1 Tests functions

1.1 Sinuses

Functions $-\sin\frac{\pi x}{a}$ and $-\sin\frac{\pi x}{b}$ are convex on square $Q=[0,1]^2$ when $a,b\geq 1$. Therefore, a function $-A\sin\frac{\pi x}{a}-B\sin\frac{\pi y}{b}$ is convex for all $A,B\geq 0$ as cone combination of convex function.

Functions x^n are convex and monotonously non-decreasing on $[0,1]$ for all $n\in\mathbb{N}$ that's why functions $\left(-A\sin\frac{\pi x}{a}-B\sin\frac{\pi y}{b}+A+B+D\right)^n$ are convex for all $D\geq 0$.

Therefore, following function is convex:

$$f(x,y)=-A_1\sin\frac{\pi x}{a_1}-B_1\sin\frac{\pi x}{b_1}+\sum_{n=2}^{N}\left(-A_n\sin\frac{\pi x}{a_n}-B_n\sin\frac{\pi y}{b_n}+A_n+B_n+D_n\right)^n,$$

where $A_i,B_i.D_i\geq 0$ and $a_i,b_i\geq 1$ for all $i=\overline{1,n}$.

The function f is differentiable infinite times and we can use it to test the method.

Let's take $a_1=\cdots=a_n=a$ and $b_1=\cdots=b_n=b$:

$$f(x,y)=-A_1\sin\frac{\pi x}{a}-B_1\sin\frac{\pi x}{b}+\sum_{n=2}^{N}\left(-A_n\sin\frac{\pi x}{a}-B_n\sin\frac{\pi y}{b}+A_n+B_n+D_n\right)^n,$$

where $A_i,B_i.D_i\geq 0$ for all $i=\overline{1,n}$ and $a,b\geq 1$.

Then functions derivative is:

$$f'_x(x,y)=\left(-A_1-\sum_{n=2}^{N}nA_n\left(-A_n\sin\frac{\pi x}{a}-B_n\sin\frac{\pi y}{b}+A_n+B_n+D_n\right)^{n-1}\right)\cdot$$
$$\cdot\frac{\pi}{a}\cos\frac{\pi x}{a}$$

$$f'_y(x,y)=\frac{\pi}{b}\left(-B_1-\sum_{n=2}^{N}nB_n\left(-A_n\sin\frac{\pi x}{a}-B_n\sin\frac{\pi y}{b}+A_n+B_n+D_n\right)^{n-1}\right)\cdot$$

$$\cdot \frac{\pi}{b} \cos \frac{\pi y}{b}$$

$$f''_{xy}(x,y) = \left(\sum_{n=2}^{N} n(n-1) A_n B_n \left(-A_n \sin \frac{\pi x}{a} - B_n \sin \frac{\pi y}{b} + A_n + B_n + D_n \right)^{n-2} \right) \cdot$$

$$\cdot \frac{\pi^2}{ab} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}$$

Using written above expressions we can give estimates for derivatives:

$$|f'_x|\Big|_{x=x_0} \geq \left(A_1 + \sum_{n=2}^{N} n A_n D_n^{n-1} \right) \frac{\pi}{a} \left| \cos \frac{\pi x_0}{a} \right|$$

$$|f'_x|\Big|_{y=y_0} \geq \left(B_1 + \sum_{n=2}^{N} n B_n D_n^{n-1} \right) \frac{\pi}{b} \left| \cos \frac{\pi y_0}{b} \right|$$

$$|f''_{xy}| \leq \frac{\pi^2}{ab} \left(\sum_{n=2}^{N} n(n-1) A_n B_n \left(A_n + B_n + D_n \right)^{n-2} \right)$$

Also we know solution of this task:

$$x^* = \begin{cases} 1, \text{if } a \geq 2, \\ \frac{a}{2}, \text{else} \end{cases} \tag{1}$$

$$y^* = \begin{cases} 1, \text{if } b \geq 2, \\ \frac{b}{2}, \text{else} \end{cases} \tag{2}$$

and task's value:

$$f^* = f(x^*, y^*).$$

We will use a and b from $[1, 2]$ and $N = 2$. That's why we can find task's value easy:

$$f^* = -A_1 - B_1 + \sum_{n=2}^{N} D_n^n$$

Also we will use $N = 2$.

1.2 Lipschitz continuous gradient

Let's consider following function:

$$f(x) = x^2 \cdot \begin{cases} \frac{3}{2}, \text{if } x < 0, \\ 1, \text{if } x \geq 0 \end{cases} \tag{3}$$

Function $f(x)$ is convex and has a Lipschitz continuous derivative but it is not twice differentiable at zero:

$$f'(x) = x \cdot \begin{cases} 3, \text{if } x < 0, \\ 2, \text{if } x \geq 0 \end{cases} \tag{4}$$

Let's find L for derivative:

$$\frac{|f'(x_1) - f'(x_2)|}{|x_1 - x_2|} = \begin{cases} 3, \text{if } x_1, x_2 < 0, \\ 2, \text{if } x_1, x_2 \geq 0 \end{cases} \leq 3 \tag{5}$$

Let $x_1 < 0, x_2 \geq 0$:

$$\frac{|f'(x_1) - f'(x_2)|}{|x_1 - x_2|} = \frac{|3x_1 - 2x_2|}{|x_1 - x_2|} \leq \frac{|x_1|}{|x_1 - x_2|} + 2 \leq 3$$

As a result, we have:

$$L = 3$$

Let's consider following function:

$$g(x, y) = af(x) + bf(y) + \phi(x, y),$$

where $a, b \geq 0$ - constants, ϕ is a convex function with a Lipschitz continuous derivative. Therefore, g has a Lipschitz continuous derivative with a Lipschitz constant $3(a + b) + L_\phi$.

Let's consider following functions:

$$g(x, y) = af(x) + bf(y) + (Ax + By)^2,$$

where A, B and $a, b \geq 0$ are constants.

Then g is convex and has a Lipschitz continuous derivative with a constant $3(a + b) + 2(|A_1| + |B_1|)^2 + |A_2| + |B_2|$ but it is not twice differentiable at zero.

$$g'_x(x,y) = \psi(x)x + 2A_1(A_1x + B_1y),$$
$$g'_y(x,y) = \psi(y)y + 2B_1(A_1x + B_1y),$$

where

$$\psi(x) = \begin{cases} 3, \text{if } x < 0, \\ 2, \text{if } x \geq 0 \end{cases} \tag{6}$$

Then a point (0,0) is optimal. Also it is only optimal point.

Let's consider minimization task for g on square $Q = [x_1, x_2] \times [y_1, y_2]$ such as $0 \in Q$. Estimates for derivatives:

$$|g'_x(x,y)|\Big|_{x=x_0} = |(\psi(x_0) + 2A_1^2)x_0 + 2A_1B_1y| \geq$$

$$\geq \begin{cases} 0, \text{if } -\frac{(\psi(x_0)+A_1^2)x_0}{2A_1B_1} \in [y_1, y_2] \ , \\ |(\psi(x_0) + 2A_1^2)x_0 + 2A_1B_1y_2|, \text{if } |(\psi(x_0) + 2A_1^2)x_0 + 2A_1B_1y| < 0, \\ |(\psi(x_0) + 2A_1^2)x_0 + 2A_1B_1y_1|, \text{if } |(\psi(x_0) + 2A_1^2)x_0 + 2A_1B_1y| > 0, \end{cases} \tag{7}$$

For g'_y we have similar estimate:

$$|g'_y(x,y)|\Big|_{y=y_0} \geq$$

$$\geq \begin{cases} 0, \text{if } -\frac{(\psi(y_0)+2B_1^2)y_0}{2A_1B_1} \in [x_1, x_2] \ , \\ |(\psi(y_0) + 2B_1^2)y_0 + 2A_1B_1x_2|, \text{if } |(\psi(y_0) + 2B_1^2)y_0 + 2A_1B_1x| < 0, \\ |(\psi(y_0) + 2B_1^2)y_0 + 2A_1B_1x_1|, \text{if } |(\psi(y_0) + 2B_1^2)y_0 + 2A_1B_1x| > 0, \end{cases} \tag{8}$$