AMR: Nonlinear Control Methods

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Course Announcements

- **No class** on Tuesday 14th and 21st October
- **Next class** on 28th October
- Today's topics:
    - Feedback Linearization
    - Robust Control (Sliding Mode)
    - Adaptive Control

Controllability: Linear vs Nonlinear Systems

Linear Systems

$$\dot{x} = Ax + bu$$

Controllability:

$$c = \{b, Ab, A^2 b, \ldots, A^{n-1} b\}$$

$$|c| \neq 0 \rightarrow \text{Controllable}$$

Nonlinear Systems

$$\dot{x} = f(x) + g(x)u$$

Controllability via Lie Brackets

$$c = \{ad_f^0 g, ad_f^1 g, \ldots, ad_f^{n-1} g\}$$

Lie Brackets

Definition (Lie Bracket)

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Notation

$$ad_f^0 g = g$$
$$ad_f^1 g = [f, g]$$
$$ad_f^2 g = [f, [f, g]]$$
$$\vdots$$

Controllability Distribution

$$c = \{g, [f, g], [f, [f, g]], \ldots\}$$

Example System Definition

Consider the nonlinear system:

$$\dot{x} = f(x) + g(x)u$$

Where:

$$x = [x_1, x_2]^T$$

$$f(x) = \begin{bmatrix} x_2 \\ x_1^2 + 2x_1x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$$

Jacobian Calculations

Jacobian of g(x)

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 0 \\ 0 & 2x_2 \end{bmatrix}$$

Jacobian of f(x)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2x_1 + 2x_2 & 2x_1 \end{bmatrix}$$

Lie Bracket Calculation

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

$$\begin{aligned}
[f, g] &= \begin{bmatrix} 2x_1 & 0 \\ 0 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1^2 + 2x_1 x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2x_1 + 2x_2 & 2x_1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 x_2 \\ 2x_1^2 x_2 + 4x_1 x_2^2 + 2x_2^3 \end{bmatrix} - \begin{bmatrix} x_2^2 \\ 2x_1^3 + 2x_1^2 x_2 + 2x_1 x_2^2 \end{bmatrix}
\end{aligned}$$

Controllability Matrix

$$C = \begin{bmatrix} g & [f, g] \end{bmatrix} = \begin{bmatrix} x_1^2 & 2x_1x_2 - x_2^2 \\ x_2^2 & 2x_1^2x_2 - 2x_1^3 \end{bmatrix}$$

Feedback Linearizability Condition

$$|C| \neq 0 \quad \text{and} \quad \text{Distribution is Involutive}$$

Involutivity Condition

The set $\{g, [f, g], \ldots, ad_f^{n-2}g\}$ must be involutive.

Robust Control Problem

Real System with Uncertainties

$$\dot{x} = f(x, u) + \Delta \quad \text{(with unknown uncertainties)}$$

Example (Spring-Damper System)

$$\ddot{x} + a\dot{x} + bx^3 + d(x) = u$$

- $a\dot{x}$: damping term
- bx^3: nonlinear spring
- $d(x)$: unknown disturbance

Control Objective

Goal

Find control u such that $x \to x_d$

Nominal Model

$$\ddot{x} + \hat{a}\dot{x} + \hat{b}x^3 = u$$

Proposed Controller

$$u = \ddot{x}_d + \hat{a}\dot{x} + \hat{b}x^3 + k_d(\dot{x}_d - \dot{x}) + k_p(x_d - x) + V_r$$

Error Dynamics

Define tracking error: $e = x_d - x$
Substituting controller into system dynamics:

$$\ddot{x}_d - \ddot{x} + k_d\dot{e} + k_p e = (a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x) - V_r$$

Which gives:

$$\ddot{e} + k_d\dot{e} + k_p e = \Delta$$

Where $\Delta = (a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x) - V_r$

State Space Representation

Define state vector:

$$z = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}, \quad \dot{z} = Az + bw$$

Where:

$$A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$w = (a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x) - V_r$$

State equations:

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -k_d z_2 - k_p z_1 - w$$

Lyapunov Stability Analysis

Lyapunov Function Candidate

$$V = \frac{1}{2} z^T P z$$

Time Derivative

$$\begin{aligned}
\dot{V} &= \frac{1}{2}[\dot{z}^T P z + z^T P \dot{z}] \\
&= \frac{1}{2}[(z^T A^T + w^T b^T) P z + z^T P (A z + b w)] \\
&= \frac{1}{2} z^T [A^T P + P A] z + z^T P b w
\end{aligned}$$

Robust Control Design

Uncertainty Bounds

- $|\hat{a} - a| < \alpha$
- $|\hat{b} - b| < \beta$
- $|d(x)| < r$
- $\beta > 0$

Bound on Total Uncertainty

$$|(a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x)| < \alpha|\dot{x}| + \beta|x|^3 + r = f(x)$$

Robust Control Law

Robust Term Design

$$V_r = \frac{z^T Pb}{|z^T Pb|} f(x)$$

This ensures:

$$z^T Pb[(a - \hat{a})\dot{x} + (b - \hat{b})x^3 + d(x) - V_r] < 0$$

Stability Guarantee

With this choice of V_r, we get $\dot{V} < 0$, guaranteeing asymptotic stability.

Sliding Mode Control Approach

Sliding Surface

$$S = \dot{e} + \lambda e$$

Control Objective

Design control such that $S \to 0$

Convergence Property

If $S \to 0$, then $\dot{e} + \lambda e = 0 \Rightarrow e \to 0$ exponentially

Sliding Mode Stability

Lyapunov Function

$$v = S^T S$$

Reaching Condition

$$\dot{v} = -\eta |S|$$

This implies:

$$S\dot{S} = -\eta |S| \Rightarrow \dot{S} = -\eta \, \mathrm{sgn}(S)$$

General System Example

Consider system:

$$\dot{x} = f(x) + u$$

Sliding surface derivative:

$$\dot{S} = \dot{e} + \lambda \dot{e} = \ddot{x}_d - \ddot{x} + \lambda \dot{e}$$

Substitute dynamics:

$$\dot{S} = \ddot{x}_d - f(x) - u + \lambda \dot{e}$$

Sliding Mode Control Law

From reaching condition:

$$\ddot{x}_d - f(x) - u + \lambda e = -\eta \operatorname{sgn}(S)$$

Solving for control:

$$u = \ddot{x}_d - f(x) + \lambda e + \eta \operatorname{sgn}(S)$$

Robust Sliding Mode Control

For system with uncertainty:

$$\dot{x} = \hat{f}(x) + \Delta + u, \quad \|\Delta\| < F$$

Robust control law:

$$u = \ddot{x}_d - \hat{f}(x) + \lambda e + (\eta + F)\,\mathrm{sgn}(S)$$

This guarantees:

$$[\ddot{x}_d - \hat{f}(x) - \Delta - u + \lambda e]\,\mathrm{sgn}(S) < -\eta$$

Adaptive Control Problem

Example (Pendulum System)

$$I\ddot{\theta} + b\dot{\theta} + C\sin\theta = \tau$$

- I: unknown inertia
- b: unknown damping
- C: unknown gravitational term

Control Objective

Design τ and parameter update laws to track desired trajectory θ_d

Adaptive Control Law

Proposed Controller

$$\tau = \hat{I}[\ddot{\theta}_d + k_d(\dot{\theta}_d - \dot{\theta}) + k_p(\theta_d - \theta)] + \hat{b}\dot{\theta} + \hat{C}\sin\theta$$

Where:

- $\hat{I}$, $\hat{b}$, $\hat{C}$: parameter estimates
- k_d, k_p: feedback gains

Error Dynamics

Subtracting actual system from control law:

$$(I-\hat{I})\ddot{\theta}+(b-\hat{b})\dot{\theta}+(C-\hat{C})\sin\theta-\hat{I}[\dot{e}+k_d\dot{e}+k_pe]=0$$

Define parametric uncertainties:

$$p=\begin{bmatrix} I & b & C \end{bmatrix}^T, \quad Y=\begin{bmatrix} \ddot{\theta} & \dot{\theta} & \sin\theta \end{bmatrix}^T$$

Then:

$$p^TY=I\ddot{\theta}+b\dot{\theta}+C\sin\theta$$

Error System Representation

Define states:

$$x_1 = e, \quad x_2 = \dot{e}$$

State equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -k_p x_1 - k_d x_2 + I^{-1}[p - \hat{p}]^T Y$$

In matrix form:

$$\dot{x} = Ax + bw$$

Where:

$$A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w = I^{-1}[p - \hat{p}]^T Y$$

Composite Lyapunov Function

Lyapunov Candidate

$$V(x, p - \hat{p}) = \frac{1}{2}x^T S x + \frac{1}{2\gamma}[p - \hat{p}]^T [p - \hat{p}]$$

Where:

- $S > 0$: symmetric positive definite matrix
- $\gamma > 0$: adaptation gain

Lyapunov Analysis

$$\dot{V} = \frac{1}{2}\dot{x}^T S x + \frac{1}{2}x^T S \dot{x} + \frac{1}{\gamma}[p - \hat{p}]^T(-\dot{\hat{p}})$$
$$= \frac{1}{2}(x^T A^T + w^T b^T) S x + \frac{1}{2}x^T S(Ax + bw) - \frac{1}{\gamma}[p - \hat{p}]^T \dot{\hat{p}}$$

Assuming $\dot{p} = 0$ (constant parameters):

$$\dot{V} = \frac{1}{2}x^T(A^T S + SA)x + x^T S b w - \frac{1}{\gamma}[p - \hat{p}]^T \dot{\hat{p}}$$

Parameter Update Law

To ensure $\dot{V} < 0$, choose:

$$x^T S b I^{-1} [p - \hat{p}]^T Y = \frac{1}{\gamma} [p - \hat{p}]^T \dot{\hat{p}}$$

This suggests the update law:

$$\dot{\hat{p}} = \gamma x^T S b I^{-1} Y$$

Final Adaptive Control Scheme

Control Law

$$\tau = \hat{I}[\ddot{\theta}_d + k_d\dot{e} + k_p e] + \hat{b}\dot{\theta} + \hat{C}\sin\theta$$

Parameter Adaptation

$$\hat{p}(t) = \hat{p}(0) + \gamma\int_0^t x^T S b I^{-1} Y \, d\tau$$

Where:

$$\hat{p} = \begin{bmatrix}\hat{I} & \hat{b} & \hat{C}\end{bmatrix}^T, \quad Y = \begin{bmatrix}\ddot{\theta} & \dot{\theta} & \sin\theta\end{bmatrix}^T$$

Summary

- **Feedback Linearization**: Uses Lie brackets for nonlinear controllability analysis
- **Robust Control**: Handles bounded uncertainties using Lyapunov methods
- **Sliding Mode**: Provides robustness through discontinuous control
- **Adaptive Control**: Estimates unknown parameters online for better tracking

Key Takeaway

These methods provide powerful tools for controlling nonlinear robotic systems in the presence of uncertainties and unknown parameters.

Questions?