STAT/MATH 395 A - PROBABILITY II – UW
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HW2 : Continuous Random Variables (2) – Solutions

Problem 1. Let Z be a random variable that follows a standard normal distribution $\mathcal{N}(0,1)$. Show that

(a) $\mathbb{E}[Z] = 0$

Answer. The probability density function of Z is :

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The expected value of Z is :

$$\begin{aligned}\mathbb{E}[Z] &= \int_{-\infty}^{\infty} \underbrace{x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}_{\text{odd function on } \mathbb{R}} \, dx \\ &= 0\end{aligned}$$

(b) $\text{Var}(Z) = 1$.

Answer. The variance of Z is : $\text{Var}(Z) = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 = \mathbb{E}[Z^2]$. We have that:

$$\mathbb{E}[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \, e^{-x^2/2} \, dx$$

Let us perform the following integration by parts : $\begin{array}{ll} u = x & dv = xe^{-x^2/2}\,dx \\ du = dx & v = -e^{-x^2/2} \end{array}$

$$\begin{aligned}\mathbb{E}[Z^2] &= \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{-xe^{-x^2/2}\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} -e^{-x^2/2} \, dx \right\} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \\ &= \int_{-\infty}^{\infty} f_Z(x) \, dx \\ &= 1 \quad \text{since } f_Z \text{ is a probability density function}\end{aligned}$$

Let X be a random variable that follows a normal distribution $\mathcal{N}(\mu, \sigma^2)$. Using a relevant transformation of Z, show that

(c) $\mathbb{E}[X] = \mu$

Answer. Since Z is a standard normal random variable, we know that $X = \mu + \sigma Z$ follows a normal distribution $\mathcal{N}(\mu, \sigma^2)$. The expected value of X is then :

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\mu + \sigma Z] \\ &= \mu + \sigma \mathbb{E}[Z] \quad \text{by linearity of the expectation} \\ &= \mu + \sigma \cdot 0 \\ &= \mu\end{aligned}$$

(d) $\mathrm{Var}(X) = \sigma^2$.

Answer. The variance is given by :

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}(\mu + \sigma Z) \\ &= \sigma^2 \,\mathrm{Var}(Z) \\ &= \sigma^2 \cdot 1 \\ &= \sigma^2\end{aligned}$$

Problem 2. The systolic blood pressure in the population is usually modeled by a normal distribution with mean 120 mmHg (millimeters of mercury) and standard deviation 8 mmHg. Using the standard normal table (click here), answer the following questions :

(a) What is the probability that a randomly selected individual has a blood pressure below 104?

Answer. Let X be the systolic blood pressure of an individual in mmHg.

$$\begin{aligned}\mathbb{P}(X < 104) &= \mathbb{P}\left(\underbrace{\frac{X-120}{8}}_{Z\sim\mathcal{N}(0,1)} \leq \frac{104-120}{8}\right) \\ &= \Phi\left(-2\right) \\ &= 1 - \Phi\left(2\right) \quad \text{symmetry property of } \Phi \\ &\approx 1 - 0.9772 \\ &\approx 0.0228\end{aligned}$$

(b) What is the probability that a randomly selected individual has a blood pressure above 130?

Answer.

$$\begin{aligned}
\mathbb{P}(X > 130) &= \mathbb{P}\left(\underbrace{\frac{X-120}{8}}_{Z\sim\mathcal{N}(0,1)} \le \frac{130-120}{8}\right) \\
&= 1 - \Phi\left(\frac{5}{4}\right) \\
&\approx 1 - 0.8944 \\
&\approx 0.1056
\end{aligned}$$

(c) What is the probability that a randomly selected individual has a blood pressure between 108 and 134?

Answer.

$$\begin{aligned}
\mathbb{P}(108 \le X \le 134) &= \mathbb{P}\left(\frac{108-120}{8} \le \underbrace{\frac{X-120}{8}}_{Z\sim\mathcal{N}(0,1)} \le \frac{134-120}{8}\right) \\
&= \Phi\left(\frac{7}{4}\right) - \Phi\left(\frac{-3}{2}\right) \\
&= \Phi\left(\frac{7}{4}\right) - \left(1 - \Phi\left(\frac{3}{2}\right)\right) \\
&\approx 0.9599 - (1 - 0.9332) \\
&\approx 0.8931
\end{aligned}$$

(d) Below which blood pressure do we find one third of the population?

Answer. We are asked the quantile value x of X such that $F_X(x) = 1/3$. Let us find the value of $z = (x - 120)/8$ in the standard normal table corresponding to 1/3. You note that values below 0.5 are not available in the table. However using the symmetry property of Φ, you can look for the value corresponding to $1 - 1/3 = 2/3$, that is approximately 0.43. The desired z value is thus the opposite value, that is -0.43. Finally, the quantile value is $x = 8\cdot(-0.43)+120 = 116.56$. In words, one third of the population has a blood pressure below 116.56 mmHg.

(e) Above which blood pressure do we find 5% of the population?

Answer. If 5% of the population have a blood pressure above x, that means that 95% of the population have a blood pressure below x. Therefore we are asked the quantile value x of X such that $F_X(x) = 0.95$. Let us find the value of $z = (x - 120)/8$ in the standard normal table corresponding to

0.95 : this is approximately 1.645 (the average of 1.64 and 1.65). Therefore, the quantile value is $x = 8 \cdot 1.645 + 120 \approx 133.16$. In words, 5% of the population have a blood pressure above 133.16 mmHg.

Problem 3. Customers arrive at a bakery at an average rate of 35 per hour.

(a) What is the probability that the next customer comes in in the next 4 minutes?

Answer. Let X be the waiting time, in minutes, before the next customer enters the bakery. X follows an exponential distribution with parameter λ equal to the average rate of customers per minute, that is $\lambda = 35/60 = 7/12$. Therefore the desired probability is given by :

$$\begin{aligned} \mathbb{P}(X < 4) &= \int_0^4 \lambda\, e^{-\lambda\, x}\, dx \\ &= -e^{-\lambda\, x}\Big|_0^4 \\ &= 1 - e^{-7/3} \\ &\approx 0.903 \end{aligned}$$

(b) What is the probability that you wait between 2 and 6 minutes before the next customer comes in?

Answer.

$$\begin{aligned} \mathbb{P}(2 \leq X \leq 6) &= \int_2^6 \lambda\, e^{-\lambda\, x}\, dx \\ &= -e^{-\lambda\, x}\Big|_2^6 \\ &= e^{-7/6} - e^{-7/2} \\ &\approx 0.281 \end{aligned}$$

(c) Actually, in the morning customers arrive at an average rate of 45 per hour. And in the afternoon, customers arrive at an average rate of 30 per hour. The bakery is open between 7:00 and noon in the morning and then from 1:00 to 7:00 in the afternoon. Assume that 70 customers arrived in an interval of 90 minutes, what is the probability that they came in the morning?

Answer. Let M and A be the events that customers arrive in the morning and in the afternoon, respectively. Note that the bakery is open during 5 hours in the morning and 6 hours in the afternoon. Therefore $\mathbb{P}(M) = 5/11$ and $\mathbb{P}(A) = 6/11$. Next, let Y be the number of customers that arrive in an interval of 90 minutes. In the morning, Y follows a Poisson distribution with parameter $\lambda_M = 45 \cdot 1.5 = 67.5$ while in the afternoon, Y follows a Poisson distribution with parameter $\lambda_A = 30 \cdot 1.5 = 45$. Thus the question translates into the computation of the conditional probability $\mathbb{P}(M|Y = 70)$. This can be computed using the Bayes' formula:

$$\begin{aligned}
\mathbb{P}(M|Y = 70) &= \frac{\mathbb{P}(Y = 70|M)\mathbb{P}(M)}{\mathbb{P}(Y = 70|M)\mathbb{P}(M) + \mathbb{P}(Y = 70|A)\mathbb{P}(A)} \\
&= \frac{67.5^{70}/70! \cdot e^{-67.5} \cdot 5/11}{67.5^{70}/70! \cdot e^{-67.5} \cdot 5/11 + 45^{70}/70! \cdot e^{-45} \cdot 6/11}
\end{aligned}$$

Problem 4. Let X be a random variable that follows an exponential distribution with parameter $\lambda > 0$.

(a) Give the cumulative distribution function F_X of X.

Answer. The probability density function of X is :

$$f_X(x) = \lambda\, e^{-\lambda\, x} \mathbb{1}_{[0,\infty)}(x)$$

Hence, the cdf of X is given by :

- If $x < 0$,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt = \int_{-\infty}^{x} 0\, dt = 0$$

- If $x \geq 0$,

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} f_X(t)\, dt \\
&= F_X(0) + \int_{0}^{x} \lambda\, e^{-\lambda\, t}\, dt \\
&= 0 - e^{-\lambda\, t}\Big|_{0}^{x} \\
&= 1 - e^{-\lambda x}
\end{aligned}$$

In a nutshell,

$$F_X(x) = \left(1 - e^{-\lambda x}\right) \mathbb{1}_{[0,\infty)}(x)$$

(b) Give the quantile function of X, that is : compute the inverse function of F_X.

Answer. Let $\alpha \in (0,1)$ be a probability value. Since F_X is bijective on $[0,\infty)$, the quantile of X of order α is defined as the value x such that the following holds :

$$F_X(x) = \alpha$$

Therefore, for a fixed value of α, we want to solve the above equation for x :

$$\begin{aligned} F_X(x) = \alpha &\Leftrightarrow 1 - e^{-\lambda x} = \alpha \\ &\Leftrightarrow e^{-\lambda x} = 1 - \alpha \\ &\Leftrightarrow -\lambda x = \ln(1-\alpha) \\ &\Leftrightarrow x = \frac{-\ln(1-\alpha)}{\lambda} \end{aligned}$$

Hence, the quantile function of X is given by :

$$F_X^{-1}(\alpha) = \frac{-\ln(1-\alpha)}{\lambda} \quad \text{for } \alpha \in (0,1)$$

(c) *Application.* Assume that calls arrive at an average rate of 12 per hour. Find the duration of a phone call with a probability at least 80%.

Answer. Let X be the duration of a phone call in hours. X follows an exponential distribution with parameter $\lambda = 12$. The question translates into finding the quantile of X of probability level $\alpha = 0.8$:

$$F_X^{-1}(0.8) = \frac{-\ln(1-0.8)}{12} \approx 0.134 \text{ hour } \approx 8.05 \text{ minutes.}$$