CALCULUS

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- When we plot function values, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure. In doing so, we are assuming that we are working with a ***continuous function***, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between.
- Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

① Continuity at a Point

Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

- The function f is **continuous at c** if

$$\lim_{x \to c} f(x) = f(c)$$

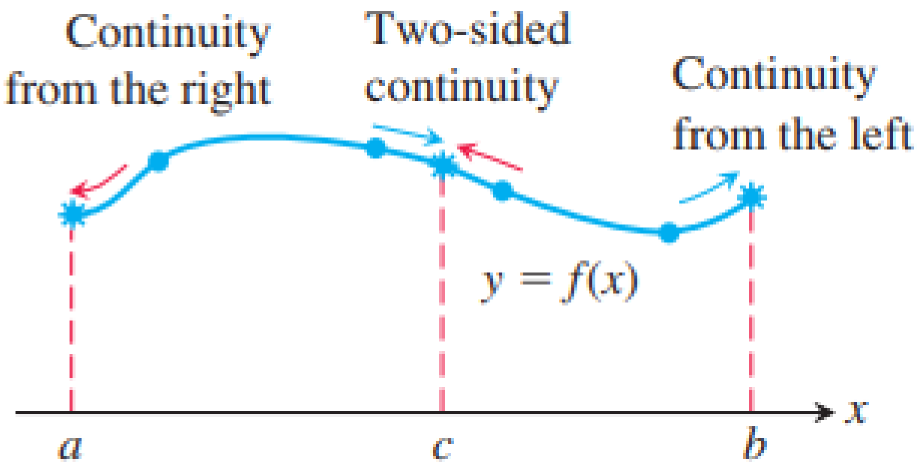
- The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c)$$

- The function f is **left-continuous at c (or continuous from the left)** if

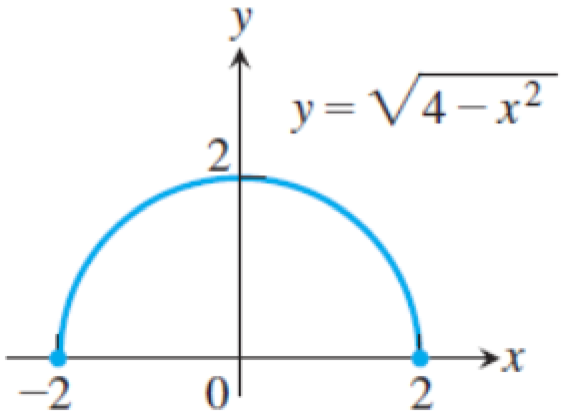
$$\lim_{x \to c^-} f(x) = f(c)$$

- A function is **continuous over a closed interval [*a*, *b*]** if it is right-continuous at *a*, left-continuous at *b*, and continuous at all interior points of the interval.



- This definition applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved. If a function is not continuous at point *c* of its domain, we say that *f* is **discontinuous at *c***, and that *f* has a discontinuity at *c*.
- Note that a function *f* can be continuous, right-continuous, or left-continuous only at a point *c* for which $f(c)$ is defined.

Example 1 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2,2]$.



Example 2 Let

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x \neq 3; \\ 5, & x = 3. \end{cases}$$

Show that $f(x)$ is continuous at $x = 3$.

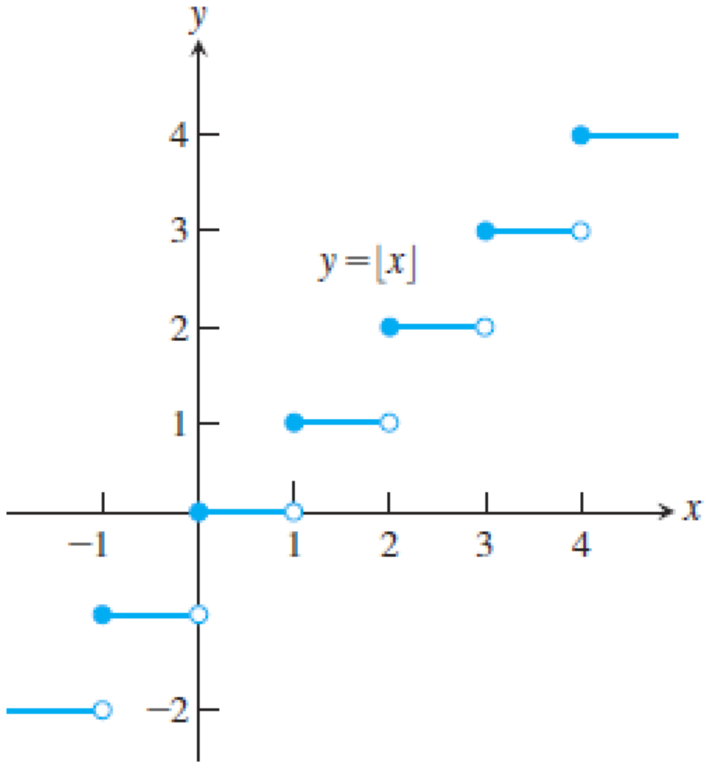
➢ Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (*c* lies in the domain of *f*).

2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).

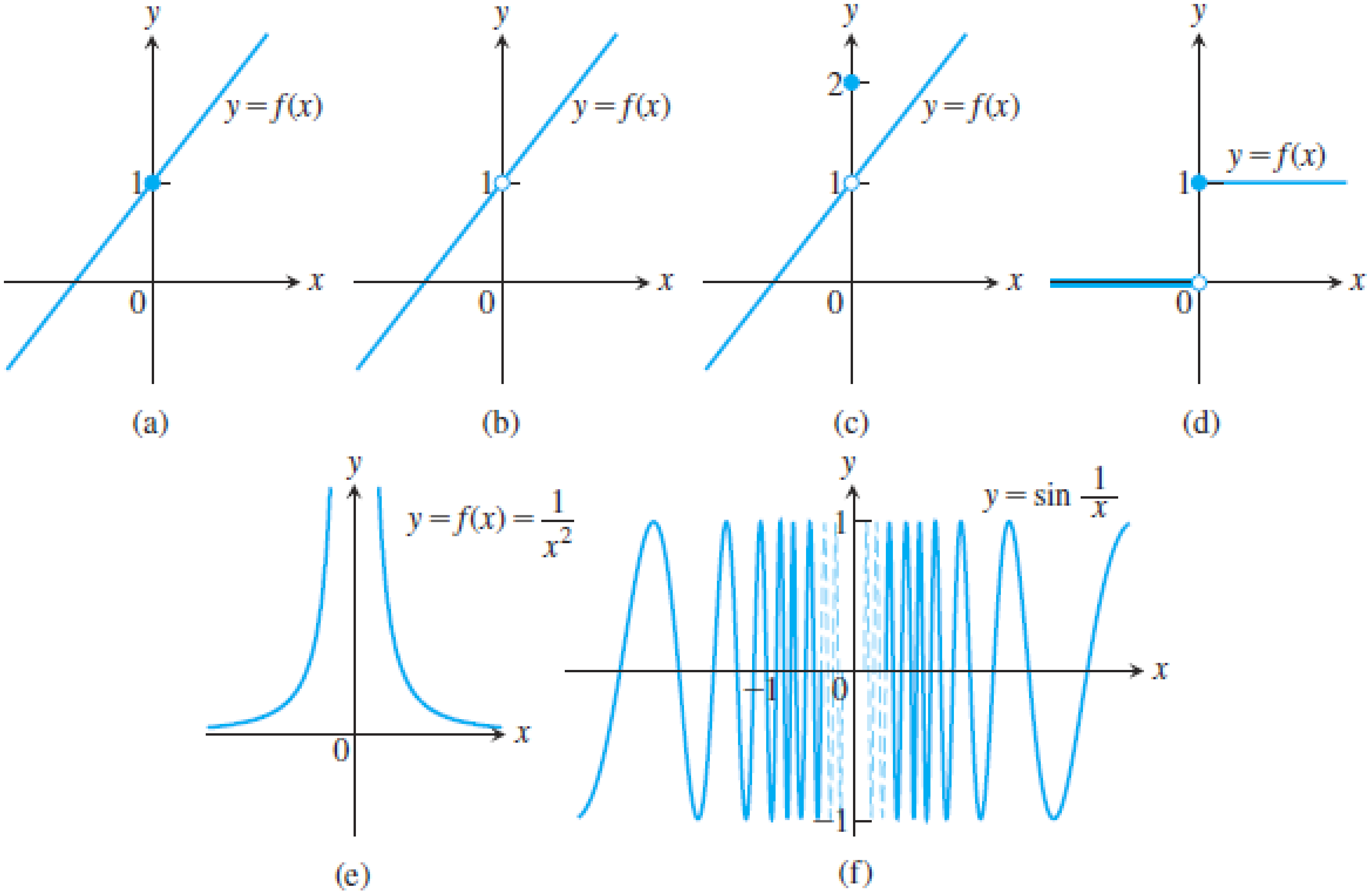
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).



Example 3:

The **greatest integer function** $y = \lfloor x \rfloor$ is discontinuous at every integer n, because the left-hand and right-hand limits are not equal as $x \to n$. But It is continuous at every real number other than the integers.

Examples of Continuous functions and Discontinuous functions at $x = 0$.



Example 4 Show that $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0;\\ 0, & x=0.\end{cases}$ is discontinuous at $x=0$.

Example 5 For what values of a and b is

$$f(x)=\begin{cases}-2, & x\leq -1;\\ ax+b, & -1<x<1;\\ 3, & x\geq 1.\end{cases}$$

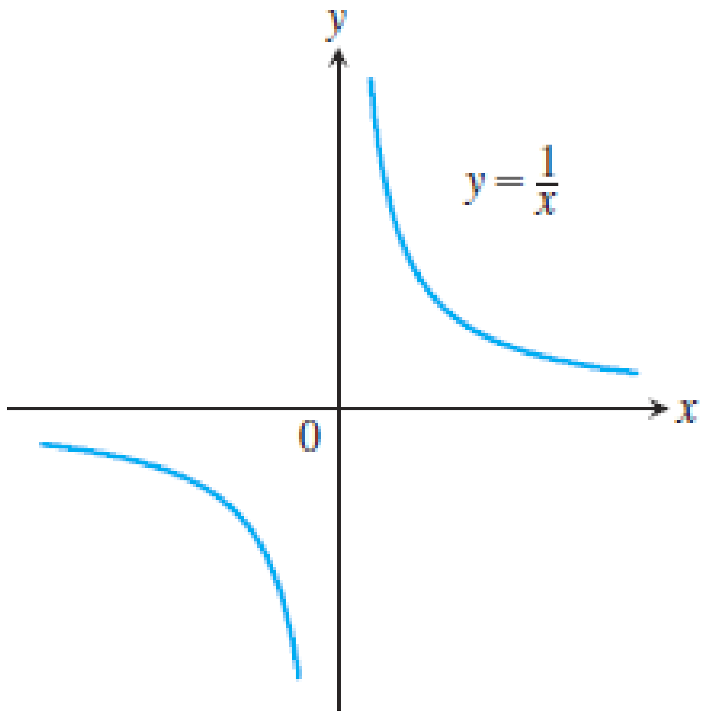
continuous at every point x?

② Continuous Functions

A function that is continuous at every point in its domain. If a function is discontinuous at one or more points of its domain, then it is a discontinuous function.

Example 6:

The function $y = 1/x$ is a continuous function because it is continuous at every point of its domain. The point $x = 0$ is not in the domain of the function f, so f is not continuous on any interval containing $x = 0$.



THEOREM 8—Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:*	$f + g$
2. *Differences:*	$f - g$
3. *Constant multiples:*	$k \cdot f$, for any number k
4. *Products:*	$f \cdot g$
5. *Quotients:*	f/g, provided $g(c) \neq 0$
6. *Powers:*	f^n, n a positive integer
7. *Roots:*	$\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer

- Algebraic combinations of continuous functions are continuous wherever they are defined.
- The following are the examples of continuous functions.

(a) polynomial functions: $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n.$

(b) rational functions: $f(x)=\dfrac{g(x)}{h(x)}$, $g(x)$ and $h(x)\neq 0$ are polynomial functions.

(c) power functions: $f(x)=x^a,\ a\in\mathbb{R}.$

(d) exponential functions: $f(x)=a^x,\ a>0,$ and $a\neq 1.$

(e) trigonometric functions: $\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x,\ \csc x.$

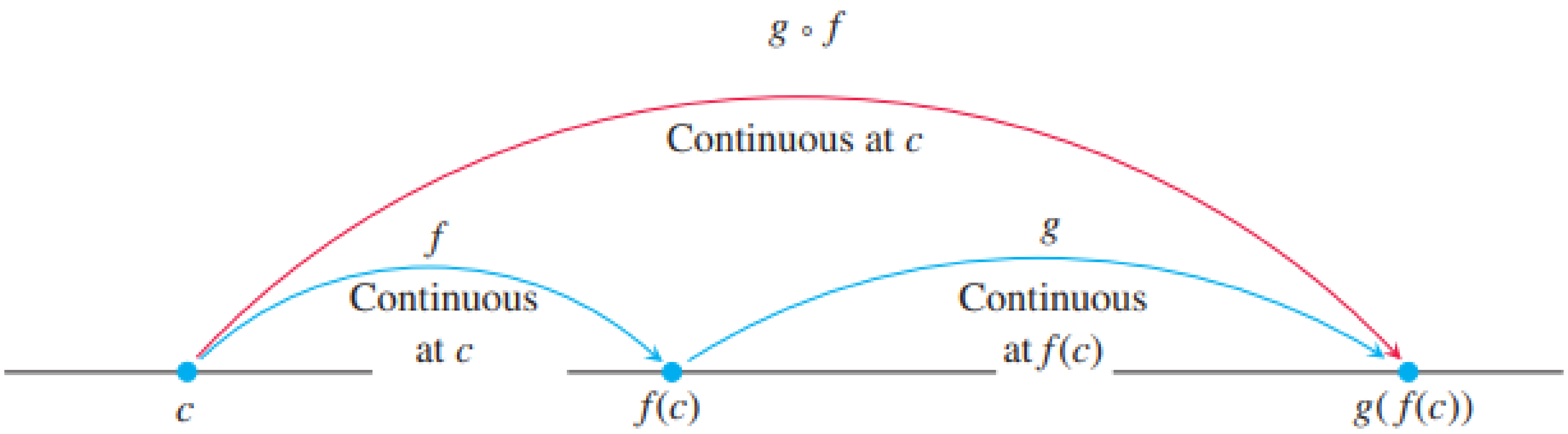
(f) sqaure root function: $f(x)=\sqrt{g(x)},\ g(x)\geq 0$ is a polynomial function.

③ Continuity of Compositions of functions

THEOREM 9 – Compositions of Continuous Functions

If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$.

- All compositions of continuous functions are continuous. Therefore, the limit of $g \circ f$ as $x \to c$ is $g(f(\mathrm{c}))$.



Example 7 Show that the following functions are continuous on their natural domains.

(a) $y=\sqrt{x^2-2x-5}$

(b) $y=\dfrac{x^{\frac{2}{3}}}{1+x^4}$

(c) $y=\left|\dfrac{x-2}{x^2-2}\right|$

(d) $y=\left|\dfrac{x\sin x}{x^2+2}\right|$

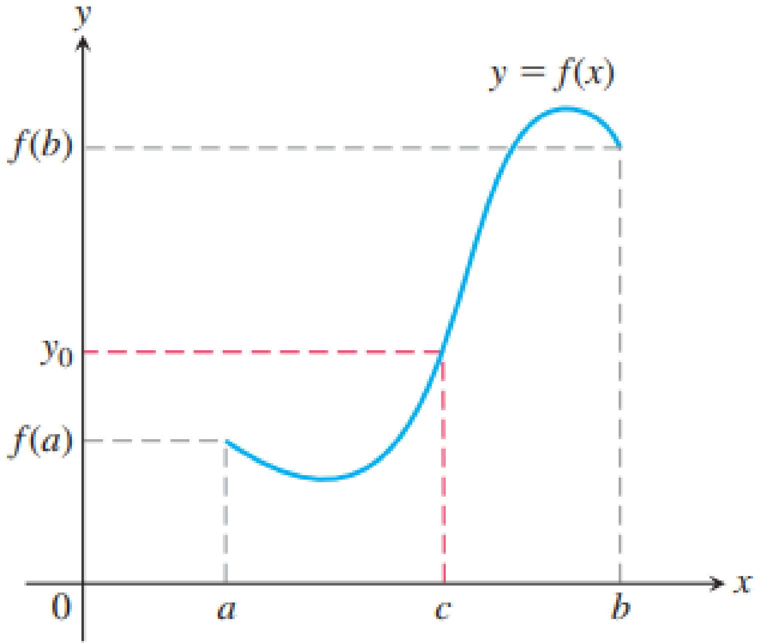
Example 8 Find the following limits.

(a) $\lim\limits_{x\to\frac{\pi}{2}}\cos\left(2x+\sin\left(\dfrac{3\pi}{2}+x\right)\right)$;

(b) $\lim\limits_{x\to 0}\sqrt{x+1}\,\mathrm{e}^{\tan x}$.

④ Intermediate Value Theorem for Continuous Functions

- If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Note: The Intermediate Value Theorem tell us if $f(x)$ is continuous on $[a, b]$, and $f(a)\,f(b) < 0$, then $f(x)$ has a zero/root in (a, b).

Example 9

Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Example 10

Use the **Intermediate Value Theorem** to prove that the equation

$$\sqrt{2x+5} = 4 - x^2$$

has a solution.

⑤ Continuous Extension to a Point

If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, we can define a new function $F(x)$ by

$$F(x) = \begin{cases} f(x), & x \neq c; \\ L, & x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the continuous extension of f to $x = c$.

For example, $f(x) = \sin x/x$, $L = 1$.

Example 11 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find the extension.