AMR: Examples

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Example 1: System Definition

Four-Wheel Mecanum Mobile Robot

- **Industrial Robots**: Use independent wheel control
- **Objective**: Design Adaptive Controller to handle parameter uncertainty
- **State Vector** $\dot{x}$: Robot body velocity $\dot{x} = [V_x, V_y, \omega]^T$ (Body Frame $\{X_R, Y_R\}$)
- **Input Vector** $\dot{\phi}$: Wheel angular velocity $\dot{\phi} = [\dot{\phi}_1, \dot{\phi}_2, \dot{\phi}_3, \dot{\phi}_4]^T$

Geometric Parameters

- R: Wheel Radius
- $2a$: Longitudinal separation between wheels
- $2b$: Lateral separation between wheels
- $\delta = 45^\circ$: Mecanum wheel roller angle

Fundamental Kinematic Constraints

Pure Rolling Assumption

- **No Slippage** for velocity kinematics
- **Rolling Condition**: Because of rolling, no longitudinal force ($F_{Wlong} = 0$)
- **Mecanum Configuration**: Rollers at $\delta = 45^\circ$

Kinematic Mapping (Forward)

The transformation from wheel velocity to body velocity:

$$\dot{x}_{3\times 1} = T_{3\times 4}\dot{\phi}_{4\times 1}$$

Local Wheel Velocities Analysis

Wheel 1 Velocity Components

The velocity components of wheel contact point relative to robot's center:

$$V_{xw1} = V_x - a\omega = R\dot{\phi}_1 \cos\delta$$
$$V_{yw1} = V_y + b\omega = R\dot{\phi}_1 \sin\delta$$

Pure Rolling Constraint

- Links longitudinal speed to angular velocity: $R\dot{\phi} = V_{xw1}$
- No slipping condition ensures kinematic consistency

Example

For Mecanum Wheels ($\delta = 45^\circ$)

$$\cos\delta = \sin\delta = \frac{\sqrt{2}}{2}$$

Inverse Kinematic Solution

Finding the Pseudo-Inverse $T^\dagger$

Goal: Solve $\dot{\phi} = T^\dagger \dot{x}$

Start with:

$$T^T \dot{x} = T^T T \dot{\phi}$$

Solve for $\dot{\phi}$:

$$(T^T T)^{-1} T^T \dot{x} = \dot{\phi}$$

Define pseudo-inverse:

$$T^\dagger = (T^T T)^{-1} T^T$$

Notation

In notes: T^t represents the pseudo-inverse $(T^T T)^{-1} T^T$

Explicit Inverse Kinematic Matrix

Transformation Matrix $T^\dagger$

$$T^\dagger = \frac{1}{R}\begin{bmatrix} 1 & -1 & -(a+b) \\ 1 & 1 & -(a+b) \\ 1 & -1 & a+b \\ 1 & 1 & a+b \end{bmatrix}$$

Individual Wheel Equations

Derived angular velocity for wheel 1:

$$\dot{\phi}_1 = -V_x + V_y - (a+b)\omega$$

Similarly for other wheels:

$$\dot{\phi}_2 = V_x + V_y - (a+b)\omega$$
$$\dot{\phi}_3 = -V_x + V_y + (a+b)\omega$$
$$\dot{\phi}_4 = V_x + V_y + (a+b)\omega$$

Dynamic Model Formulation

General Equation of Motion (EOM)

$$M\ddot{q} + C\dot{q} + G(q) = B\tau$$

Assumptions:

- $C(q) = 0$ (No Coriolis/centrifugal forces)
- $G(q) = 0$ (No gravitational effects on flat surface)

Simplified Model:

$$M\ddot{q} = B\tau$$

Input Distribution Matrix

$$B = T \quad \text{(From kinematics, [Not Always])}$$

Generalized Forces:

$$\sum F_X = m\dot{V}_x$$

Mass Matrix and Uncertainty

Generalized Mass Matrix Structure

$$M = \begin{bmatrix} m+\hat{m} & 0 & 0 \\ 0 & m+\hat{m} & 0 \\ 0 & 0 & I+\hat{I} \end{bmatrix} = M_R$$

Uncertainty Components

- M_R: Total Mass Matrix ($\hat{M} = M_R$)
- $M_{L(Robot)} + M_{T(Truck)} \Rightarrow \hat{M} = M_R$
- $M = M_R + M'_T$ (where M'_T is uncertain)
- $\hat{M}$: What we know (nominal model)
- M': Uncertain component

Control Design Challenge

Goal: Estimation and compensation of M' (uncertain mass components)

Feedback Linearization Control (FLC)

Perfect Model Assumption

FLC is used when we have perfect model knowledge ($M = \hat{M}$)

Desired Closed-Loop Behavior

Define error: $e = q_d - q$
Desired error dynamics:

$$\ddot{q}_d - \ddot{q} + k_d(\dot{q}_d - \dot{q}) + k_p(q_d - q) = 0$$

Closed-loop error dynamics (no uncertainties):

$$M\ddot{e} + k_d\dot{e} + k_p e = 0$$

Stability Condition

If $k_p > 0$, $k_d > 0$, then error dynamics are asymptotically stable

FLC Control Command

Control Law Derivation

From simplified dynamics: $M\ddot{q} = B\tau$
Substitute desired acceleration:

$$\tau_{FLC} = B^{-1}[M\ddot{q}_d + k_d\dot{e} + k_p e]$$

Components Analysis

- $M\ddot{q}_d$: Feedforward term based on desired acceleration
- $k_d\dot{e}$: Derivative feedback for damping
- $k_p e$: Proportional feedback for stiffness

Practical Limitation

Requires exact knowledge of M and B - rarely true in practice!

Adaptive Control Framework

Motivation

Adaptive control is necessary when M is uncertain

Adaptive Control Law Structure

Use nominal estimate $\hat{M}$ from nominal model $\hat{M}\ddot{q} = B\tau$:

$$\tau_{Adaptive} = B^{-1}\hat{M}[\ddot{q}_d + k_d\dot{e} + k_p e]$$

Closed-Loop Analysis

Apply $\tau_{Adaptive}$ to true system ($M\ddot{q} = B\tau$):

$$\ddot{q} = M^{-1}\hat{M}[\ddot{q}_d + k_d\dot{e} + k_p e]$$

Using $\ddot{e} = \ddot{q}_d - \ddot{q}$:

$$\ddot{e} + k_d\dot{e} + k_p e = [(M^{-1}\hat{M}) - I]\ddot{q}$$

Uncertainty Analysis and Adaptation

Closed-Loop Uncertainty

$$\ddot{e} + k_d \dot{e} + k_p e = \underbrace{\left[(M^{-1}\hat{M}) - I\right]}_{\text{Uncertainty}} \ddot{q}$$

The term $[(M^{-1}\hat{M}) - I]\ddot{q}$ represents uncertainty in closed-loop form.

Adaptation Rule

- Found using Lyapunov stability analysis
- Online parameter estimation
- Guarantees bounded errors and stability

Use Case

Adaptive control is good for **Parametric Uncertainty**:

- M is usually parametric
- Not C, G (structural uncertainties require robust control)

Example 2: Robust Control Design

System with Bounded Disturbance

$$\ddot{x} + a\dot{x} - cx^3 + d(t) = u$$

where $|d(t)| < \beta$ (bounded disturbance)

Robust Control Application

- Good for system uncertainty with limits (disturbance)
- Does not require parameter estimation
- Handles worst-case disturbances

Objective

Design stabilizing control law u to remove bounded disturbance $d(t)$

Lyapunov Candidate Selection

Energy-Based Lyapunov Function

$$V = \frac{1}{2}\dot{x}^2 + \frac{c}{4}x^4$$

Properties:

- $V(0,0) = 0$
- $V(x,\dot{x}) > 0$ for $(x,\dot{x}) \neq (0,0)$
- Similar to mechanical energy (kinetic + potential)

Time Derivative Computation

Using chain rule:

$$\dot{V} = \dot{x}\ddot{x} + c\dot{x}x^3$$

Substitute $\ddot{x}$ from EOM:

$$\ddot{x} = u - a\dot{x} + cx^3 - d$$

Stability Analysis

Lyapunov Derivative

$$\dot{V} = \dot{x}[(u - d) - a\dot{x} + cx^3 - cx^3] = (u - d)\dot{x} - a\dot{x}^2$$

Simplified:

$$\dot{V} = u\dot{x} - d\dot{x} - a\dot{x}^2$$

Stability Condition

For asymptotic stability, require $\dot{V} < 0$:

$$u\dot{x} - d\dot{x} - a\dot{x}^2 < 0$$

Rearrange:

$$u\dot{x} - d\dot{x} < a\dot{x}^2$$

Control Bound Derivation

Using Triangle Inequality

$$|u\dot{x} - d\dot{x}| < a\dot{x}^2$$

Apply triangle inequality:

$$|u\dot{x}| + |d\dot{x}| < a\dot{x}^2$$

Use disturbance bound $|d| < \beta$:

$$|u\dot{x}| + \beta|\dot{x}| < a\dot{x}^2$$

Final Control Bound

$$|u| < \frac{a\dot{x}^2 - \beta|\dot{x}|}{|\dot{x}|}$$

This ensures $\dot{V} < 0$ and asymptotic stability.