Mathematical Finance Assignment-1 Solutions

Questions:

1. Suppose you are tossing a coin until you hit heads. Further, suppose the probability that head turns up is p and X is a random variable counts the number of tosses. (If you get heads after 10 throws, then number of tosses is 10).

   Find the probability distribution of X and prove that for any integers $m, n \geq 0$
   $$\mathbb{P}[X > m + n \mid X > n] = \mathbb{P}[X > m].$$

   Solution: Let's find the probability $\mathbb{P}[X = k]$ where $k = 1, 2, \ldots$. If it took us k tosses to get a head, we must have tossed $k - 1$ tails before this. Thus, the possibility of this happening is
   $$\mathbb{P}[X = k] = p(1-p)^{k-1}, \tag{1}$$
   which is our probability distribution. Further, let's calculate $\mathbb{P}[X > k]$ for $k = 0, 1, \ldots$. This can be computed using the geometric sum of the series
   $$\begin{aligned}\mathbb{P}[X > k] &= \mathbb{P}[X = k+1] + \mathbb{P}[X = k+2] + \ldots \\ &= p(1-p)^k + p(1-p)^{k+1} + \ldots \\ &= \frac{p(1-p)^k}{1-(1-p)} = (1-p)^k. \end{aligned} \tag{2}$$
   Using equation 2, we can directly compute
   $$\begin{aligned}\mathbb{P}[X > m + n \mid X > n] &= \frac{\mathbb{P}[X > m+n]}{\mathbb{P}[X > n]} \\ &= \frac{(1-p)^{m+n}}{(1-p)^n} = (1-p)^m \\ &= \mathbb{P}[X > m].\end{aligned}$$
   Hence, proved.

2. Let X be a Poisson random variable with parameter λ. Calculate the mean and variance of X.
   Solution: Let X be a Poisson random variable with parameter λ, the probability mass function (pmf) of X is given by
   $$\mathbb{P}[X = k] = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots.$$

Then, the mean $\mathbb{E}[X]$ is given by

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=0}^{\infty} k\mathbb{P}[X=k] \\ &= 0 + \sum_{k=1}^{\infty} k\mathbb{P}[X=k] \\ &= \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!} \\ &= \lambda \sum_{k=1}^{\infty} = \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} \\ &= \lambda. \end{aligned} \tag{3}$$

And the variance $\mathrm{Var}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$ can be computed by computing $\mathbb{E}\left[X^2\right]$ as follows

$$\begin{aligned}\mathbb{E}\left[X^2\right] &= \sum_{k=0}^{\infty} k^2\mathbb{P}[X=k] \\ &= 0 + \sum_{k=0}^{\infty} k^2\mathbb{P}[X=k] \\ &= \sum_{k=1}^{\infty} k\frac{e^{-\lambda}\lambda^k}{(k-1)!} \\ &= \lambda\left[\sum_{k=0}^{\infty}(k+1)\frac{e^{-\lambda}\lambda^k}{(k)!}\right] \\ &= \lambda\left[\sum_{k=0}^{\infty} k\frac{e^{-\lambda}\lambda^k}{(k)!} + \sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^k}{(k)!}\right] \\ &= \lambda[\lambda+1] = \lambda^2 + \lambda. \end{aligned} \tag{4}$$

Since $\mathbb{E}[X] = \lambda$, $\mathrm{Var}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = \lambda$ as well.

3.* Let X be a normal random variable with mean μ and variance σ^2, show that $X = \sigma Z + \mu$ where Z is the standard normal random variable (mean 0 and variance 1).
Solution: In statistics, we say that two random variables X and Y are 'equal' if the distribution of X and Y are the same. (If $X = Y$, it is not necessary that X and Y be exactly the same random variable. Think about it!)

Let's denote $\sigma Z + \mu$ by Y for ease of notation. Now, according to the discussion above, we need to show that X and Y have the same distribution.

We know that the distribution of X is given by

$$\mathbb{P}\left[X \leq x\right] = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt.$$

Now, we need to compute the distribution of Y,

$$\mathbb{P}\left[Y \leq y\right] = \mathbb{P}\left[\sigma Z + \mu \leq y\right] = \mathbb{P}\left[Z \leq \frac{y-\mu}{\sigma}\right].$$

Now, we know the distribution of the standard normal distribution (You should know this by heart!) and so we can complete the calculation as follows

$$\begin{aligned}\mathbb{P}\left[Y \leq y\right] = \mathbb{P}\left[Z \leq \frac{y-\mu}{\sigma}\right] &= \int_{-\infty}^{\frac{y-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\ &= \int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du. \quad \{u = \sigma t + \mu\}\end{aligned}$$

Clearly, the distribution of X and Y are the same, and hence, $X = Y$.

The other way to solve this question is to show that X and Y have the same densities which is essentially the same approach but requires a more rigorous proof so I have not presented it here.

This property has very useful consequences, say $X1$ and X_2 are two normal random variables with mean μ_1, μ_2 and variances σ_1^2, σ_2^2. Using the above property, we can show that the sum $X = X_1 + X_2$ of these random variables is also a normal random variable with mean $\mu_1 + \mu_2$ with variance $(\sigma_1 + \sigma_2)^2$.