Math Camp - Day 3

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Finding Maxima and Minima

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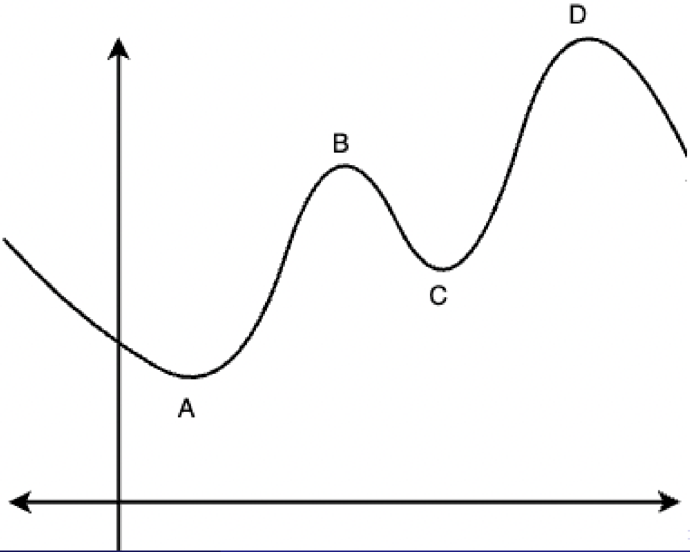
- Why else do we actually care about derivatives?
- Derivatives are useful to "optimize" functions: find the maxima (highest points) and minima (lowest points)
- Data science: The essence of fitting a model to data is finding the parameters that maximize fit or minimize error!
- Game theory and behaviour: Which response optimizes my utility?
- Regression:
  - Ordinary Least Squares **minimizes** squared error: $e'e$
  - Maximum Likelihood Estimation **maximizes** a likelihood function.

Finding Maxima and Minima

- For any function, a high point is called a maximum (plural maxima) and a low point is called a minimum (plural minima).
- Recall graphical example from yesterday
- At the extremum, drawing a tangent line (line that touches the function only at the extremum) will have a slope zero
- Intuition: A maximum happens when the slope of a function goes from positive to negative, passing through zero. A minimum happens when the slope of a function goes from negative to positive, passing through zero. The derivative is the slope of the function!
- An extremum is **local** if it is the largest or smallest value in its neighborhood (some interval of the domain of the function), and **global** if it is the largest or smallest value across all points of the function

Identifying extrema



Higher Order Derivatives

- The n^{th}-**order derivative** is the rate of change of the $n-1^{th}$-order derivative.
- The first-order derivative is the rate of change (slope) of the function. If the function is distance, the derivative is speed
- The second-order derivative is the rate of change of the instantaneous rate of change. If the function is distance, the second-order derivative is acceleration.
- The third-order derivative is the rate of change of the second-order derivative, and so on.

Repeated Derivatives and Polynomials

- Polynomials are a class of function that have interesting properties when repeatedly taking derivative
- When taking kth order derivative of an nth order polynomial, the highest term in the derivative will be x^{n-k}

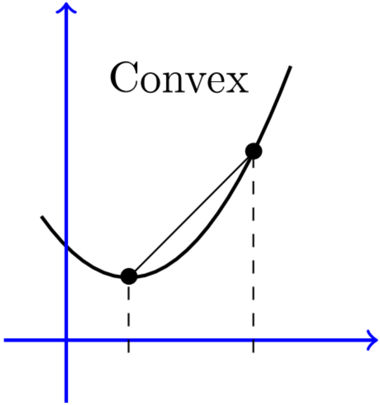
$$f(x) = 2x^{\mathbf{4}} \rightarrow f'(x) = 8x^{\mathbf{3}} \rightarrow f''(x) = 24x^{\mathbf{2}} \rightarrow f'''(x) = 48x \rightarrow f''''(x) = 48 \rightarrow f'''''(x) = 0$$

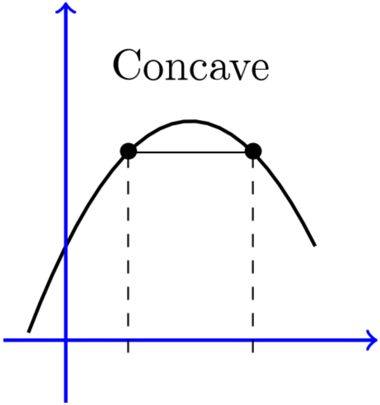
Critical Points and Inflection Points

- A **critical point** is any point x^* such that either $f'(x^*) = 0$ or $f'(x^*)$ doesn't exist.
- A critical point, called a **stationary point** when the slope is flat, can be a local minimum, a local maximum, or an inflection point.
- **Inflection points** are points at which the graph of the function changes from concave (bending down) to convex (bending up) or vice versa.

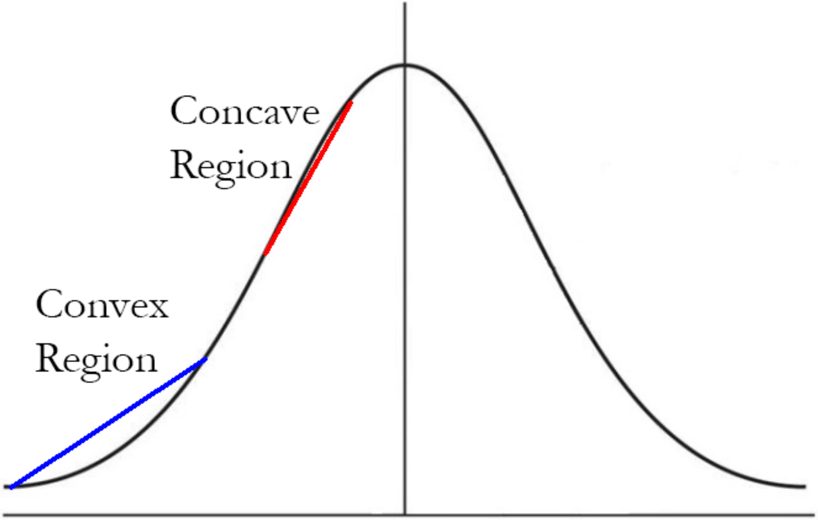
Concave and Convex

- **Secant line**: A line that intersects the curve, locally, at two points
- Concave: The secant line is under the curve
- Convex: The secant line is over the curve

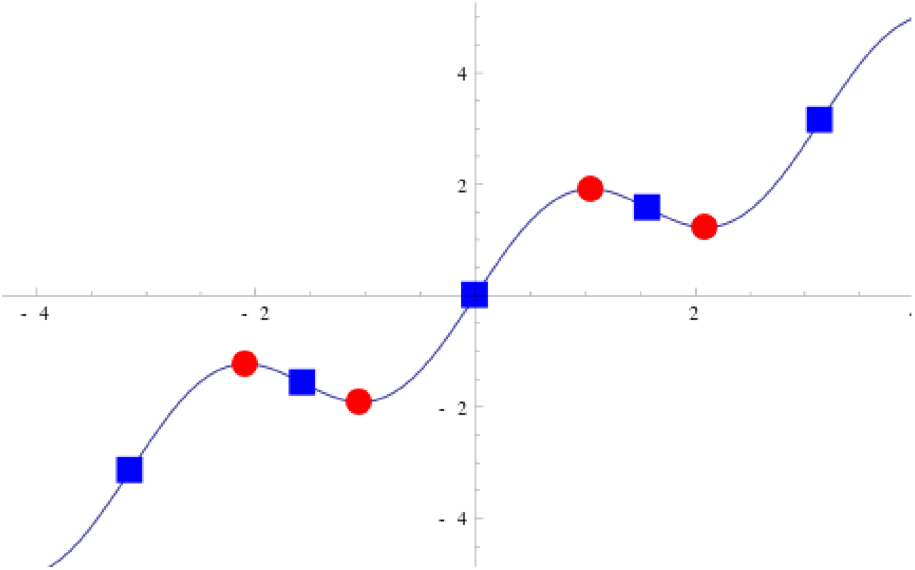




Concave and Convex



Critical Points and Inflection Points



Method to Find Minima and Maxima

1. Given a function $f(x)$, we wish to find a minimum or maximum.
2. First, find $f'(x)$.
3. **First Order Condition**: Set $f'(x^*) = 0$ and solve for all x_*. These are stationary points of the function. Any minimum or maximum that exists will be among these.

   Recall first derivative is slope.To go from negative to positive, it has to pass through zero (intermediate value theorem).

Is it a Minimum or Maximum?

4. If we are unsure whether this is a minimum, maximum, or inflection point, we have a mathematical tool to check:

   Find $f''(x)$ – the second derivative.

5. **Second Order Condition**: Solve the second derivative at the point of interest x^*, say for $x = 3$ solve $f''(3)$.
   - If $f''(x^*) < 0$, $f(x)$ has a local maximum at x^*.
   - If $f''(x^*) > 0$, $f(x)$ has a local minimum at x^*.
   - If $f''(x^*) = 0$, x^* may be an inflection point – inflection points are not of interest to us in optimization, so we will leave out rules for proceeding here

Worked Example

- Given the function $f(x) = -3x^2 + 5x - 2$, find any stationary points, and characterize them as minima, maxima, or inflection points
- First, take derivative $f'(x) = -6x + 5dx$ (split terms, power rule – not too tough).
- Set first order condition $-6x + 5 = 0$ and solve for x: $-6x = -5 \rightarrow x = 5/6$. One stationary point.
- Take second derivative $f''(x) = -6dx$
- Second derivative is less than 0 at all points, doesn't matter what x is, so the stationary point is a maximum.

tl;dr

To find the extremum, set the first derivative to zero and solve for x. To determine if it is a maximum or minimum, take the sign of the second derivative at the point.

Introduction to Integrals

Introduction to Integrals

- An integral, also called an anti-derivative, is **the Area under the Curve**.
- Basic notation: $F(x) = \int f(x)dx$
- What do these terms mean?

Defining notation

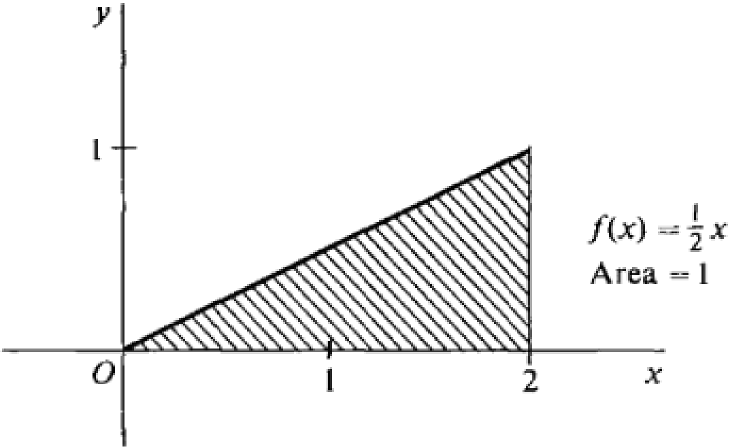
From "Calculus Made Easy" (1910):

The preliminary terror, which chokes off most [people] from even attempting to learn how to [integrate], can be abolished once for all by simply stating what is the meaning of the two principal symbols that are used in [integrating]:

(1) d which merely means "a little bit of.". **Thus dx means a little bit of x**. You will find that these bits may be considered to be indefinitely small.
(2) $\int$ which is merely a long S, and may be called "the sum of".

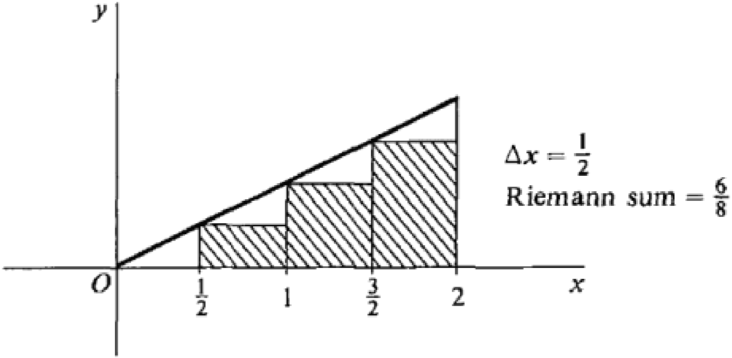
Thus, $\int dx$ **means the sum of all the little bits of x**. Now any fool can see that if x is considered as made up of a lot of little bits, each of which is called dx, if you add them up together you get the sum of all the dx's, (which is the same thing as the whole of x).

Taking an integral without even realizing it



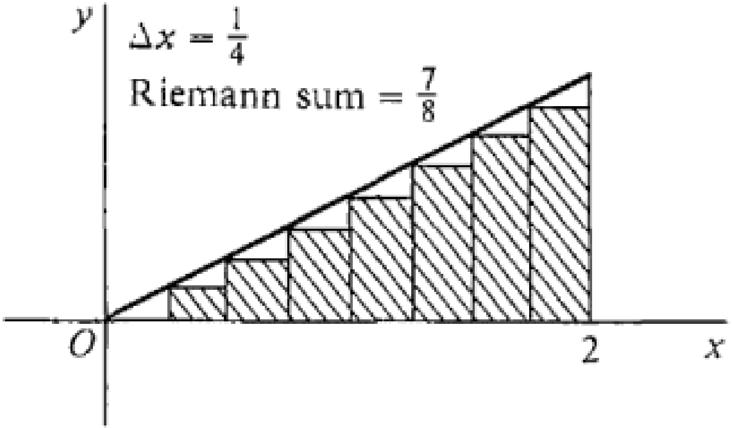
Intuition of integral

What if we didn't have an analytical formula like $area = \frac{b \times h}{2}$? Use **Riemann sums** to measure sections – approximating each portion of the function with a rectangle.



Intuition of integrals

Can't get the answer exactly right; will always count too much or too little! But by shrinking rectangles, can get closer to correct.



Reimann Sum and definite integral

- Thus, the integral (area) is the limit of a sum of (infinitely) many very small rectangles
- In other words, "the sum of a little bit of x", like the text said.
- $\int f(x)dx = \lim_{\Delta x \to 0} \sum_{i}^{n} f(x_i^*)\Delta x_i$
- Plain English: As we make the rectangles narrower and narrower, let's sum up the value of the function ($y = f(x)$) and multiply it by the width of the rectangle (Δx) to get the combined area.

Definite and Indefinite Integrals

- There are two types of integrals: **definite** integrals (which measure the area under the curve in a fixed range of x) and **indefinite** integrals (which measure the area under the curve across every value of x).
- Indefinite integral: $\int f(x)dx = F(x)$
- Definite integral: $\int_a^b f(x)dx = F(b) - F(a)$
- Definite integrals are called the **fundamental theorem of calculus**.
- Typically indefinite integrals are represented as functions, not solved, while definite integrals are represented as numerical solutions.

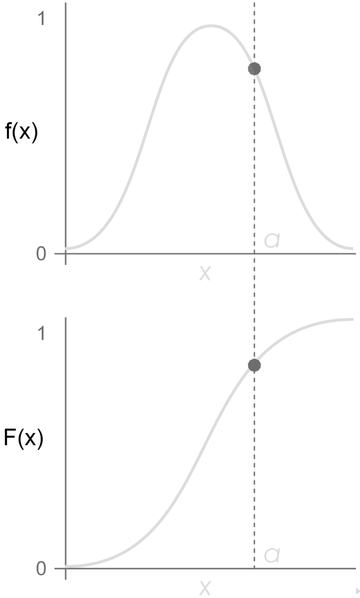
Definite and indefinite integrals, right triangle

- Consider our right triangle example, $f(x) = x$. You can draw this function to show that it will produce a line with slope 1, which forms a right triangle with respect to the x-axis, as we saw in the images above.
- What's the area under the function? Recall that base and height are both equal to x in this example.
  $\int xdx = F(x) = \frac{bh}{2} = \frac{x^2}{2}$.
- What's the area under the function from $x = 0$ to $x = 2$?
  $F(2) = \frac{2*2}{2} = 2$
- What's the area under the function from $x = 1$ to $x = 2$? Uh oh! The area here isn't just a triangle! Take definite integral!
  $$\int_1^2 xdx = F(2) - F(1) = \frac{2 \times 2}{2} - \frac{1 \times 1}{2} = 1.5$$

Relationship between integrals and derivatives

- Integral and derivative are ALMOST inverse operations:
- $\underbrace{F(x)}_{\text{integral}} \xrightarrow{\text{derivative}} \underbrace{f(x)}_{\text{function}} \xrightarrow{\text{derivative}} \underbrace{f'(x)}_{\text{derivative}}$
- $\underbrace{f'(x)}_{\text{derivative}} \xrightarrow{\text{integrate}} \underbrace{f(x)}_{\text{function}} \xrightarrow{\text{integrate}} \underbrace{F(x)}_{\text{integral}}$
- Why? Integral is area under the curve (AUC): made by adding up little bits of the curve itself.
- Rate of change of the AUC is, thus, equal to the value of the little bits – the original function!
- Rate of change of the original function is the original function's slope!

Intuition of integrals



Constant of integration

- Why ALMOST? The **constant of integration**.
- Consider the function
$$f(x) = \frac{x^2}{2} + 5$$
- Its derivative is $f'(x) = x\,dx$. Why? Derivative of constant term is 0, and so the $+5$ term drops away.
- Now, integrate this function:
$$\int f'(x) = \int xdx = \frac{x^2}{2}$$
- What happened to the $+5$ term? Why didn't it come back?

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$$\int f'(x) = \int xdx = \frac{x^2}{2}$$
- What happened to the $+5$ term? Why didn't it come back?
- For all indefinite integrals, must add a term $+C$ to represent the unknown and unknowable constant term in order to ensure integration and derivatives are inverse operators.

Constant of integration and definite integrals

- We can ignore the constant of integration in a definite integral.
- Why? Because the same constant term exists at both points, so $F(b) - F(a)$ will remove the constant anyway.
- e.g. $f(x) = 3x \rightarrow F(x) = \dfrac{3x^2}{2} + C$

$$\int_2^4 f(x) = F(4) - F(2) = \frac{3(4)^2}{2} + C - (\frac{3(2)^2}{2} + C)$$

Computing Integrals

Power rule 1	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
Power rule 2	$\int x^{-1} dx = ln\|x\| + C$
Exponential rule 1	$\int e^x dx = e^x + C$
Exponential rule 2	$\int a^x dx = \frac{a^x}{ln(a)} + C$
Logarithm rule 1	$\int ln(x) dx = xln(x) - x + C$

All these rules are the reverse of associated derivative rules! Useful to remember first power rule and first exponential rule, but generally OK to look these up as needed!

Computing Integrals

Definite integrals 1	$\int_a^b f(x)dx = -\int_b^a f(x)dx$
Empty definite integral	$\int_a^a f(x)dx = 0$
Piecewise definite int.	$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
Linear rule	$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$
Integration by parts	$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

u-substitution

One powerful integration rule is **u-substitution**, which is the integration inverse of the derivative chain rule. Reminder:

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)dx$$

The **u-substitution** rule is:

If $\int f(x)dx$ cannot be solved, choose u such that $\int f(x)dx = \int udu$, solve, and substitute x for u

u-substitution, huh?

Solve:

$$\int (x^4 + 3)^3 (x^3) dx$$

u-substitution:

$$\text{Let } u = x^4 + 3 \text{ then } \frac{du}{dx} = 4x^3 \text{ and } du = 4x^3 dx$$

So:

$$\begin{aligned}
x^3 dx &= \frac{1}{4} du \\
\int (x^4 + 3)^3 (x^3) dx &= \int u^3 (\frac{1}{4} du) \\
&= \frac{1}{4} \int u^3 du \\
&= \frac{1}{4} (\frac{u^4}{4} + C) = \frac{1}{16}(u^4 + C) \\
&= \frac{1}{16}((x^4 + 3)^4 + C)
\end{aligned}$$

Conclusion

Tomorrow

- Scalars, Vectors, and Matrices, oh my!