CALCULUS

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- One of the most important applications of the derivative is its use as a tool for finding the optimal (best) solutions to problems.
- In this chapter we apply derivatives to find extreme values of functions, to determine and analyze shapes of graphs, and to solve equations numerically. We also introduce the idea of recovering a function from its derivative.
- The key to many of these applications is the Mean Value Theorem, which connects the derivative and the average change of a function.

① Absolute (global) Maximum and Minimum

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leqslant f(c) \text{ for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

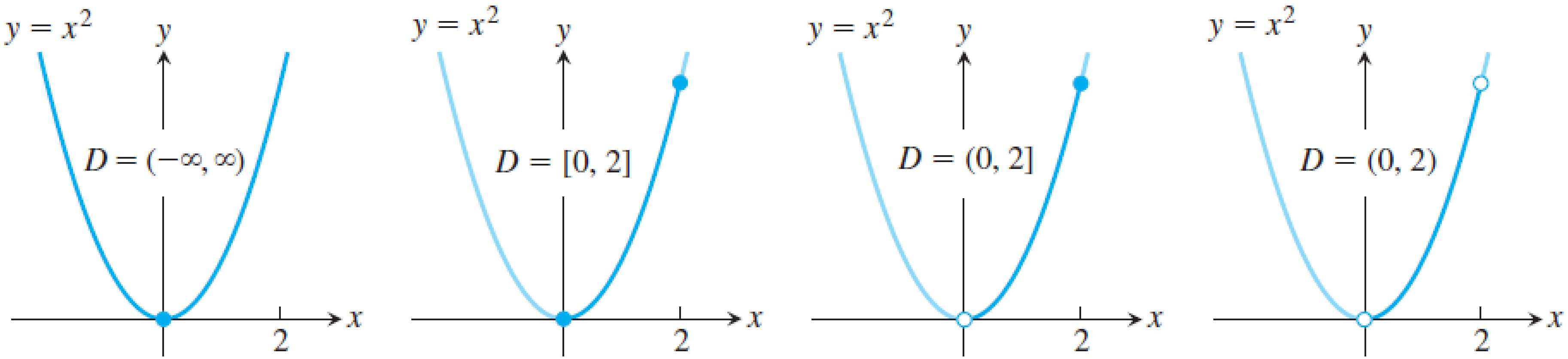
$$f(x) \geqslant f(c) \text{ for all } x \text{ in } D.$$

- Maximum and minimum values are called **extreme values** of the function f.
  Absolute maxima or minima are also referred to as **global maxima** or **minima**.

Example 1

The absolute extrema of the following functions on their domains can be seen in the following Figure. Each function has the same defining equation, $y = x^2$, but the domains vary.

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema



(a) abs min only (b) abs max and min (c) abs max only (d) no max or min

- Some of the functions do not have a maximum or a minimum value. The following theorem asserts that a function which is *continuous* over (or on) a finite *closed* interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval.

THEOREM 1 – The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leqslant f(x) \leqslant M$ for every other x in $[a, b]$.

- The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are essential. Without them, the conclusion of the theorem need not hold.

- Figure 4.3 illustrates possible locations for the absolute extrema of a continuous function on a closed interval [a, b].

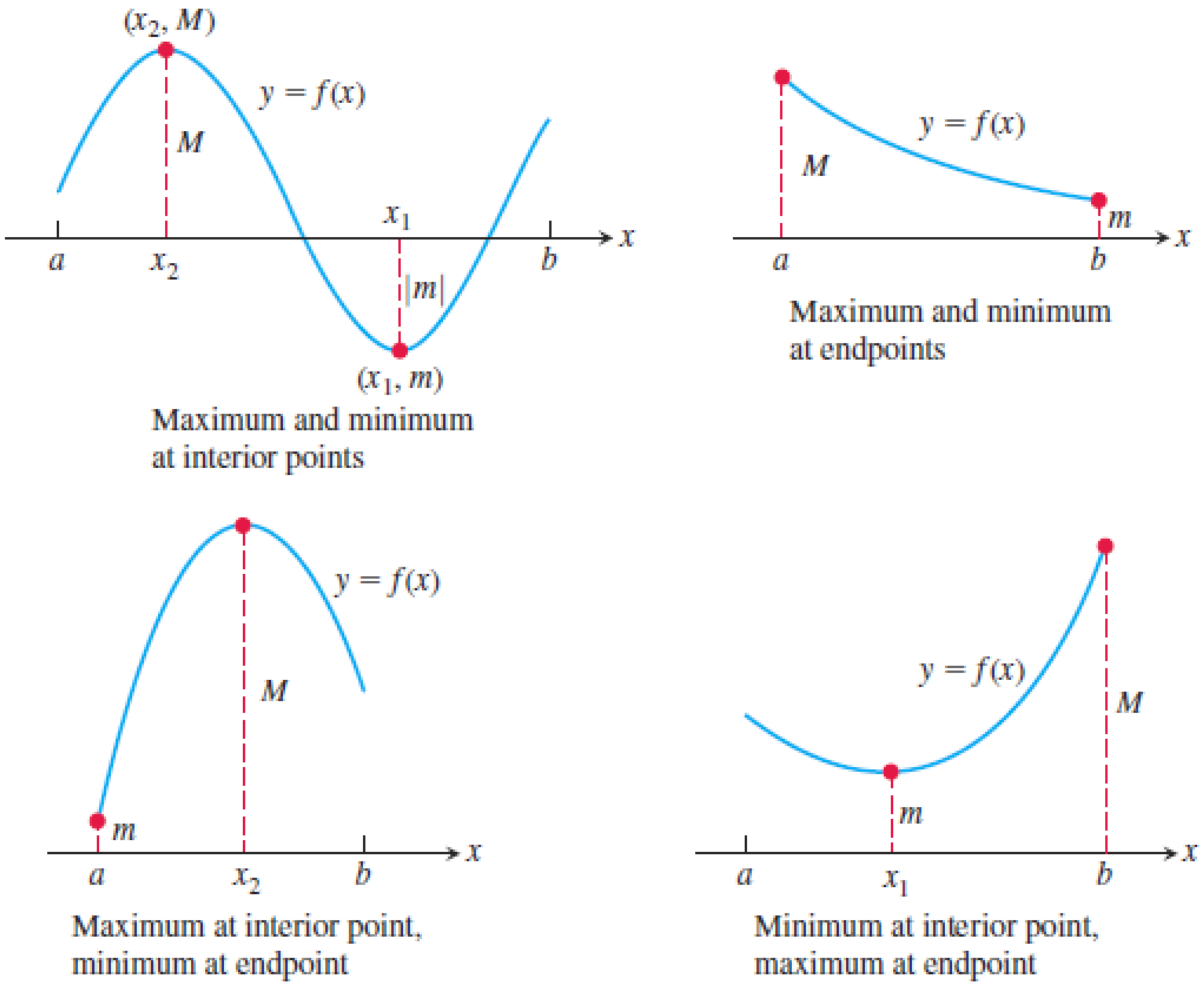


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

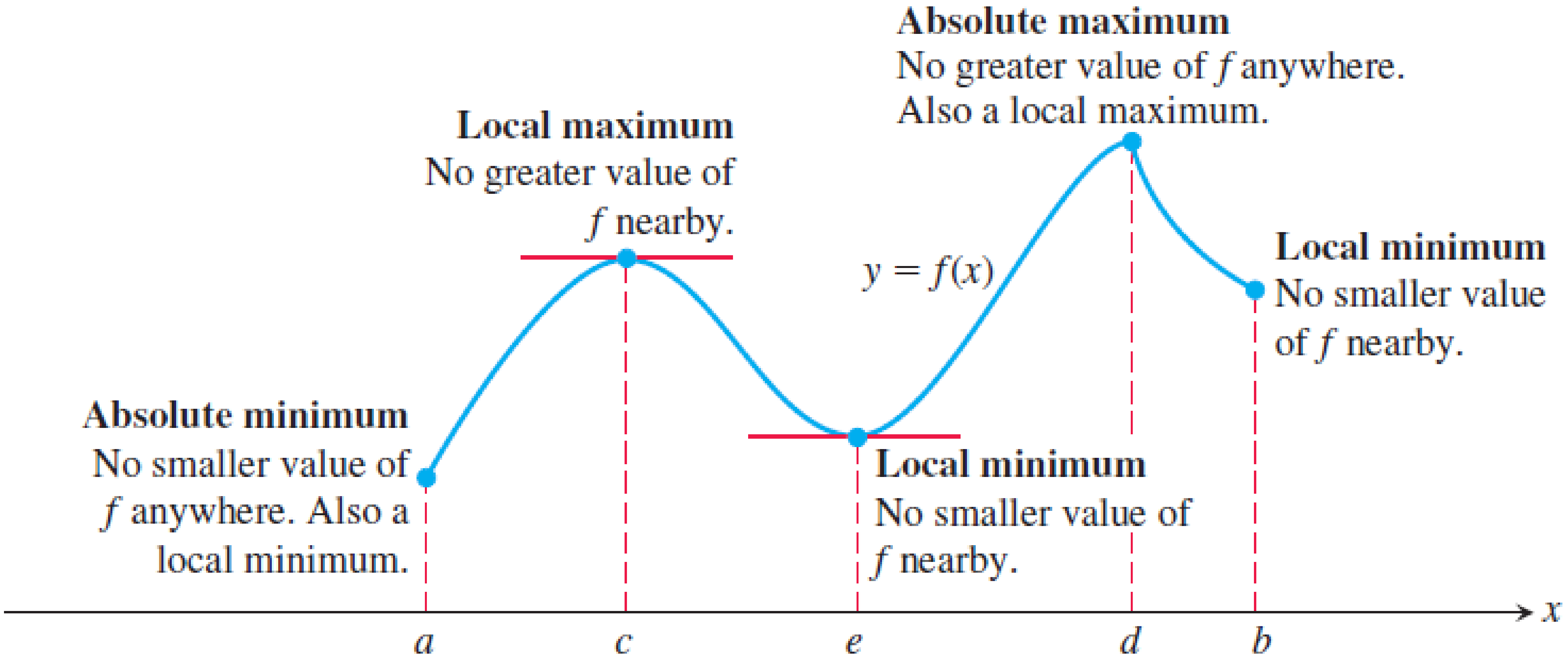
② Local (Relative) Extreme Values

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leqslant f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geqslant f(c)$ for all $x \in D$ lying in some open interval containing c.

- If the domain of f is the closed interval $[a, b]$, then f has a local maximum at the endpoint $x = a$ if $f(x) \leq f(a)$ for all x in some half-open interval $[a, a + \delta)$, $\delta > 0$. Likewise, f has a local maximum at an interior point $x = c$ if $f(x) \leq f(c)$ for all x in some open interval $(c - \delta, c + \delta)$, $\delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \leq f(b)$ for all x in some half-open interval $(b - \delta, b]$, $\delta > 0$.

- In Figure 4.5, the function f has local maxima at c and d and local minima at a, e, and b. Local extrema are also called **relative extrema**. Some functions can have infinitely many local extrema, even over a finite interval. One example is the function $f(x) = \sin(1/x)$ on the interval $(0, 1)$.



③ Finding Extrema

THEOREM 2 — Fermat's Theorem (The First Derivative Theorem)

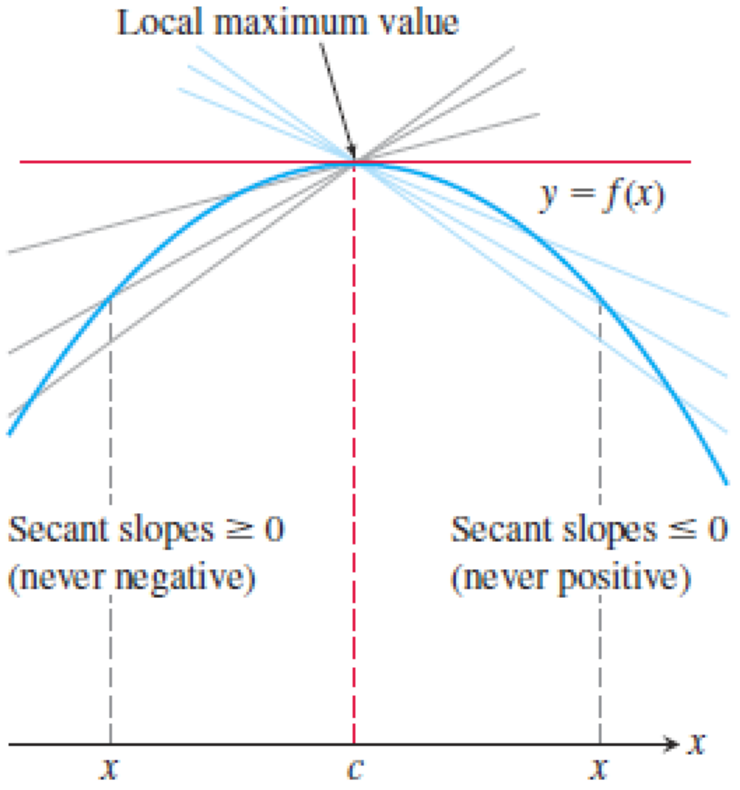
If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $\boldsymbol{f'(c) = 0}$.

Proof: The two-sided limit at $x = c$ is:

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$= \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$$

- This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities.

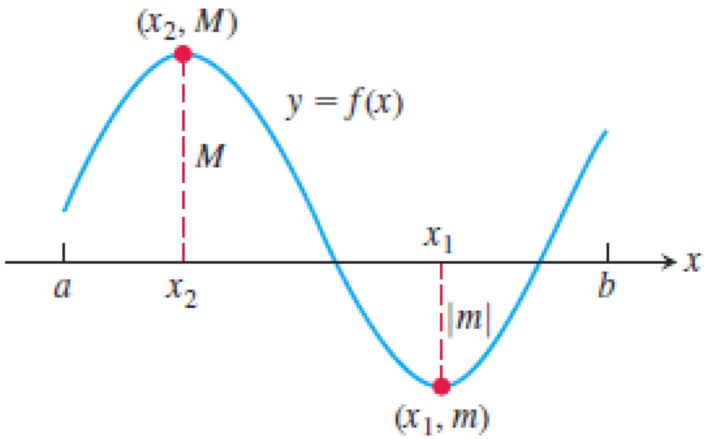


- **Fermat's Theorem** indicates that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined.
- The only places where a function f can possibly have a local extreme value are

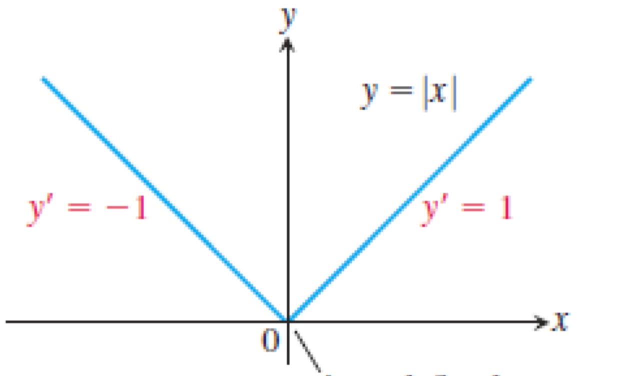
(1) interior points where $f' = 0$.

(2) interior points where f' is undefined.

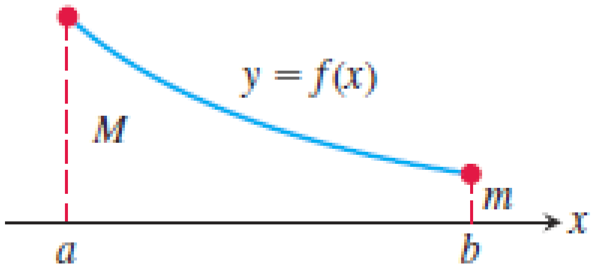
(3) endpoints of the domain of f.



Maximum and minimum at interior points



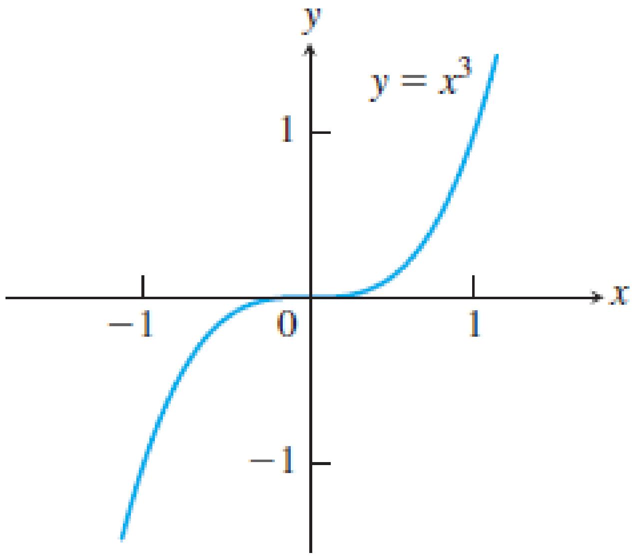
y' not defined at $x = 0$: right-hand derivative $\neq$ left-hand derivative



Maximum and minimum at endpoints

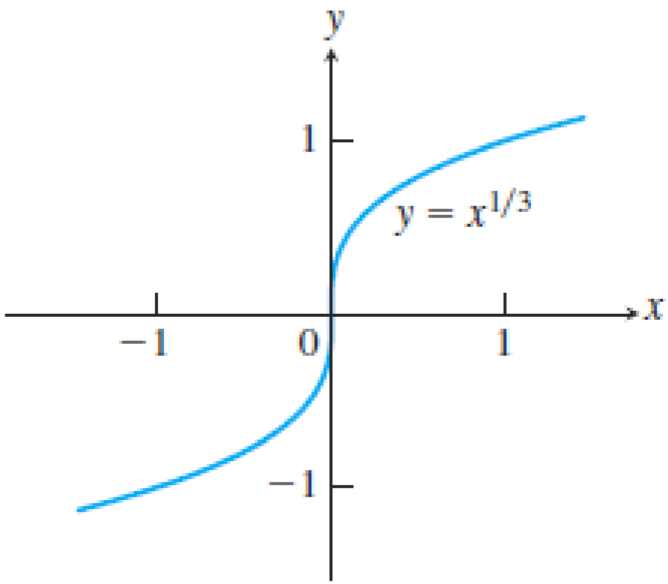
DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

- However, a function may have a critical point at $x = c$ without having a local extreme value there. For instance, both of the functions $y = x^3$ and $y = x^{1/3}$ have critical points at the origin, but neither function has a local extreme value at the origin. Instead, each function has a ***point of inflection*** there.



$y = x^3$: $f'(0) = 0$ and $f''(0) = 0$.

$y = x^{1/3}$: $f'(0)$ is undefined.



- Most problems that ask for extreme values call for finding the extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints.

Finding the Absolute Extrema of a Continuous Function *f* on a Finite Closed Interval

1. Find all critical points of f on the interval.

2. Evaluate f at all critical points and endpoints.

3. Take the largest and smallest of these values.

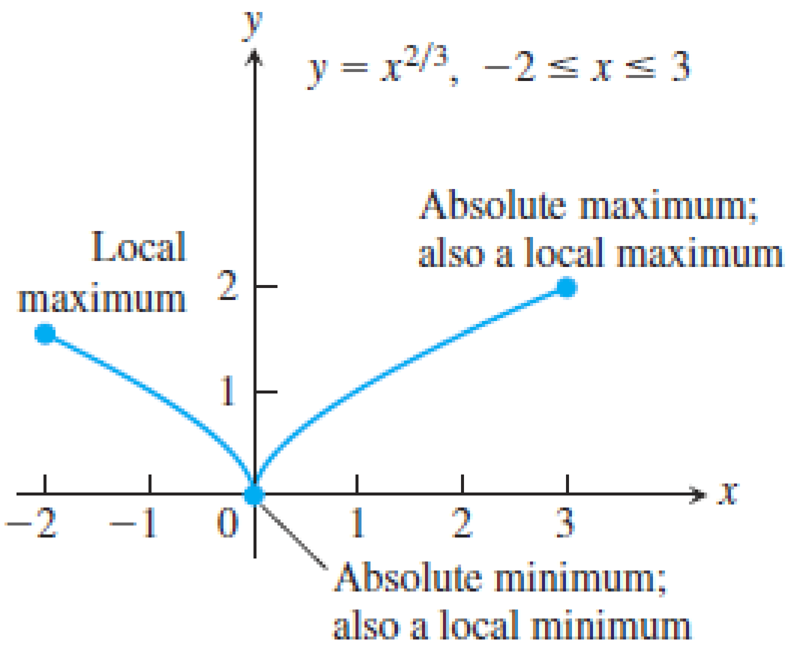
Example 2 Find the absolute maximum and minimum values on $[-2, 1]$:

(a) $f(x) = x^2$ (b) $g(t) = 8t - t^4$

Example 3

Find the absolute maximum and minimum values of

$f(x) = x^{2/3}$ on $[-2, 3]$.



Skill Practice 1

Determine all critical points for each function:

(a) $f(x) = 6x^2 - x^3$ (b) $g(x) = (2x - x^2)^{1/2}$

Skill Practice 2

Find the critical points for the function $f(x) = x^{2/3}(x^2 - 4)$. Then find the function value at each critical point and identify extreme values (absolute and local).