide the sphere into thin, solid disks of thickness *dx* whose moment of inertia we know from Table 9.2, case (f). We'll integrate over these to find the total moment of inertia.



Example 9.11 Uniform sphere with radius R, axis through center

Find the moment of inertia of a solid sphere of uniform mass density (like a billiard ball) about an axis through its center.

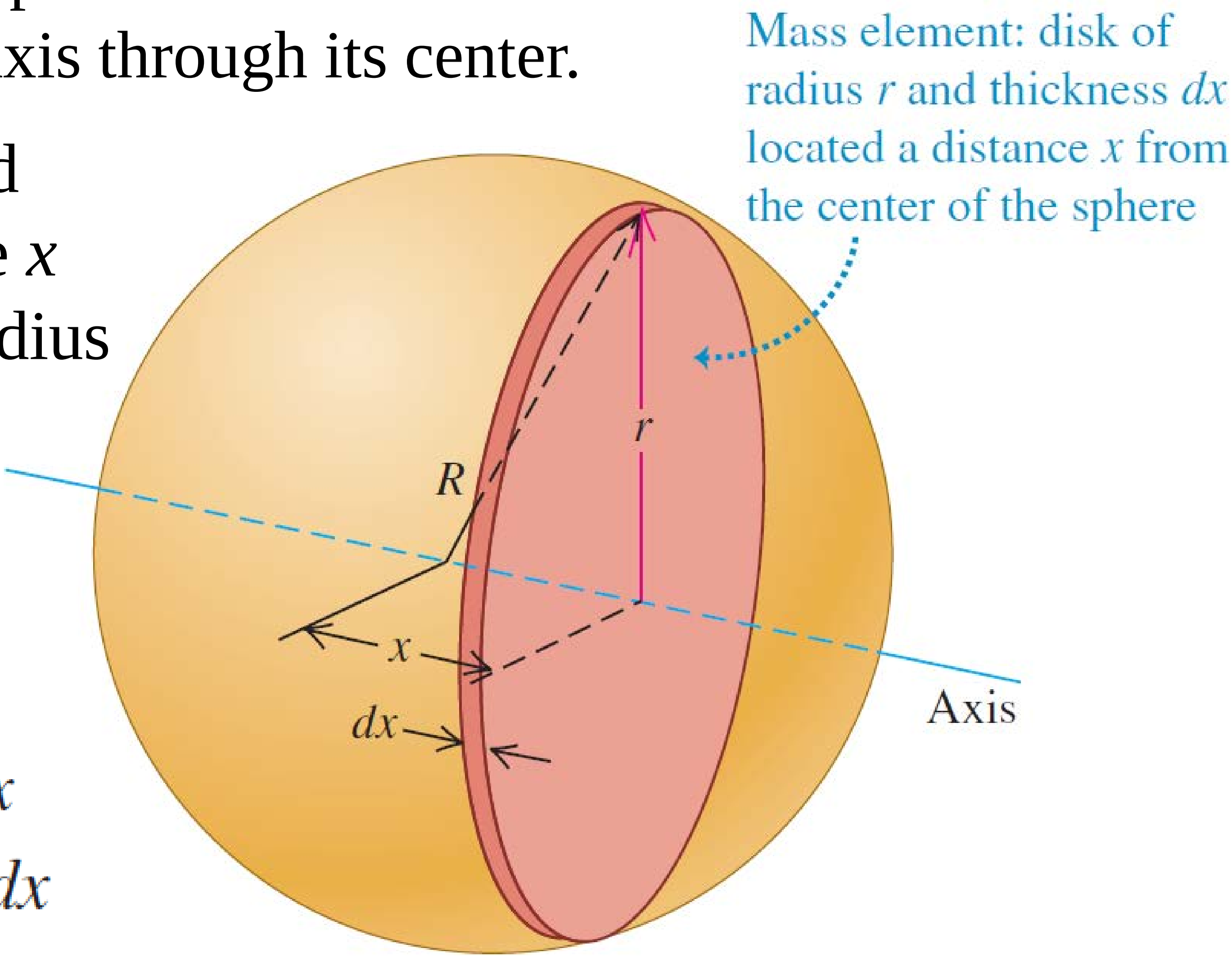
The radius and hence the volume and mass of a disk depend on its distance *x* from the center of the sphere. The radius *r* of the disk shown in Fig. 9.23 is:

$$r = \sqrt{R^2 - x^2}$$

The volume & mass of the slice:

$$dV = \pi r^2\, dx = \pi(R^2 - x^2)\, dx$$

$$dm = \rho\, dV = \pi\rho(R^2 - x^2)\, dx$$



Example 9.11 Uniform sphere with radius R, axis through center

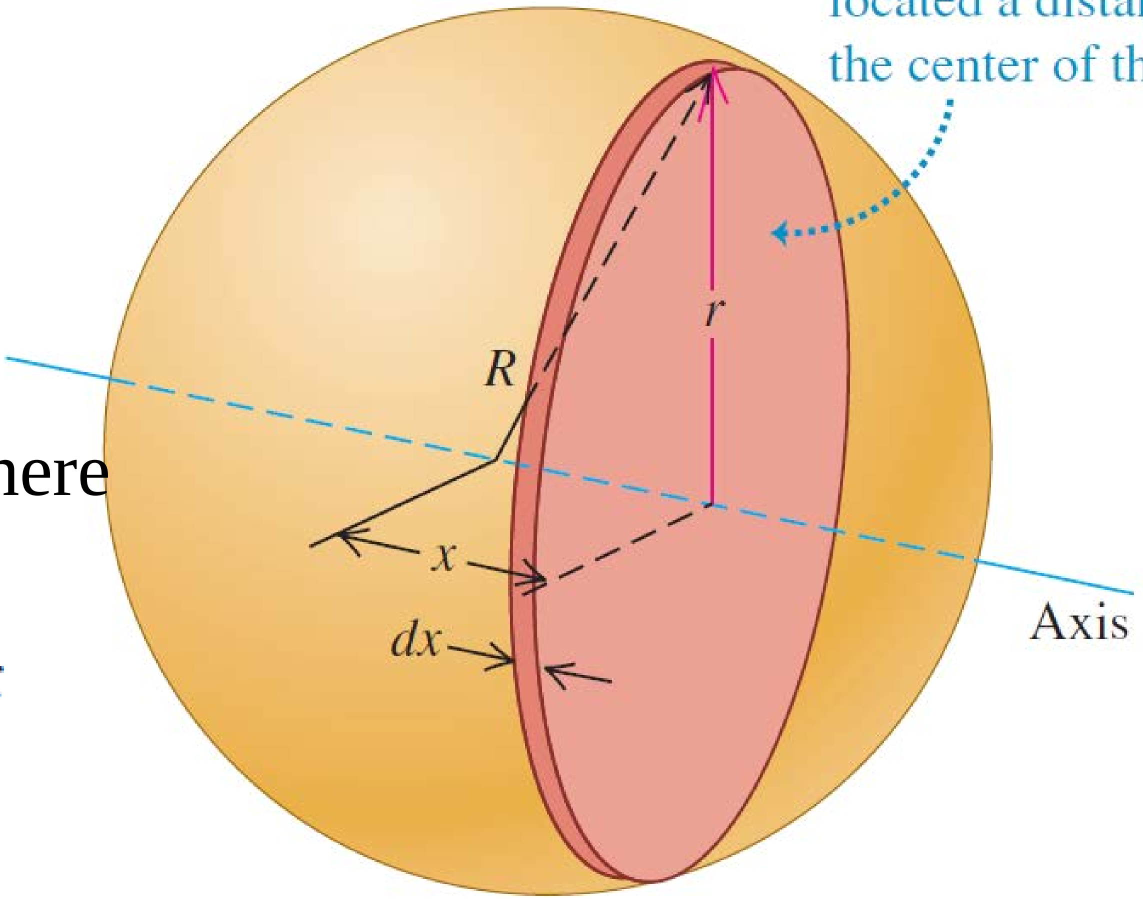
From Table 9.2, case (f),

$$dI = \frac{1}{2} r^2 \, dm$$

$$= \frac{1}{2}(R^2 - x^2)[\pi\rho(R^2 - x^2)\, dx]$$

Integrating from $x = 0$ to R gives the moment of inertia of the right hemisphere

$$I = (2)\frac{\pi\rho}{2} \int_0^R (R^2 - x^2)^2 \, dx$$



Example 9.11 Uniform sphere with radius R, axis through center

$$I = (2)\frac{\pi\rho}{2}\int_0^R (R^2 - x^2)^2\,dx$$

Carrying out the integration, we find

$$I = \frac{8\pi\rho R^5}{15}$$

Again we need to calculate ρ

The volume of the sphere is $V = 4\pi R^3/3$

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$$

$$I = \left(\frac{8\pi R^5}{15}\right)\left(\frac{3M}{4\pi R^3}\right) = \tfrac{2}{5}MR^2$$

Table 9.2, case (h)

Summary

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (9.3)$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2} \qquad (9.5),\ (9.6)$$

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \qquad (9.11)$$

(constant α_z only)

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \qquad (9.10)$$

(constant α_z only)

$$\omega_z = \omega_{0z} + \alpha_z t \qquad (9.7)$$

(constant α_z only)

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad (9.12)$$

(constant α_z only)

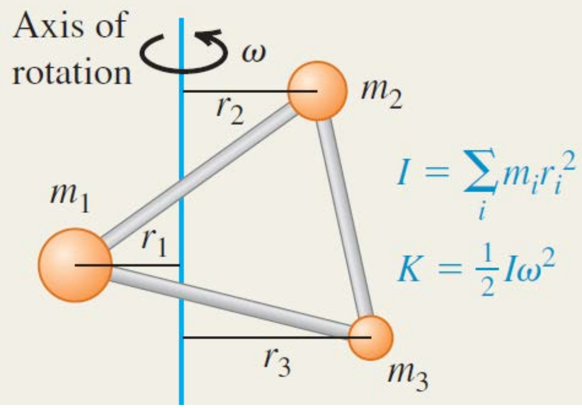
$$v = r\omega$$

$$a_{\tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$a_{\mathrm{rad}} = \frac{v^2}{r} = \omega^2 r$$

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2$$

$$K = \tfrac{1}{2}I\omega^2$$



$$I_P = I_{\mathrm{cm}} + Md^2$$