Chapter 4 Oversimplified

💡 Please use this only as a reference.

Everything is up to your own interpretation, and they are all based from the book and the homework assignments.

Space

💡 The **space** R^n consists of all column vectors v with n components.

Just imagine a box (three components) any point inside that box is some kind of vector.

e.g.

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in R^3 as it has 3 components

Subspace

💡 A **subspace** is a set of vectors that (including the zero vector) that satisfies:

- $v + w$ is in the subspace (given that v and w are in the subspace)
- cv is in the subspace (any scalar multiplication of v)
- which also means $cv + dw$ is in the subspace (all linear combinations of v and w)

It is important to remember that a subspace **must contain the zero vector (the origin),** otherwise it is not a subspace.

Column Space

💡 Column space (of a matrix) is all the possible linear combinations of the columns (of the matrix).

The system
$Ax = b$ is solvable only if b is in the column space of A

How to find the Column Space

- To find the column space, you must row reduce (elimination) the matrix. (In order to acquire the pivots, all components under the pivots must be zero).

$$e.g.\ A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix}$$

$$\rightarrow U = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- from here you can see that there is only **two pivots (col 1 and col 2)**, meaning that out of the 4 columns, only two of them make up a space.
- The rest of the two can be expressed in terms of the first two.
- Hence, the column space of the matrix is the span of the first two columns:

$$\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}$$

- Check:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix} \text{(col 3)}$$

$$(-2) \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix} \text{(col 4)}$$

Nullspace

💡 A nullspace is a space that contains all solutions x that produces $Ax = 0$.

Basically, space of vectors (in R^n) that multiplies the matrix and produces 0.

How to find the Nullspace

- The nullspace is simply the (possible) solutions to the x's that produce the zero vector.

$$Ax = 0$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Similar to finding the column space, we must first turn the matrix into an echelon form.

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Here, you can see that there is only two pivots (col 1 and col 3, or the variables x_1 and x_3). The rest of the variables (x_2, x_4, x_5) are **free variables.**
- When finding the nullspace, it is better to find the Row Reduced Echelon Form of the matrix (condition that all pivots are 1's with zeros above and below)

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Now, you need to express the pivot variables in terms of the free variables, similar to $Ax = b \rightarrow x = A^{-1}b$

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_3 + 2x_4 + 3x_5 &= 0 \end{aligned}$$

$$\rightarrow \begin{aligned} x_1 &= -2x_2 \\ x_3 &= -2x_4 - 3x_5 \end{aligned}$$

- The free variables can be expressed in terms of themselves since they are dependent on the pivot variables, and do not have their own solutions.

$$\begin{aligned} x_1 &= -2x_2 \\ x_2 &= x_2 \\ x_3 &= -2x_4 - 3x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

- The nullspace is the columns of the free variables:

$$\mathrm{Null}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Any one of these columns (or vectors) multiply matrix A to produce the zero vector. (You can check).

Complete Solution

$$x = x_p + x_n$$

- where x_p is particular solution for $Ax_p = b$ and x_n is special solution (nullspace) for $Ax_n = 0$.
- You can reason that when a matrix does not have a full set of pivots, in other words, there are free variables **dependent** on the pivot variables, the free variables do not contribute to the final answer (b). So they are assumed to produce 0.
- Finding the Complete Solution is similar to finding the nullspace. Only you need to substitute the zero vector with the final answer.

$$Ax = b$$

$$\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

- You should augment A and b for easier elimination to the Row Reduced Echelon Form.

$$\begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 3 \\ 2 & 3 & 5 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Here x_1 and x_2 are pivot variables, while the rest are free variables.
- Now similar to finding the nullspace, express the pivot variables in terms of the free variables and the real numbers (components of b)

$$\begin{aligned} x_1 + x_3 - 2x_4 &= 4 \\ x_2 + x_3 + 2x_4 &= -1 \end{aligned}$$

$$\begin{aligned} x_1 &= 4 - x_3 + 2x_4 \\ x_2 &= -1 - x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \quad \rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

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and this is the complete solution for x where $\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ is the x_p, and $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} +$ $x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ is the x_n

Independence

💡 The columns of a matrix are **linearly independent** when only $x = 0$ produces $Ax = 0$.

Or the matrix has a Full Rank, or in other words, a full set of pivots and **no free variables**.

That is, simply, when all the columns of a matrix are independent of one another. Every column is included in the Column Space.