Answer Key: Exercise 3

Math Bootcamp

Instructions

- *Please show your work! You may lose points by simply writing in the answer. If the problem requires you to execute commands in `R`, please include the code you used to get your answers. Please also include the `.R` file that contains your code. If you are not sure if work needs to be shown for a particular problem, please ask.*

Question 1

1. *Find the following limits:*

   Since all of these functions are continuous at $x = a$ or $y = a$, we simply solve for $f(a)$.

   (a) $\lim_{x\to4}\left[x^2 - 6x + 4\right] = (42 - 6(4) + 4) = -4$

   (b) $\lim_{x\to0}\left[\frac{x-25}{x+5}\right] = \left(\frac{4^2}{3(4)-2}\right) = \frac{8}{5}$

   (c) $\lim_{x\to4}\left[\frac{x^2}{3x-2}\right] = \left(\frac{0-25}{0+5}\right) = -5$

   (d) $\lim_{y\to0}\left[\frac{y^4}{y-1}\right] = \lim_{y\to0}\left[\frac{(y-1)(y+1)(y^2+1)}{y-1}\right] = 4$

2. *Find the following infinite limits:*

   Since these are infinite limits, we can solve for each of these using L'Hospital's rule:

$$\lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] \text{ to } \lim_{x\to\infty}\left[\frac{f(x)'}{g(x)'}\right]$$

   (a) $\lim_{x\to\infty}\left[\frac{9x^2}{x^2+3}\right]$

$$= \lim_{x\to\infty}\left[\frac{18x}{2x}\right]$$
$$= \lim_{x\to\infty}[9] = 9$$

   (b) $\lim_{x\to\infty}\left[\frac{3x-4}{x+3}\right]$

$$= \lim_{x\to\infty}\left[\frac{3}{1}\right]$$
$$= 3$$

   (c) $\lim_{x\to\infty}\left[\frac{2^x}{2^x+1}\right]$

$$= \lim_{x\to\infty}\left[\frac{(log2)2^x}{(log2)2^x}\right]$$
$$= 1$$

3. *Calculate the following derivatives:*

   We'll compute most of these derivatives using the power rule, $\frac{d}{dx}x^n = nx^{n-1}$.

   (a) $\frac{d}{dx}3x^{\frac{1}{3}}$

$$\frac{d3x^{\frac{1}{3}}}{dx} = x^{-}\frac{2}{3} = \frac{1}{(\sqrt[3]{x})^2}$$

   (b) $\frac{d}{dy}\left(y^3 + 3y^2 - 12\right)$

$$\frac{dy^3 + 3y^2 - 12}{dy} = 3y^2 + 6y$$

   (c) $\frac{d}{dx}(x^2+1)(x^3-1)$

$$\frac{d(x^2+1)(x^3-1)}{dx} = \frac{d(x^5 - x^2 + x^3 - 1)}{dx} = 5x^4 - 2x - 3x^2$$

   (d) $\frac{d}{dy}\exp[y^2 - 3y + 2]$

$$\frac{d}{dy}e^{[y^2-3y+2]} = \left(e^{y^2-3y+2}\right)(2y-3)$$

   (e) $\frac{d}{dx}\log(2\pi x^2)$

   We need to use the chain rule for this one,

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) :$$

$$\frac{d}{dx}\log(2\pi x^2) = \frac{1}{2\pi x^2}(4\pi x) = \frac{2}{x}$$

   (f) $\frac{d}{dx}\left(\frac{1}{100}x^{25} - \frac{1}{10}x^{0.25}\right)$

$$\frac{d}{dx}\frac{1}{100}x^{25} - \frac{1}{10}x^{0.25} = \frac{1}{4}x^{24} - \frac{1}{40}x^{-\frac{3}{4}}$$

Question 2

Calculate the area of the function $f(x) = 4x2 + 12x - 18$ that lies above the x-axis and over the domain [-10, 10].

In order to find the area of the function $f(x) = (4x2 + 12x - 18)$, within the domain [10,-10], we must first find the roots of the function in this interval to know where the function lies above the x-axis:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12 \pm \sqrt{144 + 288}}{8} \\
&= -4.098076, 1.098076
\end{aligned}
$$

You can round to -4.1 and 1.1 for ease if desired. We know that this is a quadratic form open upwards, so we need to integrate outwards from these two points ignoring the interval between:

$$
\int_{-10}^{-4.1} (4x^2 + 12x - 18)dx = \frac{4}{3}x^3 + 6x^2 - 18x \bigg|_{-10}^{-4.1}
$$

$$
\begin{aligned}
&= \left(\frac{4}{3}(-4.1)^3 + 6(-4.1)^2 - 18(-4.1)\right) - \left(\frac{4}{3}(-10)^3 + 6(10)^2 - 18(10)\right) \\
&= (-118.9547) - (-1753.3330) = 1634.378
\end{aligned}
$$

$$
\int_{10}^{1.1} (4x^2 + 12x - 18)dx = \frac{4}{3}x^3 + 6x^2 - 18x \bigg|_{10}^{1.1}
$$

$$
\begin{aligned}
&= \left(\frac{4}{3}(10)^3 + 6(10)^2 - 18(10)\right) - \left(\frac{4}{3}(1.1)^3 + 6(1.1)^2 - 18(1.1)\right) \\
&= (1753.330) - (-10.76533) = 1764.098
\end{aligned}
$$

So the total area under curve above the x-axis between -10 and 10 is 1634.378 + 1764.098 = 3398.476.

Question 3

Obtain the first, second, and third derivatives of the following functions:

1. $f(x) = 5x^4 + 3x^3 - 11x^2 + x - 7$

$$
\begin{aligned}
f(x) &= 5x^4 + 3x^3 - 11x^2 + x - 7 \\
f'(x) &= 20x^3 + 9x^2 - 22x + 1 \\
f''(x) &= 60x^2 + 18x - 22 \\
f'''(x) &= 120x + 18
\end{aligned}
$$

2. $f(y) = \sqrt{y} + \frac{1}{y^{\frac{7}{2}}}$

$$\begin{aligned}
f(y) &= y^{\frac{1}{2}} + y^{-\frac{7}{2}} \\
f'(y) &= \frac{1}{2}y^{-\frac{1}{2}} - \frac{7}{2}y^{-\frac{9}{2}} \\
f''(y) &= -\frac{1}{4}y^{-\frac{3}{2}} + \frac{63}{4}y^{-\frac{11}{2}} \\
f'''(y) &= \frac{3}{8}y^{-\frac{5}{2}} - \frac{693}{8}y^{-\frac{13}{2}}
\end{aligned}$$

3. $h(u) = \log(u) + k$

$$\begin{aligned}
h(u) &= log(u) + k \\
h'(u) &= u^{-1} \\
h''(u) &= -u^{-2} \\
h'''(u) &= 2u^{-3}
\end{aligned}$$

4. $h(z) = 111z^3 - 121z$

$$\begin{aligned}
h(z) &= 111z^3 - 121z \\
h'(z) &= 333z^2 - 121 \\
h''(z) &= 666z \\
h'''(z) &= 666
\end{aligned}$$