# Math Camp - Day 1

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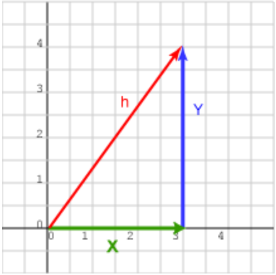
# Scalars and Vectors

# Scalars and Vectors

- A **scalar** is a number or element that has no inherent dimension to it. We are familiar with these:
  - 2, $\sqrt{13}$, $\frac{3}{7}$, Canada
- A **vector** combines one or more scalars, and can be interepreted geometrically as a position (i.e. having magnitude and direction):
  - $x = 1, 4, 5$
  - $y = 2, -1$
- We can refer to an individual scalar inside a vector by using subscripts:
  - $x_1 = 1, x_2 = 4, x_3 = 5$
  - $y_1 = 2, y_2 = -1$
- The **dimension** or **dimensionality** of a vector is the number of components in the vector. For $x$, it is 3.
- The **zero** vector is a vector that has length 0, i.e. all elements are 0.

# Vector Length

- Continuing with the geometric interpretation of a vector, you can think of a vector of dimensionality 2 as having an “x” and “y” coordinate in space.
- The **length** of a vector refers to how far it is from the origin in $n$-space, where $n$ is the dimension of the vector.
- Length can be measured using any distance function, but the most common is **Euclidean**: $||x|| = \sum x_i^2$



# More on Length

- A **zero vector** is a vector whose components are all 0, thus giving the vector length 0.
- A **normalized vector** is a vector whose components have been scaled so that its total length is 1. This preserves the shape and geometry of the vector, but discards the scale of the vector.
- A vector's length is also called a **norm** of the vector. Norms are referred by the power used in their distance function, so the Euclidean norm is the $\ell_2$ norm $||x||_2$.
- Other distances include the **Manhattan** or **taxicab** distance, $\ell_1$: $||x||_1 = \sum |x_i|$

# Vector Math

- To **add** or **subtract** two vectors, both vectors must have the same dimension. You proceed by adding or subtracting element-wise:
  - $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$
  - $\mathbf{a}$ - $\mathbf{b} = (a_1 - b_1, a_2 - b_2, ..., a_n - b_n)$
- To add a scalar to vector, simply add it to each element: $\mathbf{a} + \mathbf{3} = (a_1 + 3, a_2 + 3, ..., a_n + 3)$
- Vectors can also be **multiplied** or **divided** by scalars:
  - $cx = (cx_1, cx_2, ..., cx_n)$
  - $\frac{x}{c} = (\frac{x_1}{c}, \frac{x_2}{c}, ..., \frac{x_n}{c})$
- To multiply two vectors, we use the **element-wise product** or **dot product** – the dot operator is written explicitly:

$$a \cdot b = \sum_i^n a_i b_i$$

# Matrices

# Matrices

- If a **vector** is a series of scalars, then a **matrix** is a series of vectors. A matrix is represented as a 2D table of numbers or variables.
- If a vector is one point in some space, then a matrix is a series of points in space.
- Matrices are named with capital letters $\mathbf{X}$, while vectors are named with lower case letters $\mathbf{x}$.
- A matrix is made up of **rows** (horizontal) and **columns** (vertical)
- A matrix has two dimensions: $n \times m$ where $n$ is the number of rows and $m$ is the number of columns
- A value in a matrix is subscripted $X_{nm}$ where $n$ is the row number from the top and $m$ is the column from the left.

$$X_{n \times m} = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nm} \end{pmatrix}$$

- A matrix that has one column is a **column vector**, and a matrix that has one row is a **row vector**

# More Terms

- The **main diagonal** or **major diagonal** is the diagonal of the matrix heading from the top left to the bottom right
- The **minor diagonal** is the diagonal of the matrix heading from the bottom left to the top right.
- A **square** matrix is a matrix that has an equal number of columns and rows, i.e., $m = n$.
- A **zero** matrix is a square matrix in which all elements are 0.

# Special Types of Matrices

- A **diagonal** matrix is a square matrix in which all elements other than those on the main diagonal are zero.
- An **identity** matrix is a diagonal matrix in which all elements on the main diagonal are 1.

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- An **idempotent** matrix A is a matrix with the property $AA = A$. That is, when you multiply it by itself, it returns the original matrix.

# Special Types of Matrices

- A **lower triangular** matrix has non-zero elements only on or below the main diagonal, while an **upper triangular** matrix has non-zero elements only on or above the main diagonal.

$$L_{3\times 3} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad U_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

- A **submatrix** of a given element is the matrix that remains when we take out the row and column in which the element is (so it has one fewer column and row than the original).

# Special Types of Matrices

- A **symmetric** matrix is a square matrix in which the elements are symmetric about the main diagonal, or more formally one in which $a_{ij} = a_{ji}$.

$$X_{3\times3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

- A **permutation** matrix is one in which there is only a single value of 1 in any row and column, with all other elements 0.
- The identity matrix is a trivial permutation matrix that does not permute the elements.

$$P_{3\times3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Special Types of Matrices

- A **singular** matrix is one that has a determinant of 0.
- A **nonsingular** matrix has a determinant that is not 0.
- Nonsingular matrices have inverses.

# Matrix Operations

# Matrix Transposition

- The **transpose** of a matrix $\mathbf{X}$ is the the matrix where the rows and columns of $\mathbf{X}$ are switched.
- To find the transpose B of a matrix A, rewrite each element in B so that $b_{ji} = a_{ij}$.
- Notation: $A^T$ or $A'$
- Dimensionality: The transpose of an $n \times m$ matrix is an $m \times n$ matrix.
- All matrices have transposes.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ -1 & 6 & 2 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & -1 \\ 3 & 6 \\ 0 & 2 \end{pmatrix}$$

# Matrix and Vector Transpose Properties

| | |
|---|---|
| Inverse | $(A^T)^T = A$ |
| Additive property | $(A+B)^T = A^T + B^T$ |
| Multiplicative property | $(AB)^T = B^T A^T$ |
| Scalar multiplication | $(cA)^T = cA^T$ |

# Matrix Addition and Subtraction

- Two matrices can only be added to or subtracted from one another when they have the same dimensions.
- Simple: add (or subtract) the corresponding elements of the two matrices
- As a precondition, both matrices have to be of the same dimensions.
- If $A + B = C$, $c_{ij} = a_{ij} + b_{ij}$
- If $A - B = C$, $c_{ij} = a_{ij} - b_{ij}$

# Matrix Multiplication

- Scalar multiplication
  - Multiply each individual element of the matrix by the scalar to find the product.
  - $C = rA$, where each $c_{ij} = ra_{ij}$.
- Matrix multiplication
  - To multiply two matrices, they must be **conformable**
  - Conformability means the number of columns in the first matrix must match the number of rows in the second. The only matrices that can be multiplied, thus are $n \times m$ and $m \times k$ matrices.
  - Can a $3 \times 2$ matrix be multiplied by a $2 \times 3$ matrix? Yes
  - Can a $3 \times 2$ matrix be multiplied by a $2 \times 4$ matrix? Yes
  - Can a $3 \times 2$ matrix be multiplied by a $3 \times 3$ matrix? No
  - Always check for conformability before multiplying.
  - The dimension of the resulting matrix will be $n \times k$, meaning it will be the number of rows from the first matrix, and the number of columns from the second.

# Matrix Multiplication

- The element $n, m$ of the result matrix will be the sum of each element in column $n$ of the first matrix by each element of column $m$ of the second matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$

# Final notes on multiplication

- Can any matrix be multiplied by itself? No.

  Consider a $3 \times 4$ matrix.
- Can any matrix be multiplied by its transpose? Yes.

  If $A$ is $3 \times 4$ then $A'$ is $4 \times 3$
- Is Matrix Multiplication associative? No.

  $AB \neq BA$
- Is Matrix Multiplication associative? Yes

  $A(BC)eq(AB)C$ if the matrices are conformable
- Is Matrix Multiplication distributive? Yes:

  $A(B + C) = AB + AC$ if the matrices are conformable.

# Determinant and Inverse of a Matrix

# Determinant of a Matrix

- The **determinant** of a matrix (written $|A|$) is a commonly used function that summarizes elements of a matrix as a scalar: **a determinant will just be a number**.
- Determinants are defined only for square matrices ($n \times n$).
- For $A_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow |A| = (a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})$
- In $2 \times 2$ case, main diagonal minus off diagonal.

# Determinant of a Matrix

- For $n \times n$ matrices, the nightmare begins. **Laplace expansion** will be used.

$$B_{3\times 3} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

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- We need to calculate minors (determinants of a submatrix) and cofactors (the minor attached to an appropriate sign).

$$M_{11} = det(\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}) = (b_{22} \cdot b_{33}) - (b_{32} \cdot b_{23})$$

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- The sign of the cofactor for row $i$ and column $j$ is $-1^{(i+j)}$
- The determinant of an $n \times n$ matrix, where $n > 2$, is the sum of the products of each element and its cofactor for any row or column, by convention the first row

$$det(B) = \sum_{i=1}^{m} (-1)^{1+i} b_{1i} M_{1i}$$

# Trace of a Matrix

- The **trace** of a square matrix is the sum of the diagonal elements:
  $\sum_i^n a_{ii}$
- Trace is more or less analogous to the derivative of the determinant, but OK to not worry about this for now

# Conclusion

# Tomorrow

- Matrix Inverses
- Matrices as systems of equations
- Gradients
- Hessian, Jacobian
- Matrix Calculus and OLS