Chapter 6 Capacitors and Inductors

Introduction

Resistor: a passive element which dissipates energy only

Capacitors and inductors can neither generate nor dissipate energy but **store energy**, which can be retrieved at a later time. They are called *storage elements*.

Capacitors

A capacitor consists of two conducting plates separated by an insulator (or dielectric).

It is a passive element that stores energy in the electric field that exists between its plates.

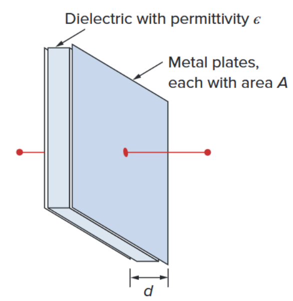


Figure 6.1
A typical capacitor.

when a voltage source v is connected to the capacitor, the source deposits a positive charge q on plate and negative charge $-q$ on the other plate.

$$q = Cv$$

the amount of charge q is proportional to the applied voltage v

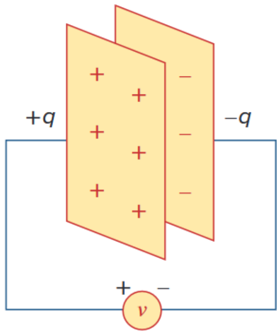


Figure 6.2
A capacitor with applied voltage *v*.

> Capacitance is the ratio of charge on plate to the voltage difference between the plates, or in other words, the amount of **charge stored per unit voltage difference** in a capacitor

$$C = \frac{\epsilon A}{d}$$

where A is the surface area of each plate, and d is the distance between the plates, and ϵ is permittivity of the insulator, or the dielectric material.

1. Larger the surface area of the plates, greater the capacitance
2. Smaller the distance between plates, greater the capacitance
3. Higher the permittivity of the material, greater the capacitance

if $vi > 0$, the capacitor is being charged

if $vi < 0$, the capacitor is being discharged

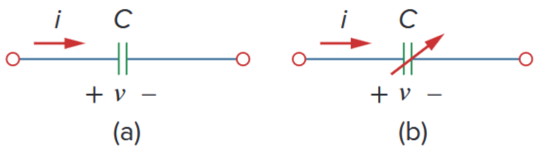


Figure 6.3
Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

A capacitor is an **open circuit** to d.c.

Current-Voltage Relationship of Capacitor

$$i = \frac{dq}{dt} \quad \text{and} \quad q = Cv$$

relating current and capacitance by differentiating both sides of the second equation:

$$q = Cv \quad \Longrightarrow \quad i = C\frac{dv}{dt}$$

Time Equation of Voltage

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$

where $i(\tau)$ is the time equation of the current

Instantaneous Power

the instantaneous power delivered to the capacitor:

$$p = vi = Cv\,\frac{dv}{dt}$$

The Energy Stored

$$w = \frac{1}{2}Cv^2 = \frac{q^2}{2C}$$

▼ Example Questions

Example 6.2

The voltage across a 5-μF capacitor is

$$v(t) = 10 \cos 6000t \text{ V}$$

Calculate the current through it.

Solution:

By definition, the current is

$$\begin{aligned} i(t) &= C\frac{dv}{dt} = 5 \times 10^{-6}\frac{d}{dt}(10 \cos 6000t) \\ &= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A} \end{aligned}$$

Example 6.3

Determine the voltage across a 2-μF capacitor if the current through it is

$$i(t) = 6e^{-3000t} \text{ mA}$$

Assume that the initial capacitor voltage is zero.

Solution:

Since $v = \frac{1}{C}\int_0^t i\,dt + v(0)$ and $v(0) = 0$,

$$v = \frac{1}{2 \times 10^{-6}} \int_0^t 6e^{-3000t}\,dt \cdot 10^{-3}$$
$$= \frac{3 \times 10^3}{-3000} e^{-3000t}\Big|_0^t = (1 - e^{-3000t}) \text{ V}$$

Example 6.4

Determine the current through a 200-μF capacitor whose voltage is shown in Fig. 6.9.

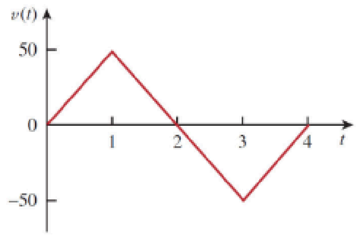


Figure 6.9
For Example 6.4.

Solution:
The voltage waveform can be described mathematically as

$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Since $i = C\,dv/dt$ and $C = 200\ \mu$F, we take the derivative of v to obtain

$$i(t) = 200 \times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 10 \text{ mA} & 0 < t < 1 \\ -10 \text{ mA} & 1 < t < 3 \\ 10 \text{ mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Thus the current waveform is as shown in Fig. 6.10.

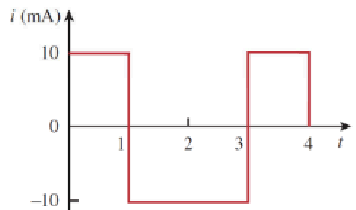


Figure 6.10
For Example 6.4.

Practice Problem 6.4

An initially uncharged 1-mF capacitor has the current shown in Fig. 6.11 across it. Calculate the voltage across it at $t = 2$ ms and $t = 5$ ms.

Answer: 100 mV, 400 mV.

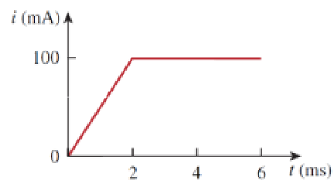


Figure 6.11
For Practice Prob. 6.4.

$i = C\frac{dv}{dt}$ $\quad\quad$ $v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$

Find time equation of current first from the graph:

$$0 \leq t < 2: \qquad i(\tau) = \frac{(100-0)\times 10^{-3}}{(2-0)\times 10^{-3}}(\tau - 0) - 0 = 50\tau$$

integrate (since “initially uncharged”, $v(0) = 0$):

$$v(t) = \frac{1}{1\times 10^{-3}}\int_0^t 50\tau \, d\tau + 0 \Rightarrow 10^3\left(25\tau^2|_0^t\right)$$

$$\Rightarrow v(t) = 25t^2 \times 10^3$$

$$v(2\times 10^{-3}) = 25(2\times 10^{-3})^2 \times 10^3 = 100 \times 10^{-3}\ \mathrm{V}$$

do the same for the second part:

$$2 \leq t < 5: \qquad i(\tau) = 100 \times 10^{-3}$$

$$v(t) = 10^3 \int_2^t 100 \times 10^{-3}\, d\tau + 100 \times 10^{-3}$$

$$\Rightarrow 10^3\left((100\times 10^{-3})\tau|_2^t\right) + 100\times 10^{-3}$$

$$\Rightarrow 10^3\left((100\times 10^{-3})t - 200\times 10^{-6}\right) + 100\times 10^{-3}$$

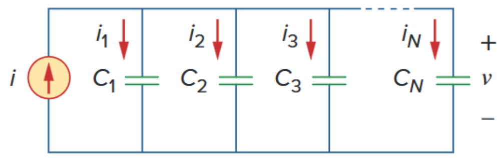
$$\Rightarrow v(t) = 100t - 100\times 10^{-3}$$

$$\Rightarrow v(5\times 10^{-3}) = (500-100)\times 10^{-3} = 400\times 10^{-3}\ \mathrm{V}$$

Series and Parallel Capacitors

Parallel Capacitor

$$C_{\text{eq}} = C_1 + C_2 + C_3 + \ldots$$



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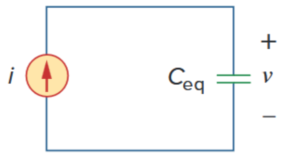


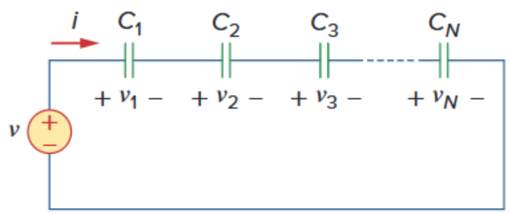
(b)

The equivalent in parallel is the sum of the individual capacitances.

Capacitors in SERIES share the SAME TOTAL CHARGE (q)

Series Capacitor

$$C_{\text{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$$



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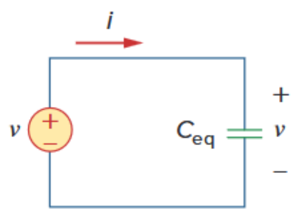


(b)

The equivalent in series is the reciprocal of the sum of reciprocals of the individual capacitances.

Capacitors in PARALLEL share the SAME VOLTAGE (v)

▼ Examples

Find the voltage across each of the capacitors in Fig. 6.20.

Practice Problem 6.7

Answer: $v_1 = 75$ V, $v_2 = 75$ V, $v_3 = 25$ V, $v_4 = 50$ V.

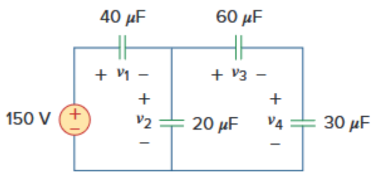


Figure 6.20
For Practice Prob. 6.7.

$$C_{\text{eq}} = \frac{40 \times (20 + \frac{60\times 30}{90})}{40 + 20 + \frac{60\times 30}{90}} = 20\mu\text{F}$$

$$q = Cv = 20 \times 10^{-6}(150) = 3 \times 10^{-3} = 3 \text{ mC}$$

capacitors in series share the same total charge so:

$$v = \frac{q}{C} \Rightarrow v_1 = \frac{3 \times 10^{-3}}{20 \times 10^{-6}} = 75\text{V}$$

capacitors in parallel share the same voltage so:

$$v_2 = 75\text{V}$$

again, capacitors in series share the same total charge:

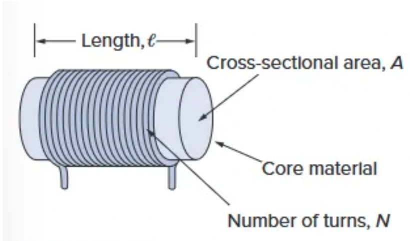
$$q = \frac{60 \times 30}{90} \times 10^{-6}(75) = 1.5 \times 10^{-3} \text{ mC}$$

$$v_3 = \frac{1.5 \times 10^{-3}}{60 \times 10^{-6}} = 25\text{V} \qquad v_4 = 50\text{V}$$

Inductors

An ***inductor*** consists a coil of conducting wire

An inductor is a passive element that stores energy in its magnetic field.



$$v = L\,\frac{di}{dt}$$

Inductance is the property whereby an inductor exhibits *opposition to the change of current* following through it, measured in henrys (H).

An inductor acts like a **short circuit** to d.c.

Current-Voltage Relationship of Inductor

Time Equation of Current

$$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau + i(t_0)$$

Instantaneous Power

$$p = vi = \left(L\frac{di}{dt}\right)i$$

The Energy Stored

$$w = \frac{1}{2}L\,i^2$$

▼ Example Questions

Practice Problem 6.8

If the current through a 1-mH inductor is $i(t) = 90 \sin(200t)$ mA, find the terminal voltage and the energy stored.

Answer: 18 cos (200t) mV, 4.05 $\sin^2$ (200t) μJ.

$$\frac{di}{dt} = 18000t\cos(200t)$$

$$v(t) = L\frac{di}{dt} = 1 \times 10^{-3}(18000\cos(200t)) = 18\cos(200t)$$

$$w = \frac{1}{2}L\,i^2 = \frac{1}{2}(1 \times 10^{-3})(8100\sin^2(200t) = 4.05\sin^2(200t)$$

Example 6.9

Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also, find the energy stored at $t = 5$ s. Assume $i(v) > 0$.

Solution:

Since $i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)$ and $L = 5$ H,

$$i = \frac{1}{5}\int_0^t 30\tau^2\,d\tau + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

The power $p = vi = 60t^5$, and the energy stored is then

$$w = \int p\,dt = \int_0^5 60t^5\,dt = 60\frac{t^6}{6}\Big|_0^5 = 156.25 \text{ kJ}$$

Alternatively, we can obtain the energy stored using Eq. (6.24), by writing

$$w\Big|_0^5 = \frac{1}{2}Li^2(5) - \frac{1}{2}Li(0) = \frac{1}{2}(5)(2 \times 5^3)^2 - 0 = 156.25 \text{ kJ}$$

as obtained before.

Example 6.10

Consider the circuit in Fig. 6.27(a). Under dc conditions, find: (a) i, v_C, and i_L, (b) the energy stored in the capacitor and inductor.

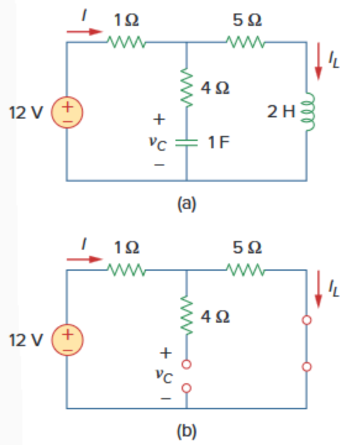


Figure 6.27
For Example 6.10.

Solution:

(a) Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit, as in Fig. 6.27(b). It is evident from Fig. 6.27(b) that

$$i = i_L = \frac{12}{1+5} = 2 \text{ A}$$

The voltage v_C is the same as the voltage across the 5-Ω resistor. Hence,

$$v_C = 5i = 10 \text{ V}$$

(b) The energy in the capacitor is

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50 \text{ J}$$

and that in the inductor is

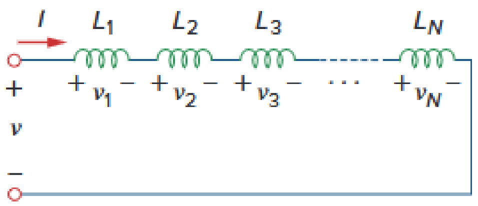
$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4 \text{ J}$$

Series and Parallel Inductors

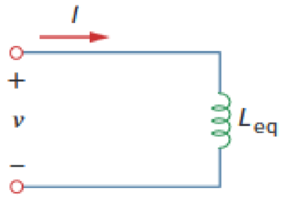
Series Inductor

$$L_{\text{eq}} = L_1 + L_2 + L_3 + \ldots$$

the equivalent in series is the sum of the individual inductors



(a)

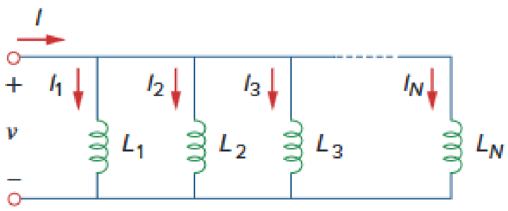


(b)

Parallel Inductor

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots$$

the equivalent in parallel is the reciprocal of the sum of the reciprocals of individual inductors



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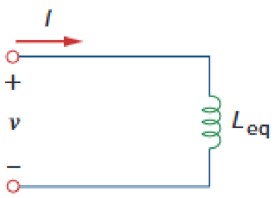

(b)

Important Characteristics of Basic Elements

TABLE 6.1

Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit

▼ Example

Example 6.12

For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.

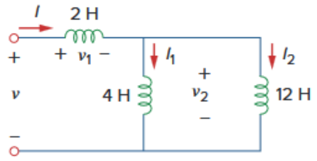

Figure 6.33
For Example 6.12.

Solution:

(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \| 12 = 2 + 3 = 5 \text{ H}$$

Thus,

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

and

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2\, dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\, dt + 5 \text{ mA}$$

$$= -3e^{-10t}\Big|_0^t + 5 \text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA}$$

Similarly,

$$i_2(t) = \frac{1}{12}\int_0^t v_2\, dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\, dt - 1 \text{ mA}$$

$$= -e^{-10t}\Big|_0^t - 1 \text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA}$$

Note that $i_1(t) + i_2(t) = i(t)$.