One Method for Convex Optimization on Square

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Method Description

Task:

$$\min_{(x,y)} \{f(x,y) | (x,y) \in Q\},$$

where f is a convex function, $Q = [a,b] \times [c,d] \in \mathbb{R}^2$.

One iteration

- Find minimum with accuracy δ on central horizontal segment in square and calculate gradient at this point

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- Find minimum with accuracy δ on central horizontal segment in square and calculate gradient at this point
- Choose rectangle which anti-gradient looks in
- Analogically for vertical segment in the rectangle

Plan

1. Strategy for segment accuracy
2. Convergence
3. Convergence
4. Tests

Strategies

True Gradient

Let f be convex and has L-Lipschitz continuous gradient.

$$\operatorname{sign} f_y'(x_0) = \operatorname{sign} f_y'(x_{current})$$

For this it is sufficient:

$$|f_y'(x_0) - f_y'(x_{current})| \leq |f_y'(x_0)|$$

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True Gradient

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For this it is sufficient:

$$|f'_y(x_0) - f'_y(x_{current})| \leq |f'_y(x_0)|$$

$$\boxed{\delta < \frac{|f'_y(x_0)|}{L}}$$

Strategies
Current Gradient

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For this it is sufficient:

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Strategies
Small Gradient

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Small Gradient

Let f be convex and has L-Lipschitz continuous gradient. Then for accuracy on function ϵ following condition in point $\mathbf{x}$ is sufficient:

$$\|\nabla f(\mathbf{x})\| \leq \frac{\epsilon}{a\sqrt{2}},$$

where a is size of current square.

Convergence

Function f is convex. Size of square Q is equal to a. One takes a center of a current square as approximate solution.

Estimate through Lipschitz function constant

Let function f be L_f-Lipschitz continuous. Then for accuracy ϵ on function it is sufficient:

$$N = \left\lceil \log_2 \frac{L_f a}{\sqrt{2}\epsilon} \right\rceil . \quad (1)$$

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Estimate through Lipschitz gradient constant

Let function f have L_g-Lipschitz continuous gradient. Moreover, point with zero derivative is **internal point**. Then for accuracy ϵ on function it is sufficient:

$$N = \left\lceil \frac{1}{2} \log_2 \frac{L_g a^2}{4\epsilon} \right\rceil . \quad (2)$$

Convergence

If following inequallity is met

$$\frac{2L_f^2}{L_g} \geq \epsilon$$

then the estimate through L_g is better than through L_f.

- Quadratic Function

$$f(x,y) = (Ax + By)^2 + Cy^2 + Dx + Ey$$

$$(x^*, y^*) \in Q$$

$$L_f = \max_{(x,y)\in Q} \|\nabla f(x,y)\|$$

$$L_g = \max |\lambda\left(H(f)\right)|$$

Experiments

Iterations Number

Theoretical Iteration Number through function constant equals 40
Theoretical Iteration Number through gradient constant equals 20

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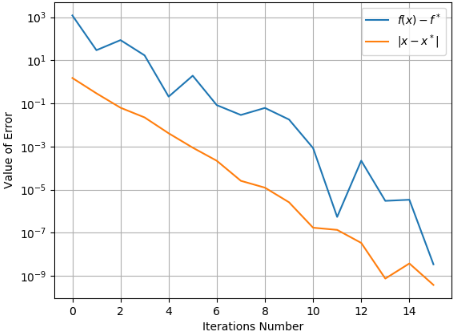

$$f(x, y) = (x + y)^2 + x^2, Q = [1, 2]^2, (x^*, y^*) = (1, 1)$$

Theoretical Iteration Number through function constant 30
Theoretical Iteration Number through gradient constant 14

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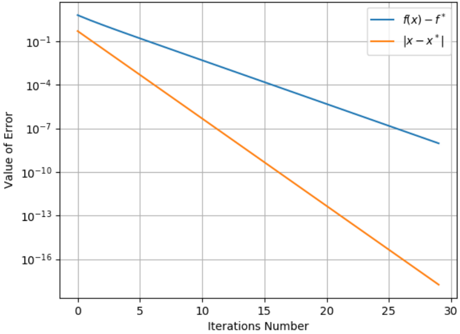

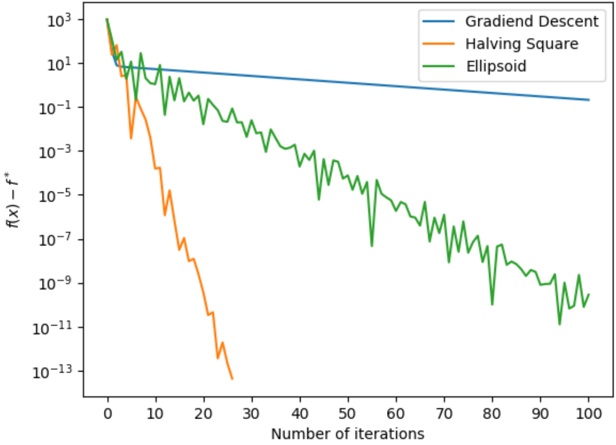

Gradiend Descent
Halving Square
Ellipsoid
$f(x) - f^*$
10^{3}
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{-13}
0 10 20 30 40 50 60 70 80 90 100
Number of iterations

Work time

Gradiend Descent 4.5 ms
Halving Square 20.9 ms
Ellipsoid 28.3 ms

Summary

- Strategy for solution on segment
- Convergence Results
- Experiments for this method
  - Iterations number
  - Comparison with different methods