SIAM J. APPL. MATH.
Vol. 50, No 5, pp. 1496-1506, October 1990

014

RESONANCES AND POWER SERIES SOLUTIONS OF THE FORCED VAN DER POL OSCILLATOR*

MOHAMMAD B. DADFAR† AND JAMES F. GEER‡

Abstract. To the first approximation, the nonresonant steady-state motion of the forced van der Pol oscillator is a combination of a free and a forced response. When the forcing amplitude increases beyond a certain critical value, which depends on the forcing frequency, the free response will decay and the motion is represented by just a particular solution that has the same frequency as the forcing term. In the first part of this paper, a general expression is determined for all of the higher-order resonant frequencies associated with a formal multiple timescale perturbation expansion of the solution. These frequencies are shown to be dense on the real axis. Then a formal power series expansion is developed for a particular solution in terms of the small damping parameter ε, which is shown to be valid when the forcing frequency is not close to a certain subset of the resonant frequencies. Using Padé approximants, information is obtained about the location and nature of the singularities in the complex ε-plane that limit the convergence of the series. Three pairs of singularities whose locations change as the forcing amplitude varies are discovered. In particular, when the forcing amplitude exceeds a certain value, a pair of singularities along the imaginary axis becomes dominant. The emergence of these singularities as the dominant ones is discussed in relation to the quenching of the free oscillation in the total response.

Key words. forced oscillations resonances, forced van der Pol oscillator, Padé approximants, power series, radius of convergence, steady-state motion

AMS(MOS) subject classifications. 70K30, 41A58, 58F30

1. Introduction. We wish to study the effects of the forcing frequency and the forcing amplitude on the formal power series solution in the damping parameter to the forced van der Pol oscillator

$$u''+\tilde{\varepsilon}(u^2-1)u'+\omega_0^2u=\tilde{K}\cos(\omega\tau). \tag{1.1}$$

Here $\tilde{\varepsilon}$ is the damping parameter, ω_0 is the undamped natural (primary resonant) frequency, $\tilde{K}$ is the forcing amplitude, ω is the forcing frequency, and "$'$" denotes differentiation with respect to time τ. Introducing the nondimensional variables

$$t=\omega\tau,\quad \rho=\frac{\omega}{\omega_0},\quad \varepsilon=\frac{\tilde{\varepsilon}}{\omega_0},\quad K=\frac{\tilde{K}}{\omega_0^2},$$

(1.1) becomes

$$\rho^2\ddot{u}+\varepsilon\rho(u^2-1)\dot{u}+u=K\cos t, \tag{1.2}$$

where the dots denote differentiation with respect to t.

Power series solutions to the free van der Pol oscillator have been studied in detail by several investigators (see [1] and [2] and the references cited therein). The effects of the forcing frequency and the forcing amplitude on the formal power series solution of a simpler forced system, i.e., a simple pendulum with an oscillating support, have been discussed in [3]. In particular, for this case, we have shown that forcing the system at a subharmonic resonance seems to have virtually no effect on the convergence of the power series solution, whereas forcing it at a superharmonic resonance dramatically affects the usefulness of the series. Low-order approximations to the solution of

* Received by the editors October 30, 1988; accepted for publication (in revised form) July 7, 1989.
† Department of Computer Science, Bowling Green State University, Bowling Green, Ohio 43403.
‡ Department of Systems Science, Thomas J. Watson School of Engineering, Applied Science and Technology, State University of New York, Binghamton, New York 13901.

the forced van der Pol oscillation have been calculated and analyzed by several investigators (see, e.g., the many references cited in [7]). In the regular case, i.e., when ε is small, if the amplitude of the forcing term is small and the forcing frequency is near the primary resonance, it is known (see, e.g., [7]) that stable periodic solutions exist only for a narrow band of frequencies around the natural frequency. However, this band increases in width as the forcing amplitude increases. As the forcing frequency moves away from the natural frequency, it is possible to construct an approximate solution in the form of the sum of a free oscillation and a forced oscillation having the same frequency as the forcing term. In this case, as the amplitude of the forcing term increases beyond a certain value, the amplitude of the free oscillation decays in time, leaving only (to the first approximation) the forcing frequency present in the solution, i.e., "frequency entrainment" or "synchronization" can occur.

As the damping parameter ε becomes large, the free van der Pol equation exhibits relaxation oscillations, and the corresponding forced oscillations can easily become entrained to a periodic input. However, for certain values of the parameters, it is known that two different subharmonics may co-exist. In particular, several interesting phenomena associated with the large ε limit of the forced van der Pol equation have been discussed by asymptotic methods in [7], [11], and [12], by numerical methods in [5] and [6], and qualitatively in [10]. (See, e.g., [9] for a recent survey of methods and results for the singular case.) In addition, chaotic solutions of the forced van der Pol relaxation oscillator have been studied in [8] and [10], whereas chaotic solutions for small values of ε are discussed in [4].

In this paper, we will be concerned with the construction and analysis of a perturbation solution to equation (1.2) that will be formally valid for small values of ε. One of the goals of our analysis is to recast the series in such a manner as to make it valid for values of ε as large as possible. In particular, we will be sensitive to possible effects of the forcing amplitude and frequency on the convergence of the series, especially in light of the remarks of the preceding paragraphs.

In § 2 we determine and discuss briefly all of the high-order resonant frequencies associated with a multiple timescale analysis of (1.2), which is formally valid for small values of the damping parameter. We show that the set of all of these frequencies is dense on the real line and hence there is a resonant frequency of some order arbitrarily close to any prescribed forcing frequency. In § 3 we develop a formal power series solution for a particular solution of (1.2) and then analyze and improve the convergence of this series in § 4. In particular, the singularities in the complex ε-plane that can limit the convergence of the series are identified and their movement is determined as a function of the forcing amplitude. We then discuss our results in § 5.

2. Multiple time scales solution. We now employ the method of multiple time scales (see, e.g., [14]) and seek a solution to (1.2) for small ε in the form

$$u=\sum_{j=0}^{\infty} u_j\varepsilon^j, \tag{2.1}$$

where each u_j is a function of the time scales $T_k=\varepsilon^k t$, $k=0,1,2,\cdots$. Substituting (2.1) into (1.2), using the relation

$$\frac{d}{dt}=D_0+\varepsilon D_1+\varepsilon^2 D_2+\cdots, \qquad D_k=\frac{\partial}{\partial T_k},$$

and then equating coefficients of like powers of ε on each side of the resulting expression, we obtain the sequence of equations

$$
\begin{aligned}
&\rho^2 D_0^2 u_0 + u_0 = K \cos (T_0), \\
&\rho^2 D_0^2 u_j + u_j = f_j, \qquad j \geqq 1,
\end{aligned} \tag{2.2}
$$

where f_j involves only the u_m with $m<j$. In particular, $f_1 = D_0 u_0 - (D_0 u_0) u_0^2 - 2 D_0 D_1 u_0$.

Using the symbolic computation system MACSYMA [13], we have computed the f_j explicitly through $j=9$ and also have solved the resulting equations for the u_j with $j \leqq 9$, using the standard techniques associated with this method [14]. In particular, using the fact that $u_0 = A \cos (X) + B \sin (X) + \{K/(1-\rho^2)\} \cos (T_0)$, where $X = T_0/\rho$, we see that f_1 involves the sine and/or cosine of T_0, X, $3X$, $2X \pm T_0$, $X \pm 2T_0$, and $3T_0$. From the explicit terms we have obtained, we can conjecture the general form of the f_j, and hence of the u_j, and then prove that this relation holds for all j by induction. The result is that f_j consists of a linear combination of terms, each of which is either the sine or cosine with argument T_0, T_0/ρ, or

$$
kT_0 \pm \frac{(2n+1-k)T_0}{\rho}, \qquad k = 0, 1, \cdots, 2n+1, \quad n = 0, 1, \cdots, j. \tag{2.3}
$$

Thus, it follows that the resonant frequencies of order j correspond to those values of ρ for which the quantity in (2.3) is equal to $\pm T_0/\rho$, i.e.,

$$
\rho = \frac{2n+2-k}{k}, \qquad k = 1, \cdots, 2n+1, \quad n = 0, 1, \cdots, j. \tag{2.4}
$$

From (2.4) we see that if a particular frequency is resonant at a specific order $\tilde{j}$, then it is a resonant frequency at all orders $j \geqq \tilde{j}$. Also, if ρ is a resonant frequency, then so is $1/\rho$. The resonant frequencies determined from (2.4) for values of j up to $j=7$ are presented in Table 1.

From (2.4), it follows that a forcing frequency ω that is a rational multiple of the natural frequency ω_0 is a resonant frequency of some order j for the van der Pol oscillator. To see this, we need only show that any rational number, say r, can be expressed in the form (2.4) for some appropriate choice of n and k. But if we let $r = p/q$, where p and q are positive integers, we can set $n = p+q-1$ and $k = 2q$ in (2.4) and find that $\rho = p/q = r$. Thus, it follows that the set of all resonant frequencies of the van der Pol oscillator are dense on the real line.

TABLE 1

Resonant frequencies (divided by ω_0) for the forced van der Pol oscillator corresponding to different orders of approximation in ε. For any particular order j, the resonant frequencies at that order are those shown on the corresponding line of the table, as well as those on all of the previous lines.

Order (j)	New resonant frequencies (p)
0	1
1	$3, \frac{1}{3}$
2	$5, 2, \frac{1}{2}, \frac{1}{5}$
3	$7, \frac{5}{3}, \frac{3}{5}, \frac{1}{7}$
4	$9, 4, \frac{7}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{7}, \frac{1}{4}, \frac{1}{9}$
5	$11, \frac{7}{5}, \frac{5}{7}, \frac{1}{11}$
6	$13, 6, \frac{11}{3}, \frac{5}{2}, \frac{9}{5}, \frac{4}{3}, \frac{3}{4}, \frac{5}{9}, \frac{2}{5}, \frac{3}{11}, \frac{1}{6}, \frac{1}{13}$
7	$15, \frac{13}{3}, \frac{11}{5}, \frac{9}{7}, \frac{7}{9}, \frac{5}{11}, \frac{3}{13}, \frac{1}{15}$

In Fig. 1, we have tried to indicate the almost "fractal-like" nature of the locations of the resonant frequencies. At each resonant frequency ω (horizontal axis) we have plotted a vertical line whose length decreases with j, where j is the lowest order for which ω is a resonant frequency. If any subinterval of that depicted in Fig. 1 is selected and a similar plot is made (but for a larger maximum value of j), the resulting figure closely resembles Fig. 1. We will comment further on these frequencies when we discuss our results in § 5.

3. A particular power series solution for small ε. We have carried out many of the details of the multiple time series analysis described in the previous section using both symbolic and numerical computations. The detailed results and observations of our investigations will be reported elsewhere. However, for our present purposes, we find it more convenient to concentrate on just a particular solution to (1.2), which can be computed using just a standard regular perturbation method. This solution, which we will construct below, can be thought of as a solution to (1.2) corresponding to a special set of initial conditions for which the free oscillation component vanishes for each order of approximation. This solution will be sufficient to help us investigate some of the effects of the forcing amplitude K and forcing frequency ρ on the convergence (and hence validity) of the perturbation solution.

We represent a particular solution u of (1.2) in the form

$$u = u(t, \varepsilon) = \sum_{j=0}^{\infty} u_j(t)\varepsilon^j \tag{3.1}$$

where each u_j depends on t, K, and ρ. Substituting (3.1) into (1.2) and collecting coefficients of like powers of ε, we obtain the sequence of ordinary differential equations:

$$\rho^2 \ddot{u}_j + u_j = f_j, \qquad j = 0, 1, \cdots \tag{3.2}$$

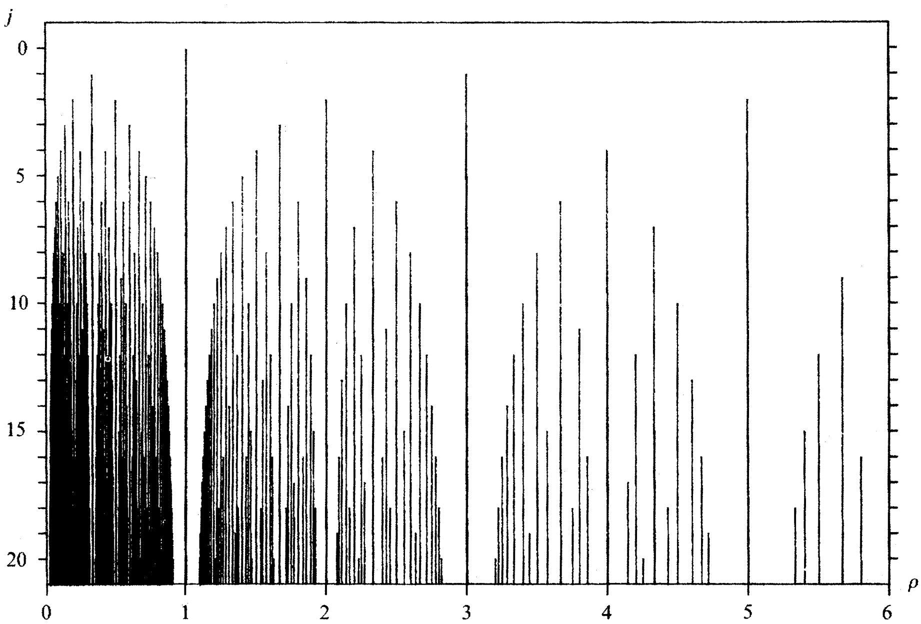

FIG. 1. *Plot of the locations of the resonant frequencies (divided by ω_0) of the forced van der Pol equation (horizontal axis) and an indication of the lowest-order j (up to $j = 20$) for which that frequency is resonant. A vertical line of length $1-(j/21)$ is drawn above each resonant frequency.*

with

$$f_0 = K \cos(t), \quad f_j = \rho \dot{u}_{j-1} - \rho \sum_{i=0}^{j-1} \sum_{k=0}^{i} u_k u_{i-k} \dot{u}_{j-(i+1)}, \quad j = 1, 2, \cdots. \tag{3.3}$$

We now represent each u_j and f_j as a linear combination of $\sin(nt)$ and $\cos(nt)$, $n = 1, 2, \cdots$, i.e.,

$$\begin{aligned} u_j(t) &= C_j + \sum_{n=1}^{N_j} \alpha_{j,n} \sin(nt) + \beta_{j,n} \cos(nt), \qquad j = 0, 1, \cdots, \\ f_j(t) &= c_j + \sum_{n=1}^{N_j} \gamma_{j,n} \sin(nt) + \delta_{j,n} \cos(nt), \qquad j = 0, 1, \cdots \end{aligned} \tag{3.4}$$

where N_j, C_j, c_j, $\alpha_{j,n}$, $\beta_{j,n}$, $\gamma_{j,n}$, and $\delta_{j,n}$ are unknown constants. Substituting (3.4) into (3.2), we obtain the relations

$$\alpha_{j,n} = \frac{\gamma_{j,n}}{1-(n\rho)^2} \quad \text{and} \quad \beta_{j,n} = \frac{\delta_{j,n}}{1-(n\rho)^2}, \quad j = 0, 1, \cdots \tag{3.5}$$

and also find that $\alpha_{j,n} = \beta_{j,n} = \gamma_{j,n} = \delta_{j,n} = 0$, $j = 0, 1, \cdots$ when n is even, i.e., u_j and f_j are functions of odd multiples of t only. From (3.5), we see immediately that ρ cannot be allowed to assume any of the superharmonic resonances $1/n$, where n is odd. Thus the solution we are about to compute will not be valid for these forcing frequencies. For such frequencies, a multiple timescale analysis is required.

In addition, we observe that $u_{2j}(t)$ and $f_{2j}(t)$ consist of only $\cos(nt)$, whereas $u_{2j+1}(t)$ and $f_{2j+1}(t)$ involve only $\sin(nt)$, $j = 0, 1, \cdots$ and $n = 1, 3, \cdots$, i.e.,

$$u_{2j}(t) = \sum_{k=0}^{2j} \beta_{2j,2k+1} \cos((2k+1)t), \qquad u_{2j+1}(t) = \sum_{k=0}^{2j+1} \alpha_{2j+1,2k+1} \sin((2k+1)t), \qquad j = 0, 1, \cdots. \tag{3.6}$$

Using a FORTRAN program, we have computed the coefficients of $u_j(t)$, $j = 0, 1, \cdots, 49$, which appear to be sufficient to analyze the actual solution [3]. From the specific form of the coefficients $u_j(t)$, the following relations hold:

$$\begin{aligned} &\text{for } \pi/2 \leqq t \leqq \pi: \quad u_j(t) = (-1)^{j+1} u_j(\pi - t), \quad j = 0, 1, \cdots, \\ &\text{for } \pi \leqq t \leqq 2\pi: \quad u_j(t) = -u_j(t-\pi), \qquad j = 0, 1, \cdots. \end{aligned} \tag{3.7}$$

Similar relations exist for the coefficients $\dot{u}_j(t)$ and $\ddot{u}_j(t)$. Therefore, we perform our analysis of the series only for t in the interval $0 \leqq t \leqq \pi/2$.

4. Analysis and improvement of the series solutions. As a representative, nonresonant case, we set $\rho = \sqrt{2}$, which appears to be sufficient so as not to generate higher-order resonances through the 50 terms we will consider. Then for different values of K we compute the series solution $u_j(t)$ from (3.6).

To begin our investigation of the behavior of the resulting power series solution and, in particular, to find out for what values of the parameter ε the above series converges, we plot the limit cycles in the phase plane for different values of ε. Figure 2 shows some representative cases for $K = 1.5$. We observe that for $\varepsilon \leqq 0.9$ the limit cycles are simple and smooth, illustrating (as we will show below) that our solution represents an actual periodic motion for these values of the parameter. For larger values of ε, the limit cycles are not smooth and hence the series solution (3.1) does not appear to approximate the actual solution for $\varepsilon > 0.9$.

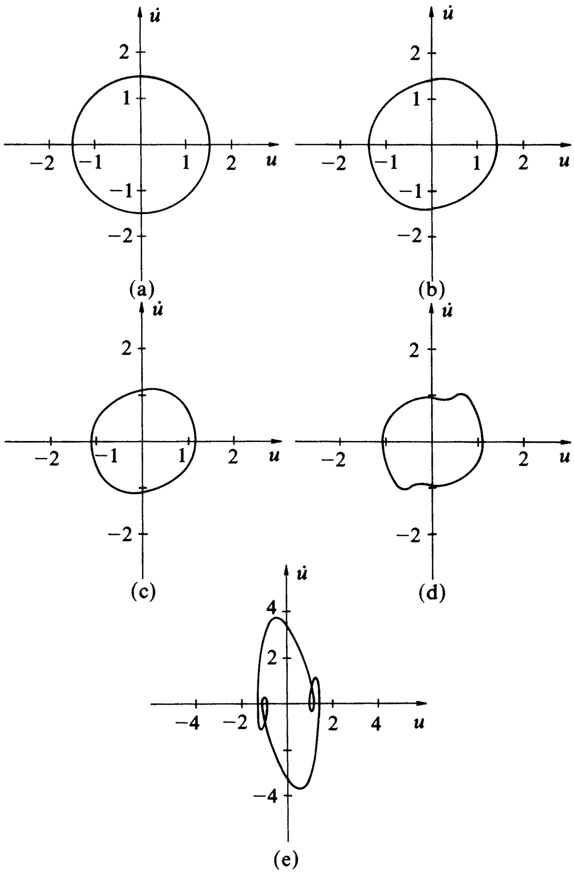

FIG. 2. *Phase plane plots for the limit cycle* $u(t, \varepsilon)$ *using the first* 50 *terms of the series* (3.1) *with* $\rho = \sqrt{2}$ *and* $K = 1.5$ *for* (a) $\varepsilon = 0$, (b) $\varepsilon = 0.5$, (c) $\varepsilon = 0.9$, (d) $\varepsilon = 0.95$, *and* (e) $\varepsilon = 1.0$.

To obtain some insight into the range of validity of the power series solution, for each K we find the locations of the singularities of (3.1) that lie nearest the origin in the complex ε-plane by using [24/24] Padé approximants for $0 \leqq t \leqq \pi/2$. (Since the odd terms for $t = 0$ and even terms for $t = \pi/2$, respectively, are zero, we use [12/12] Padé approximants in the ε^2-plane when $t = 0$ or $t = \pi/2$.) Figure 3 shows the location of the singularities of $u(t, \varepsilon)$ near the origin in the first and second quadrants of the complex ε-plane for some selected values of K with $t = \pi/5$. The distance R from the origin of the singularity (or singularities, if there exist more than one) closest to the origin serves to estimate the radius of convergence of the power series solution [15]. If we represent the singularity closest to the origin as $R\,e^{\pm i\beta}$, then for $\rho = \sqrt{2}$ and $K = 1.5$, the parameters R and β are estimated from the Padé approximants to be

$$R \approx 0.91 \quad \text{and} \quad \beta \approx 0.42,$$

where β is the argument of the singularity with respect to the positive real axis. For several other values of t we investigated in the range $0 \leqq t \leqq \pi/2$, the values of R and β are virtually identical to those just stated. Thus, our Padé analysis at discrete times seems to indicate that the radius of convergence of our series solution does not vary with t (although it does vary with K and ρ, as we will demonstrate). This is in contrast to the free van der Pol oscillator [2], for which a Padé analysis clearly indicates that the radius of convergence does vary with t.

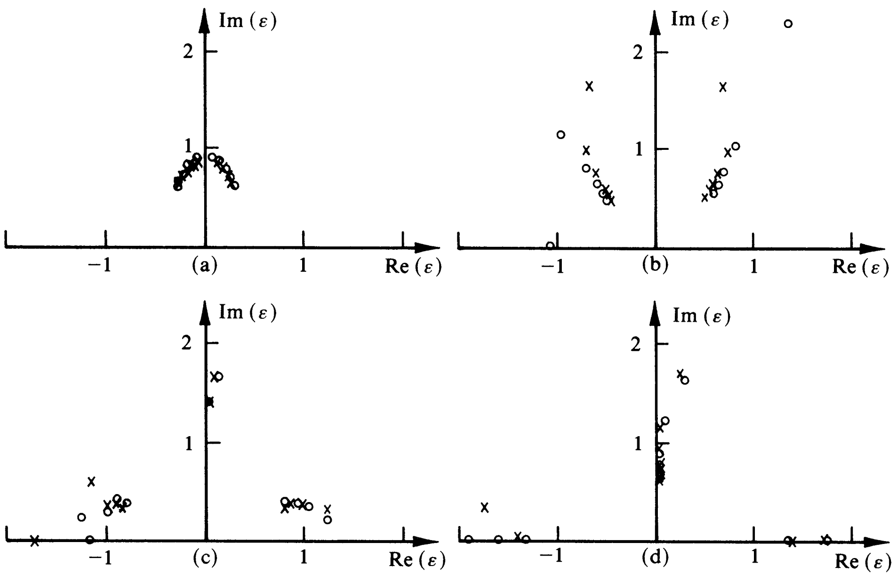

FIG. 3. *The location of the zeros* (oooo) *and poles* (x's) *in the upper half complex* ε*-plane of the* [24/24] *Padé approximants to the series* (3.1) *for* $\rho=\sqrt{2}$, $t=\pi/5$ *and* (a) $K=0.5$, (b) $K=1.0$, (c) $K=1.5$, *and* (d) $K=2.0$.

To investigate the effects of the external amplitude K on the series solution, we find the parameters R and β for different values of K. Table 2 shows some of these parameters in the first quadrant of the complex ε-plane for $\rho=\sqrt{2}$. For $K<\sqrt{3}\approx 1.732$

TABLE 2
The parameters R *and* β *of the nearest singularities of the power series* (3.1) *for* $\rho=\sqrt{2}$ *in the first quadrant of the complex* ε*-plane using* [24/24] *Padé approximants. For* $K=\sqrt{3}$, *the singularities closest to the origin of the first and second types and that of the third type are located about the same distance from the origin.*

K	R	β
0.25	0.71	1.40
0.50	0.73	1.22
0.75	0.75	1.03
1.00	0.78	0.84
1.25	0.83	0.64
1.50	0.91	0.42
1.60	0.96	0.33
1.70	1.03	0.24
$\sqrt{3}\approx 1.732$	1.06	0.21
$\sqrt{3}\approx 1.732$	(1.05)	(1.57)
1.75	1.01	1.57
1.80	0.93	1.57
1.85	0.85	1.57
2.00	0.63	1.57
2.50	0.27	1.57
3.00	0.16	1.57
3.50	0.11	1.57

as K increases, the values of R increases, and β decreases. Thereafter as K increases, the value of R decreases, whereas the argument β remains fixed at $\beta = 1.57$. For $K \geqq 4$, the [24/24] Padé approximants do not give us meaningful parameters. Presumably, we could avoid these numerical difficulties by replacing ε with another parameter related to ε, as in [3].

For all of the cases we have investigated, it appears that there are three pairs of branchpoint singularities in the complex ε-plane, with the zeros and poles of the Padé approximants simulating branchcuts. The first pair of the branchpoints (in the first and fourth quadrants of the complex ε-plane) and the second pair (in the second and third quadrants) are symmetric with respect to the imaginary axis. The parameters $\tilde{R}$ and $\tilde{\beta}$ of the second pair of branchpoints in the upper half of the complex ε-plane are related to those of the first pair by the relations

$$\tilde{R} = R \quad \text{and} \quad \tilde{\beta} = \pi - \beta. \tag{4.1}$$

For small values of K, these two pairs of singularities are close to each other and located near the imaginary axis (see Fig. 3(a)). As K increases, they move away from the origin as well as away from the imaginary axis.

The third pair of the singularities is located very close to or on the imaginary axis and moves toward the origin along the imaginary axis as K increases. In particular, for $K > \sqrt{3}$, these singularities dominate the other two pairs and become the singularities nearest to the origin and thus limit the convergence of our power series solution (3.1).

The general solution to (1.2) can be thought of as the combination of the particular solution resulting from the forcing term and a homogeneous solution for the free equation. The power series solution we have computed is periodic and is an estimate of a particular (forced) solution only. However, as shown in [14], for small values of ε, with $\rho = \sqrt{2}$ and for $K > K^* = \sqrt{2}|1 - \rho^2| \approx 1.4142$, which we will refer to as the critical value of K, the free oscillation term will, to the first approximation, decay and therefore our calculated power series is expected to represent the steady-state motion. We observe that the local maximum of R occurs when $K \approx \sqrt{3}$, which is close to this critical value.

We note that for $K > K^*$, where the power series (3.1) can represent the actual motion, the radius of convergence of the series is small and therefore, the domain of the validity of the power series is limited to small values of ε, e.g., $\varepsilon \leqq 1.0$. As we have shown in [1]-[3], by using the information obtained from Padé approximants for the series (3.1), we can transform the series into a new power series that can be used for a larger range of ε. For example, for $K = 3$, the dominant singularities are located very close to the imaginary axis as a pair of complex conjugate points at $\varepsilon = R\, e^{\pm i\beta}$ with $R = 0.157$ and $\beta = 1.57$. Using specific values for these parameters, the transformation

$$\delta(\varepsilon) = \frac{\varepsilon}{(R^4 - 2R^2\varepsilon^2 \cos 2\beta + \varepsilon^4)^{1/4}} = \frac{\varepsilon}{(R^2 + \varepsilon^2)^{1/2}} \tag{4.2}$$

maps the dominant singularities to infinity while the origin remains fixed. Points close to the dominant singularities are mapped far from the origin and hence outside the unit circle in the transformed plane. In the complex δ-plane, u can be approximated by

$$u = u(t, \delta) = \sum_{j=0}^{49} \tilde{u}_j(t)\delta^j + O(\delta^{50}) \tag{4.3}$$

where the $\tilde{u}_j(t)$'s are the revised coefficients that can be determined from (3.1), (4.2) and the u_j in a straightforward manner. Using [24/24] Padé approximants, we find that, for $K=3$, the radius of convergence of the transformed series (4.3) is

$$R_\delta \cong 1.00. \tag{4.4}$$

Thus, the new power series can be used to approximate the periodic solution for all values of ε [1]-[3], whereas the original series (3.1) is only valid for $\varepsilon < 0.157$. Similar transformation can be performed for other values of K, with corresponding improvement in the effective radius of convergence.

5. Discussion. We have computed a power series representation of a particular solution to the forced van der Pol oscillator, when the forcing frequency is not near a certain subset of the resonant frequencies. In our study we have concentrated on this particular solution, whereas the actual solution consists of a combination of both the free (homogeneous) and forced (particular) solutions. As discussed in [14], for small values of ε, and with a small external force, the free oscillation term persists in time. However, as the amplitude of the external force increases beyond a critical value K^*, the free term will decay. The value of K^* to the first order is $K^* = \sqrt{2}|1-\rho^2|$. For $K > K^*$, our computed power series is expected to estimate the actual motion. In Fig. 4 we have plotted the distance from the origin of the singularities of the first and second types for $\rho = \sqrt{2}$ as K varies between zero and about three, and have also plotted the distance from the origin of the third type of singularity. We observe that the point of intersection of these two curves occurs at $K \approx \sqrt{3}$, which is close to the first-order critical forcing amplitude $K^* = \sqrt{2}$. For $K < \sqrt{3}$, the location of the first and second

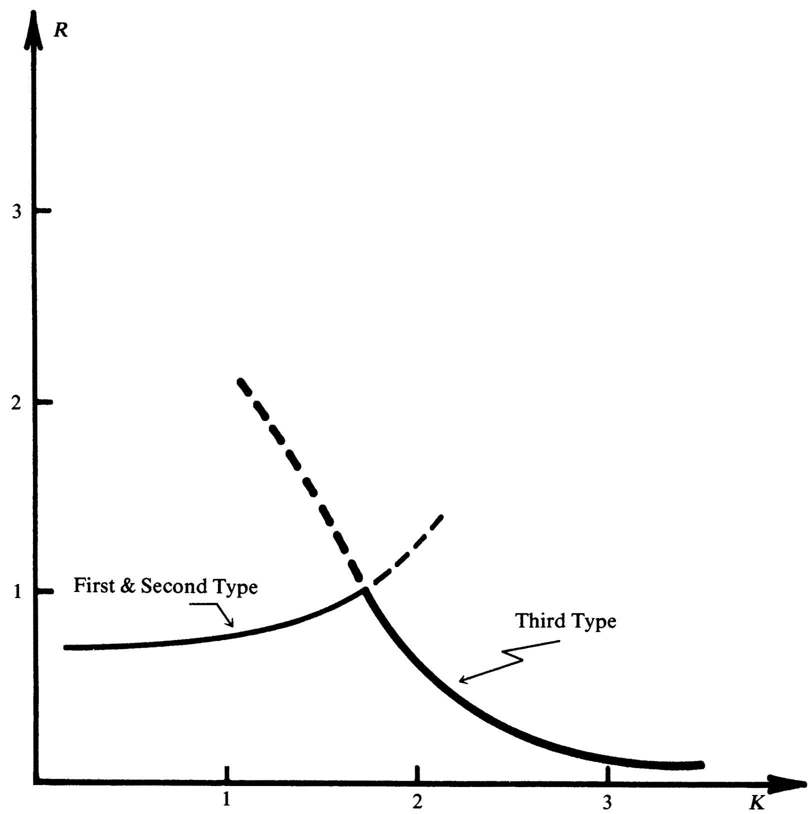

FIG. 4. *The distance from the origin of the nearest singularities of the first and second type and the third type singularities in the complex ε-plane using* [24/24] *Padé approximations to the series* (3.1).

types of singularities limits the convergence of the series, whereas for $K > \sqrt{3}$ the third type limits the convergence. It is interesting to conjecture that this "crossover" point (i.e., $K = \sqrt{3}$ in this case) is related somehow to the quenching of the free oscillation component of a more general solution. At present, however, we are unable to state precisely such a relationship.

In Fig. 5, we have plotted the parameters R and β for the dominant singularities as a function of ρ for several different values of K. For $\rho = 1.0$ (primary resonance), apparently there is a local minimum for the parameter R. As ρ increases above $\rho = 1$, R increases monotonically. Therefore, it appears that the higher-order subharmonic resonances, that we determined in § 1, do not have a significant effect on the values of R and hence on the convergence of the series. For $\rho > 3$, the values of R appear to be essentially independent of the forcing amplitude K.

In the neighborhood of the primary resonance, we notice that the value of β stays fixed at $\pi/2$ and then, as ρ increases, drops to a smaller value and thereafter increases, approaching $\pi/2$ as ρ increases. Table 2 and Fig. 5 illustrate these phenomena.

As we can see from (3.5), our power series solution is not valid at the superharmonic resonances $\rho = 1/3, 1/5, \cdots$. However, we have computed and analyzed the power series solution for other values of ρ below the primary resonance. In the region $0 < \rho < 1$,

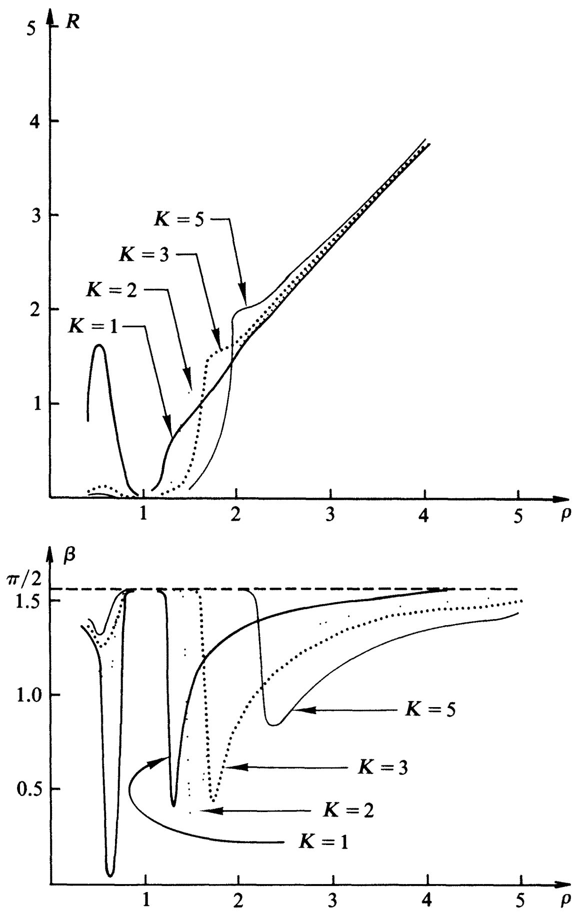

FIG. 5. *The parameters R and β of the dominant singularities of the power series* (3.1) *in the first quadrant of the complex ε-plane using* [24/24] *Padé approximations for* $K = 1, 2, 3,$ *and* 5.

it appears that the curves shown in Fig. 5 are oscillatory in nature, in contrast to the essentially monotone behavior for $\rho > 1$. In particular, it appears that there exists a local maximum for the parameter R in the neighborhood of $\rho = 0.5$, which is a second-order resonant frequency (see Table 1), whereas β seems to have a local minimum near $\rho = 0.6$, which is a third-order resonant frequency.

REFERENCES

[1] C. M. ANDERSEN AND J. F. GEER, *Power series expansions for the frequency and period of the limit cycle of the van der Pol equation*, SIAM J. Appl. Math., 42 (1982), pp. 678-693.

[2] M. B. DADFAR, J. F. GEER, AND C. M. ANDERSEN, *Perturbation analysis of the limit cycle of the free van der Pol equation*, SIAM J. Appl. Math., 44 (1984), pp. 881-895.

[3] M. B. DADFAR AND J. F. GEER, *Power series solution to a simple pendulum with oscillating support*, SIAM J. Appl. Math., 47 (1987), pp. 737-750.

[4] A. S. DMITRIEV, *Chaos in a driven self-oscillating system*, Ivuz Radiof, 26 (1983), pp. 1081-1086. (In Russian.)

[5] E. M. EL-ABBASSY AND E. M. JAMES, *Stable subharmonics of the forced van der Pol equation*, IMA J. Appl. Math., 31 (1983), pp. 269-279.

[6] J. E. FLAHERTY AND F. C. HOPPENSTEADT, *Frequency entrainment of the forced van der Pol oscillator*, Stud. Appl. Math., 58 (1978), pp. 5-15.

[7] J. GRASMAN, E. J. M. VELING, AND G. M. WILLEMS, *Relaxation oscillations governed by a van der Pol equation with periodic forcing term*, SIAM J. Appl. Math., 31 (1976), pp. 667-676.

[8] J. GRASMAN, H. NIJMEIJER, AND E. J. M. VELING, *Singular perturbations and a mapping on an interval for the forced van der Pol oscillator*, Phys. D, 13 (1984), pp. 195-210.

[9] J. GRASMAN, *Asymptotic Methods for Relaxation Oscillations and Applications*, Applied Mathematical Sciences 63, Springer-Verlag, Berlin, New York, 1987.

[10] M. Levi, *Qualitative analysis of the periodically forced relaxation oscillations*, Mem. Amer. Math. Soc., 244 (1981), pp. 1-147.

[11] J. E. LITTLEWOOD, *On non-linear differential equations of the second order: III* $y'' - k(1-y^2)y' + y = b\mu k \cos(\mu + \alpha)$ *for large k, and its generalizations*, Acta Math., 97 (1957), pp. 267-308.

[12] ———, *On the number of stable periods of a differential equation of the van der Pol type*, IEEECT, 7 (1960), pp. 535-542.

[13] MACSYMA *Reference Manual*, Version Eleven, MACSYMA Group, Symbolics, Inc., Cambridge, MA, 1985.

[14] A. H. NAYFEH AND D. T. MOOK, *Nonlinear Oscillations*, John Wiley, New York, 1979.

[15] M. VAN DYKE, *Analysis and improvement of perturbation series*, Quart. J. Mech. Appl. Math., 27 (1974), pp. 423-450.