

1 Description Of A Task

Let's consider a following task:

$$\min_{(x,y)} \{f(x,y) | (x,y) \in Q\},$$

where f is a convex function, Q - is a square on the plane.

Let's consider a following method. One solves task of minimization for a function $g(x) = f(x, y_0 = \frac{a}{2})$ on a segment $[0, a]$ with an accuracy δ on function. After that one calculates a sub-gradient in a received point and chooses the rectangle which the sub-gradient "does not look" in. Similar actions are repeated for a vertical segment. As a result we have the square decreased twice. Let's find a possible value of error δ_0 for task on segment and a sufficient iteration's number N to solve the initial task with accuracy ϵ on function.

Let's describe an algorithm formally. See pseudo-code [1](#).

Algorithm 1 Algorithm of the method

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1: function METHOD(convex function  $f$ , square  $Q = [a, b] \times [c, d]$ )
     $x_0 := solve(g = f(\cdot, \frac{c+d}{2}), [a, b], \delta)$ 
     $g = subgradient(f, (x_0, \frac{c+d}{2}))$ 
2:   if  $g[1] > 0$  then
3:      $Q := [a, b] \times [c, \frac{c+d}{2}]$ 
4:   else
5:      $Q := [a, b] \times [\frac{c+d}{2}, d]$ 
6:   end if
     $y_0 := solve(g = f(\frac{a+b}{2}, \cdot), [c, d], \delta)$ 
     $g := subgradient(f, (\frac{a+b}{2}, y_0))$ 
7:   if  $g[0] > 0$  then
8:      $Q := [a, \frac{a+b}{2}] \times [c, d]$ 
9:   else
10:     $Q := [\frac{a+b}{2}, b] \times [c, d]$ 
11:   end if
12:   if StopRec() == False then
13:     Method( $f$ ,  $Q$ )
14:   end if return  $(\frac{a+b}{2}, \frac{c+d}{2})$ 
15: end function

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2 Algorithm correctness

Let's \mathbf{x}_0 is solution of the task on segment, Q_1 is choosed rectangle, Q_2 is not choosed rectangle.

2.1 Zero Error

Lemma 1. *If the optimization task on segment is solved with zero error and the f is convex and differentiable at a point-solution, rectangle with solution of initial task was choosed correct.*

Proof. From sub-gradient definition, $\mathbf{x}^* \in \{x | (\mathbf{g}(\mathbf{x}_0), \mathbf{x}_0 - \mathbf{x}^*) \geq 0\}$. Lemma's statement follows from it and a fact that the first (or the second for vertical segment) gradient's component in point \mathbf{x}_0 is zero. \square

The method does not work for all convex functions even for zero error on segment. Let's consider following example.

Example 1.

$$f(x, y) = \max\{x - 2y, y - 2x\}, Q = [-1, 1]^2 \quad (1)$$

Function f is convex as maximum of affine functions on x and y . A solution of task on horizontal segment $[-1, 1] \times \{0\}$ is point $(0, 0)$. Its sub-differential is

$$\partial f(0, 0) = \text{conv} \{(1, -2)^\top, (-2, 1)^\top\}.$$

So if one takes a subgradient $(1, -2)^\top$ then a bottom rectangle will be choosed. But optimal value is point $(1, 1)$ and there is not it in choosed rectangle. Therefore, this method cannot give a solution of initial task with error less than $\frac{1}{2}$.

2.2 Nonzero Error

Theorem 2.1. *Let's the f has continuous derivative on the square. Then there is a neighbourhood of a solution of optimization task on segment such as a choice of rectangle will not change if one use any point from the neighbourhood.*

Proof. Let's consider a case when we work with horizontal segment. Case with vertical segment is considered analogously. Then we are interesting in $f'_y(x_0, y_0)$. If \mathbf{x}_0 is not solution of initial task, then $f'_y(x_0, y_0) \neq 0$.

From a continuity of the derivative:

$$\lim_{\delta \rightarrow 0} f'_y(x_0 + \delta, y_0) = f'_y(x_0, y_0)$$

Therefore,

$$\exists \delta_0 : \forall \mathbf{x}_\delta \in B_{\delta_0}(x_0) \times y_0 \Rightarrow \text{sign}(f'_y(\mathbf{x}_\delta)) = \text{sign}(f'_y(\mathbf{x}_0))$$

From it and lemma 1 theorem's statement follows. □

3 Error's Value

Let's find possible value of error's value δ_0 . Rectangles are defined correctly for a horizontal optimization task, if:

$$\forall \delta : |\delta| < \delta_0 \Rightarrow f'_y(\mathbf{x}_0) f'_y(x_0 + \delta, y_0) > 0 \quad (2)$$

Analogically, for a vertical segment:

$$\forall \delta : |\delta| < \delta_0 \Rightarrow f'_x(\mathbf{x}_0) f'_x(x_0, y_0 + \delta) > 0 \quad (3)$$

Theorem 3.1. *Let's function f is convex and differentiable and current rectangle is $[a, b] \times [c, d]$.*

For horizontal segment: *There is f''_{xy} on the segment. Rectangle is defined in point $(x_0 + \delta, y_0)$ correctly if one meet following condition:*

$$\delta_0 < \frac{|f'_y(\mathbf{x}_0)|}{\max_{t \in [a, b]} |f''_{xy}(t, y_0)|} \quad (4)$$

For vertical segment: *There is f''_{yx} on the segment. Rectangle is defined in point $(x_0, y_0 + \delta)$ correctly if one meet following condition:*

$$\delta_0 < \frac{|f'_x(\mathbf{x}_0)|}{\max_{t \in [c, d]} |f''_{yx}(x_0, t)|} \quad (5)$$

Proof. Let's prove this statement for horizontal segment.

Rewrite the condition (1) using Taylor formula:

$$\forall \delta : |\delta| \leq \delta_0 \Rightarrow f'_y(\mathbf{x}_0) (f'_y(\mathbf{x}_0) + f''_{xy}(\mathbf{x}_0 + (\theta\delta, 0)^\top) \delta) > 0,$$

where $\theta \in (0, 1)$

Using the written above inequality we have a following inequality for δ_0 :

$$\delta_0 < \frac{|f'_y(\mathbf{x}_0)|}{\max_{\theta \in [-1, 1]} |f''_{xy}(x_0 + \theta\delta_0, y_0)|}$$

It and an obvious inequality $\max_{\theta \in [-1, 1]} |f''_{xy}(x_0 + \theta\delta_0, y_0)| < \max_{t \in [a, b]} |f''_{xy}(t, y_0)|$ proves (3). Inequality (4) are proved similar. \square

Theorem 3.2. *Let's function f is convex and has L -Lipschitz continuous gradient.*

And on horizontal segment $\exists M_1 : \forall \mathbf{x} \Rightarrow |f'_y(\mathbf{x})| > M_1$. Then rectangle is defined correctly if the possible value of error is not more $\frac{M_1}{L}$.

And on vertical segment $\exists M_2 : \forall \mathbf{x} \Rightarrow |f'_x(\mathbf{x})| > M_2$. Then rectangle is defined correctly if the possible value of error is not more $\frac{M_2}{L}$.

Proof. Condition (1) is met if there is a derivative $f'_y(x_0 + \delta, y_0)$ in a neighbourhood of $f'_y(\mathbf{x}_0)$ with radius $|f'_y(\mathbf{x}_0)|$:

$$|f'_y(\mathbf{x}_0) - f'_y(x_0 + \delta, y_0)| < |f'_y(\mathbf{x}_0)|$$

The L -Lipschitz continuity gives following inequality:

$$|f'_y(\mathbf{x}_0) - f'_y(x_0 + \delta, y_0)| \leq L|\delta|$$

Theorem's estimate for vertical segment follows from two written above inequality. Inequality for vertical segment is proved similarly. \square

4 Number of iterations

Following estimates are correct if each iterations was correct (a rectangle is selected correctly on each iterations).

Theorem 4.1. *If function f is convex and L -Lipschitz continuous, then for to solve initial task with accuracy ϵ on function one has to make following iteration's number:*

$$N = \left\lfloor \log_2 \frac{La}{\sqrt{2}\epsilon} + 1 \right\rfloor,$$

where a is a size of the initial square Q

Proof.

$$|f(\mathbf{x}^*) - f(\mathbf{x})| < L|\mathbf{x}^* - \mathbf{x}|$$

After N iterations we have a square with size $\frac{a}{2^N}$. That's why if we choose a squares center as proximal solution we have following estimates:

$$|\mathbf{x}^* - \mathbf{x}| \leq \frac{a}{\sqrt{2}} 2^{-N}$$

$$|f(\mathbf{x}^*) - f(\mathbf{x})| < La2^{-N}$$

Therefore, for accuracy epsilon following number of iterations is sufficient:

$$N > \log_2 \frac{La}{\sqrt{2}\epsilon}$$

□

But any convex function is locally Lipschitz continuous at all $x \in \text{int } Q$. Therefore, we have following theorem.

Theorem 4.2. *If function f is convex and a solution $x^* \in \text{int } Q$, then for to solve initial task with accuracy ϵ on function one has to make following iteration's number:*

$$N = \left\lfloor \log_2 \max \left\{ \frac{a}{\epsilon_0(x^*)}, \log \frac{La}{\sqrt{2}\epsilon} \right\} + 1 \right\rfloor,$$

where a is a size of the initial square Q , $\epsilon_0(x^*)$ is size of neighbourhood of x^* which f is L -Lipschitz continuous in.

If f is Lipschitz continuous then there is following inequality:

$$|f(x_1) - f(x_2)| < L|\mathbf{x}_1 - \mathbf{x}_2|$$

In particular, if x_0 is a squares center and \mathbf{x}^* is a solution then

$$\Delta f = f(x_0) - f(x^*) < L|\mathbf{x}_0 - \mathbf{x}^*| \leq L\frac{a}{2}$$

Using this inequality and written above theorems 4.1 and 4.2 we have following results:

Theorem 4.3. *If function f is convex and L -Lipschitz continuous, then for to solve initial task with accuracy ϵ on function one has to make following iteration's number:*

$$N = \left\lceil \log_2 \frac{\Delta f}{\epsilon} + 1 \right\rceil,$$

where a is a size of the initial square Q

Theorem 4.4. *If function f is convex and a solution $x^* \in \text{int } Q$, then for to solve initial task with accuracy ϵ on function one has to make following iteration's number:*

$$N = \left\lceil \log_2 \max \left\{ \frac{a}{\epsilon_0(\mathbf{x}^*)}, \log \frac{\Delta f}{\epsilon} \right\} + 1 \right\rceil,$$

where a is a size of the initial square Q , $\epsilon_0(x^*)$ is size of neighbourhood of x^* which f is L -Lipschitz continuous in.