CALCULUS

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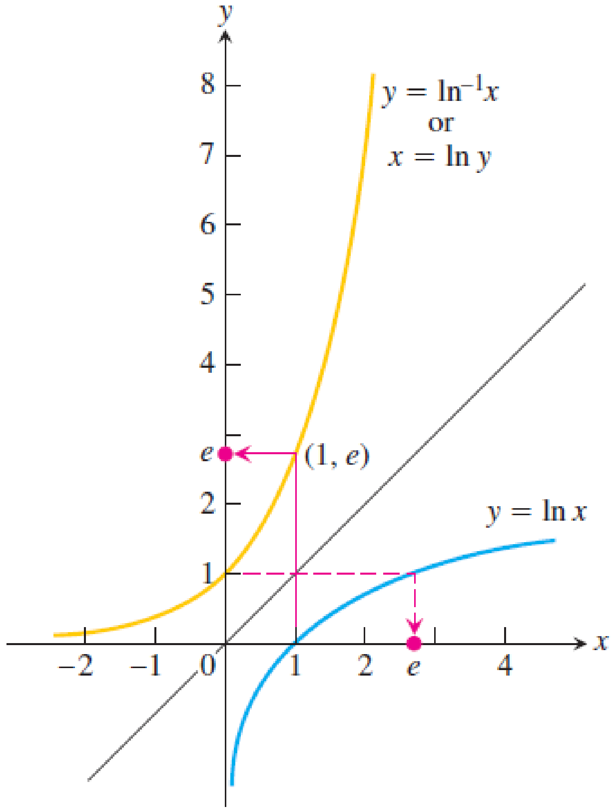
- Having developed the natural logarithmic function ln$x$, we introduce its inverse, the exponential function exp x = e^x. We study its properties and compute its derivative and integral. Finally, we introduce general exponential functions, a^x, and general logarithmic functions, $\log_a x$.

① The Inverse of ln *x* and the Number *e*

- The function ln$x$, being an increasing function of x with domain (0, ∞) and range (–∞, ∞), has an inverse $\ln^{-1}x$ with domain (–∞, ∞) and range (0, ∞). The graph of $\ln^{-1}x$ is the graph of ln$x$ reflected across the line $y = x$.



- The function $\ln^{-1} x$ is usually denoted as $\exp(x)$. We can show that $\exp(x)$ is an exponential function with base e.

DEFINITION

For every real number x, we define the **natural exponential function** to be

$$e^{x} = \exp(x)$$

- The number e was defined to satisfy the equation $\ln(e) = 1$, so $e = \exp(1)$.

  We can raise the number e to a power r in the usual algebraic way:

$$e^{2} = e \cdot e\,, \qquad e^{-2} = \frac{1}{e^{2}}\,, \qquad e^{1/2} = \sqrt{e}$$

$$e^{r} = \exp(r)\,, \qquad lne^{r} = rlne = r$$

② Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \quad (x > 0)$$

$$\ln e^x = x \quad (x \in R)$$

Example 1

Solve the equation $e^{2x-6} = 4$ for x.

Example 2

A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$.

What is the value of m?

③ The Derivative and Integral of e^x

- If u is any differentiable function of x, then

$$\ln(e^x) = x \quad \rightarrow \quad \frac{d}{dx} e^x = e^x$$

- Chain Rule $\Rightarrow$

$$\frac{d}{dx} e^u = e^x \frac{du}{dx}$$

Example 3 Find derivatives of the exponential.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(5\mathrm{e}^{x}\right)$; (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}\right)$; (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin x}\right)$; (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sqrt{3x+1}}\right)$; (e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^{2x}\right)$.

- The general antiderivative of the exponential function:

$$\int e^u du = e^u + C$$

Example 4 Evaluate

(a) $\int_0^{\ln 2} \mathrm{e}^{3x}\mathrm{d}x$; (b) $\int_0^{\frac{\pi}{2}} \mathrm{e}^{\sin x}\cos x\,\mathrm{d}x$.

④ Laws of Exponents

THEOREM 3

For all real numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1}e^{x_2} = e^{x_1+x_2}$

2. $e^{-x} = \dfrac{1}{e^x}$

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$

4. $(e^{x_1})^r = e^{rx_1}$

⑤ The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can express a^x as $(e^{\ln a})^x = e^{x\ln a}$. We therefore use the function e^x to define the other exponential functions, which allow us to raise *any* positive number to an irrational exponent.

DEFINITION For any numbers $a > 0$ and x, the **exponential function with base a** is

$$a^x = e^{x\ln a}.$$

The derivative: $(a^x)' = (e^{x\ln a})' = e^{x\ln a} \cdot \ln a = a^x \ln a$

The general antiderivative: $\int a^x dx = \frac{a^x}{\ln a} + C$

- The definition of the general exponential function enables us to make sense of raising any positive number to a real power n, rational or irrational. That is, we can define the power function $y = x^n$ for any exponent n.

DEFINITION

For any $x > 0$ and any real number n, $x^n = e^{n\ln x}$.

General Power Rule for Derivatives

For any $x > 0$ and any real number n, $(x^n)' = nx^{n-1}$

Example 5 Find y' if $y = x^x$, $x > 0$.

⑥ The Number *e* Expressed as a Limit

We have defined the number e as the number for which $\ln e = 1$, or equivalently, the value exp (1). We see that e is an important constant for the logarithmic and exponential functions, but what is its numerical value? The next theorem shows one way to calculate e as a limit.

THEOREM 4 – The Number *e* as a Limit

The number e can be calculated as the limit

$$\mathrm{e} = \lim_{x \to 0} (1+x)^{\frac{1}{x}} = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x .$$

Proof: **For $f(x) = \ln x$**

$$f'(1) = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

⑦ The Derivative of a^u

If u is a differentiable function of x, then there is

$$\frac{d}{dx}e^u = \frac{de^u}{du}\frac{du}{dx} = e^u\frac{du}{dx}$$

$$Thus: \quad \frac{d}{dx}(a^u) = \frac{d}{dx}(e^{ulna}) = (a^u lna)\frac{du}{dx}. \qquad Then: \quad \int a^u\,du = \frac{a^u}{lna} + C$$

Example 6 Find derivatives and integrals

(a) $\left(3^{\sin x}\right)'$

(b) $\int 2^{\sin x}\cos x\mathrm{d}x$

⑧ Logarithms with Base *a*

- If a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of x with base a** and denote it by $\log_a x$.

DEFINITION

For any positive number $a \neq 1$, $\log_a x$ is the inverse function of a^x.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a (a^x) = x \qquad (x \in R)$$

Rules for Logarithms with Base *a*

$$\log_a(xy) = \log_a x + \log_a y\,,\quad \log_a(x/y) = \log_a x - \log_a y\,,\quad \log_a(x^y) = y\log_a x$$

⑨ Derivatives of $\log_a x$

Note that $log_a x = \frac{lnx}{lna}$, we get

$$\frac{d}{dx}(log_a x) = \frac{1}{lna}\frac{1}{x} \quad and \quad \frac{d}{dx}(log_a u) = \frac{1}{lna}\frac{1}{u}\frac{du}{dx}$$

Example 8 Find the derivative of

(a) $y = \log_{10}(3x+1)$

(b) $y = \log_4(2x^2+1)$

(c) $y = \log_2(x^3+2x+1)$