College Algebra and Trigonometry

Prof. Liang ZHENG

Fall 2024

① Divide Polynomials using Long Division

Example 1:

Use long division to divide:

$$(6x^3 - 5x^2 - 3) \div (3x + 2)$$

Quotient: $2x^2 - 3x + 2$

Remainder: -7

Division Algorithm

Suppose that $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and the degree of $d(x)$ is less than or equal to the degree of $f(x)$. Then there exists unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

where the degree of $r(x)$ is less than $d(x)$, or $r(x)$ is the zero polynomial.

***Note*: The polynomial $f(x)$ is the dividend, $d(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.**

Example 2:

Use long division to divide:

$$(3x^4 + 2x^3 + 4x^2 + x - 5) \div (x^2 + 2)$$

Answer: $3x^2 + 2x - 2 + \frac{-3x-1}{x^2+2}$

Example 3:

Use long division to divide:

$$(2x^2 + 3x - 14) \div (x - 2)$$

Answer: $2x + 7$

② Divide Polynomials using Synthetic division

- **When dividing polynomials where the divisor is a binomial of the form $(x - c)$ and c is a constant, we can use synthetic division.**

Example 4:

Use synthetic division to divide: $(2x^3 - 10x^2 - 5) \div (x - 4)$

Answer: $2x^2 - 2x - 8 + \frac{-37}{x-4}$

Example 5:

Use synthetic division to divide: $(x^4 + 4x^3 - 2x + 18) \div (x + 2)$

Answer: $x^3 + 2x^2 - 4x + 6 + \frac{6}{x+2}$

③ Apply the Remainder and Factor Theorems

- **Consider the special case of the division algorithm where $f(x)$ is the dividend and $(x - c)$ is the divisor.**

$$f(x) = (x - c) \cdot q(x) + r$$

Note that the remainder r must be a constant.

- **Then we have:** $f(c) = (c - c) \cdot q(x) + r = r$

Remainder Theorem

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Example 6:

$f(x) = x^4 + 6x^3 - 12x^2 - 30x + 35$, use the remainder theorem to evaluate:

(a) $f(2)$ (b) $f(-7)$

Example 7:

Use the remainder theorem to determine if the given number c is a zero of the polynomial:

(a) $f(x) = 2x^4 - 4x^2 - 13x - 9$; $c = 4$

(b) $f(x) = x^3 + x^2 - 3x - 3$; $c = \sqrt{3}$

(c) $f(x) = x^3 + x + 10$; $c = 1 + 2i$

Factor Theorem

Let $f(x)$ be a polynomial.

1) **If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.**
2) **If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.**

Example 8:

Use the factor theorem to determine if the given polynomials are factors of $f(x) = x^4 - x^3 - 11x^2 + 11x + 12$.

a) $x - 3$ b) $x + 2$

Example 9:

a) Factor $f(x) = 3x^3 + 25x^2 + 42x - 40$, given that -5 is a zero of $f(x)$.

b) Find all the zeros of $f(x)$.

Example 10:

Write a polynomial $f(x)$ of degree 3 that the zeros $\frac{1}{2}$, $\sqrt{6}$, and $-\sqrt{6}$.

① Apply the Rational Zero Theorem

Rational Zero Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients and $a_n \neq 0$, and if p/q (written in lowest terms) is a rational zero of f , then

- **p is a factor of the constant term a_0 .**
- **q is a factor of the leading coefficient a_n .**

Question: What if $a_n = 0$?

Note:

1) This theorem does not guarantee the existence of rational zeros.

2) But it is important because it limits the search to find rational zeros (if they exist) to a finite number of choices.

Example 1:

List all possible rational zeros of

$$f(x) = -2x^5 + 3x^2 - 2x + 10$$

Example 2:

Find the zeros of $f(x) = x^3 - 4x^2 + 3x + 2$

Example 3:

Find the zeros and multiplicities of

$$f(x) = 2x^4 + 5x^3 - 2x^2 - 11x - 6$$

Example 4:

Find the zeros of $f(x) = x^4 - 2x^2 - 3$

② Apply the Fundamental Theorem of Algebra

Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x)$ has at least one complex zero.

- **It was first proved by German mathematician Carl Friedrich Gauss.**

Linear Factorization Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $n \geq 1$ and $a_n \neq 0$, then $f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$ where $c_1, c_2, \ldots c_n$ are complex numbers.

***Note*: The complex numbers $c_1, c_2, \ldots c_n$ are not necessarily unique.**

Number of Zeros of a Polynomial

If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x)$ has exactly n complex zeros provided that each zero is counted by its multiplicity.

Conjugate Zeros Theorem

If $f(x)$ is a polynomial with real coefficients and if $a + bi$ ($b \neq 0$) is a zero of $f(x)$, then its conjugate $a - bi$ is also a zero of $f(x)$.

Example 5:

Given $f(x) = x^4 - 6x^3 + 28x^2 - 18x + 75$, and that $3 - 4i$ is a zero of $f(x)$.

a) **Find the remaining zeros.**

b) **Factor $f(x)$ as a product of linear factors.**

Example 6:

a) **Find a third-degree polynomial $f(x)$ with integer coefficients and with zeros of $2/3$ and $4+2i$.**

b) **Find a polynomial $g(x)$ of lowest degree with zeros of -2 (multiplicity 1) and 4 (multiplicity 3), and satisfying $g(0) = 256$.**

③ Apply Descartes' Rule of Signs

$$2x^6 - 3x^4 - x^3 + 5x^2 - 6x - 4 \qquad \text{(3 sign changes)}$$

positive to negative | negative to positive | positive to negative

Question: What if $a_0 = 0$?

Descartes' Rule of Signs

Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term. Then,

1. The number of *positive* real zeros is either
   - the same as the number of sign changes in $f(x)$ or
   - less than the number of sign changes in $f(x)$ by a positive even integer.
2. The number of *negative* real zeros is either
   - the same as the number of sign changes in $f(-x)$ or
   - less than the number of sign changes in $f(-x)$ by a positive even integer.

Example 7:

a) **Determine the number of possible positive and negative real zeros.**

$$f(x) = x^5 - 6x^4 + 12x^3 - 12x^2 + 11x - 6$$

b) **Find all the zeros.**

④ Find Upper and Lower Bounds

Definition of Upper and Lower Bounds

- **A real number b is called an upper bound of the real zeros of a polynomial if all real zeros are less than or equal to b.**
- **A real number a is called an lower bound of the real zeros of a polynomial if all real zeros are greater than or equal to a.**

Upper and Lower Bound Theorem for the Real Zeros

Let $f(x)$ be a polynomial of degree $n \geq 1$ with real coefficients and a positive leading coefficient. Further, suppose that $f(x)$ is divided by $(x-c)$.

a) **If $c > 0$ and if both the remainder and the coefficients of the quotient are nonnegative, then c is an upper bound for the real zeros of $f(x)$.**

b) **If $c < 0$ and the coefficients of the quotient and the remainder alternate in sign (with 0 being considered either positive or negative as needed), then c is a lower bound for the real zeros of $f(x)$.**

Example 9:

Given $f(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$

a) Determine if the theorem identifies 2 as the upper bound of for the real zeros of $f(x)$.

b) Determine if the theorem identifies -3 as the lower bound of for the real zeros of $f(x)$.

Example 10:

Find the zeros and their multiplicities.

$$f(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$$