CALCULUS

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- In this section we consider the limit of general Riemann sums as the norm of the partitions of a closed interval $[a, b]$ approaches zero. This limiting process leads us to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

① Definition of the Definite integral

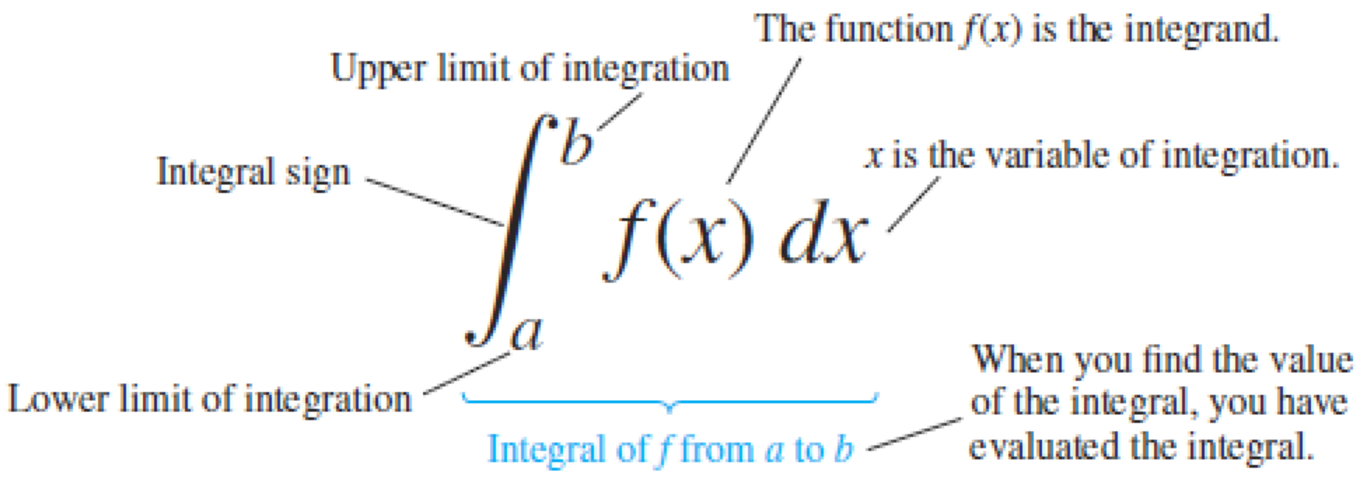
DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** if

$$J = \int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

provided that this limit exists for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and any choice of c_k in $[x_{k-1}, x_k]$.

5.3 The Definite integral

- The component parts in the integral symbol also have names:



- When the definite integral exists, we say that the Riemann sums of f on $[a, b]$ **converge** to the definite integral $J = \int_a^b f(x)dx$ and that f is **integrable** over $[a, b]$.
- The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable, meaning:

$$J = \int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

② Integrable and Nonintegrable Functions

THEOREM 1— Integrability of Continuous Functions

If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

Example 1 The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

has no Riemann integral over [0, 1] .

- Different choices for the points c_k results in different limits for the corresponding Riemann sums (S_P).

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

$$= \begin{cases} 1, & if\ c_k\ is\ rational \\ 0, & if\ c_k\ is\ irrational \end{cases}$$

③ Properties of Definite Integrals (Rules satisfied by definite integrals)

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. Max-Min Inequality: If f has maximum and minimum values on $[a, b]$, then

$$f_{min} \cdot (b-a) \le \int_a^b f(x)dx \le f_{max} \cdot (b-a)$$

7. Domination: If $f(x) \ge g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

If $f(x) \ge 0$ on $[a, b]$, then $\int_a^b f(x)dx \ge 0$ special case ($g(x) = 0$)

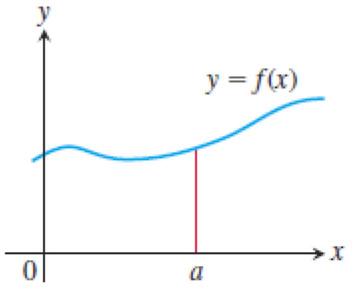
Example 2

Show that the value of

$$\int_0^1 \sqrt{1+\cos x}\,\mathrm{d}x$$

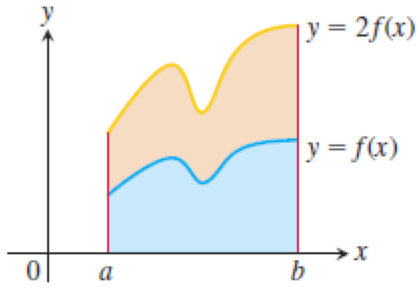
is less than or equal to $\sqrt{2}$.

Geometric Interpretations of Rules 2-7



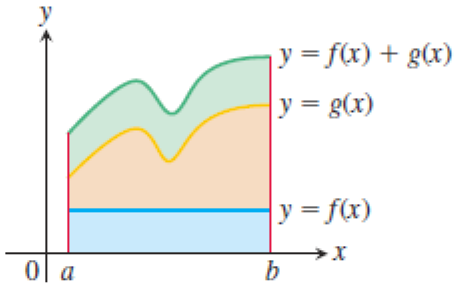
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



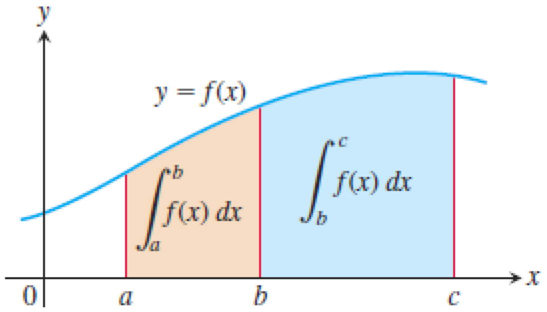
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



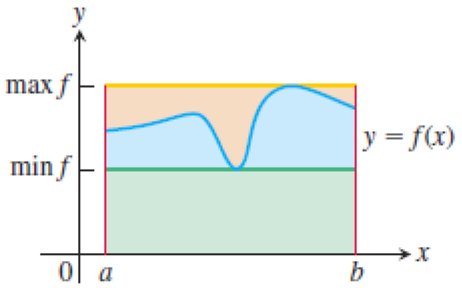
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



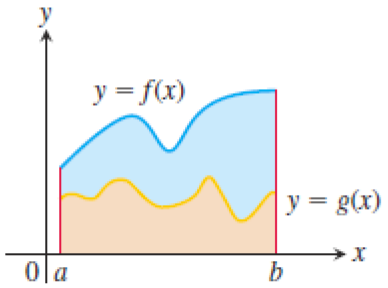
(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$$



(f) *Domination:*

If $f(x) \ge g(x)$ on $[a, b]$ then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

④ Area under the Graph of a Nonnegative Function

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)dx$$

Example 3

Find the area under $y = x$ over the interval $[0, b]$, where $b > 0$.

Question:

$$\int_a^b x^2\,\mathrm{d}x = ? \qquad (a < b)$$

⑤ Average Value of a Continuous Function

DEFINITION If f is integrable on $[a, b]$, then its average value on $[a, b]$, which is also called its **mean**, is

$$Avg(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 4

Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$.

Skill Practice 1

Evaluate the following definite integrals:

(a) $\int_0^2 12x dx$ (b) $\int_1^3 3x^2 dx$ (c) $\int_0^{\sqrt{2}} (t-\sqrt{2}) dt$

Skill Practice 2

Use a definite integral to find the area of the region between the curve $y = x^2/2 + 1$ and the x-axis on the interval $[0, 2]$.