CALCULUS

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- Expressions such as "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" look something like ordinary numbers. We say that they have the *form* of a number. But values cannot be assigned to them in a way that is consistent with the usual rules to add and multiply numbers. We call these expressions "indeterminate forms." Although they are not numbers, these indeterminate forms play a useful role in summarizing the limiting behavior of a function.
- John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or q. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital, a French nobleman who wrote the earliest introductory differential calculus text, where the rule first appeared in print.

① Indeterminate Form $\frac{0}{0}$

THEOREM 5 — L'Hopital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

- To find $\lim_{x \to a} \frac{f(x)}{g(x)}$ by L'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $\frac{0}{0}$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example 1 Find the following limits.

(a) $\lim\limits_{x\to0}\dfrac{3x-\sin x}{x}$ (b) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-1}{x}$ (c) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-1-x/2}{x^2}$ (d) $\lim\limits_{x\to0}\dfrac{x-\sin x}{x^3}$

Example 2 Find:

$$\lim_{x\to0}\frac{1-cosx}{x^2+x}$$

- L'Hôpital's Rule can be applied to one-side limits as well.

Example 3 Find:

(a) $\lim\limits_{x\to0^+}\dfrac{\sin x}{x^2}$ (b) $\lim\limits_{x\to0^-}\dfrac{\sin x}{x^2}$ (c) $\lim\limits_{x\to0^-}\dfrac{\sin x}{x^2+x}$.

② Indeterminate Forms $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$

- More advanced treatments of calculus prove that L'Hôpital's Rule applies to the indeterminate form $\frac{\infty}{\infty}$, as well as to $\frac{0}{0}$.
- If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

- In the notation $x \to a$, a may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

Example 4 Find the limits of the ∞/∞ forms.

$$(a)\ \lim_{x\to\pi/2}\frac{secx}{1+tanx},\qquad (b)\lim_{x\to\infty}\frac{lnx}{2\sqrt{x}},\qquad (c)\ \lim_{x\to\infty}\frac{e^x}{x^2}$$

Example 5 Find the limits of the ∞·0 forms.

$$(a)\ \lim_{x\to\infty}\left(xsin\frac{1}{x}\right)\qquad (b)\ \lim_{x\to0^+}\sqrt{x}lnx$$

Example 6 Find the limits of the ∞−∞ form.

$$(a)\ \lim_{x\to0}\left(\frac{1}{sinx}-\frac{1}{x}\right)\qquad (b)\ \lim_{x\to1^+}\left(\frac{1}{lnx}-\frac{1}{x-1}\right)$$

③ Indeterminate Powers

- Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit.
- This procedure is justified by the continuity of the exponential function and Theorem 10 in Section 2.5 (Limits of Continuous Functions), and it is formulated as follows. (The formula is also valid for one-sided limits.)

If $\lim_{x \to a} ln f(x) = L$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{ln f(x)} = e^L$$

Here a may be either finite or infinite.

Example 7 Use l'Hôpital's Rule to show that:

$$\lim_{x\to 0^+}(1+x)^{1/x} = e$$

Example 8 Find the limit:

$$\lim_{x\to\infty} x^{1/x}$$

Example 9 Find the limit:

$$\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^x$$

Skill Practice 1 Use l'Hôpital's Rule and the Method in Chapter 2 to determine:

$(a)\ \lim_{t\to 0}\frac{sin5t}{3t}$ $\qquad (b)\ \lim_{x\to 0}\frac{1-cosx}{3x^2}$

Skill Practice 2 Use l'Hôpital's Rule to find:

$(a)\ \lim_{x\to 0}\frac{5^x-1}{3^x-1}$ $\qquad (b)\ \lim_{h\to 0}\frac{e^h-(1+h)}{h^2}$

Skill Practice 3 Find the limits:

$(a)\ \lim_{x\to 0}(e^x+x)^{\frac{1}{x}}$ $\qquad (b)\ \lim_{x\to\infty}\frac{3^x-2^x}{4^x-3^x}$