Math Camp - Day 1

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Why do we need math?

- Essential skill for reading papers – formal language
- Comprehend, use, and develop statistical models
- Analytical reasoning
- Precise expression of magnitude and undercertainty of relationships

Math Camp Goals

- **Spaced Repetition**
- De-rusting: Revisit maths from secondary school and undergraduate
- First introductions: Introduce new concepts to students with less formal math background.
- Identify your own weaknesses: practice before getting underwater in methods sequence
- Homework: Self-teach using online resources (Khan Academy, Math Stack Exchange, Cross Validated)
- Meet your cohort and form connections that will be helpful during methods sequence

Expectations

- **COVID Safety**: Masks are to be worn in class at all times. Masks are to be properly worn (covering mouth and nose). Face shields are not an adequate substitute for masks. Follow social distancing norms, and all other TCD policies. If you must eat or drink, do so briefly or exit the room. Class will only proceed if students are observing COVID safety.

Expectations

- **We are all adults**.
- You do not need my permission to go to the WC or exit to get a breath of fresh air, but please do so quietly.
- Silence mobile phones and laptops.

- Math Camp is designed like a exam review session: we will cover a lot of material quickly. If you have a question, please interrupt to ask – if you don't understand something, you probably are not the only one and we all benefit from you speaking up.

Variables and Constants

Variables and Constants

- A **variable** is a concept or a measure that takes different values in a given set:
  - Age of people in this room
  - GDP per capita, infant mortality in a dataset of countries
  - Scores of last GAA Football Championships
  - Temperature readings
  - Vote shares
- A **constant** is a concept or measure that has a single value for a given set.
  - π (pi): The ratio of a circle's circumference its diameter
  - Also used informally for something that could be a variable, but which does not vary e.g. Gender in a dataset of Taoiseach.

Level of Measurement

- A level of measurement describes the sorts of operations we can do on a variable's class of data. Some models or rules will apply differently depending on level of measurement.
- **Nominal** (also called **categorical**) level measurement does not establish mathematical relationships among the values.
  - Religion, gender, political party, country of origin, etc.
- **Ordinal** level measurement does imply mathematical relationships among the values. The values can be placed in an order. But we cannot precisely interpret the "distance" between two values the variable might take.
  - Likert scale: "Do you approve of Boris Johnson's job performance? Strongly approve, approve, neither approve nor disapprove, disapprove, strongly disapprove"
  - Education: "Ph.D., Master's, Bachelor's, Vocational Cert, Leaving Cert, Junior Cert"

- **Interval** level measures have meaningful distances between values
  - POLITY score (How Democratic/Authoritarian a country is): from -10 to 10
  - Temperature: 20° celsius is 10° hotter than 10° celsius
- **Ratio** level variables are interval level variables that can additionally be compared as multiples of one another: these have a true "zero" point.
  - Height: Someone who is 2m tall is twice as tall as someone who is 1m tall.
  - GDP per capita: Irish GDP per capita (70,145 Euro) is 2.6x Spanish GDP per capita (26,692 Euro)

Sets

Sets

- A **set** is just a collection of elements
  - $A = \{1, 2, 3, 4\}$
  - $B = \{a, b, c\}$
  - $Counties = \{Carlow, Cavan, Clare, Cork, ...\}$
  - $Even\ numbers = \{2, 4, 6, ...\}$
- The number of elements in a set is its **cardinality**, often written $|A|$. Here, A has a cardinality of 4.

Table: Numerical sets used in Mathematics:

Notation	Meaning
$\mathbb{N}$	Natural Numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational Numbers
$\mathbb{R}$	Real Numbers

Types of Sets

- Finite set: Every member of the set can be written
- Infinite set: Every member of the set can be described, but written.
- Bounded set: There are a minimum or maximum numeric value.
- Unbounded set: The set is missing at least one bound.
- Empty (null) set: $\emptyset$
- Universal Set: U – used to express a set that contains every possible element of a given domain or problem.

Types of Subsets

- Subset: A $\subseteq$ B if every element of A is in B. A set is a subset of itself.
- Proper subset: A $\subset$ B if A is a subset of B and A is not B.
- Cardinality: The number of elements in a set, $|A|$
- Power set: Every possible subset of a set A, including both A and $\emptyset$.

Power Set: Worked Example

Suppose a class has three students: Aoife, Ciara, and Nicole. The set of student is $\{Aoife, Ciara, Nicole\}$. At the end of the class, we are interested in knowing which students earned a passing mark. The power set allows us to list every possible outcome for this question.

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$$
\begin{aligned}
PowerSet = \{ & \\
& \{\emptyset\}, \{Aoife\}, \{Ciara\}, \{Nicole\}, \{Aoife, Ciara\}, \\
& \{Aoife, Nicole\}, \{Ciara, Nicole\}, \{Aoife, Ciara, Nicole\} \\
\} &
\end{aligned}
$$

(Side note: What is the cardinality of the power set? 2^N, where N is the cardinality of the set: in this case, $2^3 = 8$)

Set Operators

- "In" (membership): An element is in a set? $1 \in A$
- "Not In": An element is not in a set: $1 \notin B$
- Difference between sets: $A \,/\, B$ (Everything in set A and not in set B)
- Complement (the opposite of a set): A^c or A' (Everything in the universal set U but not in set A).
- Intersection (elements common to two sets): $A \cap B$
- Union (elements in either of two sets): $A \cup B$
- Mutually exclusive: When two sets do not share any elements in common.
- Partition: Dividing a set into two or more mutually exclusive subsets.
- Cartesian product: $A \times B$. Combining elements of one set with elements of another as pairs.

Worked Example: Partitions

Germany is about to have elections to its Bundestag (national legislature). The set of all parties in the Bundestag is:
$\{CDU, SPD, AfD, FDP, Left, Green\}$.

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$\{CDU, SPD, AfD, FDP, Left, Green\}$.

We might partition these parties into three subsets: Left-Progressive ($\{SPD, Left, Green\}$), Liberal-Conservative ($\{CDU, FDP\}$), and Far-Right Populist ($\{AfD\}$).

Every party is in exactly one subset.

Worked Example: Cartesian Products

Suppose you own 3 shirts (black, white, and blue) and 3 pairs of trousers (jeans, khakis, shorts). What are all of the possible outfits you can make? Take a Cartesian Product of these sets

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$$
\begin{aligned}
&\text{Shirts} \times \text{Trousers} = \{ \\
&\qquad (\text{black}, \text{jeans}), (\text{black}, \text{khakis}), (\text{black}, \text{shorts}), \\
&\qquad (\text{white}, \text{jeans}), (\text{white}, \text{khakis}), (\text{white}, \text{shorts}), \\
&\qquad (\text{blue}, \text{jeans}), (\text{blue}, \text{khakis}), (\text{blue}, \text{shorts}) \\
&\}
\end{aligned}
$$

The cardinality of a Cartesian Product $|A \times B| = |A| \times |B|$: in this case, you have nine possible outfits..

Algebra Review

Basic Properties of Arithmetic

Basic arithmetic operations include: addition, subtraction, multiplication, division, exponentiation, logarithms, and factorials.

Each operation has some basic properties.

- Addition and Multiplication are **commutative**:
  - $a + b = b + a$
  - $a \times b = b \times a$
- Addition and Multiplication are **associative**:
  - $(a + b) + c = a + (b + c)$
  - $(a \times b) \times c = a \times (b \times c)$

Note that neither subtraction nor division are commutative or associative.

Basic Properties of Arithmetic

- Addition and Multiplication have **identity properties**:
  - $x + 0 = x$
  - $x \times 1 = x$
  - Note that identity is different for each operation.
- **Distributive** property allows us to combine addition and multiplication:
  - $a \times (b + c) = (a \times b) + (a \times c)$
- **Inverses**:
  - $x + (-x) = 0$ (Negation)
  - $x \times x^{-1} = 1$ (Reciprocal)

Order of Operations

When we cannot rely on associativity and commutativity, we need to evaluate operations in a particular order: *PEMDAS*

- **P**arentheses (or **B**rackets)
- **E**xponents
- **M**ultiplication
- **D**ivision
- **A**ddition
- **S**ubtraction

When dealing with multiple parentheses, work "inside out". When dealing with multiple operations otherwise, work left to right.

Worked Order of Operations Example

$$
\begin{aligned}
((3+2)/5-1)*6+3 &= (5/5-1)*6+3 \\
&= (1-1)*6+3 \\
&= 0*6+3 \\
&= 3
\end{aligned}
$$

Fractions

Basic definition

A fraction is a ratio (result of the division of) two numbers or variables.

- $\frac{a}{b}$. In this fraction, a is the **numerator** and b is the **denominator**
- $\frac{a}{b} = a \times \frac{1}{b}$
- $\frac{a}{1} = a$

Simplification by cancellation can be used when a whole term of a fraction can be reduced:

- $\frac{10x}{2x} = \frac{5x}{x} = 5$

Splitting fractions

The numerator can be spit into multiple terms as long as the denominator is not:

- $\dfrac{3+2x}{3x} = \dfrac{3}{3x} + \dfrac{2x}{3x} = \dfrac{1}{x} + \dfrac{2}{3}$
- $\dfrac{5}{3x+2} \neq \dfrac{5}{3x} + \dfrac{5}{2}$

Adding fractions

To add two fractions, the fractions must have a common denominator:

- $\frac{a}{4} + \frac{b}{2} \neq \frac{a+b}{4}$ (Can't pick one denominator)
- $\frac{a}{4} + \frac{b}{2} \neq \frac{a+b}{6}$ (Can't add denominators)
- $\frac{a}{4} + \frac{b}{2} \neq \frac{a+b}{8}$ (Can't multiply denominators)
- $\frac{a}{4} + \frac{b}{2} = \frac{a}{4} + \frac{2b}{4} = \frac{a+2b}{4}$

Multiplying fractions

To multiply two fractions, multiply numerator by numerator and denominator by denominator:

- $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$
- $\frac{a}{b} \times c = \frac{ac}{b}$

Fractions in numerators or denominators

When the numerator is a fraction, move the top denominator to the bottom:

- $\dfrac{\frac{a}{b}}{c} = \dfrac{a}{bc}$
- $\dfrac{\frac{x}{4}}{2y} = \dfrac{x}{(4)(2y)} = \dfrac{x}{8y}$

When the denominator is a fraction, move the bottom denominator to the top:

- $\dfrac{a}{\frac{b}{c}} = \dfrac{ac}{b}$
- $\dfrac{x}{\frac{y}{2}} = x \times \dfrac{2}{y} = \dfrac{2x}{y}$

Proportions, Percentages, and Percentage Points

Proportions and Percentages

- A **ratio** is the result of dividing two numbers: if Beth has 400 shares of a company and Sean has 200 shares, Beth has $\frac{400}{200} = 2$ times as many shares as Sean.
- A **proportion** is the ratio of the amount of interest to the whole amount. Beth has $\frac{400}{400+200} = \frac{400}{600} = 0.666...$ of the shares.
- To obtain a **percentage**, multiply a proportion by 100: Beth has $0.666... \times 100\% = 66.6...\%$ of the shares, or else she has $2 \times 100\% = 200\%$ Sean's number of shares.
- Subtract 1 from a ratio or 100% from a percentage to compare to a baseline: Beth has $200\% - 100\% = 100\%$ more shares than Sean
- Note that the proportion 0.5 is not the same as the percentage 0.5% – in fact, it is 100 times larger!

Comparing Two Proportions

It can sometimes be difficult to compare two proportions or percentages. Consider the 2020 Dáil election, in which Sinn Féin earned 24.5% of all votes, up from 13.8% in the 2016 Dáil election. How can we compare these two numbers?

- **Percentage points**: $24.5\% - 13.8\% = 10.7\%$. Sinn Féin earned 10.7% more votes in percentage points.
- **Percentage**: $\frac{24.5\%}{13.8\%} = 1.77 * 100\% = 177\%$. Sinn Féin earned 177% their prior vote tally as a ratio, or 77% more votes than their prior vote tally.

Exponents and Logarithms

Exponents

Given a term a^b, we say that a is the base and b is the exponent. Exponentiation refers to sequential multiplication of a base:

- $a^2 = a \times a$
- $a^3 = a \times a \times a$
- $a^0 = 1 \qquad$ (This is true for any base a)
- $a^{-1} = \dfrac{1}{a}$
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{1}{2}} = \sqrt{a}$
- Even powers of negative numbers are positive ($a^2 >= 0 \ \forall \ a \in \mathbb{R}$)

Exponent arithmetic

- $a^n + a^m \neq a^{(n+m)}$. *There is no general way to add same-base different-exponent terms.*
- $a^n \times a^m = a^{(n+m)}$
- $a^n \times b^n = (ab)^n$.*Why? Try* $3^2 \times 2^2$, *think associative rule*
- $\dfrac{a^n}{a^m} = a^{(n-m)}$
- $(a^n)^m = a^{(n \times m)}$

Logarithms

The logarithm function $log_a b$ finds the value x for which $a^x = b$. We call the a term the **base**, the b term the **argument**, and the x term the **exponent** or **logarithm**.

- $\log_2 4 = 2$ because $2^2 = 4$
- $\log_{10} 1000 = 3$ because $10^3 = 1000$
- $\log_a 1 = 0$ for any a.
- $a^{\log_a x} = x$

Two common logarithms are the base-10 logarithm (sometimes just written *log* with no base) and the natural logarithm (written *ln* or log_e). For the natural logarithm, e (Euler's number) $= 2.718...$

Restrictions on Logarithms

- Logarithm bases must be positive numbers greater than 1. Why? Because if base is negative, function would not be defined for many values of the argument: $log_{-2}(0.5)$
- Logarithm arguments must not be negative. Why? Consider $log_2(n) = x$, which implies $2^x = n$. For what value of x could n be negative? Even if x is $-\infty$, then $2^{-\infty} \sim 0$.
- A logarithm argument of 0 is not exactly defined, but $\lim_{x \to 0} log(x) = -\infty$
- The result of a logarithm can be any number, positive or negative.

Logarithm arithmetic

Logarithms have the property that they transform multiplication into addition, division into subtraction, and exponentiation into multiplication, as follows:

- $\log_a mn = \log_a m + \log_a n$
- $\log_a \frac{m}{n} = \log_a m - \log_a n$
- $\log_a x^n = n \times \log_a x$

Logarithms have the effect of "shrinking" large numbers or differences and "expanding" small numbers or differences – we will see this tomorrow.

Sums and Sequences

Arithmetic Notation: Sequences

Sometimes we want to repeatedly add or multiply elements of a set or sequence. We can do this using the Σ (sigma sum) and $\prod$ (product) operations.

Consider the following:

$$\sum_{k=1}^{5} 2k + 1$$

Unpacking this:

- $\sum$: Repeatedly add all the values of the equation to the right of this symbol.
- $\sum_{k=1}^{5}$: For the purposes of the equation to the right, we will consider the values 1, 2, 3, 4, 5 for the variable k
- $2k + 1$: The equation to be solved.

Sigma Summation

$$
\begin{aligned}
\sum_{k=1}^{5} 2k+1 = & \quad (2(1)+1)+ \\
& \quad (2(2)+1)+ \\
& \quad (2(3)+1)+ \\
& \quad (2(4)+1)+ \\
& \quad (2(5)+1) \\
= & \quad 3+5+7+9+11 \\
= & \quad 35
\end{aligned}
$$

Sigma Summation

We can also sum over the elements of a set, as follows:

$$\begin{aligned}\sum_{k \in (5,10,15)} k^2 = & \quad (5^2) + (10^2) + (15^2) \\ = & \quad 25 + 100 + 225 = 350\end{aligned}$$

Summation rules

There are many useful identities for common summations.

- $\sum_{i=1}^{n} C = C \times n$
- $\sum_{i=a}^{b} C = C \times (b - a)$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{n} C \times f(n) = C \times \sum_{n} f(n)$

Inequalities

Inequalities

An inequality is an equation where the two sides are not equal, such as $x < 5$.

- Recall that the "mouth" of the inequality faces the larger number. $x < 5$ means "x is less than 5" while $x > 5$ means "x is greater than 5".
- The four inequality operators are $<$, $\leq$, $>$, and $\geq$.
- Inequalities with "looser" conditions than the stated inequality are true: If $x > 3$ then $x > 2$ is implied, but $x > 4$ is not
- Adding or subtracting numbers from both sides of an inequality will not change the inequality
- Multiplying or dividing by positive numbers on both sides will not change the inequality
- Multiplying or dividing by negative numbers reverses the sign of the inequality

Solving Equations

Factoring

Factoring involves rearranging the terms in an equation to make further manipulation possible or to reveal something of interest. The goal is to make the expression simpler.

- Combining like terms.
  - $\alpha + \alpha^2 + 4\alpha - 6\alpha^2 + 18\alpha^3 = 18\alpha^3 - 5\alpha^2 + 5\alpha$

  A term is "like" if it has both the same variable in its base and the same exponent.

- Separating or factoring out a common term
  - $3\alpha + 4\alpha^2 = \alpha(3 + 4\alpha)$
- Factoring quadratic polynomials
  - $x^2 + x - 12 = (x + 4)(x - 3)$
- Factoring fractions
  - $\frac{2x^2+10x+12}{4x+8} = \frac{x+3}{2}$

FOIL Expansion

FOIL is a mnemonic device to simplify the multiplication of two binomials:

$$(a+b)(c+d) = \underbrace{ac}_{\text{first}} + \underbrace{ad}_{\text{outside}} + \underbrace{bc}_{\text{inside}} + \underbrace{bd}_{\text{last}}.$$

- **F**irst: $({\color{blue}a}+b)({\color{blue}c}+d) = ac$
- **O**utside: $({\color{blue}a}+b)(c+{\color{blue}d}) = ad$
- **I**nside: $(a+{\color{blue}b})({\color{blue}c}+d) = bc$
- **L**ast: $(a+{\color{blue}b})(c+{\color{blue}d}) = bd$

$$(x+4)(x-8) = x^2 - 8x + 4x - 32 = x^2 - 4x - 32$$

Solution tips

- Isolate the variable of interest
- Combine like terms where possible
- Make use of identities
- Operate on both sides of the equation
- Check your answer

Solve for x:

$$(x+1)^2 = 9$$

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Solve for x:

$$(x+1)^2 = 9$$

$$\begin{aligned} (x+1)^2 &= 9 \\ \sqrt{(x+1)^2} &= \sqrt{9} \\ (x+1) &= 3 \\ x &= 3-1 \\ x &= 2 \end{aligned}$$

Quadratic equation

A quadratic function is a function of the form $ax^2 + bx + c$, which contains a quadratic term (ax^2), a linear term (bx), and a constant term (c). We often seek to find the answer to the equation:

$$ax^2 + bx + c = 0$$

The solution can be obtained using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic equation practice

Solve:

$$6y^2 - y - 5 = 0$$

Quadratic equation practice

Solve:

$$6y^2 - y - 5 = 0$$

Note: $a = 6, b = -1, c = -5$

$$\begin{aligned} y =& \frac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-5)}}{2(6)} \\ & \frac{1 \pm \sqrt{1 + 120}}{12} \\ & \frac{1 \pm 11}{12} \\ & \frac{12}{12} = 1 \text{ and } \frac{-10}{12} = \frac{-5}{6} \end{aligned}$$

Conclusion

Tomorrow

- Functions
- Exponents and Logarithms Recap
- Limits
- Intro to Derivatives