CALCULUS

Prof. Liang ZHENG

Spring 2025

- In this section we introduce a more convenient notation for working with sums that have a large number of terms. After describing this notation and its properties, we consider what happens as the number of terms approaches infinity.

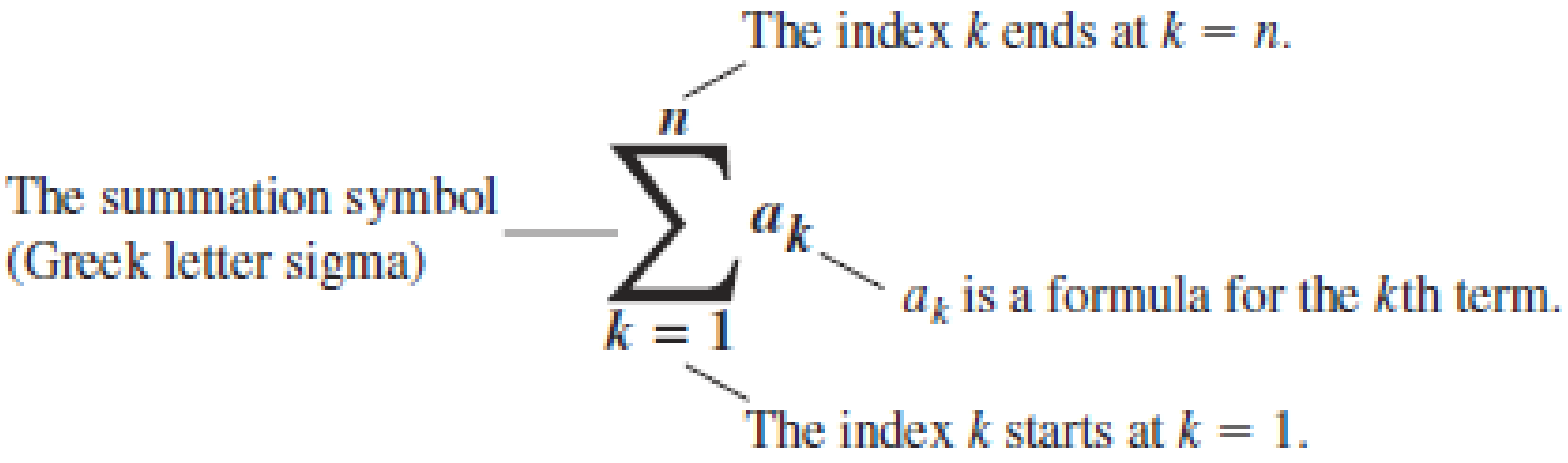
① Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n.$$

The Greek letter Σ stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ).

5.2 Sigma Notation and limits of Finite Sums



Thus, we can write the squares of the numbers 1 through 10 as

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2=\sum_{k=1}^{10}k^2.$$

and the sum of $f(i)$ for integers i from 1 to 100 as

$$f(1)+f(2)+f(3)+\cdots+f(100)=\sum_{i=1}^{100}f(i).$$

Example 1 Express the following sums in sigma notation.

(a) $1+2+3+4+5$;

(b) $(-1)^1 1+(-1)^2 2+(-1)^3 3$;

(c) $\dfrac{1}{1+1}+\dfrac{2}{2+1}$;

(d) $\dfrac{4^2}{4-1}+\dfrac{5^2}{5-1}$.

A sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1+2+3+4+5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1)+(-1)^2(2)+(-1)^3(3)$	$-1+2-3=-2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1}+\frac{2}{2+1}$	$\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1}+\frac{5^2}{5-1}$	$\frac{16}{3}+\frac{25}{4}=\frac{139}{12}$

Example 2 Express the sum $1+3+5+7+9$ in sigma notation.

Algebra Rules for Finite Sums

Algebra Rules for Finite Sums

1. *Sum Rule:* $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

2. *Difference Rule:* $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

3. *Constant Multiple Rule:* $$\sum_{k=1}^{n} c a_k = c \cdot \sum_{k=1}^{n} a_k$$ (Any number c)

4. *Constant Value Rule:* $$\sum_{k=1}^{n} c = n \cdot c$$ (Any number c)

Example 3 Demonstrate the use of the algebra rules.

(a) $\sum_{k=1}^{n}(3k-k^2)$; (b) $\sum_{k=1}^{n}(-a_k)$; (c) $\sum_{k=1}^{3}(k+4)$; (d) $\sum_{k=1}^{n}\frac{1}{n}$.

Example 4 Show that the sum of the first *n* integers is:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

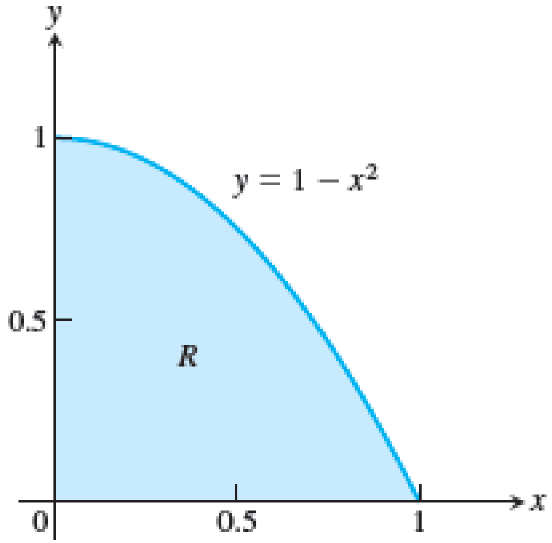
- The sums of the squares and cubes of the first *n* integers are:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

② Limits of Finite Sums

Example 5

Find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1-x^2$, and between the vertical lines $x = 0$ and $x = 1$, by using lower sum approximations.



Find $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n}$, where $f\left(\frac{k}{n}\right)=1-\left(\frac{k}{n}\right)^2$.

Example 6 Find the limit.

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k(k+1)}\right]$$

③ Riemann Sums

- The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a Riemann sum, which underlies the theory of the definite integral that will be presented in the next section.

- We begin with an arbitrary bounded function f defined on a closed interval $[a, b]$ (shown in Fig. 5.8). Then we choose $n-1$ points $\{x_1, x_2, x_3, \ldots, x_{n-1}\}$ between a and b that are in the increasing order, so that $a < x_1 < x_2 < \cdots < x_{n-1} < b$.
- Let $x_0 = a$ and $x_n = b$, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

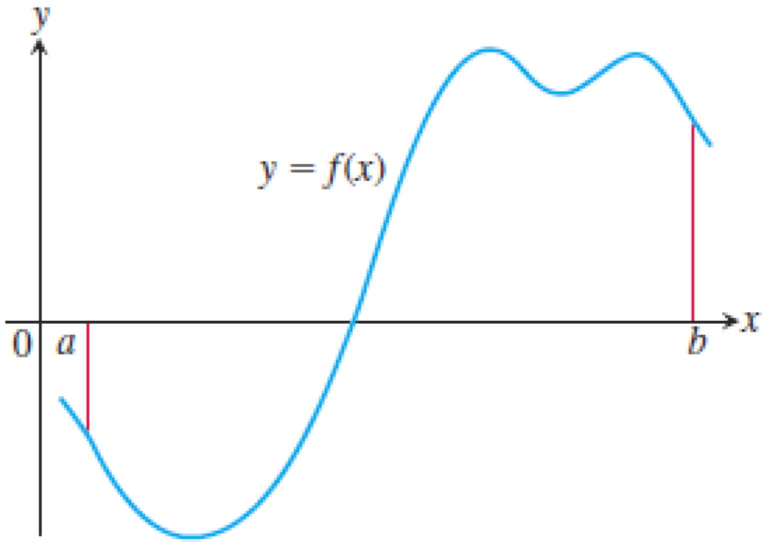


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

- The set of all of these points,

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\},$$

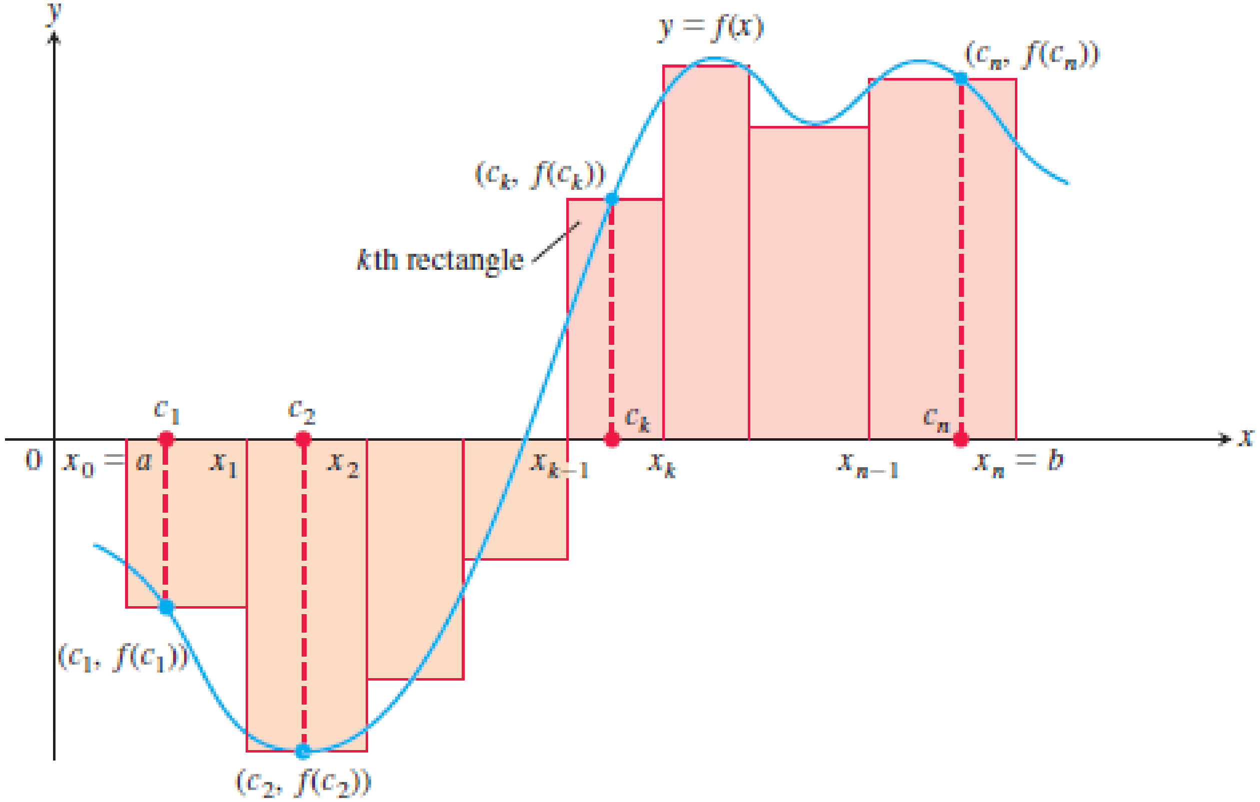
is called a **partition** of $[a, b]$.

- The partition P divides $[a, b]$ into the n closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

- The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then their common width, which we call Δx, is equal to $(b-a)/n$.
- In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k.

- Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$.



- On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. When $f(c_k) > 0$, the product $f(c_k)\Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k . When $f(c_k) < 0$, the product $f(c_k)\Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.
- Finally, we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

The sum S_P is called a **Riemann sum for *f* on the interval [*a*, *b*]**.

- There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the subinterval.

Skill Practice 1

(a) $\sum_{k=8}^{88} k$ (b) $\sum_{k=1}^{n}\left(\frac{1}{n}+2n\right)$ (c) $\sum_{k=1}^{100}\frac{1}{k(k+1)}$

Skill Practice 2

Using mathematical induction to show that the sums of the squares and cubes of the first n integers are:

(a) $\sum_{k=1}^{n} k^2=\frac{n(n+1)(2n+1)}{6}$ (b) $\sum_{k=1}^{n} k^3=\left(\frac{n(n+1)}{2}\right)^2$