

LOWER BOUND ON GROWTH OF NON-ELEMENTARY SUBGROUPS IN RELATIVELY HYPERBOLIC GROUPS

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ABSTRACT. This paper proves that in a non-elementary relatively hyperbolic group, the logarithm growth rate of any non-elementary subgroup has a linear lower bound by the logarithm of the size of the corresponding generating set. As a consequence, any non-elementary subgroup has uniform exponential growth.

1. INTRODUCTION

1.1. Results and background. Let S be a finite symmetric generating set of a group H and d_S the corresponding word metric. Denote

$$\forall n \in \mathbb{N} \cup \{0\}, S^{\leq n} := \{h \in H : d_S(1, h) \leq n\}.$$

The (logarithm) *growth rate* of H with respect to S is defined to be the following limit

$$\omega(H, S) := \lim_{n \rightarrow \infty} \frac{\log \#(S^{\leq n})}{n}$$

which exists since $\#(S^{\leq n+m}) \leq \#(S^{\leq n}) \cdot \#(S^{\leq m})$. In what follows, we always consider finitely generated groups.

The spectrum of growth rates of a group H has attracted lots of research interests:

$$\Omega(H) := \{\omega(H, S) : \#S < \infty, \langle S \rangle = H\}.$$

For a group with exponential growth, the question of Gromov [15] whether $\Omega(H)$ admits the infimum 0 was open for twenty years and answered negatively by Wilson [25] (see [2] also). He constructed the first examples of groups with non-uniform exponential growth so that a sequence of two-element generating sets with growth rates tending to 0.

A group H has *uniform exponential growth* if $\inf \Omega(H) > 0$. If a group has uniform exponential growth, it is quite interesting to ask whether $\Omega(H)$ obtains the minimum. Sambusetti [24] showed that the answer was again negative for the free products of any two non-Hopfian groups which are a special class of relatively hyperbolic groups. However, the recent work by Fujiwara-Sela [11] obtains a positive answer for hyperbolic groups by showing the set $\Omega(H)$ is well-ordered, so $\Omega(H)$ admits a minimum. This settles a question of de la Harpe. The starting point of

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their arguments relies on the fact due to Arzhantseva-Lysenok [1] that the growth rate $\omega(H, S)$ is lower bounded by a linear function of the size of S .

Noting the simple fact $\omega(H, S) \leq \log \#S$, the work of [11] and [1] seem to suggest worthy understanding the following set

$$\Theta(H) := \left\{ \frac{\omega(H, S)}{\log \#S} : \#S < \infty, \langle S \rangle = H \right\}.$$

Of course, $\Theta(H) \subset [0, 1]$. A number of inquiries could be made about the nature of $\Theta(H)$. For instance, could the set $\Theta(H)$ always be infinite? If it is infinite, what are the accumulation points of the set $\Theta(H)$? The purpose of this paper is not to give complete answers to these questions. Instead, we collect here a few simple observations to motivate further investigations.

A group H has *purely exponential growth* if $\frac{1}{C} \exp(n\omega) \leq \#S^n \leq C \exp(n\omega)$ for some $C > 0$ independent of $n \geq 1$. This class of groups includes (relatively) hyperbolic groups and many other groups (see [5] and [27] for relevant discussions). By taking $T_n := S^n$, one sees¹ that

$$\frac{\omega(H, T_n)}{\log \#T_n} \rightarrow 1, \text{ as } n \rightarrow \infty$$

Thus, the upper bound 1 is an accumulation point for any group with purely exponential growth. On the other hand, the growth tightness [13] of free groups implies that $1 \in \Theta(H)$ if and only if H is a free group. Thus it is interesting to ask whether there exist examples with $\Theta(H) \subset [0, 1 - \epsilon]$ for some $\epsilon > 0$.

The examples of Wilson also imply $0 \in \overline{\Omega(H)}$ for certain non-uniform exponential growth groups H . Analogous to the question of uniform exponential growth, we can ask for which groups $\Theta(H)$ admits a positive infimum. In fact, $\inf \Theta(H) > 0$ has been obtained for hyperbolic groups in [1].

The main result of this paper is a generalization of the previous results of Arzhantseva-Lysenok [1] to the class of relatively hyperbolic groups. Since the official introduction in the Gromov 1987 monograph [14], this class of groups has been well-studied in last thirty years, see [9], [3], [21], [8], [12]. The important examples include Gromov-hyperbolic groups, geometrically finite Kleinian groups (with variable negative curvature), infinitely-ended groups, small cancellation quotients of free products, limit groups, to name just a few.

Our main theorem establishes the positive lower bound on $\Theta(H)$ for any non-elementary subgroup in a relatively hyperbolic group. By definition, a subgroup H is called *non-elementary* if its limit set contains at least 3 points. See §2.2 for details.

Theorem 1.1. *Assume that G is a non-elementary relatively hyperbolic group. Then there exists a constant $\kappa = \kappa(G) \in (0, 1]$ such that for any non-elementary subgroup H with a finite symmetric generating set S , we have*

$$\omega(H, S) \geq \kappa \cdot \log \#S.$$

Remark. Wilson's example exhibits a sequence of 2-generator sets with growth rate tending 0. This shows that the non-elementary assumption of H is necessary: indeed, any group H can be realized as the maximal parabolic subgroup in a free product of H with any nontrivial group.

¹The authors learnt this fact from Alex Furman.

A group G has *uniform uniform exponential growth* if every finitely generated subgroup has uniform exponential growth. Xie [26] has proved that relatively hyperbolic groups have uniform exponential growth. As a direct corollary of Theorem 1.1, we obtain a strengthening of Xie's theorem.

Theorem 1.2. *Any non-elementary subgroup H of a non-elementary relatively hyperbolic group G has uniform exponential growth.*

Remark. Recall that non-elementary relatively hyperbolic groups G are growth tight, so non-Hopfian ones cannot realize its infimum of $\Omega(G)$ (see [28] [24]). It is thus interesting to know whether Fujiwara-Sela's result [11] can generalize to torsion-free toral relatively hyperbolic groups [16, 17].²

1.2. Connection with other works. It has been recent interests to study the product set growth in various classes of groups, starting in free groups [23], hyperbolic groups and acylindrical hyperbolic groups [7], free product of groups [4], and so on. We refer the reader to [4] for further references and connection with approximate groups.

To be precise, let S be any set in a group G subject to the condition S do not generate a “small” subgroup. The Helfgott type growth (in the terminology of [4]) wishes to have the following

$$\sharp(S^3) \geq c \cdot (\sharp S)^{1+\kappa}$$

for some universal $c, \kappa > 0$ depending only on G . By induction, it is easy to see that if a group G has the Helfgott type growth, then Theorem 1.1 holds for this group G . In this sense, Theorem 1.1 could be understood as asymptotic version of product set growth. Indeed, our proof boils down to a similar product growth with high powers

$$\forall i \in \mathbb{N}, \sharp(S^{i\kappa}) \geq (\sharp S)^i$$

for a universal $\kappa > 0$. Even though, our Theorem 1.1 cannot be deduced directly from the result [7, Theorem 1.9]. Their result does provide certain product set growth only assuming the acylindrical action on hyperbolic spaces. However, the large displacement assumption imposed there on S is hard to verify in practice.

Very recently, Kropholler-Lyman-Ng obtained independently Theorem 1.2 as [20, Proposition 4.12] during our writing of this paper. Similar to us, they made a variant of Xie's result as Lemma 3.1, and then run the remaining argument in [26] to get Theorem 1.2.

To conclude the introduction, let us mention briefly the proof of main theorems. We follow closely the strategy of [1] which appears to us quite robust. On the other hand, we have to deal with several difficulties from the relative case. They are resolved largely by adapting the work of Xie [26] (see Lemma 3.1) and by a strengthening of Koubi's result [19] (see Lemma 3.2). We believe that Lemma 3.2 has independent interest and admits further applications.

Structure of the paper. This paper is organized as follows. Section 2 recalls standard materials in Gromov's hyperbolic geometry, Bowditch-Gromov's definition of relatively hyperbolic groups. As mentioned above, the work of Xie and Koubi are properly adapted and strengthened in Section 3. A notion of loxodromic

²This question has been answered for *equationally Noetherian* relatively hyperbolic groups by Fujiwara's preprint [10] (posted on 2 Mar. 2021, one day earlier than ours on arXiv). One ingredient is Theorem 1.2 which was also obtained by him independently and simultaneously.

elements with large injectivity is introduced in Section 4 to streamline the strategy of Arzhantseva-Lysenok. The proof of Theorem 1.1 is then completed in Section 5.

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2. PRELIMINARY

Consider an isometric action of G on a metric space (X, d) . Let $S \subset G$ be a set of isometries. Denote $\ell_x(S) := \max_{s \in S} \{d(x, sx)\}$ for a given point $x \in X$. For a subset $A \subset X$, define

$$\ell_A(S) := \inf_{x \in A} \ell_x(S).$$

Note that $x \in X \mapsto \ell_x(S) \in \mathbb{R}$ is a continuous non-negative function.

2.1. Hyperbolic spaces and Loxodromic elements. Define the Gromov product

$$\forall x, y, o \in X, \langle x, y \rangle_o = \frac{d(x, o) + d(y, o) - d(x, y)}{2}.$$

A geodesic metric space X is called *hyperbolic* if any geodesic triangle is δ -thin: if $d(o, p) = d(o, q) \leq \langle x, y \rangle_o$ for two points $p \in [o, x]$, $q \in [o, y]$, then $d(p, q) \leq \delta$. Then for any $x, y, z, o \in X$, we have

$$\langle x, y \rangle_o \geq \min\{\langle x, z \rangle_o, \langle z, y \rangle_o\} - \delta.$$

Assume that a finitely generated group G acts properly by isometry on a proper hyperbolic space X . Then the induced action of G on the Gromov boundary ∂X of X is a convergence group action. Thus, any infinite order element fixes at least one but at most two points in ∂X . So the elements in G are classified into three nonexclusive classes: *elliptic isometry* with finite order elements, *parabolic isometry* with only fixed point and *loxodromic isometry* with exactly two fixed points. See [3] for a detailed discussion about convergence group actions and relevant notions.

Equivalently, an isometry g on a proper hyperbolic space X is loxodromic if it admits a (λ, c) -quasi-geodesic γ for some $\lambda, c > 0$ so that $\gamma, g\gamma$ have finite Hausdorff distance. Such quasi-geodesics shall be referred to as (λ, c) -quasi-axis.

Lemma 2.1. [6, Lemma 9.2.2] *If g is an isometry satisfying*

$$d(o, go) \geq 2\langle o, g^2 o \rangle_{go} + 6\delta$$

for some point $o \in X$, then g is loxodromic.

Lemma 2.2. [1, Lemma 1] *Let x_1, x_2, \dots, x_k for $k \geq 3$ be points in a δ -hyperbolic space such that for any $2 \leq i \leq k-2$, we have*

$$\langle x_{i-1}, x_{i+1} \rangle_{x_i} + \langle x_i, x_{i+2} \rangle_{x_{i+1}} \leq d(x_i, x_{i+1}) - 3\delta$$

Then

$$d(x_1, x_k) \geq \sum_{i=1}^{k-1} d(x_i, x_{i+1}) - 2 \sum_{i=2}^{k-1} (\langle x_{i-1}, x_{i+1} \rangle_{x_i} + \delta).$$

The following immediate corollary will be actually used.

Corollary 2.3. *Under the assumption of Lemma 2.2, if*

$$\langle x_{i-1}, x_{i+1} \rangle_{x_i} + \langle x_i, x_{i+2} \rangle_{x_{i+1}} \leq d(x_i, x_{i+1})/4 - \delta$$

then

$$d(x_1, x_k) \geq \frac{1}{2} \sum_{i=1}^{k-1} d(x_i, x_{i+1}).$$

Lemma 2.4. *If g, h are two isometries satisfying*

$$\frac{1}{4} \min\{d(go, o), d(ho, o)\} \geq \max\{\langle go, h^{-1}o \rangle_o, \langle g^{-1}o, ho \rangle_o\} + \delta$$

for some point $o \in X$. Then

- (1) gh is loxodromic.
- (2) there exist constants $\lambda, c > 0$ depending only on δ such that the concatenated path $\bigcup_{i \in \mathbb{Z}} (gh)^i ([o, go] \cdot g[o, ho])$ is a (λ, c) -quasi-geodesic.
- (3) there exists a constant $C = C(\delta) > 0$ such that $|\ell_X(gh) - d(o, gho)| \leq C$.

Proof. The proof uses the well-known fact that long local geodesics are global quasi-geodesics. To be precise, applying Lemma 2.2 to the points $x_1 = o, x_2 = go, x_3 = gho, \dots, x_{2n+1} = (gh)^n o$, we have

$$d(x_1, x_{2n+1}) \geq \sum_{k=1}^{2n} d(x_i, x_{i+1}) - 2 \sum_{k=2}^{2n} (\langle x_{i-1}, x_{i+1} \rangle_{x_i} + \delta) \geq \frac{1}{2} \sum_{k=1}^{2n} d(x_i, x_{i+1}).$$

The proof is completed. \square

Define the asymptotic translation length of an isometry g as follows

$$\tau(g) := \lim_{n \rightarrow \infty} \frac{d(o, g^n o)}{n}$$

for some (thus any) point $o \in X$.

Lemma 2.5. [6, Proposition 10. 6.4] *If g is a loxodromic element, then $|\ell_X(g) - \tau(g)| \leq 16\delta$.*

We say that an element $g \in G$ preserves the orientation of the bi-infinite quasi-geodesic γ if $\alpha, g\alpha$ has finite Hausdorff distance for any half-ray α of γ . It is clear that a loxodromic element preserves the orientation of any quasi-axis.

Lemma 2.6. *If g preserves the orientation of (λ, c) -quasi-axis γ , then there exists a constant C depending on λ, c, δ with the following property. For any $x \in \gamma$, there exists $y \in \gamma$ such that $\langle x, gx \rangle_y < C$ and $\langle y, gy \rangle_{gx} < C$.*

The following lemma is well-known with proof included for completeness.

Lemma 2.7. *There exists a constant $C = C(\lambda, c, \delta)$ for any $\lambda, c > 0$ with the following property. If a loxodromic element g admits a (λ, c) -quasi-axis γ , then for any $x \in \gamma$, we have $|\ell_X(g) - d(x, gx)| \leq C$.*

Proof. By Morse Lemma, any two (λ, c) -quasi-axes $\gamma, g\gamma$ have bounded Hausdorff distance depending only on λ, c, δ . Thus, the inclusion of γ into $\cup_{i \in \mathbb{Z}} g^i \gamma$ is a quasi-isometry with constants depending λ, c, δ only. We can thus assume that the quasi-axis γ is $\langle g \rangle$ -invariant.

Note that the shortest projection $\pi_\gamma(\cdot)$ to a (λ, c) -quasi-geodesic γ is C -contracting for a constant $C = C(\lambda, c, \delta)$:

$$(\forall z, w \in X, \text{diam}(\{\pi_\gamma(z), \pi_\gamma(w)\}) > C) \implies (\max\{d(\pi_\gamma(z), \gamma), d(\pi_\gamma(w), \gamma)\} \leq C).$$

We then derive the following for any $z, w \in X$:

$$\text{diam}(\{\pi_\gamma(z), \pi_{g\gamma}(w)\}) + d(z, \pi_\gamma(z)) + d(w, \pi_\gamma(w)) \leq d(z, w) + 4C.$$

Let $o \in X$ so that $d(o, go) = \ell_X(g)$. We apply the above inequality for $z = o, w = go$. Using the fact that γ is $\langle g \rangle$ -invariant, we obtain $d(o, \gamma) \leq 2C$: indeed, if not, we would have $d(\pi_\gamma(o), g\pi_\gamma(o)) < d(o, go)$. Thus, by taking its projection $\pi_\gamma(o)$, it suffices to prove the conclusion by assuming that o lies on γ .

Let $x \in \gamma$. By Lemma 2.6, we can assume up to translation that $\langle o, go \rangle_x \leq C$ and then $\langle x, gx \rangle_{go} \leq C$. Thus, $d(x, [o, go]) \leq C$ and $d(go, [x, gx]) \leq C$. Consequently,

$$\begin{aligned} |d(o, go) - d(x, gx)| &\leq |d(o, x) + d(x, go) - d(x, go) - d(go, gx)| + 4C \\ &\leq 4C. \end{aligned}$$

The proof is complete. \square

2.2. Elementary subgroups. Recall that G acts properly on a proper hyperbolic space X . The *limit set* ΛH of a subgroup H is the set of accumulation points in ∂X of any H -orbit in X . A subgroup H in G is called *elementary* if its limit set contains at most two points. See [3] for relevant discussion.

If ΛH consists of only one point p , then H is called *parabolic subgroup* and p is called a *parabolic point*. It is a well-known fact that in a convergence group action, a loxodromic element cannot fix a parabolic point. The (maximal) parabolic group plays the key role in definition 2.10 of relatively hyperbolic groups given in the next subsection. In the remainder of this subsection, we first consider the elementary subgroup with exactly two limit points.

Let γ be a quasi-axis for a loxodromic element h . The coarse stabilizer of the axis defined as follows

$$E(h) = \{g \in G : \exists r > 0, \gamma \subset N_r(g\gamma), g\gamma \subset N_r(\gamma)\}$$

gives the maximal elementary subgroup containing h . Note that the following index at most 2 subgroup

$$E^+(h) := \{g \in G : \exists n > 0, gh^n g^{-1} = h^n\}$$

is precisely the set of orientation-preserving elements in $E(h)$.

Denote $E^-(h) = E(h) \setminus E^+(h)$. Let $E^*(h)$ be the torsion group of $E^+(h)$.

Lemma 2.8. *For a loxodromic element h , the following statements hold:*

- (1) $[E(h) : \langle h \rangle] < \infty$, and $E(h)$ is a contracting subgroup with bounded intersection.
- (2) $E(h) = \{g \in G : \exists n > 0, (gh^n g^{-1} = h^n) \vee (gh^n g^{-1} = h^{-n})\}$.
- (3) $E^*(h)$ is a finite normal subgroup of $E(h)$.
- (4) $g^2 \in E^*(h)$ for any $g \in E^-(h)$.

Proof. The first two statements are [27, Lemma 2.11]. The last two statements follow from [1, Lemma 4] where only the assertion (1) is used in the proof. \square

Lemma 2.9. *Let $h \in G$ be a loxodromic element admitting a (λ, c) -quasi-axis α . Then there exists $D = D(\lambda, c, \delta)$ such that $d(go, o) \leq D$ for any $g \in E^*(h)$ and $o \in \alpha$.*

Proof. By Morse Lemma, there exists a constant $D > 3\delta$ depending only on λ, c, δ such that any two of $g\alpha, \alpha, g^{-1}\alpha$ have Hausdorff distance at most D . Since $g \in E^*(h) \subset E^+(h)$ preserves the orientation of α , up to taking inverse of g , we can

assume further that $d(go, [o, \alpha_+]_\alpha) \leq D$ and $d(g^{-1}o, [o, \alpha_-]_\alpha) \leq D$. Let $x, y \in \alpha$ such that $d(go, x), d(g^{-1}o, y) \leq D$. Since x, y are on the opposite sides of o on α , we have $d(o, [x, y]) \leq D$. Thus, $\langle x, y \rangle_o \leq D$ and then $\langle go, g^{-1}o \rangle_o \leq 3D$. If $d(o, go) > 4D$ was assumed, then $d(o, go) \geq 2d(o, g^2o) + 6\delta$. By Lemma 2.1, g is loxodromic. This is a contradiction, so $d(o, go) \leq 4D$. \square

2.3. Relatively hyperbolic groups. The notion of a relatively hyperbolicity has a number of equivalent formulation (see [9], [3], [21], [8], [12] etc). See [18] for a survey of their equivalence. In this paper we define a relatively hyperbolic group which admits a cusp-uniform action on a hyperbolic space.

Definition 2.10. Suppose G admits a proper and isometric action on a proper hyperbolic space (X, d) such that G does not fix a point in the Gromov boundary ∂X . Denote by \mathcal{P} the set of maximal parabolic subgroups in G . Assume that there is a G -invariant system of disjoint (open) horoballs \mathbb{U} centered at parabolic points of G such that the action of G on the complement called *neutered space*

$$X(\mathbb{U}) := X \setminus \mathcal{U}$$

is co-compact where $\mathcal{U} := \bigcup_{U \in \mathbb{U}} U$. Then the pair (G, \mathcal{P}) is said to be *relatively hyperbolic*, and the action of G on X is called *cusp-uniform*.

We fix a G -invariant system \mathbb{U} of horoballs and a neutered space $X(\mathbb{U})$ on which G acts co-compactly. The following result is proved by [1, Lemma 6] in hyperbolic groups. In the relative case, we follow closely their arguments.

Lemma 2.11. *Let h be a loxodromic element in G so that for some point $o \in X$ and $\lambda, c > 0$, the path $\alpha = \cup_{n \in \mathbb{Z}} [h^n o, h^{n+1} o]$ is a (λ, c) -quasi-geodesic in X . Then for any given $\theta > 0$ there exists $N = N(\lambda, c, \delta, \theta)$ independent of the point o such that for any $f \notin E(h)$, we have*

$$\text{diam}(\alpha \cap N_R(f\alpha)) \leq N \cdot d(o, ho)$$

where $R := \theta \cdot d(o, ho)$.

Proof. First of all, since h is a loxodromic element and cannot fix any parabolic point, we obtain that α cannot be contained inside any horoball $U \in \mathbb{U}$. Thus, the $\langle h \rangle$ -invariant set $\alpha \cap X(\mathbb{U})$ is a non-empty unbounded set. Namely, for any $x \in \alpha \cap X(\mathbb{U})$ and any $i \in \mathbb{Z}$, we have $h^i x \in \alpha$.

We argue by contradiction. Assume that $\text{diam}(\alpha \cap N_R(f\alpha)) > N \cdot d(o, ho)$ for a constant N determined below. Let $z, w, z', w' \in \alpha$ such that $d(z, w) = \text{diam}(\alpha \cap N_R(f\alpha))$ and $d(z, fz'), d(w, fw') \leq R$.

By hyperbolicity, $[z, w]_\alpha$ and $[z', w']_\alpha$ contain subpaths β_1, β_2 respectively such that $\beta_1, f\beta_2$ have Hausdorff distance at most $C = C(\lambda, c, \delta) > 0$ and for $i = 1, 2$, we have

$$\text{diam}(\beta_i) \geq \text{diam}(\alpha \cap N_R(f\alpha)) - 2R \geq (N - \theta)d(o, ho).$$

Since α is a (λ, c) -quasi-geodesic, there exists a monotone increasing function $N' = N'(\lambda, c, N, \theta) > 0$ such that β_1 contains at least $(N' + 1)$ translates of $[o, ho]$. Moreover, $N' = N'(\lambda, c, N, \theta) \rightarrow \infty$ as $N \rightarrow \infty$. Thus, β_1 contains $(N' + 1)$ points $x, hx, \dots, h^{N'}x \in X(\mathbb{U})$. Let $y \in \beta_2$ be a point so that $d(x, fy) \leq C$.

Assume that C also satisfies the conclusion of Lemma 2.7. With Lemma 2.5, for $1 \leq i \leq N'$, we have

$$|d(x, h^i x) - \tau(h^i)| \leq C + 16\delta, \quad |d(fy, fh^i y) - \tau(h^i)| \leq C + 16\delta$$

so

$$|d(x, h^i x) - d(fy, fh^i y)| \leq 2C + 32\delta$$

Thus, $d(h^i x, fh^i y) \leq 3C + 32\delta$ for each $1 \leq i \leq N'$.

Set $N(x, y) = \#\{g \in G : d(x, gy) \leq C\} + 1$. Since G acts cocompactly on the C -neighborhood of $X(\mathbb{U})$, we see that $N(x, y)$ over $x, y \in N_C(X(\mathbb{U}))$ is uniformly bounded above by a constant N_0 .

Choose $N > 0$ such that $N' = N'(\lambda, c, N, \theta) \geq N_0$, and consequently, we obtain $h^{-i}fh^i = h^{-j}fh^j$ for $1 \leq i \neq j \leq N'$. So $f \in E(h)$ contradicts with the assumption. The result is proved. \square

At last, let us mention the following result of Osin which holds for loxodromic elements in any acylindrical action on hyperbolic spaces.

Lemma 2.12. [22, Lemma 6.8] *There exists a finite number N_0 such that $\#E^\star(g) \leq N_0$ for any loxodromic element $g \in G$.*

3. SHORT LOXODROMIC ELEMENTS

The goal of this section is to provide short loxodromic elements.

Let $S = S^{-1}$ be a symmetric generating set of a non-elementary group H . Recall $S^{\leq n_0} := \{h \in H : d_S(1, h) \leq n_0\}$.

The following is a variant of [26, Lemma 5.3].

Lemma 3.1. *For any $M > 0$, there exists a positive integer $n_0 = n_0(M) > 0$ such that for any finite symmetric generating set S of H , we have*

$$\ell_X(S^{\leq n_0}) > M.$$

Proof. Let \mathbb{U} be a M -separated G -invariant system of horoballs centered at the parabolic points. Recall that the action of G on $X(\mathbb{U})$ is proper and co-compact. Let $K \subset X(\mathbb{U})$ be a compact set such that $\cup_{g \in G} g(K) = X(\mathbb{U})$. Fix a point $p \in K$ and denote $a = \text{diam}(K)$ depending on M . The proper action implies the set

$$A = \{g \in G : d(g(p), p) \leq 2a + M\}$$

is a finite set. Since G is finitely generated, up to increasing the value of a , we can assume that A generates G .

Consider the finite set \mathbb{H} of conjugates of H which is generated by some finite set $S' \subset A$. Since H is infinite, the proper action of H on X implies that for every $H' \in \mathbb{H}$, there is some $g_{H'} \in H'$ with $d(g_{H'}(p), p) > M + 2a$. Since A generates G and \mathbb{H} is finite, then the integer

$$n_0 := \max\{d_S(1, g_{H'}) : S' \subset A, H' := \langle S' \rangle \in \mathbb{H}\}$$

is finite.

Now let S be a finite generating set of H . If $\ell_X(S) > M$, then we are done: $\ell_X(S^{n_0}) \geq \ell_X(S) > M$. If there is some $x \in X$ with $\ell_x(S) \leq M$, then $x \in X(\mathbb{U})$. Indeed, assume that $x \in U$ for some $U \in \mathbb{U}$. By definition of $\ell_x(S) \leq M$ we have $d(s(x), x) \leq M$ for all $s \in S$. The M -separation of \mathbb{U} implies $s(U) = U$ for all $s \in S$ and so the center of U would be fixed by the non-elementary subgroup H : a contradiction. Hence, it follows that $x \in X(\mathbb{U})$.

Recalling that $\cup_{g \in G} g(K) = X(\mathbb{U})$, we choose $g \in G$ with $g(x) \in K$. We now show $S' := \{gsg^{-1} : s \in S\} \subset A$. Indeed, for each $s \in S$, we have

$$\begin{aligned} d(p, gsg^{-1}(p)) &\leq d(p, g(x)) + d(g(x), gs(x)) + d(gs(x), gsg^{-1}(p)) \\ &\leq d(p, g(x)) + d(x, s(x)) + d(g(x), p) \leq 2a + M. \end{aligned}$$

Since $S' \subset A$ generates $H' := gHg^{-1} \in \mathbb{H}$, by the definition of n_0 , there is some integer $1 \leq k \leq n_0$ such that $d_{S'}(1, gH') = k$. Thus,

$$gH' = (gs_1g^{-1}) \cdots (gs_kg^{-1}) = g(s_1 \cdots s_k)g^{-1}$$

for $s_i \in S \cup S^{-1}$. Now by triangle inequality, we have

$$\begin{aligned} d(g^{-1}gH'g(x), x) &= d(gH'g(x), g(x)) \\ &\geq d(gH'(p), p) - d(gH'(p), gH'g(x)) - d(g(x), p) \\ &= d(gH'(p), p) - d(p, g(x)) - d(g(x), p) \\ &> M + 2a - a - a = M. \end{aligned}$$

Since $g^{-1}gH'g = s_1 \cdots s_k \in S^{\leq n_0}$, it follows that $\ell_x(S^{\leq n_0}) > M$. \square

The following result improves Proposition 3.2 of [19].

Lemma 3.2. *Let X be a δ -hyperbolic geodesic metric space, and H a group of isometries of X with a finite symmetric generating set S . If $\ell_X(S) > 28\delta$, then H contains a loxodromic element $b \in S^{\leq 2}$. Moreover, there exists a constant $C = C(\delta) > 0$ such that*

$$d(o, bo) \geq \ell_X(S) - C$$

for some point $o \in X$.

Proof. Let $o \in X$ such that $\ell_X(S) + \delta > \ell_o(S) \geq \ell_X(S)$. Set $L_0 = 4\delta$ and then $\ell_o(S) > 7L_0$. Denote by S_0 the (non-empty) set of elements $s \in S$ so that

$$d(o, so) \geq \ell_o(S) - 2L_0 - \delta.$$

Let $t \in S$ such that $\ell_o(S) = d(o, to)$, and $m \in [o, to]$ so that $d(o, m) = L_0$.

The main observation is as follows.

Claim. *There exists an isometry $s \in S_0$ such that s is either loxodromic with $\langle o, s^2o \rangle_{so} \leq L_0$ or satisfies*

$$\max\{\langle to, so \rangle_o, \langle t^{-1}o, s^{-1}o \rangle_o\} \leq L_0.$$

Proof of the Claim. Assume to the contrary that for all $s \in S_0$, we have

$$(1) \quad \max\{\langle to, so \rangle_o, \langle t^{-1}o, s^{-1}o \rangle_o\} > L_0.$$

Moreover, each $s \in S_0$ is either non-loxodromic or loxodromic with $\langle o, s^2o \rangle_{so} > L_0$. If $s \in S_0$ is non-loxodromic, by Lemma 2.1, we have

$$\langle o, s^2o \rangle_{so} \geq d(o, so)/2 - 3\delta \geq (\ell_o(S) - 2L_0 - \delta)/2 - 3\delta \geq L_0.$$

Hence, for each $s \in S_0$, we have $\langle o, s^2o \rangle_{so} \geq L_0$. In particular, $\langle t^{-1}o, to \rangle_o \geq L_0$.

By (1), assume that $\langle t^\star o, s^\star o \rangle_o > L_0$ for $\star \in \{1, -1\}$. Let $m_1, m_2 \in [o, s^\star o]$ for $s \in S_0$ so that $d(o, m_1) = d(s^\star o, m_2) = L_0$. By hyperbolicity,

$$\langle s^\star o, to \rangle_o \geq \min\{\langle s^\star o, t^\star o \rangle_o, \langle t^{-1}o, to \rangle_o\} - \delta \geq L_0 - \delta$$

which by the δ -thin triangle property implies $d(m, m_1) \leq 3\delta$. Using again δ -thin triangle with $\langle o, (s^\star)^2 o \rangle_{s^\star o} \geq L_0$, we obtain that $d(m_2, s^\star m_1) \leq \delta$.

We shall derive $\ell_m(S) < \ell_X(S)$, which is a contradiction. Indeed, for each $s \in S_0$,

$$\begin{aligned} d(m, s^*m) &\leq 2d(m, m_1) + d(m_1, s^*m_1) \leq 7\delta + d(m_1, m_2) \\ &\leq 7\delta + d(o, s^*o) - 2L_0 \\ &\leq \ell_o(S) - 2L_0 + 7\delta \leq \ell_o(S) - \delta. \end{aligned}$$

The definition of $s \in S \setminus S_0$ gives $d(o, so) \leq \ell_o(S) - 2L_0 - \delta$ and thus

$$d(m, sm) \leq 2d(o, m) + d(o, so) \leq 2L_0 + d(o, so) < \ell_o(S) - \delta$$

We obtained the contradiction $\ell_m(S) \leq \ell_o(S) - \delta < \ell_X(S)$. The proof of the claim is now complete. \square

By the above claim, there exists $s \in S_0$ such that either

$$\langle so, s^{-1}o \rangle_o + \delta \leq L_0 + \delta \leq \frac{1}{4}d(o, so)$$

or

$$\max\{\langle to, so \rangle_o, \langle t^{-1}o, s^{-1}o \rangle_o\} + \delta \leq L_0 + \delta \leq \frac{1}{4} \min\{d(o, so), d(o, to)\}$$

The proof is then completed by Lemma 2.4. \square

4. SHORT LOXODROMIC ELEMENTS WITH LARGE INJECTIVITY

Let $\lambda_0 = \lambda_0(\delta)$, $c_0 = c_0(\delta)$, $C_0 = C_0(\delta)$ be given by Lemma 3.2. Let $n_0 = n_0(28\delta)$ be given by Lemma 3.1 so that $\ell_X(S^{<n_0}) > 28\delta$. Thus, S contains a loxodromic element $b \in H$ and there exist a point $o \in X$ such that the path

$$(2) \quad \alpha = \bigcup_{n \in \mathbb{Z}} b^n [o, bo]$$

is a (λ_0, c_0) -quasi-axis for b .

By hyperbolicity, any quadrilateral with (λ_0, c_0) -quasi-geodesic sides is C -thin for some $C = C(\lambda_0, c_0, \delta) > \max\{C_0, \delta\}$: any side is contained in the C -neighborhood of the other three sides.

Since $\#(S^{2n_0}) \geq \#S$, it suffices to prove Theorem 1.1 assuming the generating set S with

$$\ell_X(S) > \max\{28\delta, 2C\}.$$

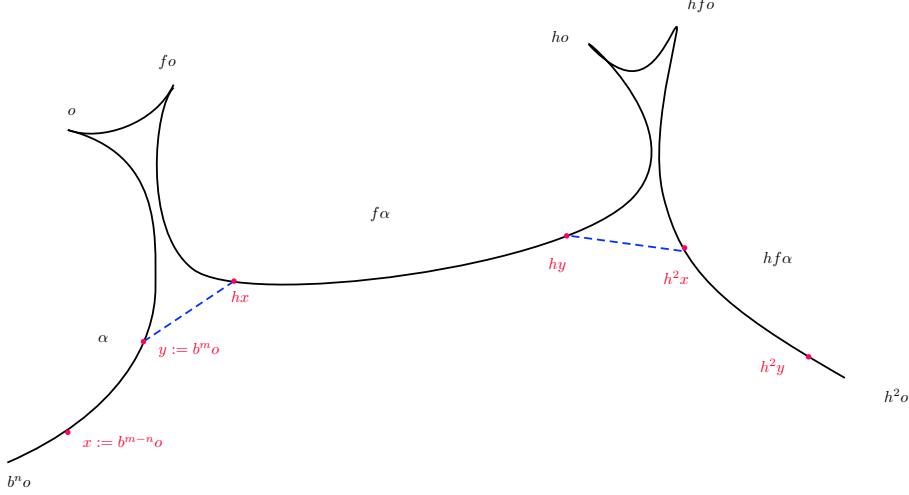
By Lemma 3.2, we have

$$(3) \quad d(o, bo) \geq \ell_X(S) - C_0 \geq C.$$

Since H is not virtually cyclic, S contains an element f such that $f \notin E(b)$. Indeed, if not, any $f \in S$ would fix the set of fixed points of b so it follows from $H = \langle S \rangle$ that the limit set of H consists of two points. By the subgroup classification in (the convergence action of) G , we obtain that H would be virtually cyclic. This is a contradiction.

4.1. Loxodromic elements raising to power. In the remainder of this section, we assume that $f \in S \setminus E(b)$. Consider the element $h := fb^n$ for $n \geq 0$.

Consider the points $x = b^{m-n}o, y = b^{-m}o$ for $0 \leq m \leq n$ on α , and then $hx = fb^m o \in f\alpha$. Consider two quasi-geodesics α and $f\alpha$ connected by a geodesic $[o, fo]$. To get a quasi-axis of h , we shall use the next lemma to truncate the part containing $[o, fo]$ of α and $f\alpha$ at the points y and hx .

FIGURE 1. Truncate the quadrilaterals y, o, fo, hx and hy, ho, hfo, h^2x

Lemma 4.1. *For any $\theta > 0$, there exists $n = n(\theta, \delta), m = m(\theta, \delta) > 0$ so that $h = fb^n$ for any $n > n(\theta, \delta)$ enjoys the following property.*

If $\theta d(o, bo) \geq d(o, fo)$, then $\langle x, hx \rangle_y, \langle y, hy \rangle_{hx} \leq C$ and $d(y, hx) \geq \theta d(o, bo)$.

Proof. Consider the quadrilateral formed by the subpaths $[x, o]_\alpha, [o, fo], f[o, b^m o]_\alpha$, and the geodesic $[x, hx]$ as depicted in Figure 1.

With the inequality (3), the (λ_0, c_0) -quasi-geodesicity of α in (2) gives

$$(4) \quad \begin{aligned} \forall n \geq 1, n \cdot d(o, bo) &\geq d(o, b^n o) \geq \lambda_0^{-1} n \cdot d(o, bo) - c_0 \\ &\geq \lambda_0^{-1} n \cdot d(o, fo) - (c_0 + nC_0/\lambda_0). \end{aligned}$$

Set $N = N(\lambda_0, c_0, \delta, \theta)$ given by Lemma 2.11. By (4), we can choose the least integer $m = m(\lambda_0, c_0, \delta) > N$ and then the least $n = n(\lambda_0, c_0, \theta, \delta)$ so that for any $n \geq n(\theta, \delta)$,

$$(5) \quad d(o, b^m o), d(o, b^{n-2m} o) \geq \max\{Nd(o, bo), 10C\}.$$

If denote $R := \theta d(o, bo) \geq d(o, fo)$, then

$$d(y, f\alpha) > R.$$

Indeed, assume to the contrary that $d(y, f\alpha) \leq R$. Since $d(o, f\alpha) \leq d(o, fo) \leq R$, by Lemma 2.11, the diameter of $\alpha \cap N_R(f\alpha)$ is at most $Nd(o, bo)$. However, $\alpha \cap N_R(f\alpha)$ contains two points $y = b^{-m} o, o$ with distance at least $Nd(o, bo)$ by Eq. 5. This is a contradiction. Thus, $d(y, f\alpha) > R$ is proved. The C -thin quadrilateral property then implies $d(y, [x, hx]) \leq C$, so we obtain $\langle x, hx \rangle_y \leq d(y, [x, hx]) \leq C$.

By symmetry, we can run the above argument for the quadrilateral with vertices y, o, fo, hy and obtain $\langle y, hy \rangle_{hx} \leq C$.

Note that $d(y, hx) \geq d(y, f\alpha) \geq R = \theta d(o, bo)$. The proof is complete. \square

Lemma 4.2. *There exist constants $n_1 = n_1(\delta), m_1 = m_1(\delta), \lambda = \lambda(\delta), c = c(\delta) > 0$ such that for any $n \geq n_1$, the element $h := fb^n$ is loxodromic with a (λ, c) -quasi-axis*

β defined as follows

$$(6) \quad \beta := \bigcup_{i \in \mathbb{Z}} h^i ([x, y]_\alpha [y, hx])$$

where $x = b^{m_1-n}o, y = b^{-m_1}o$.

Proof. The inequality (4) implies that the following constant θ depends only on δ :

$$\theta := \max \left\{ 1, \frac{d(o, fo)}{d(o, bo)}, \frac{10C}{d(o, bo)} \right\}.$$

If $m_1 := m(\delta, \theta)$ is given by Lemma 4.1, then for $n \geq m_1$, we have

$$\max\{\langle x, hx \rangle_y, \langle y, hy \rangle_x\} \leq C.$$

Note that $d(x, y) = d(o, b^{n-2m_1}o) \geq 10C$ by Eq. (5) and $d(y, hx) \geq \theta d(o, bo) \geq 10C$. Therefore,

$$\langle x, hx \rangle_y + \langle y, hy \rangle_{hx} \leq \frac{1}{4}d(y, hx) - \delta.$$

so the assumption of Corollary 2.3 is verified for the sequence of points

$$\dots, h^{-i}x, h^{-i}y, \dots, x, y, hx, hy, \dots, h^jx, h^jy, \dots$$

Hence, there exist $\lambda = \lambda(\delta), c = c(\delta) > 0$ such that β is a (λ, c) -quasi-geodesic. This proves that h is loxodromic. \square

4.2. Large injectivity. The crucial property in constructing free subgroups is the following property of a loxodromic isometry $h = fb^n$. Recall that the point $o \in X$ is provided by Lemma 3.2 so that the inequality (3) holds.

Definition 4.3. A loxodromic element h has *injective radius* $L > 0$ if $E(h)$ contains a finite subgroup F with $[F : E^*(h)] \leq 2$ so that $E = \langle h \rangle F$ and for any $g \in E(h) \setminus F$, we have $\ell_X(g) > L \cdot d(o, bo)$.

Let $N_0 > 0$ be given by Lemma 2.12 so that $\#E^*(h) \leq N_0$, and $D = D(\lambda, c, \delta)$ be given by Lemma 2.9.

Lemma 4.4. For any $L > 0$ there exists $n_2 = n_2(L, \delta) \geq n_1$ such that the loxodromic element $h = fb^n$ for $n \geq n_2$ has injective radius L . Precisely,

- (1) $\#F \leq 2N_0$ and $\ell_z(F) \leq 2D$ for any $z \in \beta$.
- (2) For any $g \in E(h) \setminus F$, we have $\ell_X(g) > L \cdot d(o, bo)$.
- (3) for any $g \in E(h) \setminus F$, there exist $i \in \mathbb{Z}$ and $t \in F$ such that $g = h^i t$.

Proof. We keep the same notation as in the proofs of Lemma 4.1 and Lemma 4.2. For any $h = fb^n$ with $n \geq n_1$, the path β in (6) is a (λ, c) -quasi-geodesic, where the constants $n_1, \lambda, c > 0$ depend only on δ .

Let $m_1 = m_1(\delta)$ given by Lemma 4.2. Denote $x = b^{m_1-n}o, y = b^{-m_1}o$. Then

$$(7) \quad \begin{aligned} d(y, hx) &\leq d(o, fo) + 2d(o, b^{m_1}o), \\ d(x, y) &= d(o, b^{n-2m_1}o). \end{aligned}$$

By Lemma 2.7, it suffices to prove the statement (2) by placing the basepoint z to the point $y = b^{-m_1}o$ at β . Since β is a (λ, c) -quasi-geodesic, by increasing $n_1 = n_1(\delta)$, we can assume

$$(8) \quad \forall i \neq 0 \in \mathbb{Z}, d(y, h^i y) > 2D + L \cdot d(o, bo).$$

We first consider elements $g \in E^+(h)$ and prove the corresponding statements (2-3).

Denote $\beta_0 = [x, y]_\alpha[y, hx]$ the fundamental domain for the action of $\langle h \rangle$ on β . By hyperbolicity, the finite Hausdorff distance $d_H(\beta, g\beta) < \infty$ implies a uniform constant $R = R(\delta) > 0$ so that $d_H(\beta, g\beta) \leq R$.

By (3), the constant θ defined as follows depends on δ only:

$$\theta := \frac{R}{d(o, bo)} \leq R/C.$$

Let $N = N(\lambda, c, \delta, \theta)$ be given by Lemma 2.11.

By Eq. (7), the least integer $n_2 \geq \max\{N, n_1\}$ such that for any $h = fb^n$ with $n > n_2$,

$$(9) \quad d(x, y) \geq d(y, hx) + 2n_2 d(o, bo)$$

depends only on δ .

Note that β is contained in the union $\bigcup_{i \in \mathbb{Z}} h^i \alpha$ and $\bigcup_{i \in \mathbb{Z}} h^i [y, hx]$. By (9), the path $g[x, y]_\alpha$ contains a subpath α_0 of diameter at least $n_2 d(o, bo)$ which is contained in the R -neighborhood $h^i \alpha$. Then there exist a subpath α_1 of α such that $d_H(g\alpha_0, h^i \alpha_1) \leq R$. Since $n_2 > N$, Lemma 2.11 implies $g^{-1} h^i \in E(b)$.

We claim that $t := g^{-1} h^i \in E(h) \cap E(b)$ is of finite order. If not, then $E(h) \cap E(b)$ is an infinite subgroup. Thus, $E(h) \cap E(b)$ act co-compactly on the quasi-axis of both h and b , so the (λ, c) -quasi-axis of b is preserved by h up to finite Hausdorff distance. Hence, we obtain $h \in E(b)$ and then $f \in E(b)$. This is a contradiction.

Therefore, the finite order element $t \in E(b)$ preserves a (λ, c) -quasi-axis β of h . By Lemma 2.9, we have $d(z, tz) < D$. So far, we have verified the assertions (2-3) for $g \in E^+(h)$.

To complete the proof, it remains to consider elements $g \in E^-(h)$. If $d(y, gy) > D$ for all $g \in E^-(h)$, then we are done by setting $F := E^*(h)$ and $\ell_X(F) \leq D$ by Lemma 2.9. Otherwise, let $r \in E^-(h)$ so that $d(y, ry) \leq D$. Thus, $F := \langle E^*(h), r \rangle$ has order at most $2N_0$ and $\ell_y(F) \leq \ell_y(E^*(h)) + d(y, ry) \leq 2D$.

Since $E^+(h)$ is of index 2 in $E(h)$, we write $g = h^i tv$ for some $t \in E^*(h)$. If $i \neq 0$, one deduce from (8) that

$$d(y, gy) \geq d(y, h^i y) - d(y, ty) - d(y, ry) \geq L d(o, bo).$$

The result is proved. \square

5. PROOF OF THEOREM 1.1

We resume the constants λ_0, c_0, C_0, C depending only on δ at in Section 4 and the results obtained there under the assumption $\ell_X(S) > \max\{28\delta, 2C\}$. Then $b \in S^{\leq 2}$ is a loxodromic element given by Lemma 3.2, and $f \in S \setminus E(b)$ exists due to the fact that H is a non-elementary subgroup.

Let $m_1 = m_1(\delta) \geq 2$ given by Lemma 4.2 and make the reference point at $y = b^{-m_1} o$ on the quasi-axis β in (6). Note that $d(o, bo) = d(y, by)$.

Set $L := 4(m_1 + 1) \geq 10$. By (3), we have $d(o, so) \leq C_0 + d(o, bo)$ for any $s \in S$, and thus

$$(10) \quad d(y, sy) \leq (2m_1 + 1)d(o, bo) + C_0 < \frac{L}{2} d(y, by)$$

which yields for any $c := s^{-1}s'$ with $s \neq s' \in S$,

$$(11) \quad d(y, cy) < L \cdot d(y, by) = L \cdot d(o, bo).$$

Let $n_2 = n_2(L, \delta) > n_1$ and F be the finite subgroup in $E(h)$ given by Lemma 4.4. The following result holds for any integer $n \geq n_2$ and $h = fb^n$.

Lemma 5.1. *Choose a largest subset S_0 of S such that $sF \neq s'F$ for any $s \neq s'$. Then for any $s \neq s' \in S_0$, $s^{-1}s' \notin E(h)$.*

Proof. By Lemma 4.4, the inequality (11) implies that $s^{-1}s'$ must be contained in F so $sF = s'F$. This contradicts to the choice of S_0 consisting of different left F -coset representatives. \square

Choose the least integer $n_3 \geq n_2$ such that the last inequality in (13) holds for any $n \geq n_3$:

$$(12) \quad d(y, hy) = d(y, fb^n y) = d(b^{m_1} o, fb^{n+m_1} o)$$

$$(13) \quad \geq d(o, b^n o) - 2d(o, b^{m_1} o) - d(o, fo) > Ld(o, bo).$$

Construct the free bases. Let us now fix $h = fb^{n_3}$ throughout the proof. Let F be the finite subgroup in $E(h)$ by Lemma 4.4. Let S_0 be a largest subset of S such that $sF \neq s'F$ for any $s \neq s'$.

For $\theta = 1$, let $m_2 = m(1, \delta), k = n(1, \delta)$ given by Lemma 4.1.

We define the free base as follows:

$$T = \{th^k t^{-1} : t \in S_0\}.$$

If set $\kappa := 2 + k(n_3 + 1)$, then $d_S(1, th^k t^{-1}) \leq \kappa$ and $T \subset S^{\leq \kappa}$.

The goal is the following.

Lemma 5.2. *The set T generates a free subgroup of rank $\#T$ in H .*

Proof. Let W be a non-empty reduced word over $T \cup T^{-1}$ written as follows

$$\begin{aligned} W &= (s_1 \cdot h^{i_1 k} \cdot s_1^{-1})(s_2 \cdot h^{i_2 k} \cdot s_2^{-1}) \cdots (s_l \cdot h^{i_l k} \cdot s_l^{-1}) \\ &= s_1 \cdot (h^{i_1 k} \cdot c_1 \cdot h^{i_2 k} \cdot c_2 \cdots c_{l-1} h^{i_l k}) \cdot s_l^{-1} \end{aligned}$$

First of all, let β_j be the subpath of β starting from y to $h^{i_j k} y$ consisting of $i_j \cdot k$ copies of $[y, hy]_\beta$. Let $p_j = [y, c_j y]$ be a geodesic labeled by c_j .

We choose z_j, w_j on β_j so that the initial subpath of β_j until z_j contains exactly m_2 copies of $[y, hy]_\beta$, and the terminal path starting at w_j contains exactly m_2 copies of $[y, hy]_\beta$. To be precise, set $z_j = h^{m_2 k} y, w_j = h^{i_j - m_2} y$.

Furthermore if $j = 1$, we let z_1 be the initial point of β_1 ; if $j = l$, let w_l be the initial point of β_l .

We now properly translate β_j and p_j for $1 \leq j \leq l$ so that β_1 originates at y , and then the terminal points of β_j followed by the initial points of p_j in a manner produces the following concatenated path:

$$\gamma = \beta_1 \cdot p_1 \cdot \beta_2 \cdot p_2 \cdot \beta_3 \cdots p_{l-1} \cdot \beta_l.$$

(We refer the reader to Figure 1 for similar illustration of cutting out quadrilaterals, where x, y, hx, hy, h^2x, h^2y should be marked as $z_1, w_1, z_2, w_2, z_3, w_3$ etc.)

By abuse of language, the translated points of z_j, w_j on β_j are still denoted by z_j, w_j , so we have plotted a sequence of points $z_1, w_1, z_2, w_2, \dots, z_l, w_l$ on γ . By the choice of z_1, w_l , the path γ starts at z_1 and ends at w_l , labeled by the word $s_1^{-1} W s_l$.

The key construction is then to cut quadrilaterals off γ along $[w_j, z_{j+1}]$ and verify that $\{z_1, w_1, z_2, w_2, \dots, z_l, w_l\}$ is a quasi-geodesic.

To truncate the quadrilaterals, we apply Lemma 4.1 to $\beta_j, c_j, \beta_{j+1}, c_{j+1}$ in order for $1 \leq j \leq l$. For concreteness, set $j = 1$.

Noting $d(y, hy) > Ld(y, cy)$ by (13) and (11), Lemma 4.1 gives

$$\langle z_1, z_2 \rangle_{w_2}, \langle w_1, w_2 \rangle_{z_2} \leq C$$

and together with Lemma 4.4.(2),

$$d(w_1, z_2) \geq \theta d(y, hy) \geq Ld(y, by).$$

By the inequality (3) that $d(y, by) \geq C \geq \delta$, we thus derive

$$(14) \quad \langle z_1, z_2 \rangle_{w_2}, \langle w_1, w_2 \rangle_{z_2} \leq d(w_1, z_2)/4 - \delta$$

Similarly, since $d(z_2, w_2) = d(y, h^{i_2 k - 2m_2} y) \geq Ld(y, by) \geq 10C$, we have

$$(15) \quad \langle w_1, w_2 \rangle_{z_2}, \langle z_2, z_3 \rangle_{w_2} \leq d(z_2, w_2)/4 - \delta$$

In conclusion, the inequalities (14) and (15) verifying the assumption of Corollary 2.3 hold for every four consecutive points in $z_1, w_1, z_2, w_2, \dots, z_l, w_l$. Thus,

$$d(z_1, w_l) \geq \frac{1}{2} \sum_{1 \leq j \leq l} d(z_j, w_j) \geq Ld(y, by)$$

By (10), we have $d(y, s_1 y) + d(y, s_l y) < Ld(y, by)$. Thus, $d(o, Wo) = d(z_1, w_1) - d(y, s_1 y) - d(y, s_l y) > 0$. Hence, any non-empty reduced word W is mapped a non-trivial isometry, so T generates a free subgroup of rank $\#T$. \square

We now finish the proof of Theorem 1.1. Summarizing the above discussion, for each generating set S of H , we constructed a finite set $T \subset S^{\leq \kappa}$ satisfying

$$\#T \geq \frac{1}{2N_0} \#S$$

so that $\langle T \rangle$ is a free group of rank $\#T$. Thus,

$$\#(S^{\leq n\kappa}) \geq (2\#T - 1)^n \geq \left(\frac{\#S - N_0}{N_0}\right)^n$$

and there exists $c_0 > 0$ such that $\omega(H, S) \geq c_0$ for any finite symmetric set S .

Choose the least integer $M = M(N_0) > 0$ such that $\#S/N_0 \geq 1 + \sqrt{\#S}$ for any $\log \#S > M$. In this case, we thus obtain

$$\omega(H, S) \geq \frac{1}{2\kappa} \log(\#S).$$

Otherwise, $\log \#S \leq M$, we have

$$\omega(H, S) \geq c_0 \geq \frac{c_0}{M} \log(\#S).$$

The proof of Theorem 1.2 is finished.

REFERENCES

1. G. N. Arzhantseva and I. G. Lysenok, *A lower bound on the growth of word hyperbolic groups*, J. London Math. Soc. (2) **73** (2006), no. 1, 109–125. MR 2197373
2. Laurent Bartholdi, *A Wilson group of non-uniformly exponential growth*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 7, 549–554. MR 1981466
3. B. Bowditch, *Relatively hyperbolic groups*, Int. J. Algebra Comput. (2012), no. 22, p1250016.
4. J. O. Button, *Explicit Helfgott type growth in free products and in limit groups*, J. Algebra **389** (2013), 61–77. MR 3065992
5. M. Coornaert, *Mesures de Patterson-Sullivan sur le bord d’un espace hyperbolique au sens de Gromov*, Pac. J. Math. (1993), no. 2, 241–270.

6. M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, Berlin, 1990, Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], With an English summary. MR 1075994
7. Thomas Delzant and Markus Steenbock, *Product set growth in groups and hyperbolic geometry*, J. Topol. **13** (2020), no. 3, 1183–1215. MR 4100130
8. C. Drutu and M. Sapir, *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), no. 5, 959–1058, With an appendix by D. Osin and M. Sapir.
9. B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. **8** (1998), no. 5, 810–840.
10. Koji Fujiwara, *The rates of growth in an acylindrically hyperbolic group*, 2021, arXiv:2103.01430.
11. Koji Fujiwara and Zlil Sela, *The rates of growth in a hyperbolic group*, 2020, arXiv:2002.10278.
12. S. Gersten, *Subgroups of small cancellation groups in dimension 2*, J. London Math. Soc. **54** (1996), 261–283.
13. R. Grigorchuk and P. de la Harpe., *On problems related to growth, entropy and spectrum in group theory*, J. Dyn. Control Syst. **3** (1997), no. 1, 51 – 89.
14. M. Gromov, *Hyperbolic groups*, Essays in group theory (S Gersten, editor), vol. 1, pp. 75–263, Springer New York, 1987.
15. Mikhael Gromov, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques [Mathematical Texts], vol. 1, CEDIC, Paris, 1981, Edited by J. Lafontaine and P. Pansu. MR 682063
16. Daniel Groves, *Limit groups for relatively hyperbolic groups. II. Makanin-Razborov diagrams*, Geom. Topol. **9** (2005), 2319–2358. MR 2209374
17. ———, *Limit groups for relatively hyperbolic groups. I. The basic tools*, Algebr. Geom. Topol. **9** (2009), no. 3, 1423–1466. MR 2530123
18. G. Hruska, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. **10** (2010), 1807–1856.
19. Malik Koubi, *Croissance uniforme dans les groupes hyperboliques*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 5, 1441–1453. MR 1662255
20. Robert Kropholler, Rylee Alanza Lyman, and Thomas Ng, *Extensions of hyperbolic groups have locally uniform exponential growth*, 2020.
21. D. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, vol. 179, Mem. Amer. Math. Soc., 2006.
22. ———, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 851–888.
23. S. R. Safin, *Powers of subsets of free groups*, Mat. Sb. **202** (2011), no. 11, 97–102. MR 2907200
24. A. Sambusetti, *Growth tightness of free and amalgamated products*, Ann. Sci. École Norm. Sup. série **35** (2002), no. 4, 477 – 488.
25. John S. Wilson, *On exponential growth and uniformly exponential growth for groups*, Invent. Math. **155** (2004), no. 2, 287–303. MR 2031429
26. X. Xie, *Growth of relatively hyperbolic groups*, Proc. Amer. Math. Soc. **135** (2007), 695–704.
27. W. Yang, *Statistically convex-cocompact actions of groups with contracting elements*, arXiv:1612.03648, to appear in IMRN.
28. ———, *Growth tightness for groups with contracting elements*, Math. Proc. Cambridge Philos. Soc **157** (2014), 297 – 319.

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