Protected: On the Polynomial System of a Hash Function's Subfunction

by Jan Ferdinand Sauer Jan 1, 2000 0 Comment

When designing a symmetric cryptographic primitive, conventional wisdom has it that

- the "simpler" an algebraic model of the primitive, the less secure, and
- a primitive's algebraic model becomes "complex" by mixing operations from different algebras, like finite fields, or vector spaces.

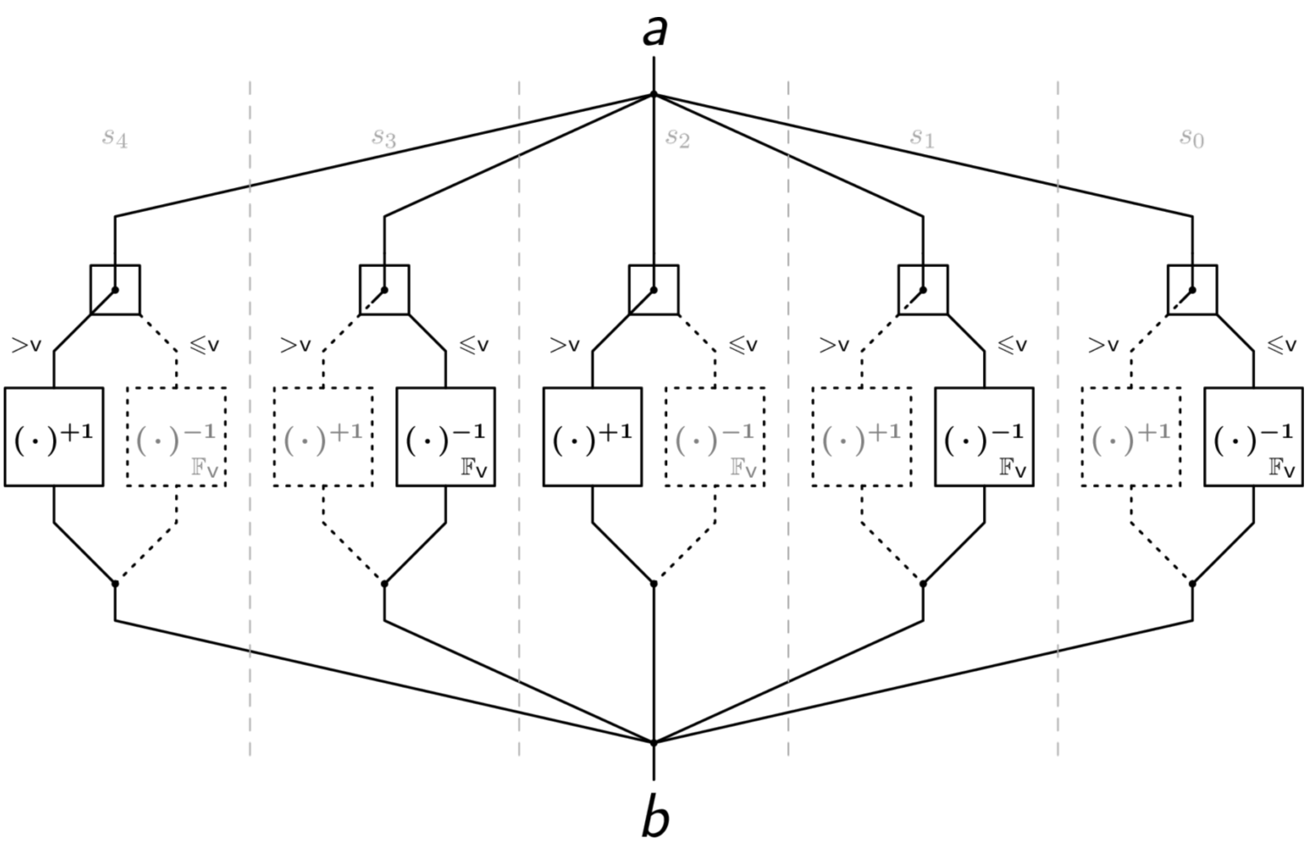
For example, take the Addition-Rotation-XOR (ARX) design principle. Adding two n-bit strings modulo 2^n is one of the simplest possible operations for a computer. However, for Gröbner basis analysis of the primitive, we need a polynomial f^+ modeling the same operation, and the polynomial's coefficients need to come from a field. Unfortunately, the field-native addition of two elements from $\mathbb{F}_{2^n}$ corresponds to XOR. Representing addition of bit strings modulo 2^n requires a polynomial of quite high degree. That's bad for algebraic cryptanalysis.

Another consequence is that ARX primitives are no suitable candidates when low arithmetic complexity is required, which includes most modern Zero-Knowledge Proof Systems. This is one of the reasons why hash functions like Rescue Prime [1,3] and Poseidon [2] have emerged: simple algebraic representability. Roughly summarized, they're designed to have polynomial representation simple enough – meaning few and low-degree polynomials – for efficient use in Zero-Knowledge Proof Systems, but complex enough to thwart the most efficient algebraic analysis methods. Most notably, all operations can be easily modeled over the same finite field, counter to the second piece of "conventional wisdom" above.

Plookup Hash's Subfunction "Bars"

The recently proposed Plookup-friendly Hash function [?] can be seen as a mix of those two paradigms: all subfunctions are either linear or can be expressed as low-degree polynomials over some finite field, but not always over the same field. The algebraically most interesting subfunction, "Bars," is the primary defense against Gröbner basis attacks. It essentially consists of the following steps:

1. Variable-base decomposition of the input
2. Conditional application of a small S-Box
3. Variable-base composition



Visualization of Plookup Hash's subfunction "Bars"

Variable-base decomposition Decomposition takes a field element and returns a tuple. Giving intuition with an example, take **a** = 318. Fix-base decomposing **a** with base 10, we can write $\mathsf{a} = 3 \cdot (10 \cdot 10) + 1 \cdot (10) + 8$, and decomposition thus results in $(3, 1, 8)$. Choosing a variable base instead, say $(s_2, s_1, s_0) = (13, 9, 12)$, we can write $\mathsf{a} = 2 \cdot (9 \cdot 12) + 8 \cdot (12) + 6$, and this variable-base decomposition thus results in $(2, 8, 6)$. More formally, given base $(s_{n-1}, \ldots, s_0)$, we can decompose like this:

$$a_0 = \mathsf{a} \bmod s_0,$$
$$a_i = \frac{\mathsf{a} - \sum_{j<i} a_j \prod_{k<j} s_k}{\prod_{j<i} s_j} \bmod s_i.$$

Conditional S-Box application A permutation is applied to each a_i that is smaller or equal to a threshold value **v**. For the Plookup Hash instance this post is concerned with, **v** is prime, and the permutation amounts to inversion of s_i as an element of finite field $\mathbb{F}_\mathsf{v}$. Picking up the example from above, let $\mathsf{v} = 7$ and $(a_2, a_1, a_0) = (2, 8, 6)$. The S-Box is applied to a_2 and a_0, but not to a_1, because it is greater than **v**. The multiplicative inverse of 2 in $\mathbb{F}_7$ is 4, that of 6 is 6. The result is $(b_2, b_1, b_0) = (4, 8, 6)$. A more formal description of this step is

$$b_i = \text{S-Box}(a_i) = \begin{cases} a_i & \text{if } a_i > \mathsf{v}, \\ \mathsf{inv}_{\mathbb{F}_\mathsf{v}}(a_i) & \text{if } a_i \leqslant \mathsf{v}. \end{cases}$$

Variable-base composition Going back from a tuple of elements to one field element, composition is the inverse of the first step. Finishing up the running example, composition gives $\mathsf{b} = 4 \cdot (9 \cdot 12) + 8 \cdot (12) + 6 = 534$. Formally, we have

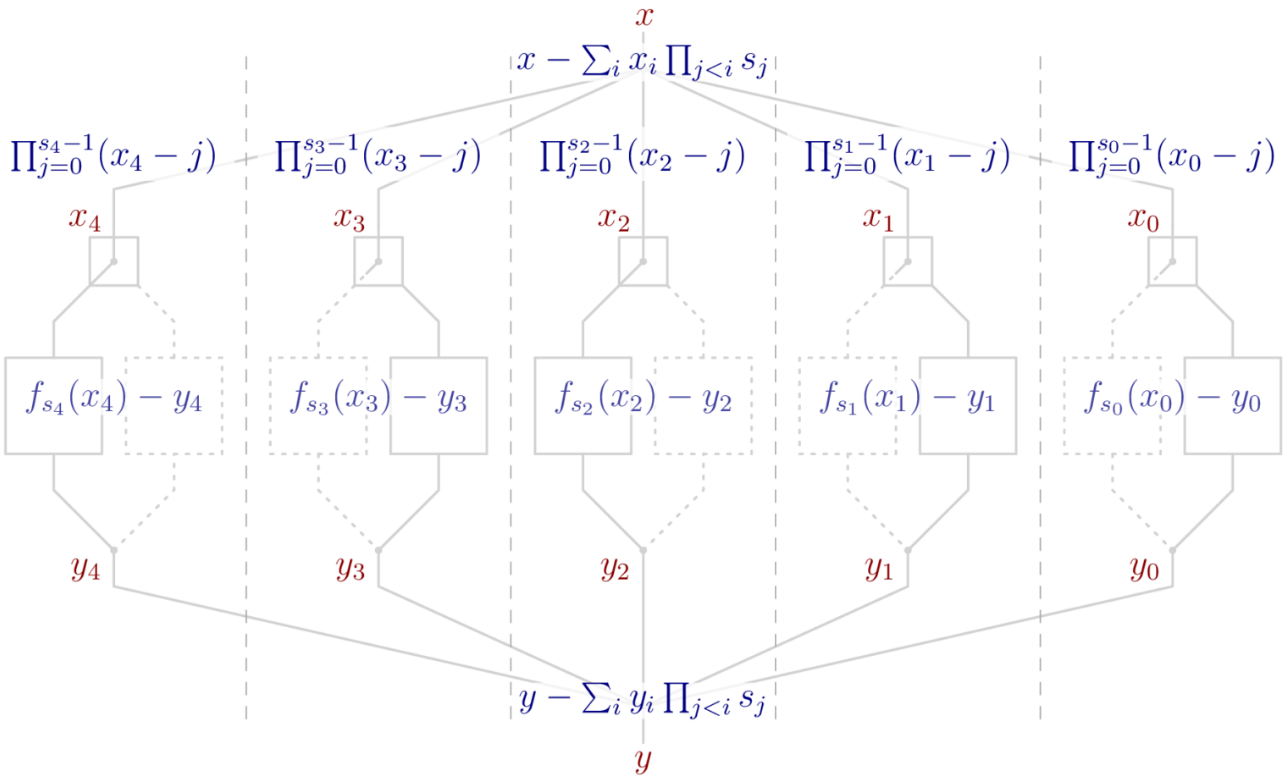
$$\mathsf{b} = \sum_i b_i \prod_{j<i} s_j.$$

A Polynomial System for “Bars”

Building a polynomial model for Bars is most straightforwardly done by modeling each of the steps. Let’s start with the decomposition of input x, where x is a variable. We introduce one new variable for each decomposition-basis element s_i. Decomposition can then be written as $x - \sum_i x_i \prod_{j<i} s_j$, which is a linear polynomial. However, this polynomial does not guarantee that x_i can only take values in the range $[0, s_i[$. We therefore need to add additional polynomial constraints that can only be satisfied if x_i is in the specified range. The polynomial we’re looking for is $\prod_{j=0}^{s_i-1}(x_i - j)$: it is zero if and only if $0 \leqslant x_i < s_i$. Summarizing, we introduce n polynomials, one of degree s_i for each element in the decomposition’s variable base.

For composition, we introduce another n many variables y_i for the output of the S-Boxes, and variable y for the output of Bars. Then, composition can be written as $y - \sum_i y_i \prod_{j<i} s_j$, much like before. The condition $0 \leqslant y_i \leqslant s_i$ will be implicitly ensured by the next and last step.

All that remains is linking the x_i with the y_i by modeling the conditional application of the S-Box. I have tried cleverly applying the Bézout identity, but the resulting polynomials were of considerably higher degree than just bluntly applying Lagrange interpolation. As a result, getting the polynomials for this step is easy but not pretty: create a list with s_i many entries $(i, \text{S-Box}(i))$ and univariably interpolate this list using variable x_i, resulting in polynomial $f_{s_i}(x_i)$. Then, $f_{s_i}(x_i) - y_i$ correctly relates variables x_i and y_i. In total, this means adding another n polynomials of degrees $s_i - 1$, respectively.



The variables (red) and polynomials (blue) modeling Bars

Computing the Gröbner Basis

Summarizing the polynomial system, we have introduced $2n+2$ variables, as well as $2n+2$ polynomials. Two of these polynomials – composition and decomposition – are linear, and for each s_i, we have one polynomial of degree s_i and one of degree $s_i – 1$.

Assuming this polynomial system is a regular sequence – and my experiments indicate this to be the case – the Macaulay bound helps with upper bounding the complexity of the Gröbner basis computation. For a set of polynomials $\{f_0, \ldots, f_{m-1}\}$, the Macaulay bound is

$$1 + \sum_i \deg(f_i) – 1.$$

Plugging in the numbers of the just built polynomial system modeling subfunction Bars, the Macaulay bound is $1 + \sum_i s_i – 1 + \sum_i s_i – 2 = 1 – 3n + 2\sum_i s_i$. The suggested parameters for the BN254 curve are $n = 27$ and each of the s_i lies in the range $[668, 703]$. This means the Macaulay bound is enormous: 37408 to be exact.

Computing the Gröbner basis for a regular sequence that has a big Macaulay bound is, generally speaking, very hard. Unfortunately, we currently only have worst-case bounds, but no way to (tightly) estimate the complexity of computing the Gröbner basis for a *concrete* polynomial system. Instead, we perform tests on toy examples and see if everything holds up.

Let's build the polynomial system for Bars based on some toy-sized parameters: $p = 47$, $v = 5$, $n = 2$, and $s_0 = s_1 = 7$. We'll have to deal with 6 variables and 6 equations, and the Macaulay bound for the resulting system is 23. This is not nothing, but it's also not very big: the polynomial system of 6 full rounds of Poseidon (with state size 2) has the same Macaulay bound, and the system for 5 rounds of Rescue (also state size 2) has comparable Macaulay bound 21. For reference, I've put the resulting polynomial system $\mathcal{F}_{(47,5)}^{[7,7]}$ below, but reading it is not necessary for the rest of this post.

Computing the associated Gröbner basis (in degrevlex order) is surprisingly fast, and the degree of the highest degree polynomial appearing in any intermediary step is 7. That's not even close to the Macaulay bound! So why is the computation so much easier than anticipated? There are some further observations to be made, for which we first need to go on a slight tangent.

Vectors of Origin

For polynomial sequence $\mathcal{F} = (f_0, \ldots, f_{m-1})$, any element g from its Gröbner basis is a (non-linear) combination of the input elements: $g = \sum_i h_i \cdot f_i$ or, using vector notation, $g = \boldsymbol{h} \cdot \mathcal{F}$. I call $\boldsymbol{h}$ the *vector of origin* for g because it tells us where g originates from in relation to $\mathcal{F}$. The set of vectors of origin describing the entire Gröbner basis is packed with information, and not all of it is easy to interpret. But something that's glaringly obvious in any vector of origin is a zero-entry. A zero in position i in vector of origin $\boldsymbol{h}$ means that f_i was unnecessary for computing g.

I've used the tool VooDoo to compute the vectors of origin for the polynomial system $\mathcal{F}_{(47,5)}^{[7,7]}$. For clarity of reading, any big polynomial is replaced by ●. Here's what the vectors look like:

$$\begin{gathered}(1,0,0,0,0,0),\\(0,\bullet,\bullet,0,0,0),\\(0,\bullet,\bullet,0,0,0),\\(0,\bullet,\bullet,0,0,0),\\(0,0,0,\bullet,\bullet,0),\\(0,0,0,\bullet,\bullet,0),\\(0,0,0,\bullet,\bullet,0),\\(0,0,0,0,0,1)\end{gathered}$$

See the pattern? It appears that computing the Gröbner basis for $\mathcal{F}_{(47,5)}^{[7,7]}$ can be reduced to computing on multiple smaller systems. After all, the polynomials describing one S-Box are pretty much completely independent from the polynomials describing another S-Box. Dividing and conquering is a strategy that is usually impossible for computing a Gröbner basis, but in this specific case, it's back on the table: the Macaulay bound for the sub-system describing one S-Box is $1 + s_i – 1 + s_i – 2$, amounting to 12 in our toy example. This doesn't fully explain the discrepancy to the observed highest degree of 7 yet, suggesting that more interesting riddles are waiting to be solved.

Above analysis doesn't mean that using subfunction Bars is unsuitable for preventing Gröbner basis attacks. However, looking at the entire polynomial system and relying on the complexity upper bound for the Gröbner basis computation implied by the Macaulay bound is probably too optimistic.

References

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2. Grassi, L., Khovratovich, D., Rechberger, C., Roy, A., Schofnegger, M.: Poseidon: A New Hash Function for Zero-Knowledge Proof Systems. In: USENIX Security. USENIX Association (2020)

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Appendix: The Polynomial System

Below, you can find the polynomial system for the toy-sized parameters from above. Note how the polynomials corresponding to the S-Boxes are identical, which is due to $s_0 = s_1$ and does not generally hold. The same is true for the polynomials ensuring $0 \leqslant x_i < s_i$. However, the similarities between the decomposition polynomial and the composition polynomial is inherent.

$$\mathcal{F}_{(47,5)}^{[7,7]} = \begin{cases} x - 7x_1 - x_0 & \text{decomposition} \\ x_0^7 - 21x_0^6 - 13x_0^5 + 17x_0^4 - 21x_0^3 + 22x_0^2 + 15x_0 & 0 \leqslant x_0 < s_0 \\ 18x_0^6 + 4x_0^5 - 15x_0^4 - 17x_0^3 - 15x_0^2 - 21x_0 - y_0 & y_0 = \text{S-Box}(x_0) \\ x_1^7 - 21x_1^6 - 13x_1^5 + 17x_1^4 - 21x_1^3 + 22x_1^2 + 15x_1 & 0 \leqslant x_1 < s_1 \\ 18x_1^6 + 4x_1^5 - 15x_1^4 - 17x_1^3 - 15x_1^2 - 21x_1 - y_1 & y_1 = \text{S-Box}(x_1) \\ y - 7y_1 - y_0 & \text{composition} \end{cases}$$

The associated reduced Gröbner basis reads as follows.

$$\mathcal{G}_{(47,5)}^{[7,7]} = \begin{cases} x - 7x_1 - x_0 & \text{decomposition} \\ \left.\begin{array}{l} x_0^2 - y_0^2 - 5x_0 + 5y_0 \\ x_0y_0^2 - y_0^3 - 5x_0y_0 + 5y_0^2 + 6x_0 - 6y_0 \\ y_0^5 - 16y_0^4 - 5y_0^3 + 10x_0y_0 - 16y_0^2 - 19x_0 - 2y_0 \end{array}\right\} & \text{S-Box } s_0 \\ \left.\begin{array}{l} x_1^2 - y_1^2 - 5x_1 + 5y_1 \\ x_1y_1^2 - y_1^3 - 5x_1y_1 + 5y_1^2 + 6x_1 - 6y_1 \\ y_1^5 - 16y_1^4 - 5y_1^3 + 10x_1y_1 - 16y_1^2 - 19x_1 - 2y_1 \end{array}\right\} & \text{S-Box } s_1 \\ y - 7y_1 - y_0 & \text{composition} \end{cases}$$

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