CALCULUS

Prof. Liang ZHENG

Spring 2025

- One of the most important applications of the derivative is its use as a tool for finding the optimal (best) solutions to problems.
- In this chapter we apply derivatives to find extreme values of functions, to determine and analyze shapes of graphs, and to solve equations numerically. We also introduce the idea of recovering a function from its derivative.
- The key to many of these applications is the Mean Value Theorem, which connects the derivative and the average change of a function.

① Absolute (global) Maximum and Minimum

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leqslant f(c) \quad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geqslant f(c) \quad \text{for all } x \text{ in } D.$$

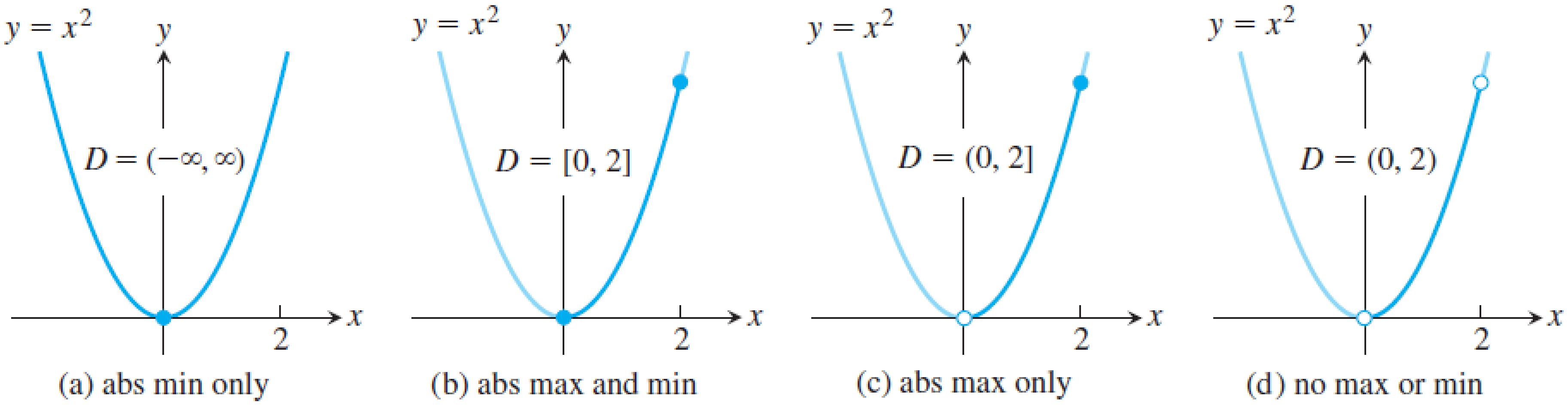
- Maximum and minimum values are called **extreme values** of the function f.

  Absolute maxima or minima are also referred to as **global maxima** or minima.

Example 1

The absolute extrema of the following functions on their domains can be seen in the following Figure. Each function has the same defining equation, $y = x^2$, but the domains vary.

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema



- Some of the functions do not have a maximum or a minimum value. The following theorem asserts that a function which is *continuous* over (or on) a finite *closed* interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval.

THEOREM 1 – The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leqslant f(x) \leqslant M$ for every other x in $[a, b]$.

- The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are essential. Without them, the conclusion of the theorem need not hold.

4.1 Extreme Values of Functions on Closed Intervals

- Figure 4.3 illustrates the possible locations for the absolute extrema of a continuous function on a closed interval [a, b].

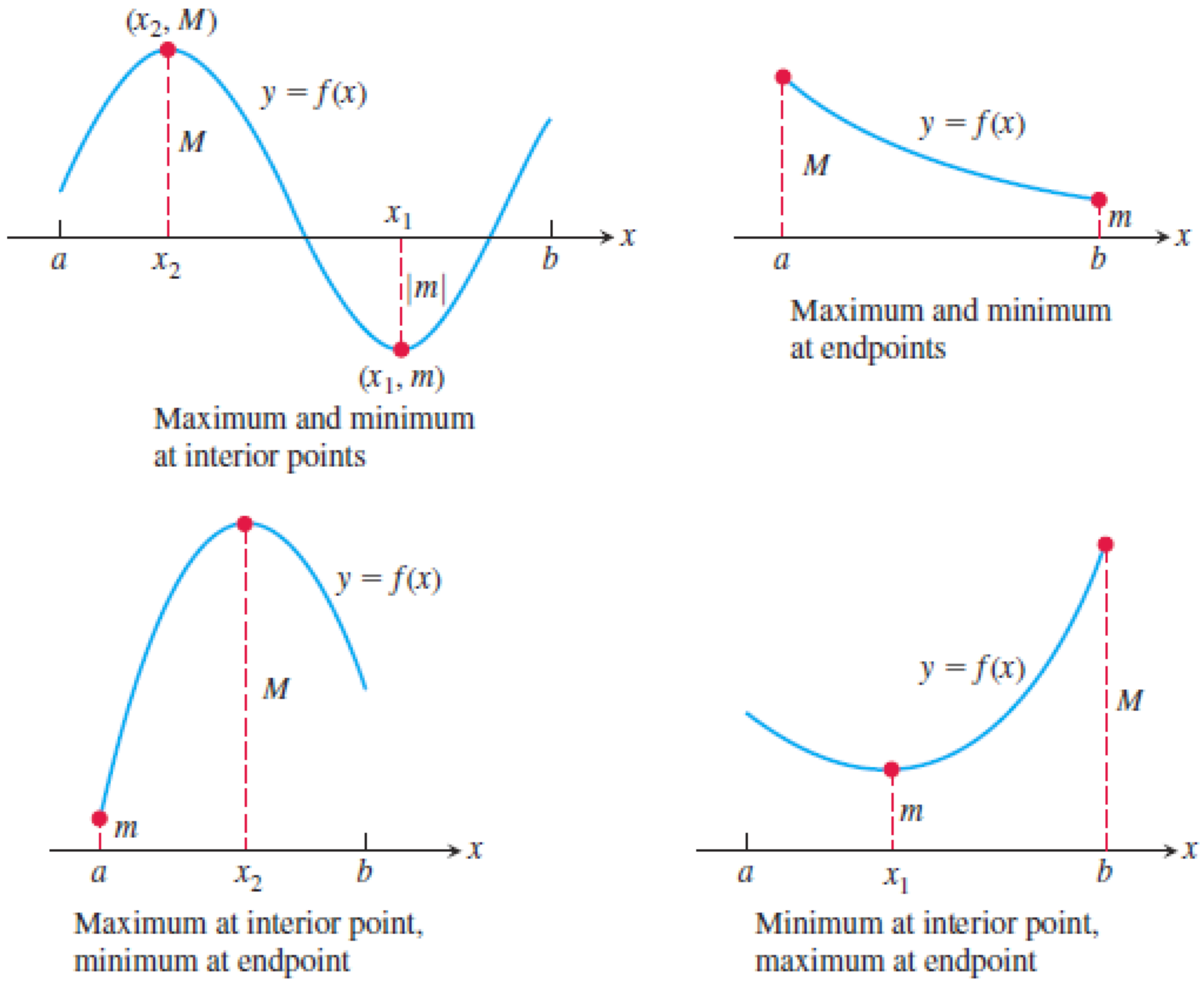


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

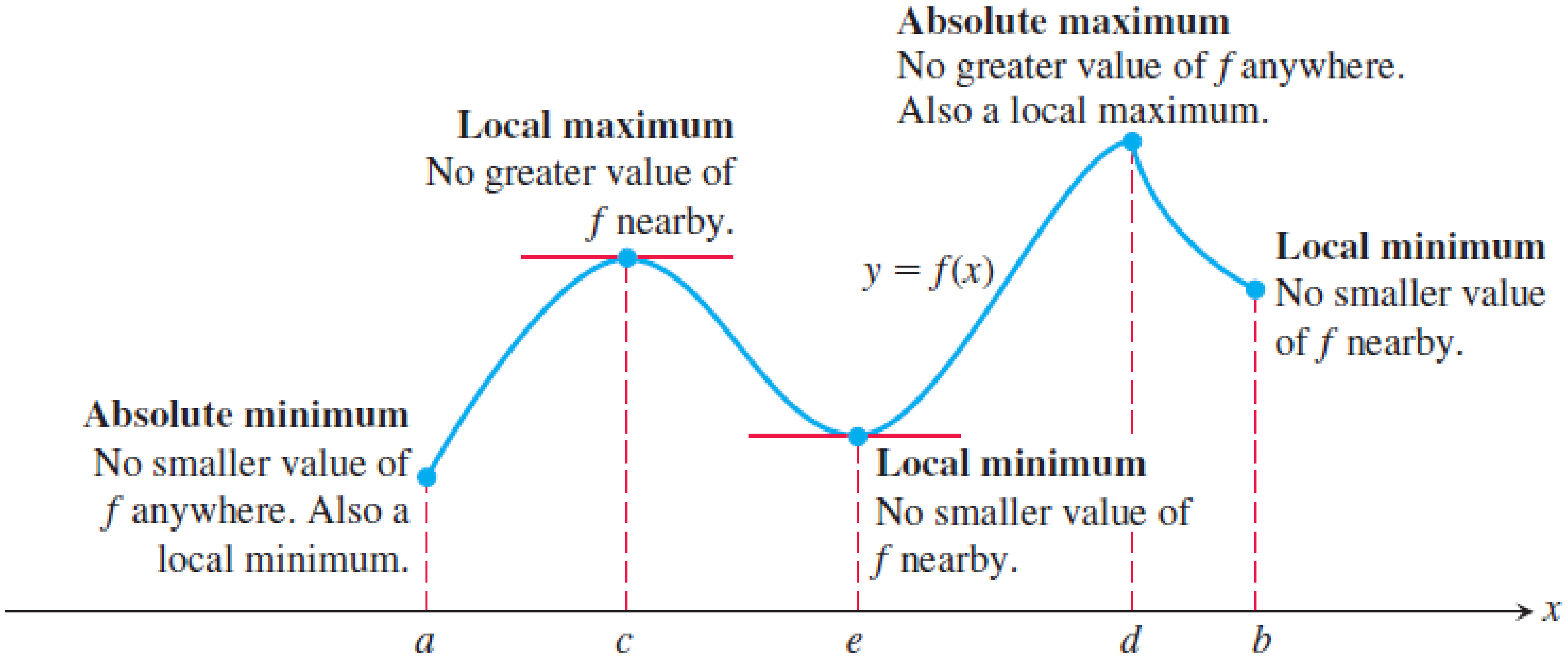
② Local (Relative) Extreme Values

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leqslant f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geqslant f(c)$ for all $x \in D$ lying in some open interval containing c.

- If the domain of f is the closed interval $[a, b]$, then f has a local maximum at the endpoint $x = a$ if $f(x) \leq f(a)$ for all x in some half-open interval $[a, a + \delta)$, $\delta > 0$. Likewise, f has a local maximum at an interior point $x = c$ if $f(x) \leq f(c)$ for all x in some open interval $(c - \delta, c + \delta)$, $\delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \leq f(b)$ for all x in some half-open interval $(b - \delta, b]$, $\delta > 0$.

- In the below figure, the function f has local maxima at c and d and local minima at a, e, and b. Local extrema are also called **relative extrema**. Some functions can have infinitely many local extrema, even over a finite interval. One example is the function $f(x) = \sin(1/x)$ on the interval $(0, 1)$.

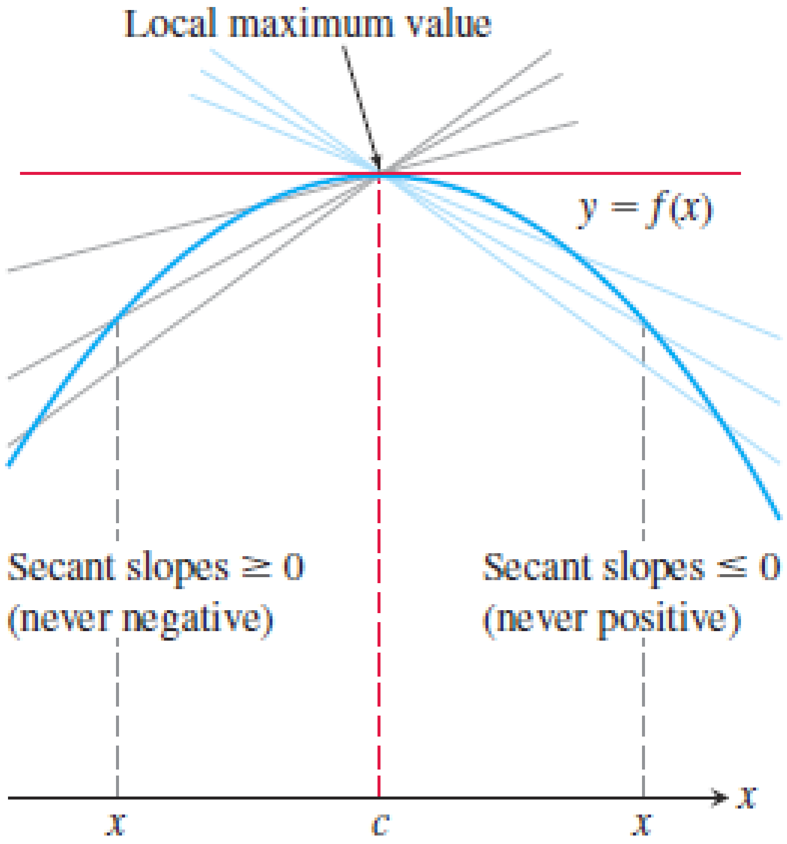


③ Finding Extrema

THEOREM 2—The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$



Proof: The two-sided limit at $x = c$ is:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

- $f(c)$ is an interior local maximum or minimum

  $\Rightarrow$ $f'(c) = 0$ or $f'(c)$ is undefined.

- The only places where a function f can possibly have a local extreme value are

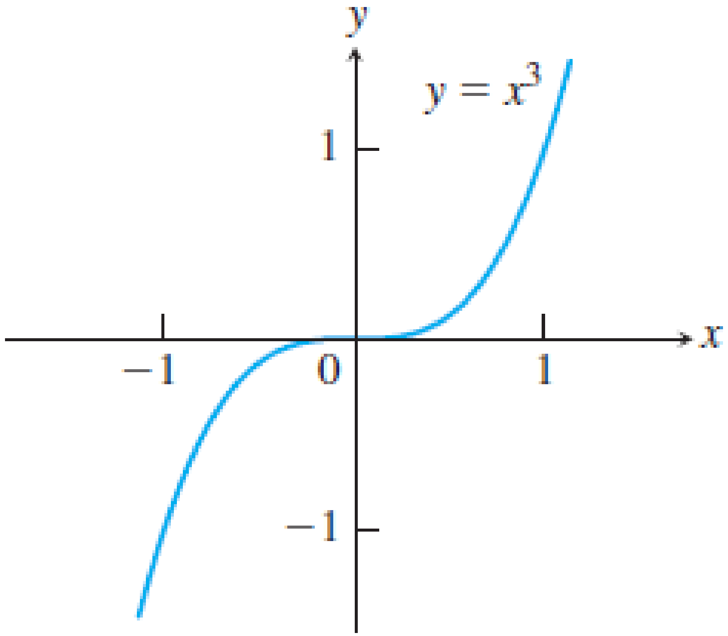
1. interior points where $f' = 0$,

2. interior points where f' is undefined,

3. endpoints of the domain of f.

- $f'(c) = 0 \quad \not\Rightarrow \quad f(c)$ is an interior local maximum or minimum.

For example, $f(x) = x^3$ on $x = 0$ is neither a local maximum nor a local maximum, but a **point of inflection**.



DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

- Most problems that ask for extreme values call for finding the extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints.

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f on the interval.

2. Evaluate f at all critical points and endpoints.

3. Take the largest and smallest of these values.

Example 2

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

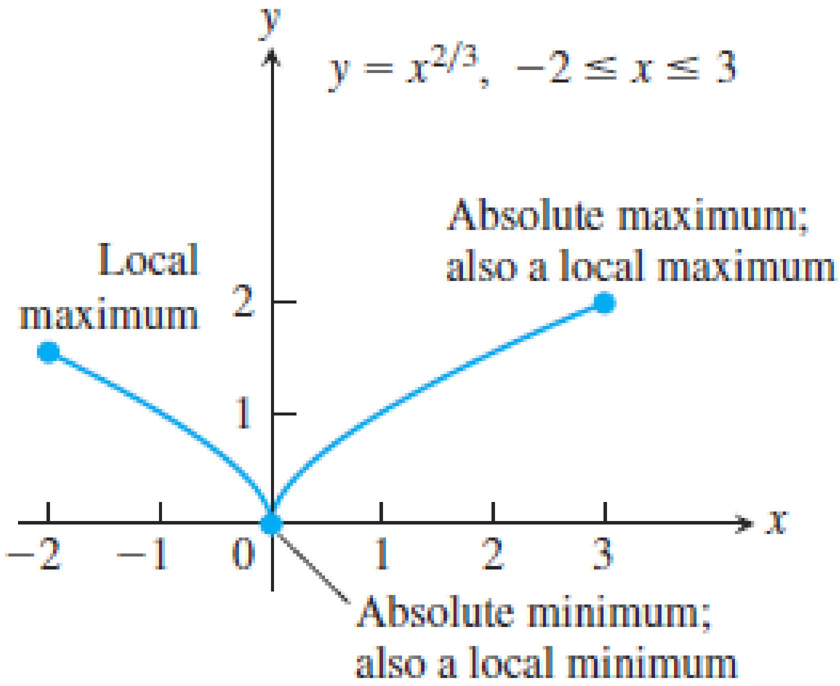
Example 3

Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Example 4

Find the absolute maximum and minimum values

of $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$.



Skill Practice 1

Determine all critical points for each function:

(a) $f(x) = 6x^2 - x^3$ (b) $g(x) = (2x - x^2)^{1/2}$

Skill Practice 2

Find the critical points and domain endpoints for each function. Then find the value of the function at each of these points and identify extreme values (absolute and local).

(a) $y = x^{2/3}(x^2 - 4)$ (b) $g(x) = x^2/(x-2)$